Alma Mater Studiorum—Università di Bologna

Dottorato di ricerca in Matematica
Ciclo XXXIII

Settore Concorsuale: 01/A2-GEOMETRIA E ALGEBRA
Settore Scientifico Disciplinare: MAT/02-ALGEBRA

Finite irreducible modules over the conformal superalgebras K_4' and CK_6

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Esame finale anno 2021

ABSTRACT

In this thesis we classify all finite irreducible modules over the conformal superalgebra K_4' by means of their correspondence with irreducible finite conformal modules over the annihilation superalgebra associated with K_4'. We obtain that degenerate Verma modules over the annihilation superalgebra associated with K_4' are part of infinite complexes and the number of these complexes is infinite; we compute the homology of these complexes with techniques of spectral sequences and provide an explicit realization of all irreducible quotients. We prove a technical result, stated by Boyallian, Kac and Liberati, on singular vectors of degenerate Verma modules over the annihilation superalgebra associated with CK_6. We start the computation of the homology of the diagram of infinite complexes of degenerate Verma modules for CK_6 found by Boyallian, Kac and Liberati.

Contents

Introduction

In this thesis we study the finite irreducible representations over the conformal superalgebras K_4' and CK_6.

Finite simple conformal superalgebras were completely classified in [FK] and consist in the list: $\mathrm{Cur}\,\mathfrak{g}$, where $\mathfrak{g}$ is a simple finite−dimensional Lie superalgebra, $W_n (n \geq 0)$, $S_{n,b}$, $\tilde{S}_n$ $(n \geq 2,\ b \in \mathbb{C})$, $K_n (n \geq 0,\ n \neq 4)$, K_4', CK_6. The finite irreducible modules over the conformal superalgebras $\mathrm{Cur}\,\mathfrak{g}$, K_0, K_1 were studied in [CK1]. Boyallian, Kac, Liberati and Rudakov classified all finite irreducible modules over the conformal superalgebras of type W and S in [BKLR]; Boyallian, Kac and Liberati classified all finite irreducible modules over the conformal superalgebras of type K_N in [BKL1]. The classification of all finite irreducible modules over the conformal superalgebras of type K_N, for $N \leq 4$, had been previously studied also by Cheng and Lam in [CL]. Finally a classification of all finite irreducible modules over the conformal superalgebra CK_6 was obtained in [BKL2] and [MZ] with different approaches. For $N = 4$ the conformal superalgebra K_4 is not simple; the derived algebra K_4' is instead a simple conformal superalgebra.

A conformal superalgebra R is a left $\mathbb{Z}_2$−graded $\mathbb{C}[\partial]$−module, endowed with $\mathbb{C}$−bilinear products $(a_{(n)}b)$, defined for all $a, b \in R$ and for all $n \geq 0$, that satisfy some properties (see Definition 1.5). The products $(a_{(n)}b)$ are called n−products. It is possible to associate with a conformal superalgebra R a Lie superalgebra as follows. We consider $\tilde{R} = R[y, y^{-1}]$, that is a left $\mathbb{Z}_2$−graded $\mathbb{C}[\tilde{\partial}]$−module, where $\tilde{\partial} = \partial + \partial_y$. We give $\tilde{R}$ a structure of conformal superalgebra with the definition of n−products starting from the n−products in R. We take the quotient $\tilde{R}/\tilde{\partial}\tilde{R}$, on which the 0−product is a Lie bracket; the Lie superalgebra $\tilde{R}/\tilde{\partial}\tilde{R}$ is denoted by $\mathrm{Lie}\,R$. We call annihilation superalgebra $\mathcal{A}(R)$ associated with R the subalgebra of $\mathrm{Lie}\,R$ generated by the monomials with nonnegative powers of y. The annihilation superalgebra has a fundamental role since the study of the finite modules over R reduces to the study of a class of modules over it, the so−called *finite conformal* modules.

Given a $\mathbb{Z}$−graded Lie superalgebra $\mathfrak{g}$ and a $\mathfrak{g}$−module V, we call singular vectors of V the vectors that are annihilated by $\mathfrak{g}_{>0}$; the set of singular vectors of V is denoted by $\mathrm{Sing}\,V$. Moreover if F is a $\mathfrak{g}_{\geq 0}$-module, we denote by $\mathrm{Ind}(F)$ the generalized Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\geq 0})} F$, that is isomorphic, as a vector space, to $U(\mathfrak{g}_{<0}) \otimes F$ via the Poincaré−Birkhoff−Witt Theorem. The $\mathbb{Z}$−grading of $\mathfrak{g}$ induces a $\mathbb{Z}$−grading on $U(\mathfrak{g}_{<0})$ and $\mathrm{Ind}(F)$. We will invert the sign of the degree, so that we have a $\mathbb{Z}_{\geq 0}$−grading on $U(\mathfrak{g}_{<0})$ and $\mathrm{Ind}(F)$. We will say that an element $v \in U(\mathfrak{g}_{<0})_k$ is homogeneous of degree k. Analogously an element $m \in U(\mathfrak{g}_{<0})_k \otimes F$ is homogeneous of degree k. The study of irreducible finite conformal modules over the annihilation superalgebras associated with the conformal superalgebras of type W, S, K is related to the study of singular vectors of the generalized Verma modules:

Theorem 0.1 ([KR2],[CL]). *Let $\mathfrak{g}$ be the annihilation superalgebra associated with a conformal superalgebra of type W, S, or K. Then:*

1. *if F is an irreducible $\mathfrak{g}_{\geq 0}$−module of finite dimension, $\mathfrak{g}_{>0}$ acts trivially on it and* $\mathrm{Ind}(F)$ *has a unique maximal submodule;*

2. *the map* $F \mapsto I(F)$*, where* $I(F)$ *is the quotient of* $\mathrm{Ind}(F)$ *with respect to its maximal submodule, is a bijective correspondence between irreducible* $\mathfrak{g}_0-$*modules of finite dimension and irreducible finite conformal* $\mathfrak{g}-$*modules;*

3. *the* $\mathfrak{g}-$*module* $\mathrm{Ind}(F)$ *is irreducible if and only if the* $\mathfrak{g}_0-$*module* F *is irreducible and* $\mathrm{Sing}(\mathrm{Ind}(F)) = \mathrm{Sing}(F)$*.*

Let us recall the definition of the contact Lie superalgebra. Let $\bigwedge(N)$ be the Grassmann superalgebra in the N odd indeterminates $\xi_1, ..., \xi_N$. Let t be an even indeterminate and $\bigwedge(1,N) = \mathbb{C}[t,t^{-1}] \otimes \bigwedge(N)$. We consider the Lie superalgebra of derivations of $\bigwedge(1,N)$:

$$W(1,N) = \left\{ D = a\partial_t + \sum_{i=1}^{N} a_i \partial_i \mid a, a_i \in \bigwedge(1,N) \right\},$$

where $\partial_t = \frac{\partial}{\partial t}$ and $\partial_i = \frac{\partial}{\partial \xi_i}$ for all $i \in \{1, ..., N\}$.
We consider the contact form $\omega = dt - \sum_{i=1}^{N} \xi_i d\xi_i$. The contact Lie superalgebra $K(1,N)$ is defined as follows:

$$K(1,N) = \{ D \in W(1,N) \mid D\omega = f_D \omega \ for\ some\ f_D \in \bigwedge(1,N) \}.$$

It is possible to associate with the Lie superalgebra $K(1,N)$ the conformal superalgebra K_N and the annihilation superalgebra is $\mathcal{A}(K_N) = K(1,N)_+$, that is one of the simple infinite$-$dimensional Lie superalgebras classified by Kac in [K2].
The annihilation superalgebra $\mathfrak{g} := \mathcal{A}(K_4')$ associated with the simple conformal superalgebra K_4' is instead an extension of $K(1,4)_+$ by a $1-$dimensional center $\mathbb{C}C$.
On $\mathfrak{g}$ we consider the standard grading, whose depth is 2. In the description of $\mathfrak{g}$ we use that $K(1,4)_+$ is isomorphic to $\bigwedge(1,4)_+ = C[t] \otimes \bigwedge(4)$ via the isomorphism

$$\bigwedge(1,4)_+ \longrightarrow K(1,4)_+$$
$$f \longmapsto 2f\partial_t + (-1)^{p(f)} \sum_{i=1}^{N} (\xi_i \partial_t f + \partial_i f)(\xi_i \partial_t + \partial_i).$$

It follows that:

$$\mathfrak{g}_0 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathbb{C}t \oplus \mathbb{C}C \cong \langle e_x, f_x, h_x \rangle \oplus \langle e_y, f_y, h_y \rangle \oplus \mathbb{C}t \oplus \mathbb{C}C,$$

where

$$e_x = \frac{-\xi_1\xi_3 - \xi_2\xi_4 - i\xi_1\xi_4 + i\xi_2\xi_3}{2}, \qquad e_y = \frac{-\xi_1\xi_3 + \xi_2\xi_4 + i\xi_1\xi_4 + i\xi_2\xi_3}{2},$$
$$f_x = \frac{\xi_1\xi_3 + \xi_2\xi_4 - i\xi_1\xi_4 + i\xi_2\xi_3}{2}, \qquad f_y = \frac{\xi_1\xi_3 - \xi_2\xi_4 + i\xi_1\xi_4 + i\xi_2\xi_3}{2},$$
$$h_x = -i\xi_1\xi_2 + i\xi_3\xi_4, \qquad h_y = -i\xi_1\xi_2 - i\xi_3\xi_4,$$

t is a grading element in $\mathfrak{g}$ and C is a central element. We denote by $\mathfrak{g}_0^{ss}$ the semisimple part of $\mathfrak{g}_0$.
We have that:

$$g_0^{ss} = \langle e_x, f_x, h_x \rangle \oplus \langle e_y, f_y, h_y \rangle \cong \langle x_1\partial_{x_2}, x_2\partial_{x_1}, x_1\partial_{x_1} - x_2\partial_{x_2} \rangle \oplus \langle y_1\partial_{y_2}, y_2\partial_{y_1}, y_1\partial_{y_1} - y_2\partial_{y_2} \rangle.$$

We will identify the irreducible $\mathfrak{g}_0^{ss}-$module of highest weight (m,n) with respect to h_x, h_y with the space of homogeneous polynomials of degree m in the variables x_1, x_2, and degree n in the

variables y_1, y_2.
The following isomorphism of $\mathfrak{g}_0^{ss}-$modules holds:

$$\mathfrak{g}_{-1} \cong \langle x_1, x_2 \rangle \otimes \langle y_1, y_2 \rangle,$$
$$\xi_2 + i\xi_1 \leftrightarrow x_1 y_1, \;\; \xi_2 - i\xi_1 \leftrightarrow x_2 y_2, \;\; -\xi_4 + i\xi_3 \leftrightarrow x_1 y_2, \;\; \xi_4 + i\xi_3 \leftrightarrow x_2 y_1. \tag{1}$$

The space $\mathfrak{g}_{-2}$ is a $\mathfrak{g}_0-$module of dimension 1 and we denote by Θ its generator $-1/2$.
Since $\text{Ind}(F) \cong U(\mathfrak{g}_{<0}) \otimes F$, it follows that $\text{Ind}(F) \cong \mathbb{C}[\Theta] \otimes \bigwedge(4) \otimes F$. Indeed, if we denote by η_i the image in $U(\mathfrak{g})$ of $\xi_i \in \bigwedge(4)$, for all $i \in \{1,2,3,4\}$, in $U(\mathfrak{g})$ we have that $\eta_i^2 = \Theta$, for all $i \in \{1,2,3,4\}$: since $[\xi_i, \xi_i] = -1$ in $\mathfrak{g}$, it follows that $\eta_i \eta_i = -\eta_i \eta_i - 1$ in $U(\mathfrak{g})$. Motivated by (1), we will use the following notation

$$w_{11} = \eta_2 + i\eta_1, \;\; w_{22} = \eta_2 - i\eta_1, \;\; w_{12} = -\eta_4 + i\eta_3, \;\; w_{21} = \eta_4 + i\eta_3. \tag{2}$$

Let F be a finite$-$dimensional irreducible $\mathfrak{g}_0-$module. We study the action of $\mathfrak{g}$ on a generalized Verma module $\text{Ind}(F)$ using the so$-$called $\lambda-$action of the elements in $\bigwedge(4) \subset \mathfrak{g}$ on $\text{Ind}(F)$, where the $\lambda-$action of an element $f \in \bigwedge(4)$ is defined as $f_\lambda(g \otimes v) = \sum_{j \geq 0} \frac{\lambda^j}{j!}(t^j f).(g \otimes v)$, for $g \otimes v \in \text{Ind}(F)$. We find an explicit form of the $\lambda-$action and show in particular that the elements $(t^j f)_{j \geq 4}$ act trivially on the elements $g \otimes v$ with $g \in \bigwedge(4)$, $v \in F$.
The $\lambda-$action has a fundamental role in the study of singular vectors since the property of a vector $\vec{m} \in \text{Ind}(F)$ of being singular can be rewritten in terms of conditions on the derivatives with respect to λ of the $\lambda-$action. Using this action, we obtain the following classification of singular vectors of the generalized Verma modules. We denote by $\mu = (m, n, \mu_t, \mu_c)$ the highest weight of the irreducible finite$-$dimensional $\mathfrak{g}_0-$module F, written with respect to h_x, h_y, t and C. We denote by $F(m, n, \mu_t, \mu_c)$ the irreducible finite$-$dimensional $\mathfrak{g}_0-$module F of highest weight μ, when it is necessary to specify the highest weight, analogously we denote by $M(m, n, \mu_t, \mu_c)$ the generalized Verma module $\text{Ind}(F) \cong U(\mathfrak{g}_{<0}) \otimes F$, with F irreducible finite$-$dimensional $\mathfrak{g}_0-$module of highest weight μ, when we need to specify the highest weight of F. We say that a vector $\vec{m} \in \text{Ind}(F)$ is a highest weight singular vector if it is a singular vector and it is annihilated by e_x and e_y. The following results about singular vectors are presented also in [BC].

Theorem 0.2. *Let F be an irreducible finite$-$dimensional $\mathfrak{g}_0-$module, with highest weight μ. A vector $\vec{m} \in \text{Ind}(F)$ is a non trivial highest weight singular vector of degree 1 if and only if $\vec{m}$ is (up to a scalar) one of the following vectors:*

a: $\mu = (m, n, -\frac{m+n}{2}, \frac{m-n}{2})$ *with* $m, n \in \mathbb{Z}_{\geq 0}$,

$$\vec{m}_{1a} = w_{11} \otimes x_1^m y_1^n;$$

b: $\mu = (m, n, 1 + \frac{m-n}{2}, -1 - \frac{m+n}{2})$, *with* $m \in \mathbb{Z}_{>0}$, $n \in \mathbb{Z}_{\geq 0}$,

$$\vec{m}_{1b} = w_{21} \otimes x_1^m y_1^n - w_{11} \otimes x_1^{m-1} x_2 y_1^n;$$

c: $\mu = (m, n, 2 + \frac{m+n}{2}, \frac{n-m}{2})$, *with* $m, n \in \mathbb{Z}_{>0}$,

$$\vec{m}_{1c} = w_{22} \otimes x_1^m y_1^n - w_{12} \otimes x_1^{m-1} x_2 y_1^n - w_{21} \otimes x_1^m y_1^{n-1} y_2 + w_{11} \otimes x_1^{m-1} x_2 y_1^{n-1} y_2;$$

d: $\mu = (m, n, 1 + \frac{n-m}{2}, 1 + \frac{m+n}{2})$, *with* $m \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{>0}$,

$$\vec{m}_{1d} = w_{12} \otimes x_1^m y_1^n - w_{11} \otimes x_1^m y_1^{n-1} y_2.$$

Theorem 0.3. *Let F be an irreducible finite−dimensional $\mathfrak{g}_0$−module, with highest weight μ. A vector $\vec{m} \in \mathrm{Ind}(F)$ is a non trivial highest weight singular vector of degree 2 if and only if $\vec{m}$ is (up to a scalar) one of the following vectors:*

a: $\mu = (0, n, 1 - \frac{n}{2}, -1 - \frac{n}{2})$ *with* $n \in \mathbb{Z}_{\geq 0}$,

$$\vec{m}_{2a} = w_{11}w_{21} \otimes y_1^n;$$

b: $\mu = (m, 0, 1 - \frac{m}{2}, 1 + \frac{m}{2})$ *with* $m \in \mathbb{Z}_{\geq 0}$,

$$\vec{m}_{2b} = w_{11}w_{12} \otimes x_1^m;$$

c: $\mu = (m, 0, 2 + \frac{m}{2}, -\frac{m}{2})$ *with* $m \in \mathbb{Z}_{>1}$,

$$\vec{m}_{2c} = w_{22}w_{21} \otimes x_1^m + (w_{11}w_{22} + w_{21}w_{12}) \otimes x_1^{m-1}x_2 - w_{11}w_{12} \otimes x_1^{m-2}x_2^2;$$

d: $\mu = (0, n, 2 + \frac{n}{2}, \frac{n}{2})$ *with* $n \in \mathbb{Z}_{>1}$,

$$\vec{m}_{2d} = w_{22}w_{12} \otimes y_1^n - (w_{22}w_{11} + w_{21}w_{12}) \otimes y_1^{n-1}y_2 - w_{11}w_{21} \otimes y_1^{n-2}y_2^2.$$

Theorem 0.4. *Let F be an irreducible finite−dimensional $\mathfrak{g}_0$−module, with highest weight μ. A vector $\vec{m} \in \mathrm{Ind}(F)$ is a non trivial highest weight singular vector of degree 3 if and only if $\vec{m}$ is (up to a scalar) one of the following vectors:*

a: $\mu = (1, 0, \frac{5}{2}, -\frac{1}{2})$,

$$\vec{m}_{3a} = w_{11}w_{22}w_{21} \otimes x_1 + w_{21}w_{12}w_{11} \otimes x_2;$$

b: $\mu = (0, 1, \frac{5}{2}, \frac{1}{2})$,

$$\vec{m}_{3b} = w_{11}w_{22}w_{12} \otimes y_1 + w_{12}w_{21}w_{11} \otimes y_2.$$

Moreover, there are no singular vectors of degree greater than 3.
Between two Verma modules $M(m, n, \mu_t, \mu_c)$ and $M(\widetilde{m}, \widetilde{n}, \widetilde{\mu}_t, \widetilde{\mu}_c)$ there exists a morphism of $\mathfrak{g}$−modules if and only if there exists a singular vector $\vec{m}$ of highest weight (m, n, μ_t, μ_c) in $M(\widetilde{m}, \widetilde{n}, \widetilde{\mu}_t, \widetilde{\mu}_c)$. The morphism of $\mathfrak{g}$−modules is constructed as follows:

$$M(m, n, \mu_t, \mu_c) \longrightarrow M(\widetilde{m}, \widetilde{n}, \widetilde{\mu}_t, \widetilde{\mu}_c)$$
$$v_\mu \longmapsto \vec{m}$$

where v_μ is a highest weight vector in $F(m, n, \mu_t, \mu_c)$. Since, as a $\mathfrak{g}$−module, $M(m, n, \mu_t, \mu_c)$ is generated by v_μ, the morphism is completely determined.

Using the classification of singular vectors we find the sequences in Figure 4.1; we observe that the diagram is similar to the one obtained for $E(3, 6)$ and $E(3, 8)$ (see [KR1],[KR2],[KR3],[KR4]). Each point represents the generalized Verma module $M(m, n, \mu_t, \mu_c)$, where (m, n, μ_t, μ_c) is determined by its position with respect to the axes $m = 0, n = 0$ and μ_t, μ_c by the quadrant. The arrows represent the morphisms of $\mathfrak{g}$−modules constructed as before. We study the realization of irreducible modules. Due to Theorem 0.1 we know that $M(m, n, \mu_t, \mu_c)$ admits a unique maximal submodule and it is irreducible if and only if it does not contain nontrivial singular vectors. Therefore, from Theorems 0.2,0.3,0.4, it follows that $M(m, n, \mu_t, \mu_c)$ is irreducible if (m, n, μ_t, μ_c) is different from:

a) $(m, n, -\frac{m+n}{2}, \frac{m-n}{2})$ with $m, n \in \mathbb{Z}_{\geq 0}$,

b) $(m, n, 1 + \frac{m-n}{2}, -1 - \frac{m+n}{2})$ with $m, n \in \mathbb{Z}_{\geq 0}$,

c) $(m, n, 2 + \frac{m+n}{2}, \frac{n-m}{2})$ with $m, n \in \mathbb{Z}_{\geq 0}$, $(m, n) \neq (0, 0)$,

d) $(m, n, 1 + \frac{n-m}{2}, 1 + \frac{m+n}{2})$ with $m, n \in \mathbb{Z}_{\geq 0}$.

If (m, n, μ_t, μ_c) is one of the weights in the previous list, we know that $M(m, n, \mu_t, \mu_c)$ has a unique maximal submodule. The purpose in this case is to realize the irreducible quotients.
By construction, if $\nabla : M(m, n, \mu_t, \mu_c) \longrightarrow M(\widetilde{m}, \widetilde{n}, \widetilde{\mu}_t, \widetilde{\mu}_c)$ is a morphism, the kernel of ∇ is the maximal submodule of $M(m, n, \mu_t, \mu_c)$ since the quotient of the Verma module with respect to $\operatorname{Ker} \nabla$ is isomorphic to $\operatorname{Im} \nabla$ that is an irreducible module. If $M(m, n, \mu_t, \mu_c)$ is a Verma module represented in Figure 4.1, with $(m, n, \mu_t, \mu_c) \neq (0, 0, 0, 0), (0, 0, 2, 0)$, then there exist two morphisms $\nabla : M(m, n, \mu_t, \mu_c) \longrightarrow M(\widetilde{m}, \widetilde{n}, \widetilde{\mu}_t, \widetilde{\mu}_c)$ and $\widehat{\nabla} : M(\hat{m}, \hat{n}, \hat{\mu}_t, \hat{\mu}_c) \longrightarrow M(m, n, \mu_t, \mu_c)$ constructed as before. If the sequence is exact in $M(m, n, \mu_t, \mu_c)$, then $\operatorname{Ker} \nabla = \operatorname{Im} \widehat{\nabla}$ is the unique irreducible submodule of $M(m, n, \mu_t, \mu_c)$. In the points in which the sequence is not exact, we study the quotient of $\operatorname{Ker} \nabla$ with respect to $\operatorname{Im} \widehat{\nabla}$. Therefore we study the homology of the complexes using spectral sequences, following [KR1]. We obtain the following result.

Theorem 0.5. *The sequences in Figure 4.1 are complexes and are exact in each module except for $M(0, 0, 0, 0)$ and $M(1, 1, 3, 0)$. The spaces of homology in $M(0, 0, 0, 0)$ and $M(1, 1, 3, 0)$ are isomorphic to the trivial representation.*

We use Theorem 0.5 in order to compute the size of the irreducible quotients $I(m, n, \mu_t, \mu_c)$ of $M(m, n, \mu_t, \mu_c)$. For a $S(\mathfrak{g}_{-2})$−module V, we define its size as (see [KR1]):

$$\operatorname{size}(V) = \frac{1}{4} \operatorname{rk}_{S(g_{-2})} V.$$

Proposition 0.6. *The size of the irreducible quotients $I(m, n, \mu_t, \mu_c)$ takes the following values:*

A) $\operatorname{size}(I(m, n, -\frac{m+n}{2}, \frac{m-n}{2})) = 2mn + m + n,$

B) $\operatorname{size}(I(m, n, 1 + \frac{m-n}{2}, -1 - \frac{m+n}{2})) = 2(m+1)(n-1) + n - 1 + 3m + 3 + 2 = 2mn + m + 3n + 2,$

C) $\operatorname{size}(I(m, n, \frac{m+n}{2} + 2, \frac{n-m}{2})) = 2(m + 1)(n + 1) + m + n + 2 = 2mn + 3m + 3n + 4,$

D) $\operatorname{size}(I(m, n, 1 + \frac{n-m}{2}, 1 + \frac{m+n}{2})) = 2mn + n + 3m + 2.$

In [BKL2] the authors classified all singular vectors over the conformal superalgebra CK_6, introduced in [CK2], and therefore all finite irreducible modules over the conformal superalgebra CK_6. In [MZ] the authors classify all irreducible modules of finite type over the conformal superalgebra CK_6 using a different approach. In [BKL2] the classification of singular vectors of highest weight of CK_6 is based on a technical lemma whose proof is missing. We know that the annihilation superalgebra associated with CK_6, that we denote by $\mathfrak{g}$, is a subalgebra of $K(1, 6)_+$ isomorphic to the exceptional Lie superalgebra $E(1, 6)$ (see [BKL2],[CK3],[CK2]). On $\mathfrak{g}$ we consider the standard grading. The space $\mathfrak{g}_0$ is isomorphic to $\mathfrak{so}(6) \oplus \mathbb{C}t$. Following [BKL2], we fix the Cartan subalgebra of $\mathfrak{so}(6)$ spanned by the following elements:

$$H_1 = -i\xi_1\xi_2, \; H_2 = -i\xi_3\xi_4, \; H_3 = -i\xi_5\xi_6,$$

and we set $h_1 := H_1 - H_2$, $h_2 := H_2 - H_3$, $h_3 := H_2 + H_3$. Let $\lambda = n_1\lambda_1 + n_2\lambda_2 + n_3\lambda_3$ be a dominant weight, where the λ_i's are the fundamental weights of $\mathfrak{so}(6)$ extended by $\lambda_i(t) = 0$. We use the notation $F(\mu_t, \lambda)$ to denote the irreducible $\mathfrak{g}_0-$module of highest weight λ with respect to $\mathfrak{so}(6)$ and weight μ_t with respect to t. The space $\mathfrak{g}_{-1}$ is an irreducible $\mathfrak{g}_0-$module of dimension 6; $\mathfrak{g}_{-2}$ is an irreducible $\mathfrak{g}_0-$module of dimension 1, we call Θ its generator $-1/2$. Let F be an irreducible $\mathfrak{g}_0-$module of finite dimension; the Verma module $\mathrm{Ind}(F) \cong \mathbb{C}[\Theta] \otimes \bigwedge(6) \otimes F$. Indeed, we denote by η_i the image in $U(\mathfrak{g})$ of $\xi_i \in \bigwedge(6)$, for all $i \in \{1,2,3,4,5,6\}$. In $U(\mathfrak{g})$ we have that $\eta_i^2 = \Theta$, for all $i \in \{1,2,3,4,5,6\}$: since $[\xi_i, \xi_i] = -1$ in $\mathfrak{g}$, it follows that $\eta_i\eta_i = -\eta_i\eta_i - 1$ in $U(\mathfrak{g})$. Let $I = (i_1, ..., i_k)$ be an ordered subset of $\{1,2,3,4,5,6\}$; we use the notation $\eta_I = \eta_{i_1}...\eta_{i_k}$. For $\xi_I \in \bigwedge(6)$ we indicate with $\overline{\xi_I}$ its Hodge dual in $\bigwedge(6)$, i.e. the unique monomial such that $\overline{\xi_I}\xi_I = \xi_1\xi_2\xi_3\xi_4\xi_5\xi_6$; we denote by $\overline{\eta_I}$ the image in $U(\mathfrak{g}_{<0})$ of $\overline{\xi_I}$. Then we extend by linearity the definition of Hodge dual to elements $\sum_I \alpha_I\eta_I \in U(\mathfrak{g}_{<0})$ and we set $\overline{\Theta^k\eta_I} = \Theta^k\overline{\eta_I}$. Let T be the vector space isomorphism $T : \mathrm{Ind}(F) \to \mathrm{Ind}(F)$ that is defined by $T(g \otimes v) = \overline{g} \otimes v$. It follows that, for a vector $\vec{m} \in \mathrm{Ind}(F)$, $T(\vec{m})$ can be written as:

$$T(\vec{m}) = \sum_{k=0}^{N} \Theta^k \Big(\sum_I \eta_I \otimes v_{I,k} \Big). \tag{3}$$

The following technical lemma (Lemma 4.4 in [BKL2]) is used in [BKL2] to classify singular vectors.

Lemma 0.7. *Let $\vec{m} \in \mathrm{Ind}(F)$ be a singular vector, such that $T(\vec{m})$ is written as in (3). The degree of $\vec{m}$ with respect to Θ is at most 2. Moreover, $T(\vec{m})$ has the following form:*

$$T(\vec{m}) = \Theta^2 \Big(\sum_{|I|\geq 5} \eta_I \otimes v_{I,2} \Big) + \Theta^1 \Big(\sum_{|I|\geq 3} \eta_I \otimes v_{I,1} \Big) + \Big(\sum_{|I|\geq 1} \eta_I \otimes v_{I,0} \Big).$$

We prove this lemma with arguments analogous to the arguments used for K_4'.
As before, there exists a morphism of $\mathfrak{g}-$modules between two Verma modules $M(\mu_t, \lambda)$ and $M(\widetilde{\mu}_t, \widetilde{\lambda})$ if and only if there exists a singular vector $\vec{m}$ of highest weight (μ_t, λ) in $M(\widetilde{\mu}_t, \widetilde{\lambda})$. The morphism of $\mathfrak{g}-$modules is constructed as follows:

$$M(\mu_t, \lambda) \longrightarrow M(\widetilde{\mu}_t, \widetilde{\lambda})$$
$$v_\mu \longmapsto \vec{m}$$

where v_μ is a highest weight vector of $F(\mu_t, \lambda)$. Since, as a $\mathfrak{g}-$module, $M(\mu_t, \lambda)$ is generated by v_μ, the morphism is completely determined. In [BKL2] such morphisms are represented as in Figure 6.1; the diagram is similar to the one obtained for $E(5, 10)$ (see [KR3], [R], [CC], [CCK2]). Each point represents the generalized Verma module $M(\mu_t, \lambda)$, where the weight is determined by the position with respect to the axes and the quadrant. The arrows represent the morphisms of $\mathfrak{g}-$modules constructed as before. In [BKL2] the irreducible quotients of Verma modules are not explicitly realized. We begin the study of the homology of the complexes in Figure 6.1. We call $M_A(\mu_t, n_1\lambda_1 + n_2\lambda_2 + n_3\lambda_3)$ the modules represented in the first quadrant in Figure 6.1, $M_B(\mu_t, n_1\lambda_1 + n_2\lambda_2 + n_3\lambda_3)$ the modules represented in the second quadrant and $M_C(\mu_t, n_1\lambda_1 + n_2\lambda_2 + n_3\lambda_3)$ the modules represented in the third quadrant. We find an explicit expression for morphisms in the first quadrant. We use arguments of spectral sequences to compute the homology of the first quadrant and we use an argument of conformal duality (see [CCK1]) to obtain the homology for the third quadrant. In particular we prove the following result.

Proposition 0.8. *As a $\mathfrak{g}-$module, the homology space is 0 for the modules $M_A(-n_1 - \frac{n_3}{2}, n_1\lambda_1 + n_3\lambda_3)$ if $(n_1, n_3) \neq (0,0)$, is isomorphic to the trivial representation for $M_A(0,0)$. The homology space is 0 for the modules $M_C(n_1 + \frac{n_2}{2} + 4, n_1\lambda_1 + n_2\lambda_2)$ if $(n_1, n_2) = (0,0)$ or $n_1 > 0$ and it is isomorphic to the trivial representation for $M_C(5, \lambda_1)$.*

We will compute the homology for $M_B(\mu_t, n_1\lambda_1 + n_2\lambda_2 + n_3\lambda_3)$ and $M_C(\frac{n_2}{2} + 4, n_2\lambda_2)$ in the future.

Chapter 1

Preliminaries on conformal superalgebras

We recall some notions on conformal superalgebras. For further details see [K1, Chapter 2], [D], [BKLR], [BKL1].
Let $\mathfrak{g}$ be a Lie superalgebra; a formal distribution with coefficients in $\mathfrak{g}$, or equivalently a $\mathfrak{g}-$valued formal distribution, in the indeterminate z is an expression of the following form:

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1},$$

with $a_n \in \mathfrak{g}$ for every $n \in \mathbb{Z}$. We denote the vector space of formal distributions with coefficients in $\mathfrak{g}$ in the indeterminate z by $\mathfrak{g}[[z, z^{-1}]]$. We denote by $\mathrm{Res}(a(z)) = a_0$ the coefficient of z^{-1} of $a(z)$. The vector space $\mathfrak{g}[[z, z^{-1}]]$ has a natural structure of $\mathbb{C}[\partial_z]-$module. We define for all $a(z) \in \mathfrak{g}[[z, z^{-1}]]$ its derivative:

$$\partial_z a(z) = \sum_{n\in\mathbb{Z}} (-n-1) a_n z^{-n-2}.$$

A formal distribution with coefficients in $\mathfrak{g}$ in the indeterminates z and w is an expression of the following form:

$$a(z, w) = \sum_{m,n\in\mathbb{Z}} a_{m,n} z^{-m-1} w^{-n-1},$$

with $a_{m,n} \in \mathfrak{g}$ for every $m, n \in \mathbb{Z}$. We denote the vector space of formal distributions with coefficients in $\mathfrak{g}$ in the indeterminates z and w by $\mathfrak{g}[[z, z^{-1}, w, w^{-1}]]$. Given two formal distributions $a(z) \in \mathfrak{g}[[z, z^{-1}]]$ and $b(w) \in \mathfrak{g}[[w, w^{-1}]]$, we define the commutator $[a(z), b(w)]$:

$$[a(z), b(w)] = \left[\sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \sum_{m\in\mathbb{Z}} b_m w^{-m-1} \right] = \sum_{m,n\in\mathbb{Z}} [a_n, b_m] z^{-n-1} w^{-m-1}.$$

Definition 1.1. Two formal distributions $a(z), b(z) \in \mathfrak{g}[[z, z^{-1}]]$ are called local if:

$$(z-w)^N [a(z), b(w)] = 0 \; for \; some \; N \gg 0.$$

We call $\delta-$function the following formal distribution with coefficients in $\mathfrak{g}$ in the indeterminates z and w:

$$\delta(z-w) = z^{-1} \sum_n \left(\frac{w}{z}\right)^n.$$

See Corollary 2.2 in [K1] for the following equivalent condition of locality.

Proposition 1.2. *Two formal distributions $a(z), b(z) \in \mathfrak{g}[[z, z^{-1}]]$ are local if and only if $[a(z), b(w)]$ can be expressed as a finite sum of the form:*

$$[a(z), b(w)] = \sum_j (a(w)_{(j)}b(w)) \frac{\partial_w^j}{j!}\delta(z-w),$$

where the coefficients $(a(w)_{(j)}b(w)) := \mathrm{Res}_z(z-w)^j[a(z), b(w)]$ are formal distributions in the indeterminate w.

Definition 1.3 (Formal Distribution Superalgebra)**.** Let $\mathfrak{g}$ be a Lie superalgebra and $\mathcal{F}$ a family of mutually local $\mathfrak{g}-$valued formal distributions in the indeterminate z. The pair $(\mathfrak{g}, \mathcal{F})$ is called a *formal distribution superalgebra* if the coefficients of all formal distributions in $\mathcal{F}$ span $\mathfrak{g}$.

We define the $\lambda-$bracket between two formal distributions $a(z), b(z) \in \mathfrak{g}[[z, z^{-1}]]$ as the generating series of the $(a(z)_{(j)}b(z))$'s:

$$[a(z)_\lambda b(z)] = \sum_{j\geq 0} \frac{\lambda^j}{j!}(a(z)_{(j)}b(z)). \tag{1.1}$$

Definition 1.4 (Conformal superalgebra)**.** A *conformal superalgebra* R is a left $\mathbb{Z}_2-$graded $\mathbb{C}[\partial]-$module endowed with a $\mathbb{C}-$linear map, called $\lambda-$bracket, $R\otimes R \to \mathbb{C}[\lambda]\otimes R$, $a\otimes b \mapsto [a_\lambda b]$, that satisfies the following properties for all $a, b, c \in R$:

(1) $[\partial a_\lambda b] = -\lambda[a_\lambda b], \quad [a_\lambda \partial b] = (\lambda+\partial)[a_\lambda b]$;

(2) $[a_\lambda b] = -(-1)^{p(a)p(b)}[b_{-\lambda-\partial}a]$;

(3) $[a_\lambda[b_\mu c]] = [[a_\lambda b]_{\lambda+\mu}c] + (-1)^{p(a)p(b)}[b_\mu[a_\lambda c]]$;

where $p(a)$ denotes the parity of the element $a \in R$ and $p(\partial a) = p(a)$ for all $a \in R$.

We call $n-$products the coefficients $(a_{(n)}b)$ that appear in $[a_\lambda b] = \sum_{j\geq 0}\frac{\lambda^j}{j!}(a_{(j)}b)$ and give an equivalent definition of conformal superalgebra.

Definition 1.5 (Conformal superalgebra)**.** A *conformal superalgebra* R is a left $\mathbb{Z}_2-$graded $\mathbb{C}[\partial]-$module endowed with a $\mathbb{C}-$bilinear product $(a_{(n)}b) : R\otimes R \to R$, defined for every $n \geq 0$, that satisfies the following properties for all $a, b, c \in R$, $m, n \geq 0$:

(1) $(a_{(n)}b) = 0,\ for\ n \gg 0$;

(2) $((\partial a)_{(n)}b) = -n(a_{(n-1)}b)$;

(3) $(a_{(n)}b) = -(-1)^{p(a)p(b)}\sum_{j\geq 0}(-1)^{j+n}\frac{\partial^j}{j!}(b_{(n+j)}a)$;

(4) $(a_{(m)}(b_{(n)}c)) = \sum_{j=0}^{m}\binom{m}{j}((a_{(j)}b)_{(m+n-j)}c) + (-1)^{p(a)p(b)}(b_{(n)}(a_{(m)}c))$;

where $p(\partial a) = p(a)$ for all $a \in R$.

Using conditions (2) and (3) it is easy to show that for all $a, b \in R$, $n \geq 0$:

$$(a_{(n)}\partial b) = \partial(a_{(n)}b) + n(a_{(n-1)}b).$$

Due to this relation and (2), the map $\partial : R \to R$, $a \mapsto \partial a$ is a derivation with respect to the 0−product.

Remark 1.6. A formal distribution superalgebra, endowed with $\lambda-$bracket (1.1), satisfies conditions (1), (2), (3) of conformal superalgebras, for a proof see Proposition 2.3 in [K1].

We say that a conformal superalgebra R is *finite* if it is finitely generated as a $\mathbb{C}[\partial]-$module. An *ideal* I of R is a $\mathbb{C}[\partial]-$submodule of R such that $a_{(n)}b \in I$ for every $a \in R$, $b \in I$, $n \geq 0$. A conformal superalgebra R is *simple* if it has no non-trivial ideals and the $\lambda-$bracket is not identically zero. We denote by R' the *derived subalgebra* of R, i.e. the $\mathbb{C}-$span of all $n-$products.

Definition 1.7. A module M over a conformal superalgebra R is a $\mathbb{Z}_2-$graded $\mathbb{C}[\partial]-$module endowed with $\mathbb{C}-$linear maps $R \to \operatorname{End}_{\mathbb{C}} M$, $a \mapsto a^M_{(n)}$, defined for every $n \geq 0$, that satisfy the following properties for all $a, b \in R$, $v \in M$, $m, n \geq 0$:

(1) $a^M_{(n)}v = 0 \ for \ n \gg 0$;

(2) $(\partial a)^M_{(n)}v = [\partial^M, a^M_{(n)}]v = -na^M_{(n-1)}v$;

(3) $[a^M_{(m)}, b^M_{(n)}]v = \sum_{j=0}^m \binom{m}{j}(a_{(j)}b)^M_{(m+n-j)}v$.

A module M is called *finite* if it is a finitely generated $\mathbb{C}[\partial]-$module.
We can construct a conformal superalgebra starting from a formal distribution superalgebra $(\mathfrak{g}, \mathcal{F})$. Let $\overline{\mathcal{F}}$ be the closure of $\mathcal{F}$ under all the $n-$products, ∂_z and linear combinations. By Dong's Lemma, $\overline{\mathcal{F}}$ is still a family of mutually local distributions (see [K1]). It turns out that $\overline{\mathcal{F}}$ is a conformal superalgebra. We will refer to it as the conformal superalgebra associated with $(\mathfrak{g}, \mathcal{F})$.
Let us recall the construction of the annihilation superalgebra associated with a conformal superalgebra R. Let $\widetilde{R} = R[y, y^{-1}]$, set $p(y) = 0$ and $\widetilde{\partial} = \partial + \partial_y$. We define the following $n-$products on $\widetilde{R}$, for all $a, b \in R$, $f, g \in \mathbb{C}[y, y^{-1}]$, $n \geq 0$:

$$(af_{(n)}bg) = \sum_{j \in \mathbb{Z}_+} (a_{(n+j)}b)\Big(\frac{\partial_y^j}{j!}f\Big)g.$$

In particular if $f = y^m$ and $g = y^n$ we have for all $k \geq 0$:

$$(ay^m{}_{(k)}by^n) = \sum_{j \in \mathbb{Z}_+} \binom{m}{j}(a_{(k+j)}b)y^{m+n-j}.$$

We observe that $\widetilde{\partial}\widetilde{R}$ is a two sided ideal of $\widetilde{R}$ with respect to the $0-$product. The quotient $\operatorname{Lie} R := \widetilde{R}/\widetilde{\partial}\widetilde{R}$ has a structure of Lie superalgebra with the bracket induced by the $0-$product, i.e. for all $a, b \in R$, $f, g \in \mathbb{C}[y, y^{-1}]$:

$$[af, bg] = \sum_{j \in \mathbb{Z}_+} (a_{(j)}b)\Big(\frac{\partial_y^j}{j!}f\Big)g. \tag{1.2}$$

The images in $\operatorname{Lie} R$ of elements $ay^m \in \widetilde{R}$ are often denoted by a_m.

Definition 1.8. The annihilation superalgebra $\mathcal{A}(R)$ of a conformal superalgebra R is the subalgebra of $\operatorname{Lie} R$ spanned by all elements a_n with $n \geq 0$ and $a \in R$.
The extended annihilation superalgebra $\mathcal{A}(R)^e$ of a conformal superalgebra R is the Lie superalgebra $\mathbb{C}\partial \ltimes \mathcal{A}(R)$. The semidirect sum $\mathbb{C}\partial \ltimes \mathcal{A}(R)$ is the vector space $\mathbb{C}\partial \oplus \mathcal{A}(R)$ endowed with the structure of Lie superalgebra given by the bracket:

$$[\partial, ay^m] = -\partial_y(ay^m) = -may^{m-1},$$

for all $a \in R$.

For all $a \in R$ we consider the following formal power series in $\mathcal{A}(R)[[\lambda]]$:

$$a_\lambda = \sum_{n\geq 0} \frac{\lambda^n}{n!} a_n.$$

For all $a, b \in R$, we have: $[a_\lambda, b_\mu] = [a_\lambda b]_{\lambda+\mu}$ and $(\partial a)_\lambda = -\lambda a_\lambda$ (for a proof see [CCK1]).

Proposition 1.9 ([CK1]). *A module over a conformal superalgebra R is the same as a module over the Lie superalgebra $\mathcal{A}(R)^e$ such that $a_\lambda m \in \mathbb{C}[\lambda] \otimes M$ for all $a \in R$, $m \in M$, i.e. for every $a \in R$, $m \in M$ there exists $n_0 \in \mathbb{N}$ such that $a_n.m = 0$ for all $n \geq n_0$.*

Proposition 1.9 reduces the study of modules over a conformal superalgebra R to the study of a class of modules over its (extended) annihilation superalgebra.

Proposition 1.10 ([BKL1]). *Let $\mathfrak{g}$ be the annihilation superalgebra of a conformal superalgebra R. Assume that $\mathfrak{g}$ satisfies the following conditions:*

L1 *$\mathfrak{g}$ is $\mathbb{Z}-$graded with finite depth d;*

L2 *There exists an element whose centralizer in $\mathfrak{g}$ is contained in $\mathfrak{g}_0$;*

L3 *There exists an element $\Theta \in \mathfrak{g}_{-d}$ such that $\mathfrak{g}_{i-d} = [\Theta, \mathfrak{g}_i]$, for all $i \geq 0$.*

Finite modules over R are the same as modules V over $\mathfrak{g}$, called finite conformal, that satisfy the following properties:

1. *For every $v \in V$, there exists $j_0 \in \mathbb{Z}$, $j_0 \geq -d$, such that $\mathfrak{g}_j.v = 0$ when $j \geq j_0$;*
2. *V is finitely generated as a $\mathbb{C}[\Theta]-$module.*

Remark 1.11. We point out that condition L2 is automatically satisfied when $\mathfrak{g}$ contains a grading element, i.e. an element $a \in \mathfrak{g}$ such that $[a, b] = \deg(b)b$ for all $b \in \mathfrak{g}$.

Let $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ be a $\mathbb{Z}-$graded Lie superalgebra. We will use the notation $\mathfrak{g}_{>0} = \oplus_{i>0}\mathfrak{g}_i$, $\mathfrak{g}_{<0} = \oplus_{i<0}\mathfrak{g}_i$ and $\mathfrak{g}_{\geq 0} = \oplus_{i\geq 0}\mathfrak{g}_i$. We denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$.

Definition 1.12. Let F be a $\mathfrak{g}_{\geq 0}-$module. The *generalized Verma module* associated with F is the $\mathfrak{g}-$module $\mathrm{Ind}(F)$ defined by:

$$\mathrm{Ind}(F) := \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\geq 0}}(F) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\geq 0})} F.$$

We will identify $\mathrm{Ind}(F)$ with $U(\mathfrak{g}_{<0}) \otimes F$ as vector spaces via the Poincaré−Birkhoff−Witt Theorem. The $\mathbb{Z}-$grading of $\mathfrak{g}$ induces a $\mathbb{Z}-$grading on $U(\mathfrak{g}_{<0})$ and $\mathrm{Ind}(F)$. We will invert the sign of the degree, so that we have a $\mathbb{Z}_{\geq 0}-$grading on $U(\mathfrak{g}_{<0})$ and $\mathrm{Ind}(F)$. We will say that an element $v \in U(\mathfrak{g}_{<0})_k$ is homogeneous of degree k. Analogously an element $m \in U(\mathfrak{g}_{<0})_k \otimes F$ is homogeneous of degree k.

Proposition 1.13. *Let $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ be a $\mathbb{Z}-$graded Lie superalgebra. If F is an irreducible finite−dimensional $\mathfrak{g}_{\geq 0}-$module, then* $\mathrm{Ind}(F)$ *has a unique maximal submodule. We denote by* $\mathrm{I}(F)$ *the quotient of* $\mathrm{Ind}(F)$ *by the unique maximal submodule.*

Proof. First we point out that a submodule $V \neq \{0\}$ of $\mathrm{Ind}(F)$ is proper if and only if it does not contain nontrivial elements of degree 0. Indeed, if V contains an element $v_0 \neq 0$ of degree 0, then it contains $1 \otimes F = \mathfrak{g}_{\geq 0}.v_0$, due to irreducibility of F. Therefore $\mathfrak{g}_{<0}.F = \mathrm{Ind}(F) \subseteq V$. The union S of all proper submodules is still a proper submodule of $\mathrm{Ind}(F)$, since S does not contain nontrivial elements of degree 0, thus S is the unique maximal proper submodule. □

Definition 1.14. Given a $\mathfrak{g}-$module V, we call *singular vectors* the elements of:

$$\mathrm{Sing}(V) = \{v \in V \mid \mathfrak{g}_{>0}.v = 0\}.$$

In the case $V = \mathrm{Ind}(F)$, for a $\mathfrak{g}_{\geq 0}-$module F, we will call *trivial singular vectors* the elements of $\mathrm{Sing}(V)$ that lie in $1 \otimes F$ and *nontrivial singular vectors* the nonzero elements of $\mathrm{Sing}(V)$ that do not lie in $1 \otimes F$.

Theorem 1.15 ([KR2],[CL])**.** *Let $\mathfrak{g}$ be a Lie superalgebra that satisfies L1, L2, L3, then:*

1. *if F is an irreducible finite−dimensional $\mathfrak{g}_{\geq 0}-$module, then $\mathfrak{g}_{>0}$ acts trivially on it;*
2. *the map $F \mapsto \mathrm{I}(F)$ is a bijective map between irreducible finite−dimensional $\mathfrak{g}_0-$modules and irreducible finite conformal $\mathfrak{g}-$modules;*
3. *the $\mathfrak{g}-$module $\mathrm{Ind}(F)$ is irreducible if and only if the $\mathfrak{g}_0-$module F is irreducible and $\mathrm{Ind}(F)$ has no nontrivial singular vectors.*

We recall the notion of duality for conformal modules (see for further details [BKLR], [CCK1]). Let R be a conformal superalgebra and M a conformal module over R.

Definition 1.16. The conformal dual M^* of M is defined by:

$$M^* = \{f_\lambda : M \to \mathbb{C}[\lambda] \mid f_\lambda(\partial m) = \lambda f_\lambda(m),\ \forall m \in M\}.$$

The structure of $\mathbb{C}[\partial]-$module is given by $(\partial f)_\lambda(m) = -\lambda f_\lambda(m)$, for all $f \in M^*$, $m \in M$. The $\lambda-$action of R is given, for all $a \in R$, $m \in M$, $f \in M^*$, by:

$$(a_\lambda f)_\mu(m) = -(-1)^{p(a)p(f)} f_{\mu-\lambda}(a_\lambda m).$$

Definition 1.17. Let $T : M \to N$ be a morphism of $R-$modules, i.e. a linear map such that for all $a \in R$ and $m \in M$:

i: $T(\partial m) = \partial T(m)$,

ii: $T(a_\lambda m) = a_\lambda T(m)$.

The dual morphism $T^* : N^* \to M^*$ is defined, for all $f \in N^*$ and $m \in M$, by:

$$[T^*(f)]_\lambda (m) = -f_\lambda (T(m)).$$

Theorem 1.18 ([BKLR], Proposition 2.6)**.** *Let R be a conformal superalgebra and M, N $R-$modules. Let $T : M \longrightarrow N$ be a homomorphism of $R-$modules such that $N/\operatorname{Im} T$ is a finitely generated torsion−free $\mathbb{C}[\partial]-$module. Then the standard map $\Psi : N^*/\operatorname{Ker} T^* \longrightarrow (M/\operatorname{Ker} T)^*$, given by $[\Psi(\overline{f})]_\lambda(\overline{m}) = f_\lambda(T(m))$ (where by the bar we denote the corresponding class in the quotient), is an isomorphism of $R-$modules.*

We denote by $\mathbf{F}$ the functor that maps a conformal module M over a conformal superalgebra R to its conformal dual M^* and maps a morphism between conformal modules $T : M \to N$ to its dual $T^* : N^* \to M^*$.

Proposition 1.19. *The functor $\boldsymbol{F}$ is exact if we consider only morphisms $T : M \to N$, where $N/\operatorname{Im} T$ is a finitely generated torsion free $\mathbb{C}[\partial]-$module.*

Proof. Let us consider an exact short sequence of conformal modules:

$$0 \to M \xrightarrow{d_1} N \xrightarrow{d_2} P \to 0.$$

Therefore we know that $d_2 \circ d_1 = 0$, d_1 is injective, d_2 is surjective and $\operatorname{Ker} d_2 = \operatorname{Im} d_1$. We consider the dual of this sequence:

$$0 \to P^* \xrightarrow{d_2^*} N^* \xrightarrow{d_1^*} M^* \to 0.$$

By Theorem 1.18 and Remark 3.11 in [CCK1], we know that d_1^* is surjective and d_2^* is injective. We have to show that $\operatorname{Ker} d_1^* = \operatorname{Im} d_2^*$. Let us first show that $\operatorname{Ker} d_1^* \supset \operatorname{Im} d_2^*$. Let $\beta \in \operatorname{Im} d_2^* \subset N^*$. We have $\beta = d_2^*(\alpha)$ for some $\alpha \in P^*$. We have for all $m \in M$:

$$\left[d_1^*(\beta)\right]_\lambda (m) = -\beta_\lambda(d_1(m)) = \alpha_\lambda(d_2(d_1(m))) = 0.$$

Let us now show that $\operatorname{Ker} d_1^* \subset \operatorname{Im} d_2^*$. Let $\beta \in \operatorname{Ker} d_1^* \subset N^*$. We have for all $m \in M$:

$$0 = \left[d_1^*(\beta)\right]_\lambda (m) = -\beta_\lambda(d_1(m)).$$

Since $\operatorname{Ker} d_2 = \operatorname{Im} d_1$, this condition tells that β vanishes on $\operatorname{Ker} d_2$. We also know that for every $p \in P$, $p = d_2(n_p)$, for some $n_p \in N$. We define $\alpha \in P^*$ as follows, for all $p \in P$:

$$\alpha_\lambda(p) = \alpha_\lambda(d_2(n_p)) = \begin{cases} -\beta_\lambda(n_p) & \text{if } p \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let us show that α actually lies in P^*. For every $p \in P$:

$$\alpha_\lambda(\partial p) = \alpha_\lambda(\partial d_2(n_p)) = \alpha_\lambda(d_2(\partial n_p)) = \begin{cases} -\beta_\lambda(\partial n_p) & \text{if } \partial p \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\beta \in N^*$, we know that $-\beta_\lambda(\partial n_p) = -\lambda\beta_\lambda(n_p)$. Therefore $\alpha_\lambda(\partial p) = \lambda\alpha_\lambda(p)$.
We have for all $n \in N$ that:

$$\left[d_2^*(\alpha)\right]_\lambda (n) = -\alpha_\lambda(d_2(n)) = \beta_\lambda(n).$$

□

Chapter 2

The conformal superalgebra K_4'

We recall the notion of the contact Lie superalgebra. Let $\bigwedge(N)$ be the Grassmann superalgebra in the N odd indeterminates $\xi_1, ..., \xi_N$. Let t be an even indeterminate and $\bigwedge(1,N) = \mathbb{C}[t,t^{-1}] \otimes \bigwedge(N)$. We consider the Lie superalgebra of derivations of $\bigwedge(1,N)$:

$$W(1,N) = \Big\{ D = a\partial_t + \sum_{i=1}^{N} a_i \partial_i \mid a, a_i \in \bigwedge(1,N) \Big\},$$

where $\partial_t = \frac{\partial}{\partial t}$ and $\partial_i = \frac{\partial}{\partial \xi_i}$ for every $i \in \{1, ..., N\}$.
Let us consider the contact form $\omega = dt - \sum_{i=1}^{N} \xi_i d\xi_i$. The contact Lie superalgebra $K(1,N)$ is defined by:

$$K(1,N) = \{ D \in W(1,N) \mid D\omega = f_D \omega \ for\ some\ f_D \in \bigwedge(1,N) \}.$$

We denote by $K'(1,N)$ the derived algebra $[K(1,N), K(1,N)]$ of $K(1,N)$. Analogously, let $\bigwedge(1,N)_+ = \mathbb{C}[t] \otimes \bigwedge(N)$. We consider the Lie superalgebra of derivations of $\bigwedge(1,N)_+$:

$$W(1,N)_+ = \Big\{ D = a\partial_t + \sum_{i=1}^{N} a_i \partial_i \mid a, a_i \in \bigwedge(1,N)_+ \Big\}.$$

The Lie superalgebra $K(1,N)_+$ is defined by:

$$K(1,N)_+ = \{ D \in W(1,N)_+ \mid D\omega = f_D \omega \ for\ some\ f_D \in \bigwedge(1,N)_+ \}.$$

One can define on $\bigwedge(1,N)$ a Lie superalgebra structure as follows: for all $f, g \in \bigwedge(1,N)$ we let:

$$[f,g] = \Big(2f - \sum_{i=1}^{N} \xi_i \partial_i f\Big)(\partial_t g) - (\partial_t f)\Big(2g - \sum_{i=1}^{N} \xi_i \partial_i g\Big) + (-1)^{p(f)}\Big(\sum_{i=1}^{N} \partial_i f \partial_i g\Big). \tag{2.1}$$

We recall that $K(1,N) \cong \bigwedge(1,N)$ as Lie superalgebras via the following map (see [CK3]):

$$\begin{aligned} \textstyle\bigwedge(1,N) &\longrightarrow K(1,N) \\ f &\longmapsto 2f\partial_t + (-1)^{p(f)} \sum_{i=1}^{N} (\xi_i \partial_t f + \partial_i f)(\xi_i \partial_t + \partial_i). \end{aligned}$$

From now on we will always identify elements of $K(1,N)$ with elements of $\bigwedge(1,N)$ and we will omit the symbol $\wedge$ between the ξ_i's. We consider on $K(1,N)$ the standard grading, i.e. for every $t^m \xi_{i_1} \cdots \xi_{i_s} \in K(1,N)$ we have $\deg(t^m \xi_{i_1} \cdots \xi_{i_s}) = 2m + s - 2$.
Now we want to realize $K(1,N)_+$ as the annihilation superalgebra of a conformal superalgebra.

In order to do this, we construct a formal distribution superalgebra using the following family of formal distributions:

$$\mathcal{F}=\Big\{A(z):=\sum_{m\in\mathbb{Z}}(At^m)z^{-m-1}=A\delta(t-z),\ \forall A\in\wedge(N)\Big\}.$$

Note that the set of all the coefficients of formal distributions in $\mathcal{F}$ spans $\bigwedge(1,N)$.

Proposition 2.1. *For all $A(z),B(z)\in\mathcal{F}$, $A(z)$ and $B(z)$ are local. For all $A,B\in\bigwedge(N)$, $A=\xi_{i_1}\cdots\xi_{i_r}$ and $B=\xi_{j_1}\cdots\xi_{j_s}$, the $n-$products are given by:*

$$\begin{aligned}&(A(z)_{(0)}B(z))=(r-2)\partial_z(AB)(z)+(-1)^r\sum_{i=1}^{N}(\partial_iA\,\partial_iB)(z); \qquad (2.2)\\&(A(z)_{(1)}B(z))=(r+s-4)(AB)(z);\\&(A(z)_{(n)}B(z))=0\ for\ n>1.\end{aligned}$$

For all $A,B\in\bigwedge(N)$, $n\geq 0$, all other $n-$products can be found by linearity and the relations:

$$\begin{aligned}((\partial_zA)(z)_{(n)}B(z))&=-n(A(z)_{(n-1)}B(z)),\\(A(z)_{(n)}\partial_zB(z))&=\partial_z(A(z)_{(n)}B(z))+n(A(z)_{(n-1)}B(z)).\end{aligned}$$

The closure of $\mathcal{F}$ under all $n-$products and ∂_z is $\bar{\mathcal{F}}=\mathbb{C}[\partial_z]\mathcal{F}$.

Proof. It is sufficient to show the result for all $A(z),B(z)\in\mathcal{F}$ with A,B monomials in $\bigwedge(N)$ and use linearity. Let $A,B\in\bigwedge(N)$, $A=\xi_{i_1}\cdots\xi_{i_r}$ and $B=\xi_{j_1}\cdots\xi_{j_s}$; we show that the formal distributions

$$A(z)=\sum_{m\in\mathbb{Z}}(At^m)z^{-m-1}\ and\ B(z)=\sum_{n\in\mathbb{Z}}(Bt^n)z^{-n-1}$$

are local. Indeed, we have:

$$\begin{aligned}&[A(z),B(w)]=\\&=\sum_{m,n\in\mathbb{Z}}[At^m,Bt^n]z^{-m-1}w^{-n-1}\\&=\sum_{m,n\in\mathbb{Z}}\Big((n\,(2-r)-m\,(2-s))\,AB\,t^{m+n-1}+(-1)^r\sum_{i=1}^{N}\partial_iA\,\partial_iB\,t^{m+n}\Big)z^{-m-1}w^{-n-1}\\&=\sum_{m,n\in\mathbb{Z}}(n\,(2-r)-m\,(2-s))\,AB\,t^{m+n-1}\,\frac{z^{-m-1}}{w^{n+1}}+\sum_{m,n\in\mathbb{Z}}(-1)^r\sum_{i=1}^{N}\partial_iA\,\partial_iB\,t^{m+n}\,\frac{z^{-m-1}}{w^{n+1}}.\end{aligned}$$

We set $h=m+n-1$ in the first series and $l=m+n$ in the second series. We obtain:

$$\begin{aligned}&[A(z),B(w)]=\\&=\sum_{h,m\in\mathbb{Z}}((h-m+1)\,(2-r)-m\,(2-s))\,AB\,t^h\,\frac{z^{-m-1}}{w^{-(m-h-2)}}+\sum_{l,m\in\mathbb{Z}}(-1)^r\sum_{i=1}^{N}\partial_iA\,\partial_iB\,t^l\,\frac{z^{-m-1}}{w^{-(m-l-1)}}\\&=\sum_{h,m\in\mathbb{Z}}(h+1)(2-r)AB\,t^hw^{-h-2}z^{-m-1}w^m+\sum_{h,m\in\mathbb{Z}}m(r-2+s-2)AB\,t^hw^{-h-1}z^{-m-1}w^{m-1}\\&\quad+\sum_{l,m\in\mathbb{Z}}(-1)^r\sum_{i=1}^{N}\partial_iA\,\partial_iB\,t^lw^{-l-1}z^{-m-1}w^m\end{aligned}$$

$$=(r-2)\partial_w((AB)(w))\delta(z-w)+(r+s-4)(AB)(w)\partial_w\delta(z-w)+(-1)^r\sum_{i=1}^{N}(\partial_i A\,\partial_i B)(w)\delta(z-w)$$

$$=\Big((r-2)\,\partial_w\,(AB)\,(w)+(-1)^r\sum_{i=1}^{N}(\partial_i A\,\partial_i B)\,(w)\Big)\delta(z-w)+(r+s-4)\,(AB)\,(w)\partial_w\delta(z-w).$$

Therefore, for all $A,B\in\bigwedge(N)$, $A=\xi_{i_1}\cdots\xi_{i_r}$ and $B=\xi_{j_1}\cdots\xi_{j_s}$, the $n-$products are given by:

$$\begin{aligned}
&(A(z)_{(0)}B(z))=(r-2)\partial_z(AB)(z)+(-1)^r\sum_{i=1}^{N}(\partial_i A\,\partial_i B)(z);\\
&(A(z)_{(1)}B(z))=(r+s-4)(AB)(z);\\
&(A(z)_{(n)}B(z))=0 \ for\ n>1.
\end{aligned}$$

For all $A,B\in\bigwedge(N)$, $n\geq 0$, all other $n-$products can be found by linearity and the relations:

$$\begin{aligned}
((\partial_z A)(z)_{(n)}B(z))&=-n(A(z)_{(n-1)}B(z)),\\
(A(z)_{(n)}\partial_z B(z))&=\partial_z(A(z)_{(n)}B(z))+n(A(z)_{(n-1)}B(z)).
\end{aligned}$$

Hence the closure of $\mathcal{F}$ under all $n-$products and ∂_z is $\bar{\mathcal{F}}=\mathbb{C}[\partial_z]\mathcal{F}$. □

The closure $\bar{\mathcal{F}}$ is the conformal superalgebra associated with the formal distribution superalgebra $(K(1,N)$,$\mathcal{F})$.

Proposition 2.2. *The conformal superalgebra $\bar{\mathcal{F}}=\mathbb{C}[\partial_z]\mathcal{F}$ is a free $\mathbb{C}[\partial_z]-$module.*

Proof. We have that the set of all elements of type $A\delta(t-z)$, where $A=\xi_{i_1}...\xi_{i_r}$ is a monomial in $\bigwedge(N)$, is a basis of $\mathcal{F}$. Let us consider a finite linear combination, with coefficients in $\mathbb{C}[\partial_z]$, of elements of this basis:

$$\sum_{i=1}^{s}P_i(\partial_z)A_i\delta(t-z)=0,$$

where $A_i\in\bigwedge(N)$, $P_i(\partial_z)\in\mathbb{C}[\partial_z]$ for every $1\leq i\leq s$. From linear independence of the A_i's, we obtain for every $1\leq i\leq s$:

$$P_i(\partial_z)\delta(t-z)=0.$$

Therefore every coefficient P_i must be 0. □

We will identify $\bar{\mathcal{F}}=\mathbb{C}[\partial_z]\otimes\mathcal{F}$ with $K_N:=\mathbb{C}[\partial]\otimes\bigwedge(N)$. We identify ∂_z with ∂ and every $A(z)\in\mathcal{F}$ with $A\in\bigwedge(N)$. We will refer to K_N as the conformal superalgebra associated with $K(1,N)$. For all monomials $f,g\in\bigwedge(N)$, $f=\xi_{i_1}...\xi_{i_r}$ and $g=\xi_{j_1}...\xi_{j_s}$, the $\lambda-$bracket is given by:

$$[f_\lambda g]=((r-2)\partial(fg)+(-1)^r\sum_{i=1}^{N}(\partial_i f)(\partial_i g))+\lambda(r+s-4)fg, \tag{2.3}$$

by Proposition 2.1. In [BKL1] it is shown that the annihilation superalgebra of K_N is $\mathcal{A}(K_N)=K(1,N)_+$ and that it satisfies conditions L1, L2, L3. Thus, the study of finite irreducible modules over the conformal superalgebra K_N is reduced to the study of singular vectors of Verma modules on $K(1,N)_+$.

For $N=4$, K_N is not simple. The derived superalgebra K_4' is one of the finite simple conformal superalgebras completely classified in [FK]. Our aim is to study all finite irreducible modules over the conformal superalgebra K_4'.
Let V be a vector space and $\{b_i\}_{i\in I}$ a basis of it. An element $v\in V$ can be uniquely expressed as $v=\sum_i c_i b_i$. The support of v is $\operatorname{Supp} v=\{b_i : c_i\neq 0\}$.

Proposition 2.3. *The element $\xi_1\xi_2\xi_3\xi_4\notin K_4'$. More precisely:*

$$K_4'=\langle \partial^k\xi_{i_1}\cdots\xi_{i_r},\ \partial^l\xi_1\xi_2\xi_3\xi_4,\ \textit{for } 0\leq r<4,\ k\in\mathbb{Z}_{\geq 0},\ l\in\mathbb{Z}_{>0}\rangle.$$

Proof. By Proposition 2.2, we know that $\left\{\partial^k\xi_{i_1}\cdots\xi_{i_r},\ \text{for } k\in\mathbb{Z}_{\geq 0},\ 0\leq r\leq 4\right\}$ is a basis for K_4. We first show that $\xi_1\xi_2\xi_3\xi_4\notin K_4'$. Since the $j-$products are bilinear maps, it is sufficient to show that $\xi_1\xi_2\xi_3\xi_4$ does not belong to $\operatorname{Supp}(f_{(j)}g)$, for any $f=\xi_{i_1}\cdots\xi_{i_r}$, $g=\xi_{j_1}\cdots\xi_{j_s}\in\bigwedge(4)$.
The element $\xi_1\xi_2\xi_3\xi_4$ does not belong to $\operatorname{Supp}(f_{(0)}g)$. Indeed it does not belong to the support of $(-1)^r(\sum_{i=1}^4(\partial_i f)(\partial_i g))$, because, for all $1\leq i\leq 4$, $\xi_1\xi_2\xi_3\xi_4\notin\operatorname{Supp}((\partial_i f)(\partial_i g))$. Clearly it does not belong to the support of $(r-2)\partial(fg)$.
The element $\xi_1\xi_2\xi_3\xi_4$ does not belong to $\operatorname{Supp}(f_{(1)}g)$. Indeed if $\xi_1\xi_2\xi_3\xi_4\in\operatorname{Supp}((r+s-4)fg)$, then $r+s=4$, that is a contradiction.
Every monomial $\partial^k f\in K_4\setminus\mathbb{C}\xi_1\xi_2\xi_3\xi_4$ lies in K_4', indeed:

1. if $k>0$, then $\partial^k f=\left(-\frac{1}{2}{}_{(0)}\partial^{k-1}f\right)$;
2. if $k=0$, then there exists a $\xi_i\in\bigwedge(4)$ such that $\xi_i f\neq 0$. We have $f=-(\xi_i\,{}_{(0)}\xi_i f)$.

Therefore, we have:

$$K_4'=\langle \partial^k\xi_{i_1}\cdots\xi_{i_r},\ \partial^l\xi_1\xi_2\xi_3\xi_4,\ \text{for } 0\leq r<4,\ k\in\mathbb{Z}_{\geq 0},\ l\in\mathbb{Z}_{>0}\rangle.$$

□

Proposition 2.4. *The element $t^{-1}\xi_1\xi_2\xi_3\xi_4\notin K'(1,4)$. More precisely:*

$$K'(1,4)=\langle t^k\xi_{i_1}\cdots\xi_{i_r},\ t^l\xi_1\xi_2\xi_3\xi_4,\ \textit{for } 0\leq r<4,\ k,l\in\mathbb{Z},\ l\neq -1\rangle.$$

Proof. We know that $\left\{t^k\xi_{i_1}\cdots\xi_{i_r},\ \text{for } k\in\mathbb{Z},\ 0\leq r\leq 4\right\}$ is a basis for $K(1,4)$. Let us first show that $t^{-1}\xi_1\xi_2\xi_3\xi_4\notin K'(1,4)$. Since the bracket (2.1) is bilinear, it is sufficient to prove that $t^{-1}\xi_1\xi_2\xi_3\xi_4$ does not belong to $\operatorname{Supp}[f,g]$ for any f,g monomials of $K(1,4)$. Let $f=at^k\xi_{i_1}\cdots\xi_{i_r}$ and $g=bt^l\xi_{j_1}\cdots\xi_{j_s}\in K(1,4)$, with $a,b\in\mathbb{C}$. We have the following possibilities.

1. Let us suppose that $t^{-1}\xi_1\xi_2\xi_3\xi_4=(2-r)f\partial_t g$. Hence, we have:

$$t^k\partial_t(t^l)=t^{-1};\quad a\cdot b\cdot l=\frac{1}{2-r}\quad\text{and}\quad \xi_{i_1}\cdots\xi_{i_r}\cdot\xi_{j_1}\cdots\xi_{j_s}=\xi_1\xi_2\xi_3\xi_4.$$

Then we obtain $l=-k$ and $k\neq 0$. Indeed the power k of t^k cannot be 0, since $\partial_t t^l=t^{-1}$ is impossible for a Laurent polynomial. Therefore:

$$\begin{aligned}[f,g]&=\frac{2-r}{2-r}t^k t^{-1-k}\xi_{i_1}\cdots\xi_{i_r}\cdot\xi_{j_1}\cdots\xi_{j_s}-(kt^{k-1}\xi_{i_1}\cdots\xi_{i_r})(2-s)\frac{t^{-k}}{-k(2-r)}\xi_{j_1}\cdots\xi_{j_s}\\&=t^{-1}\xi_1\xi_2\xi_3\xi_4+\frac{2-s}{2-r}t^{-1}\xi_1\xi_2\xi_3\xi_4\\&=\frac{4-r-s}{2-r}t^{-1}\xi_1\xi_2\xi_3\xi_4=0,\quad\text{since } r+s=4.\end{aligned}$$

2. Due to antisymmetry of $[f,g]$, we have $t^{-1}\xi_1\xi_2\xi_3\xi_4 \neq (2-s)(\partial_t f)g$.
3. The element $t^{-1}\xi_1\xi_2\xi_3\xi_4$ does not belong to the support of $(-1)^{p(f)}(\sum_{i=1}^{4}(\partial_i f)(\partial_i g))$. Indeed, for all $1 \leq i \leq 4$, $(\partial_i f)(\partial_i g) \neq t^{-1}\xi_1\xi_2\xi_3\xi_4$.

Every monomial $f \in K(1,4) \setminus \mathbb{C}t^{-1}\xi_1\xi_2\xi_3\xi_4$ lies in $K'(1,4)$, indeed:

1. if $\deg(f) \neq 0$, then $f = \frac{[t,f]}{\deg(f)}$;
2. if $\deg(f) = 0$, then f is either equal to $\alpha\xi_i\xi_j = -\alpha[\xi_k\xi_i\xi_j, \xi_i\xi_j]$ for some $1 \leq i < j \leq 4$, $\alpha \in \mathbb{C}$ and $k \neq i,j$ or it is equal to $\alpha t = -\alpha[t\xi_1, \xi_1]$ for some $\alpha \in \mathbb{C}$.

Therefore, we have:

$$K'(1,4) = \langle t^k\xi_{i_1}\cdots\xi_{i_r},\ t^l\xi_1\xi_2\xi_3\xi_4,\ \text{for } 0 \leq r < 4,\ k,l \in \mathbb{Z},\ l \neq -1\rangle.$$

□

2.1 The annihilation superalgebra of K_4'

Motivated by Proposition 1.10 and Theorem 1.15, we want to understand the structure of $\mathcal{A}(K_4')$. Let us recall some notions on central extensions of Lie superalgebras.

Definition 2.5. Let $\mathfrak{g}$ be a Lie superalgebra. A 2−*cocycle* on $\mathfrak{g}$ is a bilinear map $\psi : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ that satisfies the following conditions:

1. $\psi(a,b) = -(-1)^{p(a)p(b)}\psi(b,a)$,
2. $(-1)^{p(a)p(c)}\psi(a,[b,c]) + (-1)^{p(a)p(b)}\psi(b,[c,a]) + (-1)^{p(a)p(c)}\psi(c,[a,b]) = 0$,

for all $a,b,c \in \mathfrak{g}$. The set of all 2−cocycles on $\mathfrak{g}$ is denoted by $Z^2(\mathfrak{g},\mathbb{C})$.

Remark 2.6. We denote the set of linear maps $\mathfrak{g} \to \mathbb{C}$ by $C^1(\mathfrak{g},\mathbb{C})$, we call its elements 1−*cochains*. For every 1−cochain $f \in C^1(\mathfrak{g},\mathbb{C})$, it is possible to construct a 2−cocycle δf on $\mathfrak{g}$. For all $a,b \in \mathfrak{g}$ we define:

$$\delta f(a,b) = f([a,b]).$$

It is a straightforward verification that δf is a 2−cocycle on $\mathfrak{g}$. The map $\delta : C^1(\mathfrak{g},\mathbb{C}) \to Z^2(\mathfrak{g},\mathbb{C})$, $f \to \delta f$, is called *coboundary operator*.

Definition 2.7. We denote by $B^2(\mathfrak{g},\mathbb{C})$ the image of $\delta : C^1(\mathfrak{g},\mathbb{C}) \to Z^2(\mathfrak{g},\mathbb{C})$. Two 2−cocycles $\psi_1, \psi_2 \in Z^2(\mathfrak{g},\mathbb{C})$ are *cohomologous* when $\psi_1 - \psi_2 \in B^2(\mathfrak{g},\mathbb{C})$. We denote by $H^2(\mathfrak{g},\mathbb{C})$ the quotient $\frac{Z^2(\mathfrak{g},\mathbb{C})}{B^2(\mathfrak{g},\mathbb{C})}$.

Definition 2.8. A Lie superalgebra $\hat{\mathfrak{g}}$ is a *central extension* of $\mathfrak{g}$ by a one−dimensional center $\mathbb{C}C$ if there exist two morphisms $i : \mathbb{C}C \to \hat{\mathfrak{g}}$ and $s : \hat{\mathfrak{g}} \to \mathfrak{g}$ such that the following sequence is exact:

$$0 \to \mathbb{C}C \xrightarrow{i} \hat{\mathfrak{g}} \xrightarrow{s} \mathfrak{g} \to 0,$$

and $\mathrm{Ker}(s)$ lies in the center of $\hat{\mathfrak{g}}$.

Definition 2.9. Two central extentions $\hat{\mathfrak{g}}_1$ and $\hat{\mathfrak{g}}_2$ of $\mathfrak{g}$ by a one−dimensional center $\mathbb{C}C$ are isomorphic if there exists an isomorphism of Lie superalgebras $\Phi : \hat{\mathfrak{g}}_1 \to \hat{\mathfrak{g}}_2$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C}C & \xrightarrow{i_1} & \hat{\mathfrak{g}}_1 & \xrightarrow{s_1} & \mathfrak{g} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ 0 & \longrightarrow & \mathbb{C}C & \xrightarrow{i_2} & \hat{\mathfrak{g}}_2 & \xrightarrow{s_2} & \mathfrak{g} & \xrightarrow{d} & 0. \end{array}$$

Proposition 2.10. *There is a bijection between central extensions of* $\mathfrak{g}$ *by a one−dimensional center and elements of* $H^2(\mathfrak{g}, \mathbb{C})$*. If* $\psi \in Z^2(\mathfrak{g}, \mathbb{C})$ *the corresponding central extension is, up to isomorphisms,* $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}C$ *where:*

$$[C, a] = 0 \quad \textit{and} \quad [a, b]_{\hat{\mathfrak{g}}} = [a, b]_{\mathfrak{g}} + \psi(a, b)C,$$

for all $a, b \in \mathfrak{g}$*.*

Proof. From the definition it follows directly that a central extension $\hat{\mathfrak{g}} \cong \mathfrak{g} \oplus \mathbb{C}i(C)$ as vector spaces and we have the following relation between the bracket $[\cdot, \cdot]_{\hat{\mathfrak{g}}}$ in $\hat{\mathfrak{g}}$ and the bracket $[\cdot, \cdot]_{\mathfrak{g}}$ in $\mathfrak{g}$ for all $a, b \in \mathfrak{g}$, $\alpha, \beta \in \mathbb{C}$:

$$[a + \alpha i(C), b + \beta i(C)]_{\hat{\mathfrak{g}}} = [a, b]_{\mathfrak{g}} + \psi(a, b)i(C),$$

where $\psi : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is a 2−cocycle.
Conversely, given $\psi \in C^2(\mathfrak{g}, \mathbb{C})$, we can construct a central extension $\hat{\mathfrak{g}}$ of $\mathfrak{g}$. We define $\hat{\mathfrak{g}} := \mathfrak{g} \oplus \mathbb{C}C$. For all $a, b \in \mathfrak{g}$, $\alpha, \beta \in \mathbb{C}$, we set $i(\alpha C) := \alpha C$, $s(a + \alpha C) := a$ and $[a + \alpha C, b + \beta C]_{\hat{\mathfrak{g}}} := [a, b]_{\mathfrak{g}} + \psi(a, b)C$. It follows directly from the definition of 2−cocycles that it is a central extension.
Finally we show that two isomorphic central extensions $\hat{\mathfrak{g}}_1 \cong \mathfrak{g} \oplus \mathbb{C}C$ and $\hat{\mathfrak{g}}_2 \cong \mathfrak{g} \oplus \mathbb{C}C$ correspond to cohomologous 2−cocycles. Since $\hat{\mathfrak{g}}_1$ and $\hat{\mathfrak{g}}_2$ are isomorphic, we have an isomorphism $\Phi : \hat{\mathfrak{g}}_1 \to \hat{\mathfrak{g}}_2$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{C}C & \xrightarrow{i_1} & \hat{\mathfrak{g}}_1 & \xrightarrow{s_1} & \mathfrak{g} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\
0 & \longrightarrow & \mathbb{C}C & \xrightarrow{i_2} & \hat{\mathfrak{g}}_2 & \xrightarrow{s_2} & \mathfrak{g} & \xrightarrow{d} & 0.
\end{array}$$

Thus for all $a \in \mathfrak{g}$, $\alpha \in \mathbb{C}$:

$$\Phi(a + \alpha C) = a + \rho(a)C + \alpha C, \tag{2.4}$$

where $\rho \in C^1(\mathfrak{g}, \mathbb{C})$.
We call ψ_1(resp. ψ_2) the 2−cocycle that corresponds to $\hat{\mathfrak{g}}_1$(resp. $\hat{\mathfrak{g}}_2$). We have for all $a, b \in \mathfrak{g}$:

$$\begin{aligned}
\Phi([a, b]_{\hat{\mathfrak{g}}_1}) &= \Phi([a, b]_{\mathfrak{g}} + \psi_1(a, b)C) \\
&= [a, b]_{\mathfrak{g}} + \left(\rho([a, b]_{\mathfrak{g}}) + \psi_1(a, b)\right) C.
\end{aligned}$$

But from the fact that Φ is an isomorphism we also have:

$$\begin{aligned}
\Phi([a, b]_{\hat{\mathfrak{g}}_1}) &= [\Phi(a), \Phi(b)]_{\hat{\mathfrak{g}}_2} \\
&= [a + \rho(a)C, b + \rho(b)C]_{\hat{\mathfrak{g}}_2} \\
&= [a, b]_{\mathfrak{g}} + \psi_2(a, b)C.
\end{aligned}$$

Therefore, $\delta\rho + \psi_1 = \psi_2$.
Analogously, if $\psi_1, \psi_2 \in Z^2(\mathfrak{g}, \mathbb{C})$ are cohomologous, i.e. $\psi_1 - \psi_2 = \delta\eta \in B^2(\mathfrak{g}, \mathbb{C})$, then we can construct an isomorphism between the central extensions defined by ψ_1 and ψ_2 as in (2.4) letting $\rho := \eta$. □

The following proposition is the main result of this section.

Proposition 2.11. *The following is a surjective morphism of Lie superalgebras:*

$$\begin{aligned}
\phi : \mathrm{Lie}\, K_4' &\longrightarrow K'(1, 4) \\
P(\xi)y^m &\longmapsto P(\xi)t^m \ \textit{if} \ P(\xi) \neq \xi_1\xi_2\xi_3\xi_4
\end{aligned}$$

$$\partial\xi_1\xi_2\xi_3\xi_4 y^m \longmapsto -m\xi_1\xi_2\xi_3\xi_4 t^{m-1}.$$

The Lie superalgebra $\mathrm{Lie}\,K_4'$ *is a central extension of* $K'(1,4)$ *by a one−dimensional center.* *The annihilation superalgebra of* K_4' *is a central extension of* $K(1,4)_+$ *by a one−dimensional center* $\mathbb{C}C$*:*

$$\mathcal{A}(K_4') = K(1,4)_+ \oplus \mathbb{C}C.$$

The extension is given by a $2-$*cocycle* $\psi \in Z^2(K(1,4)_+, \mathbb{C})$ *whose non−trivial entries are computed, using bilinearity and antisimmetry of* ψ*, from:*

$$\psi(1, \xi_1\xi_2\xi_3\xi_4) = -2,$$
$$\psi(\xi_i, \partial_i\xi_1\xi_2\xi_3\xi_4) = -1.$$

We need a lemma in order to prove Proposition 2.11.

Lemma 2.12. *The element* $\partial\xi_1\xi_2\xi_3\xi_4 y^0 \in \mathrm{Lie}\,K_4'$ *is central.*

Proof. We have, for all $py^l \in \mathrm{Lie}\,K_4'$, with $p \in K_4'$:

$$\left[\partial\xi_1\xi_2\xi_3\xi_4 y^0, py^l\right] = \binom{0}{0}\left((\partial\xi_1\xi_2\xi_3\xi_4)_0 p\right) y^l = 0.$$

In the last equality we used the fact that $((\partial\xi_1\xi_2\xi_3\xi_4)_0 p)$ is computed as the restriction of the $0-$product in $\mathrm{Lie}\,K_4$, for which we can use the relation $(\partial a_{(n)} b) = -n(a_{(n-1)} b)$. □

We set the following notation. Given a proposition P, we will use χ_P:

$$\chi_P = \begin{cases} 1 & \text{if } P \text{ is true,} \\ 0 & \text{if } P \text{ is false.} \end{cases}$$

Remark 2.13. From the definition of $\mathrm{Lie}\,K_4'$, for all $a \in K_4'$ and $m \in \mathbb{Z}$, we have that $\partial a y^m = -m a y^{m-1}$. We showed that $\xi_1\xi_2\xi_3\xi_4 \notin K_4'$. Hence, every class of equivalence of a monomial $\partial^k P(\xi) y^n \in \mathrm{Lie}\,K_4'$ has a unique representative of the type:

$$(-1)^k \frac{n!}{(n-k)!} P(\xi) y^{n-k} \; if \;\; P(\xi) \neq \xi_1\xi_2\xi_3\xi_4,$$

or of the type:

$$(-1)^{k-1} \frac{n!}{(n-(k-1))!} \partial\xi_1\xi_2\xi_3\xi_4 y^{n-k+1} \; if \;\; P(\xi) = \xi_1\xi_2\xi_3\xi_4.$$

Therefore the set $\left\{\xi_{i_1} \cdots \xi_{i_r} y^k, \, \partial\xi_1\xi_2\xi_3\xi_4 y^m, \; for \; k, m \in \mathbb{Z}, \; r \neq 4\right\}$ is a basis for $\mathrm{Lie}\,K_4'$.

Proof of Proposition 2.11. Observe that ϕ is well defined due to Remark 2.13 and Proposition 2.4. It is clear from its definition that ϕ is surjective.
We prove that ϕ is a morphism of Lie superalgebras. We have to distinguish four cases:

1. Let $f = Q(\xi) y^h$ and $g = \widetilde{Q}(\xi) y^l$ in $\mathrm{Lie}\,K_4'$, with $Q(\xi) = \xi_{i_1} \cdots \xi_{i_r}$, $\widetilde{Q}(\xi) = \xi_{j_1} \cdots \xi_{j_s}$, $r < 4$, $s < 4$, $Q(\xi) \cdot \widetilde{Q}(\xi) \neq \pm\xi_1\xi_2\xi_3\xi_4$ and $h, l \in \mathbb{Z}$. In $\mathrm{Lie}\,K_4'$ we have, using bracket (1.2) and $n-$products (2.2):

$$[f, g] = \sum_{j \in \mathbb{Z}_+} \binom{h}{j} (Q_{(j)} \widetilde{Q}) y^{h+l-j}$$

$$\begin{aligned}
&= \binom{h}{0}(Q_{(0)}\widetilde{Q})y^{h+l} + \binom{h}{1}(Q_{(1)}\widetilde{Q})y^{h+l-1} \\
&= (r-2)\partial(Q\widetilde{Q})y^{h+l} + (-1)^r\sum_{i=1}^{4}\partial_i Q\,\partial_i\widetilde{Q}\,y^{h+l} + h(r+s-4)Q\widetilde{Q}\,y^{h+l-1} \\
&= -(r-2)(Q\widetilde{Q})(h+l)y^{h+l-1} + (-1)^r\sum_{i=1}^{4}\partial_i Q\,\partial_i\widetilde{Q}\,y^{h+l} + h(r+s-4)Q\widetilde{Q}\,y^{h+l-1} \\
&= ((2-r)l + h(s-2))Q\widetilde{Q}\,y^{h+l-1} + (-1)^r\sum_{i=1}^{4}\partial_i Q\,\partial_i\widetilde{Q}\,y^{h+l}.
\end{aligned}$$

In $K'(1,4)$ we have, using bracket (2.1):

$$\begin{aligned}
[\phi(f),\phi(g)] &= \big[t^h Q(\xi), t^l\widetilde{Q}(\xi)\big] \\
&= (2-r)t^h Q(\xi) l t^{l-1}\widetilde{Q}(\xi) - h t^{h-1}Q(\xi)(2-s)t^l\widetilde{Q}(\xi) + (-1)^r\sum_{i=1}^{4}\partial_i Q\,\partial_i\widetilde{Q}\,t^{h+l} \\
&= ((2-r)l + h(s-2))Q\widetilde{Q}\,t^{h+l-1} + (-1)^r\sum_{i=1}^{4}\partial_i Q\,\partial_i\widetilde{Q}\,t^{h+l} = \phi([f,g]).
\end{aligned}$$

2. Let $f = Q(\xi)y^h$ and $g = \widetilde{Q}(\xi)y^l$ in Lie K_4', with $Q(\xi) = \xi_{i_1}\cdots\xi_{i_r}$, $\widetilde{Q}(\xi) = \xi_{j_1}\cdots\xi_{j_s}$, $r < 4$, $s < 4$, $Q(\xi)\cdot\widetilde{Q}(\xi) = \xi_1\xi_2\xi_3\xi_4$ and $h, l \in \mathbb{Z}$. In Lie K_4' we have, using bracket (1.2) and $n-$products (2.2):

$$\begin{aligned}
[f,g] &= \sum_{j\in\mathbb{Z}_+}\binom{h}{j}(Q_{(j)}\widetilde{Q})y^{h+l-j} \\
&= \binom{h}{0}(Q_{(0)}\widetilde{Q})y^{h+l} + \binom{h}{1}(Q_{(1)}\widetilde{Q})y^{h+l-1} \\
&= (r-2)\partial(\xi_1\xi_2\xi_3\xi_4)y^{h+l}.
\end{aligned}$$

In $K'(1,4)$ we have, using bracket (2.1):

$$\begin{aligned}
[\phi(f),\phi(g)] &= \big[t^h Q(\xi), t^l\widetilde{Q}(\xi)\big] \\
&= (2-r)t^h Q(\xi) l t^{l-1}\widetilde{Q}(\xi) - h t^{h-1}Q(\xi)(2-s)t^l\widetilde{Q}(\xi) \\
&= ((2-r)l + h(4-r-2))\xi_1\xi_2\xi_3\xi_4 t^{h+l-1} \\
&= (2-r)(l+h)\xi_1\xi_2\xi_3\xi_4 t^{h+l-1} = \phi\big((r-2)\partial\,(\xi_1\xi_2\xi_3\xi_4)\,y^{h+l}\big).
\end{aligned}$$

3. Let $f = \partial\xi_1\xi_2\xi_3\xi_4 y^m$ and $g = \partial\xi_1\xi_2\xi_3\xi_4 y^l$ in Lie K_4', with $m, l \in \mathbb{Z}$. In Lie K_4' we have, using bracket (1.2) and $n-$products (2.2):

$$[f,g] = \sum_{j\in\mathbb{Z}_+}\binom{h}{j}(\partial\xi_1\xi_2\xi_3\xi_4\,{}_{(j)}\partial\xi_1\xi_2\xi_3\xi_4)y^{h+l-j} = 0.$$

Indeed, for every $j \geq 0$, the $j-$product $(\partial\xi_1\xi_2\xi_3\xi_4\,{}_{(j)}\partial\xi_1\xi_2\xi_3\xi_4)$ is computed in terms of $(\xi_1\xi_2\xi_3\xi_4\,{}_{(j)}\xi_1\xi_2\xi_3\xi_4) = 0$ for all $j \geq 0$. On the other hand in $K'(1,4)$ we have, using bracket (2.1):

$$[\phi(f),\phi(g)] = \big[-m\xi_1\xi_2\xi_3\xi_4 t^{m-1}, -l\xi_1\xi_2\xi_3\xi_4 t^{l-1}\big] = 0.$$

4. Let $f=\partial\xi_1\xi_2\xi_3\xi_4 y^m$ and $g=P(\xi)y^l$ in $\mathrm{Lie}\,K_4'$, with $P(\xi)\neq\xi_1\xi_2\xi_3\xi_4$, $m,l\in\mathbb{Z}$. First, we point out that $(\partial\xi_1\xi_2\xi_3\xi_4\,{}_{(j)}P(\xi))=-j(\xi_1\xi_2\xi_3\xi_4\,{}_{(j-1)}P(\xi))=0$ for all $j>2$. In $\mathrm{Lie}\,K_4'$ we have, using bracket (1.2) and $n-$products (2.2):

$$
\begin{aligned}
[f,g]&=\big[\partial\xi_1\xi_2\xi_3\xi_4 y^m,P(\xi)y^l\big]\\
&=(\partial\xi_1\xi_2\xi_3\xi_4\,{}_{(0)}P(\xi))y^{m+l}+m(\partial\xi_1\xi_2\xi_3\xi_4\,{}_{(1)}P(\xi))y^{m+l-1}+\binom{m}{2}(\partial\xi_1\xi_2\xi_3\xi_4\,{}_{(2)}P(\xi))y^{m+l-2}\\
&=0-m(\xi_1\xi_2\xi_3\xi_4\,{}_{(0)}P(\xi))y^{m+l-1}+\binom{m}{2}(\partial\xi_1\xi_2\xi_3\xi_4\,{}_{(2)}P(\xi))y^{m+l-2}\\
&=-2m\,\partial\xi_1\xi_2\xi_3\xi_4\,\chi_{P(\xi)\in\mathbb{C}}\,y^{m+l-1}-m\sum_{i=1}^N\partial_i(\xi_1\xi_2\xi_3\xi_4)\partial_i\,(P(\xi))\;y^{m+l-1}\\
&-2\binom{m}{2}\partial(4+0-4)\,\xi_1\xi_2\xi_3\xi_4\,\chi_{P(\xi)\in\mathbb{C}}\,y^{m+l-2}\\
&=-2m\,\partial\xi_1\xi_2\xi_3\xi_4\,\chi_{P(\xi)\in\mathbb{C}}\,y^{m+l-1}-m\sum_{i=1}^N\partial_i(\xi_1\xi_2\xi_3\xi_4)\partial_i(P(\xi))\,y^{m+l-1}.
\end{aligned}
$$

In $K'(1,4)$ we have, using bracket (2.1):

$$
\begin{aligned}
[\phi(f),\phi(g)]&=\big[-m\xi_1\xi_2\xi_3\xi_4 t^{m-1},P(\xi)t^l\big]\\
&=-m(-2l-2(m-1))\,t^{m+l-2}\xi_1\xi_2\xi_3\xi_4\,\chi_{P(\xi)\in\mathbb{C}}-m\sum_{i=1}^N\partial_i(\xi_1\xi_2\xi_3\xi_4)\,\partial_i\,(P(\xi))\;t^{m+l-1}\\
&=\phi([f,g]).
\end{aligned}
$$

The previous computations imply that the kernel of the map $\phi:\mathrm{Lie}\,K_4'\longrightarrow K'(1,4)$ is $\mathrm{Ker}\,\phi=\langle\partial\xi_1\xi_2\xi_3\xi_4\rangle$ and so the following sequence is exact:

$$0\to\langle\partial\xi_1\xi_2\xi_3\xi_4\rangle\xrightarrow{i}\mathrm{Lie}\,K_4'\xrightarrow{\phi}K'(1,4)\to 0.$$

By Lemma 2.12 the Lie superalgebra $\mathrm{Lie}\,K_4'$ is therefore a central extension of $K'(1,4)$ by the one$-$dimensional center $\langle\partial\xi_1\xi_2\xi_3\xi_4\rangle$.
In particular, we point out that $\phi:\mathrm{Lie}\,K_4'/\mathbb{C}\partial\xi_1\xi_2\xi_3\xi_4\to K'(1,4)$ is an isomorphism. In the previous computations we computed all the possible brackets between monomials in $\mathrm{Lie}\,K_4'$, therefore in particular all the possible brackets between monomials in $\mathcal{A}(K_4')$. We point out that the central element $\partial\xi_1\xi_2\xi_3\xi_4$ lies in the support of $[f,g]$, with f and g monomials in $\mathcal{A}(K_4')$, only in the case (2) of the previous computations for $h=l=0$ and $r\neq 2$. In particular:

$$[\xi_i,\partial_i\xi_1\xi_2\xi_3\xi_4]=-\partial\xi_1\xi_2\xi_3\xi_4$$

and

$$[1,\xi_1\xi_2\xi_3\xi_4]=-2\partial\xi_1\xi_2\xi_3\xi_4.$$

Hence, $\mathcal{A}(K_4')$ is a central extension of $K(1,4)_+$ by a one$-$dimensional center $\langle\partial\xi_1\xi_2\xi_3\xi_4\rangle$ and the extension is given by a 2$-$cocycle $\psi\in Z^2(K(1,4)_+,\mathbb{C})$ whose non$-$trivial entries are computed, using bilinearity and antisimmetry of ψ, from:

$$
\begin{aligned}
&\psi(1,\xi_1\xi_2\xi_3\xi_4)=-2,\\
&\psi(\xi_i,\partial_i\xi_1\xi_2\xi_3\xi_4)=-1.
\end{aligned}
$$

□

Chapter 3

Verma modules

In this chapter we study the action of $\mathfrak{g} := \mathcal{A}(K'_4) = K(1,4)_+ \oplus \mathbb{C}C$ on a Verma module $\mathrm{Ind}(F)$, where F is a finite−dimensional irreducible $\mathfrak{g}_{\geq 0}$−module, on which $\mathfrak{g}_{>0}$ acts trivially. The grading on $\mathfrak{g}$ is the standard grading of $K(1,4)_+$ and C has degree 0. We have:

$$\begin{aligned}
\mathfrak{g}_{-2} &= \langle 1 \rangle \,, \\
\mathfrak{g}_{-1} &= \langle \xi_1, \xi_2, \xi_3, \xi_4 \rangle \,, \\
\mathfrak{g}_{0} &= \langle \{C, t, \xi_i \xi_j \ 1 \leq i < j \leq 4\} \rangle \,.
\end{aligned}$$

Remark 3.1. The annihilation superalgebra $\mathfrak{g}$ satisfies conditions L1, L2, L3. Indeed:

1. L1 is obvious.
2. The element t is a grading element, i.e. $[t, a] = \deg(a)a$ for all $a \in \mathfrak{g}$. Hence, by Remark 1.11, t satisfies condition L2.
3. The element Θ of L3 is chosen as $-\frac{1}{2} \in \mathfrak{g}_{-2}$. Indeed for all $m, s \in \mathbb{Z}_{\geq 0}$ such that $2m + s - 2 \geq -2$, the element $t^m \xi_{i_1} \cdots \xi_{i_s} \in \mathfrak{g}_{2m+s-2}$ satisfies $t^m \xi_{i_1} \cdots \xi_{i_s} = -\frac{1}{m+1}[\Theta, t^{m+1} \xi_{i_1} \cdots \xi_{i_s}]$. We also have $C = [\Theta, \xi_1 \xi_2 \xi_3 \xi_4]$.

Remark 3.2. Since $\mathrm{Ind}(F) \cong U(\mathfrak{g}_{<0}) \otimes F$, it follows that $\mathrm{Ind}(F) \cong \mathbb{C}[\Theta] \otimes \bigwedge(4) \otimes F$. Indeed, let us denote by η_i the image in $U(\mathfrak{g})$ of $\xi_i \in \bigwedge(4)$, for all $i \in \{1,2,3,4\}$. In $U(\mathfrak{g})$ we have that $\eta_i^2 = \Theta$, for all $i \in \{1,2,3,4\}$: since $[\xi_i, \xi_i] = -1$ in $\mathfrak{g}$, we have $\eta_i \eta_i = -\eta_i \eta_i - 1$ in $U(\mathfrak{g})$.

From now on it is always assumed that F is a finite−dimensional irreducible $\mathfrak{g}_{\geq 0}$−module. Let us focus on $\mathfrak{g}_0 = \langle \{C, t, \xi_i \xi_j \quad 1 \leq i < j \leq 4\} \rangle \cong \mathfrak{so}(4) \oplus \mathbb{C}E_{00} \oplus \mathbb{C}C$, where $\mathfrak{so}(4)$ is the Lie algebra of skew−symmetric matrices and $E_{00} := t$. As in [BKL1], we denote $F_{i,j} := -\xi_i \xi_j$, $F_{i,j}$ corresponds to $E_{i,j} - E_{j,i} \in \mathfrak{so}(4)$. We take as a basis of a Cartan subalgebra $\mathfrak{h}$ the following (cf. [Kn] pag.83):

$$H_1 = iF_{1,2}, \ H_2 = iF_{3,4}.$$

We call $h_1 := H_1 - H_2$, $h_2 := H_1 + H_2$. Let $\varepsilon_j \in \mathfrak{h}^*$ be such that $\varepsilon_j(H_k) = \delta_{j,k}$. The set of roots is $\Delta = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 + \varepsilon_2, -(\varepsilon_1 - \varepsilon_2), -(\varepsilon_1 + \varepsilon_2)\}$, the set of positive roots is $\Delta^+ = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 + \varepsilon_2\}$. We have the following root decomposition:

$$\mathfrak{so}(4) = \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha) \ \text{with} \ \mathfrak{g}_\alpha = \mathbb{C}E_\alpha,$$

where the E_α's are:

$$\begin{aligned}
E_{\varepsilon_1 - \varepsilon_2} &= F_{1,3} + F_{2,4} + iF_{1,4} - iF_{2,3}, \\
E_{\varepsilon_1 + \varepsilon_2} &= F_{1,3} - F_{2,4} - iF_{1,4} - iF_{2,3},
\end{aligned}$$

$$E_{-(\varepsilon_1-\varepsilon_2)} = F_{1,3} + F_{2,4} - iF_{1,4} + iF_{2,3},$$
$$E_{-(\varepsilon_1+\varepsilon_2)} = F_{1,3} - F_{2,4} + iF_{1,4} + iF_{2,3}.$$

We will use the following notation:

$$\alpha_{1,2} = \frac{(E_{\varepsilon_1-\varepsilon_2} + E_{\varepsilon_1+\varepsilon_2})}{2}, \tag{3.1}$$
$$\beta_{1,2} = \frac{(E_{\varepsilon_1-\varepsilon_2} - E_{\varepsilon_1+\varepsilon_2})}{2}. \tag{3.2}$$

The set $\{\alpha_{1,2}, \beta_{1,2}\}$ is a basis of the upper Borel subalgebra $B_{\mathfrak{so}(4)}$.
We will write the weights $\mu = (m, n, \mu_0, \mu_c)$ of weight vectors of $\mathfrak{g}_0-$modules with respect to action of the vectors h_1, h_2, E_{00} and C.

Remark 3.3. Since C is central, by Schur's lemma, C acts as a scalar on F, and so we will denote also this scalar by C.

We will study the action of $\mathfrak{g}$ on $\mathrm{Ind}(F)$ using the $\lambda-$action notation:

$$f_\lambda(g \otimes v) = \sum_{j\geq 0} \frac{\lambda^j}{j!}(t^j f).(g \otimes v),$$

for $f \in \bigwedge(4)$, $g \in U(\mathfrak{g}_{<0})$ and $v \in F$. In order to find an explicit formula for $f_\lambda(g \otimes v)$ with $f \in \bigwedge(4)$, $g \in U(\mathfrak{g}_{<0})$, $v \in F$, we need some lemmas. We will denote by capital letters ordered sets $I = (i_1, i_2, \cdots i_k)$ of integers that lie in $\{1, 2, 3, 4\}$. By abuse of notation, we will denote by $I \cap J$ (resp. $I \setminus J$) the increasingly ordered set whose elements are the elements of the intersection of the underlying sets of I and J (resp. the elements of the difference of the underlying sets of I and J). We will say $I \subseteq J$ when the underlying set of I is contained in the underlying set of J. Analogously we will denote by I^c the increasingly ordered set whose elements are the elements of the complement of the underlying set of I. Given $I = (i_1, i_2, \cdots i_k)$, we will use the notation ξ_I (resp. η_I) to denote the element $\xi_{i_1}\xi_{i_2}\cdots\xi_{i_k} \in \bigwedge(4)$ (resp. the element $\eta_{i_1}\eta_{i_2}\cdots\eta_{i_k} \in U(\mathfrak{g}_{<0})$) and we will denote $|\xi_I| = |I| = k$ (resp. $|\eta_I| = |I| = k$). We will denote $\xi_* = \xi_1\xi_2\xi_3\xi_4$ (resp. $\eta_* = \eta_1\eta_2\eta_3\eta_4$). Given $I = (i_1, i_2, \cdots i_k)$ and $I^c = (j_{k+1}, j_{k+2}, \cdots j_4)$, we will denote by ε_I the sign of the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & k & k+1 & \cdots & 4 \\ i_1 & i_2 & \cdots & i_k & j_{k+1} & \cdots & j_4 \end{pmatrix}.$$

We will also use the following notation, for $a \in \mathbb{C}$, $I = (i_1, i_2, \cdots i_k)$:

$$\partial_I \eta_S = \partial_{i_1}\partial_{i_2}\ldots\partial_{i_k}\eta_S \qquad \partial_I \xi_S = \partial_{i_1}\partial_{i_2}\ldots\partial_{i_k}\xi_S;$$
$$\partial_{a\xi_I}\eta_S = a\partial_I\eta_S \qquad \partial_{a\xi_I}\xi_S = a\partial_I\xi_S;$$
$$\partial_\emptyset \eta_S = \eta_S \qquad \partial_\emptyset \xi_S = \xi_S.$$

Given monomials $\xi_I \in \bigwedge(4)$ and $\eta_J \in U(\mathfrak{g}_{<0})$, we will use the following notation:

$$\xi_I \star \eta_J = \chi_{I\cap J=\emptyset}\eta_I\eta_J,$$
$$\eta_J \star \xi_I = \chi_{I\cap J=\emptyset}\eta_J\eta_I.$$

We extend this notation by linearity to elements $\sum_I \xi_I \in \bigwedge(4)$ and $\sum_J \eta_J \in U(\mathfrak{g}_{<0})$.
We observe that in $\mathfrak{g}$, by (2.1) and Proposition 2.11:

$$[t^m\xi_I, \xi_r] = -mt^{m-1}\xi_I\xi_r + (-1)^{|I|}t^m\partial_r\xi_I + \psi(t^m\xi_I, \xi_r)C.$$

In particular:

$$[t^m\xi_I, \xi_r] = -mt^{m-1}\xi_I\xi_r + (-1)^{|I|}t^m\partial_r\xi_I + \chi_{m=0}\chi_{r=I^c}\varepsilon_I C. \tag{3.3}$$

Lemma 3.4. *Let $\xi_I \in \Lambda(4)$, $\eta_Q \in U(\mathfrak{g}_{<0})$, $v \in F$ and $m \geq 3$. We have:*

$$(t^m \xi_I).(\eta_Q \otimes v) = -\chi_{m=3}\chi_{|I|=0}\, 6\, \partial_{I^c}\eta_Q \otimes Cv.$$

Proof. We can always assume, without loss of generality, that $\eta_Q = \eta_J \eta_K$ with $I \cap J = \emptyset$, $K \subseteq I$. We first point out that $(t^m\xi_I).(\eta_Q \otimes v) = 0$ when $m > 3$ because $\deg(t^m \xi_I) = 2m + |I| - 2 > 4 \geq \deg(\eta_Q)$.
Let us show, using (3.3), the thesis for $m = 3$, $I, K = \emptyset$ and $J = (1, 2, 3, 4)$:

$$\begin{aligned}
(t^3).(\eta_1\eta_2\eta_3\eta_4 \otimes v) &= -3(t^2\xi_1)\eta_2\eta_3\eta_4 \otimes v - 3\eta_1(t^2\xi_2)\eta_3\eta_4 \otimes v - 3\eta_1\eta_2(t^2\xi_3)\eta_4 \otimes v - 3\eta_1\eta_2\eta_3(t^2\xi_4) \otimes v\\
&= 6(t\xi_1\xi_2)\eta_3\eta_4 \otimes v - 6\eta_2(t\xi_1\xi_3)\eta_4 \otimes v\\
&= -6(\xi_1\xi_2\xi_3)\eta_4 \otimes v\\
&= -6 \otimes Cv.
\end{aligned}$$

If $m = 3$, $|I| > 0$, $(t^3\xi_I).(\eta_Q \otimes v) = 0$ because $\deg(t^3\xi_I) = 2m + |I| - 2 > 4 \geq \deg(\eta_Q)$. If $m = 3$, $|I| = 0$ and $|Q| \neq 4$, $(t^3).(\eta_Q \otimes v) = 0$ because $\deg(t^3) = 4 > \deg(\eta_Q)$. □

Now we study the term of degree 0 in λ of the $\lambda-$action.

Lemma 3.5. *Let I, J, K with $I \cap J = \emptyset$, $K \subseteq I$. We have:*

$$\begin{aligned}
\xi_I.(\eta_J\eta_K \otimes v) = &\sum_{L\subseteq K} (-1)^{|I|(|J|+|K|)+|L|(|L|-1)/2-|L|(|K|-|L|)}\eta_J(\partial_L\eta_K)(\partial_L\xi_I) \otimes v\\
&+ \chi_{|I|=3}\, \varepsilon_I\, \partial_{I^c}(\eta_J)\eta_K \otimes Cv.
\end{aligned}$$

Proof. From repeated applications of (3.3) it follows:

$$\xi_I.(\eta_J\eta_K) \otimes v = (-1)^{|I||J|}\eta_J\xi_I\eta_K \otimes v + \chi_{|I|=3}\, \varepsilon_I\, \partial_{I^c}(\eta_J)\eta_K \otimes Cv. \tag{3.4}$$

Indeed, from (3.3), if $|I| = 1$, 2, then ξ_I commutes with every ξ_r such that $r \in J$ and formula (3.4) is straightforward. In the case $|I| = 3$ and $J = I^c$, using (3.3), we have:

$$\xi_I.(\eta_{I^c}\eta_K) \otimes v = -\eta_{I^c}\xi_I\eta_K \otimes v + \chi_{|I|=3}\, \varepsilon_I\, \partial_{I^c}(\eta_J)\eta_K \otimes Cv.$$

Finally for $|I| = 3, 4$ and $|J| = 0$, formula (3.4) is immediate.
The rest of the proof is the same as the proof of Lemma A.2 in [BKL1] and it is done by induction on $|K|$ using formula (3.4). □

Lemma 3.6. *Let $f = \xi_I \in \Lambda(4)$, $g = \eta_L \in U(\mathfrak{g}_{<0})$. We have:*

$$\begin{aligned}
f.(g \otimes v) =&(-1)^{p(f)}(|f| - 2)\Theta(\partial_f g) \otimes v + \sum_{i=1}^{4} \partial_{(\partial_i f)}(\xi_i \star g) \otimes v + (-1)^{p(f)} \sum_{i<j} \partial_{(\partial_i\partial_j f)} g \otimes F_{i,j} v\\
&+ \chi_{|I|=3}\, \varepsilon_I\, \partial_{I^c}(g) \otimes Cv.
\end{aligned}$$

Proof. The proof is analogous to the proof in [BKL1] of Lemma A.3, and it is based on Lemma 3.5. The extra term in C is due to the additional term of Lemma 3.5, which is not present in Lemma A.2 of [BKL1]. □

Now we study the term of degree 1 in λ of the $\lambda-$action.

Lemma 3.7. *Let $f = \xi_I \in \bigwedge(4)$, $g = \eta_L \in U(\mathfrak{g}_{<0})$. We have:*

$$\begin{aligned} tf.(g \otimes v) =&(-1)^{p(f)}(\partial_f g) \otimes E_{00}v + (-1)^{p(f)+p(g)} \sum_{i=1}^{4}((\partial_f(\partial_i g)) \star \xi_i) \otimes v \\ &+ \sum_{i \neq j} \left(\partial_{\partial_i f}(\partial_j g) \otimes F_{i,j}v\right) + \chi_{|I|=2}\, \varepsilon_I\, \partial_{I^c}(g) \otimes Cv. \end{aligned}$$

Proof. Without loss of generality we can suppose that $g = \eta_J \eta_K$ with $I \cap J = \emptyset$, $K \subseteq I$. Let us prove that:

$$\begin{aligned} t\xi_I.(\eta_J\eta_K \otimes v) =&(-1)^{|I||J|}\eta_J(t\xi_I)\eta_K \otimes v + \sum_{j=1}^{4}(-1)^{|I||J|-|I|+|J|}(\partial_j\eta_J)(\xi_I\xi_j)\eta_K \otimes v \\ &+ \chi_{|I|=2}\, \varepsilon_I(\partial_{I^c}\eta_J)\eta_K \otimes Cv. \end{aligned} \tag{3.5}$$

The formula is the same as the relation proved for $K(1,N)_+$ in the proof of Lemma A.4 of [BKL1], except for an additional term in C. We point out that a term with C is involved only if $|I| = 2$ and $|J| = 2$. Let us prove (3.5) by induction on $|J|$. If $|J| = 0$, (3.5) is straightforward. Let us consider $\eta_{\widetilde{J}} = \eta_J\eta_s$ with $\widetilde{J} \cap I = \emptyset$ and $s \notin J$. We have, using (3.5) for η_J, that:

$$\begin{aligned} t\xi_I.(\eta_J\eta_s\eta_K \otimes v) =&(-1)^{|I||J|}\eta_J(t\xi_I)\eta_s\eta_K \otimes v + \sum_{j=1}^{4}(-1)^{|I||J|-|I|+|J|}(\partial_j\eta_J)(\xi_I\xi_j)\eta_s\eta_K \otimes v \\ &+ \chi_{|I|=2}\, \varepsilon_I(\partial_{I^c}\eta_J)\eta_s\eta_K \otimes Cv. \end{aligned}$$

Notice that, since we are supposing $\eta_{\widetilde{J}} = \eta_J\eta_s$ with $\widetilde{J} \cap I = \emptyset$ and $s \notin J$, the term $\chi_{|I|=2}\, \varepsilon_I(\partial_{I^c}\eta_J)\eta_s\eta_K \otimes Cv$ is 0 because if $|I| = 2$, then $|J| < 2$. We have, using (3.3), that:

$$\begin{aligned} t\xi_I.(\eta_J\eta_s\eta_K \otimes v) =&(-1)^{|I||J|}\eta_J(t\xi_I)\eta_s\eta_K \otimes v + \sum_{j=1}^{4}(-1)^{|I||J|-|I|+|J|}(\partial_j\eta_J)(\xi_I\xi_j)\eta_s\eta_K \otimes v \\ =&(-1)^{|I|(|J|+1)}\eta_J\eta_s(t\xi_I)\eta_K \otimes v - (-1)^{|I||J|}\eta_J(\xi_I\xi_s)\eta_K \otimes v \\ &+ \sum_{j=1}^{4}(-1)^{|I||J|-|I|+|J|+|I|+1}(\partial_j\eta_J)\eta_s(\xi_I\xi_j)\eta_K \otimes v \\ &- (-1)^{|J|}\chi_{|I|=2}\chi_{|J|=1}\varepsilon_I(\partial_{I^c}\eta_J\eta_s)\eta_K \otimes Cv. \end{aligned}$$

We observe that:

$$\begin{aligned} -(-1)^{|I||J|}\eta_J(\xi_I\xi_s)\eta_K \otimes v &= (-1)^{|I||J|+1+|J|}(\partial_s\eta_{\widetilde{J}})(\xi_I\xi_s)\eta_K \otimes v \\ &= (-1)^{|I||\widetilde{J}|-|I|+|\widetilde{J}|}(\partial_s\eta_{\widetilde{J}})(\xi_I\xi_s)\eta_K \otimes v. \end{aligned}$$

Therefore:

$$\begin{aligned} t\xi_I.(\eta_J\eta_s\eta_K \otimes v) =&(-1)^{|I|(|\widetilde{J}|)}\eta_{\widetilde{J}}(t\xi_I)\eta_K \otimes v + \sum_{j=1}^{4}(-1)^{|I||\widetilde{J}|-|I|+|\widetilde{J}|}(\partial_j\eta_{\widetilde{J}})(\xi_I\xi_j)\eta_K \otimes v \\ &+ \chi_{|I|=2}\varepsilon_I(\partial_{I^c}\eta_{\widetilde{J}})\eta_K \otimes Cv. \end{aligned}$$

Hence, formula (3.5) is proved. The rest of the proof is analogous to the proof of Lemma A.4 in [BKL1] and it is based on (3.5). □

Now we study the term of degree 2 in λ of the $\lambda-$action.

Lemma 3.8. *Let $f=\xi_I \in \bigwedge(4)$, $g=\eta_L \in U(\mathfrak{g}_{<0})$. We have:*

$$\Big(\frac{1}{2}t^2 f\Big).(g\otimes v)=(-1)^{p(f)}\Big(\sum_{i<j}\partial_f(\partial_i\partial_j g)\otimes F_{i,j}v\Big)-\chi_{|I|=1}\,\varepsilon_I\,\partial_{I^c}g\otimes Cv.$$

Proof. As before, without loss of generality, we can suppose that $g=\eta_J\eta_K$ with $I\cap J=\emptyset$, $K\subseteq I$. Let us prove that:

$$\Big(\frac{1}{2}t^2\xi_I\Big).(\eta_J\eta_K\otimes v)= \tag{3.6}$$
$$=-\chi_{I=K}\sum_{i,j\in J,i<j}(-1)^{|I||J|+|I|(|I|+1)/2}\big((\partial_i\partial_j\eta_J)(\partial_I\eta_K)(\xi_i\xi_j)\otimes v\big)-\chi_{|I|=1}\,\varepsilon_I\,\partial_{I^c}(\eta_J)\eta_K\otimes Cv.$$

In order to prove (3.6), we need to prove the following:

$$\Big(\frac{1}{2}t^2\xi_I\Big).(\eta_J\eta_K\otimes v)= \tag{3.7}$$
$$\sum_{S\subseteq J,|S|=0}\mathrm{sgn}_S\,\frac{1}{2}(\partial_S\eta_J)(t^2\xi_I\xi_S)\eta_K\otimes v+\sum_{S\subseteq J,|S|=1}\mathrm{sgn}_S(\partial_S\eta_J)(t\xi_I\xi_S)\eta_K\otimes v$$
$$+\sum_{S\subseteq J,|S|=2}\big(\mathrm{sgn}_S(\partial_S\eta_J)(\xi_I\xi_S)\eta_K\otimes v\big)-\chi_{|I|=1}\,\varepsilon_I\,\partial_{I^c}(\eta_J)\eta_K\otimes Cv,$$

where $\mathrm{sgn}_S=\pm1$ and $\mathrm{sgn}_J=\pm1$. The numbers sgn_S and sgn_J will be computed explicitly later. We prove (3.7) by induction on $|J|$. If $|J|=0$, (3.7) is straightforward. Let us consider $\eta_{\widetilde{J}}=\eta_J\eta_r$ with $\widetilde{J}\cap I=\emptyset$ and $r\notin J$. We have, using (3.7) for η_J, that:

$$\Big(\frac{1}{2}t^2\xi_I\Big).(\eta_J\eta_r\eta_K\otimes v)=$$
$$=\sum_{S\subseteq J,|S|=0}\mathrm{sgn}_S\,\frac{1}{2}(\partial_S\eta_J)(t^2\xi_I\xi_S)\eta_r\eta_K\otimes v+\sum_{S\subseteq J,|S|=1}\mathrm{sgn}_S(\partial_S\eta_J)(t\xi_I\xi_S)\eta_r\eta_K\otimes v$$
$$+\sum_{S\subseteq J,|S|=2}\mathrm{sgn}_S(\partial_S\eta_J)(\xi_I\xi_S)\eta_r\eta_K\otimes v-\chi_{|I|=1}\,\varepsilon_I\,\partial_{I^c}(\eta_J)\eta_r\eta_K\otimes Cv.$$

Notice that, since we are supposing $\eta_{\widetilde{J}}=\eta_J\eta_r$ with $\widetilde{J}\cap I=\emptyset$ and $r\notin J$, the term $-\chi_{|I|=1}\,\varepsilon_I\,\partial_{I^c}(\eta_J)\eta_r\eta_K\otimes Cv$ is 0 because if $|I|=1$, then $|J|<3$. We have, using (3.3), that:

$$\Big(\frac{1}{2}t^2\xi_I\Big).(\eta_J\eta_r\eta_K\otimes v)=$$
$$=\sum_{S\subseteq J,|S|=0}\mathrm{sgn}_S\,\frac{1}{2}(\partial_S\eta_J)(t^2\xi_I\xi_S)\eta_r\eta_K\otimes v+\sum_{S\subseteq J,|S|=1}\mathrm{sgn}_S(\partial_S\eta_J)(t\xi_I\xi_S)\eta_r\eta_K\otimes v$$
$$+\sum_{S\subseteq J,|S|=2}\mathrm{sgn}_S(\partial_S\eta_J)(\xi_I\xi_S)\eta_r\eta_K\otimes v$$
$$=\sum_{S\subseteq J,|S|=0}\mathrm{sgn}_S\,\frac{1}{2}(\partial_S\eta_J)\eta_r(t^2\xi_I\xi_S)\eta_K\otimes v+\sum_{S\subseteq J,|S|=0}\mathrm{sgn}_S(\partial_S\eta_J)(t\xi_I\xi_S\xi_r)\eta_K\otimes v$$
$$+\sum_{S\subseteq J,|S|=1}\mathrm{sgn}_S(\partial_S\eta_J)\eta_r(t\xi_I\xi_S)\eta_K\otimes v+\sum_{S\subseteq J,|S|=1}\mathrm{sgn}_S(\partial_S\eta_J)(\xi_I\xi_S\xi_r)\eta_K\otimes v$$

$$+\sum_{S\subseteq J,|S|=2}\big(\operatorname{sgn}_S(\partial_S\eta_J)\eta_r(\xi_I\xi_S)\eta_K\otimes v\big)+\chi_{|I|=1}\,\varepsilon_I\operatorname{sgn}_J\,\partial_{I^c}(\eta_J\eta_r)\eta_K\otimes Cv$$
$$=\sum_{S\subseteq J,|S|=0}\operatorname{sgn}_S\frac{1}{2}(\partial_S\eta_J\eta_r)(t^2\xi_I\xi_S)\eta_K\otimes v+\sum_{S\subseteq J,|S|=0}\operatorname{sgn}_S(\partial_r\eta_{\widetilde{J}})(t\xi_I\xi_S\xi_r)\eta_K\otimes v$$
$$+\sum_{S\subseteq J,|S|=1}\operatorname{sgn}_S(\partial_S\eta_J\eta_r)(t\xi_I\xi_S)\eta_K\otimes v+\sum_{S\subseteq J,|S|=1}\operatorname{sgn}_S(\partial_S\partial_r\eta_{\widetilde{J}})(\xi_I\xi_S\xi_r)\eta_K\otimes v$$
$$+\sum_{S\subseteq J,|S|=2}\big(\operatorname{sgn}_S(\partial_S\eta_J\eta_r)(\xi_I\xi_S)\eta_K\otimes v\big)+\chi_{|I|=1}\,\varepsilon_I\operatorname{sgn}_J\,\partial_{I^c}(\eta_J\eta_r)\eta_K\otimes Cv$$
$$=\sum_{S\subseteq\widetilde{J},|S|=0}\operatorname{sgn}_S\frac{1}{2}(\partial_S\eta_{\widetilde{J}})(t^2\xi_I\xi_S)\eta_K\otimes v+\sum_{S\subseteq\widetilde{J},|S|=1}\operatorname{sgn}_S(\partial_S\eta_{\widetilde{J}})(t\xi_I\xi_S)\eta_K\otimes v$$
$$+\sum_{S\subseteq\widetilde{J},|S|=2}\big(\operatorname{sgn}_S(\partial_S\eta_{\widetilde{J}})(\xi_I\xi_S)\eta_K\otimes v\big)+\chi_{|I|=1}\,\varepsilon_I\operatorname{sgn}_J\,\partial_{I^c}(\eta_{\widetilde{J}})\eta_K\otimes Cv.$$

We now compute explicitly the sign of the term $\partial_{I^c}(\eta_{\widetilde{J}})\eta_K\otimes Cv$. Hence we consider I with $|I|=1$ and $|\widetilde{J}|=3$, that is the only case in which there is a term involving C. For $I=(i)$ and $\widetilde{J}=(j,k,l)=I^c$ we have:

$$\Big(\frac{1}{2}t^2\xi_i\Big).(\eta_j\eta_k\eta_l\eta_K\otimes v)=$$
$$=-(t\xi_i\xi_j)\eta_k\eta_l\eta_K\otimes v+\eta_j(t\xi_i\xi_k)\eta_l\eta_K\otimes v-\eta_j\eta_k(t\xi_i\xi_l)\eta_K\otimes v-\eta_j\eta_k\eta_l\Big(\frac{1}{2}t^2\xi_i\Big)\eta_K\otimes v$$
$$=(\xi_i\xi_j\xi_k)\eta_l\eta_K\otimes v+\eta_k(\xi_i\xi_j\xi_l)\eta_K\otimes v-\eta_k\eta_l(t\xi_i\xi_j)\eta_K\otimes v$$
$$-\eta_j(\xi_i\xi_k\xi_l)\eta_K\otimes v+\eta_j\eta_l(t\xi_i\xi_k)\eta_K\otimes v-\eta_j\eta_k(t\xi_i\xi_l)\eta_K\otimes v-\eta_j\eta_k\eta_l\Big(\frac{1}{2}t^2\xi_i\Big)\eta_K\otimes v$$
$$=-\eta_l(\xi_i\xi_j\xi_k)\eta_K\otimes v+\varepsilon_I\,\eta_K\otimes Cv+\eta_k(\xi_i\xi_j\xi_l)\eta_K\otimes v-\eta_k\eta_l(t\xi_i\xi_j)\eta_K\otimes v$$
$$-\eta_j(\xi_i\xi_k\xi_l)\eta_K\otimes v+\eta_j\eta_l(t\xi_i\xi_k)\eta_K\otimes v-\eta_j\eta_k(t\xi_i\xi_l)\eta_K\otimes v-\eta_j\eta_k\eta_l\Big(\frac{1}{2}t^2\xi_i\Big)\eta_K\otimes v$$
$$=\sum_{i=0}^{2}\sum_{S\subseteq\widetilde{J},|S|=i}\operatorname{sgn}_{S,i}\frac{1}{2}\frac{2}{(2-i)!}(\partial_S\eta_{\widetilde{J}})(t^{2-i}\xi_I\xi_S)\eta_K\otimes v-\varepsilon_I\,(\partial_{I^c}\eta_{\widetilde{J}})\eta_K\otimes Cv,$$

where $\operatorname{sgn}_{S,i}=\pm1$. Hence, we proved (3.7). We notice that in (3.7) the terms

$$\sum_{S\subseteq J,|S|=0}\operatorname{sgn}_S\frac{1}{2}(\partial_S\eta_J)(t^2\xi_I\xi_S)\eta_K\otimes v+\sum_{S\subseteq J,|S|=1}\operatorname{sgn}_S(\partial_S\eta_J)(t\xi_I\xi_S)\eta_K\otimes v$$

are actually zero, since $\deg(t^2\xi_I\xi_S)>\deg(\eta_K)$ and $\deg(t\xi_I\xi_S)>\deg(\eta_K)$. Hence (3.7) reduces to:

$$\Big(\frac{1}{2}t^2\xi_I\Big).(\eta_J\eta_K\otimes v)=\sum_{S\subseteq J,|S|=2}\big(\operatorname{sgn}_S(\partial_S\eta_J)(\xi_I\xi_S)\eta_K\otimes v\big)-\chi_{|I|=1}\varepsilon_I\,(\partial_{I^c}\eta_J)\eta_K\otimes Cv.$$

In the proof of Lemma A.5 in [BKL1], the number sgn_S for $|S|=2$ is computed explicitly, in particular it is shown that it is equal to $-(-1)^{|I||J|}$. It follows that (3.7) reduces to:

$$\Big(\frac{1}{2}t^2\xi_I\Big).(\eta_J\eta_K\otimes v)=-(-1)^{|I||J|}\sum_{i<j}\big((\partial_i\partial_j\eta_J)(\xi_I\xi_i\xi_j)\eta_K\otimes v\big)-\chi_{|I|=1}\varepsilon_I\,(\partial_{I^c}\eta_J)\eta_K\otimes Cv.\quad(3.8)$$

Formula (3.6) can be proved using (3.8), (3.3) and induction on $|K|$. The proof is similar to the proof of (3.7). Finally, the rest of the proof is analogous to the proof of Lemma A.5 in [BKL1] and it is based on (3.6). □

The previous lemmas can be summarized in the following result.

Proposition 3.9. *Let $f=\xi_I\in\bigwedge(4)$, $g=\eta_L\in U(\mathfrak{g}_{<0})$. The $\lambda-$action has the following expression:*

$$\begin{aligned}
f_\lambda(g\otimes v) =&(-1)^{p(f)}(|f|-2)\Theta(\partial_f g)\otimes v+\sum_{i=1}^{4}\partial_{(\partial_i f)}(\xi_i\star g)\otimes v\\
&+(-1)^{p(f)}\sum_{i<j}\left(\partial_{(\partial_i\partial_j f)}g\otimes F_{i,j}v\right)+\chi_{|I|=3}\,\varepsilon_I\,\partial_{I^c}(g)\otimes Cv\\
&+\lambda\Big((-1)^{p(f)}(\partial_f g)\otimes E_{00}v+(-1)^{p(f)+p(g)}\sum_{i=1}^{4}((\partial_f(\partial_i g))\star\xi_i)\otimes v\\
&+\sum_{i\neq j}\left(\partial_{\partial_i f}(\partial_j g)\otimes F_{i,j}v\right)+\chi_{|I|=2}\,\varepsilon_I\,\partial_{I^c}g\otimes Cv\Big)\\
&+\lambda^2\Big((-1)^{p(f)}\sum_{i<j}\left(\partial_f(\partial_i\partial_j g)\otimes F_{i,j}v\right)-\chi_{|I|=1}\,\varepsilon_I\,\partial_{I^c}g\otimes Cv\Big)\\
&+\lambda^3\left(-\chi_{|I|=0}\,\partial_{I^c}g\otimes Cv\right).
\end{aligned}$$

Now we introduce the definition of Hodge dual. For $\eta_I\in\bigwedge(4)$ we indicate with $\overline{\eta_I}$ its Hodge dual in $U(\mathfrak{g}_{<0})$, i.e. the unique monomial such that $\overline{\eta_I}\star\xi_I=\eta_1\eta_2\eta_3\eta_4$. Then we extend by linearity the definition of Hodge dual to elements $\sum_I\alpha_I\eta_I\in U(\mathfrak{g}_{<0})$ and we set $\overline{\Theta^k\eta_I}=\Theta^k\overline{\eta_I}$.
We recall Lemma 4.2 from [BKL1].

Lemma 3.10. *For $f=\xi_I\in\bigwedge(4)$, $g=\eta_L\in U(\mathfrak{g}_{<0})$, $i\in\{1,2,3,4\}$, we have:*

$$\overline{\partial_i g}=\overline{g}\star\xi_i=(-1)^{|\overline{g}|}\xi_i\star\overline{g}, \tag{3.9}$$

$$\overline{\partial_f g}=(-1)^{(|f|(|f|-1)/2)+|f||\overline{g}|}f\star\overline{g}, \tag{3.10}$$

$$\overline{\xi_i\star g}=-(-1)^{|\overline{g}|}\partial_i\overline{g}, \tag{3.11}$$

$$\overline{g\star\xi_i}=-\partial_i\overline{g}. \tag{3.12}$$

Proposition 3.11. *Let T be the vector space isomorphism $T:\mathrm{Ind}(F)\rightarrow\mathrm{Ind}(F)$ that is defined by $T(g\otimes v)=\overline{g}\otimes v$. Let $f=\xi_I\in\bigwedge(4)$, $g=\eta_L\in U(\mathfrak{g}_{<0})$. We have:*

$$\begin{aligned}
&T\circ f_\lambda\circ T^{-1}(\overline{g}\otimes v)=\\
&=(-1)^{(|f|(|f|+1)/2)+|f||\overline{g}|}\Big\{(|f|-2)\Theta(f\star\overline{g})\otimes v-(-1)^{p(f)}\sum_{i=1}^{4}((\partial_i f)\star(\partial_i\overline{g}))\otimes v\\
&-\sum_{r<s}\left((\partial_r\partial_s f)\star\overline{g}\otimes F_{r,s}v\right)+\chi_{|I|=3}\,\varepsilon_I\,\xi_{I^c}\star\overline{g}\otimes Cv\\
&+\lambda\Big[f\star\overline{g}\otimes E_{00}v-(-1)^{p(f)}\sum_{i=1}^{4}\partial_i((f\xi_i)\star\overline{g})\otimes v+(-1)^{p(f)}\sum_{i\neq j}\left(((\partial_i f)\xi_j)\star\overline{g}\otimes F_{i,j}v\right)\\
&+\chi_{|I|=2}\,\varepsilon_I\xi_{I^c}\star\overline{g}\otimes Cv)\Big]\\
&+\lambda^2\Big[-\sum_{i<j}\left((f\xi_i\xi_j\star\overline{g})\otimes F_{i,j}v\right)-\chi_{|I|=1}\,\varepsilon_I\xi_{I^c}\star\overline{g}\otimes Cv)\Big]+\lambda^3\Big[-\chi_{|I|=0}\xi_*\star\overline{g}\otimes Cv\Big]\Big\}.
\end{aligned}$$

Proof. The proof is analogous to the proof of Theorem 4.3 in [BKL1]. We consider the vector space isomorphism $T:\mathrm{Ind}(F)\rightarrow\mathrm{Ind}(F)$ that is defined by $T(g\otimes v)=\overline{g}\otimes v$. The formula in the statement is the expression for $T \circ f_\lambda \circ T^{-1}(\overline{g} \otimes v)$ for $f = \xi_I \in \bigwedge(4)$, $\overline{g} = \overline{\eta_L} \in U(g_{<0})$. The thesis for the terms that do not involve C can be shown like in the proof of Theorem 4.3 in [BKL1]. We focus on the terms in C.

1. We compute $\overline{\chi_{|I|=3}\, \varepsilon_I\, \partial_{I^c}(g)} = \chi_{|I|=3}\, \varepsilon_I\, \overline{\partial_{I^c}(g)}$. Using (3.9), we obtain that $\overline{\partial_{I^c} g} = (-1)^{|\overline{g}|} \xi_{I^c} \star \overline{g}$.

2. We compute $\overline{\chi_{|I|=2}\, \varepsilon_I\, \partial_{I^c} g} = \chi_{|I|=2}\, \varepsilon_I\, \overline{\partial_{I^c} g}$. Using (3.10), we obtain that $\overline{\partial_{I^c} g} = (-1)^{2\frac{1}{2}+2|\overline{g}|} \xi_{I^c} \star \overline{g} = -\xi_{I^c} \star \overline{g}$.

3. We compute $\overline{-\chi_{|I|=1}\, \varepsilon_I\, \partial_{I^c} g} = -\chi_{|I|=1}\, \varepsilon_I\, \overline{\partial_{I^c} g}$. Using (3.10), we obtain that $\overline{\partial_{I^c} g} = (-1)^{3\frac{2}{2}+3|\overline{g}|} \xi_{I^c} \star \overline{g} = (-1)^{1+|\overline{g}|} \xi_{I^c} \star \overline{g}$.

4. We compute $\overline{-\chi_{|I|=0}\, \partial_{I^c} g} = -\chi_{|I|=0}\, \overline{\partial_{I^c} g}$. Using (3.10), we obtain that $\overline{\partial_{I^c} g} = (-1)^{4\frac{3}{2}+4|\overline{g}|} \xi_{I^c} \star \overline{g} = \xi_{I^c} \star \overline{g}$.

Hence the formula is proved. □

In the following lemma we give a recursive formula in order to compute, for $f = \xi_I \in \bigwedge(4)$ and $g = \eta_L \in U(g_{<0})$, the $\lambda-$action $f_\lambda(\Theta^k g \otimes v)$ starting from $f_\lambda(\Theta^{k-1} g \otimes v)$. This recursive formula holds both for formula in Proposition 3.9 and for the formula in Proposition 3.11.

Lemma 3.12. *Let $f = \xi_I \in \bigwedge(4)$, $g = \eta_L \in U(\mathfrak{g}_{<0})$ and $k \in \mathbb{Z}_{>0}$. We have:*

$$f_\lambda(\Theta^k g \otimes v) = (\Theta + \lambda)(f_\lambda \Theta^{k-1} g \otimes v) - \chi_{|I|=4} \varepsilon_I \Theta^{k-1} g \otimes Cv.$$

Proof. We have by (2.1) and Proposition 2.11:

$$\begin{aligned} f_\lambda(\Theta^k g \otimes v) &= \sum_{l \geq 0} \frac{\lambda^j}{j!} (t^j f).(\Theta^k g \otimes v) \\ &= \sum_{l \geq 0} \frac{\lambda^j}{j!} \Theta(t^j f) \Theta^{k-1} g \otimes v + \sum_{l \geq 0} \frac{\lambda^j}{j!} (j t^{j-1} f) \Theta^{k-1} g \otimes v - \chi_{|f|=4} \varepsilon_I \Theta^{k-1} g \otimes Cv \\ &= (\Theta + \lambda)(f_\lambda \Theta^{k-1} g \otimes v) - \chi_{|f|=4} \varepsilon_I \Theta^{k-1} g \otimes Cv. \end{aligned}$$

For $|f| \neq 4$ the formula reduces to:

$$f_\lambda(\Theta^k g \otimes v) = (\Theta + \lambda)^k f_\lambda(g \otimes v).$$

□

Chapter 4

Singular vectors

The aim of this chapter is to classify all the singular vectors of Verma modules on $\mathfrak{g}$.

Remark 4.1. From the definition of the $\lambda-$action we deduce that $\vec{m} \in \mathrm{Ind}(F)$ is a highest weight singular vector if and only if the following hold:

S1 $\frac{d^2}{d\lambda^2}(f_\lambda \vec{m}) = 0$ for all $f \in \bigwedge(4)$;

S2 $\frac{d}{d\lambda}(f_\lambda \vec{m})_{|\lambda=0} = 0$ for all $f = \xi_I \in \bigwedge(4)$ such that $|I| \geq 1$;

S3 $(f_\lambda \vec{m})_{|\lambda=0} = 0$ for all $f = \xi_I \in \bigwedge(4)$ such that $|I| \geq 3$ or $f \in B_{so(4)}$.

Indeed condition **S1** is equivalent to

$$\sum_{j \geq 0} j(j-1) \frac{\lambda^{j-2}}{j!} (t^j f).\vec{m} = 0,$$

for all $f \in \bigwedge(4)$, that is $(t^j f).\vec{m} = 0$ for all $f \in \bigwedge(4)$ and $j \geq 2$.
Condition **S2** is equivalent to $(tf).\vec{m} = 0$ for all $f = \xi_I \in \bigwedge(4)$ such that $|I| \geq 1$.
Condition **S3** is equivalent to $f.\vec{m} = 0$ for all $f = \xi_I \in \bigwedge(4)$ such that $|I| \geq 3$ or $f \in B_{\mathfrak{so}(4)}$.
Therefore **S1**, **S2**, **S3** are equivalent to the conditions $\mathfrak{g}_{>0}.\vec{m} = 0$ and $B_{\mathfrak{so}(4)}.\vec{m} = 0$, i.e. $\vec{m}$ is a highest weight singular vector.

Remark 4.2. We denote by $\mathfrak{g}_0^{ss}$ the semisimple part of $\mathfrak{g}_0$. We introduce the following notation:

$$e_x = \frac{E_{\varepsilon_1 - \varepsilon_2}}{2}, \quad f_x = -\frac{E_{-(\varepsilon_1 - \varepsilon_2)}}{2}, \quad h_x = H_1 - H_2,$$

and

$$e_y = \frac{E_{\varepsilon_1 + \varepsilon_2}}{2}, \quad f_y = -\frac{E_{-(\varepsilon_1 + \varepsilon_2)}}{2}, \quad h_y = H_1 + H_2.$$

We have that:

$$\mathfrak{g}_0^{ss} = \langle e_x, f_x, h_x \rangle \oplus \langle e_y, f_y, h_y \rangle \cong \langle x_1 \partial_{x_2}, x_2 \partial_{x_1}, x_1 \partial_{x_1} - x_2 \partial_{x_2} \rangle \oplus \langle y_1 \partial_{y_2}, y_2 \partial_{y_1}, y_1 \partial_{y_1} - y_2 \partial_{y_2} \rangle.$$

By direct computations, we obtain the following result.

Lemma 4.3. *As $\mathfrak{g}_0^{ss}-$modules:*

$$\mathfrak{g}_{-1} \cong \langle x_1, x_2 \rangle \otimes \langle y_1, y_2 \rangle.$$

The isomorphism is given by:

$$\xi_2 + i\xi_1 \leftrightarrow x_1 y_1, \ \ \xi_2 - i\xi_1 \leftrightarrow x_2 y_2, \ \ -\xi_4 + i\xi_3 \leftrightarrow x_1 y_2, \ \ \xi_4 + i\xi_3 \leftrightarrow x_2 y_1.$$

Motivated by the previous lemma, we will use the notation

$$w_{11}=\eta_2+i\eta_1,\ w_{22}=\eta_2-i\eta_1,\ w_{12}=-\eta_4+i\eta_3,\ w_{21}=\eta_4+i\eta_3. \tag{4.1}$$

We point out that $[w_{11},w_{22}]=4\Theta$, $[w_{12},w_{21}]=-4\Theta$ and all other brackets between the w's are 0. We will identify the irreducible $\mathfrak{g}_0^{ss}-$module of highest weight (m,n) with respect to h_x,h_y with the space of homogeneous polynomials of degree m in the variables x_1,x_2, and of degree n in the variables y_1,y_2.
The aim of this chapter is to solve equations **S1**, **S2**, **S3** in order to obtain the following classification of singular vectors. We recall that the highest weight of F is always written with respect to the action of h_x, h_y, E_{00} and C.

Theorem 4.4. *Let F be an irreducible finite−dimensional $\mathfrak{g}_0-$module, with highest weight μ. A vector in $\vec{m}\in\mathrm{Ind}(F)$ is a non trivial highest weight singular vector of degree 1 if and only if $\vec{m}$ is (up to a scalar) one of the following vectors:*

a: $\mu=(m,n,-\frac{m+n}{2},\frac{m-n}{2})$ *with* $m,n\in\mathbb{Z}_{\geq 0}$,

$$\vec{m}_{1a}=w_{11}\otimes x_1^m y_1^n;$$

b: $\mu=(m,n,1+\frac{m-n}{2},-1-\frac{m+n}{2})$, *with* $m\in\mathbb{Z}_{>0}$, $n\in\mathbb{Z}_{\geq 0}$,

$$\vec{m}_{1b}=w_{21}\otimes x_1^m y_1^n-w_{11}\otimes x_1^{m-1}x_2y_1^n;$$

c: $\mu=(m,n,2+\frac{m+n}{2},\frac{n-m}{2})$, *with* $m,n\in\mathbb{Z}_{>0}$,

$$\vec{m}_{1c}=w_{22}\otimes x_1^m y_1^n-w_{12}\otimes x_1^{m-1}x_2y_1^n-w_{21}\otimes x_1^m y_1^{n-1}y_2+w_{11}\otimes x_1^{m-1}x_2y_1^{n-1}y_2;$$

d: $\mu=(m,n,1+\frac{n-m}{2},1+\frac{m+n}{2})$, *with* $m\in\mathbb{Z}_{\geq 0}$, $n\in\mathbb{Z}_{>0}$,

$$\vec{m}_{1d}=w_{12}\otimes x_1^m y_1^n-w_{11}\otimes x_1^m y_1^{n-1}y_2.$$

Theorem 4.5. *Let F be an irreducible finite−dimensional $\mathfrak{g}_0-$module, with highest weight μ. A vector $\vec{m}\in\mathrm{Ind}(F)$ is a non trivial highest weight singular vector of degree 2 if and only if $\vec{m}$ is (up to a scalar) one of the following vectors:*

a: $\mu=(0,n,1-\frac{n}{2},-1-\frac{n}{2})$ *with* $n\in\mathbb{Z}_{\geq 0}$,

$$\vec{m}_{2a}=w_{11}w_{21}\otimes y_1^n;$$

b: $\mu=(m,0,1-\frac{m}{2},1+\frac{m}{2})$ *with* $m\in\mathbb{Z}_{\geq 0}$,

$$\vec{m}_{2b}=w_{11}w_{12}\otimes x_1^m;$$

c: $\mu=(m,0,2+\frac{m}{2},-\frac{m}{2})$ *with* $m\in\mathbb{Z}_{>1}$,

$$\vec{m}_{2c}=w_{22}w_{21}\otimes x_1^m+(w_{11}w_{22}+w_{21}w_{12})\otimes x_1^{m-1}x_2-w_{11}w_{12}\otimes x_1^{m-2}x_2^2;$$

d: $\mu=(0,n,2+\frac{n}{2},\frac{n}{2})$ *with* $n\in\mathbb{Z}_{>1}$,

$$\vec{m}_{2d}=w_{22}w_{12}\otimes y_1^n-(w_{22}w_{11}+w_{21}w_{12})\otimes y_1^{n-1}y_2-w_{11}w_{21}\otimes y_1^{n-2}y_2^2.$$

Theorem 4.6. *Let F be an irreducible finite−dimensional $\mathfrak{g}_0$−module, with highest weight μ. A vector $\vec{m} \in \mathrm{Ind}(F)$ is a non trivial highest weight singular vector of degree 3 if and only if $\vec{m}$ is (up to a scalar) one of the following vectors:*

a: $\mu = (1, 0, \frac{5}{2}, -\frac{1}{2})$,

$$\vec{m}_{3a} = w_{11}w_{22}w_{21} \otimes x_1 + w_{21}w_{12}w_{11} \otimes x_2;$$

b: $\mu = (0, 1, \frac{5}{2}, \frac{1}{2})$,

$$\vec{m}_{3b} = w_{11}w_{22}w_{12} \otimes y_1 + w_{12}w_{21}w_{11} \otimes y_2.$$

Theorem 4.7. *There are no singular vectors of degree greater than 3.*

Remark 4.8. Let us call $M(\mu_1, \mu_2, \mu_3, \mu_4)$ the Verma module $\mathrm{Ind}(F(\mu_1, \mu_2, \mu_3, \mu_4))$, where $F(\mu_1, \mu_2, \mu_3, \mu_4)$ is the irreducible $\mathfrak{g}_0$−module with highest weight $(\mu_1, \mu_2, \mu_3, \mu_4)$. We call a Verma module *degenerate* if it is not irreducible. We point out that, given $M(\mu_1, \mu_2, \mu_3, \mu_4)$ and $M(\widetilde{\mu}_1, \widetilde{\mu}_2, \widetilde{\mu}_3, \widetilde{\mu}_4)$ Verma modules, we can construct a non trivial morphism of $\mathfrak{g}$−modules from the former to the latter if and only if there exists a highest weight singular vector $\vec{m}$ in $M(\widetilde{\mu}_1, \widetilde{\mu}_2, \widetilde{\mu}_3, \widetilde{\mu}_4)$ of highest weight $(\mu_1, \mu_2, \mu_3, \mu_4)$. The map is uniquely determined by:

$$\nabla : M(\mu_1, \mu_2, \mu_3, \mu_4) \longrightarrow M(\widetilde{\mu}_1, \widetilde{\mu}_2, \widetilde{\mu}_3, \widetilde{\mu}_4)$$
$$v_\mu \longmapsto \vec{m},$$

where v_μ is a highest weight vector of $F(\mu_1, \mu_2, \mu_3, \mu_4)$. If $\vec{m}$ is a singular vector of degree d, we say that ∇ is a morphism of degree d.

We use Remark 4.8 to construct the maps in Figure 4.1 of all possible morphisms in the case of K'_4. The maps will be described in detail in chapter 5. From Theorems 4.4, 4.5 and 4.6 it follows that the module $M(0, 0, 2, 0)$ does not contain non trivial singular vectors, hence it is irreducible due to Theorem 1.15.

Proposition 4.9. *The module $M(0, 0, 2, 0)$ is irreducible and it is isomorphic to the coadjoint representation of $K(1, 4)_+$, where we consider the restricted dual, i.e. $K(1, 4)^*_+ = \oplus_{j \in \mathbb{Z}} (K(1, 4)_{+j})^*$.*

Proof. Due to Theorem 1.15, the module $M(0, 0, 2, 0)$ is irreducible since it does not contain non trivial singular vectors. We point out that, since C acts as the scalar 0 on $M(0, 0, 2, 0)$, the action of $\mathfrak{g}$ on $M(0, 0, 2, 0)$ is determined by the action of $K(1, 4)_+$.
We first want to show explicitly that $K(1, 4)^*_+$ is an irreducible $K(1, 4)_+$−module.
We recall that the action on the restricted dual is given, for every $g, v \in K(1, 4)_+$ and $f \in K(1, 4)^*_+$, by:

$$(g.f)(v) = -(-1)^{p(g)p(f)} f([g, v]),$$

where $p(g)$ (resp. $p(f)$) denotes the parity of g (resp. f) and the bracket is given by (2.1).
Since we are considering the restricted dual, a basis of $K(1, 4)^*_+$ is given by Θ^* and the elements $(t^s \xi_{i_1} \cdots \xi_{i_p})^*$ with $s \geq 0$. We first show that $K(1, 4)_+.\Theta^* = K(1, 4)^*_+$. Then we will show that, given $0 \neq x \in K(1, 4)^*_+$, we have that $\Theta^* \in K(1, 4)_+.x$.
We show, by induction on p, that $(\xi_{i_1} \cdots \xi_{i_p})^*$ lies in $K(1, 4)_+.\Theta^*$, in particular $\xi_{i_1}. \cdots (\xi_{i_p}.(\Theta^*)) = \beta(\xi_{i_1} \cdots \xi_{i_p})^*$ for a scalar $\beta \in \mathbb{C} \setminus \{0\}$ that is not needed explicitly.
Indeed, we have, using bracket (2.1), that for every $i \in \{1, 2, 3, 4\}$ and for every monomial a in $K(1, 4)_+$:

$$(\xi_i.\Theta^*)(a) = -\Theta^*([\xi_i, a]) = \begin{cases} 0 \text{ if } a \neq \alpha\xi_i, \text{ for every } \alpha \in \mathbb{C} \setminus \{0\}, \\ -(\Theta^*)(2\alpha\Theta) = -2\alpha \text{ if } a = \alpha\xi_i \text{ for some } \alpha \in \mathbb{C} \setminus \{0\}. \end{cases}$$

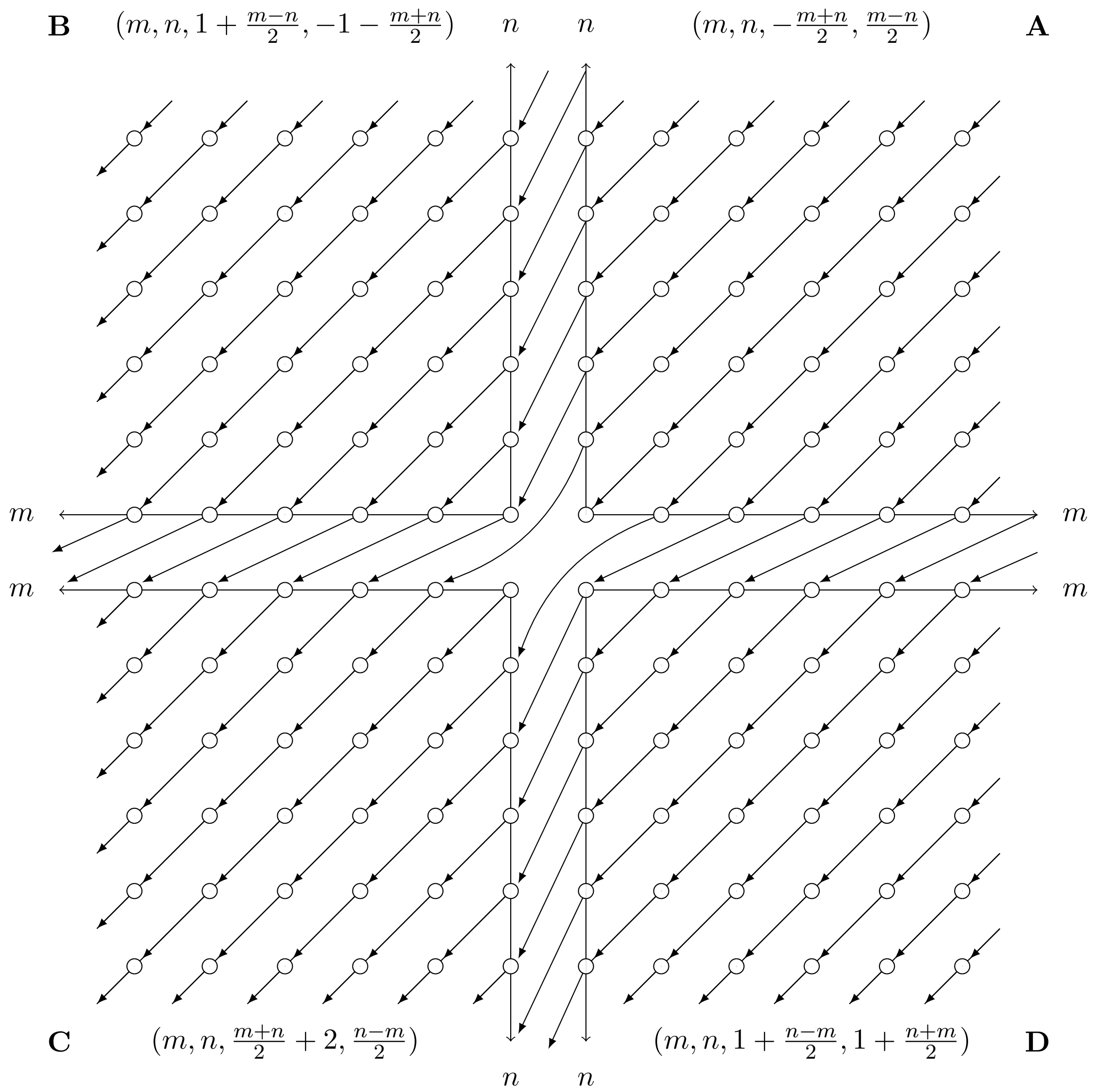
Then $\xi_i^* = -\frac{1}{2}(\xi_i.\Theta^*)$.
Now we show that $\xi_{i_1}.\cdots(\xi_{i_p}.(\Theta^*)) = \beta(\xi_{i_1}\cdots\xi_{i_p})^*$ for a scalar $\beta \in \mathbb{C}\setminus\{0\}$, using that $\xi_{i_1}.\cdots(\xi_{i_q}.(\Theta^*)) = \gamma_q(\xi_{i_1}\cdots\xi_{i_q})^*$, for every $q < p$ and for a scalar $\gamma_q \in \mathbb{C}\setminus\{0\}$. Indeed we have, using bracket (2.1), that for every monomial a in $K(1,4)_+$:

$$(\xi_{i_1}.\cdots(\xi_{i_p}.(\Theta^*)))(a) = (\xi_{i_1}.\gamma_{p-1}(\xi_{i_2}\cdots\xi_{i_p})^*)(a)$$
$$= (-1)^p\gamma_{p-1}(\xi_{i_2}\cdots\xi_{i_p})^*([\xi_{i_1},a]) = \begin{cases} 0 \text{ if } a \neq \alpha\xi_{i_1}\xi_{i_2}\cdots\xi_{i_p}, \text{ for every } \alpha\in\mathbb{C}\setminus\{0\}\,, \\ -(-1)^p\gamma_{p-1}\alpha \text{ if } a = \alpha\xi_{i_1}\xi_{i_2}\cdots\xi_{i_p} \text{ for some } \alpha\in\mathbb{C}\setminus\{0\}\,. \end{cases}$$

We now show, by induction on s, that, for fixed p, $(t^s\xi_{i_1}\cdots\xi_{i_p})^*$ lies in $K(1,4)_+.\Theta^*$; in particular $(t^s\xi_{i_1}\cdots\xi_{i_p})^*$ is obtained, up to a scalar, repeating $s-$times the application of Θ on $\xi_{i_1}.\cdots(\xi_{i_p}.(\Theta^*))$. Indeed we have, using bracket (2.1), that for every monomial a in $K(1,4)_+$:

$$(\Theta.(\xi_{i_1}.\cdots(\xi_{i_p}.(\Theta^*))))(a) = (\Theta.\beta(\xi_{i_1}\cdots\xi_{i_p})^*)(a)$$
$$= -\beta(\xi_{i_1}\cdots\xi_{i_p})^*([\Theta,a]) = \begin{cases} 0 \text{ if } a \neq \alpha t\xi_{i_1}\cdots\xi_{i_p} \text{ for every } \alpha\in\mathbb{C}\setminus\{0\}\,, \\ \beta\alpha \text{ if } a = \alpha t\xi_{i_1}\cdots\xi_{i_p} \text{ for some } \alpha\in\mathbb{C}\setminus\{0\}\,, \end{cases} = \beta(t\xi_{i_1}\cdots\xi_{i_p})^*(a).$$

Figure 4.1



Now we show that $(t^s\xi_{i_1}\cdots\xi_{i_p})^*$ lies in $K(1,4)_+.\Theta^*$ using that $(t^q\xi_{i_1}\cdots\xi_{i_p})^* = \gamma_q(\underbrace{\Theta.\cdots(\Theta}_{q-times}.(\xi_{i_1}\cdots\xi_{i_p})^*)$, for every $q<s$ and for $\gamma_q \in \mathbb{C}\setminus\{0\}$. Indeed we have, using bracket (2.1), that for every monomial a in $K(1,4)_+$:

$$\begin{aligned}(\underbrace{\Theta.\cdots(\Theta}_{s-times}.(\xi_{i_1}\cdots\xi_{i_p})^*))(a) &= -\gamma_{s-1}(t^{s-1}\xi_{i_1}...\xi_{i_p})^*([\Theta,a]) \\ &= \begin{cases} 0 \text{ if } a\neq \alpha t^s\xi_{i_1}...\xi_{i_p}, \text{ for every } \alpha\in\mathbb{C}\setminus\{0\}, \\ \gamma_{s-1}s\alpha \text{ if } a=\alpha t^s\xi_{i_1}...\xi_{i_p} \text{ for some } \alpha\in\mathbb{C}\setminus\{0\}, \end{cases} \\ &= s\gamma_{s-1}(t^s\xi_{i_1}\cdots\xi_{i_p})^*.\end{aligned}$$

Now we show that, given $0\neq x\in K(1,4)_+^*$, we have $\Theta^*\in K(1,4)_+.x$. Let us write x as a finite linear combination of elements in the basis:

$$x=\sum_{s,I}\alpha_{s,I}(t^s\xi_I)^*.$$

We choose one of the monomial of maximum degree among the $(t^s\xi_I)^*$'s in Supp(x), i.e. one monomial with maximum value of $2s+|I|-2$. Let us call this monomial $(t^{s_{max}}\xi_{I_{max}})^*$. We observe that, using bracket (2.1), for every monomial a in $K(1,4)_+$ we have:

$$\begin{aligned}(t^{s_{max}+1}\xi_{I_{max}}.(t^{s_{max}}\xi_{I_{max}})^*)(a) &= -(-1)^{|I_{max}|^2}(t^{s_{max}}\xi_{I_{max}})^*([t^{s_{max}+1}\xi_{I_{max}},a]) \\ &= \begin{cases} 0 \text{ if } a\neq\alpha\Theta \text{ for every } \alpha\in\mathbb{C}\setminus\{0\}, \\ -(-1)^{|I_{max}|^2}(s_{max}+1)\alpha \text{ if } a=\alpha\Theta \text{ for some } \alpha\in\mathbb{C}\setminus\{0\}. \end{cases}\end{aligned}$$

Therefore $\Theta^*=\gamma t^{s_{max}+1}\xi_{I_{max}}.(t^{s_{max}}\xi_{I_{max}})^*$ for some $\gamma\in\mathbb{C}\setminus\{0\}$. We also have, using bracket (2.1), that for every monomial a in $\mathfrak{g}$ and $(t^s\xi_I)^*\neq(t^{s_{max}}\xi_{I_{max}})^*$ in Supp(x):

$$(t^{s_{max}+1}\xi_{I_{max}}.(t^s\xi_I)^*)(a) = -(-1)^{|I_{max}||I|}(t^s\xi_I)^*([t^{s_{max}+1}\xi_{I_{max}},a])=0.$$

Indeed if $\deg(t^s\xi_I)<\deg(t^{s_{max}}\xi_{I_{max}})$, then

$$(t^s\xi_I)^*([t^{s_{max}+1}\xi_{I_{max}},a])=0,$$

since $\deg([t^{s_{max}+1}\xi_{I_{max}},a])\geq\deg(t^{s_{max}}\xi_{I_{max}})$.
If $\deg(t^s\xi_I)=\deg(t^{s_{max}}\xi_{I_{max}})$, then

$$(t^s\xi_I)^*([t^{s_{max}+1}\xi_{I_{max}},a])=0,$$

since $\deg([t^{s_{max}+1}\xi_{I_{max}},a])=\deg(t^{s_{max}}\xi_{I_{max}})$ only for $a=\Theta$, but $(t^s\xi_I)\neq t^{s_{max}}\xi_{I_{max}}$.
Therefore $\Theta^*=\gamma t^{s_{max}+1}\xi_{I_{max}}.x$.
We now show that $M(0,0,2,0)$ is isomorphic to the coadjoint representation of $K(1,4)_+$.
Indeed, we notice that a morphism of $K(1,4)_+-$modules $\Phi: M(0,0,2,0)\rightarrow K(1,4)_+^*$ must satisfy, for all $m\in M(0,0,2,0)$ and $a,b\in K(1,4)_+$:

$$\Phi(b.m)(a)=-\Phi(m)([b,a]),$$

where the bracket is given by (2.1). We take the morphism of modules Φ such that $\Phi(v)=\Theta^*$, where v is a highest weight vector in $F(0,0,2,0)$. Let us show that this map is surjective. We show, by induction on p, that $(\xi_{i_1}\cdots\xi_{i_p})^*$ lies in $\operatorname{Im}\Phi$, in particular $\Phi(\xi_{i_1}\cdots\xi_{i_p}v)=\beta(\xi_{i_1}\cdots\xi_{i_p})^*$ for a scalar $\beta\in\mathbb{C}\setminus\{0\}$ that is not needed explicitly.

Indeed, we have, using bracket (2.1), that for every $i \in \{1,2,3,4\}$ and for every monomial a in $K(1,4)_+$:

$$\Phi(\xi_i v)(a) = -\Phi(v)([\xi_i,a]) = -\Theta^*([\xi_i,a]) = \begin{cases} 0 \text{ if } a \neq \alpha\xi_i, \\ -(\Theta^*)(2\alpha\Theta) = -2\alpha \text{ if } a = \alpha\xi_i \text{ for some } \alpha \in \mathbb{C}\setminus\{0\}\,. \end{cases}$$

Then $\xi_i^* = -\frac{1}{2}\Phi(\xi_i.v) \in \mathrm{Im}(\Phi)$. Now we show that $\Phi(\xi_{i_1}\cdots\xi_{i_p}v) = \beta(\xi_{i_1}\cdots\xi_{i_p})^*$ for a scalar $\beta \in \mathbb{C}\setminus\{0\}$, using that $\Phi(\xi_{i_1}\cdots\xi_{i_q}v) = \gamma_q(\xi_{i_1}\cdots\xi_{i_q})^*$, for every $q < p$ and for a scalar $\gamma_q \in \mathbb{C}\setminus\{0\}$. Indeed we have, using bracket (2.1), that for every monomial a in $K(1,4)_+$:

$$\begin{aligned}&\Phi(\xi_{i_1}\cdots\xi_{i_p}v)(a) = -\Phi(\xi_{i_2}\cdots\xi_{i_p}v)([\xi_{i_1},a])\\ &= -\gamma_{p-1}(\xi_{i_2}\cdots\xi_{i_p})^*([\xi_{i_1},a]) = \begin{cases} 0 \text{ if } a \neq \alpha\xi_{i_1}\cdots\xi_{i_p}, \\ \gamma_{p-1}\alpha \text{ if } a = \alpha\xi_{i_1}\cdots\xi_{i_p} \text{ for some } \alpha \in \mathbb{C}\setminus\{0\}\,, \end{cases} = \gamma_{p-1}(\xi_{i_1}\cdots\xi_{i_p})^*.\end{aligned}$$

We now show, by induction on s, that, for fixed p, $(t^s\xi_{i_1}\cdots\xi_{i_p})^*$ lies in $\mathrm{Im}\,\Phi$; in particular $(t^s\xi_{i_1}\cdots\xi_{i_p})^* = \beta\Phi(\Theta^s\xi_{i_1}\cdots\xi_{i_p}v)$ for a scalar $\beta \in \mathbb{C}\setminus\{0\}$.
Indeed we have, using bracket (2.1), that for every monomial a in $K(1,4)_+$:

$$\begin{aligned}\Phi(\Theta\xi_{i_1}\cdots\xi_{i_p}v)(a) &= -\Phi(\xi_{i_1}\cdots\xi_{i_p}v)([\Theta,a]) = -\beta(\xi_{i_1}\cdots\xi_{i_p})^*([\Theta,a])\\ &= \begin{cases} 0 \text{ if } a \neq \alpha t\xi_{i_1}\cdots\xi_{i_p} \\ \beta\alpha \text{ if } a = \alpha t\xi_{i_1}\cdots\xi_{i_p} \text{ for some } \alpha \in \mathbb{C}\setminus\{0\} \end{cases} = \beta(t\xi_{i_1}\cdots\xi_{i_p})^*(a).\end{aligned}$$

Now we show that $(t^s\xi_{i_1}\cdots\xi_{i_p})^*$ lies in $\mathrm{Im}\,\Phi$ using that $(t^q\xi_{i_1}\cdots\xi_{i_p})^* = \gamma_q\Phi(\Theta^q\xi_{i_1}\cdots\xi_{i_p}v)$, for every $q < s$ and for $\gamma_q \in \mathbb{C}\setminus\{0\}$. Indeed we have, using bracket (2.1), that for every monomial a in $K(1,4)_+$:

$$\begin{aligned}\Phi(\Theta^s\xi_{i_1}\cdots\xi_{i_p}v)(a) &= -\Phi(\Theta^{s-1}\xi_{i_1}\cdots\xi_{i_p}v)([\Theta,a]) = -\gamma_{s-1}(t^{s-1}\xi_{i_1}\cdots\xi_{i_p})^*([\Theta,a])\\ &= \begin{cases} 0 \text{ if } a \neq \alpha t^s\xi_{i_1}\cdots\xi_{i_p}, \\ \gamma_{s-1}s\alpha \text{ if } a = \alpha t^s\xi_{i_1}\cdots\xi_{i_p} \text{ for some } \alpha \in \mathbb{C}\setminus\{0\} \end{cases} = \gamma_{s-1}s(t^s\xi_{i_1}\cdots\xi_{i_p})^*.\end{aligned}$$

Now we show that Φ is injective. We know that $(K(1,4)_+)_{>0}.v = 0$ and also $(\mathfrak{sl}_2\oplus\mathfrak{sl}_2\oplus\mathbb{C}C).v = 0$. Then we analyze what happens when we compute $\Phi(x.v)$ for $x = \alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3 + \alpha_4\xi_4 + \beta\Theta \in (K(1,4)_+)_{<0}$.

$$\begin{aligned}\Phi(x.v)(a) &= -\Theta^*([x,a]) = -\Theta^*([\alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3 + \alpha_4\xi_4 + \beta\Theta, a])\\ &= -2\alpha_1\xi_1^*(a) - 2\alpha_2\xi_2^*(a) - 2\alpha_3\xi_3^*(a) - 2\alpha_4\xi_4^*(a) - 2\beta t^*(a).\end{aligned}$$

But ξ_1^*, ξ_2^*, ξ_3^*, ξ_4^* and t^* are linearly independent, therefore, in order to have $\Phi(x.v) = 0$, we need $x = 0$. The same argument can be iterated for $x_1.(x_2.(x_3....(x_r))).v$, with x_i's $\in (K(1,4)_+)_{<0}$. Hence, Φ is injective. □

In order to prove Theorems 4.4, 4.5 and 4.6, we need some lemmas.

Remark 4.10. We point out that, by Remark 4.1, a vector $\vec{m} \in \mathrm{Ind}(F)$ is a highest weight singular vector if and only if it satisfies **S1**, **S2**, **S3**. Since T, defined as in Proposition 3.11, is an isomorphism and $\vec{m} = T^{-1}\overline{\vec{m}}$, the fact that $\vec{m} \in \mathrm{Ind}(F)$ satisfies **S1**, **S2**, **S3** is equivalent to impose conditions **S1**, **S2**, **S3** for $T \circ f_\lambda \circ T^{-1}\overline{\vec{m}}$, using the expression given by Proposition 3.11.
Therefore in the following Lemmas we will consider a vector $T(\vec{m}) \in \mathrm{Ind}(F)$ and we will impose that the expression for $(T\circ f_\lambda \circ T^{-1})T(\vec{m}) = (T\circ f_\lambda)\vec{m}$ given by Proposition 3.11 satisfies conditions **S1**, **S2**, **S3**. We will have that $\vec{m}$ is a highest weight singular vector.

Motivated by Remark 4.10, in the following lemmas we consider $\vec{m} \in U(\mathfrak{g}_{<0})$ and we will use the expression for the $\lambda-$action of Proposition 3.11 for $T(\vec{m})$. We consider a vector $\vec{m} \in \mathrm{Ind}(F)$ such that:

$$T(\vec{m}) = \sum_{k=0}^{N} \sum_{I} \Theta^k \eta_I \otimes v_{I,k}, \tag{4.2}$$

with $v_{I,k} \in F$ for all k. For all k, we will denote $v_{*,k} = v_{(1,2,3,4),k}$. In this notation we consider the sets I always in increasing order. For instance, if $T(\vec{m}) = \eta_1\eta_2 \otimes u - 3\eta_2\eta_1 \otimes w$, we write it as $\eta_1\eta_2 \otimes v_{(1,2),0}$ where $v_{(1,2),0} = u + 3w$.
In order to make clearer how the $\lambda-$action of Proposition 3.11 works for a vector $\vec{m} \in \mathrm{Ind}(F)$, such that $T(\vec{m})$ is written as in (4.2), let us see the following example.

Example 4.11. Let $T(\vec{m}) = \Theta^2\eta_1\eta_3 \otimes v_{(1,3),2} + \eta_2 \otimes v_{(2),0}$ and $f = \xi_2$. Using Proposition 3.11 and Lemma 3.12, we have:

$$\begin{aligned}
&(T \circ f_\lambda)\vec{m} = \\
&= -(\lambda+\Theta)^2\Big\{ -\Theta(\xi_2 \star \eta_1\eta_3) \otimes v_{(1,3),2} + \sum_{i=1}^{4}((\partial_i\xi_2) \star (\partial_i\eta_1\eta_3)) \otimes v_{(1,3),2} \\
&\quad + \lambda\big[\xi_2 \star \eta_1\eta_3 \otimes E_{00}v_{(1,3),2} + \sum_{i=1}^{4} \partial_i((\xi_2\xi_i) \star \eta_1\eta_3) \otimes v_{(1,3),2} - \sum_{i\neq j}\big(((\partial_i\xi_2)\xi_j) \star \eta_1\eta_3 \otimes F_{i,j}v_{(1,3),2}\big)\big] \\
&\quad + \lambda^2\big[-\sum_{i<j}\big((\xi_2\xi_i\xi_j \star \eta_1\eta_3) \otimes F_{i,j}v_{(1,3),2}\big) - \varepsilon_{(2)}\,\xi_{(2)^c} \star \eta_1\eta_3 \otimes Cv_{(1,3),2}\big]\Big\} \\
&\quad - \Theta(\xi_2 \star \eta_2) \otimes v_{(2),0} + \sum_{i=1}^{4}((\partial_i\xi_2) \star (\partial_i\eta_2)) \otimes v_{(2),0} + \lambda\big[\xi_2 \star \eta_2 \otimes E_{00}v_{(2),0} + \sum_{i=1}^{4}\partial_i((\xi_2\xi_i) \star \eta_2) \otimes v_{(2),0} \\
&\quad - \sum_{i\neq j}\big(((\partial_i\xi_2)\xi_j) \star \eta_2 \otimes F_{i,j}v_{(2),0}\big)\big] + \lambda^2\big[-\sum_{i<j}\big((\xi_2\xi_i\xi_j \star \eta_2) \otimes F_{i,j}v_{(2),0}\big) - \varepsilon_{(2)}\,\xi_{(2)^c} \star \eta_2 \otimes Cv_{(2),0}\big] \\
&= -(\lambda+\Theta)^2\big\{ +\Theta\eta_1\eta_2\eta_3 \otimes v_{(1,3),2} + \lambda\big[-\eta_1\eta_2\eta_3 \otimes E_{00}v_{(1,3),2} + \partial_4((\xi_2\xi_4) \star \eta_1\eta_3) \otimes v_{(1,3),2} \\
&\quad - ((\partial_2\xi_2)\xi_4) \star \eta_1\eta_3 \otimes F_{2,4}v_{(1,3),2}\big]\big\} + 1 \otimes v_{(2),0} + \lambda\big[-\sum_{2\neq j}\xi_j \star \eta_2 \otimes F_{2,j}v_{(2),0}\big] + \lambda^2\eta_* \otimes Cv_{(2),0} \\
&= -(\lambda+\Theta)^2\big\{\Theta\eta_1\eta_2\eta_3 \otimes v_{(1,3),2} + \lambda\big[-\eta_1\eta_2\eta_3 \otimes E_{00}v_{(1,3),2} + \eta_1\eta_2\eta_3 \otimes v_{(1,3),2} - \eta_1\eta_3\eta_4 \otimes F_{2,4}v_{(1,3),2}\big]\big\} \\
&\quad + 1 \otimes v_{(2),0} + \lambda\big[\eta_1\eta_2 \otimes F_{1,2}v_{(2),0} - \eta_2\eta_3 \otimes F_{2,3}v_{(2),0} - \eta_2\eta_4 \otimes F_{2,4}v_{(2),0}\big] + \lambda^2\eta_* \otimes Cv_{(2),0}.
\end{aligned}$$

Lemma 4.12. *Let $\vec{m} \in \mathrm{Ind}(F)$ be a singular vector such that $T(\vec{m})$ is written as in formula (4.2). The degree of $\vec{m}$ in Θ is at most 3.*

Proof. Using Proposition 3.11, Lemma 3.12 and Remark 4.10, condition **S1** for $f = 1$ reduces to:

$$\begin{aligned}
0 =& \frac{d^2}{d\lambda^2}((T \circ 1_\lambda)\vec{m}) = \sum_{k=2}^{N}\sum_{I} k(k-1)(\lambda+\Theta)^{k-2}\Bigg[(-2)\Theta\eta_I \otimes v_{I,k} + \lambda\left(\eta_I \otimes E_{00}v_{I,k} - (4-|I|)\eta_I \otimes v_{I,k}\right) \\
&+ \lambda^2\Bigg(-\sum_{i<j}\xi_i\xi_j \star \eta_I \otimes F_{i,j}v_{I,k}\Bigg) + \lambda^3\left(-\chi_{|I|=0}\eta_1\eta_2\eta_3\eta_4 \otimes Cv_{I,k}\right)\Bigg] \\
&+ 2\sum_{k=1}^{N}\sum_{I} k(\lambda+\Theta)^{k-1}\Bigg[\eta_I \otimes E_{00}v_{I,k} - (4-|I|)\eta_I \otimes v_{I,k} \\
&- 2\lambda\Bigg(\sum_{i<j}\xi_i\xi_j \star \eta_I \otimes F_{i,j}v_{I,k}\Bigg) + 3\lambda^2\left(-\chi_{|I|=0}\eta_1\eta_2\eta_3\eta_4 \otimes Cv_{I,k}\right)\Bigg]
\end{aligned}$$

$$+\sum_{k=0}^{N}\sum_{I}(\lambda+\Theta)^k\left[-2\sum_{i<j}\xi_i\xi_j\star\eta_I\otimes F_{i,j}v_{I,k}+6\lambda\left(-\chi_{|I|=0}\eta_1\eta_2\eta_3\eta_4\otimes Cv_{I,k}\right)\right].$$

Now we write Θ as $\Theta+\lambda-\lambda$ and consider this expression as a polynomial in the indeterminates $\Theta+\lambda$ and λ. The coefficients of $(\lambda+\Theta)^s\lambda^3$, with $s\geq 0$, are:

$$\sum_I(s+2)(s+1)\chi_{|I|=0}\eta_*\otimes Cv_{I,s+2}=0. \tag{4.3}$$

We consider the coefficients of $(\lambda+\Theta)^s\lambda^2$ with $s\geq 1$ and obtain:

$$-\sum_I\sum_{i<j}(s+2)(s+1)\xi_i\xi_j\star\eta_I\otimes F_{i,j}v_{I,s+2}-6\sum_I(s+1)\chi_{|I|=0}\eta_*\otimes Cv_{I,s+1}=0.$$

Therefore, using (4.3), we obtain that for $s\geq 1$:

$$\sum_I\sum_{i<j}\xi_i\xi_j\star\eta_I\otimes F_{i,j}v_{I,s+2}=0. \tag{4.4}$$

Now we look at the coefficients of $(\lambda+\Theta)^s\lambda$ with $s\geq 2$ and obtain:

$$\begin{aligned}&\sum_I(s+2)(s+1)(2\eta_I\otimes v_{I,s+2}+\eta_I\otimes E_{00}v_{I,s+2}-(4-|I|)\eta_I\otimes v_{I,s+2})\\&-4\sum_I\sum_{i<j}(s+1)\xi_i\xi_j\star\eta_I\otimes F_{i,j}v_{I,s+1}-6\sum_I\chi_{|I|=0}\eta_*\otimes Cv_{I,s}=0.\end{aligned}$$

Therefore, using (4.3) and (4.4), we obtain that for $s\geq 2$:

$$\sum_I(2\eta_I\otimes v_{I,s+2}+\eta_I\otimes E_{00}v_{I,s+2}-(4-|I|)\eta_I\otimes v_{I,s+2})=0. \tag{4.5}$$

Finally we look at the coefficients of $(\lambda+\Theta)^s$ with $s\geq 3$ and obtain:

$$\begin{aligned}&\sum_I(s+1)(s)(-2\eta_I\otimes v_{I,s+1})+2(s+1)(\eta_I\otimes E_{00}v_{I,s+1}-(4-|I|)\eta_I\otimes v_{I,s+1})\\&-2\sum_I\sum_{i<j}\xi_i\xi_j\star\eta_I\otimes F_{i,j}v_{I,s}=0.\end{aligned}$$

Therefore, using (4.4) and (4.5), we obtain that for $s\geq 3$:

$$\begin{aligned}&\sum_I(s+1)(s)(-2\eta_I\otimes v_{I,s+1})+2(s+1)(\eta_I\otimes E_{00}v_{I,s+1}-(4-|I|)\eta_I\otimes v_{I,s+1})\\&=\sum_I(s+1)(s)(-2\eta_I\otimes v_{I,s+1})+2(s+1)(-2\eta_I\otimes v_{I,s+1})\\&=\sum_I(s+1)(s+2)(-2\eta_I\otimes v_{I,s+1})=0.\end{aligned}$$

Hence, $v_{I,k}=0$ for all $k\geq 4$. □

Therefore, we proved that, for a singular vector $\vec{m}\in\mathrm{Ind}(F)$, $T(\vec{m})$ has the following form:

$$T(\vec{m})=\Theta^3(\sum_I\eta_I\otimes v_{I,3})+\Theta^2(\sum_I\eta_I\otimes v_{I,2})+\Theta(\sum_I\eta_I\otimes v_{I,1})+(\sum_I\eta_I\otimes v_{I,0}). \tag{4.6}$$

Now for a vector $\vec{m} \in \mathrm{Ind}(F)$, such that $T(\vec{m})$ is as in 4.6, we write the $\lambda-$action in the following way, using Proposition 3.11, Lemma 3.12 and Remark 4.10 for $f = \xi_L \in \bigwedge(4)$:

$$\begin{aligned}
&(T \circ f_\lambda)\vec{m} \qquad (4.7)\\
=&b_0(f) + G_1(f) + \lambda\big[B_0(f) - a_0(f) - G_2(f)\big] + (\lambda+\Theta)\big[a_0(f) + b_1(f) + 2G_2(f)\big]\\
&+ (\lambda+\Theta)^2\big[a_1(f) + b_2(f) + 3G_3(f)\big] + (\lambda+\Theta)^3\big[a_2(f) + b_3(f)\big] + (\lambda+\Theta)^4 a_3(f)\\
&+ \lambda(\lambda+\Theta)\big[B_1(f) - a_1(f) - 3G_3(f)\big] + \lambda^2(\lambda+\Theta)C_1(f) + \lambda^3(\lambda+\Theta)D_1(f)\\
&+ \lambda(\lambda+\Theta)^2\big[B_2(f) - a_2(f)\big] + \lambda^2(\lambda+\Theta)^2 C_2(f) + \lambda^3(\lambda+\Theta)^2 D_2(f)\\
&+ \lambda(\lambda+\Theta)^3\big[B_3(f) - a_3(f)\big] + \lambda^2(\lambda+\Theta)^3 C_3(f) + \lambda^3(\lambda+\Theta)^3 D_3(f)\\
&+ \lambda^2\big[C_0(f) + G_3(f)\big] + \lambda^3 D_0(f),
\end{aligned}$$

where the coefficients $a_p(f)$, $b_p(f)$, $B_p(f)$, $C_p(f)$, $D_p(f)$, $G_p(f)$ depend on f for every $0 \le p \le 3$. The following is their explicit expression:

$$a_p(f) = \sum_I (-1)^{(|f|(|f|+1)/2)+|f||I|}\Big[(|f|-2)(f \star \eta_I) \otimes v_{I,p}\Big];$$

$$\begin{aligned}
b_p(f) = \sum_I (-1)^{(|f|(|f|+1)/2)+|f||I|}\Big[&-(-1)^{p(f)}\sum_{i=1}^{4}((\partial_i f) \star (\partial_i \eta_I)) \otimes v_{I,p} - \sum_{r<s}((\partial_r\partial_s f) \star \eta_I \otimes F_{r,s} v_{I,p})\\
&+ \chi_{|L|=3}\, \varepsilon_L\, \xi_{L^c} \star \eta_I \otimes C v_{I,p}\Big];
\end{aligned}$$

$$\begin{aligned}
B_p(f) = \sum_I (-1)^{(|f|(|f|+1)/2)+|f||I|}\Big[&f \star \eta_I \otimes E_{00} v_{I,p} - (-1)^{p(f)}\sum_{i=1}^{4}\partial_i((f\xi_i) \star \eta_I) \otimes v_{I,p}\\
&+ (-1)^{p(f)}(\sum_{i\neq j}((\partial_i f)\xi_j) \star \eta_I \otimes F_{i,j} v_{I,p}) + \chi_{|L|=2}\, \varepsilon_L\, \xi_{L^c} \star \eta_I \otimes C v_{I,p})\Big];
\end{aligned}$$

$$C_p(f) = \sum_I (-1)^{(|f|(|f|+1)/2)+|f||I|}\Big[-\sum_{i<j}(f\xi_i\xi_j \star \eta_I \otimes F_{i,j} v_{I,p}) - \chi_{|L|=1}\, \varepsilon_L\, \xi_{L^c} \star \eta_I \otimes C v_{I,p})\Big];$$

$$D_p(f) = \sum_I (-1)^{(|f|(|f|+1)/2)+|f||I|}\Big[-\chi_{|L|=0}\, \xi_* \star \eta_I \otimes C v_{I,p}\Big];$$

$$G_p(f) = -\sum_I \chi_{|L|=4} \varepsilon_L \eta_I \otimes C v_{I,p}.$$

We will write a_p instead of $a_p(f)$ when it is clear from the context on which f the coefficient depends. Analogously we will write b_p, B_p, C_p, D_p, G_p instead of $b_p(f)$, $B_p(f)$, $C_p(f)$, $D_p(f)$, $G_p(f)$.

Proposition 4.13. *Let $\vec{m} \in \mathrm{Ind}(F)$, such that $T(\vec{m})$ is as in formula (4.6). Using notation (4.7), we have that:*

1. *condition **S1** is equivalent to the following system of relations $\forall f = \xi_I \in \bigwedge(4)$:*

$$\begin{aligned}
&D_3 = 0,\\
&D_2 = 0,\\
&C_3 = 0,\\
&C_1 + 2B_2 + a_2 + 3b_3 = 0,\\
&D_1 + a_3 = 0,\\
&C_2 - 3a_3 = 0,
\end{aligned}$$

$$\begin{aligned}
&B_3 + 2a_3 = 0,\\
&D_0 + C_1 + B_2 + b_3 = 0,\\
&C_0 + B_1 + b_2 + G_3 = 0,
\end{aligned}$$

2. *condition* ***S2*** *is equivalent to the following system of relations for all* $f = \xi_I \in \bigwedge(4)$ *with* $|I| \geq 1$*:*

$$\begin{aligned}
&B_0 + b_1 + G_2 = 0,\\
&B_1 + a_1 + 2b_2 + 3G_3 = 0,\\
&2a_2 + B_2 + 3b_3 = 0,\\
&3a_3 + B_3 = 0,
\end{aligned}$$

3. *condition* ***S3*** *is equivalent to the following system of relations for all* $f = \xi_I \in \bigwedge(4)$ *with* $|I| \geq 3$ *or* $f \in B_{\mathfrak{so}(4)}$*:*

$$\begin{aligned}
&b_0 + G_1 = 0,\\
&a_0 + b_1 + 2G_2 = 0,\\
&a_1 + b_2 + 3G_3 = 0,\\
&a_2 + b_3 = 0,\\
&a_3 = 0.
\end{aligned}$$

Proof. We compute $\frac{d^2}{d\lambda^2}((T \circ f_\lambda)\vec{m})$ and $\frac{d}{d\lambda}((T \circ f_\lambda)\vec{m})$ using notation (4.7). We have that:

$$\begin{aligned}
\frac{d}{d\lambda}((T \circ f_\lambda)\vec{m}) =& B_0 + b_1 + G_2 + \lambda\big[2C_0 + B_1 - a_1 - G_3\big] + \lambda^2\big[3D_0 + C_1\big] + \lambda^3 D_1\\
&+ (\lambda+\Theta)\big[B_1 + a_1 + 2b_2 + 3G_3\big] + (\lambda+\Theta)^2\big[2a_2 + B_2 + 3b_3\big] + (\lambda+\Theta)^3\big[3a_3 + B_3\big]\\
&+ \lambda(\lambda+\Theta)\big[2C_1 + 2B_2 - 2a_2\big] + \lambda^2(\lambda+\Theta)\big[3D_1 + 2C_2\big] + 2\lambda^3(\lambda+\Theta)D_2\\
&+ \lambda(\lambda+\Theta)^2\big[3B_3 - 3a_3 + 2C_2\big] + \lambda^2(\lambda+\Theta)^2\big[3D_2 + 3C_3\big] + 3\lambda^3(\lambda+\Theta)^2 D_3\\
&+ 2\lambda(\lambda+\Theta)^3 C_3 + 3\lambda^2(\lambda+\Theta)^3 D_3,
\end{aligned}$$

and

$$\begin{aligned}
\frac{d^2}{d\lambda^2}((T \circ f_\lambda)\vec{m}) =& 2C_0 + 2B_1 + 2b_2 + 2G_3 + \lambda\big[6D_0 + 4C_1 + 2B_2 - 2a_2\big] + \lambda^2\big[6D_1 + 2C_2\big] + 2\lambda^3 D_2\\
&+ (\lambda+\Theta)\big[2C_1 + 4B_2 + 2a_2 + 6b_3\big] + \lambda(\lambda+\Theta)\big[6D_1 + 8C_2 + 6B_3 - 6a_3\big]\\
&+ \lambda^2(\lambda+\Theta)\big[12D_2 + 6C_3\big] + 6\lambda^3(\lambda+\Theta)D_3\\
&+ (\lambda+\Theta)^2\big[2C_2 + 6a_3 + 6B_3\big] + \lambda(\lambda+\Theta)^2\big[12C_3 + 6D_2\big] + 18\lambda^2(\lambda+\Theta)^2 D_3\\
&+ 2(\lambda+\Theta)^3 C_3 + 6\lambda(\lambda+\Theta)^3 D_3.
\end{aligned}$$

We consider these expressions as polynomials in the variables λ and $\lambda + \Theta$. Condition **S1** reduces to the following system of relations $\forall f = \xi_I \in \bigwedge(4)$:

$$\begin{aligned}
&C_3 = 0,\\
&D_2 = 0,\\
&D_3 = 0,\\
&C_2 + 3a_3 + 3B_3 = 0,
\end{aligned}$$

$$\begin{aligned}
&C_1 + 2B_2 + a_2 + 3b_3 = 0,\\
&3D_1 + 4C_2 + 3B_3 - 3a_3 = 0,\\
&3D_0 + 2C_1 + B_2 - a_2 = 0,\\
&3D_1 + C_2 = 0,\\
&C_0 + B_1 + b_2 + G_3 = 0.
\end{aligned}$$

These conditions are equivalent to the relations in (1). Condition **S2** states that for all $f = \xi_I \in \bigwedge(4)$ with $|I| \geq 1$:

$$\frac{d}{d\lambda}((T \circ f_\lambda)\vec{m})_{|\lambda=0} = B_0 + b_1 + G_2 + \Theta\big[B_1 + a_1 + 2b_2 + 3G_3\big] + \Theta^2\big[2a_2 + B_2 + 3b_3\big] + \Theta^3\big[3a_3 + B_3\big] = 0.$$

Therefore condition **S2** reduces to the following system of relations for all $f = \xi_I \in \bigwedge(4)$ with $|I| \geq 1$:

$$\begin{aligned}
&B_0 + b_1 + G_2 = 0,\\
&B_1 + a_1 + 2b_2 + 3G_3 = 0,\\
&2a_2 + B_2 + 3b_3 = 0,\\
&3a_3 + B_3 = 0.
\end{aligned}$$

Condition **S3** states that for all $f = \xi_I \in \bigwedge(4)$ such that $|I| \geq 3$ or $f \in B_{\mathfrak{so}(4)}$:

$$((T \circ f_\lambda)\vec{m})_{|\lambda=0} = b_0 + G_1 + \Theta\big[a_0 + b_1 + 2G_2\big] + \Theta^2\big[a_1 + b_2 + 3G_3\big] + \Theta^3\big[a_2 + b_3\big] + \Theta^4 a_3 = 0.$$

Therefore condition **S3** reduces to the following system of relations for all $f = \xi_I \in \bigwedge(4)$ with $|I| \geq 3$ or $f \in B_{\mathfrak{so}(4)}$:

$$\begin{aligned}
&b_0 + G_1 = 0,\\
&a_0 + b_1 + 2G_2 = 0,\\
&a_1 + b_2 + 3G_3 = 0,\\
&a_2 + b_3 = 0,\\
&a_3 = 0.
\end{aligned}$$

□

Let us show some other reductions on singular vectors.

Lemma 4.14. *Let $\vec{m} \in \mathrm{Ind}(F)$ be a singular vector, such that $T(\vec{m})$ is written as in formula (4.6). For all I we have that $v_{I,3} = 0$.*

Proof. By Proposition 4.13, we know that $\forall f = \xi_L \in \bigwedge(4)$ with $|f| \geq 1$ we have $2a_3 + B_3 = 0$ and $3a_3 + B_3 = 0$. Therefore $a_3 = 0$ for all $f = \xi_L \in \bigwedge(4)$ with $|f| \geq 1$. Let us suppose that there exists $v_{I,3} \neq 0$ with $0 \leq |I| \leq 3$. Let I_0 be a set of minimal cardinality with this property. We have that:

$$0 = a_3 = \sum_{|I| \geq |I_0|} (-1)^{(|f|(|f|+1)/2) + |f||I|}(|f| - 2)(f \star \eta_I \otimes v_{I,3}).$$

We choose $f = \xi_{I_0^c}$ if $|I_0^c| \neq 2$, otherwise we choose $f = \xi_s$ with $s \notin I_0$. In the first case we have:

$$0 = a_3(f) = \sum_{|I| \geq |I_0|} (-1)^{(|f|(|f|+1)/2) + |f||I|}(|I_0^c| - 2)(\xi_{I_0^c} \star \eta_I \otimes v_{I,3})$$

$$= (-1)^{(|I_0^c|(|I_0^c|+1)/2)+|I_0^c||I_0|}(|I_0^c|-2)\varepsilon_{I_0^c}(\eta_* \otimes v_{I_0,3}).$$

This implies that $v_{I_0,3}=0$. In the second case, we have:

$$\begin{aligned} 0 = a_3(f) &= -\sum_{|I|\geq|I_0|}(-1)^{(|f|(|f|+1)/2)+|f||I|}(\xi_s \star \eta_I \otimes v_{I,3}) \\ &= -(-1)^{1+|I_0|}(\xi_s \star \eta_{I_0} \otimes v_{I_0,3}) - (-1)^{s-1}(\eta_* \otimes v_{(s)^c,3}) + \sum_{j\in I_0} \operatorname{sgn}_{s,j} \eta_{(j)^c} \otimes v_{(s,j)^c,3}, \end{aligned}$$

where $\operatorname{sgn}_{s,j}=\pm 1$ depends on s and j and is not needed explicitly. From linear independence of $\xi_s \star \eta_{I_0}$, η_* and $\eta_{(j)^c}$ for $j \in I_0$, we obtain that $v_{I_0,3}=0$. Hence, for all I with $|I|\leq 3$ we have that $v_{I,3}=0$.
We now show that $v_{*,3}=0$. For $f=\xi_{i_0}$ we know by Proposition 4.13 that $D_0+C_1+B_2+b_3=0$, $C_1+2B_2+a_2+3b_3=0$ and $2a_2+B_2+3b_3=0$. We take a linear combination of these equations and we obtain that $D_0+a_2+b_3=0$ for $f=\xi_{i_0}$. Since $D_0(\xi_{i_0})=0$, we have:

$$0 = a_2+b_3 = -(\sum_I (-1)^{1+|I|}\xi_{i_0} \star \eta_I \otimes v_{I,2}) + (-1)^{i_0}\eta_{i_0^c} \otimes v_{*,3} = 0.$$

Using linear independence of $\eta_{i_0^c}$ and the $\xi_{i_0} \star \eta_I$'s we obtain that $v_{*,3}=0$. □

Lemma 4.15. *Let $\vec{m} \in \operatorname{Ind}(F)$ be a singular vector, such that $T(\vec{m})$ is written as in formula* (4.6)*. For all I we have that $v_{I,2}=0$.*

Proof. By Proposition 4.13 we know that $D_0+C_1+B_2+b_3=0$, $C_1+2B_2+a_2+3b_3=0$ for all $f=\xi_L$ and $2a_2+B_2+3b_3=0$ for all $f=\xi_L$ with $|f|\geq 1$. In Lemma 4.14, we proved that $v_{I,3}=0$ for all I, therefore $b_3=0$. We know that $D_0=0$ for all $f=\xi_L$ with $|f|\geq 1$. Hence for all f such that $|f|\geq 1$, we have:

$$\begin{cases} C_1+B_2=0, \\ C_1+2B_2+a_2=0, \\ 2a_2+B_2=0. \end{cases}$$

This implies that $C_1=B_2=a_2=0$ if $|f|\geq 1$. The proof is now analogous to Lemma 4.14, using that $a_2=0$, and we deduce that $v_{I,2}=0$ for I such that $|I|\leq 3$.
We now show that $v_{*,2}=0$. By Proposition 4.13 we know that $b_0(f)=0$ for all $f=\xi_L$ with $|f|=3$. We choose $f=\xi_1\xi_2\xi_3$ and obtain:

$$\begin{aligned} 0 =& b_0(\xi_1\xi_2\xi_3) \\ =& \sum_I \sum_{i=1}^4 (-1)^{3|I|}(\partial_i \xi_1\xi_2\xi_3) \star (\partial_i \eta_I) \otimes v_{I,0} - \sum_I \sum_{r<s} (-1)^{3|I|}(\partial_r\partial_s \xi_1\xi_2\xi_3) \star \eta_I \otimes F_{r,s} v_{I,0} \\ &+ \sum_I (-1)^{3|I|}\xi_4 \star \eta_I \otimes C v_{I,0}. \end{aligned}$$

In the previous equation, the terms that contain η_4 only are:

$$\eta_4 \otimes C v_{\emptyset,0}.$$

Therefore $v_{\emptyset,0}=0$, if $C\neq 0$. If $C=0$ the $\lambda-$action in Proposition 3.11 reduces to the $\lambda-$action found in Theorem 4.3 of [BKL1]. In Lemma B.4 of [BKL1] it is shown that $v_{\emptyset,0}=0$. We use that

$v_{\emptyset,0}=0$ in order to prove the thesis. Indeed by Proposition 4.13 we have that $C_0+B_1+b_2=0$, $B_1+a_1+2b_2=0$ for all $1\leq|f|<4$, and so $C_0-a_1-b_2=0$. We choose $f=\xi_{i_0}$ and we have:

$$\begin{aligned}a_1(\xi_{i_0})=&-\sum_I(-1)^{1+|I|}\xi_{i_0}\star\eta_I\otimes v_{I,1},\\ C_0(\xi_{i_0})=&-\sum_I\sum_{i<j}(-1)^{1+|I|}\xi_{i_0}\xi_i\xi_j\star\eta_I\otimes F_{i,j}v_{I,0}+\\ &-\sum_I(-1)^{1+|I|}\varepsilon_{(i_0)}\xi_{(i_0)^c}\star\eta_I\otimes Cv_{I,0},\\ b_2(\xi_{i_0})=&(-1)^{1+4}(-1)^{i_0-1}\eta_{(i_0)^c}\otimes v_{*,2}=(-1)^{i_0}\eta_{(i_0)^c}\otimes v_{*,2}.\end{aligned}$$

The terms in $\eta_{(i_0)^c}$ of $C_0-a_1-b_2=0$ are:

$$\varepsilon_{(i_0)}\eta_{(i_0)^c}\otimes Cv_{\emptyset,0}-(-1)^{i_0}\eta_{(i_0)^c}\otimes v_{*,2}=0.$$

Since $v_{\emptyset,0}=0$, we have that $v_{*,2}=0$. □

Lemma 4.16. *Let $\vec{m}\in\operatorname{Ind}(F)$ be a singular vector, such that $T(\vec{m})$ is written as in formula (4.6). For all I such that $|I|\leq 2$, we have that $v_{I,1}=0$.*

Proof. By Proposition 4.13 and Lemmas 4.14 and 4.15, we know that $a_1(f)=0$ for all $f=\xi_L$, $|L|\geq 3$. Then, from an analogous argument to the one used in Lemma 4.14, for all I such that $|I|\leq 1$, we have that $v_{I,1}=0$.
Now let us show that for all I such that $|I|=2$, we have that $v_{I,1}=0$. By Proposition 4.13 and Lemmas 4.14 and 4.15, we know that $B_0+b_1=0$ for all $f=\xi_L\in\bigwedge(4)$ with $|L|\geq 1$. We choose $f=\xi_a$ and set $(a)^c=(b,c,d)$. We have:

$$\begin{aligned}b_1(\xi_a)=&+\sum_{|I|\geq 2}(-1)^{1+|I|}(\partial_a\eta_I)\otimes v_{I,1}\\ =&\sum_{i<j,\,i,j\neq a}(-1)^{1+2}(\partial_a\eta_{(i,j)^c})\otimes v_{(i,j)^c,1}+\sum_{i\neq a}(-1)^{1+3}(\partial_a\eta_{(i)^c})\otimes v_{(i)^c,1}+(-1)^{1+4}\partial_a\xi_*\otimes v_{*,1},\\ B_0(\xi_a)=&\sum_{|I|}(-1)^{1+|I|}\xi_a\star\eta_I\otimes E_{00}v_{I,0}+\sum_{i\neq a}\sum_{|I|}(-1)^{1+|I|}\partial_i(\xi_a\xi_i\star\eta_I)\otimes_{I,0}\\ &-\sum_{a\neq j}\sum_{|I|}(-1)^{1+|I|}\xi_j\star\eta_I\otimes F_{a,j}v_{I,0}.\end{aligned}$$

The terms in η_d of $B_0(\xi_a)$ are:

$$\eta_d\otimes F_{a,d}v_{\emptyset,0}.$$

We have shown in Lemma 4.15 that $v_{\emptyset,0}=0$. Therefore, taking the terms of b_1 in η_d, we obtain:

$$(\partial_a\eta_{(b,c)^c})\otimes v_{(b,c)^c,1}=0.$$

Hence we have $v_{(b,c)^c,1}=0$. □

By Lemmas 4.14, 4.15 and 4.16, for a singular vector $\vec{m}\in U(\mathfrak{g}_{<0})$, $T(\vec{m})$ has the following form:

$$T(\vec{m})=\Theta(\sum_{|I|\geq 3}\eta_I\otimes v_{I,1})+(\sum_{|I|\geq 1}\eta_I\otimes v_{I,0}).\tag{4.8}$$

Therefore, from (4.8), we have that there can only be singular vectors of degree 3, 2 and 1. Hence we have showed Theorem 4.7. Following the notation used in [BKL1], we rewrite (4.8) in the following way: for $|I|=3$, η_I will be written as $\eta_{(i)^c}$, where $(i)^c=I$, $v_{I,1}$ will be renamed as $v_{i,1}$ and $v_{I,0}$ will be renamed as v_i, so that they depend on one index; for $|I|=2$, η_I will be written as $\eta_{(i,j)^c}$, where $(i,j)^c=I$, and $v_{I,0}$ will be renamed as $v_{i,j}$. In particular, from (4.8), for the singular vectors $\vec{m}$ of degree 3, 2 and 1, $T(\vec{m})$ have respectively the form:

degree 3 $T(\vec{m})=\Theta(\sum_i \eta_{(i)^c}\otimes v_{i,1})+(\sum_i \eta_i\otimes v_{i,0})$,

degree 2 $T(\vec{m})=\Theta\eta_*\otimes v_*+(\sum_{i<j}\eta_{(i,j)^c}\otimes v_{i,j})$,

degree 1 $T(\vec{m})=(\sum_i \eta_{(i)^c}\otimes v_i)$.

By Proposition 4.13 and Lemmas 4.14, 4.15, 4.16 we obtain the following result.

Proposition 4.17. *Let $\vec{m}\in \mathrm{Ind}(F)$, such that $T(\vec{m})$ is written as in formula* (4.8)*. Using notation* (4.7)*, we have that:*

1. *condition **S1** reduces to the following system of relations for all $f=\xi_I\in\bigwedge(4)$:*

$$\begin{aligned} &C_1=0,\\ &D_1=0,\\ &D_0=0,\\ &C_0+B_1=0; \end{aligned}$$

2. *condition **S2** reduces to the following system of relations for all $f=\xi_I\in\bigwedge(4)$ with $|I|\geq 1$:*

$$\begin{aligned} &B_0+b_1=0,\\ &B_1+a_1=0; \end{aligned}$$

3. *condition **S3** reduces to the following system of relations for all $f=\xi_I\in\bigwedge(4)$ with $|I|\geq 3$ or $f\in B_{\mathfrak{so}(4)}$:*

$$\begin{aligned} &b_0+G_1=0,\\ &a_0+b_1=0,\\ &a_1=0. \end{aligned}$$

4.1 Vectors of degree 2

The aim of this section is to classify all singular vectors of degree 2. We have that for a singular vector $\vec{m}$ of degree 2

$$T(\vec{m})=\Theta\eta_*\otimes v_*+(\sum_{i<j}\eta_{(i,j)^c}\otimes v_{i,j}). \tag{4.9}$$

We will assume that $v_{i,j}=-v_{j,i}$ for all i,j. We write $\vec{m}$ in the following way:

$$\begin{aligned} \vec{m}=&(\eta_2+i\eta_1)(\eta_4+i\eta_3)\otimes w_1+(\eta_2+i\eta_1)(\eta_4-i\eta_3)\otimes w_2+(\eta_2-i\eta_1)(\eta_4+i\eta_3)\otimes w_3 \\ &+(\eta_2-i\eta_1)(\eta_4-i\eta_3)\otimes w_4+(\eta_2+i\eta_1)(\eta_2-i\eta_1)\otimes w_5+(\eta_4+i\eta_3)(\eta_4-i\eta_3)\otimes w_6+\Theta\otimes w_7 \\ =&(-\eta_1\eta_3+i\eta_1\eta_4+i\eta_2\eta_3+\eta_2\eta_4)\otimes w_1+(\eta_1\eta_3+i\eta_1\eta_4-i\eta_2\eta_3+\eta_2\eta_4)\otimes w_2+ \\ &(\eta_1\eta_3-i\eta_1\eta_4+i\eta_2\eta_3+\eta_2\eta_4)\otimes w_3+(-\eta_1\eta_3-i\eta_1\eta_4-i\eta_2\eta_3+\eta_2\eta_4)\otimes w_4+ \end{aligned} \tag{4.10}$$

$$(2\Theta + 2i\eta_1\eta_2) \otimes w_5 + (2\Theta + 2i\eta_3\eta_4) \otimes w_6 + \Theta \otimes w_7.$$

Then, keeping in mind the relation between $\vec{m}$ and $T(\vec{m})$, we have:

$$\begin{aligned} v_{1,2} &= 2iw_5, \\ v_{1,3} &= w_1 - w_2 - w_3 + w_4, \\ v_{1,4} &= iw_1 + iw_2 - iw_3 - iw_4, \\ v_{2,3} &= iw_1 - iw_2 + iw_3 - iw_4, \\ v_{2,4} &= -w_1 - w_2 - w_3 - w_4, \\ v_{3,4} &= 2iw_6, \\ v_* &= 2w_5 + 2w_6 + w_7. \end{aligned} \tag{4.11}$$

Indeed, let us show for example one of the previous equations. In (4.9), let us consider $\eta_{(1,3)^c} \otimes v_{1,3} = \eta_2\eta_4 \otimes v_{1,3}$. We have that $\eta_2\eta_4$ is the Hodge dual of $-\eta_1\eta_3$. In (4.10), the terms in $\eta_1\eta_3$ are:

$$-\eta_1\eta_3 \otimes w_1 + \eta_1\eta_3 \otimes w_2 + \eta_1\eta_3 \otimes w_3 - \eta_1\eta_3 \otimes w_4,$$

therefore $v_{1,3} = w_1 - w_2 - w_3 + w_4$.
In the following lemma we write explicitly the relations of Proposition 4.17 for a vector as in formula (4.9).

Lemma 4.18. *Let $\vec{m} \in \mathrm{Ind}(F)$, such that $T(\vec{m})$ is written as in formula* (4.9). *We have that:*
1) *condition* ***S1*** *reduces to the following relation for $f = 1$:*

$$-\sum_{i<j} (\xi_i\xi_j \star \eta_{(i,j)^c} \otimes F_{i,j} v_{i,j}) + \eta_* \otimes E_{00} v_* = 0; \tag{4.12}$$

2) *condition* ***S2*** *reduces to the following relation for all $f = \xi_L \in \bigwedge(4)$ with $|L| = 1, 2$:*

$$\begin{aligned} &\sum_{i<j} \Big[f \star \eta_{(i,j)^c} \otimes E_{00} v_{i,j} - (-1)^{p(f)} \sum_{l=1}^{4} \partial_l ((f\xi_l) \star \eta_{(i,j)^c}) \otimes v_{i,j} + (-1)^{p(f)} (\sum_{k \neq l} ((\partial_k f)\xi_l) \star \eta_{(i,j)^c} \otimes F_{k,l} v_{i,j}) \\ &+ \chi_{|L|=2}\, \varepsilon_L\, \xi_{L^c} \star \eta_{(i,j)^c} \otimes C v_{i,j}) \Big] - (-1)^{p(f)} \sum_{i=1}^{4} ((\partial_i f) \star (\partial_i \eta_*)) \otimes v_* - \sum_{r<s} ((\partial_r \partial_s f) \star \eta_* \otimes F_{r,s} v_*) = 0; \end{aligned} \tag{4.13}$$

3) *condition* ***S3*** *reduces to the following system of relations.*
For $f \in B_{\mathfrak{so}(4)}$:

$$\sum_{r<s} ((\partial_r \partial_s f) \star \eta_* \otimes F_{r,s} v_*) = 0. \tag{4.14}$$

For $f = \xi_L$ with $|L| \geq 3$ or $f \in B_{\mathfrak{so}(4)}$:

$$\begin{aligned} &\sum_{i<j} (-1)^{(|f|(|f|+1)/2)} \Big[- (-1)^{p(f)} \sum_{l=1}^{4} ((\partial_l f) \star (\partial_l \eta_{(i,j)^c})) \otimes v_{i,j} - \sum_{r<s} ((\partial_r \partial_s f) \star \eta_{(i,j)^c} \otimes F_{r,s} v_{i,j}) \\ &+ \chi_{|L|=3}\, \varepsilon_L\, \xi_{L^c} \star \eta_{(i,j)^c} \otimes C v_{i,j} \Big] - \chi_{|L|=4} \varepsilon_L \eta_* \otimes C v_* = 0. \end{aligned} \tag{4.15}$$

Lemma 4.19. *Let $\vec{m} \in \mathrm{Ind}(F)$, such that $T(\vec{m})$ is written as in formula (4.9). Then the relations of Lemma 4.18 reduce to the following equations.*
For every $f \in B_{\mathfrak{so}(4)}$ we have:

$$f.v_* = 0. \tag{4.16}$$

For every $a \in \{1,2,3,4\}$:

$$-v_* = \sum_{j \neq a} (-1)^{a+j} F_{j,a} v_{j,a}. \tag{4.17}$$

For every $a \neq b \in \{1,2,3,4\}$:

$$0 = E_{00} v_{a,b} - v_{a,b} - \sum_{j \neq a,b} (-1)^{a+j} F_{a,j} v_{j,b}. \tag{4.18}$$

For every $a, b \in \{1,2,3,4\}$ with $(c,d) = (a,b)^c$:

$$0 = -F_{a,b} v_* + (-1)^{a+b} E_{00} v_{a,b} - \sum_{j \neq a,b} (-1)^{b+j} F_{a,j} v_{j,b} + \sum_{j \neq a,b} (-1)^{a+j} F_{b,j} v_{j,a} - \varepsilon_{(a,b)} \varepsilon_{(c,d)} C v_{c,d}. \tag{4.19}$$

Moreover:

$$0 = E_{00} v_* + \sum_{i<j} (-1)^{i+j} F_{i,j} v_{i,j}, \tag{4.20}$$

and

$$0 = (\sum_{i<j} F_{(i,j)^c} \otimes v_{i,j}) - C v_*. \tag{4.21}$$

For every $a, b, c \in \{1,2,3,4\}$ and $d = (a,b,c)^c$, we have:

$$0 = (-1)^{b+c} v_{b,c} + (-1)^{a+c} F_{a,b} v_{a,c} - (-1)^{a+b} F_{a,c} v_{a,b} + \varepsilon_{(a,b,c)} (-1)^{a+d} C v_{a,d}. \tag{4.22}$$

Finally:

$$\begin{aligned}
&\alpha_{1,2}(v_{1,2}) = -i v_{1,3} + v_{2,3}, && \beta_{1,2}(v_{1,2}) = -v_{1,4} - i v_{2,4}, \\
&\alpha_{1,2}(v_{1,3}) = i v_{1,2}, && \beta_{1,2}(v_{1,3}) = -i v_{3,4}, \\
&\alpha_{1,2}(v_{1,4}) = -v_{3,4}, && \beta_{1,2}(v_{1,4}) = v_{1,2}, \\
&\alpha_{1,2}(v_{2,3}) = -v_{1,2}, && \beta_{1,2}(v_{2,3}) = v_{3,4}, \\
&\alpha_{1,2}(v_{2,4}) = -i v_{3,4}, && \beta_{1,2}(v_{2,4}) = i v_{1,2}, \\
&\alpha_{1,2}(v_{3,4}) = v_{1,4} + i v_{2,4}, && \beta_{1,2}(v_{3,4}) = i v_{1,3} - v_{2,3},
\end{aligned} \tag{4.23}$$

where $\alpha_{1,2}$ and $\beta_{1,2}$ are defined by (3.1) *and* (3.2).

Proof. Equation (4.14) for $f \in B_{\mathfrak{so}(4)}$ is equivalent to Equation (4.16). Indeed $B_{\mathfrak{so}(4)} = \langle \alpha_{1,2} = F_{1,3} - iF_{2,3}, \beta_{1,2} = F_{2,4} + iF_{1,4} \rangle$ and we obtain:

$$\begin{aligned}
0 =& \eta_* \otimes (-F_{1,3} + iF_{2,3}) v_*, \\
0 =& \eta_* \otimes (-F_{2,4} - iF_{1,4}) v_*.
\end{aligned}$$

Thus this implies that v_* must be a highest weight vector, when it is nonzero.
We consider Equation (4.13) with $f = \xi_a$:

$$\sum_{i<j} \left[\xi_a \star \eta_{(i,j)^c} \otimes E_{00}v_{i,j} + \sum_{l=1}^{4} \partial_l((\xi_a\xi_l) \star \eta_{(i,j)^c}) \otimes v_{i,j} - \sum_{a\neq l} \xi_l \star \eta_{(i,j)^c} \otimes F_{a,l}v_{i,j}\right] + \partial_a \eta_* \otimes v_* = 0, \tag{4.24}$$

and, considering the terms in $\eta_{(a)^c}$, we obtain:

$$\begin{aligned}
0 &= -\sum_{l<a} \xi_l \star \eta_{(l,a)^c} \otimes F_{a,l}v_{l,a} - \sum_{a<l} \xi_l \star \eta_{(a,l)^c} \otimes F_{a,l}v_{a,l} + \partial_a\eta_* \otimes v_* \\
&= -\sum_{l<a}(-1)^{l-1}\eta_{(a)^c} \otimes F_{a,l}v_{l,a} - \sum_{a<l}(-1)^{l}\eta_{(a)^c} \otimes F_{a,l}v_{a,l} + \partial_a\eta_* \otimes v_* \\
&= \sum_{l\neq a}(-1)^{l-1}\eta_{(a)^c} \otimes F_{l,a}v_{l,a} + (-1)^{a-1}\eta_{(a)^c} \otimes v_*.
\end{aligned}$$

Hence, considering the coefficient $\eta_{(a)^c}$, we obtain as in [BKL1]:

$$-v_* = \sum_{l\neq a}(-1)^{a+l}F_{l,a}v_{l,a}.$$

Analogously, if $b \neq a \in \{1,2,3,4\}$, considering the coefficient of $\eta_{(b)^c}$ in Equation (4.24), we obtain if $a < b$:

$$\begin{aligned}
0 &= \eta_a\eta_{(a,b)^c} \otimes E_{00}v_{a,b} - \eta_a\eta_{(a,b)^c} \otimes v_{a,b} - \sum_{a\neq l,\, l<b} \xi_l \star \eta_{(l,b)^c} \otimes F_{a,l}v_{l,b} - \sum_{a\neq l,\, l>b} \xi_l \star \eta_{(b,l)^c} \otimes F_{a,l}v_{b,l} \\
&= (-1)^{a-1}\eta_{(b)^c} \otimes E_{00}v_{a,b} - (-1)^{a-1}\eta_{(b)^c} \otimes v_{a,b} - \sum_{a\neq l,\, l<b} (-1)^{l-1}\eta_{(b)^c} \otimes F_{a,l}v_{l,b} - \sum_{a\neq l,\, l>b} (-1)^{l}\eta_{(b)^c} \otimes F_{a,l}v_{b,l} \\
&= (-1)^{a-1}\eta_{(b)^c} \otimes E_{00}v_{a,b} - (-1)^{a-1}\eta_{(b)^c} \otimes v_{a,b} + \sum_{l\neq a,b} (-1)^{l}\eta_{(b)^c} \otimes F_{a,l}v_{l,b},
\end{aligned}$$

and if $b < a$:

$$\begin{aligned}
0 &= \eta_a\eta_{(b,a)^c} \otimes E_{00}v_{b,a} - \eta_a\eta_{(b,a)^c} \otimes v_{b,a} - \sum_{a\neq l,\, l<b} \xi_l \star \eta_{(l,b)^c} \otimes F_{a,l}v_{l,b} - \sum_{a\neq l,\, l>b} \xi_l \star \eta_{(b,l)^c} \otimes F_{a,l}v_{b,l} \\
&= (-1)^{a}\eta_{(b)^c} \otimes E_{00}v_{b,a} - (-1)^{a}\eta_{(b)^c} \otimes v_{b,a} - \sum_{a\neq l,\, l<b} (-1)^{l-1}\eta_{(b)^c} \otimes F_{a,l}v_{l,b} - \sum_{a\neq l,\, l>b} (-1)^{l}\eta_{(b)^c} \otimes F_{a,l}v_{b,l} \\
&= (-1)^{a}\eta_{(b)^c} \otimes E_{00}v_{b,a} - (-1)^{a}\eta_{(b)^c} \otimes v_{b,a} + \sum_{l\neq a,b} (-1)^{l}\eta_{(b)^c} \otimes F_{a,l}v_{l,b}.
\end{aligned}$$

Hence:

$$0 = E_{00}v_{a,b} - v_{a,b} - \sum_{l\neq a,b}(-1)^{a+l}F_{a,l}v_{l,b}.$$

Equation (4.13) for $\xi_a\xi_b$ with $(c,d) = (a,b)^c$, is:

$$\begin{aligned}
0 =& \eta_* \otimes F_{a,b}v_* - (-1)^{a+b}\eta_* \otimes E_{00}v_{a,b} \\
&+ \sum_{i<j}\sum_{l\neq k}(\partial_l(\xi_a\xi_b)\xi_k) \star \eta_{(i,j)^c} \otimes F_{l,k}v_{i,j} + \varepsilon_{(a,b)}(\xi_c\xi_d) \star \eta_{(c,d)^c} \otimes Cv_{c,d}.
\end{aligned}$$

The coefficient of $-\eta_*$ is:

$$0 = -F_{a,b}v_* + (-1)^{a+b}E_{00}v_{a,b} - \sum_{j\neq a,b}(-1)^{b+j}F_{a,j}v_{j,b} + \sum_{j\neq a,b}(-1)^{a+j}F_{b,j}v_{j,a} - \varepsilon_{(a,b)}\varepsilon_{(c,d)}Cv_{c,d}.$$

Equation (4.12), since the terms in C are not involved, reduces, as in [BKL1], to:

$$0 = E_{00}v_* + \sum_{i<j}(-1)^{i+j}F_{i,j}v_{i,j}.$$

Equation (4.15) for $f = \xi_a\xi_b\xi_c$ with $d = (a,b,c)^c$ reduces to:

$$0 = \sum_{i<j}\sum_{l=1}^{4}\partial_l(\xi_a\xi_b\xi_c)\star\partial_l(\eta_{(i,j)^c})\otimes v_{i,j} - \sum_{i<j}\sum_{r<s}\partial_r\partial_s(\xi_a\xi_b\xi_c)\star\eta_{(i,j)^c}\otimes F_{r,s}v_{i,j} + \sum_{i<j}\varepsilon_{(a,b,c)}\xi_d\star\eta_{(i,j)^c}\otimes Cv_{i,j}.$$

From the terms in $(-1)^a\eta_{(a)^c}$ we have:

$$(-1)^{b+c}v_{b,c} + (-1)^{a+c}F_{a,b}v_{a,c} - (-1)^{a+b}F_{a,c}v_{a,b} + \varepsilon_{(a,b,c)}(-1)^{a+d}Cv_{a,d} = 0.$$

From the terms in $(-1)^{b-1}\eta_{(b)^c}$ and the terms in $(-1)^c\eta_{(c)^c}$, we obtain the same equation for b,a,c and c,a,b.
Equation (4.15) for $f = \xi_*$ is:

$$\begin{aligned}0 =& -\sum_{i<j}\sum_{l=1}^{4}\partial_l(\xi_1\xi_2\xi_3\xi_4)\star\partial_l(\eta_{(i,j)^c})\otimes v_{i,j} - \sum_{i<j}\sum_{r<s}\partial_r\partial_s(\xi_1\xi_2\xi_3\xi_4)\star\eta_{(i,j)^c}\otimes F_{r,s}v_{i,j} - \eta_*\otimes Cv_*\\ =&\eta_*\otimes(\sum_{i<j}F_{(i,j)^c}\otimes v_{i,j}) - \eta_*\otimes Cv_*.\end{aligned}$$

Equation (4.15) for $f = \alpha_{1,2}\in B_{\mathfrak{so}(4)}$ is:

$$\begin{aligned}0 =& -\sum_{i<j}\sum_{l=1}^{4}\partial_l(-\xi_1\xi_3 + i\xi_2\xi_3)\star\partial_l(\eta_{(i,j)^c})\otimes v_{i,j} - \sum_{i<j}\sum_{r<s}\partial_r\partial_s(-\xi_1\xi_3 + i\xi_2\xi_3)\star\eta_{(i,j)^c}\otimes F_{r,s}v_{i,j}\\ =& -(\eta_1 - i\eta_2)\eta_4\otimes v_{1,2} - \eta_3\eta_4\otimes\alpha_{1,2}v_{1,2} - i\eta_3\eta_4\otimes v_{1,3} - \eta_2\eta_4\otimes\alpha_{1,2}v_{1,3}\\ &+ \eta_1\eta_2\otimes v_{1,4} - \eta_2\eta_3\otimes\alpha_{1,2}v_{1,4} + \eta_3\eta_4\otimes v_{2,3} - \eta_1\eta_4\otimes\alpha_{1,2}v_{2,3}\\ &+ i\eta_1\eta_2\otimes v_{2,4} - \eta_1\eta_3\otimes\alpha_{1,2}v_{2,4} + (-\eta_2\eta_3 - i\eta_1\eta_3)\otimes v_{3,4} - \eta_1\eta_2\otimes\alpha_{1,2}v_{3,4}\\ =&\eta_1\eta_2\otimes(v_{1,4} + iv_{2,4} - \alpha_{1,2}v_{3,4}) + \eta_1\eta_3\otimes(-\alpha_{1,2}v_{2,4} - iv_{3,4})\\ &+ \eta_1\eta_4\otimes(-v_{1,2} - \alpha_{1,2}v_{2,3}) + \eta_2\eta_3\otimes(-\alpha_{1,2}v_{1,4} - v_{3,4})\\ &+ \eta_2\eta_4\otimes(-\alpha_{1,2}v_{1,3} + iv_{1,2}) + \eta_3\eta_4\otimes(-\alpha_{1,2}v_{1,2} - iv_{1,3} + v_{2,3}).\end{aligned}$$

From the previous equation we obtain relations (4.23) for $\alpha_{1,2}$.
Equation (4.15) for $f = \beta_{1,2}\in B_{\mathfrak{so}(4)}$ is:

$$\begin{aligned}0 =& -\sum_{i<j}\sum_{l=1}^{4}\partial_l(-\xi_2\xi_4 - i\xi_1\xi_4)\star\partial_l(\eta_{(i,j)^c})\otimes v_{i,j} - \sum_{i<j}\sum_{r<s}\partial_r\partial_s(-\xi_2\xi_4 - i\xi_1\xi_4)\star\eta_{(i,j)^c}\otimes F_{r,s}v_{i,j}\\ =& -(\eta_2 + i\eta_1)(-\eta_3)\otimes v_{1,2} - \eta_3\eta_4\otimes\beta_{1,2}v_{1,2} + i\eta_1\eta_2\otimes v_{1,3} - \eta_2\eta_4\otimes\beta_{1,2}v_{1,3}\\ &- \eta_3\eta_4\otimes v_{1,4} - \eta_2\eta_3\otimes\beta_{1,2}v_{1,4} - \eta_1\eta_2\otimes v_{2,3} - \eta_1\eta_4\otimes\beta_{1,2}v_{2,3}\\ &- i\eta_3\eta_4\otimes v_{2,4} - \eta_1\eta_3\otimes\beta_{1,2}v_{2,4} + (-i\eta_2\eta_4 + \eta_1\eta_4)\otimes v_{3,4} - \eta_1\eta_2\otimes\beta_{1,2}v_{3,4}\\ =&\eta_1\eta_2\otimes(iv_{1,3} - v_{2,3} - \beta_{1,2}v_{3,4}) + \eta_1\eta_3\otimes(iv_{1,2} - \beta_{1,2}v_{2,4})\end{aligned}$$

$$+ \eta_1\eta_4 \otimes (-\beta_{1,2}v_{2,3} + v_{3,4}) + \eta_2\eta_3 \otimes (v_{1,2} - \beta_{1,2}v_{1,4})$$
$$+ \eta_2\eta_4 \otimes (-iv_{3,4} - \beta_{1,2}v_{1,3}) + \eta_3\eta_4 \otimes (-\beta_{1,2}v_{1,2} - v_{1,4} - iv_{2,4}).$$

From the previous equation we obtain relations (4.23) for $\beta_{1,2}$.

□

Lemma 4.20. *There are no singular vectors $\vec{m}$ of degree 2, such that $T(\vec{m})$ is written as in formula (4.9) with $v_* \neq 0$.*

Proof. Let $T(\vec{m}) \in \mathrm{Ind}(F)$ be as in formula (4.9). We show that relations of Lemma 4.19 lead to $v_* = 0$. Let $a, b \in \{1, 2, 3, 4\}$ with $a < b$ and $(a, b)^c = (c, d)$.
We consider Equation (4.18) for a, b, multiplied by $-(-1)^{a+b}$, plus Equation (4.18), for reversed role of a and b, multiplied by $(-1)^{a+b}$; we get:

$$0 = -2(-1)^{a+b}E_{00}v_{a,b} + 2(-1)^{a+b}v_{a,b} + \sum_{j\neq a,b}(-1)^{b+j}F_{a,j}v_{j,b} - \sum_{j\neq a,b}(-1)^{a+j}F_{b,j}v_{j,a}.$$

We compare this with Equation (4.19) and obtain:

$$F_{a,b}v_* = (-1)^{a+b+1}E_{00}v_{a,b} + 2(-1)^{a+b}v_{a,b} - Cv_{c,d}, \tag{4.25}$$

since for $a < b$ we have that $\varepsilon_{(a,b)}\varepsilon_{(c,d)} = 1$.
Equation (4.19) reduces to:

$$0 = -F_{a,b}v_* + (-1)^{a+b}E_{00}v_{a,b} + \sum_{j\neq a,b}[(-1)^{b+j}F_{a,j}v_{b,j} - (-1)^{a+j}F_{b,j}v_{a,j}] - Cv_{c,d}. \tag{4.26}$$

We insert equations (4.22) for j, a, b, where we denote by $h = h(j) = (j, a, b)^c$, into this and obtain:

$$\begin{aligned} F_{a,b}v_* =& (-1)^{a+b}E_{00}v_{a,b} - 2(-1)^{a+b}v_{a,b} - \sum_{j\neq a,b}\varepsilon_{(j,a,b)}((-1)^{h+j}Cv_{j,h}) - Cv_{c,d} \qquad (4.27)\\ =& (-1)^{a+b}E_{00}v_{a,b} - 2(-1)^{a+b}v_{a,b} - \sum_{j<a \text{ or } j>b}((-1)^{j}Cv_{j,h}) - \sum_{a<j<b}((-1)^{j+1}Cv_{j,h}) - Cv_{c,d}. \end{aligned}$$

Combining (4.27) and (4.25), we get:

$$(-1)^{a+b}2E_{00}v_{a,b} = 4(-1)^{a+b}v_{a,b} + \sum_{j<a \text{ or } j>b}((-1)^{j}Cv_{j,h}) + \sum_{a<j<b}((-1)^{j+1}Cv_{j,h}).$$

We substitute this in 2(4.27) and obtain:

$$2F_{a,b}v_* = -\sum_{j<a \text{ or } j>b}((-1)^{j}Cv_{j,h}) - \sum_{a<j<b}((-1)^{j+1}Cv_{j,h}) - 2Cv_{c,d}.$$

This reduces to the following, for every $a < b$:

$$F_{a,b}v_* = 0. \tag{4.28}$$

From these conditions it follows that v_* is killed also by the negative roots, so $F = \langle v_* \rangle$ has dimension 1 and $\mathfrak{so}(4)$ acts trivially on it. Moreover all the $v_{a,b}$'s are multiple of v_* since $F = \langle v_* \rangle$. From (4.17), $-v_* = \sum_{j\neq a}(-1)^{a+j}F_{j,a}v_{j,a}$ for every $1 \leq a \leq 4$ then, since all the $v_{a,b}$'s are multiple of v_*, then $v_* = 0$ that is a contradiction since in our hypothesis $v_* \neq 0$. □

By Lemma 4.20 we know that, for a singular vector $\vec{m}$ of degree 2, $T(\vec{m})$ is of the form:

$$T(\vec{m}) = (\sum_{i<j} \eta_{(i,j)^c} \otimes v_{i,j}). \tag{4.29}$$

By Lemma 4.19 we obtain the following result.

Lemma 4.21. *Let $\vec{m} \in \mathrm{Ind}(F)$, such that $T(\vec{m})$ is written as in formula (4.29). Then the relations of Lemma 4.18 reduce to the following equations.*
For all $a \in \{1,2,3,4\}$:

$$0 = \sum_{j \neq a} (-1)^{a+j} F_{j,a} v_{j,a}. \tag{4.30}$$

For all $a, b \in \{1,2,3,4\}$:

$$0 = E_{00} v_{a,b} - v_{a,b} - \sum_{j \neq a,b} (-1)^{a+j} F_{a,j} v_{j,b}. \tag{4.31}$$

For all $a, b \in \{1,2,3,4\}$ with $(a,b)^c = (c,d)$:

$$0 = (-1)^{a+b} E_{00} v_{a,b} - \sum_{j \neq a,b} (-1)^{b+j} F_{a,j} v_{j,b} + \sum_{j \neq a,b} (-1)^{a+j} F_{b,j} v_{j,a} - \varepsilon_{(a,b)} \varepsilon_{(c,d)} C v_{c,d}. \tag{4.32}$$

For every $a, b, c \in \{1,2,3,4\}$ and $d = (a,b,c)^c$:

$$0 = (-1)^{b+c} v_{b,c} + (-1)^{a+c} F_{a,b} v_{a,c} - (-1)^{a+b} F_{a,c} v_{a,b} + \varepsilon_{(a,b,c)} (-1)^{a+d} C v_{a,d}. \tag{4.33}$$

Moreover:

$$0 = \sum_{i<j} (-1)^{i+j} F_{i,j} v_{i,j}, \tag{4.34}$$

and

$$0 = \sum_{i<j} F_{(i,j)^c} v_{i,j}. \tag{4.35}$$

Finally:

$$\begin{aligned}
\alpha_{1,2}(v_{1,2}) &= -iv_{1,3} + v_{2,3}, & \beta_{1,2}(v_{1,2}) &= -v_{1,4} - iv_{2,4}, \\
\alpha_{1,2}(v_{1,3}) &= iv_{1,2}, & \beta_{1,2}(v_{1,3}) &= -iv_{3,4}, \\
\alpha_{1,2}(v_{1,4}) &= -v_{3,4}, & \beta_{1,2}(v_{1,4}) &= v_{1,2}, \\
\alpha_{1,2}(v_{2,3}) &= -v_{1,2}, & \beta_{1,2}(v_{2,3}) &= v_{3,4}, \\
\alpha_{1,2}(v_{2,4}) &= -iv_{3,4}, & \beta_{1,2}(v_{2,4}) &= iv_{1,2}, \\
\alpha_{1,2}(v_{3,4}) &= v_{1,4} + iv_{2,4}, & \beta_{1,2}(v_{3,4}) &= iv_{1,3} - v_{2,3},
\end{aligned} \tag{4.36}$$

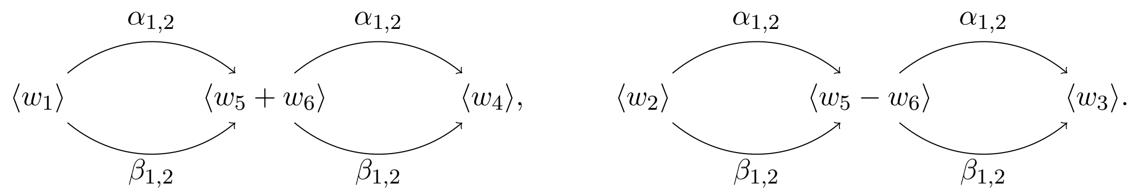
where $\alpha_{1,2}$ and $\beta_{1,2}$ are defined by (3.1) *and* (3.2).

Remark 4.22. Relations (4.36) for $\alpha_{1,2}$ and $\beta_{1,2}$ are equivalent to the following relations, in which we use notation (4.11):

$$\begin{aligned}
\alpha_{1,2}(w_1) &= -w_5 - w_6, & \beta_{1,2}(w_1) &= w_5 + w_6, \\
\alpha_{1,2}(w_2) &= w_5 - w_6, & \beta_{1,2}(w_2) &= w_5 - w_6,
\end{aligned} \tag{4.37}$$

$$\alpha_{1,2}(w_3) = 0, \qquad \beta_{1,2}(w_3) = 0,$$
$$\alpha_{1,2}(w_4) = 0, \qquad \beta_{1,2}(w_4) = 0,$$
$$\alpha_{1,2}(w_5) = w_3 - w_4, \qquad \beta_{1,2}(w_5) = w_3 + w_4,$$
$$\alpha_{1,2}(w_6) = -w_3 - w_4, \qquad \beta_{1,2}(w_6) = -w_3 + w_4.$$

We represent these relations with the following drawings:



Proof of Theorem 4.5. Throughout this proof μ_0 will denote the highest weight of F with respect to E_{00}, μ_1 (resp. μ_2) will denote the highest weight of F with respect to H_1(resp. H_2) and $m = \mu_1 - \mu_2$ (resp. $n = \mu_1 + \mu_2$) will denote the highest weight of F with respect to h_x (resp. h_y).

1) We suppose $w_5 = w_6 = 0$. Therefore $w_3 = w_4 = 0$. Indeed, by Equations (4.37), $\alpha_{1,2}(w_5 + w_6) = -2w_4 = 0$ and $\alpha_{1,2}(w_5 - w_6) = 2w_3 = 0$.
We have the following subcases.

1a) We suppose $w_1 \neq 0$ and $w_2 = 0$. From (4.37), we have that w_1 is a highest weight vector. Then from (4.11):

$$\begin{aligned} v_{1,2} &= 0, \\ v_{1,3} &= w_1, \\ v_{1,4} &= iw_1, \\ v_{2,3} &= iw_1, \\ v_{2,4} &= -w_1, \\ v_{3,4} &= 0. \end{aligned}$$

Equation (4.31) for $a = 1, b = 3$ reduces to:

$$\begin{aligned} 0 =& E_{00}v_{1,3} - v_{1,3} + F_{1,2}v_{2,3} - F_{1,4}v_{3,4} \\ =& E_{00}(w_1) - w_1 + F_{1,2}(iw_1). \end{aligned}$$

Therefore:

$$0 = (E_{00} + H_1)w_1 - w_1.$$

We obtain $\mu_0 = 1 - \mu_1$. Equation (4.33) for $a = 1, b = 2, c = 3$ reduces to:

$$\begin{aligned} 0 =& - v_{2,3} + F_{1,2}v_{1,3} + F_{1,3}v_{1,2} - Cv_{1,4} \\ =& - iw_1 + F_{1,2}w_1 - iCw_1. \end{aligned}$$

Therefore:

$$0 = (H_1 + C)w_1 + w_1.$$

We obtain $C = -1 - \mu_1$. Equation (4.33) for $a = 3, b = 1, c = 4$ reduces to:

$$0 = - v_{1,4} + F_{1,3}v_{3,4} + F_{3,4}v_{1,3} - Cv_{2,3}$$

$$= -\, iw_1 + F_{3,4}w_1 - iCw_1.$$

Therefore:

$$0 = (H_2 + C)w_1 + w_1.$$

We obtain $\mu_2 = -1-(-1-\mu_1) = \mu_1$. Therefore the weight of w_1 with respect to h_x, h_y, E_{00}, C is $(0, n, 1 - \frac{n}{2}, -1 - \frac{n}{2})$ for $n \in \mathbb{Z}_{\geq 0}$.
All the other equations of Lemma 4.21 are easily verified by this choice of $v_{1,2}, v_{1,3}, v_{1,4}, v_{2,3}, v_{2,4}, v_{3,4}$.
The singular vector obtained, written using notation (4.1) is:

$$\vec{m}_{2a} = w_{11}w_{21} \otimes y_1^n,$$

in $M(0, n, 1 - \frac{n}{2}, -1 - \frac{n}{2})$ for $n \in \mathbb{Z}_{\geq 0}$.

1b) We suppose $w_1 = 0$ and $w_2 \neq 0$. From (4.37), we have that w_2 is a highest weight vector. Then from (4.11) we have:

$$\begin{aligned} v_{1,2} &= 0, \\ v_{1,3} &= -w_2, \\ v_{1,4} &= iw_2, \\ v_{2,3} &= -iw_2, \\ v_{2,4} &= -w_2, \\ v_{3,4} &= 0. \end{aligned}$$

Using (4.11) and Equation (4.31) for $a = 1, b = 3$ we obtain:

$$\begin{aligned} 0 =& E_{00}v_{1,3} - v_{1,3} + F_{1,2}v_{2,3} - F_{1,4}v_{3,4} \\ =& E_{00}(-w_2) + w_2 + F_{1,2}(-iw_2). \end{aligned}$$

Therefore:

$$0 = (E_{00} + H_1)w_2 - w_2.$$

We obtain $\mu_0 = 1 - \mu_1$. Using (4.11) and Equation (4.33) for $a = 1, b = 2, c = 3$ we obtain:

$$\begin{aligned} 0 =& -\, v_{2,3} + F_{1,2}v_{1,3} + F_{1,3}v_{1,2} - Cv_{1,4} \\ =& iw_2 - F_{1,2}w_2 - iCw_2. \end{aligned}$$

Therefore:

$$0 = (H_1 - C)w_2 + w_2.$$

We obtain $C = 1 + \mu_1$. Using (4.11) and Equation (4.33) for $a = 3, b = 1, c = 4$ we obtain:

$$\begin{aligned} 0 =& -\, v_{1,4} + F_{1,3}v_{3,4} + F_{3,4}v_{1,3} - Cv_{2,3} \\ =& -\, iw_2 - F_{3,4}w_2 + iCw_2. \end{aligned}$$

Therefore:

$$0 = (H_2 + C)w_2 - w_2.$$

We obtain $\mu_2 = 1 - (1 + \mu_1) = -\mu_1$. Therefore the weight of w_1 with respect to h_x, h_y, E_{00}, C is $(m, 0, 1 - \frac{m}{2}, 1 + \frac{m}{2})$ with $m \in \mathbb{Z}_{\geq 0}$.
All the other equations of Lemma 4.21 are easily verified by this choice of $v_{1,2}, v_{1,3}, v_{1,4}, v_{2,3}, v_{2,4}, v_{3,4}$.
The singular vector obtained, written using notation (4.1), is:

$$\vec{m}_{2b} = w_{11}w_{12} \otimes x_1^m,$$

in $M(m, 0, 1 - \frac{m}{2}, 1 + \frac{m}{2})$ with $m \in \mathbb{Z}_{\geq 0}$.

1c) We suppose $w_1 \neq 0$ and $w_2 \neq 0$. From (4.37), we have that both vectors are a highest weight vector of F. Using (4.11) and Equation (4.33) for $a = 3, b = 2, c = 4$ we obtain:

$$\begin{aligned} 0 =& v_{2,4} + F_{2,3}v_{3,4} - F_{3,4}v_{2,3} - Cv_{1,3} \\ =& - w_1 - w_2 + H_2(-w_1 + w_2) + C(-w_1 + w_2). \end{aligned}$$

Using (4.11) and Equation (4.33) for $a = 4, b = 1, c = 3$ reduces to:

$$\begin{aligned} 0 =& v_{1,3} - F_{1,4}v_{3,4} + F_{3,4}v_{1,4} - Cv_{2,4} \\ =& w_1 - w_2 + F_{3,4}(iw_1 + iw_2) - C(-w_1 - w_2) \\ =& w_1 - w_2 + H_2w_1 + H_2w_2 + Cw_1 + Cw_2. \end{aligned}$$

We take the sum of these 2 equations and get:

$$0 = -2w_2 + 2H_2w_2 + 2Cw_2.$$

Hence we obtain $\mu_2 = -C + 1$. We take the difference of these 2 equations and get:

$$0 = 2w_1 + 2H_2w_1 + 2Cw_1.$$

Hence we obtain $\mu_2 = -C - 1$. This is impossible.

2) Let us analyze the case $w_7 = 0$ and its subcases. From (4.11) we know $w_5 + w_6 = 0$. We suppose $w_5 \neq 0$ (the case $w_5 = 0$ leads to **1)**, then from $\alpha_{1,2}(w_5) = w_3 - w_4$ and $\alpha_{1,2}(w_6) = -w_3 - w_4$ we deduce that $w_4 = 0$; we also know that $w_2 \neq 0$ since $\alpha_{1,2}(w_2) = 2w_5$. Let us split the problem in the following subcases.

2a) We suppose $w_1 = w_4 = w_7 = 0$ and $w_2 \neq 0$, $w_3 \neq 0$, $w_5 + w_6 = 0$. By Remark 4.22 we have:

$$\begin{aligned} &\alpha_{1,2}(w_2) = 2w_5 && \beta_{1,2}(w_2) = 2w_5, \\ &\alpha_{1,2}(w_3) = 0 && \beta_{1,2}(w_3) = 0, \\ &\alpha_{1,2}(w_5) = w_3 && \beta_{1,2}(w_5) = w_3, \\ &\alpha_{1,2}(w_6) = -w_3 && \beta_{1,2}(w_6) = -w_3. \end{aligned}$$

Therefore w_3 is a highest weight vector. Equations (4.11) reduce to:

$$\begin{aligned} v_{1,2} &= 2iw_5, \\ v_{1,3} &= -w_2 - w_3, \\ v_{1,4} &= iw_2 - iw_3, \\ v_{2,3} &= -iw_2 + iw_3, \\ v_{2,4} &= -w_2 - w_3, \\ v_{3,4} &= -2iw_5. \end{aligned}$$

Let us compute the weight of w_3. Using (4.11) and Equation (4.31) for $a = 1, b = 3$ we obtain:

$$\begin{aligned} 0 =& E_{00}v_{1,3} - v_{1,3} + F_{1,2}v_{2,3} - F_{1,4}v_{3,4} \\ =& E_{00}(-w_2 - w_3) + w_2 + w_3 + F_{1,2}(-iw_2 + iw_3) - F_{1,4}(-2iw_5). \end{aligned}$$

Using (4.11) and Equation (4.31) for $a = 2, b = 3$ we obtain:

$$0 = E_{00}v_{2,3} - v_{2,3} + F_{2,1}v_{1,3} + F_{2,4}v_{3,4}$$

$$=E_{00}(-iw_2+iw_3)+iw_2-iw_3+F_{2,1}(-w_2-w_3)+F_{2,4}(-2iw_5).$$

We take the last equation minus the previous multiplied by i and get:

$$\begin{aligned}0=&E_{00}(-iw_2+iw_3+iw_2+iw_3)+iw_2-iw_3-iw_2-iw_3-F_{1,2}(-w_2-w_3)\\&+F_{1,2}(-w_2+w_3)+iF_{1,4}(-2iw_5)+F_{2,4}(-2iw_5).\end{aligned}$$

Therefore:

$$0=2iE_{00}(w_3)-2iw_3+2F_{1,2}w_3-2i\beta_{1,2}w_5,$$

that is equivalent to:

$$0=E_{00}w_3-w_3-H_1w_3-w_3.$$

We obtain $\mu_0=\mu_1+2$. Let us consider equation (4.33) for $a=1,b=2,c=4$, using (4.11) we obtain:

$$\begin{aligned}0=&v_{2,4}-F_{1,2}v_{1,4}+F_{1,4}v_{1,2}-Cv_{1,3}\\=&-w_2-w_3-F_{1,2}(iw_2-iw_3)+F_{1,4}2iw_5-C(-w_2-w_3).\end{aligned}$$

Let us now consider equation (4.33) for $a=2,b=1,c=4$:

$$\begin{aligned}0=&-v_{1,4}-F_{1,2}v_{2,4}-F_{2,4}v_{1,2}-Cv_{2,3}\\=&-iw_2+iw_3-F_{1,2}(-w_2-w_3)+(-1)^3F_{2,4}(2iw_5)-C(-iw_2+iw_3).\end{aligned}$$

We take the sum of last multiplied by i and the previous, we get:

$$\begin{aligned}0=&w_2-w_3-w_2-w_3-F_{1,2}(-iw_2-iw_3+iw_2-iw_3)\\&-iF_{2,4}(2iw_5)+F_{1,4}2iw_5-C(+w_2-w_3-w_2-w_3)\\=&-2w_3+2iF_{1,2}w_3+\beta_{1,2}2w_5+2Cw_3\\=&-2w_3+2iF_{1,2}w_3+2w_3+2Cw_3=+2H_1w_3+2Cw_3.\end{aligned}$$

We obtain $C=-\mu_1$. Now let us consider equation (4.33) for $a=3,b=1,c=4$, using (4.11) we obtain:

$$\begin{aligned}0=&-v_{1,4}+F_{1,3}v_{3,4}+F_{3,4}v_{1,3}-Cv_{2,3}\\=&-(iw_2-iw_3)+F_{1,3}(-2iw_5)+F_{3,4}(-w_2-w_3)-C(-iw_2+iw_3).\end{aligned}$$

Let us consider equation (4.33) for $a=3,b=2,c=4$, using (4.11) we obtain:

$$\begin{aligned}0=&v_{2,4}+F_{2,3}v_{3,4}-F_{3,4}v_{2,3}-Cv_{1,3}\\=&-w_2-w_3+F_{2,3}(-2iw_5)-F_{3,4}(-iw_2+iw_3)-C(-w_2-w_3).\end{aligned}$$

We compute the difference between the first and the second multiplied by i:

$$\begin{aligned}0=&-iw_2+iw_3+iw_2+iw_3+F_{1,3}(-2iw_5)-iF_{2,3}(-2iw_5)\\&+F_{3,4}(-w_2-w_3+w_2-w_3)-C(-iw_2+iw_3+iw_2+iw_3)\\=&2iw_3-2iw_3-2F_{3,4}w_3-2iCw_3=-2i(-H_2+C)w_3.\end{aligned}$$

We obtain $\mu_2=C$. Hence, the weight of w_3 with respect to h_x,h_y,E_{00},C is $(m,0,\frac{m}{2}+2,-\frac{m}{2})$. From $\alpha_{1,2}(w_5)+\beta_{1,2}(w_5)=E_{\varepsilon_1-\varepsilon_2}w_5=2w_3$ and $\alpha_{1,2}(w_2)+\beta_{1,2}(w_2)=E_{\varepsilon_1-\varepsilon_2}w_2=4w_5$, we have:

$$w_5=-\frac{1}{2m}E_{-(\varepsilon_1-\varepsilon_2)}w_3=-w_6,\qquad w_2=\frac{1}{4m(m-1)}E_{-(\varepsilon_1-\varepsilon_2)}E_{-(\varepsilon_1-\varepsilon_2)}w_3.$$

From this we also know that $m > 1$.
All the other equations of Lemma 4.21 are easily verified by this choice of $v_{1,2}, v_{1,3}, v_{1,4}, v_{2,3}, v_{2,4}, v_{3,4}$.
The singular vector obtained, written using notation (4.1), is:

$$\vec{m}_{2c} = w_{22}w_{21} \otimes x_1^m + (w_{11}w_{22} + w_{21}w_{12}) \otimes x_1^{m-1}x_2 - w_{11}w_{12} \otimes x_1^{m-2}x_2^2,$$

in $M(m, 0, \frac{m}{2} + 2, -\frac{m}{2})$ with $m > 1$.

2b) We suppose that $w_3 = w_4 = w_7 = 0$ and $w_1 \neq 0$, $w_2 \neq 0$, $w_5 \neq 0$, $w_5 + w_6 = 0$. Using (4.11), Equation (4.33) for $a = 1, b = 3, c = 4$ reduces to:

$$\begin{aligned} 0 =& - v_{3,4} - F_{1,3}v_{1,4} - F_{1,4}v_{1,3} - Cv_{1,2} \\ =& - (-2iw_5) - F_{1,3}(iw_1 + iw_2) - F_{1,4}(w_1 - w_2) - C(2iw_5). \end{aligned}$$

Using (4.11), Equation (4.33) for $a = 2, b = 3, c = 4$ reduces to:

$$\begin{aligned} 0 =& - v_{3,4} + F_{2,3}v_{2,4} + F_{2,4}v_{2,3} - Cv_{1,2} \\ =& - (-2iw_5) + F_{2,3}(-w_1 - w_2) + F_{2,4}(iw_1 - iw_2) - C(2iw_5). \end{aligned}$$

We take the sum of these two equations and get:

$$\begin{aligned} 0 =& 4iw_5 - F_{1,3}(iw_1 + iw_2) + F_{2,3}(-w_1 - w_2) \\ & - F_{1,4}(w_1 - w_2) + F_{2,4}(iw_1 - iw_2) - 4iCw_5 \\ =& 4iw_5 + i(F_{1,3} - iF_{2,3})w_1 - i(F_{1,3} - iF_{2,3})w_2 \\ & + i(F_{2,4} + iF_{1,4})w_1 - i(F_{2,4} + iF_{1,4})w_2 - 4iCw_5 \\ =& 4iw_5 - 4iw_5 - 4iCw_5 = -4iCw_5. \end{aligned}$$

We obtain $C = 0$. For $C = 0$, the $\lambda-$action of Proposition 3.11 reduces to the action found in Theorem 4.3 of [BKL1]; in that case it was shown that there are no singular vectors of degree 2.

2c) We suppose $w_1 = w_3 = w_4 = w_7 = 0$ and $w_2 \neq 0$, $w_5 \neq 0$, $w_5 + w_6 = 0$. Using (4.11), Equation (4.33) for $a = 1, b = 3, c = 4$ reduces to:

$$\begin{aligned} 0 =& - v_{3,4} - F_{1,3}v_{1,4} - F_{1,4}v_{1,3} - Cv_{1,2} \\ =& - (-2iw_5) - F_{1,3}(iw_2) - F_{1,4}(-w_2) - C(2iw_5). \end{aligned}$$

Using (4.11), Equation (4.33) for $a = 2, b = 3, c = 4$ reduces to:

$$\begin{aligned} 0 =& - v_{3,4} + F_{2,3}v_{2,4} + F_{2,4}v_{2,3} - Cv_{1,2} \\ =& - (-2iw_5) + F_{2,3}(-w_2) + F_{2,4}(-iw_2) - C(2iw_5). \end{aligned}$$

We take the sum and get:

$$\begin{aligned} 0 =& 4iw_5 - F_{1,3}(+iw_2) + F_{2,3}(-w_2) - F_{1,4}(-w_2) + F_{2,4}(-iw_2) - 4iCw_5 \\ =& 4iw_5 - i(F_{1,3} - iF_{2,3})w_2 - i(F_{2,4} + iF_{1,4})w_2 - 4iCw_5 \\ =& 4iw_5 - 4iw_5 - 4iCw_5 = -4iCw_5. \end{aligned}$$

We obtain $C = 0$. For $C = 0$, the $\lambda-$action of Proposition 3.11 reduces to the action found in Theorem 4.3 of [BKL1]; in that case it was shown that there are no singular vectors of degree 2.

2d) We suppose that $w_4 = w_7 = 0$ and $w_1 \neq 0$, $w_2 \neq 0$, $w_3 \neq 0$, $w_5 \neq 0$, $w_5 + w_6 = 0$. From (4.37), we have that w_1 and w_3 are highest weight vectors, therefore they are multiples.
Using (4.11), Equation (4.33) for $a = 1, b = 3, c = 4$ reduces to:

$$\begin{aligned}0 =& -v_{3,4} - F_{1,3}v_{1,4} - F_{1,4}v_{1,3} - Cv_{1,2}\\ =& -(-2iw_5) - F_{1,3}(iw_1 + iw_2 - iw_3) - F_{1,4}(w_1 - w_2 - w_3) - C(2iw_5).\end{aligned}$$

Using (4.11), Equation (4.33) for $a = 2, b = 3, c = 4$ reduces to:

$$\begin{aligned}0 =& -v_{3,4} + F_{2,3}v_{2,4} + F_{2,4}v_{2,3} - Cv_{1,2}\\ =& -(-2iw_5) + F_{2,3}(-w_1 - w_2 - w_3) + F_{2,4}(iw_1 - iw_2 + iw_3) - C(2iw_5).\end{aligned}$$

We take the sum of these equations and get:

$$\begin{aligned}0 =& 4iw_5 - F_{1,3}(iw_1 + iw_2 - iw_3) + F_{2,3}(-w_1 - w_2 - w_3)\\ &- F_{1,4}(w_1 - w_2 - w_3) + F_{2,4}(iw_1 - iw_2 + iw_3) - 4iCw_5\\ =& 4iw_5 - i(F_{1,3} - iF_{2,3})w_1 - i(F_{1,3} - iF_{2,3})w_2 + i(F_{2,4} + iF_{1,4})w_1 - i(F_{2,4} + iF_{1,4})w_2\\ &+ i(F_{1,3} + F_{2,4} - iF_{1,4} + iF_{2,3})w_3 - 4iCw_5\\ =& 4iw_5 - 4iw_5 + iE_{-(\varepsilon_1-\varepsilon_2)}w_3 - 4iCw_5 = -4iCw_5 + iE_{-(\varepsilon_1-\varepsilon_2)}w_3.\end{aligned}$$

We take the difference and get:

$$\begin{aligned}0 =& -F_{1,3}(iw_1 + iw_2 - iw_3) - F_{2,3}(-w_1 - w_2 - w_3)\\ &- F_{1,4}(w_1 - w_2 - w_3) - F_{2,4}(iw_1 - iw_2 + iw_3)\\ =& -i(F_{1,3} + iF_{2,3} - iF_{1,4} + F_{2,4})w_1 - i(F_{1,3} + iF_{2,3} + iF_{1,4} - F_{2,4})w_2\\ &+ i(F_{1,3} - iF_{2,3})w_3 - i(F_{2,4} + iF_{1,4})w_3\\ =& -iE_{-(\varepsilon_1-\varepsilon_2)}w_1 - iE_{-(\varepsilon_1+\varepsilon_2)}w_2.\end{aligned}$$

Since w_1 is a highest weight vector and w_2 is not, these two terms are linearly independent unless they are both 0, in particular $E_{-(\varepsilon_1-\varepsilon_2)}w_1$. But we know that w_3 is a multiple of w_1, then $-4iCw_5 + iE_{-(\varepsilon_1-\varepsilon_2)}w_3 = -4iCw_5 = 0$. We obtain $C = 0$. For $C = 0$, the $\lambda-$action of Proposition 3.11 reduces to the action found in Theorem 4.3 of [BKL1]; in that case it was shown that there are no singular vectors of degree 2.

3) Let us now analyze the case $w_7 \neq 0$, with $w_5 = w_6 \neq 0$ and all its subcases. In particular we know that this implies $w_1 \neq 0$ because $\alpha_{1,2}(w_1) = -w_5 - w_6$, and $w_3 = 0$ since $\alpha_{1,2}(w_5 - w_6) = 2w_3$.

3a) We suppose $w_2 = w_3 = 0$ and $w_1 \neq 0$, $w_4 \neq 0$, $w_7 \neq 0$, $w_5 = w_6 \neq 0$. By Remark 4.22 we have:

$$\begin{aligned}&\alpha_{1,2}(w_1) = -2w_5 && \beta_{1,2}(w_1) = 2w_5,\\ &\alpha_{1,2}(w_4) = 0 && \beta_{1,2}(w_4) = 0,\\ &\alpha_{1,2}(w_5) = -w_4 && \beta_{1,2}(w_5) = w_4,\\ &\alpha_{1,2}(w_6) = -w_4 && \beta_{1,2}(w_6) = w_4.\end{aligned}$$

We have that w_4 is a highest weight vector and Equations (4.11) reduce to:

$$\begin{aligned}v_{1,2} &= 2iw_5,\\ v_{1,3} &= w_1 + w_4,\\ v_{1,4} &= iw_1 - iw_4,\end{aligned}$$

$$\begin{aligned} v_{2,3} &= iw_1 - iw_4, \\ v_{2,4} &= -w_1 - w_4, \\ v_{3,4} &= 2iw_5. \end{aligned}$$

Let us compute the weight of w_4. Using (4.11), Equation (4.31) for $a = 1, b = 3$ reduces to:

$$\begin{aligned} 0 =& E_{00}v_{1,3} - v_{1,3} + F_{1,2}v_{2,3} - F_{1,4}v_{3,4} \\ =& E_{00}(w_1 + w_4) - w_1 - w_4 + F_{1,2}(iw_1 - iw_4) - F_{1,4}(2iw_5). \end{aligned}$$

Using (4.11), Equation (4.31) for $a = 2, b = 3$ reduces to:

$$\begin{aligned} 0 =& E_{00}v_{2,3} - v_{2,3} + F_{2,1}v_{1,3} + F_{2,4}v_{3,4} \\ =& E_{00}(iw_1 - iw_4) - iw_1 + iw_4 - F_{1,2}(w_1 + w_4) + F_{2,4}(2iw_5). \end{aligned}$$

We take the sum of the first equation and the second multiplied by i and get:

$$\begin{aligned} 0 =& E_{00}(w_1 + w_4 - w_1 + w_4) - w_1 - w_4 + w_1 - w_4 \\ &+ F_{1,2}(iw_1 - iw_4 - iw_1 - iw_4) - 2iF_{1,4}(w_5) - 2F_{2,4}(w_5) \\ =& E_{00}(2w_4) - 2w_4 + F_{1,2}(-2iw_4) - 2\beta_{1,2}(w_5) \\ =& E_{00}(2w_4) - 2w_4 + F_{1,2}(-2iw_4) - 2w_4 \\ =& 2E_{00}w_4 - 2H_1w_4 - 4w_4. \end{aligned}$$

We obtain $\mu_0 = \mu_1 + 2$. Let us consider Equation (4.33) for $a = 1, b = 2, c = 4$, using (4.11) we obtain:

$$\begin{aligned} 0 =& v_{2,4} - F_{1,2}v_{1,4} + F_{1,4}v_{1,2} - Cv_{1,3} \\ =& - w_1 - w_4 - F_{1,2}(iw_1 - iw_4) + F_{1,4}2iw_5 - C(w_1 + w_4). \end{aligned}$$

Let us now take Equation (4.33) for $a = 2, b = 1, c = 4$, using (4.11) we obtain:

$$\begin{aligned} 0 =& - v_{1,4} - F_{1,2}v_{2,4} - F_{2,4}v_{1,2} - Cv_{2,3} \\ =& - iw_1 + iw_4 - F_{1,2}(-w_1 - w_4) - F_{2,4}(2iw_5) - C(iw_1 - iw_4). \end{aligned}$$

We take the difference between the last and the previous multiplied by i and get:

$$\begin{aligned} 0 =& - iw_1 + iw_4 + iw_1 + iw_4 - F_{1,2}(-w_1 - w_4 + w_1 - w_4) \\ &- F_{2,4}(2iw_5) - iF_{1,4}2iw_5 - C(iw_1 - iw_4 - iw_1 - iw_4) \\ =& 2iw_4 + 2F_{1,2}w_4 - 2i\beta_{1,2}w_5 + 2iCw_4 \\ =& 2F_{1,2}w_4 + 2iCw_4. \end{aligned}$$

Therefore:

$$0 = H_1w_4 - Cw_4.$$

We obtain $C = \mu_1$. Using (4.11), Equation (4.33) for $a = 3, b = 1, c = 4$ reduces to:

$$\begin{aligned} 0 =& - v_{1,4} + F_{1,3}v_{3,4} + F_{3,4}v_{1,3} - Cv_{2,3} \\ =& - (iw_1 - iw_4) + F_{1,3}(2iw_5) + F_{3,4}(w_1 + w_4) - C(iw_1 - iw_4). \end{aligned}$$

Using (4.11), Equation (4.33) for $a = 3, b = 2, c = 4$ reduces to:

$$0 = v_{2,4} + F_{2,3}v_{3,4} - F_{3,4}v_{2,3} - Cv_{1,3}$$

$$= - w_1 - w_4 + F_{2,3}(2iw_5) - F_{3,4}(iw_1 - iw_4) - C(w_1 + w_4).$$

We compute the difference between the first and the second multiplied by i:

$$\begin{aligned}0 =& - iw_1 + iw_4 + iw_1 + iw_4 + F_{1,3}(2iw_5) - iF_{2,3}(2iw_5)\\ &+ F_{3,4}(w_1 + w_4 - w_1 + w_4) + C(-iw_1 + iw_4 + iw_1 + iw_4)\\ =&2iw_4 - 2iw_4 + 2F_{3,4}w_4 + 2iCw_4.\end{aligned}$$

Therefore:

$$0 = H_2 w_4 - Cw_4.$$

We obtain $\mu_2 = C$. Hence the weight of w_4 with respect to h_x, h_y, E_{00}, C is $(0, n, \frac{n}{2} + 2, \frac{n}{2})$. From $\alpha_{1,2}(w_5) - \beta_{1,2}(w_5) = E_{\varepsilon_1+\varepsilon_2} w_5 = -2w_4$, $\alpha_{1,2}(w_1) - \beta_{1,2}(w_1) = E_{\varepsilon_1+\varepsilon_2} w_1 = -4w_5$ and $2w_5 + 2w_6 + w_7 = 0$, we have:

$$\begin{aligned}w_5 &= \frac{1}{2n} E_{-(\varepsilon_1+\varepsilon_2)} w_4 = w_6,\\ w_7 &= -4w_5,\\ w_1 &= \frac{1}{4n(n-1)} E_{-(\varepsilon_1+\varepsilon_2)} E_{-(\varepsilon_1+\varepsilon_2)} w_4.\end{aligned}$$

From this we know that $n \geq 2$. All the other equations of Lemma 4.21 are easily verified by this choice of $v_{1,2}, v_{1,3}, v_{1,4}, v_{2,3}, v_{2,4}, v_{3,4}$. The singular vector obtained, written using notation (4.1), is:

$$\vec{m}_{2d} = w_{22} w_{12} \otimes y_1^n - (w_{22} w_{11} + w_{21} w_{12}) \otimes y_1^{n-1} y_2 - w_{11} w_{21} \otimes y_1^{n-2} x_2^2,$$

in $M(0, n, \frac{n}{2} + 2, \frac{n}{2})$ with $n > 1$.

3b) We suppose $w_2 = w_3 = w_4 = 0$ and $w_1 \neq 0$, $w_7 \neq 0$, $w_5 = w_6 \neq 0$. Using (4.11), Equation (4.33) for $a = 1, b = 3, c = 4$ reduces to:

$$\begin{aligned}0 =& - v_{3,4} - F_{1,3} v_{1,4} - F_{1,4} v_{1,3} - Cv_{1,2}\\ =& - (2iw_5) - F_{1,3}(iw_1) - F_{1,4}(w_1) - C(2iw_5).\end{aligned}$$

Using (4.11), Equation (4.33) for $a = 2, b = 3, c = 4$ reduces to:

$$\begin{aligned}0 =& - v_{3,4} + F_{2,3} v_{2,4} + F_{2,4} v_{2,3} - Cv_{1,2}\\ =& - (2iw_5) + F_{2,3}(-w_1) + F_{2,4}(iw_1) - C(2iw_5).\end{aligned}$$

We take the sum of these two equations and obtain:

$$\begin{aligned}0 =& - 4iw_5 - i(F_{1,3} - iF_{2,3})(w_1) + i(F_{2,4} + iF_{1,4})w_1 - C4iw_5\\ =& - 4iw_5 - i\alpha_{1,2}(w_1) + i\beta_{1,2}(w_1) - C4iw_5\\ =& - 4iw_5 + 2iw_5 + 2iw_5 - 4iCw_5 = -4iCw_5.\end{aligned}$$

We obtain $C = 0$. For $C = 0$, the $\lambda-$action of Proposition 3.11 reduces to the action found in Theorem 4.3 of [BKL1]; in that case it was shown that there are no singular vectors of degree 2.

3c) We suppose $w_3 = w_4 = 0$ and $w_1 \neq 0, w_2 \neq 0$, $w_7 \neq 0$, $w_5 = w_6 \neq 0$. Using (4.11), Equation (4.33) for $a = 1, b = 3, c = 4$ reduces to:

$$\begin{aligned} 0 = &- v_{3,4} - F_{1,3}v_{1,4} - F_{1,4}v_{1,3} - Cv_{1,2} \\ = &- (2iw_5) - F_{1,3}(iw_1 + iw_2) - F_{1,4}(w_1 - w_2) - C(2iw_5). \end{aligned}$$

Using (4.11), Equation (4.33) for $a = 2, b = 3, c = 4$ reduces to:

$$\begin{aligned} 0 = &- v_{3,4} + F_{2,3}v_{2,4} + F_{2,4}v_{2,3} - Cv_{1,2} \\ = &- (2iw_5) + F_{2,3}(-w_1 - w_2) + F_{2,4}(iw_1 - iw_2) - C(2iw_5). \end{aligned}$$

We take the sum and obtain:

$$\begin{aligned} 0 = &- 4iw_5 - i(F_{1,3} - iF_{2,3})(w_1) + i(F_{2,4} + iF_{1,4})w_1 \\ &- i(F_{1,3} - iF_{2,3})w_2 - i(F_{2,4} + iF_{1,4})w_2 - C4iw_5 \\ = &- 4iw_5 - i\alpha_{1,2}(w_1) + i\beta_{1,2}(w_1) - i\alpha_{1,2}(w_2) - i\beta_{1,2}(w_2) - C4iw_5 \\ = &- 4iw_5 + 2iw_5 + 2iw_5 - 4iCw_5 = -4iCw_5. \end{aligned}$$

We obtain $C = 0$. For $C = 0$, the $\lambda-$action of Proposition 3.11 reduces to the action found in Theorem 4.3 of [BKL1]; in that case it was shown that there are no singular vectors of degree 2.

3d) We suppose $w_3 = 0$ and $w_1 \neq 0$, $w_2 \neq 0$, $w_4 \neq 0$, $w_7 \neq 0$, $w_5 = w_6 \neq 0$. We have that w_2 and w_4 are multiples because, from (4.37), we know that they are both highest weight vectors. Using (4.11), Equation (4.33) for $a = 1, b = 3, c = 4$ reduces to:

$$\begin{aligned} 0 = &- v_{3,4} - F_{1,3}v_{1,4} - F_{1,4}v_{1,3} - Cv_{1,2} \\ = &- (2iw_5) - F_{1,3}(iw_1 + iw_2 - iw_4) - F_{1,4}(w_1 - w_2 + w_4) - C(2iw_5). \end{aligned}$$

Using (4.11), Equation (4.33) for $a = 2, b = 3, c = 4$ reduces to:

$$\begin{aligned} 0 = &- v_{3,4} + F_{2,3}v_{2,4} + F_{2,4}v_{2,3} - Cv_{1,2} \\ = &- (2iw_5) + F_{2,3}(-w_1 - w_2 - w_4) + F_{2,4}(iw_1 - iw_2 - iw_4) - C(2iw_5). \end{aligned}$$

We take the sum and obtain:

$$\begin{aligned} 0 = &- 4iw_5 - i(F_{1,3} - iF_{2,3})(w_1) + i(F_{2,4} + iF_{1,4})w_1 - i(F_{1,3} - iF_{2,3})w_2 \\ &- i(F_{2,4} + iF_{1,4})w_2 + i(F_{1,3} - F_{2,4} + iF_{1,4} + iF_{2,3})w_4 - C4iw_5 \\ = &- 4iw_5 - i\alpha_{1,2}(w_1) + i\beta_{1,2}(w_1) - i\alpha_{1,2}(w_2) - i\beta_{1,2}(w_2) \\ &+ i(F_{1,3} - F_{2,4} + iF_{1,4} + iF_{2,3})w_4 - C4iw_5 \\ = &- 4iw_5 + 2iw_5 + 2iw_5 - 4iCw_5 + iE_{-(\varepsilon_1+\varepsilon_2)}w_4 = -4iCw_5 + iE_{-(\varepsilon_1+\varepsilon_2)}w_4. \end{aligned}$$

We take the difference and obtain:

$$\begin{aligned} 0 = &- i(F_{1,3} + iF_{2,3})w_1 + i(-F_{2,4} + iF_{1,4})w_1 - i(F_{1,3} + iF_{2,3})w_2 - i(-F_{2,4} + iF_{1,4})w_2 + \\ &+ i(F_{1,3} + F_{2,4} + iF_{1,4} - iF_{2,3})w_4 \\ = &- iE_{-(\varepsilon_1-\varepsilon_2)}w_1 - iE_{-(\varepsilon_1+\varepsilon_2)}w_2. \end{aligned}$$

Since w_2 is a highest weight vector and w_1 is not, these two terms are linearly independent, unless they are both 0, in particular $E_{-(\varepsilon_1+\varepsilon_2)}w_2 = 0$. But we know that w_4 is a multiple of w_2, then $-4iCw_5 + iE_{-(\varepsilon_1+\varepsilon_2)}w_4 = -4iCw_5 = 0$. We obtain $C = 0$. For $C = 0$, the $\lambda-$action of Proposition 3.11 reduces to the action found in Theorem 4.3 of [BKL1]; in that case it was shown that there are no singular vectors of degree 2.

4) Let us now analyze the case $w_7 \neq 0$ (hence $w_5 \neq -w_6$), with $w_5 \neq w_6$ and all its subcases. In particular we know that this implies $w_1 \neq 0$ because $\alpha_{1,2}(w_1) = -w_5 - w_6$ and $w_2 \neq 0$ because $\alpha_{1,2}(w_2) = w_5 - w_6$. We have the following subcases.

4a) We suppose $w_3 = w_4 = 0$ and $w_1 \neq 0$, $w_2 \neq 0$, $w_5 \neq 0$, $w_6 \neq 0$, $w_5 \neq \pm w_6$.
Using (4.11), Equation (4.33) for $a = 1, b = 3, c = 4$ reduces to:

$$\begin{aligned} 0 =& - v_{3,4} - F_{1,3}v_{1,4} - F_{1,4}v_{1,3} - Cv_{1,2} \\ =& - (2iw_6) - F_{1,3}(iw_1 + iw_2) - F_{1,4}(w_1 - w_2) - C(2iw_5). \end{aligned}$$

Using (4.11), Equation (4.33) for $a = 2, b = 3, c = 4$ reduces to:

$$\begin{aligned} 0 =& - v_{3,4} + F_{2,3}v_{2,4} + F_{2,4}v_{2,3} - Cv_{1,2} \\ =& - (2iw_6) + F_{2,3}(-w_1 - w_2) + F_{2,4}(iw_1 - iw_2) - C(2iw_5). \end{aligned}$$

We take the sum and obtain:

$$\begin{aligned} 0 =& - 4iw_6 - C4iw_5 - i(F_{1,3} - iF_{2,3})(w_1) - i(F_{1,3} - iF_{2,3})(w_2) \\ &+ i(F_{2,4} + iF_{1,4})(w_1) - i(F_{2,4} + iF_{1,4})(w_2) \\ =& - 4iw_6 - C4iw_5 + i(w_5 + w_6) - i(w_5 - w_6) + i(w_5 + w_6) - i(w_5 - w_6) \\ =& - C4iw_5 = 0. \end{aligned}$$

We obtain $C = 0$. For $C = 0$, the $\lambda-$action of Proposition 3.11 reduces to the action found in Theorem 4.3 of [BKL1]; in that case it was shown that there are no singular vectors of degree 2.

4b) We suppose that $w_4 = 0$ and $w_1 \neq 0$, $w_2 \neq 0$, $w_3 \neq 0$, $w_5 \neq 0$, $w_6 \neq 0$, $w_5 \neq \pm w_6$.
Using (4.11), Equation (4.33) for $a = 1, b = 3, c = 4$ reduces to:

$$\begin{aligned} 0 =& - v_{3,4} - F_{1,3}v_{1,4} - F_{1,4}v_{1,3} - Cv_{1,2} \\ =& - (2iw_6) - F_{1,3}(iw_1 + iw_2 - iw_3) - F_{1,4}(w_1 - w_2 - w_3) - C(2iw_5). \end{aligned}$$

Using (4.11), Equation (4.33) for $a = 2, b = 3, c = 4$ reduces to:

$$\begin{aligned} 0 =& - v_{3,4} + F_{2,3}v_{2,4} + F_{2,4}v_{2,3} - Cv_{1,2} \\ =& - (2iw_6) + F_{2,3}(-w_1 - w_2 - w_3) + F_{2,4}(iw_1 - iw_2 + iw_3) - C(2iw_5). \end{aligned}$$

We take the sum and obtain:

$$\begin{aligned} 0 =& - 4iw_6 - C4iw_5 - i(F_{1,3} - iF_{2,3})(w_1) - i(F_{1,3} - iF_{2,3})(w_2) + \\ &+ i(F_{2,4} + iF_{1,4})(w_1) - i(F_{2,4} + iF_{1,4})(w_2) + i(F_{1,3} + F_{2,4} - iF_{1,4} + iF_{2,3})w_3 \\ =& - 4iw_6 - C4iw_5 + i(w_5 + w_6) - i(w_5 - w_6) + i(w_5 + w_6) - i(w_5 - w_6) + iE_{-(\varepsilon_1 - \varepsilon_2)}w_3 \\ =& - C4iw_5 + iE_{-(\varepsilon_1 - \varepsilon_2)}w_3 = 0. \end{aligned}$$

Using (4.11), Equation (4.33) for $a = 4, b = 1, c = 2$ reduces to:

$$\begin{aligned} 0 =& - v_{1,2} + F_{1,4}v_{2,4} + F_{2,4}v_{1,4} - Cv_{3,4} \\ =& - (2iw_5) + F_{1,4}(-w_1 - w_2 - w_3) + F_{2,4}(iw_1 + iw_2 - iw_3) - C(2iw_6). \end{aligned}$$

Using (4.11), Equation (4.33) for $a = 3, b = 1, c = 2$ reduces to:

$$0 = - v_{1,2} - F_{1,3}v_{2,3} - F_{2,3}v_{1,3} - Cv_{3,4}$$

$$= -(2iw_5) - F_{1,3}(iw_1 - iw_2 + iw_3) - F_{2,3}(w_1 - w_2 - w_3) - C(2iw_6).$$

We take the sum and obtain:

$$\begin{aligned}0 =& -4iw_5 - C4iw_6 + i(F_{2,4} + iF_{1,4})w_1 + i(F_{2,4} + iF_{1,4})w_2 - i(F_{1,3} - iF_{2,3})w_1\\ &+ i(F_{1,3} - iF_{2,3})w_2 - i(F_{1,3} + F_{2,4} - iF_{1,4} + iF_{2,3})w_3\\ =& -4iw_5 - C4iw_6 + i(w_5 + w_6) + i(w_5 - w_6) + i(w_5 + w_6) + i(w_5 - w_6) - iE_{-(\varepsilon_1-\varepsilon_2)}w_3\\ =& -C4iw_6 - iE_{-(\varepsilon_1-\varepsilon_2)}w_3.\end{aligned}$$

Hence we know that:

$$\begin{aligned}0 &= -C4iw_5 + iE_{-(\varepsilon_1-\varepsilon_2)}w_3,\\ 0 &= -C4iw_6 - iE_{-(\varepsilon_1-\varepsilon_2)}w_3.\end{aligned}$$

From this we obtain that $w_5 + w_6 = 0$ that is a contradiction.

4c) We suppose that $w_3 = 0$ and $w_1 \neq 0$, $w_2 \neq 0$, $w_4 \neq 0$, $w_5 \neq 0$, $w_6 \neq 0$, $w_5 \neq \pm w_6$. Using (4.11), Equation (4.33) for $a = 1, b = 3, c = 4$ reduces to:

$$\begin{aligned}0 =& -v_{3,4} - F_{1,3}v_{1,4} - F_{1,4}v_{1,3} - Cv_{1,2}\\ =& -(2iw_6) - F_{1,3}(iw_1 + iw_2 - iw_4) - F_{1,4}(w_1 - w_2 + w_4) - C(2iw_5).\end{aligned}$$

Using (4.11), Equation (4.33) for $a = 2, b = 3, c = 4$ reduces to:

$$\begin{aligned}0 =& -v_{3,4} + F_{2,3}v_{2,4} + F_{2,4}v_{2,3} - Cv_{1,2}\\ =& -(2iw_6) + F_{2,3}(-w_1 - w_2 - w_4) + F_{2,4}(iw_1 - iw_2 - iw_4) - C(2iw_5).\end{aligned}$$

We take the sum and obtain:

$$\begin{aligned}0 =& -4iw_6 - C4iw_5 - i(F_{1,3} - iF_{2,3})(w_1) - i(F_{1,3} - iF_{2,3})(w_2)+\\ &+ i(F_{2,4} + iF_{1,4})(w_1) - i(F_{2,4} + iF_{1,4})(w_2) + i(F_{1,3} - F_{2,4} + iF_{1,4} + iF_{2,3})w_4\\ =& -4iw_6 - C4iw_5 + i(w_5 + w_6) - i(w_5 - w_6) + i(w_5 + w_6) - i(w_5 - w_6) + iE_{-(\varepsilon_1+\varepsilon_2)}w_4\\ =& -C4iw_5 + iE_{-(\varepsilon_1+\varepsilon_2)}w_4.\end{aligned}$$

Using (4.11), Equation (4.33) for $a = 4, b = 1, c = 2$ reduces to:

$$\begin{aligned}0 =& -v_{1,2} + F_{1,4}v_{2,4} + F_{2,4}v_{1,4} - Cv_{3,4}\\ =& -(2iw_5) + F_{1,4}(-w_1 - w_2 - w_4) + F_{2,4}(iw_1 + iw_2 - iw_4) - C(2iw_6).\end{aligned}$$

Using (4.11), Equation (4.33) for $a = 3, b = 1, c = 2$ reduces to:

$$\begin{aligned}0 =& -v_{1,2} - F_{1,3}v_{2,3} - F_{2,3}v_{1,3} - Cv_{3,4}\\ =& -(2iw_5) - F_{1,3}(iw_1 - iw_2 - iw_4) - F_{2,3}(w_1 - w_2 + w_4) - C(2iw_6).\end{aligned}$$

We take the sum and obtain:

$$\begin{aligned}0 =& -4iw_5 - C4iw_6 + i(F_{2,4} + iF_{1,4})w_1 + i(F_{2,4} + iF_{1,4})w_2 - i(F_{1,3} - iF_{2,3})w_1 + i(F_{1,3} - iF_{2,3})w_2\\ &+ i(F_{1,3} - F_{2,4} + iF_{1,4} + iF_{2,3})w_4\\ =& -4iw_5 - C4iw_6 + i(w_5 + w_6) + i(w_5 - w_6) + i(w_5 + w_6) + i(w_5 - w_6) + iE_{-(\varepsilon_1+\varepsilon_2)}w_4\\ =& -C4iw_6 + iE_{-(\varepsilon_1+\varepsilon_2)}w_4.\end{aligned}$$

Hence we know that:

$$\begin{aligned}0 &= -C4iw_5 + iE_{-(\varepsilon_1+\varepsilon_2)}w_4,\\ 0 &= -C4iw_6 + iE_{-(\varepsilon_1+\varepsilon_2)}w_4.\end{aligned}$$

From this we have that $w_5 - w_6 = 0$ that is a contradiction.

4d) We suppose $w_1 \neq 0$, $w_2 \neq 0$, $w_3 \neq 0$, $w_4 \neq 0$, $w_5 \neq 0$, $w_6 \neq 0$, $w_5 \neq \pm w_6$.
Using (4.11), Equation (4.33) for $a = 1, b = 3, c = 4$ reduces to:

$$\begin{aligned}0 =& - v_{3,4} - F_{1,3}v_{1,4} - F_{1,4}v_{1,3} - Cv_{1,2} \\ =& - (2iw_6) - F_{1,3}(iw_1 + iw_2 - iw_3 - iw_4) - F_{1,4}(w_1 - w_2 - w_3 + w_4) - C(2iw_5).\end{aligned}$$

Using (4.11), Equation (4.33) for $a = 2, b = 3, c = 4$ reduces to:

$$\begin{aligned}0 =& - v_{3,4} + F_{2,3}v_{2,4} + F_{2,4}v_{2,3} - Cv_{1,2} \\ =& - (2iw_6) + F_{2,3}(-w_1 - w_2 - w_3 - w_4) + F_{2,4}(iw_1 - iw_2 + iw_3 - iw_4) - C(2iw_5).\end{aligned}$$

We take the sum and obtain:

$$\begin{aligned}0 =& - 4iw_6 - C4iw_5 - i(F_{1,3} - iF_{2,3})(w_1) - i(F_{1,3} - iF_{2,3})(w_2)+ \\ &+ i(F_{2,4} + iF_{1,4})(w_1) - i(F_{2,4} + iF_{1,4})(w_2) + i(F_{1,3} - F_{2,4} + iF_{1,4} + iF_{2,3})w_4+ \\ &+ i(F_{1,3} + F_{2,4} - iF_{1,4} + iF_{2,3})w_3 \\ =& - 4iw_6 - C4iw_5 + i(w_5 + w_6) - i(w_5 - w_6) + i(w_5 + w_6) - i(w_5 - w_6) \\ &+ iE_{-(\varepsilon_1-\varepsilon_2)}w_3 + iE_{-(\varepsilon_1+\varepsilon_2)}w_4 \\ =& - C4iw_5 + iE_{-(\varepsilon_1-\varepsilon_2)}w_3 + iE_{-(\varepsilon_1+\varepsilon_2)}w_4 = 0.\end{aligned}$$

Hence we know that:

$$0 = -C4iw_5 + iE_{-(\varepsilon_1-\varepsilon_2)}w_3 + iE_{-(\varepsilon_1+\varepsilon_2)}w_4.$$

Now we apply $\alpha_{1,2}$ and $\beta_{1,2}$, we call the weight of w_3 and w_4 (they are multiples, because they are highest weight vectors from (4.37)) with respect to $H_1 - H_2$ and $H_1 + H_2$ respectively m and n. We have:

$$\begin{aligned}0 &= -4C(w_3 - w_4) - 2mw_3 - 2nw_4, \\ 0 &= -4C(w_3 + w_4) - 2mw_3 + 2nw_4.\end{aligned}$$

From the sum of these $-8Cw_3 = 4mw_3$, from the difference $8Cw_4 = +4nw_4$. This leads to $m = -2C$, $n = 2C$. The weight should be dominant, then C must be 0, but for $C = 0$ the $\lambda-$action of Proposition 3.11 reduces to the action found in Theorem 4.3 of [BKL1]; in that case it was shown that there are no singular vectors of degree 2.

4e) We suppose $w_5 = 0$ and $w_1 \neq 0$, $w_2 \neq 0$, $w_6 \neq 0$, we deduce from $\alpha_{1,2}(w_5) = w_3 - w_4 = 0$ and $\beta_{1,2}(w_5) = w_3 + w_4 = 0$ that $w_3 = w_4 = 0$.
Using (4.11), Equation (4.33) for $a = 4, b = 1, c = 2$ reduces to:

$$\begin{aligned}0 =& - v_{1,2} + F_{1,4}v_{2,4} + F_{2,4}v_{1,4} - Cv_{3,4} \\ =& F_{1,4}(-w_1 - w_2) + F_{2,4}(iw_1 + iw_2) - C(2iw_6).\end{aligned}$$

Using (4.11), Equation (4.33) for $a = 3, b = 1, c = 2$ reduces to:

$$\begin{aligned}0 =& - v_{1,2} - F_{1,3}v_{2,3} - F_{2,3}v_{1,3} - Cv_{3,4} \\ =& - F_{1,3}(iw_1 - iw_2) - F_{2,3}(w_1 - w_2) - C(2iw_6).\end{aligned}$$

We take the sum and obtain:

$$\begin{aligned}0 =& - 4iCw_6 - i(F_{1,3} - iF_{2,3})w_1 + i(F_{1,3} - iF_{2,3})w_2 + i(F_{2,4} + iF_{1,4})(w_1) \\ &+ i(F_{2,4} + iF_{1,4})(w_2)\end{aligned}$$

$$= -4iCw_6 + iw_6 - iw_6 + iw_6 - iw_6 = -4iCw_6.$$

We obtain $C = 0$. For $C = 0$, the $\lambda-$action of Proposition 3.11 reduces to the action found in Theorem 4.3 of [BKL1]; in that case it was shown that there are no singular vectors of degree 2.

4f) We suppose that $w_6 = 0$ and $w_1 \neq 0$, $w_2 \neq 0$, $w_5 \neq 0$. We deduce, from $\alpha_{1,2}(w_6) = -w_3 - w_4 = 0$ and $\beta_{1,2}(w_6) = -w_3 + w_4 = 0$, that $w_3 = w_4 = 0$.
Using (4.11), Equation (4.33) for $a = 1, b = 3, c = 4$ reduces to:

$$\begin{aligned} 0 =& - v_{3,4} - F_{1,3}v_{1,4} - F_{1,4}v_{1,3} - Cv_{1,2} \\ =& - F_{1,3}(iw_1 + iw_2) - F_{1,4}(w_1 - w_2) - C(2iw_5). \end{aligned}$$

Using (4.11), Equation (4.33) for $a = 2, b = 3, c = 4$ reduces to:

$$\begin{aligned} 0 =& - v_{3,4} + F_{2,3}v_{2,4} + F_{2,4}v_{2,3} - Cv_{1,2} \\ =& + F_{2,3}(-w_1 - w_2) + F_{2,4}(iw_1 - iw_2) - C(2iw_5). \end{aligned}$$

We take the sum and obtain:

$$\begin{aligned} 0 =& - C4iw_5 - i(F_{1,3} - iF_{2,3})(w_1) - i(F_{1,3} - iF_{2,3})(w_2)+ \\ &+ i(F_{2,4} + iF_{1,4})(w_1) - i(F_{2,4} + iF_{1,4})(w_2) \\ =& - C4iw_5 + iw_5 - iw_5 + w_5 - iw_5 \\ =& - C4iw_5. \end{aligned}$$

We obtain $C = 0$. For $C = 0$, the $\lambda-$action of Proposition 3.11 reduces to the action found in Theorem 4.3 of [BKL1]; in that case it was shown that there are no singular vectors of degree 2.

□

4.2 Vectors of degree 3

The aim of this section is to classify all singular vectors of degree 3.
We have that, for a singular $\vec{m}$ vector of degree 3, $T(\vec{m})$ is of the form:

$$T(\vec{m}) = \Theta(\sum_i \eta_{(i)^c} \otimes v_{i,1}) + (\sum_i \eta_i \otimes v_{i,0}). \tag{4.38}$$

We write $\vec{m}$ as:

$$\begin{aligned} \vec{m} =&(\eta_2 + i\eta_1)(\eta_2 - i\eta_1)(\eta_4 + i\eta_3) \otimes w_1 + (\eta_2 + i\eta_1)(\eta_2 - i\eta_1)(\eta_4 - i\eta_3) \otimes w_2+ \\ &(\eta_4 + i\eta_3)(\eta_4 - i\eta_3)(\eta_2 + i\eta_1) \otimes w_3 + (\eta_4 + i\eta_3)(\eta_4 - i\eta_3)(\eta_2 - i\eta_1) \otimes w_4+ \\ &\Theta(\eta_2 + i\eta_1) \otimes w_5 + \Theta(\eta_2 - i\eta_1) \otimes w_6 + \Theta(\eta_4 + i\eta_3) \otimes w_7 + \Theta(\eta_4 - i\eta_3) \otimes w_8 \\ =&(2\Theta i\eta_3 + 2\Theta\eta_4 - 2\eta_1\eta_2\eta_3 + 2i\eta_1\eta_2\eta_4) \otimes w_1 + (-2i\Theta\eta_3 + 2\Theta\eta_4 + 2\eta_1\eta_2\eta_3 + 2i\eta_1\eta_2\eta_4) \otimes w_2+ \\ &(2i\Theta\eta_1 + 2\Theta\eta_2 - 2\eta_1\eta_3\eta_4 + 2i\eta_2\eta_3\eta_4) \otimes w_3 + (-2i\Theta\eta_1 + 2\Theta\eta_2 + 2\eta_1\eta_3\eta_4 + 2i\eta_2\eta_3\eta_4) \otimes w_4+ \\ &\Theta(\eta_2 + i\eta_1) \otimes w_5 + \Theta(\eta_2 - i\eta_1) \otimes w_6 + \Theta(\eta_4 + i\eta_3) \otimes w_7 + \Theta(\eta_4 - i\eta_3) \otimes w_8. \end{aligned} \tag{4.39}$$

Keeping in mind the relation between $\vec{m}$ and $T(\vec{m})$, we have:

$$v_{1,0} = 2iw_3 + 2iw_4, \tag{4.40}$$

$$\begin{aligned}
v_{2,0} &= 2w_3 - 2w_4,\\
v_{3,0} &= 2iw_1 + 2iw_2,\\
v_{4,0} &= 2w_1 - 2w_2,\\
v_{1,1} &= -2iw_3 + 2iw_4 - iw_5 + iw_6,\\
v_{2,1} &= 2w_3 + 2w_4 + w_5 + w_6,\\
v_{3,1} &= -2iw_1 + 2iw_2 - iw_7 + iw_8,\\
v_{4,1} &= 2w_1 + 2w_2 + w_7 + w_8.
\end{aligned}$$

Indeed, let us show for example one of the previous equations. In (4.38), let us consider $\eta_2 \otimes v_{2,0}$. We have that η_2 is the Hodge dual of $-\eta_1\eta_3\eta_4$. In (4.39), the terms in $\eta_1\eta_3\eta_4$ are:

$$-2\eta_1\eta_3\eta_4 \otimes w_3 + 2\eta_1\eta_3\eta_4 \otimes w_4,$$

hence $v_{2,0} = 2w_3 - 2w_4$. Analogously for $v_{1,0}, v_{3,0}$ and $v_{4,0}$. Moreover in (4.38), let us consider, for example, $\Theta\eta_{(1)^c} \otimes v_{1,1} = \Theta\eta_2\eta_3\eta_4 \otimes v_{1,1}$. We have that $\Theta\eta_2\eta_3\eta_4$ is the Hodge dual of $-\Theta\eta_1$. In (4.39), the terms in $\Theta\eta_1$ are:

$$2i\Theta\eta_1 \otimes w_3 - 2i\Theta\eta_1 \otimes w_4 + i\Theta\eta_1 \otimes w_5 - i\Theta\eta_1 \otimes w_6,$$

hence $v_{1,1} = -2iw_3 + 2iw_4 - iw_5 + iw_6$. Analogously for $v_{2,1}, v_{3,1}$ and $v_{4,1}$.
In the following lemma we write explicitly the relations of Proposition 4.17 for a vector as in formula (4.38).

Lemma 4.23. *Let $\vec{m} \in \operatorname{Ind} F$ be a vector, such that $T(\vec{m})$ is written as in formula* (4.38).
1) *Condition **S1** reduces to the following relation for $f = \xi_L$ with $|L| = 0, 1$:*

$$\begin{aligned}
0 = &\sum_i \Big[-\sum_{l<k} (f\xi_l\xi_k \star \eta_i \otimes F_{l,k}v_{i,0}) - \chi_{|L|=1}\, \varepsilon_L\, \xi_{L^c} \star \eta_i \otimes Cv_{i,0}) \Big] \qquad (4.41)\\
&+ \sum_i \Big[f \star \eta_{(i)^c} \otimes E_{00}v_{i,1} - (-1)^{p(f)} \sum_{l=1}^{4} \partial_l((f\xi_l) \star \eta_{(i)^c}) \otimes v_{i,1} + (-1)^{p(f)} (\sum_{l\neq k} ((\partial_l f)\xi_k) \star \eta_{(i)^c} \otimes F_{l,k}v_{i,1})\\
&+ \chi_{|L|=2}\, \varepsilon_L\, \xi_{L^c} \star \eta_{(i)^c} \otimes Cv_{i,1}) \Big].
\end{aligned}$$

2) *Condition **S2** reduces to the following system of relations.*
For $f = \xi_L$ with $|L| = 1, 2, 3$:

$$\begin{aligned}
0 = &\sum_i \Big[f \star \eta_i \otimes E_{00}v_{i,0} - (-1)^{p(f)} \sum_{l=1}^{4} \partial_l((f\xi_l) \star \eta_i) \otimes v_{i,0} + (-1)^{p(f)} (\sum_{l\neq k} ((\partial_l f)\xi_k) \star \eta_i \otimes F_{l,k}v_{i,0}) \qquad (4.42)\\
&+ \chi_{|L|=2}\, \varepsilon_L\, \xi_{L^c} \star \eta_i \otimes Cv_{i,0}) \Big] + \sum_i \Big[-(-1)^{p(f)} \sum_{l=1}^{4} ((\partial_l f) \star (\partial_l \eta_{(i)^c})) \otimes v_{i,1}\\
&- \sum_{r<s} ((\partial_r\partial_s f) \star \eta_{(i)^c} \otimes F_{r,s}v_{i,1}) + \chi_{|L|=3}\, \varepsilon_L\, \xi_{L^c} \star \eta_{(i)^c} \otimes Cv_{i,1} \Big].
\end{aligned}$$

For $f = \xi_L$ with $|L| = 1$:

$$0 = \sum_i \Big[f \star \eta_{(i)^c} \otimes E_{00}v_{i,1} - (-1)^{p(f)} \sum_{l=1}^{4} \partial_l((f\xi_l) \star \eta_{(i)^c}) \otimes v_{i,1} + (-1)^{p(f)} (\sum_{l\neq k} ((\partial_l f)\xi_k) \star \eta_{(i)^c} \otimes F_{l,k}v_{i,1}) \qquad (4.43)$$

$$+ \chi_{|L|=2}\,\varepsilon_L\,\xi_{L^c} \star \eta_{(i)^c} \otimes Cv_{i,1})\Big] + \sum_i \Big[(|f|-2)(f \star \eta_{(i)^c}) \otimes v_{i,1}\Big].$$

*3) Condition **S3** reduces to the following system of relations.*
For $f = \xi_L$ with $|L| = 3, 4$ or $f \in B_{\mathfrak{so}(4)}$:

$$0 = \sum_i (-1)^{(|f|(|f|+1)/2)+|f|}\Big[-(-1)^{p(f)}\sum_{l=1}^{4}((\partial_l f) \star (\partial_l \eta_i)) \otimes v_{i,0} - \sum_{r<s}((\partial_r\partial_s f) \star \eta_i \otimes F_{r,s}v_{i,0}) \tag{4.44}$$
$$+ \chi_{|L|=3}\,\varepsilon_L\,\xi_{L^c} \star \eta_i \otimes Cv_{i,0}\Big] - \sum_i \chi_{|L|=4}\varepsilon_L \eta_{(i)^c} \otimes Cv_{i,1}.$$

For $f = \xi_L$ with $|L| = 3$ or $f \in B_{\mathfrak{so}(4)}$:

$$0 = \sum_i \Big[(|f|-2)(f \star \eta_i) \otimes v_{i,0}\Big] + \sum_i \Big[-(-1)^{p(f)}\sum_{l=1}^{4}((\partial_l f) \star (\partial_l \eta_{(i)^c})) \otimes v_{i,1} \tag{4.45}$$
$$- \sum_{r<s}((\partial_r\partial_s f) \star \eta_{(i)^c} \otimes F_{r,s}v_{i,1}) + \chi_{|L|=3}\,\varepsilon_L\,\xi_{L^c} \star \eta_{(i)^c} \otimes Cv_{i,1}\Big].$$

Lemma 4.24. *Let $\vec{m} \in \operatorname{Ind} F$ be a vector such that $T(\vec{m})$ is written as in formula (4.38). The relations of Lemma 4.23 reduce to the following equations.*
For all $i \in \{1, 2, 3, 4\}$:

$$v_{i,1} = (-1)^{i+1}2Cv_{i,0}. \tag{4.46}$$

For all $r \neq s \in \{1, 2, 3, 4\}$:

$$E_{00}v_{r,0} - 2v_{r,0} + F_{s,r}v_{s,0} = 0. \tag{4.47}$$

Moreover C (resp. E_{00}) acts as multiplication by $\pm\frac{1}{2}$ (resp. $\frac{5}{2}$) on F.
For all $a, b, c \in \{1, 2, 3, 4\}$ with $d = (a, b, c)^c$:

$$v_{c,0} + F_{a,c}v_{a,0} + F_{b,c}v_{b,0} = 0, \tag{4.48}$$
$$F_{b,c}v_{d,0} - \varepsilon_{(a,b,c)}Cv_{a,0} = 0. \tag{4.49}$$

For all $a, b, c \in \{1, 2, 3, 4\}$ with $d = (a, b, c)^c$:

$$-\varepsilon_{(a,b,c)}v_{d,0} + (-1)^c F_{a,b}v_{c,1} - (-1)^b F_{a,c}v_{b,1} + (-1)^a F_{b,c}v_{a,1} + C\varepsilon_{(a,b,c)}(-1)^d v_{d,1} = 0. \tag{4.50}$$

For all $a, b \in \{1, 2, 3, 4\}$ with $(c, d) = (a, b)^c$, if we let $k = c, d$ and $s \neq a, b, k$:

$$E_{00}v_{k,0} - v_{k,0} + F_{a,k}v_{a,0} + F_{b,k}v_{b,0} + (-1)^{\chi_{a<k<b}}F_{a,b}v_{s,1} = 0, \tag{4.51}$$
$$- F_{b,c}v_{d,0} + F_{b,d}v_{c,0} + \varepsilon_{(a,b)}Cv_{a,0} + (-1)^{a+b-1}(-1)^{\chi_{c<a<d}}v_{a,1} + (-1)^{\chi_{c<a<d}}F_{a,b}v_{b,1} = 0. \tag{4.52}$$

For all $a, b, c \in \{1, 2, 3, 4\}$ with $a < b < c$ and $(d) = (a, b, c)^c$:

$$0 =(-1)^d E_{00}v_{d,0} + (-1)^d F_{a,d}v_{a,0} + (-1)^d F_{b,d}v_{b,0} + (-1)^d F_{c,d}v_{c,0} \tag{4.53}$$
$$+ (-1)^{c-1}F_{a,b}v_{c,1} - (-1)^{b-1}F_{a,c}v_{b,1} + (-1)^{a-1}F_{b,c}v_{a,1} - Cv_{d,1}.$$

For all $a \in \{1,2,3,4\}$ *with* $(b,c,d)=(a)^c$*:*

$$(-1)^a v_{a,1} - (-1)^a E_{00} v_{a,1} + (-1)^b F_{a,b} v_{b,1} + (-1)^c F_{a,c} v_{c,1} + (-1)^d F_{a,d} v_{d,1} = 0. \tag{4.54}$$

For all $a \in \{1,2,3,4\}$ *with* $(b,c,d)=(a)^c$*:*

$$-F_{b,c} v_{d,0} + F_{b,d} v_{c,0} - F_{c,d} v_{b,0} + E_{00} v_{a,1} - v_{a,1} = 0. \tag{4.55}$$

For all $a \in \{1,2,3,4\}$ *with* $(b,c,d)=(a)^c$*:*

$$\begin{aligned}0 =&(-1)^a F_{b,c} v_{d,0} + (-1)^{a-1} F_{b,d} v_{c,0} + (-1)^a F_{c,d} v_{b,0} + \varepsilon_{(a)} (-1)^{a-1} \otimes C v_{a,0} + (-1)^{a-1} E_{00} v_{a,1} \\ &+ (-1)^b F_{a,b} v_{b,1} + (-1)^c F_{a,c} v_{c,1} + (-1)^d F_{a,d} v_{d,1}.\end{aligned} \tag{4.56}$$

Finally:

$$\begin{aligned}
&\alpha_{1,2}(v_{1,0}) = -v_{3,0}, && \beta_{1,2}(v_{1,0}) = -i v_{4,0}, \\
&\alpha_{1,2}(v_{2,0}) = i v_{3,0}, && \beta_{1,2}(v_{2,0}) = -v_{4,0}, \\
&\alpha_{1,2}(v_{3,0}) = v_{1,0} - i v_{2,0}, && \beta_{1,2}(v_{3,0}) = 0, \\
&\alpha_{1,2}(v_{4,0}) = 0 && \beta_{1,2}(v_{4,0}) = i v_{1,0} + v_{2,0},
\end{aligned} \tag{4.57}$$

where $\alpha_{1,2}$ *and* $\beta_{1,2}$ *are defined by* (3.1) *and* (3.2)*.*

Proof. We consider the difference between (4.41) and (4.43) for $f=\xi_b$. We denote by $(a,c,d)=(b)^c$. We have that:

$$-\xi_b \star \eta_{(b)^c} \otimes v_{b,1} = \Big(-\sum_{i=1}^4 \sum_{l<k} \xi_b \xi_l \xi_k \star \eta_i \otimes F_{l,k} v_{i,0} - \varepsilon_{(b)} (\xi_a \xi_c \xi_d) \star \eta_b \otimes C v_{b,0} \Big).$$

It is equivalent to:

$$\xi_b \star \eta_{(b)^c} \otimes v_{b,1} = \sum_{l<k,l,k\neq b} \xi_b \xi_l \xi_k \star \eta_{(b,l,k)^c} \otimes F_{l,k} v_{(b,l,k)^c,0} - \varepsilon_{(b)} \eta_b \eta_a \eta_c \eta_d \otimes C v_{b,0}. \tag{4.58}$$

Let us focus on Equation (4.42) for $f=\xi_s$ with $s \neq b$. We have:

$$0 = \sum_{i=1}^4 \partial_s \eta_{(i)^c} \otimes v_{i,1} + \sum_{i=1}^4 \xi_s \star \eta_i \otimes E_{00} v_{i,0} + \sum_{i=1}^4 \sum_{l=1}^4 \partial_l ((\xi_s \xi_l) \star \eta_i) \otimes v_{i,0} - \sum_{i=1}^4 \sum_{l\neq s} \xi_l \star \eta_i \otimes F_{s,l} v_{i,0}. \tag{4.59}$$

The terms in $\eta_{(s,b)^c}$ of this equation are:

$$\partial_s \eta_{(b)^c} \otimes v_{b,1} - \sum_{l\neq s,b} \xi_l \star \eta_{(s,b,l)^c} \otimes F_{s,l} v_{(s,b,l)^c,0} = 0.$$

We take the sum over $s \neq b$ and, as in [BKL1], using (4.58) we obtain:

$$\begin{aligned}
0 =& \sum_{s\neq b} \xi_s \star \big(\partial_s \eta_{(b)^c}\big) \otimes v_{b,1} - \sum_{s\neq b} \sum_{l\neq s,b} (\xi_s \xi_l) \star \eta_{(s,b,l)^c} \otimes F_{s,l} v_{(s,b,l)^c,0} \\
=& 3\eta_{(b)^c} \otimes v_{b,1} - 2\big(\sum_{s<l,s,l\neq b} (\xi_s \xi_l) \star \eta_{(s,b,l)^c} \otimes F_{s,l} v_{(s,b,l)^c,0} \big) \\
=& (3-2)\eta_{(b)^c} \otimes v_{b,1} - 2\varepsilon_{(b)} \eta_{(b)^c} \otimes C v_{b,0}
\end{aligned}$$

$$=\eta_{(b)^c} \otimes (v_{b,1} - 2\varepsilon_{(b)} C v_{b,0}).$$

That is:

$$\begin{aligned}
v_{1,1} &= 2Cv_{1,0},\\
v_{2,1} &= -2Cv_{2,0},\\
v_{3,1} &= 2Cv_{3,0},\\
v_{4,1} &= -2Cv_{4,0}.
\end{aligned}$$

Given $r \neq s \in \{1,2,3,4\}$, the terms in $\eta_s\eta_r$ of (4.59) are:

$$\eta_s\eta_r \otimes E_{00}v_{r,0} + \sum_{l\neq s,r} \partial_l((\xi_s\xi_l)\star\eta_r)\otimes v_{r,0} + \eta_s\eta_r \otimes F_{s,r}v_{s,0} = 0.$$

This condition is equivalent to:

$$E_{00}v_{r,0} - 2v_{r,0} + F_{s,r}v_{s,0} = 0.$$

Using (4.46), Equation (4.44) for $f = \xi_*$ is:

$$\begin{aligned}
0 = &- \sum_{i=1}^{4}\sum_{l=1}^{4} \partial_l(\xi_*)\star\partial_l(\eta_i)\otimes v_{i,0} - \sum_{r<s}\sum_{i=1}^{4} \partial_r\partial_s(\xi_*)\star\eta_i \otimes F_{r,s}v_{i,0} - C\sum_i \eta_{(i)^c}\otimes v_{i,1}\\
= &- \eta_2\eta_3\eta_4 \otimes v_{1,0} + \eta_1\eta_3\eta_4 \otimes v_{2,0} - \eta_1\eta_2\eta_4 \otimes v_{3,0} + \eta_1\eta_2\eta_3 \otimes v_{4,0} + \eta_3\eta_4\eta_1 \otimes F_{1,2}v_{1,0} + \eta_3\eta_4\eta_2 \otimes F_{1,2}v_{2,0}\\
&- \eta_2\eta_4\eta_1 \otimes F_{1,3}v_{1,0} - \eta_2\eta_4\eta_3 \otimes F_{1,3}v_{3,0} + \eta_2\eta_3\eta_1 \otimes F_{1,4}v_{1,0} + \eta_2\eta_3\eta_4 \otimes F_{1,4}v_{4,0} + \eta_1\eta_4\eta_2 \otimes F_{2,3}v_{2,0}\\
&+ \eta_1\eta_4\eta_3 \otimes F_{2,3}v_{3,0} + \eta_1\eta_2\eta_3 \otimes F_{3,4}v_{3,0} + \eta_1\eta_2\eta_4 \otimes F_{3,4}v_{4,0} - \eta_1\eta_3\eta_2 \otimes F_{2,4}v_{2,0} - \eta_1\eta_3\eta_4 \otimes F_{2,4}v_{4,0}\\
&- C(\eta_1\eta_2\eta_3)\otimes(-2C)v_{4,0} - C(\eta_1\eta_3\eta_4)\otimes(-2C)v_{2,0} - C(\eta_1\eta_2\eta_4)\otimes(2C)v_{3,0} - C(\eta_2\eta_3\eta_4)\otimes(2C)v_{1,0}.
\end{aligned}$$

The coefficient of $\eta_1\eta_2\eta_3$ is:

$$v_{4,0} + F_{1,4}v_{1,0} + F_{3,4}v_{3,0} + F_{2,4}v_{2,0} + 2C^2v_{4,0} = 0. \tag{4.60}$$

Using (4.47), we obtain:

$$v_{4,0} + 2v_{4,0} - E_{00}v_{4,0} + 2v_{4,0} - E_{00}v_{4,0} + 2v_{4,0} - E_{00}v_{4,0} + 2C^2v_{4,0} = 0.$$

This is equivalent to:

$$E_{00}v_{4,0} = \frac{7+2C^2}{3}v_{4,0}. \tag{4.61}$$

The coefficient of $\eta_1\eta_2\eta_4$ is:

$$-v_{3,0} - F_{1,3}v_{1,0} - F_{2,3}v_{2,0} + F_{3,4}v_{4,0} - 2C^2v_{3,0} = 0. \tag{4.62}$$

Using (4.47), we obtain:

$$-v_{3,0} - 2v_{3,0} + E_{00}v_{3,0} - 2v_{3,0} + E_{00}v_{3,0} - 2v_{3,0} + E_{00}v_{3,0} - 2C^2v_{3,0} = 0.$$

This is equivalent to:

$$E_{00}v_{3,0} = \frac{7+2C^2}{3}v_{3,0}. \tag{4.63}$$

The coefficient of $\eta_1\eta_3\eta_4$ is:

$$v_{2,0} + F_{1,2}v_{1,0} - F_{2,3}v_{3,0} - F_{2,4}v_{4,0} + 2C^2 v_{2,0} = 0. \tag{4.64}$$

Using (4.47), we obtain:

$$v_{2,0} + 2v_{2,0} - E_{00}v_{2,0} + 2v_{2,0} - E_{00}v_{2,0} + 2v_{2,0} - E_{00}v_{2,0} + 2C^2 v_{2,0} = 0.$$

This is equivalent to:

$$E_{00}v_{2,0} = \frac{7+2C^2}{3} v_{2,0}. \tag{4.65}$$

The coefficient of $\eta_2\eta_3\eta_4$ is:

$$-v_{1,0} + F_{1,2}v_{2,0} + F_{1,3}v_{3,0} + F_{1,4}v_{4,0} - 2C^2 v_{1,0} = 0. \tag{4.66}$$

Using (4.47) we obtain:

$$-v_{1,0} - 2v_{1,0} + E_{00}v_{1,0} - 2v_{1,0} + E_{00}v_{1,0} - 2v_{1,0} + E_{00}v_{1,0} - 2C^2 v_{1,0} = 0.$$

This is equivalent to:

$$E_{00}v_{1,0} = \frac{7+2C^2}{3} v_{1,0}. \tag{4.67}$$

Therefore E_{00} acts as $\frac{7+2C^2}{3}$.
Let us analyze Equation (4.44) for $f = \xi_a\xi_b\xi_c$. We denote $(d) = (a,b,c)^c$. We obtain:

$$\sum_{i=1}^{4}\sum_{l=1}^{4} \partial_l(\xi_a\xi_b\xi_c) \star \partial_l(\eta_i) \otimes v_{i,0} - \sum_{r<s} \partial_r\partial_s(\xi_a\xi_b\xi_c) \star \eta_i \otimes F_{r,s}v_{i,0} + \sum_i \varepsilon_{(a,b,c)}\xi_d \star \eta_i \otimes Cv_{i,0} = 0.$$

Looking at the coefficient of $\eta_i\eta_j$ for every $i,j \in \{a,b,c,d\}$, we obtain:

$$\begin{aligned} v_{c,0} + F_{a,c}v_{a,0} + F_{b,c}v_{b,0} &= 0, \\ F_{b,c}v_{d,0} - \varepsilon_{(a,b,c)}Cv_{a,0} &= 0. \end{aligned}$$

Equation (4.48), for $a = 2, b = 3, c = 1$, is:

$$v_{1,0} - F_{1,3}v_{3,0} - F_{1,2}v_{2,0} = 0.$$

Using (4.47) and the value of E_{00}, we get:

$$\begin{aligned} 0 =& v_{1,0} - F_{1,3}v_{3,0} - F_{1,2}v_{2,0} \\ =& v_{1,0} - 2\frac{1+2C^2}{3}v_{1,0} = \frac{1-4C^2}{3}v_{1,0}. \end{aligned}$$

From this we have that $C = \pm\frac{1}{2}$ and E_{00} acts as $\frac{5}{2}$.
Equation (4.45) for $f = \xi_a\xi_b\xi_c$, with $d = (a,b,c)^c$, reduces to:

$$0 = \eta_a\eta_b\eta_c\eta_d \otimes v_{d,0} - \sum_i^4 \sum_{r<s} \partial_r\partial_s(\xi_a\xi_b\xi_c) \star \eta_{(i)^c} \otimes F_{r,s}v_{i,1} + \varepsilon_{(a,b,c)}\xi_d \star \eta_{(d)^c} \otimes Cv_{d,1}.$$

The coefficient of η_* is:

$$-\varepsilon_{(a,b,c)}v_{d,0} + (-1)^c F_{a,b}v_{c,1} - (-1)^b F_{a,c}v_{b,1} + (-1)^a F_{b,c}v_{a,1} + C\varepsilon_{(a,b,c)}(-1)^d v_{d,1} = 0.$$

Equation (4.42) for $f = \xi_a\xi_b$, with $(c,d) = (a,b)^c$, reduces to:

$$\begin{aligned} 0 =& \eta_a\eta_b\eta_c \otimes E_{00}v_{c,0} + \eta_a\eta_b\eta_d \otimes E_{00}v_{d,0} - \sum_i \sum_{l=1}^4 \partial_l((\xi_a\xi_b\xi_l) \star \eta_i) \otimes v_{i,0} + \\ &+ \sum_i \sum_{l \neq k} (\partial_l(\xi_a\xi_b)\xi_k) \star \eta_i \otimes F_{l,k}v_{i,0} + \varepsilon_{(a,b)} \sum_i \xi_c\xi_d \star \eta_i \otimes Cv_{i,0} + \\ &- \sum_i \sum_{l=1}^4 \partial_l(\xi_a\xi_b) \star \partial_l(\eta_{(i)^c}) \otimes v_{i,1} - \sum_i \sum_{r<s} \partial_r\partial_s(\xi_a\xi_b) \star \eta_{(i)^c} \otimes F_{r,s}v_{i,1}. \end{aligned}$$

The coefficient of $\eta_a\eta_b\eta_k$ for $k = c, d$ and $s \neq a, b, k$ is:

$$E_{00}v_{k,0} - v_{k,0} + F_{a,k}v_{a,0} + F_{b,k}v_{b,0} + (-1)^{\chi_{a<k<b}} F_{a,b}v_{s,1} = 0.$$

The coefficient of $\eta_a\eta_c\eta_d$ is:

$$-F_{b,c}v_{d,0} + F_{b,d}v_{c,0} + \varepsilon_{(a,b)}Cv_{a,0} + (-1)^{a+b-1}(-1)^{\chi_{c<a<d}} v_{a,1} + (-1)^{\chi_{c<a<d}} F_{a,b}v_{b,1} = 0.$$

From the coefficient of $\eta_b\eta_c\eta_d$ we obtain the same equation with reversed roles of a and b. Equation (4.42) for $f = \xi_a\xi_b\xi_c$, with $a < b < c$ and $d = (a,b,c)^c$, reduces to:

$$\begin{aligned} 0 =& \eta_a\eta_b\eta_c\eta_d \otimes E_{00}v_{d,0} - \sum_i \sum_{l \neq k} (\partial_l(\xi_a\xi_b\xi_c)\xi_k) \star \eta_i \otimes F_{l,k}v_{i,0} \\ &- \sum_i \sum_{r<s} \partial_r\partial_s(\xi_a\xi_b\xi_c) \star \eta_{(i)^c} \otimes F_{r,s}v_{i,1} + \varepsilon_{(a,b,c)}\eta_d\eta_{(d)^c} \otimes Cv_{d,1}. \end{aligned}$$

The coefficient of η_* is:

$$\begin{aligned} 0 =& (-1)^d E_{00}v_{d,0} + (-1)^d F_{a,d}v_{a,0} + (-1)^d F_{b,d}v_{b,0} + (-1)^d F_{c,d}v_{c,0} + \\ &+ (-1)^{c-1} F_{a,b}v_{c,1} - (-1)^{b-1} F_{a,c}v_{b,1} + (-1)^{a-1} F_{b,c}v_{a,1} - Cv_{d,1}. \end{aligned}$$

Equation (4.43) for $f = \xi_a$ with $(b,c,d) = (a)^c$, reduces to:

$$(-1)^a v_{a,1} - (-1)^a E_{00}v_{a,1} + (-1)^b F_{a,b}v_{b,1} + (-1)^c F_{a,c}v_{c,1} + (-1)^d F_{a,d}v_{d,1} = 0.$$

Equation (4.41) for $f = 1$ reduces to:

$$0 = -\sum_i \sum_{l<k} \xi_l\xi_k \star \eta_i \otimes F_{l,k}v_{i,0} + \sum_i \eta_{(i)^c} \otimes E_{00}v_{i,1} - \sum_i \sum_{l=1}^4 \partial_l(\xi_l \star \eta_{(i)^c}) \otimes v_{i,1}.$$

We obtain that for all $a \in \{1,2,3,4\}$, with $(b,c,d) = (a)^c$, the coefficient of $\eta_{(a)^c}$ is:

$$-F_{b,c}v_{d,0} + F_{b,d}v_{c,0} - F_{c,d}v_{b,0} + E_{00}v_{a,1} - v_{a,1} = 0.$$

Condition (4.41) for $f = \xi_a$, with $(a)^c = (b,c,d)$, reduces to:

$$0 = -\sum_i \sum_{l<k} \xi_a\xi_l\xi_k \star \eta_i \otimes F_{l,k}v_{i,0} - \varepsilon_{(a)}\eta_b\eta_c\eta_d\eta_a \otimes Cv_{a,0} + \xi_a \star \eta_{(a)^c} \otimes E_{00}v_{a,1} - \sum_i \sum_{k \neq a} \xi_k \star \eta_{(i)^c} \otimes F_{a,k}v_{i,1}.$$

The coefficient of η_* is:

$$0 = (-1)^a F_{b,c}v_{d,0} + (-1)^{a-1} F_{b,d}v_{c,0} + (-1)^a F_{c,d}v_{b,0} + \varepsilon_{(a)}(-1)^{a-1} \otimes Cv_{a,0} + (-1)^{a-1} E_{00}v_{a,1}$$

$$+ (-1)^b F_{a,b} v_{b,1} + (-1)^c F_{a,c} v_{c,1} + (-1)^d F_{a,d} v_{d,1}.$$

Let us analyze Equation (4.44) for $f = \alpha_{1,2}$:

$$\begin{aligned}
0 =& - b_0(-\xi_1\xi_3 + i\xi_2\xi_3) \\
=& - \sum_i \sum_{l=1}^{4} \partial_l(-\xi_1\xi_3 + i\xi_2\xi_3) \star \partial_l(\eta_i) \otimes v_{i,0} - \sum_i \sum_{r<s} \partial_r\partial_s(-\xi_1\xi_3 + i\xi_2\xi_3) \star \eta_i \otimes F_{r,s} v_{i,0} \\
=& \eta_3 \otimes v_{1,0} - \eta_1 \otimes \alpha_{1,2} v_{1,0} - i\eta_3 \otimes v_{2,0} - \eta_2 \otimes \alpha_{1,2} v_{2,0} + (-\eta_1 + i\eta_2) \otimes v_{3,0} - \eta_3 \otimes \alpha_{1,2} v_{3,0} - \eta_4 \otimes \alpha_{1,2} v_{4,0} \\
=& \eta_1 \otimes (-\alpha_{1,2} v_{1,0} - v_{3,0}) + \eta_2 \otimes (i v_{3,0} - \alpha_{1,2} v_{2,0}) + \eta_3 \otimes (v_{1,0} - i v_{2,0} - \alpha_{1,2} v_{3,0}) + \eta_4 \otimes (-\alpha_{1,2} v_{4,0}).
\end{aligned}$$

Therefore, we have:

$$\begin{aligned}
\alpha_{1,2}(v_{1,0}) &= -v_{3,0}, \\
\alpha_{1,2}(v_{2,0}) &= i v_{3,0}, \\
\alpha_{1,2}(v_{3,0}) &= v_{1,0} - i v_{2,0}, \\
\alpha_{1,2}(v_{4,0}) &= 0.
\end{aligned}$$

For $f = \beta_{1,2}$, Equation (4.44) reduces to:

$$\begin{aligned}
0 =& - b_0(-\xi_2\xi_4 - i\xi_1\xi_4) \\
=& - \sum_i \sum_{l=1}^{4} \partial_l(-\xi_2\xi_4 - i\xi_1\xi_4) \star \partial_l(\eta_i) \otimes v_{i,0} - \sum_i \sum_{r<s} \partial_r\partial_s(-\xi_2\xi_4 - i\xi_1\xi_4) \star \eta_i \otimes F_{r,s} v_{i,0} \\
=& i\eta_4 \otimes v_{1,0} - \eta_1 \otimes \beta_{1,2} v_{1,0} + \eta_4 \otimes v_{2,0} - \eta_2 \otimes \beta_{1,2} v_{2,0} - \eta_3 \otimes \beta_{1,2} v_{3,0} + (-\eta_2 - i\eta_1) \otimes v_{4,0} - \eta_4 \otimes \beta_{1,2} v_{4,0} \\
=& \eta_1 \otimes (-\beta_{1,2} v_{1,0} - i v_{4,0}) + \eta_2 \otimes (-v_{4,0} - \beta_{1,2} v_{2,0}) + \eta_3 \otimes (-\beta_{1,2} v_{3,0}) + \eta_4 \otimes (i v_{1,0} + v_{2,0} - \beta_{1,2} v_{4,0}).
\end{aligned}$$

Therefore, we have:

$$\begin{aligned}
\beta_{1,2}(v_{1,0}) &= -i v_{4,0}, \\
\beta_{1,2}(v_{2,0}) &= -v_{4,0}, \\
\beta_{1,2}(v_{3,0}) &= 0, \\
\beta_{1,2}(v_{4,0}) &= i v_{1,0} + v_{2,0}.
\end{aligned}$$

Equation (4.45) for $f = \alpha_{1,2}$ reduces to:

$$\begin{aligned}
0 =& - b_1(-\xi_1\xi_3 + i\xi_2\xi_3) \\
=& - \sum_i \sum_{l=1}^{4} \partial_l(-\xi_1\xi_3 + i\xi_2\xi_3) \star \partial_l(\eta_{(i)^c}) \otimes v_{i,1} - \sum_i \sum_{r<s} \partial_r\partial_s(-\xi_1\xi_3 + i\xi_2\xi_3) \star \eta_{(i)^c} \otimes F_{r,s} v_{i,1} \\
=& \eta_1\eta_2\eta_4 \otimes v_{1,1} - \eta_2\eta_3\eta_4 \otimes \alpha_{1,2} v_{1,1} + i\eta_1\eta_2\eta_4 v_{2,1} - \eta_1\eta_3\eta_4 \otimes \alpha_{1,2} v_{2,1} + (-\eta_2\eta_3\eta_4 - i\eta_1\eta_3\eta_4) \otimes v_{3,1} \\
&- \eta_1\eta_2\eta_4 \otimes \alpha_{1,2} v_{3,1} - \eta_1\eta_2\eta_3 \otimes \alpha_{1,2} v_{4,1}.
\end{aligned}$$

Therefore, we have:

$$\begin{aligned}
\alpha_{1,2} v_{1,1} &= -v_{3,1}, \\
\alpha_{1,2} v_{2,1} &= -i v_{3,1}, \\
\alpha_{1,2} v_{3,1} &= v_{1,1} + i v_{2,1}, \\
\alpha_{1,2} v_{4,1} &= 0.
\end{aligned}$$

These relations are coherent with the previous equations and (4.46).
Equation (4.45) for $f = \beta_{1,2}$ reduces to:

$$\begin{aligned} 0 =& - b_1(-\xi_2\xi_4 - i\xi_1\xi_4) \\ =& - \sum_i \sum_{l=1}^{4} \partial_l(-\xi_2\xi_4 - i\xi_1\xi_4) \star \partial_l(\eta_{(i)^c}) \otimes v_{i,1} - \sum_i \sum_{r<s} \partial_r\partial_s(-\xi_2\xi_4 - i\xi_1\xi_4) \star \eta_{(i)^c} \otimes F_{r,s}v_{i,1} \\ =& - i\eta_1\eta_2\eta_3 \otimes v_{1,1} - \eta_2\eta_3\eta_4 \otimes \beta_{1,2}v_{1,1} + \eta_1\eta_2\eta_3 \otimes v_{2,1} - \eta_1\eta_3\eta_4 \otimes \beta_{1,2}v_{2,1} - \eta_1\eta_2\eta_4 \otimes \beta_{1,2}v_{3,1} \\ &+ (-\eta_1\eta_3\eta_4 + i\eta_2\eta_3\eta_4)v_{4,1} - \eta_1\eta_2\eta_3 \otimes \beta_{1,2}v_{4,1}. \end{aligned}$$

Therefore, we have:

$$\begin{aligned} \beta_{1,2}v_{1,1} &= iv_{4,1}, \\ \beta_{1,2}v_{2,1} &= -v_{4,1}, \\ \beta_{1,2}v_{3,1} &= 0, \\ \beta_{1,2}v_{4,1} &= -iv_{1,1} + v_{2,1}. \end{aligned}$$

These relations are coherent with the previous equations and (4.46). □

Remark 4.25. Let us point out that relations (4.46) are equivalent to the following, using notation (4.40):

$$\begin{aligned} -2iw_3 + 2iw_4 - iw_5 + iw_6 &= 2C(2iw_3 + 2iw_4), \\ 2w_3 + 2w_4 + w_5 + w_6 &= -2C(2w_3 - 2w_4), \\ -2iw_1 + 2iw_2 - iw_7 + iw_8 &= 2C(2iw_1 + 2iw_2), \\ 2w_1 + 2w_2 + w_7 + w_8 &= -2C(2w_1 - 2w_2). \end{aligned}$$

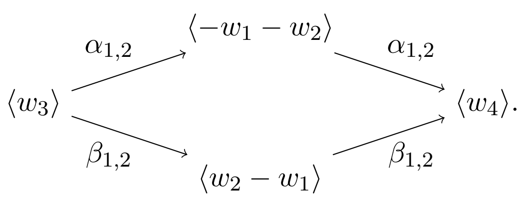
Thus, we obtain:

$$\begin{aligned} w_5 &= -(2 + 4C)w_3, \\ w_6 &= -(2 - 4C)w_4, \\ w_7 &= -(2 + 4C)w_1, \\ w_8 &= -(2 - 4C)w_2. \end{aligned} \tag{4.68}$$

Equations (4.57) are equivalent to the following, using notation (4.40):

$$\begin{aligned} \alpha_{1,2}(w_1) &= w_4, & \beta_{1,2}(w_1) &= -w_4, \\ \alpha_{1,2}(w_2) &= w_4, & \beta_{1,2}(w_2) &= w_4, \\ \alpha_{1,2}(w_3) &= -w_1 - w_2, & \beta_{1,2}(w_3) &= -w_1 + w_2, \\ \alpha_{1,2}(w_4) &= 0, & \beta_{1,2}(w_4) &= 0. \end{aligned} \tag{4.69}$$

We represent these relations with the following drawings:



Proof of Theorem 4.6. Let us analyze the following cases.

1) Let us suppose $w_4 = 0$. This splits in the following subcases.

1a) We suppose $w_2 = w_4 = 0$ and $w_1 \neq 0$, $w_3 \neq 0$. By Equations (4.69), we have that w_1 is a highest weight vector. Let us compute its weight. From (4.40) we have:

$$\begin{aligned} v_{1,0} &= 2iw_3, \\ v_{2,0} &= 2w_3, \\ v_{3,0} &= 2iw_1, \\ v_{4,0} &= 2w_1, \\ v_{1,1} &= 2C(2iw_3), \\ v_{2,1} &= -2C(2w_3), \\ v_{3,1} &= 2C(2iw_1), \\ v_{4,1} &= -2C(2w_1). \end{aligned}$$

By Equation (4.47) for $a = 3, b = 4$ we obtain:

$$F_{3,4}v_{3,0} = 2v_{4,0} - E_{00}v_{4,0}.$$

It is equivalent to:

$$F_{3,4}(2iw_1) = 4w_1 - E_{00}2w_1 = (4-5)w_1.$$

Therefore, we have $H_2w_1 = -\frac{1}{2}w_1$.
By Equation (4.49) for $a = 3, b = 1, c = 2$, we obtain:

$$F_{1,2}v_{4,0} - Cv_{3,0} = 0.$$

It is equivalent to:

$$F_{1,2}(2w_1) - C(2iw_1) = 0.$$

Therefore, we have $H_1w_1 = -Cw_1$. Since $H_1 + H_2$ acts as a non negative integer on w_1, we obtain $C = -\frac{1}{2}$, and the highest weight of w_1 with respect to h_x, h_y, E_{00}, C is $(1, 0, \frac{5}{2}, -\frac{1}{2})$.
By Equations (4.69) we know that $\alpha_{1,2}(w_3) + \beta_{1,2}(w_3) = E_{\varepsilon_1-\varepsilon_2}w_3 = -2w_1$. Hence $w_3 = \frac{1}{2}E_{-(\varepsilon_1-\varepsilon_2)}w_1$. Moreover, by Equations (4.68) and $C = -\frac{1}{2}$, we have $w_5 = w_6 = w_7 = w_8 = 0$.
All the other equations of Lemma 4.24 are verified by this choice of $v_{1,0}, v_{2,0}, v_{3,0}, v_{4,0}, v_{1,1}, v_{2,1}, v_{3,1}, v_{4,1}$.
We have therefore obtained, using notation (4.1), the following singular vector in $M(1, 0, \frac{5}{2}, -\frac{1}{2})$:

$$\vec{m}_{3a} = w_{11}w_{22}w_{21} \otimes x_1 + w_{21}w_{12}w_{11} \otimes x_2.$$

1b) We suppose $w_1 = w_4 = 0$ and $w_2 \neq 0$, $w_3 \neq 0$. By Equations (4.69), we have that w_2 is a highest weight vector, let us compute its weight. From (4.40) we have:

$$\begin{aligned} v_{1,0} &= 2iw_3, \\ v_{2,0} &= 2w_3, \\ v_{3,0} &= 2iw_2, \\ v_{4,0} &= -2w_2, \\ v_{1,1} &= 2C(2iw_3). \\ v_{2,1} &= -2C(2w_3). \\ v_{3,1} &= 2C(2iw_2). \end{aligned}$$

$$v_{4,1} = -2C(-2w_2).$$

By Equation (4.47) for $a = 3, b = 4$ we obtain:

$$F_{3,4}v_{3,0} = 2v_{4,0} - E_{00}v_{4,0}.$$

It is equivalent to:

$$F_{3,4}(2iw_2) = -4w_2 - E_{00}(-2w_2) = (-4 + 5)w_2.$$

Therefore we have $iF_{3,4}w_2 = \frac{1}{2}w_2$.
By Equation (4.49) for $a = 3, b = 1, c = 2$, we obtain:

$$F_{1,2}v_{4,0} - Cv_{3,0} = 0.$$

It is equivalent to:

$$F_{1,2}(-2w_2) - C(2iw_2) = 0.$$

Therefore we have $H_1w_2 = Cw_2$.
Since $H_1 - H_2$ acts as a non negative integer on w_2, we obtain $C = \frac{1}{2}$, and the highest weight of w_2 with respect to h_x, h_y, E_{00}, C is $(0, 1, \frac{5}{2}, \frac{1}{2})$.
By Equations (4.69) we know that $\alpha_{1,2}(w_3) - \beta_{1,2}(w_3) = E_{\varepsilon_1+\varepsilon_2}w_3 = -2w_2$. Hence $w_3 = \frac{1}{2}E_{-(\varepsilon_1+\varepsilon_2)}w_2$. Moreover, by Equations (4.68) and $C = \frac{1}{2}$, we have that $w_5 = -2E_{-(\varepsilon_1+\varepsilon_2)}w_2$ and $w_6 = w_7 = w_8 = 0$. All the other equations of Lemma 4.24 are verified by this choice of $v_{1,0}, v_{2,0}, v_{3,0}, v_{4,0}, v_{1,1}, v_{2,1}, v_{3,1}, v_{4,1}$.
We have therefore obtained, using notation (4.1), the following singular vector in $M(0, 1, \frac{5}{2}, \frac{1}{2})$:

$$\vec{m}_{3b} = w_{11}w_{22}w_{12} \otimes y_1 + w_{12}w_{21}w_{11} \otimes y_2.$$

1c) We suppose $w_4 = 0$ and $w_1 \neq 0$, $w_2 \neq 0$, $w_3 \neq 0$. By Equations (4.69), we know that w_1 and w_2 are highest weight vectors.
We consider equations (4.49) for $a = 3, b = 1, c = 2$ and, using (4.40), we obtain:

$$\begin{aligned} 0 =& F_{1,2}v_{4,0} - Cv_{3,0} \\ =& F_{1,2}(2w_1 - 2w_2) - C(2iw_1 + 2iw_2). \end{aligned}$$

We consider equations (4.49) for $a = 4, b = 1, c = 2$ and, using (4.40), we obtain:

$$\begin{aligned} 0 =& F_{1,2}v_{3,0} + Cv_{4,0} \\ =& F_{1,2}(2iw_1 + 2iw_2) + C(2w_1 - 2w_2). \end{aligned}$$

We take the sum between the first multiplied by i and the second, and the difference between the second and the first multiplied by i:

$$\begin{aligned} 0 &= 4H_1w_1 + 4Cw_1 \\ 0 &= 4H_1w_2 - 4Cw_2. \end{aligned}$$

This leads to $C = 0$. But, for $C = 0$, the $\lambda-$action of Proposition 3.11 reduces to the action found in Theorem 4.3 of [BKL1]; in that case it was shown that there are no singular vectors of degree 3.

1d) We suppose $w_3 = w_4 = 0$. Then, by Equations (4.69), we have that $0 = \beta_{1,2}(w_3) = -w_1 + w_2$ and $0 = \alpha_{1,2}(w_3) = -w_1 - w_2$. Hence we get $w_1 = w_2 = 0$. By Equations (4.68) we know also that $w_5 = w_6 = w_7 = w_8 = 0$. Therefore we obtain the trivial vector.

1e) We suppose $w_1 = w_2 = w_4 = 0$ and $w_3 \neq 0$. By Equations (4.68) we know also that $w_6 = w_7 = w_8 = 0$ and $w_5 = -(2+4C)w_3$. By Equations (4.69), we know that w_3 and w_5 are highest weight vectors.
We know that Equation (4.48) for $a = 2, b = 3, c = 1$ reduces to the following, using (4.40):

$$\begin{aligned} 0 =& v_{1,0} - F_{1,2}v_{2,0} - F_{1,3}v_{3,0} \\ =& 2iw_3 - 2F_{1,2}w_3. \end{aligned}$$

Therefore $H_1 w_3 = -w_3$, but this is impossible for a highest weight vector, since we know that both $h_x = H_1 - H_2$ and $h_y = H_1 + H_2$ act as nonnegative scalars on w_3, therefore $H_1 = \frac{h_x + h_y}{2}$ acts as a nonnegative scalar on w_3.

2) Let us suppose that $w_4 \neq 0$. By Equations (4.69), we have that $w_1 \neq 0$, $w_2 \neq 0$, $w_3 \neq 0$ and that w_4 is a highest weight vector.
By Equation (4.48) for $a = 1, b = 3, c = 2$ we obtain, using (4.40):

$$\begin{aligned} 0 =& - v_{2,0} + F_{2,3}v_{3,0} - F_{1,2}v_{1,0} \\ =& - 2w_3 + 2w_4 + F_{2,3}(2iw_1 + 2iw_2) - F_{1,2}(2iw_3 + 2iw_4). \end{aligned}$$

By Equation (4.48) for $a = 2, b = 3, c = 1$ we obtain, using (4.40):

$$\begin{aligned} 0 =& v_{1,0} - F_{1,2}v_{2,0} - F_{1,3}v_{3,0} \\ =& 2iw_3 + 2iw_4 - F_{1,2}(2w_3 - 2w_4) - F_{1,3}(2iw_1 + 2iw_2). \end{aligned}$$

We take the sum of the second and the first multiplied by i:

$$\begin{aligned} 0 =& 4iw_4 + (-F_{1,3} + iF_{2,3})(2iw_1 + 2iw_2) + 4F_{1,2}w_4 \\ =& 4iw_4 - 2iw_4 - 2iw_4 + 4F_{1,2}w_4. \end{aligned}$$

Therefore $H_1 w_4 = 0$. By Equation (4.49) for $a = 1, b = 3, c = 4$ we obtain, using (4.40):

$$\begin{aligned} 0 =& F_{3,4}v_{2,0} - Cv_{1,0} \\ =& F_{3,4}(2w_3 - 2w_4) - C(2iw_3 + 2iw_4). \end{aligned}$$

By Equation (4.49) $a = 2, b = 3, c = 4$ we obtain, using (4.40):

$$\begin{aligned} 0 =& F_{3,4}v_{1,0} + Cv_{2,0} \\ =& F_{3,4}(2iw_3 + 2iw_4) + C(2w_3 - 2w_4). \end{aligned}$$

We take the difference between the second and the first multiplied by i:

$$0 = 4H_2w_4 - 4Cw_4.$$

Therefore $H_2 w_4 = Cw_4$. Therefore $(H_1 - H_2)w_4 = -H_2w_4 = -Cw_4$ and $(H_1 + H_2)w_4 = H_2w_4 = Cw_4$. Hence $C = 0$. But, for $C = 0$, the $\lambda-$action of Proposition 3.11 reduces to the action found in Theorem 4.3 of [BKL1]; in that case it was shown that there are no singular vectors of degree 3.

□

4.3 Vectors of degree 1

The aim of this section is to classify singular vectors of degree 1. Let us consider a vector $\vec{m} \in \mathrm{Ind}(F)$ of degree 1 such that $T(\vec{m})$ is of the form:

$$T(\vec{m}) = \sum_i \eta_{(i)^c} \otimes v_i. \tag{4.70}$$

We write $\vec{m}$ as:

$$\vec{m} = \sum_{l=1}^{2} \left((\eta_{2l} - i\eta_{2l-1}) \otimes w_l + (\eta_{2l} + i\eta_{2l-1}) \otimes \widetilde{w}_l \right). \tag{4.71}$$

Hence for $l = 1, 2$:

$$\begin{aligned} v_{2l} &= w_l + \widetilde{w}_l, \\ v_{2l-1} &= i(w_l - \widetilde{w}_l). \end{aligned} \tag{4.72}$$

Indeed, let us show one of these relations. In (4.70), let us consider $\eta_{(1)^c} \otimes v_1$. We have that $\eta_{(1)^c} = \eta_{(2,3,4)}$ is the Hodge dual of $-\eta_1$. In (4.71), the terms in η_1 are $-i\eta_1 \otimes w_1 + i\eta_1 \otimes \widetilde{w}_1$. Hence $v_1 = i(w_1 - \widetilde{w}_1)$. The other relations in (4.72) are obtained analogously.
In the following lemma we write explicitly the relations of Proposition 4.17 for a vector as in formula (4.70).

Lemma 4.26. *Let $\vec{m} \in \mathrm{Ind}(F)$ such that $T(\vec{m})$ is written as in formula (4.70). Then relations of Proposition 4.17 reduce to the following equations.*
1) *Condition **S2** reduces to the following relation for $f = \xi_L$ with $|L| = 1$:*

$$0 = \sum_i \Big[f \star \eta_{(i)^c} \otimes E_{00} v_i - (-1)^{p(f)} \sum_{l=1}^{4} \partial_l ((f\xi_l) \star \eta_{(i)^c}) \otimes v_i + (-1)^{p(f)} (\sum_{l \neq k} ((\partial_l f)\xi_k) \star \eta_{(i)^c} \otimes F_{l,k} v_i) \tag{4.73}$$

$$+ \chi_{|L|=2}\, \varepsilon_L\, \xi_{L^c} \star \eta_{(i)^c} \otimes C v_i) \Big].$$

2) *Condition **S3** reduces to the following relation for $f = \xi_L$ with $|L| = 3$ or $f \in B_{\mathfrak{so}(4)}$:*

$$\begin{aligned} 0 = \sum_i \Big[&- (-1)^{p(f)} \sum_{l=1}^{4} ((\partial_l f) \star (\partial_l \eta_{(i)^c})) \otimes v_i - \sum_{r<s} ((\partial_r \partial_s f) \star \eta_{(i)^c} \otimes F_{r,s} v_i) \\ &+ \chi_{|L|=3}\, \varepsilon_L\, \xi_{L^c} \star \eta_{(i)^c} \otimes C v_i \Big]. \end{aligned} \tag{4.74}$$

Lemma 4.27. *Let $\vec{m} \in \mathrm{Ind}(F)$ such that $T(\vec{m})$ is written as in formula (4.70). Then relations of Lemma 4.26 reduce to the following equations.*
For all $a \in \{1, 2, 3, 4\}$:

$$0 = (-1)^a E_{00} v_a - \sum_{k \neq a} (-1)^k F_{a,k} v_k. \tag{4.75}$$

For all a, b, c and $d = (a, b, c)^c$:

$$0 = (-1)^c F_{a,b} v_c - (-1)^b F_{a,c} v_b + (-1)^a F_{b,c} v_a + \varepsilon_{(a,b,c)} (-1)^d C v_d. \tag{4.76}$$

Finally:

$$\begin{aligned} \alpha_{1,2}(v_1) &= -v_3, & \beta_{1,2}(v_1) &= iv_4, \\ \alpha_{1,2}(v_2) &= -iv_3, & \beta_{1,2}(v_2) &= -v_4, \\ \alpha_{1,2}(v_3) &= v_1 + iv_2, & \beta_{1,2}(v_3) &= 0, \\ \alpha_{1,2}(v_4) &= 0, & \beta_{1,2}(v_4) &= -iv_1 + v_2, \end{aligned} \tag{4.77}$$

where $\alpha_{1,2}$ *and* $\beta_{1,2}$ *are defined by* (3.1) *and* (3.2).

Proof. Let us consider Equation (4.73) for $a \in \{1, 2, 3, 4\}$. From coefficient of $-\eta_*$, we obtain:

$$0 = (-1)^a E_{00} v_a - \sum_{k \neq a} (-1)^k F_{a,k} v_k.$$

Let us consider Equation (4.74) for $f = \xi_a \xi_b \xi_c$ with $d = (a, b, c)^c$. We have:

$$0 = -\sum_{r<s} \sum_{i=1}^{4} \partial_r \partial_s (\xi_a \xi_b \xi_c) \star \eta_{(i)^c} \otimes F_{r,s} v_i + \varepsilon_{(a,b,c)} \eta_d \eta_a \eta_b \eta_c \otimes C v_d.$$

It is equivalent to:

$$0 = \eta_c \eta_{(c)^c} \otimes F_{a,b} v_c - \eta_b \eta_{(b)^c} \otimes F_{a,c} v_b + \eta_a \eta_{(a)^c} \otimes F_{b,c} v_a + \varepsilon_{(a,b,c)} \eta_d \eta_{(d)^c} \otimes C v_d,$$

that is:

$$0 = (-1)^c F_{a,b} v_c - (-1)^b F_{a,c} v_b + (-1)^a F_{b,c} v_a + \varepsilon_{(a,b,c)} (-1)^d C v_d.$$

Let us consider Equation (4.74) for $f = \alpha_{1,2}$. We have:

$$\begin{aligned} 0 =& -\sum_i \sum_{l=1}^{4} \partial_l (-\xi_1 \xi_3 + i\xi_2 \xi_3) \star \partial_l (\eta_{(i)^c}) \otimes v_i - \sum_i \sum_{r<s} \partial_r \partial_s (-\xi_1 \xi_3 + i\xi_2 \xi_3) \star \eta_{(i)^c} \otimes F_{r,s} v_i \\ =& -(\xi_1 - i\xi_2) \star (-\eta_2 \eta_4) \otimes v_1 - \eta_2 \eta_3 \eta_4 \otimes \alpha_{1,2} v_1 - (\xi_1 - i\xi_2) \star (-\eta_1 \eta_4) \otimes v_2 - \eta_1 \eta_3 \eta_4 \otimes \alpha_{1,2} v_2 \\ & - (-\xi_3) \star (\eta_2 \eta_4) \otimes v_3 - (i\xi_3) \star (-\eta_1 \eta_4) \otimes v_3 - \eta_1 \eta_2 \eta_4 \otimes \alpha_{1,2} v_3 - \eta_1 \eta_2 \eta_3 \otimes \alpha_{1,2} v_4 \\ =& \eta_1 \eta_2 \eta_4 \otimes (v_1 + iv_2 - \alpha_{1,2} v_3) + \eta_2 \eta_3 \eta_4 \otimes (-v_3 - \alpha_{1,2} v_1) + \eta_1 \eta_3 \eta_4 \otimes (-iv_3 - \alpha_{1,2} v_2) + \eta_1 \eta_2 \eta_3 \otimes (-\alpha_{1,2} v_4). \end{aligned}$$

Therefore:

$$\begin{aligned} \alpha_{1,2}(v_1) &= -v_3, \\ \alpha_{1,2}(v_2) &= -iv_3, \\ \alpha_{1,2}(v_3) &= v_1 + iv_2, \\ \alpha_{1,2}(v_4) &= 0. \end{aligned}$$

Let us consider Equation (4.74) for $f = \beta_{1,2}$. We have:

$$\begin{aligned} 0 =& -\sum_i \sum_{l=1}^{4} \partial_l (-\xi_2 \xi_4 - i\xi_1 \xi_4) \star \partial_l (\eta_{(i)^c}) \otimes v_i - \sum_i \sum_{r<s} \partial_r \partial_s (-\xi_2 \xi_4 - i\xi_1 \xi_4) \star \eta_{(i)^c} \otimes F_{r,s} v_i \\ =& -(\xi_2 + i\xi_1) \star (\eta_2 \eta_3) \otimes v_1 - \eta_2 \eta_3 \eta_4 \otimes \beta_{1,2} v_1 - (\xi_2) \star (\eta_1 \eta_3) \otimes v_2 - \eta_1 \eta_3 \eta_4 \otimes \beta_{1,2} v_2 - \eta_1 \eta_2 \eta_4 \otimes \beta_{1,2} v_3 \\ & - (-\xi_4) \star (-\eta_1 \eta_3) \otimes v_4 - (-i\xi_4) \star (\eta_2 \eta_3) \otimes v_4 - \xi_1 \xi_2 \xi_3 \otimes \beta_{1,2} v_4 \\ =& \eta_1 \eta_2 \eta_3 \otimes (-iv_1 + v_2 - \beta_{1,2} v_4) + \eta_1 \eta_2 \eta_4 \otimes (-\beta_{1,2} v_3) + \eta_1 \eta_3 \eta_4 \otimes (-v_4 - \beta_{1,2} v_2) + \eta_2 \eta_3 \eta_4 \otimes (iv_4 - \beta_{1,2} v_1). \end{aligned}$$

Therefore:

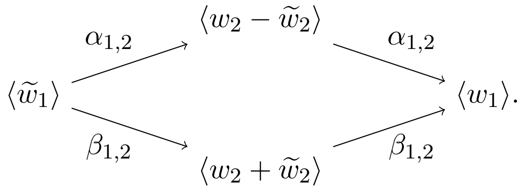
$$\begin{aligned}\beta_{1,2}(v_1) &= iv_4,\\ \beta_{1,2}(v_2) &= -v_4,\\ \beta_{1,2}(v_3) &= 0,\\ \beta_{1,2}(v_4) &= -iv_1 + v_2.\end{aligned}$$

□

Remark 4.28. Using notation (4.72), we obtain that Equations (4.77) are equivalent to:

$$\alpha_{1,2}(w_1) = 0, \tag{4.78}$$
$$\alpha_{1,2}(\widetilde{w}_1) = w_2 - \widetilde{w}_2, \tag{4.79}$$
$$\alpha_{1,2}(w_2) = w_1 = -\alpha_{1,2}(\widetilde{w}_2), \tag{4.80}$$
$$\beta_{1,2}(w_1) = 0, \tag{4.81}$$
$$\beta_{1,2}(\widetilde{w}_1) = -(w_2 + \widetilde{w}_2), \tag{4.82}$$
$$\beta_{1,2}(w_2) = w_1 = \beta_{1,2}(\widetilde{w}_2). \tag{4.83}$$

We represent these relations with the following drawings:



Using notation (4.72), Equations (4.75) can be written in the following way, as in [BKL1] (see Lemma B.6):

$$\begin{aligned}2(E_{00}+H_1)\widetilde{w}_1 &= E_{-(\varepsilon_1-\varepsilon_2)}\widetilde{w}_2 - E_{-(\varepsilon_1+\varepsilon_2)}w_2, \\ 2(E_{00}+H_2)\widetilde{w}_2 &= E_{-(\varepsilon_1+\varepsilon_2)}w_1 - E_{\varepsilon_1-\varepsilon_2}\widetilde{w}_1,\end{aligned} \tag{4.84}$$
$$\begin{aligned}2(E_{00}-H_1)w_1 &= E_{\varepsilon_1-\varepsilon_2}w_2 - E_{\varepsilon_1+\varepsilon_2}\widetilde{w}_2, \\ 2(E_{00}-H_2)w_2 &= E_{\varepsilon_1+\varepsilon_2}\widetilde{w}_1 - E_{-(\varepsilon_1-\varepsilon_2)}w_1.\end{aligned} \tag{4.85}$$

Proof of Theorem 4.4. Throughout this proof μ_0 will denote the highest weight of F with respect to E_{00}, μ_1 (resp. μ_2) will denote the highest weight of F with respect to H_1 (resp. H_2) and $m=\mu_1-\mu_2$ (resp. $n=\mu_1+\mu_2$) will denote the highest weight of F with respect to h_x (resp. h_y).
Let us first observe that, by Equation (4.80), we have that if $w_1 \neq 0$ then $w_2 \neq 0$. Therefore the following three cases are possible.

1. $w_1 = w_2 = 0$,
2. $w_1 \neq 0$ and $w_2 \neq 0$,
3. $w_1 = 0$ and $w_2 \neq 0$.

1) We suppose that $w_1 = w_2 = 0$.
By Equation (4.79), we obtain that if $\widetilde{w}_2 \neq 0$, then $\widetilde{w}_1 \neq 0$. Hence, there are two subcases.

1a) Let us suppose that $\widetilde{w}_1 \neq 0$ and $\widetilde{w}_2 = 0$. By Equations (4.78) and (4.81), we know that $\widetilde{w}_1$ is a highest weight vector. Let us compute its weight. By (4.72) we know that $v_1 = -i\widetilde{w}_1, v_2 = \widetilde{w}_1, v_3 = 0, v_4 = 0$.
Equation (4.76) for $a = 1, b = 3, c = 4$ reduces, using (4.72), to:

$$0 = -F_{3,4}(-i\widetilde{w}_1) + C\widetilde{w}_1. \tag{4.86}$$

Then $H_2\widetilde{w}_1 = -C\widetilde{w}_1$. By (4.84), $2(E_{00} + H_1)\widetilde{w}_1 = 0$, that means $\mu_0 + \mu_1 = 0$. Hence the weight of $\widetilde{w}_1$ with respect to h_x, h_y, E_{00}, C is $(m, n, -\frac{m+n}{2}, \frac{m-n}{2})$ with $m, n \in \mathbb{Z}_{\geq 0}$.
All the other equations of Lemma 4.27 are verified by this choice of v_1, v_2, v_3, v_4.
We have therefore obtained, using notation (4.1), the following singular vector in $M(m, n, -\frac{m+n}{2}, \frac{m-n}{2})$ with $m, n \in \mathbb{Z}_{\geq 0}$:

$$\vec{m}_{1a} = w_{11} \otimes x_1^m y_1^n.$$

1b) We suppose that $\widetilde{w}_1 \neq 0$ and $\widetilde{w}_2 \neq 0$. By (4.80) and (4.83), we know that $\widetilde{w}_2$ is a highest weight vector, let us compute its weight.
By (4.72) we know that $v_1 = -i\widetilde{w}_1, v_2 = \widetilde{w}_1, v_3 = -i\widetilde{w}_2, v_4 = \widetilde{w}_2$.
Equation (4.76) for $a = 1, b = 2, c = 3$ reduces, using (4.72), to:

$$\begin{aligned} 0 =& - F_{1,2}(i(w_2 - \widetilde{w}_2)) - F_{1,3}(w_1 + \widetilde{w}_1) - F_{2,3}(i(w_1 - \widetilde{w}_1)) + C(w_2 + \widetilde{w}_2) \\ =& iF_{1,2}\widetilde{w}_2 - F_{1,3}\widetilde{w}_1 + iF_{2,3}\widetilde{w}_1 + C\widetilde{w}_2. \end{aligned} \tag{4.87}$$

Equation (4.76) for $a = 1, b = 2, c = 4$ reduces, using (4.72), to:

$$\begin{aligned} 0 =& F_{1,2}(w_2 + \widetilde{w}_2) - F_{1,4}(w_1 + \widetilde{w}_1) - F_{2,4}(i(w_1 - \widetilde{w}_1)) + C(i(w_2 - \widetilde{w}_2)) \\ =& F_{1,2}\widetilde{w}_2 - F_{1,4}\widetilde{w}_1 + iF_{2,4}\widetilde{w}_1 - iC\widetilde{w}_2. \end{aligned} \tag{4.88}$$

We consider a linear combination of (4.87) and (4.88) and obtain:

$$\begin{aligned} 0 =& 2H_1\widetilde{w}_2 + 2C\widetilde{w}_2 - \alpha_{1,2}\widetilde{w}_1 - \beta_{1,2}\widetilde{w}_1 \\ =& 2H_1\widetilde{w}_2 + 2C\widetilde{w}_2 + 2\widetilde{w}_2, \end{aligned}$$

that is equivalent to:

$$H_1\widetilde{w}_2 = -C\widetilde{w}_2 - \widetilde{w}_2. \tag{4.89}$$

Therefore $\mu_1 = -1 - C$.
By Equation (4.84) we have:

$$2(E_{00} + H_2)\widetilde{w}_2 = -E_{\varepsilon_1 - \varepsilon_2}\widetilde{w}_1 = (-\alpha_{1,2} - \beta_{1,2})\widetilde{w}_1 = 2\widetilde{w}_2.$$

Therefore $\mu_0 + \mu_2 = 1$. The highest weight of $\widetilde{w}_2$ with respect to h_x, h_y, E_{00}, C is $(m, n, 1 + \frac{m-n}{2}, -\frac{m+n}{2} - 1)$. We point out that $m \in \mathbb{Z}_{>0}$ and $n \in \mathbb{Z}_{\geq 0}$. Indeed by (4.79) and (4.82), we have that $(\alpha_{1,2} + \beta_{1,2})(\widetilde{w}_1) = E_{\varepsilon_1 - \varepsilon_2}(\widetilde{w}_1) = -2\widetilde{w}_2 \neq 0$ and therefore $\widetilde{w}_1 = \frac{1}{2m}E_{-(\varepsilon_1 - \varepsilon_2)}\widetilde{w}_2$.
All the other equations of Lemma 4.27 are verified by this choice of v_1, v_2, v_3, v_4.
We have therefore obtained, using notation (4.1), the following singular vector in $M(m, n, 1 + \frac{m-n}{2}, -\frac{m+n}{2} - 1)$ with $m \in \mathbb{Z}_{>0}$, $n \in \mathbb{Z}_{\geq 0}$:

$$\vec{m}_{1b} = w_{21} \otimes x_1^m y_1^n - w_{11} \otimes x_1^{m-1} x_2 y_1^n.$$

2) We suppose that $w_1 \neq 0$ and $w_2 \neq 0$. Then by (4.80) we have that $\widetilde{w}_2 \neq 0$ and by (4.83) we have that $\widetilde{w}_1 \neq 0$. By (4.78) and (4.81) we know that w_1 is a highest weight vector. Equation (4.76) for $a = 1, b = 3, c = 4$ reduces, using (4.72), to:

$$\begin{aligned} 0 =& F_{1,3}(w_2 + \widetilde{w}_2) + F_{1,4}(i(w_2 - \widetilde{w}_2)) - F_{3,4}(i(w_1 - \widetilde{w}_1)) + C(w_1 + \widetilde{w}_1) \\ =& F_{1,3}w_2 + F_{1,3}\widetilde{w}_2 + iF_{1,4}w_2 - iF_{1,4}\widetilde{w}_2 - iF_{3,4}w_1 + iF_{3,4}\widetilde{w}_1 + Cw_1 + C\widetilde{w}_1. \end{aligned} \tag{4.90}$$

Equation (4.76) for $a = 2, b = 3, c = 4$ reduces, using (4.72), to:

$$\begin{aligned} 0 =& F_{2,3}(w_2 + \widetilde{w}_2) + F_{2,4}(i(w_2 - \widetilde{w}_2)) + F_{3,4}(w_1 + \widetilde{w}_1) + C(i(w_1 - \widetilde{w}_1)) \\ =& F_{2,3}w_2 + F_{2,3}\widetilde{w}_2 + iF_{2,4}w_2 - iF_{2,4}\widetilde{w}_2 + F_{3,4}w_1 + F_{3,4}\widetilde{w}_1 + Ciw_1 - iC\widetilde{w}_1. \end{aligned} \tag{4.91}$$

Using a linear combination of (4.90) and (4.91), we obtain:

$$\begin{aligned} 0 =& (F_{1,3} + iF_{1,4} - iF_{2,3} + F_{2,4})w_2 + (F_{1,3} - iF_{1,4} - iF_{2,3} - F_{2,4})\widetilde{w}_2 - 2iF_{3,4}w_1 + 2Cw_1 \\ =& E_{\varepsilon_1-\varepsilon_2}w_2 + E_{\varepsilon_1+\varepsilon_2}\widetilde{w}_2 - 2H_2w_1 + 2Cw_1 \\ =& (\alpha_{1,2} + \beta_{1,2})w_2 + (\alpha_{1,2} - \beta_{1,2})\widetilde{w}_2 - 2H_2w_1 + 2Cw_1 \\ =& 2w_1 - 2w_1 - 2H_2w_1 + 2Cw_1. \end{aligned}$$

Therefore $\mu_2 = C$. By Equation (4.85) we have:

$$2(E_{00} - H_1)w_1 = E_{\varepsilon_1-\varepsilon_2}w_2 - E_{\varepsilon_1+\varepsilon_2}\widetilde{w}_2 = (\alpha_{1,2} + \beta_{1,2})w_2 - (\alpha_{1,2} - \beta_{1,2})\widetilde{w}_2 = 4w_1.$$

Therefore $\mu_0 - \mu_1 = 2$.
Now we want to express $w_2, \widetilde{w}_1, \widetilde{w}_2$ in function of w_1.
By (4.80) and (4.83), $\alpha_{1,2}(w_2) + \beta_{1,2}(w_2) = E_{\varepsilon_1-\varepsilon_2}w_2 = 2w_1$. Therefore $w_2 = \frac{-1}{2\mu_1-2C}E_{-(\varepsilon_1-\varepsilon_2)}w_1$.
By (4.80) and (4.83), $\alpha_{1,2}(\widetilde{w}_2) - \beta_{1,2}(\widetilde{w}_2) = E_{\varepsilon_1+\varepsilon_2}\widetilde{w}_2 = -2w_1$. Therefore $\widetilde{w}_2 = \frac{1}{2\mu_1+2C}E_{-(\varepsilon_1+\varepsilon_2)}w_1$.
By (4.78) and (4.81), $\alpha_{1,2}(\widetilde{w}_1) + \beta_{1,2}(\widetilde{w}_1) = E_{\varepsilon_1-\varepsilon_2}\widetilde{w}_1 = -2\widetilde{w}_2$ and, by (4.79) and (4.82), $\alpha_{1,2}(\widetilde{w}_1) - \beta_{1,2}(\widetilde{w}_1) = E_{\varepsilon_1+\varepsilon_2}\widetilde{w}_1 = 2w_2$. We obtain:

$$\widetilde{w}_1 = \frac{1}{(2\mu_1 + 2C)(2\mu_1 - 2C)}E_{-(\varepsilon_1-\varepsilon_2)}E_{-(\varepsilon_1+\varepsilon_2)}w_1.$$

Finally, the highest weight of w_1 with respect to h_x, h_y, E_{00}, C is $(m, n, \frac{m+n}{2} + 2, \frac{n-m}{2})$ with $m, n \in \mathbb{Z}_{>0}$.
All the other equations of Lemma 4.27 are verified by this choice of $w_1, w_2, \widetilde{w}_1, \widetilde{w}_2$ and hence of v_1, v_2, v_3, v_4.
We have therefore obtained, using notation (4.1), the following singular vector in $M(m, n, \frac{m+n}{2} + 2, \frac{n-m}{2})$ with $m, n \in \mathbb{Z}_{>0}$:

$$\vec{m}_{1c} = w_{22} \otimes x_1^m y_1^n - w_{12} \otimes x_1^{m-1}x_2y_1^n - w_{21} \otimes x_1^m y_1^{n-1}y_2 + w_{11} \otimes x_1^{m-1}x_2y_1^{n-1}y_2.$$

3) We suppose that $w_1 = 0$ and $w_2 \neq 0$. By (4.78) and (4.81), $\widetilde{w}_1 \neq 0$, since $(\alpha_{1,2} - \beta_{1,2})\widetilde{w}_1 = 2w_2 \neq 0$. Hence there are 2 subcases.

3a) We suppose that $w_1 = 0$, $w_2 \neq 0$, $\widetilde{w}_1 \neq 0$ and $\widetilde{w}_2 = 0$. In this case, from (4.80) and (4.83), it follows that w_2 is a highest weight vector. Let us compute its weight. Equation (4.76) for $a = 1, b = 2, c = 3$ reduces, using (4.72), to:

$$\begin{aligned} 0 =& -F_{1,2}(i(w_2 - \widetilde{w}_2)) - F_{1,3}(w_1 + \widetilde{w}_1) - F_{2,3}(i(w_1 - \widetilde{w}_1)) + C(w_2 + \widetilde{w}_2) \\ =& -iF_{1,2}w_2 - F_{1,3}\widetilde{w}_1 + iF_{2,3}\widetilde{w}_1 + Cw_2. \end{aligned} \tag{4.92}$$

Equation (4.76) for $a=1, b=2, c=4$ reduces, using (4.72), to:

$$\begin{aligned}0 =&F_{1,2}(w_2+\widetilde{w}_2)-F_{1,4}(w_1+\widetilde{w}_1)-F_{2,4}(i(w_1-\widetilde{w}_1))+C(i(w_2-\widetilde{w}_2))\\ =&F_{1,2}w_2-F_{1,4}\widetilde{w}_1+iF_{2,4}\widetilde{w}_1+iCw_2.\end{aligned} \tag{4.93}$$

Considering a linear combination of (4.92) and (4.93), we obtain:

$$\begin{aligned}0=&-2H_1w_2-\alpha_{1,2}\widetilde{w}_1+\beta_{1,2}\widetilde{w}_1+2Cw_2\\ =&-2H_1w_2-2w_2+2Cw_2.\end{aligned}$$

Therefore $\mu_1=-1+C$.
By (4.85) we have:

$$2(E_{00}-H_2)w_2=E_{\varepsilon_1+\varepsilon_2}\widetilde{w}_1=(\alpha_{1,2}-\beta_{1,2})\widetilde{w}_1=2w_2.$$

Therefore $\mu_0-\mu_2=1$.
Hence the highest weight of w_2 with respect to h_x, h_y, E_{00}, C is $(m,n,\frac{n-m}{2}+1,\frac{m+n}{2}+1)$, with $m\in\mathbb{Z}_{\geq 0}$, $n\in\mathbb{Z}_{>0}$. Indeed $n>0$ since, by (4.79) and (4.82), $(\alpha_{1,2}-\beta_{1,2})(\widetilde{w}_1)=E_{\varepsilon_1+\varepsilon_2}(\widetilde{w}_1)=2w_2\neq 0$. Hence we obtain that $\widetilde{w}_1=\frac{-1}{2n}E_{-(\varepsilon_1+\varepsilon_2)}w_2$.
All the other equations of Lemma 4.27 are verified by this choice of $w_1,w_2,\widetilde{w}_1,\widetilde{w}_2$ and hence of v_1,v_2,v_3,v_4.
We have therefore obtained, using notation (4.1), the following singular vector in $M(m,n,\frac{n-m}{2}+1,\frac{m+n}{2}+1)$, with $m\in\mathbb{Z}_{\geq 0}$, $n\in\mathbb{Z}_{>0}$:

$$\vec{m}_{1d}=w_{12}\otimes x_1^m y_1^n-w_{11}\otimes x_1^m y_1^{n-1}y_2.$$

3b) We suppose that $w_1=0$, $w_2\neq 0$, $\widetilde{w}_1\neq 0$, $\widetilde{w}_2\neq 0$. By (4.79) and (4.82), w_2 and $\widetilde{w}_2$ are highest weight vectors. Let us compute their highest weight.
Equation (4.76) for $a=1, b=2, c=3$ reduces, using (4.72), to:

$$\begin{aligned}0=&-F_{1,2}(i(w_2-\widetilde{w}_2))-F_{1,3}(w_1+\widetilde{w}_1)-F_{2,3}(i(w_1-\widetilde{w}_1))+C(w_2+\widetilde{w}_2)\\ &-iF_{1,2}w_2+iF_{1,2}\widetilde{w}_2-F_{1,3}\widetilde{w}_1+iF_{2,3}\widetilde{w}_1+Cw_2+C\widetilde{w}_2.\end{aligned} \tag{4.94}$$

Equation (4.76) for $a=1, b=2, c=4$ reduces, using (4.72), to:

$$\begin{aligned}0=&F_{1,2}(w_2+\widetilde{w}_2)-F_{1,4}(w_1+\widetilde{w}_1)-F_{2,4}(i(w_1-\widetilde{w}_1))+C(i(w_2-\widetilde{w}_2))\\ &F_{1,2}w_2+F_{1,2}\widetilde{w}_2-F_{1,4}\widetilde{w}_1+iF_{2,4}\widetilde{w}_1+iCw_2-iC\widetilde{w}_2.\end{aligned} \tag{4.95}$$

Considering linear combinations of (4.94) and (4.95), we obtain:

$$\begin{aligned}0=&-2H_1w_2-\alpha_{1,2}\widetilde{w}_1+\beta_{1,2}\widetilde{w}_1+2Cw_2\\ =&-2H_1w_2-2w_2+2Cw_2,\end{aligned}$$

that is

$$H_1w_2=-w_2+Cw_2;$$

and:

$$0=2H_1\widetilde{w}_2-\alpha_{1,2}\widetilde{w}_1-\beta_{1,2}\widetilde{w}_1+2C\widetilde{w}_2,$$

that is:

$$H_1\widetilde{w}_2=-\widetilde{w}_2-C\widetilde{w}_2.$$

This implies that $C=0$. But, for $C=0$, the $\lambda-$action of Proposition 3.11 reduces to the action found in Theorem 4.3 of [BKL1] where the vectors of degree 1 were classified, but this case was ruled out.

□

Chapter 5

Homology

In this chapter we study the homology of the complexes in Figure 4.1. The main result is the following Theorem:

Theorem 5.1. *The sequences in Figure 4.1 are complexes and they are exact in each module except for $M(0,0,0,0)$ and $M(1,1,3,0)$. The homology spaces in $M(0,0,0,0)$ and $M(1,1,3,0)$ are isomorphic to the trivial representation.*

Lemma 5.2. *Let $\nabla : M(\mu_1,\mu_2,\mu_3,\mu_4) \longrightarrow M(\widetilde{\mu}_1,\widetilde{\mu}_2,\widetilde{\mu}_3,\widetilde{\mu}_4)$ be a morphism represented in Figure 4.1 and constructed as in Remark 4.8. Then $\operatorname{Im}\nabla$ is an irreducible $\mathfrak{g}-$submodule of $M(\widetilde{\mu}_1,\widetilde{\mu}_2,\widetilde{\mu}_3,\widetilde{\mu}_4)$.*

Proof. By Theorems 4.4, 4.5, 4.6 and Remark 4.8, we know that $M(\widetilde{\mu}_1,\widetilde{\mu}_2,\widetilde{\mu}_3,\widetilde{\mu}_4)$ contains a unique, up to scalars, highest weight nontrivial singular vector, that we call $\vec{m}$. By construction of ∇, $\operatorname{Im}\nabla$ is the $\mathfrak{g}-$submodule of $M(\widetilde{\mu}_1,\widetilde{\mu}_2,\widetilde{\mu}_3,\widetilde{\mu}_4)$ generated by $\vec{m}$. In particular it is straightforward that $\mathfrak{g}_0\vec{m}$ is an irreducible finite−dimensional $\mathfrak{g}_0-$module on which $\mathfrak{g}_{>0}$ acts trivially, since $\vec{m}$ is singular. The $\mathfrak{g}-$module $\operatorname{Im}\nabla = \mathfrak{g}\vec{m}$ is therefore isomorphic to $\operatorname{Ind}(\mathfrak{g}_0\vec{m})$. Hence, due to Theorem 1.15, $\operatorname{Im}\nabla$ is an irreducible $\mathfrak{g}-$module since in $M(\widetilde{\mu}_1,\widetilde{\mu}_2,\widetilde{\mu}_3,\widetilde{\mu}_4)$ there is only the highest weight nontrivial singular vector $\vec{m}$ that is trivial for $\operatorname{Ind}(\mathfrak{g}_0\vec{m})$. □

Remark 5.3. Using Theorem 5.1 we are able to realize all irreducible quotients of Verma modules. We have that:

- If $(\mu_1,\mu_2,\mu_3,\mu_4)$ is not among the weights that occur in Theorems 4.4, 4.5, 4.6, then $M(\mu_1,\mu_2,\mu_3,\mu_4)$ is irreducible, due to Theorem 1.15, since it does not contain nontrivial singular vectors.

- If $(\mu_1,\mu_2,\mu_3,\mu_4)$ is among the weights that occur in Theorems 4.4, 4.5, 4.6, then $M(\mu_1,\mu_2,\mu_3,\mu_4)$ is degenerate. We denote its irreducible quotient by $I(\mu_1,\mu_2,\mu_3,\mu_4)$. By Remark 4.8, we know that from each $M(\mu_1,\mu_2,\mu_3,\mu_4)$ in Figure 4.1, except for $M(0,0,0,0)$, we can construct a morphism ∇ to another Verma module $M(\widetilde{\mu}_1,\widetilde{\mu}_2,\widetilde{\mu}_3,\widetilde{\mu}_4)$. Due to Lemma 5.2, $\operatorname{Ker}\nabla$ is the maximal submodule of $M(\mu_1,\mu_2,\mu_3,\mu_4)$ because $M(\mu_1,\mu_2,\mu_3,\mu_4)/\operatorname{Ker}\nabla \cong \operatorname{Im}\nabla$ is irreducible. Therefore $I(\mu_1,\mu_2,\mu_3,\mu_4) \cong M(\mu_1,\mu_2,\mu_3,\mu_4)/\operatorname{Ker}\nabla$ and $\operatorname{Im}\nabla$ is an explicit realization for $I(\mu_1,\mu_2,\mu_3,\mu_4)$.
  If $M(\mu_1,\mu_2,\mu_3,\mu_4)$ is a Verma module represented in Figure 4.1, with $(\mu_1,\mu_2,\mu_3,\mu_4) \neq (0,0,0,0), (0,0,2,0)$, then there exist two morphisms $\nabla : M(\mu_1,\mu_2,\mu_3,\mu_4) \longrightarrow M(\widetilde{\mu}_1,\widetilde{\mu}_2,\widetilde{\mu}_3,\widetilde{\mu}_4)$ and $\widehat{\nabla} : M(\hat{\mu}_1,\hat{\mu}_2,\hat{\mu}_3,\hat{\mu}_4) \longrightarrow M(\mu_1,\mu_2,\mu_3,\mu_4)$ constructed as in Remark 4.8. Due to Theorem 5.1, if $(\mu_1,\mu_2,\mu_3,\mu_4) \neq (0,0,0,0)$ and $(1,1,3,0)$, the submodule $\operatorname{Ker}\nabla = \operatorname{Im}\widehat{\nabla}$ is irreducible and it is the unique submodule of $M(\mu_1,\mu_2,\mu_3,\mu_4)$; in this case $I(\mu_1,\mu_2,\mu_3,\mu_4)$ is also isomorphic to the Cokernel of the map $\widehat{\nabla}$ that ends in $M(\mu_1,\mu_2,\mu_3,\mu_4)$.
  In the case of $M(0,0,0,0)$, by Remark 4.8, we have a morphism $\nabla : M(1,1,-1,0) \to M(0,0,0,0)$. By Theorem 5.1, $M(0,0,0,0)/\operatorname{Im}\nabla$ is irreducible and therefore $I(0,0,0,0) \cong M(0,0,0,0)/\operatorname{Im}\nabla$.

5.1 The morphisms

In order to show Theorem 5.1, we start by writing explicitly the morphisms that occur in Figure 4.1. We recall the notation introduced in Remark 4.2 for the basis of $\mathfrak{g}_0^{ss}$:

$$e_x = \frac{E_{\varepsilon_1-\varepsilon_2}}{2}, \quad f_x = -\frac{E_{-(\varepsilon_1-\varepsilon_2)}}{2}, \quad h_x = H_1 - H_2,$$

and

$$e_y = \frac{E_{\varepsilon_1+\varepsilon_2}}{2}, \quad f_y = -\frac{E_{-(\varepsilon_1+\varepsilon_2)}}{2}, \quad h_y = H_1 + H_2.$$

We have that:

$$\begin{aligned}\mathfrak{g}_0 =&\langle e_x, f_x, h_x\rangle \oplus \langle e_y, f_y, h_y\rangle \oplus \mathbb{C}t \oplus \mathbb{C}C\\ \cong&\langle x_1\partial_{x_2}, x_2\partial_{x_1}, x_1\partial_{x_1} - x_2\partial_{x_2}\rangle \oplus \langle y_1\partial_{y_2}, y_2\partial_{y_1}, y_1\partial_{y_1} - y_2\partial_{y_2}\rangle\\ &\oplus \mathbb{C}(-\frac{1}{2}(x_1\partial_{x_1} + x_2\partial_{x_2} + y_1\partial_{y_1} + y_2\partial_{y_2})) \oplus \mathbb{C}(\frac{1}{2}(x_1\partial_{x_1} + x_2\partial_{x_2}) - \frac{1}{2}(y_1\partial_{y_1} + y_2\partial_{y_2})).\end{aligned}$$

Let us recall the notation (4.1):

$$w_{11} = \eta_2 + i\eta_1, \ w_{22} = \eta_2 - i\eta_1, \ w_{12} = -\eta_4 + i\eta_3, \ w_{21} = \eta_4 + i\eta_3.$$

We point out that

$$[w_{11}, w_{22}] = 4\Theta, \quad [w_{12}, w_{21}] = -4\Theta \tag{5.1}$$

and all other brackets between the w's are 0. Moreover in $U(\mathfrak{g}_{<0})$ we have:

$$w_{11}^2 = w_{22}^2 = w_{12}^2 = w_{21}^2 = 0. \tag{5.2}$$

Indeed for example $w_{11}^2 = (\eta_2 + i\eta_1)(\eta_2 + i\eta_1) = \Theta + i\eta_2\eta_1 + i\eta_1\eta_2 - \Theta = 0$.
We introduce the following $\mathfrak{g}_0-$modules:

$$\begin{aligned}V_A &= \mathbb{C}\left[x_1, x_2, y_1, y_2\right],\\ V_B &= \mathbb{C}\left[\partial_{x_1}, \partial_{x_2}, y_1, y_2\right]_{[1,-1]},\\ V_C &= \mathbb{C}\left[\partial_{x_1}, \partial_{x_2}, \partial_{y_1}, \partial_{y_2}\right]_{[2,0]},\\ V_D &= \mathbb{C}\left[x_1, x_2, \partial_{y_1}, \partial_{y_2}\right]_{[1,1]}.\end{aligned}$$

The subscripts $[i,j]$ mean that t acts on V_X, for $X = A, B, C, D$, as $-\frac{1}{2}(x_1\partial_{x_1}+x_2\partial_{x_2}+y_1\partial_{y_1}+y_2\partial_{y_2})$ plus $i\,\mathrm{Id}$ and C acts on V_X, for $X = A, B, C, D$, as $\frac{1}{2}(x_1\partial_{x_1} + x_2\partial_{x_2}) - \frac{1}{2}(y_1\partial_{y_1} + y_2\partial_{y_2})$ plus $j\,\mathrm{Id}$; the subscript $[i,j]$ is assumed to be $[0,0]$ when it is omitted, i.e. for $X = A$.
The elements of $\mathfrak{g}_0^{ss}$ act on V_X, for $X = A, B, C, D$, in the standard way:

$$x_i\partial_{x_j}x_k = \chi_{j=k}x_i, \quad x_i\partial_{x_j}.\partial_{x_k} = -\chi_{i=k}\partial_{x_j}, \quad x_i\partial_{x_j}y_k = 0, \quad x_i\partial_{x_j}.\partial_{y_k} = 0;$$

$$y_i\partial_{y_j}y_k = \chi_{j=k}y_i, \quad y_i\partial_{y_j}.\partial_{y_k} = -\chi_{i=k}\partial_{y_j}, \quad y_i\partial_{y_j}x_k = 0, \quad y_i\partial_{y_j}.\partial_{x_k} = 0.$$

We introduce the following bigrading:

$$V_X^{m,n} := \{f \in V_X \,:\, (x_1\partial_{x_1} + x_2\partial_{x_2}).f = mf \text{ and } (y_1\partial_{y_1} + y_2\partial_{y_2}).f = nf\}. \tag{5.3}$$

The $V_X^{m,n}$'s are irreducible $\mathfrak{g}_0-$modules, in particular we point out that, for $m, n \in \mathbb{Z}_{\geq 0}$, $V_A^{m,n}$ is the irreducible $\mathfrak{g}_0-$module determined by coordinates (m,n) in quadrant **A** of Figure 4.1, $V_B^{-m,n}$

is the irreducible $\mathfrak{g}_0-$module determined by coordinates (m,n) in quadrant **B**, $V_C^{-m,-n}$ is the irreducible $\mathfrak{g}_0-$module determined by coordinates (m,n) in quadrant **C** and $V_D^{m,-n}$ is the irreducible $\mathfrak{g}_0-$module determined by coordinates (m,n) in quadrant **D**. Therefore for $m,n\in\mathbb{Z}_{\geq 0}$:

$$V_A^{m,n}\cong F\Big(m,n,-\frac{m+n}{2},\frac{m-n}{2}\Big),$$
$$V_B^{-m,n}\cong F\Big(m,n,1+\frac{m-n}{2},-\frac{m+n}{2}-1\Big),$$
$$V_C^{-m,-n}\cong F\Big(m,n,\frac{m+n}{2}+2,\frac{n-m}{2}\Big),$$
$$V_D^{m,-n}\cong F\Big(m,n,1+\frac{n-m}{2},\frac{m+n}{2}+1\Big).$$

We have that $V_X=\oplus_{m,n}V_X^{m,n}$ is the direct sum of all the irreducible $\mathfrak{g}_0-$modules in quadrant **X**. We denote by $M_X^{m,n}=U(\mathfrak{g}_{<0})\otimes V_X^{m,n}$; we point out that, for $m,n\in\mathbb{Z}_{\geq 0}$, $M_A^{m,n}$ is the Verma module represented in Figure 4.1 in quadrant **A** with coordinates (m,n), $M_B^{-m,n}$ is the Verma module represented in quadrant **B** with coordinates (m,n), $M_C^{-m,-n}$ is the Verma module represented in quadrant **C** with coordinates (m,n), $M_D^{m,-n}$ is the Verma module represented in quadrant **D** with coordinates (m,n). Moreover we denote by $M_X=\oplus_{m,n\in\mathbb{Z}}M_X^{m,n}$ the direct sum of all Verma modules in the quadrant **X** of Figure 4.1.
We follow the notation in [KR1] and define, for every $u\in U(\mathfrak{g}_{<0})$ and $\phi\in\mathrm{Hom}(V_X,V_Y)$, the map $u\otimes\phi:M_X\longrightarrow M_Y$ by:

$$(u\otimes\phi)(u'\otimes v)=u'\,u\otimes\phi(v), \tag{5.4}$$

for every $u'\otimes v\in U(\mathfrak{g}_{<0})\otimes V_X$. From this definition it is clear that the map $u\otimes\phi$ commutes with the action of $\mathfrak{g}_{<0}$.
We consider, for $j=1,2$, the map $\partial_{x_j}:V_X\longrightarrow V_X$ that is the derivation by x_j for $X=A,D$ and the multiplication by ∂_{x_j} for $X=B,C$. We define analogously, for $j=1,2$, the map $\partial_{y_j}:V_X\longrightarrow V_X$, that is the derivation by y_j for $X=A,B$ and the multiplication by ∂_{y_j} for $X=C,D$. We will often write, by abuse of notation, ∂_{x_j} instead of $1\otimes\partial_{x_j}:M_X\longrightarrow M_X$.
We define the maps $\Delta^+:M_X\longrightarrow M_X$, $\Delta^-:M_X\longrightarrow M_X$, $\nabla:M_X\longrightarrow M_X$ as follows:

$$\Delta^+=w_{11}\otimes\partial_{x_1}+w_{21}\otimes\partial_{x_2}, \tag{5.5}$$
$$\Delta^-=w_{12}\otimes\partial_{x_1}+w_{22}\otimes\partial_{x_2}, \tag{5.6}$$
$$\nabla=\Delta^+\partial_{y_1}+\Delta^-\partial_{y_2}=w_{11}\otimes\partial_{x_1}\partial_{y_1}+w_{21}\otimes\partial_{x_2}\partial_{y_1}+w_{12}\otimes\partial_{x_1}\partial_{y_2}+w_{22}\otimes\partial_{x_2}\partial_{y_2}. \tag{5.7}$$

In particular,

$$\nabla_{|M_X^{m,n}}:M_X^{m,n}\longrightarrow M_X^{m-1,n-1};$$

by abuse of notation we will write ∇ instead of $\nabla_{|M_X^{m,n}}$. We will show that the map ∇ is the explicit expression of the morphisms of degree 1 in Figure 4.1.

Remark 5.4. By (5.1) and (5.2) it is straightforward that $(\Delta^+)^2=0$, $(\Delta^-)^2=0$ and $\Delta^+\Delta^-+\Delta^-\Delta^+=0$.

Remark 5.5. We point out that $\nabla:M_X^{m,n}\longrightarrow M_X^{m-1,n-1}$ is constructed so that $\nabla(v)$, for v highest weight vector in $V_X^{m,n}$, is the highest weight singular vector of degree 1 in $M_X^{m-1,n-1}$, classified in Theorem 4.4. In particular:

a: let $\nabla:M_A^{m,n}\longrightarrow M_A^{m-1,n-1}$. The highest weight vector in $V_A^{m,n}$ is $x_1^m y_1^n$. We have:

$$\nabla(x_1^m y_1^n)=w_{11}\otimes mnx_1^{m-1}y_1^{n-1},$$

that is the highest weight singular vector $\vec{m}_{1a}$ of $M(m-1,n-1,-\frac{m+n-2}{2},\frac{m-n}{2})$ found in Theorem 4.4.

b: Let $\nabla : M_B^{-m,n} \longrightarrow M_B^{-m-1,n-1}$. The highest weight vector in $V_B^{-m,n}$ is $\partial_{x_2}^m y_1^n$. We have:

$$\nabla(\partial_{x_2}^m y_1^n) = w_{11} \otimes n\partial_{x_1}\partial_{x_2}^m y_1^{n-1} + w_{21} \otimes n\partial_{x_2}^{m+1} y_1^{n-1},$$

that is the highest weight singular vector $\vec{m}_{1b}$ of $M(m+1, n-1, 1+\frac{m-n+2}{2}, -\frac{m+n}{2}-1)$ found in Theorem 4.4.

c: Let $\nabla : M_C^{-m,-n} \longrightarrow M_C^{-m-1,-n-1}$. The highest weight vector in $V_C^{-m,-n}$ is $\partial_{x_2}^m \partial_{y_2}^n$. We have:

$$\nabla(\partial_{x_2}^m \partial_{y_2}^n) = w_{11} \otimes \partial_{x_1}\partial_{x_2}^m \partial_{y_1}\partial_{y_2}^n + w_{21} \otimes \partial_{x_2}^{m+1}\partial_{y_1}\partial_{y_2}^n + w_{12} \otimes \partial_{x_1}\partial_{x_2}^m \partial_{y_2}^{n+1} + w_{22} \otimes \partial_{x_2}^{m+1}\partial_{y_2}^{n+1},$$

that is the highest weight singular vector $\vec{m}_{1c}$ of $M(m+1, n+1, \frac{m+n+2}{2}+2, \frac{n-m}{2})$ found in Theorem 4.4.

d: Let $\nabla : M_D^{m,-n} \longrightarrow M_D^{m-1,-n-1}$. The highest weight vector in $V_D^{m,-n}$ is $x_1^m \partial_{y_2}^n$. We have:

$$\nabla(x_1^m \partial_{y_2}^n) = w_{11} \otimes m x_1^{m-1}\partial_{y_1}\partial_{y_2}^n + w_{12} \otimes m x_1^{m-1}\partial_{y_2}^{n+1},$$

that is the highest weight singular vector $\vec{m}_{1d}$ of $M(m-1, n+1, 1+\frac{n-m+2}{2}, \frac{m+n}{2}+1)$ found in Theorem 4.4.

The following is straightforward.

Lemma 5.6. *Let $u \otimes \phi$ be a map as in (5.4). Let us suppose that $u \otimes \phi = \sum_i u_i \otimes \phi_i$ where $\{u_i\}_i$ and $\{\phi_i\}_i$ are bases of dual $\mathfrak{g}_0-$modules and u_i is the dual of ϕ_i for all i. Then $u \otimes \phi$ commutes with $\mathfrak{g}_0$.*

Lemma 5.7. *Let us consider a map $u \otimes \phi \in U(\mathfrak{g}_{<0}) \otimes \mathrm{Hom}(V_X, V_Y)$. In order to show that $u \otimes \phi$ commutes with $\mathfrak{g}_0$, it is sufficient to show that $wu \otimes \phi(v) = u \otimes \phi(w.v)$ for all $v \in V_X$, $w \in \mathfrak{g}_0$.*

Proof. Let $w \in \mathfrak{g}_0$. We have, for every $u_{i_1}u_{i_2}\dots u_{i_k} \otimes v \in U(\mathfrak{g}_{<0}) \otimes V_X$:

$$w.(u_{i_1}u_{i_2}\dots u_{i_k} \otimes v) = u_{i_1}u_{i_2}\dots u_{i_k} \otimes w.v + \sum \widetilde{u}_{i_1}\widetilde{u}_{i_2}\dots\widetilde{u}_{i_k} \otimes v.$$

Hence we have that for a map $u \otimes \phi \in U(\mathfrak{g}_{<0}) \otimes \mathrm{Hom}(V_X, V_Y)$:

$$(u \otimes \phi)(w.(u_{i_1}u_{i_2}\dots u_{i_k} \otimes v)) = u_{i_1}u_{i_2}\dots u_{i_k}u \otimes \phi(w.v) + \sum \widetilde{u}_{i_1}\widetilde{u}_{i_2}\dots\widetilde{u}_{i_k}u \otimes \phi(v).$$

On the other hand we have:

$$\begin{aligned} w.(u \otimes \phi)(u_{i_1}u_{i_2}\dots u_{i_k} \otimes v) &= w.(u_{i_1}u_{i_2}\dots u_{i_k}u \otimes \phi(v)) \\ &= u_{i_1}u_{i_2}\dots u_{i_k}wu \otimes \phi(v) + \sum \widetilde{u}_{i_1}\widetilde{u}_{i_2}\dots\widetilde{u}_{i_k}u \otimes \phi(v). \end{aligned}$$

Therefore, in order to show that $u \otimes \phi$ commutes with $\mathfrak{g}_0$, it is sufficient to show that $wu \otimes \phi(v) = u \otimes \phi(w.v)$ for all $v \in V_X$, $w \in \mathfrak{g}_0$. □

Lemma 5.8. *Let $\Phi : M_X \to M_Y$ be a linear map. Let us suppose that Φ commutes with $\mathfrak{g}_{\leq 0}$ and that $\Phi(v)$ is a singular vector for every v highest weight vector in $V_X^{m,n}$ and for all $m, n \in \mathbb{Z}$. Then Φ is a morphism of $\mathfrak{g}-$modules.*

Proof. Due to Proposition 2.3 in [KR1], it is sufficient to show that $\mathfrak{g}_{>0}\Phi(w) = 0$ for every $w \in V_X$, in order to prove that Φ commutes with $\mathfrak{g}_{>0}$.
We know that $\mathfrak{g}_{>0}\Phi(v) = 0$ for v highest weight vector in $V_X^{m,n}$ for all $m, n \in \mathbb{Z}$. Let v be the highest weight vector in $V_X^{m,n}$, f one among f_x, f_y, e one among e_x, e_y and $g_+ \in \mathfrak{g}_{>0}$. We have that:

$$g_+.\Phi(f.v) = g_+.(f.\Phi(v)) = f.(g_+.\Phi(v)) + [g_+, f].\Phi(v) = 0.$$

This can be iterated and we obtain that $\mathfrak{g}_{>0}.\Phi(w) = 0$ for all $w \in V_X^{m,n}$. Hence $\mathfrak{g}_{>0}.\Phi(w) = 0$ for all $w \in V_X$. □

Proposition 5.9. *The map* ∇*, defined in* (5.7)*, is a morphism of* $\mathfrak{g}-$*modules and* $\nabla^2 = 0$.

Proof. The map $\nabla : M_X \to M_X$ commutes with $\mathfrak{g}_{<0}$ by (5.4). By Remark 5.5 and Lemmas 5.6, 5.8 it follows that ∇ is a morphism of $\mathfrak{g}-$modules. The property $\nabla^2 = 0$ follows from the fact that ∇ is a map between Verma modules that contain only highest weight singular vectors of degree 1, by Theorems 4.4, 4.5, 4.6. □

By Remark 5.5 and Proposition 5.9, it follows that, for all $m, n \in \mathbb{Z}_{\geq 0}$:

i: the maps $\nabla : M_A^{m,n} \longrightarrow M_A^{m-1,n-1}$ are the morphisms represented in Figure 4.1 in quadrant **A**;

ii: the maps $\nabla : M_B^{-m,n} \longrightarrow M_B^{-m-1,n-1}$ are the morphisms represented in Figure 4.1 in quadrant **B**;

iii: the maps $\nabla : M_C^{-m,-n} \longrightarrow M_C^{-m-1,-n-1}$ are the morphisms represented in Figure 4.1 in quadrant **C**;

iv: the maps $\nabla : M_D^{m,-n} \longrightarrow M_D^{m-1,-n-1}$ are the morphisms represented in Figure 4.1 in quadrant **D**.

We introduce the following notation:

$$\begin{aligned}
V_{A'} &= \oplus_{m\in\mathbb{Z}} V_A^{m,0} = \mathbb{C}\left[x_1, x_2\right],\\
V_{B'} &= \oplus_{m\in\mathbb{Z}} V_B^{m,0} = \mathbb{C}\left[\partial_{x_1}, \partial_{x_2}\right]_{[1,-1]},\\
V_{C'} &= \oplus_{m\in\mathbb{Z}} V_C^{m,0} = \mathbb{C}\left[\partial_{x_1}, \partial_{x_2}\right]_{[2,0]},\\
V_{D'} &= \oplus_{m\in\mathbb{Z}} V_D^{m,0} = \mathbb{C}\left[x_1, x_2\right]_{[1,1]}.
\end{aligned}$$

We denote $M_{X'} = U(\mathfrak{g}_{<0}) \otimes V_{X'}$. We point out that $M_{X'}$ is the direct sum of Verma modules of Figure 4.1 in quadrant **X** that lie on the axis $n = 0$. We consider the map $\tau_1 : M_{A'} \longrightarrow M_{D'}$ that is the identity. We have that:

$$\begin{aligned}
[t, \tau_1] &= \tau_1,\\
[C, \tau_1] &= \tau_1.
\end{aligned} \tag{5.8}$$

We call $\nabla_2 : M_{A'} \longrightarrow M_{D'}$ the map

$$\Delta^-\Delta^+\tau_1 = w_{11}w_{12} \otimes \partial_{x_1}^2 + w_{11}w_{22} \otimes \partial_{x_1}\partial_{x_2} + w_{21}w_{12} \otimes \partial_{x_1}\partial_{x_2} + w_{21}w_{22} \otimes \partial_{x_2}^2.$$

We consider the map $\tau_2 : M_{B'} \longrightarrow M_{C'}$ that is the identity. We have that:

$$\begin{aligned}
[t, \tau_2] &= \tau_2,\\
[C, \tau_2] &= \tau_2.
\end{aligned} \tag{5.9}$$

By abuse of notation, we also call $\nabla_2 : M_{B'} \longrightarrow M_{C'}$ the map

$$\Delta^-\Delta^+\tau_2 = w_{11}w_{12} \otimes \partial_{x_1}^2 + w_{11}w_{22} \otimes \partial_{x_1}\partial_{x_2} + w_{21}w_{12} \otimes \partial_{x_1}\partial_{x_2} + w_{21}w_{22} \otimes \partial_{x_2}^2.$$

We observe that $M_{X'} = \oplus_{m\in\mathbb{Z}} M_X^{m,0}$ for $X = A, B, C, D$. We will denote $M_{X'}^m = M_X^{m,0}$. We have that, for every $m \geq 2$:

$$\nabla_{2|M_{A'}^m} : M_{A'}^m \longrightarrow M_{D'}^{m-2}$$

and

$$\nabla_{2|M_{B'}^{-m}} : M_{B'}^{-m} \longrightarrow M_{C'}^{-m-2}.$$

By abuse of notation we will also write ∇_2 instead of $\nabla_{2|M^m_{A'}}$ and $\nabla_{2|M^{-m}_{B'}}$. We will show that the map ∇_2 is the explicit expression of the morphisms of degree 2 in Figure 4.1 from the quadrant **A** to the quadrant **D** and from the quadrant **B** to the quadrant **C**.

Remark 5.10. i: The map $\nabla_2 : M^m_{A'} \longrightarrow M^{m-2}_{D'}$ is constructed so that $\nabla_2(v)$, for v highest weight vector in $V^m_{A'}$, is the highest weight singular vector of degree 2 in $M^{m-2}_{D'}$, classified in Theorem 4.5. Indeed, the highest weight vector in $V^m_{A'}$ is x_1^m and we have:

$$\nabla_2(x_1^m) = w_{11}w_{12} \otimes m(m-1)x_1^{m-2},$$

that is the highest weight singular vector $\vec{m}_{2b}$ of $M(m-2, 0, 1-\frac{m-2}{2}, 1+\frac{m-2}{2})$ found in Theorem 4.5.

ii: The map $\nabla_2 : M^{-m}_{B'} \longrightarrow M^{-m-2}_{C'}$ is constructed so that $\nabla_2(v)$, for v highest weight vector in $V^{-m}_{B'}$, is the highest weight singular vector of degree 2 in $M^{-m-2}_{C'}$, classified in Theorem 4.5. Indeed, the highest weight vector in $V^{-m}_{B'}$ is $\partial^m_{x_2}$ and we have:

$$\nabla_2(\partial^m_{x_2}) = w_{11}w_{12} \otimes \partial^2_{x_1}\partial^m_{x_2} + (w_{11}w_{22} + w_{21}w_{12}) \otimes \partial_{x_1}\partial^{m+1}_{x_2} + w_{21}w_{22} \otimes \partial^{m+2}_{x_2},$$

that is the highest weight singular vector $\vec{m}_{2c}$ of $M(m+2, 0, 2+\frac{m+2}{2}, -\frac{m+2}{2})$ found in Theorem 4.5.

Proposition 5.11. *The map $\nabla_2 : M_{A'} \longrightarrow M_{D'}$ (resp. $\nabla_2 : M_{B'} \longrightarrow M_{C'}$) is a morphism of $\mathfrak{g}-$modules and $\nabla_2\nabla = \nabla\nabla_2 = 0$.*

Proof. The map ∇_2 commutes with $\mathfrak{g}_{<0}$ by (5.4). By Remark 5.10 and Lemmas 5.6, 5.8 it follows that ∇_2 is a morphism of $\mathfrak{g}-$modules. Finally, $\nabla_2\nabla = \nabla\nabla_2 = 0$ follows from the fact that due to Theorem 4.6, there are no highest weight singular vectors of degree 3 in the codomain of $\nabla_2\nabla$ and $\nabla\nabla_2$. ☐

By Remark 5.10 and Proposition 5.11, it follows that, for every $m \geq 2$, the maps $\nabla_2 : M^m_{A'} \longrightarrow M^{m-2}_{D'}$ are the morphisms represented in Figure 4.1 from the quadrant **A** to the quadrant **D** and the maps $\nabla_2 : M^{-m}_{B'} \longrightarrow M^{-m+2}_{C'}$ are the morphisms from the quadrant **B** to the quadrant **C**.
We now define the map $\tau_3 : V^{0,0}_A \longrightarrow V^{0,0}_C$ that is the identity. We have that:

$$\begin{aligned} &[t, \tau_3] = 2\tau_3, \\ &[C, \tau_3] = 0. \end{aligned} \tag{5.10}$$

We define the map $\nabla_3 : M^{0,1}_A \longrightarrow M^{-1,0}_C$ as follows, using definition (5.4), for every $m \in M^{0,1}_A$:

$$\nabla_3(m) = \Delta^- \circ (w_{11}w_{21} \otimes \tau_3) \circ (1 \otimes \partial_{y_1})(m) + \Delta^- \circ ((w_{12}w_{21} + w_{11}w_{22}) \otimes \tau_3) \circ (1 \otimes \partial_{y_2})(m).$$

Remark 5.12. The map $\nabla_3 : M^{0,1}_A \longrightarrow M^{-1,0}_C$ is constructed so that $\nabla_3(v)$, for v highest weight vector in $V^{0,1}_A$, is the highest weight singular vector of degree 3 in $M^{-1,0}_C$, classified in Theorem 4.6. Indeed, the highest weight vector in $V^{0,1}_A$ is y_1 and we have:

$$\nabla_3(y_1) = w_{11}w_{21}w_{12} \otimes \partial_{x_1} + w_{11}w_{21}w_{22} \otimes \partial_{x_2} = w_{21}w_{12}w_{11} \otimes \partial_{x_1} - w_{11}w_{22}w_{21} \otimes \partial_{x_2},$$

that is the highest weight singular vector $\vec{m}_{3a}$ of $M(1, 0, \frac{5}{2}, -\frac{1}{2})$ found in Theorem 4.6.

Proposition 5.13. *The map ∇_3 is a morphism of $\mathfrak{g}-$modules and $\nabla_3\nabla = \nabla\nabla_3 = 0$.*

Proof. First we show that the map ∇_3 is a morphism of $\mathfrak{g}-$modules. It commutes with $\mathfrak{g}_{<0}$ due to (5.4). Due to Lemmas 5.7 and 5.8, it is sufficient to show that $wu\otimes\nabla_3(v)=u\otimes\nabla_3(w.v)$ for every $w\in\mathfrak{g}_0$, $v\in V_A^{0,1}$.
We have, for every $v\in V_A^{0,1}$:

$$\begin{aligned}
&h_x.\nabla_3(v)=\\
&=\Delta^-h_x(w_{11}w_{21}\otimes\tau_3)(1\otimes\partial_{y_1})(v)+\Delta^-h_x((w_{12}w_{21}+w_{11}w_{22})\otimes\tau_3)(1\otimes\partial_{y_2})(v)\\
&=\Delta^-(w_{11}w_{21}\otimes\tau_3)h_x(1\otimes\partial_{y_1})(v)+\Delta^-((w_{12}w_{21}+w_{11}w_{22})\otimes\tau_3)h_x(1\otimes\partial_{y_2}))(v)=\nabla_3(h_x.v);\\
&e_x.\nabla_3(v)=\\
&=\Delta^-e_x(w_{11}w_{21}\otimes\tau_3)(1\otimes\partial_{y_1})(v)+\Delta^-e_x((w_{12}w_{21}+w_{11}w_{22})\otimes\tau_3)(1\otimes\partial_{y_2})(v)\\
&=\Delta^-(w_{11}w_{21}\otimes\tau_3)e_x(1\otimes\partial_{y_1})(v)+\Delta^-((w_{12}w_{21}+w_{11}w_{22})\otimes\tau_3)e_x(1\otimes\partial_{y_2})(v)=\nabla_3(e_x.v);\\
&f_x.\nabla_3(v)=\\
&=\Delta^-f_x(w_{11}w_{21}\otimes\tau_3)(1\otimes\partial_{y_1})(v)+\Delta^-f_x((w_{12}w_{21}+w_{11}w_{22})\otimes\tau_3)(1\otimes\partial_{y_2})(v)\\
&=\Delta^-(w_{11}w_{21}\otimes\tau_3)f_x(1\otimes\partial_{y_1})(v)+\Delta^-((w_{12}w_{21}+w_{11}w_{22})\otimes\tau_3)f_x(1\otimes\partial_{y_2})(v)=\nabla_3(f_x.v);\\
&h_y.\nabla_3(v)=\\
&=-\Delta^-(w_{11}w_{21}\otimes\tau_3)(1\otimes\partial_{y_1})(v)+\Delta^-(2w_{11}w_{21}\otimes\tau_3)(1\otimes\partial_{y_1})(v)\\
&\quad+\Delta^-(w_{11}w_{21}\otimes\tau_3)(-1\otimes\partial_{y_1})(v)+\Delta^-(w_{11}w_{21}\otimes\tau_3)(1\otimes\partial_{y_1}))(h_y.v)\\
&\quad-\Delta^-((w_{12}w_{21}+w_{11}w_{22})\otimes\tau_3)(1\otimes\partial_{y_2})(v)+\Delta^-((w_{12}w_{21}+w_{11}w_{22})\otimes\tau_3)(1\otimes\partial_{y_2})(v)\\
&\quad+\Delta^-((w_{12}w_{21}+w_{11}w_{22})\otimes\tau_3)(1\otimes\partial_{y_2})(h_y.v)=\nabla_3(h_y.v);\\
&e_y.\nabla_3(v)=\\
&=\Delta^+(w_{11}w_{21}\otimes\tau_3)(1\otimes\partial_{y_1})(v)+\Delta^-(w_{11}w_{21}\otimes\tau_3)(-1\otimes\partial_{y_2})(v)\\
&\quad+\Delta^+((w_{12}w_{21}+w_{11}w_{22})\otimes\tau_3)(1\otimes\partial_{y_2})(v)+\Delta^-(2w_{11}w_{21})\otimes\tau_3)(1\otimes\partial_{y_2})+\nabla_3(e_y.v)\\
&=\Delta^-(w_{11}w_{21}\otimes\tau_3)(1\otimes\partial_{y_2})(v)+w_{12}w_{21}w_{11}\otimes\partial_{x_1}\tau_3\partial_{y_2}(v)\\
&\quad+w_{11}w_{22}w_{11}\otimes\partial_{x_1}\tau_3\partial_{y_2}(v)+w_{11}w_{22}w_{21}\otimes\partial_{x_2}\tau_3\partial_{y_2}(v)+\nabla_3(e_y.v)\\
&=\Delta^-(w_{11}w_{21}\otimes\tau_3)(1\otimes\partial_{y_2})(v)+w_{21}w_{11}w_{12}\otimes\partial_{x_1}\tau_3\partial_{y_2}(v)-w_{11}w_{21}w_{22}\otimes\partial_{x_2}\tau_3\partial_{y_2}(v)+\nabla_3(e_y.v)\\
&=\Delta^-(w_{11}w_{21}\otimes\tau_3)(1\otimes\partial_{y_2})(v)-\Delta^-(w_{11}w_{21}\otimes\tau_3)(1\otimes\partial_{y_2})(v)+\nabla_3(e_y.v)=\nabla_3(e_y.v);\\
&f_y.\nabla_3(v)=\\
&=\Delta^-((w_{12}w_{21}+w_{11}w_{22})\otimes\tau_3)(1\otimes\partial_{y_1})(v)+\Delta^-(2w_{12}w_{22}\otimes\tau_3)(1\otimes\partial_{y_2})(v)\\
&\quad+\Delta^-((w_{12}w_{21}+w_{11}w_{22}\otimes\tau_3)(-1\otimes\partial_{y_1})(v)+\nabla_3(f_y.v)=\nabla_3(f_y.v).
\end{aligned}$$

It is straightforward, using (5.10), that ∇_3 commutes with t and C.
Finally $\nabla_3\nabla=\nabla\nabla_3=0$ since there are no singular vectors of degree 4 due to Theorem 4.7. □

Let us define the maps $\widetilde{\Delta}^+:M_X\longrightarrow M_X$ and $\widetilde{\Delta}^-:M_X\longrightarrow M_X$ as follows:

$$\widetilde{\Delta}^+=w_{11}\otimes\partial_{y_1}+w_{12}\otimes\partial_{y_2}, \tag{5.11}$$

$$\widetilde{\Delta}^-=w_{21}\otimes\partial_{y_1}+w_{22}\otimes\partial_{y_2}. \tag{5.12}$$

We point out that the morphism ∇, defined in (5.7), can be expressed also by:

$$\nabla=\widetilde{\Delta}^+\partial_{x_1}+\widetilde{\Delta}^-\partial_{x_2}.$$

Remark 5.14. By (5.1) and (5.2) it is straightforward that $(\widetilde{\Delta}^+)^2=0$, $(\widetilde{\Delta}^-)^2=0$ and $\widetilde{\Delta}^+\widetilde{\Delta}^-+\widetilde{\Delta}^-\widetilde{\Delta}^+=0$.

We introduce the following notation:

$$V_{A''}=\oplus_{n\in\mathbb{Z}}V_A^{0,n}=\mathbb{C}\left[y_1,y_2\right],$$

$$\begin{aligned}V_{B''} &= \oplus_{n\in\mathbb{Z}} V_B^{0,n} = \mathbb{C}\,[y_1, y_2]_{[1,-1]}\,,\\ V_{C''} &= \oplus_{n\in\mathbb{Z}} V_C^{0,n} = \mathbb{C}\,[\partial_{y_1}, \partial_{y_2}]_{[2,0]}\,,\\ V_{D''} &= \oplus_{n\in\mathbb{Z}} V_D^{0,n} = \mathbb{C}\,[\partial_{y_1}, \partial_{y_2}]_{[1,1]}\,.\end{aligned}$$

We denote $M_{X''} = U(\mathfrak{g}_{<0}) \otimes V_{X''}$. We point out that $M_{X''}$ is the direct sum of Verma modules of Figure 4.1 in quadrant **X** that lie on the axis $m = 0$. We consider the map $\widetilde{\tau}_1 : M_{A''} \longrightarrow M_{B''}$ that is the identity. We have that:

$$\begin{aligned}[t, \widetilde{\tau}_1] &= \widetilde{\tau}_1,\\ [C, \widetilde{\tau}_1] &= -\widetilde{\tau}_1.\end{aligned} \tag{5.13}$$

We call $\widetilde{\nabla}_2 : M_{A''} \longrightarrow M_{B''}$ the map

$$\widetilde{\Delta}^-\widetilde{\Delta}^+\widetilde{\tau}_1 = w_{11}w_{21} \otimes \partial^2_{y_1} + w_{12}w_{21} \otimes \partial_{y_1}\partial_{y_2} + w_{11}w_{22} \otimes \partial_{y_1}\partial_{y_2} + w_{12}w_{22} \otimes \partial^2_{y_2}.$$

We consider the map $\widetilde{\tau}_2 : M_{D''} \longrightarrow M_{C''}$ that is the identity. We have that:

$$\begin{aligned}[t, \widetilde{\tau}_2] &= \widetilde{\tau}_2,\\ [C, \widetilde{\tau}_2] &= -\widetilde{\tau}_2.\end{aligned} \tag{5.14}$$

By abuse of notation, we also call $\widetilde{\nabla}_2 : M_{D''} \longrightarrow M_{C''}$ the map

$$\widetilde{\Delta}^-\widetilde{\Delta}^+\widetilde{\tau}_2 = w_{11}w_{21} \otimes \partial^2_{y_1} + w_{12}w_{21} \otimes \partial_{y_1}\partial_{y_2} + w_{11}w_{22} \otimes \partial_{y_1}\partial_{y_2} + w_{12}w_{22} \otimes \partial^2_{y_2}.$$

We have that, for every $n \geq 2$:

$$\widetilde{\nabla}_{2\;|M^n_{A''}} : M^n_{A''} \longrightarrow M^{n-2}_{B''}$$

and

$$\widetilde{\nabla}_{2\;|M^{-n}_{D''}} : M^{-n}_{D''} \longrightarrow M^{-n-2}_{C''}.$$

By abuse of notation we will also write $\widetilde{\nabla}_2$ instead of $\widetilde{\nabla}_{2\;|M^n_{A''}}$ and $\widetilde{\nabla}_{2\;|M^{-n}_{D''}}$. We will show that the map $\widetilde{\nabla}_2$ is the explicit expression of the morphisms of degree 2 in Figure 4.1 from the quadrant **A** to the quadrant **B** and from the quadrant **D** to the quadrant **C**.

Remark 5.15. i: The map $\widetilde{\nabla}_2 : M^n_{A''} \longrightarrow M^{n-2}_{B''}$ is constructed so that $\widetilde{\nabla}_2(v)$, for v highest weight vector in $V^n_{A''}$, is the highest weight singular vector of degree 2 in $M^{n-2}_{B''}$, classified in Theorem 4.5. Indeed, the highest weight vector in $V^n_{A''}$ is y^n_1 and we have:

$$\widetilde{\nabla}_2(y^n_1) = w_{11}w_{21} \otimes n(n-1)y^{n-2}_1,$$

that is the highest weight singular vector $\vec{m}_{2a}$ of $M(0, n-2, 1-\frac{n-2}{2}, -1-\frac{n-2}{2})$ found in Theorem 4.5.

ii: The map $\widetilde{\nabla}_2 : M^{-n}_{B''} \longrightarrow M^{-n-2}_{C''}$ is constructed so that $\widetilde{\nabla}_2(v)$, for v highest weight vector in $V^{-n}_{B''}$, is the highest weight singular vector of degree 2 in $M^{-n-2}_{C''}$, classified in Theorem 4.5. Indeed, the highest weight vector in $V^{-n}_{B''}$ is $\partial^n_{y_2}$ and we have:

$$\widetilde{\nabla}_2(\partial^n_{y_2}) = w_{11}w_{21} \otimes \partial^2_{y_1}\partial^n_{y_2} + (w_{11}w_{22} + w_{12}w_{21}) \otimes \partial_{y_1}\partial^{n+1}_{y_2} + w_{12}w_{22} \otimes \partial^{n+2}_{y_2},$$

that is the highest weight singular vector $\vec{m}_{2d}$ of $M(0, n+2, 2+\frac{n+2}{2}, -\frac{n+2}{2})$ found in Theorem 4.5.

Proposition 5.16. *The map* $\widetilde{\nabla}_2 : M_{A''} \longrightarrow M_{B''}$ *(resp.* $\widetilde{\nabla}_2 : M_{D''} \longrightarrow M_{C''}$*) is a morphism of* $\mathfrak{g}-$*modules and* $\widetilde{\nabla}_2\nabla = \nabla\widetilde{\nabla}_2 = 0$.

Proof. The map $\widetilde{\nabla}_2$ commutes with $\mathfrak{g}_{<0}$ by (5.4). By Remark 5.15 and Lemmas 5.6, 5.8 it follows that $\widetilde{\nabla}_2$ is a morphism of $\mathfrak{g}-$modules. Finally, $\widetilde{\nabla}_2\nabla = \nabla\widetilde{\nabla}_2 = 0$ follows from the fact that due to Theorem 4.6, there are no highest weight singular vectors of degree 3 in the codomain of $\widetilde{\nabla}_2\nabla$ and $\nabla\widetilde{\nabla}_2$. □

We recall the definition of the map $\tau_3 : V_A^{0,0} \longrightarrow V_C^{0,0}$ that is the identity. We have already pointed out that:

$$[t, \tau_3] = 2\tau_3,$$
$$[C, \tau_3] = 0.$$

We define the map $\widetilde{\nabla}_3 : M_A^{1,0} \longrightarrow M_C^{0,-1}$ as follows, using definition (5.4), for every $m \in M_A^{1,0}$:

$$\widetilde{\nabla}_3(m) = \widetilde{\Delta}^- \circ (w_{11}w_{12} \otimes \tau_3) \circ (1 \otimes \partial_{x_1})(m) + \widetilde{\Delta}^- \circ ((w_{21}w_{12} + w_{11}w_{22}) \otimes \tau_3) \circ (1 \otimes \partial_{x_2})(m).$$

Remark 5.17. The map $\widetilde{\nabla}_3 : M_A^{1,0} \longrightarrow M_C^{0,-1}$ is constructed so that $\widetilde{\nabla}_3(v)$, for v highest weight vector in $V_A^{1,0}$, is the highest weight singular vector of degree 3 in $M_C^{0,-1}$, classified in Theorem 4.6. Indeed, the highest weight vector in $V_A^{1,0}$ is x_1 and we have:

$$\widetilde{\nabla}_3(x_1) = w_{11}w_{12}w_{21} \otimes \partial_{y_1} + w_{11}w_{12}w_{22} \otimes \partial_{y_2} = w_{12}w_{21}w_{11} \otimes \partial_{y_1} - w_{11}w_{22}w_{12} \otimes \partial_{y_2},$$

that is the highest weight singular vector $\vec{m}_{3b}$ of $M(0, 1, \frac{5}{2}, \frac{1}{2})$ found in Theorem 4.6.

Proposition 5.18. *The map* $\widetilde{\nabla}_3$ *is a morphism of* $\mathfrak{g}-$*modules and* $\widetilde{\nabla}_3\nabla = \nabla\widetilde{\nabla}_3 = 0$.

Proof. First we show that the map ∇_3 is a morphism of $\mathfrak{g}-$modules. It commutes with $\mathfrak{g}_{<0}$ due to (5.4). Due to Lemmas 5.7 and 5.8, it is sufficient to show that $wu \otimes \widetilde{\nabla}_3(v) = u \otimes \widetilde{\nabla}_3(w.v)$ for every $w \in \mathfrak{g}_0$, $v \in V_A^{1,0}$. We have, for every $v \in V_A^{1,0}$:

$$\begin{aligned}
h_x.\widetilde{\nabla}_3(v) &= \widetilde{\nabla}_3(h_x.v) - \widetilde{\Delta}^-(w_{11}w_{12} \otimes \tau_3)(1 \otimes \partial_{x_1})(v) + 2\widetilde{\Delta}^-(w_{11}w_{12} \otimes \tau_3)(1 \otimes \partial_{x_1})(v) \\
&\quad - \widetilde{\Delta}^-(w_{11}w_{12} \otimes \tau_3)(1 \otimes \partial_{x_1})(v) - \widetilde{\Delta}^-((w_{21}w_{12} + w_{11}w_{22}) \otimes \tau_3)(1 \otimes \partial_{x_2})(v) \\
&\quad + \widetilde{\Delta}^-((w_{21}w_{12} + w_{11}w_{22}) \otimes \tau_3)(1 \otimes \partial_{x_2})(v) = \widetilde{\nabla}_3(h_x.v); \\
h_y.\widetilde{\nabla}_3(v) &= \widetilde{\nabla}_3(h_y.v) + \widetilde{\Delta}^-(w_{11}w_{12} - w_{11}w_{12} \otimes \tau_3)(1 \otimes \partial_{x_1})(v) \\
&\quad + \widetilde{\Delta}^-((w_{21}w_{12} + w_{11}w_{22} - w_{21}w_{12} - w_{11}w_{22}) \otimes \tau_3)(1 \otimes \partial_{x_2})(v) = \widetilde{\nabla}_3(h_y.v); \\
e_x.\widetilde{\nabla}_3(v) &= \widetilde{\nabla}_3(e_x.v) + \widetilde{\Delta}^+(w_{11}w_{12} \otimes \tau_3)(1 \otimes \partial_{x_1})(v) + \widetilde{\Delta}^-(w_{11}w_{12} \otimes \tau_3)(-1 \otimes \partial_{x_2})(v) \\
&\quad + \widetilde{\Delta}^+((w_{21}w_{12} + w_{11}w_{22}) \otimes \tau_3)(1 \otimes \partial_{x_2})(v) + \widetilde{\Delta}^-(2w_{11}w_{12} \otimes \tau_3)(1 \otimes \partial_{x_2})(v) \\
&= \widetilde{\nabla}_3(e_x.v) + \widetilde{\Delta}^-(w_{11}w_{12} \otimes \tau_3)(1 \otimes \partial_{x_2})(v) + w_{21}w_{12}w_{11} \otimes \partial_{y_1}\tau_3\partial_{x_2} \\
&\quad + w_{11}w_{22}w_{11} \otimes \partial_{y_1}\tau_3\partial_{x_2} + w_{11}w_{22}w_{12} \otimes \partial_{y_2}\tau_3\partial_{x_2} \\
&= \widetilde{\nabla}_3(e_x.v) + \widetilde{\Delta}^-(w_{11}w_{12} \otimes \tau_3)(1 \otimes \partial_{x_2})(v) + w_{12}w_{11}w_{21} \otimes \partial_{y_1}\tau_3\partial_{x_2} - 4\Theta w_{11} \otimes \partial_{y_1}\tau_3\partial_{x_2} \\
&\quad + 4\Theta w_{11} \otimes \partial_{y_1}\tau_3\partial_{x_2} - w_{11}w_{12}w_{22} \otimes \partial_{y_2}\tau_3\partial_{x_2} \\
&= \widetilde{\nabla}_3(e_x.v) + \widetilde{\Delta}^-(w_{11}w_{12} \otimes \tau_3)(1 \otimes \partial_{x_2})(v) - \widetilde{\Delta}^-(w_{11}w_{12} \otimes \tau_3)(1 \otimes \partial_{x_2})(v) = \widetilde{\nabla}_3(e_x.v); \\
f_x.\widetilde{\nabla}_3(v) &= \widetilde{\nabla}_3(f_x.v) + \widetilde{\Delta}^-((w_{21}w_{12} + w_{11}w_{22}) \otimes \tau_3)(1 \otimes \partial_{x_1})(v) + \widetilde{\Delta}^-(2w_{21}w_{22} \otimes \tau_3)(1 \otimes \partial_{x_2})(v) \\
&\quad + \widetilde{\Delta}^-((w_{21}w_{12} + w_{11}w_{22}) \otimes \tau_3)(-1 \otimes \partial_{x_1})(v) = \widetilde{\nabla}_3(f_x.v); \\
e_y.\widetilde{\nabla}_3(v) &= \widetilde{\nabla}_3(e_y.v) + \widetilde{\Delta}^-((w_{21}w_{11} + w_{11}w_{21}) \otimes \tau_3)(1 \otimes \partial_{x_2})(v) = \widetilde{\nabla}_3(e_y.v);
\end{aligned}$$

$$f_y.\widetilde{\nabla}_3(v) = \widetilde{\nabla}_3(f_y.v) + \widetilde{\Delta}^-((w_{22}w_{12} + w_{12}w_{22}) \otimes \tau_3)(1 \otimes \partial_{x_2})(v) = \widetilde{\nabla}_3(f_y.v).$$

It is straightforward, using (5.10), that $\widetilde{\nabla}_3$ commutes with t and C.
Finally $\widetilde{\nabla}_3\nabla = \nabla\widetilde{\nabla}_3 = 0$ since there are no singular vectors of degree 4 due to Theorem 4.7. □

5.2 Preliminaries on spectral sequences

For the proof of Theorem 5.1 we will use the theory of spectral sequences. Therefore we recall some notions about this theory; for further details see [KR1, Appendix] and [M, Chapter XI]. We follow the notation used in [KR1].
Let A be a module with a filtration:

$$... \subset F_{p-1}A \subset F_pA \subset F_{p+1}A \subset ..., \tag{5.15}$$

where $p \in \mathbb{Z}$. A filtration is called *convergent above* if $A = \cup_p F_pA$. Let us suppose that A is endowed with a differential $d : A \longrightarrow A$ such that:

$$d^2 = 0 \quad \text{and} \quad d(F_pA) \subset F_{p-s+1}A, \tag{5.16}$$

for fixed s and every p in $\mathbb{Z}$. The classical case studied in [M, Chapter XI, Section 3] corresponds to $s = 1$. We will need the case $s = 0$.
The filtration (5.15) induces a filtration on the module $H(A)$ of the homology spaces of A; indeed, for every $p \in \mathbb{Z}$, $F_pH(A)$ is defined as the image of $H(F_pA)$ under the injection $F_pA \longrightarrow A$.

Definition 5.19. Let $E = \{E_p\}_{p\in\mathbb{Z}}$ be a family of modules. A differential $d : E \longrightarrow E$ of degree $-r \in \mathbb{Z}$ is a family of homorphisms $\{d_p : E_p \longrightarrow E_{p-r}\}_{p\in\mathbb{Z}}$ such that $d_p \circ d_{p+r} = 0$ for all $p \in \mathbb{Z}$. We denote by $H(E) = H(E, d)$ the homology of E under the differential d that is the family $\{H_p(E, d)\}_{p\in\mathbb{Z}}$, where:

$$H_p(E, d) = \frac{\mathrm{Ker}(d_p : E_p \longrightarrow E_{p-r})}{\mathrm{Im}(d_{p+r} : E_{p+r} \longrightarrow E_p)}.$$

Definition 5.20 (Spectral sequence)**.** A *spectral sequence* $E = \{(E^r, d^r)\}_{r\in\mathbb{Z}}$ is a sequence of families of modules with differential (E^r, d^r) as in definition 5.19, such that, for all r, d^r has degree $-r$ and:

$$H(E^r, d^r) \cong E^{r+1}.$$

Proposition 5.21. *Let A be a module with a filtration as in* (5.15) *and differential as in* (5.16)*. Therefore it is uniquely determined a spectral sequence, as in definition 5.20, $E = \{(E^r, d^r)\}_{r\in\mathbb{Z}}$ such that:*

$$H(E^r, d^r) \cong E^{r+1}, \tag{5.17}$$

$$E^r_p \cong F_pA/F_{p-1}A \quad \textit{for} \quad r \leq s - 1, \tag{5.18}$$

$$d^r = 0 \quad \textit{for} \quad r < s - 1, \tag{5.19}$$

$$d^{s-1} = \mathrm{Gr}\, d, \tag{5.20}$$

$$E^s_p \cong H(F_pA/F_{p-1}A). \tag{5.21}$$

Proof. For the proof see [KR1, Appendix]. □

We point out, that for our purposes, (5.21) is important, because it states that E^s is isomorphic to the homology of the module $\mathrm{Gr}\, A$ with respect to the differential induced by d.

Remark 5.22. Let $\{(E^r, d^r)\}_{r\in\mathbb{Z}}$ be a spectral sequence as in definition 5.20. We know that $E^1_p \cong H_p(E^0, d^0)$. We denote $E^1_p \cong C^0_p/B^0_p$, where $C^0_p = \operatorname{Ker} d^0_p$ and $B^0_p = \operatorname{Im} d^0_{p+r}$. Analogously $E^2_p \cong H_p(E^1, d^1)$ and $E^2_p \cong C^1_p/B^1_p$, where $C^1_p/B^0_p = \operatorname{Ker} d^1_p$, $B^1_p/B^0_p = \operatorname{Im} d^1_{p+r}$ and $B^1_p \subset C^1_p$. Thus, we obtain:

$$B^0_p \subset B^1_p \subset B^2_p \subset ... \subset C^2_p \subset C^1_p \subset C^0_p.$$

Definition 5.23. Let A be a module with a filtration as in (5.15) and differential as in (5.16). Let $\{(E^r, d^r)\}_{r\in\mathbb{Z}}$ be the spectral sequence determined by Proposition 5.21. We define E^∞_p as:

$$E^\infty_p = \frac{\bigcap_r C^r_p}{\bigcup_r B^r_p}.$$

Let B be a module with a filtration as in (5.15). We say that the spectral sequence converges to B if, for all p:

$$E^\infty_p \cong F_pB/F_{p-1}B.$$

Proposition 5.24. *Let A be a module with a filtration as in* (5.15) *and differential as in* (5.16). *Let us suppose that $\cup_p F_pA = A$ and, for some N, $F_{-N}A = 0$. Then the spectral sequence converges to the homology of A, that is:*

$$E^\infty_p \cong F_pH(A)/F_{p-1}H(A).$$

Proof. For the proof see [KR1, Appendix]. □

Remark 5.25. Let A be a module with a filtration as in (5.15) and differential as in (5.16). We moreover suppose that $A = \oplus_{n\in\mathbb{Z}} A_n$ is a $\mathbb{Z}-$graded module and $d : A_n \longrightarrow A_{n-1}$ for all $n \in \mathbb{Z}$. Therefore the filtration (5.15) induces a filtration on each A_n. The family $\{F_pA_n\}_{p,n\in\mathbb{Z}}$ is indexed by (p, n). It is customary to write the indices as (p, q), where p is the degree of the filtration and $q = n - p$ is the complementary degree. The filtration is called *bounded below* if, for all $n \in \mathbb{Z}$, there exists a $s = s(n)$ such that $F_sA_n = 0$.
In this case the spectral sequence $E = \{(E^r, d^r)\}_{r\in\mathbb{Z}}$, determined as in Proposition 5.21, is a family of modules $E^r = \{E^r_{p,q}\}_{p,q\in\mathbb{Z}}$ indexed by (p, q), where $E^r_p = \sum_{p,q\in\mathbb{Z}} E^r_{p,q}$, with the differential $d^r = \{d^r_{p,q} : E_{p,q} \longrightarrow E_{p-r,q+r-1}\}_{p,q\in\mathbb{Z}}$ of bidegree $(-r, r-1)$ such that $d_{p,q} \circ d_{p+r,q-r+1} = 0$ for all $p, q \in \mathbb{Z}$. Equations (5.17), (5.18) ,(5.19), (5.20) and (5.21) can be written so that the role of q is explicit. For instance, Equation (5.17) can be written as:

$$H_{p,q}(E^r, d^r) = \frac{\operatorname{Ker}(d^r_{p,q} : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1})}{\operatorname{Im}(d^r_{p+r,q-r+1} : E^r_{p+r,q-r+1} \longrightarrow E^r_{p,q})} \cong E^{r+1}_{p,q}.$$

for all $p, q \in \mathbb{Z}$. Equation (5.21) can be written as $E^s_{p,q} \cong H(F_pA_{p+q}/F_{p-1}A_{p+q})$ for all $p, q \in \mathbb{Z}$.

We now recall some results on spectral sequences of bicomplexes; for further details see [KR1] and [M, Chapter XI, Section 6].

Definition 5.26 (Bicomplex)**.** A bicomplex K is a family $\{K_{p,q}\}_{p,q\in\mathbb{Z}}$ of modules endowed with two families of differentials, defined for all integers p, q, d' and d'' such that

$$d' : K_{p,q} \longrightarrow K_{p-1,q}, \quad d'' : K_{p,q} \longrightarrow K_{p,q-1}$$

and $d'^2 = d''^2 = d'd'' + d''d' = 0$.

We can also think K as a $\mathbb{Z}-$bigraded module where $K = \sum_{p,q \in \mathbb{Z}} K_{p,q}$. A bicomplex K as in Definition 5.26 can be represented by the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & \downarrow{\scriptstyle d''} & & \downarrow{\scriptstyle d''} & & \downarrow{\scriptstyle d''} & & \\
\cdots & \xrightarrow{d'} & K_{p+1,q+1} & \xrightarrow{d'} & K_{p,q+1} & \xrightarrow{d'} & K_{p-1,q+1} & \xrightarrow{d'} & \cdots \\
 & & \downarrow{\scriptstyle d''} & & \downarrow{\scriptstyle d''} & & \downarrow{\scriptstyle d''} & & \\
\cdots & \xrightarrow{d'} & K_{p+1,q} & \xrightarrow{d'} & K_{p,q} & \xrightarrow{d'} & K_{p-1,q} & \xrightarrow{d'} & \cdots \\
 & & \downarrow{\scriptstyle d''} & & \downarrow{\scriptstyle d''} & & \downarrow{\scriptstyle d''} & & \\
\cdots & \xrightarrow{d'} & K_{p+1,q-1} & \xrightarrow{d'} & K_{p,q-1} & \xrightarrow{d'} & K_{p-1,q-1} & \xrightarrow{d'} & \cdots \\
 & & \downarrow{\scriptstyle d''} & & \downarrow{\scriptstyle d''} & & \downarrow{\scriptstyle d''} & & .
\end{array} \tag{5.22}$$

Definition 5.27 (Second homology)**.** Let K be a bicomplex. The *second homology* of K is the homology computed with respect to d'', i.e.:

$$H''_{p,q}(K) = \frac{\operatorname{Ker}(d'' : K_{p,q} \longrightarrow K_{p,q-1})}{d''(K_{p,q+1})}.$$

The second homology of K is a bigraded complex with differential $d' : H''_{p,q}(K) \longrightarrow H''_{p-1,q}(K)$ induced by the original d'.
Its homology is defined as:

$$H'_p H''_q(K) = \frac{\operatorname{Ker}(d' : H''_{p,q}(K) \longrightarrow H''_{p-1,q})}{d'(H''_{p+1,q}(K))},$$

and it is a bigraded module.

Definition 5.28 (First homology)**.** Let K be a bicomplex. The *first homology* of K is the homology computed with respect to d', i.e.:

$$H'_{p,q}(K) = \frac{\operatorname{Ker}(d' : K_{p,q} \longrightarrow K_{p-1,q})}{d'(K_{p+1,q})}.$$

The first homology of K is a bigraded complex with differential $d'' : H'_{p,q}(K) \longrightarrow H'_{p,q-1}(K)$ induced by the original d''.
Its homology is defined as:

$$H''_q H'_p(K) = \frac{\operatorname{Ker}(d'' : H'_{p,q}(K) \longrightarrow H'_{p,q-1})}{d''(H'_{p,q+1}(K))},$$

and it is a bigraded module.

Definition 5.29 (Total complex)**.** A bicomplex K defines a single complex $T = Tot(K)$:

$$T_n = \sum_{p+q=n} K_{p,q}, \quad d = d' + d'' : T_n \longrightarrow T_{n-1}.$$

From the properties of d' and d'', it follows that $d^2 = 0$.

We point out that T_n is the sum of the modules of the secondary diagonal in diagram (5.22). We have that:

$$\cdots \xrightarrow{d} T_{n+1} \xrightarrow{d} T_n \xrightarrow{d} T_{n-1} \xrightarrow{d} \cdots .$$

The first filtration F' of $T = Tot(K)$ is defined as:

$$(F'_p T)_n = \sum_{h \leq p} K_{h,n-h}.$$

The associated spectral sequence E' is called *first spectral sequence.* Analogously we can define the second filtration and the *second spectral sequence.*

Proposition 5.30. *Let (K, d', d'') be a bicomplex with total differential d. The first spectral sequence $E' = \{(E'^r, d^r)\}$, $E'^r = \sum_{p,q} E'^r_{p,q}$ has the property:*

$$(E'^0, d^0) \cong (K, d''), \quad (E'^1, d^1) \cong (H(K, d''), d'), \quad E'^2_{p,q} \cong H'_p H''_q(K).$$

The second spectral sequence $E'' = \{(E''^r, \delta^r)\}$, $E''^r = \sum_{p,q} E''^r_{p,q}$ has the property:

$$(E''^0, \delta^0) \cong (K, d'), \quad (E''^1, \delta^1) \cong (H(K, d'), d''), \quad E''^2_{p,q} \cong H''_q H'_p(K).$$

If the first filtration is bounded below and convergent above, then the first spectral sequence converges to the homology of T with respect to the total differential d.
If the second filtration is bounded below and convergent above, then the second spectral sequence converges to the homology of T with respect to the total differential d.

Proof. See [M, Chapter XI]. □

5.3 Computation of the homology

The aim of this section is to prove Theorem 5.1. Following [KR1], let us consider the filtration of $U(\mathfrak{g}_{<0})$ defined as follows: for all $i \geq 0$, $F_i U(\mathfrak{g}_{<0})$ is the subspace of $U(\mathfrak{g}_{<0})$ spanned by elements with at most i terms of $\mathfrak{g}_{<0}$. Therefore:

$$\mathbb{C} = F_0 U(\mathfrak{g}_{<0}) \subset F_1 U(\mathfrak{g}_{<0}) \subset \ldots \subset F_{i-1} U(\mathfrak{g}_{<0}) \subset F_i U(\mathfrak{g}_{<0}) \subset \ldots ,$$

where $F_i U(\mathfrak{g}_{<0}) = \mathfrak{g}_{<0} F_{i-1} U(\mathfrak{g}_{<0}) + F_{i-1} U(\mathfrak{g}_{<0})$. We call $F_i M_X = F_i U(\mathfrak{g}_{<0}) \otimes V_X$. We have that $\nabla F_i M_X \subset F_{i+1} M_X$. Hence M_X is a filtered complex with the bigrading induced by (5.3) and differential ∇.
We can apply Propositions 5.21 and 5.24 to our complex (M_X, ∇) and obtain a spectral sequence $\{(E^i, \nabla^i)\}$ such that $E^0 = H(\operatorname{Gr} M_X)$, $E^{i+1} \cong H(E^i, \nabla^i)$ and $E^\infty \cong \operatorname{Gr} H(M_X)$.
Therefore we first study $\operatorname{Gr} M_X$.

Remark 5.31. We observe that $\mathfrak{g}$ contains a copy of $W(1,0) = \langle p(t)\partial_t \rangle$ via the injective morphism:

$$W(1,0) \longrightarrow \mathfrak{g}$$
$$p(t)\partial_t \longrightarrow \frac{p(t)}{2}$$

Indeed, let us prove that this injective map is a morphism of Lie superalgebras. In $\mathfrak{g}$:

$$\Big[\frac{p(t)}{2}, \frac{q(t)}{2}\Big] = \frac{1}{2} p(t)\partial_t q(t) - \frac{1}{2}\partial_t p(t) q(t).$$

In particular, we point out that $\mathfrak{g}_{-2}$ is contained in this copy of $W(1,0)$.

We consider the standard filtration on $W(1,0) = L^W_{-1} \supset L^W_0 \supset L^W_1 \dots$.

Lemma 5.32. *For all $i \geq 0$ and $j \geq -1$:*

$$L^W_j F_i M_X \subset F_{i-j} M_X. \tag{5.23}$$

Proof. We point out that $L^W_j \subseteq \bigoplus_{k\geq j} \mathfrak{g}_{2k}$, since $p(t)\partial_t \in L^W_{\deg(p(t))-1}$ corresponds to $\frac{p(t)}{2} \in \mathfrak{g}$ and $\deg(\frac{p(t)}{2}) = 2\deg(p(t)) - 2$.
Let us fix j and show the thesis by induction on i. It is clear that $L^W_j F_0 M_X \subset F_{-j} M_X$. Indeed let $w_j \in L^W_j$, $v \in F_0 M_X$, then:

$$w_j.v \in \begin{cases} F_0 M_X & \text{if} \quad j \geq 0; \\ F_1 M_X & \text{if} \quad j = -1. \end{cases} \tag{5.24}$$

Now let us suppose that the thesis holds for i. Let $w_j \in L^W_j$ and $u_1 u_2 \dots u_r \otimes v \in F_{i+1} M_X$, with $r \leq i+1$ and $u_1, u_2, \dots, u_r \in \mathfrak{g}_{<0}$. We moreover suppose that, for some N, $u_s = \Theta$ for all $s \leq N$ and $u_s \in \mathfrak{g}_{-1}$ for all $s > N$. We have:

$$w_j u_1 u_2 \dots u_r \otimes v = (-1)^{p(w_j)p(u_1)} u_1 w_j u_2 \dots u_r \otimes v + [w_j, u_1] u_2 \dots u_r \otimes v.$$

Using the hypothesis of induction, we know that $u_1 w_j u_2 \dots u_r \otimes v \in F_{r-j} M_X \subset F_{i+1-j} M_X$. Let us focus on $[w_j, u_1] u_2 \dots u_r \otimes v$. We have two possibilities: $[w_j, u_1] \in \oplus_{k\geq j} \mathfrak{g}_{2k-2}$ if $u_1 = \Theta$ or $[w_j, u_1] \in \oplus_{k\geq j} \mathfrak{g}_{2k-1}$ if $u_1 \in \mathfrak{g}_{-1}$.
In the case $u_1 = \Theta$, $[w_j, u_1] \in L^W_{j-1}$ and, by hypothesis of induction, $[w_j, u_1] u_2 \dots u_r \otimes v \in F_{r-j} M_X \subset F_{i+1-j} M_X$.
In the case $u_1 \in \mathfrak{g}_{-1}$, we have that $\deg([w_j, u_1] u_2 \dots u_r) \geq 2j - 1 - r + 1$ and, by our assumption, $u_2, \dots, u_r \in \mathfrak{g}_{-1}$. Therefore $[w_j, u_1] u_2 \dots u_r \otimes v \in F_{r-2j} M_X \subset F_{i+1-j} M_X$. □

By (5.23), we know, since $W(1,0) \cong \operatorname{Gr} W(1,0)$, that the action of $W(1,0)$ on M_X descends on $\operatorname{Gr} M_X$.
We point out that, using the Poincaré−Birkhoff−Witt Theorem, we have $\operatorname{Gr} U(\mathfrak{g}_{<0}) \cong S(\mathfrak{g}_{-2}) \otimes \bigwedge(\mathfrak{g}_{-1})$; indeed we have already noticed that in $U(\mathfrak{g}_{<0})$, for all $i \in \{1,2,3,4\}$, $\eta_i^2 = \Theta$.
We define:

$$\mathcal{W} = W(1,0) + \mathfrak{g}_0 = W(1,0) \oplus \mathfrak{g}_0^{ss} \oplus \mathbb{C}C,$$

that is a Lie subalgebra of $\mathfrak{g}$. On $\mathcal{W}$ we consider the filtration $\mathcal{W} = L^{\mathcal{W}}_{-1} \supset L^{\mathcal{W}}_0 \supset L^{\mathcal{W}}_1 \dots$, where $L^{\mathcal{W}}_0 = L^W_0 \oplus \mathfrak{g}_0^{ss} \oplus \mathbb{C}C$ and $L^{\mathcal{W}}_k = L^W_k$ for all $k > 0$. Therefore, as $\mathcal{W}$−modules:

$$\operatorname{Gr} M_X = \operatorname{Gr} U(\mathfrak{g}_{<0}) \otimes V_X \cong S(\mathfrak{g}_{-2}) \otimes \bigwedge(\mathfrak{g}_{-1}) \otimes V_X.$$

From (5.23), it follows that $L^{\mathcal{W}}_1 = L^W_1$ annihilates $G_X := \bigwedge(\mathfrak{g}_{-1}) \otimes V_X$. Therefore, as $\mathcal{W}$−modules:

$$\operatorname{Gr} M_X \cong S(\mathfrak{g}_{-2}) \otimes (\bigwedge(\mathfrak{g}_{-1}) \otimes V_X) \cong \operatorname{Ind}^{\mathcal{W}}_{L^{\mathcal{W}}_0}(\bigwedge(\mathfrak{g}_{-1}) \otimes V_X).$$

We observe that $\operatorname{Gr} M_X$ is a complex with the morphism induced by ∇, that we still call ∇. Indeed $\nabla F_i M_X \subset F_{i+1} M_X$ for all i, therefore it is well defined the induced morphism

$$\nabla : \operatorname{Gr}_i M_X = F_i M_X / F_{i-1} M_X \longrightarrow \operatorname{Gr}_{i+1} M_X = F_{i+1} M_X / F_i M_X,$$

that has the same formula as ∇ defined in (5.7), apart from the fact that the multiplication by the w's must be seen as multiplication in $\operatorname{Gr} U(\mathfrak{g}_{<0})$ instead of $U(\mathfrak{g}_{<0})$.
Therefore we have that (G_X, ∇) is a subcomplex of $(\operatorname{Gr} M_X, \nabla)$: indeed it is sufficient to restrict

∇ to G_X; the complex $(\mathrm{Gr}\, M_X, \nabla)$ is obtained from (G_X, ∇) extending the coefficients to $S(\mathfrak{g}_{-2})$. We point out that also the homology spaces $H^{m,n}(G_X)$ are annihilated by $L_1^{\mathcal{W}}$. Therefore, as $\mathcal{W}-$modules:

$$H^{m,n}(\mathrm{Gr}\, M_X) \cong S(\mathfrak{g}_{-2}) \otimes H^{m,n}(G_X) \cong \mathrm{Ind}_{L_0^{\mathcal{W}}}^{\mathcal{W}}(H^{m,n}(G_X)). \tag{5.25}$$

From (5.25) and Proposition 5.24, it follows that:

Proposition 5.33. *If $H^{m,n}(G_X) = 0$, then $H^{m,n}(\mathrm{Gr}\, M_X) = 0$ and therefore $H^{m,n}(M_X) = 0$.*

5.3.1 Homology of complexes G_X

Motivated by Proposition 5.33, in this section we study the homology of the complexes G_X's. We denote by $G_{X'} := \bigwedge(\mathfrak{g}_{-1}) \otimes V_{X'}$.
Let us consider the evaluation maps from V_X to $V_{X'}$ that map $y_1, y_2, \partial_{y_1}, \partial_{y_2}$ to zero and are the identity on all other elements. We can compose these maps with ∇_2 when $X = A, B$ and obtain new maps, that we still call ∇_2, from G_A to $G_{D'}$ and from G_B to $G_{C'}$ respectively.
We consider also the map from $G_{A'}$ to G_D (resp. from $G_{B'}$ to G_C) that is the composition of $\nabla_2 : G_{A'} \longrightarrow G_{D'}$ (resp.$\nabla_2 : G_{B'} \longrightarrow G_{C'}$) and the inclusion of $G_{D'}$ into G_D (resp. $G_{C'}$ into G_C); we will call also this composition ∇_2. We define:

$$\begin{aligned} G_{A^\circ} &= \mathrm{Ker}(\nabla_2 : G_A \longrightarrow G_{D'}), \quad & G_{D^\circ} &= \mathrm{CoKer}(\nabla_2 : G_{A'} \longrightarrow G_D), \\ G_{B^\circ} &= \mathrm{Ker}(\nabla_2 : G_B \longrightarrow G_{C'}), \quad & G_{C^\circ} &= \mathrm{CoKer}(\nabla_2 : G_{B'} \longrightarrow G_C). \end{aligned}$$

Remark 5.34. The map ∇ is still defined on G_{X° since $\nabla\nabla_2 = \nabla_2\nabla = 0$.
The bigrading (5.3) induces a bigrading also on the G_{X°'s. We point out that $G_A^{m,n} = G_{A^\circ}^{m,n}$ for $n > 0$, $G_D^{m,n} = G_{D^\circ}^{m,n}$ for $n < 0$, $G_B^{m,n} = G_{B^\circ}^{m,n}$ for $n > 0$ and $G_C^{m,n} = G_{C^\circ}^{m,n}$ for $n < 0$.
The complexes (G_{X°, ∇) start or end at the axes of Figure 4.1. Thus for us:

$$H^{m,n}(G_{A^\circ}) = \begin{cases} \frac{\mathrm{Ker}(\nabla : G_{A^\circ}^{m,n} \longrightarrow G_{A^\circ}^{m-1,n-1})}{\mathrm{Im}(\nabla : G_{A^\circ}^{m+1,n+1} \longrightarrow G_{A^\circ}^{m,n})} & \text{for } m > 0, n > 0; \\ \frac{G_{A^\circ}^{m,n}}{\mathrm{Im}(\nabla : G_{A^\circ}^{m+1,n+1} \longrightarrow G_{A^\circ}^{m,n})} & \text{for } m = 0 \text{ or } n = 0; \end{cases}$$

$$H^{m,n}(G_{D^\circ}) = \begin{cases} \frac{\mathrm{Ker}(\nabla : G_{D^\circ}^{m,n} \longrightarrow G_{D^\circ}^{m-1,n-1})}{\mathrm{Im}(\nabla : G_{D^\circ}^{m+1,n+1} \longrightarrow G_{D^\circ}^{m,n})} & \text{for } m > 0, n < 0; \\ \frac{G_{D^\circ}^{m,n}}{\mathrm{Im}(\nabla : G_{D^\circ}^{m+1,n+1} \longrightarrow G_{D^\circ}^{m,n})} & \text{for } m = 0; \\ \mathrm{Ker}(\nabla : G_{D^\circ}^{m,n} \longrightarrow G_{D^\circ}^{m-1,n-1}) & \text{for } n = 0; \end{cases}$$

$$H^{m,n}(G_{B^\circ}) = \begin{cases} \frac{\mathrm{Ker}(\nabla : G_{B^\circ}^{m,n} \longrightarrow G_{B^\circ}^{m-1,n-1})}{\mathrm{Im}(\nabla : G_{B^\circ}^{m+1,n+1} \longrightarrow G_{B^\circ}^{m,n})} & \text{for } m < 0, n > 0; \\ \mathrm{Ker}(\nabla : G_{B^\circ}^{m,n} \longrightarrow G_{B^\circ}^{m-1,n-1}) & \text{for } m = 0; \\ \frac{G_{B^\circ}^{m,n}}{\mathrm{Im}(\nabla : G_{B^\circ}^{m+1,n+1} \longrightarrow G_{B^\circ}^{m,n})} & \text{for } n = 0; \end{cases}$$

$$H^{m,n}(G_{C^\circ}) = \begin{cases} \frac{\mathrm{Ker}(\nabla : G_{C^\circ}^{m,n} \longrightarrow G_{C^\circ}^{m-1,n-1})}{\mathrm{Im}(\nabla : G_{C^\circ}^{m+1,n+1} \longrightarrow G_{C^\circ}^{m,n})} & \text{for } m < 0, n < 0; \\ \mathrm{Ker}(\nabla : G_{C^\circ}^{m,n} \longrightarrow G_{C^\circ}^{m-1,n-1}) & \text{for } m = 0 \text{ or } n = 0. \end{cases}$$

Remark 5.35. The following relations are straightforward from the definition of the G_{X°'s and Remark 5.34:

$$\begin{aligned} H^{m,n}(G_A) &= H^{m,n}(G_{A^\circ}) & &\text{for } m > 0, n \geq 0; \\ H^{m,n}(G_D) &= H^{m,n}(G_{D^\circ}) & &\text{for } m > 0, n \leq 0; \\ H^{m,n}(G_B) &= H^{m,n}(G_{B^\circ}) & &\text{for } m < 0, n \geq 0; \\ H^{m,n}(G_C) &= H^{m,n}(G_{C^\circ}) & &\text{for } m < 0, n \leq 0. \end{aligned}$$

Motivated by Remark 5.35 and Proposition 5.33, we study the homology of the complexes G_{X°'s.
Now we introduce an additional bigrading as follows:

$$\begin{aligned}(V_X)_{[p,q]} &= \{f \in V_X \, : \, y_1\partial_{y_1} f = pf \text{ and } y_2\partial_{y_2} f = qf\}, \\ (G_X)_{[p,q]} &= \wedge(\mathfrak{g}_{-1}) \otimes (V_X)_{[p,q]}.\end{aligned} \tag{5.26}$$

The definition can be extended also to G_{X°. We point out that this new bigrading is related to the bigrading (5.3) by the equation $p+q=n$.
We have that $d' := \Delta^+\partial_{y_1} : (G_X)_{[p,q]} \longrightarrow (G_X)_{[p-1,q]}$, $d'' := \Delta^-\partial_{y_2} : (G_X)_{[p,q]} \longrightarrow (G_X)_{[p,q-1]}$ and $d = d' + d'' : \oplus_m G_X^{m,n} \longrightarrow \oplus_m G_X^{m,n-1}$.
We know, by Remark 5.4, that $d'^2 = d''^2 = d'd'' + d''d' = 0$. Therefore $\oplus_m G_X^m$ and $\oplus_m G_{X^\circ}^m$, with the bigrading (5.26), are bicomplexes with differentials d', d'' and total differential $\nabla = d' + d''$.
Now let:

$$\wedge_+^i = \wedge^i \langle w_{11}, w_{21}\rangle \quad \text{and} \quad \wedge_-^i = \wedge^i\langle w_{12}, , w_{22}\rangle.$$

We point out that $\wedge_+^i$ and $\wedge_-^i$ are isomorphic as $\langle x_1\partial_{x_1} - x_2\partial_{x_2}, x_1\partial_{x_2}, x_2\partial_{x_1}\rangle$−modules; therefore, in the following results, we will often write $\wedge^i$ when we are speaking of the $\langle x_1\partial_{x_1} - x_2\partial_{x_2}, x_1\partial_{x_2}, x_2\partial_{x_1}\rangle$−module isomorphic to $\wedge_+^i$ and $\wedge_-^i$.
We introduce the following notation, for all $\alpha, \beta \in \mathbb{Z}$:

$$G_A(\alpha,\beta)_{[p,q]} = \wedge_+^{\alpha-p}\wedge_-^{\beta-q} \otimes \mathbb{C}[x_1,x_2]\, y_1^p y_2^q, \qquad G_B(\alpha,\beta)_{[p,q]} = \wedge_+^{\alpha-p}\wedge_-^{\beta-q} \otimes \mathbb{C}[\partial_{x_1},\partial_{x_2}]\, y_1^p y_2^q,$$
$$G_D(\alpha,\beta)_{[p,q]} = \wedge_+^{\alpha-p}\wedge_-^{\beta-q} \otimes \mathbb{C}[x_1,x_2]\, \partial_{y_1}^{-p}\partial_{y_2}^{-q}, \qquad G_C(\alpha,\beta)_{[p,q]} = \wedge_+^{\alpha-p}\wedge_-^{\beta-q} \otimes \mathbb{C}[\partial_{x_1},\partial_{x_2}]\, \partial_{y_1}^{-p}\partial_{y_2}^{-q}.$$

From now on we will use the notation $\wedge_\pm^i[x_1,x_2]$ (resp. $\wedge_\pm^i[\partial_{x_1},\partial_{x_2}]$) for $\wedge_\pm^i \otimes \mathbb{C}[x_1,x_2]$ (resp. $\wedge_\pm^i \otimes \mathbb{C}[\partial_{x_1},\partial_{x_2}]$).
We have that, as $\langle x_1\partial_{x_1} - x_2\partial_{x_2}, x_1\partial_{x_2}, x_2\partial_{x_1}\rangle$−modules, $G_X = \oplus_{\alpha,\beta} G_X(\alpha,\beta)$, where $G_X(\alpha,\beta) = \oplus_{p,q} G_X(\alpha,\beta)_{[p,q]}$.
Analogously we can define $G_{X^\circ}(\alpha,\beta)_{[p,q]}$ and, as $\langle x_1\partial_{x_1} - x_2\partial_{x_2}, x_1\partial_{x_2}, x_2\partial_{x_1}\rangle$−modules we have: $G_{X^\circ} = \oplus_{\alpha,\beta} G_{X^\circ}(\alpha,\beta)$, where $G_{X^\circ}(\alpha,\beta) = \oplus_{p,q} G_{X^\circ}(\alpha,\beta)_{[p,q]}$.
The $G_X(\alpha,\beta)$'s and $G_{X^\circ}(\alpha,\beta)$'s are bicomplexes, with the bigrading (5.26) and differentials $d' = \Delta^+\partial_{y_1}$ and $d'' = \Delta^-\partial_{y_2}$.
The computation of homologies of G_X and G_{X° can be reduced to the computation for $G_X(\alpha,\beta)$ and $G_{X^\circ}(\alpha,\beta)$.
To prove the following results we will use Proposition 5.30.
In the following lemmas we compute the homology of the $G_{X^\circ}(\alpha,\beta)$'s. We start with the homology of the $G_{X^\circ}(\alpha,\beta)$'s when either α or β do not lie in $\{0,1,2\}$.

Lemma 5.36. *Let us suppose that $\alpha > 2$ or $\beta > 2$. Let $k = max(\alpha,\beta)$. Then as $\langle x_1\partial_{x_1} - x_2\partial_{x_2}, x_1\partial_{x_2}, x_2\partial_{x_1}\rangle$−modules:*

$$H^{m,n}(G_{A^\circ}(\alpha,\beta)) = H^{m,n}(G_A(\alpha,\beta)) \cong \begin{cases} 0 & \text{if } m>0 \text{ or } m=0,\ n<k, \\ \wedge^{\alpha+\beta-n} & \text{if } m=0,\ n\geq k; \end{cases}$$

$$H^{m,n}(G_{B^\circ}(\alpha,\beta)) = H^{m,n}(G_B(\alpha,\beta)) \cong \begin{cases} 0 & \text{if } m<0 \text{ or } m=0,\ n<k-2, \\ \wedge^{\alpha+\beta-n-2} & \text{if } m=0,\ n\geq k-2. \end{cases}$$

Let us suppose that $\alpha < 0$ or $\beta < 0$. Let $k = min(\alpha,\beta)$. Then, as $\langle x_1\partial_{x_1} - x_2\partial_{x_2}, x_1\partial_{x_2}, x_2\partial_{x_1}\rangle$−modules:

$$H^{m,n}(G_{D^\circ}(\alpha,\beta)) = H^{m,n}(G_D(\alpha,\beta)) \cong \begin{cases} 0 & \text{if } m>0 \text{ or } m=0,\ n>k, \\ \wedge^{\alpha+\beta-n} & \text{if } m=0,\ n\leq k; \end{cases}$$

$$H^{m,n}(G_{C^\circ}(\alpha,\beta)) = H^{m,n}(G_C(\alpha,\beta)) \cong \begin{cases} 0 & \text{if } m<0 \text{ or } m=0,\ n>k-2, \\ \bigwedge^{\alpha+\beta-n-2} & \text{if } m=0,\ n\le k-2. \end{cases}$$

Proof. We first observe that if $\alpha > 2$ or $\beta > 2$ (resp. $\alpha < 0$ or $\beta < 0$), then $G_{X^\circ}(\alpha,\beta) = G_X(\alpha,\beta)$ for $X = A, B$ (resp. $X = C, D$), since they are different only when $p+q=0$, that does not occur here. We use the theory of spectral sequences of bicomplexes.
We prove the statement in the case $\beta > 2$ for $X = A, B$ and $\beta < 0$ for $X = C, D$; the case $\alpha > 2$ for $X = A, B$ and $\alpha < 0$ for $X = C, D$ can be proved analogously using the second spectral sequence instead of the first one.

Case A) Let us consider $G_{A^\circ}(\alpha,\beta)$ with the differential $d'' = \Delta^-\partial_{y_2}$:

$$\ldots \xleftarrow{\Delta^-\partial_{y_2}} \textstyle\bigwedge_+^{\alpha-p}\bigwedge_-^{\beta-q+1}[x_1,x_2]\,y_1^p y_2^{q-1} \xleftarrow{\Delta^-\partial_{y_2}} \bigwedge_+^{\alpha-p}\bigwedge_-^{\beta-q}[x_1,x_2]\,y_1^p y_2^q$$
$$\xleftarrow{\Delta^-\partial_{y_2}} \textstyle\bigwedge_+^{\alpha-p}\bigwedge_-^{\beta-q-1}[x_1,x_2]\,y_1^p y_2^{q+1} \xleftarrow{\Delta^-\partial_{y_2}} \ldots .$$

This complex is the tensor product of $\bigwedge_+^{\alpha-p} y_1^p$ and the following complex, since $\bigwedge_+^{\alpha-p} y_1^p$ is not involved in d'':

$$0 \xleftarrow{\Delta^-\partial_{y_2}} \textstyle\bigwedge_-^2[x_1,x_2]\,y_2^{\beta-2} \xleftarrow{\Delta^-\partial_{y_2}} \bigwedge_-^1[x_1,x_2]\,y_2^{\beta-1} \xleftarrow{\Delta^-\partial_{y_2}} \bigwedge_-^0[x_1,x_2]\,y_2^{\beta} \xleftarrow{\Delta^-\partial_{y_2}} 0.$$

This complex is exact except for the right end, in which the homology space is $\mathbb{C}y_2^\beta$. Let us analyze in detail.

i: Let us consider the map $\Delta^-\partial_{y_2} : \bigwedge_-^0[x_1,x_2]\,y_2^\beta \longrightarrow \bigwedge_-^1[x_1,x_2]\,y_2^{\beta-1}$. We compute the kernel. Let $p(x_1,x_2)y_2^\beta \in \bigwedge_-^0[x_1,x_2]\,y_2^\beta$. We have:

$$\Delta^-\partial_{y_2}(p(x_1,x_2)y_2^\beta) = w_{12}\otimes\partial_{x_1}p(x_1,x_2)\beta y_2^{\beta-1} + w_{22}\otimes\partial_{x_2}p(x_1,x_2)\beta y_2^{\beta-1}.$$

It is zero if and only if p is constant, therefore the kernel is $\mathbb{C}y_2^\beta$.

ii: Let us consider the map $\Delta^-\partial_{y_2} : \bigwedge_-^1[x_1,x_2]\,y_2^{\beta-1} \longrightarrow \bigwedge_-^2[x_1,x_2]\,y_2^{\beta-2}$. We compute the kernel. Let $w_{12}\otimes p_1(x_1,x_2)y_2^{\beta-1} + w_{22}\otimes p_2(x_1,x_2)y_2^{\beta-1} \in \bigwedge_-^1[x_1,x_2]\,y_2^{\beta-1}$. We have:

$$\begin{aligned}&\Delta^-\partial_{y_2}(w_{12}\otimes p_1(x_1,x_2)y_2^{\beta-1} + w_{22}\otimes p_2(x_1,x_2)y_2^{\beta-1})\\ &\quad= w_{12}w_{22}\otimes\partial_{x_2}p_1(x_1,x_2)(\beta-1)y_2^{\beta-2} + w_{22}w_{12}\otimes\partial_{x_1}p_2(x_1,x_2)(\beta-1)y_2^{\beta-2}.\end{aligned}$$

This is zero if and only if $\partial_{x_2}p_1(x_1,x_2) = \partial_{x_1}p_2(x_1,x_2)$, that means that $p_1(x_1,x_2) = \int\partial_{x_1}p_2(x_1,x_2)dx_2$. Hence an element of the kernel is:

$$w_{12}\otimes\Big(\int\partial_{x_1}p_2(x_1,x_2)dx_2\Big)y_2^{\beta-1} + w_{22}\otimes p_2(x_1,x_2)y_2^{\beta-1} = \Delta^-\partial_{y_2}\Big(\Big(\int p_2(x_1,x_2)dx_2\Big)\frac{y_2^\beta}{\beta}\Big).$$

Thus at this point the sequence is exact.

iii: We consider the map $\Delta^-\partial_{y_2} : \bigwedge_-^2[x_1,x_2]\,y_2^{\beta-2} \longrightarrow 0$. We have that:

$$w_{12}w_{22}\otimes p(x_1,x_2)y_2^{\beta-2} = \Delta^-\partial_{y_2}\Big(w_{12}\otimes\Big(\int p(x_1,x_2)dx_2\Big)\frac{y_2^{\beta-1}}{\beta-1}\Big).$$

Since the original complex was the tensor product with $\bigwedge_+^{\alpha-p} y_1^p$, we have that the non zero homology group is $\bigwedge_+^{\alpha-p} y_1^p y_2^\beta$. $E'^1_{p,q}(G_{A^\circ}(\alpha,\beta))$ survives only for $q=\beta$. Now we should compute its homology with respect to d', but the $E'^1_{p,q}(G_{A^\circ}(\alpha,\beta))$'s do not involve x_1, x_2, so the differentials d''s are zero and we have $E'^2 = E'^1$. Moreover, for a one−row spectral sequence, we know that $E'^2 = ... = E'^\infty$ since, for all $r \geq 2$ and all $p \in \mathbb{Z}$, $d^r_{p,\beta}$ has bidegree $(-r, r-1)$, i.e. $d^r_{p,\beta} : E^r_{p,\beta} \longrightarrow E^r_{p-r,\beta+r-1} = 0$, $d^r_{p+r,\beta-r+1} : E^r_{p+r,\beta-r+1} = 0 \longrightarrow E^r_{p,\beta}$. We have:

$$E'^\infty_{p,q}(G_A(\alpha,\beta)) = \begin{cases} 0 & \text{if } q \neq \beta, \\ \bigwedge_+^{\alpha-p} y_1^p y_2^\beta & \text{if } q = \beta. \end{cases}$$

We observe that the first filtration $(F'_p(G_A(\alpha,\beta)))_n = \sum_{h \leq p} (G_A(\alpha,\beta))_{[h,n-h]}$ is bounded below, since $F'_{-1} = 0$, and it is convergent above. Therefore by Proposition 5.30:

$$\sum_m H^{m,n}(G_A(\alpha,\beta)) \cong \sum_{p+q=n} E'^\infty_{p,q}(G_A(\alpha,\beta)) = E'^\infty_{n-\beta,\beta}(G_A(\alpha,\beta)) \cong \bigwedge_+^{\alpha+\beta-n} y_1^{n-\beta} y_2^\beta.$$

Since there are no x_1's and x_2's involved, this means that $H^{m,n}(G_A(\alpha,\beta)) = 0$ if $m \neq 0$ and $H^{0,n}(G_A(\alpha,\beta)) = \bigwedge_+^{\alpha+\beta-n} y_1^{n-\beta} y_2^\beta \cong \bigwedge^{\alpha+\beta-n}$ as $\langle x_1\partial_{x_1} - x_2\partial_{x_2}, x_1\partial_{x_2}, x_2\partial_{x_1}\rangle$−modules.

Case D) In the case of $G_D(\alpha,\beta)$, using the same argument, when $\beta < 0$ we obtain:

$$E'^\infty_{p,q}(G_D(\alpha,\beta)) = \begin{cases} 0 & \text{if } q \neq \beta, \\ \bigwedge_+^{\alpha-p} \partial_{y_1}^{-p} \partial_{y_2}^{-\beta} & \text{if } q = \beta. \end{cases}$$

Therefore:

$$\sum_m H^{m,n}(G_D(\alpha,\beta)) \cong \sum_{p+q=n} E'^\infty_{p,q}(G_D(\alpha,\beta)) = E'^\infty_{n-\beta,\beta}(G_D(\alpha,\beta)) \cong \bigwedge_+^{\alpha+\beta-n} \partial_{y_1}^{-n+\beta} \partial_{y_2}^{-\beta}.$$

Since there are no x_1's and x_2's involved, this means that $H^{m,n}(G_D(\alpha,\beta)) = 0$ if $m \neq 0$ and $H^{0,n}(G_D(\alpha,\beta)) = \bigwedge_+^{\alpha+\beta-n} \partial_{y_1}^{-n+\beta} \partial_{y_2}^{-\beta} \cong \bigwedge^{\alpha+\beta-n}$ as $\langle x_1\partial_{x_1} - x_2\partial_{x_2}, x_1\partial_{x_2}, x_2\partial_{x_1}\rangle$−modules.

Case B) In the case of $G_B(\alpha,\beta)$ when $\beta > 2$ we have the following complex with the differential $d'' = \Delta^-\partial_{y_2}$:

$$\leftarrow \bigwedge_+^{\alpha-p} \bigwedge_-^{\beta-q+1} [\partial_{x_1}, \partial_{x_2}]\, y_1^p y_2^{q-1} \xleftarrow{\Delta^-\partial_{y_2}} \bigwedge_+^{\alpha-p} \bigwedge_-^{\beta-q} [\partial_{x_1}, \partial_{x_2}]\, y_1^p y_2^q \xleftarrow{\Delta^-\partial_{y_2}} \bigwedge_+^{\alpha-p} \bigwedge_-^{\beta-q-1} [\partial_{x_1}, \partial_{x_2}]\, y_1^p y_2^{q+1}.$$

This complex is the tensor product of $\bigwedge_+^{\alpha-p} y_1^p$ and the following complex, since $\bigwedge_+^{\alpha-p} y_1^p$ is not involved in d'':

$$0 \xleftarrow{\Delta^-\partial_{y_2}} \bigwedge_-^2 [\partial_{x_1}, \partial_{x_2}]\, y_2^{\beta-2} \xleftarrow{\Delta^-\partial_{y_2}} \bigwedge_-^1 [\partial_{x_1}, \partial_{x_2}]\, y_2^{\beta-1} \xleftarrow{\Delta^-\partial_{y_2}} \bigwedge_-^0 [\partial_{x_1}, \partial_{x_2}]\, y_2^{\beta} \xleftarrow{\Delta^-\partial_{y_2}} 0.$$

This complex is exact except for the left end, in which the homology space is $\mathbb{C}\bigwedge_-^2 y_2^{\beta-2}$. Let us analyze in detail.

i: Let us consider the map $\Delta^-\partial_{y_2} : \bigwedge_-^0 [\partial_{x_1}, \partial_{x_2}]\, y_2^\beta \longrightarrow \bigwedge_-^1 [\partial_{x_1}, \partial_{x_2}]\, y_2^{\beta-1}$. We compute the kernel. Let $p(\partial_x, \partial_{x_2}) y_2^\beta \in \bigwedge_-^0 [\partial_{x_1}, \partial_{x_2}]\, y_2^\beta$. We have:

$$\Delta^-\partial_{y_2}(p(\partial_{x_1}, \partial_{x_2}) y_2^\beta) = w_{12} \otimes \partial_{x_1} p(\partial_{x_1}, \partial_{x_2}) \beta y_2^{\beta-1} + w_{22} \otimes \partial_{x_2} p(\partial_x, \partial_{x_2}) \beta y_2^{\beta-1}.$$

It is zero if and only if $\partial_{x_1} p(\partial_{x_1}, \partial_{x_2}) = \partial_{x_2} p(\partial_{x_1}, \partial_{x_2}) = 0$, that is $p = 0$, therefore the kernel is 0.

ii: Let us consider the map $\Delta^-\partial_{y_2} : \bigwedge_-^1 [\partial_{x_1}, \partial_{x_2}]\, y_2^{\beta-1} \longrightarrow \bigwedge_-^2 [\partial_{x_1}, \partial_{x_2}]\, y_2^{\beta-2}$. We compute the kernel. Let $w_{12} \otimes p_1(\partial_{x_1}, \partial_{x_2}) y_2^{\beta-1} + w_{22} \otimes p_2(\partial_{x_1}, \partial_{x_2}) y_2^{\beta-1} \in \bigwedge_-^1 [\partial_{x_1}, \partial_{x_2}]\, y_2^{\beta-1}$. We have:

$$\begin{aligned}\Delta^-\partial_{y_2}(w_{12} \otimes p_1(\partial_{x_1}, \partial_{x_2}) y_2^{\beta-1} &+ w_{22} \otimes p_2(\partial_{x_1}, \partial_{x_2}) y_2^{\beta-1}) \\ &= w_{12}w_{22} \otimes \partial_{x_2} p_1 (\beta-1) y_2^{\beta-2} + w_{22}w_{12} \otimes \partial_{x_1} p_2 (\beta-1) y_2^{\beta-2}.\end{aligned}$$

This is zero if and only if $\partial_{x_2} p_1(\partial_{x_1}, \partial_{x_2}) = \partial_{x_1} p_2(\partial_{x_1}, \partial_{x_2})$, that means that $p_1(x_1, x_2) = \frac{\partial_{x_1} p_2(x_1,x_2)}{\partial_{x_2}}$ (in particular p_2 must contain at least one ∂_{x_2}). Therefore an element of the kernel is:

$$w_{12} \otimes \frac{\partial_{x_1} p_2(\partial_{x_1}, \partial_{x_2})}{\partial_{x_2}} y_2^{\beta-1} + w_{22} \otimes p_2(\partial_x, \partial_y) y_2^{\beta-1} = \Delta^-\partial_{y_2}\Big(\frac{p_2(\partial_{x_1}, \partial_{x_2})}{\partial_{x_2}} \frac{y_2^\beta}{\beta}\Big).$$

At this point the sequence is exact.

iii: Let us consider the map $\Delta^-\partial_{y_2} : \bigwedge_-^2 [\partial_{x_1}, \partial_{x_2}]\, y_2^{\beta-2} \longrightarrow 0$. Obviously every element $w_{12}w_{22} \otimes p(\partial_{x_1}, \partial_y) y_2^{\beta-2} \in \bigwedge_-^2 [\partial_{x_1}, \partial_{x_2}]\, y_2^{\beta-2}$ lies in the kernel. If p contains at least one ∂_{x_1}, then:

$$w_{12}w_{22} \otimes p(\partial_{x_1}, \partial_{x_2}) y_2^{\beta-2} = \Delta^-\partial_{y_2}\Big(-w_{22} \otimes \frac{p(\partial_{x_1}, \partial_{x_2})}{\partial_{x_1}} \frac{y_2^{\beta-1}}{\beta-1}\Big).$$

If p contains at least one ∂_{x_2}, then:

$$-w_{12}w_{22} \otimes p(\partial_{x_1}, \partial_{x_2}) y_2^{\beta-2} = \Delta^-\partial_{y_2}\Big(w_{12} \otimes \frac{p(\partial_{x_1}, \partial_{x_2})}{\partial_{x_2}} \frac{y_2^{\beta-1}}{\beta-1}\Big).$$

If p is constant, it does not belong to the image of $\Delta^-\partial_{y_2}$, then the homology group is isomorphic to $\mathbb{C}\bigwedge_-^2 y_2^{\beta-2}$.

Since the original complex was the tensor product with $\bigwedge_+^{\alpha-p} y_1^p$ we have that the non zero homology group is $\bigwedge_+^{\alpha-p}\bigwedge_-^2 y_1^p y_2^{\beta-2}$. The space $E'^1_{p,q}(G_{B^\circ}(\alpha,\beta))$ survives only for $q = \beta - 2$. We have that $E'^1_{p,q} \cong E'^2_{p,q}$ because the map d' is 0 on the $E'^1_{p,q}$'s (the image of the map d' always involves elements of positive degree in ∂_{x_1} or ∂_{x_2} that are 0 in $E'^1_{p,q}$ for the previous computation).
Since we have a one row spectral sequence, we have $E'^2 = ... = E'^\infty$. We have:

$$E'^\infty_{p,q}(G_B(\alpha,\beta)) = \begin{cases} 0 & \text{if } q \neq \beta-2, \\ \bigwedge_+^{\alpha-p}\bigwedge_-^2 y_1^p y_2^{\beta-2} & \text{if } q = \beta-2. \end{cases}$$

We observe that the first filtration $(F'_p(G_B(\alpha,\beta)))_n = \sum_{h\leq p}(G_B(\alpha,\beta))_{[h,n-h]}$ is bounded below, since $F'_{-1} = 0$, and it is convergent above. Therefore by Proposition 5.30:

$$\sum_m H^{m,n}(G_B(\alpha,\beta)) \cong \sum_{p+q=n} E'^\infty_{p,q}(G_B(\alpha,\beta)) = E'^\infty_{n-\beta+2,\beta-2}(G_B(\alpha,\beta)) \cong \bigwedge_+^{\alpha+\beta-n-2}\bigwedge_-^2 y_1^{n-\beta+2} y_2^{\beta-2}.$$

Since there are no ∂_{x_1}'s and ∂_{x_2}'s involved, this means that $H^{m,n}(G_B(\alpha,\beta)) = 0$ if $m \neq 0$ and $H^{0,n}(G_B(\alpha,\beta)) = \bigwedge_+^{\alpha+\beta-n-2}\bigwedge_-^2 y_1^{n-\beta+2} y_2^{\beta-2} \cong \bigwedge^{\alpha+\beta-n-2}$ as $\langle x_1\partial_{x_1} - x_2\partial_{x_2}, x_1\partial_{x_2}, x_2\partial_{x_1}\rangle$−modules.

Case C) In the case of $G_C(\alpha,\beta)$, using the same argument, when $\beta < 0$ we obtain:

$$E'^\infty_{p,q}(G_C(\alpha,\beta)) = \begin{cases} 0 & \text{if } q \neq \beta-2, \\ \bigwedge_+^{\alpha-p}\bigwedge_-^2 \partial_{y_1}^{-p}\partial_{y_2}^{-\beta+2} & \text{if } q = \beta-2. \end{cases}$$

Therefore:

$$\sum_m H^{m,n}(G_C(\alpha,\beta)) \cong \sum_{p+q=n} E'^{\infty}_{p,q}(G_C(\alpha,\beta)) = E'^{\infty}_{n-\beta+2,\beta-2}(G_C(\alpha,\beta)) \cong \bigwedge_+^{\alpha+\beta-n-2}\bigwedge_-^2 \partial_{y_1}^{-n+\beta-2}\partial_{y_2}^{-\beta+2}.$$

Since there are no ∂_{x_1}'s and ∂_{x_2}'s involved, this means that $H^{m,n}(G_C(\alpha,\beta)) = 0$ if $m \neq 0$ and $H^{0,n}(G_C(\alpha,\beta)) = \bigwedge_+^{\alpha+\beta-n-2}\partial_{y_1}^{-n+\beta-2}\partial_{y_2}^{-\beta+2} \cong \bigwedge^{\alpha+\beta-n-2}$ as $\langle x_1\partial_{x_1} - x_2\partial_{x_2}, x_1\partial_{x_2}, x_2\partial_{x_1}\rangle$−modules. □

In Lemma 5.36 we computed the homology of the $G_{X^\circ}(\alpha,\beta)$'s in the case that either α or β do not belong to $\{0,1,2\}$. In order to compute the homology of the $G_{X^\circ}(\alpha,\beta)$'s in the case that both α and β belong to $\{0,1,2\}$, we need the following remark and lemmas.

Remark 5.37. We introduce some notation that will be used in the following lemmas. Let $0 < \beta \leq 2$. Let us define:

$$\widetilde{G}_A(\alpha,\beta)_{[p,q]} = \begin{cases} \bigwedge_+^{\alpha-p}\bigwedge_-^{\beta-q}[x_1,x_2] & \text{if } p \geq 0, q \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We have an isomorphism of bicomplexes $\gamma : G_A(\alpha,\beta)_{[p,q]} \longrightarrow \widetilde{G}_A(\alpha,\beta)_{[p,q]}$ which is the valuating map that values y_1 and y_2 in 1 and is the identity on all other elements. We consider on $\widetilde{G}_A(\alpha,\beta)$ the differentials $d' = \Delta^+$ and $d'' = \Delta^-$ induced by $\Delta^+\partial_{y_1}$ and $\Delta^-\partial_{y_2}$ for $G_A(\alpha,\beta)$. We also define:

$$G_{D'}(\alpha,\beta)_{[p,q]} = \begin{cases} \bigwedge_+^{\alpha+1}\bigwedge_-^{\beta+1}[x_1,x_2] & \text{if } p = q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The following is a commutative diagram:

$$\begin{array}{ccc} G_A(\alpha,\beta) & \xrightarrow{\nabla_2} & G_{D'}(\alpha,\beta) \\ \gamma \big\downarrow & & \big\downarrow id \\ \widetilde{G}_A(\alpha,\beta) & \xrightarrow{\Delta^-\Delta^+\tau_1} & G_{D'}(\alpha,\beta). \end{array}$$

We have that $\widetilde{G}_{A^\circ}(\alpha,\beta) := \mathrm{Ker}(\Delta^-\Delta^+\tau_1 : \widetilde{G}_A(\alpha,\beta) \longrightarrow G_{D'}(\alpha,\beta))$ is isomorphic, as a bicomplex, to $G_{A^\circ}(\alpha,\beta)$. Its diagram is the same of $\widetilde{G}_A(\alpha,\beta)$ except for $p = q = 0$. The diagram of $\widetilde{G}_A(\alpha,\beta)$ is the following, respectively for $\alpha = 0$, $\alpha = 1$, $\alpha \geq 2$:

$$\begin{array}{c} \alpha = 0 \\ \bigwedge_+^0\bigwedge_-^0[x_1,x_2] \\ \downarrow \\ \cdots \\ \downarrow \\ \bigwedge_+^0\bigwedge_-^\beta[x_1,x_2], \end{array} \quad \begin{array}{ccc} & \alpha = 1 & \\ \bigwedge_+^1\bigwedge_-^0[x_1,x_2] & \leftarrow & \bigwedge_+^0\bigwedge_-^0[x_1,x_2] \\ \downarrow & & \downarrow \\ \cdots & & \cdots \\ \downarrow & & \downarrow \\ \bigwedge_+^1\bigwedge_-^\beta[x_1,x_2] & \leftarrow & \bigwedge_+^0\bigwedge_-^\beta[x_1,x_2], \end{array} \quad \begin{array}{ccccc} & & \alpha \geq 2 & & \\ \bigwedge_+^2\bigwedge_-^0[x_1,x_2] & \leftarrow & \bigwedge_+^1\bigwedge_-^0[x_1,x_2] & \leftarrow & \bigwedge_+^0\bigwedge_-^0[x_1,x_2] \\ \downarrow & & \downarrow & & \downarrow \\ \cdots & & \cdots & & \cdots \\ \downarrow & & \downarrow & & \downarrow \\ \bigwedge_+^2\bigwedge_-^\beta[x_1,x_2] & \leftarrow & \bigwedge_+^1\bigwedge_-^\beta[x_1,x_2] & \leftarrow & \bigwedge_+^0\bigwedge_-^\beta[x_1,x_2], \end{array}$$

where the horizontal maps are d' and the vertical maps are d''. The diagram of $\widetilde{G}_{A^\circ}(\alpha,\beta)$ is analogous to this, except for $p = q = 0$, where $\bigwedge_+^\alpha\bigwedge_-^\beta[x_1,x_2]$ is substituted by $\mathrm{Ker}(\Delta^-\Delta^+ : \bigwedge_+^\alpha\bigwedge_-^\beta[x_1,x_2] \longrightarrow \bigwedge_+^{\alpha+1}\bigwedge_-^{\beta+1}[x_1,x_2])$, that we shortly call $\mathrm{Ker}(\Delta^-\Delta^+)$ in the next diagram.

The E'^1 spectral sequence of $\widetilde{G}_{A^\circ}(\alpha,\beta)$, i.e. the homology with respect to Δ^-, is the following, respectively for $\alpha = 0$, $\alpha = 1$, $\alpha \geq 2$, $\beta = 1$ and $\alpha = 0$, $\alpha = 1$, $\alpha \geq 2$, $\beta = 2$ (the computation is analogous to Lemma 5.36):

$$\begin{array}{c}\alpha=0,\beta=1\\ \bigwedge_+^0\\ \downarrow\\ \frac{\mathrm{Ker}(\Delta^-\Delta^+)}{\mathrm{Im}(\Delta^-)},\end{array}\qquad \begin{array}{ccc} & \alpha=1,\beta=1 & \\ \bigwedge_+^1 & \leftarrow & \bigwedge_+^0\\ \downarrow & & \downarrow\\ \frac{\mathrm{Ker}(\Delta^-\Delta^+)}{\mathrm{Im}(\Delta^-)} & \leftarrow & \frac{\bigwedge_+^0\bigwedge_-^1[x_1,x_2]}{\mathrm{Im}(\Delta^-)},\end{array}\qquad \begin{array}{ccccc} & & \alpha\geq 2,\beta=1 & & \\ \bigwedge_+^2 & \leftarrow & \bigwedge_+^1 & \leftarrow & \bigwedge_+^0\\ \downarrow & & \downarrow & & \downarrow\\ \frac{\bigwedge_+^2\bigwedge_-^1[x_1,x_2]}{\mathrm{Im}(\Delta^-)} & \leftarrow & \frac{\bigwedge_+^1\bigwedge_-^1[x_1,x_2]}{\mathrm{Im}(\Delta^-)} & \leftarrow & \frac{\bigwedge_+^0\bigwedge_-^1[x_1,x_2]}{\mathrm{Im}(\Delta^-)}.\end{array}$$

$$\begin{array}{c}\alpha=0,\beta=2\\ \bigwedge_+^0\\ \downarrow\\ 0\\ \downarrow\\ \frac{\mathrm{Ker}(\Delta^-\Delta^+)}{\mathrm{Im}(\Delta^-)},\end{array}\qquad \begin{array}{ccc} & \alpha=1,\beta=2 & \\ \bigwedge_+^1 & \leftarrow & \bigwedge_+^0\\ \downarrow & & \downarrow\\ 0 & \leftarrow & 0\\ \downarrow & & \downarrow\\ \frac{\mathrm{Ker}(\Delta^-\Delta^+)}{\mathrm{Im}(\Delta^-)} & \leftarrow & \frac{\bigwedge_+^0\bigwedge_-^2[x_1,x_2]}{\mathrm{Im}(\Delta^-)},\end{array}\qquad \begin{array}{ccccc} & & \alpha\geq 2,\beta=2 & & \\ \bigwedge_+^2 & \leftarrow & \bigwedge_+^1 & \leftarrow & \bigwedge_+^0\\ \downarrow & & \downarrow & & \downarrow\\ 0 & \leftarrow & 0 & \leftarrow & 0\\ \downarrow & & \downarrow & & \downarrow\\ \frac{\bigwedge_+^2\bigwedge_-^2[x_1,x_2]}{\mathrm{Im}(\Delta^-)} & \leftarrow & \frac{\bigwedge_+^1\bigwedge_-^2[x_1,x_2]}{\mathrm{Im}(\Delta^-)} & \leftarrow & \frac{\bigwedge_+^0\bigwedge_-^2[x_1,x_2]}{\mathrm{Im}(\Delta^-)}.\end{array}$$

We have that, in the diagram of the E'^1 spectral sequence, only the rows for $q=0$ and $q=\beta$ are different from 0. The previous diagram will be the first step in Lemma 5.40 for the computation of the homology of the $\widetilde{G}_{A^\circ}(\alpha,\beta)$'s when $\alpha,\beta\in\{0,1,2\}$.
Analogously we define, for $0\leq\beta<2$:

$$\widetilde{G}_C(\alpha,\beta)_{[p,q]}=\begin{cases}\bigwedge_+^{\alpha-p}\bigwedge_-^{\beta-q}[\partial_{x_1},\partial_{x_2}] & \text{if } p\leq 0, q\leq 0,\\ 0 & \text{otherwise.}\end{cases}$$

We have an isomorphism of bicomplexes $\gamma: G_C(\alpha,\beta)_{[p,q]}\longrightarrow\widetilde{G}_C(\alpha,\beta)_{[p,q]}$ which is the valuating map that values ∂_{y_1} and ∂_{y_2} in 1 and is the identity on all other elements. We consider on $\widetilde{G}_C(\alpha,\beta)$ the differentials $d'=\Delta^+$ and $d''=\Delta^-$ induced by $\Delta^+\partial_{y_1}$ and $\Delta^-\partial_{y_2}$ for $G_C(\alpha,\beta)$. We also define:

$$G_{B'}(\alpha,\beta)_{[p,q]}=\begin{cases}\bigwedge_+^{\alpha-1}\bigwedge_-^{\beta-1}[\partial_{x_1},\partial_{x_2}] & \text{if } p=q=0,\\ 0 & \text{otherwise.}\end{cases}$$

We have the following commutative diagram:

$$\begin{array}{ccc} G_{B'}(\alpha,\beta) & \xrightarrow{\ \nabla_2\ } & G_C(\alpha,\beta)\\ id\downarrow & & \downarrow\gamma\\ G_{B'}(\alpha,\beta) & \xrightarrow{\ \Delta^-\Delta^+\tau_2\ } & \widetilde{G}_C(\alpha,\beta).\end{array}$$

We have that $\widetilde{G}_{C^\circ}(\alpha,\beta):=\mathrm{CoKer}(\Delta^-\Delta^+\tau_2: G_{B'}(\alpha,\beta)\longrightarrow\widetilde{G}_C(\alpha,\beta))$ is isomorphic, as a bicomplex, to $G_{C^\circ}(\alpha,\beta)$. Its diagram is the same of $\widetilde{G}_C(\alpha,\beta)$ except for $p=q=0$. In the following diagram we shortly write $\mathrm{CoKer}(\Delta^-\Delta^+)$ for:

$$\mathrm{CoKer}(\Delta^-\Delta^+:\bigwedge_+^{\alpha-1}\bigwedge_-^{\beta-1}[\partial_{x_1},\partial_{x_2}]\longrightarrow\bigwedge_+^{\alpha}\bigwedge_-^{\beta}[\partial_{x_1},\partial_{x_2}]).$$

The diagram of the bicomplex $\widetilde{G}_{C^\circ}(\alpha,\beta)$ is the following, respectively for $\alpha=2$, $\alpha=1$ and $\alpha\leq 0$:

$$\begin{array}{cccccc}
\alpha=2 & & \alpha=1 & & \alpha\le 0 & \\
\mathrm{CoKer}(\Delta^-\Delta^+), & \bigwedge_+^2\bigwedge_-^\beta[\partial_{x_1},\partial_{x_2}] \leftarrow & \mathrm{CoKer}(\Delta^-\Delta^+), & \bigwedge_+^2\bigwedge_-^\beta[\partial_{x_1},\partial_{x_2}] \leftarrow & \bigwedge_+^1\bigwedge_-^\beta[\partial_{x_1},\partial_{x_2}] \leftarrow & \bigwedge_+^0\bigwedge_-^\beta[\partial_{x_1},\partial_{x_2}] \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\bigwedge_+^2\bigwedge_-^2[\partial_{x_1},\partial_{x_2}], & \bigwedge_+^2\bigwedge_-^2[\partial_{x_1},\partial_{x_2}] \leftarrow & \bigwedge_+^1\bigwedge_-^2[\partial_{x_1},\partial_{x_2}], & \bigwedge_+^2\bigwedge_-^2[\partial_{x_1},\partial_{x_2}] \leftarrow & \bigwedge_+^1\bigwedge_-^2[\partial_{x_1},\partial_{x_2}], \leftarrow & \bigwedge_+^0\bigwedge_-^2[\partial_{x_1},\partial_{x_2}],
\end{array}$$

where the horizontal maps are d' and the vertical maps are d''.
In the following diagram we shortly write $\mathrm{Ker}(\Delta^-)_{i,j}$ for:

$$\mathrm{Ker}(\Delta^- : \bigwedge_+^i\bigwedge_-^j[\partial_{x_1},\partial_{x_2}] \longrightarrow \bigwedge_+^i\bigwedge_-^{j+1}[\partial_{x_1},\partial_{x_2}]),$$

and we shortly write $\frac{\mathrm{Ker}(\Delta^-)}{\mathrm{Im}(\Delta^-\Delta^+)}$ for:

$$\frac{\mathrm{Ker}(\Delta^- : \bigwedge_+^\alpha\bigwedge_-^\beta[\partial_{x_1},\partial_{x_2}] \longrightarrow \bigwedge_+^\alpha\bigwedge_-^{\beta+1}[\partial_{x_1},\partial_{x_2}])}{\Delta^-\Delta^+ : \bigwedge_+^{\alpha-1}\bigwedge_-^{\beta-1}[\partial_{x_1},\partial_{x_2}] \longrightarrow \bigwedge_+^\alpha\bigwedge_-^\beta[\partial_{x_1},\partial_{x_2}]}.$$

The E'^1 spectral sequence of $\widetilde{G}_{C^\circ}(\alpha,\beta)$ is the following, respectively for $\alpha=2$, $\alpha=1$, $\alpha\le 0$, $\beta=1$ and $\alpha=2$, $\alpha=1$, $\alpha\le 0$, $\beta=0$ (the computation is analogous to Lemma 5.36):

$$\begin{array}{cccccc}
\alpha=2,\beta=1 & \alpha=1,\beta=1 & & & \alpha\le 0,\beta=1 & \\
\frac{\mathrm{Ker}(\Delta^-)}{\mathrm{Im}(\Delta^-\Delta^+)} & \mathrm{Ker}(\Delta^-)_{2,1} \leftarrow & \frac{\mathrm{Ker}(\Delta^-)}{\mathrm{Im}(\Delta^-\Delta^+)} & \mathrm{Ker}(\Delta^-)_{2,1} \leftarrow & \mathrm{Ker}(\Delta^-)_{1,1} \leftarrow & \mathrm{Ker}(\Delta^-)_{0,1} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\bigwedge_+^2\bigwedge_-^2, & \bigwedge_+^2\bigwedge_-^2 \leftarrow & \bigwedge_+^1\bigwedge_-^2, & \bigwedge_+^2\bigwedge_-^2 \leftarrow & \bigwedge_+^1\bigwedge_-^2 \leftarrow & \bigwedge_+^0\bigwedge_-^2.
\end{array}$$

$$\begin{array}{cccccc}
\alpha=2,\beta=0 & \alpha=1,\beta=0 & & & \alpha\le 0,\beta=0 & \\
\frac{\mathrm{Ker}(\Delta^-)}{\mathrm{Im}(\Delta^-\Delta^+)} & \mathrm{Ker}(\Delta^-)_{2,0} \leftarrow & \frac{\mathrm{Ker}(\Delta^-)}{\mathrm{Im}(\Delta^-\Delta^+)} & \mathrm{Ker}(\Delta^-)_{2,0} \leftarrow & \mathrm{Ker}(\Delta^-)_{1,0} \leftarrow & \mathrm{Ker}(\Delta^-)_{0,0} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 \leftarrow & 0 & 0 \leftarrow & 0 \leftarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\bigwedge_+^2\bigwedge_-^2 & \bigwedge_+^2\bigwedge_-^2 \leftarrow & \bigwedge_+^1\bigwedge_-^2, & \bigwedge_+^2\bigwedge_-^2 \leftarrow & \bigwedge_+^1\bigwedge_-^2 \leftarrow & \bigwedge_+^0\bigwedge_-^2.
\end{array}$$

We have that only the rows $q=0$ and $q=\beta-2$ are different from 0. We point out that, since $\beta<2$:

$$\frac{\mathrm{Ker}(\Delta^-)}{\mathrm{Im}(\Delta^-\Delta^+)} \cong \frac{\Delta^-(\bigwedge_+^\alpha\bigwedge_-^{\beta-1}[\partial_{x_1},\partial_{x_2}])}{\Delta^-\Delta^+(\bigwedge_+^{\alpha-1}\bigwedge_-^{\beta-1}[\partial_{x_1},\partial_{x_2}])} \cong \mathrm{CoKer}(\Delta^-(\bigwedge_+^{\alpha-1}\bigwedge_-^{\beta-1}[\partial_{x_1},\partial_{x_2}]) \xrightarrow{\Delta^+} \Delta^-(\bigwedge_+^\alpha\bigwedge_-^{\beta-1}[\partial_{x_1},\partial_{x_2}])).$$

The isomorphism holds because $\beta<2$ and we know, by Lemma 5.36, that

$$0 \xrightarrow{\Delta^-} \bigwedge_-^0[\partial_{x_1},\partial_{x_2}] \xrightarrow{\Delta^-} \bigwedge_-^1[\partial_{x_1},\partial_{x_2}] \xrightarrow{\Delta^-} \bigwedge_-^2[\partial_{x_1},\partial_{x_2}] \xrightarrow{\Delta^-} 0$$

is exact except for the right end.
The previous diagram will be the first step in Lemma 5.40 for the computation of the homology of the $\widetilde{G}_{C^\circ}(\alpha,\beta)$'s when $\alpha,\beta\in\{0,1,2\}$.

The following two technical lemmas will be used in the proof of Lemma 5.40 for the computation of the homology of the $G_{X^\circ}(\alpha,\beta)$'s when $\alpha,\beta \in \{0,1,2\}$.

Lemma 5.38. *Let $0 \le \beta \le 2$. Let us consider the complex $S(\alpha,\beta)$ defined as follows:*

$$S(\alpha,\beta)_\alpha \xleftarrow{\Delta^+} \ldots \xleftarrow{\Delta^+} \Delta^-(\wedge_+^k \wedge_-^\beta[x_1,x_2]) \xleftarrow{\Delta^+} \ldots \xleftarrow{\Delta^+} \Delta^-(\wedge_+^0 \wedge_-^\beta[x_1,x_2]),$$

where $S(\alpha,\beta)_\alpha = \mathrm{Ker}(\Delta^-(\wedge_+^\alpha \wedge_-^\beta[x_1,x_2]) \xrightarrow{\Delta^+} \Delta^-(\wedge_+^{\alpha+1}\wedge_-^\beta[x_1,x_2]))$. The homology spaces of the complex $S(\alpha,\beta)$, from left to right, are respectively isomorphic to:

$$H_\alpha(S(\alpha,\beta)) \cong \wedge^{\alpha+\beta+1},\ \ldots,\ H_k(S(\alpha,\beta)) \cong \wedge^{k+1+\beta},\ \ldots,\ H_0(S(\alpha,\beta)) \cong \wedge^{\beta+1}.$$

Proof. We first focus on $0 < \beta \le 2$. In order to make the proof more clear, we show the statement for $\beta = 1$ that is more significant; the proof for $\beta = 2$ is analogous. We observe that, due to the definition of $S(\alpha,1)$, $H_i(S(\alpha,1)) = H_i(S(\alpha+1,1))$ for $0 \le i \le \alpha$, then it is sufficient to compute it for large α. We take $\alpha > 2$. For sake of simplicity, we choose $\alpha = 3$. We point out that the complex $S(3,1)$ reduces to:

$$0 \xleftarrow{\Delta^+} \Delta^-(\wedge_+^2\wedge_-^1[x_1,x_2]) \xleftarrow{\Delta^+} \Delta^-(\wedge_+^1\wedge_-^1[x_1,x_2]) \xleftarrow{\Delta^+} \Delta^-(\wedge_+^0\wedge_-^1[x_1,x_2]).$$

In this case the thesis reduces to show that:

$$H_3(S(3,1)) \cong 0, \quad H_2(S(3,1)) \cong 0, \quad H_1(S(3,1)) \cong 0, \quad H_0(S(3,1)) \cong \wedge_+^2.$$

We use that the complex $S(3,1)$ is isomorphic, via Δ^-, to the complex:

$$0 \xleftarrow{\Delta^+} \frac{\wedge_+^2\wedge_-^1[x_1,x_2]}{\mathrm{Im}(\Delta^-)} \xleftarrow{\Delta^+} \frac{\wedge_+^1\wedge_-^1[x_1,x_2]}{\mathrm{Im}(\Delta^-)} \xleftarrow{\Delta^+} \frac{\wedge_+^0\wedge_-^1[x_1,x_2]}{\mathrm{Im}(\Delta^-)},$$

that is exactly the row for $q = 0$ in the diagram of the E'^1 spectral sequence of $\widetilde{G}_{A^\circ}(3,1)$ in Remark 5.37. In particular, since $\alpha = 3$, this is the row for $q = 0$ and values of p respectively $0, 1, 2$ and 3 from the left to the right. The fact that the two complexes are isomorphic follows from $\beta = 1 > 0$ and the fact that by Lemma 5.36 we know that

$$0 \xrightarrow{\Delta^-} \wedge_-^0[x_1,x_2] \xrightarrow{\Delta^-} \wedge_-^1[x_1,x_2] \xrightarrow{\Delta^-} \wedge_-^2[x_1,x_2] \xrightarrow{\Delta^-} 0$$

is exact except for the left end.
We have that, since $E'^2(G_{A^\circ}(3,1))$ has two nonzero rows for $q = 0$ and $q = 1$ (see the diagram in Remark 5.37), the differentials are all zero except for $d^r_{p,q}$ for $r = \beta + 1 = 2$, $q = 0$, $1 < p \le 3$. Indeed $1 < p \le 3$ because:

$$d^2_{p,0} : E'^2_{p,0} \longrightarrow E'^2_{p-2,1}$$

and we have that $E'^2_{p-2,1} = 0$ if $p - 2 < 0$ and $E'^2_{p,0} = 0$ if $p > 3$.
Since the homologies of $G_{A^\circ}(3,1)$ and $\widetilde{G}_{A^\circ}(3,1)$ are isomorphic, by Lemma 5.36 we have:

$$\sum_{p+q=n} E'^\infty_{p,q}(\widetilde{G}_{A^\circ}(3,1)) = \begin{cases} 0 & \text{if } n < 3, \\ \wedge_+^1 & \text{if } n = 3. \end{cases} \tag{5.27}$$

From this relation we obtain that $d^2_{p,0}$, for $1 < p \le 3$, must be an isomorphism. Indeed, let us first show that $d^2_{p,0}$, for $1 < p \le 3$, is surjective. We have:

$$d^2_{p,0} : E'^2_{p,0}(\widetilde{G}_{A^\circ}(3,1)) \longrightarrow E'^2_{p-2,1}(\widetilde{G}_{A^\circ}(3,1)). \tag{5.28}$$

We know that $E'^2_{p-2,1}(\widetilde{G}_{A^\circ}(3,1)) \cong \wedge_+^{\alpha+\beta-p+1} = \wedge_+^{5-p}$ using an argument similar to Lemma 5.36. But, by (5.27), we know that for $n = p-1 < 3$:

$$\sum_{\widetilde{p}+\widetilde{q}=p-1} E'^{\infty}_{\widetilde{p},\widetilde{q}}(\widetilde{G}_{A^\circ}(3,1)) = 0.$$

Moreover $d^r = 0$ for $r > 2$ and $d^2_{p-2,1} = 0$. Therefore $d^2_{p,0}$ must be surjective.
Let us see that $d^2_{p,0}$ is injective. If $p < 3$, then $E'^{\infty}_{p,0}(\widetilde{G}_{A^\circ}(3,1)) = 0$ since it appears in the sum

$$\sum_{\widetilde{p}+\widetilde{q}=p} E'^{\infty}_{\widetilde{p},\widetilde{q}}(\widetilde{G}_{A^\circ}(3,1)) = 0,$$

by (5.27). Moreover

$$d^2_{p+2,-1} : E'^2_{p+2,-1}(\widetilde{G}_{A^\circ}(3,1)) = 0 \longrightarrow E'^2_{p,0}(\widetilde{G}_{A^\circ}(3,1))$$

is identically 0.
Hence $\mathrm{Ker}(d^2_{p,0}) = 0$.
If $p = 3$, we know, by (5.27), that

$$\sum_{\widetilde{p}+\widetilde{q}=p} E'^{\infty}_{\widetilde{p},\widetilde{q}}(\widetilde{G}_{A^\circ}(3,1)) \cong \wedge^1_+ \tag{5.29}$$

and $E'^{\infty}_{p,0}(\widetilde{G}_{A^\circ}(3,1))$ appears in this sum.
Moreover we know that

$$E'^2_{2,1}(\widetilde{G}_{A^\circ}(3,1)) = E'^{\infty}_{2,1}(\widetilde{G}_{A^\circ}(3,1)) \cong \wedge^1_+,$$

since $d^r = 0$, when $r > 2$, $d^2_{4,0} = d^2_{2,1} = 0$ and $E'^2_{2,1}(\widetilde{G}_{A^\circ}(3,1)) \cong \wedge^1_+$ due to a reasoning similar to Lemma 5.36.
Since $E'^{\infty}_{2,1}(\widetilde{G}_{A^\circ}(3,1))$ also appears in the sum (5.29), we conclude that $E'^{\infty}_{p,0}(\widetilde{G}_{A^\circ}(3,1)) = 0$.
But

$$d^2_{p+2,-1} : E'^2_{p+2,-1}(\widetilde{G}_{A^\circ}(3,1)) = 0 \longrightarrow E'^2_{p,0}(\widetilde{G}_{A^\circ}(3,1))$$

is identically 0, then $\mathrm{Ker}(d^2_{p,0}) = 0$.
Therefore we obtain from the isomorphism that $E'^2_{p,0}(\widetilde{G}_{A^\circ}(3,1)) \cong \wedge^{5-p}_+$.
Hence:

$$H_3(S(3,1)) \cong 0, \quad H_2(S(3,1)) \cong 0, \quad H_1(S(3,1)) \cong 0, \quad H_0(S(3,1)) \cong \wedge^2_+.$$

We now prove the statement in the case $\beta = 0$. Due to the definition of $S(\alpha,0)$, $H_i(S(\alpha,0)) = H_i(S(\alpha+1,0))$ for $0 \leq i \leq \alpha$, then it is sufficient to compute it for large α. For sake of simplicity, we choose $\alpha = 2$. We point out that the complex $S(2,0)$ reduces to:

$$\Delta^-(\wedge^2_+\wedge^0_-[x_1,x_2]) \xleftarrow{\Delta^+} \Delta^-(\wedge^1_+\wedge^0_-[x_1,x_2]) \xleftarrow{\Delta^+} \Delta^-(\wedge^0_+\wedge^0_-[x_1,x_2]).$$

In this case the thesis reduces to show that:

$$H_2(S(2,0)) \cong 0, \quad H_1(S(2,0)) \cong \wedge^2_+, \quad H_0(S(2,0)) \cong \wedge^1_+.$$

We compute the homology spaces by direct computations.

i: Let us compute $H_0(S(2,0))$. We take $p(x_1,x_2) \in \bigwedge_+^0 \bigwedge_-^0[x_1,x_2]$; an element in $\Delta^-(\bigwedge_+^0\bigwedge_-^0[x_1,x_2])$ has the following form:

$$P := w_{12} \otimes \partial_{x_1} p + w_{22} \otimes \partial_{x_2} p.$$

Hence:

$$\Delta^+(P) = w_{12}w_{11} \otimes \partial_{x_1}^2 p + w_{12}w_{21} \otimes \partial_{x_1}\partial_{x_2} p + w_{22}w_{11} \otimes \partial_{x_1}\partial_{x_2} p + w_{22}w_{21} \otimes \partial_{x_2}^2 p.$$

Therefore P lies in the kernel if and only if $\partial_{x_1}^2 p = \partial_{x_1}\partial_{x_2} p = \partial_{x_2}^2 p = 0$, that is $p = ax_1 + bx_2$, for $a, b \in \mathbb{C}$. Thus $H_0(S(2,0)) \cong \bigwedge^1$.

ii: Let us compute $H_1(S(2,0))$. We take $w_{11}p(x_1,x_2) + w_{21}q(x_1,x_2) \in \bigwedge_+^1\bigwedge_-^0[x_1,x_2]$; an element in $\Delta^-(\bigwedge_+^1\bigwedge_-^0[x_1,x_2])$ has the following form:

$$P := w_{11}w_{12} \otimes \partial_{x_1} p + w_{11}w_{22} \otimes \partial_{x_2} p + w_{21}w_{12} \otimes \partial_{x_1} q + w_{21}w_{22} \otimes \partial_{x_2} q.$$

Hence:

$$\Delta^+(P) = w_{11}w_{12}w_{21} \otimes \partial_{x_1}\partial_{x_2} p + w_{11}w_{22}w_{21} \otimes \partial_{x_2}^2 p + w_{21}w_{12}w_{11} \otimes \partial_{x_1}^2 q + w_{21}w_{22}w_{11} \otimes \partial_{x_1}\partial_{x_2} q.$$

Therefore P lies in the kernel if and only if:

$$\begin{cases} \partial_{x_1}\partial_{x_2} p - \partial_{x_1}^2 q = 0, \\ \partial_{x_2}^2 p - \partial_{x_1}\partial_{x_2} q = 0. \end{cases}$$

We obtain that:

$$\begin{cases} \partial_{x_1} q = \int \partial_{x_1}\partial_{x_2} p dx_1 = \partial_{x_2} p + Q_2(x_2), \\ \partial_{x_1} q = \int \partial_{x_2}^2 p dx_2 = \partial_{x_2} p + Q_1(x_1), \end{cases}$$

where $Q_1(x_1)$ (resp. $Q_2(x_2)$) is a polynomial expression costant in x_2 (resp. costant in x_1). Therefore, if P lies in the kernel then $\partial_{x_1} q = \partial_{x_2} p + a$, with $a \in \mathbb{C}$. Let us consider an element of the kernel, we obtain that:

$$\begin{aligned} P =& w_{11}w_{12} \otimes \partial_{x_1} p + w_{11}w_{22} \otimes \partial_{x_2} p + w_{21}w_{12} \otimes (\partial_{x_2} p + a) + w_{21}w_{22} \otimes \int \partial_{x_2}^2 p dx_1 \\ =& \Delta^+\big(- w_{12} \otimes p - w_{22} \otimes \int \partial_{x_2} p dx_1\big) + w_{21}w_{12} \otimes a = \Delta^+\big(\Delta^-\big(- \int p dx_1\big)\big) + w_{21}w_{12} \otimes a. \end{aligned}$$

We point out that $w_{21}w_{12} \otimes a$ does not lie in the image of the map $\Delta^-(\bigwedge_+^0\bigwedge_-^0[x_1,x_2]) \xrightarrow{\Delta^+} \Delta^-(\bigwedge_+^1\bigwedge_-^0[x_1,x_2])$, because $w_{21}w_{12} \otimes a = \Delta^+(-w_{12} \otimes ax_2)$ but $-w_{12} \otimes ax_2 \notin \Delta^-(\bigwedge_+^0\bigwedge_-^0[x_1,x_2])$. Thus $H_1(S(2,0)) \cong \bigwedge^2$.

iii: Let us compute $H_2(S(2,0))$. We take $w_{11}w_{21}p(x_1,x_2) \in \bigwedge_+^2\bigwedge_-^0[x_1,x_2]$; an element in $\Delta^-(\bigwedge_+^2\bigwedge_-^0[x_1,x_2])$ has the following form:

$$P := w_{11}w_{21}w_{12} \otimes \partial_{x_1} p + w_{11}w_{21}w_{22} \otimes \partial_{x_2} p.$$

We point out that:

$$P = \Delta^+(-w_{11}w_{12} \otimes \int \partial_{x_1} p dx_2 - w_{11}w_{22} \otimes p) = \Delta^+(\Delta^-(-w_{11} \otimes \int p dx_2)).$$

Therefore every element of $\Delta^-(\bigwedge_+^2\bigwedge_-^0[x_1,x_2])$ lies in the image of the map $\Delta^-(\bigwedge_+^1\bigwedge_-^0[x_1,x_2]) \xrightarrow{\Delta^+} \Delta^-(\bigwedge_+^2\bigwedge_-^0[x_1,x_2])$. Thus $H_0(S(2,0)) \cong 0$.

□

Lemma 5.39. *Let $0 \leq \beta \leq 2$. Let us consider the complex $T(\alpha,\beta)$ defined as follows:*

$$\Delta^-(\wedge_+^2\wedge_-^{\beta-1}[\partial_{x_1},\partial_{x_2}]) \xleftarrow{\Delta^+} \dots \xleftarrow{\Delta^+} \Delta^-(\wedge_+^k\wedge_-^{\beta-1}[\partial_{x_1},\partial_{x_2}]) \xleftarrow{\Delta^+} \dots \xleftarrow{\Delta^+} \xleftarrow{\Delta^+} \mathrm{CoKer}(\Delta^-(\wedge_+^{\alpha-1}\wedge_-^{\beta-1}[\partial_{x_1},\partial_{x_2}]) \xrightarrow{\Delta^+} \Delta^-(\wedge_+^{\alpha}\wedge_-^{\beta-1}[\partial_{x_1},\partial_{x_2}])). \tag{5.30}$$

The homology spaces of the complex $T(\alpha,\beta)$, from left to right, are respectively isomorphic to:

$$H_2(T(\alpha,\beta)) \cong \wedge_+^{\beta-1}, \ \dots, \ H_k(T(\alpha,\beta)) \cong \wedge_+^{k+\beta-3}, \ \dots, \ H_\alpha(T(\alpha,\beta)) \cong \wedge_+^{-1+\alpha+\beta-2}.$$

Proof. We first point out that the statement is obvious for $\beta=0$ since in this case the complex is trivial and the homology spaces are obviously trivial.
We now focus on $\beta=1$. The complex $T(\alpha,\beta)$, due to its construction, has the property that $H_i(T(\alpha,\beta)) = H_i(T(\alpha-1,\beta))$ for $\alpha \leq i \leq 2$; then we can compute the homology for small α. Let us take $\alpha<0$.
For sake of simplicity we focus on $\alpha=-1$. In this case the complex $T(-1,1)$ reduces to:

$$\Delta^-(\wedge_+^2\wedge_-^0[\partial_{x_1},\partial_{x_2}]) \xleftarrow{\Delta^+} \Delta^-(\wedge_+^1\wedge_-^0[\partial_{x_1},\partial_{x_2}]) \xleftarrow{\Delta^+} \Delta^-(\wedge_+^0\wedge_-^0[\partial_{x_1},\partial_{x_2}]) \xleftarrow{\Delta^+} 0. \tag{5.31}$$

The thesis reduces to:

$$H_2(T(-1,1)) \cong \wedge_+^0, \ H_1(T(-1,1)) \cong 0, \ H_0(T(-1,1)) \cong 0, \ H_{-1}(T(-1,1)) \cong 0.$$

In order to prove the thesis, we use that the complex $T(-1,1)$ is isomorphic, via Δ^-, to the row for $q=0$ in the diagram of the E'^1 spectral sequence of $\widetilde{G}_{C^\circ}(-1,1)$ in Remark 5.37, that is:

$$\mathrm{Ker}(\Delta^-)_{2,1} \xleftarrow{\Delta^+} \mathrm{Ker}(\Delta^-)_{1,1} \xleftarrow{\Delta^+} \mathrm{Ker}(\Delta^-)_{0,1} \xleftarrow{\Delta^+} 0,$$

where we shortly write $\mathrm{Ker}(\Delta^-)_{i,j}$ for:

$$\mathrm{Ker}(\Delta^- : \wedge_+^i\wedge_-^j[\partial_{x_1},\partial_{x_2}] \longrightarrow \wedge_+^i\wedge_-^{j+1}[\partial_{x_1},\partial_{x_2}]).$$

We point out that in this case, the spaces $\mathrm{Ker}(\Delta^-)_{2,1}$, $\mathrm{Ker}(\Delta^-)_{1,1}$ and $\mathrm{Ker}(\Delta^-)_{0,1}$ correspond respectively to the valus of $p=-3,-2,-1$ and $q=0$ in the diagram of the E'^1 spectral sequence of $\widetilde{G}_{C^\circ}(-1,1)$ (see Remark 5.37).
The fact that the two complexes are isomorphic follows from $\beta=1<2$ and the fact that by Lemma 5.36 we know that

$$0 \xrightarrow{\Delta^-} \wedge_-^0[\partial_{x_1},\partial_{x_2}] \xrightarrow{\Delta^-} \wedge_-^1[\partial_{x_1},\partial_{x_2}] \xrightarrow{\Delta^-} \wedge_-^2[\partial_{x_1},\partial_{x_2}] \xrightarrow{\Delta^-} 0$$

is exact except for the right end.
In this case the complex E'^1 of $\widetilde{G}_{C^\circ}(-1,1)$ has two nonzero rows, for $q=0$ and $q=\beta-2=-1$, and the differentials $d^r_{p,q}$ are all zero except for $r=2$, $q=-1$ and $-2<p\leq 0$. Indeed:

$$d^2_{p,-1} : E'^2_{p,-1} \longrightarrow E'^2_{p-2,0},$$

where $E'^2_{p,-1}=0$ if $p>0$ and $E'^2_{p-2,0}=0$ if $p-2<-3$.
We know by Lemma 5.36 that:

$$\sum_{p+q=n} E'^{\infty}_{p,q}(\widetilde{G}_{C^\circ}(-1,1)) = \begin{cases} 0 & \text{if } n>-3 \\ \wedge_+^1 & \text{if } n=-3. \end{cases} \tag{5.32}$$

From this we obtain that $d^r_{p,q}$ for $r=2$, $q=-1$ and $-2<p\leq 0$ must be an isomorphism. Indeed, let us first show that $d^2_{p,q}$ for $q=-1$ and $-2<p\leq 0$ is injective. We have that:

$$d^2_{p,-1} : E'^2_{p,-1}(\widetilde{G}_{C^\circ}(-1,1)) \longrightarrow E'^2_{p-2,0}(\widetilde{G}_{C^\circ}(-1,1)). \tag{5.33}$$

We obtain that $E'^2_{p,-1}(\widetilde{G}_{C^\circ}(-1,1)) \cong \wedge_+^{-p-1}$ using an argument similar to Lemma 5.36.
We know, by (5.32), that for $n=p-1>-3$:

$$\sum_{\tilde{p}+\tilde{q}=p-1} E'^\infty_{\tilde{p},\tilde{q}}(\widetilde{G}_{C^\circ}(-1,1)) = 0.$$

Hence $E'^\infty_{p,-1}(\widetilde{G}_{C^\circ}(-1,1)) = 0$.
Moreover $d^r=0$ for $r>2$ and $d^2_{p+2,-2}=0$ since its domain is 0. Therefore $d^2_{p,-1}$ must be injective.
Let us see that $d^2_{p,-1}$ is surjective.
If $p-2>-3$, then $E'^\infty_{p-2,0}(\widetilde{G}_{C^\circ}(-1,1))$ appears in the sum

$$\sum_{\tilde{p}+\tilde{q}=p-2} E'^\infty_{\tilde{p},\tilde{q}}(\widetilde{G}_{C^\circ}(-1,1)) = 0,$$

by (5.32).
Therefore $E'^\infty_{p-2,0}(\widetilde{G}_{C^\circ}(-1,1)) = 0$.
But we know that

$$d^2_{p-2,0} : E'^2_{p-2,0}(\widetilde{G}_{C^\circ}(-1,1)) \longrightarrow E'^2_{p-4,1}(\widetilde{G}_{C^\circ}(-1,1)) = 0$$

is identically 0 because the codomain is 0. Hence $d^2_{p,-1}$ must be surjective.
If $p-2=-3$, then $E'^\infty_{p-2,0}(\widetilde{G}_{C^\circ}(-1,1))$ appears in the sum

$$\sum_{\tilde{p}+\tilde{q}=\alpha-2} E'^\infty_{\tilde{p},\tilde{q}}(\widetilde{G}_{C^\circ}(-1,1)) = \wedge_+^1, \tag{5.34}$$

by (5.32).
We know that $E'^2_{-2,-1}(\widetilde{G}_{C^\circ}(-1,1)) = E'^\infty_{-2,-1}(\widetilde{G}_{C^\circ}(-1,1)) \cong \wedge_+^1$, since $d^r=0$, when $r>2$, $d^2_{0,-2} = d^2_{-2,-1} = 0$ and $E'^2_{-2,-1}(\widetilde{G}_{C^\circ}(-1,1)) \cong \wedge_+^1$ due to an argument similar to Lemma 5.36. Since $E'^\infty_{-2,-1}(\widetilde{G}_{C^\circ}(-1,1))$ also appears in the sum (5.34), we conclude that $E'^\infty_{p-2,0}(\widetilde{G}_{C^\circ}(-1,1)) = 0$.
We point out that

$$d^2_{p-2,0} : E'^2_{p-2,0}(\widetilde{G}_{C^\circ}(-1,1)) \longrightarrow E'^2_{p-4,1}(\widetilde{G}_{C^\circ}(-1,1)) = 0$$

is identically 0 because the codomain is 0. Therefore $d^2_{p,-1}$ must be surjective.
Hence, since $d^2_{p,-1}$ is an isomorphism, we obtain that

$$E'^2_{p-2,0}(\widetilde{G}_{C^\circ}(-1,1)) \cong \wedge_+^{-1-p}.$$

Then $E'^2_{s,0}(\widetilde{G}_{C^\circ}(-1,1)) \cong \wedge_+^{-s-3}$.
Therefore we obtain:

$$H_2(T(-1,1)) \cong \wedge_+^0,\ H_1(T(-1,1)) \cong 0,\ H_0(T(-1,1)) \cong 0,\ H_{-1}(T(-1,1)) \cong 0.$$

We finally prove the statement in the case $\beta = 2$. Due to the definition of $T(\alpha, 2)$, $H_i(T(\alpha, 2)) = H_i(T(\alpha - 1, 2))$ for $\alpha \leq i \leq 2$, then it is sufficient to compute it for small α. For sake of simplicity, we choose $\alpha = 0$. We point out that the complex $T(0, 2)$ reduces to:

$$\Delta^-(\wedge^2_+\wedge^1_-[\partial_{x_1}, \partial_{x_2}]) \xleftarrow{\Delta^+} \Delta^-(\wedge^1_+\wedge^1_-[\partial_{x_1}, \partial_{x_2}]) \xleftarrow{\Delta^+} \Delta^-(\wedge^0_+\wedge^1_-[\partial_{x_1}, \partial_{x_2}]).$$

In this case the thesis reduces to show that:

$$H_2(T(0, 2)) \cong \wedge^1_+, \quad H_1(T(0, 2)) \cong \wedge^0_+, \quad H_0(T(0, 2)) \cong 0.$$

We compute the homology spaces by direct computations.

i: Let us compute $H_0(T(0, 2))$. We take $w_{12} \otimes p(\partial_{x_1}, \partial_{x_2}) + w_{22} \otimes q(\partial_{x_1}, \partial_{x_2}) \in \wedge^0_+\wedge^1_-[\partial_{x_1}, \partial_{x_2}]$; an element in $\Delta^-(\wedge^0_+\wedge^1_-[\partial_{x_1}, \partial_{x_2}])$ has the following form:

$$P := w_{12}w_{22} \otimes \partial_{x_2}p + w_{22}w_{12} \otimes \partial_{x_1}q.$$

Hence:

$$\Delta^+(P) = w_{12}w_{22}w_{11} \otimes \partial_{x_1}\partial_{x_2}p + w_{12}w_{22}w_{21} \otimes \partial^2_{x_2}p + w_{22}w_{12}w_{11} \otimes \partial^2_{x_1}q + w_{22}w_{12}w_{21} \otimes \partial_{x_1}\partial_{x_2}q.$$

Therefore P lies in the kernel if and only if:

$$\begin{cases} p\partial_{x_1}\partial_{x_2} - q\partial^2_{x_1} = 0, \\ p\partial^2_{x_2} - q\partial_{x_1}\partial_{x_2} = 0. \end{cases}$$

The previous equations are equivalent to $p\partial_{x_2} - q\partial_{x_1} = 0$. Hence P lies in the kernel if and only if $P = 0$. Thus $H_0(T(0, 2)) \cong 0$.

ii: Let us compute $H_1(T(0, 2))$. We take $w_{11}w_{12} \otimes p_1 + w_{11}w_{22} \otimes p_2 + w_{21}w_{12} \otimes q_1 + w_{21}w_{22} \otimes q_2 \in \wedge^1_+\wedge^1_-[\partial_{x_1}, \partial_{x_2}]$; an element in $\Delta^-(\wedge^1_+\wedge^1_-[\partial_{x_1}, \partial_{x_2}])$ has the following form:

$$P := w_{11}w_{12}w_{22} \otimes \partial_{x_2}p_1 + w_{11}w_{22}w_{12} \otimes \partial_{x_1}p_2 + w_{21}w_{12}w_{22} \otimes \partial_{x_2}q_1 + w_{21}w_{22}w_{12} \otimes \partial_{x_1}q_2.$$

Hence:

$$\begin{aligned}\Delta^+(P) =& w_{11}w_{12}w_{22}w_{21} \otimes \partial^2_{x_2}p_1 + w_{11}w_{22}w_{12}w_{21} \otimes \partial_{x_1}\partial_{x_2}p_2 + w_{21}w_{12}w_{22}w_{11} \otimes \partial_{x_1}\partial_{x_2}q_1 \\ &+ w_{21}w_{22}w_{12}w_{11} \otimes \partial^2_{x_1}q_2.\end{aligned}$$

Therefore P lies in the kernel if and only if:

$$\partial^2_{x_2}p_1 - \partial_{x_1}\partial_{x_2}p_2 - \partial_{x_1}\partial_{x_2}q_1 + \partial^2_{x_1}q_2 = 0. \tag{5.35}$$

We notice that if $p_1 \neq 0$, then (5.35) reduces to:

$$p_1 = \frac{\partial_{x_1}\partial_{x_2}p_2 + \partial_{x_1}\partial_{x_2}q_1 - \partial^2_{x_1}q_2}{\partial^2_{x_2}}.$$

Hence if P lies in the kernel and $p_1 \neq 0$ then:

$$P = \Delta^+\big(w_{12}w_{22} \otimes \big(q_1 - \frac{q_2\partial_{x_1}}{\partial_{x_2}}\big)\big) = \Delta^+(\Delta^-\big(w_{12} \otimes \frac{q_1}{\partial_{x_2}} + w_{22} \otimes \frac{q_2}{\partial_{x_2}}\big).$$

Thus if P lies in the kernel and $p_1 \neq 0$, then $P = 0$ in homology.
If P lies in the kernel, $p_1 = 0$ and $p_2 \neq 0$, then (5.35) reduces to:

$$p_2 = \frac{-\partial_{x_1}\partial_{x_2}q_1 + \partial_{x_1}^2 q_2}{\partial_{x_1}\partial_{x_2}} = -q_1 + \frac{\partial_{x_1}q_2}{\partial_{x_2}}.$$

Therefore if P lies in the kernel, $p_1 = 0$ and $p_2 \neq 0$ then:

$$\begin{aligned} P &= w_{11}w_{22}w_{12} \otimes \partial_{x_1}(-q_1 + \frac{\partial_{x_1}q_2}{\partial_{x_2}}) + w_{21}w_{12}w_{22} \otimes \partial_{x_2}q_1 + w_{21}w_{22}w_{12} \otimes \partial_{x_1}q_2 \\ &= \Delta^+\big(-w_{22}w_{12} \otimes q_1 + w_{22}w_{12}\frac{\partial_{x_1}q_2}{\partial_{x_2}}\big) = \Delta^+\big(-w_{22}w_{12} \otimes q_1\big) + \Delta^+\big(\Delta^-\big(w_{22}\frac{q_2}{\partial_{x_2}}\big)\big). \end{aligned}$$

Hence in homology $P = \Delta^+(-w_{22}w_{12} \otimes q_1)$. If q_1 has at least degree 1 in $\partial_{x_1}, \partial_{x_2}$ then $P = 0$ in homology: indeed, for example, if q_1 has at least degree 1 in ∂_{x_2} then $P = \Delta^+(-w_{22}w_{12} \otimes q_1) = \Delta^+\Delta^-\big(w_{12} \otimes \frac{q_1}{\partial_{x_2}}\big)$. Otherwise, if P lies in the kernel, $p_1 = 0$, $p_2 \neq 0$ and $q_1 = a \in \mathbb{C}$, then in homology $P = \Delta^+(-w_{22}w_{12} \otimes a) \neq 0$ since $-w_{22}w_{12} \otimes a \notin \Delta^-(\bigwedge_+^0 \bigwedge_-^1[\partial_{x_1}, \partial_{x_2}])$.
If P lies in the kernel, $p_1 = 0$ and $p_2 = 0$, then (5.35) reduces to:

$$\partial_{x_1}\partial_{x_2}q_1 = \partial_{x_1}^2 q_2,$$

and therefore $P = w_{21}w_{12}w_{22} \otimes \partial_{x_2}q_1 + w_{21}w_{22}w_{12} \otimes \partial_{x_1}q_2 = 0$.
Therefore $H_1(T(0,2)) \cong \bigwedge^0$.

iii: Let us compute $H_2(T(0,2))$. We take $w_{11}w_{21}w_{12} \otimes p(\partial_{x_1}, \partial_{x_2}) + w_{11}w_{21}w_{22} \otimes q(\partial_{x_1}, \partial_{x_2}) \in \bigwedge_+^2 \bigwedge_-^1[\partial_{x_1}, \partial_{x_2}]$; an element in $\Delta^-(\bigwedge_+^2 \bigwedge_-^1[\partial_{x_1}, \partial_{x_2}])$ has the following form:

$$P := w_{11}w_{21}w_{12}w_{22} \otimes \partial_{x_2}p + w_{11}w_{21}w_{22}w_{12} \otimes \partial_{x_1}q.$$

We point out that

$$P = \Delta^+(w_{11}w_{12}w_{22} \otimes p - w_{21}w_{22}w_{12} \otimes q),$$

but $w_{11}w_{12}w_{22} \otimes p - w_{21}w_{22}w_{12} \otimes q \in \Delta^-(\bigwedge_+^1 \bigwedge_-^1[\partial_{x_1}, \partial_{x_2}])$ if and only if both p and q have at least degree 1 in $\partial_{x_1}, \partial_{x_2}$. Indeed, for example, if p has at least degree 1 in ∂_{x_2} and q has at least degree 1 in ∂_{x_1}:

$$w_{11}w_{12}w_{22} \otimes p - w_{21}w_{22}w_{12} \otimes q = \Delta^-\big(w_{11}w_{12} \otimes \frac{p}{\partial_{x_2}} - w_{21}w_{22} \otimes \frac{q}{\partial_{x_1}}\big).$$

Therefore P does not lie in $\Delta^+\Delta^-(\bigwedge_+^1 \bigwedge_-^1[\partial_{x_1}, \partial_{x_2}])$ if and only if $P = w_{11}w_{21}w_{12}w_{22} \otimes a\partial_{x_2} + w_{11}w_{21}w_{22}w_{12} \otimes b\partial_{x_1}$ for $a, b \in \mathbb{C}$. Thus $H_2(T(0,2)) \cong \bigwedge^1$.

□

Now using Remark 5.37 and Lemmas 5.38, 5.39, we are able to compute the homology of the $G_{X^\circ}(\alpha, \beta)$'s when $\alpha, \beta \in \{0, 1, 2\}$.

Lemma 5.40. *If $0 \leq \alpha \leq \beta \leq 2$ then, as $\langle x_1\partial_{x_1} - x_2\partial_{x_2}, x_1\partial_{x_2}, x_2\partial_{x_1}\rangle$−modules:*

$$H^{m,n}(G_{A^\circ}(\alpha, \beta)) \cong \begin{cases} \bigwedge^{\alpha+\beta-n} & \text{if } m = 0, \ n \geq \beta, \\ \bigwedge^{\alpha+\beta-n+1} & \text{if } m = 1, \ 0 \leq n \leq \alpha, \\ 0 & \text{otherwise}; \end{cases}$$

$$H^{m,n}(G_{D^\circ}(\alpha,\beta)) \cong \begin{cases} \bigwedge^{\alpha+\beta-n} & \textit{if } m=0, \ \ n \leq 0, \\ 0 & \textit{otherwise}; \end{cases}$$

$$H^{m,n}(G_{B^\circ}(\alpha,\beta)) \cong \begin{cases} \bigwedge^{\alpha+\beta-n-2} & \textit{if } m=0, \ \ n \geq 0, \\ 0 & \textit{otherwise}; \end{cases}$$

If $0 \leq \beta \leq \alpha \leq 2$ *then, as* $\langle x_1\partial_{x_1} - x_2\partial_{x_2}, x_1\partial_{x_2}, x_2\partial_{x_1}\rangle$*−modules:*

$$H^{m,n}(G_{C^\circ}(\alpha,\beta)) \cong \begin{cases} \bigwedge^{\alpha+\beta-n-2} & \textit{if } m=0, \ \ n \leq \beta-2, \\ \bigwedge^{-1+\alpha+\beta-n-2} & \textit{if } m=-1, \ \ \alpha-2 \leq n \leq 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Analogously if $0 \leq \beta \leq \alpha \leq 2$ *then, as* $\langle x_1\partial_{x_1} - x_2\partial_{x_2}, x_1\partial_{x_2}, x_2\partial_{x_1}\rangle$*−modules:*

$$H^{m,n}(G_{A^\circ}(\alpha,\beta)) \cong \begin{cases} \bigwedge^{\alpha+\beta-n} & \textit{if } m=0, \ \ n \geq \alpha, \\ \bigwedge^{\alpha+\beta-n+1} & \textit{if } m=1, \ \ 0 \leq n \leq \beta, \\ 0 & \textit{otherwise}; \end{cases}$$

$$H^{m,n}(G_{D^\circ}(\alpha,\beta)) \cong \begin{cases} \bigwedge^{\alpha+\beta-n} & \textit{if } m=0, \ \ n \leq 0, \\ 0 & \textit{otherwise}; \end{cases}$$

$$H^{m,n}(G_{B^\circ}(\alpha,\beta)) \cong \begin{cases} \bigwedge^{\alpha+\beta-n-2} & \textit{if } m=0, \ \ n \geq 0, \\ 0 & \textit{otherwise}; \end{cases}$$

If $0 \leq \alpha \leq \beta \leq 2$ *then, as* $\langle x_1\partial_{x_1} - x_2\partial_{x_2}, x_1\partial_{x_2}, x_2\partial_{x_1}\rangle$*−modules:*

$$H^{m,n}(G_{C^\circ}(\alpha,\beta)) \cong \begin{cases} \bigwedge^{\alpha+\beta-n-2} & \textit{if } m=0, \ \ n \leq \alpha-2, \\ \bigwedge^{-1+\alpha+\beta-n-2} & \textit{if } m=-1, \ \ \beta-2 \leq n \leq 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We prove the statement in the case $0 \leq \alpha \leq \beta \leq 2$ for $X = A, B, D$ and $0 \leq \beta \leq \alpha \leq 2$ for $X = C$ using the theory of spectral sequences for bicomplexes; the case $0 \leq \beta \leq \alpha \leq 2$ for $X = A, B, D$ and $0 \leq \alpha \leq \beta \leq 2$ for $X = C$ can be proved analogously using the second spectral sequence instead of the first.

Case A) Let us first consider $G_{A^\circ}(0,0) = \mathrm{Ker}(\nabla_2 : \bigwedge_+^0\bigwedge_-^0[x_1,x_2]y_1^0y_2^0 \longrightarrow \bigwedge_+^1\bigwedge_-^1[x_1,x_2])$. We have that $G_{A^\circ}(0,0) = \mathbb{C} + \langle x_1, x_2\rangle$, since an element $p(x_1,x_2) \in \bigwedge_+^0\bigwedge_-^0[x_1,x_2]y_1^0y_2^0$ goes to 0 if and only if $\partial_{x_1}\partial_{x_1}p = \partial_{x_1}\partial_{x_2}p = \partial_{x_2}\partial_{x_2}p = 0$. In this case the statement is true indeed, since we are focusing on $\alpha = \beta = 0$, then $p = q = 0$. Therefore $G_{A^\circ}^{m,n}(0,0) = 0$ when $n \neq 0$, $G_{A^\circ}^{1,0}(0,0) = \langle x_1, x_2\rangle$, $G_{A^\circ}^{0,0}(0,0) = \mathbb{C}$ and, from

$$\xrightarrow{\nabla} G_{A^\circ}^{2,1}(0,0) = 0 \xrightarrow{\nabla} G_{A^\circ}^{1,0}(0,0) = \langle x_1, x_2\rangle \to 0,$$

we have that $H^{1,0}(G_{A^\circ}(0,0)) \cong \bigwedge^1$. From the sequence

$$\xrightarrow{\nabla} G_{A^\circ}^{1,1}(0,0) = 0 \xrightarrow{\nabla} G_{A^\circ}^{0,0}(0,0) = \mathbb{C} \to 0,$$

we deduce that $H^{0,0}(G_{A^\circ}(0,0)) \cong \bigwedge^0$. Then we can assume $\beta > 0$. As in Remark 5.37 we consider:

$$\widetilde{G}_A(\alpha,\beta)_{[p,q]} = \begin{cases} \bigwedge_+^{\alpha-p}\bigwedge_-^{\beta-q}[x_1,x_2] & \text{if } p \geq 0, q \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We consider on this space the differentials $d' = \Delta^+$ and $d'' = \Delta^-$ induced by $\Delta^+\partial_{y_1}$ and $\Delta^-\partial_{y_2}$ for $G_A(\alpha, \beta)$. As in Remark 5.37, the E'^1 spectral sequence of $\widetilde{G}_{A^\circ}(\alpha, \beta)$, i.e. the homology with respect to Δ^-, is the following:

$$\begin{array}{ccc} \bigwedge_+^\alpha & \leftarrow \cdots \leftarrow & \bigwedge_+^0 \\ \downarrow & & \downarrow \\ 0 & \cdots & 0 \\ \downarrow & & \downarrow \\ \frac{\mathrm{Ker}(\Delta^-\Delta^+)}{\mathrm{Im}(\Delta^-)} & \leftarrow \cdots \leftarrow & \frac{\bigwedge_+^0\bigwedge_-^\beta[x_1,x_2]}{\mathrm{Im}(\Delta^-)}. \end{array}$$

We have that only the rows for $q = 0$ and $q = \beta$ are different from 0. We observe that d' is 0 on the row $q = \beta$. Moreover $d^r_{p,q}$ is 0 for $r \geq 2$ because either the domain or the codomain of these maps are 0, since $\alpha \leq \beta$. Therefore $E'^2 = ... = E'^\infty$.
We need to compute E'^2 for $q = 0$. We apply Lemma 5.38 to compute the homology for the row $q = 0$. We point out that the isomorphism in (5.28) of Lemma 5.38 was induced by ∇, that decreases the degree in x_1, x_2 by 1. Therefore $E'^2_{p,0}(\widetilde{G}_{A^\circ}(\alpha, \beta)) \cong \bigwedge_+^{\alpha+\beta-p+1}$ is formed by elements with representatives of degree 1 in x_1, x_2.
Hence we have that if $n \geq \beta > \alpha$:

$$\sum_{p+q=n} E'^\infty_{p,q}(\widetilde{G}_{A^\circ}(\alpha, \beta)) = E'^\infty_{n-\beta,\beta}(\widetilde{G}_{A^\circ}(\alpha, \beta)) = E'^2_{n-\beta,\beta}(\widetilde{G}_{A^\circ}(\alpha, \beta)) \cong \bigwedge_+^{\alpha+\beta-n}.$$

Indeed in this sum there is not the possibility $(p, q) = (p, 0)$ with $p \leq \alpha < \beta$. We have that $H^{0,n}(\widetilde{G}_{A^\circ}(\alpha, \beta)) \cong \bigwedge_+^{\alpha+\beta-n}$, if $n \geq \beta > \alpha$.
If $0 \leq n \leq \alpha < \beta$:

$$\sum_{p+q=n} E'^\infty_{p,q}(\widetilde{G}_{A^\circ}(\alpha, \beta)) = E'^\infty_{n,0}(\widetilde{G}_{A^\circ}(\alpha, \beta)) = E'^2_{n,0}(\widetilde{G}_{A^\circ}(\alpha, \beta)) \cong \bigwedge_+^{\alpha+\beta-n+1} \ (\textit{in } x_1, x_2 \textit{ degree } 1). \tag{5.36}$$

We have the $H^{1,n}(\widetilde{G}_{A^\circ}(\alpha, \beta)) \cong \bigwedge_+^{\alpha+\beta-n+1}$, if $0 \leq n \leq \alpha$.
If $n = \alpha = \beta$ we have both terms in the sum, but one is represented by elements of degree 1 in x_1, x_2, the others by elements of degree 0, then the result is unchanged.

Case D) We define:

$$\widetilde{G}_D(\alpha, \beta)_{[p,q]} = \begin{cases} \bigwedge_+^{\alpha-p}\bigwedge_-^{\beta-q}[x_1, x_2] & \text{if } p \leq 0, q \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We have an isomorphism of bicomplexes $\gamma : G_D(\alpha, \beta)_{[p,q]} \longrightarrow \widetilde{G}_D(\alpha, \beta)_{[p,q]}$ which is the valuating map that values ∂_{y_1} and ∂_{y_2} in 1 and is the identity on all other elements. We consider on $\widetilde{G}_D(\alpha, \beta)$ the differentials $d' = \Delta^+$ and $d'' = \Delta^-$ induced by $\Delta^+\partial_{y_1}$ and $\Delta^-\partial_{y_2}$ for $G_D(\alpha, \beta)$. We also define:

$$G_{A'}(\alpha, \beta)_{[p,q]} = \begin{cases} \bigwedge_+^{\alpha-1}\bigwedge_-^{\beta-1}[x_1, x_2] & \text{if } p = q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We have the following commutative diagram:

$$\begin{array}{ccc} G_{A'}(\alpha,\beta) & \xrightarrow{\nabla_2} & G_D(\alpha,\beta) \\ id \downarrow & & \downarrow \gamma \\ G_{A'}(\alpha,\beta) & \xrightarrow{\Delta^-\Delta^+\tau_1} & \widetilde{G}_D(\alpha,\beta). \end{array}$$

We have that $\widetilde{G}_{D^\circ}(\alpha,\beta) := \mathrm{CoKer}(\Delta^-\Delta^+\tau_1 : G_{A'}(\alpha,\beta) \longrightarrow \widetilde{G}_D(\alpha,\beta))$ is isomorphic, as a bicomplex, to G_{D°. Its diagram is the same of $\widetilde{G}_D$ except for $p = q = 0$ (upper right point in the following diagram), where instead of $\bigwedge_+^\alpha \bigwedge_-^\beta [x_1, x_2]$ there is $\mathrm{CoKer}(\Delta^-\Delta^+ : \bigwedge_+^{\alpha-1}\bigwedge_-^{\beta-1}[x_1,x_2] \longrightarrow \bigwedge_+^\alpha\bigwedge_-^\beta[x_1,x_2])$.
Moreover we observe that $G_{A'}(0,0) = 0$, then $G_{D^\circ}(0,0) = G_D(0,0)$ and we can use the same argument of Lemma 5.36. We then assume $\beta > 0$, the diagram of $G_D(\alpha,\beta)$ is:

$$\begin{array}{ccc} \bigwedge_+^2\bigwedge_-^\beta[x_1,x_2] \leftarrow & \cdots \leftarrow & \bigwedge_+^\alpha\bigwedge_-^\beta[x_1,x_2] \\ \downarrow & & \downarrow \\ \cdots & \cdots & \cdots \\ \downarrow & & \downarrow \\ \bigwedge_+^2\bigwedge_-^2[x_1,x_2] \leftarrow & \cdots \leftarrow & \bigwedge_+^\alpha\bigwedge_-^2[x_1,x_2], \end{array}$$

where the horizontal maps are d' and the vertical maps are d''. In the following diagram we shortly write $\frac{\mathrm{Ker}(\Delta^-)}{\mathrm{Im}(\Delta^-\Delta^+)}$ for the space:

$$\frac{\mathrm{Ker}(\Delta^- : \bigwedge_+^\alpha\bigwedge_-^\beta[x_1,x_2] \longrightarrow \bigwedge_+^\alpha\bigwedge_-^{\beta+1}[x_1,x_2])}{\mathrm{Im}(\Delta^-\Delta^+ : \bigwedge_+^{\alpha-1}\bigwedge_-^{\beta-1}[x_1,x_2] \longrightarrow \bigwedge_+^\alpha\bigwedge_-^\beta[x_1,x_2])}.$$

In the following diagram we also shortly write $\mathrm{Ker}(\Delta^-)_{i,j}$ for:

$$\mathrm{Ker}(\Delta^- : \bigwedge_+^i\bigwedge_-^j[x_1,x_2] \longrightarrow \bigwedge_+^i\bigwedge_-^{j+1}[x_1,x_2]).$$

The E'^1 spectral sequence of $\widetilde{G}_{D^\circ}(\alpha,\beta)$ is (the computation is analogous to Lemma 5.36):

$$\begin{array}{ccc} \mathrm{Ker}(\Delta^-)_{2,\beta} \ \leftarrow & \cdots \leftarrow & \frac{\mathrm{Ker}(\Delta^-)}{\mathrm{Im}(\Delta^-\Delta^+)} \\ \downarrow & & \downarrow \\ 0 & \cdots & 0 \\ \downarrow & & \downarrow \\ 0 & \leftarrow \ 0 \ \leftarrow & 0. \end{array}$$

We observe that, since $\beta > 0$:

$$\frac{\mathrm{Ker}(\Delta^-)}{\mathrm{Im}(\Delta^-\Delta^+)} \cong \frac{\Delta^-(\bigwedge_+^\alpha\bigwedge_-^{\beta-1}[x_1,x_2])}{\Delta^-\Delta^+(\bigwedge_+^{\alpha-1}\bigwedge_-^{\beta-1}[x_1,x_2])} \cong \mathrm{CoKer}(\Delta^-(\bigwedge_+^{\alpha-1}\bigwedge_-^{\beta-1}[x_1,x_2]) \xrightarrow{\Delta^+} \Delta^-(\bigwedge_+^\alpha\bigwedge_-^{\beta-1}[x_1,x_2])).$$

The non zero row of the previous diagram is isomorphic, via Δ^-, to the following complex:

$$\Delta^-(\bigwedge_+^2\bigwedge_-^{\beta-1}[x_1,x_2]) \xleftarrow{\Delta^+} \ldots \xleftarrow{\Delta^+} (\mathrm{CoKer}(\Delta^-(\bigwedge_+^{\alpha-1}\bigwedge_-^{\beta-1}[x_1,x_2]) \xrightarrow{\Delta^+} \Delta^-(\bigwedge_+^\alpha\bigwedge_-^{\beta-1}[x_1,x_2]))). \tag{5.37}$$

The fact that the two complexes are isomorphic follows from $\beta > 0$ and the fact that by Lemma 5.36 we know that

$$0 \xrightarrow{\Delta^-} \bigwedge_-^0[x_1,x_2] \xrightarrow{\Delta^-} \bigwedge_-^1[x_1,x_2] \xrightarrow{\Delta^-} \bigwedge_-^2[x_1,x_2] \xrightarrow{\Delta^-} 0$$

is exact except for the left end.
We observe that we can compute the homology of the complex (5.37) using homology of $S(2,\beta-1)$ and Lemma 5.38. Indeed this complex is different from $S(2,\beta-1)$ only at the right end (using the direction of the previous). Indeed the left end is $\Delta^-(\bigwedge_+^2\bigwedge_-^{\beta-1}[x_1,x_2]) \cong \mathrm{Ker}(\Delta^-(\bigwedge_+^2\bigwedge_-^{\beta-1}[x_1,x_2]) \xrightarrow{\Delta^+} \Delta^-(\bigwedge_+^3\bigwedge_-^{\beta-1}[x_1,x_2]=0))$ that is the left hand of $S(2,\beta-1)$.
The homology at the right end of our complex (5.37) is:

$$\mathrm{Ker}\,\Big(\Delta^+\Big(\frac{\Delta^-(\bigwedge_+^{\alpha}\bigwedge_-^{\beta-1}[x_1,x_2])}{\Delta^-\Delta^+(\bigwedge_+^{\alpha-1}\bigwedge_-^{\beta-1}[x_1,x_2])}\Big)\Big) \cong \frac{\mathrm{Ker}(\Delta^+(\Delta^-(\bigwedge_+^{\alpha}\bigwedge_-^{\beta-1}[x_1,x_2])))}{\Delta^-\Delta^+(\bigwedge_+^{\alpha-1}\bigwedge_-^{\beta-1}[x_1,x_2])} \cong H_\alpha(S(2,\beta-1)).$$

Therefore we can use the homology groups of $S(2,\beta-1)$ and obtain that the homology spaces for complex (5.37) are isomorphic, respectively from left to right, to:

$$\textstyle\bigwedge_+^{2+\beta} \quad \dots \quad \bigwedge_+^{\alpha+\beta}.$$

We conclude because $E'^2_{n,0}(\widetilde{G}_{D^\circ}(\alpha,\beta)) = E'^\infty_{n,0}(\widetilde{G}_{D^\circ}(\alpha,\beta)) \cong \bigwedge_+^{\alpha-n+\beta}$ and:

$$\sum_{p+q=n} E'^\infty_{p,q}(\widetilde{G}_{D^\circ}(\alpha,\beta)) = E'^\infty_{n,0}(\widetilde{G}_{D^\circ}(\alpha,\beta)) \cong \textstyle\bigwedge_+^{\alpha-n+\beta}.$$

Case C) Let us first consider $G_{C^\circ}(2,2) = \mathrm{CoKer}(\nabla_2 : \bigwedge_+^1\bigwedge_-^1[\partial_{x_1},\partial_{x_2}] \longrightarrow \bigwedge_+^2\bigwedge_-^2[\partial_{x_1},\partial_{x_2}])$. We have that $G_{C^\circ}(0,0) = \mathbb{C} + \langle\partial_{x_1},\partial_{x_2}\rangle$, since an element $p(\partial_{x_1},\partial_{x_2}) \in \bigwedge_+^1\bigwedge_-^1[\partial_{x_1},\partial_{x_2}]$ goes to an element with degree increased by 2 in $\partial_{x_1},\partial_{x_2}$. In this case the statement is true. Indeed, since we are focusing on $\alpha=\beta=2$, therefore $p=q=0$. Hence $G^{m,n}_{C^\circ}(2,2)=0$ when $n\neq 0$, $G^{-1,0}_{C^\circ}(0,0) = \langle\partial_{x_1},\partial_{x_2}\rangle$, $G^{0,0}_{C^\circ}(2,2)=\mathbb{C}$. From the sequence

$$\xrightarrow{\nabla} G^{0,1}_{C^\circ}(2,2) = 0 \xrightarrow{\nabla} G^{-1,0}_{C^\circ}(2,2) = \langle\partial_{x_1},\partial_{x_2}\rangle \to 0,$$

we have that $H^{-1,0}(G_{C^\circ}(0,0)) \cong \bigwedge^1$. From the sequence

$$\xrightarrow{\nabla} G^{1,1}_{C^\circ}(2,2) = 0 \xrightarrow{\nabla} G^{0,0}_{C^\circ}(2,2) = \mathbb{C} \to 0,$$

we have that $H^{0,0}(G_{C^\circ}(2,2)) \cong \bigwedge^2$. Hence we focus on $\beta<2$.
As in Remark 5.37, we consider:

$$\widetilde{G}_C(\alpha,\beta)_{[p,q]} = \begin{cases} \bigwedge_+^{\alpha-p}\bigwedge_-^{\beta-q}[\partial_{x_1},\partial_{x_2}] & \text{if } p\le 0, q\le 0,\\ 0 & \text{otherwise.}\end{cases}$$

We consider on this space the differentials $d'=\Delta^+$ and $d''=\Delta^-$ induced by $\Delta^+\partial_{y_1}$ and $\Delta^-\partial_{y_2}$ for $G_C(\alpha,\beta)$.

As in Remark 5.37, the E'^1 spectral sequence of $\widetilde{G}_{C^\circ}(\alpha,\beta)$ is:

$$\begin{array}{ccccc} \mathrm{Ker}(\Delta^-)_{2,\beta} & \leftarrow & \cdots \leftarrow & \frac{\mathrm{Ker}(\Delta^-)}{\mathrm{Im}(\Delta^-\Delta^+)} \\ \downarrow & & & \downarrow \\ 0 & & \cdots & 0 \\ \downarrow & & & \downarrow \\ \bigwedge_+^2\bigwedge_-^2 & \leftarrow & \cdots \leftarrow & \bigwedge_+^{\alpha}\bigwedge_-^2. \end{array}$$

We have that only the rows $q = 0$ and $q = \beta - 2$ are different from 0. We observe that d' is 0 on the row $q = \beta - 2$. Moreover $d^r_{p,q}$ is 0 for $r \geq 2$ because either the domain or the codomain of these maps are 0, since $2 - \alpha \leq 2 - \beta$. Therefore $E'^2 = ... = E'^\infty$.
We need to compute E'^2 for $q = 0$.

We can apply Lemma 5.39 to compute the homology for the row $q = 0$. We observe that the isomorphism in (5.33) of Lemma 5.39 is induced by ∇ that increases the degree in $\partial_{x_1}, \partial_{x_2}$ by 1. Thus the elements of $E'^2_{p,0}(\widetilde{G}_{C^\circ}(\alpha,\beta))$ are represented by elements with degree 1 in $\partial_{x_1}, \partial_{x_2}$.
Therefore we have that if $0 \leq \beta \leq \alpha \leq 2$ and $\alpha - 2 \leq n \leq 0$:

$$\sum_m H^{m,n}(\widetilde{G}_{C^\circ}(\alpha,\beta)) = \sum_{p+q=n} E'^\infty_{p,q}(\widetilde{G}_{C^\circ}(\alpha,\beta)) = E'^\infty_{n,0}(\widetilde{G}_{C^\circ}(\alpha,\beta)) \cong$$
$$\textstyle\bigwedge_+^{-1+\alpha+\beta-n-2} \quad (\textit{degree } 1 \textit{ in } \partial_{x_1}, \partial_{x_2}).$$

Then $H^{-1,n}(\widetilde{G}_{C^\circ}(\alpha,\beta)) \cong \bigwedge_+^{-1+\alpha+\beta-n-2}$. We then have that if $0 \leq \beta \leq \alpha \leq 2$ and $n \leq \beta - 2$:

$$\sum_m H^{m,n}(\widetilde{G}_{C^\circ}(\alpha,\beta)) = \sum_{p+q=n} E'^\infty_{p,q}(\widetilde{G}_{C^\circ}(\alpha,\beta)) = E'^\infty_{n-\beta+2,\beta-2}(\widetilde{G}_{C^\circ}(\alpha,\beta)) \cong \textstyle\bigwedge_+^{\alpha+\beta-n-2}.$$

Hence $H^{0,n}(\widetilde{G}_{C^\circ}(\alpha,\beta)) \cong \bigwedge_+^{\alpha+\beta-n-2}$. Finally if $0 \leq \beta \leq \alpha \leq 2$ and $n = \beta - 2 = \alpha - 2$, both the terms appear in the sum, but the degree with respect to $\partial_{x_1}, \partial_{x_2}$ is in one case 1 and in the other 0, then we have the same result.

Case B) We define:

$$\widetilde{G}_B(\alpha,\beta)_{[p,q]} = \begin{cases} \bigwedge_+^{\alpha-p}\bigwedge_-^{\beta-q}[\partial_{x_1}, \partial_{x_2}] & \text{if } p \geq 0, q \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We have an isomorphism of bicomplexes $\gamma : G_B(\alpha,\beta)_{[p,q]} \longrightarrow \widetilde{G}_B(\alpha,\beta)_{[p,q]}$ which is the valuating map that values y_1 and y_2 in 1 and is the identity on all other elements. We consider on $\widetilde{G}_B(\alpha,\beta)$ the differentials $d' = \Delta^+$ and $d'' = \Delta^-$ induced by $\Delta^+\partial_{y_1}$ and $\Delta^-\partial_{y_2}$ for $G_B(\alpha,\beta)$. We also define:

$$G_{C'}(\alpha,\beta)_{[p,q]} = \begin{cases} \bigwedge_+^{\alpha+1}\bigwedge_-^{\beta+1}[\partial_{x_1}, \partial_{x_2}] & \text{if } p = q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We have the following commutative diagram:

$$\begin{array}{ccc} G_B(\alpha,\beta) & \xrightarrow{\quad\nabla_2\quad} & G_{C'}(\alpha,\beta) \\ \gamma \big\downarrow & & \big\downarrow id \\ \widetilde{G}_B(\alpha,\beta) & \xrightarrow{\Delta^-\Delta^+\tau_2} & G_{C'}(\alpha,\beta). \end{array}$$

If we consider $G_{C'}(\alpha,2) = 0$, then $G_{B^\circ}(\alpha,2) = G_B(\alpha,2)$ and we can use the same argument we used in Lemma 5.36, then we focus on $\beta < 2$.
We have that $\widetilde{G}_{B^\circ}(\alpha,\beta) = \mathrm{Ker}(\Delta^-\Delta^+\tau_2 : \widetilde{G}_B(\alpha,\beta) \longrightarrow G_{C'}(\alpha,\beta))$ is isomorphic, as a bicomplex, to $G_{B^\circ}(\alpha,\beta)$, its diagram is the same of $\widetilde{G}_B(\alpha,\beta)$ except for $p = q = 0$ (lower left point in the following diagram). In the following diagram we shortly write $\mathrm{Ker}(\Delta^-\Delta^+)$ for:

$$\mathrm{Ker}(\Delta^-\Delta^+ : \textstyle\bigwedge_+^{\alpha}\bigwedge_-^{\beta}[\partial_{x_1}, \partial_{x_2}] \longrightarrow \bigwedge_+^{\alpha+1}\bigwedge_-^{\beta+1}[\partial_{x_1}, \partial_{x_2}]).$$

The diagram of $\widetilde{G}_{B^\circ}(\alpha,\beta)$ is:

$$\begin{array}{ccc} \wedge_+^\alpha\wedge_-^0[\partial_{x_1},\partial_{x_2}] \leftarrow & \cdots \leftarrow & \wedge_+^0\wedge_-^0[\partial_{x_1},\partial_{x_2}] \\ \downarrow & & \downarrow \\ \cdots & \cdots & \cdots \\ \downarrow & & \downarrow \\ \mathrm{Ker}(\Delta^-\Delta^+) \leftarrow & \cdots \leftarrow & \wedge_+^0\wedge_-^\beta[\partial_{x_1},\partial_{x_2}], \end{array}$$

where the horizontal maps are d' and the vertical maps are d''. The $E^{'1}$ spectral sequence of $\widetilde{G}_{B^\circ}(\alpha,\beta)$ is:

$$\begin{array}{ccccc} 0 & \leftarrow & 0 & \leftarrow & 0 \\ \downarrow & & & & \downarrow \\ 0 & & \cdots & & 0 \\ \downarrow & & & & \downarrow \\ \frac{\mathrm{Ker}(\Delta^-\Delta^+)}{\mathrm{Im}(\Delta^-)} & \leftarrow & \cdots & \leftarrow & \frac{\wedge_+^0\wedge_-^\beta[\partial_{x_1},\partial_{x_2}]}{\mathrm{Im}(\Delta^-)}. \end{array}$$

Since $\beta < 2$ and, by Lemma 5.36, the following complex is exact except for the right end:

$$0 \xrightarrow{\Delta^-} \wedge_-^0[\partial_{x_1},\partial_{x_2}] \xrightarrow{\Delta^-} \wedge_-^1[\partial_{x_1},\partial_{x_2}] \xrightarrow{\Delta^-} \wedge_-^2[\partial_{x_1},\partial_{x_2}] \xrightarrow{\Delta^-} 0;$$

then we observe that:

$$\frac{\wedge_+^k\wedge_-^\beta[\partial_{x_1},\partial_{x_2}]}{\mathrm{Im}(\Delta^-)} \cong \frac{\wedge_+^k\wedge_-^\beta[\partial_{x_1},\partial_{x_2}]}{\mathrm{Ker}(\Delta^- : \wedge_+^k\wedge_-^\beta[\partial_{x_1},\partial_{x_2}] \longrightarrow \wedge_+^k\wedge_-^{\beta+1}[\partial_{x_1},\partial_{x_2}])} \cong \Delta^-(\wedge_+^k\wedge_-^\beta[\partial_{x_1},\partial_{x_2}]).$$

Moreover, since $\beta < 2$:

$$\frac{\mathrm{Ker}(\Delta^-\Delta^+ : \wedge_+^\alpha\wedge_-^\beta[\partial_{x_1},\partial_{x_2}] \longrightarrow \wedge_+^{\alpha+1}\wedge_-^{\beta+1}[\partial_{x_1},\partial_{x_2}])}{\mathrm{Im}(\Delta^- : \wedge_+^\alpha\wedge_-^{\beta-1}[\partial_{x_1},\partial_{x_2}] \longrightarrow \wedge_+^\alpha\wedge_-^\beta[\partial_{x_1},\partial_{x_2}])} \cong$$
$$\frac{\mathrm{Ker}(\Delta^-\Delta^+ : \wedge_+^\alpha\wedge_-^\beta[\partial_{x_1},\partial_{x_2}] \longrightarrow \wedge_+^{\alpha+1}\wedge_-^{\beta+1}[\partial_{x_1},\partial_{x_2}])}{\mathrm{Ker}(\Delta^- : \wedge_+^\alpha\wedge_-^\beta[\partial_{x_1},\partial_{x_2}] \longrightarrow \wedge_+^\alpha\wedge_-^{\beta+1}[\partial_{x_1},\partial_{x_2}])} \cong$$
$$\mathrm{Ker}(\Delta^+ : \Delta^-(\wedge_+^\alpha\wedge_-^\beta[\partial_{x_1},\partial_{x_2}]) \longrightarrow \Delta^-(\wedge_+^{\alpha+1}\wedge_-^\beta[\partial_{x_1},\partial_{x_2}])).$$

The non zero row of the $E^{'1}$ spectral sequence of $\widetilde{G}_{B^\circ}(\alpha,\beta)$ is therefore isomorphic to:

$$\mathrm{Ker}(\Delta^+ : \Delta^-(\wedge_+^\alpha\wedge_-^\beta[\partial_{x_1},\partial_{x_2}]) \longrightarrow \Delta^-(\wedge_+^{\alpha+1}\wedge_-^\beta[\partial_{x_1},\partial_{x_2}])) \xleftarrow{\Delta^+} \ldots \xleftarrow{\Delta^+} \Delta^-(\wedge_+^0\wedge_-^\beta[\partial_{x_1},\partial_{x_2}]).$$

We observe that we can compute its homology using the homology of $T(0,\beta+1)$. Indeed this complex is different from $T(0,\beta+1)$ only at the right end (using the direction of the previous complex). Indeed the homology of the left end is the same as in $T(0,\beta+1)$, since we are considering only the Kernel. The right end of our complex corresponds in $T(0,\beta+1)$ to:

$$\mathrm{CoKer}(\Delta^-(\wedge_+^{-1}\wedge_-^\beta[\partial_{x_1},\partial_{x_2}]) = 0 \xrightarrow{\Delta^+} \Delta^-(\wedge_+^0\wedge_-^\beta[\partial_{x_1},\partial_{x_2}])) \cong \Delta^-(\wedge_+^0\wedge_-^\beta[\partial_{x_1},\partial_{x_2}]).$$

Since $E_{n,0}^{'2}(\widetilde{G}_{B^\circ}(\alpha,\beta)) = E_{n,0}^{'\infty}(\widetilde{G}_{B^\circ}(\alpha,\beta))$, if $n \geq 0$, we have:

$$\sum_{p+q=n} E_{p,q}^{'\infty}(\widetilde{G}_{B^\circ}(\alpha,\beta)) = E_{n,0}^{'\infty}(\widetilde{G}_{B^\circ}(\alpha,\beta)) \cong \wedge_+^{\alpha-n+\beta-2}.$$

□

We now sum up the information of Lemmas 5.36 and 5.40 in the following result about the homology of the G_{X°'s.
Following [KR1], we introduce the notation $P(n,t,c)$ that denotes the irreducible $\langle y_1\partial_{y_1}-y_2\partial_{y_2}, y_1\partial_{y_2}, y_2\partial_{y_1}\rangle \oplus \mathbb{C}t\oplus\mathbb{C}C-$module of highest weight (n,t,c) with respect to $y_1\partial_{y_1}-y_2\partial_{y_2}, t, C$ when $n\in\mathbb{Z}_{\geq 0}$ and $P(n,t,c)=0$ when $n<0$.
Moreover we call $Q(i,n,t,c)$ the irreducible $\mathfrak{g}_0-$module of highest weight (i,n,t,c) with respect to $x_1\partial_{x_1}-x_2\partial_{x_2}, y_1\partial_{y_1}-y_2\partial_{y_2}, t, C$ when $i,n\in\mathbb{Z}_{\geq 0}$ and $Q(i,n,t,c)=0$ when $n<0$ or $i<0$.
Moreover, for $i\in\{0,1,2\}$, we will denote by r_i the remainder i mod 2, that is $r_i=0$ for $i=0,2$ and $r_i=1$ for $i=1$.
Using Lemmas 5.36, 5.40 and the fact that $G_{X^\circ}=\oplus_{\alpha,\beta}G_{X^\circ}(\alpha,\beta)$, we obtain the following result.

Proposition 5.41. *As* $\mathfrak{g}_0-$*modules:*

$$H^{m,n}(G_{A^\circ})\cong\begin{cases}\sum_{i=0}^2 Q\big(r_i, n-i, -i-\frac{1}{2}n, -\frac{1}{2}n\big) & if\ \ m=0,\ \ n\geq 0,\\ \sum_{i=0}^2 Q\big(r_i, i-n-1, -i-\frac{1}{2}n+\frac{1}{2}, -\frac{1}{2}n+\frac{1}{2}\big) & if\ \ m=1,\ \ 0\leq n\leq 1,\\ 0 & otherwise.\end{cases}$$

$$H^{m,n}(G_{D^\circ})\cong\begin{cases}\sum_{i=0}^2 Q\big(r_i, -n+i, -i-\frac{1}{2}n+1, -\frac{1}{2}n+1\big) & if\ \ m=0,\ \ n\leq 0,\\ 0 & otherwise.\end{cases}$$

$$H^{m,n}(G_{B^\circ})\cong\begin{cases}\sum_{i=0}^2 Q\big(r_i, n+2-i, -i-\frac{1}{2}n-1, -\frac{1}{2}n-1\big) & if\ \ m=0,\ \ n\geq 0,\\ 0 & otherwise.\end{cases}$$

$$H^{m,n}(G_{C^\circ})\cong\begin{cases}\sum_{i=0}^2 Q\big(r_i, -n-2+i, -i-\frac{1}{2}n, -\frac{1}{2}n\big) & if\ \ m=0,\ \ n\leq 0,\\ \sum_{i=0}^2 Q\big(r_i, n+2-i-1, -i-\frac{1}{2}n-\frac{1}{2}, -\frac{1}{2}n-\frac{1}{2}\big) & if\ \ m=-1,\ \ -1\leq n\leq 0,\\ 0 & otherwise.\end{cases}$$

Proof. This result follows directly from Lemmas 5.36, 5.40 and the decomposition $G_{X^\circ}=\oplus_{\alpha,\beta}G_{X^\circ}(\alpha,\beta)$.
Let us see it explicitly for $X=A$. The proof is analogous for $X=B,C,D$.
From the decomposition $G_{A^\circ}=\oplus_{\alpha,\beta}G_{A^\circ}(\alpha,\beta)$ we obtain that:

$$H^{m,n}(G_{A^\circ})=\sum_{\alpha,\beta}H^{m,n}(G_{A^\circ}(\alpha,\beta)). \tag{5.38}$$

We first point out that, from the definition of the $G_{A^\circ}(\alpha,\beta)$'s, the element $y_1\partial_{y_1}-y_2\partial_{y_2}$ acts on elements of $H^{m,n}(G_{A^\circ}(\alpha,\beta))$ as multiplication by $\alpha-\beta$.
By Lemmas 5.36 and 5.40 we obtain that the RHS of (5.38) is 0 for $m>1$.
For $m=0$, Equation (5.38) reduces to:

$$\begin{aligned}H^{0,n}(G_{A^\circ})=&H^{0,n}(G_{A^\circ}(0,n))+H^{0,n}(G_{A^\circ}(1,n-1))+...+H^{0,n}(G_{A^\circ}(n-1,1))+H^{0,n}(G_{A^\circ}(n,0))\\ &+H^{0,n}(G_{A^\circ}(1,n))+H^{0,n}(G_{A^\circ}(2,n-1))+...H^{0,n}(G_{A^\circ}(n-1,2))+H^{0,n}(G_{A^\circ}(n,1))\\ &+H^{0,n}(G_{A^\circ}(2,n))+H^{0,n}(G_{A^\circ}(3,n-1))...+H^{0,n}(G_{A^\circ}(n-1,3))+H^{0,n}(G_{A^\circ}(n,2)).\end{aligned} \tag{5.39}$$

We point out that the RHS of (5.39) is the sum of three irreducible $\mathfrak{g}_0-$modules that we call M_0, M_1 and M_2, that are defined as follows. As vector spaces:

$$M_0:=H^{0,n}(G_{A^\circ}(0,n))+H^{0,n}(G_{A^\circ}(1,n-1))+...+H^{0,n}(G_{A^\circ}(n,0))\cong\wedge^0\otimes P\Big(n,-\frac{1}{2}n,-\frac{1}{2}n\Big).$$

Indeed by Lemmas 5.36 and 5.40:

$$\begin{array}{c} H^{0,n}(G_{A^\circ}(n,0)) = \bigwedge^0 \otimes y_1^n \\ \downarrow y_2\partial_{y_1} \\ H^{0,n}(G_{A^\circ}(n-1,1)) = \bigwedge^0 \otimes y_1^{n-1}y_2 \\ \vdots \\ H^{0,n}(G_{A^\circ}(1,n-1)) = \bigwedge^0 \otimes y_1 y_2^{n-1} \\ \downarrow y_2\partial_{y_1} \\ H^{0,n}(G_{A^\circ}(0,n)) = \bigwedge^0 \otimes y_2^n. \end{array}$$

Therefore, as a $\mathfrak{g}_0-$module, $M_0 \cong Q\Big(0, n, -\frac{1}{2}n, -\frac{1}{2}n\Big)$. Moreover as vector spaces:

$$M_1 := H^{0,n}(G_{A^\circ}(1,n)) + H^{0,n}(G_{A^\circ}(2,n-1)) + ... + H^{0,n}(G_{A^\circ}(n,1)) \cong \bigwedge^1 \otimes P\Big(n, -\frac{1}{2}n, -\frac{1}{2}n\Big).$$

Indeed by Lemmas 5.36 and 5.40:

$$\begin{array}{c} H^{0,n}(G_{A^\circ}(n,1)) \cong \bigwedge^1_- \otimes y_1^n \\ \downarrow y_2\partial_{y_1} \\ H^{0,n}(G_{A^\circ}(n-1,2)) \cong \bigwedge^1_- \otimes y_1^{n-1}y_2 \\ \vdots \\ H^{0,n}(G_{A^\circ}(3,n-2)) \cong \bigwedge^1_- \otimes y_1^3 y_2^{n-3} \\ \downarrow y_2\partial_{y_1} \\ H^{0,n}(G_{A^\circ}(2,n-1)) \cong \bigwedge^1_- \otimes y_1^2 y_2^{n-2} \\ \downarrow y_2\partial_{y_1} \\ H^{0,n}(G_{A^\circ}(1,n)) \cong \bigwedge^1_- \otimes y_1 y_2^{n-1}. \end{array}$$

Indeed let us observe that, by Lemma 5.36, $H^{0,n}(G_{A^\circ}(n,1)) \cong \bigwedge^1_- \otimes y_1^n = \langle w_{12} \otimes y_1^n, w_{22} \otimes y_1^n \rangle$. We have:

$$\begin{aligned} (y_1\partial_{y_1} - y_2\partial_{y_2}).(w_{12} \otimes y_1^n) &= (n-1)w_{12} \otimes y_1^n, \\ y_2\partial_{y_1}.(w_{12} \otimes y_1^n) &= w_{12} \otimes n y_1^{n-1} y_2 \in H^{0,n}(G_{A^\circ}(n-1,2)), \\ y_1\partial_{y_2}.(w_{12} \otimes y_1^n) &= w_{11} \otimes y_1^n = \nabla\Big(\frac{x_1 y_1^{n+1}}{n+1}\Big) = 0 \quad \text{in} \quad H^{0,n}(G_{A^\circ}(n+1,0)). \end{aligned}$$

Analogously:

$$\begin{aligned} (y_1\partial_{y_1} - y_2\partial_{y_2}).(w_{22} \otimes y_1^n) &= (n-1)w_{22} \otimes y_1^n, \\ y_2\partial_{y_1}.(w_{22} \otimes y_1^n) &= w_{22} \otimes n y_1^{n-1} y_2 \in H^{0,n}(G_{A^\circ}(n-1,2)), \\ y_1\partial_{y_2}.(w_{22} \otimes y_1^n) &= w_{21} \otimes y_1^n = \nabla\Big(\frac{x_2 y_1^{n+1}}{n+1}\Big) = 0 \quad \text{in} \quad H^{0,n}(G_{A^\circ}(n+1,0)). \end{aligned}$$

Moreover let us show explicitly that

$$H^{0,n}(G_{A^\circ}(3,n-2)) \xrightarrow{y_2\partial_{y_1}} H^{0,n}(G_{A^\circ}(2,n-1)).$$

Indeed, by Lemmas 5.36 and 5.40, $H^{0,n}(G_{A^\circ}(3,n-2)) \cong \bigwedge^1_- \otimes y_1^3 y_2^{n-3}$ and $H^{0,n}(G_{A^\circ}(2,n-1)) \cong \bigwedge^1_+ \otimes y_1 y_2^{n-1}$. We have that:

$$y_2\partial_{y_1}.(w_{12} \otimes y_1^3 y_2^{n-3}) = w_{12} \otimes 3 y_1^2 y_2^{n-2}.$$

But $w_{12}\otimes y_1^2y_2^{n-2}=-w_{11}\otimes\frac{2y_1y_2^{n-1}}{n-1}$ in $H^{0,n}(G_{A^\circ}(2,n-1))$ since:

$$\nabla\Big(\frac{x_1y_1^2y_2^{n-1}}{n-1}\Big)=w_{11}\otimes\frac{2y_1y_2^{n-1}}{n-1}+w_{12}\otimes y_1^2y_2^{n-2}.$$

Analogously for $w_{22}\otimes y_1^3y_2^{n-3}$ we have that:

$$y_2\partial_{y_1}.(w_{22}\otimes y_1^3y_2^{n-3})=w_{22}\otimes 3y_1^2y_2^{n-2}=-w_{21}\otimes\frac{6y_1y_2^{n-1}}{n-1}\quad\text{in}\quad H^{0,n}(G_{A^\circ}(2,n-1)).$$

Finally by Lemmas 5.36 and 5.40, $H^{0,n}(G_{A^\circ}(1,n))\cong\bigwedge_+^1\otimes y_2^n\cong\bigwedge_-^1\otimes y_1y_2^{n-1}$. Therefore:

$$y_2\partial_{y_1}.(w_{12}\otimes y_1y_2^{n-1})=w_{12}\otimes y_2^n=\nabla\Big(\frac{x_1y_2^{n+1}}{n+1}\Big)=0\quad\text{in}\quad H^{0,n}(G_{A^\circ}(0,n+1)),$$
$$y_2\partial_{y_1}.(w_{22}\otimes y_1y_2^{n-1})=w_{22}\otimes y_2^n=\nabla\Big(\frac{x_2y_2^{n+1}}{n+1}\Big)=0\quad\text{in}\quad H^{0,n}(G_{A^\circ}(0,n+1)).$$

Hence, as a $\mathfrak{g}_0-$module, $M_1\cong Q\Big(1,n-1,-1-\frac{1}{2}n,-\frac{1}{2}n\Big)$. Finally as vector spaces:

$$M_2:=H^{0,n}(G_{A^\circ}(2,n))+H^{0,n}(G_{A^\circ}(3,n-1))+...+H^{0,n}(G_{A^\circ}(n,2))\cong\wedge^2\otimes P\Big(n,-\frac{1}{2}n,-\frac{1}{2}n\Big).$$

Indeed by Lemmas 5.36 and 5.40:

$$\begin{array}{c} H^{0,n}(G_{A^\circ}(n,2))\cong\bigwedge_2^-\otimes y_1^n\\ \big\downarrow y_2\partial_{y_1}\\ H^{0,n}(G_{A^\circ}(n-1,3))\cong\bigwedge_2^-\otimes y_1^{n-1}y_2\\ \vdots\\ H^{0,n}(G_{A^\circ}(3,n-1))\cong\bigwedge_2^-\otimes y_1^3y_2^{n-3}\\ \big\downarrow y_2\partial_{y_1}\\ H^{0,n}(G_{A^\circ}(2,n))\cong\bigwedge_2^-\otimes y_1^2y_2^{n-2}.\end{array}$$

Indeed let us observe that, by Lemma 5.36 and 5.40, $H^{0,n}(G_{A^\circ}(n,2))=\bigwedge_-^2\otimes y_1^n=\langle w_{12}w_{22}\otimes y_1^n\rangle$. We have:

$$\begin{aligned}(y_1\partial_{y_1}-y_2\partial_{y_2}).(w_{12}w_{22}\otimes y_1^n)&=(n-2)w_{12}w_{22}\otimes y_1^n,\\ y_2\partial_{y_1}.(w_{12}w_{22}\otimes y_1^n)&=w_{12}w_{22}\otimes ny_1^{n-1}y_2\in H^{0,n}(G_{A^\circ}(n-1,3)),\\ y_1\partial_{y_2}.(w_{12}w_{22}\otimes y_1^n)&=w_{11}w_{22}\otimes y_1^n+w_{12}w_{21}\otimes y_1^n\\ &=\nabla\Big(-w_{22}\otimes\frac{x_1y_1^{n+1}}{n+1}+w_{12}\otimes x_2\frac{y_1^{n+1}}{n+1}\Big)=0\quad\text{in}\quad H^{0,n}(G_{A^\circ}(n+1,1)).\end{aligned}$$

Moreover let us show explicitly that

$$H^{0,n}(G_{A^\circ}(3,n-1))\xrightarrow{y_2\partial_{y_1}}H^{0,n}(G_{A^\circ}(2,n)).$$

Indeed, by Lemmas 5.36 and 5.40, $H^{0,n}(G_{A^\circ}(3,n-1))\cong\bigwedge_-^2\otimes y_1^3y_2^{n-3}$ and $H^{0,n}(G_{A^\circ}(2,n))\cong\bigwedge_+^2\otimes y_2^n$. We have that:

$$y_2\partial_{y_1}.(w_{12}w_{22}\otimes y_1^3y_2^{n-3})=w_{12}w_{22}\otimes 3y_1^2y_2^{n-2}.$$

But $w_{12}w_{22}\otimes y_1^2y_2^{n-2}=-w_{21}w_{11}\otimes \frac{2y_2^n}{n(n-1)}$ in $H^{0,n}(G_{A^\circ}(2,n))$ since:

$$\nabla\Big(w_{12}\otimes \frac{x_2y_1^2y_2^{n-1}}{n-1}+w_{21}\otimes\frac{2x_1y_1y_2^n}{n(n-1)}\Big)=w_{12}w_{22}\otimes y_1^2y_2^{n-2}+w_{21}w_{11}\otimes\frac{2y_2^n}{n(n-1)}.$$

Finally $y_2\partial_{y_1}$ acts trivially on $H^{0,n}(G_{A^\circ}(2,n))$:

$$y_2\partial_{y_1}.(w_{12}w_{22}\otimes y_1^2y_2^{n-2})=w_{12}w_{22}\otimes 2y_1y_2^{n-1}=\nabla\Big(w_{12}\otimes\frac{2x_2y_1y_2^n}{n}\Big)=0 \quad \text{in} \quad H^{0,n}(G_{A^\circ}(1,n)).$$

Therefore, as a $\mathfrak{g}_0$−module, $M_2\cong Q\Big(0,n-2,-2-\frac{1}{2}n,-\frac{1}{2}n\Big)$.
Now let us focus on $m=1$. We notice that, by Lemma 5.40, $H^{1,n}(G_{A^\circ}(\alpha,\beta))\neq 0$ only for $0\leq\alpha\leq\beta\leq 2$ and $0\leq n\leq\alpha$ or $0\leq\beta\leq\alpha\leq 2$ and $0\leq n\leq\beta$; in these cases $H^{1,n}(G_{A^\circ}(\alpha,\beta))\cong\bigwedge^{\alpha+\beta-n+1}$. Therefore we have that $H^{1,n}(G_{A^\circ}(\alpha,\beta))=0$ if $n\geq 2$. Indeed for $n=2$ we obtain $\alpha=\beta=2$ and $\bigwedge^{\alpha+\beta-n+1}\cong\bigwedge^3=0$. The case $n>2$ is ruled out by conditions $0\leq\alpha\leq\beta\leq 2$ and $0\leq n\leq\alpha$ or $0\leq\beta\leq\alpha\leq 2$ and $0\leq n\leq\beta$. Hence we focus on $n=0$ and $n=1$.
Let $n=0$. We have that Equation (5.38) reduces to:

$$H^{1,0}(G_{A^\circ})=H^{1,0}(G_{A^\circ}(0,0))+H^{1,0}(G_{A^\circ}(1,0))+H^{1,0}(G_{A^\circ}(0,1)). \tag{5.40}$$

We point out that the RHS of (5.40) is the sum of two irreducible $\mathfrak{g}_0$−modules M_1 and M_2 that are defined as follows. We define:

$$M_1:=H^{1,0}(G_{A^\circ}(0,0)).$$

By relation (5.36) in the proof of Lemma 5.40, as a $\mathfrak{g}_0$−module:

$$H^{1,0}(G_{A^\circ}(0,0))\cong Q\Big(1,0,-\frac{1}{2},\frac{1}{2}\Big)$$

Moreover:

$$M_2:=H^{1,0}(G_{A^\circ}(1,0))+H^{1,0}(G_{A^\circ}(0,1)).$$

By relation (5.36) in the proof of Lemma 5.40, as a $\mathfrak{g}_0$−module:

$$H^{1,0}(G_{A^\circ}(1,0))+H^{1,0}(G_{A^\circ}(0,1))\cong Q\Big(0,1,-\frac{3}{2},\frac{1}{2}\Big)$$

Finally, let $n=1$. We have that:

$$H^{1,1}(G_{A^\circ})=H^{1,1}(G_{A^\circ}(1,1)). \tag{5.41}$$

By relation (5.36) in the proof of Lemma 5.40, as a $\mathfrak{g}_0$−module:

$$H^{1,1}(G_{A^\circ}(1,1))\cong Q\big(0,0,-2,0\big).$$

□

5.3.2 Homology of complexes M_X

We are now able to compute the homology of the complexes M_X's.

Proposition 5.42.

$$\begin{aligned} H^{m,n}(M_A) &= 0 \quad \textit{for all } (m,n) \neq (0,0),(1,1),\\ H^{m,n}(M_B) &= 0 \quad \textit{for all } (m,n),\\ H^{m,n}(M_C) &= 0 \quad \textit{for all } (m,n) \neq (0,0),(-1,-1),\\ H^{m,n}(M_D) &= 0 \quad \textit{for all } (m,n). \end{aligned}$$

Proof. By Remarks 5.34, 5.35 and Proposition 5.41 we know that:

$$\begin{aligned} H^{m,n}(G_A) = H^{m,n}(G_{A^\circ}) &= 0 \quad \text{if } m>1 \text{ or } (m=1 \text{ and } n \geq 2),\\ H^{m,n}(G_D) = H^{m,n}(G_{D^\circ}) &= 0 \quad \text{if } m>0 \text{ and } n \leq 0,\\ H^{m,n}(G_B) = H^{m,n}(G_{B^\circ}) &= 0 \quad \text{if } m<0 \text{ and } n \geq 0,\\ H^{m,n}(G_C) = H^{m,n}(G_{C^\circ}) &= 0 \quad \text{if } m<-1 \text{ or } (m=-1 \text{ and } n \leq -2). \end{aligned}$$

Therefore we obtain, by Proposition 5.33, that:

$$\begin{aligned} H^{m,n}(M_A) &= 0 \quad \text{if } m>1 \text{ or } (m=1 \text{ and } n \geq 2),\\ H^{m,n}(M_D) &= 0 \quad \text{if } m>0 \text{ and } n \leq 0,\\ H^{m,n}(M_B) &= 0 \quad \text{if } m<0 \text{ and } n \geq 0,\\ H^{m,n}(M_C) &= 0 \quad \text{if } m<-1 \text{ or } (m=-1 \text{ and } n \leq -2). \end{aligned}$$

Let us analyze the modules $H^{m,n}(G_{X^\circ})$ for $m=0$. We have, by Proposition 5.41, that $H^{0,n}(G_{A^\circ}) \cong H^{0,n-2}(G_{B^\circ})$ as $\mathfrak{g}_0-$modules for $n \geq 2$, indeed:

$$\begin{aligned} H^{0,n}(G_{A^\circ}) &\cong \sum_{i=0}^{2} Q\Big(r_i, n-i, -i-\frac{1}{2}n, -\frac{1}{2}n\Big),\\ H^{0,n-2}(G_{B^\circ}) &\cong \sum_{i=0}^{2} Q\Big(r_i, n-i, -i-\frac{1}{2}n, -\frac{1}{2}n\Big). \end{aligned}$$

By Remark 5.34, we know that

$$\begin{aligned} H^{0,n}(G_{A^\circ}) &= \frac{G^{0,n}_{A^\circ}}{\operatorname{Im}(\nabla : G^{1,n+1}_{A^\circ} \to G^{0,n}_{A^\circ})},\\ H^{0,n-2}(G_{B^\circ}) &= \operatorname{Ker}(\nabla : G^{0,n-2}_{B^\circ} \to G^{-1,n-3}_{B^\circ}) \quad \text{for } n \geq 3,\\ H^{0,0}(G_{B^\circ}) &= G^{0,0}_{B^\circ} \qquad \text{for } n=2. \end{aligned}$$

We want to show that the map induced by $\widetilde{\nabla}_2$ between $H^{0,n}(G_{A^\circ})$ and $H^{0,n-2}(G_{B^\circ})$, for $n \geq 2$, is an isomorphism.
Indeed the kernel of the map induced by $\widetilde{\nabla}_2$ between $H^{0,n}(G_{A^\circ})$ and $H^{0,n-2}(G_{B^\circ})$, for $n \geq 2$, is actually isomorphic to

$$\frac{\operatorname{Ker}(\widetilde{\nabla}_2 : G^{0,n}_{A^\circ} \to G^{0,n-2}_{B^\circ})}{\operatorname{Im}(\nabla : G^{1,n+1}_{A^\circ} \to G^{0,n}_{A^\circ})} = \frac{\operatorname{Ker}(\widetilde{\nabla}_2 : G^{0,n}_{A} \to G^{0,n-2}_{B})}{\operatorname{Im}(\nabla : G^{1,n+1}_{A} \to G^{0,n}_{A})} = H^{0,n}(G_A).$$

Moreover the image of the map induced by $\widetilde{\nabla}_2$ between $H^{0,n}(G_{A^\circ})$ and $H^{0,n-2}(G_{B^\circ})$, for $n \geq 2$, is

$$\operatorname{Im}(\widetilde{\nabla}_2 : G^{0,n}_{A^\circ} \to G^{0,n-2}_{B^\circ}) = \operatorname{Im}(\widetilde{\nabla}_2 : G^{0,n}_{A} \to G^{0,n-2}_{B}).$$

Therefore if we show that $\widetilde{\nabla}_2$ induces an isomorphism between $H^{0,n}(G_{A^\circ})$ and $H^{0,n-2}(G_{B^\circ})$ for $n \geq 2$, we can conclude that as $\mathfrak{g}_0-$ modules:

$$\frac{\mathrm{Ker}(\widetilde{\nabla}_2 : G_{A^\circ}^{0,n} \to G_{B^\circ}^{0,n-2})}{\mathrm{Im}(\nabla : G_{A^\circ}^{1,n+1} \to G_{A^\circ}^{0,n})} \cong 0;$$

$$\mathrm{Im}(\widetilde{\nabla}_2 : G_{A^\circ}^{0,n} \to G_{B^\circ}^{0,n-2}) \cong \mathrm{Ker}(\nabla : G_{B^\circ}^{0,n-2} \to G_{B^\circ}^{-1,n-3}) = \mathrm{Ker}(\nabla : G_{B}^{0,n-2} \to G_{B}^{-1,n-3}) \quad \text{for } n \geq 3;$$

$$\mathrm{Im}(\widetilde{\nabla}_2 : G_{A^\circ}^{0,n} \to G_{B^\circ}^{0,n-2}) \cong G_{B^\circ}^{0,0} = \mathrm{Ker}(\nabla_2 : G_{B}^{0,0} \to G_{C}^{-2,0}) \quad \text{for } n = 2.$$

This means that if we show that the map induced by $\widetilde{\nabla}_2$ between $H^{0,n}(G_{A^\circ})$ and $H^{0,n-2}(G_{B^\circ})$, for $n \geq 2$, is an isomorphism, then we obtain $H^{0,n}(G_A) = 0$ and $H^{0,n-2}(G_B) = 0$ for $n \geq 2$. Hence by Proposition 5.33 we obtain that $H^{0,n}(M_A) = 0$ and $H^{0,n-2}(M_B) = 0$ for $n \geq 2$.
Thus, let us show that the induced map is an isomorphism. It is sufficient to show that the images of highest weight vectors in $H^{0,n}(G_{A^\circ})$ are different from 0. By Proposition 5.41 we know that the highest weight vectors in $H^{0,n}(G_{A^\circ})$ are $y_1^n, w_{12} \otimes y_1^n, w_{12}w_{22} \otimes y_1^n$. We have:

$$\begin{aligned} &\widetilde{\nabla}_2(y_1^n) = w_{11}w_{21} \otimes n(n-1)y_1^{n-2}, \\ &\widetilde{\nabla}_2(w_{12} \otimes y_1^n) = w_{12}w_{11}w_{21} \otimes n(n-1)y_1^{n-2}, \\ &\widetilde{\nabla}_2(w_{12}w_{22} \otimes y_1^n) = w_{12}w_{22}w_{11}w_{21} \otimes n(n-1)y_1^{n-2}. \end{aligned}$$

By Proposition 5.41 we have that $H^{0,1}(G_{A^\circ}) \cong H^{-1,0}(G_{C^\circ})$ as $\mathfrak{g}_0-$modules, indeed:

$$\begin{aligned} H^{0,1}(G_{A^\circ}) &\cong Q\Big(0,1,-\frac{1}{2},-\frac{1}{2}\Big) + Q\Big(1,0,-\frac{3}{2},-\frac{1}{2}\Big), \\ H^{-1,0}(G_{C^\circ}) &\cong Q\Big(0,1,-\frac{1}{2},-\frac{1}{2}\Big) + Q\Big(1,0,-\frac{3}{2},-\frac{1}{2}\Big). \end{aligned}$$

With an analogous argument, in order to obtain that $H^{0,1}(M_A) = H^{-1,0}(M_C) = 0$, it is sufficient to show that the map induced by ∇_3 between $H^{0,1}(G_{A^\circ})$ and $H^{-1,0}(G_{C^\circ})$ is an isomorphism.
We show that the map induced by ∇_3 is different from 0 on highest weight vectors in $H^{0,1}(G_{A^\circ})$. By Proposition 5.41 we know that the highest weight vectors in $H^{0,1}(G_{A^\circ})$ are $y_1, w_{12} \otimes y_1$. We have:

$$\begin{aligned} &\nabla_3(y_1) = w_{11}w_{21}w_{12}\partial_{x_1} + w_{11}w_{21}w_{22}\partial_{x_2}, \\ &\nabla_3(w_{12} \otimes y_1) = w_{12}w_{11}w_{21}w_{12}\partial_{x_1} + w_{12}w_{11}w_{21}w_{22}\partial_{x_2} = w_{12}w_{11}w_{21}w_{22}\partial_{x_2}. \end{aligned}$$

By Proposition 5.41 it follows that $H^{0,n}(G_{D^\circ}) \cong H^{0,n-2}(G_{C^\circ})$ as $\mathfrak{g}_0-$modules for $n \leq 0$, indeed:

$$\begin{aligned} H^{0,n}(G_{D^\circ}) &\cong \sum_{i=0}^{2} Q\Big(r_i, -n+i, -i-\frac{1}{2}n+1, -\frac{1}{2}n+1\Big), \\ H^{0,n-2}(G_{C^\circ}) &\cong \sum_{i=0}^{2} Q\Big(r_i, -n+i, -i-\frac{1}{2}n+1, -\frac{1}{2}n+1\Big). \end{aligned}$$

With an analogous argument, in order to obtain that $H^{0,n}(M_D) = H^{0,n-2}(M_C) = 0$ for $n \leq 0$, it is sufficient to show that the map induced by $\widetilde{\nabla}_2$ between $H^{0,n}(G_{D^\circ})$ and $H^{0,n-2}(G_{C^\circ})$ is an isomorphism for $n \leq 0$.
We show that the map induced by $\widetilde{\nabla}_2$ is different from 0 on highest weight vectors in $H^{0,n}(G_{D^\circ})$. By Proposition 5.41 we know that the highest weight vectors in $H^{0,n}(G_{D^\circ})$ are $\partial_{y_2}^{-n}, w_{11} \otimes \partial_{y_2}^{-n}, w_{11}w_{21} \otimes \partial_{y_2}^{-n}$. We have:

$$\widetilde{\nabla}_2(\partial_{y_2}^{-n}) = w_{11}w_{21} \otimes \partial_{y_1}^2 \partial_{y_2}^{-n} + w_{11}w_{22} \otimes \partial_{y_1}\partial_{y_2}^{-n+1} + w_{12}w_{21} \otimes \partial_{y_1}\partial_{y_2}^{-n+1} + w_{12}w_{22} \otimes \partial_{y_2}^{-n+2},$$

$$\widetilde{\nabla}_2(w_{11}\otimes\partial_{y_2}^{-n}) = w_{11}w_{12}w_{21}\otimes\partial_{y_1}\partial_{y_2}^{-n+1} + w_{11}w_{12}w_{22}\otimes\partial_{y_2}^{-n+2},$$
$$\widetilde{\nabla}_2(w_{11}w_{21}\otimes\partial_{y_2}^{-n}) = w_{11}w_{21}w_{12}w_{22}\otimes\partial_{y_2}^{-n+2}.$$

Finally, by Proposition 5.41 we have that $H^{1,0}(G_{A^\circ})\cong H^{0,-1}(G_{C^\circ})$ as $\mathfrak{g}_0-$modules, indeed:

$$H^{1,0}(G_{A^\circ})\cong Q\Big(1,0,-\frac{1}{2},\frac{1}{2}\Big)+Q\Big(0,1,-\frac{3}{2},\frac{1}{2}\Big),$$
$$H^{0,-1}(G_{C^\circ})\cong Q\Big(1,0,-\frac{1}{2},\frac{1}{2}\Big)+Q\Big(0,1,-\frac{3}{2},\frac{1}{2}\Big).$$

With an analogous argument, in order to obtain that $H^{1,0}(M_A)=H^{0,-1}(M_C)=0$, it is sufficient to show that the map induced by $\widetilde{\nabla}_3$ between $H^{1,0}(G_{A^\circ})$ and $H^{0,-1}(G_{C^\circ})$ is an isomorphism.
We show that the map induced by $\widetilde{\nabla}_3$ is different from 0 on highest weight vectors in $H^{1,0}(G_{A^\circ})$. By Proposition 5.41 we know that the highest weight vectors in $H^{1,0}(G_{A^\circ})$ are $x_1, w_{11}\otimes x_2 - w_{22}\otimes x_1$. We have:

$$\widetilde{\nabla}_3(x_1) = w_{11}w_{12}w_{21}\partial_{y_1} + w_{11}w_{12}w_{22}\partial_{y_2},$$
$$\widetilde{\nabla}_3(w_{11}\otimes x_2 - w_{22}\otimes x_1) = w_{11}w_{21}w_{12}w_{22}\partial_{y_2} - w_{22}w_{11}w_{12}w_{21}\partial_{y_1}.$$

□

Let us now focus on the remaining four cases.

Proposition 5.43.

$$H^{0,0}(M_C)\cong 0,$$
$$H^{-1,-1}(M_C)\cong\mathbb{C}.$$

In order to prove Proposition 5.43, we need the following results and the theory of spectral sequences. So far we have shown that $E^0(M_C)^{0,0} = H^{0,0}(\mathrm{Gr}\,M_C) = S(\mathfrak{g}_{-2})\otimes H^{0,0}(G_C)$ and $E^0(M_C)^{-1,-1} = H^{-1,-1}(\mathrm{Gr}\,M_C) = S(\mathfrak{g}_{-2})\otimes H^{-1,-1}(G_C)$ as $\mathcal{W}-$modules.

Lemma 5.44. *Let*

$$\xi = iw_{11}w_{21}\Delta^-\partial_{y_1} + (iw_{12}w_{21}+iw_{11}w_{22})\Delta^-\partial_{y_2}.$$

be an element in $M_C^{-1,-1}$. The following hold:

1. $\nabla\xi = 0,$
2. $\mathfrak{g}_0.\xi = 0,$
3. $(t\xi_1+it\xi_2).\xi\in\mathrm{Im}\,\nabla,\ (\xi_1\xi_3\xi_4+i\xi_2\xi_3\xi_4).\xi\in\mathrm{Im}\,\nabla,$
4. $\xi\notin\mathrm{Im}\,\nabla,$
5. *$[\xi]$ is a basis for the $\mathfrak{g}_0-$module $H^{-1,-1}(G_C)\cong\mathbb{C}$.*

Proof. **1)** Let us show that $\nabla\xi=0$.

$$\begin{aligned}\nabla\xi =& iw_{11}w_{21}w_{12}w_{21}\otimes\partial_{x_1}\partial_{y_1}^2\partial_{x_2} + iw_{11}w_{21}w_{22}w_{11}\otimes\partial_{x_1}\partial_{y_1}^2\partial_{x_2} + iw_{12}w_{21}w_{12}w_{11}\otimes\partial_{x_1}^2\partial_{y_1}\partial_{y_2}\\ &+ iw_{12}w_{21}w_{12}w_{21}\otimes\partial_{x_1}\partial_{y_1}\partial_{y_2}\partial_{x_2} + iw_{12}w_{21}w_{22}w_{11}\otimes\partial_{x_1}\partial_{y_1}\partial_{y_2}\partial_{x_2} + iw_{11}w_{22}w_{12}w_{21}\otimes\partial_{x_1}\partial_{y_1}\partial_{y_2}\partial_{x_2}\\ &+ iw_{11}w_{22}w_{12}w_{11}\otimes\partial_{x_1}^2\partial_{y_1}\partial_{y_2}\end{aligned}$$

$$
\begin{aligned}
=& - 4i\Theta w_{11}w_{21} \otimes \partial_{x_1} \otimes \partial_{y_1}^2\partial_{x_2} + i4\Theta w_{11}w_{21} \otimes \partial_{x_1}\partial_{y_1}^2\partial_{x_2} - 4i\Theta w_{12}w_{11} \otimes \partial_{x_1}^2\partial_{y_1}\partial_{y_2} \\
& - 4i\Theta w_{12}w_{21} \otimes \partial_{x_1} \otimes \partial_{y_1}\partial_{y_2}\partial_{x_2} - iw_{21}w_{12}w_{22}w_{11} \otimes \partial_{x_1}\partial_{y_1}\partial_{y_2}\partial_{x_2} - 4i\Theta w_{22}w_{11} \otimes \partial_{x_1}\partial_{y_1}\partial_{y_2}\partial_{x_2} \\
& - iw_{22}w_{11}w_{12}w_{21} \otimes \partial_{x_1}\partial_{y_1}\partial_{y_2}\partial_{x_2} + 4i\Theta w_{12}w_{21} \otimes \partial_{x_1}\partial_{y_1}\partial_{y_2}\partial_{x_2} - iw_{11}w_{22}w_{11}w_{12} \otimes \partial_{x_1}^2\partial_{y_1}\partial_{y_2} \\
=& - 4i\Theta w_{12}w_{11} \otimes \partial_{x_1}^2\partial_{y_1}\partial_{y_2} + iw_{22}w_{11}w_{12}w_{21} \otimes \partial_{x_1}\partial_{y_1}\partial_{y_2}\partial_{x_2} - iw_{22}w_{11}w_{12}w_{21} \otimes \partial_{x_1}\partial_{y_1}\partial_{y_2}\partial_{x_2} \\
& + 4i\Theta w_{12}w_{11} \otimes \partial_{x_1}^2\partial_{y_1}\partial_{y_2} = 0.
\end{aligned}
$$

2) Let us show that $\mathfrak{g}_0.\xi = 0$.

$$
\begin{aligned}
x_1\partial_{x_2}.\xi =& - iw_{11}w_{21}w_{12} \otimes \partial_{x_2}\partial_{y_1} + iw_{11}w_{21}w_{12} \otimes \partial_{x_2}\partial_{y_1} - iw_{12}w_{21}w_{12}\partial_{x_2}\partial_{y_2} + iw_{12}w_{11}w_{12} \otimes \partial_{x_1}\partial_{y_2} \\
& + iw_{12}w_{11}w_{22} \otimes \partial_{x_2}\partial_{y_2} + iw_{12}w_{21}w_{12} \otimes \partial_{x_2}\partial_{y_2} - iw_{11}w_{22}w_{12} \otimes \partial_{x_2}\partial_{y_2} + iw_{11}w_{12}w_{12} \otimes \partial_{x_1}\partial_{y_2} = 0; \\
x_2\partial_{x_1}.\xi =& iw_{11}w_{21}w_{22} \otimes \partial_{x_1}\partial_{y_1} - iw_{11}w_{21}w_{22} \otimes \partial_{x_1}\partial_{y_1} + iw_{22}w_{21}w_{12} \otimes \partial_{x_1}\partial_{y_2} \\
& + iw_{12}w_{21}w_{22} \otimes \partial_{x_1}\partial_{y_2} - iw_{12}w_{21}w_{22} \otimes \partial_{x_1}\partial_{y_2} + iw_{21}w_{22}w_{12} \otimes \partial_{x_1}\partial_{y_2} = 0; \\
y_2\partial_{y_1}.\xi =& iw_{12}w_{21}w_{12} \otimes \partial_{x_1}\partial_{y_1} + iw_{11}w_{22}w_{12} \otimes \partial_{x_1}\partial_{y_1} + iw_{12}w_{21}w_{22} \otimes \partial_{x_2}\partial_{y_1} \\
& - iw_{12}w_{21}w_{12} \otimes \partial_{x_1}\partial_{y_1} - iw_{12}w_{21}w_{22} \otimes \partial_{x_2}\partial_{y_1} - iw_{11}w_{22}w_{12} \otimes \partial_{x_1}\partial_{y_1} = 0; \\
y_1\partial_{y_2}.\xi =& - iw_{11}w_{21}w_{12} \otimes \partial_{x_1}\partial_{y_2} - iw_{11}w_{21}w_{22} \otimes \partial_{x_2}\partial_{y_2} + iw_{11}w_{21}w_{12} \otimes \partial_{x_1}\partial_{y_2} \\
& + iw_{12}w_{21}w_{11} \otimes \partial_{x_1}\partial_{y_2} + iw_{11}w_{21}w_{22} \otimes \partial_{x_2}\partial_{y_2} + iw_{11}w_{21}w_{12} \otimes \partial_{x_1}\partial_{y_2} \\
& + iw_{11}w_{22}w_{11} \otimes \partial_{x_1}\partial_{y_2} \\
=& - iw_{21}w_{12}w_{11} \otimes \partial_{x_1}\partial_{y_2} - 4i\Theta w_{11} \otimes \partial_{x_1}\partial_{y_2} + iw_{11}w_{21}w_{12} \otimes \partial_{x_1}\partial_{y_2} \\
& + 4i\Theta w_{11} \otimes \partial_{x_1}\partial_{y_2} = 0.
\end{aligned}
$$

The fact that $t.\xi = C.\xi = 0$ is a straightforward computation.
3) We point out that $(t\xi_1 + it\xi_2)$ and $\xi_1\xi_3\xi_4 + i\xi_2\xi_3\xi_4$ are the lowest weight vectors of $\mathfrak{g}_1$ (see the Appendix).
We compute $(t\xi_1 + it\xi_2).\xi$ in three parts separately. Let us denote $\xi = m_1 + m_2 + m_3$, where

$$
\begin{aligned}
m_1 &= iw_{11}w_{21}\Delta^-\partial_{y_1}, \\
m_2 &= iw_{12}w_{21}\Delta^-\partial_{y_2}, \\
m_3 &= iw_{11}w_{22}\Delta^-\partial_{y_2}.
\end{aligned}
$$

We will use the following relations that come from bracket (2.1) and Proposition 2.11:

$$
\begin{aligned}
&[t\xi_1 + it\xi_2, w_{11}] = -2i(t + H_1), && [t\xi_1 + it\xi_2, w_{22}] = 0, \\
&[t\xi_1 + it\xi_2, w_{12}] = -2iy_2\partial_{y_1}, && [t\xi_1 + it\xi_2, w_{21}] = -2ix_2\partial_{x_1}.
\end{aligned}
$$

We have:

$$
\begin{aligned}
(t\xi_1 + it\xi_2).(m_1) =& 2(t + H_1)w_{21}\Delta^-\partial_{y_1} - iw_{11}(-2ix_2\partial_{x_1})\Delta^-\partial_{y_1} + iw_{11}w_{21}(t\xi_1 + it\xi_2)\Delta^-\partial_{y_1} \\
=& 2(t + H_1)w_{21}\Delta^-\partial_{y_1} - 2w_{11}(x_2\partial_{x_1})\Delta^-\partial_{y_1} + 2w_{11}w_{21}(y_2\partial_{y_1}) \otimes \partial_{x_1}\partial_{y_1} \\
=& 2w_{21}(t + H_1)\Delta^-\partial_{y_1} - 2w_{21}\Delta^-\partial_{y_1} - 2w_{11}(x_2\partial_{x_1})\Delta^-\partial_{y_1} \\
=& - 2w_{21}w_{12}\partial_{x_1}\partial_{y_1} + 2w_{21}w_{12}(t + H_1)\partial_{x_1}\partial_{y_1} - 4w_{21}w_{22}\partial_{x_2}\partial_{y_1} \\
& + 2w_{21}w_{22}(t + H_1)\partial_{x_2}\partial_{y_1} - 2w_{21}\Delta^-\partial_{y_1} - 2w_{11}(x_2\partial_{x_1})\Delta^-\partial_{y_1} \\
=& - 2w_{21}w_{12}\partial_{x_1}\partial_{y_1} + 4w_{21}w_{21}\partial_{x_1}\partial_{y_1} - 4w_{21}w_{22} \otimes \partial_{x_2}\partial_{y_1} \\
& + 6w_{21}w_{22}\partial_{x_2}\partial_{y_1} - 2w_{21}\Delta^-\partial_{y_1} - 2w_{11}(x_2\partial_{x_1})\Delta^-\partial_{y_1} \\
=& 2w_{21}w_{12}\partial_{x_1}\partial_{y_1} + 2w_{21}w_{22}\partial_{x_2}\partial_{y_1} - 2w_{21}w_{12}\partial_{x_1}\partial_{y_1} \\
& - 2w_{21}w_{22}\partial_{x_2}\partial_{y_1} - 2w_{11}(x_2\partial_{x_1})\Delta^-\partial_{y_1} \\
=& - 2w_{11}w_{22}\partial_{x_1}\partial_{y_1} + w_{11}w_{22} \otimes \partial_{x_1}\partial_{y_1} = 0;
\end{aligned}
$$

$$\begin{aligned}(t\xi_1+it\xi_2).(m_2)=&(2y_2\partial_{y_1})w_{21}\Delta^-\partial_{y_2}-iw_{12}(-2ix_2\partial_{x_1})\Delta^-\partial_{y_2}+iw_{12}w_{21}(-2iy_2\partial_{y_1})\otimes\partial_{x_1}\partial_{y_2}\\=&w_{21}(2y_2\partial_{y_1})\Delta^-\partial_{y_2}+2w_{22}\Delta^-\partial_{y_2}-2w_{12}(x_2\partial_{x_1})\Delta^-\partial_{y_2}-2w_{12}w_{21}\partial_{x_1}\partial_{y_1}\\=&-2w_{21}w_{12}\partial_{x_1}\partial_{y_1}-2w_{21}w_{22}\partial_{x_2}\partial_{y_1}+2w_{22}w_{12}\partial_{x_1}\partial_{y_2}-2w_{12}w_{21}\partial_{x_1}\partial_{y_1};\\(t\xi_1+it\xi_2).(m_3)=&2(t+H_1)w_{22}\Delta^-\partial_{y_2}+iw_{11}w_{22}(-2iy_2\partial_{y_1})\partial_{x_1}\partial_{y_2}\\=&2w_{22}(t+H_1)\Delta^-\partial_{y_2}-4w_{22}\Delta^-\partial_{y_2}-2w_{11}w_{22}\partial_{x_1}\partial_{y_1}\\=&-2w_{22}w_{12}\partial_{x_1}\partial_{y_2}+6w_{22}w_{12}\partial_{x_1}\partial_{y_2}-4w_{22}w_{12}\partial_{x_1}\partial_{y_2}-2w_{11}w_{22}\partial_{x_1}\partial_{y_1}\\=&-2w_{11}w_{22}\partial_{x_1}\partial_{y_1}.\end{aligned}$$

Then we have that:

$$\begin{aligned}(t\xi_1+it\xi_2).\xi=&-2w_{21}w_{12}\partial_{x_1}\partial_{y_1}-2w_{21}w_{22}\partial_{x_2}\partial_{y_1}+2w_{22}w_{12}\partial_{x_1}\partial_{y_2}\\&-2w_{12}w_{21}\partial_{x_1}\partial_{y_1}-2w_{11}w_{22}\partial_{x_1}\partial_{y_1}\\=&-2w_{21}w_{12}\partial_{x_1}\partial_{y_1}-2w_{21}w_{22}\partial_{x_2}\partial_{y_1}+2w_{22}w_{12}\partial_{x_1}\partial_{y_2}\\&+2w_{21}w_{12}\partial_{x_1}\partial_{y_1}+8\Theta\otimes\partial_{x_1}\partial_{y_1}-8\Theta\otimes\partial_{x_1}\partial_{y_1}+2w_{22}w_{11}\partial_{x_1}\partial_{y_1}\\=&\nabla(2w_{22}\otimes 1).\end{aligned}$$

We compute $(\xi_1\xi_3\xi_4+i\xi_2\xi_3\xi_4).\xi$ in three parts separately. We will shortly write g_1 instead of $(\xi_1\xi_3\xi_4+i\xi_2\xi_3\xi_4)$. We will use the following relations that come from bracket (2.1) and Proposition 2.11:

$$[g_1,w_{11}]=2C+2H_2,\qquad [g_1,w_{22}]=0,$$
$$[g_1,w_{12}]=2y_2\partial_{y_1},\qquad [g_1,w_{21}]=-2x_2\partial_{x_1}.$$

We have that:

$$\begin{aligned}g_1.m_1=&i(2C+2H_2)w_{21}\Delta^-\partial_{y_1}-iw_{11}(-2x_2\partial_{x_1})\Delta^-\partial_{y_1}+iw_{11}w_{21}g_1\Delta^-\partial_{y_1}\\=&iw_{21}(2C+2H_2)\Delta^-\partial_{y_1}+2iw_{21}\Delta^-\partial_{y_1}+2iw_{11}(x_2\partial_{x_1})\Delta^-\partial_{y_1}+iw_{11}w_{21}(2y_2\partial_{y_1})\otimes\partial_{x_1}\partial_{y_1}\\=&-2iw_{21}w_{12}\partial_{x_1}\partial_{y_1}+iw_{21}w_{12}(2C+2H_2)\otimes\partial_{x_1}\partial_{y_1}+iw_{21}w_{22}(2C+2H_2)\otimes\partial_{x_2}\partial_{y_1}\\&+2iw_{21}\Delta^-\partial_{y_1}+2iw_{11}w_{22}\otimes\partial_{x_1}\partial_{y_1}-2iw_{11}w_{22}\otimes\partial_{x_1}\partial_{y_1}+iw_{11}w_{21}\otimes(2y_2\partial_{y_1})\partial_{x_1}\partial_{y_1}\\=&-2iw_{21}w_{12}\partial_{x_1}\partial_{y_1}-2iw_{21}w_{22}\otimes\partial_{x_2}\partial_{y_1}+2iw_{21}w_{12}\partial_{x_1}\partial_{y_1}+2iw_{21}w_{22}\partial_{x_2}\partial_{y_1}\\&+2iw_{11}w_{22}\otimes\partial_{x_1}\partial_{y_1}-2iw_{11}w_{22}\partial_{x_1}\partial_{y_1}=0;\\g_1.m_2=&(2iy_2\partial_{y_1})w_{21}\Delta^-\partial_{y_2}-iw_{12}(-2x_2\partial_{x_1})\Delta^-\partial_{y_2}+iw_{12}w_{21}(g_1)\Delta^-\partial_{y_2}\\=&2iw_{22}\Delta^-\partial_{y_2}+2iw_{21}w_{12}\otimes(y_2\partial_{y_1})\partial_{x_1}\partial_{y_2}+2iw_{21}w_{22}\otimes(y_2\partial_{y_1})\partial_{x_2}\partial_{y_2}+2iw_{12}w_{22}\partial_{x_1}\partial_{y_2}\\&-2iw_{12}w_{22}\otimes\partial_{x_1}\partial_{y_2}+iw_{12}w_{21}(g_1)\Delta^-\partial_{y_2}\\=&2iw_{22}\Delta^-\partial_{y_2}-2iw_{21}w_{12}\otimes\partial_{x_1}\partial_{y_1}-2iw_{21}w_{22}\otimes\partial_{x_2}\partial_{y_1}+2iw_{12}w_{22}\otimes\partial_{x_1}\partial_{y_2}-2iw_{12}w_{22}\otimes\partial_{x_1}\partial_{y_2}\\&+iw_{12}w_{21}(g_1)\Delta^-\partial_{y_2}\\=&2iw_{22}w_{12}\partial_{x_1}\partial_{y_2}-2iw_{21}w_{12}\partial_{x_1}\partial_{y_1}-2iw_{21}w_{22}\partial_{x_2}\partial_{y_1}-2iw_{12}w_{21}\partial_{x_1}\partial_{y_1}\\=&2iw_{22}w_{12}\partial_{x_1}\partial_{y_2}-2iw_{21}w_{22}\partial_{x_2}\partial_{y_1}+8i\Theta\otimes\partial_{x_1}\partial_{y_1};\\g_1.m_3=&i(2C+2H_2)w_{22}\Delta^-\partial_{y_2}+iw_{11}w_{22}(g_1)\Delta^-\partial_{y_2}\\=&iw_{22}(2C+2H_2)\Delta^-\partial_{y_2}+iw_{11}w_{22}(2y_2\partial_{y_1})\partial_{x_1}\partial_{y_2}\\=&iw_{22}w_{12}(2C+2H_2)\partial_{x_1}\partial_{y_2}-2iw_{22}w_{12}\partial_{x_1}\partial_{y_2}-2iw_{11}w_{22}\partial_{x_1}\partial_{y_1}\\=&2iw_{22}w_{12}\partial_{x_1}\partial_{y_2}-2iw_{22}w_{12}\partial_{x_1}\partial_{y_2}-2iw_{11}w_{22}\partial_{x_1}\partial_{y_1}\\=&2iw_{22}w_{11}\partial_{x_1}\partial_{y_1}-8i\Theta\otimes\partial_{x_1}\partial_{y_1}.\end{aligned}$$

Then we have:

$$(\xi_1\xi_3\xi_4+i\xi_2\xi_3\xi_4).\xi=2iw_{22}w_{12}\partial_{x_1}\partial_{y_2}-2iw_{21}w_{22}\partial_{x_2}\partial_{y_1}+8i\Theta\otimes\partial_{x_1}\partial_{y_1}+2iw_{22}w_{11}\partial_{x_1}\partial_{y_1}-8i\Theta\otimes\partial_{x_1}\partial_{y_1}$$

$$=\nabla(2iw_{22}\otimes 1).$$

4) Let us show that $\xi \notin \operatorname{Im}\nabla$. Let us consider $\nabla : M(0,0,2,0) \longrightarrow M(1,1,3,0)$. Since $M(0,0,2,0)$ is irreducible and $\nabla \neq 0$, the map ∇ is injective. Therefore, if $\xi = \nabla(v)$, then $t.v = 0$ because of injectivity and the fact that $t.\xi = 0$. Let us take $v \in M(0,0,2,0)$ whose weight is 0 with respect to t. Then:

$$\begin{aligned} v =&\alpha_1\Theta\otimes 1+\alpha_2 w_{11}w_{22}\otimes 1+\alpha_3 w_{11}w_{12}\otimes 1+\alpha_4 w_{11}w_{21}\otimes 1\\ &+\alpha_5 w_{22}w_{12}\otimes 1+\alpha_6 w_{22}w_{21}\otimes 1+\alpha_7 w_{12}w_{21}\otimes 1.\end{aligned}$$

If $\xi = \nabla(v)$, from injectivity and the fact that $(x_1\partial_{x_1} - x_2\partial_{x_2}).\xi = 0$, we obtain that $(x_1\partial_{x_1} - x_2\partial_{x_2}).v = 0$, that is:

$$(x_1\partial_{x_1} - x_2\partial_{x_2}).v = 2\alpha_3 w_{11}w_{21}\otimes 1 - 2\alpha_6 w_{22}w_{21}\otimes 1 = 0.$$

We deduce that $\alpha_3 = \alpha_6 = 0$. Similarly, if $\xi = \nabla(v)$, from injectivity and the fact that $(y_1\partial_{y_1} - y_2\partial_{y_2}).\xi = 0$, we obtain $(y_1\partial_{y_1} - y_2\partial_{y_2}).v = 0$, that is:

$$(y_1\partial_{y_1} - y_2\partial_{y_2}).v = 2\alpha_4 w_{11}w_{21}\otimes 1 - 2\alpha_5 w_{22}w_{12}\otimes 1 = 0.$$

We deduce that $\alpha_4 = \alpha_5 = 0$. Hence:

$$v = \alpha_1\Theta\otimes 1+\alpha_2 w_{11}w_{22}\otimes 1+\alpha_7 w_{12}w_{21}\otimes 1.$$

We compute ∇v.

$$\begin{aligned}\nabla v =&\alpha_1\Theta w_{11}\otimes\partial_{x_1}\partial_{y_1}+\alpha_1\Theta w_{21}\otimes\partial_{x_2}\partial_{y_1}+\alpha_1\Theta w_{12}\otimes\partial_{x_1}\partial_{y_2}+\alpha_1\Theta w_{22}\otimes\partial_{x_2}\partial_{y_2}\\ &+\alpha_2 w_{11}w_{22}w_{11}\otimes\partial_{x_1}\partial_{y_1}+\alpha_2 w_{11}w_{22}w_{21}\otimes\partial_{x_2}\partial_{y_1}+\alpha_2 w_{11}w_{22}w_{12}\otimes\partial_{x_1}\partial_{y_2}\\ &+\alpha_7 w_{12}w_{21}w_{11}\otimes\partial_{x_1}\partial_{y_1}+\alpha_7 w_{12}w_{21}w_{12}\otimes\partial_{x_1}\partial_{y_2}+\alpha_7 w_{12}w_{21}w_{22}\otimes\partial_{x_2}\partial_{y_2}.\end{aligned}$$

The terms in $\partial_{x_1}\partial_{y_1}$ in ∇v and ξ should be the same, then we have:

$$\alpha_1\Theta w_{11}+\alpha_2 w_{11}w_{22}w_{11}+\alpha_7 w_{12}w_{21}w_{11} = +iw_{11}w_{21}w_{12}.$$

Therefore:

$$\begin{cases}\alpha_1+4\alpha_2-4\alpha_7=0,\\ -\alpha_7=i.\end{cases}$$

The terms in $\partial_{x_2}\partial_{y_2}$ in ∇v and ξ should be the same, then we have:

$$\alpha_1\Theta w_{22}\otimes\partial_{x_2}\partial_{y_2}+\alpha_7 w_{12}w_{21}w_{22}\otimes\partial_{x_2}\partial_{y_2} = iw_{12}w_{21}w_{22}\otimes\partial_{x_2}\partial_{y_2}.$$

$$\begin{cases}\alpha_1=0,\\ \alpha_7=i.\end{cases}$$

This leads to a contradiction.
Let us show also that $[\xi]\neq 0$ in $H^{-1,-1}(G_C)$ because ξ does not lie in the image of $\nabla : G_C^{0,0} \longrightarrow G_C^{-1,-1}$. Indeed if ξ lies in the image of $\nabla : G_C^{0,0} \longrightarrow G_C^{-1,-1}$, therefore $[\xi]=0$ in $H^{-1,-1}(M_C)$ since $H^{-1,-1}(\mathrm{Gr}_C) = S(\mathfrak{g}_{-2})\otimes H^{-1,-1}(G_C)$ is the first step of the spectral sequence; this is a contradiction.
5) By Proposition 5.41 we know that $H^{-1,-1}(G_C)\cong\mathbb{C}$ as a $\mathfrak{g}_0-$module. From the previous properties we know that $0\neq[\xi]\in H^{-1,-1}(G_C)$, hence $[\xi]$ is a basis for the $\mathfrak{g}_0-$module $H^{-1,-1}(G_C)$.
□

Corollary 5.45. *ξ is a secondary singular vector in $M(1,1,3,0)$, i.e. a singular vector in the quotient $M(1,1,3,0)/\operatorname{Im}\nabla$.*

Lemma 5.46. *Let*

$$\lambda = \frac{1}{2} i w_{11} w_{21} w_{12} w_{22} \otimes 1 + i\Theta w_{12} w_{21} \otimes 1 + i\Theta w_{11} w_{22} \otimes 1$$

be an element in $M_C^{0,0} = M(0,0,2,0)$. The following hold:

1. *$x_1\partial_{x_2}.\lambda = 0$ and $y_1\partial_{y_2}.\lambda = 0$,*
2. *λ is a basis for the $\mathfrak{g}_0-$module $H^{0,0}(G_C) \cong \mathbb{C}$.*

Proof. Let us prove that $x_1\partial_{x_2}.\lambda = 0$ and $y_1\partial_{y_2}.\lambda = 0$. We have:

$$\begin{aligned} x_1\partial_{x_2}.\lambda =& i w_{12} w_{11} \Theta \otimes 1 + i\Theta w_{11} w_{12} \otimes 1 = 0, \\ y_1\partial_{y_2}.\lambda =& + \frac{1}{2} i w_{11} w_{21} w_{12} w_{21} \otimes 1 + i\Theta w_{11} w_{21} \otimes 1 + i\Theta w_{11} w_{21} \otimes 1 = 0. \end{aligned}$$

We point out that $\nabla\lambda$ is a cycle in $\operatorname{Gr} M_C$ since $\lambda \in F_4 M_C$ and $\nabla\lambda \in F_4 M_C$. Indeed in M_C:

$$\begin{aligned} \nabla\lambda =& \frac{1}{2} i w_{11} w_{21} w_{12} w_{22} w_{11} \otimes \partial_{x_1}\partial_{y_1} + \frac{1}{2} i w_{11} w_{21} w_{12} w_{22} w_{21} \otimes \partial_{x_2}\partial_{y_1} + i\Theta w_{12} w_{21} w_{11} \otimes \partial_{x_1}\partial_{y_1} \\ &+ i\Theta w_{12} w_{21} w_{22} \otimes \partial_{x_2}\partial_{y_2} + i\Theta w_{12} w_{21} w_{12} \otimes \partial_{x_1}\partial_{y_2} + i\Theta w_{11} w_{22} w_{21} \otimes \partial_{x_2}\partial_{y_1} \\ &+ i\Theta w_{11} w_{22} w_{12} \otimes \partial_{x_1}\partial_{y_2} + i\Theta w_{11} w_{22} w_{11} \otimes \partial_{x_1}\partial_{y_1} \\ =& 2 i w_{11} w_{21} w_{12} \Theta \otimes \partial_{x_1}\partial_{y_1} + 2 i w_{11} w_{21} w_{22} \Theta \otimes \partial_{x_2}\partial_{y_1} + i\Theta w_{12} w_{21} w_{11} \otimes \partial_{x_1}\partial_{y_1} \\ &+ i\Theta w_{12} w_{21} w_{22} \otimes \partial_{x_2}\partial_{y_2} + i\Theta w_{12} w_{21} w_{12} \otimes \partial_{x_1}\partial_{y_2} + i\Theta w_{11} w_{22} w_{21} \otimes \partial_{x_2}\partial_{y_1} \\ &+ i\Theta w_{11} w_{22} w_{12} \otimes \partial_{x_1}\partial_{y_2} + i\Theta w_{11} w_{22} w_{11} \otimes \partial_{x_1}\partial_{y_1} \\ =& i\Theta w_{11} w_{21} w_{12} \otimes \partial_{x_1}\partial_{y_1} + i\Theta w_{11} w_{21} w_{22} \otimes \partial_{x_2}\partial_{y_1} + i\Theta w_{12} w_{21} w_{22} \otimes \partial_{x_2}\partial_{y_2} \\ &+ i\Theta w_{12} w_{21} w_{12} \otimes \partial_{x_1}\partial_{y_2} + i\Theta w_{11} w_{22} w_{12} \otimes \partial_{x_1}\partial_{y_2} \\ =& \Theta\xi. \end{aligned}$$

Moreover $[\lambda]$ lies in $H^{0,0}(G_C)$ since the terms of λ that include Θ are in $F_3 M$, the other is in $F_4 M_C$. By Proposition 5.41 we know that $H^{0,0}(G_C) \cong \mathbb{C}$ as a $\mathfrak{g}_0-$module. From the previous computations we know that $0 \neq [\lambda] \in H^{0,0}(G_C)$, hence $[\lambda]$ is a basis for the $\mathfrak{g}_0-$module $H^{0,0}(G_C)$. We have also that $\nabla[\lambda] = \Theta[\xi]$.

□

Proof of Proposition 5.43. By (5.25) and Lemmas 5.44, 5.46 we know that as $\mathcal{W}-$modules

$$\begin{aligned} E^0(M_C)^{0,0} &= H^{0,0}(\operatorname{Gr} M_C) \cong S(\mathfrak{g}_{-2}) \otimes H^{0,0}(G_C) = S(\mathfrak{g}_{-2}) \otimes \langle [\lambda] \rangle, \\ E^0(M_C)^{-1,-1} &= H^{-1,-1}(\operatorname{Gr} M_C) \cong S(\mathfrak{g}_{-2}) \otimes H^{-1,-1}(G_C) = S(\mathfrak{g}_{-2}) \otimes \langle [\xi] \rangle. \end{aligned}$$

By Lemma 5.46, the morphism $\nabla^{(0)} : E^0(M_C)^{0,0} \longrightarrow E^0(M_C)^{-1,-1}$ maps $[\lambda]$ to $\Theta[\xi]$. Therefore $\nabla^{(0)}$ is injective and $E^1(M_C)^{0,0} \cong 0$, $E^1(M_C)^{-1,-1} \cong \mathbb{C}$.
Thus $E^\infty(M_C)^{0,0} \cong E^1(M_C)^{0,0} = 0$ and $E^\infty(M_C)^{-1,-1} \cong E^1(M_C)^{-1,-1} \cong \mathbb{C}$ as $\mathcal{W}-$modules, and hence as $\mathfrak{g}-$modules. □

Now we focus on the two remaining cases of M_A.

Proposition 5.47.

$$\begin{aligned} H^{0,0}(M_A) &\cong \mathbb{C}, \\ H^{1,1}(M_A) &\cong 0. \end{aligned}$$

Remark 5.48. By straightforward computation we show that $H^{0,0}(M_A) \cong M_A^{0,0}/\operatorname{Im}\nabla \cong \mathbb{C}$. Indeed $\operatorname{Im}\nabla$ is the $\mathfrak{g}-$module generated by the singular vector $w_{11} \otimes 1$ and we have that:

$$\begin{aligned}
&x_2\partial_{x_1}.(w_{11} \otimes 1) = w_{21} \otimes 1,\\
&y_2\partial_{y_1}.(w_{11} \otimes 1) = w_{12} \otimes 1,\\
&y_2\partial_{y_1}.(x_2\partial_{x_1}.(w_{11} \otimes 1)) = w_{22} \otimes 1,\\
&w_{12}.(w_{21} \otimes 1) + w_{21}.(w_{12} \otimes 1) = -4\Theta \otimes 1.
\end{aligned}$$

Therefore the only elements that do not lie in the image of ∇ are those of $F(0,0,0,0)$.

In order to prove Proposition 5.47 we need the following result.

Lemma 5.49. *Let*

$$s = (w_{11} \otimes x_2 - w_{21} \otimes x_1)y_2 - (w_{12} \otimes x_2 - w_{22} \otimes x_1)y_1$$

be an element in $M_A^{1,1}$. *The following hold:*

1. $\nabla[s] = 0$ *in* $\operatorname{Gr} M_A$,
2. s *is a highest weight vector of weight (0,0,-2,0),*
3. s *is a basis for the* $\mathfrak{g}_0-$*module* $H^{1,1}(G_A) \cong \mathbb{C}$.

Proof. 1. Let us show that $\nabla[s] = 0$ in $\operatorname{Gr} M_A$.

$$\nabla s = w_{11}w_{22} \otimes 1 - w_{21}w_{12} \otimes 1 - w_{12}w_{21} \otimes 1 + w_{22}w_{11} \otimes 1 = 8\Theta \otimes 1 \in F_1M_A.$$

Since $s \in F_1M_A$, then $\nabla[s] = 0$ in $\operatorname{Gr} M_A$.

2. Let us show that s is a highest weight vector of weight (0,0,-2,0). We have:

$$\begin{aligned}
&x_1\partial_{x_2}.s = w_{11} \otimes x_1y_2 - w_{11} \otimes x_1y_2 - w_{12} \otimes x_1y_1 + w_{12} \otimes x_1y_1 = 0,\\
&x_2\partial_{x_1}.s = w_{21} \otimes x_2y_2 - w_{21} \otimes x_2y_2 - w_{22} \otimes x_2y_1 + w_{22} \otimes x_2y_1 = 0,\\
&y_1\partial_{y_2}.s = w_{11} \otimes x_2y_1 - w_{21} \otimes x_1y_1 - w_{11} \otimes x_2y_1 + w_{21} \otimes x_1y_1 = 0,\\
&y_2\partial_{y_1}.s = w_{12} \otimes x_2y_2 - w_{22} \otimes x_1y_2 - w_{12} \otimes x_2y_2 + w_{22} \otimes x_1y_2 = 0,\\
&t.s = \big(-1 - \frac{1}{2} - \frac{1}{2}\big)s = -2s,\\
&C.s = 0.
\end{aligned}$$

3. It follows from the facts that s lies in $G_A^{1,1}$, $\nabla[s] = 0$ in $\operatorname{Gr} M_A$, the space $H^{1,1}(G_A)$ is one$-$dimensional by Proposition 5.41 and s does not lie in $\operatorname{Im}\nabla$, where $\nabla : G_A^{2,2} \longrightarrow G_A^{1,1}$. Indeed let us see that $s \notin \operatorname{Im}\nabla$. Since $t.s = -2s$, it should come from an element in $G_A^{2,2}$ of weight -2 with respect to t, that is an element v, where $v \in F(2,2,-2,0)$. But:

$$\nabla v = w_{11} \otimes \partial_{x_1}\partial_{y_1}v + w_{21} \otimes \partial_{x_2}\partial_{y_1}v + w_{12} \otimes \partial_{x_1}\partial_{y_2}v + w_{22} \otimes \partial_{x_2}\partial_{y_2}v.$$

Then v should satisfy the following identities:

$$\begin{cases}
\partial_{x_1}\partial_{y_1}v = x_2y_2,\\
\partial_{x_2}\partial_{y_1}v = -x_1y_2,\\
\partial_{x_1}\partial_{y_2}v = -x_2y_1,\\
\partial_{x_2}\partial_{y_2}v = x_1y_1.
\end{cases}$$

This is impossible.

□

Proof of Proposition 5.47. By (5.25), Remark 5.48 and Lemma 5.49, we know that as $\mathcal{W}-$modules

$$E^0(M_A)^{0,0} = H^{0,0}(\mathrm{Gr}\, M_A) \cong S(\mathfrak{g}_{-2}) \otimes H^{0,0}(G_A) = S(\mathfrak{g}_{-2}) \otimes \langle 1 \rangle,$$
$$E^0(M_A)^{1,1} = H^{1,1}(\mathrm{Gr}\, M_A) \cong S(\mathfrak{g}_{-2}) \otimes H^{1,1}(G_A) = S(\mathfrak{g}_{-2}) \otimes \langle [s] \rangle.$$

By Lemma 5.49, the morphism $\nabla^{(0)} : E^0(M_A)^{1,1} \longrightarrow E^0(M_A)^{0,0}$ maps $[s]$ to $8\Theta \otimes [1]$. Therefore $\nabla^{(0)}$ is injective and $E^1(M_A)^{1,1} \cong 0$. Thus $E^\infty(M_A)^{1,1} \cong E^1(M_A)^{1,1} = 0$ as $\mathcal{W}-$modules, and hence as $\mathfrak{g}-$modules. □

Remark 5.50. We point out that for $C = 0$, the study of finite irreducible modules over K'_4 reduces to the study of finite irreducible modules over K_4, already studied in [BKL1]. In particular, for $C = 0$, the diagram of maps between degenerate modules reduces to the diagonal $m = n$ in the quadrants A and C of Figure 4.1. For K_4 the homology had been already computed in [BKL1, Propositions 6.2, 6.4] using de Rham complexes. Propositions 5.43 and 5.47 are coherent with the results of [BKL1, Propositions 6.2, 6.4] for K_4.

5.4 Size

The aim of this section is to compute the size of the modules $I(m, n, \mu_t, \mu_C)$. For a $S(\mathfrak{g}_{-2})-$module V, we define its size as (see [KR1]):

$$\mathrm{size}(V) = \frac{1}{4}\, \mathrm{rk}_{S(g_{-2})} V.$$

Proposition 5.51. A) $\mathrm{size}(I(m, n, -\frac{m+n}{2}, \frac{m-n}{2})) = 2mn + m + n,$

B) $\mathrm{size}(I(m, n, 1 + \frac{m-n}{2}, -1 - \frac{m+n}{2})) = 2(m+1)(n-1) + n - 1 + 3m + 3 + 2 = 2mn + m + 3n + 2,$

C) $\mathrm{size}(I(m, n, \frac{m+n}{2} + 2, \frac{n-m}{2})) = 2(m+1)(n+1) + m + n + 2 = 2mn + 3m + 3n + 4,$

D) $\mathrm{size}(I(m, n, 1 + \frac{n-m}{2}, 1 + \frac{m+n}{2})) = 2mn + n + 3m + 2.$

In order to prove Proposition 5.51 we need some preliminary results.

Remark 5.52. A consequence of results in [CCK1] on conformal duality is that, in the case of K'_4, the conformal dual of $M = \mathrm{Ind}(F)$, where $F = F(m, n, \mu_t, \mu_C)$ is an irreducible $\mathfrak{g}_0-$module, corresponds to the shifted dual $\mathrm{Ind}(F^\vee)$, where $F^\vee \cong F(m, n, -\mu_t + 2, -\mu_C)$.

We will say that $I(m, n, \mu_t, \mu_C)$ is of type X if $M(m, n, \mu_t, \mu_C)$ is represented in quadrant **X** in Figure 4.1.

Remark 5.53. We point out that it is sufficient to compute the size for modules $I(m, n, -\frac{m+n}{2}, \frac{m-n}{2})$ of type A and $I(m, n, 1 + \frac{n-m}{2}, 1 + \frac{m+n}{2})$ of type D and use conformal duality, since conformal dual modules have the same size.
Let us show that the module $I(m, n, \frac{m+n}{2} + 2, \frac{n-m}{2})$ of type C is the conformal dual of $I(m+1, n+1, -\frac{m+n+2}{2}, \frac{m-n}{2})$ of type A, , when $(m, n) \neq (0, 0)$. Indeed, by Remark 4.8, we have the following dual maps:

$$\nabla^{m+1,n+1} : M\Big(m+1, n+1, -\frac{m+n+2}{2}, \frac{m-n}{2}\Big) \longrightarrow M\Big(m, n, -\frac{m+n}{2}, \frac{m-n}{2}\Big),$$
$$\nabla^{m,n} : M\Big(m, n, \frac{m+n}{2} + 2, \frac{n-m}{2}\Big) \longrightarrow M\Big(m+1, n+1, \frac{m+n+2}{2} + 2, \frac{n-m}{2}\Big).$$

We use Remark 5.52 and Theorem 1.18 with $T := \nabla^{m,n}$, $M := M\Big(m, n, \frac{m+n}{2} + 2, \frac{n-m}{2}\Big)$ and $N := M\Big(m+1, n+1, \frac{m+n+2}{2} + 2, \frac{n-m}{2}\Big)$.
We point out that we can apply Theorem 1.18, because we know that $M(m+1, n+1, \frac{m+n+2}{2} + 2, \frac{n-m}{2}) / \operatorname{Im}(\nabla^{m,n})$ is a finitely generated torsion−free $\mathbb{C}[\Theta]$−module.
Indeed, by Propositions 5.42 and 5.43, the complex of type C is exact in $M(m+1, n+1, \frac{m+n+2}{2} + 2, \frac{n-m}{2})$ when $(m+1, n+1) \neq (1,1)$. Therefore:

$$\frac{M(m+1, n+1, \frac{m+n+2}{2} + 2, \frac{n-m}{2})}{\operatorname{Im}(\nabla^{m,n})} = \frac{M(m+1, n+1, \frac{m+n+2}{2} + 2, \frac{n-m}{2})}{\operatorname{Ker}(\nabla^{m+1,n+1})} \cong \operatorname{Im}(\nabla^{m+1,n+1}).$$

But $\operatorname{Im}(\nabla^{m+1,n+1})$ is a submodule of the free module $M(m+2, n+2, \frac{m+n+4}{2} + 2, \frac{n-m}{2})$, thus it is torsion−free as a $\mathbb{C}[\Theta]$−module.
We have that $M / \operatorname{Ker} T = M(m, n, \frac{m+n}{2} + 2, \frac{n-m}{2}) / \operatorname{Ker}(\nabla^{m,n}) \cong I(m, n, \frac{m+n}{2} + 2, \frac{n-m}{2})$ is the dual of $N^* / \operatorname{Ker} T^* \cong \operatorname{Im} T^* \cong I(m+1, n+1, -\frac{m+n+2}{2}, \frac{m-n}{2})$.
Using the same argument, it is possible to show that the module $I(m, n, 1 + \frac{m-n}{2}, -1 - \frac{m+n}{2})$ of type B is the conformal dual of $I(m+1, n-1, 1 + \frac{n-m-2}{2}, 1 + \frac{m+n}{2})$ of type D.

5.4.1 The character

We now introduce the notion of character, that will be used for the computation of the size.
Let s be an indeterminate. We define the character of a $\mathfrak{g}$−module V, following [KR1], as:

$$\operatorname{ch} V = \operatorname{tr}_V s^{-t}.$$

The character is a Laurent series in the indeterminate s; the coefficient of s^k is the dimension of the eigenspace of V of eigenvalue k with respect to the action of $-t \in \mathfrak{g}_0$.

Remark 5.54. Let V be a $\mathfrak{g}$−module and W a $\mathfrak{g}$−submodule of V. It is straightforward that $\operatorname{ch} V/W = \operatorname{ch} V - \operatorname{ch} W$.

We now compute directly the character of a Verma module $M(m, n, \mu_t, \mu_C) = U(\mathfrak{g}_{<0}) \otimes F(m, n, \mu_t, \mu_C)$ using the fact that $-t$ acts on elements of $\mathfrak{g}_{-2}$ as the multiplication by 2 and on elements of $\mathfrak{g}_{-1}$ as the multiplication by 1. We have, if $-1 < s < 1$:

$$\operatorname{ch} M(m, n, \mu_t, \mu_C) = s^{-\mu_t} \dim F(m, n, \mu_t, \mu_C) \cdot \frac{(1+s)^4}{1-s^2},$$

where we used the facts that:

$$1 \cdot \binom{4}{0} + s \cdot \binom{4}{1} + s^2 \cdot \binom{4}{2} + s^3 \cdot \binom{4}{3} + s^4 \cdot \binom{4}{4} = (1+s)^4$$

and if $|s|^2 < 1$:

$$\sum_{k=0}^{\infty} s^{2k} = \frac{1}{1-s^2}.$$

For the computation of the size of a $\mathfrak{g}$−module V we use that:

$$\operatorname{size}(V) = \frac{1}{4} \lim_{s \to 1} (1-s^2) \operatorname{ch} V. \tag{5.42}$$

Proposition 5.55. *The character of* $I(m,n,-\frac{m+n}{2},-\frac{n-m}{2})$ *of type A is, if* $(m,n)\neq(0,0)$*:*

$$\operatorname{ch} I\Big(m,n,-\frac{m+n}{2},\frac{m-n}{2}\Big) = s^{\frac{m+n}{2}}\frac{(1+s)^4}{1-s^2}\Big(\frac{2}{(1+s)^3}+\frac{m+n-1}{(1+s)^2}+\frac{mn}{1+s}\Big).$$

The character of $I\Big(m,n,1+\frac{n-m}{2},1+\frac{m+n}{2}\Big)$ *of type D is:*

$$\begin{aligned}\operatorname{ch} I(m,n,1+\frac{n-m}{2},1+\frac{m+n}{2}) =& s^{-1-\frac{n-m}{2}}\frac{(1+s)^4}{1-s^2}\Big[\Big(\frac{-2}{(1+s)^3}+\frac{3+n-m}{(1+s)^2}+\frac{mn+2m}{1+s}\Big)\\ &-(-1)^{n+1}s^{n+1}\Big(\frac{-2}{(1+s)^3}+\frac{-m-n+1}{(1+s)^2}+\frac{m+n+1}{1+s}\Big)\Big]\\ &+s^{\frac{m+n+2}{2}}\frac{(1+s)^4}{1-s^2}(-1)^{n+1}\Big(\frac{2}{(1+s)^3}+\frac{m+n+1}{(1+s)^2}\Big).\end{aligned}$$

Proof. We can compute the character of modules $I(m,n,\mu_t,\mu_C)$ using the character of $M(m,n,\mu_t,\mu_C)$. Let us now focus on the case $I(m,n,-\frac{m+n}{2},-\frac{n-m}{2})$ of type A. By Propositions 5.42 and 5.47, the following is an exact sequence, if $(m,n)\neq(0,0)$:

$$\begin{aligned}&\ldots\longrightarrow M\Big(m+j,n+j,-\frac{m+n+2j}{2},\frac{m-n}{2}\Big)\longrightarrow\ldots\longrightarrow M\Big(m+1,n+1,-\frac{m+n+2}{2},\frac{m-n}{2}\Big)\\ &\longrightarrow M\Big(m,n,-\frac{m+n}{2},\frac{m-n}{2}\Big)\longrightarrow I(m,n,-\frac{m+n}{2},\frac{m-n}{2})\longrightarrow 0.\end{aligned}$$

Hence, using Remark 5.54:

$$\operatorname{ch} I\Big(m,n,-\frac{m+n}{2},\frac{m-n}{2}\Big) = s^{\frac{m+n}{2}}\frac{(1+s)^4}{1-s^2}\sum_{j=0}^{\infty}(-1)^j s^j(j+m+1)(j+n+1).$$

We use the following identity, that holds if $|s|<1$ and is a consequence of the binomial series:

$$\sum_{j=0}^{\infty}(-1)^j s^j\binom{j+m}{m}=\frac{1}{(1+s)^{m+1}}.$$

We use the fact that:

$$\begin{aligned}(j+m+1)(j+n+1) &= (j+1)(j+n+1)+m(j+n+1)\\ &= (j+1)(j+1)+(j+1)n+m(j+1)+mn\\ &= (j+2)(j+1)+(j+1)(m+n-1)+mn.\end{aligned}$$

Therefore:

$$\operatorname{ch} I\Big(m,n,-\frac{m+n}{2},\frac{m-n}{2}\Big) = s^{\frac{m+n}{2}}\frac{(1+s)^4}{1-s^2}\Big(\frac{2}{(1+s)^3}+\frac{m+n-1}{(1+s)^2}+\frac{mn}{1+s}\Big).$$

Now we compute the character for modules $I(m,n,1+\frac{n-m}{2},1+\frac{m+n}{2})$ of type D.
By Proposition 5.42 the following is an exact sequence:

$$\begin{aligned}&\to M\Big(m+n+2+j,j,-\frac{m+n+2+2j}{2},\frac{m+n+2}{2}\Big)\to\cdots\to M\Big(m+n+2,0,-\frac{m+n+2}{2},\frac{m+n+2}{2}\Big)\\ &\to M\Big(m+n,0,1+\frac{-m-n}{2},1+\frac{m+n}{2}\Big)\to M(m+n-1,1,1+\frac{-m-n+2}{2},1+\frac{m+n}{2})\to\ldots\\ &\qquad\to M\Big(m,n,1+\frac{n-m}{2},1+\frac{m+n}{2}\Big)\to I\Big(m,n,1+\frac{n-m}{2},1+\frac{m+n}{2}\Big)\to 0,\end{aligned}$$

where the first row is composed of modules of type A; then the complex changes and the following terms are of type D. Hence, using Remark 5.54:

$$\mathrm{ch}\, I\Big(m, n, 1+\frac{n-m}{2}, 1+\frac{m+n}{2}\Big) = s^{-1-\frac{n-m}{2}} \frac{(1+s)^4}{1-s^2} \sum_{j=0}^{n} (-1)^j (j+m+1)(n-j+1) s^j$$
$$+ s^{\frac{m+n+2}{2}} \frac{(1+s)^4}{1-s^2} \sum_{i=0}^{\infty} (-1)^{n+1+i} (i+m+n+2+1)(i+1) s^j.$$

We use that:

$$\sum_{j=0}^{n} (-1)^j (j+m+1)(n-j+1) s^j =$$
$$= \sum_{j=0}^{\infty} (-1)^j s^j (j+m+1)(n-j+1) - \sum_{j=n+1}^{\infty} (-1)^j (j+m+1)(n-j+1) s^j.$$

Let us compute the first series; we have:

$$\begin{aligned}(j+m+1)(n-j+1) &= (j+1)(n-j+1) + m(n-j+1)\\ &= (j+1)(-j+1) + n(j+1) - m(j-1) + mn\\ &= -(j+1)(j+2) + 3(j+1) + n(j+1) - m(j+1) + 2m + mn.\end{aligned}$$

Then, if $|s| < 1$:

$$\sum_{j=0}^{\infty} (-1)^j (j+m+1)(n-j+1) s^j = \frac{-2}{(1+s)^3} + \frac{3+n-m}{(1+s)^2} + \frac{mn+2m}{1+s}.$$

Let us compute the second series; we have:

$$-\sum_{j=n+1}^{\infty} (-1)^j (j+m+1)(n-j+1) s^j = -\sum_{k=0}^{\infty} (-1)^{k+n+1} s^{k+n+1} (k+n+1+m+1)(n-k-n-1+1).$$

We use that:

$$\begin{aligned}&(k+n+1+m+1)(n-k-n-1+1) =\\ &\qquad = (k+n+2+m)(-k) = -k^2 - k(m+n+2)\\ &\qquad = -k(k+1) + k - (k+1)(m+n+2) + m+n+2\\ &\qquad = -(k+2)(k+1) + 2(k+1) + k+1 - (k+1)(m+n+2) + m+n+1.\end{aligned}$$

Then, if $|s| < 1$:

$$-\sum_{j=n+1}^{\infty} (-1)^j s^j (j+m+1)(n-j+1) s^j = -\sum_{k=0}^{\infty} (-1)^{k+n+1} s^{k+n+1} (k+n+m+2)(-k) =$$
$$-(-1)^{n+1} s^{n+1} \Big(\frac{-2}{(1+s)^3} + \frac{-m-n+1}{(1+s)^2} + \frac{m+n+1}{1+s}\Big).$$

Finally, we have:

$$\mathrm{ch}\, I(m, n, 1+\frac{n-m}{2}, 1+\frac{m+n}{2}) = s^{-1-\frac{n-m}{2}} \frac{(1+s)^4}{1-s^2} \Big[\Big(\frac{-2}{(1+s)^3} + \frac{3+n-m}{(1+s)^2} + \frac{mn+2m}{1+s}\Big)$$

$$- (-1)^{n+1} s^{n+1} \Big(\frac{-2}{(1+s)^3} + \frac{-m-n+1}{(1+s)^2} + \frac{m+n+1}{1+s} \Big) \Big]$$
$$+ s^{\frac{m+n+2}{2}} \frac{(1+s)^4}{1-s^2} (-1)^{n+1} \Big(\frac{2}{(1+s)^3} + \frac{m+n+1}{(1+s)^2} \Big).$$

□

Proof of Proposition 5.51. We first focus on $I(0,0,0,0)$ of type A. We have that size$(I(0,0,0,0)) = 0$. Indeed the following is an exact sequence:

$$\ldots \longrightarrow M\Big(m+j, n+j, -\frac{m+n+2j}{2}, -\frac{n-m}{2}\Big) \longrightarrow \ldots \longrightarrow M\Big(m+1, n+1, -\frac{m+n+2}{2}, -\frac{n-m}{2}\Big)$$
$$\stackrel{\nabla}{\to} M(0,0,0,0) \stackrel{\phi}{\to} I(0,0,0,0) \longrightarrow 0,$$

where ϕ is the projection to the quotient, $I(0,0,0,0) \cong \frac{M(0,0,0,0)}{\operatorname{Im} \nabla}$, $\operatorname{Ker} \phi = \operatorname{Im} \nabla$. Therefore using the same computation for case A, we find size$(I(0,0,0,0)) = 0$.
Now let us compute the size of $I(0,0,2,0)$ of type C. Since $M(0,0,2,0)$ is irreducible, we know that size$(I(0,0,2,0)) =$ size$(M(0,0,2,0)) = 4$.
Finally the size of $I(m,n,\mu_t,\mu_C)$ of type A for $(m,n) \neq (0,0)$ and of type D follows directly from Proposition 5.55 and (5.42). The size of $I(m,n,\mu_t,\mu_C)$ of type C for $(m,n) \neq (0,0)$ and of type B follows from Remark 5.53. □

Chapter 6

The conformal superalgebra CK_6

6.1 Singular vectors

In this chapter we recall the definition of the conformal superalgebra CK_6. We recall some definitions and notation from [BKL2]. From Chapter 2 we know that the conformal superalgebra of type K is:

$$K_N = \mathbb{C}[\partial] \otimes \bigwedge(N).$$

The $\lambda-$bracket for $f, g \in \bigwedge(N)$, $f = \xi_{i_1} \cdots \xi_{i_r}$ and $g = \xi_{j_1} \cdots \xi_{j_s}$, is given by:

$$[f_\lambda g] = \big((r-2)\partial(fg) + (-1)^r \sum_{i=1}^{N} (\partial_i f)(\partial_i g)\big) + \lambda(r+s-4)fg.$$

The associated annihilation superalgebra is:

$$\mathcal{A}(K_N) = K(1,N)_+.$$

We will identify $K(1,N)_+$ with $\bigwedge(1,N)_+$ using the following isomorphism of Lie superalgebras introduced in Chapter 2:

$$\bigwedge(1,N)_+ \longrightarrow K(1,N)_+$$
$$f \longmapsto 2f\partial_t + (-1)^{p(f)} \sum_{i=1}^{N} (\xi_i \partial_t f + \partial_i f)(\xi_i \partial_t + \partial_i).$$

We recall that on $K(1,N)_+$ the bracket is given by (2.1). We consider on $K(1,N)_+$ the $\mathbb{Z}-$grading $\deg(t^s \xi_{i_1} \xi_{i_2} ... \xi_{i_k}) = 2s + k - 2$. We set $\xi_1 \xi_2 ... \xi_N = \xi_*$. We focus on $N = 6$. Analogously to the case of K_4', we will use capital letters to denote ordered sets $I = (i_1, i_2, ..., i_k)$ of distinct integers in $\{1,2,3,4,5,6\}$. Given I and J ordered sets, the definitions of $I \cap J$, $I \setminus J$ and I^c are analogous to the definitions given in the case of K_4' (see Chapter 3).
Following [BKL2], for $\xi_I \in \bigwedge(6)$ we define the modified Hodge dual ξ_I^* to be the unique monomial such that $\xi_I \xi_I^* = \xi_*$ (notice that the definition of modified Hodge dual differs for a sign from the definition of Hodge dual given in the case of K_4'). We can extend the definition of the modified Hodge dual to monomials $t^k \xi_I \in \bigwedge(1,N)_+$ letting $(t^k \xi_I)^* = t^k \xi_I^*$. For $f = \xi_I$ we set $|f| = |I|$.
The conformal superalgebra CK_6 is a subalgebra of K_6 defined by (see construction in [CK2]):

$$CK_6 = \mathbb{C}[\partial] - \text{span}\left\{ f - i(-1)^{\frac{|f|(|f|+1)}{2}} (-\partial)^{3-|f|} f^*,\ f \in \bigwedge(6), 0 \le |f| \le 3 \right\}.$$

We introduce the linear operator $A : K(1,6)_+ \longrightarrow K(1,6)_+$ given for monomials with d odd variables by:

$$A(f) = (-1)^{\frac{d(d+1)}{2}} \left(\frac{d}{dt}\right)^{3-d} f^*,$$

and extended by linearity. The annihilation superalgebra associated with CK_6 is the subalgebra of $K(1,6)_+$ given by the image of $Id - iA$; it is isomorphic to the exceptional Lie superalgebra $E(1,6)$ (see [BKL2],[CK3],[CK2]). The map A preserves the $\mathbb{Z}-$grading, then $E(1,6)$ inherits the $\mathbb{Z}-$grading. The homogeneous components of non$-$positive degree of $E(1,6)$ and $K(1,6)_+$ coincide and are:

$$\begin{aligned} &E(1,6)_{-2} = \langle 1 \rangle, \\ &E(1,6)_{-1} = \langle \xi_1, \xi_2, ..., \xi_6 \rangle, \\ &E(1,6)_0 = \langle t, \xi_i\xi_j,\ 1 \leq i,j \leq 6 \rangle. \end{aligned}$$

Following the notation used in [BKL2], from now on we will denote $E_{00} := t$, $F_{i,j} := -\xi_i\xi_j$, $\Theta = -\frac{1}{2}$, $\mathfrak{g} := E(1,6)$.

Let us focus on $\mathfrak{g}_0 = \langle t, \xi_i\xi_j \quad 1 \leq i < j \leq 6 \rangle \cong \mathfrak{so}(6) \oplus \mathbb{C}E_{00}$. We point out that t is a grading element for $\mathfrak{g}$. Following [BKL2], we consider the following as basis of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{so}(6)$:

$$H_1 = iF_{1,2},\ H_2 = iF_{3,4},\ H_3 = iF_{5,6}.$$

We set $h_1 := H_1 - H_2$, $h_2 := H_2 - H_3$, $h_3 := H_2 + H_3$. Let $\varepsilon_j \in \mathfrak{h}^*$ such that $\varepsilon_j(H_k) = \delta_{j,k}$. The roots are $\Delta = \{\pm\varepsilon_l \pm \varepsilon_j, 1 \leq l < j \leq 3\}$, the simple positive roots are $\Delta^+ = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_2 + \varepsilon_3\}$. The root decomposition is:

$$\mathfrak{so}(6) = \mathfrak{h} \oplus (\oplus_{\alpha\in\Delta}\mathfrak{g}_\alpha) \quad \text{with} \quad \mathfrak{g}_\alpha = \mathbb{C}E_\alpha,$$

where the E_α's are, for $1 \leq l < j \leq 3$:

$$\begin{aligned} E_{\varepsilon_l - \varepsilon_j} &= F_{2l-1,2j-1} + F_{2l,2j} + iF_{2l-1,2j} - iF_{2l,2j-1}, \\ E_{\varepsilon_l + \varepsilon_j} &= F_{2l-1,2j-1} - F_{2l,2j} - iF_{2l-1,2j} - iF_{2l,2j-1}, \\ E_{-(\varepsilon_l - \varepsilon_j)} &= F_{2l-1,2j-1} + F_{2l,2j} - iF_{2l-1,2j} + iF_{2l,2j-1}, \\ E_{-(\varepsilon_l + \varepsilon_j)} &= F_{2l-1,2j-1} - F_{2l,2j} + iF_{2l-1,2j} + iF_{2l,2j-1}. \end{aligned}$$

We define for $1 \leq l < j \leq 3$:

$$\begin{aligned} \alpha_{l,j} &= \frac{1}{2}(E_{\varepsilon_l - \varepsilon_j} + E_{\varepsilon_l + \varepsilon_j}), \\ \beta_{l,j} &= \frac{1}{2}(E_{\varepsilon_l - \varepsilon_j} - E_{\varepsilon_l + \varepsilon_j}). \end{aligned}$$

The upper Borel subalgebra $B_{\mathfrak{so}_6}$ is:

$$B_{\mathfrak{so}_6} = \langle \alpha_{l,j}, \beta_{l,j},\ 1 \leq l < j \leq 3 \rangle.$$

Remark 6.1. By straightforward computation, it is possible to show that $\mathfrak{g}_1$ is the sum of two irreducible $\mathfrak{g}_0-$modules and the following are corresponding lowest weight vectors in $E(1,6)_1$:

$$\begin{aligned} &v_1 = t\xi_1 + it\xi_2, \\ &v_2 = -\xi_1\xi_3\xi_5 - i\xi_2\xi_4\xi_6 + \xi_2\xi_4\xi_5 + i\xi_1\xi_3\xi_6 - \xi_1\xi_4\xi_6 - i\xi_2\xi_3\xi_5 - \xi_2\xi_3\xi_6 - i\xi_1\xi_4\xi_5. \end{aligned}$$

Let F be a finite−dimensional irreducible $\mathfrak{g}_0$−module, such that $\mathfrak{g}_{>0}$ acts trivially on it; we have that $\mathrm{Ind}(F) \cong \mathbb{C}[\Theta] \otimes \bigwedge(6) \otimes F$. Indeed, let us denote by η_i the image in $U(\mathfrak{g})$ of $\xi_i \in \bigwedge(6)$, for all $i \in \{1,2,3,4,5,6\}$. In $U(\mathfrak{g})$ we have that $\eta_i^2 = \Theta$, for all $i \in \{1,2,3,4,5,6\}$: since $[\xi_i, \xi_i] = -1$ in $\mathfrak{g}$, we have $\eta_i\eta_i = -\eta_i\eta_i - 1$ in $U(\mathfrak{g})$. We describe the action of $\mathfrak{g}$ on $\mathrm{Ind}(F)$ using the λ−action notation, i.e.

$$f_\lambda(g \otimes v) = \sum_{j \geq 0} \frac{\lambda^j}{j!}(ft^j).(g \otimes v),$$

with $f \in \bigwedge(6)$, $g \in U(\mathfrak{g}_{<0})$ and $v \in F$. Given $\xi_I \in \bigwedge(6)$ and $\eta_J \in U(\mathfrak{g}_{<0})$, we define:

$$\xi_I \star \eta_J = \chi_{I \cap J = \emptyset}\eta_I\eta_J,$$

$$\eta_J \star \xi_I = \chi_{I \cap J = \emptyset}\eta_J\eta_I.$$

We extend the definition of modified Hodge dual to the elements of $U(\mathfrak{g}_{<0})$ in the following way: for $\eta_I \in U(\mathfrak{g}_{<0})$, we let η_I^* to be the unique monomial such that $\xi_I \star \eta_I^* = \eta_*$. We define the Hodge dual of elements of $\bigwedge(6)$ (resp. $U(\mathfrak{g}_{<0})$) in the following way: for $\xi_I \in \bigwedge(6)$ (resp. $\eta_I \in U(\mathfrak{g}_{<0})$), we let $\overline{\xi_I}$ (resp. $\overline{\eta_I}$) to be the unique monomial such that $\overline{\xi_I}\xi_I = \xi_*$ (resp. $\overline{\eta_I} \star \xi_I = \eta_*$). Then we extend by linearity the definition of Hodge dual to elements $\sum_I \alpha_I\eta_I \in U(\mathfrak{g}_{<0})$ and we set $\overline{\Theta^k\eta_I} = \Theta^k\overline{\eta_I}$. We point out that for $g \in U(\mathfrak{g}_{<0})$, $\overline{g} = (-1)^{|g|}g^*$.
Due to the fact that the homogeneous components of non−positive degree of $E(1,6)$ are the same as those of $K(1,6)_+$, the λ−action is given by restricting the λ−action in Theorem 4.1 in [BKL1]. We define T the isomorphism $T : \mathrm{Ind}(F) \to \mathrm{Ind}(F)$, $g \otimes v \mapsto \overline{g} \otimes v$. We recall the following result proved in [BKL1, Theorem 4.3].

Proposition 6.2 ([BKL1])**.** *Let $f = \xi_I \in \bigwedge(6)$ and $g = \eta_L \in U(\mathfrak{g}_{<0})$.*

$$\begin{aligned}
&T \circ f_\lambda \circ T^{-1}(\overline{g} \otimes v)\\
=&(-1)^{(|f|(|f|+1)/2)+|f||\overline{g}|}\Bigg\{(|f|-2)\Theta(f \star \overline{g}) \otimes v - (-1)^{p(f)}\sum_{i=1}^{6}(\partial_i f) \star (\partial_i \overline{g}) \otimes v - \sum_{r<s}(\partial_r\partial_s f) \star \overline{g} \otimes F_{r,s}v\\
&+\lambda\Big(f \star \overline{g} \otimes E_{00}v - (-1)^{p(f)}\sum_{i=1}^{6}\partial_i((f\xi_i) \star \overline{g}) \otimes v + (-1)^{p(f)}\sum_{i \neq j}((\partial_i f)\xi_j) \star \overline{g} \otimes F_{i,j}v\Big) +\\
&-\lambda^2\sum_{i<j}(f\xi_i\xi_j) \star \overline{g} \otimes F_{i,j}v\Bigg\}.
\end{aligned}$$

The following Theorem holds both for the λ−action and the action described in Proposition 6.2.

Lemma 6.3. *Let $f \in \bigwedge(6)$, $g \in U(\mathfrak{g}_{<0})$ and $k \geq 0$, the following holds:*

$$f_\lambda\big(\Theta^k g \otimes v\big) = (\Theta + \lambda)^k(f_\lambda g \otimes v).$$

Proof. The proof is analogous to Lemma 3.12. □

Let $\vec{m} \in \mathrm{Ind}(F)$, with F irreducible $\mathfrak{g}_0$−module. From [BKL2] we know that $\vec{m}$ is a highest weight singular vector if and only if:

S1: For $f \in \bigwedge(6)$, with $0 \leq |f| \leq 3$:

$$\frac{d^2}{d\lambda^2}\left(f_\lambda\vec{m} - i(-1)^{\frac{|f|(|f|+1)}{2}}\lambda^{3-|f|}\left(f^*{}_\lambda\vec{m}\right)\right) = 0.$$

S2: For $f \in \bigwedge(6)$, with $1 \leq |f| \leq 3$:

$$\frac{d}{d\lambda}\left(f_\lambda \vec{m} - i(-1)^{\frac{|f|(|f|+1)}{2}} \lambda^{3-|f|}\left(f^*{}_\lambda \vec{m}\right)\right)_{|\lambda=0} = 0.$$

S3: For f, with $|f| = 3$ or $f \in B_{\mathfrak{so}_6}$:

$$\left(f_\lambda \vec{m} - i(-1)^{\frac{|f|(|f|+1)}{2}} \lambda^{3-|f|}\left(f^*{}_\lambda \vec{m}\right)\right)_{|\lambda=0} = 0.$$

Remark 6.4. We point out that, by the previous conditions, a vector $\vec{m} \in \mathrm{Ind}(F)$ is a highest weight singular vector if and only if it satisfies **S1**, **S2**, **S3**. Since T, defined as in Proposition 6.2, is an isomorphism and $\vec{m} = T^{-1}\overline{\vec{m}}$, the fact that $\vec{m} \in \mathrm{Ind}(F)$ satisfies **S1**, **S2**, **S3** is equivalent to impose conditions **S1**, **S2**, **S3** for $(T \circ (f_\lambda - i(-1)^{\frac{|f|(|f|+1)}{2}} \lambda^{3-|f|} f^*{}_\lambda) \circ T^{-1})\overline{\vec{m}}$, using the expression given by Proposition 6.2.
Therefore in the following Lemmas we will consider a vector $T(\vec{m}) \in \mathrm{Ind}(F)$ and we will impose that the expression for $(T \circ (f_\lambda - i(-1)^{\frac{|f|(|f|+1)}{2}} \lambda^{3-|f|} f^*{}_\lambda) \circ T^{-1})T(\vec{m}) = (T \circ (f_\lambda - i(-1)^{\frac{|f|(|f|+1)}{2}} \lambda^{3-|f|} f^*{}_\lambda))\vec{m}$ given by Proposition 6.2 satisfies conditions **S1**, **S2**, **S3**. We will have that $\vec{m}$ is a highest weight singular vector.

Motivated by Remark 6.4, in the following lemmas we consider $\vec{m} \in U(\mathfrak{g}_{<0})$ and we will use the expression for the $\lambda-$action of Proposition 6.2 for $T(\vec{m})$. We consider a singular vector $\vec{m} \in \mathrm{Ind}(F)$ such that:

$$T(\vec{m}) = \sum_{k=0}^{N} \Theta^k \left(\sum_I \eta_I \otimes v_{I,k}\right). \tag{6.1}$$

In [BKL2] the following Lemma is stated (Lemma 4.4 in [BKL2]), even if the proof is missing.

Lemma 6.5. *Let $\vec{m} \in \mathrm{Ind}(F)$ be a singular vector, such that $T(\vec{m})$ is written as in* (6.1). *Then the degree of $\vec{m}$ with respect to Θ is at most 2. Moreover, $T(\vec{m})$ has the following form:*

$$T(\vec{m}) = \Theta^2\left(\sum_{|I|\geq 5} \eta_I \otimes v_{I,2}\right) + \Theta^1\left(\sum_{|I|\geq 3} \eta_I \otimes v_{I,1}\right) + \left(\sum_{|I|\geq 1} \eta_I \otimes v_{I,0}\right).$$

The rest of this section is dedicated to the proof of Lemma 6.5.

Lemma 6.6. *A singular vector $\vec{m} \in \mathrm{Ind}(F)$, such that $T(\vec{m})$ is written as in* (6.1), *has degree at most 4 with respect to Θ.*

Proof. By Remark 6.4, condition **S1** $f = \xi_1$ is:

$$\frac{d^2}{d\lambda^2}\left(T(\xi_1{}_\lambda \vec{m} + i\lambda^2(\xi_2\xi_3\xi_4\xi_5\xi_6{}_\lambda \vec{m}))\right) = 0.$$

Using Proposition 6.2 and Lemma 6.3, the previous equation is:

$$\frac{d^2}{d\lambda^2}\left\{\sum_{k=0}^{N}\sum_I (\lambda+\Theta)^k(-1)^{1+|I|}\left[-\Theta\xi_1 \star \eta_I \otimes v_{I,k} + \partial_1\eta_I \otimes v_{I,k}\right.\right. \tag{6.2}$$
$$\left. + \lambda\left(\xi_1 \star \eta_I \otimes E_{00}v_{I,k} + \sum_{i=1}^{6}\partial_i(\xi_1\xi_i \star \eta_I) \otimes v_{I,k} - \sum_{j\neq 1}\xi_j \star \eta_I \otimes F_{1,j}v_{I,k}\right) - \lambda^2 \sum_{i<j}\xi_1\xi_i\xi_j \star \eta_I \otimes F_{i,j}v_{I,k}\right]$$

$$+ i\lambda^2 \sum_{k=0}^{N} \sum_{I} (-1)^{1+|I|} (\lambda+\Theta)^k \Big[3\Theta \xi_2\xi_3\xi_4\xi_5\xi_6 \star \eta_I) \otimes v_{I,k}$$

$$+ \sum_{i=1}^{6} \partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6) \star \partial_i \eta_I \otimes v_{I,k} - \sum_{r<s} \partial_r\partial_s(\xi_2\xi_3\xi_4\xi_5\xi_6) \star \eta_I \otimes F_{r,s} v_{I,k}$$

$$+ \lambda\Big(\xi_2\xi_3\xi_4\xi_5\xi_6 \star \eta_I \otimes E_{00} v_{I,k} + \sum_{i=1}^{6} \partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6\xi_i \star \eta_I) \otimes v_{I,k} - \sum_{i\neq j}((\partial_i \xi_2\xi_3\xi_4\xi_5\xi_6)\xi_j) \star \eta_I \otimes F_{i,j} v_{I,k} \Big) \Big] \Big\}$$

$$= \sum_{k=0}^{N} \sum_{I} (\lambda+\Theta)^k (-1)^{1+|I|} \Big(-2 \sum_{i<j} \xi_1\xi_i\xi_j \star \eta_I \otimes F_{i,j} v_{I,k} \Big) +$$

$$+ 2 \sum_{k=1}^{N} \sum_{I} k(\lambda+\Theta)^{k-1} (-1)^{1+|I|} \Big[\xi_1 \star \eta_I \otimes E_{00} v_{I,k} + \sum_{i=1}^{6} \partial_i(\xi_1\xi_i \star \eta_I \otimes v_{I,k}) - \sum_{j\neq 1} \xi_j \star \eta_I \otimes F_{1,j} v_{I,k}$$

$$- 2\lambda \sum_{i<j} \xi_1\xi_i\xi_j \star \eta_I \otimes F_{i,j} v_{I,k} \Big] + \sum_{k=2}^{N} \sum_{I} k(k-1)(\lambda+\Theta)^{k-2} (-1)^{1+|I|} \Big[-\Theta\xi_1 \star \eta_I \otimes v_{I,k} + \partial_1 \eta_I \otimes v_{I,k}$$

$$+ \lambda\Big(\xi_1 \star \eta_I \otimes E_{00} v_{I,k} + \sum_{i=1}^{6} \partial_i(\xi_1\xi_i \star \eta_I) \otimes v_{I,k} - \sum_{j\neq 1} \xi_j \star \eta_I \otimes F_{1,j} v_{I,k} \Big) - \lambda^2 \sum_{i<j} \xi_1\xi_i\xi_j \star \eta_I \otimes F_{i,j} v_{I,k} \Big]$$

$$+ \sum_{k=0}^{N} \sum_{I} (-1)^{1+|I|} 2i(\lambda+\Theta)^k \Big[3\Theta \xi_2\xi_3\xi_4\xi_5\xi_6 \star \eta_I \otimes v_{I,k} + \sum_{i=1}^{6} \partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6) \star \partial_i \eta_I \otimes v_{I,k}$$

$$- \sum_{r<s} \partial_r\partial_s(\xi_2\xi_3\xi_4\xi_5\xi_6) \star \eta_I \otimes F_{r,s} v_{I,k} +$$

$$+ \lambda\Big(\xi_2\xi_3\xi_4\xi_5\xi_6 \star \eta_I \otimes E_{00} v_{I,k} + \sum_{i=1}^{6} \partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6\xi_i \star \eta_I) \otimes v_{I,k} - \sum_{i\neq j}(\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6)\xi_j) \star \eta_I \otimes F_{i,j} v_{I,k} \Big) \Big] +$$

$$+ 2 \sum_{k=1}^{N} \sum_{I} (-1)^{1+|I|} 2i\lambda k(\lambda+\Theta)^{k-1} \Big[3\Theta \xi_2\xi_3\xi_4\xi_5\xi_6 \star \eta_I \otimes v_{I,k} + \sum_{i=1}^{6} \partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6) \star \partial_i \eta_I \otimes v_{I,k}$$

$$- \sum_{r<s} \partial_r\partial_s(\xi_2\xi_3\xi_4\xi_5\xi_6) \star \eta_I \otimes F_{r,s} v_{I,k}$$

$$+ \lambda\Big(\xi_2\xi_3\xi_4\xi_5\xi_6 \star \eta_I \otimes E_{00} v_{I,k} + \sum_{i=1}^{6} \partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6\xi_i \star \eta_I) \otimes v_{I,k} - \sum_{i\neq j}(\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6)\xi_j) \star \eta_I \otimes F_{i,j} v_{I,k} \Big) \Big] +$$

$$+ \sum_{k=2}^{N} \sum_{I} (-1)^{1+|I|} i\lambda^2 k(k-1)(\lambda+\Theta)^{k-2} \Big[3\Theta \xi_2\xi_3\xi_4\xi_5\xi_6 \star \eta_I \otimes v_{I,k} + \sum_{i=1}^{6} \partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6) \star \partial_i \eta_I \otimes v_{I,k}$$

$$- \sum_{r<s} \partial_r\partial_s(\xi_2\xi_3\xi_4\xi_5\xi_6) \star \eta_I \otimes F_{r,s} v_{I,k}$$

$$+ \lambda\Big(\xi_2\xi_3\xi_4\xi_5\xi_6 \star \eta_I \otimes E_{00} v_{I,k} + \sum_{i=1}^{6} \partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6\xi_i \star \eta_I) \otimes v_{I,k} - \sum_{i\neq j}(\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6)\xi_j) \star \eta_I \otimes F_{i,j} v_{I,k} \Big) \Big] +$$

$$+ \sum_{k=0}^{N} \sum_{I} (-1)^{1+|I|} 4i\lambda(\lambda+\Theta)^k \Big[\xi_2\xi_3\xi_4\xi_5\xi_6 \star \eta_I \otimes E_{00} v_{I,k} + \sum_{i=1}^{6} \partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6\xi_i \star \eta_I) \otimes v_{I,k}$$

$$- \sum_{i\neq j} \partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6)\xi_j \star \eta_I \otimes F_{i,j} v_{I,k} \Big] +$$

$$+\sum_{k=1}^{N}\sum_{I}(-1)^{1+|I|}2i\lambda^2k(\lambda+\Theta)^{k-1}\Big[\xi_2\xi_3\xi_4\xi_5\xi_6\star\eta_I\otimes E_{00}v_{I,k}+\sum_{i=1}^{6}\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6\xi_i\star\eta_I)\otimes v_{I,k}$$
$$-\sum_{i\neq j}(\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6)\xi_j)\star\eta_I\otimes F_{i,j}v_{I,k}\Big].$$

We consider the previous expression as a polynomial expression in λ and $\lambda+\Theta$. Let us consider the terms of Equation (6.2) in $\lambda^3(\lambda+\Theta)^{k-2}$, for all k. We have:

$$\sum_{k=2}^{N}\sum_{I}(-1)^{1+|I|}(i\lambda^3k(k-1)(\lambda+\Theta)^{k-2})\Big[-3\xi_2\xi_3\xi_4\xi_5\xi_6\star\eta_I\otimes v_{I,k}+$$
$$\xi_2\xi_3\xi_4\xi_5\xi_6\star\eta_I\otimes E_{00}v_{I,k}+\sum_{i=1}^{6}\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6\xi_i\star\eta_I)\otimes v_{I,k}-\sum_{i\neq j}(\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6)\xi_j)\star\eta_I\otimes F_{i,j}v_{I,k}\Big].$$

Equivalently, the coefficient of $\lambda^3(\lambda+\Theta)^s$, for $s\geq 0$ fixed, is:

$$\sum_{I}(-1)^{1+|I|}\Big[-3\xi_2\xi_3\xi_4\xi_5\xi_6\star\eta_I\otimes v_{I,s+2}+\xi_2\xi_3\xi_4\xi_5\xi_6\star\eta_I\otimes E_{00}v_{I,s+2} \tag{6.3}$$
$$+\sum_{i=1}^{6}\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6\xi_i\star\eta_I)\otimes v_{I,s+2}-\sum_{i\neq j}(\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6)\xi_j)\star\eta_I\otimes F_{i,j}v_{I,s+2}\Big]=0.$$

Let us consider the terms of Equation (6.2) in $\lambda^2(\lambda+\Theta)^s$, for all s:

$$\sum_{k=2}^{N}\sum_{I}k(k-1)\lambda^2(\lambda+\Theta)^{k-2}(-1)^{1+|I|}\Big[-\sum_{i<j}\xi_1\xi_i\xi_j\star\eta_I\otimes F_{i,j}v_{I,k}\Big]+$$
$$+2\sum_{k=1}^{N}\sum_{I}(-1)^{1+|I|}2i\lambda^2k(\lambda+\Theta)^{k-1}\Big[-3\xi_2\xi_3\xi_4\xi_5\xi_6\star\eta_I\otimes v_{I,k}+$$
$$+\xi_2\xi_3\xi_4\xi_5\xi_6\star\eta_I\otimes E_{00}v_{I,k}+\sum_{i=1}^{6}\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6\xi_i\star\eta_I)\otimes v_{I,k}-\sum_{i\neq j}(\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6)\xi_j)\star\eta_I\otimes F_{i,j}v_{I,k}\Big]+$$
$$+\sum_{k=2}^{N}\sum_{I}(-1)^{1+|I|}i\lambda^2k(k-1)(\lambda+\Theta)^{k-1}[3\xi_2\xi_3\xi_4\xi_5\xi_6\star\eta_I\otimes v_{I,k}]+$$
$$+\sum_{k=2}^{N}\sum_{I}(-1)^{1+|I|}i\lambda^2k(k-1)(\lambda+\Theta)^{k-2}\Big[\sum_{i=1}^{6}\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6)\star\partial_i\eta_I\otimes v_{I,k}$$
$$-\sum_{r<s}\partial_r\partial_s(\xi_2\xi_3\xi_4\xi_5\xi_6)\star\eta_I\otimes F_{r,s}v_{I,k}\Big]+$$
$$+\sum_{k=1}^{N}\sum_{I}(-1)^{1+|I|}2i\lambda^2k(\lambda+\Theta)^{k-1}\Big[\xi_2\xi_3\xi_4\xi_5\xi_6\star\eta_I\otimes E_{00}v_{I,k}+\sum_{i=1}^{6}\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6\xi_i\star\eta_I)\otimes v_{I,k}$$
$$-\sum_{i\neq j}(\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6)\xi_j)\star\eta_I\otimes F_{i,j}v_{I,k}\Big].$$

Then if we look actually at the coefficient of $\lambda^2(\lambda+\Theta)^s$, for $s\geq 1$ fixed, we obtain:

$$\sum_{I}(s+1)(s+2)(-1)^{1+|I|}\Big[-\sum_{i<j}\xi_1\xi_i\xi_j\star\eta_I\otimes F_{i,j}v_{I,s+2}\Big]+$$

$$+2\sum_I(-1)^{1+|I|}2i(s+1)\Big[-3\xi_2\xi_3\xi_4\xi_5\xi_6\star\eta_I\otimes v_{I,s+1}+\xi_2\xi_3\xi_4\xi_5\xi_6\star\eta_I\otimes E_{00}v_{I,s+1}$$
$$+\sum_{i=1}^{6}\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6\xi_i\star\eta_I)\otimes v_{I,s+1}-\sum_{i\neq j}(\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6)\xi_j)\star\eta_I\otimes F_{i,j}v_{I,s+1}\Big]+$$
$$+\sum_I(-1)^{1+|I|}is(s+1)[3\xi_2\xi_3\xi_4\xi_5\xi_6\star\eta_I\otimes v_{I,s+1}]+$$
$$+\sum_I(-1)^{1+|I|}i(s+1)(s+2)\Big[\sum_{i=1}^{6}\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6)\star\partial_i\eta_I\otimes v_{I,s+2}$$
$$-\sum_{r<\widetilde{s}}\partial_r\partial_{\widetilde{s}}(\xi_2\xi_3\xi_4\xi_5\xi_6)\star\eta_I\otimes F_{r,\widetilde{s}}v_{I,s+2}\Big]+$$
$$+\sum_I(-1)^{1+|I|}2i(s+1)\Big[\xi_2\xi_3\xi_4\xi_5\xi_6\star\eta_I\otimes E_{00}v_{I,s+1}+\sum_{i=1}^{6}\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6\xi_i\star\eta_I)\otimes v_{I,s+1}$$
$$-\sum_{i\neq j}(\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6)\xi_j)\star\eta_I\otimes F_{i,j}v_{I,s+1}\Big]=0.$$

Using (6.3), we obtain that the expression in the second and third rows is zero, and the last two rows equal to $-\sum_I(-1)^{1+|I|}2i(s+1)[-3\xi_2\xi_3\xi_4\xi_5\xi_6\star\eta_I\otimes v_{I,s+1}]$, then we get, for $s\geq 1$:

$$\sum_I(s+2)(-1)^{1+|I|}\Big[-\sum_{i<j}\xi_1\xi_i\xi_j\star\eta_I\otimes F_{i,j}v_{I,s+2}\Big]+\sum_I(-1)^{1+|I|}is[3\xi_2\xi_3\xi_4\xi_5\xi_6\star\eta_I\otimes v_{I,s+1}]+$$
$$+\sum_I(-1)^{1+|I|}i(s+2)\Big[\sum_{i=1}^{6}\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6)\star\partial_i\eta_I\otimes v_{I,s+2}-\sum_{r<\widetilde{s}}\partial_r\partial_{\widetilde{s}}(\xi_2\xi_3\xi_4\xi_5\xi_6)\star\eta_I\otimes F_{r,\widetilde{s}}v_{I,s+2}\Big]+$$
$$-\sum_I(-1)^{1+|I|}2i[-3\xi_2\xi_3\xi_4\xi_5\xi_6\star\eta_I\otimes v_{I,s+1}]=0.$$

That is for $s\geq 1$:

$$\sum_I(-1)^{1+|I|}\Big[-\sum_{i<j}\xi_1\xi_i\xi_j\star\eta_I\otimes F_{i,j}v_{I,s+2}\Big]+\sum_I(-1)^{1+|I|}i[3\xi_2\xi_3\xi_4\xi_5\xi_6\star\eta_I\otimes v_{I,s+1}]+$$
$$+\sum_I(-1)^{1+|I|}i\Big[\sum_{i=1}^{6}\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6)\star\partial_i\eta_I\otimes v_{I,s+2}-\sum_{r<\widetilde{s}}\partial_r\partial_{\widetilde{s}}(\xi_2\xi_3\xi_4\xi_5\xi_6)\star\eta_I\otimes F_{r,\widetilde{s}}v_{I,s+2}\Big]=0. \tag{6.4}$$

Let us consider the terms of Equation (6.2) in $\lambda(\lambda+\Theta)^s$, for all s. We have:

$$2\sum_{k=1}^{N}\sum_I k(\lambda+\Theta)^{k-1}\lambda(-1)^{1+|I|}\Big[-2\sum_{i<j}\xi_1\xi_i\xi_j\star\eta_I\otimes F_{i,j}v_{I,k}\Big]+$$
$$+\sum_{k=2}^{N}\sum_I k(k-1)(\lambda+\Theta)^{k-2}\lambda(-1)^{1+|I|}\Big[\xi_1\star\eta_I\otimes v_{I,k}+\xi_1\star\eta_I\otimes E_{00}v_{I,k}+\sum_{i=1}^{6}\partial_i(\xi_1\xi_i\star\eta_I)\otimes v_{I,k}$$
$$-\sum_{j\neq 1}\xi_j\star\eta_I\otimes F_{1,j}v_{I,k}\Big]+$$
$$+\sum_{k=0}^{N}\sum_I(-1)^{1+|I|}2i(\lambda+\Theta)^k\lambda\Big[-3\xi_2\xi_3\xi_4\xi_5\xi_6\star\eta_I\otimes v_{I,k}+\xi_2\xi_3\xi_4\xi_5\xi_6\star\eta_I\otimes E_{00}v_{I,k}$$

$$+\sum_{i=1}^{6}\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6\xi_i\star\eta_I)\otimes v_{I,k}-\sum_{i\neq j}\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6)\xi_j\star\eta_I\otimes F_{i,j}v_{I,k}\Big]+$$

$$+\sum_{k=1}^{N}\sum_I(-1)^{1+|I|}4i\lambda k(\lambda+\Theta)^k[3\xi_2\xi_3\xi_4\xi_5\xi_6\xi_I\otimes v_{I,k}]+$$

$$+\sum_{k=1}^{N}\sum_I(-1)^{1+|I|}4i\lambda k(\lambda+\Theta)^{k-1}\Big[\sum_{i=1}^{6}\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6)\star\partial_i\eta_I\otimes v_{I,k}-\sum_{r<s}\partial_r\partial_s(\xi_2\xi_3\xi_4\xi_5\xi_6)\star\eta_I\otimes F_{r,s}v_{I,k}\Big]+$$

$$+\sum_{k=0}^{N}\sum_I(-1)^{1+|I|}4i\lambda(\lambda+\Theta)^k\Big[\xi_2\xi_3\xi_4\xi_5\xi_6\star\eta_I\otimes E_{00}v_{I,k}+\sum_{i=1}^{6}\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6\xi_i\star\eta_I)\otimes v_{I,k}$$

$$-\sum_{i\neq j}\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6)\xi_j\star\eta_I\otimes F_{i,j}v_{I,k}\Big].$$

If we look actually at coefficients of $\lambda(\lambda+\Theta)^s$, for $s\geq 2$ fixed, we obtain:

$$2\sum_I(s+1)(-1)^{1+|I|}\Big[-2\sum_{i<j}\xi_1\xi_i\xi_j\star\eta_I\otimes F_{i,j}v_{I,s+1}\Big]+$$

$$+\sum_I(s+1)(s+2)(-1)^{1+|I|}\Big[\xi_1\star\eta_I\otimes v_{I,s+2}+\xi_1\star\eta_I\otimes E_{00}v_{I,s+2}+\sum_{i=1}^{6}\partial_i(\xi_1\xi_i\star\eta_I)\otimes v_{I,s+2}$$

$$-\sum_{j\neq 1}\xi_j\star\eta_I\otimes F_{1,j}v_{I,s+2}\Big]+$$

$$+\sum_I(-1)^{1+|I|}2i\Big[-3\xi_2\xi_3\xi_4\xi_5\xi_6\star\eta_I\otimes v_{I,s}+\xi_2\xi_3\xi_4\xi_5\xi_6\star\eta_I\otimes E_{00}v_{I,s}$$

$$+\sum_{i=1}^{6}\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6\xi_i\star\eta_I)\otimes v_{I,s}-\sum_{i\neq j}(\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6)\xi_j)\star\eta_I\otimes F_{i,j}v_{I,s}\Big]+$$

$$+\sum_I(-1)^{1+|I|}4is[3\xi_2\xi_3\xi_4\xi_5\xi_6\star\eta_I\otimes v_{I,s}]+$$

$$+\sum_I(-1)^{1+|I|}4i(s+1)\Big[\sum_{i=1}^{6}\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6)\star\partial_i\eta_I\otimes v_{I,s+1}-\sum_{r<\widetilde{s}}\partial_r\partial_{\widetilde{s}}(\xi_2\xi_3\xi_4\xi_5\xi_6)\star\eta_I\otimes F_{r,\widetilde{s}}v_{I,s+1}\Big]+$$

$$+\sum_I(-1)^{1+|I|}4i\Big[\xi_2\xi_3\xi_4\xi_5\xi_6\star\eta_I\otimes E_{00}v_{I,s}+\sum_{i=1}^{6}\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6\xi_i\star\eta_I)\otimes v_{I,s}$$

$$-\sum_{i\neq j}(\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6)\xi_j)\star\eta_I\otimes F_{i,j}v_{I,s}\Big]=0.$$

We use (6.3) to point out that the sum in the fourth and fifth rows is zero. Moreover, again due to (6.3), the sum of the terms in the last two rows equals to $-\sum_I(-1)^{1+|I|}4i[-3\xi_2\xi_3\xi_4\xi_5\xi_6\star\eta_I\otimes v_{I,s}]=\sum_I(-1)^{1+|I|}4i[3\xi_2\xi_3\xi_4\xi_5\xi_6\star\eta_I\otimes v_{I,s}]$. The sum of this term and terms from the first, sixth, seventh rows is zero due to (6.4).

The remaining terms give the following condition for $s\geq 2$:

$$\sum_I(-1)^{1+|I|}\Big[\xi_1\star\eta_I\otimes v_{I,s+2}+\xi_1\star\eta_I\otimes E_{00}v_{I,s+2}+\sum_{i=1}^{6}\partial_i(\xi_1\xi_i\star\eta_I)\otimes v_{I,s+2} \tag{6.5}$$

$$-\sum_{j\neq 1}\xi_j\star\eta_I\otimes F_{1,j}v_{I,s+2}\Big]=0.$$

Let us consider the terms of Equation (6.2) in $(\lambda+\Theta)^s$, for all s. We have:

$$\sum_{k=0}^{N}\sum_{I}(\lambda+\Theta)^k(-1)^{1+|I|}\Big(-2\sum_{i<j}\xi_1\xi_i\xi_j\star\eta_I\otimes F_{i,j}v_{I,k}\Big)+$$
$$2\sum_{k=1}^{N}\sum_{I}k(\lambda+\Theta)^{k-1}(-1)^{1+|I|}\Big[\xi_1\star\eta_I\otimes E_{00}v_{I,k}+\sum_{i=1}^{6}\partial_i(\xi_1\xi_i\xi_I\otimes v_{I,k})-\sum_{j\neq 1}\xi_j\star\eta_I\otimes F_{1,j}v_{I,k}\Big]$$
$$+\sum_{k=2}^{N}\sum_{I}k(k-1)(\lambda+\Theta)^{k-1}(-1)^{1+|I|}[-\xi_1\star\eta_I\otimes v_{I,k}]+\sum_{k=2}^{N}\sum_{I}k(k-1)(\lambda+\Theta)^{k-2}(-1)^{1+|I|}[\partial_1\eta_I\otimes v_{I,k}]$$
$$+\sum_{k=0}^{N}\sum_{I}(-1)^{1+|I|}2i(\lambda+\Theta)^{k+1}[3\xi_2\xi_3\xi_4\xi_5\xi_6\star\eta_I\otimes v_{I,k}]+$$
$$+\sum_{k=0}^{N}\sum_{I}(-1)^{1+|I|}2i(\lambda+\Theta)^{k}\Big[\sum_{i=1}^{6}\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6)\star\partial_i(\eta_I)\otimes v_{I,k}-\sum_{r<\tilde{s}}\partial_r\partial_{\tilde{s}}(\xi_2\xi_3\xi_4\xi_5\xi_6)\star\eta_I\otimes F_{r,\tilde{s}}v_{I,k}\Big].$$

If we look actually at coefficients of $(\lambda+\Theta)^s$, for $s\geq 3$ fixed, we obtain:

$$\sum_{I}(-1)^{1+|I|}\Big(-2\sum_{i<j}\xi_1\xi_i\xi_j\star\eta_I\otimes F_{i,j}v_{I,s}\Big)+$$
$$+2\sum_{I}(s+1)(-1)^{1+|I|}\Big[\xi_1\star\eta_I\otimes E_{00}v_{I,s+1}+\sum_{i=1}^{6}\partial_i(\xi_1\xi_i\xi_I)\otimes v_{I,s+1}-\sum_{j\neq 1}\xi_j\star\eta_I\otimes F_{1,j}v_{I,s+1}\Big]$$
$$+\sum_{I}s(s+1)(-1)^{1+|I|}[-\xi_1\star\eta_I\otimes v_{I,s+1}]+\sum_{I}(s+1)(s+2)(-1)^{1+|I|}[\partial_1\eta_I\otimes v_{I,s+2}]$$
$$+\sum_{I}(-1)^{1+|I|}2i[3\xi_2\xi_3\xi_4\xi_5\xi_6\star\eta_I\otimes v_{I,s-1}]+$$
$$+\sum_{I}(-1)^{1+|I|}2i\Big[\sum_{i=1}^{6}\partial_i(\xi_2\xi_3\xi_4\xi_5\xi_6)\star\partial_i(\eta_I)\otimes v_{I,s}-\sum_{r<\widetilde{s}}\partial_r\partial_{\widetilde{s}}(\xi_2\xi_3\xi_4\xi_5\xi_6)\star\eta_I\otimes F_{r,\widetilde{s}}v_{I,s}\Big]=0.$$

Using (6.4) we get that the sum of terms from the first row and the last two rows is zero. Using (6.5) we obtain that the sum of terms from the second row equals to $-2\sum_I(s+1)(-1)^{1+|I|}[\xi_1\star\eta_I\otimes v_{I,s+1}]$. We obtain that for $s\geq 3$:

$$-2\sum_{I}(s+1)(-1)^{1+|I|}[\xi_1\star\eta_I\otimes v_{I,s+1}]+\sum_{I}s(s+1)(-1)^{1+|I|}[-\xi_1\star\eta_I\otimes v_{I,s+1}]$$
$$+\sum_{I}(s+1)(s+2)(-1)^{1+|I|}[\partial_1\eta_I\otimes v_{I,s+2}]=0.$$

That is for $s\geq 3$:

$$-\sum_{I}(-1)^{1+|I|}[\xi_1\star\eta_I\otimes v_{I,s+1}]+\sum_{I}(-1)^{1+|I|}[\partial_1\eta_I\otimes v_{I,s+2}]=0. \tag{6.6}$$

If we look at terms of (6.6) involving η_K with $|K|\leq 5$ and $1\in K$, we get:

$$-\sum_{|I|\leq 4,1\notin I}(-1)^{1+|I|}[\xi_1\star\eta_I\otimes v_{I,s+1}]=0.$$

From this we get, using linear independence of the terms $\xi_1\star\eta_I$ for different I's, that $v_{I,k}=0$ when $|I|\leq 4$, $1\notin I$ and $k\geq 4$. (Since we could have chosen at the beginning a general ξ_i instead of

ξ_1, condition $1 \notin I$ is not necessary). If we look at terms of (6.6) involving η_K with $|K| = 4$ and $1 \notin K$, we get:

$$\partial_1 \eta_{K \cup \{1\}} \otimes v_{K \cup \{1\}, s+2} = 0.$$

From this we obtain that $v_{I,k} = 0$ when $|I| = 5$, $1 \in I$ and $k \geq 5$. (Since we could have chosen at the beginning a general ξ_i instead of ξ_1, condition $1 \in I$ is not necessary). The terms of (6.6) involving η_1^* are:

$$\eta_1^* \otimes v_{*,s+2} = 0.$$

From this we obtain $v_{*,k} = 0$ if $k \geq 5$. □

Therefore, for a singular vector $\vec{m}$, $T(\vec{m})$ has the following form:

$$T(\vec{m}) = \Theta^4(\sum_I \eta_I \otimes v_{I,4}) + \Theta^3(\sum_I \eta_I \otimes v_{I,3}) + \Theta^2(\sum_I \eta_I \otimes v_{I,2}) + \Theta^1(\sum_I \eta_I \otimes v_{I,1}) + (\sum_I \eta_I \otimes v_{I,0}). \tag{6.7}$$

Following [BKL2], we write the $\lambda-$action in the following way, using Proposition 6.2 and Lemma 6.3:

$$\begin{aligned}(T \circ f_\lambda)(\vec{m}) =& b_0 + \lambda(B_0 - a_0) + \lambda^2 C_0 + (\lambda + \Theta)[a_0 + b_1] + (\lambda + \Theta)\lambda(B_1 - a_1) + (\lambda + \Theta)\lambda^2 C_1 \\ &+ (\lambda + \Theta)^2[a_1 + b_2] + (\lambda + \Theta)^2\lambda(B_2 - a_2) + (\lambda + \Theta)^2\lambda^2 C_2 \\ &+ (\lambda + \Theta)^3[a_2 + b_3] + (\lambda + \Theta)^3\lambda(B_3 - a_3) + (\lambda + \Theta)^3\lambda^2 C_3 \\ &+ (\lambda + \Theta)^4[a_3 + b_4] + (\lambda + \Theta)^4\lambda(B_4 - a_4) + (\lambda + \Theta)^4\lambda^2 C_4 + (\lambda + \Theta)^5 a_4,\end{aligned}$$

where the coefficients a, b, B, C depend on f and are explicitly defined as follows. For all $0 \leq p \leq 4$ we let:

$$a_p(f) = \sum_I (-1)^{(|f|(|f|+1)/2)+|f||I|} \Big[(|f| - 2)(f \star \eta_I) \otimes v_{I,p}\Big]; \tag{6.8}$$

$$b_p(f) = \sum_I (-1)^{(|f|(|f|+1)/2)+|f||I|} \Big[-(-1)^{p(f)} \sum_{i=1}^{6} (\partial_i f) \star (\partial_i \eta_I) \otimes v_{I,p} - \sum_{r<s} (\partial_r \partial_s f) \star \eta_I \otimes F_{r,s} v_{I,p}\Big];$$

$$\begin{aligned}B_p(f) = &\sum_I (-1)^{(|f|(|f|+1)/2)+|f||I|} \Big[f \star \eta_I \otimes E_{00} v_{I,p} - (-1)^{p(f)} \sum_{i=1}^{6} \partial_i (f\xi_i \star \eta_I) \otimes v_{I,p} \\ &+ (-1)^{p(f)} \sum_{i \neq j} ((\partial_i f)\xi_j) \star \eta_I \otimes F_{i,j} v_{I,p})\Big];\end{aligned}$$

$$C_p(f) = \sum_I (-1)^{(|f|(|f|+1)/2)+|f||I|} \Big[-\sum_{i<j} (f\xi_i\xi_j) \star \eta_I \otimes F_{i,j} v_{I,p}\Big].$$

We also have:

$$\begin{aligned}(T \circ f^*_\lambda)(\vec{m}) =& bd_0 + \lambda(Bd_0 - ad_0) + \lambda^2 Cd_0 + (\lambda + \Theta)[ad_0 + bd_1] + (\lambda + \Theta)\lambda(Bd_1 - ad_1) \\ &+ (\lambda + \Theta)\lambda^2 Cd_1 + (\lambda + \Theta)^2[ad_1 + bd_2] + (\lambda + \Theta)^2\lambda(Bd_2 - ad_2) + (\lambda + \Theta)^2\lambda^2 Cd_2 \\ &+ (\lambda + \Theta)^3[ad_2 + bd_3] + (\lambda + \Theta)^3\lambda(Bd_3 - ad_3) + (\lambda + \Theta)^3\lambda^2 Cd_3 \\ &+ (\lambda + \Theta)^4[ad_3 + bd_4] + (\lambda + \Theta)^4\lambda(Bd_4 - ad_4) + (\lambda + \Theta)^4\lambda^2 Cd_4 + (\lambda + \Theta)^5 ad_4,\end{aligned} \tag{6.9}$$

where we mean that $ad_p = a_p(f^*)$, $bd_p = b_p(f^*)$, $Bd_p = B_p(f^*)$, $Cd_p = C_p(f^*)$.

Proof of Lemma 6.5. Let us analyze condition **S2** for $f = \xi_j$:

$$\frac{d}{d\lambda}\Big(T(\xi_j\,{}_\lambda\vec{m} + i\lambda^2(\xi_j^*\,{}_\lambda\vec{m}))\Big)_{|\lambda=0} = 0.$$

Using notation (6.8), we obtain:

$$B_0 + b_1 + \Theta[B_1 + a_1 + 2b_2] + \Theta^2[2a_2 + B_2 + 3b_3] + \Theta^3[3a_3 + B_3 + 4b_4] + \Theta^4[4a_4 + B_4] = 0.$$

Therefore:

$$\begin{cases} 4a_4 + B_4 = 0, \\ 3a_3 + B_3 + 4b_4 = 0, \\ 2a_2 + B_2 + 3b_3 = 0, \\ B_1 + a_1 + 2b_2 = 0, \\ B_0 + b_1 = 0. \end{cases}$$

Let us analyze explicitly equation $B_1 + a_1 + 2b_2 = 0$ for $f = \xi_j$.
The coefficient of the terms of $B_1 + a_1 + 2b_2 = 0$ that contain η_j only are:

$$-v_{\emptyset,1} + E_{00}v_{\emptyset,1} + 5(-v_{\emptyset,1}) = 0. \tag{6.10}$$

Now we analyze $(B_0 + b_1)(\xi_j) = 0$ and we consider the coefficient of the terms that involve η_j only. We get:

$$E_{00}v_{\emptyset,0} - 5v_{\emptyset,0} = 0. \tag{6.11}$$

We focus on $(2a_2 + B_2 + 3b_3)(\xi_j) = 0$ and consider the coefficient of the terms that involve η_j only. We get:

$$-2v_{\emptyset,2} + E_{00}v_{\emptyset,2} - 5v_{\emptyset,2} = 0. \tag{6.12}$$

Now let us analyze $B_0(\xi_j) + b_1(\xi_j) = 0$. The coefficient of the terms in 1 only come from b_1 when $g = \eta_j$ and we obtain:

$$v_{(j),1} = 0. \tag{6.13}$$

Now let us analyze $(B_1 + a_1 + 2b_2)(\xi_j) = 0$. The coefficient of the terms in 1 only come from b_2 when $g = \eta_j$ and we obtain:

$$v_{(j),2} = 0. \tag{6.14}$$

Let us analyze condition **S2** for $f = \xi_i\xi_j\xi_k$. We have:

$$\frac{d}{d\lambda}\big(T(\xi_i\xi_j\xi_k\,{}_\lambda\vec{m} - i\lambda^0((\xi_i\xi_j\xi_k)^*_\lambda\vec{m}))\big)_{|\lambda=0} = 0.$$

Using notation (6.8) and (6.9), we obtain:

$$\begin{aligned} &B_0 + b_1 + \Theta[B_1 + a_1 + 2b_2] + \Theta^2[2a_2 + B_2 + 3b_3] + \Theta^3[3a_3 + B_3 + 4b_4] + \Theta^4[4a_4 + B_4] \\ &- i\left\{Bd_0 + bd_1 + \Theta[Bd_1 + ad_1 + 2bd_2] + \Theta^2[2ad_2 + Bd_2 + 3bd_3] + \Theta^3[3ad_3 + Bd_3 + 4bd_4] \right. \\ &\left. + \Theta^4[4ad_4 + Bd_4]\right\} = 0. \end{aligned}$$

Therefore we obtain:

$$\begin{cases} 4a_4 + B_4 - i(4ad_4 + Bd_4) = 0, \\ 3a_3 + B_3 + 4b_4 - i(3ad_3 + Bd_3 + 4bd_4) = 0, \\ 2a_2 + B_2 + 3b_3 - i(2ad_2 + Bd_2 + 3bd_3) = 0, \\ B_1 + a_1 + 2b_2 - i(Bd_1 + ad_1 + 2bd_2) = 0, \\ B_0 + b_1 - i(Bd_0 + bd_1) = 0. \end{cases}$$

Let us now analyze condition **S3** for $f = \xi_i\xi_j\xi_k$. We have

$$\big(T(\xi_i\xi_j\xi_k\,{}_\lambda\vec{m} - i\lambda^0((\xi_i\xi_j\xi_k)^*\,{}_\lambda\vec{m}))\big)_{|\lambda=0} = 0.$$

Using notation (6.8) and (6.9), we obtain:

$$\begin{aligned} &b_0 + \Theta[a_0 + b_1] + \Theta^2[a_1 + b_2] + \Theta^3[a_2 + b_3] + \Theta^4[a_3 + b_4] + \Theta^5 a_4 \\ &- i\left\{bd_0 + \Theta[ad_0 + bd_1] + \Theta^2[ad_1 + bd_2] + \Theta^3[ad_2 + bd_3] + \Theta^4[ad_3 + bd_4] + \Theta^5 ad_4\right\} = 0. \end{aligned}$$

Therefore we obtain:

$$\begin{cases} a_4 - iad_4 = 0, \\ a_3 + b_4 - i(ad_3 + bd_4) = 0, \\ a_2 + b_3 - i(ad_2 + bd_3) = 0, \\ a_1 + b_2 - i(ad_1 + bd_2) = 0, \\ a_0 + b_1 - i(ad_0 + bd_1) = 0, \\ b_0 - ibd_0 = 0. \end{cases}$$

Hence for $f = \xi_i\xi_j\xi_k$, combining, by **S2**, $B_1 + a_1 + 2b_2 - i(Bd_1 + ad_1 + 2bd_2) = 0$ and, by **S3**, $a_1 + b_2 - i(ad_1 + bd_2) = 0$, we obtain $B_1 - a_1 - i(Bd_1 - ad_1) = 0$. We take the coefficient of $\eta_i\eta_j\eta_k$ and obtain:

$$E_{00}v_{\emptyset,1} - 3v_{\emptyset,1} - v_{\emptyset,1} = 0.$$

But we also have by (6.10) that $E_{00}v_{\emptyset,1} - 6v_{\emptyset,1} = 0$, thus $v_{\emptyset,1} = 0$.
Moreover for $f = \eta_i\eta_j\eta_k$ we know, by **S3**, that $a_0 + b_1 - i(ad_0 + bd_1) = 0$ and, by **S2**, that $B_0 + b_1 - i(Bd_0 + bd_1) = 0$. Hence $a_0 - B_0 - i(ad_0 - Bd_0) = 0$. We take the coefficient of $\eta_i\eta_j\eta_k$ and obtain:

$$0 = v_{\emptyset,0} - (E_{00}v_{\emptyset,0} - 3v_{\emptyset,0}) = -E_{00}v_{\emptyset,0} + 4v_{\emptyset,0}.$$

But by (6.11) we have $E_{00}v_{\emptyset,0} - 5v_{\emptyset,0} = 0$, thus $v_{\emptyset,0} = 0$.
Finally for $f = \xi_i\xi_j\xi_k$ we know, by **S3**, that $a_2 + b_3 - i(ad_2 + bd_3) = 0$ and, by **S2**, that $2a_2 + B_2 + 3b_3 - i(2ad_2 + Bd_2 + 3bd_3) = 0$; therefore $-a_2 + B_2 - i(-ad_2 + Bd_2) = 0$. We take the coefficient of $\eta_i\eta_j\eta_k$ and obtain:

$$0 = -v_{\emptyset,2} + E_{00}v_{\emptyset,2} - 3v_{\emptyset,2} = E_{00}v_{\emptyset,2} - 4v_{\emptyset,2}.$$

But by (6.12) we have $E_{00}v_{\emptyset,2} - 7v_{\emptyset,2} = 0$, thus $v_{\emptyset,2} = 0$.
So far we have shown that, for all $i \in \{1,2,3,4,5,6\}$, $v_{\emptyset,0} = v_{\emptyset,1} = v_{\emptyset,2} = v_{(i),1} = v_{(i),2} = 0$.
Let us now show that $v_{(jl),1} = 1$. By condition **S2** for $f = \xi_j$ we know $B_0 + b_1 = 0$. We take the coefficient of η_l and obtain:

$$0 = -\eta_l \otimes v_{(jl),1} + \eta_l \otimes F_{j,l}v_{\emptyset,0}.$$

Therefore $v_{(jl),1} = 0$.

Let us analyze the condition **S1** for $|f| = 0$:

$$
\begin{aligned}
0 = &\frac{d^2}{d\lambda^2}(T(1_\lambda \vec{m} - i(-1)^0\lambda^3((\xi_*)_\lambda \vec{m}))) = \\
&2C_0 + 2B_1 + 2b_2 + \lambda[4C_1 + 2B_2 - 2a_2] + 2\lambda^2 C_2 + (\lambda+\Theta)[2C_1 + 4B_2 + 2a_2 + 6b_3] \\
&+ (\lambda+\Theta)\lambda[8C_2 + 6B_3 - 6a_3] + (\lambda+\Theta)\lambda^2 6C_3 + (\lambda+\Theta)^2[2C_2 + 6B_3 + 6a_3 + 12b_4] \\
&+ (\lambda+\Theta)^2\lambda[12C_3 + 12B_4 - 12a_4] + (\lambda+\Theta)^2\lambda^2 12C_4 + (\lambda+\Theta)^3[8B_4 + 12a_4 + 2C_3] \\
&+ (\lambda+\Theta)^3\lambda 16C_4 + (\lambda+\Theta)^4 2C_4 \\
&- 6i\lambda\left\{bd_0 + \lambda(Bd_0 - ad_0) + \lambda^2 Cd_0 + (\lambda+\Theta)[ad_0 + bd_1] + (\lambda+\Theta)\lambda(Bd_1 - ad_1) + (\lambda+\Theta)\lambda^2 Cd_1\right. \\
&+(\lambda+\Theta)^2[ad_1 + bd_2] + (\lambda+\Theta)^2\lambda(Bd_2 - ad_2) + (\lambda+\Theta)^2\lambda^2 Cd_2 \\
&+(\lambda+\Theta)^3[ad_2 + bd_3] + (\lambda+\Theta)^3\lambda(Bd_3 - ad_3) + (\lambda+\Theta)^3\lambda^2 Cd_3 \\
&\left.+(\lambda+\Theta)^4[ad_3 + bd_4] + (\lambda+\Theta)^4\lambda(Bd_4 - ad_4) + (\lambda+\Theta)^4\lambda^2 Cd_4 + (\lambda+\Theta)^5 ad_4\right\} \\
&- 6i\lambda^2\left\{Bd_0 + bd_1 + \lambda[2Cd_0 + Bd_1 - ad_1] + \lambda^2 Cd_1 + (\lambda+\Theta)[Bd_1 + ad_1 + 2bd_2]\right. \\
&+(\lambda+\Theta)\lambda[2Cd_1 + 2Bd_2 - 2ad_2] + (\lambda+\Theta)\lambda^2 2Cd_2 + (\lambda+\Theta)^2[Bd_2 + 2ad_2 + 3bd_3] \\
&+(\lambda+\Theta)^2\lambda[2Cd_2 + 3Bd_3 - 3ad_3] + (\lambda+\Theta)^2\lambda^2 3Cd_3 + (\lambda+\Theta)^3[Bd_3 + 3ad_3 + 4bd_4] \\
&+(\lambda+\Theta)^3\lambda[4Bd_4 - 4ad_4 + 2Cd_3] + (\lambda+\Theta)^3\lambda^2 4Cd_4 \\
&\left.+(\lambda+\Theta)^4[Bd_4 + 4ad_4] + (\lambda+\Theta)^4\lambda 2Cd_4\right\} \\
&- i\lambda^3\left\{2Cd_0 + 2Bd_1 + 2bd_2 + \lambda[4Cd_1 + 2Bd_2 - 2ad_2] + 2\lambda^2 Cd_2 + (\lambda+\Theta)[2Cd_1 + 4Bd_2 + 2ad_2 + 6bd_3]\right\} \\
&+(\lambda+\Theta)\lambda[8Cd_2 + 6Bd_3 - 6ad_3] + (\lambda+\Theta)\lambda^2 6Cd_3 + (\lambda+\Theta)^2[2Cd_2 + 6Bd_3 + 6ad_3 + 12bd_4]\} \\
&+(\lambda+\Theta)^2\lambda[12Cd_3 + 12Bd_4 - 12ad_4] + (\lambda+\Theta)^2\lambda^2 12Cd_4 + (\lambda+\Theta)^3[8Bd_4 + 12ad_4 + 2Cd_3]\} \\
&+(\lambda+\Theta)^3\lambda 16Cd_4 + (\lambda+\Theta)^4 2Cd_4.\}
\end{aligned}
$$

We consider this expression as sum of polynomials in the variables λ and $\lambda + \Theta$. The condition reduces to the following system of equations:

$$C_4 = 0, \tag{6.15}$$
$$C_3 + 4B_4 + 6a_4 = 0, \tag{6.16}$$
$$C_2 + 3a_3 + 3B_3 + 6b_4 = 0, \tag{6.17}$$
$$C_1 + a_2 + 2B_2 + 3b_3 = 0, \tag{6.18}$$
$$ad_3 + bd_4 = 0, \tag{6.19}$$
$$2Bd_4 + 3ad_4 = 0, \tag{6.20}$$
$$Cd_4 = 0, \tag{6.21}$$
$$8C_4 - 3iad_2 - 3ibd_3 = 0, \tag{6.22}$$
$$Bd_3 + ad_3 + 2bd_4 = 0, \tag{6.23}$$
$$-5Cd_3 - 8Bd_4 + 3ad_4 = 0, \tag{6.24}$$
$$Cd_4 = 0, \tag{6.25}$$
$$2C_3 + 2(B_4 - a_4) - iad_1 - ibd_2 = 0, \tag{6.26}$$
$$2C_4 - 2iBd_2 - iad_2 - 6ibd_3 = 0, \tag{6.27}$$
$$-5Cd_2 - 6Bd_3 + 2ad_3 - 2bd_4 = 0, \tag{6.28}$$
$$-10Cd_3 - 4(Bd_4 - ad_4) = 0, \tag{6.29}$$
$$Cd_4 = 0, \tag{6.30}$$

$$4C_2+3B_3-3a_3-3iad_0-3ibd_1=0, \tag{6.31}$$
$$C_3-2iBd_1-2ibd_2=0, \tag{6.32}$$
$$-10Cd_1-8Bd_2+5ad_2-3bd_3=0, \tag{6.33}$$
$$-10Cd_2-3Bd_3+3ad_3=0, \tag{6.34}$$
$$2C_1+B_2-a_2-3ibd_0=0, \tag{6.35}$$
$$C_2-6iBd_0+3iad_0-3ibd_1=0, \tag{6.36}$$
$$-10Cd_0-4Bd_1+3ad_1-bd_2=0, \tag{6.37}$$
$$5Cd_1+Bd_2-ad_2=0, \tag{6.38}$$
$$Cd_2=0, \tag{6.39}$$
$$C_0+B_1+b_2=0. \tag{6.40}$$

We know by (6.37) that:

$$bd_2=-10Cd_0-4Bd_1+3ad_1=-4Bd_1+3ad_1.$$

Indeed $Cd_0=0$ since $f^*=\xi_*$. Using this relation we have that Equations (6.16), (6.26), (6.32) reduce to:

$$\begin{cases} C_3+4B_4+6a_4=0,\\ 2C_3+2(B_4-a_4)-iad_1-ibd_2=2C_3+2(B_4-a_4)-iad_1-i(-4Bd_1+3ad_1)=0,\\ C_3-2iBd_1-2ibd_2=C_3-2iBd_1-2i(-4Bd_1+3ad_1)=0.\end{cases}$$

This can be rewritten as:

$$\begin{cases} C_3+4B_4+6a_4=0,\\ 2C_3+2(B_4-a_4)-4iad_1+4iBd_1=0,\\ C_3-6iad_1+6iBd_1=0.\end{cases}$$

Now we consider the following linear combinations of the three equations:

$$\begin{cases} 3B_4+7a_4+2iad_1-2iBd_1=0,\\ B_4-a_4+4iad_1-4iBd_1=0.\end{cases}$$

Now ad_1 and Bd_1 involve only terms in η_* with $v_{\emptyset,1}$ that is 0. So $a_4(1)=0$. Then, using that $|f|=0$:

$$a_4(f)=\sum_I \operatorname{sgn}(-2\eta_I)\otimes v_{I,4}=0,$$

where sgn $=\pm1$ and is not needed explicitly here. Using linear independence of distinct η_I's, we get $v_{I,4}=0$.

Now let us consider Equations (6.17), (6.31), (6.36):

$$\begin{cases} C_2+3a_3+3B_3+6b_4=0,\\ 4C_2+3B_3-3a_3-3iad_0-3ibd_1=0,\\ C_2-6iBd_0+3iad_0-3ibd_1=0.\end{cases}$$

We observe that $b_4(1)=0$, ad_0 and Bd_0 involve only terms with $v_{\emptyset,0}$ that is 0, bd_1 involves only terms with $v_{\emptyset,1}, v_{I,1}$ where $|I|=1,2$, that are zero. Then these equations reduce to:

$$\begin{cases} C_2+3a_3+3B_3=0,\\ 4C_2+3B_3-3a_3=0,\\ C_2=0.\end{cases}$$

From this we have that $a_3(1) = 0$. As before we deduce $v_{I,3} = 0$.
Thus we have shown that, for a singular vector $\vec{m}$, $T(\vec{m})$ has the following form:

$$T(\vec{m}) = \Theta^2\Big(\sum_{|I|\geq 2}\eta_I \otimes v_{I,2}\Big) + \Theta^1\Big(\sum_{|I|\geq 3}\eta_I \otimes v_{I,1}\Big) + \Big(\sum_{|I|\geq 1}\eta_I \otimes v_{I,0}\Big).$$

This means that there are singular vectors $\vec{m}$ of at most degree 8 and, in particular, $T(\vec{m})$ has the following form:

$$\begin{aligned}
T(\vec{m}) &= \Theta^2 \sum_{|I|=2} \eta_I \otimes v_{I,2} \quad \text{degree } 8,\\
T(\vec{m}) &= \Theta^2 \sum_{|I|=3} \eta_I \otimes v_{I,2} \quad \text{degree } \ 7,\\
T(\vec{m}) &= \Theta^2 \sum_{|I|=4} \eta_I \otimes v_{I,2} \quad \text{degree } 6,\\
T(\vec{m}) &= \Theta^2 \sum_{|I|=5} \eta_I \otimes v_{I,2} + \Theta \sum_{|I|=3} \eta_I \otimes v_{I,1} + \sum_{|I|=1} \eta_I \otimes v_{I,0} \quad \text{degree } 5,\\
T(\vec{m}) &= \Theta^2 \sum_{|I|=6} \eta_I \otimes v_{I,2} + \Theta \sum_{|I|=4} \eta_I \otimes v_{I,1} + \sum_{|I|=2} \eta_I \otimes v_{I,0} \quad \text{degree } 4,\\
T(\vec{m}) &= \Theta \sum_{|I|=5} \eta_I \otimes v_{I,1} + \sum_{|I|=3} \eta_I \otimes v_{I,0} \quad \text{degree } 3,\\
T(\vec{m}) &= \Theta \sum_{|I|=6} \eta_I \otimes v_{I,1} + \sum_{|I|=4} \eta_I \otimes v_{I,0} \quad \text{degree } 2,\\
T(\vec{m}) &= \sum_{|I|=5} \eta_I \otimes v_{I,0} \quad \text{degree } 1.
\end{aligned}$$

If we look at vectors of degree 8,7 and 6 we can use the relation $(B_1 + a_1 + 2b_2)(f) = 0$ from **S2** for $f = \xi_j$. In both these three cases it reduces to $b_2(f) = 0$ since there are no $v_{I,1}$'s involved. We get that:

$$b_2(\xi_j) = \sum_I \mathrm{sgn}_I\, \partial_j \eta_I \otimes v_{I,2} \ \text{ for } \ |I| = 2, 3, 4,$$

where sgn $= \pm 1$ and is not needed explicitly here. By linear independence we get $v_{I,2} = 0$ for $|I| = 2, 3, 4$. □

6.2 Homology

In [BKL2], Boyallian, Kac and Liberati completely classified the highest weight singular vectors for CK_6, using the reduction found in Lemma 6.5. Using an analog of Remark 4.8, they obtain the morphisms between degenerate modules for CK_6 represented in Figure 6.1. We point out that the Verma modules represented in Figure 6.1 are all degenerate except for the one represented at the origin of the third quadrant.
The aim of this section is to compute the homology of the complexes in Figure 6.1 in the first and third quadrant. The computation of the homology for the second quadrant will be done in the future. The first step for the computation of the homology is to find an explicit expression for the maps in Figure 6.1.

Remark 6.7. From now on we will often use the following isomorphism of Lie algebras between $\mathfrak{so}(6)$ and $\mathfrak{sl}(4)=\langle x_i\partial_j, x_i\partial_i - x_j\partial_j, 1\leq i\neq j\leq 4\rangle$ given by:

$$\begin{aligned}
\Psi:\mathfrak{sl}(4) &\longrightarrow \mathfrak{so}(6)\\
-2x_3\partial_2 &\longmapsto E_{-(\varepsilon_1-\varepsilon_2)},\\
2x_2\partial_3 &\longmapsto E_{(\varepsilon_1-\varepsilon_2)},\\
x_2\partial_2 - x_3\partial_3 &\longmapsto h_1,\\
-2x_2\partial_1 &\longmapsto E_{-(\varepsilon_2-\varepsilon_3)},\\
2x_1\partial_2 &\longmapsto E_{(\varepsilon_2-\varepsilon_3)},\\
x_1\partial_1 - x_2\partial_2 &\longmapsto h_2,\\
-2x_4\partial_3 &\longmapsto E_{-(\varepsilon_2+\varepsilon_3)},\\
2x_3\partial_4 &\longmapsto E_{(\varepsilon_2+\varepsilon_3)},\\
x_3\partial_3 - x_4\partial_4 &\longmapsto h_3,
\end{aligned}$$

and extended uniquely to a Lie algebra isomorphism. We will call $\mathfrak{g}_0^{ss}=\mathfrak{so}(6)\cong\mathfrak{sl}(4)$.

Remark 6.8. By a straightforward computation, it is possible to show that $\mathfrak{g}_{-1}$ is an irreducible $\mathfrak{g}_0^{ss}$−module of highest weight (1,0,0) with respect to h_1,h_2,h_3. In particular $\mathfrak{g}_{-1}$ is isomorphic to $\bigwedge^2((\mathbb{C}^4)^*)$ and the isomorphism is given by:

$$\begin{aligned}
\xi_2+i\xi_1 &\longleftrightarrow \partial_{x_3}\wedge\partial_{x_4}, & \xi_2-i\xi_1 &\longleftrightarrow -\partial_{x_1}\wedge\partial_{x_2},\\
\xi_4+i\xi_3 &\longleftrightarrow -\partial_{x_2}\wedge\partial_{x_4}, & \xi_4-i\xi_3 &\longleftrightarrow -\partial_{x_1}\wedge\partial_{x_3},\\
\xi_6+i\xi_5 &\longleftrightarrow \partial_{x_1}\wedge\partial_{x_4}, & \xi_6-i\xi_5 &\longleftrightarrow -\partial_{x_2}\wedge\partial_{x_3}.
\end{aligned}\qquad (6.41)$$

Motivated by relations (6.41), from now on we will use the notation:

$$\begin{aligned}
w_{34} &= \eta_2+i\eta_1, & w_{12} &= -\eta_2+i\eta_1,\\
w_{24} &= -(\eta_4+i\eta_3), & w_{13} &= -\eta_4+i\eta_3,\\
w_{14} &= \eta_6+i\eta_5, & w_{23} &= -\eta_6+i\eta_5.
\end{aligned}\qquad (6.42)$$

We point out that $[w_{34},w_{12}]=-4\Theta$, $[w_{24},w_{13}]=4\Theta$, $[w_{14},w_{23}]=-4\Theta$ and all the other brackets between the w's are zero.

Let

$$\lambda_1=\varepsilon_1,\quad \lambda_2=\frac{\varepsilon_1+\varepsilon_2-\varepsilon_3}{2},\quad \lambda_3=\frac{\varepsilon_1+\varepsilon_2+\varepsilon_3}{2},$$

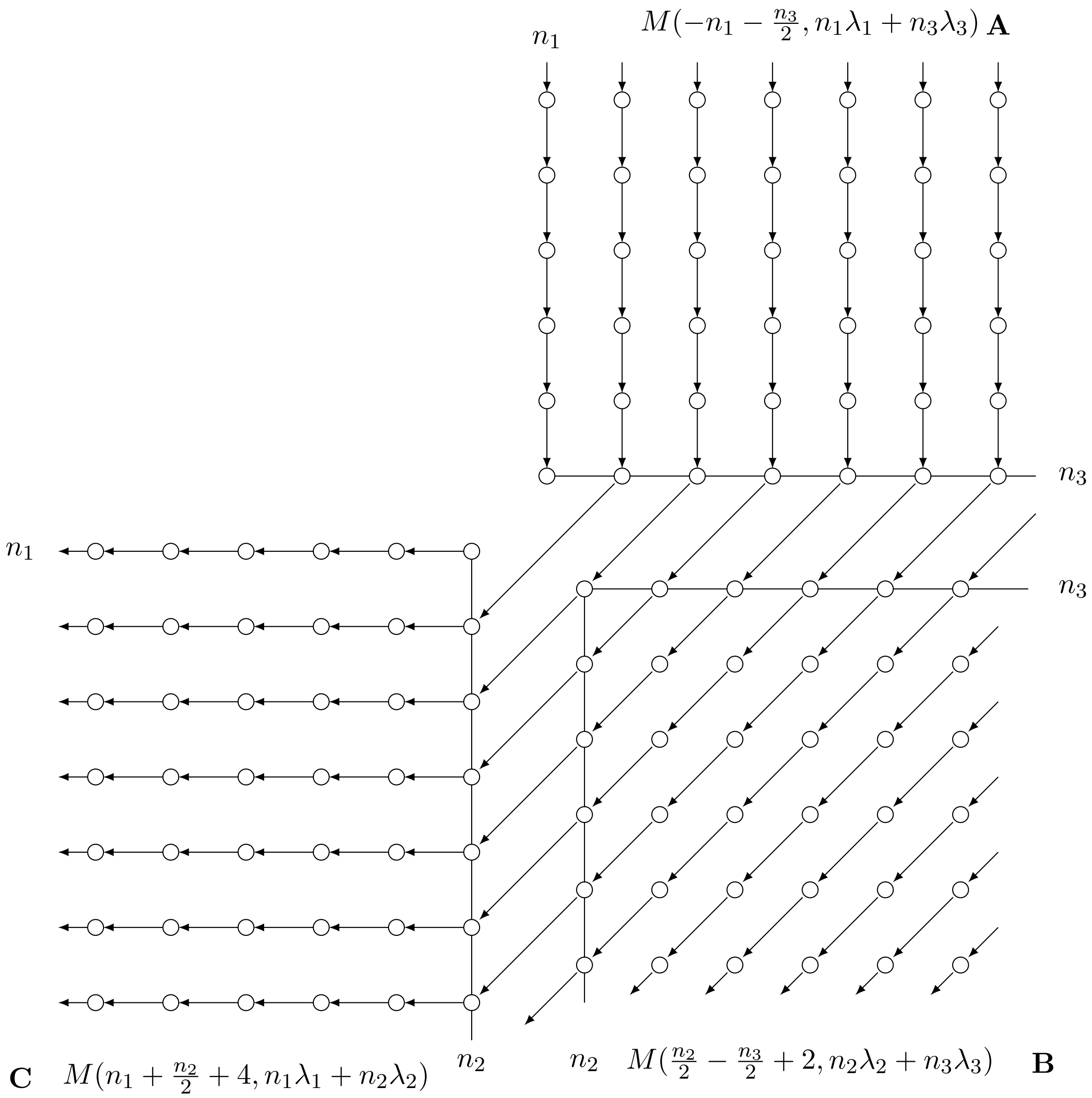
be the fundamental weights of $\mathfrak{so}(6)$ extended by $\lambda_i(t)=0$. We denote by $\lambda=n_1\lambda_1+n_2\lambda_2+n_3\lambda_3$ a dominant weight. Therefore n_1,n_2,n_3 are the weights with respect to h_1,h_2,h_3. We use the notation $F(n_1,n_2,n_3)$ to indicate the irreducible $\mathfrak{so}(6)$−module of highest weight λ.

Following [BKL2], the Verma modules are denoted by $M(n_0,n_1\lambda_1+n_2\lambda_2+n_3\lambda_3)$ where n_0 is the weight with respect to the central element t.

For the degenerate modules represented in Figure 6.1, we will use the following notation:

$$\begin{aligned}
M_A^{n_1,n_3} &:= M\left(-n_1-\frac{n_3}{2}, n_1\lambda_1+n_3\lambda_3\right)=U(\mathfrak{g}_{<0})\otimes V_A^{n_1,n_3},\\
M_B^{n_2,n_3} &:= M\left(\frac{n_2}{2}-\frac{n_3}{2}+2, n_2\lambda_2+n_3\lambda_3\right)=U(\mathfrak{g}_{<0})\otimes V_B^{n_2,n_3},\\
M_C^{n_1,n_2} &:= M\left(n_1+\frac{n_2}{2}+4, n_1\lambda_1+n_2\lambda_2\right)=U(\mathfrak{g}_{<0})\otimes V_C^{n_1,n_2},
\end{aligned}$$

Figure 6.1



where the modules M_A's are represented in the first quadrant, the modules M_B's in the second quadrant, the modules M_C's in the third quadrant and, as $\mathfrak{so}(6)-$modules, $V_A^{n_1,n_3} \cong F(n_1, 0, n_3)$, $V_B^{n_2,n_3} \cong F(0, n_2, n_3)$, $V_C^{n_1,n_2} \cong F(n_1, n_2, 0)$. The element t acts as multiplication by $-n_1 - \frac{n_3}{2}$ on $V_A^{n_1,n_3}$, as multiplication by $\frac{n_2}{2} - \frac{n_3}{2} + 2$ on $V_B^{n_2,n_3}$ and as multiplication by $n_1 + \frac{n_2}{2} + 4$ on $V_C^{n_1,n_2}$.

Remark 6.9. We will think $V_A^{n_1,n_3}$ as the irreducible submodule of

$$\mathrm{Sym}^{n_1}(\wedge^2((\mathbb{C}^4)^*)) \otimes \mathrm{Sym}^{n_3}((\mathbb{C}^4)^*)$$

generated by the highest weight vector $v_\lambda := (\partial_{x_3} \wedge \partial_{x_4})^{n_1} \partial_4^{n_3}$ where $\{\partial_1, \partial_2, \partial_3, \partial_4\}$ is a basis for $(\mathbb{C}^4)^*$.

We will think $V_B^{n_2,n_3}$ as the irreducible submodule of

$$\mathrm{Sym}^{n_2}(\mathbb{C}^4) \otimes \mathrm{Sym}^{n_3}((\mathbb{C}^4)^*)$$

generated by the highest weight vector $v_\lambda := x_1^{n_2} \partial_4^{n_3}$ where $\{x_1, x_2, x_3, x_4\}$ is the standard basis of $\mathbb{C}^4$ and $\{\partial_1, \partial_2, \partial_3, \partial_4\}$ is a basis for $(\mathbb{C}^4)^*$.

We will think $V_C^{n_1,n_2}$ as the irreducible submodule of

$$\mathrm{Sym}^{n_1}(\bigwedge^2(\mathbb{C}^4)) \otimes \mathrm{Sym}^{n_2}(\mathbb{C}^4)$$

generated by the highest weight vector $v_\lambda := (x_1 \wedge x_2)^{n_1} x_1^{n_2}$ where $\{x_1, x_2, x_3, x_4\}$ is the standard basis of $\mathbb{C}^4$ and $\{x_i \wedge x_j\}$ is a basis for $\bigwedge^2(\mathbb{C}^4)$.
We observe that t acts as $\frac{x_1\partial_{x_1}+x_2\partial_{x_2}+x_3\partial_{x_3}+x_4\partial_{x_4}}{2}$ on vectors of $V_A^{n_1,n_3}$, as $\frac{x_1\partial_{x_1}+x_2\partial_{x_2}+x_3\partial_{x_3}+x_4\partial_{x_4}}{2}+2$ on vectors of $V_B^{n_2,n_3}$ and as $\frac{x_1\partial_{x_1}+x_2\partial_{x_2}+x_3\partial_{x_3}+x_4\partial_{x_4}}{2}+4$ on vectors of $V_C^{n_1,n_2}$.

In [BKL2], Boyallian, Kac and Liberati completely classified the highest weight singular vectors for CK_6, using the reduction found in Lemma 6.5; they obtain the following classification.

Theorem 6.10 ([BKL2] Theorem 4.1)**.** *Let F be an irreducible finite−dimensional $\mathfrak{g}_0$−module of highest weight $\mu = (n_0, n_1\lambda_1 + n_2\lambda_2 + n_3\lambda_3)$. Therefore a vector in $\mathrm{Ind}(F)$ is a nontrivial highest weight singular vector if and only if it is (up to a scalar) one of the following:*

(a) $\mu = (\frac{9}{2}, \lambda_2)$,

$$\begin{aligned}\vec{m}_{5a} =&\Theta^2\big[w_{34}\otimes x_3 + w_{24}\otimes x_2 + w_{14}\otimes x_1\big]\\ &+\frac{\Theta w_{34}}{4}\big[w_{23}w_{14} - w_{14}w_{23} + w_{24}w_{13} - w_{13}w_{24}\big]\otimes x_3\\ &+\frac{\Theta w_{24}}{4}\big[w_{23}w_{14} - w_{14}w_{23} - w_{34}w_{12} + w_{12}w_{34}\big]\otimes x_2\\ &+\frac{\Theta w_{14}}{4}\big[w_{24}w_{13} - w_{13}w_{24} - w_{34}w_{12} + w_{12}w_{34}\big]\otimes x_1 + \Theta w_{34}w_{24}w_{14}\otimes x_4\\ &+ i\frac{w_{34}}{16}\big(w_{13}w_{24} - w_{24}w_{13}\big)\big(-w_{14}w_{23} + w_{23}w_{14}\big)\otimes x_3\\ &+ i\frac{w_{24}}{16}\big(w_{34}w_{12} - w_{12}w_{34}\big)\big(w_{23}w_{14} - w_{14}w_{23}\big)\otimes x_2\\ &+ i\frac{w_{14}}{16}\big(w_{34}w_{12} - w_{12}w_{34}\big)\big(w_{24}w_{13} - w_{13}w_{24}\big)\otimes x_1;\end{aligned}$$

(b) $\mu = (\frac{n_2}{2} + 4, n_2\lambda_2)$, *with* $n_2 \geq 2$,

$$\begin{aligned}\vec{m}_{3b} =&\frac{w_{13}}{2}\big(w_{14}w_{23} - w_{23}w_{14} + w_{12}w_{34} - w_{34}w_{12}\big)\otimes x_3x_1^{n_2-1}\\ &+\frac{w_{24}}{2}\big(w_{23}w_{14} - w_{14}w_{23} + w_{12}w_{34} - w_{34}w_{12}\big)\otimes x_4x_2x_1^{n_2-2}\\ &+\frac{w_{23}}{2}\big(w_{13}w_{24} - w_{24}w_{13} + w_{12}w_{34} - w_{34}w_{12}\big)\otimes x_3x_2x_1^{n_2-2}\\ &+\frac{w_{14}}{2}\big(w_{24}w_{13} - w_{13}w_{24} - w_{34}w_{12} + w_{12}w_{34}\big)\otimes x_4x_1^{n_2-1}\\ &+\frac{w_{12}}{2}\big(w_{14}w_{23} - w_{23}w_{14} + w_{24}w_{13} - w_{13}w_{24}\big)\otimes x_2x_1^{n_2-1}\\ &+\frac{w_{34}}{2}\big(w_{23}w_{14} - w_{14}w_{23} + w_{24}w_{13} - w_{13}w_{24}\big)\otimes x_3x_4x_1^{n_2-2}\\ &+ w_{34}w_{24}w_{14}\otimes x_4^2x_1^{n_2-2} + w_{12}w_{24}w_{23}\otimes x_2^2x_1^{n_2-2}\\ &- w_{34}w_{13}w_{23}\otimes x_3^2x_1^{n_2-2} - w_{12}w_{13}w_{14}\otimes x_1^{n_2};\end{aligned}$$

(c) $\mu = (-\frac{n_3}{2} + 2, n_3\lambda_3)$, *with* $n_3 \geq 0$,

$$\vec{m}_{3c} = w_{34}w_{24}w_{14}\otimes \partial_4^{n_3};$$

(d) $\mu = (n_1 + \frac{n_2}{2} + 4, n_1\lambda_1 + n_2\lambda_2)$, *with* $n_1 \geq 1$, $n_2 \geq 0$,

$$\vec{m}_{1d} =$$

$$= -w_{12} \otimes (x_1 \wedge x_2)^{n_1} x_1^{n_2} - w_{13} \otimes (x_1 \wedge x_3)(x_1 \wedge x_2)^{n_1-1} x_1^{n_2}$$
$$+ w_{23} \otimes \frac{1}{2(n_1+n_2+1)} \big[2n_2(x_1 \wedge x_2)^{n_1} x_3 x_1^{n_2-1} - 2n_2(x_1 \wedge x_3)(x_1 \wedge x_2)^{n_1-1} x_2 x_1^{n_2-1}$$
$$- 2(n_1+1)(x_2 \wedge x_3)(x_1 \wedge x_2)^{n_1-1} x_1^{n_2}\big] +$$
$$+ w_{34} \otimes \frac{1}{2(n_1+n_2+1)} \big[- 2n_2(x_1 \wedge x_4)(x_1 \wedge x_2)^{n_1-1} x_3 x_1^{n_2-1}$$
$$+ 2(n_1-1)(x_1 \wedge x_4)(x_2 \wedge x_3)(x_1 \wedge x_2)^{n_1-2} x_1^{n_2} + 2n_2(x_1 \wedge x_3)(x_1 \wedge x_2)^{n_1-1} x_4 x_1^{n_2-1}$$
$$- 4(x_3 \wedge x_4)(x_1 \wedge x_2)^{n_1-1} x_1^{n_2} - 2(n_1-1)(x_1 \wedge x_3)(x_2 \wedge x_4)(x_1 \wedge x_2)^{n_1-2} x_1^{n_2}\big]$$
$$- w_{24} \otimes \frac{1}{2(n_1+n_2+1)} \big[- 2n_2(x_1 \wedge x_2)^{n_1} x_4 x_1^{n_2-1} + 2n_2(x_1 \wedge x_4)(x_1 \wedge x_2)^{n_1-1} x_2 x_1^{n_2-1}$$
$$+ (2n_1+2)(x_2 \wedge x_4)(x_1 \wedge x_2)^{n_1-1} x_1^{n_2}\big] - w_{14} \otimes (x_1 \wedge x_4)(x_1 \wedge x_2)^{n_1-1} x_1^{n_2};$$

(e) $\mu = (\frac{n_2}{2} - \frac{n_3}{2} + 2, n_2\lambda_2 + n_3\lambda_3)$, *with* $n_2 \geq 1$, $n_3 \geq 0$,

$$\vec{m}_{1e} = w_{34} \otimes x_1^{n_2-1} x_3 \partial_4^{n_3} + w_{24} \otimes x_1^{n_2-1} x_2 \partial_4^{n_3} + w_{14} \otimes x_1^{n_2} \partial_4^{n_3};$$

(f) $\mu = (-n_1 - \frac{n_3}{2}, n_1\lambda_1 + n_3\lambda_3)$, *with* $n_1 \geq 0$, $n_3 \geq 0$,

$$\vec{m}_{1f} = w_{34} \otimes (\partial_3 \wedge \partial_4)^{n_1} \partial_4^{n_3}.$$

We point out that, by Theorem 6.10, there are only nontrivial singular vectors of degree 5, 3 and 1.

We define, using (5.4), the following map between the modules M_A in the first quadrant:

$$\nabla_A : M\left(-n_1 - \frac{n_3}{2}, n_1\lambda_1 + n_3\lambda_3\right) \longrightarrow M\left(-(n_1-1) - \frac{n_3}{2}, (n_1-1)\lambda_1 + n_3\lambda_3\right) \tag{6.43}$$
$$\nabla_A = w_{34} \otimes \partial_{3,4} + w_{24} \otimes \partial_{2,4} + w_{14} \otimes \partial_{1,4} + w_{12} \otimes \partial_{1,2} + w_{23} \otimes \partial_{2,3} + w_{13} \otimes \partial_{1,3},$$

where $\partial_{i,j}$ denotes the derivative with respect to the element $\partial_i \wedge \partial_j$. We assume that $\partial_{i,j} = -\partial_{j,i}$ for all i, j.

Remark 6.11. The map ∇_A is constructed so that it sends the highest weight vector $(\partial_3 \wedge \partial_4)^{n_1} \partial_4^{n_3}$ of $M(-n_1 - \frac{n_3}{2}, n_1\lambda_1 + n_3\lambda_3)$ to:

$$\vec{m} = w_{34} \otimes n_1 (\partial_3 \wedge \partial_4)^{n_1-1} \partial_4^{n_3}$$

that is the highest weight singular vector of $M\left(-(n_1-1) - \frac{n_3}{2}, (n_1-1)\lambda_1 + n_3\lambda_3\right)$ found in Theorem 6.10.

Lemma 6.12. *The map* ∇_A*, defined in* (6.43)*, is a morphism of* $\mathfrak{g}-$*modules and* $\nabla_A^2 = 0$.

Proof. The map ∇_A commutes with $\mathfrak{g}_{<0}$ by (5.4). By Remark 6.11 and Lemmas 5.6, 5.8 it follows that ∇_A is a morphism of $\mathfrak{g}-$modules. The property $\nabla_A^2 = 0$ follows from the fact that ∇_A is a map between Verma modules that contain only highest weight singular vectors of degree 1 and there are no singular vectors of degree 2, by Theorem 6.10. □

We call ∇_3 the $\mathfrak{g}-$morphisms from M_A^{0,n_3} to M_B^{0,n_3-2}, for all $n_3 > 1$, that map the highest weight vector $\partial_{x_4}^{n_3}$ of V_A^{0,n_3} to the highest weight singular vector of degree 3 of M_B^{0,n_3-2} found in Theorem 6.10. We point out that $\nabla_3 \nabla_A : M_A^{1,n_3} \longrightarrow M_B^{0,n_3-2}$ is 0 since there are no highest weight singular vectors of degree 4 due to Theorem 6.10. The morphisms ∇_3 are represented from the first to the second quadrant in Figure 6.1.

We call ∇_5 the $\mathfrak{g}-$morphism, from $M_A^{0,1}$ to $M_C^{0,1}$, that maps the highest weight vector ∂_{x_4} of $V_A^{0,1}$

to the highest weight singular vector of degree 5 of $M_C^{0,1}$ found in Theorem 6.10. We point out that $\nabla_5 \nabla_A : M_A^{1,1} \longrightarrow M_C^{0,1}$ is 0 since there are no highest weight singular vectors of degree 6 due to Theorem 6.10. The morphism ∇_5 is represented from the first to the third quadrant in Figure 6.1. We now compute the homology for the first quadrant. We call $M_A = \oplus_{n_1,n_3} M_A^{n_1,n_3}$ and $V_A = \oplus_{n_1,n_3} V_A^{n_1,n_3}$.
Following [KR1], let us consider the filtration on $U(\mathfrak{g}_{<0})$ as follows: for all $i \geq 0$, $F_i U(\mathfrak{g}_{<0})$ is the subspace of $U(\mathfrak{g}_{<0})$ spanned by elements with at most i terms of $\mathfrak{g}_{<0}$. Therefore:

$$\mathbb{C} = F_0 U(\mathfrak{g}_{<0}) \subset F_1 U(\mathfrak{g}_{<0}) \subset ... \subset F_{i-1} U(\mathfrak{g}_{<0}) \subset F_i U(\mathfrak{g}_{<0}) \subset ... ,$$

where $F_i U(\mathfrak{g}_{<0}) = \mathfrak{g}_{<0} F_{i-1} U(\mathfrak{g}_{<0}) + F_{i-1} U(\mathfrak{g}_{<0})$. We call $F_i M_A = F_i U(\mathfrak{g}_{<0}) \otimes V_A$. We have that $\nabla_A F_i M_A \subset F_{i+1} M_A$ and the filtration is bounded below. Then we can use the theory of spectral sequences; we first study $\operatorname{Gr} M_A$. We consider the subalgebra $\mathfrak{g}_{\bar{0}}$ of $\mathfrak{g}$ given by the even elements, that, since the grading is consistent, is $\mathfrak{g}_{\bar{0}} = \oplus_{i \geq -1} \mathfrak{g}_{2i}$. On $\mathfrak{g}_{\bar{0}}$ we consider the filtration $\mathfrak{g}_{\bar{0}} = L_{-1} \supset L_0 = \oplus_{i \geq 0} \mathfrak{g}_{2i} \supset L_1 = \oplus_{i \geq 1} \mathfrak{g}_{2i} \dots$.

Lemma 6.13. *For all $j \geq -1$ and $i \geq 0$, we have:*

$$L_j F_i M_A \subset F_{i-j} M_A. \tag{6.44}$$

Proof. The proof is analogous to Lemma 5.32. □

By (6.44), we know, since $\mathfrak{g}_{\bar{0}} \cong \operatorname{Gr} \mathfrak{g}_{\bar{0}}$, that the action of $\mathfrak{g}_{\bar{0}}$ on M_A descends on $\operatorname{Gr} M_A$.
We point out that, using the Poincaré–Birkhoff–Witt Theorem, we have $\operatorname{Gr} U(\mathfrak{g}_{<0}) \cong S(\mathfrak{g}_{-2}) \otimes \bigwedge(\mathfrak{g}_{-1})$; indeed we have already noticed that in $U(\mathfrak{g}_{<0})$, for all $i \in \{1,2,3,4,5,6\}$, $\eta_i^2 = \Theta$.
Therefore, as $\mathfrak{g}_{\bar{0}}$–modules:

$$\operatorname{Gr} M_A = \operatorname{Gr} U(\mathfrak{g}_{<0}) \otimes V_A \cong S(\mathfrak{g}_{-2}) \otimes \bigwedge(\mathfrak{g}_{-1}) \otimes V_A.$$

From (6.44), it follows that L_1 annihilates $G_A := \bigwedge(\mathfrak{g}_{-1}) \otimes V_A$. Therefore, as $\mathfrak{g}_{\bar{0}}$–modules:

$$\operatorname{Gr} M_A \cong S(\mathfrak{g}_{-2}) \otimes (\bigwedge(\mathfrak{g}_{-1}) \otimes V_A) \cong \operatorname{Ind}_{L_0}^{\mathfrak{g}_{\bar{0}}} (\bigwedge(\mathfrak{g}_{-1}) \otimes V_A).$$

We observe that $\operatorname{Gr} M_A$ is a complex with the morphism induced by ∇_A, that we still call ∇_A. Indeed $\nabla_A F_i M_A \subset F_{i+1} M_A$ for all i, therefore it is well defined the induced morphism

$$\nabla_A : \operatorname{Gr}_i M_A = F_i M_A / F_{i-1} M_A \longrightarrow \operatorname{Gr}_{i+1} M_A = F_{i+1} M_A / F_i M_A,$$

that has the same formula as ∇_A defined in (6.43), apart from the fact that the multiplication by the w's must be seen as multiplication in $\operatorname{Gr} U(\mathfrak{g}_{<0})$ instead of $U(\mathfrak{g}_{<0})$.
Therefore we have that (G_A, ∇_A) is a subcomplex of $(\operatorname{Gr} M_A, \nabla_A)$: indeed it is sufficient to restrict ∇_A to G_A; the complex $(\operatorname{Gr} M_A, \nabla_A)$ is obtained from (G_A, ∇_A) extending the coefficients to $S(\mathfrak{g}_{-2})$.
We point out that also the homology spaces $H^{m,n}(G_A)$ are annihilated by L_1. Therefore, as $\mathfrak{g}_{\bar{0}}$–modules:

$$H^{m,n}(\operatorname{Gr} M_A) \cong S(\mathfrak{g}_{-2}) \otimes H^{m,n}(G_A) \cong \operatorname{Ind}_{L_0}^{\mathfrak{g}_{\bar{0}}} (H^{m,n}(G_A)). \tag{6.45}$$

From (6.45) and Proposition 5.24, it follows that:

Proposition 6.14. *If $H^{m,n}(G_A) = 0$, then $H^{m,n}(\operatorname{Gr} M_A) = 0$ and therefore $H^{m,n}(M_A) = 0$.*

We introduce the notation, for all $n_3 \geq 0$:

$$V_{A'}^{n_3} = V_A^{0,n_3},$$
$$V_{B'}^{n_3} = V_B^{0,n_3}.$$

We will call $V_{A'} = \oplus_{n_3} V_{A'}^{n_3}$, $V_{B'} = \oplus_{n_3} V_{B'}^{n_3}$, $G_{A'} = \bigwedge(\mathfrak{g}_{-1}) \otimes V_{A'}$ and $G_{B'} = \bigwedge(\mathfrak{g}_{-1}) \otimes V_{B'}$.
Let us consider the evaluation map from V_A to $V_{A'}$, that maps $\partial_1 \wedge \partial_2, \partial_1 \wedge \partial_3, \partial_1 \wedge \partial_4, \partial_2 \wedge \partial_3, \partial_2 \wedge \partial_4, \partial_3 \wedge \partial_4$ to zero and is the identity on all other elements; we can compose this map with ∇_3 and obtain a new map, that we still call ∇_3, from G_A to $G_{B'}$. Analogously we can consider the inclusion of $G_{B'}$ into G_B and compose the map with ∇_3; we obtain a map from $G_{A'}$ to G_B that we still call ∇_3. We define:

$$G_{A^\circ} = \mathrm{Ker}(\nabla_3 : G_A \longrightarrow G_{B'}), \quad G_{B^\circ} = \mathrm{CoKer}(\nabla_3 : G_{A'} \longrightarrow G_B).$$

The map ∇_A is still defined on G_{A° since $\nabla_3 \nabla_A = 0$.

Remark 6.15. From its definition, it is obvious that $G_{A^\circ}^{n_1,n_3} = G_A^{n_1,n_3}$ if $n_1 > 0$. Therefore:

$$H^{n_1,n_3}(G_A) = H^{n_1,n_3}(G_{A^\circ}).$$

Remark 6.16. Let us focus on some technical computations. We point out that, using Remark 6.7:

$$\frac{-t + i\xi_1\xi_2 + i\xi_3\xi_4 + i\xi_5\xi_6}{2} = \frac{-t - (h_1 + \frac{h_2+h_3}{2}) - h_3}{2} = -\frac{x_1\partial_1 + x_2\partial_2 + x_3\partial_3 - x_4\partial_4}{2}.$$

Using bracket (2.1), we obtain:

$$\left[\frac{-t + i\xi_1\xi_2 + i\xi_3\xi_4 + i\xi_5\xi_6}{2}, \xi_2 - i\xi_1\right] = \xi_2 - i\xi_1, \qquad \left[\frac{-t + i\xi_1\xi_2 + i\xi_3\xi_4 + i\xi_5\xi_6}{2}, \xi_2 + i\xi_1\right] = 0,$$
$$\left[\frac{-t + i\xi_1\xi_2 + i\xi_3\xi_4 + i\xi_5\xi_6}{2}, \xi_4 - i\xi_3\right] = \xi_4 - i\xi_3, \qquad \left[\frac{-t + i\xi_1\xi_2 + i\xi_3\xi_4 + i\xi_5\xi_6}{2}, \xi_4 + i\xi_3\right] = 0,$$
$$\left[\frac{-t + i\xi_1\xi_2 + i\xi_3\xi_4 + i\xi_5\xi_6}{2}, \xi_6 - i\xi_5\right] = \xi_6 - i\xi_5, \qquad \left[\frac{-t + i\xi_1\xi_2 + i\xi_3\xi_4 + i\xi_5\xi_6}{2}, \xi_6 + i\xi_5\right] = 0.$$

We also have, using Remark 6.7:

$$\frac{-t - i\xi_1\xi_2 - i\xi_3\xi_4 - i\xi_5\xi_6}{2} = \frac{-t + (h_1 + \frac{h_2+h_3}{2}) + h_3}{2} = -x_4\partial_4.$$

Using bracket (2.1), we obtain:

$$\left[\frac{-t - i\xi_1\xi_2 - i\xi_3\xi_4 - i\xi_5\xi_6}{2}, \xi_2 - i\xi_1\right] = 0, \qquad \left[\frac{-t - i\xi_1\xi_2 - i\xi_3\xi_4 - i\xi_5\xi_6}{2}, \xi_2 + i\xi_1\right] = \xi_2 + i\xi_1,$$
$$\left[\frac{-t - i\xi_1\xi_2 - i\xi_3\xi_4 - i\xi_5\xi_6}{2}, \xi_4 - i\xi_3\right] = 0, \qquad \left[\frac{-t - i\xi_1\xi_2 - i\xi_3\xi_4 - i\xi_5\xi_6}{2}, \xi_4 + i\xi_3\right] = \xi_4 + i\xi_3,$$
$$\left[\frac{-t - i\xi_1\xi_2 - i\xi_3\xi_4 - i\xi_5\xi_6}{2}, \xi_6 - i\xi_5\right] = 0, \qquad \left[\frac{-t - i\xi_1\xi_2 - i\xi_3\xi_4 - i\xi_5\xi_6}{2}, \xi_6 + i\xi_5\right] = \xi_6 + i\xi_5.$$

Motivated by Remark 6.16, we introduce an additional bigrading:

$$(V_A)_{[p,q]} = \left\{ f \in V_A \, : \, 2\left(-\frac{x_1\partial_1 + x_2\partial_2 + x_3\partial_3 - x_4\partial_4}{2}\right).f = pf \, , \, (-2x_4\partial_4).f = qf \right\}, \qquad (6.46)$$

$$(G_A)_{[p,q]} = \bigwedge(\mathfrak{g}_{-1}) \otimes (V_A)_{|[p,q]}.$$

We observe that, for elements in $(V_A^{n_1,n_3})_{|[p,q]}$, we have $p+q = 2n_1+n_3$ that is the eigenvalue of $-2t$ on $V_A^{n_1,n_3}$. The definitions can be extended also to G_{A°.
We define $d' := w_{12} \otimes \partial_{1,2} + w_{23} \otimes \partial_{2,3} + w_{13} \otimes \partial_{1,3}$ and $d'' = w_{34} \otimes \partial_{3,4} + w_{24} \otimes \partial_{2,4} + w_{14} \otimes \partial_{1,4}$, so that $d' + d'' = \nabla_A$.
Using Remark 6.16 and notation (6.42), it is an easy check that $d' : (G_A)_{|[p,q]} \longrightarrow (G_A)_{|[p-2,q]}$ and $d'' : (G_A)_{|[p,q]} \longrightarrow (G_A)_{|[p,q-2]}$.
By Remark 6.8, it follows that that $(d')^2 = (d'')^2 = d'd'' + d''d' = 0$. We point out that:

$$\nabla_A : \oplus_{n_1+\frac{n_3}{2}=k} G_A^{n_1,n_3} \longrightarrow \oplus_{n_1+\frac{n_3}{2}=k-1} G_A^{n_1,n_3}.$$

Therefore $\oplus_{n_1+\frac{n_3}{2}=k} G_A^{n_1,n_3}$ is a bicomplex with bigrading (6.46), differentials d', d'' and total differential $\nabla_A = d' + d''$, the same holds for $\oplus_{n_1+\frac{n_3}{2}=k} G_{A^\circ}^{n_1,n_3}$. Now let:

$$\bigwedge_+^i = \bigwedge^i\langle w_{34}, w_{24}, w_{14}\rangle \text{ and } \bigwedge_-^i = \bigwedge^i\langle w_{12}, w_{13}, w_{23}\rangle.$$

We define:

$$G_A(\alpha,\beta)_{[p,q]} = \bigwedge_-^{\frac{\alpha-p}{2}} \bigwedge_+^{\frac{\beta-q}{2}} (V_A)_{[p,q]}.$$

We point out that $\alpha - p$ and $\beta - q$ are always even, indeed α (resp. β) is the eigenvalue of $2\left(-\frac{x_1\partial_1+x_2\partial_2+x_3\partial_3-x_4\partial_4}{2}\right)$ (resp. $-2x_4\partial_4$) on elements in $\bigwedge_-^{\frac{\alpha-p}{2}} \bigwedge_+^{\frac{\beta-q}{2}} (V_A)_{[p,q]}$ and the elements of $\bigwedge_-^{\frac{\alpha-p}{2}}$ (resp. $\bigwedge_+^{\frac{\beta-q}{2}}$) have even eigenvalue with respect to $2\left(-\frac{x_1\partial_1+x_2\partial_2+x_3\partial_3-x_4\partial_4}{2}\right)$ (resp. $-2x_4\partial_4$), due to Remark 6.16.
If α is even, only even values of p occur. If α is odd, only odd values of p occur. The values of β and q are always even.
We point out that $\alpha + \beta \geq 0$. Indeed this value represents the eigenvalue of $-2t$ on elements in $G_A(\alpha,\beta)_{[p,q]}$ and $-2t$ has non negative eigenvalues on V_A and eigenvalue 2 for elements in $\mathfrak{g}_{-1}$. Moreover $\beta \geq 0$ since $(V_A)_{[p,q]} \neq 0$ only for $q \geq 0$.
We have that $G_A = \oplus_{\alpha,\beta} G_A(\alpha,\beta)$, where $G_A(\alpha,\beta) = \oplus_{p,q} G_A(\alpha,\beta)_{[p,q]}$. By Remark 6.16, it follows that $\nabla_A : G_A(\alpha,\beta) \to G_A(\alpha,\beta)$. The same definition holds for $G_{A^\circ}(\alpha,\beta)_{[p,q]}$. The computation of homologies of G_A and G_{A° can be reduced to the computation for $G_A(\alpha,\beta)$ and $G_{A^\circ}(\alpha,\beta)$.

Lemma 6.17. *Let α,β be such that $\beta \geq 0$, $\alpha+\beta \geq 0$.*
As $\langle x_1\partial_1 - x_2\partial_2, x_1\partial_2, x_2\partial_1, x_2\partial_2 - x_3\partial_3, x_2\partial_3, x_3\partial_2, x_1\partial_3, x_3\partial_1\rangle$−modules:

$$\frac{G_A^{n_1=0,n_3=\alpha+\beta}(\alpha,\beta)}{\operatorname{Im}\nabla_A} \cong \bigwedge_-^0 \bigwedge_+^0 \otimes (V_A^{0,\alpha+\beta})_{[\alpha,\beta]},$$
$$H^{n_1,n_3}(G_A(\alpha,\beta)) = 0 \quad \textit{for } (n_1,n_3) \neq (0,\alpha+\beta).$$

Proof. We split the proof in different cases.

A) We first analyze the case of α odd. We point out that p can assume only odd values. We modify the bigrading in order to obtain a new bigrading and use the theory of spectral sequences for bicomplexes.
For every p,q we denote by $\widetilde{p} = \frac{p+1}{2}, \widetilde{q} = \frac{q}{2}$. We will denote $(\widetilde{V}_A)_{[\widetilde{p},\widetilde{q}]} = (V_A)_{[p,q]}$. We denote $\widetilde{G}_A(\alpha,\beta)_{[\widetilde{p},\widetilde{q}]} = G_A(\alpha,\beta)_{[p,q]}$. We now have that $d' : \widetilde{G}_A(\alpha,\beta)_{[\widetilde{p},\widetilde{q}]} \longrightarrow \widetilde{G}_A(\alpha,\beta)_{[\widetilde{p}-1,\widetilde{q}]}$ and $d'' : \widetilde{G}_A(\alpha,\beta)_{[\widetilde{p},\widetilde{q}]} \longrightarrow \widetilde{G}_A(\alpha,\beta)_{[\widetilde{p},\widetilde{q}-1]}$. We have still that $G_A(\alpha,\beta) = \oplus_{\widetilde{p},\widetilde{q}} \widetilde{G}_A(\alpha,\beta)_{[\widetilde{p},\widetilde{q}]}$. We split into four subcases.

1. We first consider the case $\beta > 6$ and $\alpha + \beta > 6$. We use the theory of spectral sequences of bicomplexes. Let us consider $G_A(\alpha, \beta)$ with the differential d'':

$$\xleftarrow{d''} \bigwedge_{-}^{\frac{\alpha-(2\widetilde{p}-1)}{2}} \bigwedge_{+}^{\frac{\beta-2\widetilde{q}}{2}+1} \otimes (\widetilde{V}_A)_{[\widetilde{p},\widetilde{q}-1]} \xleftarrow{d''} \bigwedge_{-}^{\frac{\alpha-(2\widetilde{p}-1)}{2}} \bigwedge_{+}^{\frac{\beta-2\widetilde{q}}{2}} \otimes (\widetilde{V}_A)_{[\widetilde{p},\widetilde{q}]} \xleftarrow{d''} \bigwedge_{-}^{\frac{\alpha-(2\widetilde{p}-1)}{2}} \bigwedge_{+}^{\frac{\beta-2\widetilde{q}}{2}-1} \otimes (\widetilde{V}_A)_{[\widetilde{p},\widetilde{q}+1]} \xleftarrow{d''} .$$

It is the tensor product of $\bigwedge_{-}^{\frac{\alpha-(2\widetilde{p}-1)}{2}}$ and the following complex, since $\bigwedge_{-}^{\frac{\alpha-(2\widetilde{p}-1)}{2}}$ is not involved in d'':

$$0 \xleftarrow{d''} \bigwedge_{+}^{3} \otimes (\widetilde{V}_A)_{[\widetilde{p},\frac{\beta}{2}-3]} \xleftarrow{d''} \bigwedge_{+}^{2} \otimes (\widetilde{V}_A)_{[\widetilde{p},\frac{\beta}{2}-2]} \xleftarrow{d''} \bigwedge_{+}^{1} \otimes (\widetilde{V}_A)_{[\widetilde{p},\frac{\beta}{2}-1]} \xleftarrow{d''} \bigwedge_{+}^{0} \otimes (\widetilde{V}_A)_{[\widetilde{p},\frac{\beta}{2}]} \xleftarrow{d''} 0.$$

We have that condition $\beta > 6$ assures that for $\widetilde{q}$ the value $\frac{\beta}{2} - 3$ is acceptable since it is positive. This complex is exact except for the right end, let us analyze in detail.

i: Let us consider the map $d'' : \bigwedge_{+}^{0} \otimes (\widetilde{V}_A)_{[\widetilde{p},\frac{\beta}{2}]} \longrightarrow \bigwedge_{+}^{1} \otimes (\widetilde{V}_A)_{[\widetilde{p},\frac{\beta}{2}-1]}$. We compute the kernel. Let $f \in \bigwedge_{+}^{0} \otimes (\widetilde{V}_A)_{[\widetilde{p},\frac{\beta}{2}]}$. We have:

$$d''(f) = w_{34} \otimes \partial_{3,4} f + w_{24} \otimes \partial_{2,4} f + w_{14} \otimes \partial_{1,4} f.$$

It is zero if and only if $\partial_{3,4} f = \partial_{2,4} f = \partial_{1,4} f = 0$.
ii: Let us consider the map $d'' : \bigwedge_{+}^{1} \otimes (\widetilde{V}_A)_{[\widetilde{p},\frac{\beta}{2}-1]} \longrightarrow \bigwedge_{+}^{2} \otimes (\widetilde{V}_A)_{[\widetilde{p},\frac{\beta}{2}-2]}$. We compute the kernel. Let $v = w_{34} \otimes p_1 + w_{24} \otimes p_2 + w_{14} \otimes p_3 \in \bigwedge_{+}^{1} \otimes (\widetilde{V}_A)_{[\widetilde{p},\frac{\beta}{2}-1]}$. We have:

$$\begin{aligned} d''(v) =& w_{34} w_{24} \otimes \partial_{2,4} p_1 + w_{34} w_{14} \partial_{1,4} \otimes p_1 + w_{24} w_{34} \otimes \partial_{3,4} p_2 \\ &+ w_{24} w_{14} \otimes \partial_{1,4} p_2 + w_{14} w_{34} \otimes \partial_{3,4} p_3 + w_{14} w_{24} \partial_{2,4} p_3. \end{aligned}$$

This is zero if and only if:

$$\begin{cases} \partial_{2,4} p_1 - \partial_{3,4} p_2 = 0, \\ \partial_{1,4} p_1 - \partial_{3,4} p_3 = 0, \\ \partial_{1,4} p_2 - \partial_{2,4} p_3 = 0. \end{cases}$$

That means that $p_1 = \int \partial_{3,4} p_2 d_{2,4}$, $p_3 = \int \partial_{1,4} p_2 d_{2,4}$, where by $\int p d_{i,j}$ we mean a primitive of p considered as a function in the indeterminate $\partial_i \wedge \partial_j$. Hence, an element of the kernel is:

$$w_{34} \otimes \int \partial_{3,4} p_2 d_{2,4} + w_{24} \otimes p_2 + w_{14} \otimes \int \partial_{1,4} p_2 d_{2,4} = d'' \left(\int p_2 d_{2,4} \right).$$

Thus at this point the sequence is exact.
iii: Let us consider the map $d'' : \bigwedge_{+}^{2} \otimes (\widetilde{V}_A)_{[\widetilde{p},\frac{\beta}{2}-2]} \longrightarrow \bigwedge_{+}^{3} \otimes (\widetilde{V}_A)_{[\widetilde{p},\frac{\beta}{2}-3]}$. We compute the kernel. Let $v = w_{34} w_{24} \otimes p_1 + w_{34} w_{14} \otimes p_2 + w_{24} w_{14} \otimes p_3 \in \bigwedge_{+}^{2} \otimes (\widetilde{V}_A)_{[\widetilde{p},\frac{\beta}{2}-2]}$. We have:

$$d''(v) = w_{34} w_{24} w_{14} \otimes \partial_{1,4} p_1 + w_{34} w_{14} w_{24} \otimes \partial_{2,4} p_2 + w_{24} w_{14} w_{34} \otimes \partial_{3,4} p_3.$$

Therefore $\partial_{1,4} p_1 - \partial_{2,4} p_2 + \partial_{3,4} p_3 = 0$, that is equivalent to $p_3 = \int (-\partial_{1,4} p_1 + \partial_{2,4} p_2) d_{3,4}$. In that case:

$$w_{34} w_{24} \otimes p_1 + w_{34} w_{14} \otimes p_2 + w_{24} w_{14} \otimes p_3 = d'' \left(-w_{24} \otimes \int p_1 d_{3,4} - w_{14} \otimes \int p_2 d_{3,4} \right).$$

Thus at this point the sequence is exact.

iv: Let us consider the map $d'': \Lambda^3_+ \otimes (\widetilde{V}_A)_{[\widetilde{p},\frac{\beta}{2}-3]} \longrightarrow 0$. We have that $\Lambda^3_+ \otimes (\widetilde{V}_A)_{[\widetilde{p},\frac{\beta}{2}-3]} \ni w_{34}w_{24}w_{14} \otimes f = d''\big(w_{34}w_{24} \otimes \int f d_{1,4}\big)$. Thus at this point the sequence is exact.

In the following diagram we use the notation $K_{\widetilde{p}} := \Big\{f \in (\widetilde{V}_A)_{[\widetilde{p},\frac{\beta}{2}]}, \mid \partial_{3,4}f = \partial_{2,4}f = \partial_{1,4}f = 0\Big\}$.
The following is the diagram of the $E^{'1}$ spectral sequence, where the horizontal maps are d' and the vertical maps are d'':

$$\begin{array}{ccccccc}
\Lambda^3_-\Lambda^0_+ \otimes K_{\frac{\alpha-5}{2}} & \longleftarrow & \Lambda^2_-\Lambda^0_+ \otimes K_{\frac{\alpha-3}{2}} & \longleftarrow & \Lambda^1_-\Lambda^0_+ \otimes K_{\frac{\alpha-1}{2}} & \longleftarrow & \Lambda^0_-\Lambda^0_+ \otimes K_{\frac{\alpha+1}{2}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0.
\end{array}$$

The only nonzero row is:

$$0 \overset{d'}{\longleftarrow} \Lambda^3_-\Lambda^0_+ \otimes K_{\frac{\alpha+1}{2}-3} \overset{d'}{\longleftarrow} \Lambda^2_-\Lambda^0_+ \otimes K_{\frac{\alpha+1}{2}-2} \overset{d'}{\longleftarrow} \Lambda^1_-\Lambda^0_+ \otimes K_{\frac{\alpha+1}{2}-1} \overset{d'}{\longleftarrow} \Lambda^0_-\Lambda^0_+ \otimes K_{\frac{\alpha+1}{2}} \overset{d'}{\longleftarrow} 0.$$

We point out that condition $\alpha+\beta > 6$ allows $\widetilde{p}$ to arrive to the value $p = \frac{\alpha+1}{2} - 3$. Indeed, since for all p, q we have $p + q \geq 0$, then $\widetilde{p} + \widetilde{q} \geq \frac{1}{2}$. The condition $\widetilde{p} + \widetilde{q} = \frac{\alpha+1}{2} - 3 + \frac{\beta}{2} > \frac{1}{2}$ is satisfied. We can compute the homology of this row and, with an analogous reasoning to the previous one, we get that the only nonzero row of $E^{'2}$ is:

$$0 \overset{d'}{\longleftarrow} 0 \overset{d'}{\longleftarrow} 0 \overset{d'}{\longleftarrow} 0 \overset{d'}{\longleftarrow} \Lambda^0_-\Lambda^0_+ \otimes K_{\frac{\alpha+1}{2}} \cap \tilde{K}_{\frac{\beta}{2}} \overset{d'}{\longleftarrow} 0.$$

where $\tilde{K}_{\widetilde{q}} = \Big\{f \in (\widetilde{V}_A)_{[\frac{\alpha+1}{2},\widetilde{q}]}, \mid \partial_{1,2}f = \partial_{1,3}f = \partial_{2,3}f = 0\Big\}$. Since for a one row spectral sequence we have that $E^{'2} = E^{'\infty}$, we have:

$$\oplus_{2(n_1+\frac{n_3}{2})=n} H^{n_1,n_3}(G_A(\alpha,\beta)) \cong \sum_{\widetilde{p}+\widetilde{q}=\frac{n+1}{2}} E^{'\infty}_{\widetilde{p},\widetilde{q}} = E^{'\infty}_{\frac{\alpha+1}{2},\frac{\beta}{2}} = \Lambda^0_-\Lambda^0_+ \otimes K_{\frac{\alpha+1}{2}} \cap \tilde{K}_{\frac{\beta}{2}}.$$

Since in $K_{\frac{\alpha+1}{2}} \cap \tilde{K}_{\frac{\beta}{2}}$ we have only elements with $\partial_1^{a_1}\partial_2^{a_2}\partial_3^{a_3}\partial_4^{a_4}$, we have:

$$H^{0,n_3}(G_A(\alpha,\beta)) \cong \Lambda^0_-\Lambda^0_+ \otimes (V_A^{0,n_3})_{[n_3-\beta,\beta]}.$$

2) We now consider the case $\beta > 6$ and $\alpha + \beta \leq 6$, that is $0 \leq \alpha + \beta := h \leq 6$. The computation of $E^{'1}$ is analogous to the previous case and we obtain the same diagram, but the only nonzero row is now:

$$0 \overset{d'}{\longleftarrow} \Lambda_-^{\frac{h-1}{2}}\Lambda^0_+ \otimes K_{\frac{\alpha+1}{2}-\frac{h-1}{2}} \overset{d'}{\longleftarrow} \cdots \overset{d'}{\longleftarrow} \Lambda^1_-\Lambda^0_+ \otimes K_{\frac{\alpha+1}{2}-1} \overset{d'}{\longleftarrow} \Lambda^0_-\Lambda^0_+ \otimes K_{\frac{\alpha+1}{2}} \overset{d'}{\longleftarrow} 0.$$

Indeed condition $0 \leq \alpha+\beta := h$ allows $\widetilde{p}$ to be $\frac{\alpha+1}{2} - \frac{h-1}{2}$. Since for all p, q we have $p+q \geq 0$, then $\widetilde{p} + \widetilde{q} \geq 1$ because it is an integer and it is strictly greater than 0. For $\widetilde{p} = \frac{\alpha+1}{2} - \frac{h-1}{2}$ and $\widetilde{q} = \frac{\beta}{2}$ we have $\widetilde{p} + \widetilde{q} = \frac{\alpha+1}{2} - \frac{h-1}{2} + \frac{\beta}{2} = 1$.
The condition $\widetilde{p} + \widetilde{q} = 1$ means $p + q = 1$, therefore in $K_{\frac{\alpha+1}{2}-\frac{h-1}{2}}$ we have only elements for which $2n_1 + n_3 = 1$, that is $n_1 = 0, n_3 = 1$. This means that there are only elements constant in $\partial_i \wedge \partial_j$, that lie in the image of d'. Indeed for $h = 3$, $\gamma \in K_{\frac{\alpha+1}{2}-\frac{h-1}{2}}$:

$$w_{12} \otimes \gamma = d'(\gamma(\partial_1 \wedge \partial_2)),$$

$$w_{13} \otimes \gamma = d'(\gamma(\partial_1 \wedge \partial_3)),$$
$$w_{23} \otimes \gamma = d'(\gamma(\partial_2 \wedge \partial_3)).$$

For $h = 5$, $\gamma \in K_{\frac{\alpha+1}{2} - \frac{h-1}{2}}$:

$$w_{12}w_{13} \otimes \gamma = d'(-w_{13} \otimes \gamma(\partial_1\partial_2)),$$
$$w_{12}w_{23} \otimes \gamma = d'(w_{12} \otimes \gamma(\partial_2\partial_3)),$$
$$w_{13}w_{23} \otimes \gamma = d'(w_{13} \otimes \gamma(\partial_2\partial_3)).$$

Then we can conclude as in the previous case.

3) We now focus on $0 \leq \beta \leq 6$ and $\alpha + \beta > 6$. We use the first spectral sequence and obtain the product of $\bigwedge_-^{\frac{\alpha-(2\tilde{p}-1)}{2}}$ with the complex:

$$0 \xleftarrow{d''} \bigwedge_+^{\frac{\beta}{2}} \otimes (\widetilde{V}_A)_{[\tilde{p},0]} \xleftarrow{d''} \bigwedge_+^{\frac{\beta}{2}-1} \otimes (\widetilde{V}_A)_{[\tilde{p},1]} \xleftarrow{d''} \dots \xleftarrow{d''} \bigwedge_+^{0} \otimes (\widetilde{V}_A)_{[\tilde{p},\frac{\beta}{2}]} \xleftarrow{d''} 0.$$

In $\bigwedge_+^{\frac{\beta}{2}} \otimes (\widetilde{V}_A)_{[\tilde{p},0]}$ the sequence is exact since $\tilde{q} = 0$ implies that the polynomials are constant in $\partial_1 \wedge \partial_4$, $\partial_2 \wedge \partial_4$, $\partial_3 \wedge \partial_4$, thus they lie in the image of d''. Therefore we obtain the following diagram for E'^1, where the horizontal maps are d' and the vertical maps are d'':

$$\begin{array}{ccccccc}
\bigwedge_-^3\bigwedge_+^0 \otimes K_{\frac{\alpha-5}{2}} & \longleftarrow & \bigwedge_-^2\bigwedge_+^0 \otimes K_{\frac{\alpha-3}{2}} & \longleftarrow & \bigwedge_-^1\bigwedge_+^0 \otimes K_{\frac{\alpha-1}{2}} & \longleftarrow & \bigwedge_-^0\bigwedge_+^0 \otimes K_{\frac{\alpha+1}{2}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0.
\end{array}$$

In particular the only nonzero row is:

$$0 \xleftarrow{d'} \bigwedge_-^3\bigwedge_+^0 \otimes K_{\frac{\alpha-5}{2}} \xleftarrow{d'} \bigwedge_-^2\bigwedge_+^0 \otimes K_{\frac{\alpha-3}{2}} \xleftarrow{d'} \bigwedge_-^1\bigwedge_+^0 \otimes K_{\frac{\alpha-1}{2}} \xleftarrow{d'} \bigwedge_-^0\bigwedge_+^0 \otimes K_{\frac{\alpha+1}{2}} \xleftarrow{d'} 0.$$

We can conclude in the same way.

4) We now focus on $0 \leq \beta \leq 6$ and $0 \leq \alpha + \beta = h \leq 6$. We can use the same reasoning as before, we use the first spectral sequence and obtain the product of $\bigwedge_-^{\frac{\alpha-(2\tilde{p}-1)}{2}}$ with the complex:

$$0 \xleftarrow{d''} \bigwedge_+^{\frac{\beta}{2}} \otimes (\widetilde{V}_A)_{[\tilde{p},0]} \xleftarrow{d''} \bigwedge_+^{\frac{\beta}{2}-1} \otimes (\widetilde{V}_A)_{[\tilde{p},1]} \xleftarrow{d''} \dots \xleftarrow{d''} \bigwedge_+^{0} \otimes (\widetilde{V}_A)_{[\tilde{p},\frac{\beta}{2}]} \xleftarrow{d''} 0.$$

Again in $\bigwedge_+^{\frac{\beta}{2}} \otimes (\widetilde{V}_A)_{[\tilde{p},0]}$ the sequence is exact since $\tilde{q} = 0$ implies that the polynomials are constant in $\partial_1 \wedge \partial_4$, $\partial_2 \wedge \partial_4$, $\partial_3 \wedge \partial_4$, thus they lie in the image of d''. Therefore we obtain the following diagram for E'^1, where the horizontal maps are d' and the vertical maps are d'':

$$\begin{array}{ccccccccc}
\bigwedge_-^{\frac{h-1}{2}}\bigwedge_+^0 \otimes K_{\frac{\alpha+1}{2}-\frac{h-1}{2}} & \leftarrow & \cdots & \leftarrow & \bigwedge_-^k\bigwedge_+^0 \otimes K_{\frac{\alpha+1-2k}{2}} & \leftarrow & \cdots & \leftarrow & \bigwedge_-^0\bigwedge_+^0 \otimes K_{\frac{\alpha+1}{2}} \\
\downarrow & & & & \downarrow & & & & \downarrow \\
0 & \leftarrow & \cdots & \leftarrow & 0 & \leftarrow & \cdots & \leftarrow & 0 \\
\downarrow & & & & \downarrow & & & & \downarrow \\
\vdots & & & & \vdots & & & & \vdots \\
\downarrow & & & & \downarrow & & & & \downarrow \\
0 & \leftarrow & \cdots & \leftarrow & 0 & \leftarrow & \cdots & \leftarrow & 0.
\end{array}$$

The only nonzero row is:

$$0 \xleftarrow{d'} \Lambda_-^{\frac{h-1}{2}}\Lambda_+^0 \otimes K_{\frac{\alpha+1}{2}-\frac{h-1}{2}} \xleftarrow{d'} \cdots \xleftarrow{d'} \Lambda_-^1\Lambda_+^0 \otimes K_{\frac{\alpha+1}{2}-1} \xleftarrow{d'} \Lambda_-^0\Lambda_+^0 \otimes K_{\frac{\alpha+1}{2}} \xleftarrow{d'} 0.$$

We conclude as in case 2).

B) We now analyze the case of α even. We point out that p can assume only even values. We modify the bigrading in order to obtain a new bigrading and use the theory of spectral sequences. For every p,q we denote by $\widetilde{p} = \frac{p}{2}, \widetilde{q} = \frac{q}{2}$. We will denote $(\widetilde{V}_A)_{[\widetilde{p},\widetilde{q}]} = (V_A)_{[p,q]}$. We denote $\widetilde{G}_A(\alpha,\beta)_{[\widetilde{p},\widetilde{q}]} = G_A(\alpha,\beta)_{[p,q]}$. We now have that $d' : \widetilde{G}_A(\alpha,\beta)_{[\widetilde{p},\widetilde{q}]} \longrightarrow \widetilde{G}_A(\alpha,\beta)_{[\widetilde{p}-1,\widetilde{q}]}$ and $d'' : \widetilde{G}_A(\alpha,\beta)_{[\widetilde{p},\widetilde{q}]} \longrightarrow \widetilde{G}_A(\alpha,\beta)_{[\widetilde{p},\widetilde{q}-1]}$. We have still that $G_A(\alpha,\beta) = \oplus_{\widetilde{p},\widetilde{q}}\widetilde{G}_A(\alpha,\beta)_{[\widetilde{p},\widetilde{q}]}$. We split into four subcases.

1) We first consider the case $\beta > 6$ and $\alpha+\beta > 6$. We use the theory of spectral sequences of bicomplexes. Let us consider $G_A(\alpha,\beta)$ with the differential d'':

$$\xleftarrow{d''} \Lambda_-^{\frac{\alpha-2\widetilde{p}}{2}}\Lambda_+^{\frac{\beta-2\widetilde{q}}{2}+1} \otimes (\widetilde{V}_A)_{[\widetilde{p},\widetilde{q}-1]} \xleftarrow{d''} \Lambda_-^{\frac{\alpha-2\widetilde{p}}{2}}\Lambda_+^{\frac{\beta-2\widetilde{q}}{2}} \otimes (\widetilde{V}_A)_{[\widetilde{p},\widetilde{q}]} \xleftarrow{d''} \Lambda_-^{\frac{\alpha-2\widetilde{p}}{2}}\Lambda_+^{\frac{\beta-2\widetilde{q}}{2}-1} \otimes (\widetilde{V}_A)_{[\widetilde{p},\widetilde{q}+1]} \xleftarrow{d''} .$$

It is the tensor product of $\Lambda_-^{\frac{\alpha-2\widetilde{p}}{2}}$ and the following complex, since $\Lambda_-^{\frac{\alpha-2\widetilde{p}}{2}}$ is not involved in d'':

$$0 \xleftarrow{d''} \Lambda_+^3 \otimes (\widetilde{V}_A)_{[\widetilde{p},\frac{\beta}{2}-3]} \xleftarrow{d''} \Lambda_+^2 \otimes (\widetilde{V}_A)_{[\widetilde{p},\frac{\beta}{2}-2]} \xleftarrow{d''} \Lambda_+^1 \otimes (\widetilde{V}_A)_{[\widetilde{p},\frac{\beta}{2}-1]} \xleftarrow{d''} \Lambda_+^0 \otimes (\widetilde{V}_A)_{[\widetilde{p},\frac{\beta}{2}]} \xleftarrow{d''} 0.$$

Note that β is always even. We have that condition $\beta > 6$ assures that for $\widetilde{q}$ the value $\frac{\beta}{2}-3$ is acceptable since it is positive. This complex is exact except for the right end, the computations are the same as in case A1.
In the following diagram we use the notation $K_{\widetilde{p}} := \left\{ f \in (\widetilde{V}_A)_{[\widetilde{p},\frac{\beta}{2}]}, \mid \partial_{3,4}f = \partial_{2,4}f = \partial_{1,4}f = 0 \right\}$.
The following is the diagram of the $E^{'1}$ spectral sequence, where the horizontal maps are d' and the vertical maps are d'':

$$\begin{array}{ccccccc}
\Lambda_-^3\Lambda_+^0 \otimes K_{\frac{\alpha-6}{2}} & \longleftarrow & \Lambda_-^2\Lambda_+^0 \otimes K_{\frac{\alpha-4}{2}} & \longleftarrow & \Lambda_-^1\Lambda_+^0 \otimes K_{\frac{\alpha-2}{2}} & \longleftarrow & \Lambda_-^0\Lambda_+^0 \otimes K_{\frac{\alpha}{2}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0.
\end{array}$$

The only nonzero row is:

$$0 \xleftarrow{d'} \Lambda_-^3\Lambda_+^0 \otimes K_{\frac{\alpha}{2}-3} \xleftarrow{d'} \Lambda_-^2\Lambda_+^0 \otimes K_{\frac{\alpha}{2}-2} \xleftarrow{d'} \Lambda_-^1\Lambda_+^0 \otimes K_{\frac{\alpha}{2}-1} \xleftarrow{d'} \Lambda_-^0\Lambda_+^0 \otimes K_{\frac{\alpha}{2}} \xleftarrow{d'} 0.$$

We point out that condition $\alpha+\beta > 6$ allows $\widetilde{p}$ to arrive to the value $p = \frac{\alpha}{2}-3$. Indeed, since for all p,q we have $p+q \geq 0$, then $\widetilde{p}+\widetilde{q} \geq 0$. The condition $\widetilde{p}+\widetilde{q} = \frac{\alpha}{2}-3+\frac{\beta}{2} > 0$ is satisfied. We can compute the homology of this row and, with an analogous reasoning to the previous one, we get that the only nonzero row of $E^{'2}$ is:

$$0 \xleftarrow{d'} 0 \xleftarrow{d'} 0 \xleftarrow{d'} 0 \xleftarrow{d'} \Lambda_-^0\Lambda_+^0 \otimes K_{\frac{\alpha}{2}} \cap \tilde{K}_{\frac{\beta}{2}} \xleftarrow{d'} 0,$$

where $\tilde{K}_{\widetilde{q}} = \left\{ f \in (\widetilde{V}_A)_{[\frac{\alpha}{2},\widetilde{q}]}, \mid \partial_{1,2} f = \partial_{1,3} f = \partial_{2,3} f = 0 \right\}$. Since for a one row spectral sequence $E^{'2} = E^{'\infty}$, we have:

$$\oplus_{2(n_1+\frac{n_3}{2})=n} H^{n_1,n_3}(G_A(\alpha,\beta)) \cong \sum_{\widetilde{p}+\widetilde{q}=\frac{n}{2}} E^{'\infty}_{\widetilde{p},\widetilde{q}} = E^{'\infty}_{\frac{\alpha}{2},\frac{\beta}{2}} = \wedge^0_- \wedge^0_+ \otimes K_{\frac{\alpha}{2}} \cap \tilde{K}_{\frac{\beta}{2}}.$$

Since in $K_{\frac{\alpha}{2}} \cap \tilde{K}_{\frac{\beta}{2}}$ we have only elements with $\partial_1^{a_1}\partial_2^{a_2}\partial_3^{a_3}\partial_4^{a_4}$, we have:

$$H^{0,n_3}(G_A(\alpha,\beta)) \cong \wedge^0_- \wedge^0_+ \otimes (V_A^{0,n_3})_{[n_3-\beta,\beta]}.$$

2) We now consider the case $\beta > 6$ and $\alpha+\beta \leq 6$, that is $0 \leq \alpha+\beta := h \leq 6$. The computation of $E^{'1}$ is analogous to the previous case and we obtain the same diagram, but the only nonzero row is now:

$$0 \xleftarrow{d'} \wedge_-^{\frac{h}{2}} \wedge^0_+ \otimes K_{\frac{\alpha}{2}-\frac{h}{2}} \xleftarrow{d'} \cdots \xleftarrow{d'} \wedge^1_- \wedge^0_+ \otimes K_{\frac{\alpha}{2}-1} \xleftarrow{d'} \wedge^0_- \wedge^0_+ \otimes K_{\frac{\alpha}{2}} \xleftarrow{d'} 0.$$

Indeed condition $0 \leq \alpha+\beta := h$ allows $\widetilde{p}$ to be $\frac{\alpha}{2}-\frac{h}{2}$. Since for all p,q we have $p+q \geq 0$, then $\widetilde{p}+\widetilde{q} \geq 0$. For $\widetilde{p} = \frac{\alpha}{2}-\frac{h}{2}$ and $\widetilde{q} = \frac{\beta}{2}$ we have $\widetilde{p}+\widetilde{q} = 0$.
The condition $\widetilde{p}+\widetilde{q} = 0$ means $p+q=0$, therefore in $K_{\frac{\alpha}{2}-\frac{h}{2}}$ we have only elements for which $2n_1+n_3 = 0$, that is $n_1 = 0, n_3 = 0$. This means that in $\in K_{\frac{\alpha}{2}-\frac{h}{2}}$ there are only constant elements, that lie in the image of d', since for $h=2$, $\gamma \in K_{\frac{\alpha}{2}-\frac{h}{2}}$:

$$\begin{aligned} w_{12} \otimes \gamma &= d'(\gamma(\partial_1 \wedge \partial_2)), \\ w_{13} \otimes \gamma &= d'(\gamma(\partial_1 \wedge \partial_3)), \\ w_{23} \otimes \gamma &= d'(\gamma(\partial_2 \wedge \partial_3)). \end{aligned}$$

For $h=4$, $\gamma \in K_{\frac{\alpha}{2}-\frac{h}{2}}$:

$$\begin{aligned} w_{12}w_{13} \otimes \gamma &= d'(-w_{13} \otimes \gamma(\partial_1 \wedge \partial_2)), \\ w_{12}w_{23} \otimes \gamma &= d'(w_{12} \otimes \gamma(\partial_2 \wedge \partial_3)), \\ w_{13}w_{23} \otimes \gamma &= d'(w_{13} \otimes \gamma(\partial_2 \wedge \partial_3)). \end{aligned}$$

Then we can conclude as in the previous case.

3) We now focus on $0 \leq \beta \leq 6$ and $\alpha+\beta > 6$. We use the first spectral sequence and obtain the product of $\wedge_-^{\frac{\alpha-2\widetilde{p}}{2}}$ with the complex:

$$0 \xleftarrow{d''} \wedge_+^{\frac{\beta}{2}} \otimes (\widetilde{V}_A)_{[\widetilde{p},0]} \xleftarrow{d''} \wedge_+^{\frac{\beta}{2}-1} \otimes (\widetilde{V}_A)_{[\widetilde{p},1]} \xleftarrow{d''} \ldots \xleftarrow{d''} \wedge^0_+ \otimes (\widetilde{V}_A)_{[\widetilde{p},\frac{\beta}{2}]} \xleftarrow{d''} 0.$$

In $\wedge_+^{\frac{\beta}{2}} \otimes (\widetilde{V}_A)_{[\widetilde{p},0]}$ the sequence is exact since $\widetilde{q} = 0$ implies that the polynomials are constant in $\partial_1 \wedge \partial_4$, $\partial_2 \wedge \partial_4$, $\partial_3 \wedge \partial_4$, thus they lie in the image of d''. Therefore we obtain the following diagram for $E^{'1}$, where the horizontal maps are d' and the vertical maps are d'':

$$\begin{array}{ccccccc}
\wedge^3_- \wedge^0_+ \otimes K_{\frac{\alpha-6}{2}} & \longleftarrow & \wedge^2_- \wedge^0_+ \otimes K_{\frac{\alpha-4}{2}} & \longleftarrow & \wedge^1_- \wedge^0_+ \otimes K_{\frac{\alpha-2}{2}} & \longleftarrow & \wedge^0_- \wedge^0_+ \otimes K_{\frac{\alpha}{2}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0.
\end{array}$$

In particular the only nonzero row is:

$$0 \xleftarrow{d'} \Lambda_-^3\Lambda_+^0 \otimes K_{\frac{\alpha}{2}-3} \xleftarrow{d'} \Lambda_-^2\Lambda_+^0 \otimes K_{\frac{\alpha}{2}-2} \xleftarrow{d'} \Lambda_-^1\Lambda_+^0 \otimes K_{\frac{\alpha}{2}-1} \xleftarrow{d'} \Lambda_-^0\Lambda_+^0 \otimes K_{\frac{\alpha}{2}} \xleftarrow{d'} 0.$$

We can conclude in the same way.

4) We now focus on $0 \leq \beta \leq 6$ and $0 \leq \alpha+\beta = h \leq 6$. We can use the same reasoning as before, we use the first spectral sequence and obtain the product of $\Lambda_-^{\frac{\alpha-2\widetilde{p}}{2}}$ with the complex:

$$0 \xleftarrow{d''} \Lambda_+^{\frac{\beta}{2}} \otimes (\widetilde{V}_A)_{[\widetilde{p},0]} \xleftarrow{d''} \Lambda_+^{\frac{\beta}{2}-1} \otimes (\widetilde{V}_A)_{[\widetilde{p},1]} \xleftarrow{d''} \dots \xleftarrow{d''} \Lambda_+^0 \otimes (\widetilde{V}_A)_{[\widetilde{p},\frac{\beta}{2}]} \xleftarrow{d''} 0.$$

Again in $\Lambda_+^{\frac{\beta}{2}} \otimes (\widetilde{V}_A)_{[\widetilde{p},0]}$ the sequence is exact since $\widetilde{q} = 0$ implies that the polynomials are constant in $\partial_1 \wedge \partial_4$, $\partial_2 \wedge \partial_4$, $\partial_3 \wedge \partial_4$, thus they lie in the image of d''. Therefore we obtain the following diagram for E'^1, where the horizontal maps are d' and the vertical maps are d'':

$$\begin{array}{ccccccccc}
\Lambda_-^{\frac{h}{2}}\Lambda_+^0 \otimes K_{\frac{\alpha-h}{2}} & \leftarrow\cdots\leftarrow & & & \Lambda_-^{k}\Lambda_+^0 \otimes K_{\frac{\alpha-2k}{2}} & \leftarrow\cdots\leftarrow & & & \Lambda_-^{0}\Lambda_+^0 \otimes K_{\frac{\alpha}{2}} \\
\downarrow & & & & \downarrow & & & & \downarrow \\
0 & \leftarrow & \cdots & \leftarrow & 0 & \leftarrow & \cdots & \leftarrow & 0 \\
\downarrow & & & & \downarrow & & & & \downarrow \\
\vdots & & & & \vdots & & & & \vdots \\
\downarrow & & & & \downarrow & & & & \downarrow \\
0 & \leftarrow & \cdots & \leftarrow & 0 & \leftarrow & \cdots & \leftarrow & 0.
\end{array}$$

The only nonzero row is:

$$0 \xleftarrow{d'} \Lambda_-^{\frac{h}{2}}\Lambda_+^0 \otimes K_{\frac{\alpha}{2}-\frac{h}{2}} \xleftarrow{d'} \cdots \xleftarrow{d'} \Lambda_-^1\Lambda_+^0 \otimes K_{\frac{\alpha}{2}-1} \xleftarrow{d'} \Lambda_-^0\Lambda_+^0 \otimes K_{\frac{\alpha}{2}} \xleftarrow{d'} 0.$$

We conclude as in case 2).

□

By Lemma 6.17 and decomposition $G_A = \oplus_{\alpha,\beta} G_A(\alpha,\beta)$ we obtain the following result.

Lemma 6.18. *As $\mathfrak{g}_0-$modules:*

$$H^{n_1,n_3}(G_A) \cong 0 \quad \textit{if } n_1 > 0.$$

$$\frac{G_A^{0,n_3}}{\operatorname{Im}\nabla_A} \cong \oplus_{\alpha+\beta=n_3}\Lambda_-^0\Lambda_+^0 \otimes (V_A^{0,n_3})_{[\alpha,\beta]} = V_A^{0,n_3}.$$

Proposition 6.19. *As $\mathfrak{g}_0-$modules:*

$$H^{n_1,n_3}(G_{A^\circ}) \cong \begin{cases} V_A^{0,0} \cong \mathbb{C} & \textit{if } n_1 = n_3 = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. By Lemma 6.18 and Remark 6.15 we obtain that $H^{n_1,n_3}(G_{A^\circ}) \cong 0$ if $n_1 > 0$.
In Lemma 6.18 we computed $\frac{G_A^{0,n_3}}{\operatorname{Im}(\nabla_A : G_A^{1,n_3} \longrightarrow G_A^{0,n_3})}$, but we are interested in the homology of G_{A°. We have that the Kernel of the map induced by ∇_3 between $H^{0,n_3}(G_A)$ and $H^{0,n_3-2}(G_B)$, with $n_3 > 1$, is actually isomorphic to $\frac{\operatorname{Ker}(\nabla_3 : G_A^{0,n_3} \to G_B^{0,n_3-2})}{\operatorname{Im}(\nabla_A : G_A^{1,n_3} \longrightarrow G_A^{0,n_3})} \cong \operatorname{Ker}(\nabla_3 : G_A^{0,n_3} \to G_B^{0,n_3-2}) \cap V_A^{0,n_3}$. We show that it is 0. Indeed it is sufficient to show that ∇_3 restricted to V_A^{0,n_3} is injective, but this comes from the fact that $\nabla_3(\partial_4^{n_3}) \neq 0$, where $\partial_4^{n_3}$ is a highest weight of V_A^{0,n_3}. The same argument holds for $H^{0,1}(G_A)$ and ∇_5. □

Finally we can show the following result, that follows from Propositions 6.19 and 6.14.

Proposition 6.20. *As* $\mathfrak{g}_0-$*modules:*

$$H^{n_1,n_3}(M_A) \cong \begin{cases} \mathbb{C} & \textit{if } n_1 = n_3 = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. By Propositions 6.19 and 6.14 we obtain that, for $(n_1, n_3) \neq (0,0)$, $H^{n_1,n_3}(M_A) \cong 0$.
We point out that the singular vectors that determine the maps ∇_A for $n_3 = 0$ are singular vectors also in the case of K_6 (see [BKL2, Theorem 4.1, Remark 4.2] and [BKL1, Theorem 5.1]). Since the maps $\nabla_A : M_A^{n_1,0} \to M_A^{n_1-1,0}$ are completely determined by the image of v, highest weight vector of $V_A^{n_1,0}$, and, due to equivariance, the action of $\mathfrak{g}_{\leq 0}$ on v, we obtain that for $n_3 = 0$ the maps coincide with the maps in the case of K_6. The homology in this case was computed in [BKL1]. It was shown that it is different from zero only for $n_1 = n_3 = 0$ and $H^{0,0}(M_A) \cong \mathbb{C}$. □

Now we focus on the third quadrant.

Remark 6.21. A consequence of results in [CCK1] on conformal duality is that, in the case of CK_6, the conformal dual of $\mathrm{Ind}(F)$, where $F = F(n_0, n_1\lambda_1 + n_2\lambda_2 + n_3\lambda_3)$ is an irreducible $\mathfrak{g}_0-$module, corresponds to the shifted dual $\mathrm{Ind}(F^\vee)$, where $F^\vee \cong F(-n_0 + 4, n_1\lambda_1 + n_3\lambda_2 + n_2\lambda_3)$. We will use the results about duality for shifted duals.

Proposition 6.22. *As* $\mathfrak{g}_0-$*modules:*

$$H^{n_1,n_2}(M_C) = \begin{cases} \mathbb{C} & if \;\; (n_1, n_2) = (1,0), \\ 0 & if \;\; (n_1, n_2) = (0,0) \;\; or \;\; n_1 > 0. \end{cases}$$

Proof. We point out that the singular vectors that determine the maps ∇_C for $n_2 = 0$ are singular vectors also in the case of K_6 (see [BKL2, Theorem 4.1, Remark 4.2] and [BKL1, Theorem 5.1]). Since the maps $\nabla_C : M_C^{n_1,0} \to M_C^{n_1+1,0}$ are completely determined by the image of v, highest weight vector of $V_C^{n_1,0}$, and, due to equivariance, the action of $\mathfrak{g}_{\leq 0}$ on v, we obtain that for $n_2 = 0$ the maps coincide with the maps in the case of K_6. The homology in this case was computed in [BKL1]. We know that:

$$H^{n_1,0}(M_C) = \begin{cases} \mathbb{C} & \text{for } n_1 = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We can use duality to compute the remaining homology spaces for the third quadrant for $n_1 > 0$ and $n_2 > 0$. Indeed we have that, for $n_1 > 0$, $n_2 > 0$, the maps

$$M_C^{n_1-1,n_2} \xrightarrow{\nabla_C} M_C^{n_1,n_2} \xrightarrow{\nabla_C} M_C^{n_1+1,n_2}$$

are dual to:

$$M_A^{n_1+1,n_3} \xrightarrow{\nabla_A} M_A^{n_1,n_3} \xrightarrow{\nabla_A} M_A^{n_1-1,n_3},$$

where $n_3 = n_2$. We showed that the previous sequence is exact in $M_A^{n_1,n_3}$ and $M_A^{n_1-1,n_3}$. therefore we have that $\frac{M_A^{n_1,n_2}}{\mathrm{Im}(\nabla_A)} \cong \frac{M_A^{n_1,n_2}}{\mathrm{Ker}(\nabla_A)}$ is isomorphic to a submodule of a free module, hence it is a finitely generated torsion free $\mathbb{C}[\Theta]-$module. The same holds for $\frac{M_A^{n_1-1,n_2}}{\mathrm{Im}(\nabla_A)}$. Hence, by Remark 6.21 and Proposition 1.19, we obtain exactness in $M_C^{n_1,n_2}$ for $n_1 > 0$, $n_2 > 0$. □

Appendix A

Appendix

Let $\mathfrak{g} = \mathcal{A}(K_4')$. Let us show that $\mathfrak{g}_{>0}$ is generated by $\mathfrak{g}_1 = \langle t\xi_l, \xi_i\xi_j\xi_k,\ 1 \leq l,i,j,k \leq 4,\ i<j<k \rangle$. It is straightforward that $\mathfrak{g}_1^2 = \mathfrak{g}_2 = \langle \xi_1\xi_2\xi_3\xi_4, t^2, t\xi_i\xi_j \rangle$, indeed:

$$\begin{aligned} \xi_1\xi_2\xi_3\xi_4 &= [t\xi_1, \xi_2\xi_3\xi_4], \\ t^2 &= -[t\xi_1, t\xi_1], \\ t\xi_i\xi_j &= -[t\xi_k, \xi_k\xi_i\xi_j]. \end{aligned}$$

Now, by induction, we show that if $\mathfrak{g}_1^{i-1} = \mathfrak{g}_{i-1}$ then $\mathfrak{g}_1^i = \mathfrak{g}_i$ for $i > 2$. If $i = 2k-1$, then $\mathfrak{g}_i = \langle t^l \xi_{j_1} \cdots \xi_{j_s} \rangle$ with $2l+s-2 = i$ and s odd, that is $s=1$ or $s=3$. We have:

$$t^l\xi_{j_1}\cdots\xi_{j_s} = -[t\xi_p, t^{l-1}\xi_p\xi_{j_1}\xi_{j_s}] \ \text{ for } \ p \in \{1,2,3,4\} \setminus \{j_1, ..., j_s\}.$$

If $i = 2k$, then $\mathfrak{g}_i = \langle t^l \xi_{j_1} \cdots \xi_{j_s} \rangle$ with $2l+s-2 = i$ and s even. We have that:

$$\begin{aligned} &t^{k+1} = -[t^k\xi_1, t\xi_1], \\ &t^l\xi_{j_1}\xi_{j_2} = -[t\xi_p, t^{l-1}\xi_p\xi_{j_1}\xi_{j_2}] \ \text{ for } \ p \in \{1,2,3,4\} \setminus \{j_1, j_2\}, \\ &t^l\xi_1\xi_2\xi_3\xi_4 = \frac{1}{-l-1}[t^l\xi_1\xi_2\xi_3, t\xi_4]. \end{aligned}$$

We point out that in this last case $-l-1 \neq 0$ since l is non$-$negative.
Finally we show that $\mathfrak{g}_1 = S_1 \oplus S_2$, with $S_1 = \langle t\xi_1, t\xi_2, t\xi_3, t\xi_4 \rangle$ and $S_2 = \langle \xi_1\xi_2\xi_3, \xi_1\xi_2\xi_4, \xi_1\xi_3\xi_4, \xi_2\xi_3\xi_4 \rangle$ that are irreducible $\mathfrak{g}_0-$modules. It is obvious that they are modules, indeed t and C act as scalars and:

$$[\xi_i\xi_j, t\xi_k] = \begin{cases} 0 \ \text{ if } \ k \neq i,j, \\ t\xi_j \ \text{ if } \ k = i, \\ -t\xi_i \ \text{ if } \ k = j. \end{cases}$$

From the last computation it is also clear that S_1 is irreducible. Let $I = (i,j)$ and J with $|J| = 3$. We have that:

$$[\xi_I, \xi_J] = \chi_{|I\cap J|=1}\partial_{I\cap J}\xi_I\partial_{I\cap J}\xi_J.$$

From the last computation it is also clear that S_2 is irreducible. Finally we show that $t\xi_1 + it\xi_2$ is a lowest weight vector of S_1 and $\xi_1\xi_3\xi_4 + i\xi_2\xi_3\xi_4$ is a lowest weight vector of S_2. Indeed:

$$\begin{aligned} [H_1, t\xi_1 + it\xi_2] &= [-i\xi_1\xi_2, t\xi_1 + it\xi_2] = -it\xi_2 - t\xi_1, \\ [H_2, t\xi_1 + it\xi_2] &= [-i\xi_3\xi_4, t\xi_1 + it\xi_2] = 0, \end{aligned}$$

$$
\begin{aligned}
[E_{-(\varepsilon_1-\varepsilon_2)}, t\xi_1 + it\xi_2] &= [-\xi_1\xi_3 - \xi_2\xi_4 + i\xi_1\xi_4 - i\xi_2\xi_3, t\xi_1 + it\xi_2] \\
&= -t\xi_3 - it\xi_4 + it\xi_4 + t\xi_3 = 0, \\
[E_{-(\varepsilon_1+\varepsilon_2)}, t\xi_1 + it\xi_2] &= [-\xi_1\xi_3 + \xi_2\xi_4 - i\xi_1\xi_4 - i\xi_2\xi_3, t\xi_1 + it\xi_2] \\
&= -t\xi_3 + it\xi_4 - it\xi_4 + t\xi_3 = 0, \\
[H_1, \xi_1\xi_3\xi_4 + i\xi_2\xi_3\xi_4] &= [-i\xi_1\xi_2, \xi_1\xi_3\xi_4 + i\xi_2\xi_3\xi_4] = -i\xi_2\xi_3\xi_4 - \xi_1\xi_3\xi_4, \\
[H_2, \xi_1\xi_3\xi_4 + i\xi_2\xi_3\xi_4] &= [-i\xi_3\xi_4, \xi_1\xi_3\xi_4 + i\xi_2\xi_3\xi_4] = 0, \\
[E_{-(\varepsilon_1-\varepsilon_2)}, \xi_1\xi_3\xi_4 + i\xi_2\xi_3\xi_4] &= [-\xi_1\xi_3 - \xi_2\xi_4 + i\xi_1\xi_4 - i\xi_2\xi_3, \xi_1\xi_3\xi_4 + i\xi_2\xi_3\xi_4] \\
&= -i\xi_1\xi_2\xi_4 + \xi_2\xi_1\xi_3 + \xi_1\xi_2\xi_3 - i\xi_2\xi_1\xi_4 = 0, \\
[E_{-(\varepsilon_1+\varepsilon_2)}, \xi_1\xi_3\xi_4 + i\xi_2\xi_3\xi_4] &= [-\xi_1\xi_3 + \xi_2\xi_4 - i\xi_1\xi_4 - i\xi_2\xi_3, \xi_1\xi_3\xi_4 + i\xi_2\xi_3\xi_4] \\
&= -i\xi_1\xi_2\xi_4 - \xi_2\xi_1\xi_3 - \xi_1\xi_2\xi_3 - i\xi_2\xi_1\xi_4 = 0.
\end{aligned}
$$

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Acknowledgments

First of all I would like to thank my supervisors prof. Nicoletta Cantarini and prof. Fabrizio Caselli. They guided me during these three years of PhD and answered, with patience, to my several doubts; their suggestions and advices were fundamental for this thesis.

I am deeply grateful to professor Victor Kac, that hosted me at MIT; his precious suggestions and comments guided my research and were crucial for this thesis.

I thank my colleagues of PhD, whose support has been fundamental, especially in the most difficult moments of these years. In particular I want to thank the colleagues of the 'last compartment' of the eighth floor: they always had time to listen, give advices and have a laugh; this journey of three years has been a mix of emotions and their support made it easier and funnier.

Finally, I want to thank my parents and my sister Annalisa for their support and for always believing in my abilities, even when I didn't believe it myself.