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The Ehrhart Theory of Matroid Polytopes

Presentata da

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Introduction

Since its introduction in 1935 by Whitney [Whi35], matroid theory has grown very prolifically and has proven extremely useful both as a language and as a tool, not only in pure combinatorics but in other branches of mathematics as well. The fact that many topological invariants of the complement of a complex hyperplane arrangement are encoded in the intersection lattice, viz. its underlying matroid, serves as an example of a contact point between topology and matroid theory [OS80]. Moreover, in the recent years such contact points, in this case between matroid theory and algebraic geometry, have provided a deeper understanding (and proofs, of course) of facts that were observed many years ago by renowned mathematicians and for which we lacked a proper demonstration beyond some easy and/or particular cases [AHK18, BHM^{+}20].

Both of the aforementioned examples have something in common: a geometric point of view of matroids. This idea of the *geometry of matroids* has also been explored in the recent years in the spirit of *discrete* (rather than algebraic or topological) geometry. More precisely, the understanding of matroids as certain convex polytopes provided a very rich and profound source of theorems and tools for both geometry *and* combinatorics.

When talking about polytopes (particularly, *lattice* polytopes) one of the most popular and interesting invariants is the so-called *Ehrhart polynomial* [Ehr62]. These polynomials capture much of the geometry, the combinatorics and the arithmetics of the polytope. Such an invariant, which is as simple as it could be, i.e. a polynomial with rational coefficients, is in fact *not that simple* to understand for polytopes in their full generality.

A reasonable question to ask is if there is some interpretation for the coefficients of the Ehrhart polynomials that arise from matroid polytopes. However, it is already a challenging problem to understand even particular coefficients of such polynomials. The volumes of matroid polytopes, which happen to be the leading coefficients of the Ehrhart polynomials, are already very enigmatic. There exist interpretations of the volumes for the matroids belonging to the very restrictive class of representable matroids [GGMS87] and a somewhat obscure formula for the general case [ABD10] which does not seem to be very useful in practice. These issues already give a tiny hint of how difficult the problem of understanding Ehrhart polynomials of matroid polytopes is expected to be.

In a 2007 article (published two years thereafter) De Loera, Haws and Köppe conjectured that the Ehrhart polynomial of matroid polytopes had only positive coefficients [DHK09]. Since then, this conjecture was approached with different methods and ideas without success. Much of the resistance of this problem to be solved was, in part, because the community (myself included) believed the conjecture to be true. In fact, Castillo and Liu in 2015 conjectured a stronger version of that statement and they were able to provide very good evidence pointing to its truthfulness [CL18].

The ultimate aim of this thesis is to provide a counterexample to such conjectures. Through the way we will prove many positive results that are interesting on their own. In particular, we give a combinatorial formula for the Ehrhart polynomial of hypersimplices, generalizing a result of Laplace. This had been classified as an open problem in the highly influential book *Enumerative Combinatorics* by R. Stanley [Sta12].

Our exposition is based on the following five articles, all of which were written in the course of the last two years, [Fer21a, Fer21b, Fer21c, Fer21d, FJS21], listed in order of appearance.

- L. Ferroni, *Hypersimplices are Ehrhart positive*, J. Comb. Theory Ser. A. 178:14, 2021.
- L. Ferroni, *On the Ehrhart polynomial of minimal matroids*, Discrete Comput. Geom., 2021.
- L. Ferroni, *Integer point enumeration on independence polytopes and half-open hypersimplices*, Discrete Math., 344(8):112446, 2021.
- L. Ferroni, *Matroids are not Ehrhart positive*, submitted.
- L. Ferroni, K. Jochemko, B. Schröter, *Ehrhart polynomials of rank two matroids*, preprint.

We have included in this thesis some (few) additional results that do not appear within the above articles. They are mainly related to *independence* matroid polytopes and their Ehrhart theory. For instance, we show that there exist independence matroid polytopes that are not Ehrhart positive. In particular, we rule out the possibility of these polytopes to be any better than the basis polytopes of matroids from the Ehrhart theoretic perspective.

Outline

We now describe how this thesis is organized. The main matter of it is concentrated in Chapters 3 and 4, which contain essentially all of the contributions of the papers mentioned above, and some additional counterpart results for independence polytopes.

In Chapter 1 we state and prove all the preliminary of matroids that we need, so that all of our exposition is self-contained from the point of view of matroids and matroid polytopes. We assume a minimum of familiarity with convex polytopes, though.

In Chapter 2 we review the basic facts of Ehrhart theory, without giving the proofs, and establish some very basic results for the case of matroid polytopes.

In Chapter 3 we study the Ehrhart theory of uniform matroids with an emphasis on giving both a proof of the Ehrhart positivity for all hypersimplices and, even more remarkably, an explicit combinatorial formula for each of the coefficients of such polynomials. Then we carry out a proof of the Ehrhart positivity of half-open hypersimplices and of the independence polytope of uniform matroids.

In Chapter 4 we do the following:

- In Section 4.1 we introduce minimal matroids and link them with the geometric point of view of the operation of circuit-hyperplane relaxation for matroids.
- In Section 4.2 we establish a formula for the Ehrhart polynomials of all sparse paving matroids, a class of objects that is conjectured to dominate among all matroids.
- In Section 4.3 we give an explicit construction of matroids that fail to be Ehrhart positive. We prove that for $k \geq 3$ there exists a matroid of rank k that is not Ehrhart positive.
- In Section 4.4 we address the case $k = 2$, and find an explicit formula for the Ehrhart polynomial of rank two matroids, and prove their Ehrhart positivity.

In Chapter 5 we end this dissertation by discussing some related open problems and conjectures on the Ehrhart theory of matroids.

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Al sueño que se hace realidad.

Contents

CHAPTER 1

The geometry of matroids

1.1 Matroids as combinatorial structures

The study of matroids was initiated by Whitney [Whi35] with the aim of abstracting the notion of linear dependence. Since the first decades, the theory of matroids has evolved and established links with other areas of mathematics. A fundamental characteristic of matroids is that they can be defined in several different but equivalent ways. In this section we will review some of the most basic definitions and results, following the pace and the notation of Oxley's book [Oxl11].

Definition 1.1.1. A matroid M is a pair $(E, \mathcal{B})$ where E is a finite set and $\mathcal{B}$ is a family of subsets of E, i.e. $\mathcal{B} \subseteq 2^E$, satisfying the following two properties:

- $\mathcal{B} \neq \varnothing$.
- If B_1 and B_2 are in $\mathcal{B}$ and $x \in B_1 \smallsetminus B_2$, then there is an element $y \in B_2 \smallsetminus B_1$ such that $(B_1 \smallsetminus \{x\}) \cup \{y\} \in \mathcal{B}$.

If $M = (E, \mathcal{B})$ is a matroid, the members of $\mathcal{B}$ are referred to as the *bases* of the matroid.

The second of the above properties is usually called the *basis-exchange-property*. Also, it is customary to identify a matroid M with its groundset E, and talk about the *subsets of* M or the *elements of* M. The number of elements of E is what we refer to as the *cardinality* of the matroid M. The terminology on matroid theory and the prototype of matroid come from notions of linear algebra.

Example 1.1.2. Let V be a vector space over a field $\mathbb{F}$. Consider a finite list of vectors $E = \{v_1, \dots, v_n\} \subseteq V$. Let W be the subspace of V spanned by the elements of E. It is clear that W is finite dimensional over $\mathbb{F}$, so let us assume that $\dim W = k$. Consider

$$\mathcal{B} = \{B \subseteq E : \{v_i\}_{i \in B} \text{ is a basis of } W\}.$$

It is clear that $\mathcal{B} \neq \varnothing$ (when $k = 0$ then $\mathcal{B} = \{\varnothing\} \neq \varnothing$). Also let us verify that the basis-exchange-property holds. If we pick $B_1 \neq B_2$ in $\mathcal{B}$ and an element $v \in B_1 \smallsetminus B_2$, we have that $\dim \langle B_1 \smallsetminus \{v\}\rangle = k - 1$. Since $\dim \langle B_2 \rangle = k$, by a standard linear algebra reasoning, we see that there has to be at least one vector $w \in B_2$ such that w does not lie in $\langle B_1 \smallsetminus \{v\}\rangle$. Hence, we get that $\dim \langle (B_1 \smallsetminus \{v\}) \cup \{w\}\rangle = k$ and thus $(B_1 \smallsetminus \{v\}) \cup \{w\} \in \mathcal{B}$.

The matroids that Example 1.1.2 provides are known in the literature as *representable matroids*. It is clear that all the bases in the above example have the same cardinality. This phenomenom translates to matroids in their full generality.

Lemma 1.1.3 *Let $M = (E, \mathcal{B})$ be a matroid. All the members of $\mathcal{B}$ have the same cardinality.*

Proof. Let us assume that there exist two bases $B_1 \neq B_2$ of M satisfying $|B_1| > |B_2|$ and, among all such pairs of bases, let us choose one that minimizes $|B_1 \smallsetminus B_2|$. Since $|B_1| > |B_2|$, in particular we have that $B_1 \smallsetminus B_2 \neq \varnothing$. Thus, choosing $x \in B_1 \smallsetminus B_2$ and using the basis-exchange-property, we can find an element $y \in B_2 \smallsetminus B_1$ such that $(B_1 \smallsetminus \{x\}) \cup \{y\} \in \mathcal{B}$. Calling $B_3 = (B_1 \smallsetminus \{x\}) \cup \{y\}$, we may observe that $|B_3| > |B_2|$ and that $|B_3 \smallsetminus B_2| < |B_1 \smallsetminus B_2|$. This contradicts our choice of B_1 and B_2. □

Since all the bases of a matroid have the same cardinality, it is customary to refer to this cardinality as the *rank* of the matroid.

Example 1.1.4. Consider $E = \{1, \dots, n\}$ and $\mathcal{B} = \{B \subseteq E : |B| = k\}$. It is straightforward to verify that $M = (E, \mathcal{B})$ is a matroid. This matroid is called the *uniform matroid* of cardinality n and rank k. We denote this matroid by $U_{k,n}$.

Observe that the uniform matroid $U_{k,n}$ is in fact representable. If we take the vector space $V = \mathbb{R}^k$ and n vectors in *general position* in V, the construction of Example 1.1.2 yields our matroid $U_{k,n}$. To be more explicit, we have an *isomorphism* between $U_{k,n}$ and the matroid we just outlined how to construct. We can make this statement precise.

Definition 1.1.5. Let $M_1 = (E_1, \mathcal{B}_1)$ and $M_2 = (E_2, \mathcal{B}_2)$ two matroids. We say that M_1 and M_2 are *isomorphic* if there exists a bijection $\varphi : E_1 \to E_2$ with the property that $B \in \mathcal{B}_1$ if and only if $\varphi(B) \in \mathcal{B}_2$.

Observe that in our prototype of matroid, i.e. the matroids described by Example 1.1.2, we have a notion of *linear independence*. This can be translated to the general case as follows.

Definition 1.1.6. Let $M = (E, \mathcal{B})$ be a matroid. A subset $I \subseteq E$ is said to be *independent* if $I \subseteq B$ for some $B \in \mathcal{B}$. When a set is not independent we say it is *dependent*. We denote the family of all independent subsets of M by $\mathcal{I}(M)$.

The family of all the independence subsets is a rich combinatorial object, possessing a nice property. Recall that an *(abstract) simplicial complex* on E is a family $\Delta \subseteq 2^E$

with the property that for every $F \in \Delta$ and $G \subseteq F$, it holds $G \in \Delta$. A simplicial complex is *pure* when all its elements of maximal cardinality have the same number of elements.

Proposition 1.1.7 *Let $M = (E, \mathcal{B})$ be a matroid and let $\mathcal{I} = \mathcal{I}(M)$. Then*

(a) $\mathcal{I}$ is a pure simplicial complex.

(b) If I_1 and I_2 members of $\mathcal{I}$ satisfy $|I_1| < |I_2|$, then there is an element $e \in I_2 \smallsetminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

Proof. The fact that $\mathcal{I}$ is a simplicial complex is immediate, and its pureness follows from Lemma 1.1.3. Assume that (b) does not hold. Choose I_1 and I_2 in $\mathcal{I}$ such that $|I_1| < |I_2|$ and that for all $e \in I_2 \smallsetminus I_1$, it is $I_1 \cup \{e\} \notin \mathcal{I}$. By definition there exist B_1 and B_2 in $\mathcal{B}$ such that $I_1 \subseteq B_1$ and $I_2 \subseteq B_2$. Assume that B_2 is chosen so that $|B_2 \smallsetminus (I_2 \cup B_1)|$ is minimal. Because of our choice of I_1 and I_2 it must be:

$$I_2 \smallsetminus I_1 = I_2 \smallsetminus B_1.$$

Now, observe that

- $B_2 \smallsetminus (I_2 \cup B_1) = \varnothing$. To prove this, assume that this set was nonempty and pick an element x in it. By the basis-exchange-property we must have an element $y \in B_1 \smallsetminus B_2$ such that $B_3 = (B_2 \smallsetminus \{x\}) \cup \{y\} \in \mathcal{B}$. However, in that scenario it would be $|B_3 \smallsetminus (I_2 \cup B_1)| < |B_2 \smallsetminus (I_2 \cup B_1)|$, and thus contradicting our choice of B_2. This also proves that $B_2 \smallsetminus B_1 = I_2 \smallsetminus B_1$.

- $B_1 \smallsetminus (I_1 \cup B_2) = \varnothing$. To prove this, assume that this set was nonempty and pick an element x in it. By the basis-exchange-property we must have an element $y \in B_2 \smallsetminus B_1$ such that $B_3 = (B_1 \smallsetminus \{x\}) \cup \{y\} \in \mathcal{B}$. This implies that $I_1 \cup \{y\} \subseteq B_3$, so that $I_1 \cup \{y\} \in \mathcal{I}$. Since we said that $y \in B_2 \smallsetminus B_1$ and we proved in the preceding bullet that $B_2 \smallsetminus B_1 = I_2 \smallsetminus B_1$, it follows that $y \in I_1 \smallsetminus I_2$. We have contradicted our choice of I_1 and I_2. This proves that $B_1 \smallsetminus B_2 = I_1 \smallsetminus B_2$.

Observe now that

$$B_1 \smallsetminus B_2 = I_1 \smallsetminus B_2 \subseteq I_1 \smallsetminus I_2.$$

Also, since $|B_1| = |B_2|$, it must be $|B_1 \smallsetminus B_2| = |B_2 \smallsetminus B_1|$. Therefore:

$$|I_1 \smallsetminus I_2| \geq |B_1 \smallsetminus B_2| = |B_2 \smallsetminus B_1| = |I_2 \smallsetminus B_1| = |I_2 \smallsetminus I_1|.$$

But the inequality $|I_1 \smallsetminus I_2| \geq |I_2 \smallsetminus I_1|$ implies that $|I_1| \geq |I_2|$ which contradicts $|I_1| < |I_2|$. □

Remark 1.1.8. It can be proved that the simplicial complex of independent subsets of a matroid is *shellable* [Bjö92]. Moreover, (b) in the above proposition guarantees that all the vertex-induced subcomplexes of $\mathcal{I}$ are shellable. This property characterizes the simplicial complexes coming from matroids.

When one already has a matroid, it is possible to construct new matroids in a very intuitive way. That is the spirit of the following result.

Lemma 1.1.9 *Let $M = (E, \mathcal{B})$ a matroid with family of independent sets $\mathcal{I}$. Let $A \subseteq E$ be a subset. If we denote $\mathcal{I}'$ the set*

$$\mathcal{I}' = \{I \in \mathcal{I} : I \cap A = \varnothing\},$$

then $\mathcal{I}'$ is the family of independent subsets of a matroid $M' = (E \smallsetminus A, \mathcal{B}')$, where $\mathcal{B}'$ are the maximal members of $\mathcal{I}'$ with respect to set-inclusion.

Proof. Let us assume that A consists of only one element, say $A = \{e\}$. Let us choose two distinct elements B_1' and B_2' in $\mathcal{B}'$. Since the B_1' and B_2' are in $\mathcal{I}$, we have that $e \notin B_1'$ and $e \notin B_2'$. Since B_1' and B_2' are independent in M, there are bases B_1 and B_2 such that $B_1' \subseteq B_1$ and $B_2' \subseteq B_2$. Let us pick $x \in B_1' \smallsetminus B_2'$. There are two possibilities:

- $e \in B_1$. This implies that $B_1' = B_1 \smallsetminus \{e\}$ because of the maximality of the elements of $\mathcal{B}'$ described in the statement. In particular, since B_2' was also maximal, we have that $|B_2'| = |B_1'|$ and that $e \in B_2$. In this case, we have that $x \in B_1' \smallsetminus B_2' = B_1 \smallsetminus B_2$. The basis-exchange-property in M gives us an element $y \in B_2 \smallsetminus B_1 = B_2' \smallsetminus B_1'$ such that $(B_1 \smallsetminus \{x\}) \cup \{y\} \in \mathcal{B}$. Since this basis of M does not contain the element e, we have the basis-exchange-property for B_1' and B_2'.

- $e \notin B_1$. Reasoning as above, we get that $e \notin B_2$, and again we have that $B_1 \smallsetminus B_2 = B_1' \smallsetminus B_2'$ and that $B_2 \smallsetminus B_1 = B_2' \smallsetminus B_1'$, and we can repeat the same proof as above.

If A has more than one element, we can delete one at a time and proceed inductively as above. □

The operation on matroids described in the preceding Lemma is called the *deletion*. We say that the matroid M' constructed above is obtained from M by *deleting A* and we denote $M' = M \smallsetminus A$. Notice that if we delete the complement of $A \subseteq E$, we end up obtaining a matroid on the groundset A. This is what we call the *restriction of M to A*, and we denote it by $M|_A$.

Another property that Example 1.1.2 possesses is that we can define a notion of *dimension* to every subset of the groundset E. This concept can be generalized to arbitrary matroids.

Proposition 1.1.10 *Let $M = (E, \mathcal{B})$ be a matroid. The function $\operatorname{rk} : 2^E \to \mathbb{Z}_{\geq 0}$ given by*

$$\operatorname{rk}(A) = \max_{B \in \mathcal{B}} |A \cap B|,$$

satisfies the following three properties:

(a) If $A \subseteq E$, then $\operatorname{rk}(A) \leq |A|$.

(b) If $A_1 \subseteq A_2 \subseteq E$*, then* $\mathrm{rk}(A_1) \leq \mathrm{rk}(A_2)$.

(c) If $A_1, A_2 \subseteq E$*, then*

$$\mathrm{rk}(A_1) + \mathrm{rk}(A_2) \geq \mathrm{rk}(A_1 \cup A_2) + \mathrm{rk}(A_1 \cap A_2).$$

Proof. Notice that (a) and (b) are immediate from the definition. We claim that, for every $A \subseteq E$, it is

$$\mathrm{rk}(A) = \text{rank of } M|_A.$$

Indeed, if we denote by $\mathcal{I}$ the independent subsets of M and $\mathcal{I}'$ the independent subsets of $M|_A$, we have:

$$\begin{aligned}
\text{rank of } M|_A &= \max_{I' \in \mathcal{I}'} |I'| \\
&= \max_{\substack{I \in \mathcal{I} \\ I \cap (E \setminus A) = \varnothing}} |I| \\
&= \max_{B \in \mathcal{B}} |B \setminus (E \setminus A)| \\
&= \max_{B \in \mathcal{B}} |B \cap A| \\
&= \mathrm{rk}(A).
\end{aligned}$$

Now, to prove (c), fix $A_1, A_2 \subseteq E$ and consider the matroids $M|_{A_1}$, $M|_{A_2}$, $M|_{A_1 \cap A_2}$ and $M|_{A_1 \cup A_2}$. Let us fix $B_{A_1 \cap A_2}$ a basis of $M|_{A_1 \cap A_2}$. It is clear that $B_{A_1 \cap A_2}$ is independent in $M|_{A_1 \cup A_2}$ so that we can find a basis $B_{A_1 \cup A_2}$ of this matroid that contains that set. Now, $B_{A_1 \cup A_2} \cap A_1$ and $B_{A_1 \cup A_2} \cap A_2$ are independent in $M|_{A_1}$ and $M|_{A_2}$ respectively. Thus, we have that

$$\begin{aligned}
\mathrm{rk}(A_1) + \mathrm{rk}(A_2) &\geq |B_{A_1 \cup A_2} \cap A_1| + \mathrm{rk}(A_2) \\
&\geq |B_{A_1 \cup A_2} \cap A_1| + |B_{A_1 \cup A_2} \cap A_2| \\
&= |B_{A_1 \cup A_2} \cap (A_1 \cup A_2)| + |B_{A_1 \cup A_2} \cap (A_1 \cap A_2)| \\
&= \mathrm{rk}(A_1 \cup A_2) + |B_{A_1 \cap A_2}| \\
&= \mathrm{rk}(A_1 \cup A_2) + \mathrm{rk}(A_1 \cap A_2)
\end{aligned}$$

□

The function rk provided by the above proposition is what we call the *rank function* of the matroid. Notice that $A \subseteq E$ is independent if and only if $\mathrm{rk}(A) = |A|$.

Example 1.1.11. Let G be a graph with set of edges E. We allow G to possess parallel edges and loops. We say that $B \subseteq E$ is a *spanning forest of* G if B does not contain any cycle, all the vertices of G belong to at least one of the edges of B, and B is a maximal set satisfying the preceding two properties. Let us call $\mathcal{B}$ the family of all spanning forests of G. It is can be proved that $M = (E, \mathcal{B})$ is a matroid. All matroids arising with such a construction are called *graphic*.

The graph-theoretic framework allows us to motivate an abstract version of several notions for arbitrary matroids.

Definition 1.1.12. Let $M = (E, \mathcal{B})$ be a matroid. We say that $C \subseteq E$ is a *circuit* if C is dependent but all proper subsets of C are independent. We denote by $\mathcal{C}(M)$ the family of all circuits of M.

Proposition 1.1.13 *Let $M = (E, \mathcal{B})$ be a matroid and let $\mathcal{C} = \mathcal{C}(M)$. Then*

(a) $\varnothing \notin \mathcal{C}$.

(b) If C_1 and C_2 are in $\mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.

(c) If $C_1 \neq C_2$ are in $\mathcal{C}$ and $e \in C_1 \cap C_2$, then there is a circuit $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \smallsetminus \{e\}$.

Proof. Observe that (a) and (b) are immediate from the definitions. Observe also that if C is a circuit, then $\mathrm{rk}(C) = |C| - 1$. To prove (c) it suffices to proceed by contradiction. Assuming that $(C_1 \cup C_2) \smallsetminus \{e\}$ contains no circuit yields that this set is independent. Thus, we obtain that

$$\begin{aligned}
\mathrm{rk}((C_1 \cup C_2) \smallsetminus \{e\}) &= |(C_1 \cup C_2) \smallsetminus \{e\}| \\
&= |C_1 \cup C_2| - 1 \\
&= |C_1| + |C_2| - |C_1 \cap C_2| - 1 \\
&= \mathrm{rk}(C_1) + 1 + \mathrm{rk}(C_2) + 1 - |C_1 \cap C_2| - 1 \\
&= \mathrm{rk}(C_1) + \mathrm{rk}(C_2) + 1 - |C_1 \cap C_2|
\end{aligned}$$

But, notice that Proposition 1.1.10 gives us:

$$\mathrm{rk}(C_1) + \mathrm{rk}(C_2) \geq \mathrm{rk}(C_1 \cup C_2) + \mathrm{rk}(C_1 \cap C_2),$$

so that

$$\mathrm{rk}((C_1 \cup C_2) \smallsetminus e) + |C_1 \cap C_2| - 1 \geq \mathrm{rk}(C_1 \cup C_2) + \mathrm{rk}(C_1 \cap C_2).$$

However, since $C_1 \cap C_2$ is a proper subset of C_1, then it has to be independent, so that $|C_1 \cap C_2| = \mathrm{rk}(C_1 \cap C_2)$. Hence, we obtained:

$$\mathrm{rk}((C_1 \cup C_2) \smallsetminus e) - 1 \geq \mathrm{rk}(C_1 \cup C_2),$$

which gives the desired contradiction, since $\mathrm{rk}(C_1 \cup C_2) \geq \mathrm{rk}((C_1 \cup C_2) \smallsetminus \{e\})$. $\square$

Example 1.1.14. Observe that in a matroid coming from a graph as in Example 1.1.11, a circuit of the matroid corresponds to a simple cycle of the graph. This also justifies the terminology we have introduced.

Remark 1.1.15. If I is an independent set of a matroid and $x \notin I$, there is at most one circuit C contained in $I \cup \{x\}$. Indeed, if we have two circuits $C_1, C_2 \subseteq I \cup \{x\}$, both of them must contain x, because I was independent. Thus $x \in C_1 \cap C_2$ and hence property (c) above implies that there is a circuit $C_3 \subseteq (C_1 \cup C_2) \smallsetminus \{x\} \subseteq I$, which is impossible.

Matroids admit a notion of *closure*. If one has a set A and an element $x \notin A$ it may happen that either $\mathrm{rk}(A \cup \{x\}) = \mathrm{rk}(A)$ or $\mathrm{rk}(A \cup \{x\}) = \mathrm{rk}(A) + 1$. In the first case, we say that x is in the closure of A. This corresponds to the notion of *span* for a list of vectors in a vector space. Also, this serves to motivate the following definition.

Definition 1.1.16. Let $M = (E, \mathcal{B})$ be a matroid. A subset $F \subseteq E$ is said to be a *flat* if $\mathrm{rk}(F \cup \{e\}) > \mathrm{rk}(F)$ for all $e \notin F$. The family of all flats of M will be denoted by $\mathcal{F}(M)$. If F is a flat of rank $\mathrm{rk}(E) - 1$, we say that F is a *hyperplane*. The family of all hyperplanes of M will be denoted by $\mathcal{H}(M)$.

Remark 1.1.17. Every subset $A \subseteq E$ is contained in a unique inclusion-minimal flat. Just by taking $\overline{A} = \{x \in E : \mathrm{rk}(A \cup \{x\}) = \mathrm{rk}(A)\}$, one obtains such a flat. This flat is referred to as the *flat spanned by* A or *the closure of* A.

Remark 1.1.18. The set $\mathcal{F}$ of all flats of a matroid $M = (E, \mathcal{B})$ can be seen as a poset with ther order given by the set-inclusion. This poset happens to be a lattice with two particular properties: it is atomic and it is modular. Such lattices are called *geometric*, and provide another way of talking about matroids.

We state and prove now a technical result that we will need to prove results regarding matroid polytopes. It was originally proved in [Bru69]. We include a somewhat different proof here.

Proposition 1.1.19 (Symmetric-exchange-property) *Let $M = (E, \mathcal{B})$ be a matroid of rank k. If B_1 and B_2 are in $\mathcal{B}$ and $x \in B_1 \smallsetminus B_2$, then there is an element $y \in B_2 \smallsetminus B_1$ such that $(B_1 \smallsetminus \{x\}) \cup \{y\} \in \mathcal{B}$ and $(B_2 \smallsetminus \{y\}) \cup \{x\} \in \mathcal{B}$.*

Proof. Since $B_2 \cup \{x\}$ is dependent, it contains a circuit C. Moreover, by Remark 1.1.15 this circuit is unique. Now, observe that since C is a circuit, we have that $\mathrm{rk}((B_1 \cup C) \smallsetminus \{x\}) = \mathrm{rk}(B_1 \cup C) = k$. In particular, since the rank of $B_1 \smallsetminus \{x\}$ is $k - 1$ and the rank of this set joined with $C \smallsetminus \{x\}$ is k, we can find a basis B_3 such that $B_1 \smallsetminus \{x\} \subseteq B_3 \subseteq (B_1 \cup C) \smallsetminus \{x\}$. It follows that $B_3 = (B_1 \smallsetminus \{x\}) \cup \{y\}$ for some $y \in C \smallsetminus \{x\}$. Now, observe that by our choice of C and its uniqueness, we have that $(B_2 \smallsetminus \{y\}) \cup \{x\}$ is independent, because it does not contain any circuits. Since the cardinality of this independent set is k, it is a basis. $\square$

Duality of Matroids

A fundamental operation of matroids is that of duality. The main motivation comes from the orthogonality of vector spaces when the matroid is representable, and from the *graph-theoretic duality* when the matroid comes from a (planar) graph.

Theorem 1.1.20 *Let $M = (E, \mathcal{B})$ be a matroid. Let $\mathcal{B}^*$ be the family given by all the complements of the members of $\mathcal{B}$. Then $M^* = (E, \mathcal{B}^*)$ is a matroid.*

Proof. Since $\mathcal{B}$ is non-empty, so has to be $\mathcal{B}^*$. Let us pick B_1^* and B_2^* distinct members of $\mathcal{B}^*$, and let $x \in B_1^* \smallsetminus B_2^*$. By calling B_1 and B_2 the complements of these two sets, we have that $B_1, B_2 \in \mathcal{B}$ and that $x \in B_2 \smallsetminus B_1$, so by the symmetric-exchange-property we can find an element $y \in B_1 \smallsetminus B_2$ such that both $B_3 = (B_1 \smallsetminus \{y\}) \cup \{x\} \in \mathcal{B}$ and $B_4 = (B_2 \smallsetminus \{x\}) \cup \{y\} \in \mathcal{B}$. It follows that $y \in B_2^* \smallsetminus B_1^*$ and that $B_3^* = E \smallsetminus B_3$ satisfies $B_3^* = (B_1^* \smallsetminus \{x\}) \cup \{y\}$, so $(E, \mathcal{B}^*)$ is indeed a matroid. □

The matroid M^* given by the above theorem is what we call the *dual* of M. Obviously $(M^*)^* = M$, so that duality is an involution and the terminology is justified. Observe that if M has cardinality n and rank k, then the rank of M^* is $n-k$.

Example 1.1.21. Consider the uniform matroid $U_{k,n}$ of rank k and cardinality n. Its dual is exactly $U_{k,n}^* = U_{n-k,n}$. Observe that the matroids $U_{k,2k}$ are self-dual.

Proposition 1.1.22 *Let $M = (E, \mathcal{B})$ be a matroid of rank k, and let M^* be its dual. Suppose that $X \subseteq E$. Then*

(a) $\operatorname{rk}(X) = k$ *if and only if $E \smallsetminus X$ is independent in M^*.*

(b) X is a hyperplane of M if and only if $E \smallsetminus X$ is a circuit of M^*.

(c) X is circuit of M if and only if $E \smallsetminus X$ is a hyperplane of M^*.

Proof. Let us prove (a). The condition $\operatorname{rk}(X) = k$ in M holds if and only if X contains a basis B of M. This, in turn, is equivalent to $E \smallsetminus X \subseteq E \smallsetminus B$, which happens if and only if $E \smallsetminus X$ is independent in M^*.

Notice that (c) is just (b) applied to the matroid M^* and its dual M. Observe that X is a hyperplane of M is equivalent to X being a flat of rank $k-1$, and this in turn is equivalent to the condition that X has rank $k-1$ but $X \cup \{e\}$ has rank k for all $e \notin X$, and using (a) this is equivalent to $E \smallsetminus X$ being dependent and $E \smallsetminus (X \cup \{e\})$ being independent in M^* for all $e \notin X$. The last assertion is equivalent to saying that all proper subsets of $E \smallsetminus X$ are independent in M^* but $E \smallsetminus X$ is dependent or, more simply, that $E \smallsetminus X$ is a circuit of M^*. □

We introduce more terminology that we will refer to in the sequel.

Definition 1.1.23. Let $M = (E, \mathcal{B})$ be a matroid and fix $x, y \in E$.

- We say that x is a *loop* if $\operatorname{rk}(\{e\}) = 0$.
- We say that x is a *coloop* if x is a loop of M^*.
- We say that x and y are *parallel* if $\{x, y\}$ is a circuit.

Remark 1.1.24. It is not difficult to prove that x is a loop of M if and only if x is not contained in any basis of M and that x is a coloop if and only if x is contained in *every* basis of M.

Connectivity

Unlike the previous notions we have established for matroids, the concept of *connected matroid* is not actually a way of abstracting *connected graphs* but *biconnected graphs*.

To justify the necessity of looking at biconnectedness for graphs, notice that if G_1 is a graph consisting on three vertices $\{v_1, v_2, v_3\}$ and two edges $\{v_1, v_2\}$ and $\{v_2, v_3\}$ and G_2 is a graph on four vertices $\{w_1, w_2, w_3, w_4\}$ and two edges $\{w_1, w_2\}$ and $\{w_3, w_4\}$, then the matroids associated to G_1 and to G_2 are isomorphic, although it is clear that G_1 is connected but G_2 is not.

Thus, we will state a definition of connected matroid that captures the notion of being biconnected when restricted to graphic matroids. Recall that a connected graph is biconnected when the graph obtained by the deletion of any vertex and its incident edges remains connected.

Definition 1.1.25. A *connected matroid* is a matroid $M = (E, \mathcal{B})$ such that for every two distinct elements $x, y \in E$ there is a circuit C that contains both x and y.

Remark 1.1.26. Let G be a loopless graph without isolated vertices and suppose that G has at least 3 vertices. Then G is biconnected if and only if, for every pair of distinct edges of G, there is a cycle containing both. This fact is what motivates the above definition. The proof of this fact is a standard graph-theoretic exercise.

The following property, which is another important property of the circuits of a matroid, essentially states that a certain relation on the elements of a matroid is transitive.

Proposition 1.1.27 *Let $M = (E, \mathcal{B})$ be a matroid and let $x, y, z \in E$ be distinct elements. If there is a circuit C_1 containing x and y, and there is a circuit C_2 containing y and z, then there is a circuit C_3 containing x and z.*

Proof. [Oxl11, Proposition 4.1.2] or [Wel76, pg. 68]. □

It is possible to define an equivalence relation between the elements of a matroid as follows. We say that $x \sim y$ if either $x = y$ or there exists a circuit C of M such that $x, y \in C$. The reflexivity and symmetry are trivial, and the transitivity follows from the last Proposition. The equivalence classes under this relation are called the *connected components* of the matroid. Hence a matroid is *disconnected* when there are two or more connected components.

Definition 1.1.28. Let $M = (E, \mathcal{B})$ be a matroid. We say that $A \subseteq E$ is *inseparable* if the matroid $M|_A$ is connected. If A is not inseparable, we say that it is *separable*.

It is not difficult to see that an inseparable subset of a matroid is contained in one of its connected components. In other words, all the maximal inseparable subsets are exactly the connected components of M.

This discussion also motivates us to introduce another basic operation on matroids that provides another way of understanding matroid connectivity.

Definition 1.1.29. Let $M_1 = (E_1, \mathcal{B}_1)$ and $M_2 = (E_2, \mathcal{B}_2)$ be two matroids. Consider the disjoint union $E = E_1 \sqcup E_2$, and the family:

$$\mathcal{B} = \{B_1 \sqcup B_2 : B_1 \in \mathcal{B}_1 \text{ and } B_2 \in \mathcal{B}_2\},$$

then $M = (E, \mathcal{B})$ is a matroid. We call M the *direct sum* of M_1 and M_2 and we write $M = M_1 \oplus M_2$.

The proof of the fact that the above object is well-defined requires only a straightforward use of the definitions. Notice that $M = M_1 \oplus M_2$ implies that the rank of M is the sum of the ranks of M_1 and M_2.

Lemma 1.1.30 *A matroid M is disconnected if and only if there exists two matroids M_1 and M_2 such that*

$$M = M_1 \oplus M_2.$$

Proof. If $M = M_1 \oplus M_2$, let us choose two elements $x \in M_1$ and $y \in M_2$ and assume there exists a circuit C of M containing both. Since $C \smallsetminus \{x\}$ is independent, then it is contained in a basis B of M. Thus, we have that $C \smallsetminus \{x\} \subseteq B_1 \sqcup B_2$ for some bases B_1 of M_1 and B_2 of M_2. Since this is a disjoint union, we must have $C \smallsetminus \{x\} \subseteq B_1$ or $C \smallsetminus \{x\} \subseteq B_2$. Since $y \in C$, it has to be $C \smallsetminus \{x\} \subseteq B_2$. We can prove in an entirely analogous way that $C \smallsetminus \{y\}$ is contained in a basis B_1' of M_1. This implies that $C \smallsetminus \{x, y\} \subseteq B_1' \cap B_2 = \varnothing$. Hence, $C = \{x, y\}$, which yields a contradiction because $\{x\}$ and $\{y\}$ are thus independent, and so would be their union because the independent subsets of a direct sum are obtained that way. This proves that such a circuit C cannot exist, and M is disconnected.

Conversely, assume that M is disconnected, so that there are two elements x, y that lie in different connected components. Let us say that T_1 is the connected component containing x. Consider $T_2 = E \smallsetminus T_1$. Let us prove that $M = M|_{T_1} \oplus M|_{T_2}$. The equality between the groundsets is clear. It remains to prove that the set of bases of M and $M|_{T_1} \oplus M|_{T_2}$ coincide.

- A basis of $M|_{T_1} \oplus M|_{T_2}$ is the disjoint union of a basis B_1 of $M|_{T_1}$ and a basis B_2 of $M|_{T_2}$. We claim that $B_1 \cup B_2$ is independent in M. Let us assume the opposite. Then it must contain a circuit $C \subseteq B_1 \cup B_2$, and given that B_1 and B_2 are both independent in M, in particular C has to contain at least one element of B_1 and one element of B_2. This says that these two elements are in the same connected component in M, which is clearly impossible.

  Now, let us prove that $B_1 \cup B_2$ is indeed a basis of M. Choose a basis B of M containing $B_1 \cup B_2$. If it was possible to choose an element $e \in B \smallsetminus (B_1 \cup B_2)$, then we would have two possibilities according on $e \in T_1$ or $e \in T_2$. If $e \in T_1$, then $B_1 \cup e$ would be an independent set contradicting that B_1 was a basis. The other case is entirely analogous. This proves that all the bases of $M|_{T_1} \oplus M|_{T_2}$ are bases of M.

- Fix a basis B of M and consider $B_1 = B \cap T_1$ and $B_2 = B \cap T_2$. Clearly $B = B_1 \sqcup B_2$. Observe that B_1 is independent in $M|_{T_1}$, so that we can find a basis B_1' of $M|_{T_1}$ containing it. Analogously, we can find a basis B_2' of $M|_{T_2}$ containing it. Since $B_1' \sqcup B_2'$ is a basis of M by the preceding bullet, and it contains $B_1 \sqcup B_2$ which is also a basis of M, we must have $B_1' \sqcup B_2' = B_1 \sqcup B_2$, and hence $B_1' = B_1$ and $B_2' = B_2$. This proves that all the bases of M are the disjoint union of a basis of $M|_{T_1}$ and a basis of $M|_{T_2}$. □

It is natural to ask if every matroid can be decomposed uniquely as a direct sum of connected matroids. That is the subject of the following proposition.

Proposition 1.1.31 *Let M be a matroid and let $T_1, \dots, T_s$ be its connected components. Then*

$$M = \bigoplus_{i=1}^{s} M|_{T_i}.$$

Moreover, if $N_1, \dots, N_r$ are connected matroids such that $M = \bigoplus_{i=1}^{r} N_i$, then $r = s$ and the N_i's are a permutation of the $M|_{T_i}$'s.

Proof. This result is a straightforward extension of Lemma 1.1.30, and the proof is a direct consequence of the proof of that Lemma. □

Also, we have the following result.

Proposition 1.1.32 *Let $M = (E, \mathcal{B})$ be a matroid. A subset $A \subseteq E$ is separable if and only if it admits a partition into non-empty sets $A = A_1 \sqcup A_2$ such that*

$$\mathrm{rk}(A) = \mathrm{rk}(A_1) + \mathrm{rk}(A_2).$$

Proof. If $M|_A$ is disconnected, in particular we can write $M|_A = M|_{A_1} \oplus M|_{A_2}$ for some sets A_1 and A_2 such that $A_1 \sqcup A_2 = A$. It follows that $\mathrm{rk}(A_1) + \mathrm{rk}(A_2) = \mathrm{rk}(A)$.

Conversely, if there exists such a partition, one can prove directly that $M|_{A_1} \oplus M|_{A_2} = M|_A$, which is thus a disconnected matroid. □

Another property of the connected components that we will use is the following.

Proposition 1.1.33 *Let $M = (E, \mathcal{B})$ be a matroid. Two elements x, y belong the same connected component if and only if there exists a basis $B \in \mathcal{B}$ such that $(B \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.*

Proof. ($\Rightarrow$) By definition, two elements lie in the same connected component if and only if there is a circuit containing both of them. Let us assume that $\{x, y\} \subseteq C \in \mathcal{C}(M)$. Since $C \setminus \{y\}$ is independent, in particular it is contained in some basis B. Hence $y \in C \subseteq B \cup \{y\}$. Also, we have that $x \in C \subseteq B \cup \{y\}$, so that $x \in B$. Consider $X = (B \setminus \{x\}) \cup \{y\}$ and $Y = C$. By the properties of the rank, we have:

$$\mathrm{rk}(X) + \mathrm{rk}(Y) \geq \mathrm{rk}(X \cup Y) + \mathrm{rk}(X \cap Y),$$

which translates into:

$$\mathrm{rk}(X) + \mathrm{rk}(C) \geq \mathrm{rk}(B \cup \{y\}) + \mathrm{rk}(C \setminus \{x\}),$$

and using that C is a circuit, we see that $\mathrm{rk}(C \smallsetminus \{x\}) = \mathrm{rk}(C)$, so that in fact:

$$\mathrm{rk}(X) \geq \mathrm{rk}(B \cup \{y\}) = k,$$

from where it follows that X is a set of cardinality k and rank k or, in other words, a basis.

($\Leftarrow$) If $X = (B \smallsetminus \{x\}) \cup \{y\}$ is a basis of M, then $X \cup \{x\}$ is dependent and thus contains a circuit C which, in turn, contains x. Now, if $y \notin C$, we have that $C \subseteq (X \cup \{x\}) \smallsetminus \{y\} = B$, which is impossible since B was a basis. Hence it follows that $y \in C$, so that C is a circuit containing both x and y. $\square$

1.2 Matroids as polytopes

The basis polytope

Having established the basic combinatorics of matroids, we will move into the world of polytopes. In this subsection we will be following [FS05].

Definition 1.2.1. Let $M = (E, \mathcal{B})$ be a matroid. Assume that $|E| = n$ and label its elements as $\{1, \dots, n\}$. For each $A \subseteq E$ consider the point $e_A \in \mathbb{R}^n$ given by

$$e_A = \sum_{i \in A} e_i,$$

where e_i is the i-th canonical vector of $\mathbb{R}^n$.

- We define the *basis polytope* of M as the polytope

$$\mathcal{P}(M) = \text{convex hull } \{e_B : B \in \mathcal{B}\}.$$

- We define the *independence polytope* of M as the polytope

$$\mathcal{P}_{\mathsf{I}}(M) = \text{convex hull } \{e_I : I \in \mathcal{I}\},$$

  where $\mathcal{I}$ stands for the family all the independent sets of M.

Observe that different labelings for the elements of the groundset E yield different embeddings of the polytopes in $\mathbb{R}^n$. As we will be primarily focused on Ehrhart polynomials, we can assume that the groundset *is* $E = \{1, \dots, n\}$ and forget about the labelings. We will customarily assume that.

Also, notice that $\mathcal{P}(M)$ is obtained from $\mathcal{P}_{\mathsf{I}}(M)$ by intersecting with the hyperplane:

$$H = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = \mathrm{rk}(M) \right\}.$$

In other words, $\mathcal{P}(M)$ is a face of $\mathcal{P}_{\mathsf{I}}(M)$.

Let us introduce now a basic polytope that will appear repeatedly throughout this thesis.

Definition 1.2.2. The *hypersimplex* $\Delta_{k,n}$ is the polytope defined by

$$\Delta_{k,n} = \left\{ x \in [0,1]^n : \sum_{i=1}^{n} x_i = k \right\}.$$

Observe that if M is a matroid of rank k and cardinality n, the polytope $\mathcal{P}(M)$ is contained in the hypersimplex $\Delta_{k,n}$. This is because all the vertices e_B of $\mathcal{P}(M)$ have sum of coordinates equal to k and obviously lie in the hypercube $[0,1]^n$.

Example 1.2.3. Let us consider the uniform matroid $U_{2,4} = (E, \mathcal{B})$ where $E = \{1,2,3,4\}$ and $\mathcal{B}$ consists of all subsets of E of cardinality 2. If we denote $B_1 = \{1,2\}$, $B_2 = \{1,3\}$, $B_3 = \{1,4\}$, $B_4 = \{2,3\}$, $B_5 = \{2,4\}$ and $B_6 = \{3,4\}$, the six bases of M, we have the corresponding points in $\mathbb{R}^4$:

$$\begin{aligned} e_{B_1} &= (1,1,0,0),\\ e_{B_2} &= (1,0,1,0),\\ e_{B_3} &= (1,0,0,1),\\ e_{B_4} &= (0,1,1,0),\\ e_{B_5} &= (0,1,0,1),\\ e_{B_6} &= (0,0,1,1). \end{aligned}$$

The polytope $\mathcal{P}(U_{2,4})$ is given by the convex hull of $\{e_{B_1}, e_{B_2}, e_{B_3}, e_{B_4}, e_{B_5}, e_{B_6}\}$. As we will see below, this is exactly the hypersimplex $\Delta_{2,4}$.

Something that is immediate from the definition is that both the basis polytope and the independence polytope of a matroid are the convex hull of a finite set of points with 0/1-coordinates. This points are in fact *vertices* of the polytopes. This is a general fact.

Proposition 1.2.4 *Let $p_1, \ldots, p_s \in \mathbb{R}^n$ be points such that the coordinates of each p_j are either* 0 *or* 1*. Consider the polytope:*

$$\mathcal{P} = \text{convex hull}\,\{p_1, \ldots, p_s\}.$$

Then the vertices of $\mathcal{P}$ are exactly $p_1, \ldots, p_s$.

Proof. Let us assume, without loss of generality, that p_s is not a vertex of $\mathcal{P}$. It follows that $\mathcal{P} = \text{convex hull}\,\{p_1, \ldots, p_{s-1}\}$. Hence, we can write p_s as a convex combination of $p_1, \ldots, p_{s-1}$:

$$p_s = \sum_{i=1}^{s-1} \lambda_i p_i,$$

where $0 \le \lambda_i \le 1$ for each $1, \ldots, s-1$ and $\sum_{i=1}^{s-1} \lambda_i = 1$. Let us assume that p_s has a 1 on its j-th coordinate. We have:

$$1 = \langle p_s, e_j \rangle$$

$$= \sum_{i=1}^{s-1} \lambda_i \langle p_i, e_j \rangle .$$

Since the sum of all λ_i's was one, it follows that $\langle p_i, e_j \rangle = 1$ for all i's such that $\lambda_i > 0$. Hence, it follows that

$$\lambda_i > 0 \Longrightarrow \langle p_i, e_j \rangle = 1 \text{ for all } j \text{ such that } \langle p_s, e_j \rangle = 1.$$

Analogously, we can prove that

$$\lambda_i > 0 \Longrightarrow \langle p_i, e_j \rangle = 0 \text{ for all } j \text{ such that } \langle p_s, e_j \rangle = 0.$$

And both of these statements can be summarized in:

$$\lambda_i > 0 \Longrightarrow \langle p_i, e_j \rangle = \langle p_s, e_j \rangle \text{for all } j,$$

and the expression on the right says that $p_i = p_s$. This is impossible, because there is at least one $\lambda_i > 0$, and for any such i, the above implies that $p_i = p_s$ which contradicts that the p_i's were all distinct. □

Polytopes having all of its vertices with coordinates 0 or 1 are called $0/1$-*polytopes*. Observe that both the basis polytope and the independence polytope of a matroid are $0/1$-polytopes.

Proposition 1.2.5 *The basis polytope of the uniform matroid $U_{k,n}$ coincides with the hypersimplex $\Delta_{k,n}$.*

Proof. Consider the affine space generated by the vertices of $\mathcal{P}(U_{k,n})$. Since the sum of coordinates of all the vertices of $\mathcal{P}(U_{k,n})$ is k, we know that its codimension is at least 1. Consider a vector u orthogonal to that affine space. We claim that u is parallel to $(1, \dots, 1)$. Indeed, observe that for each $1 \leq i < j \leq n$, we can choose two $0/1$-vectors with exactly k ones that differ only in the positions i and j. In other words, there are two vertices of $\mathcal{P}(U_{k,n})$ whose difference is $e_i - e_j$. Since u is orthogonal to this difference, we get that $u_i = u_j$. Since i and j were arbitrary, we obtain that all the coordinates of u are equal. This proves that u is parallel to $(1, \dots, 1)$. Hence, the affine space generated by the vertices of $\mathcal{P}(U_{k,n})$ is exactly the set of points of equation $\sum_{i=1}^{n} x_i = k$. This proves that $\mathcal{P}(U_{k,n}) = \Delta_{k,n}$. □

Now let us establish the geometric counterpart of the operations of dualization and direct sum. Both of these operations have nice interpretations for the basis polytope.

Proposition 1.2.6 *Let M and N be two matroids on n elements. The following are equivalent:*

(a) M and N are the dual of each other.

(b) $\mathcal{P}(M)$ and $\mathcal{P}(N)$ are obtained by an involution of the form:

$$\mathcal{P}(M) = (1, \dots, 1) - \mathcal{P}(N).$$

Proof. Observe that if M and N are the dual of each other, to every vertex e_B of $\mathcal{P}(M)$ corresponds a vertex $e_{E\smallsetminus B}$ of $\mathcal{P}(N)$ which is clearly obtained by the involution of (b).

Conversely, let B be a basis of M. By the involution of (b) we obtain the basis $E \smallsetminus B$ on N, and this is indeed a bijection. This implies that $N = M^*$. □

For the operation of direct sum, we also get a nice interpretation for the independence polytope.

Proposition 1.2.7 *Let M_1 and M_2 be two matroids. Then*

$$\mathcal{P}(M_1 \oplus M_2) = \mathcal{P}(M_1) \times \mathcal{P}(M_2),$$

and

$$\mathcal{P}_{\mathrm{I}}(M_1 \oplus M_2) = \mathcal{P}_{\mathrm{I}}(M_1) \times \mathcal{P}_{\mathrm{I}}(M_2).$$

Proof. The first is evident from the definitions. The second too, because the independence subsets of $M_1 \oplus M_2$ are exactly the disjoint union of an independent subset of M_1 and an independent subset of M_2. □

We have the following fundamental result, which is due to Gel'fand, Goresky, MacPherson and Serganova [GGMS87]. It gives a complete characterization of the polytopes that can arise as the basis polytope of a matroid.

Theorem 1.2.8 *A polytope $\mathcal{P} \subseteq \mathbb{R}^n$ is the basis polytope of a matroid if and only if it satisfies the following two conditions:*

- *$\mathcal{P}$ is a $0/1$-polytope.*
- *All the edges of $\mathcal{P}$ are parallel to $e_i - e_j$ for some $i \neq j$.*

Proof. ($\Rightarrow$) Let $\mathcal{P}$ be the basis polytope of the matroid $M = (E, \mathcal{B})$ of rank k. By Proposition 1.2.4 we automatically have that $\mathcal{P}$ is a $0/1$-polytope. Notice also that all the points of $\mathcal{P}$ have sum of coordinates equal to k, because in particular all of its vertices do so by Lemma 1.1.3. Let us prove that all the edges of $\mathcal{P}$ are of the claimed form.

Fix e_{B_1} and e_{B_2} adjacent vertices on the polytope. The edge determined by these two vertices admits a supporting hyperplane H. In other words, there exists $u \in \mathbb{R}^n$ and a half-space H_u^+ defined by an inequality:

$$\langle x, u\rangle \leq 1,$$

and a hyperplane H_u, given by

$$\langle x, u\rangle = 1,$$

with the property that $\mathcal{P} \subseteq H_u^+$ and that $\mathcal{P} \cap H_u = \operatorname{conv}(\{e_{B_1}, e_{B_2}\})$.

Since $B_1 \neq B_2$, by the symmetric-exchange-property, for any $i \in B_1 \smallsetminus B_2$ we can find $j \in B_2 \smallsetminus B_1$ such that

$$B_3 := (B_1 \smallsetminus \{i\}) \cup \{j\} \in \mathcal{B},$$

$$B_4 := (B_2 \smallsetminus \{j\}) \cup \{i\} \in \mathcal{B}.$$

Let us assume that $B_2 \neq B_3$. Since it obviously is $B_3 \neq B_1$, we have that $\langle e_{B_3}, u\rangle < 1$, and in particular

$$\left\langle \frac{e_{B_3} + e_{B_4}}{2}, u \right\rangle = \frac{1}{2}\langle e_{B_3}, u\rangle + \frac{1}{2}\langle e_{B_4}, u\rangle < 1$$

But, on the other hand,

$$\left\langle \frac{e_{B_3} + e_{B_4}}{2}, u \right\rangle = \left\langle \frac{e_{B_1} + e_{B_2}}{2}, u \right\rangle = 1.$$

This contradiction implies that it has to be $B_2 = B_3$, so that B_1 and B_2 differ by just one element and, in fact:

$$e_{B_1} - e_{B_2} = e_i - e_j,$$

as desired.

($\Leftarrow$) Let us assume that $\mathcal{P}$ is a $0/1$-polytope with all the edges of the desired form. Since all the vertices of $\mathcal{P}$ can be associated to a certain subset of $E = \{1, \dots, n\}$ by taking the indices of the coordinates that are equal to 1, we can construct a family $\mathcal{B} \subseteq 2^E$. We must prove that this is indeed the set of bases of a matroid. To this end, fix B_1 and B_2 in $\mathcal{B}$ and consider an element $i \in B_1 \smallsetminus B_2$. Observe that if we consider all the vertices adjacent to e_{B_1} in the polytope, we can cone over the vertex e_{B_1} and, in particular, every point of the polytope can be written as a non-negative combination of the generators of this cone. In particular, we can write:

$$e_{B_2} - e_{B_1} = \sum_{e_B \text{ adjacent to } e_{B_1}} \lambda_B (e_B - e_{B_1}),$$

where each λ_B is non-negative. Since $i \in B_1 \smallsetminus B_2$, by equating the i-th coordinates in the above equality, we get that there is at least one e_B adjacent to e_{B_1} such that $\lambda_B \neq 0$ and $i \notin B$. Hence, by the property on the edges, there exists j such that $e_B - e_{B_1} = e_j - e_i$. By equating the j-th coordinates we get that $j \in B_2 \smallsetminus B_1$, and also $(B_1 \smallsetminus \{i\}) \cup \{j\} = B \in \mathcal{B}$ which is then the set of bases of a matroid. $\square$

Theorem 1.2.9 *Let $M = (E, \mathcal{B})$ be a matroid such that $|E| = n$. The dimension of its basis polytope is given by*

$$\dim \mathcal{P}(M) = n - c(M),$$

where $c(M)$ denotes the number of connected components of M. In particular, M is connected if and only if $\dim \mathcal{P}(M) = n - 1$.

Proof. By definition $\dim \mathcal{P}(M)$ is the dimension of the least affine space containing $\mathcal{P}(M)$. In particular, since this coincides with the dimension of the following linear space:

$$\operatorname{span}\{e_{B_1} - e_{B_2} : e_{B_1} \text{ and } e_{B_2} \text{ are adjacent in } \mathcal{P}(M)\},$$

we have the following chain of equalities:

$$\begin{aligned}
\dim \mathcal{P}(M) &= \dim \operatorname{span} \{e_{B_1} - e_{B_2} : e_{B_1} \text{ and } e_{B_2} \text{ are adjacent in } \mathcal{P}(M)\} \\
&= \dim \operatorname{span} \left\{e_i - e_j : \text{for some } B_1 \in \mathcal{B} \text{ it is } (B_1 \smallsetminus \{i\}) \cup \{j\} \in \mathcal{B}\right\} \\
&= \dim \operatorname{span} \left\{e_i - e_j : i \text{ and } j \text{ lie in the same connected component}\right\} \\
&= \sum_{\substack{T \text{ connected} \\ \text{component of } M}} (|T| - 1) \\
&= n - c(M)
\end{aligned}$$

where we used Proposition 1.1.33. □

So far we have a description of the polytope by means of its vertices. It is also desirable to be able to express $\mathcal{P}(M)$ as the solution set of a system of linear inequalities. That is what we do in the following result. Observe that since the loops of a matroid are exactly the elements that do not belong to any basis, if M has a set of loops L, then

$$\mathcal{P}(M) = \mathcal{P}(M \smallsetminus L) \times \{(\underbrace{0, \dots, 0}_{|L|})\}.$$

In particular, we can restrict to loopless matroids.

Theorem 1.2.10 *Let $M = (E, \mathcal{B})$ be a loopless matroid of rank k and cardinality n. Let $\mathcal{F}(M)$ denote the set of all flats of M. Then*

$$\mathcal{P}(M) = \left\{ x \in \Delta_{k,n} : \sum_{i \in F} x_i \leq \operatorname{rk}(F) \textit{ for all inseparable } F \in \mathcal{F}(M) \right\}.$$

Proof. Consider any face Q of the polytope $\mathcal{P}(M)$. There exists a hyperplane H_u and a corresponding half-space H_u^+ such that H_u is defined by the equation:

$$\langle u, x \rangle = 1,$$

and H_u^+ by the inequality

$$\langle u, x \rangle \leq 1,$$

having the property that $\mathcal{P}(M) \subseteq H_u^+$ and $Q = \mathcal{P}(M) \cap H$. Since u is normal to the face Q, and all the edges of that face have the form $e_i - e_j$ for some $i \neq j$, then in particular $\langle u, e_i - e_j \rangle = 0$ for all such i, j. This imposes conditions of the form $u_i = u_j$. Conversely, for a vector to be normal to Q it suffices to satisfy the condition $u_i = u_j$ for all $i \neq j$ such that $e_i - e_j$ appears as an edge. In other words, it is possible to choose u to be a 0/1-vector, by putting a 1 on the i-th coordinate if in Q there is an edge of the form $e_i - e_j$, and a 0 if there is not such an edge. In particular, we can adjust our half-spaces to be described by an inequality of the form:

$$\sum_{i \in A} x_i \leq r,$$

for some $r \in \mathbb{R}$ and $A \subseteq E$, or the form:

$$\sum_{i \in A} x_i \geq r'.$$

for some $r' \in \mathbb{R}$ and $A \subseteq E$. Observe that the second case is just:

$$\sum_{i \in E \setminus A} x_i \leq r,$$

for $r = k - r'$, since all our points x lie in the hypersimplex $\Delta_{k,n}$. In either case, what we have is that for each face Q, we can choose the defining inequality to be of the form:

$$\sum_{i \in S} x_i \leq m_S$$

for some subset $S \subseteq E$ and some real value $m_S \geq 0$. We can further determine m_S by noticing that the expression on the left is maximized at some vertex of the polytope. In other words,

$$m_S = \max_{B \in \mathcal{B}} \sum_{i \in S \cap B} e_i = \max_{B \in \mathcal{B}} |B \cap S| = \operatorname{rk}(S).$$

Hence, we have proved:

$$\mathcal{P}(M) = \left\{ x \in \Delta_{k,n} : \sum_{i \in S} x_i \leq \operatorname{rk}(S) \text{ for all } S \subseteq E \right\}. \tag{1.1}$$

This is because $\supseteq$ follows from what we have just showed, and $\subseteq$ is clear since all the vertices e_B of $\mathcal{P}(M)$ satisfy all such inequalities.

Also, we can reduce the list of inequalities as follows. For a subset $S \subseteq E$, pick the least flat F containing S. We have that $\operatorname{rk}(F) = \operatorname{rk}(S)$. Hence, the inequality $\sum_{i \in F} x_i \leq \operatorname{rk}(F)$ trivially implies the inequality $\sum_{i \in S} x_i \leq \operatorname{rk}(S)$. Now, observe that if F is a separable flat, we can partition it into two non-empty parts as $F = A_1 \sqcup A_2$ such that $\operatorname{rk}(F) = \operatorname{rk}(A_1) + \operatorname{rk}(A_2)$. Since M is loopless, the rank of both A_1 and A_2 is at least one, so that $\operatorname{rk}(A_1) < \operatorname{rk}(F)$ and $\operatorname{rk}(A_2) < \operatorname{rk}(F)$. Now, A_1 and A_2 are contained in some flats $F_1 \subseteq F$ and $F_2 \subseteq F$. This implies that the inequality for the flat F is obtained by those of F_1 and F_2. We can inductively reduce only to inseparable flats. $\square$

Remark 1.2.11. Observe that if F is an inseparable flat of M, then since $M|_F$ is connected, in particular $M|_F$ does not have coloops. A flat with this property is called a *cyclic flat*. This is equivalent for F to be a union of circuits. It is known (see for instance [Oxl11, Exercise 2.1.13]) that if F is a cyclic flat of M, then $E \setminus F$ is a cyclic flat of M^*.

Remark 1.2.12. In [FS05, Proposition 2.6] Feichtner and Sturmfels introduced the notion of *flacet*. A flacet of M is a flat $F \in \mathcal{F}(M)$ such that $M|_F$ and $M^*|_{E \setminus F}$ are inseparable. Since by Proposition 1.2.6 there is an rigid transformation mapping $\mathcal{P}(M)$ to $\mathcal{P}(M^*)$ it is not difficult to see that one can reduce the inequalities description of Theorem 1.2.10 by only using the flacets instead of all inseparable flats.

Independence polytopes

Now we will approach independence polytopes, which were defined as the convex hull of the indicator vectors of the independent sets of a matroid. Our goal is to establish a version of Theorem 1.2.8 for these polytopes and a corresponding description using inequalities. A standard reference is Volume B of Schrijver's book *Combinatorial Optimization* [Sch03].

Let us first start with a result concerning the dimension of this polytope whenever the matroid does not contain loops.

Proposition 1.2.13 *If $M = (E, \mathcal{B})$ is a loopless matroid such that $|E| = n$, then*

$$\dim \mathcal{P}_{\mathsf{I}}(M) = n.$$

Proof. We proceed by induction in n. For $n = 1$ the conclusion is true, because we have just one element e which by hypothesis is not a loop, and hence the matroid has two independent sets: $\varnothing$ and $\{e\}$. They determine a segment in $\mathbb{R}$ which of course has dimension 1. Now, for an arbitrary M, if it is disconnected, we can write $M = M_1 \oplus M_2$ and use Proposition 1.2.7 and the induction hypothesis, because

$$\begin{aligned}\dim \mathcal{P}_{\mathsf{I}}(M) &= \dim \mathcal{P}_{\mathsf{I}}(M_1) + \dim \mathcal{P}_{\mathsf{I}}(M_2)\\ &= |M_1| + |M_2|\\ &= n.\end{aligned}$$

If M is connected, then we know that $\mathcal{P}(M)$ has dimension $n - 1$. Since this polytope is a facet of $\mathcal{P}_{\mathsf{I}}(M)$ and the origin is a vertex of this polytope lying outside such facet, we get that its dimension is n, as desired. □

Now, let us establish the edge directions of an independence polytope.

Proposition 1.2.14 *Let M be a matroid and $\mathcal{P}_{\mathsf{I}}(M)$ its independence polytope. All the edges of $\mathcal{P}_{\mathsf{I}}(M)$ are of the following form:*

- *$e_i - e_j$ for $i \neq j$.*
- *e_i or $-e_i$.*

Proof. Let us assume now that I and J are independent subsets of M such that e_I and e_J are adjacent in $\mathcal{P}_{\mathsf{I}}(M)$. We have two possibilities:

- If $\operatorname{rk}(I) < \operatorname{rk}(I \cup J)$, then there is an element $j \in J \smallsetminus I$ such that $I \cup \{j\}$ is independent. But observe that we have the following equality:

$$e_I + e_J = e_{I\cup\{j\}} + e_{J\smallsetminus\{j\}},$$

and since the involved vectors are all vertices of $\mathcal{P}_{\mathsf{I}}(M)$ and we assumed that e_I and e_J were adjacent, we must have that $I \cup \{j\} = J$ and $J \smallsetminus \{j\} = I$. Hence $e_I - e_J = -e_j$.

- If $\mathrm{rk}(I) = \mathrm{rk}(I \cup J)$. If $\mathrm{rk}(J) < \mathrm{rk}(I \cup J)$ we can further proceed as above, so we may also assume that $\mathrm{rk}(I) = \mathrm{rk}(J) = \mathrm{rk}(I \cup J)$. Consider the matroid $M' = M|_{I \cup J}$. We have that I and J are bases of M'. By the symmetric-exchange-property, for any $j \in J \smallsetminus I$ there exists $i \in I \smallsetminus J$ such that $(I \smallsetminus \{i\}) \cup \{j\}$ and $(J \smallsetminus \{j\}) \cup \{i\}$ are bases of M'. In particular, these two sets are independent on M, and we have:

$$e_I + e_J = e_{(I \smallsetminus \{i\}) \cup \{j\}} + e_{(J \smallsetminus \{j\}) \cup \{i\}},$$

  and by the same reasoning that we used above, it follows that $(I \smallsetminus \{i\}) \cup \{j\} = J$, and thus $e_I - e_J = e_i - e_j$. □

Now that we know what the edge directions are, we may obtain a description using inequalities for $\mathcal{P}_{\mathsf{I}}(M)$. By the same reasoning as for the basis polytope, we can restrict only to loopless matroids.

Theorem 1.2.15 *Let $M = (E, \mathcal{B})$ be a loopless matroid of cardinality n and rank k. The independence polytope $\mathcal{P}_{\mathsf{I}}(M)$ is given by*

$$\mathcal{P}_{\mathsf{I}}(M) = \left\{ x \in \mathbb{R}^n_{\geq 0} : \sum_{i \in F} x_i \leq \mathrm{rk}(F) \textit{ for all inseparable } F \in \mathcal{F}(M) \right\}.$$

Proof. The proof carries out exactly as in Theorem 1.2.10. Essentially, since the edge directions are $e_i - e_j$ and $\pm e_i$, for every face of $\mathcal{P}_{\mathsf{I}}(M)$ with normal vector u, we have conditions of the form $u_i = u_j$ and $u_i = 0$. By proceeding as in the basis polytope case, we can reduce the description to inequalities of the form:

$$\sum_{i \in S} x_i \leq m_S,$$

for every subset $S \subseteq E$. The right-hand-side can be determined as

$$m_S = \max_{I \in \mathcal{I}} \sum_{i \in S \cap I} e_i = \max_{I \in \mathcal{I}} |I \cap S| = \mathrm{rk}(S),$$

so that our polytope is described by

$$\mathcal{P}_{\mathsf{I}}(M) = \left\{ x \in \mathbb{R}^n_{\geq 0} : \sum_{i \in S} x_i \leq \mathrm{rk}(S) \text{ for all } S \subseteq E \right\} \tag{1.2}$$

which is the analog of equation (1.1). Now, we can reduce only to inseparable flats exactly as in the proof of Theorem 1.2.10. □

Remark 1.2.16. The preceding result is due to Edmonds and its first proof appeared in [Edm70]. The proof within in that article depends on the interplay of the greedy algorithm and matroids. Another version of the same proof can be found in [Sch03].

Remark 1.2.17. An important fact from the algorithmic point of view is that the inequalities description of $\mathcal{P}_{\mathrm{I}}(M)$ given in Theorem 1.2.15 is *minimal*. In other words for a loopless matroid M of cardinality n, we have:

$$\#\{\text{facets of } \mathcal{P}_{\mathrm{I}}(M)\} = n + \#\{F \in \mathcal{F}(M) \text{ inseparable}\}.$$

For a proof of this fact we refer to [Sch03, Theorem 40.5]. However, it is important to emphasize that the description for the basis polytope given in Theorem 1.2.10 is not minimal, even when restricting only to flacets instead of inseparable flats. In general it may happen that some of the inequalities $x_i \geq 0$ are redundant.

Example 1.2.18. Let us consider the uniform matroid $U_{k,n}$. In Proposition 1.2.5 we proved that $\mathcal{P}(U_{k,n})$ is the hypersimplex $\Delta_{k,n}$. Notice that the flats of $U_{k,n}$ are exactly the subsets of cardinality $0, 1, \ldots, k-1$ and n. The only inseparable flats of $U_{k,n}$ are the whole groundset and the singletons. Hence:

$$\mathcal{P}_{\mathrm{I}}(U_{k,n}) = \left\{ x \in [0,1]^n : \sum_{i=1}^{n} x_i \leq k \right\}.$$

Now we are ready to state the corresponding characterization of the polytopes that arise as the independence polytope of a matroid.

Theorem 1.2.19 *A polytope $\mathcal{P} \subseteq \mathbb{R}^n$ is the independence polytope of a matroid if and only if it satisfies the following three conditions:*

- *$\mathcal{P}$ is a $0/1$-polytope.*
- *All the edges of $\mathcal{P}$ are parallel to $e_i - e_j$ for some $i \neq j$ or parallel to e_i for some i.*
- *The origin is a vertex of $\mathcal{P}$.*

Proof. ($\Rightarrow$) Assume $\mathcal{P}$ is the independence polytope of a matroid. The first condition follows from Proposition 1.2.4. The second condition follows from Proposition 1.2.14. The third is a consequence of the fact that the empty set is independent in any matroid, so that the origin has to be a vertex of $\mathcal{P}$.

($\Leftarrow$) Assume that $\mathcal{P}$ satisfies all the conditions of the statement. Clearly, the vertices of $\mathcal{P}$ give place to a family of subsets of $E = \{1, \ldots, n\}$. Let us call s the maximum sum of coordinates of a vertex of $\mathcal{P}$. Since $\mathcal{P}$ is a $0/1$-polytope, this number s is a non-negative integer. We will proceed by induction in s. If $s = 0$, then the whole polytope is actually a point, and it corresponds to a matroid consisting on n loops. If $s = 1$, then we have the origin and some points of the form e_i. Our matroid is realised as a set of parallel edges and some loops.

Assume now that we have proved that when $s = 0, \ldots, k-1$, and $k \geq 2$, then a polytope satisfying the three conditions of the statement is the independence polytope of a matroid. Assume now that our polytope $\mathcal{P}$ satisfies such conditions and has $s = k$. Consider the hyperplane H_k of equation:

$$\sum_{i=1}^{n} x_i = k,$$

and the intersection $\mathcal{Q} = \mathcal{P} \cap H_k$. It is clear that $\mathcal{Q}$ is a $0/1$-polytope with edge-directions of the form $e_i - e_j$. By Theorem 1.2.8 we know that $\mathcal{Q}$ is the basis polytope of a matroid M. Also, if we consider the vertices of $\mathcal{P}$ with sum of coordinates less or equal than $k-1$, by induction hypothesis, they determine the independence polytope of a matroid M'.

Fix any vertex e_B of $\mathcal{Q}$ and consider the set $I = B \setminus \{e\}$ for any $e \in B$. We claim that e_I is a vertex of $\mathcal{P}$. Assume e_I was not a vertex and consider the inequality:

$$\langle x, e_I \rangle \geq k - 2$$

The vertices of $\mathcal{P}$, which are obviously all of the form e_A for $A \subseteq E$, satisfy this inequality if and only if either:

- $A \subsetneq I$ is such that $|A| = k - 2$ and A is independent in M'.
- $A \supsetneq I$ is such that $|A| = k$ and A is a basis of M.

(There are not other possibilities because we are assuming that e_I is not a vertex of $\mathcal{P}$). Notice that this implies that there is an edge in $\mathcal{P}$ which is not of the form $\pm e_i$ nor $e_i - e_j$, which is a contradiction.

Hence, we obtain that e_I is a vertex of $\mathcal{P}$. Also, since $|I| = k - 1$, we have that e_I is a vertex of $\mathcal{P}_{\mathsf{I}}(M')$ and this implies that all subsets of I are vertices of $\mathcal{P}_{\mathsf{I}}(M')$ and thus of $\mathcal{P}$. Now, recall that I was an arbitrary subset of cardinality $k-1$ of a basis B of M. This proves that $\mathcal{P} = \mathcal{P}_{\mathsf{I}}(M)$, and the proof is complete. □

Generalized Permutohedra and Polymatroids

In [Pos09] Postnikov introduced a family of polytopes which generalizes the notion of matroid polytope.

Definition 1.2.20. A polytope $\mathcal{P} \subseteq \mathbb{R}^n$ is a *generalized permutohedron* if all the edges of $\mathcal{P}$ are parallel to some vector of the form $e_i - e_j$.

The terminology is justified because a generalized permutohedron is in essence a polytope obtained by deforming the regular permutohedron (i.e. the polytope having as vertices all the permutations of the vector $(1, 2, \ldots, n)$), always preserving the edge directions.

Observe that now we can restate Theorem 1.2.8 in a shorter way: matroid polytopes are exactly generalized permutohedra which are $0/1$-polytopes.

It is clear that the independence polytope of a matroid will fail to be a generalized permutohedron. However, with a clever trick, we can obtain a description of this polytope as a generalized permutohedron.

Definition 1.2.21. Let M be a matroid of rank k and cardinality n. We define the *lifted independence polytope* of M as the polytope $\widetilde{\mathcal{P}}_{\mathsf{I}}(M) \subseteq \mathbb{R}^{n+1}$ given by

$$\widetilde{\mathcal{P}}_{\mathsf{I}}(M) := \text{convex hull } \{(e_I, k - \operatorname{rk}(I)) : I \subseteq E \text{ is independent}\} \subseteq \mathbb{R}^{n+1}.$$

Two lattice polytopes $\mathcal{P}_1 \subseteq \mathbb{R}^n$ and $\mathcal{P}_2 \subseteq \mathbb{R}^m$ are said to be *integrally equivalent* when there is an affine map $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ such that its restriction to $\mathcal{P}_1$ induces a bijection $\varphi : \mathcal{P}_1 \to \mathcal{P}_2$ which preserves the lattice, i.e. the image under φ of $\mathbb{Z}^n \cap \operatorname{aff} \mathcal{P}_1$ is $\mathbb{Z}^m \cap \operatorname{aff} \mathcal{P}_2$, where $\operatorname{aff} \mathcal{P}_1$ denotes the affine space spanned by $\mathcal{P}_1$ and analogously for $\mathcal{P}_2$.

We have $\mathcal{P}_{\mathsf{I}}(M)$ and $\widetilde{\mathcal{P}}_{\mathsf{I}}(M)$ are integrally equivalent, as the projection that forgets the last coordinate fulfills the conditions. Something immediate from the definitions is that integrally equivalent polytopes have the same Ehrhart polynomial (cf. Chapter 2).

Theorem 1.2.22 *For every matroid M, the lifted independence polytope $\widetilde{\mathcal{P}}_{\mathsf{I}}(M)$ is a generalized permutohedron.*

Proof. Let us pick two adjacent vertices v and w in $\widetilde{\mathcal{P}}_{\mathsf{I}}(M)$. They are of the form:

$$v = (e_{I_1}, k - \operatorname{rk}(I_1)),$$
$$w = (e_{I_2}, k - \operatorname{rk}(I_2)),$$

for some independent sets I_1 and I_2 of M. Moreover, the vertices e_{I_1} and e_{I_2} are adjacent in $\mathcal{P}_{\mathsf{I}}(M)$. There are two cases:

- If $|I_1| = |I_2|$, then $\operatorname{rk}(I_1) = |I_1| = |I_2| = \operatorname{rk}(I_2)$. Since e_{I_1} and e_{I_2} are adjacent in $\mathcal{P}_{\mathsf{I}}(M)$ the only possibility is that $e_{I_1} - e_{I_2} = e_i - e_j$ for some i, j. In particular, $v - w = (e_i - e_j, 0)$.

- If $|I_1| \neq |I_2|$, assume without loss of generality that $|I_1| < |I_2|$. The condition of e_{I_1} and e_{I_2} being adjacent in $\mathcal{P}_{\mathsf{I}}(M)$ implies that $e_{I_2} - e_{I_1} = e_i$ for some i. This says that $I_2 = I_1 \sqcup \{i\}$. In particular $\operatorname{rk}(I_2) = \operatorname{rk}(I_1) + 1$, and then

$$v - w = (e_i, -1),$$

which has the desired form. $\square$

Remark 1.2.23. This result was stated implicitly in [ABD10]. We include it explicitly here to simplify future referencing and motivate the main results.

We now digress about a family of objects that were introduced by Edmonds and Rota in [ER66] as another generalization of matroid.

Definition 1.2.24. A *polymatroid* $P = (E, f)$ consists of a finite set E and a map $f : 2^E \to \mathbb{R}$ with the following properties:

- $f(\varnothing) = 0$.

- If $A_1 \subseteq A_2 \subseteq E$, then $f(A_1) \leq f(A_2)$.
- If $A_1, A_2 \subseteq E$, then

$$f(A_1) + f(A_2) \geq f(A_1 \cup A_2) + f(A_1 \cap A_2).$$

Remark 1.2.25. Observe that if we further require that $f(A) \leq |A|$ and that the values that f assume are integers, what we end up obtaining is in fact a matroid. This is why polymatroids are a generalization of matroids indeed.

We can associate two polytopes to any polymatroid, just as we did for matroids. Since the concepts of *bases* and *independent sets* were not defined for polymatroids, we shall approach a definition of the polytopes by using inequalities instead of giving its list of vertices as we did for matroids.

Definition 1.2.26. Let $P = (E, f)$ be a polymatroid on $E = \{1, \ldots, n\}$. We define the following two polytopes

$$\mathcal{P}(P) = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = f(E), \sum_{i \in S} x_i \leq f(S) \text{ for all } S \subseteq E \right\}.$$

$$\mathcal{P}_{\mathrm{I}}(P) = \left\{ x \in \mathbb{R}^n_{\geq 0} : \sum_{i \in S} x_i \leq f(S) \text{ for all } S \subseteq E \right\}.$$

We call $\mathcal{P}(P)$ the *basis polytope of P* and $\mathcal{P}_{\mathrm{I}}(P)$ the *independence polytope of P*.

Both objects are indeed polytopes, as they are bounded. This is because the basis polytope has each coordinate bounded by

$$0 \leq f(E) - f(E \smallsetminus \{i\}) \leq x_i \leq f(\{i\}),$$

and the independence polytope by

$$0 \leq x_i \leq f(\{i\}).$$

In either case, we want to emphasize that the coordinates of the vertices of both the basis polytope and the independence polytope of a polymatroid are nonnegative.

Notice also that by the inequalities description we obtained in equation (1.1) for the basis polytope of a matroid and equation (1.2) for the independence polytope of a matroid, it follows that when the polymatroid is a matroid, the above polytopes coincide with its basis and its independence polytope, respectively.

Recall that Theorem 1.2.8 proves the remarkable fact that the class of basis polytopes of matroids is just the class of 0/1-generalized permutohedra. We can characterize the class of basis polytopes of polymatroids in a very nice way.

Theorem 1.2.27 *A polytope $\mathcal{P}$ is the basis polytope of a polymatroid if and only if it has nonnegative coordinates and it is a generalized permutohedron.*

Proof. A proof of this fact can be found explicitly in the note [Fin15], or implicitly in [DF10] (where the result follows from the corresponding characterization for *megamatroids*). Another proof can be found in [CL20]. □

In other words, up to a translation, we have that basis polytopes of polymatroids and generalized permutohedra are the same thing. Although the study of these objects has grown especially since the appearance of Postnikov's article [Pos09], during the four preceding decades the study of polymatroids had been extensive and many results were already known for them.

CHAPTER 2

The Ehrhart polynomial

2.1 A glimpse of Ehrhart theory

In 1962 Eugène Ehrhart [Ehr62] initiated the study of the lattice-point counting for integral polytopes with the following remarkable result.

Theorem 2.1.1 (Ehrhart's Theorem) *Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a lattice polytope of dimension d. For each integer $t \geq 1$ consider the number of lattice points in $t\mathcal{P}$ (the dilation of $\mathcal{P}$ with respect to the origin):*

$$\operatorname{ehr}(\mathcal{P}, t) = \#(t\mathcal{P} \cap \mathbb{Z}^n).$$

Then $\operatorname{ehr}(\mathcal{P}, -)$ *is a polynomial in the variable t of degree d.*

The polynomial $\operatorname{ehr}(\mathcal{P}, t)$ is called *the Ehrhart polynomial of* $\mathcal{P}$. For a proof of Theorem 2.1.1, we refer to [BR15, Theorem 3.8] by Beck and Robins, which is the standard introductory bibliography to Ehrhart's theory.

Ehrhart polynomials have proven useful invariants of polytopes that encode much of its arithmetic, geometric and combinatorial information. Moreover, they appear naturally when dealing with

- Hilbert functions of finitely generated graded algebras.
- Triangulations of polytopes.
- Order polynomials of posets.
- Chromatic polynomials of graphs.
- The number of solutions to certain diophantine equations.

Some of these interactions (and many others) are explored in [BS18]. The aim of this section is to review all the results in Ehrhart theory that we will need in the sequel. All the proofs can be found in [BR15].

Basic results

We start by introducing another polynomial associated to the Ehrhart polynomial. It encodes essentially the same information but has nicer combinatorial properties as we will see below.

Definition 2.1.2. Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a lattice polytope of dimension d. Let us write $\operatorname{ehr}(\mathcal{P}, t)$ in the basis $\binom{t+d}{d}, \binom{t+d-1}{d}, \ldots \binom{t}{d}$ of the vector space of real polynomials of degree at most d. We have

$$\operatorname{ehr}(\mathcal{P}, t) = h_0^* \binom{t+d}{d} + h_1^* \binom{t+d-1}{d} + \ldots + h_{d-1}^* \binom{t+1}{d} + h_d^* \binom{t}{d}.$$

The $(d+1)$-uple $(h_0^*, \ldots, h_d^*)$ is called *the h^*-vector* of $\mathcal{P}$. The polynomial

$$h^*(\mathcal{P}, x) := h_0^* + h_1^* x + \ldots + h_{d-1}^* x^{d-1} + h_d^* x^d,$$

is called *the h^*-polynomial of* $\mathcal{P}$.

Remark 2.1.3. If we consider the *Ehrhart series* of $\mathcal{P}$, which is defined by the formal series

$$\operatorname{Ehr}(\mathcal{P}, x) := \sum_{m=0}^{\infty} \operatorname{ehr}(\mathcal{P}, m) x^m,$$

then it holds:

$$\operatorname{Ehr}(\mathcal{P}, x) = \frac{h^*(\mathcal{P}, x)}{(1-x)^{\dim \mathcal{P}+1}}.$$

In other words, the h^*-polynomial is just the numerator of the generating function of the Ehrhart polynomial evaluated on the nonnegative integers.

Example 2.1.4. Consider the polytope

$$\mathcal{T} = \text{convex hull } \{(0,0,0), (1,0,0), (0,1,0), (1,1,13)\} \subseteq \mathbb{R}^3.$$

This is called the *Reeve's tetrahedron*. It is not difficult to prove that the above is a lattice polytope of dimension 3. Thus, we know that $\operatorname{ehr}(\mathcal{T}, t)$ is a polynomial of degree 3. We can translate the above to an inequalities description of the polytope and count by hand the number of lattice points that lie in $\mathcal{T}$, $2\mathcal{T}$, $3\mathcal{T}$ and $4\mathcal{T}$ and see that these quantities are respectively 4, 22, 68 and 155. By interpolating, we can see that

$$\operatorname{ehr}(\mathcal{T}, t) = \tfrac{13}{6} t^3 + t^2 - \tfrac{1}{6} t + 1. \tag{2.1}$$

Also:

$$h^*(\mathcal{T}, x) = 12x^2 + 1.$$

Observe that the Ehrhart polynomial of $\mathcal{T}$ has a negative coefficient.

One of the main reasons that we pay attention to the h^*-polynomial is the following important result by Stanley [Sta93].

Theorem 2.1.5 *Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a lattice polytope. The Ehrhart h^*-vector of $\mathcal{P}$ has non-negative integer entries.*

Observe that in general the coefficients of the Ehrhart polynomial are rational and, moreover, they can be negative, as we have seen in Equation (2.1). Hence, in many situations it is more desirable to deal with (or to understand) the h^*-polynomial of a polytope $\mathcal{P}$ instead of its Ehrhart polynomial. There are, however, geometric interpretations for three of the coefficients of Ehrhart polynomials.

Recall that the *relative volume* of a lattice polytope $\mathcal{P}$ of dimension d in $\mathbb{R}^n$ is defined as the volume of $\mathcal{P}$ with respect to the sublattice $\operatorname{aff}(\mathcal{P}) \cap \mathbb{Z}^d$. For example, the segment with endpoints $(0,0)$ and $(3,3)$ has relative volume 3 (and *not* $\sqrt{18}$), as we must restrict ourselves to the sublattice $\mathbb{Z}(1,1)$.

Theorem 2.1.6 *Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a polytope of dimension d. Then*

$$\begin{aligned}
[t^d]\,\operatorname{ehr}(\mathcal{P},t) &= \operatorname{vol}(\mathcal{P}),\\
[t^{d-1}]\,\operatorname{ehr}(\mathcal{P},t) &= \tfrac{1}{2}\operatorname{vol}(\partial\mathcal{P}),\\
[t^0]\,\operatorname{ehr}(\mathcal{P},t) &= 1,
\end{aligned}$$

where vol *stands for relative volumes.*

As we mentioned, from the work of McMullen [McM77] it follows that there are formulas for all the coefficients in terms of relative volumes of certain faces of the polytope. However, such expressions are not easy to deal with.

Another basic result is the so-called *reciprocity*. It establishes a relation between the number of lattice points on the relative interior of the dilations of $\mathcal{P}$ and the Ehrhart polynomial of $\mathcal{P}$ evaluated at *negative* integers.

Theorem 2.1.7 (Macdonald's Reciprocity) *Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a lattice polytope of dimension d. The number of integer points in the relative interior of $t\mathcal{P}$ is given by*

$$\operatorname{ehr}(\mathcal{P}^\circ, t) := (-1)^d \operatorname{ehr}(\mathcal{P}, -t).$$

This is part of a family of *combinatorial-reciprocity* theorems, see the book [BS18] by Beck and Sanyal for a beautiful and detailed treatment of this topic. We formulate now some of the consequences of the reciprocity theorem.

Proposition 2.1.8 *Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a lattice polytope of dimension d. Then*

$$\deg h^*(\mathcal{P}, x) = m,$$

where $(d-m+1)\mathcal{P}$ is the smallest integer dilation of $\mathcal{P}$ that contains a lattice point in its relative interior.

Also, we have an interpretation for the value of the h^*-polynomial evaluated at $x = 1$.

Proposition 2.1.9 *Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a lattice polytope of dimension d. Then*

$$h^*(\mathcal{P}, 1) = \frac{1}{d!}\operatorname{vol}(\mathcal{P}).$$

2.2 Ehrhart positivity and some conjectures

Some of the principal open problems in Ehrhart theory are given by the following questions:

- Which polynomials arise as the Ehrhart polynomial of some polytope?
- When does the Ehrhart polynomial of a polytope have positive coefficients?
- Are there combinatorial interpretations for the coefficients of the h^*-polynomial of a polytope?
- When is the h^*-vector of a polytope unimodal?
- When do two polytopes have the same Ehrhart polynomial?

Providing answers to these questions is a challenging task even in low dimensions. In this dissertation we will be primarily concerned with the question regarding the positivity of the Ehrhart coefficients of polytopes.

We have established that

$$\operatorname{ehr}(\mathcal{P}, t) = \operatorname{vol}(\mathcal{P})t^d + \frac{1}{2}\operatorname{vol}(\partial\mathcal{P})t^{d-1} + \ldots + 1, \tag{2.2}$$

where vol is the function that associates to a lattice polytope its relative volume.

Equation (2.2) exhibits explicitly that the coefficients of degrees d, $d-1$ and 0 of the Ehrhart polynomial of a polytope $\mathcal{P}$ of dimension d are always positive. Unfortunately, the coefficients accompanying the terms of degrees $1, \ldots, d-2$ are not as well-understood. Although it is possible to derive general formulas for each of the coefficients of $\operatorname{ehr}(\mathcal{P}, t)$, they are of a much more complicated nature.

As we saw in equation (2.1), some of the remaining coefficients can be negative when one deals with a general lattice polytope. Even when one restricts to the family of 0/1-polytopes, it is possible to construct examples of polytopes that have negative Ehrhart coefficients [LT19].

When $\mathcal{P}$ is an integral polytope such that $\operatorname{ehr}(\mathcal{P}, t)$ has positive coefficients, we say that $\mathcal{P}$ is *Ehrhart positive*. A main reference about Ehrhart positivity is Fu Liu's survey [Liu19].

One of the main open problems in this framework was a conjecture posed in 2007 by De Loera, Haws and Köppe [DHK09].

Conjecture 2.2.1 *If M is a matroid and $\mathcal{P}(M)$ is its basis polytope, then $\mathcal{P}(M)$ is Ehrhart positive.*

In 2015 Castillo and Liu posed the following stronger conjecture [CL18].

Conjecture 2.2.2 *If $\mathcal{P}$ is an integral generalized permutohedron, then $\mathcal{P}$ is Ehrhart positive.*

Since by Theorem 1.2.22 every independence matroid polytope is integrally equivalent to an integral generalized permutohedron, Conjecture 2.2.2 implies that also all independence matroid polytopes are Ehrhart positive.

There was much evidence to support these two conjectures. In [Pos09], Postnikov gave a proof of the fact that all the members of a quite large family of generalized permutohedra, which he called $\mathcal{Y}$*-generalized permutohedra*, were Ehrhart positive. This family, although very general, does not contain the family of matroid polytopes as a subclass.

In the same paper in which they conjectured the Ehrhart positivity of generalized permutohedra, Castillo and Liu proved that the Ehrhart coefficients of degree $d-2$ and $d-3$ of a generalized permutohedron are always positive. In [CL21] they also proved that the linear coefficient is always positive. The following is a summary of their results.

Proposition 2.2.3 *Let $\mathcal{P}$ be an integral generalized permutohedron of dimension $d \geq 3$, and let $\operatorname{ehr}(\mathcal{P}, t)$ denote its Ehrhart polynomial. Then*

- $[t^1] \operatorname{ehr}(\mathcal{P}, t)$ *is positive.*
- $[t^{d-2}] \operatorname{ehr}(\mathcal{P}, t)$ *is positive.*
- $[t^{d-3}] \operatorname{ehr}(\mathcal{P}, t)$ *is positive.*

Also, if $\dim \mathcal{P} = d \leq 6$, then $\mathcal{P}$ is Ehrhart positive.

Another proof of the positivity of the linear term was independently found by Jochemko and Ravichandran in [JR21].

With a direct software computation, it was also verified that for every matroid M with at most 9 elements, both its basis polytope $\mathcal{P}(M)$ and its independence polytope $\mathcal{P}_{\mathrm{I}}(M)$ were Ehrhart positive.

More evidence was provided by two of our main results. In Chapter 3 we will prove the following.

Theorem 2.2.4 *If M is a uniform matroid, then $\mathcal{P}(M)$ and $\mathcal{P}_{\mathrm{I}}(M)$ are Ehrhart positive.*

In Chapter 4 we will also prove that

Theorem 2.2.5 *If M is a matroid of rank 2, then $\mathcal{P}(M)$ is Ehrhart positive.*

However, in spite of all the aforementioned evidence, we will disprove Conjectures 2.2.1 and 2.2.2. In fact, we will show that there exist counterexamples with any possible rank greater than 2.

Theorem 2.2.6 *For every $k \geq 3$ there exists a connected matroid of rank 3 such that $\mathcal{P}(M)$ is not Ehrhart positive.*

2.3 Basic results for matroids

The goal of this section is to establish a few very basic results on the Ehrhart theory of matroids. By doing this, we will be able to reveal some of the most fundamental facts about these polytopes and justify that sometimes we pay attention only to connected matroids.

Theorem 2.3.1 *Let $M = M_1 \oplus M_2$ be a matroid. Then*

$$\operatorname{ehr}(\mathcal{P}(M), t) = \operatorname{ehr}(\mathcal{P}(M_1), t) \cdot \operatorname{ehr}(\mathcal{P}(M_2), t),$$

and

$$\operatorname{ehr}(\mathcal{P}_{\mathsf{I}}(M), t) = \operatorname{ehr}(\mathcal{P}_{\mathsf{I}}(M_1), t) \cdot \operatorname{ehr}(\mathcal{P}_{\mathsf{I}}(M_2), t).$$

Proof. This is a consequence of Proposition 1.2.7 and the fact that the Ehrhart polynomial of a product of polytopes is the product of the Ehrhart polynomials of each of them. □

The following result is useful for example to predict what the degree of the Ehrhart polynomial of a matroid polytope will be.

Proposition 2.3.2 *Let M be a loopless matroid on n elements. Then*

(a) $\deg \operatorname{ehr}(\mathcal{P}(M), t) = n - c(M)$*, where $c(M)$ denotes the number of connected components of M.*

(b) $\deg \operatorname{ehr}(\mathcal{P}_{\mathsf{I}}(M), t) = n$.

Proof. Both statements are consequence of the fact that the degree of the Ehrhart polynomial of a polytope is its dimension. Recall that the dimension of the basis polytope of a matroid on n elements is precisely $n - c(M)$ by Theorem 1.2.9, while the dimension of the independence polytope is n (when M is loopless) by Proposition 1.2.13. □

Another operation that behaves well with Ehrhart polynomials is matroid duality.

Proposition 2.3.3 *Let M be a matroid. Then*

$$\operatorname{ehr}(\mathcal{P}(M), t) = \operatorname{ehr}(\mathcal{P}(M^*), t).$$

Proof. Proposition 1.2.6 tells us that $\mathcal{P}(M)$ and $\mathcal{P}(M^*)$ are one obtained from of the other by an involution that is an integral equivalence. Hence, their Ehrhart polynomials must coincide. □

Remark 2.3.4. Let M a matroid of cardinality n and rank k. Since the Ehrhart polynomial of $\mathcal{P}(M)$ is equal to that of $\mathcal{P}(M^*)$, when speaking about basis polytopes, it suffices to restrict ourselves only to the case on which $2k \leq n$.

It is natural to ask about the degree of the h^*-polynomials. A somewhat brief description of such degrees exists for independence polytopes of matroids.

Proposition 2.3.5 *Let M be a loopless matroid on $E = \{1, \dots, n\}$. Let m be the minimum positive integer such that $m\mathcal{P}_{\mathsf{I}}^{\circ}(M) \cap \mathbb{Z}^n \neq \varnothing$. Then*

$$m = 1 + \max_{F \in \mathcal{F}(M)} \left\lfloor \frac{|F|}{\operatorname{rk}(F)} \right\rfloor. \tag{2.3}$$

Moreover, the maximum can be taken over all inseparable flats.

Proof. Let us call $m' = \max_{F\in\mathcal{F}(M)} \left\lfloor \frac{|F|}{\operatorname{rk}(F)} \right\rfloor$ and let us call m the least integer such that $m\mathcal{P}_{\mathsf{I}}(M)$ contains an interior lattice point.

Assume that $m\mathcal{P}_{\mathsf{I}}^{\circ}(M)$ contains a lattice point $p = (p_1, \dots, p_n)$ in its interior. We claim that the point $e_E = (1, \dots, 1)$ is in the interior of the polytope too. Observe that by the description of $\mathcal{P}_{\mathsf{I}}(M)$ given in 1.2.15 and by Remark 1.2.17, we have that $x_i \geq 0$ is a facet. In particular it has to be $p_i > 0$ for each $i = 1, \dots, n$. Now, observe that if we decrease any coordinate of p while leaving it positive, the point still satisfies all the inequalities that define $\mathcal{P}_{\mathsf{I}}(M)$. This proves that $e_E \in \mathcal{P}_{\mathsf{I}}^{\circ}(M)$.

Now, the fact that $e_E \notin (m-1)\mathcal{P}_{\mathsf{I}}^{\circ}(M)$ gives;

$$|F| = \sum_{i\in F} 1 \geq (m-1)\operatorname{rk}(F) \text{ for some (inseparable) } F \in \mathcal{F}(M).$$

Choose any such a flat. Observe that $\left\lfloor \frac{|F_1|}{\operatorname{rk}(F_1)} \right\rfloor \leq m'$. So that $m - 1 \leq m'$. Also, since $e_E \in m\mathcal{P}_{\mathsf{I}}^{\circ}(M)$, we have that

$$|F| = \sum_{i\in F} 1 < m \cdot \operatorname{rk}(F) \text{ for all (inseparable) } F \in \mathcal{F}(M).$$

This yields that $m' < m$. It follows that $m' + 1 = m$, as we claimed. □

Corollary 2.3.6 *If M is a loopless matroid on n elements, the degree of the h^*-polynomial of $\mathcal{P}_{\mathsf{I}}(M)$ is exactly $n - m + 1$ where m is given by* (2.3).

Proof. It follows from Proposition 2.1.8 and the result above. □

Corollary 2.3.7 *Let M be a loopless matroid. Let m be as in* (2.3). *Then* $\operatorname{ehr}(\mathcal{P}_{\mathsf{I}}(M), t)$ *is divisible by the polynomial* $(t+1)(t+2)\cdots(t+m-1)$.

Proof. Since $\operatorname{ehr}(\mathcal{P}_{\mathsf{I}}^{\circ}(M), j) = 0$ for $j = 1, \dots, m-1$, by the Macdonald's reciprocity, we get that such values of j are zeros of $\operatorname{ehr}(\mathcal{P}_{\mathsf{I}}(M), t)$. □

CHAPTER 3

The Ehrhart polynomial of the hypersimplex

The *leit motiv* of this thesis are the Ehrhart polynomials of matroid polytopes. In some sense, the first nontrivial example of matroid polytope that one encounters is that of hypersimplices. Recall that in Proposition 1.2.5 we proved that the hypersimplex $\Delta_{k,n}$ is the basis polytope of the uniform matroid $U_{k,n}$. In this chapter we will prove that all hypersimplices are Ehrhart positive and that the same holds for the independence polytopes of all uniform matroids. Moreover, we will give a combinatorial formula for each of the coefficients of $\operatorname{ehr}(\Delta_{k,n}, t)$. This was classified as an open problem in Richard Stanley's book *Enumerative Combinatorics* [Sta12, Ch. 4, Problem 62].

It is worth noting that according to [Sta77] the calculation of the leading coefficient of these polynomials (it is, the normalized volume of the hypersimplex) dates back to Laplace, though apparently he did not do it explicitly. The leading coefficients of $\operatorname{ehr}(\Delta_{k,n}, t)$, after multiplying by $(n-1)!$, are what in the literature mathematicians call *Eulerian numbers* and they are usually denoted by $A(n-1, k-1)$. This quantity counts the number of permutations on $n-1$ elements with exactly $k-1$ descents [Sta12, GKP94]. Thanks to the fact that the second highest coefficient of an Ehrhart polynomial is half the sum of volumes of the facets of the polytope, and the facets of a hypersimplex are smaller hypersimplices, there existed also a combinatorial interpretation of the second highest coefficient in terms of Eulerian numbers. The remaining coefficients remained elusive.

There exists interpretations for the coefficients of the h^*-polynomial of the hypersimplex. In [Li12], Li introduced the half-open hypersimplices and proved a conjecture of Stanley on the interpretation of their h^*-vector. Also, more recently Early conjectured and Kim proved [Ear17, Kim20] a different combinatorial interpretation of the coefficients h^*-polynomial of all hypersimplices. We will not say much about the h^*-polynomial of hypersimplices here, but we will discuss ramifications and further directions of research in the last chapter of this dissertation.

3.1 Katzman's formula

In [Kat05] Katzman found an explicit formula for the Ehrhart polynomial of $\Delta_{k,n}$, in the context of algebras of Veronese type. In [DHK09, Lemma 29] De Loera et al. used this formula to prove that hypersimplices $\Delta_{2,n}$ have an Ehrhart polynomial with positive coefficients. Their proof is based on inequalities using properties of the binomial coefficients. We will prove Katzman's formula by using generating functions and we will use it to deduce the Ehrhart positivity of all hypersimplices.

Theorem 3.1.1 *The Ehrhart polynomial of the hypersimplex $\Delta_{k,n}$ is given by*

$$\operatorname{ehr}(\Delta_{k,n}, t) = \sum_{j=0}^{k-1} (-1)^j \binom{n}{j} \binom{(k-j)t + n - 1 - j}{n-1}. \tag{3.1}$$

It is not at all apparent from this formula that the coefficients of $\operatorname{ehr}(\Delta_{k,n}, t)$ are positive. Indeed the alternating factor $(-1)^j$ and the fact that the variable t appears inside a binomial coefficient which in turn for $j > 1$ is a polynomial with some negative coefficients do not permit us to see this fact directly. Before proving Theorem 3.1.1 we establish a useful Lemma.

Lemma 3.1.2 *If $1 \leq k \leq n-1$ and $t \geq 0$, then the coefficient of x^{kt} in the polynomial $(1 + x + x^2 + \cdots + x^t)^n$ is exactly* $\operatorname{ehr}(\Delta_{k,n}, t)$.

Proof. By definition, the polynomial $\operatorname{ehr}(\Delta_{k,n}, t)$ counts the number of elements in the set $t\Delta_{k,n} \cap \mathbb{Z}^n$. This set can be rewritten as:

$$\left\{ y \in \{0, 1, \ldots, t\}^n : \sum_{i=1}^{n} y_i = kt \right\}.$$

But notice that the coefficient of x^{kt} in the product

$$(1 + x + x^2 + \cdots + x^t)^n = \underbrace{(1 + x + x^2 + \cdots + x^t) \cdots (1 + x + x^2 + \cdots + x^t)}_{n \text{ times}}$$

is exactly the number of ways of choosing a sequence of n elements in the set $\{0, 1, \ldots, t\}$ in such a way that their sum is exactly kt. That is exactly the cardinality of our set. $\square$

Recall that if one has a (formal) power series $f(x) := \sum_{j=0}^{\infty} a_j x^j$, it is customary to use the notation $[x^\ell] f(x) := a_\ell$.

Proof of Theorem 3.1.1. We will use generating functions to compute the coefficient of x^{kt} in $(1 + x + \cdots + x^t)^n$ and then we will use the preceding Lemma. Notice that

$$[x^{kt}] \left(1 + x + \cdots + x^t\right)^n = [x^{kt}] \left(\frac{1 - x^{t+1}}{1 - x} \right)^n$$

$$= [x^{kt}]\left((1 - x^{t+1})^n \cdot \frac{1}{(1-x)^n}\right)$$

So writing $(1 - x^{t+1})^n = \sum_{j=0}^{n}(-1)^j\binom{n}{j}x^{(t+1)j}$ and $\frac{1}{(1-x)^n} = \sum_{j=0}^{\infty}\binom{n-1+j}{n-1}x^j$, the coefficient of x^{kt} in this product can be computed as a convolution:

$$\sum_{j=0}^{k-1}(-1)^j\binom{n}{j}\binom{n-1+(k-j)t-j}{n-1},$$

where the sum ends in $k-1$ since in the first of our two formal series we have $x^{(t+1)j}$ and we are computing the coefficient of x^{kt}. Also, the second binomial coefficient in our expression comes from the fact that $(t+1)j + ((k-j)t - j) = kt$. $\square$

3.2 Weighted Lah Numbers

In this section we develop some useful tools to prove the Ehrhart positivity of $\Delta_{k,n}$. We recall the definition of the *Lah numbers* (also known as *Stirling Numbers of the 3rd kind*).

Definition 3.2.1. The *Lah number* $L(n,m)$ is defined as the number of ways of partitioning the set $\{1, 2, \ldots, n\}$ in exactly m linearly ordered blocks. We will denote the set of all such partitions by $\mathcal{L}(n,m)$.

Example 3.2.2. $L(3,2) = 6$ because we have the following possible partitions:

$$\{(1,2),(3)\}, \{(2,1),(3)\},$$

$$\{(1,3),(2)\}, \{(3,1),(2)\},$$

$$\{(2,3),(1)\}, \{(3,2),(1)\}.$$

If π is a partition of $\{1,\ldots,n\}$ in m linearly ordered blocks, for any of these blocks b, we will write $b \in \pi$. So, for example $(2,3) \in \{(2,3),(1)\}$. Also, we will use the notation $|b|$ to denote the number of elements in b.

Remark 3.2.3. We have the equality $L(n,m) = \frac{n!}{m!}\binom{n-1}{m-1}$. This can be proven easily by a combinatorial argument as follows. Order the n numbers on the set in any fashion. To get the partition we can put $m-1$ divisions in any of the $n-1$ spaces between two consecutive numbers. Then divide by $m!$, the number of ways of ordering all the blocks.

There already exist some generalizations of these numbers. We will introduce a new one that we will call *weighted Lah numbers*.

Definition 3.2.4. Let π be a partition of the set $\{1,\ldots,n\}$ into m linearly ordered blocks. We define the *weight of* π by the following formula:

$$w(\pi) := \sum_{b\in\pi} w(b),$$

where $w(b)$ is the number of elements in b that are smaller (as positive integers) than the first element in b.

Example 3.2.5. Among the 6 partitions that we have seen that exist of $\{1,2,3\}$ into 2 blocks, we have:

$$w(\{(1,2),(3)\}) = 0+0 = 0, \quad w(\{(2,1),(3)\}) = 1+0 = 1,$$

$$w(\{(1,3),(2)\}) = 0+0 = 0, \quad w(\{(3,1),(2)\}) = 1+0 = 1,$$

$$w(\{(2,3),(1)\}) = 0+0 = 0, \quad w(\{(3,2),(1)\}) = 1+0 = 1.$$

Note that there are exactly 3 of these partitions of weight 0 and exactly 3 of weight 1.

Definition 3.2.6. We define the *weighted Lah Numbers* $W(\ell, n, m)$ as the number of partitions of weight ℓ of $\{1, \dots, n\}$ into exactly m linearly ordered blocks.

Example 3.2.7. Rephrasing the conclusion of the Example 3.2.5, we have that $W(0,3,2) = 3$ and $W(1,3,2) = 3$.

m \ ℓ	0	1	2	3	4
1	24	24	24	24	24
2	50	70	70	50	
3	35	50	35		
4	10	10			
5	1				

Table 3.1: $W(\ell, 5, m)$

m \ ℓ	0	1	2	3	4	5
1	120	120	120	120	120	120
2	274	404	444	404	274	
3	225	375	375	225		
4	85	130	85			
5	15	15				
6	1					

Table 3.2: $W(\ell, 6, m)$

The set of all partitions of $\{1, \dots, n\}$ into m linearly ordered blocks and weight ℓ will be denoted by $\mathcal{W}(\ell, n, m)$.

Remark 3.2.8. Observe that $W(\ell, n, m) \neq 0$ only for $0 \leq \ell \leq n-m$. This is because the maximum weight can be obtained by ordering every block in such a way that its maximum element is on the first position. Also, we have the following:

$$W(0, n, m) = \begin{bmatrix} n \\ m \end{bmatrix},$$

where the brackets denote the *(unsigned) Stirling numbers of the first kind* [GKP94]. This can be proven combinatorially by noticing that for every permutation with exactly m cycles, we can present it in a unique way as a partition of $\{1, \dots, n\}$ into m linearly ordered blocks having every block its minimum element in the first position.

Remark 3.2.9. We have symmetry, namely:

$$W(\ell, n, m) = W(n - m - \ell, n, m).$$

This equality is a consequence of the fact that for $\pi \in \mathcal{W}(\ell, n, m)$ we can associate bijectively an element $\pi' \in \mathcal{W}(n - m - \ell, n, m)$ as follows. In π interchange the positions of 1 and n, of 2 and $n - 1$, and so on. What one gets is exactly a partition of weight $n - m - \ell$.

Some recurrences

It is possible to obtain many recurrences to compute $W(\ell, n, m)$ recursively. For instance we include the following:

Proposition 3.2.10 *The following recurrence holds for $n, m \geq 2$:*

$$W(\ell, n, m) = (n-1)W(\ell - 1, n - 1, m) + \sum_{j=0}^{n-1} \binom{n-1}{j} j!W(\ell, n - 1 - j, m - 1).$$

Proof. Every $\pi \in \mathcal{W}(\ell, n, m)$ has the number 1 inside a block. If this number is *not* the first element of its block, this means that if we remove it from π we end up getting an element of $\mathcal{W}(\ell - 1, n - 1, m)$ (with every element shifted by one). Analogously, we can pick an element of $\mathcal{W}(\ell - 1, n - 1, m)$ (which we think of as having every element shifted by one) and reconstruct an element of $\mathcal{W}(\ell, n, m)$ by adjoining the element 1 in such a way that it is not the first element of a block. There are $n - 1$ possibilities of where to put the number 1 to get an element of $\mathcal{W}(\ell, n, m)$. So we get the first summand.

The remaining cases to consider are those in which 1 is the first element of its block. In this case we choose j elements to be in this block, and in every possible order of these elements, the block will always have weight 0. So the remaining $n - j - 1$ elements will have to be arranged in $m - 1$ blocks of total weight ℓ. □

Remark 3.2.11. The last proposition tells us that if we make the subtraction $W(\ell, n, m) - (n-1)W(\ell, n-1, m)$ we end up getting an expression for which the sum cancels out to give just the recurrence:

$$\begin{aligned} W(\ell, n, m) &= (n-1)W(\ell - 1, n - 1, m) + (n-1)W(\ell, n - 1, m) \\ &\quad + W(\ell, n - 1, m - 1) - (n-1)(n-2)W(\ell - 1, n - 2, m). \end{aligned}$$

We take the opportunity to state and prove two additional results that we will need in the sequel.

Proposition 3.2.12 *The following recurrence holds for $n, m \geq 2$:*

$$W(\ell, n, m) = (n-1)W(\ell, n - 1, m) + \sum_{j=0}^{n-1} \binom{n-1}{j} j!W(\ell - j, n - 1 - j, m - 1).$$

Proof. Every $\pi \in \mathcal{W}(\ell, n, m)$ has the number n inside a block. If this number is *not* the first element of its block, this means that if we remove it from π we end up getting an element of $\mathcal{W}(\ell, n-1, m)$. Analogously, we can pick an element of $\mathcal{W}(\ell, n-1, m)$ and reconstruct an element of $\mathcal{W}(\ell, n, m)$ by adjoining the element n in such a way that it is not the first element of a block. There are $n-1$ possibilities of where to put the number n to get an element of $\mathcal{W}(\ell, n, m)$. So we get the first summand.

The remaining cases to consider are those in which n is the first element of its block. In this case we choose j elements to be in this block, and in every possible order of these elements, the block will always have weight j. So the remaining $n-j-1$ elements will have to be arranged in $m-1$ blocks of total weight $\ell - j$. □

Corollary 3.2.13 *For each* $2 \le m \le n$ *and* $0 \le \ell \le n-m$ *one has:*

$$W(\ell, n, m) > (n-1)W(\ell, n-1, m).$$

Proof. From the preceding Proposition, it suffices to show that at least one term of the sum:

$$\sum_{j=0}^{n-1} \binom{n-1}{j} j! W(\ell - j, n-1-j, m-1),$$

is nonzero. Notice that taking $j = \ell$ in the above sum yields the term:

$$\binom{n-1}{\ell} \ell! W(0, n-1-\ell, m-1).$$

Notice that $W(0, n-1-\ell, m-1) > 0$ under the constraints on ℓ, n, and m. In fact, by Remark 3.2.8 we know that it is equal to an unsigned Stirling number of the first kind. □

A generating function for $W(\ell, n, m)$

We establish now a bivariate generating function for $W(\ell, n, m)$ for a fixed m.

Theorem 3.2.14 *We have the equality:*

$$W(\ell, n, m) = \frac{n!}{m!}[x^n s^\ell]\left(\tfrac{1}{(1-s)^m}\left(\log\left(\tfrac{1}{1-x}\right) - \log\left(\tfrac{1}{1-sx}\right)\right)^m\right)$$

Proof. Notice that it suffices to prove that

$$W(\ell, n, m) = \frac{n!}{m!}[x^n s^\ell]\left(\sum_{k=1}^{\infty} \frac{x^k}{k}(1 + s + \cdots + s^{k-1})\right)^m. \tag{3.2}$$

This is because using the formula for the geometric series, the sum in the parentheses can be rewritten as $\frac{1}{1-s}\left(\sum_{k=1}^{\infty} \frac{x^k}{k} - \sum_{k=1}^{\infty} \frac{(sx)^k}{k}\right)$ which in turn is just

$$\frac{1}{1-s}\left(\log\left(\frac{1}{1-x}\right) - \log\left(\frac{1}{1-sx}\right)\right)$$

which gives the desired result. Now, to prove (3.2) we proceed as follows. First notice that

$$m!W(\ell,n,m) = \sum_{\widetilde{\pi}} \sum_{\substack{(j_1,\dots,j_m)\in\mathbb{Z}^m \\ j_1+\cdots+j_m=\ell \\ 0\le j_i<|b_i|}} \prod_{i=1}^{m} (|b_i|-1)! \tag{3.3}$$

where the first sum runs over all the orderings $\widetilde{\pi} = (b_1,\dots,b_m)$ of all elements $\pi = \{b_1,\dots,b_m\} \in \mathcal{L}(n,m)$. This comes from the fact that for every such $\widetilde{\pi}$, if we choose how much weight to assign to each of the blocks, each block has its first element determined, and the remaining elements can be reordered in any fashion. Of course, this way we count every element of $\mathcal{W}(\ell,n,m)$ exactly $m!$ times. Taking out the product out of the second sum above, we get:

$$\begin{aligned}
m!W(\ell,n,m) &= \sum_{\widetilde{\pi}} \left(\prod_{i=1}^{m}(|b_i|-1)!\right) \left|\left\{(j_1,\dots,j_m)\in\mathbb{Z}^m : \sum_{i=1}^{m} j_i = \ell, 0\le j_i < |b_i|\right\}\right| \\
&= \sum_{\widetilde{\pi}} \left(\prod_{i=1}^{m}(|b_i|-1)!\right) [s^\ell] \left(\prod_{i=1}^{m}\sum_{j=0}^{|b_i|-1} s^j\right) \\
&= [s^\ell] \sum_{\widetilde{\pi}} \left(\prod_{i=1}^{m}(|b_i|-1)! \sum_{j=0}^{|b_i|-1} s^j\right)
\end{aligned}$$

Notice that the term inside the last sum does not take into account the whole element $\widetilde{\pi} = (b_1,\dots,b_m)$, but only the size $|b_i|$ of each block. Thus, if we fix the sizes $|b_1|,\dots,|b_m|$ of the blocks, we can recover exactly how many elements $\widetilde{\pi}$ have blocks of such sizes. Using multinomial coefficients, and abusing notation to write $b_i = |b_i|$:

$$\begin{aligned}
m!W(\ell,n,m) &= [s^\ell] \sum_{\substack{(b_1,\dots,b_m)\in\mathbb{Z}^m \\ b_1+\cdots+b_m=n \\ b_i\ge 0}} \binom{n}{b_1,\dots,b_m} \left(\prod_{i=1}^{m}(b_i-1)! \sum_{j=0}^{b_i-1} s^j\right) \\
&= [s^\ell] \sum_{\substack{(b_1,\dots,b_m)\in\mathbb{Z}^m \\ b_1+\cdots+b_m=n \\ b_i\ge 0}} n! \left(\prod_{i=1}^{m}\frac{1}{b_i}\sum_{j=0}^{b_i-1} s_j\right) \\
&= [s^\ell x^n] n! \left(\sum_{k=1}^{n} \frac{x^k}{k}(1+s+\cdots+s^{k-1})\right)^m,
\end{aligned}$$

which proves (3.2). $\square$

Corollary 3.2.15 *For all ℓ, n, m one has:*

$$W(\ell,n,m) = \sum_{j=0}^{\ell}\sum_{i=0}^{n-m}(-1)^{i+j}\binom{n}{j}\binom{m+\ell-j-1}{m-1}\left[\begin{matrix} j \\ j-i\end{matrix}\right]\left[\begin{matrix} n-j \\ m+i-j\end{matrix}\right].$$

Proof. From the exponential generating function of the Stirling numbers of the first kind [GKP94, pg. 351] one has:

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{\alpha!}{\beta!}[x^\alpha]\left(\log\left(\tfrac{1}{1-x}\right)\right)^\beta .$$

Now, using Theorem 3.2.14, we have the chain of equalities:

$$\begin{aligned}
W(\ell,n,m) &= \frac{n!}{m!}[x^n s^\ell]\left(\tfrac{1}{(1-s)^m}\left(\log\left(\tfrac{1}{1-x}\right)-\log\left(\tfrac{1}{1-sx}\right)\right)^m\right)\\
&= \frac{n!}{m!}[x^n s^\ell]\left(\frac{1}{(1-s)^m}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\left(\log\left(\tfrac{1}{1-x}\right)\right)^{m-k}\left(\log\left(\tfrac{1}{1-sx}\right)\right)^k\right)\\
&= n![x^n s^\ell]\left(\frac{1}{(1-s)^m}\sum_{k=0}^{m}(-1)^k\frac{\log\left(\tfrac{1}{1-x}\right)^{m-k}}{(m-k)!}\frac{\log\left(\tfrac{1}{1-sx}\right)^{k}}{k!}\right)\\
&= n![s^\ell]\left(\frac{1}{(1-s)^m}\sum_{k=0}^{m}(-1)^k\sum_{j=0}^{n}[x^{n-j}]\left(\frac{\log\left(\tfrac{1}{1-x}\right)^{m-k}}{(m-k)!}\right)[x^j]\left(\frac{\log\left(\tfrac{1}{1-sx}\right)^{k}}{k!}\right)\right)\\
&= n![s^\ell]\left(\frac{1}{(1-s)^m}\sum_{k=0}^{m}(-1)^k\sum_{j=k}^{n-m+k}[x^{n-j}]\left(\frac{\log\left(\tfrac{1}{1-x}\right)^{m-k}}{(m-k)!}\right)[x^j]\left(\frac{\log\left(\tfrac{1}{1-sx}\right)^{k}}{k!}\right)\right)\\
&= n![s^\ell]\left(\frac{1}{(1-s)^m}\sum_{k=0}^{m}(-1)^k\sum_{j=k}^{n-m+k}\frac{1}{(n-j)!}\begin{bmatrix}n-j\\ m-k\end{bmatrix}\frac{1}{j!}s^j\begin{bmatrix}j\\ k\end{bmatrix}\right)\\
&= [s^\ell]\left(\frac{1}{(1-s)^m}\sum_{k=0}^{m}\sum_{j=k}^{n-m+k}(-1)^k\binom{n}{j}s^j\begin{bmatrix}n-j\\ m-k\end{bmatrix}\begin{bmatrix}j\\ k\end{bmatrix}\right)\\
&= [s^\ell]\left(\frac{1}{(1-s)^m}\sum_{j=0}^{n}\sum_{k=j-m+n}^{j}(-1)^k\binom{n}{j}s^j\begin{bmatrix}n-j\\ m-k\end{bmatrix}\begin{bmatrix}j\\ k\end{bmatrix}\right)\\
&= [s^\ell]\left(\frac{1}{(1-s)^m}\sum_{j=0}^{n}\sum_{i=0}^{n-m}(-1)^{j-i}\binom{n}{j}\begin{bmatrix}n-j\\ m+i-j\end{bmatrix}\begin{bmatrix}j\\ j-i\end{bmatrix}s^j\right)\\
&= \sum_{j=0}^{\ell}\sum_{i=0}^{n-m}(-1)^{j-i}\binom{n}{j}\binom{m-1+\ell-j}{m-1}\begin{bmatrix}n-j\\ m+i-j\end{bmatrix}\begin{bmatrix}j\\ j-i\end{bmatrix}
\end{aligned}$$

where in the fifth equation we changed the limits from $0 \le j \le n$ to $k \le j \le n-m+k$ given that the coefficients of the first factor inside the sum are zero for degree $n-j < m-k$ and the coefficients of the second factor are zero for $j < k$. $\square$

3.3 The Ehrhart positivity of the hypersimplex

For $0 \le m \le n-1$, we will call $e_{k,n,m}$ the coefficient of t^m in the polynomial $\mathrm{ehr}(\Delta_{k,n},t)$. Our aim is to show that all these $e_{k,n,m}$ are positive.

For a, b, u integer numbers such that $u \geq 0$, we will denote $P^u_{a,b}$ the sum of all possible products of u different integer numbers chosen in the interval of integers $[a, b]$. This is:

$$P^u_{a,b} := \sum_{a \leq x_1 < \ldots < x_u \leq b} x_1 \cdots x_u.$$

It is easy to see that for $a = 1$ one gets:

$$P^u_{1,b} = \begin{bmatrix} b+1 \\ b+1-u \end{bmatrix}. \tag{3.4}$$

Lemma 3.3.1 *The following formula holds:*

$$e_{k,n,m} = \frac{1}{(n-1)!} \sum_{j=0}^{k-1} \sum_{i=0}^{n-m-1} (-1)^{i+j} \binom{n}{j} (k-j)^m \begin{bmatrix} n-j \\ m+1+i-j \end{bmatrix} \begin{bmatrix} j \\ j-i \end{bmatrix}.$$

Proof. We will work with the formula (3.1). Observe that

$$\begin{aligned} [t^m] \binom{(k-j)t+n-1-j}{n-1} &= \frac{1}{(n-1)!} [t^m] \left(((k-j)t+n-1-j) \cdots ((k-j)t+1-j)\right) \\ &= \frac{1}{(n-1)!} (k-j)^m P^{n-1-m}_{1-j,n-1-j}, \end{aligned}$$

Observe that one has the following equality:

$$\begin{aligned} P^{n-1-m}_{1-j,n-1-j} &= \sum_{i=0}^{n-m-1} P^i_{1-j,-1} P^{n-1-m-i}_{1,n-1-j} \\ &= \sum_{i=0}^{n-m-1} (-1)^i P^i_{1,j-1} P^{n-1-m-i}_{1,n-1-j}. \end{aligned}$$

Therefore, using (3.4) we have that

$$P^{n-m-1}_{1-j,n-1-j} = \sum_{i=0}^{n-m-1} (-1)^i \begin{bmatrix} j \\ j-i \end{bmatrix} \begin{bmatrix} n-j \\ m+1+i-j \end{bmatrix},$$

so, in particular,

$$[t^m] \binom{(k-j)t+n-1-j}{n-1} = \frac{1}{(n-1)!} \sum_{i=0}^{n-m-1} (-1)^i (k-j)^m \begin{bmatrix} j \\ j-i \end{bmatrix} \begin{bmatrix} n-j \\ m+1+i-j \end{bmatrix}.$$

The result follows easily from (3.1) and this last identity. □

Remark 3.3.2. If we use the shorter name

$$f_{j,n,m} := \sum_{i=0}^{n-m-1} (-1)^i \begin{bmatrix} j \\ j-i \end{bmatrix} \begin{bmatrix} n-j \\ m+1+i-j \end{bmatrix},$$

we can rewrite the formula of Lemma 3.3.1 as follows:

$$e_{k,n,m} = \frac{1}{(n-1)!}\sum_{j=0}^{k}(-1)^j \binom{n}{j} (k-j)^m f_{j,n,m}, \tag{3.5}$$

where we changed the upper limit of the sum since, when $j = k$, we are adding 0.

Now we are ready to state and prove the main result of this chapter. We establish a formula for the coefficients of the Ehrhart polynomial of all hypersimplices by using the weighted Lah numbers and the Eulerian numbers.

Theorem 3.3.3 *For all hypersimplices $\Delta_{k,n}$ and $0 \le m \le n-1$, we have that*

$$[x^m]\,\mathrm{ehr}(\Delta_{k,n}, t) = \frac{1}{(n-1)!}\sum_{\ell=0}^{k-1} W(\ell, n, m+1)A(m, k-\ell-1),$$

In particular all hypersimplices are Ehrhart positive.

Proof. From equation (3.5) we can see that

$$e_{k,n,m} = \frac{1}{(n-1)!}[x^k]F_{n,m}(x)\cdot G_m(x),$$

where $F_{n,m}(x) := \sum_{j=0}^{n}(-1)^j\binom{n}{j} f_{j,n,m}x^j$ and $G_m(x) := \sum_{j=0}^{\infty} j^m x^j$. It is a well known consequence of the *Worpitzky Identity* [GKP94] that

$$G_m(x) = \frac{1}{(1-x)^{m+1}}\sum_{j=0}^{m} A(m,j)x^{j+1},$$

where $A(m, j)$ an Eulerian number, the number of permutations of m elements with exactly j descents.

So we have that the product $F_{n,m}(x)\cdot G_m(x)$ is equal to:

$$\frac{1}{(1-x)^{m+1}}F_{n,m}(x)\sum_{j=0}^{m} A(m,j)x^{j+1}.$$

We compute the product of the first two factors. Let:

$$C_{n,m}(x) := \frac{1}{(1-x)^{m+1}}F_{n,m}(x),$$

and observe that

$$\begin{aligned}
[x^\ell]C_{n,m}(x) &= [x^\ell]\left(\frac{1}{(1-x)^{m+1}}F_{n,m}(x)\right)\\
&= \sum_{j=0}^{\ell}(-1)^j\binom{n}{j} f_{j,n,m}\binom{m+\ell-j}{m}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{\ell} \sum_{i=0}^{n-m-1} (-1)^{i+j} \binom{n}{j} \binom{m+\ell-j}{m} \left[{j \atop j-i}\right] \left[{n-j \atop m+1+i-j}\right] \\
&= W(\ell, n, m+1).
\end{aligned}
$$

where in the last step we used Corollary 3.2.15. In particular $C_{n,m}(x)$ is a polynomial, and the result now follows computing the product $C_{n,m}(x) \cdot \sum_{j=0}^{m} A(m, j) x^{j+1}$ to get the identity of the statement. □

Remark 3.3.4. As a consequence of our formula, we have:

$$
\begin{aligned}
[t^{n-1}] \operatorname{ehr}(\Delta_{k,n}, t) &= \frac{1}{(n-1)!} \sum_{\ell=0}^{k-1} W(\ell, n, n) A(n-1, k-\ell-1) \\
&= \frac{1}{(n-1)!} W(0, n, n) A(n-1, k-1) \\
&= \frac{1}{(n-1)!} A(n-1, k-1),
\end{aligned}
$$

which was Laplace's result, proved also by Stanley [Sta77].

3.4 The independence polytope of the uniform matroid

Recall that in Example 1.2.18 we showed that the independence polytope of the uniform matroid $U_{k,n}$ is given by

$$
\mathcal{P}_{\mathrm{I}}(U_{k,n}) = \left\{ x \in [0,1]^n : \sum_{i=1}^{n} x_i \leq k \right\}. \tag{3.6}
$$

By following almost the same steps of the proof of Theorem 3.1.1, we can find a formula for the Ehrhart polynomial of this polytope that looks like Katzman's formula for the hypersimplex.

Theorem 3.4.1 *The Ehrhart polynomial of $\mathcal{P}_{\mathrm{I}}(U_{k,n})$ is given by*

$$
\operatorname{ehr}(\mathcal{P}_{\mathrm{I}}(U_{k,n}), t) = \sum_{j=0}^{k-1} (-1)^j \binom{n}{j} \binom{(k-j)t + n - j}{n}. \tag{3.7}
$$

Proof. By definition we have that

$$
\operatorname{ehr}(\mathcal{P}_{\mathrm{I}}(U_{k,n}), t) = \# \left\{ y \in \{0, \ldots, t\}^n : \sum_{i=1}^{n} y_i \leq kt \right\}.
$$

In other words, we see that $\operatorname{ehr}(\mathcal{P}_{\mathsf{l}}(U_{k,n}),t)$ is just the following:

$$\operatorname{ehr}(\mathcal{P}_{\mathsf{l}}(U_{k,n}),t) = \sum_{m=0}^{kt}[x^m](1+x+\cdots+x^t)^n.$$

Thus, by reasoning as in the proof of Theorem 3.1.1, we have:

$$\begin{aligned}
\operatorname{ehr}(\mathcal{P}_{\mathsf{l}}(U_{k,n}),t) &= \sum_{m=0}^{kt}[x^m]\left(\frac{1-x^{t+1}}{1-x}\right)^n \\
&= \sum_{m=0}^{kt}\sum_{j=0}^{m}(-1)^j\binom{n}{j}\binom{n-1+(m-(t+1)j)}{n-1} \\
&= \sum_{j=0}^{kt}\sum_{m=j}^{kt}(-1)^j\binom{n}{j}\binom{n-1+(m-(t+1)j)}{n-1} \\
&= \sum_{j=0}^{kt}(-1)^j\binom{n}{j}\sum_{m=j}^{kt}\binom{n-1+(m-(t+1)j)}{n-1} \\
&= \sum_{j=0}^{kt}(-1)^j\binom{n}{j}\sum_{m=0}^{kt-j}\binom{n-1-tj+m}{n-1} \\
&= \sum_{j=0}^{kt}(-1)^j\binom{n}{j}\binom{n+(k-j)t-j}{n} \\
&= \sum_{j=0}^{k-1}(-1)^j\binom{n}{j}\binom{n+(k-j)t-j}{n},
\end{aligned}$$

where we used the Hockey-stick identity, which is stated in the appendix as Lemma A.0.1, and in the last step we changed the upper limit to avoid summing zeros. □

In [Li12], Li introduced the so-called *half-open hypersimplices* $\Delta'_{k,n}$:

$$\Delta'_{k,n} := \left\{ x \in [0,1]^{n-1} : k-1 < \sum_{i=1}^{n-1} x_i \leq k \right\}. \tag{3.8}$$

for $k > 1$ and $\Delta'_{1,n} := \Delta_{1,n}$. This was done in the context of studying the h^*-polynomial of the hypersimplex $\Delta_{k,n}$. Observe that although $\Delta'_{k,n}$ is a polytope with some missing facets, the function that counts the number of lattice points in each integral dilation of $\Delta'_{k,n}$ is still a polynomial. This can be proven by means of a simple inclusion-exclusion argument. Hence, it makes sense to talk about the Ehrhart polynomial of $\Delta'_{k,n}$, which we will denote by $\operatorname{ehr}(\Delta'_{k,n},t)$.

An interesting property of these half-open hypersimplices is that

$$\mathcal{P}_{\mathsf{l}}(U_{k,n}) = \Delta'_{1,n+1} \sqcup \Delta'_{2,n+1} \sqcup \cdots \sqcup \Delta'_{k,n+1}, \tag{3.9}$$

where the symbol $\sqcup$ stands for disjoint union. Hence, if we prove that each of these half-open hypersimplices is Ehrhart positive, we can conclude so for the independence matroid polytope of the uniform matroid $U_{k,n}$.

The Ehrhart polynomial of $\Delta'_{k,n}$ can be calculated in terms of the Ehrhart polynomials of two hypersimplices.

Proposition 3.4.2 *If* $1 < k < n-1$, *then*

$$\operatorname{ehr}(\Delta'_{k,n}, t) = \operatorname{ehr}(\Delta_{k,n}, t) - \operatorname{ehr}(\Delta_{k-1,n-1}, t).$$

Proof. Observe that the map $\pi : \Delta_{k,n} \to \mathbb{R}^{n-1}$ that forgets the last coordinate is an integral equivalence, and its image is given by

$$\pi(\Delta_{k,n}) = \left\{ x \in [0,1]^{n-1} : k-1 \leq \sum_{i=1}^{n-1} x_i \leq k \right\},$$

the set of points in $[0,1]^{n-1}$ that have sum of coordinates in the interval $[k-1, k]$.

The following decomposition

$$\pi(\Delta_{k,n}) = \Delta'_{k,n} \sqcup \Delta_{k-1,n-1},$$

yields an equality between Ehrhart polynomials:

$$\operatorname{ehr}(\pi(\Delta_{k,n}), t) = \operatorname{ehr}(\Delta'_{k,n}, t) + \operatorname{ehr}(\Delta_{k-1,n-1}, t),$$

and since π was an integral equivalence, we have that $\operatorname{ehr}(\pi(\Delta_{k,n}), t) = \operatorname{ehr}(\Delta_{k,n}, t)$, from where the result follows. $\square$

Let us now prove that half-open hypersimplices are Ehrhart positive.

Theorem 3.4.3 *Let us denote* $\operatorname{ehr}(\Delta'_{k,n}, t)$ *the Ehrhart polynomial of* $\Delta'_{k,n}$. *Then*

$$[t^m] \operatorname{ehr}(\Delta'_{k,n}, t) > 0 \textit{ for all } 1 \leq m \leq n-1.$$

Also, the constant term is 1 *for* $k = 1$ *and* 0 *for* $k > 1$.

Proof. Notice that $\Delta'_{1,n} = \Delta_{1,n}$, so the case $k = 1$ is already settled by the Ehrhart positivity of hypersimplices.

From now on, consider $1 \leq m \leq n-1$. Proposition 3.4.2 says that we have to prove:

$$e_{k,n,m} > e_{k-1,n-1,m}.$$

Since the hypersimplices $\Delta_{k,n}$ and $\Delta_{n-k,n}$ are one obtained from the other via an integral equivalence, we also know that $e_{k,n,m} = e_{n-k,n,m}$. This reasoning shows that $e_{k-1,n-1,m} = e_{n-k,n-1,m}$. So, it suffices to show that

$$e_{n-k,n,m} > e_{n-k,n-1,m}.$$

However, setting for simplicity $k' = n - k$ and using equation (3.5), the last inequality is equivalent to:

$$\frac{1}{(n-1)!}\sum_{\ell=0}^{k'-1} W(\ell,n,m{+}1)A(m,k'{-}\ell{-}1) > \frac{1}{(n-2)!}\sum_{\ell=0}^{k'-1} W(\ell,n{-}1,m{+}1)A(m,k'{-}\ell{-}1)$$

Which in turn is equivalent to prove that

$$\sum_{\ell=0}^{k'-1}\left(\frac{1}{n-1}W(\ell,n,m+1) - W(\ell,n-1,m+1)\right)A(m,k'-\ell-1) > 0$$

And as we saw in Corollary 3.2.13, the term in the parentheses is positive, and thus the proof is complete. □

Theorem 3.4.4 *The independence matroid polytope of the uniform matroid $U_{k,n}$ is Ehrhart positive.*

Proof. From the disjoint decomposition of equation (3.9) it follows that

$$\operatorname{ehr}\left(\mathcal{P}_{\mathrm{I}}(U_{k,n}), t\right) = \sum_{j=1}^{k}\operatorname{ehr}(\Delta'_{j,n+1}, t),$$

and hence, the independent term is 1, and the rest of them are positive because in each summand on the right one has such positivity. □

CHAPTER 4

Ehrhart polynomials of matroids

In Chapter 3 we discussed the case of uniform matroids and we proved that both $\mathcal{P}(M)$ and $\mathcal{P}_{\mathsf{I}}(M)$ are Ehrhart positive when M is uniform.

Observe that if M is a matroid on n elements and rank k, it holds that $\mathcal{P}(M)$ is a subpolytope of the hypersimplex $\Delta_{k,n} = \mathcal{P}(U_{k,n})$. In particular, for every nonnegative integer t, one has the inequality:

$$\operatorname{ehr}(\mathcal{P}(M), t) \leq \operatorname{ehr}(\Delta_{k,n}, t).$$

On the other hand, an analogous reasoning shows that

$$\operatorname{ehr}(\mathcal{P}_{\mathsf{I}}(M), t) \leq \operatorname{ehr}(\mathcal{P}_{\mathsf{I}}(U_{k,n}), t).$$

We in fact conjecture that these inequalities are true coefficient-wise.

Conjecture 4.0.1 *If M is a connected matroid of rank k and cardinality n, then*

- $\operatorname{ehr}(\mathcal{P}(M), t)$ *is coefficient-wise smaller than* $\operatorname{ehr}(\Delta_{k,n}, t)$.
- $\operatorname{ehr}(\mathcal{P}_{\mathsf{I}}(M), t)$ *is coefficient-wise smaller than* $\operatorname{ehr}(\mathcal{P}_{\mathsf{I}}(U_{k,n}), t)$.

This suggests that, as we have a matroid that presumably realises the maximum Ehrhart coefficient at each degree, it might as well exist a matroid that realises the minimum coefficient-wise Ehrhart polynomial when looking only at connected matroids of rank k, cardinality n. The requirement on the connectedness is put to guarantee that the Ehrhart polynomials of the matroids will all have the same degree, and thus it would make more sense to compare all of their coefficients. Notice that uniform matroids are pretty natural candidates to realise the maximum, but for the minimum we lack such a handy matroid. Experimenting with small values of n and k revealed that the matroid realising the minimum coefficients seemed always to be the (unique) matroid with the least number of bases.

In Section 4.1 we will describe our candidates which are coined *minimal matroids*, prove a formula for their Ehrhart polynomials, and prove in Corollary 4.1.9 that they are Ehrhart positive. Minimal matroids play a prominent role in the construction of counterexamples to the main Conjectures in Section 4.3. In particular, we see that although these matroids were introduced to understand if they were the minimum coefficient-wise Ehrhart, this in fact is no longer true for sufficiently large values of n and k.

In Section 4.2 we will describe the operation of circuit-hyperplane relaxations and prove a formula for the Ehrhart polynomial of all sparse paving matroids. This will prove Conjecture 4.0.1 for this class of matroids.

In Section 4.3 we will disprove the Ehrhart positivity conjectures for matroids (Conjecture 2.2.1) and hence for generalized permutohedra (Conjecture 2.2.2): for every $k \geq 3$ there exists a connected matroid of rank k that has a negative Ehrhart coefficient.

Finally, in Section 4.4 we will discuss the rank 2 case: we will find a formula for the Ehrhart polynomial of all connected matroids of rank 2, and we will prove that all of them are Ehrhart positive. Moreover, we will see that the minimal matroid and the uniform matroid realise the minimum and the maximum coefficient-wise Ehrhart polynomials.

4.1 Minimal matroids and relaxations

Minimal matroids and their polytopes

We start this section by recalling a result established independently in [Din71] and [Mur71].

Theorem 4.1.1 *If M is a connected matroid with n elements and rank k, then $|\mathcal{B}(M)| \geq k(n-k)+1$. Furthermore there is a unique (up to isomorphism) connected matroid of size n and rank k for which equality is attained.*

These matroids will be referred to as the *minimal matroids*. We proceed to a realization of them. They happen to be indeed graphic matroids.

Proposition 4.1.2 *Let $T_{k,n}$ be the graph given by a cycle of length $k+1$ where one edge is replaced with $n-k$ parallel copies. Then the matroid of $T_{k,n}$ is connected, has cardinality n, rank k and exactly $k(n-k)+1$ bases.*

Proof. We will use the name *red edges* when we refer to the $n-k$ parallel edges as in the statement. The remaining edges will be called *black edges*.

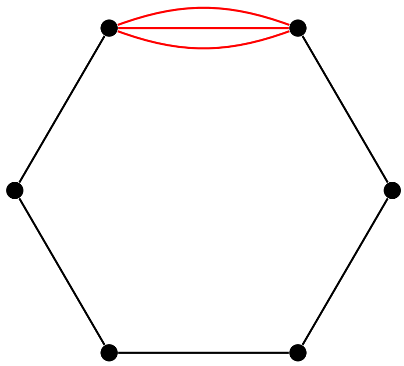


Figure 4.1: $T_{5,8}$

Observe that the matroid of $T_{k,n}$ does indeed trivially satisfy the cardinality and rank conditions: we have n elements in total and the maximal independent sets are of cardinality k. It is also straightforward to verify that this graph is biconnected and hence its matroid is connected.

Finally, since a basis of the matroid corresponds to a spanning tree on the graph, we notice that we have two kind of spanning trees: those that contain just one red edge, and those that contain none. In the first case, we can choose one among the $n-k$ red edges, and leave out one among of the k black edges. In the second case, no red edges implies that the spanning tree must consist of all black edges. Thus, $(n-k)k+1$ is the total number of spanning trees. □

Remark 4.1.3. It is clear from the uniqueness of the minimal matroid that the dual of the minimal matroid $T_{k,n}$ is isomorphic to $T_{n-k,n}$.

In all what follows we will use the name $T_{k,n}$ for the matroid of the graph $T_{k,n}$. This abuse of notation should not cause confusions.

Let us characterize all flats of the matroid $T_{k,n}$. Using the notation of the proof of Proposition 4.1.2, we see that there are two types of flats in $T_{k,n}$: those that contain a red edge (and hence all of them), and those that consist of only black edges.

We label all black edges with the numbers $\{1, 2, \ldots, k\}$ and the red ones with the numbers $\{k+1, \ldots, n\}$.

- Those flats that contain all red edges, may contain any number $m \neq k-1$ of black edges. It cannot contain exactly $k-1$, since adding the remaining edge will not increase the rank, thus contradicting the definition of flat. Hence there are $2^k - k$ such flats.

- Those flats that do not contain red edges may contain any proper subset of black edges. Hence there are $2^k - 1$ such flats.

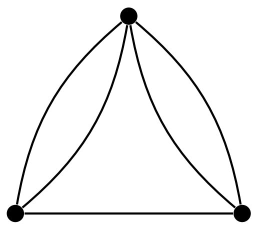
Using Proposition 1.2.10 we can formulate now a characterization of $\mathcal{P}(T_{k,n})$ using inequalities.

Proposition 4.1.4 *The polytope $\mathcal{P}(T_{k,n})$ is described with inequalities as*

$$\mathcal{P}(T_{k,n}) = \left\{ x \in [0,1]^n : \sum_{i=1}^{n} x_i = k \text{ and } \sum_{i=k+1}^{n} x_i \leq 1 \right\}.$$

Proof. Among the flats that we have just described, notice that the inseparable ones are the singletons and the set of all red edges. In other words, by using Theorem 1.2.10, the result of the statement follows. □

Remark 4.1.5. It is not true that the polytope of every connected matroid of rank k and cardinality n contains a copy of $\mathcal{P}(T_{k,n})$. For example, let M be the matroid of the following graph:



This matroid M has 8 bases, given that the graph has 8 spanning trees. It is connected, has rank 2 and cardinality 5. Also, $T_{2,5}$ has exactly 7 bases. There is no way we can delete one basis from the set $\mathcal{B}(M)$ and obtain the set of bases of a matroid isomorphic to $T_{2,5}$. At the level of polytopes, this means that no subset of 7 vertices of $\mathcal{P}(M)$ induces a polytope that is a copy of $\mathcal{P}(T_{k,n})$.

The Ehrhart polynomial of minimal matroids

In this section we give a formula for the Ehrhart polynomial of $\mathcal{P}(T_{k,n})$. Our proofs are elementary and consist of several manipulations using combinatorial identities. In the Appendix we include the proofs of some results that are used throughout our computations. We remark that alternative proofs are possible using the language of generalized hypergeometric functions and hypergeometric transformations [GKP94].

We start with our first formula for $\operatorname{ehr}(\mathcal{P}(T_{k,n}), t)$. We will denote this polynomial by $D_{k,n}(t)$. An equivalent version of this formula was found in [KMR18, Theorem 3.8].

Theorem 4.1.6 *Let $D_{k,n}(t) \in \mathbb{Q}[t]$ be the Ehrhart polynomial of $\mathcal{P}(T_{k,n})$. Then the following equality holds.*

$$D_{k,n}(t) = \sum_{j=0}^{k-1} \binom{k-1}{j} \binom{n-k-1}{j} \binom{t+n-1-j}{n-1}.$$

Proof. Using Proposition 4.1.4, this is:

$$D_{k,n}(t) = \#\left(\mathbb{Z}^n \cap t\mathcal{P}(T_{k,n})\right)$$

$$= \#\left\{x \in [0,t]^n : \sum_{i=1}^{n} x_i = tk \text{ and } \sum_{i=k+1}^{n} x_i \le t\right\}.$$

To count the number of elements of this set, we proceed as follows. Let us fix a number $0 \le j \le t$ and set the sum $\sum_{i=k+1}^{n} x_i$ to be exactly j. The number of ways to achieve this is exactly the number of ways of putting j indistinguishable balls into $n-k$ distinguishable boxes, which is just $\binom{n-k-1+j}{n-k-1}$.

Now we have to count the number of ways of putting $tk - j$ indistinguishable balls into exactly k distinguishable boxes, each of them having a capacity of t. Using Proposition A.0.3 in the appendix one has then

$$\begin{aligned} D_{k,n}(t) &= \sum_{j=0}^{t} \binom{n-k-1+j}{n-k-1}\binom{k-1+tk-(tk-j)}{k-1}. \\ &= \sum_{j=0}^{t} \binom{n-k-1+j}{j}\binom{k-1+j}{j} \qquad (4.1) \end{aligned}$$

Then, by the Double Hockey-stick identity, stated as Proposition A.0.5 in the appendix, one gets the result. □

The formula presented in the preceding Theorem, and the one of equation (4.1) are useful for computations, but do not show the positivity of the coefficients of $D_{k,n}$. A first step towards that is to notice the following factorization:

Lemma 4.1.7 *The following identity holds:*

$$D_{k,n}(t) = \binom{t+n-k}{n-k}\sum_{j=0}^{k-1} \frac{n-k}{n-k+j}\binom{t}{j}\binom{k-1}{j}$$

Proof. The proof consists only of sum manipulations starting with equation (4.1). Steps on numbered equations are justified below.

$$\begin{aligned} D_{k,n}(t) &= \sum_{j=0}^{t} \binom{n-k-1+j}{j}\binom{k-1+j}{j} \\ &= \sum_{j=0}^{t} \binom{n-k-1+j}{j}\sum_{i=0}^{k-1}\binom{k-1}{k-1-i}\binom{j}{i} \qquad (4.2) \\ &= \sum_{i=0}^{k-1}\sum_{j=0}^{t} \binom{k-1}{i}\binom{n-k-1+j}{j}\binom{j}{j-i} \\ &= \sum_{i=0}^{k-1}\sum_{j=0}^{t} \binom{k-1}{i}\binom{n-k-1+j}{j-i}\binom{n-k-1+i}{i} \qquad (4.3) \\ &= \sum_{i=0}^{k-1}\binom{k-1}{i}\binom{n-k-1+i}{i}\sum_{j=0}^{t}\binom{n-k-1+j}{n-k-1+i} \end{aligned}$$

$$= \sum_{i=0}^{k-1} \binom{k-1}{i} \binom{n-k-1+i}{i} \binom{t+n-k-1}{n-k+i} \tag{4.4}$$

$$= \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{n-k}{n-k+i} \binom{t}{j} \binom{t+n-k}{n-k} \tag{4.5}$$

where in (4.2) we used Vandermonde's Identity, in (4.3) the identity $\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$, in (4.4) the Hockey-Stick Identity (also known as the parallel summation formula [GKP94]) and in (4.5) just some simplifications (see the Appendix for the statements of some identities). □

Observe that from this Lemma we get that $D_{k,n}(t)$ can be written as a product of a polynomial with positive coefficients: $\binom{t+n-k}{n-k}$ and a remaining factor, which we will call $R_{k,n}(t)$. It is:

$$R_{k,n}(t) = \sum_{j=0}^{k-1} \frac{n-k}{n-k+j} \binom{t}{j} \binom{k-1}{j}$$

Hence, if we prove that $R_{k,n}$ has positive coefficients, then we will be able to conclude the positivity of the coefficients of $D_{k,n}$. This is done in the following Lemma.

Lemma 4.1.8

$$R_{k,n}(t) = \frac{1}{\binom{n-1}{k-1}} \sum_{j=0}^{k-1} \binom{n-k-1+j}{j} \binom{t+j}{j}$$

Proof. We have the following chain of equalities:

$$\binom{n-1}{k-1} R_{k,n}(t) = \binom{n-1}{k-1} \sum_{j=0}^{k-1} \frac{n-k}{n-k+j} \binom{t}{j} \binom{k-1}{j}$$

$$= \sum_{j=0}^{k-1} \binom{n-1}{k-1} \frac{\binom{n-k-1+j}{j}}{\binom{n-k+j}{j}} \binom{t}{j} \binom{k-1}{j}$$

$$= \sum_{j=0}^{k-1} \binom{n-1}{k-1} \binom{k-1}{k-1-j} \frac{\binom{n-k-1+j}{j}}{\binom{n-k+j}{j}} \binom{t}{j}$$

$$= \sum_{j=0}^{k-1} \binom{n-1}{k-1-j} \binom{n-k+j}{n-k} \frac{\binom{n-k-1+j}{j}}{\binom{n-k+j}{j}} \binom{t}{j} \tag{4.6}$$

$$= \sum_{j=0}^{k-1} \binom{n-1}{k-1-j} \binom{n-k-1+j}{j} \binom{t}{j} \tag{4.7}$$

where in (4.6) we used the identity $\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$. On the other hand:

$$\begin{aligned}
\sum_{j=0}^{k-1}\binom{n-k-1+j}{j}\binom{t+j}{j} &= \sum_{j=0}^{k-1}\binom{n-k-1+j}{j}\sum_{i=0}^{j}\binom{t}{i}\binom{j}{j-i} && (4.8)\\
&= \sum_{i=0}^{k-1}\sum_{j=i}^{k-1}\binom{t}{i}\binom{n-k-1+j}{j}\binom{j}{j-i}\\
&= \sum_{i=0}^{k-1}\sum_{j=i}^{k-1}\binom{t}{i}\binom{n-k-1+j}{j-i}\binom{n-k-1+i}{i}\\
&= \sum_{i=0}^{k-1}\binom{n-k-1+i}{i}\binom{t}{i}\sum_{j=i}^{k-1}\binom{n-k-1+j}{j-i}\\
&= \sum_{i=0}^{k-1}\binom{n-k-1+i}{i}\binom{t}{i}\sum_{j=0}^{k-1-i}\binom{n-k-1+i+j}{j}\\
&= \sum_{i=0}^{k-1}\binom{n-k-1+i}{i}\binom{t}{i}\binom{n-1}{n-k+i} && (4.9)
\end{aligned}$$

where in (4.8) we used Vandermonde's Identity (cf. Appendix A) and in (4.9) we used the classic Hockey Stick Identity. Observe that (4.7) and (4.9) are equal, so the result of the statement follows. □

Corollary 4.1.9 *The polynomial $D_{k,n}(t)$ is given by,*

$$D_{k,n}(t) = \frac{1}{\binom{n-1}{k-1}}\binom{t+n-k}{n-k}\sum_{j=0}^{k-1}\binom{n-k-1+j}{j}\binom{t+j}{j} \tag{4.10}$$

and in particular has positive coefficients.

Proof. The equation (4.10) is just a consequence of the preceding Lemmas. From this equality, as we said above, the positivity of the coefficients is clear as the variable appears in binomial coefficients of the form $\binom{t+a}{a}$ which are polynomials with positive coefficients. □

Remark 4.1.10. Notice that from our formula (4.10) for $D_{k,n}$ it is evident that $D_{k,n}(t-1)$ has nonnegative coefficients.

The h^*-polynomial

As a consequence of Theorem 4.1.6 we have a formula for the h^*-polynomial of $T_{k,n}$.

Corollary 4.1.11 *The h^*-polynomial of the basis polytope of $T_{k,n}$ is given by the formula:*

$$h^*(\mathcal{P}(T_{k,n}), x) = \sum_{j=0}^{k-1} \binom{k-1}{j}\binom{n-k-1}{j} x^j.$$

This polynomial is real-rooted.

Proof. It follows readily from Theorem 4.1.6 and the definition of the h^*-polynomial. The real-rootedness of this polynomial is a well known fact, see for example the Concluding Remarks in [KMR18]. □

Although the Ehrhart polynomial of $T_{k,n}$ is a bit difficult to work with, the h^*-polynomial permits us to obtain some information of the polytope $\mathcal{P}(T_{k,n})$.

Corollary 4.1.12 *The volume of the polytope $\mathcal{P}(T_{k,n})$ is given by*

$$\operatorname{vol}(\mathcal{P}(T_{k,n})) = \frac{1}{(n-1)!}\binom{n-2}{k-1}.$$

Proof. Since by Proposition 2.1.9 the normalized volume is given by $h^*(T_{n,k}, 1)$, it suffices to do the computation:

$$\begin{aligned} h^*(T_{k,n}, 1) &= \sum_{j=0}^{k-1} \binom{k-1}{j}\binom{n-k-1}{j} \\ &= \sum_{j=0}^{k-1} \binom{k-1}{k-1-j}\binom{n-k-1}{j} \\ &= \binom{n-2}{k-1}, \end{aligned}$$

where in the last step we used Vandermonde's Identity (see the appendix). □

Circuit-hyperplane relaxations

We will discuss a matroidal operation that behaves nicely with the Ehrhart polynomial of basis and independence polytopes. It will turn out that this operation behaves well with valuative invariants such as the Ehrhart polynomial, or the volume[1] . In fact, it has a very tight connection with the minimal matroids.

Recall that if M is a matroid on the ground set E of rank k and cardinality n, then a *hyperplane* of M is a coatom in the lattice of flats of M. Equivalently, a flat $F \subseteq E$ is said to be a hyperplane if $\operatorname{rk}(F) = k - 1$.

[1]Moreover, the Kazhdan-Lusztig polynomial of a matroid (introduced in [EPW16]) is a valuative invariant [AS20]. With an approach very similar to what we are about to do here with the Ehrhart theory of matroids, in a joint work with L. Vecchi we were able to explore the Kazhdan-Lusztig theory of all sparse paving matroids [FV21]. For more about valuative invariants of matroids, we recommend [DF10].

If $H \subseteq M$ is a hyperplane and a circuit, then one can *relax* the matroid M, declaring that H is a basis. More precisely:

Proposition 4.1.13 *Let M be a matroid with set of bases $\mathcal{B}$ that has a circuit-hyperplane H. Let $\widetilde{\mathcal{B}} = \mathcal{B} \cup \{H\}$. Then $\widetilde{\mathcal{B}}$ is the set of bases of a matroid $\widetilde{M}$ on the same ground set as M.*

Proof. By the definition of matroid, we only need to verify that if we pick H and a basis B of M, for every element $h \in H \setminus B$ there exists $x \in B \setminus H$ such that $(H \setminus \{h\}) \cup \{x\}$ is a basis of M.

Indeed, for such $h \in H \setminus B$, choose any $x \in B \setminus H$. Since H is a circuit, we have that $\mathrm{rk}(H) = \mathrm{rk}(H \setminus \{h\}) = |H| - 1$. Since H is a hyperplane, we have that $\mathrm{rk}(H) = |B| - 1$. In particular, we have that $|H| = |B|$. Also, notice that $H \setminus \{h\}$ is independent. We have that $\overline{H \setminus \{h\}} \subseteq \overline{H} = H$ and since $x \notin H$, it follows that adding x to $H \setminus \{h\}$ increases its rank. In other words, $(H \setminus \{h\}) \cup \{x\}$ has rank $|B|$ and cardinality $|B|$, and is thus a basis. □

The operation of declaring a circuit-hyperplane to be a basis is known in the literature by the name of *relaxation*. Many famous matroids arise as a result of this operation on another matroid. For example the *Non-Pappus matroid* is the result of relaxing a circuit-hyperplane on the *Pappus matroid*, and analogously the *Non-Fano* matroid can be obtained by a relaxation of the *Fano* matroid (for some other examples see [Oxl11]).

Of course, relaxing a circuit-hyperplane does not alter the rank of the matroid. It also preserves or increases its degree of connectivity (see [Oxl11, Propositon 8.4.2]).

Lemma 4.1.14 *Let M be a matroid with set of bases $\mathcal{B}$ and a circuit-hyperplane H. Let $\widetilde{M}$ be the relaxed matroid. Then, the set of flats $\widetilde{\mathcal{F}}$ of $\widetilde{M}$ is given by*

$$\widetilde{\mathcal{F}} = (\mathcal{F} \setminus \{H\}) \cup \{F \subseteq H : |F| = |H| - 1\},$$

where $\mathcal{F}$ is the set of flats of M.

Proof. Notice that the rank function $\widetilde{\mathrm{rk}}$ of $\widetilde{M}$ coincides with the rank function rk of M with the only exception of $\mathrm{rk}(H) + 1 = \widetilde{\mathrm{rk}}(H)$.

Let F be a flat of $\widetilde{M}$ that is not a flat of M. Then $\widetilde{\mathrm{rk}}(F \cup \{e\}) > \widetilde{\mathrm{rk}}(F)$ for all $e \notin F$. Since $F \neq H$, we have that $\widetilde{\mathrm{rk}}(F) = \mathrm{rk}(F)$. Notice that there exists an e such that $F \cup \{e\} = H$, since otherwise our inequality holds for all e but using rk instead of $\widetilde{\mathrm{rk}}$ and thus contradicting that F is not a flat of M. Then $F \subseteq H$ and $|F| = |H| - 1$, as claimed.

The reverse inclusion follows from the fact that all such sets are flats of $\widetilde{M}$. If $F \neq H$ is a flat of M, then $\mathrm{rk}(F \cup \{e\}) > \mathrm{rk}(F)$ for all $e \notin F$; in particular, by using $\widetilde{\mathrm{rk}}$ instead of rk this will still be true even if $F \cup \{e\} = H$, because in that case $\widetilde{\mathrm{rk}}(F \cup \{e\}) = 1 + \mathrm{rk}(F \cup \{e\}) > \mathrm{rk}(F) = \widetilde{\mathrm{rk}}(F)$. Also, if $F = H \setminus \{h\}$ for some $h \in H$, then clearly adding h to F will increase its rank in $\widetilde{M}$, so let us pick an element

$e \notin H$ and notice that as $F \cup \{e\} \neq H$, we have:

$$\widetilde{\mathrm{rk}}(F \cup \{e\}) = \mathrm{rk}(F \cup \{e\}) = \mathrm{rk}((H \smallsetminus \{h\}) \cup \{e\}) = \mathrm{rk}(M)$$

where in the last equality we used that $(H \smallsetminus \{h\}) \cup \{e\}$ is a basis of M, as we have seen in the proof of Proposition 4.1.13. Since $\mathrm{rk}(M) = \widetilde{\mathrm{rk}}(H) = \widetilde{\mathrm{rk}}(F) + 1$, it follows that F is indeed a flat. $\square$

This description of the flats of the relaxed matroid $\widetilde{M}$ helps us to characterize its basis polytope and its independence polytope by deleting just one inequality. Namely, the precise inequality corresponding to the flat H.

Proposition 4.1.15 *Let M be a matroid of rank k and cardinality n with a circuit-hyperplane H. Then the basis polytope of the relaxation $\widetilde{M}$ is given by*

$$\mathcal{P}(\widetilde{M}) = \left\{ x \in \mathbb{R}^n_{\geq 0} : \sum_{i=1}^{n} x_i = k \textit{ and } \sum_{i \in F} x_i \leq \mathrm{rk}(F) \textit{ for all } F \in \mathcal{F}(M) \smallsetminus \{H\} \right\},$$

and its independence matroid polytope is given by

$$\mathcal{P}_{\mathrm{I}}(\widetilde{M}) = \left\{ x \in \mathbb{R}^n_{\geq 0} : \sum_{i \in F} x_i \leq \mathrm{rk}(F) \textit{ for all } F \in \mathcal{F}(M) \smallsetminus \{H\} \right\}.$$

Proof. Using the notation of the preceding Lemma, it suffices to see that the inequalities that come from flats of $\widetilde{M}$ of the form $F = H \smallsetminus \{h\}$ with $h \in H$ are redundant. Indeed, since any such F is independent, the inequality $\sum_{i \in F} x_i \leq \mathrm{rk}(F)$ is trivially implied by the inequalities $x_i \leq 1$. $\square$

The following result states the exact relation between minimal matroids and the operation of circuit-hyperplane relaxation in the language of polytopes. We will say that a polytope $\mathcal{P}$ is obtained by *stacking* a polytope $\mathcal{Q}$ on a facet of another polytope $\mathcal{R}$ if

$$\mathcal{P} = \mathcal{Q} \cup \mathcal{R},$$

and $\mathcal{Q} \cap \mathcal{R}$ is a facet of both $\mathcal{Q}$ and $\mathcal{R}$.

Theorem 4.1.16 *Let M be a connected matroid of rank k and cardinality n with a circuit-hyperplane H and let $\widetilde{M}$ be the relaxed matroid. Then*

(a) The polytope $\mathcal{P}(\widetilde{M})$ is obtained by stacking a copy of $\mathcal{P}(T_{k,n})$ on a facet of $\mathcal{P}(M)$.

(b) The polytope $\mathcal{P}_{\mathrm{I}}(\widetilde{M})$ is obtained by stacking a polytope integrally equivalent to $\mathcal{P}(T_{k,n+1})$ on a facet of $\mathcal{P}_{\mathrm{I}}(M)$.

Proof.

(a) Notice that $\mathcal{P}(\widetilde{M})$ contains all the vertices of $\mathcal{P}(M)$ and an extra vertex corresponding to H. In the proof of Proposition 4.1.13, we have seen that the basis H of $\widetilde{M}$ has the property that $(H \smallsetminus \{h\}) \cup \{x\} \in \mathcal{B}(M) \subseteq \mathcal{B}(\widetilde{M})$ for all $h \in H$ and $x \notin H$. By Theorem 1.2.8, we know that e_H is adjacent to all the bases of that form in $\mathcal{P}(\widetilde{M})$. Let us prove that there are $k(n-k)$ such bases. Call $H = \{h_1, \ldots, h_k\}$. Since H is a circuit-hyperplane of M, if we call $\{e_1, \ldots, e_{n-k}\}$ the elements in the complement of H, we have that

$$B_{ij} := (H \smallsetminus \{h_i\}) \cup \{e_j\}$$

is a basis of M for each $1 \leq i \leq k$ and each $1 \leq j \leq n-k$. These correspond to the $k(n-k)$ vertices adjacent to H in $\mathcal{P}(\widetilde{M})$. Also, for each i and j we have that B_{ij} is adjacent with all $B_{i'j}$ and all $B_{ij'}$ for $i' \neq i$ and $j' \neq j$. All this amounts to say that if we restrict ourselves to the polytope $\mathcal{Q}$ given by the $k(n-k)+1$ vertices given by H and all the B_{ij}, it is the basis polytope of some matroid N of rank k, cardinality n and $k(n-k)+1$ bases. Since this polytope also has dimension $n-1$, such matroid N has to be connected, so that by the uniqueness of the minimal matroid we see that $N = T_{k,n}$, and what we are doing is exactly stacking $\mathcal{P}(T_{k,n})$ on a facet of $\mathcal{P}(M)$.

(b) The polytope $\mathcal{P}_{\mathsf{I}}(\widetilde{M})$ has only one extra vertex with respect to $\mathcal{P}_{\mathsf{I}}(\widetilde{M})$. By the proof of Theorem 1.2.19, we have that the vertices adjacent to H are of the form $H \smallsetminus \{h\}$ for $h \in H$ and the $k(n-k)$ vertices of the facet $\mathcal{P}(M)$ that we described in (a). To each of the vertices $e_{H \smallsetminus \{h\}}$ assign a point in $\mathbb{R}^{n+1}$ given by $(e_{H \smallsetminus \{h\}}, 1)$. To each the vertices $e_{B_{ij}}$ assign the point $(e_{B_{ij}}, 0)$, and to H assign the point $(e_H, 0)$. Observe that we have $k(n-k)+k+1 = k(n+1-k)+1$ vertices. Using the same reasoning we used in the proof of Theorem 1.2.22, we can see that it is 0/1-polytope having all of its edges of the form $e_i - e_j$, and moreover, it has dimension n. It follows that it is the basis polytope of a connected matroid of rank k and $n+1$ elements having $k(n+1-k)+1$ bases, and is thus the polytope $\mathcal{P}(T_{k,n+1})$. □

An immediate consequence of the above subdivision is that the circuit-hyperplane relaxation behaves nicely with Ehrhart polynomials.

Corollary 4.1.17 *Let M be a connected matroid of rank k and cardinality n with a circuit-hyperplane H. Let $\widetilde{M}$ be the corresponding relaxation. Then*

(a) $\operatorname{ehr}(\mathcal{P}(\widetilde{M}), t) = \operatorname{ehr}(\mathcal{P}(M), t) + \operatorname{ehr}(\mathcal{P}(T_{k,n}), t-1)$.

(b) $\operatorname{ehr}(\mathcal{P}_{\mathsf{I}}(\widetilde{M}), t) = \operatorname{ehr}(\mathcal{P}_{\mathsf{I}}(M), t) + \operatorname{ehr}(\mathcal{P}(T_{k,n+1}), t-1)$.

In particular, the circuit-hyperplane relaxation preserves Ehrhart positivity for basis polytopes and independence polytopes of matroids.

Proof. We do the proof for the basis polytope since the other is analogous. Using the notation of the proof of the preceding Theorem, we know that

$$\mathcal{P}(\widetilde{M}) = \mathcal{P}(M) \cup \mathcal{Q},$$

and that $\mathcal{P}(M) \cap \mathcal{Q}$ is a common facet of $\mathcal{P}(M)$ and $\mathcal{Q}$. So an inclusion-exclusion argument reveals now that

$$\operatorname{ehr}(\mathcal{P}(\widetilde{M}), t) = \operatorname{ehr}(\mathcal{P}(M), t) + D_{k,n}(t) - S(t),$$

where $S(t)$ is the Ehrhart polynomial of the facet of $\mathcal{Q}$ consisting of all the $k(n-k)$ bases of $T_{k,n}$ containing a red edge. It is evident from Proposition 4.1.4 that this facet of $\mathcal{Q}$ can be interpreted as

$$\left\{ x \in [0,1]^n : \sum_{i=1}^{n} x_i = k \text{ and } \sum_{i=k+1}^{n} x_i = 1 \right\},$$

and then the number of integer points in a dilation by the factor t of this facet is given by

$$S(t) = \# \left\{ x \in \mathbb{Z}^n \cap [0,t]^n : \sum_{i=1}^{n} x_i = kt \text{ and } \sum_{i=k+1}^{n} x_i = t \right\},$$

from which, using the same *balls and boxes* reasoning, exactly as in the proof of Theorem 4.1.6, we see that

$$S(t) = \binom{n-k-1+t}{n-k-1}\binom{k-1+t}{k-1},$$

and we have from equation (4.1) that $D_{k,n}(t) - S(t)$ is equal then to $D_{k,n}(t-1)$. We conclude then the Ehrhart positivity of $\mathcal{P}(M)$ and $\mathcal{P}_{\mathrm{I}}(M)$ is preserved under relaxations by Remark 4.1.10. □

Remark 4.1.18. It is worth noting that the case of the presence of a circuit-hyperplane is the only scenario in which one can add just one basis and preserve the matroid structure [Mil99]. To be precise, if $\mathcal{B}$ is the set of bases of a matroid M and H is a subset such that $\mathcal{B} \sqcup \{H\}$ is also the set of bases of a matroid, this means that H was originally a circuit-hyperplane of M. For a proof of this result one can also read [Tru82, Lemma 6].

Of course, one has an equivalent version of the above result in the language of h^*-polynomials.

Corollary 4.1.19 *If M is a matroid of rank k and cardinality n with a circuit-hyperplane H and $\widetilde{M}$ is the relaxed matroid, then*

$$h^*(\mathcal{P}(\widetilde{M}), x) = h^*(\mathcal{P}(M), x) + x \cdot h^*(\mathcal{P}(T_{k,n}), x).$$

$$h^*(\mathcal{P}_{\mathrm{I}}(\widetilde{M}), x) = h^*(\mathcal{P}_{\mathrm{I}}(M), x) + x \cdot h^*(\mathcal{P}(T_{k,n+1}), x).$$

Proof. The result follows by using that the h^*-polynomial is the numerator of the generating function of the Ehrhart polynomial. □

4.2 Sparse paving matroids

Now let us describe a class of matroids that is intimately related with the circuit-hyperplane relaxation.

Definition 4.2.1. Let M be a matroid of rank k. We say that M is *paving* if every circuit of M has cardinality at least k. We say that M is *sparse paving* if both M and its dual M^* are paving.

Observe that if a matroid M of rank k is paving, then its circuits must be all of size k or $k+1$. Also, since the hyperplanes of M are exactly the complements of circuits of M^*, when M^* is paving what we have is that all the hyperplanes of M have size exactly k or $k-1$.

If M is sparse paving, when one picks a circuit C of length k, since its rank is $k-1$, it must be contained in a hyperplane H. Thus, $k = |C| \leq |H| \in \{k-1, k\}$. Hence, the only possibility is $|H| = k$, and therefore $C = H$. In particular C is a hyperplane. Conversely, any hyperplane of size k of a sparse paving matroid is a circuit.

Lemma 4.2.2 *A matroid M of rank k is sparse paving if and only if every subset of cardinality k is either a basis or a circuit-hyperplane.*

Proof. Observe that if a matroid is such that every subset of cardinality k is either a basis or a circuit-hyperplane, then it automatically is sparse paving. This is because the existence of a circuit of size less than k is ruled out. Such a circuit can be completed to a set of cardinality k which will fail to be a basis and a circuit-hyperplane. Analogous considerations avoid the possibility of the existence of a hyperplane of size greater than k.

For the other implication, assume M is sparse paving and pick a subset A of cardinality k that is not a basis. Hence, we have that A is dependent, and thus contains a circuit C. Since M is sparse paving we have that $k \leq |C| \leq |A| = k$, and since $C \subseteq A$, it follows that $C = A$ and hence A is a circuit. Since A has cardinality k, by the considerations prior to the statement of the Lemma, it follows that A is a hyperplane. $\square$

It follows easily from the above result that uniform matroids are sparse paving and, moreover, that every sparse paving matroid can be relaxed until obtaining a uniform matroid. This is because after relaxing one circuit-hyperplane, the remaining circuit-hyperplanes are still circuit-hyperplanes of the new matroid.

Remark 4.2.3. In [MNWW11] Mayhew, Newman, Welsh and Whittle conjectured that *asymptotically all* matroids are sparse paving. There is some evidence supporting that assertion [Pv15].

As a corollary of Theorem 4.1.17 and of the fact that we have explicit formulas for $\operatorname{ehr}(\mathcal{P}(T_{k,n}), t-1)$ (Corollary 4.10), $\operatorname{ehr}(\Delta_{k,n}, t)$ (Theorem 3.1.1) and $\operatorname{ehr}(\mathcal{P}_{|}(U_{k,n}), t)$ (Theorem 3.4.1), we can deduce explicit formulas for the Ehrhart polynomial of the polytopes all sparse paving matroids.

Theorem 4.2.4 *Let M be a sparse paving matroid having n elements, rank k, and exactly λ circuit-hyperplanes. Then*

$$\begin{aligned}\operatorname{ehr}(\mathcal{P}(M), t) &= \operatorname{ehr}(\Delta_{k,n}, t) - \lambda \operatorname{ehr}(\mathcal{P}(T_{k,n}), t-1)\\ \operatorname{ehr}(\mathcal{P}_{\mathsf{I}}(M), t) &= \operatorname{ehr}(\mathcal{P}_{\mathsf{I}}(U_{k,n}), t) - \lambda \operatorname{ehr}(\mathcal{P}(T_{k,n+1}), t-1).\end{aligned}$$

Proof. It is a direct consequence of Theorem 4.1.17, since relaxing all the λ hyperplanes of M yields the uniform matroid $U_{k,n}$. □

Corollary 4.2.5 *Conjecture 4.0.1 is true for all sparse paving matroids.*

Proof. Since we already know that $\operatorname{ehr}(\mathcal{P}(T_{k,n}), t-1)$ has positive coefficients, Theorem 4.2.4 gives the result. □

Now that we have a good method to compute Ehrhart polynomials for a presumably enormous family of matroids, it is reasonable to try to search for a potential counterexample to Conjecture 2.2.1 within that family. This is what we will do in the next section.

Bounding the number of circuit-hyperplanes

The heuristics of our search will be the following. Since $\operatorname{ehr}(\mathcal{P}(T_{k,n}), t-1)$ has positive coefficients, we see in Corollary 4.2.4 that the coefficients of $\operatorname{ehr}(\mathcal{P}(M), t)$ are smaller when λ is bigger. We will try to find n and k that admit a λ sufficiently big to attain a negative coefficient for $\operatorname{ehr}(\mathcal{P}(M), t)$.

Fortunately, sparse paving matroids have a nice relation with two classes of objects that are very interesting on their own, and for which we have plenty of literature to deduce good bounds.

The *Johnson Graph* $J(n,k)$ is the graph having as vertices all the subsets of cardinality k of the set $\{1, \ldots, n\}$, and edges connecting them when their intersection is a set of cardinality $k-1$. It can be seen that $J(n,k)$ is the 1-skeleton of the hypersimplex $\Delta_{k,n}$.

The following result provides a dictionary between sparse paving matroids, a particular class of binary codes and stable subsets of the Johnson graph.

Theorem 4.2.6 *Let $\mathcal{S}$ be a subset of $\{1, \ldots, n\}$ such that all members of $\mathcal{S}$ have cardinality k. Then the following are equivalent:*

(a) $\mathcal{S}$ is the set of circuit-hyperplanes of a sparse paving matroid with n elements and rank k.

(b) $\mathcal{S}$ is a stable subset of nodes of the Johnson Graph $J(n,k)$.

(c) The set of all the indicator vectors of the elements of $\mathcal{S}$ is the set of words of a binary code such that all words have length n, constant weight k and minimum Hamming distance at least 4 (i.e. any two distinct words of the code differ in at least 4 positions).

Proof. The proof of the equivalence between (a) and (b) can be found in [BPV15, Lemma 8]. We reproduce it here for the sake of completeness. To prove that (a) $\Rightarrow$ (b), assume that M is sparse paving and that H_1 and H_2 are two circuit-hyperplanes such that $|H_1 \cap H_2| = k-1$ (i.e. that they correspond to adjacent vertices of $J(n,k)$). Since M is paving we obtain that $\operatorname{rk}(H_1 \cap H_2) = |H_1 \cap H_2| = k-1$. Also, as H_1 and H_2 are hyperplanes, we have:

$$\operatorname{rk}(H_1) + \operatorname{rk}(H_2) = 2(k-1) = 2k-2.$$

But, on the other hand, as $H_1 \cup H_2$ must have rank k, we have:

$$\operatorname{rk}(H_1 \cap H_2) + \operatorname{rk}(H_1 \cup H_2) = (k-1) + k = 2k-1.$$

All this information together implies that

$$\operatorname{rk}(H_1) + \operatorname{rk}(H_2) < \operatorname{rk}(H_1 \cup H_2) + \operatorname{rk}(H_1 \cap H_2),$$

which cannot happen for a matroid. Now, to prove that (b) implies (a), assume that $\mathcal{S}$ is a stable subset of the Johnson Graph $J(n,k)$, and consider the collection $\mathcal{B}$ of all subsets of $\{1,\dots,n\}$ of cardinality k that are not in $\mathcal{S}$. We trivially have that $\mathcal{B} \neq \varnothing$ because the Johnson graph contains edges. Assume that the basis-exchange-property does not hold. There exist two sets B_1 and B_2 in $\mathcal{B}$ and an element $x \in B_1 \smallsetminus B_2$ such that $(B_1 \smallsetminus \{x\}) \cup \{y\} \notin \mathcal{B}$ for all $y \in B_2 \smallsetminus B_1$. Observe that this implies that $|B_2 \smallsetminus B_1| > 1$, because otherwise it would be $(B_1 \smallsetminus \{x\}) \cup \{y\} = B_2 \in \mathcal{B}$ for the only $y \in B_2 \smallsetminus B_1$. Now, let us choose two distinct elements $y, z \in B_2 \smallsetminus B_1$ and consider the sets $H_1 = (B_1 \smallsetminus \{x\}) \cup \{y\}$ and $H_2 = (B_1 \smallsetminus \{x\}) \cup \{z\}$. Since H_1 and H_2 are not in $\mathcal{B}$, it holds that they are in $\mathcal{S}$. This is a contradiction, because $|H_1 \cap H_2| = |B_1 \smallsetminus \{x\}| = k-1$ which contradicts that $\mathcal{S}$ was stable.

The equivalence between (b) and (c) follows from the definitions: an edge on the Johnson graph corresponds to two words that have Hamming distance equal to 2. $\square$

A stronger form of Theorem 4.2.6 has also appeared in [JS17, Theorem 26], in the context of the study of *split matroids*, a generalization of sparse paving matroids. Now we will use the fact that there are binary codes of length n, constant weight k and minimum Hamming distance at least 4 that contain *many* words. Although there are several constructive proofs for such codes with even more words for special cases of n or k, the bounds we will use here suffice for our purposes. The statement and the proof of the following result are due to Graham and Sloane and can be found in [GS80]. We reproduce the proof here for the sake of completeness.

Theorem 4.2.7 *There exists a binary code $\mathcal{S}$ with words of length n, constant weight k, Hamming distance at least* 4*, and such that* $|\mathcal{S}| \geq \frac{1}{n}\binom{n}{k}$.

Proof. Let us denote by $\mathbb{F}_{2,k}^n$ the set of all binary words of length n and constant weight k, and by $\mathbb{Z}_n$ the set of integers modulo n. Consider the map:

$$T : \mathbb{F}_{2,k}^n \to \mathbb{Z}_n,$$

$$T(a_1, \ldots, a_n) = \sum_{i=1}^{n} (i-1)a_i \pmod{n}.$$

For each $i = 0, \ldots, n-1$ let us call $C_i = T^{-1}(\{i\})$. We claim that for each C_i, the minimum distance between two of its words is at least 4. Assume on the contrary that there are two distinct words $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ at distance less than 4. Since both words have the same weight, we have that their distance is exactly 2. Also, we see that there must exist two positions, say r and s, such that $a_r = 1$ and $b_r = 0$ and $a_s = 0$ and $b_s = 1$. But observe that

$$T(\mathbf{a}) = x + r = i \pmod{n},$$
$$T(\mathbf{b}) = x + s = i \pmod{n}$$

for a certain $x \in \mathbb{Z}_n$. This implies that $r \equiv s \pmod{n}$ which is clearly impossible. Thus, the minimum distance between words of C_i is 4. Now, since:

$$\binom{n}{k} = |\mathbb{F}_{2,k}^n| = \sum_{i=0}^{n-1} |C_i|,$$

we see that there has to be at least one i such that $|C_i| \geq \frac{1}{n}\binom{n}{k}$. $\square$

Let us show quickly how this set-up allows us to construct a counterexample to Conjectures 2.2.1 and 2.2.2.

Theorem 4.2.8 *There exists a sparse paving matroid M with 20 elements, rank 9 and having 8398 circuit-hyperplanes, and hence having Ehrhart polynomial with negative quadratic and cubic coefficients.*

Proof. By Theorem 4.2.7 there exists a binary code of length 20, constant weight 9 and Hamming distance at least 4, having at least $\frac{1}{20}\binom{20}{9} = 8398$ words. In fact, it can be proved that for the particular choice of $n = 20$ and $k = 9$ all the C_i's in the proof of Theorem 4.2.7 have cardinality 8398. In particular, by choosing for instance C_0 as our code, we have a code with 8398 words.

By the equivalence between (a) and (c) in Theorem 4.2.6, we get that there is a sparse paving matroid M with 20 elements and rank 9 that has exactly 8398 circuit-hyperplanes.

Now, using the formula of Theorem 4.1.17, we obtain that

$$\operatorname{ehr}(\mathcal{P}(M), t) = \operatorname{ehr}(\mathcal{P}(U_{9,20}), t) - 8398 \operatorname{ehr}(\mathcal{P}(T_{9,20}), t-1)$$

and we can compute this polynomial explicitly and see that its quadratic coefficient is $-\frac{142179543511}{15437822400} < 0$ and its cubic coefficient is $-\frac{4816883312963}{51459408000} < 0$. $\square$

Remark 4.2.9. Incidentally, this method provides a way of constructing $0/1$-polytopes in $\mathbb{R}^n$ that have nice properties on their edges, and which possess an exponential number of facets.

Observe that Proposition 4.1.15 and Remark 1.2.17 guarantee that

$$\#\{\text{facets of } \mathcal{P}_{\mathsf{I}}(\widetilde{M})\} = \#\{\text{facets of } \mathcal{P}_{\mathsf{I}}(M)\} - 1.$$

This is because a circuit-hyperplane is always an inseparable flat of a matroid, given that H is itself a circuit and $M|_H$ is thus connected.

Notice that the number of facets of $\mathcal{P}_{\mathsf{I}}(U_{k,n})$ is exactly $2n+1$, because by (1.2.18) it is determined by the inequalities $0 \leq x_i \leq 1$ and $\sum_{i=1}^n x_i \leq k$. In particular, we obtain that choosing a sparse paving matroid on n elements and rank $\lfloor \frac{n}{2} \rfloor$ with as many as $\frac{1}{n}\binom{n}{\lfloor \frac{n}{2}\rfloor}$ circuit-hyperplanes, we obtain a 0/1-polytope in $\mathbb{R}^n$ (an independence matroid polytope) with the following number of facets:

$$2n + 1 + \frac{1}{n}\binom{n}{\lfloor \frac{n}{2}\rfloor} \sim c\,\frac{2^n}{n^{3/2}}.$$

The best known methods to construct 0/1-polytopes with many facets are random [BP01], though there are deterministic methods that achieve a bound of $\sim 3.6^n$ facets. We can construct similarly a basis polytope with the same asymptotic number of facets using the same idea.

4.3 Counterexamples to Ehrhart positivity

We have already disproved both Conjectures 2.2.1 and 2.2.2. We will see that the strategy we used in fact allows us to find matroids of any rank $k \geq 3$ such that $\mathcal{P}(M)$ is not Ehrhart positive. This phenomenom also happens for the independence polytope.

Experimentation with several values of n and k shows that the most well-behaved coefficient for our purposes is the quadratic one. In other words, in the vast majority of cases, when a matroid is not Ehrhart positive, in particular its quadratic Ehrhart coefficient is negative.

Remark 4.3.1. It is not true that if a matroid has a negative Ehrhart coefficient, then in particular the quadratic coefficient must be negative. For example our construction yields a matroid with 22 elements, rank 7 and 7752 circuit-hyperplanes that has a negative term only on degree 3.

The idea is to give a good lower bound for $[t^2]\operatorname{ehr}(\mathcal{P}(T_{k,n}), t-1)$ and a good upper bound for $[t^2]\operatorname{ehr}(\Delta_{k,n}, t)$ that allow us to work more comfortably.

We start with a precise expression for the quadratic coefficient of the polynomial $\operatorname{ehr}(\mathcal{P}(T_{k,n}), t-1)$.

Lemma 4.3.2 *The quadratic coefficient of* $\operatorname{ehr}(\mathcal{P}(T_{k,n}), t-1)$ *is given by*

$$[t^2]\operatorname{ehr}(\mathcal{P}(T_{k,n}), t-1) = \frac{1}{\binom{n-1}{k-1}}\left(\begin{bmatrix} n-k \\ 2 \end{bmatrix} \frac{1}{(n-k)!} + \frac{1}{n-k}\sum_{j=1}^{k-1}\frac{1}{j}\binom{n-k-1+j}{j} \right).$$

Proof. This can be obtained by hand from the formula on Equation (4.10). $\square$

Using the preceding Lemma we can give a nice lower bound for the quadratic coefficient of $\operatorname{ehr}(\mathcal{P}(T_{k,n}), t-1)$, essentially by just ignoring many of the terms appearing in the expression we just obtained.

Proposition 4.3.3 *The quadratic coefficient of* $\operatorname{ehr}(\mathcal{P}(T_{k,n}), t-1)$ *satisfies:*

$$[t^2]\operatorname{ehr}(\mathcal{P}(T_{k,n}), t-1) \geq \frac{1}{k(n-1)}.$$

Proof. Observe that in the sum inside the parentheses in the formula of Lemma 4.3.2, we can pick only the term corresponding to $j = k-1$ and forget the rest. Hence,

$$\begin{aligned}
[t^2]\operatorname{ehr}(\mathcal{P}(T_{k,n}), t-1) &\geq \frac{1}{\binom{n-1}{k-1}}\left(\begin{bmatrix} n-k \\ 2 \end{bmatrix} \frac{1}{(n-k)!} + \frac{1}{(n-k)(k-1)}\binom{n-2}{k-1}\right) \\
&\geq \frac{1}{\binom{n-1}{k-1}} \cdot \frac{1}{(n-k)(k-1)}\binom{n-2}{k-1} \\
&= \frac{1}{(k-1)(n-1)} \\
&\geq \frac{1}{k(n-1)}
\end{aligned}$$

where in the second to last step we just expanded the binomial coefficients and canceled many factors. □

Remark 4.3.4. Before establishing an upper bound for the quadratic coefficient of $\operatorname{ehr}(\Delta_{k,n}, t)$, we will state some formulas that relate the Stirling numbers of the first kind with the so-called *harmonic numbers*. If we denote by $H_n^{(k)}$ the number $1 + \frac{1}{2^k} + \ldots + \frac{1}{n^k}$, and $H_n = H_n^{(1)}$, we have the following identities:

$$\frac{1}{(n-1)!}\begin{bmatrix} n \\ 1 \end{bmatrix} = 1, \tag{4.11}$$

$$\frac{1}{(n-1)!}\begin{bmatrix} n \\ 2 \end{bmatrix} = H_{n-1}, \tag{4.12}$$

$$\frac{1}{(n-1)!}\begin{bmatrix} n \\ 3 \end{bmatrix} = \frac{1}{2}\left(H_{n-1}^2 - H_{n-1}^{(2)}\right). \tag{4.13}$$

To bound $[t^2]\operatorname{ehr}(\Delta_{k,n}, t)$ we will use Theorem 3.3.3.

Proposition 4.3.5 *The quadratic coefficient of* $\operatorname{ehr}(\Delta_{k,n}, t)$ *satisfies:*

$$[t^2]\operatorname{ehr}(\Delta_{k,n}, t) \leq \frac{\binom{k+1}{2} + \binom{k}{2}}{(n-1)!} \cdot \begin{bmatrix} n \\ 3 \end{bmatrix} \leq \binom{k+1}{2} H_{n-1}^2$$

where H_{n-1} *denotes the harmonic number* $1 + \frac{1}{2} + \ldots + \frac{1}{n-1}$.

Proof. From Theorem 3.3.3 we know a formula for each of the Ehrhart coefficients of the uniform matroid $U_{k,n}$. In particular, for the quadratic term it holds:

$$[t^2]\operatorname{ehr}(\Delta_{k,n},t) = \frac{1}{(n-1)!}\left(W(k-1,n,3) + W(k-2,n,3)\right). \tag{4.14}$$

Let us prove that

$$W(\ell,n,3) \leq \binom{\ell+2}{2} W(0,n,3). \tag{4.15}$$

To this end, let us start with a partition of $\{1,\dots,n\}$ into 3 blocks having total weight 0. Consider the operation consisting of the following three steps:

- Swap the elements in the first position of the first block with the x-th smallest element of the first block.
- Swap the elements in the first position of the second block with the y-th smallest element of the second block.
- Swap the elements in the first position of the third block with the z-th smallest element of the third block.

If $(x-1)+(y-1)+(z-1) = \ell$ what we obtain is a partition of $\{1,\dots,n\}$ into three blocks having total weight ℓ. Observe that we can do this in at most $\binom{\ell+2}{2}$ ways (the number of ways of putting ℓ balls into 3 boxes). Also, in this way we can achieve all the possible partitions of weight ℓ. It is clear how to deduce the inequality (4.15) from this fact. Now, we know from Remark 3.2.8 that $W(0,n,m) = \left[{n \atop m}\right]$. If we use the formula of equation (4.14), we get the first inequality in our statement. Also, since

$$\frac{1}{(n-1)!}\left[{n \atop 3}\right] = \frac{1}{2}\left(H_{n-1}^2 - H_{n-1}^{(2)}\right) \leq \frac{1}{2}H_{n-1}^2,$$

it is easy to conclude the second inequality of the statement (we also used that $\binom{k}{2} \leq \binom{k+1}{2}$ to get a simpler form of the right-hand-side). $\square$

We can use our bounds to construct counterexamples on every rank $k \geq 3$.

Theorem 4.3.6 *If $n \geq 3589$ and $3 \leq k \leq n-3$ then there exists a matroid of rank k and cardinality n that is not Ehrhart positive. For $4 \leq k \leq n-4$ we may choose $n \geq 104$. Moreover, there exists non Ehrhart positive connected matroids with n elements for all $n \geq 19$.*

Proof. From Theorem 4.2.7 and the equivalence between (a) and (c) in Theorem 4.2.6, we have that there exists a sparse paving matroid M of rank k and cardinality n, having at least

$$\lambda = \frac{1}{n}\binom{n}{k}$$

circuit-hyperplanes. We know by Theorem 4.2.4 that

$$\operatorname{ehr}(M,t) = \operatorname{ehr}(\Delta_{k,n},t) - \lambda \operatorname{ehr}(\mathcal{P}(T_{k,n}), t-1),$$

and thus:

$$\begin{aligned}
[t^2]\operatorname{ehr}(M,t) &= [t^2]\operatorname{ehr}(\Delta_{k,n},t) - \lambda [t^2]\operatorname{ehr}(\mathcal{P}(T_{k,n}), t-1) \\
&\leq [t^2]\operatorname{ehr}(\Delta_{k,n},t) - \frac{1}{n}\binom{n}{k}[t^2]\operatorname{ehr}(\mathcal{P}(T_{k,n}), t-1) \\
&\leq \binom{k+1}{2} H_{n-1}^2 - \frac{1}{n}\binom{n}{k}\frac{1}{k(n-1)}
\end{aligned}$$

where we used Lemmas 4.3.3 and 4.3.5. It suffices to analyze when the following inequality is achieved:

$$\binom{k+1}{2} H_{n-1}^2 < \frac{1}{n}\binom{n}{k}\frac{1}{k(n-1)} \tag{4.16}$$

Let us split into some cases:

- If $k = 3$, we obtain:
$$6H_{n-1}^2 \leq \frac{1}{3n(n-1)}\binom{n}{3},$$
Since $H_{n-1}^2 \sim \log(n)^2$, we see that the right-hand-side grows much faster than the left-hand-side. In particular, the inequality holds for all $n \geq 10439$. Also, we can verify by hand the following finite cases $3589 \leq n \leq 10438$ and see that for all of them one has $[t^2]\operatorname{ehr}(M,t) < 0$. This proves that there exist counterexamples of rank 3 for all $n \geq 3589$.

- If $k = 4, 5, 6, 7, 8$, analogous considerations show that for $n \geq 104$, we can always find such counterexamples.

- If $k \geq 9$, recalling that in Remark 2.3.4 we stated that the Ehrhart polynomial of a matroid is equal to that of its dual, we can assume that $2k \leq n$ and consider a stronger version of inequality (4.16):
$$\binom{n+1}{2}(\log(n)+1)^2 < \frac{1}{n}\binom{n}{9}\frac{1}{n(n-1)},$$
which holds for all $n \geq 55$ (we used the basic inequality $H_n \leq \log(n) + 1$). By checking manually the cases $n = 20, \dots, 54$, we prove that there are counterexamples with any cardinality ≥ 20 (the case $n = 19$ is addressed below). □

An analogous procedure allows us to find matroids of any rank $k \geq 3$ such that its independence polytope has a negative Ehrhart coefficient.

Theorem 4.3.7 *There exists a matroid M of rank 3 with 4000 elements such that $\operatorname{ehr}(\mathcal{P}_{\mathrm{I}}(M), t)$ has a negative coefficient.*

The smallest counterexample

We have proved that for every $k \geq 3$ there is a (connected) matroid of rank k that is not Ehrhart positive.

Also, from all of our results it follows that for all $n \geq 20$ there is a (connected) matroid of cardinality n that is not Ehrhart positive. It is natural to ask if there are smaller counterexamples.

In fact, for small values of k and n there exist much better bounds and many precise values for the maximum number of circuit-hyperplanes that a sparse paving matroid of rank k and cardinality n can have. See for instance [BE11, Table 2] by Brouwer and Etzion. Using these values, one can prove that there exists a sparse paving matroid with 19 elements, rank 9 and having 6726 circuit-hyperplanes that is not Ehrhart positive.

We can rule out the existence of sparse paving matroids with less than 18 elements. To prove this, it suffices to give a good enough upper bound for the maximum number of circuit-hyperplanes a matroid of rank k and n elements can have.

Lemma 4.3.8 *Let M be a sparse paving matroid of rank k having n elements. Then, the number of circuit-hyperplanes λ of M satisfies:*

$$\lambda \leq \binom{n}{k} \min\left\{\frac{1}{k+1}, \frac{1}{n-k+1}\right\}. \tag{4.17}$$

Proof. If M is paving, in particular all the $\binom{n}{k-1}$ subsets of cardinality $k-1$ are independent. Let us form a bipartite graph where one of the parts has a node for each independent set of cardinality $k-1$ and the other part has the bases of the matroid M, where we put an edge connecting an independent set I with a basis B whenever $I \subseteq B$. Since an independent set I of rank $k-1$ is contained in a unique hyperplane (the flat spanned by I itself), it follows that either I is a hyperplane or $I \subsetneq H$ for a unique H hyperplane. In the latter case, $|H| \geq k$ and since M^* is paving, this implies that $|H| = k$, so that H is a circuit-hyperplane. Summarizing, each of the nodes of our graph corresponding to independent sets of cardinality $k-1$ has degree $n-k+1$ (when I is itself a hyperplane) or $n-k$ (when I is contained in a unique circuit-hyperplane). In particular, the number of edges of the whole graph is at least $(n-k)\binom{n}{k-1}$. However, by looking at the nodes corresponding to the bases, we know that each basis has degree k, so that the number of edges is exactly $k|\mathcal{B}(M)|$. Hence:

$$(n-k)\binom{n}{k-1} \leq k|\mathcal{B}(M)|,$$

which translates into

$$\left(1 - \frac{1}{n-k+1}\right)\binom{n}{k} \leq |\mathcal{B}(M)|.$$

Since the number of circuit hyperplanes is $\lambda = \binom{n}{k} - |\mathcal{B}(M)|$, it follows that

$$\lambda \leq \frac{1}{n-k+1}\binom{n}{k}.$$

Finally, using the same reasoning that we used above but with M^* instead of M, as the number of circuit-hyperplanes is the same for M and M^*, it follows also that

$$\lambda \leq \frac{1}{k+1}\binom{n}{n-k} = \frac{1}{k+1}\binom{n}{k},$$

from where one concludes the inequality of the statement. □

Corollary 4.3.9 *If M is a sparse paving matroid on $n \leq 17$ elements, then M is Ehrhart positive.*

Proof. Let us denote by $\lambda_{k,n}$ the expression on the right-hand-side of inequality (4.17). Calculating explicitly the polynomials $\operatorname{ehr}(\mathcal{P}(U_{k,n}), t) - \lambda_{k,n} \operatorname{ehr}(\mathcal{P}(T_{k,n}), t-1)$ for $1 \leq k \leq n \leq 17$, we can see that they all have positive coefficients. □

Remark 4.3.10. According to [BE11] the maximum size that a stable set in the Johnson Graph $J(18, 9)$ can have is at least 3540, which improves the bound coming from Theorem 4.2.7, $\frac{1}{18}\binom{18}{9} = 2702$. However, using the bound from Lemma 4.3.8 we get that this quantity is at most 4862. A sharper inequality using the so-called Johnson bound yields that, in fact, this quantity is less or equal than 4420. In other words, we know that the maximum number of circuit-hyperplanes that a matroid on 18 elements and rank 9 can have lies between 3540 and 4420, and this seems to be the best we can currently assert for $k = 9$ and $n = 18$ (see [AVZ00]). However:

$$\operatorname{ehr}(\mathcal{P}(U_{9,18}), t) - 4240 \operatorname{ehr}(\mathcal{P}(T_{9,18}), t-1),$$

has a negative cubic coefficient. This implies that if we could improve our 3540 to a 4240, then there would be a matroid on 18 elements that is not Ehrhart positive.

4.4 The rank 2 case

We define a partial order $\preceq$ on the ring of polynomials $\mathbb{R}[t]$ as follows. The polynomial $p(t) = \sum_{j=0}^{d} a_j t^j$ is said to be *nonnegative* if all its coefficients are nonnegative, that is, $a_j \geq 0$ for all $j \geq 0$. In this case, we write $p(t) \succeq 0$. Furthermore, we write $p(t) \succeq q(t)$ whenever $p(t) - q(t) \succeq 0$. We say that the inequality is *strict on the coefficients of positive degree* if $p(t) - q(t)$ has only positive coefficients, except for possibly the constant coefficient which may be zero.

We observe that $\preceq$ defines a partial order that is preserved under multiplication with nonnegative polynomials. That is, for $p, q, r \in \mathbb{R}[t]$ and $r(t) \succeq 0$

$$p(t) \preceq q(t) \implies p(t) \cdot r(t) \preceq q(t) \cdot r(t). \tag{4.18}$$

Note that if a matroid M has a loop, then M is a direct sum $M = M' \oplus U_{0,1}$. As loops do not change the polytope, only their embedding, we may assume from now on that all matroids that we consider are loopless. We benefit of the following fact.

Lemma 4.4.1 *Let M be a matroid of rank 2 with no loops. Then M is either connected or a direct sum of two uniform matroids of rank one. In particular, the basis polytope of the latter is a product of two simplices.*

The flats of a rank 2 matroid M are the set of all loops, the hyperplanes and the ground set. If M is loopless or connected, then the set of loops is empty. Neither the empty set nor the ground set impose a facet defining inequality in the description of Theorem 1.2.10. Thus we obtain the following simplification of Theorem 1.2.10 for a loopless matroid of rank 2 on a groundset of size n.

$$\mathcal{P}(M) = \left\{ x \in \Delta_{2,n} : \sum_{i \in H} x_i \leq 1 \text{ for all matroid hyperplanes } H \text{ of } M \right\}. \tag{4.19}$$

A key property of rank 2 loopless matroids is that they are all paving. Geometrically this is captured in the following Lemma.

Lemma 4.4.2 *Let M be a loopless matroid of rank 2 and $u \in \Delta_{2,n} \setminus \mathcal{P}(M)$. Then u violates exactly one of the inequalities*

$$\sum_{i \in H} x_i \leq 1$$

where H is a matroid hyperplane of M.

Proof. Clearly $u \in \Delta_{2,n} \setminus \mathcal{P}(M)$ has to violate at least one of the above inequalities. Suppose u satisfies

$$\sum_{i \in H} u_i > 1 \text{ and } \sum_{i \in G} u_i > 1$$

where G and H are distinct matroid hyperplanes. The intersection $G \cap H$ is empty as M has no loops. Therefore

$$2 < \sum_{i \in H} u_i + \sum_{i \in G} u_i \leq \sum_{i=1}^{n} u_i .$$

Contradicting that the coordinate sum of u is 2 whenever $u \in \Delta_{2,n}$. □

In [JS17] Joswig and Schröter introduced the class of split matroids which provides the same separation property in arbitrary rank. This class strictly contains paving matroids and thus include the loopless matroids of rank 2.

Ehrhart polynomials

We consider the polytopes

$$\mathcal{Q}_{k,n} = \left\{ x \in \Delta_{2,n} : \sum_{i=1}^{k-1} x_i \leq 1, \sum_{i=k+1}^{n} x_i \leq 1 \right\}$$

for all $1 \leq k \leq n-1$, together with their half-open version

$$\widetilde{\mathcal{Q}}_{k,n} := \left\{ x \in \Delta_{2,n} : \sum_{i=1}^{k-1} x_i \leq 1, \sum_{i=k+1}^{n} x_i < 1 \right\}.$$

Observe that for $k = 1$, $\mathcal{Q}_{1,n}$ is isomorphic to $\Delta_{1,n-1}$ and $\widetilde{\mathcal{Q}}_{1,n}$ is the empty polytope.

Remark 4.4.3. The polytope $\mathcal{Q}_{k,n}$ is the basis polytope of a rank 2 matroid of cardinality n, where the first $k-1$ elements are parallel and the last $n-k$ elements are parallel. This particular matroid is induced by a graph. This graph consists of a cycle of length three whenever $k > 1$ to which several parallel edges have been added as follows, there is one copy of one edge, $n-k$ parallel copies of another edge, and $k-1$ parallel copies of a third edge.

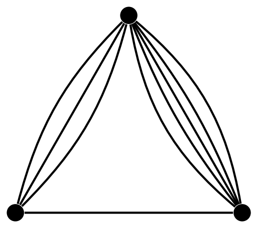


Figure 4.2: The graph of Remark 4.4.3 with $n = 9$ edges and $k = 4$.

Figure 4.2 depicts this graph for the case $n = 9$ and $k = 4$. These matroids fall into the well studied class of lattice path matroids. More precisely they are the snakes $\mathsf{S}(k-1, 2, n-k-1)$ in the notation of [KMR18].

We obtain the following formulas for the Ehrhart polynomials of the polytope $\mathcal{Q}_{k,n}$ and the half-open polytope $\widetilde{\mathcal{Q}}_{k,n}$.

Proposition 4.4.4 *For all* $1 \leq k \leq n-1$

$$\begin{aligned} \operatorname{ehr}(\mathcal{Q}_{k,n}, t) &= \binom{t+k-1}{k-1}\binom{t+n-k}{n-k} - \binom{t+n-2}{n-1}, \quad \textit{and} \\ \operatorname{ehr}(\widetilde{\mathcal{Q}}_{k,n}, t) &= \binom{t+k-1}{k-1}\binom{t+n-k-1}{n-k} - \binom{t+n-2}{n-1}. \end{aligned}$$

Proof. By definition we have

$$\begin{aligned} \operatorname{ehr}(\mathcal{Q}_{k,n}, t) &= \#(t\mathcal{Q}_{k,n} \cap \mathbb{Z}^n) \\ &= \# \left\{ x \in [0,t]^n \cap \mathbb{Z}^n : \sum_{i=1}^{n} x_i = 2t, \sum_{i=1}^{k-1} x_i \leq t, \sum_{i=k+1}^{n} x_i \leq t \right\}. \end{aligned}$$

The above expression can be interpreted as the number of ways of placing $2t$ indistinguishable balls into n distinct boxes, each of capacity t, under the additional constraints that the first $k-1$ as well as the last $n-k$ boxes together contain at most t balls. The number x_i equals the number of balls in box i in this setting.

As a first step, we ignore the capacity bound $x_k \le t$ for a moment, and count the number of ways that t balls can be placed into the first k boxes, and the remaining t balls are placed into the last $n-k+1$ boxes. There are $\binom{t+k-1}{k-1}\binom{t+n-k}{n-k}$ ways of placing $2t$ balls in such a way. (Notice that we do not over-count here, as the number of balls placed in box k in the first batch can be recovered from the balls in the boxes 1 to $k-1$, and similarly for the second batch.)

As a second step we count in how many cases we placed more than t balls in box k. In these cases the k-th box contains at least $t+1$ many balls. If we ignore $t+1$ many balls in box k, there are $\binom{t+n-2}{n-1}$ many possibilities to place the remaining $2t-(t+1)=t-1$ balls into n boxes. Subtracting this number from the above leads to the first formula.

To obtain the second formula we observe that the polytope $\mathcal{Q}_{k,n}$ is the disjoint union of $\widetilde{\mathcal{Q}}_{k,n}$ and the product of simplices

$$\left\{x \in [0,1]^n : \sum_{i=1}^{k} x_i = 1,\ \sum_{i=k+1}^{n} x_i = 1\right\} = \Delta_{1,k} \times \Delta_{1,n-k}$$

whose Ehrhart polynomial is equal to $\binom{t+k-1}{k-1}\binom{t+n-k-1}{n-k-1}$. It follows that

$$\begin{aligned}
\operatorname{ehr}(\widetilde{\mathcal{Q}}_{k,n}, t) &= \operatorname{ehr}(\mathcal{Q}_{k,n}, t) - \binom{t+k-1}{k-1}\binom{t+n-k-1}{n-k-1} \\
&= \binom{t+k-1}{k-1}\left(\binom{t+n-k}{n-k} - \binom{t+n-k-1}{n-k-1}\right) - \binom{t+n-2}{n-1} \\
&= \binom{t+k-1}{k-1}\binom{t+n-k-1}{n-k} - \binom{t+n-2}{n-1}
\end{aligned}$$

as desired. □

We observe that $\mathcal{Q}_{2,n}$ agrees with the basis polytope of the minimal matroid $T_{2,n}$. For $1 \le \ell \le n-1$ we now consider the half-open polytope

$$\mathcal{R}_{\ell,n} := \left\{x \in \Delta_{2,n} : \sum_{i=1}^{\ell} x_i > 1\right\} = \left\{x \in \Delta_{2,n} : \sum_{i=\ell+1}^{n} x_i < 1\right\}.$$

Observe that $\mathcal{R}_{1,n}$ agrees with $\widetilde{\mathcal{Q}}_{1,n}$ which is the empty polytope. Furthermore, note that each of the polytopes $\mathcal{R}_{\ell,n}$ can be decomposed as

$$\mathcal{R}_{\ell,n} = \widetilde{\mathcal{Q}}_{1,n} \sqcup \widetilde{\mathcal{Q}}_{2,n} \sqcup \cdots \sqcup \widetilde{\mathcal{Q}}_{\ell,n}\,.$$

For $1 \le a \le n$ we define the polynomials

$$P_{a,n} := \sum_{k=1}^{a} \binom{t+n-k-1}{n-k}\binom{t+k-1}{k-1}\,.$$

In particular, $P_{0,n} := 0$ for all $n \ge 0$.

As a direct consequence of Proposition 4.4.4 we obtain the following.

Corollary 4.4.5 *For all* $1 \le \ell \le n-1$ *the Ehrhart polynomial of* $\mathcal{R}_{\ell,n}$ *equals*

$$\operatorname{ehr}(\mathcal{R}_{\ell,n}, t) = P_{\ell,n}(t) - \ell \binom{t+n-2}{n-1}.$$

We are ready to state a formula for the Ehrhart polynomial of all connected rank 2 matroids.

Theorem 4.4.6 *Let M be a connected matroid of rank 2. Suppose that M has exactly s hyperplanes of sizes $a_1, \dots, a_s$. Then $s \ge 3$ and we have*

$$\operatorname{ehr}(\mathcal{P}(M), t) = \binom{2t+n-1}{n-1} - \sum_{i=1}^{s} P_{a_i,n}(t),$$

where

$$P_{a,n}(t) := \sum_{k=1}^{a} \binom{t+n-k-1}{n-k}\binom{t+k-1}{k-1}$$

for $1 \le a \le n$.

Proof. First note that a rank 2 matroid is disconnected whenever it has only $s \le 2$ hyperplanes. Moreover, the ground set of a connected matroid M of rank 2 with s hyperplanes has at least $s \ge 3$ elements, and a connected matroid on $n \ge 2$ elements is loopless. Thus formula (4.19) applies and hence the basis polytope of M is

$$\mathcal{P}(M) = \left\{ x \in \Delta_{2,n} : \sum_{i \in H} x_i \le 1 \text{ for every } H \text{ hyperplane} \right\}.$$

Furthermore, the matroid hyperplanes of a loopless rank 2 matroid partition the ground set. Now pick any hyperplane H of cardinality a_r. The subset of $\Delta_{2,n}$ that violates the inequality for H is a copy of $\mathcal{R}_{a_r,n}$ after permuting the coordinates. Moreover, Lemma 4.4.2 shows that a point in $\Delta_{2,n}$ can violate at most one inequality imposed by a hyperplane.

Therefore, by applying the formulas for the Ehrhart polynomials of Corollary 4.4.5 and for the hypersimplex, we get

$$\begin{aligned}
\operatorname{ehr}(\mathcal{P}(M), t) &= \operatorname{ehr}(\Delta_{2,n}, t) - \sum_{i=1}^{s} \operatorname{ehr}(\mathcal{R}_{a_i,n}, t) \\
&= \left(\binom{2t+n-1}{n-1} - n\binom{t+n-2}{n-1} \right) - \sum_{i=1}^{s} \left(P_{a_i,n}(t) - a_i \binom{t+n-2}{n-1} \right) \\
&= \binom{2t+n-1}{n-1} - \sum_{i=1}^{s} P_{a_i,n}(t)
\end{aligned}$$

where in the last step we used $a_1 + \cdots + a_s = n$ which is satisfied since the hyperplanes form a partition of the groundset. $\square$

Ehrhart positivity

We are going to prove that all matroids of rank 2 are Ehrhart positive. Our proof rests on the following *superadditivity* of the polynomials $P_{a,n}$.

Proposition 4.4.7 *For all nonnegative integers a, b, n such that $a + b \le n$*

$$P_{a,n} + P_{b,n} \preceq P_{a+b,n}\,.$$

Moreover, the inequality on the coefficients of positive degree is strict whenever $a, b > 0$.

Proof. There is nothing to show if $a = 0$. Thus fix numbers $1 \le a \le b$ such that $a + b \le n$. We are going to prove that

$$P_{a,n} + P_{b,n} \preceq P_{a-1,n} + P_{b+1,n}. \tag{4.20}$$

This will prove the claim since applying this inequality a times yields

$$P_{a,n} + P_{b,n} \preceq P_{a-1,n} + P_{b+1,n} \preceq P_{a-2,n} + P_{b+2,n} \preceq \cdots \preceq P_{0,n} + P_{a+b,n} = P_{a+b,n}.$$

Moreover, our proof will show that in (4.20) the inequality on the coefficients of positive degree is strict. Inequality (4.20) is equivalent to

$$P_{a,n} - P_{a-1,n} \preceq P_{b+1,n} - P_{b,n},$$

which, by definition, is equivalent to

$$\binom{t+n-a-1}{n-a}\binom{t+a-1}{a-1} \preceq \binom{t+n-b-2}{n-b-1}\binom{t+b}{b}. \tag{4.21}$$

Notice that both sides have the common factor $\binom{t+n-b-2}{n-b-1}\binom{t+a-1}{a-1}$ which has nonnegative coefficients. After canceling this factor and multiplying with the positive number $\binom{b}{b-a+1}\binom{n-a}{b-a+1}$, we obtain the inequality

$$\binom{t+n-a-1}{b-a+1}\binom{b}{b-a+1} \preceq \binom{t+b}{b-a+1}\binom{n-a}{b-a+1}. \tag{4.22}$$

Inequality (4.21) is implied by (4.22) using property (4.18). Also, notice that if we prove that (4.22) is strict for all coefficients, then (4.21) is strict for all coefficients of positive degree. This is because the polynomial $\binom{t+n-b-2}{n-b-1}\binom{t+a-1}{a-1}$ is a product of t and a polynomial with positive coefficients.

To prove this, we use the following variables $c = n - a$ and $u = b - a + 1$. Since $a + b \le n$ we have $b \le c$. Moreover, we have $1 \le u \le b$. Observe that inequality (4.22) reads

$$\binom{t+c-1}{u}\binom{b}{u} \preceq \binom{t+b}{u}\binom{c}{u}, \tag{4.23}$$

after substitution. Observe further that if $b = c$, then the inequality is automatically satisfied, and is in fact strict on all coefficients. Assume now that $b < c$, so that $c - 1 \geq b$. Notice that if we multiply twice with $u!$, the inequality to prove becomes

$$(t + c - 1) \cdots (t + c - u) \cdot \frac{b!}{(b - u)!} \preceq (t + b) \cdots (t + b - u + 1) \cdot \frac{c!}{(c - u)!}.$$

which can be rewritten as

$$\frac{(c - u)!}{c!} \cdot (t + c - 1) \cdots (t + c - u) \preceq \frac{(b - u)!}{b!} \cdot (t + b) \cdots (t + b - u + 1).$$

And this is equivalent to

$$\frac{c - u}{c} \cdot \left(\frac{t}{c - 1} + 1\right) \cdots \left(\frac{t}{c - u} + 1\right) \preceq \left(\frac{t}{b} + 1\right) \cdots \left(\frac{t}{b - u + 1} + 1\right).$$

And since $c - 1 \geq b$, and $\frac{c-u}{c} < 1$, the claim follows from property (4.18) by comparing the coefficients at each individual factor on the left with the corresponding factor on the right. □

Theorem 4.4.8 *Let M be a connected matroid of rank 2 on n elements. Then*

$$\operatorname{ehr}(\mathcal{P}(T_{2,n}), t) \preceq \operatorname{ehr}(\mathcal{P}(M), t) \preceq \operatorname{ehr}(\mathcal{P}(U_{2,n}), t)\,.$$

Moreover, the inequalities are strict on the coefficients of positive degree whenever the matroid M is neither minimal nor uniform. In particular, all the basis polytopes of matroids of rank 2 matroids are Ehrhart positive.

Proof. The minimal matroid $T_{2,n}$ has exactly three hyperplanes, of cardinalities 1, 1, and $n-2$, respectively. The uniform matroid $U_{2,n}$, on the other hand, has n hyperplanes each of cardinality 1.

Since we are under the hypothesis of M being connected, we know that M has at least $s \geq 3$ hyperplanes that partition the groundset. Assume that these hyperplanes have cardinalities $a_1, \ldots, a_s$. These numbers sum to n.

By using Theorem 4.4.6, after cancelling $\binom{2t+n-1}{n-1}$ and multiplying by -1, the inequalities to prove read

$$\underbrace{P_{1,n} + \cdots + P_{1,n}}_{n \text{ summands}} \preceq \sum_{i=1}^{s} P_{a_i,n}(t) \preceq P_{1,n} + P_{1,n} + P_{n-2,n}. \qquad (4.24)$$

The left inequality follows directly from the superadditivity in Proposition 4.4.7, since we may group the summands on the left into groups of sizes $a_1, \ldots, a_s$ and get the inequality with the expression in the middle. To prove the right inequality, we proceed by looking at inequality (4.20). Recall that $s \geq 3$, so that we can assume $1 \leq a_1 \leq a_2 \leq a_3$. By repeatedly applying (4.20) we get

$$P_{a_1,n} + P_{a_2,n} + P_{a_3,n} \preceq P_{1,n} + P_{a_1+a_2-1,n} + P_{a_3,n}$$

$$\preceq P_{1,n} + P_{1,n} + P_{a_1+a_2+a_3-2,n}.$$

Using the superadditivity again we arrive at

$$\sum_{i=1}^{s} P_{a_i,s} \preceq P_{1,n} + P_{1,n} + P_{a_1+a_2+a_3-2,n} + \sum_{i=4}^{s} P_{a_i,s} \preceq P_{1,n} + P_{1,n} + P_{n-2,n},$$

which completes the proof of the desired inequality. In particular, it follows that all connected matroids of rank 2 are Ehrhart positive. Moreover, the inequalities given in (4.24) are strict for the coefficients of positive degree by Proposition 4.4.7. This proves that the coefficients of the Ehrhart polynomial of a connected rank 2 matroid M are strictly between those of the minimal and the uniform matroid whenever the coefficient is not the constant term and M is neither $T_{2,n}$ nor $U_{2,n}$. $\square$

CHAPTER 5

Conjectures and open problems

5.1 The h^*-polynomial

In the preceding chapters we have put little to no attention to the h^*-polynomial of matroid polytopes. In [DHK09], apart from the Ehrhart positivity conjecture that we have disproved, De Loera et al. left another intriguing conjecture that we will now discuss.

A polynomial $p(t) = \sum_{j=0}^{d} a_i t^i$ of degree d is said to be *unimodal* if there is an index j such that

$$a_0 \leq \cdots \leq a_{j-1} \leq a_j \geq a_{j+1} \geq \cdots \geq a_d.$$

Also, we say that it has *no internal zeros* if whenever we choose two nonzero coefficients, say a_j and a_k with $j < k$, it holds that all the coefficients a_i for $j \leq i \leq k$ are nonzero.

A condition that is stronger than unimodality is the *log-concavity*. We say that p defined as above is *log-concave* if it has no internal zeros and for every $j = 1, \dots, d-1$ it holds:

$$a_j^2 \geq a_{j-1}a_{j+1}.$$

Many polynomials that appear naturally in combinatorics have some of the two above properties. In some cases, it is easier to prove the log-concavity than the unimodality but, in general terms, both of these two properties are usually hard to establish.

Let us state the second of De Loera et al.'s conjectures.

Conjecture 5.1.1 *The h^*-polynomial of the basis polytope of a matroid has unimodal coefficients.*

This conjecture, which has not been proved nor disproved yet, admits a stronger form, that we will support with many examples and particular cases.

Conjecture 5.1.2 *The h^*-polynomial of the basis and the independence polytope of a matroid have real roots.*

The fact that our conjecture is indeed stronger than Conjecture 5.1.1 is a consequence of the following well-known result, whose proof can be found for example in [Brä15].

Proposition 5.1.3 *Let p be a polynomial with positive coefficients that has only real roots. Then p is log-concave and, in particular, it is unimodal.*

Many families of polytopes do have h^*-polynomials that are real-rooted. For instance, consider any n-uple of positive integers $\mathbf{s} = (s_1, \dots, s_n)$, and the polytope:

$$\mathcal{P}_{\mathbf{s}} = \left\{ x \in \mathbb{R}^n : 0 \leq \frac{x_1}{s_1} \leq \frac{x_2}{s_2} \leq \cdots \leq \frac{x_n}{s_n} \leq 1 \right\}.$$

Such a polytope is called an $\mathbf{s}$-*lecture hall polytope*. It can be proved that it has integral vertices. Moreover, in [SV15] Savage and Visontai proved

Theorem 5.1.4 *The h^*-polynomials of all* $\mathbf{s}$*-lecture hall polytopes are real-rooted.*

Another example of polytopes with such property is that of *zonotopes*. Consider a list of integer vectors $L = \{v_1, \dots, v_m\} \subseteq \mathbb{R}^n$, and let $\mathcal{Z} \subseteq \mathbb{R}^n$ be the polytope defined by:

$$\mathcal{Z} = \left\{ \sum_{i=1}^{m} \lambda_i v_i : 0 \leq \lambda_i \leq 1 \text{ for all } i \right\}.$$

A polytope as above is what we call a zonotope.

Theorem 5.1.5 *All integral zonotopes are Ehrhart positive and h^*-real-rooted.*

For a proof of the Ehrhart positivity, one can read [DM12] by D'Adderio and Moci, whereas the proof of the h^*-real-rootedness is due to Beck, Jochemko and McCullough and can be found in [BJM19].

A particulary interesting zonotope that is worth-mentioning is precisely the *regular permutohedron*, i.e. the convex hull of all the points in $\mathbb{R}^n$ of the form $(\sigma(1), \dots, \sigma(n))$ for the $n!$ permutations $\sigma \in S_n$.

Evidence for the real-rootedness conjecture

As a corollary of a result proved by Wagner in [Wag92] we have the following:

Proposition 5.1.6 *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two lattice polytopes. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are h^*-real-rooted, then $\mathcal{P}_1 \times \mathcal{P}_2$ is h^*-real-rooted.*

A natural consequence of this and Proposition 1.2.7 is that we can restrict ourselves again only to connected matroids to attack Conjecture 5.1.2.

Theorem 5.1.7 *The following polytopes are h^*-real-rooted.*

(a) $\mathcal{P}(T_{k,n})$ *for all minimal matroids* $T_{k,n}$*.*

(b) $\mathcal{P}_{\mathsf{I}}(T_{k,n})$ *for* $1 \leq k \leq n \leq 200$

(c) $\mathcal{P}(U_{k,n})$ *and* $\mathcal{P}_{\mathsf{I}}(U_{k,n})$ *for all* $1 \leq k \leq n \leq 200$.

(d) $\mathcal{P}(M)$ *and* $\mathcal{P}_{\mathsf{I}}(M)$ *for all sparse paving matroids of rank* 2.

(e) $\mathcal{P}(M)$ *and* $\mathcal{P}_{\mathsf{I}}(M)$ *for all matroids with* 9 *or less elements.*

(f) $\mathcal{P}(M)$ *and* $\mathcal{P}_{\mathsf{I}}(M)$ *for all sparse paving matroids with at most* 30 *elements.*

(g) $\mathcal{P}(M)$ *for all matroids* M *of rank* 2 *and at most* 35 *elements.*

Proof. We have already stated (a) as Corollary 4.1.11. To prove (b) and (c) we use a computer and our explicit formulas of the Ehrhart polynomials of such polytopes; for $\mathcal{P}_{\mathsf{I}}(T_{k,n})$ it is necessary to use the fact that $T_{k,n}$ is obtained by relaxing a circuit-hyperplane on the matroid $U_{k-1,k} \oplus U_{1,n-k}$, and Corollary 4.1.17. The proof of (d) is ad-hoc and is included below. For (e) we employed a database of matroids and the use of a computer. To prove (f) we use a computer and the upper bound for the number of circuit-hyperplanes of a sparse-paving matroid given by Lemma 4.3.8. To prove (g), we rely on the fact that connected matroids of rank 2 and n elements are in bijection with partitions of the set $\{1, \ldots, n\}$ into at least 3 parts, and that Theorem 4.4.6 gives us a formula for the Ehrhart polynomial by knowing the cardinalities of the parts. □

Part of the proof of (d) was pointed out to the author by Fedor Petrov, in the platform MathOverflow. It is very ad-hoc in nature, and heavily relies on a very explicit formula that exists for $h^*(\Delta_{2,n}, x)$. Before giving the proof we observe the following.

Remark 5.1.8. Whenever M is a sparse paving matroid of rank 2 with λ circuit-hyperplanes, it holds $\lambda \leq \lfloor \frac{n}{2} \rfloor$. Indeed, by the equivalences of Theorem 4.2.6, the Johnson graph $J(n, 2)$ has as set of vertices all the 2-subsets of $\{1, \ldots, n\}$. A stable subset of this graph corresponds with a way of choosing 2-subsets of $\{1, \ldots, n\}$ in such a way that are pairwise disjoint. If there were more than $\lfloor \frac{n}{2} \rfloor$ such sets, there would be at least $n + 1$ elements in total, which is a contradiction.

Proof of (d) in Theorem 5.1.7. By using Corollary 4.1.19, we obtain that whenever M is a sparse paving matroid of rank k, cardinality n and having exactly λ circuit-hyperplanes, then

$$\begin{aligned} h^*(\mathcal{P}(M), x) &= h^*(\Delta_{k,n}, x) - \lambda x \cdot h^*(\mathcal{P}(T_{k,n}), t) \\ h^*(\mathcal{P}_{\mathsf{I}}(M), x) &= h^*(\mathcal{P}_{\mathsf{I}}(U_{k,n}), x) - \lambda x \cdot h^*(\mathcal{P}(T_{k,n+1}), t) \end{aligned}$$

By using the formulas of Theorem 3.1.1 and Theorem 3.4.1, we readily see that the Ehrhart polynomials of $\Delta_{2,n}$ and $\mathcal{P}_{\mathsf{I}}(U_{2,n})$ can be written as follows

$$\operatorname{ehr}(\Delta_{2,n}, t) = \binom{2t + n - 1}{n - 1} - n \binom{t + n - 2}{n - 1}$$

$$\operatorname{ehr}(\mathcal{P}_{\mathsf{I}}(U_{2,n}),t) = \binom{2t+n}{n} - n\binom{t+n-1}{n}.$$

By using a combinatorial identity, it can be proved that

$$h^*(\Delta_{2,n},x) = -nx + \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} x^j = -nx + \frac{(1+\sqrt{x})^n + (1-\sqrt{x})^n}{2}.$$

Also, by looking at the above formulas, it is possible to see that

$$h^*(\mathcal{P}_{\mathsf{I}}(U_{k,n}),x) = h^*(\Delta_{2,n+1},x) + x.$$

Recall that in Theorem 4.1.11 we established a formula for the h^*-polynomial of the basis polytope of $T_{k,n}$. When the rank $k = 2$, it assumes a very simple form:

$$h^*(\mathcal{P}(T_{2,n}),x) = (n-3)x + 1.$$

Putting all this pieces together, we obtain

$$h^*(\mathcal{P}(M),x) = -\lambda(n-3)x^2 - (\lambda+n)x + \frac{(1+\sqrt{x})^n + (1-\sqrt{x})^n}{2}$$
$$h^*(\mathcal{P}_{\mathsf{I}}(M),x) = -\lambda(n-2)x^2 - (\lambda+n)x + \frac{(1+\sqrt{x})^{n+1} + (1-\sqrt{x})^{n+1}}{2}$$

From now on, we will focus only on the h^*-polynomial of $\mathcal{P}(M)$. The proof for the independence polytope carries out exactly the same way, by just being careful with the shifting in some of the coefficients. What we have to prove is that $h^*(\mathcal{P}(M),x)$ has $\lfloor \frac{n}{2} \rfloor$ negative roots. By writing $x = -\tan^2\theta$ for $0 \leq \theta < \pi/2$, and using the identity

$$\begin{aligned}
\frac{(1+i\tan\theta)^n + (1-i\tan\theta)^n}{2} &= \frac{(\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n}{2\cos^n t} \\
&= \frac{e^{in\theta} + e^{-in\theta}}{2\cos^n\theta} \\
&= \frac{\cos n\theta}{\cos^n\theta}
\end{aligned}$$

we can change our h^*-polynomial into:

$$\begin{aligned}
h^*(\mathcal{P}(M), -\tan^2\theta) &= -\lambda(n-3)\tan^4\theta + (\lambda+n)\tan^2\theta + \frac{\cos n\theta}{\cos^n\theta} \\
&= \frac{-\lambda(n-3)\sin^4\theta\cos^{n-4}\theta + (\lambda+n)\sin^2\theta\cos^{n-2}\theta + \cos n\theta}{\cos^n\theta}.
\end{aligned}$$

The denominator is clearly positive for $0 < \theta < \pi/2$. Let us analyze the numerator. What we will prove is that evaluating it at the points $\theta_j = j\frac{\pi}{n}$ for $j = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$ gives us values that alternate in sign. This will conduct us to the result, by a mere application of the intermediate value theorem (notice that it shall be impossible to evaluate in $\frac{\pi}{2}$, but this is a minor issue that can be resolved by evaluating in a sufficiently close point, by continuity). To prove that it alternates in sign, first observe that $\cos(n\theta_j) = 1$ whenever j is even and equals -1 whenever j is odd, so that we can focus on the following expression

$$-\lambda(n-3)\sin^4\theta\cos^{n-4}\theta + (\lambda+n)\sin^2\theta\cos^{n-2}\theta.$$

By using the variable $y = \cos\theta$, we can rewrite this as follows

$$f(y) = -\lambda(n-3)(1-y^2)^2 y^{n-4} + (\lambda+n)(1-y^2)y^{n-2}.$$

Our claim is that $-1 < f(y) < 1$ for all $0 \le y \le 1$. Observe that this implies that the numerator of the expression for the h^*-polynomial alternates in sign for the θ_j's, because the summand $\cos n\theta$ will be 1 or -1 and the remaining terms will not add up a value of enough size to change the sign.

We have reduced the problem of proving the real-rootedness of $h^*(\mathcal{P}(M), x)$ to an elementary problem of maxima and minima for the differentiable function f in the compact interval $[0, 1]$. What follows is the proof of both the upper and the lower bounds. We warn the reader that although the expression for f is a polynomial, achieving these tight bounds is somewhat tricky and requires some clever manipulations. Recall that by Remark 5.1.8 we have that $\lambda \le \lfloor \frac{n}{2} \rfloor$, as this will be used several times in what follows.

To bound f we will use the following factorization:

$$\begin{aligned} f(y) &= y^{n-4}(1-y^2)\left(-\lambda(n-3)(1-y^2) + (\lambda+n)y^2\right) \\ &= y^{n-4}(1-y^2)\left(y^2 - \frac{(n-3)\lambda}{n+(n-2)\lambda}\right)(n+(n-2)\lambda). \end{aligned} \tag{5.1}$$

Let us prove first that the maximal value of f for $y \in [0, 1]$ is less than 1. Observe that the first, the second and the fourth factors are nonnegative, so that we may restrict ourselves to values of y such that the third factor is positive too. Let us consider some cases:

- Assume that $0 \le \lambda < \frac{7}{20}n$. The reason of the strange constant $\frac{7}{20}$ is that it provides a somewhat good approximation of $\frac{e}{2} - 1$ with small numerator and denominator. Consider the factors $y^{n-4}(1-y^2)$. By differentiating, we can see that its maximum is achieved at $y_0 = \sqrt{1 - \frac{2}{n-2}}$. Hence:

$$\begin{aligned} f(y) &\le y_0^{n-4}(1-y_0^2)\left(y^2 - \frac{(n-3)\lambda}{n+(n-2)\lambda}\right)(n+(n-2)\lambda) \\ &= \left(1 - \frac{2}{n-2}\right)^{\frac{n-4}{2}} \frac{2}{n-2}\left(y^2 - \frac{(n-3)\lambda}{n+(n-2)\lambda}\right)(n+(n-2)\lambda) \\ &\le \left(1 - \frac{2}{n-2}\right)^{\frac{n-4}{2}} \frac{2}{n-2}\left(1^2 - \frac{(n-3)\lambda}{n+(n-2)\lambda}\right)(n+(n-2)\lambda) \\ &= \left(1 - \frac{2}{n-2}\right)^{\frac{n-4}{2}} \frac{2}{n-2}\cdot(n+\lambda) \end{aligned}$$

$$< \left(1 - \frac{2}{n-2}\right)^{\frac{n-4}{2}} \frac{2}{n-2} \cdot \frac{27n}{20}$$

Notice that the factor $\left(1 - \frac{2}{n-2}\right)^{\frac{n-4}{2}}$ converges to $e^{-1} \approx 0.3678$. In fact, for $n \geq 177$, it is $\left(1 - \frac{2}{n-2}\right)^{\frac{n-4}{2}} < \frac{37}{100}$. So that for such values of n, we have:

$$f(y) < \frac{37}{100} \cdot \frac{1}{n-2} \cdot \frac{54n}{20} = \frac{1998}{2000} \frac{n}{n-2}$$

and adding the condition $n \geq 2000$. In fact, $\left(1 - \frac{2}{n-2}\right)^{\frac{n-4}{2}} \frac{2}{n-2} \cdot \frac{27n}{20}$ is smaller than one for all $n \geq 447$. It is also possible to verify with a computer that for $n \leq 446$ and $0 \leq \lambda \leq \frac{7n}{20}$, indeed f is bounded by above by 1.

- Let us assume now that $\frac{7n}{20} \leq \lambda \leq \frac{n}{2}$. We analyze two subcases now.

  - Assume $1 - y^2 < \frac{1}{n+\lambda}$. Then, we have the chain of inequalities:

$$\begin{aligned} f(y) &< y^{n-4} \frac{1}{n+\lambda} \left(y^2 - \frac{(n-3)\lambda}{n+(n-2)\lambda}\right)(n+(n-2)\lambda) \\ &\leq 1 \cdot \frac{1}{n+\lambda} \left(1 - \frac{(n-3)\lambda}{n+(n-2)\lambda}\right)(n+(n-2)\lambda) \\ &= \frac{1}{n+\lambda} \cdot (n+\lambda) \\ &= 1. \end{aligned}$$

  - Assume $1 - y^2 \geq \frac{1}{n+\lambda}$. Since we needed the positivity for the third factor in (5.1), we have to look at f in the following interval

$$\frac{(n-3)\lambda}{n+(n-2)\lambda} \leq y^2 \leq 1 - \frac{1}{n+\lambda}. \tag{5.2}$$

Using this, we obtain:

$$\begin{aligned} f(y) &\leq y^{n-4}(1-y^2)\left(1 - \frac{1}{n+\lambda} - \frac{(n-3)\lambda}{n+(n-2)\lambda}\right)(n+(n-2)\lambda) \\ &= y^{n-4}(1-y^2)\left(n+\lambda - \frac{n+(n-2)\lambda}{n+\lambda}\right) \\ &= y^{n-4}(1-y^2)(n+\lambda) - y^{n-4}(1-y^2)\frac{n+(n-2)\lambda}{n+\lambda} \end{aligned} \tag{5.3}$$

Let us call $f_1(y) = y^{n-4}(1-y^2)(n+\lambda)$ and $f_2(y) = y^{n-4}(1-y^2)\frac{n+(n-2)\lambda}{n+\lambda}$. We will maximize f_1 and minimize f_2 in the interval of equation (5.2). It suffices to show that the difference between that maximum and that minimum is less than one and the proof will be complete.

To maximize f_1 we proceed by differentiating as we did in the first case. We obtain:

$$
\begin{aligned}
f_1(y) &\le \left(1 - \frac{2}{n-2}\right)^{\frac{n-4}{2}} \cdot \frac{2}{n-2} \cdot (n+\lambda) \\
&\le \left(1 - \frac{2}{n-2}\right)^{\frac{n-4}{2}} \cdot \frac{2}{n-2} \cdot \frac{3n}{2} && (5.4) \\
&\le \frac{37}{100} \cdot \frac{3n}{n-2} && (5.5) \\
&= \frac{111}{100} \cdot \frac{n}{n-2} \\
&\le \frac{111}{100} \cdot \frac{2000}{1998} \\
&= 1.1122\ldots
\end{aligned}
$$

where, as in the first case, we used the bounds that come from $n \ge 2000$. Moreover, the expression on equation (5.4) is already less than $1.1122\ldots$ for all $n \ge 390$.

To minimize f_2, we observe that up to multiplication by a constant, f_2 and f_1 are the same function. Both of these functions only have one critic point in the interval (5.2), that corresponds to a maximum. Hence, the minimum of f_2 is attained at one of the two extremes of the interval.

- If $y = \sqrt{1 - \frac{1}{n+\lambda}}$. We obtain:

$$
\begin{aligned}
f_2(y) &= \left(1 - \frac{1}{n+\lambda}\right)^{\frac{n-4}{2}} \cdot \frac{1}{n+\lambda} \cdot \frac{n+(n-2)\lambda}{n+\lambda} \\
&\ge \left(1 - \frac{1}{n+\frac{7n}{20}}\right)^{\frac{n-4}{2}} \cdot \frac{1}{n+\frac{n}{2}} \cdot \frac{n+(n-2)\frac{7n}{20}}{n+\frac{n}{2}} \\
&= \left(1 - \frac{1}{n+\frac{7n}{20}}\right)^{\frac{n-4}{2}} \cdot \frac{4}{9n}\left(1 + (n-2)\frac{7}{20}\right) \\
&\ge e^{-\frac{2}{1+7/20}} \cdot \frac{28n+24}{45n} \\
&> e^{-\frac{2}{1+7/20}} \cdot \frac{28}{45} \\
&= 0.14143\cdots
\end{aligned}
$$

- If $y = \sqrt{\frac{(n-3)\lambda}{n+(n-2)\lambda}}$. We obtain:

$$
\begin{aligned}
f_2(y) &= \left(\frac{(n-3)\lambda}{n+(n-2)\lambda}\right)^{\frac{n-4}{2}} \left(\frac{n+\lambda}{n+(n-2)\lambda}\right) \frac{n+(n-2)\lambda}{n+\lambda} \\
&= \left(\frac{(n-3)\lambda}{n+(n-2)\lambda}\right)^{\frac{n-4}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(1 - \frac{\lambda + n}{n + (n-2)\lambda}\right)^{\frac{n-4}{2}} \\
&\geq \left(1 - \frac{\frac{n}{2} + n}{n + (n-2)\frac{7n}{20}}\right)^{\frac{n-4}{2}} \\
&= \left(1 - \frac{20/7 \cdot 3/2}{20/7 + (n-2)}\right)^{\frac{n-4}{2}} \\
&\geq e^{-\frac{3}{4}\cdot\frac{20}{7}} \\
&= 0.117319\ldots
\end{aligned}
$$

Now it is clear that the maximum of f_1 is at most 1.112 and the minimum of f_2 is at least 0.117. Hence, the maximum value of $f = f_1 - f_2$ in the interval defined in (5.2) is less than $0.995 < 1$, as desired.

So far we have proved that $f(y) < 1$ for all $y \in [0, 1]$. It remains to prove that $f(y) > -1$ in this interval. Observe that now we have to focus only on values such that the third factor in (5.1) is negative. In other words, by changing its sign, we have to maximize

$$
g(y) = y^{n-4}(1 - y^2)\left(\frac{(n-3)\lambda}{n + (n-2)\lambda} - y^2\right)(n + (n-2)\lambda)
$$

in the interval $\left[0, \sqrt{\frac{(n-3)\lambda}{n+(n-2)\lambda}}\right]$.

Let us factor $g(y) = u_1(y) \cdot u_2(y) \cdot (n + (n-2)\lambda)$, with:

$$
\begin{aligned}
u_1(x) &= y^{\frac{n-4}{2}}(c - y^2) \\
u_2(x) &= y^{\frac{n-4}{2}}(1 - y^2)
\end{aligned}
$$

where we use the shorthand $c = \frac{(n-3)\lambda}{n+(n-2)\lambda}$. We can bound u_1 by find its maximum via differentiating and equating to zero. In fact $u_1'(x) = c\frac{n-4}{2}y^{\frac{n-6}{2}} - \frac{n}{2}y^{\frac{n-2}{2}}$ yields that the maximum is achieved at $y^2 = c\left(1 - \frac{4}{n}\right)$. Hence:

$$
\begin{aligned}
u_1(x) &\leq c^{\frac{n-4}{4}}\left(1 - \frac{4}{n}\right)^{\frac{n-4}{4}} c\frac{4}{n} \\
&< c^{\frac{n}{4}}\frac{2}{n}
\end{aligned}
$$

where we used that the term $\left(1 - \frac{4}{n}\right)^{\frac{n-4}{4}}$ is less than $\frac{1}{2}$ for $n > 10$ (in fact, it converges to e^{-1}). By doing the same with u_2, we obtain:

$$
u_2(x) < \frac{2}{n},
$$

and combining all of our results, what we obtain is:

$$
f_2(x) = u_1(y)u_2(y) \cdot (n + (n-2)\lambda)
$$

$$
\begin{aligned}
&< c^{\frac{n-2}{4}} \frac{4}{n^2} (n + (n-2)\lambda) \\
&\leq c^{\frac{n}{4}} \frac{4}{n^2} \cdot \left(n + (n-2)\frac{n}{2}\right) \\
&= c^{\frac{n}{4}} \cdot 2 \\
&= \left(\frac{(n-3)\lambda}{n + (n-2)\lambda}\right)^{\frac{n-2}{4}} \cdot 2 \\
&\leq \left(\frac{(n-3)\lambda}{2\lambda + (n-2)\lambda}\right)^{\frac{n-2}{4}} \cdot 2 \\
&= \left(1 - \frac{3}{n}\right)^{\frac{n-2}{4}} \cdot 2 \\
&< 1,
\end{aligned}
$$

where in the last step we used that $\left(1 - \frac{3}{n}\right)^{\frac{n-2}{4}} < \frac{1}{2}$ for $n > 10$, provided that this term converges to $e^{-3/4} \approx 0.472 < \frac{1}{2}$. To finish the proof, we take care of the fact of checking that our claim on the cases $n \leq 446$. We have verified this with a computer. $\square$

5.2 Open problems

In this final section we collect several open problems and conjectures that may lead to further understanding of the Ehrhart theory of matroids. They are not intended as a comprehensive collection of open problems but rather some directions of research that we believe might very well be pursued.

Weighted Lah Numbers

We start with a conjecture that has been checked for several values using a computer. It is interesting as it provides a certain understanding of the weighted Lah numbers, which were the key to prove the Ehrhart positivity of hypersimplices.

Conjecture 5.2.1 *The polynomial $C_{n,m}$ defined by*

$$C_{n,m}(x) = \sum_{\ell=0}^{n-m-1} W(\ell, n, m+1)x^{\ell},$$

has all of its complex roots lying on the unit circle.

In Chapter 3 we obtained a formula for the coefficients of the Ehrhart polynomial of the hypersimplex. Our formula is manifestly positive and combinatorial. A question that arises is the following: is it possible to interpret these coefficients as the solution to a “nice” combinatorial problem?

Open Problem 5.2.2 *Find a combinatorial interpretation for the quantity:*

$$\sum_{\ell=0}^{k-1} W(\ell, n, m+1) A(m, k-\ell-1).$$

Ehrhart coefficients

We restate here Conjecture 4.0.1, which initially motivated us to take a look at minimal matroids. Although minimal matroids were studied first as potential candidates to realise the minimum coefficient-wise Ehrhart polynomial for connected matroids of fixed rank and cardinality, at the end of the story, they do not satisfy that property after all. This is because we have seen that $\mathcal{P}(T_{k,n})$ is Ehrhart positive, whereas there are basis polytopes of matroids with negative Ehrhart coefficients. Nevertheless, the *upper-bound conjecture* still stands.

Conjecture 5.2.3 *If M is a connected matroid of rank k and cardinality n, then*

- $\operatorname{ehr}(\mathcal{P}(M), t)$ *is coefficient-wise smaller than* $\operatorname{ehr}(\Delta_{k,n}, t)$.
- $\operatorname{ehr}(\mathcal{P}_{\mathsf{I}}(M), t)$ *is coefficient-wise smaller than* $\operatorname{ehr}(\mathcal{P}_{\mathsf{I}}(U_{k,n}), t)$.

We recall that this Conjecture has been proved for all sparse paving matroids and for all rank 2 matroids. Observe that it is not even clear that the Ehrhart polynomial of the basis polytope of the minimal matroid $T_{k,n}$ is coefficient-wise smaller than the Ehrhart polynomial of the hypersimplex $\Delta_{k,n}$.

Also, we take the opportunity to state the following problem which might be difficult to address without the help of a computer (and maybe still very hard *with* the help of a computer).

Open Problem 5.2.4 *Is there any matroid M of cardinality* 18 *such that $\mathcal{P}(M)$ is not Ehrhart positive?*

Recall that we proved that there are non Ehrhart positive matroids for each cardinality starting from 19. Also, we proved that for sparse paving matroids with 17 or less elements, the Ehrhart coefficients are always positive.

Triangulations and Positroids

A fact that is intimately related with the h^*-polynomials is the existence of *unimodular triangulations*, i.e. a division of the polytope into simplices of unit volume. Not every polytope admits such a triangulation, but when it does, the h^*-vector of the polytope coincides with the h-vector of the triangulation (for a proof, see [BR15]). It has been proved that in many instances the basis polytope of a matroid admits a unimodular triangulation and, moreover, a regular unimodular triangulation, i.e. a triangulation that is obtained by first lifting the vertices of the polytope to generic heights and then taking the triangulation induced by the lower facets of the lifted polytope.

Conjecture 5.2.5 *If M is a matroid then $\mathcal{P}(M)$ and $\mathcal{P}_{\mathsf{I}}(M)$ admit regular unimodular triangulations.*

A version of this conjecture was stated in [Haw09] and in [Bra16]. There are some matroids for which the above conjecture is known to be true. In particular, hypersimplices have such property [Sta77]. Li [Li12] and by Early and Kim [Ear17, Kim20] to give combinatorial interpretations of the h^*-polynomial of all hypersimplices. A natural question is if it is possible to deduce our formula for the coefficients of the Ehrhart polynomial using such triangulations.

Open Problem 5.2.6 *Derive a geometric proof of Theorem 3.3.3.*

An important property of hypersimplices that captures the fact that they have such a nice triangulation is that they belong to another very interesting general class of polytopes.

Definition 5.2.7. A polytope $\mathcal{P} \subseteq \mathbb{R}^n$ is said to be an *alcoved polytope* if it can be described as:

$$\mathcal{P} = \left\{ x \in \mathbb{R}^n : \begin{array}{llll} \alpha_i & \leq \quad x_i & \leq \beta_i & \text{for } 1 \leq i \leq n \\ \alpha_{ij} & \leq x_i + \cdots + x_j & \leq \beta_{ij} & \text{for } 1 \leq i < j \leq n \end{array} \right\},$$

for some $\alpha_i, \beta_i, \alpha_{ij}, \beta_{ij} \in \mathbb{Z}$.

A basic property of alcoved polytopes is that they are always lattice polytopes. They were introduced by Lam and Postnikov in [LP07]. Their original definition is slightly different but essentially equivalent to the above. We state now a fundamental characteristic of alcoved polytopes.

Theorem 5.2.8 *If $\mathcal{P}$ is an alcoved polytope, then it admits a regular unimodular triangulation.*

This motivates us to look more closely into the class of alcoved polytopes that are the basis polytope of some matroid. Matroids whose basis polytope is an alcoved polytope are called *positroids*, [Pos06, Oh11, ARW16]. Again, this was not the original definition of positroids given in [Pos06], but it is equivalent to that. The class of positroids is contained in the class of representable matroids and contains all lattice path matroids.

Open Problem 5.2.9 *Extend Li and Kim's results to all positroids and, if possible, deduce a formula as Theorem 3.3.3 for all positroids.*

A fact that follows readily from our definition of positroid is that all matroids of rank 2 are positroids. Also, as we mentioned before, all hypersimplices are positroids (this is a bit easier to verify). We conjecture that positroids are Ehrhart positive.

Conjecture 5.2.10 *Let M be a positroid. Then $\mathcal{P}(M)$ and $\mathcal{P}_{\mathsf{I}}(M)$ are Ehrhart positive.*

Given the fact that we mentioned regular unimodular triangulations, it is important to say that another well-known consequence of having regular unimodular triangulations is a partial h^*-unimodality.

Theorem 5.2.11 (Athanasiadis - Hibi - Stanley) *Let $\mathcal{P}$ be a lattice polytope of dimension n admitting a regular unimodular triangulation. Then its h^*-vector satisfies the following inequalities*

$$h^*_{\lfloor \frac{n+1}{2} \rfloor} \geq \cdots \geq h^*_{n-1} \geq h^*_n.$$
$$h^*_i \geq h^*_{n+1-i} \textit{ for all } i = 0, \ldots, \lfloor \tfrac{n+1}{2} \rfloor.$$
$$h^*_i \leq \binom{h^*_1 + i - 1}{i} \textit{ for all } i = 0, \ldots, n.$$

For a proof of this and other related results, see [Ath04]. In [BR07] Bruns and Römer proved that whenever a lattice polytope is *Gorenstein*, i.e. it has a *symmetric* h^*-vector and a regular unimodular triangulation, then it is unimodal. Recently in [HLM$^+$21] Hibi et al. gave a characterization of all graphic matroids with symmetric h^*-vector. Their result was extended by Lasoń and Michałek [LM20], and they were able to characterize all matroids with an independence or a basis polytope with a symmetric h^*-vector.

In particular, by combining some of the above results it is possible to see that all positroids with a Gorenstein basis polytope have a unimodal h^*-vector. As an extremely particular case, this allows us to recover a result by De Negri and Hibi [DH97] that establishes that all hypersimplices of the form $\Delta_{n,2n}$ have unimodal h^*-vectors.

APPENDIX A

Combinatorial identities and related results

We have collected in this final section some combinatorial results that are used throughout several proofs.

Lemma A.0.1 (Hockey-Stick identity)

$$\sum_{j=0}^{m} \binom{r+j}{j} = \binom{r+m+1}{r+1}.$$

Lemma A.0.2 (Vandermonde's Identity)

$$\sum_{j=0}^{m} \binom{r}{j}\binom{s}{m-j} = \binom{r+s}{m}.$$

Proposition A.0.3 *Let n and $c_1 \geq \ldots \geq c_k$ be nonnegative integers such that $n \geq \sum_{i=1}^{k-1} c_i$. The number of ways N of putting exactly n indistinguishable balls into k distinguishable boxes of capacities $c_1, c_2, \ldots, c_k$ is given by:*

$$N = \binom{k-1+\sum_{i=1}^{k} c_i - n}{k-1}.$$

Proof. Note that instead of thinking of putting balls in a box, we can think of *leaving free space* in a box.

The sum of free spaces in any possible distribution will be exactly $\sum_{i=1}^{k} c_i - n$. Thus we have to assign free spaces $f_1, \ldots, f_k$ to every box in such a way that their sum is:

$$f_1 + \ldots + f_k = \sum_{i=1}^{k} c_i - n,$$

and we are given the constraint $0 \leq f_i \leq c_i$, of which the inequalities $f_i \leq c_i$ are superfluous since the constraints $f_i \geq 0$ for all i (it is, *all of them*) already imply that

$$f_i \leq \sum_{i=1}^{k} c_i - n \leq c_k \leq c_i .$$

Hence we just have to count the number of ways to put $\sum_{i=1}^{k} c_i - n$ indistinguishable balls into k distinguishable boxes, which gives the desired result. □

Lemma A.0.4 (Surányi's Identity)

$$\binom{r+j}{r}\binom{s+j}{s} = \sum_k \binom{r}{k}\binom{s}{k}\binom{j+r+s-k}{r+s}$$

Proof. See [Szé85, Corollary 2]. □

Lemma A.0.5 (Double Hockey-Stick Identity)

$$\sum_{j=0}^{m}\binom{r+j}{j}\binom{s+j}{j} = \sum_k \binom{r}{k}\binom{s}{k}\binom{r+s+1+m-k}{r+s+1}$$

Proof. We proceed using Surányi's Identity:

$$\begin{aligned}
\sum_{j=0}^{m}\binom{r+j}{j}\binom{s+j}{j} &= \sum_{j=0}^{m}\binom{r+j}{r}\binom{s+j}{s} \\
&= \sum_{j=0}^{m}\sum_{k=0}^{s}\binom{r}{k}\binom{s}{k}\binom{j+r+s-k}{r+s} \\
&= \sum_{k=0}^{s}\binom{r}{k}\binom{s}{k}\sum_{j=0}^{m}\binom{j+r+s-k}{r+s} \\
&= \sum_{k=0}^{m}\binom{r}{k}\binom{s}{k}\binom{r+s+1+m-k}{r+s+1},
\end{aligned}$$

where in the last step we used the Hockey-Stick identity. □

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