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ROBUST CONTROLLERS DESIGN FOR UNKNOWN ERROR AND EXOSYSTEM: A HYBRID OPTIMIZATION AND OUTPUT REGULATION APPROACH

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Abstract

This thesis addresses the problem of robustness in control in two main topics: linear output regulation when no knowledge is assumed of the modes of the exosystem, and hybrid gradient-free optimization. A framework is presented for the solution of the first problem, in which asymptotic regulation is achieved in case of a persistence of excitation condition. The stability properties of the closed-loop system are proved under a small-gain argument with no minimum phase assumption. The second part of the thesis addresses, and proposes, a solution to the gradient-free optimization problem, solved by a discrete-time direct search algorithm. The algorithm is shown to convergence to the set of minima of a particular class of non convex functions. It is, then, applied considering it coupled with a continuous-time dynamical system. A hybrid controller is developed in order to guarantee convergence to the set of minima and stability of the interconnection of the two systems. Almost global asymptotic is proven for the proposed hybrid controller. Shown to not be robust to any bounded measurement noise, a robust solution is also proposed. The aim of this thesis is to lay the ground for a solution of the output regulation problem in case the error is unknown, but a proxy optimization function is available. A controller embedding the characteristics of the two proposed approaches, as a main solution to the aforementioned problem, will be the focus of future studies.

Sommario

Il seguente lavoro di tesi studia il problema della robustezza sotto due principali punti di vista: regolazione dell'uscita di un sistema lineare nel caso in cui le dinamiche dell'esosistema siano sconosciute, e ottimizzazione ibrida in assenza di informazioni sul gradiente e sulla funzione di costo. Un framework viene presentato per la soluzione del primo problema, in cui regolazione asintotica viene raggiunta nel caso in cui sia verificata una condizione di persistenza delle eccitazioni. La stabilità del sistema in catena chiusa viene dimostrata grazie ad una condizione di piccolo guadagno, senza alcuna assunzione di fase minima. La seconda parte della tesi tratta, e propone, una soluzione al problema di ottimizzazione senza l'utilizzo del gradiente o informazioni sulla funzione di costo. La soluzione proposta adotta un algoritmo a ricerca diretta a tempo discreto. Convergenza per una classe di funzioni non convesse è dimostrata. L'algoritmo proposto è poi applicato ad un sistema dinamico a tempo continuo. A tal proposito, un regolatore ibrido viene sviluppato per l'interconnesione dei due sistemi, dimostrando stabilità asintotica quasi globale del sistema in catena chiusa. Tuttavia, dimostrato non robusto, un trade-off tra stabilità asintotica e robustezza viene reso evidente, ed una soluzione robusta viene proposta. L'obiettivo di questo lavoro di tesi è di gettare le basi per una solutione al problema di regolazione dell'uscita nel caso in cui l'errore non sia misurabile, ma un suo proxy, sottoforma di funzione da ottimizzare, sia disponibile. Un controllore che unifichi gli approcci studiati in questi tesi sarà la soluzione principale studiata nell'immediato futuro a tale problema.

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Contents

Notation

$\mathbb{R}$	set of real numbers
$\mathbb{R}_+$	set of non negative real numbers
$\mathbb{R}_+^*$	set of strictly positive real numbers
$\mathbb{N}$	set of natural numbers zero included
$\mathbb{N}^*$	set of natural numbers zero excluded
$\mathbb{C}$	set of complex numbers
$\mathbb{Z}$	set of integer numbers
$\mathbb{Q}$	set of rational numbers
$\in$	belongs to
$\subset$	subset
$\supset$	superset
$:=$	defined as
$\forall$	for all
$\exists$	there exists
$A \cap B$	intersection of sets
$A \cup B$	union of sets
$A \setminus B$	difference of sets
$A \pm B$	set $\{a \pm b : a \in A, b \in B\}$
αA	with $a \in \mathbb{R}$, set $\{\alpha a, a \in A\}$
$\emptyset$	the empty set

$A \times B$	Cartesian product of sets
A^n	n-fold product of the set A
$\bar{A}$	closure of A
∂A	boundary A
$\text{diam}(A)$	$\sup\{d(x, y) \mid x, y \in A\}$
$\lvert S\rvert$	when S is a set, $\lvert S\rvert := \sup_{s\in S} \lvert s\rvert$
$\mathbb{B}$	open ball of radius 1
$\alpha\mathbb{B}$	open ball of radius $\alpha > 0$
$\bar{\mathbb{B}}$	closed ball of radius 1
$\alpha\bar{\mathbb{B}}$	closed ball of radius $\alpha > 0$
$E^{n\times m}$	set of matrices with n rows and m columns and coefficients in E
$\lvert \cdot \rvert$	vector or matrix induced norm
$\lvert \cdot \rvert_A$	$\inf_{a\in A} \lvert \cdot -a\rvert, distance from the set$ A
M^T	transpose matrix
M^{-1}	inverse matrix
$M^\dagger$	Moore-Penrose generalized inverse matrix
M^{-T}	$(M^{-1})^T$
$M \geq 0$	positive semi-definite matrix
$M > 0$	positive definite matrix
$\det M$	determinant of M
$\text{rank}M$	rank of M
$\sigma(M)$	spectrum of M, the set of its eigenvalues
$A \otimes B$	Kronecker product of matrices
$\text{Im}A$	image of M
$\text{Ker}A$	kernel of M
$0_{n\times m}$	matrix of dimension $n \times m$ whose entries are all zeros When $n = m$ we write 0_n and when the dimension is clear from the context the subscript is omitted and we write simply 0.
I_n	n-dimensional identity matrix. When the dimension is clear from the context the subscript is omitted and we write simply I
$\text{diag}(A_1, \ldots, A_n)$	block-diagonal matrix block diagonal elements the square matrices $A_1, \ldots, A_n$

$\mathrm{col}(A_1, \dots, A_n)$	column concatenation of the elements A_i
$\mathrm{col}(A : A \in \mathcal{A})$	column concatenation of the elements $A \in \mathcal{A}$. If $\mathcal{A}$ is indexed by the set N we also write $\mathrm{col}(A_n : n \in N)$
Hurwitz matrix	matrix with all eigenvalues having strictly negative real part than 1
simply stable matrix	matrix with all eigenvalues with zero real part and algebraic multiplicity 1
$\mathbb{HC}(n)$	subset of $\mathbb{R}^n$ of all the coefficients $(c_1, \dots, c_n)$ of a Hurwitz monic polynomial of dimension n, i.e. such that $p(\lambda) := \lambda^n + c_n\lambda^{n-1} + \cdots + c_2\lambda + c_1$ has only roots with strictly negative real part
$f : A \to B$	a function from A to B
$f\|_C$	with $f : A \to B$ and $C \subset A$, $f\|_C$ is the restriction of f to C
$f : A \rightrightarrows B$	a set-valued function from A to B
$\mathrm{dom}F$	the domain of F
$\mathrm{ran}F$	the range of F
$\mathrm{supp}F$	the support of F
$f \in \mathcal{K}$	f is a class-K function, i.e. $f : [0, a) \to \mathbb{R}_+$ ($a \in \mathbb{R}_+^*$) is continuous, strictly increasing and $f(0) = 0$
$f \in \mathcal{K}_\infty$	f is a class-K_∞ function, i.e. $f \in \mathcal{K}$ and $f(x) \to_{x\to a} \infty$ is
$f \in \mathcal{L}$	f is a class-L function, i.e. $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, strictly decreasing and $f(x) \to_{x\to\infty}= 0$
$\beta \in \mathcal{KL}$	β is a class-KL function, i.e. $\beta(\cdot, t) \in \mathcal{K}$ for each t and $\beta(s, \cdot) \in \mathcal{L}$ for each s
C^n	the set of n-time continuously differentiable functions (C^0 is the set of continuous functions)
$\beta \in \mathcal{KL}$	β is a class-KL function, i.e. $\beta(\cdot, t) \in \mathcal{K}$ for each t and $\beta(s, \cdot) \in \mathcal{L}$ for each s
$\|x\|_{t_1,t_2}$	when $x(\cdot)$ is a locally essentially bounded function defined on $\mathbb{R}$, for $t_1, t_2 \in \mathbb{R}$ we let $\|x\|_{t_1,t_2} := \mathrm{ess.sup}_{t\in[t_1,t_2]}\|x(t)\|$
$\|x\|_\infty$	when $x(\cdot)$ is a locally essentially bounded function defined on $\mathbb{R}$, we let $\|x\|_{t_1,t_2} := \mathrm{ess.sup}_{t\in\mathbb{R}}\|x(t)\|$
$x_{[i,j]}$	when $i, j \in \mathbb{N}, i \leq j$ and $x \in \mathbb{R}^n, n \geq j$, we let $x_{[i,j]} := (x_i, x_{i+1}, \dots, x_j)$

$x^{(i,j)}$	when $i, j \in \mathbb{N}, i \leq j$, and x is a j-differentiable function on $\mathbb{R}$, we let $x^{(i,j)} := x^{(i)}, x^{(i+1)}, \ldots, x^{(j)}$
$(x_n)_{n=a}^{b}$	sequence $x_a, \ldots, x_b$
$(x_n)_n$	a sequence of elements x_n indexed by $n \in \mathbb{N}$
$(x_n) \to x$	short for $\lim_{n\to\infty} x_n = x$
$\limsup x$	if $x : \mathbb{R} \to \mathbb{R}^n$, short for $\limsup_{n\to\infty} x(t)$

Introduction

It's human nature to stretch, to go, to see, to understand. Exploration is not a choice, really; it's an imperative. — Michael Collins.

The astronaut Michael Collins, one of the first men on the Moon, would see exploration as the key to the human understanding of the world. Exploration, and study and observation, of the surrounding environment has been the drive that pushed the development of humankind, since the first men that discovered fire by observing a tree struck by a lightning, to Newton theorizing gravity from a falling apple.

In particular, exploration and observation have always been the tools to make up for the lack of knowledge in the scientific fields. No less it is in the control theory field, where lack of knowledge of system parameters or of important not measurable quantities, is often compensated via identification techniques, based on elaborating observed variables or adaptive strategies, often using "exploratory" moves.

Those same approaches have started to take root also in the field of output regulation when, recently, the strong assumption of perfect knowledge of the exosystem dynamics has started to be discarded, moving toward a more robust setting. We find, for example, in this scenario the works Marino and Tomei (2003) and Marino and Santosuosso (2007), where adaptive observers have been used to asymptotically estimate the internal model's parameters, or Bin et al. (2019b), where, instead, discrete-time identification schemes are proposed in an adaptive design for multivariable linear systems. The problem, however, is far from being solved. As such, in the first part of this thesis we will tackle the problem of linear output regulation for continuous-time dynamical systems when no knowledge of the exosystem is assumed, proposing a solution based on estimating the parameters of the internal model unit through a continuous-time least-squares identifier. As preliminaries we will first recall the standard results on linear output regulation theory, to then dwell upon the framework for a system theoretic study of continuous-time identifiers, as highlighted in Bin (2019).

In a journey toward a robust output regulation framework, driven by the idea of gaining robustness through exploration, we propose a study on the class of gradient-free optimization techniques denoted *Direct Search* methods. The main characteristic of these algorithms, and the reason for their study, is that they do not require gradient information, or the expression of the objective function to be minimized, but only the measurements of the function values, taken by iteratively computing exploratory moves. Moreover, being based on exploratory steps along a set of vectors spanning the search space, their implementation is computationally very efficient. We study these class of algorithms, inherently discrete-time, interconnected to a continuous-time dynamical system. As such we propose, in Section 6, a hybrid controller implementing a particular direct search algorithm developed in Section 5, and study its robustness properties, showing that without additional assumptions related to the knowledge of the objective function, other than the classic assumptions of Direct Search algorithms, a trade-off between robustness and asymptotic convergence to the minimum is inevitable. The novelty of the algorithm proposed is that it guarantees convergence of the whole sequence of iterates even for a, particular class of, non convex functions. Moreover, contrary to the standard results on Direct Search algorithms, convergence is achieved without resorting to a full exploration of all the possible directions at each iterations (see Kolda et al. (2003)), but simply continuing the exploration along any direction providing (sufficient) decrease of the cost function. This aspect is fundamental from an implementation point of view, speeding up considerably the convergence.

The reason to study these algorithms stems from the idea of both providing alternative identifier schemes for the solution of the problem of adaptive output regulation, both to propose a solution to the output regulation problem when the regulated error is not available for measurements, but, instead, a “proxy” of the error can be measured. In particular, we want to lay the ground for the scenario in which the proxy is an objective function, measurable but possibly unknown, whose minimum is coincident with the reference signal to be tracked. As such, as for this scenario the regulated error is not locally observable at the minimum, we believe, and this will be the matter of future studies, that the proposed hybrid controller can be a valid solution, interconnected with a properly designed regulator, to tackle the output regulation problem under the hypothesis of not measurability of the error.

Part I

Adaptive Output Regulation for Linear Systems

1

Output Regulation for Linear Systems

The branch of control theory denoted *Output Regulation* addresses the problem of steering some regulated outputs in order to, possibly exactly, follow some externally generated reference signals, while the dynamics are affected by exogenous perturbations. For linear systems, the output regulation problem was firstly addressed by Francis, Wonham and Davison in the 70s (see e.g. Francis and Wonham (1975), Francis and Wonham (1976) and Davison (1976)), where the so-called internal model principle was introduced. In a nutshell it states that in order to perfectly track a reference signal and completely reject an exogenous disturbance a "copy" of the dynamic model generating the reference signal and the disturbance, or better the modes generating those signals, should be embedded in the regulator. It is thus intuitive as, from a theoretical point of view, the reference signals and the exogenous disturbances can be treated in the same way, namely considered to be both generated by a unified dynamical system, denoted *exosystem*. Based on this idea, the regulator design proposed by Davison is based on extending the plant with an *internal model unit*, able to replicate the modes of the exosystem, and stabilizing the cascade given by the internal model unit and the plant. We stress as in the classic framework developed by Davison, Francis and Wonham, and contrary to the design developed in Chapter 3, the dynamics of the exosystem are assumed to be known, as they are used in the design of the internal model unit. Nonetheless, even if perfect knowledge of the exosystem is assumed, and actually required, for the regulator design, the remarkable property of the regulator solving the linear output regulation problem is its robustness to parametric uncertainties in the plant dynamics, often referred to as *structural stability*.

In the following sections we will review the basic ingredient of the output regulation framework for linear systems. Firstly introducing the concept of steady-state and then reporting the classic design of the regulator as proposed by Davison in Davison (1976).

1.1 The Steady State

As central to the output regulation framework, we recall in this section the concept of *steady state* for linear systems.

Consider the following linear system

$$\begin{aligned} \dot{w} &= Sw \\ \dot{z} &= Fz + Gw, \end{aligned} \tag{1.1}$$

where $w \in \mathbb{R}^{n_w}$, $z \in \mathbb{R}^{n_z}$, and $S \in \mathbb{R}^{n_w \times n_w}$, $F \in \mathbb{R}^{n_z \times n_z}$ and $G \in \mathbb{R}^{n_z \times n_w}$.

The following theorem characterizes the steady state behavior of (1.1).

Theorem 1.1 (Steady state behaviour of multivariable linear systems)**.** *Assume $\sigma(S) \cap \sigma(F) = \emptyset$, then there exists a unique $\Pi \in \mathbb{R}^{n_z \times n_w}$, that solves the Sylvester equation*

$$\Pi S = F\Pi + G, \tag{1.2}$$

such that the subspace $\{(w, z) \in \mathbb{R}^{n_w} \times \mathbb{R}^{n_z} : (w, z) = (w, \Pi w)\}$ is forward invariant. Moreover, if F is Hurwitz, then

$$\lim_{t\to\infty} z(t) - \Pi w(t) = 0. \tag{1.3}$$

Proof. Existence and uniqueness of Π solving the Sylvester equation (1.2) follows from the empty intersection of the spectra of F and S.

Now, by Theorem 2.4 in Basile and Marro (1992), a subspace $\mathcal{L} \subset \mathbb{R}^n$ is forward invariant for the trajectories of the linear system $\dot{x} = Ax$, with $x \in \mathbb{R}^n$, if and only it is A-invariant, namely $Ax \in \mathcal{L}$ for all $x \in \mathcal{L}$. Thus $\{(w, z) \in \mathbb{R}^{n_w} \times \mathbb{R}^{n_z} : (w, z) = (w, \Pi w)\}$ is forward invariant if and only if

$$\begin{pmatrix} S & 0 \\ G & F \end{pmatrix} \begin{pmatrix} I_{n_w} \\ \Pi \end{pmatrix} \subset \text{Im} \begin{pmatrix} I_{n_w} \\ \Pi \end{pmatrix}.$$

The first result follows from (1.2) by noticing that

$$\begin{pmatrix} S & 0 \\ G & F \end{pmatrix} \begin{pmatrix} I_{n_w} \\ \Pi \end{pmatrix} = \begin{pmatrix} S \\ G + F\Pi \end{pmatrix} = \begin{pmatrix} S \\ \Pi S \end{pmatrix} = \begin{pmatrix} I_{n_w} \\ \Pi \end{pmatrix} Sw.$$

Assuming F Hurwitz, define $\tilde{z} = z - \Pi w$. Then, by (1.2),

$$\dot{\tilde{z}} = Fz + Gw - \Pi S w = F(\tilde{z} + \Pi w) + Gw - \Pi S w = F\tilde{z},$$

and since F is Hurwitz, it follows that $\lim_{t \to \infty} z(t) - \Pi w(t) = 0$. ■

In the output regulation framework the matrix S is, in general, assumed to be *simply stable*, i.e. with all eigenvalues on the imaginary axis, and F to be Hurwitz, thus $\sigma(S) \cap \sigma(F) = \emptyset$. In this framework, the above theorem states that a Hurwitz linear system, driven by, possibly oscillating, exogenous signals, will asymptotically converge to a subspace of its state space where it evolves as a linear combination of the exogenous signals.

The condition $\sigma(S) \cap \sigma(F) = \emptyset$ can also be interpreted as a *non-resonance* condition for the cascade interconnection of the w and x subsystems, guaranteeing forward invariance of the set $\{(w, z) \in \mathbb{R}^{n_w} \times \mathbb{R}^{n_z} : (w, z) = (w, \Pi w)\}$ by preventing the existence of solutions that would leave this set due to a *resonant* behaviour between the two subsystems.

1.2 The Linear Output Regulation Problem

Consider a linear system of the form

$$\dot{w} = Sw \tag{1.4}$$

$$\dot{x} = Ax + Bu + Pw \tag{1.5}$$

$$y_m = C_m x + Q_m w \tag{1.6}$$
$$e = C_e x + Q_e w, \tag{1.7}$$

with $w \in \mathbb{R}^{n_w}$ an exogenous input, $x \in \mathbb{R}^{n_x}$ the state, $u \in \mathbb{R}^{n_u}$ the control input, $e \in \mathbb{R}^{n_e}$ the regulation error, $y_m \in \mathbb{R}^{n_m}$ additional measurements and $n_w, n_x, n_u, n_e, n_m \in \mathbb{N}$ such that $n_u \geq n_e$. The signal $w(t)$ models exogenous disturbances, to be rejected, and reference signals, to be tracked, whose modes are defined by the matrix S, that we suppose to be simply stable.

In this framework, the linear output regulation problem reads as follows: design a linear regulator of the form

$$\dot{\mu} = A_r \mu + B_r y \tag{1.8}$$
$$u = K_r \mu + K_y y, \tag{1.9}$$

with $\mu \in \mathbb{R}^{n_r}$, for some $n_r \in \mathbb{N}$, $y := \text{col}(y_m, e)$, and A_r, B_r, K_r and K_y matrices of proper dimension, such for the closed-loop system given by (1.4)-(1.7) (1.8):

- the origin is an asymptotically stable equilibrium point when the exosystem is disconnected, i.e. when $w(t) \equiv 0$;
- for any initial condition, $\lim_{t\to\infty} e(t) = 0$.

The closed-loop system (1.4),(1.5),(1.8) can be compactly rewritten in the form

$$\dot{w} = Sw \tag{1.10}$$
$$\dot{z} = Fz + \Sigma w, \tag{1.11}$$

where $z := (x, \mu)$ and

$$F := \begin{pmatrix} A + BK_yC & BK_r \\ B_rC & A_r \end{pmatrix} \quad \Sigma := \begin{pmatrix} P + BK_yQ \\ B_rQ \end{pmatrix},$$

with $C := [C_m \ C_e]$ and $Q := [Q_m \ Q_e]$.

Notice that (1.10) has the same structure of (1.1) hence, by Theorem 1.1, if $\sigma(S) \cap \sigma(F) \neq \emptyset$ and F is Hurwitz, (1.10) will converge to a steady state of the form $(w, z) = (w, \Pi_z w)$ for some $\Pi_z \in \mathbb{R}^{n_z} \times \mathbb{R}^{n_w}$, with $n_z := n_x + n_r$. The goal is thus to build a regulator such that F is Hurwitz and at steady-state the error (1.7) is null.

The closed-loop system should thus satisfy at steady-state the following equations

$$\begin{aligned} \Pi_x S &= (A + BK_yC)\Pi_x + BK_r\Pi_r + P + BK_yQ \\ \Pi_r S &= B_rC\Pi_x + A_r\Pi_r + B_rQ \\ C_e\Pi_x &+ Q_e = 0. \end{aligned}$$

The first and last equations are usually called the *regulator*, or *Francis*, *equations*.

Davison in Davison (1976) proposed a design for such a regulator. The structure he devised, following the internal model principle of Francis and Wonham, is based on extending (1.4)-(1.7) with an *internal model unit*, and then designing a state feedback able to stabilize the cascade interconnection of the plant and the internal model unit. On a more general scenario, we consider the stabilizing feedback to be a dynamical system.

On the wake of the results of Davison, we extend the considered plant with the following dynamical system

$$\dot{\eta} = \Phi\eta + Ge, \tag{1.12}$$

with $\eta \in \mathbb{R}^{n_\eta}$, $n_\eta := n_e n_w$, and

$$\Phi := \begin{pmatrix} 0 & I_{n_e} & 0 & 0 & \cdots & 0 \\ 0 & 0 & I_{n_e} & \cdots & 0 & \\ \vdots & & & \ddots & \vdots & \\ 0 & 0 & \cdots & & I_{n_e} & \\ -c_0 I_{n_e} & -c_1 I_{n_e} & & \cdots & & -c_{n_w-1} I_{n_e} \end{pmatrix} \quad G := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I_{n_e} \end{pmatrix},$$

where $c_0, ..., c_{n_w-1}$ are the coefficients of the characteristic polynomial of S, defined as

$$p_S(\lambda) = \lambda^{n_w} + c_{n_w-1}\lambda^{n_w-1} + \cdots + c_1\lambda + c_0. \tag{1.13}$$

As such, the internal model unit embeds n_e copies of the dynamics of the exosystem, thus the name *internal model unit*.

The following controller should be defined in order to stabilize the cascade given by the plant and the internal model unit.

$$\begin{array}{rcl}\dot{\xi} & = & A_\xi \xi + B_\eta \eta + B_y y \\ u & = & K_\xi \xi + K_\eta \eta + K_y y.\end{array} \tag{1.14}$$

As such, to guarantee that a controller of the form (1.14) exists, i.e. to guarantee stabilizability of the cascade interconnection given by the plant and the internal model unit, the following *non-resonance* condition should be satisfied.

$$\operatorname{rank}\begin{pmatrix} A-\lambda I & B \\ C_e & 0 \end{pmatrix} = n_x + n_e \qquad \forall \lambda \in \sigma(S), \tag{1.15}$$

where (1.15) is expressed with respect to the spectrum of S, since the spectrum of Φ is given, by construction, by n_e copies of the spectrum of S. Hence, the condition is satisfied for Φ if only if it so for S.

We are now ready to state the sufficient and necessary conditions under which the linear output regulation problem is solved.

Theorem 1.2. *Let* (A, B) *be stabilizable,* (A, C) *be detectable, and* (1.15) *be satisfied. Then the regulator* (1.12), (1.14) *solves the linear output regulation problem for the system* (1.4)-(1.7).

Proof. Consider the closed-loop matrix given by the plant, the internal model unit and the stabilizer

$$F := \begin{pmatrix} A + BK_yC & BK_\eta & BK_\xi \\ GC_e & \Phi & 0_{n_\eta \times n_\xi} \\ B_yC & B_\eta & A_\xi \end{pmatrix}, \tag{1.16}$$

and notice that, by (1.15), we can design the stabilizer (1.14) to make it Hurwitz. As a consequence, the cascade given by the exosystem, and the plant, the internal model unit and the stabilizer has the same structure as (1.1), and thus, again, by Theorem 1.1, since F is Hurwitz and S is simply stable, by assumption, there exists $\Pi \in \mathbb{R}^{(n_x+n_\eta+n_\xi)\times n_w}$ such that the set $\{(w, x, \eta, \xi) : (w, x, \eta, \xi) = (w, \Pi_x w, \Pi_\eta w, \Pi_\xi w)\}$, with $\Pi = \operatorname{col}(\Pi_x, \Pi_\eta, \Pi_\xi)$, is forward invariant for (w, x, η, ξ). From Theorem 1.1, considering the dynamics of the internal model unit, we conclude that at steady-state

$$\Pi_\eta S = \Phi \Pi_\eta + G(C_e \Pi_x + Q_e).$$

From the structure of the matrix Φ and considering $\Phi_\eta = \operatorname{col}(\Pi_{\eta_{\eta_1}}, \Pi_{\eta_2}, ..., \Pi_{\eta_{n_w}})$, it follows that

$$\begin{aligned}\Pi_{\eta_i} S &= \Pi_{\eta_{i+1}} \qquad \forall i = 1, 2, ..., n_w - 1\\ \Pi_{\eta_{n_w}} S &= \sum_{i=1}^{n_w} c_{i-1} \Pi_{\eta_i} + G(C_e \Pi_x + Q_e).\end{aligned}$$

Hence

$$\Pi_{\eta_i} = \Pi_{\eta_1} S^{i-1}, \qquad \forall i = 2, \dots, n_w,$$

that implies

$$\begin{aligned}\Pi_{\eta_1} S^{n_w} &= \Pi_{\eta_{n_w}} S = \sum_{i=1}^{n_w} c_{i-1} \Pi_{\eta_1} S^{i-1} + G(C_e \Pi_x + Q_e) =\\ &= \Pi_{\eta_1} \left(\sum_{i=1}^{n_w} c_{i-1} S^{i-1} \right) + G(C_e \Pi_x + Q_e).\end{aligned}$$

From the expression of the characteristic polynomial of S in (1.13) and the Cayley-Hamilton Theorem, we readily conclude that $G(C_e \Pi_x + Q_e) = 0$ and thus $e = 0$. ■

We conclude this section with some remarks regarding the properties of the presented linear regulator.

As the matrices P, Q and Π play no role in the regulator design, and neither do A, B and C, if not for the stabilizer design, the above regulator is *structurally robust* to any variation of the matrices P, Q, and also A, B and C as long as stability is preserved.

Nonetheless, as already anticipated, knowledge of the exosystem modes is necessary in order to properly construct the internal model unit.

2

Continuous-time Least-Squares Identifier

Ljung, in the introduction of Ljung (1999), defines the system identification field as “dealing with the problem of building mathematical models of dynamical systems based on observed data from the systems”. The main reason to resort to identification techniques is to cope with missing knowledge of some, or all, parts of the system dynamics. The interaction between identification and control stems, indeed, from the need to robustly control plants in these scenarios (see for example Gevers (1996)). Along those lines, the main idea of the next chapter is to build a robust regulator for with an internal model unit that is adapted online thanks to an identifier. The framework presented in this chapter (reporting the results of Bin et al. (2019a) and Bin (2019) for the continuous-time case), casts the identification problem as an optimization problem.

In particular we will treat continuous-time identifiers, with a focus on least-squares identifiers, from a system theoretic point of view by defining the identifiers as dynamical systems to which it is associated a cost functional to be minimized, determining in a user-defined way a criteria upon which base the selection of the parameters of the model.

2.1 Identification framework

Let $\alpha^\star(t) \in \mathbb{R}^a$ and $\beta^\star(t) \in \mathbb{R}^b$, with $a, b \in \mathbb{N}$, be two continuous-time signals. We want to find a *model* relating α and β, namely a function $\phi : \mathbb{R}^a \to \mathbb{R}^b$ such that $\phi(\alpha^\star(t)) = \beta^\star(t)$ for all $t \in \mathbb{R}_{\geq 0}$. As in the classic system identification literature Ljung (1999), Ljung and Soderstrom (1985), we consider, for the identification problem, the class of models to be parameterized by a variable $\theta \in \mathbb{R}^d$, with $d \in \mathbb{N}$ denoted the *order* of the model, usually defined by a priori knowledge on the identification problem. We denote by $\mathcal{M}$ this class of models, and as $\Phi : \mathbb{R}^d \times \mathbb{R}^a \to \mathbb{R}^b$ the function associating to each element of $\mathbb{R}^d$ a model in $\mathcal{M}$.

Without loss of generality, we assume the signals $\alpha^\star$ and $\beta^\star$ to be the output of an autonomous system $\dot{w} = S(w)$ with state $w \in W$, with $W \subset \mathbb{R}^{n_w}$ compact and $n_w \in \mathbb{N}$, and $S : W \to W$. With slight abuse of notation, we will refer without distinction to $\alpha^\star(t)$, $\beta^\star(t)$ and $\alpha^\star(w)$, $\beta^\star(w)$.

Moreover, as it will be instrumental for the analysis in the next chapter, where the equivalent of the real signals $\alpha^\star$ and $\beta^\star$ will not be available, but in their stead we will work with an estimate of their values, we define a "corrupted" version of the signals $\alpha^\star$ and $\beta^\star$ as

$$\begin{aligned} \alpha(t) &= \alpha^\star(t) + \delta_\alpha \\ \beta(t) &= \beta^\star(t) + \delta_\beta, \end{aligned}$$

with $\delta := (\delta_\alpha, \delta_\beta)$ a bounded signal.

Once $\mathcal{M}$ and d are chosen by the user, the goal of the identifier is to find the best $\theta \in \mathbb{R}^d$ fitting the values of α and β through $\Phi(\theta, \alpha^\star) = \beta^\star$. This problem is equivalent to the minimization of the *prediction error*

$$\epsilon(w, \theta) := \beta^\star(w) - \Phi(\theta, \alpha^\star(w)). \tag{2.1}$$

We define the identifier as a continuous-time dynamical system of the form

$$\begin{aligned} \dot{z} &= f_{id}(z, \alpha, \beta) \\ \theta &= h_{id}(z), \end{aligned} \tag{2.2}$$

with $z \in \mathbb{R}^{n_z}$, $f : \mathbb{R}^{n_z} \times \mathbb{R}^a \times \mathbb{R}^b \to \mathbb{R}^{n_z}$ and $h : \mathbb{R}^{n_z} \to \mathbb{R}^d$, to which it is associated a cost functional

$$J(\theta)(t) := \int_0^t c(\epsilon(w(s), \theta))ds + \omega(\theta), \tag{2.3}$$

with $c : \mathbb{R}^b \to \mathbb{R}_{\geq 0}$ positive definite and such that $c(0) = 0$, and $\omega : \mathbb{R}^d \to \mathbb{R}$ a regularization term.

The minimization of the cost functional J defines an optimization problem based on the history of the prediction error. The design of the identifier should be done so that its output (θ) minimizes (2.3), namely to achieve θ asymptotically converging to the, possibly set-valued, map

$$\theta^\circ(t) := \arg \inf_{\theta \in \mathbb{R}^d} J(\theta)(t). \tag{2.4}$$

Denote as $z^\star$ the ideal steady state of the identifier (2.2), then the following properties should be satisfied.

Property 2.1 (Identifier requirements)**.** *Bin et al. (2019a)*

1. *Optimality: The output $\theta^\star = h(z^\star)$ is such that*

$$\theta^\star(t) \in \theta^\circ(t) \qquad \forall t \geq 0$$

2. *Stability: Define $\tilde{z} := z - z^\star$. There exist functions $\beta_{\tilde{z}} \in \mathcal{KL}$ and $\rho \in \mathcal{K}$ such that*

$$\|\tilde{z}(t)\| \leq \beta_{\tilde{z}}(\|\tilde{z}(0)\|, t) + \alpha_{\tilde{z}}\|\delta\|_[0, t) \quad \forall t \geq 0$$

3. *Regularity: There exist $T \geq 0$, $\kappa \in \mathcal{K}$, and $\varepsilon(z(t)^\star) > 0$ such that, for all $t \geq T$ and $\|z(t) - z(t)^\star\| < \varepsilon(z(t)^\star)$,*

$$\|h(z(t)) - h(z^\star(t))\| \leq \kappa(\|z(t) - z^\star(t)\|)$$

In order, the first point asks that the ideal steady state of the identifier is (one of) the solution(s) to the optimization problem defined by the functional (2.3). The second point requires an *input-to-state-stability* property of the identifier with respect to the ideal steady state with respect to the perturbation δ. The last point, denoted *regularity requirement*, can be interpreted, with the addition of requirement 1., as a detectability property of $\theta - \theta^\star$ from $z - z^\star$.

2.2 Continuous-time Least-squares Identifier

As a particular case of the identification problem formulated in the previous section, we consider the class of continuous-time least-squares identifiers. We consider the class of models, $\mathcal{M}$, to be the class of functions linearly parameterized in θ. The function Φ will thus assume the form

$$\Phi(\theta, \alpha) = \theta^T \gamma(\alpha),$$

with $\gamma : \mathbb{R}^a \to \mathbb{R}^d$ locally Lipschitz. As a consequence, we modify the cost functional, as in the classic least-squares framework, as to weight, with a forgetting factor defined by $\lambda > 0$, the history of the squared prediction errors, namely

$$J(\theta)(t) = \lambda \int_0^t e^{-\lambda(t-s)} \|\beta^\star(w(t) - \theta^T \gamma(\alpha^\star(w(t)))\|^2 ds + \theta^T \Sigma \theta, \tag{2.5}$$

with $\Sigma \in \mathbb{R}^{d \times d}$ symmetric positive semi-definite.

We define the continuous-time least-squares identifier as the dynamical system defined on the partitioned state space $\mathcal{Z} := \mathbb{R}^{d\times d} \times \mathbb{R}^d$, with partitioned state $z = (R, v)$, with $R \in \mathbb{R}^{d\times d}$ symmetric positive semi-definite and $v \in \mathbb{R}^d$ with the following dynamics

$$\begin{array}{rcl} \dot{R} &=& -\lambda R + \lambda \gamma(\alpha)\gamma(\alpha)^T \\ \dot{v} &=& -\lambda v + \lambda \gamma(\alpha)\beta \\ \theta &=& (R+\Sigma)^\dagger v, \end{array} \tag{2.6}$$

where λ is the same as in (2.5) and $\cdot^\dagger$ denotes the Moore-Penrose pseudoinverse.

We equip $\mathcal{Z}$ with the norm $|z| = |(R, v)| := \sqrt{|R|^2 + |v|^2}$. In order to obtain the differentiability of the map $t \mapsto (R(t) + \Sigma)^\dagger v(t)$, and in order to have uniqueness of solutions to the minimization problem (2.5), we define the following *persistence of excitation* (PE) property for η.

Definition 2.1. *With $\epsilon, T > 0$, the signal η is said to have the (ϵ, T)-persistency of* *exctitation property if for all $t \geq T$*

$$\det\left(\int_0^t e^{-\lambda(t-s)}\gamma(\alpha(s))\gamma(\alpha(s))^T ds + \Sigma\right) \geq \epsilon. \tag{2.7}$$

We observe that, by continuity, the PE condition (2.7) can be checked online simply by looking at $\det(R(t) + \Sigma)$. As a matter of fact as will be shown in proof of the following proposition, for any initial condition $R(0)$, the difference between $R(t) + \Sigma$ and the matrix appearing in (2.7) is asymptotically small for small δ_α. Thus the persistency of excitation can be translated as a property of the exosystem.

We also observe that if (2.7) is satisfied with ϵ_1 and T_1, then it is satisfied with ϵ_2 and T_2 for any $\epsilon_2 < \epsilon_1$ and $T_2 > T_1$.

Denote as $\mathcal{E} \subset W \times \mathcal{Z}$ the subset of the state space for which the (ϵ, T)-PE property is satisfied.

The next proposition shows how the least-squares identifier thus defined satisfies the identifier requirements.

Proposition 2.1. *Suppose W compact and pick (ϵ, T) such that the (ϵ, T)-persistence of excitation is satisfied. Then the identifier* (2.6) *satisfies the identifier requirements with restriction to the set $\mathcal{E}$.*

Proof. Consider $\delta = 0$, namely $\alpha(t) = \alpha^\star(t)$ and $\beta(t) = \beta^\star$, and define $z^\star(t) \in \mathcal{Z}$ as $z^\star = (R^\star, v^\star)$, where

$$\begin{aligned} R^\star(t) :=& \lambda \int_0^t e^{-\lambda(t-s)}\gamma(\alpha^\star(s))\gamma(\alpha^\star(s))^T ds \\ v^\star(t) :=& \lambda \int_0^t e^{-\lambda(t-s)}\gamma(\alpha^\star(s))\beta^\star(s) ds. \end{aligned}$$

Then $z^\star(t)$ is solution of (2.6) for $z(0) = 0$ and $(\alpha, \beta) = (\alpha^\star, \beta^\star)$.

To prove point 1., we notice that by differentiating (2.5) with $\alpha = \alpha^\star$ with respect to θ, we get

$$\begin{aligned} &\nabla_\theta(\mathcal{J}(\theta))(t) = \\ &= -2\lambda \int_0^t e^{-\lambda(t-s)}(\gamma(\alpha^\star(s))\varepsilon(s,\theta))ds + 2\Sigma\theta = \\ &= 2(R^\star(t) + \Sigma)\theta - v^\star(t)). \end{aligned}$$

Since the set of minimizers of (2.5) for $\alpha = \alpha^\star$ is given by $\{\theta \in \mathbb{R}^d : \nabla_\theta(\mathcal{J}(\theta))(t) =$

$0\}$, it follows that

$$\theta^\star(t) = (R^\star(t) + \Sigma)^\dagger v^\star(t).$$

minimizes (2.5), thus requirement 1. is satisfied.

Define $\tilde{z} := z - z^\star$ and

$$\Gamma(\alpha^\star(t), \delta(t)) := \sigma(\alpha^\star(t) + \delta_\alpha(t))\sigma(\alpha^\star(t) + \delta_\alpha(t))^T$$
$$\mathcal{I}(\beta^\star(t), \alpha^\star(t), \delta(t)) := \sigma(\alpha^\star(t) + \delta_\alpha(t))(\beta^\star(t) + \delta_\beta(t)).$$

Then

$$\|R(t) - R^\star(t)\| \leq e^{-\lambda t}\,\|R(0) - R^\star(0)\| + \lambda \int_0^t e^{-\lambda(t-s)}\|\Gamma(\alpha^\star(s), \delta(s)) - \Gamma(\alpha^\star, 0)\|ds \leq$$
$$\leq e^{-\lambda t}\,\|R(0) - R^\star(0)\| + \lambda \int_0^t e^{-\lambda(t-s)} L_\Gamma \|\delta(s)\|ds \leq$$
$$\leq e^{-\lambda t}\,\|R(0) - R^\star(0)\| + \lambda \int_0^t e^{-\lambda(t-s)} L_\Gamma \|\delta\|_{[0,t]}ds =$$
$$= e^{-\lambda t}\,\|R(0) - R^\star(0)\| - L_\Gamma\|\delta\|_{[0,t]} + e^{-\lambda t}L_\Gamma\|\delta\|_{[0,t]} \leq$$
$$\leq e^{-\lambda t}\,\|R(0) - R^\star(0)\| + L_\Gamma\|\delta\|_{[0,t]},$$

where L_Γ is the Lipschitz constant of Γ.

The same reasoning applies to $v(t) - v^\star(t)$. Denote as $L_\mathcal{I}$ the Lipschitz constant of $\mathcal{I}$. Then we see that

$$\|\tilde{z}(t)\| \leq e^{-\lambda t}(\|\tilde{R}(0)\| + \|\tilde{v}(0)\|) + (L_\Gamma + L_\mathcal{I})\|\delta\|_{[0,t)]} = e^{-\lambda t}\|\tilde{z}(0)\| + (L_\Gamma + L_\mathcal{I})\|\delta\|_{[0,t)]}. \tag{2.8}$$

Point 2. is readily satisfied for $\beta_{\tilde{z}}(|\tilde{z}(0)|, t) := e^{-\lambda t}\|\tilde{z}(0)\|$ and $\alpha_{\tilde{z}}(\|\delta\|_{[0,t)}) := (L_\Gamma + L_\mathcal{I})\|\delta\|_{[0,t)]}$.

Now notice that if the (ϵ, T)-persistency of excitation condition is satisfied, then $\det(R(t) + \Sigma) \geq \epsilon$ for all $t \geq T$. As such, $(R(t) + \Sigma)$ is invertible for all $t \geq T$, and thus $(R(t) + \Sigma)^\dagger v(t)$ is smooth in $z = (R, v)$. As a consequence, $(R(t) + \Sigma)^\dagger v(t)$ is locally Lipschitz in (R, v) for all $t \geq T$, proving requirement 3.

■

3

Adaptive Linear Output Regulation for Multivariable Linear Systems via Slow Identifiers

In the classic linear output regulation framework, as asymptotic regulation is inexorably lost whenever the exosystem is not perfectly known, the linear regulator is not robust with respect to any, although arbitrarily small, perturbation of the exosystem.

The general problem of designing a regulator for a linear system ensuring asymptotic regulation in the presence of uncertainties in the exosystem is still open, even though in the last decades many papers have been written on the topic. In Marino and Tomei (2003) and Marino and Santosuosso (2007) adaptive observers have been used to asymptotically estimate the internal model's parameters in the single-input single-output (SISO) case. In both papers, perfect knowledge of the plant is assumed, sacrificing robustness with respect to plant's perturbations for robustness to uncertainties in the exosystem. Multivariable linear systems have been considered in Mizumoto and Iwai (2007), under a strong minimum-phase assumption, and in Bando and Ichikawa (2006), where only state-feedback tracking is addressed. Further approaches can be found in the context of nonlinear SISO minimum-phase normal forms. In Serrani et al. (2001) an estimation law based on Lyapunov-like arguments is proposed to deal with linear uncertain exosystems. Instead of adaptation, immersion arguments have been used in Ding (2003), Marino and Tomei (2008), Isidori et al. (2012), Forte et al. (2013) and Bin et al. (2016) for linear and some classes of nonlinear exosystems. More recently, a different approach based on identification techniques has been proposed in Forte et al. (2017) for SISO normal forms, while in Marino and Tomei (2017) a hybrid adaptive observer is designed for SISO stable plants ,and an adaptive design for multivariable linear systems, based on discrete-time identification schemes, has been proposed in Bin et al. (2019b).

This chapter contains the results in Melis et al. (2019). Here we consider the output regulation problem for general multivariable linear systems, with the reference signals and the disturbances that are generated by an unknown exosystem. On the heels of Bin et al. (2019b), we augment a canonical linear regulator with an identifier that adapts the internal model on the basis of the measurable data. Differently from Bin et al. (2019b), the identifier is continuous-time, and the asymptotic properties of the regulator are obtained thanks to a time-scale separation of the two units. The identifier is designed to solve a least-squares optimization problem defined by the available measurements. Linearity and persistency of excitation ensure the existence of a unique semiglobal solution matching the parameters of the exosystem, despite possibly large deviations of the plant's state from the ideal error-zeroing steady state. The design of the stabilization and the adaptation laws turns out to be decoupled, making thus possible to handle general linear non-minimum phase systems.

3.1 Problem Formulation

We consider linear systems of the form

$$\dot{w} = Sw \tag{3.1}$$

$$\dot{x} = Ax + Bu + Pw \tag{3.2}$$

$$y_m = C_m x + Q_m w \tag{3.3}$$
$$e = C_e x + Q_e w, \tag{3.4}$$

with $w \in \mathbb{R}^{n_w}$ an exogenous input, $x \in \mathbb{R}^{n_x}$ the state, $u \in \mathbb{R}^{n_u}$ the control input, $e \in \mathbb{R}^{n_e}$ the regulation error, $y_m \in \mathbb{R}^{n_m}$ additional measurements and $n_w, n_x, n_u, n_e, n_m \in \mathbb{N}$ such that $n_u \geq n_e$. The exogenous signal $w(t)$ represents disturbances and reference signals acting on the system, whose modes are defined by the matrix S, that we do not assume to be known but we suppose to be neutrally stable.

We consider the problem of output regulation for the system (3.1)-(3.4), that is, we aim to design an output feedback regulator of the form

$$\dot{\mu} = f_c(\mu, y) \tag{3.5}$$
$$u = \gamma(\mu, y), \tag{3.6}$$

where $y := \mathrm{col}(y_m, e)$ and $\mu \in \Xi$, with Ξ an Euclidean space, such that the trajectories solution of (3.1)-(3.5) are bounded and

$$\lim_{t\to\infty} e(t) = 0. \tag{3.7}$$

In the rest of the chapter we make the following standing assumptions[1]

Assumption 3.1. *(A, B) is stabilizable, (C, A), with $C := [C_m\ C_e]$, is detectable and* $\mathrm{rank}\, B \geq \mathrm{rank}\, C_e$.

Assumption 3.2. *S is neutrally stable and the initial conditions of* (3.1) *range in a compact invariant set $W \subset \mathbb{R}^{n_w}$.*

3.2 The Regulator Structure

The regulator is composed of three different subsystems: the *internal model unit*, the *identifier* and the *stabiliser*. The internal model unit, based on the design proposed by Davison and recalled in chapter 1, is an error-driven dynamical system that, ideally, incorporates the modes of the exosystem. In this section, the identifier is a continuous-time system whose objective is to adapt the internal model unit to

[1] We observe that Assumptions 1 is also necessary for the solvability of the problem at hand.

asymptotically match the actual exosystem's parameters. If the exosystem were known, the internal model could be designed as a linear system in normal form with the same characteristic polynomial of S, Davison (1976). As we do not assume to know S, we still retain a similar structure, with the parameters defining the internal model's dynamics that are decided at runtime by the identifier. The stabiliser is a subsystem that, for each fixed value of the identifier, stabilizes the cascade interconnection of the plant and internal model unit. We detail the three subsystems in the rest of the section.

3.2.1 The Internal Model

The internal model unit is designed as the following dynamical system

$$\dot{\eta} = \Phi(\eta, z) + Ge, \tag{3.8}$$

with $\eta \in \mathbb{R}^{n_e(n_w+1)}$, $z \in \mathcal{Z}$, with $\mathcal{Z}$ an Euclidean space that will be defined in the next subsection, the state of the identifier that will adapt the internal model and

$$\Phi(\eta, z) := \begin{pmatrix} \eta_2 \\ \eta_3 \\ \vdots \\ \eta_{n_w+1} \\ \Psi(\eta, z) \end{pmatrix} \qquad G := \begin{pmatrix} 0_{n_e} \\ 0_{n_e} \\ \vdots \\ 0_{n_e} \\ I_{n_e} \end{pmatrix} \tag{3.9}$$

with $\eta_i \in \mathbb{R}^{n_e}$, $i = 1, 2, ..., n_w + 1$, such that $\eta = \mathrm{col}(\eta_1, \eta_2, ..., \eta_{n_w+1})$, and $\Psi : \mathbb{R}^{n_e(n_w+1)} \times \mathbb{R}^{n_w} \times \mathcal{Z} \to \mathbb{R}^{n_e}$ to be fixed. If the closed-loop system is stable, it reaches a steady-state in which all the variables oscillate with the same modes of the exosystem. As the dimension of η is $n_w + 1$, the Cayley-Hamilton Theorem implies that, at such steady state, η must satisfy a regression of the kind

$$\eta_{n_w+1} = (\theta^{\circ T} \otimes I_{n_e})\eta_{[1,n_w]}, \tag{3.10}$$

with $\theta^{\circ} \in \mathbb{R}^{n_w}$ matching the coefficients of the characteristic polynomial of S (modulo a change of sign). The intuition behind the proposed approach is to look at (3.10) as a *prediction error model*, asymptotically relating the state η of the internal model (measured) with the sought unknown characteristic polynomial of S.

The design of the identifier z to find θ° in (3.10), based on the continuous-time least-squares identifier studied in Section 2.2, is postponed to the next section, while for the moment we assume that we have a guess of θ° given by

$$\theta = \omega(z),$$

with $\omega : \mathcal{Z} \to \mathbb{R}^{n_w}$, to be defined later.

The design of the internal model is completed by letting

$$\Psi(\eta, z) := (p_{\mathcal{E}}(\omega(z))^T \otimes I_{n_e})\eta_{[2,n_w+1]} + \bar{\rho}(\eta, z), \tag{3.11}$$

with $p_{\mathcal{E}} : \mathbb{R}^{n_w} \to \mathcal{E}$ the projection operator onto a compact convex set $\mathcal{E} \subset \mathbb{R}^{n_w}$ to be fixed, and with $\bar{\rho} : \mathbb{R}^{n_e(n_w+1)} \times \mathcal{Z} \to \mathbb{R}^{n_e}$ a bounded function to be chosen later according to the identifier's structure.

With the choice (3.11), and for a suitable choice of $\bar{\rho}$ such that $\bar{\rho}(\eta, z)$ vanishes whenever θ equals its "ideal" value θ°, the system (3.8) with $\theta = \theta^\circ$ is able to reproduce all the modes of the exosystem, and it thus candidates as a proper internal model.

To fix the set $\mathcal{E}$ in (3.11), we first define the set

$$\mathcal{Q} := \left\{ \theta \in \mathbb{R}^{n_w} : \text{rank} \begin{pmatrix} A - \mu I & B \\ C_e & 0 \end{pmatrix} < n_x + n_e, \quad \mu \in \sigma \left(\begin{pmatrix} 0_{n_e n_w \times n_e} & I_{n_e n_w} \\ 0_{n_e \times n_e} & \theta^T \otimes I_{n_e} \end{pmatrix} \right) \right\}$$

which represents the set of $\theta \in \mathbb{R}^{n_w}$ for which the $non-resonance$ condition is not satisfied, i.e. for which the cascade (3.2), (3.8) is not stabilizable. We thus fix the set $\mathcal{E}$ as any compact convex set such that $\mathcal{E} \cap \mathcal{Q} = \emptyset$. The existence of such a set, when the transfer function of the plant has a finite number of zeros, has been proved in Bin et al. (2019b).

3.2.2 The Identifier

We approach the design of the identifier z by looking at (3.10) as a linear regression relating the steady state values of the state η, and by casting the estimation problem of the exact parameters θ° as a recursive least-squares problem in the variables η_{n_w+1} and $\eta_{[1,n_w]}$. More precisely, relating to the framework in Section 2.2, we define the *prediction error*

$$\varepsilon(t,\theta) := \eta_{n_w+1}(t) - (\theta^T \otimes I_{n_e})\eta_{[1,n_w]}(t), \tag{3.12}$$

and we associate to each signal $\eta(t)$ the *cost functional*

$$(\mathcal{J}_\eta(\theta))(t) = \lambda \int_0^t e^{-\lambda(t-s)} |\varepsilon(s,\theta)|^2 ds, \tag{3.13}$$

with $\lambda > 0$. Notice that here we assumed no regularization term, $\Sigma = 0$, as we aim at perfectly computing θ°. The parameter θ is decided by the identifier so as to minimize (3.13).

Referring to the structure in Section 2.2, we define the identifier on $\mathcal{Z} := \mathbb{R}^{n_w \times n_w} \times \mathbb{R}^{n_w}$ and with state partitioned as $z := (R, v)$, , with $R \in \mathbb{R}^{n_w \times n_w}$ symmetric positive semi-definite and $v \in \mathbb{R}^{n_w}$, whose evolution is described by the following equations

$$\begin{aligned} \dot{R} &= -\lambda R + \lambda \gamma(\eta_{[1,n_w]})\gamma(\eta_{[1,n_w]})^T \\ \dot{v} &= -\lambda v + \lambda \gamma(\eta_{[1,n_w]})\eta_{n_w+1} \\ \theta &= R^\dagger v, \end{aligned} \tag{3.14}$$

where λ is the same as in (3.13), and $\gamma : \mathbb{R}^{n_e n_w} \to \mathbb{R}^{n_w \times n_e}$ is defined as

$$\gamma(\eta_{[1,n_w]}) := \mathrm{col}(\eta_1^T \ \eta_2^T \ \cdots \ \eta_{n_w}^T).$$

We highlight that the use of the pseudoinverse operator is mandatory, even for R square, since no conditions preventing it from being singular have been stated.

We restate the *persistence of excitation* (PE) property for η.

Definition 3.1. *With $\epsilon, T > 0$, the signal η is said to have the (ϵ, T)-persistency of excitation property if for all $t \geq T$*

$$\det \int_0^t e^{-\lambda(t-s)} \gamma(\eta_{[1,n_w]}(s))\gamma(\eta_{[1,n_w]}(s))^T ds \geq \epsilon. \tag{3.15}$$

As, with respect to Definition 2.1, $\Sigma = 0$, we observe that the PE condition (3.15) can be checked online simply by looking at $\det(R(t))$.

For future readability we rewrite (3.14) in compact form

$$\begin{array}{rcl}\dot{z} &=& \lambda l(z,\eta)\\ \theta &=& \omega(z),\end{array} \tag{3.16}$$

with $l : \mathcal{Z} \times \mathbb{R}^{n_e(n_w+1)} \to \mathcal{Z}$ and $\omega : \mathcal{Z} \to \mathbb{R}^{n_w}$ defined as

$$\begin{array}{rcl} l(z,\eta) &:=& \begin{pmatrix} -R + \gamma(\eta_{[1,n_w]})\gamma(\eta_{[1,n_w]})^T \\ -v + \gamma(\eta_{[1,n_w]})\eta_{n_w+1} \end{pmatrix} \\ \omega(z) &:=& R^\dagger v. \end{array}$$

We conclude the design of the internal model unit by letting $\bar{\rho}(\eta, z)$ in (3.11) be defined as

$$\bar{\rho}(\eta, z) := \lambda \mathrm{sat}\left(\frac{\partial \Delta(\eta_{[1,n_w]}, z)}{\partial z} l(z,\eta)\right), \tag{3.17}$$

with $\mathrm{sat}(\cdot)$ any properly defined smooth saturation function and

$$\Delta(\eta_{[1,n_w]}, z) := (\omega(z)^T \otimes I_{n_e})\eta_{[1,n_w]}.$$

3.2.3 The Stabilizer

The stabilizer is a linear output feedback controller parametrized by θ, and smooth in $p_{\mathcal{E}}(\theta)$, of the form

$$\begin{array}{l}\dot{\chi} = H_\chi(p_{\mathcal{E}}(\theta))\chi + H_y(p_{\mathcal{E}}(\theta))y + H_\eta(p_{\mathcal{E}}(\theta))\eta \\ u = K_\chi(p_{\mathcal{E}}(\theta))\chi + K_y(p_{\mathcal{E}}(\theta))y + K_\eta(p_{\mathcal{E}}(\theta))\eta,\end{array} \tag{3.18}$$

with $\chi \in \mathbb{R}^{n_\chi}$, and it is designed in order to make the matrix

$$F(\theta) := \begin{pmatrix} A + BK_y(p_{\mathcal{E}}(\theta))C & BK_\eta(p_{\mathcal{E}}(\theta)) & BK_\chi(p_{\mathcal{E}}(\theta)) \\ G_\eta C_e & \Phi_\eta(\theta) & 0 \\ H_y(p_{\mathcal{E}}(\theta))C & H_\eta(p_{\mathcal{E}}(\theta)) & H_\chi(p_{\mathcal{E}}(\theta)) \end{pmatrix} \tag{3.19}$$

Hurwitz for all $\theta \in \mathbb{R}^{n_w}$, where:

$$\Phi_\eta(\theta) = \begin{pmatrix} 0_{n_e n_w \times n_e} & I_{n_e n_w} \\ 0_{n_e \times n_e} & p_{\mathcal{E}}(\theta)^T \otimes I_{n_e} \end{pmatrix}, G_\eta = \begin{pmatrix} 0_{n_e n_w \times n_e} \\ I_{n_e} \end{pmatrix} \tag{3.20}$$

Under Assumption 3.1, and by construction of $\mathcal{E}$, a stabilizer of the form (3.18) making $F(\theta)$ Hurwitz always exists. An example of synthesis can be found in Stilwell and Rugh (2000).

3.3 Asymptotic Properties of the Closed-Loop System

The closed-loop system reads as

$$\dot{z} = \lambda l(z, \xi) \tag{3.21}$$

$$\dot{w} = Sw \tag{3.22}$$

$$\dot{\xi} = F(z)\xi + P_\xi w + \lambda \rho_\xi(\eta, z), \tag{3.23}$$

where $\xi := (x, \eta, \chi)$, $F(z)$, for $\theta = R^\dagger v$ is as in (3.19), $P_\xi := \mathrm{col}(P,\ 0_{(n_w+1+n_\chi)\times n_w})$ and

$$\rho_\xi(\eta, z) := \mathrm{col}\left(0_{n_x\times 1},\ \frac{1}{\lambda}\bar{\rho}(\eta, z),\ 0_{n_\chi \times 1}\right),$$

The following proposition characterizes the asymptotic properties of the regulator.

Proposition 3.1. *Suppose Assumptions 3.1 and 3.2 are satisfied, and that the transfer function of the plant has a finite number of zeros. Then, for any compact set* $Z \times W \times \Xi \subset \mathcal{Z} \times \mathbb{R}^{n_w} \times \mathbb{R}^{n_x+n_e(n_w+1)+n_\chi}$, *there exists* $\lambda_1^\star > 0$ *such that, for all* $\lambda \in (0, \lambda_1^\star]$, *the trajectories of the closed loop* (3.21)-(3.23) *are bounded. If in addition there exists* $\theta^\circ \in \mathcal{E}$ *such that*

$$-\sum_{i=1}^{n_w} \theta_i^\circ s^{i-1} + s^{n_w} = p_S(s),$$

then for any $\epsilon > 0$ *there exists* $\lambda_2^\star \in (0, \lambda_1^\star]$ *such that, if* $\lambda \in (0, \lambda_2^\star]$, *any solution of the closed-loop system* (3.21)-(3.23) *such that, for some* $T > 0$, η *has the* $(\epsilon, T)-$*persistency of excitation property, also satisfies*

$$\lim_{t\to\infty} e(t) = 0.$$

Proof.

1) *Boundedness of trajectories of the closed loop system*

Consider the closed loop system (3.21), (3.22) and (3.23) under the change of coordinates $\xi \mapsto \tilde{\xi} = \xi - \Pi(\theta)w$, with $\Pi(\theta)$ the unique solution to the Sylvester equation $\Pi(\theta)S - F(\theta)\Pi(\theta) = P_\xi$. The matrix $\Phi(\theta)$ is smooth in $p_\mathcal{E}(\theta)$ since $F(\theta)$ is smooth in $p_\mathcal{E}(\theta)$ and, from Section 4.3 in Horn and Johnson (1994), the solution to the Sylvester equation is smooth in the elements of $F(\theta)$. In the new coordinates the closed-loop system reads as

$$\dot{z} = \lambda l(z, \eta) \tag{3.24}$$
$$\dot{w} = Sw \tag{3.25}$$
$$\dot{\tilde{\xi}} = F(\theta)\tilde{\xi} + \lambda\rho_{\tilde{\xi}}(\tilde{\eta}, w, \theta, z), \tag{3.26}$$

where:

$$\rho_{\tilde{\xi}}(\tilde{\eta}, w, \theta, z) = \\ = \frac{1}{\lambda}\bar{\rho}(\tilde{\eta} + \Pi_\eta(\theta)w, z) - \frac{\partial \Pi(\theta)}{\partial \theta} w \frac{\partial \omega(z)}{\partial z} l(z, \tilde{\eta} + \Pi_\eta(\theta)w). \tag{3.27}$$

We proceed to study the stability of (3.24)(3.25)(3.26) as proposed in Teel et al. (2003) for systems in *two time − scale averaging* form. In particular, considering separately the stability of the *boundary layer* system for $\lambda = 0$, and averaged *reduced* system, and show that for appropriate choice of λ, semiglobal practical stability of the closed-loop system (3.24)(3.25)(3.26) is achieved.

To study the boundary layer system, we impose $\lambda = 0$. As a consequence, system (3.24)(3.25)(3.26) is reduced to

$$\begin{aligned} \dot{z}_{bl} &= 0 \\ \dot{w}_{bl} &= Sw_{bl} \\ \dot{\tilde{\xi}}_{bl} &= F(\theta)\tilde{\xi}_{bl}. \end{aligned} \tag{3.28}$$

Since for the boundary layer system θ is constant, and $F(\theta)$ is Hurwitz for all $\theta \in \mathbb{R}^{n_w}$, from the design of the stabilizer in Section 3.2.3, it directly follows that

$$|\tilde{\xi}_{bl}(t)| \leq \beta_{bl}(|\tilde{\xi}_{bl}(0)|, t), \tag{3.29}$$

with

$$\beta_{bl}(|\tilde{\xi}_{bl}(0)|,t) = e^{F(\theta)t}\tilde{\xi}_{bl}(0). \tag{3.30}$$

As a consequence, Assumption 3 of Teel et al. (2003) is satisfied.

We now study the reduced system, referring to Remark 17 of Teel et al. (2003) to compute an admissible average of the reduced system. Namely, consider the dynamical system

$$\dot{z}_{av} = F_{av}(z_{av}), \tag{3.31}$$

with

$$F_{av}(z_{av}) = \lim_{T\to\infty,\lambda\to 0^+} \frac{1}{T}\int_0^T l(z_{av}, \Pi_\eta(\omega(z_{av}))w(t))dt, \tag{3.32}$$

where the input $\bar{\eta}$ has been considered at its equilibrium $\Pi_\eta(\omega(z))w(t)$ and dependency on time t has been considered only for the "fast" state variables, i.e. ξ and z.

By periodicity in t of $\Pi_\eta(\theta)w(t)$, Lipschitz continuity in θ (due to smoothness of $\Pi_\eta(\theta)$ and boundedness of the same since we are considering the projection of θ on the compact set $\mathcal{E}$) and considering the expression of $l(\cdot,\cdot)$ in (3.14), it follows from Lemma 4.6.4 in Sanders et al. (2007) that the integral in (3.32) exists and the limit is uniform in z on compact sets $\Gamma \subset \mathcal{Z}$. In particular, system (3.31) can be rewritten as

$$\dot{z}_{av} = -z_{av} + \bar{u}_{av}(z_{av}) \tag{3.33}$$

where $\bar{u}_{av}(\theta)$ is the average over t of
$\mathrm{col}(\gamma(\Pi_{\eta_{[1,d]}}(\omega(z_{av}))w(t))\gamma(\Pi_{\eta_{[1,d]}}(\omega(z_{av}))w(t))^T, \gamma(\Pi_{\eta_{[1,d]}}(\omega(z_{av}))w(t))\Pi_{\eta_{d+1}}(\omega(z_{av}))w(t))$.

The state variable z converges exponentially to

$$z^\star_{av}(t) := \int_0^t e^{-(t-s)}\bar{u}_{av}(\omega(z_{av}(s)))ds. \tag{3.34}$$

By denoting as $\tilde{z}_{av}(t) = z_{av}(t) - z^\star_{av}(t)$, Assumption 4 in Teel et al. (2003) is readily satisfied, since

$$|\tilde{z}_{av}(t)| \leq \beta_{av}(|\tilde{z}_{av}(0)|, \lambda t), \tag{3.35}$$

with

$$\beta_{av}(|\tilde{z}_{av}(0)|, \lambda t) = e^{-\lambda t}|\tilde{z}_{av}(0)|. \tag{3.36}$$

It can be easily shown that for any compact set of initial conditions $Z \times W \times \Xi \subset \mathcal{Z} \times \mathbb{R}^{n_w} \times \mathbb{R}^{n_x+n_e(n_w+1)+n_\chi}$, Assumptions 7-8 in Teel et al. (2003) are satisfied as well.
From Proposition 2 in Teel et al. (2003), Theorem 1 in Teel et al. (2003) follows. We report it next.

Theorem 3.1. *For each $\delta > 0$ there exists $\lambda_1^\star$ such that, for all $\lambda \in (0, \lambda_1^\star]$ and all initial conditions in $Z \times W \times \Xi \subset \mathcal{Z} \times \mathbb{R}^{n_w} \times \mathbb{R}^{n_x+n_e(n_w+1)+n_\chi}$, the solutions to* (3.21)(3.22)(3.23) *exist and satisfy for all $t \geq 0$*

$$|z(t) - z_{av}^\star(t)| \leq \beta_{av}(|z(0) - z_{av}(0)|, \lambda t) + \delta \tag{3.37}$$

$$|\tilde{\xi}| \leq \beta_{bl}(|\tilde{\xi}(0)|, t) + \delta \tag{3.38}$$

Boundedness of the trajectories of (3.21)(3.22)(3.23) follows by noticing that $\Pi(\theta)w(t)$, the equilibrium for the state variable ξ, is bounded for all $\theta \in \mathbb{R}^{n_w}$. Moreover, boundedness of $\Pi(\theta)w(t)$, implies that $z_{av}^\star$ is bounded as well, and as a consequence $z(t)$ is bounded.

2) *Semiglobal asymptotic stability*

Define $z^\star(t) \in \mathcal{Z}$ as

$$z^\star(t) := \lambda \int_0^t e^{-\lambda(t-s)} \bar{u}(\eta^\star) ds, \tag{3.39}$$

with $\eta^\star \in \mathbb{R}^{n_e(n_w+1)}$ and $\bar{u}(\eta^\star) := \text{col}(\gamma(\eta^\star_{[1,d]})\gamma(\eta^\star_{[1,d]})^T, \gamma(\eta^\star_{[1,d]})\eta^\star_{d+1})$. Then $z^\star(t)$ is solution of (3.14) to $z(0) = 0$ and $\eta = \eta^\star$.

By differentiation of (3.13) with respect to θ, for fixed $t \in \mathbb{R}_{\geq 0}$, we notice that

$$\nabla_\theta(\mathcal{J}_{\eta^\star}(\theta))(t) = \tag{3.40}$$
$$= -2\lambda \int_0^t e^{-\lambda(t-s)}(\gamma(\eta^\star_{[1,d]}(s))\varepsilon(s,\theta))ds = \tag{3.41}$$
$$= 2(R^\star(t)\theta - v^\star(t)) \tag{3.42}$$

Since the set of minimizers of (3.13) for $\eta = \eta^\star$ is given by $\{\theta \in \mathbb{R}^{n_\theta} : \nabla_\theta(\mathcal{J}_{\eta^\star}(\theta))(t) = 0\}$, it follows that

$$\theta^\star(t) = (R^\star(t))^\dagger v^\star(t) \tag{3.43}$$

For every $\delta \in \mathbb{R}^{n_e(n_w+1)}$, define now $\eta = \eta^\star + \delta$. It is possible to show, by Lipschitz continuity of $\bar{u}(\cdot)$ in η (following from boundedness of trajectories), that

$$|z(t) - z^\star(t)| \leq e^{-\lambda t}|z(0) - z^\star(0)| + L_{\bar{u}}\|\delta_t\|_\infty \tag{3.44}$$

with $L_{\bar{u}}$ the Lipschitz constant of $\bar{u}(\cdot)$.

From $\eta = \Pi_\eta(\theta)w$, the definition of $\Pi(\theta)$ gives

$$\begin{aligned} \Pi_{\eta_i}(\theta)S &= \Pi_{\eta_{i+1}}(\theta), \quad i = 1, ..., n_w \\ \Pi_{\eta_i}(\theta) &= \Pi_{\eta_1}(\theta)S^{i-1} \quad i = 1, ..., n_w + 1, \end{aligned} \tag{3.45}$$

By letting c_i, $i = 0, ..., n_w - 1$, be the coefficients of the characteristic polynomial of S and by the Cayley-Hamilton Theorem, we have

$$\begin{aligned} \Pi_{\eta_{n_w+1}}(\theta) &= \Pi_{\eta_1}(\theta)S^{n_w} = -\Pi_{\eta_1}(\theta)\sum_{i=0}^{n_w-1} c_i S^i = \\ &= -\sum_{i=0}^{n_w-1} c_i \Pi_{\eta_1}(\theta)S^i = -\sum_{i=0}^{n_w-1} c_i \Pi_{\eta_{i+1}}(\theta), \end{aligned}$$

The prediction error (3.12) would read in this case as

$$\begin{aligned} \varepsilon(t,\theta) = &-\sum_{i=0}^{n_w-1} c_i \Pi_{\eta_{i+1}}(\theta)w(t) - (\theta^T \otimes I_{n_e})\Pi_{\eta_{[1,n_w]}}(\theta)w(t) = \\ &-\sum_{i=0}^{n_w-1} c_i \Pi_{\eta_{i+1}}(\theta)w(t) - \sum_{i=0}^{n_w-1} \theta_{i+1}\Pi_{\eta_{i+1}}(\theta)w(t). \end{aligned} \tag{3.46}$$

It follows that (3.13) has a global solution given by

$$\theta^\circ = -\operatorname{col}(c_0, ..., c_{n_w-1}).$$

Under persistency of excitation, θ° is also the unique solution to (3.13).

Suppose now that $\theta^\circ \in \mathcal{E}$. By choosing $z^\star$ in (3.44) such that $\theta^\circ = \omega(z^\star)$, it follows that (3.14), in the coordinates $z - z^\star$, is ISS with respect to $\eta - \eta^\star$, with $\eta^\star = \Pi_\eta(\theta^\circ)w$.

From boundedness of the trajectories of (ξ, z), it follows that the term $\rho_{\tilde{\xi}}(\cdot)$ in (3.26) is bounded. Moreover, noticing that for $z = z^\star$, $\rho_{\tilde{\xi}}(\cdot) = 0$, there exists a constant $\bar{a} > 0$ such that $|\rho_{\tilde{\xi}}(\cdot)| \leq \bar{a}\,\|z - z^\star\|$ uniformly in $\tilde{\xi}^\star$, where $\tilde{\xi}^\star = \xi - \Pi(\theta^\circ)w$.

Denoting $\tilde{z}^\star = z - z^\star$, we can write the bounds

$$|\tilde{\xi}^\star(t)| \leq \beta_{\tilde{\xi}^\star}(|\tilde{\xi}^\star(0)|, t) + \alpha_{\tilde{\xi}^\star}(\|\tilde{z}^\star_t\|_\infty) \tag{3.47}$$

$$|\tilde{z}^\star(t)| \leq \beta_{\tilde{z}^\star}(|\tilde{z}^\star(0)|, t) + \alpha_{\tilde{z}^\star}(\|\tilde{\xi}^\star_t\|_\infty), \tag{3.48}$$

where

$$\beta_{\tilde{\xi}^\star}(|\tilde{\xi}^\star(0)|, t) := e^{-F(\theta)t}|\tilde{\xi}^\star(0)| \tag{3.49}$$

$$\alpha_{\tilde{\xi}^\star}(\|\tilde{z}^\star\|_\infty) := \lambda\bar{a}\int_0^\infty \|e^{F(\theta)s}\|ds\|\tilde{z}^\star_t\|_\infty \tag{3.50}$$

$$\beta_{\tilde{z}^\star}(|\tilde{z}^\star(0)|, t) := e^{-\lambda t}|\tilde{z}^\star(0)| \tag{3.51}$$

$$\alpha_{\tilde{z}^\star}(\|\tilde{\xi}^\star\|_\infty) := L_{\bar{u}}\|\tilde{\xi}^\star_t\|_\infty. \tag{3.52}$$

From Proposition 4.1 in Jiang et al. (1994) it follows that for sufficiently small $\lambda\bar{a}$ the interconnection (3.21)(3.23) is semiglobally (due to the boundedness analysis based on Teel et al. (2003)) asymptotically stable.

Since λ is a design parameter, the above condition can always be fulfilled. We can thus define as $\lambda_2^\star := \min(\lambda_1^\star, \lambda_{sm})$, with $\lambda_{sm} > 0$ the minimum $\lambda > 0$ satisfying the small-gain condition.

Then, by the definition of $\Pi(\theta)$, the structure of the identifier, and by using (3.45), we obtatin that, for $\theta = \theta^\circ$, the quantity $\Pi_e(\theta) := C_e\Pi_x(\theta) + Q_e$ fulfils

$$\Pi_e(\theta^\circ) = \Pi_{\eta_{n_w+1}}(\theta^\circ)S - \sum_{i=1}^{n_w}\theta_i^\circ\Pi_{\eta_{i+1}}(\theta^\circ) =$$

$$= -\sum_{i=1}^{n_w} (c_{i-1} + \theta_i^\circ)\Pi_{\eta_i}(\theta^\circ)S = 0,$$

hence $e \to 0$, proving the claim of Proposition 3.1.

■

Proposition 3.1 states that, in order to obtain asymptotic regulation along the persistently exciting solutions, the dynamics of the identifier have to be slow enough compared to the rest of the control system. In this respect it is worth comparing this result with the approach of Bin et al. (2019b), where the time-separation of the adaptation dynamics is obtained by means of a discrete-time identifier working on time instants that must be separated, on average, by a sufficiently large amoung of time.

We also observe that the convergence of e to zero is uniform only inside the set of the solutions for which the signals $\eta(t)$ satisfy Definition 3.1 with the same ϵ and T.

3.4 An Example

As an example of application we will consider a linear system of the form (3.2) defined by the following matrices

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 3 & 1 \\ 2 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -1 \\ 0 & 0 \\ 2 & 1 \end{pmatrix},$$
$$C_e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix},$$
$$Q_e = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix},$$

The exosystem matrix S is defined as $S = \text{blkdiag}(S_1, S_2, S_3)$, with:

$$S_1 = \gamma_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; S_2 = \gamma_2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; S_3 = 0,$$

and where $\gamma_1, \gamma_2 > 0$ are unknown parameters. The errors to be regulated are thus defined as

$$e_1 := x_1 - w_5, \qquad e_2 := x_3 - w_3.$$

We observe that the system considered here is not minimum-phase with respect to the input u and the output e, and relative to the ideal error-zeroing steady state given by the graph of Π, where Π is such that, for some $\Gamma \in \mathbb{R}^{n_m \times n_w}$, (Π, Γ) is the unique solution of the regulator equations

$$\Pi S = A\Pi + B\Gamma + P, \quad C_e\Pi + Q_e = 0.$$

As a matter of fact, changing coordinates as $x \mapsto \tilde{x} = x - \Pi w$, and letting $e = 0$, yields

$$\dot{\tilde{x}}_2 = 3\tilde{x}_2.$$

For simplicity, we will assume $C_m := \mathrm{col}(0, 1, 0)^T$ and $Q_m := 0_{1\times 5}$, and define the output of the considered system as $y := \mathrm{col}(e_1, x_2, e_2)$. The set $\mathcal{E}$ was chosen, after experimental tests on the non-resonance of the extended system (3.2)-(3.8), as $\mathcal{E} = [-3, 7] \times [-12, -3] \times [-6, 3] \times [-20, -7] \times [-7, 2]$. The stabilizer can thus be designed as the static feedback regulator $u = K(\theta)\,\mathrm{col}(y, \eta)$, with $K(\theta) := \mathrm{col}(K_y(\theta), K_\eta(\theta))$ a gain scheduling controller. For simplicity, in this example, $K(\theta) = \bar{K}$, with $\bar{K}$ constant, and $\theta(0) \in \mathcal{E}$. Figure 3.1 shows the results of the simulation of the control system obtained with $\lambda = 0.01$, $\gamma_1 = 3$, $\gamma_2 = 1$, $w(0) = \mathrm{col}(1, -1, 0, -1, 7)$ and $x(0) = (10, -2, 3)$.

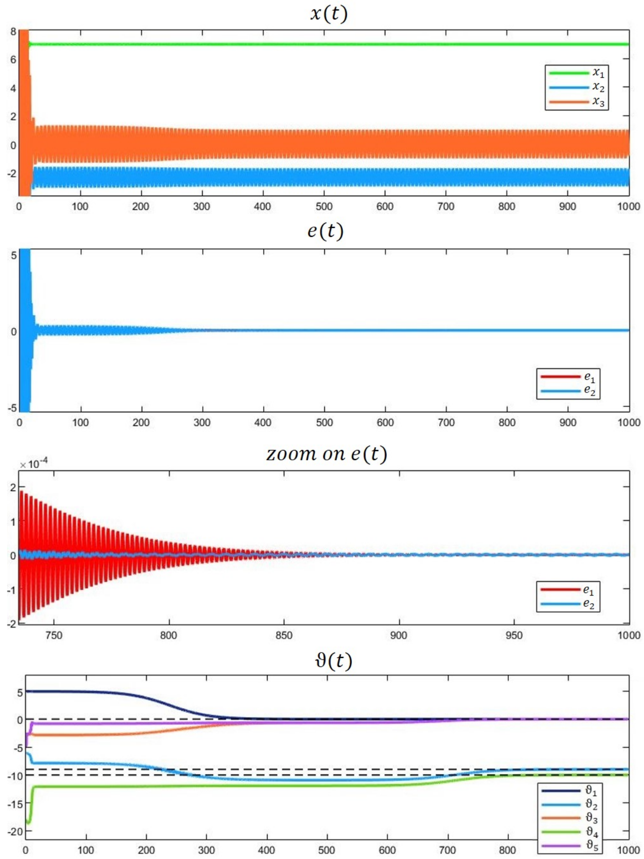


Figure 3.1: Plots of the trajectories of $x(t)$, $e(t)$, $\theta(t)$ resulting from the simulation. In the fourth plot, the black dashed lines represent the values of the coefficients of $\varphi_S(s)$.

Concluding Remarks

This first part of the thesis was aimed at the development of an adaptive output regulation controller to tackle the regulation problem when no knowledge of an exogenous system, whose signals may represent both the reference trajectory, both disturbances, is assumed. To lay the groundwork for the development in the following sections, we recalled in Section 1 the basic ingredients of the theory of output regulation for linear systems, restating, in particular, the concept of steady state for a linear system driven by an exogenous input, and the design of the controller solving the output regulation problem, when knowledge of the modes of the exosystem is assumed, proposed in Davison (1976).

On the wake of the results in Bin et al. (2017), we propose to solve the aforementioned output regulation problem by employing a continuous time identifier, in the form of least-squares, to estimate online the parameters of an internal model unit to be adapted. As a consequence, we review in Section 2 the identification framework proposed in Bin et al. (2019a), and applying it to continuous time least squares identifiers. In Section 3, at last, we develop the stated adaptive controller. In particular, by designing the identifier dynamics to depend linearly on a tunable parameter λ, we can exploit a small gain condition between the identifier and the extended plant in order to, by two-time scales separation, design the stabilizer and the identifier separately and conclude boundedness of the closed loop trajectories. Moreover, if the true model parameters belong to the restriction of considered parameter space, under a persistence of excitation condition, semiglobal asymptotic regulation to the error zeroing subspace is achieved. We highlight that the design proposed does not make any minimum-phase assumption, as evident from the reported numerical example.

The solution proposed in this thesis is just a small step towards a more general approach for robust output regulation. Even if works addressing the same problem, for both the linear and nonlinear case, with similar ideas (see e.g.Bin et al. (2019a) and Bin et al. (2017)) are recently being published, the problem is far from solved. In particular, the main future research directions that we will investigate are two-fold. One one hand, investigating new identifier schemes, in a pursuit of better and more general solutions. One candidate is the Direct Search algorithm developed in the next part of the thesis. In this case, on the wake of Bin et al. (2017), a hybrid implementation would be necessary. On the other hand, still relaying on the results developed in the next sections, if we consider the error to not be measurable, and, in its stead, to be able to measure a proxy function, whose minimum position corresponds to the error zeroing subspace, an interconnection of the adaptive regulator developed in the previous sections and the hybrid controller developed in the next ones, might candidate as possible solution to solve this problem.

Part II

Robust Direct Search Hybrid Optimization

4

Direct Search Methods

As already mentioned in the introduction to this thesis, we aim at developing a robust control framework in which the least amount of knowledge is assumed to be available. As such, we introduce in section the class of algorithms denoted Direct Search algorithms.

The main advantage of Direct Search algorithms is that they do make not use, or try to estimate, any derivative information of the objective function. Moreover no knowledge of the cost function has to be a priori assumed, apart from some structural properties. As consequence they lend themselves as a nice candidate to tackle the problem of output regulation when no knowledge of the error is available, but only a proxy function, not known, and whose minimum is the desired steady state, is available for measurement.

Not assuming any knowledge of the cost function to be optimized, Direct Search algorithm are based on exploratory moves. Namely, given a set of vectors spanning the search space, they iteratively compute steps along these directions, continuing the exploration in a direction only if the cost function value has decreased (or increased).

On one hand, not assuming or estimating derivative information, affects the speed of convergence of these algorithms, but on the other hand, as we will see in Chapter 6, grants clear robustness bounds to measurement noise affecting the objective function, as well as gives an opening to their application to non-smooth optimization problems (see for example Popovic and Teel (2004)). Moreover, due to the inherent simplicity of the Direct Search algorithms, their implementation is straightforward and computationally very efficient.

In the next sections we will first define direct search algorithms and their distinguishing features, to then introduce a broad subclass of direct search algorithms denote Generating Set Search (GSS) algorithms. The following section treats a direct search algorithm based on conjugate directions, whose interesting property is that it converges to the minimum of a convex quadratic function in a finite number of line minimizations. At last we design a GSS algorithm, exploiting the property of conjugate directions, for a particular class of continuously differentiable functions. As direct search algorithms, by definition, can at best be proven to converge to a stationary point for a general continuously differentiable objective function, we define the class of objective functions in order to remain general, but at the same time guarantee convergence to the set of minima.

4.1 Direct Search Methods

Let us consider the following optimization problem

$$\min_{x \in \mathcal{X} \subset \mathbb{R}^n} f(x), \tag{4.1}$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is the *objective function*, x the *optimization variable* and $\mathcal{X} \subset \mathbb{R}^n$ the *search space*. We propose to solve the stated optimization problem by means of Direct Search algorithm, developed in the next section.

Direct search algorithms are a class of gradient-free optimization algorithms based on comparison of the objective function values between different points of the search space.

The first appearance of the term "direct search" is attributed to Hooke and Jeeves in the 1961 paper Hooke and Jeeves (1961), where the following definition is reported

> We use the phrase "direct search" to describe sequential examination of

> trial solutions involving comparison of each trial solution with the "best" obtained up to that time together with a strategy for determining (as a function of earlier results) what the next trial solution will be. The phrase implies our preference, based on experience, for straightforward search strategies which employ no techniques of classical analysis except where there is a demonstrable advantage in doing so.

As pointed out in Lewis et al. (2000) and Kolda et al. (2003), however, this definition brought confusion in the scientific community as no distinction between "direct search" and "gradient-free" was made clear. Namely, it is not apparent if procedures estimating the gradient of the objective function from function values or fitting the collected data into model functions still belong to the class of direct search algorithms.

The answer to this question remained unclear until Trosset in Trosset (1996) gave a clear definition of "direct search" algorithms. The definition proposed by Trosset, to which we will oblige, is given in the following

Definition 4.1. *A direct search method for numerical optimizations is any algorithm that depends on the objective function only through the ranks of a countable set of function values.*

Hence direct search algorithms, at least in the unconstrained case, work with the difference between the objective function values, regardless of the actual objective function values.

Even before the definition by Hooke and Jeeves in Hooke and Jeeves (1961), direct search algorithms were already known and used. One of the first reported examples of direct search algorithms can be found in Davidon (1991), where Davidon describes a simple direct search algorithm used by Fermi and Metropolis for numerical optimization. He states:

> Enrico Fermi and Nicholas Metropolis used one of the first digital computers, the Los Alamos Maniac, to determine which values of certain theoretical parameters (phase shifts) best fit experimental data (scattering cross sections) Fermi and Metropolis (1952). They varied one theoretical parameter at a time by steps of the same magnitude, and when no such increase or decrease in any one parameter further improved the fit to the experimental data, they halved the step size and repeated

> the process until the steps were deemed sufficiently small. Their simple procedure was slow but sure, and several of us used it on the Avidac computer at the Argonne National Laboratory for adjusting six theoretical parameters to fit the pion-proton scattering data we had gathered using the University of Chicago synchrocyclotron Davidon (1991).

As simple as it is, the procedure used by Fermi and Metropolis encompasses all the characteristics of a direct search algorithm. Namely no knowledge of the objective function is assumed, nor any attempt is made to fit the objective function values to any model, or to estimate the gradient.

The algorithm works by acquiring new objective function values by exploring the search space, one variable at a time, through steps of the same magnitude. As Davidon, we will denote this quantity *step size* and identify it with the symbol Δ.

We will call the procedure of minimizing an objective function along a direction, starting from a point of the search space, a *line minimization* procedure. In this regard we distinguish between two types of line minimizations.

Definition 4.2 (Exact line minimization)**.** *Consider a direction $d \in \mathbb{R}^n$, a starting position $\bar{x} \in \mathbb{R}^n$ and a function $f : \mathbb{R}^n \to \mathbb{R}$. An exact line minimization is any procedure solving the following optimization problem*

$$\min_{\alpha \in \mathbb{R}} f(x + \alpha d),$$

and returning the value α.

Definition 4.3 (Discrete line minimization with fixed step size)**.** *Consider a direction $d \in \mathbb{R}^n$, a starting position $\bar{x} \in \mathbb{R}^n$, a function $f : \mathbb{R}^n \to \mathbb{R}$. A discrete line minimization with fixed step size $\Delta > 0$ is any procedure that solves the following optimization problem*

$$\min_{\beta \in \mathbb{Z}} f(x + \beta \Delta d)$$

and returns the value β.

The algorithm used by Fermi and Metropolis is based on discrete line minimizations, and, as it explores only the coordinate axes, it is usually denoted as *compass search* (or *coordinate search*) algorithm.

In the next sections we will introduce a class of direct search algorithms based on line minimizations, named *Generating Set Search* methods, and a particular algorithm belonging to that class, the *Recursive Smith-Powell* (or *conjugate directions*) algorithm, whose main property is to guarantee convergence the minimizer of a convex quadratic function in a finite number of (exact) line minimizations.

4.1.1 Generating Set Search (GSS) Algorithms

The convergence of the compass search algorithm stems from exploring the search space along a set of directions spanning all $\mathbb{R}^n$, the coordinate axes. Generating set search algorithms generalize this idea. The name *Generating Set Search* comes, indeed, from the set of directions along which the minimum of the objective function is searched. This set is finite and has to "generate" (positively span) all the feasible directions, i.e. $\mathbb{R}^n$ in the unconstrained case.

Definition 4.4 (Positively spanning set)**.** *Let $\mathcal{G} = \{d_1, d_2, ..., d_p\}$ be a set of $p \geq n+1$ vectors in $\mathbb{R}^n$. Then the set $\mathcal{G}$ generates (or positively spans) $\mathbb{R}^n$ if for any vector $v \in \mathbb{R}^n$, there exist $\lambda_1, \lambda_2 ..., \lambda_p \geq 0$ such that:*

$$v = \textstyle\sum_{i=1}^{p} \lambda_i d_i$$

The importance of positively spanning sets is due to the following property.

Property 4.1. *For a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ and a set of positively spanning directions $\{d_1, d_2, ..., d_p\}$, with $p \geq n+1$, for all $x \in \mathbb{R}^n$ such that $\nabla f(x) \neq 0$, there exists $i \in \{1, 2, ..., p\}$ such that $\langle \nabla f(x), d_i \rangle < 0$.*

Namely at least one direction in $\{d_1, d_2, ..., d_p\}$ is a descent direction.

A general GSS algorithm is shown in Fig. 4.1 (Kolda et al. (2003)).

Denote as $\mathcal{G}_k$ the set of directions positively spanning $\mathbb{R}^n$ at iteration k. At each iteration k of the algorithm in Fig. 4.1, a step $\Delta_k > 0$ is taken from the current iterate x_k in a direction $d_k \in \mathcal{G}_k$ and the objective function f is evaluated at that point, i.e. $f(x_k + \Delta_k d_k)$. If a point with a smaller objective function value is found, namely the following *sufficient decrease* condition is satisfied

$$f(x_k) > f(x_k + \Delta_k d_k) - \rho(\Delta_k) \quad \textit{with } d_k \in \mathcal{G}_k, \tag{4.2}$$

the search continuous in that direction, namely the next iteration starting value is taken as:

$$x_{k+1} = x_k + \Delta_k d_k, \tag{4.3}$$

Algorithm : GSS

Data: Suppose given a set $\mathcal{G}_0 := \{d_{01}, d_{02}, ..., d_{0p}\}$ of positively spanning directions in $\mathbb{R}^n$ and an intial step size $\Delta_0 > 0$. Let $\theta_{max} \in (0, 1)$ be an upper bound on the contraction parameter $\theta_k \in (0, 1)$. Let $\phi_{max} \in [1, \infty)$ be the upper bound on the expansion parameter $\phi_k \in [1, \infty)$. Let $\rho : [0, +\infty) \to \mathbb{R}$ be a continuous function such that $\rho(\Delta_k) = o(\Delta_k)$ as $\Delta_k \to 0$. Let $\beta_{max} \geq \beta_{min} > 0$ be upper and lower bounds, respectively, on the lengths of the vectors in the generating sets $\mathcal{G}_k$. Let $\kappa_{min} > 0$ be a lower bound on the cosine measure of any generating set $\mathcal{G}_k$. Let the initial position $x_{00} \in \mathbb{R}^n$ be given.

for $k \in \mathbb{N}$ **do**
- **if** $\exists d_k \in \mathcal{G}_k : \ f(x_k + \Delta_k d_k) \leq f(x_k) - \rho(\Delta_k)$ **then**
  - $x_{k+1} \longleftarrow x_k + \Delta_k d_k$
  - $\Delta_{k+1} \longleftarrow \Phi_k \Delta_k$
- **else**
  - $x_{k+1} \longleftarrow x_k$
  - $\Delta_k \longleftarrow \theta_k \Delta_k$
- **end**
- Select $\mathcal{G}_{k+1}$ such that $\forall d \in \mathcal{G}_{k+1}$ $\beta_{\min} \leq \|d\| \leq \beta_{\max}$ and $\kappa(\mathcal{G}_{k+1}) \geq \kappa_{\min}$.

end

Figure 4.1: The GSS algorithm

possibly increasing the step size. In case no member of $\mathcal{G}_k$ satisfies (4.2) we will call the iteration *unsuccessful*. In this case the step size Δ_k is reduced and the procedure restarts all over again.

Notice that the sufficient decrease condition is reduced to the classic *simple decrease* condition for $\rho(\cdot) \equiv 0$. The importance of the sufficient decrease condition over the simple decrease is related to the globalization strategy adopted, and will be clarified soon.

The *cosine measure* of $\mathcal{G}$ is defined as

$$\kappa(\mathcal{G}) := \min_{v \in \mathbb{R}^n} \max_{d \in \mathcal{G}} \frac{v^T d}{\|v\| \, \|d\|}, \tag{4.4}$$

that is the cosine of the biggest angle between a direction $d \in \mathcal{G}$ and any vector $v \in \mathbb{R}^n$.

The lower bound on $\kappa(\mathcal{G}_k)$ in Fig. 4.1 guarantees that the property of Property 4.1 is satisfied for all k. As reported in Lucidi and Sciandrone (2002), this condition can be relaxed to a lower bound on the limit of $\kappa(\mathcal{G}_k)$ for $k \to \infty$.

In Kolda et al. (2003) it has been shown that the step size parameter Δ_k is directly related to the convergence of GSS methods to a stationary point (namely a point x_k such that $\nabla f(x_k) = 0$).

Indeed let us define as $\mathcal{U} \subset \mathbb{N}$ the set of k that are unsuccessful iterations, the next result follows.

Theorem 4.1. *Kolda et al. (2003) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable, and suppose ∇f is Lipschitz continuous with constant M. Then GSS produces iterates such that for any $k \in \mathcal{U}$, we have*

$$\|\nabla f(x_k)\| \leq \kappa(\mathcal{G}_k)^{-1} \left[M\Delta_k \beta_{max} + \frac{\rho(\Delta_k)}{\Delta_k \beta_{max}} \right] \tag{4.5}$$

It follows that if

$$\lim_{k\to\infty} \Delta_k = 0, \tag{4.6}$$

then a stationary point could be reached, at least for the sequence of unsuccessful iterations. Moreover from (4.5) it is possible to notice that a stopping criterion based on a minimum $\Delta_k = \Delta_{tol}$ does not provide any guarantee that a small enough neighborhood of a stationary point has been reached.

In order to guarantee the convergence to a stationary point of the GSS algorithm, conditions have to be placed on the choice of the step size and the choice of the directions in $\mathcal{G}$ at each iteration. Namely for a too short or a too long step size, the convergence to a stationary point can be hindered, since the sequence of iterate could converge to a point different from a stationary point. The same happens if the chosen descent direction (satisfying only a simple decrease condition) is almost perpendicular to the gradient of f.

For line minimization methods in which the gradient is known, in order to choose a proper step size it is sufficient to satisfy the Wolfe conditions at each iteration,

$$\begin{aligned} f(x_k + \Delta_k d_k) \leq f(x_k) + c_1 \Delta_k \nabla f(x_k)^T \\ \nabla f(x_k + \Delta_k d_k)^T d_k \geq c_2 \nabla f(x_k)^T d_k \end{aligned} \tag{4.7}$$

with $0 < c_1 < c_2 < 1$.

The first condition, named the *Armijo* condition (or *sufficient decrease* condition, not casually with the same name as (4.2)) ensures that the steps are not too big, while the second condition, called the *curvature* condition, ensures that the steps are not too small.

The same conditions would guarantee a proper choice of the step size for GSS algorithms but the problem is that we are assuming no derivative information. In fact the second condition of (4.7) can be automatically ensured by *backtracking*, namely starting with a relatively big step size and reducing it only in case the *Armijo* condition is not satisfied (that is, in our case, only at *unsuccessfull* iterations).

The *Armijo* condition can instead be satisfied by ensuring the *sufficient decrease* condition (4.2) (or by imposing other conditions not relevant to us, refer to Kolda et al. (2003) for a complete review).

A bad choice in the descent direction can be prevented by imposing a lower bound on the cosine measure of the set of directions, as well as the sufficient decrease condition.

The next theorem wraps up the general convergence properties for GSS methods based on a sufficient decrease condition.

Theorem 4.2. *Kolda et al. (2003) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. Moreover let f be bounded below and the sublevel sets of f*

$$\mathcal{L}_{f(x_0)} := \{x \in \mathbb{R}^n : f(x) \leq f(x_0), x_0 \in \mathbb{R}^n\} \tag{4.8}$$

be compact $\forall x_0 \in \mathbb{R}^n$. Then the GSS algorithm in Fig. 4.1 produces iterates such that

$$\lim_{k\to\infty,\ k\in\mathcal{U}} \nabla f(x_k) = 0 \tag{4.9}$$

Remark 4.1. Notice that the sequence x_k, for $k \in \mathcal{U}$, does not need to converge to a single stationary point for the above result to hold. Indeed, as shown in Audet (2004), the sequence of points at unsuccessful iterations can converge to a set of stationary points. △

Remark 4.2. We report that the same result of Theorem 4.2 can be achieved requiring a simple decrease condition but adding different assumptions, see for example Coope and Price (1999) and Kolda et al. (2003). △

Theorem 4.2 can be strengthened to have the whole sequence of iterates converging to a set of stationary points under the hypothesis that, at each iteration, all the directions are checked and only the "best" direction is selected for decrease (see Theorem 3.1Fig. 4.3-4.4 in Kolda et al. (2003)).

4.1.2 Recursive Smith-Powell Method

In this section we will introduce the Recursive Smith-Powell (RSP) method, also called Conjugate Directions method.

Let us start by defining what conjugate directions are and the properties that they possess.

Definition 4.5 (Conjugate directions)**.** *Two non-zero directions (or vectors)* $d_1, d_2 \in \mathbb{R}^n$ *are conjugate with respect to a matrix* $H \in \mathbb{R}^{n\times n}$ *(or* H*-conjugate) if*

$$d_1^T H d_2 = 0 \qquad (\equiv d_2^T H d_1 = 0) \tag{4.10}$$

In the rest of the thesis we will address as *convex quadratic* any function $f : \mathbb{R}^n \to \mathbb{R}$ that can be defined as $f(x) := x^T Hx + b^T x + c$, with $c \in \mathbb{R}, b \in \mathbb{R}^{n\times 1}$ and $H \in \mathbb{R}^{n\times n}$ the Hessian matrix of f assumed to be positive definite, namely $x^T Hx > 0\ \forall x \in \mathbb{R}^n$. When clear from the context, we will refer to two, or more, directions as conjugate with respect to a convex quadratic function when they are conjugate with respect to its Hessian matrix.

The importance of conjugate directions stems from the following theorem.

Theorem 4.3. *Consider a convex quadratic function* $f_c : \mathbb{R}^n \to \mathbb{R}$ *defined as* $f_c(x) := x^T Hx + b^T x + c$*, with* $c \in \mathbb{R}, b \in \mathbb{R}^{n\times 1}$ *and* $H \in \mathbb{R}^{n\times n}$ *the Hessian matrix of* f_c *assumed to be positive definite. Then, given a set of* n *conjugate directions, with respect to* H*, namely a set* $\mathcal{G} := \{d_1, d_2, ..., d_n\} \subset \mathbb{R}^n$*, where*

$$d_i^T H d_j = 0 \qquad i \neq j,$$

a sequence of n *exact line minimizations, one per direction in* $\mathcal{G}$*, reaches exactly the minimum of* f_c*.*

Proof. First of all notice that the vectors in a set of n conjugate directions in $\mathbb{R}^n$ with respect to a positive definite symmetric matrix H, are all linearly independent. To show this, we move by contradiction.

Suppose they are not linearly independent. Then there exists a vector $\alpha =$

$\text{col}(\alpha_1, \alpha_2, ...\alpha_n) \in \mathbb{R}^n$, with $\alpha \neq \mathbf{0}$, such that

$$\sum_{k=1}^{n} \alpha_k d_k = 0. \tag{4.11}$$

Pre-multiplying by H

$$H \sum_{k=1}^{n} \alpha_k d_k = \sum_{k=1}^{n} H \alpha_k d_k = 0. \tag{4.12}$$

Taking the scalar inner product with any $d_j \in \mathcal{G}$, with $j \in \{1, 2, ..., n-1\}$, it follows that

$$\langle d_j, \sum_{k=1}^{n} \alpha_k d_k \rangle = \alpha_j \langle d_j, H d_j \rangle = 0, \tag{4.13}$$

since $\langle d_j, H d_k \rangle = 0 \; \forall j \neq k$ due to conjugacy. Since H is symmetric positive definite, it defines an inner product, and thus $\langle d_j, H d_j \rangle$ should be bigger than zero for $d_j \neq \mathbf{0}$. As a consequence, $\alpha_j = 0$.

Repeating the same reasoning for all $j \in \{1, 2, ..., n-1\}$, it follows that $\alpha = \mathbf{0}$, thus reaching a contradiction. Hence the vectors in $\mathcal{G}$ are linearly independent.

Moving back to the proof of Theorem 4.3, rewrite f_c as

$$f_c(x) = (x - x^\star)^T H (x - x^\star) + \bar{c}, \tag{4.14}$$

where $x^\star$ is the position of the minimum of f_c, $Hx^\star = -b$ and $\bar{c} = c - {x^\star}^T H x^\star$. As the directions in $\mathcal{G}$ form a basis for $\mathbb{R}^n$, we can write

$$x - x^\star = \sum_{k=1}^{n} (\alpha_k - \alpha_k^\star) d_k, \tag{4.15}$$

for some $\alpha_k, \alpha_k^\star \in \mathbb{R}$, $k = 1, 2, ..., n$. From (4.15), we can rewrite (4.14) as

$$f_c(\alpha) = \sum_{k=1}^{n} (\alpha_k - \alpha_k^\star)^2 d_k^T H d_k + \bar{c}.$$

The function f_c can thus be minimized by $\alpha_k = \alpha_k^\star$, $k = 1, 2, ..., n$, and this is equivalent to performing n exact line minimizations along the directions d_k. ■

As the result of Theorem 4.3 is surely interesting for function optimization purposes, in order to compute a set of n conjugate directions, knowledge of the Hessian matrix of the cost function should be assumed. As this is often not possible, Smith

(1962) developed an algorithm, then extended by Powell (1964), that iteratively computes a set of conjugate directions without assuming knowledge of the cost function, apart from its structure (i.e. convex quadratic), by exploiting the following *Parallel Subspace Property*.

Theorem 4.4 (Parallel Subspace Property (Fletcher (2000), Theorem Fig. 4.3-4.4.2.1))**.** *Consider a convex quadratic function $f : \mathbb{R}^n \to \mathbb{R}$ and two parallel affine subspaces $S_1, S_2 \subset \mathbb{R}^n$, generated by linearly independent directions $d_1, d_2, ..., d_k$, where $k < n$, from the points $x_1, x_2 \in \mathbb{R}^n$ respectively. That is $S_j := \{x \in \mathbb{R}^n : x = x_j + \sum_{i=1}^{k} \alpha_i d_i \ \forall \alpha_i \in \mathbb{R}\}$ for $j = 1, 2$. Denote the points which minimize f on S_1 and S_2 by $x_1^\star$ and $x_2^\star$ respectively. Then $x_2^\star - x_1^\star$ is conjugate to $d_1, d_2, ..., d_k$.*

Thus starting from one direction $d_1 \in \mathbb{R}^n$, and computing two exact line minimizations along two parallel (but not coincident) lines defined by d_1, it is possible to compute a direction $d_2 \in \mathbb{R}^n$ conjugate to d_1. The process can continue iteratively by exploring two parallel affine subspaces defined by d_1, d_2 in order to compute d_3 conjugate to d_1, d_2 and so on. That is the idea of the algorithm proposed in Smith (1962).

Notice that, since the restriction of a convex quadratic function to a k-dimensional linear subspace is convex quadratic, to find the minimum of a convex quadratic function on the linear subspace, it is enough to compute k line minimizations along the k conjugate directions defining the subspace.

The algorithm proposed by Powell in Powell (1964) (Fig. 4.2) preserves the *quadratic termination property* of Theorem 4.3, namely reaches the minimum of a convex quadratic function in a finite number of line minimizations. In particular, it is reached in n^2 line minimizations.

In case of discrete line minimizations Theorem 4.3 is, in general, not true anymore. An approximate result can be achieved in that case.

Theorem 4.5 (Discrete Theorem 4.3 Byatt et al. (2004))**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex quadratic function with minimizer $x^\star \in \mathbb{R}^n$. Then, given a step size $\Delta > 0$ and a set of conjugate directions $\mathcal{G}$ (with respect to f), a sequence of n discrete line miniminizations with fixed step size Δ converges to a point $\hat{x}^\star$ such that*

$$\|x^\star - \hat{x}^\star\| \leq \frac{1}{2} diam(P_{\mathcal{G}}(x^\star)), \tag{4.16}$$

where $P_{\mathcal{G}}(x^\star)$ is the n-parallelotope generated by the directions in $\mathcal{G}$ scaled by Δ and

Algorithm : RSP

Data: Suppose given a set $\mathcal{G} := \{d_0, ..., d_{n-1}\}$ of linearly independent directions in $\mathbb{R}^n$. Let the initial position $x_\circ \in \mathbb{R}^n$ be given.
Initialization: Let α_{n-1} be such that $x_\circ + \alpha_{n-1} d_{n-1}$ is the minimizer of (4.17) resulting from an **exact line minimization procedure** along the direction d_{n-1} from $x_\circ$.
$x_{00} \longleftarrow x_\circ + \alpha_{n-1} d_{n-1}$
for $k \in \mathbb{N}$ **do**
    for $j \in \{0, 1, ..., n-1\}$ **do**
        Compute an exact line minimization along d_j from x_{kj} to obtain α_j
        $x_{k(j+1)} \longleftarrow x_{kj} + \alpha_j d_j$
    end
    $z \longleftarrow x_{k0} + \sum_{j=0}^{n-1} \alpha_j d_j$
    for $j \in \{0, 1, ..., n-2\}$ **do**
        $d_j \longleftarrow d_{j+1}$
    end
    $d_{n-1} \longleftarrow z - x_{k0} \quad (= \sum_{j=0}^{n-1} \alpha_j d_j)$
    Compute an exact line minimization along d_{n-1} from z to obtain α_{n-1}
    $x_{(k+1)0} \longleftarrow z + \alpha_{n-1} d_{n-1}$
end

Figure 4.2: The RSP algorithm

centered at $x^\star$.

The problem, again, is that, without knowledge of the cost function, a set of conjugate directions is hard to compute, in a finite amount of line minimizations, without exploiting the Parallel Subspace Property. As this property is no more valid in the discrete setting (unless the minimum is reached exactly in each line minimization), it turns out that, without going out of the framework of direct search algorithms, e.g. by resorting to interpolation techniques, it is not possible to construct a set of conjugate directions while retaining convergence for general functions. Indeed, for example, Brodlie (1975) and Nazareth (1976) could only achieve asymptotic conjugacy for convex quadratic functions.

In Coope and Price (1999) the quadratic termination property was achieved by quadratic interpolation of the explored points and estimate of the gradient and Hessian of the convex quadratic function.

As the aim of this work is not to set apart from the direct search framework, and as asymptotic conjugacy brings no gain with respect to asymptotic convergence due to the step size reduction, we will, on the wake of Mayhew et al. (2007), retain the simplest form of the RSP as a method to compute new directions in the GSS framework, in order to, possibly, better guide the search of the minimum for convex quadratic functions or in a neighborhood small enough of the minimizers for more general functions.

4.1.3 A globally convergent GSS algorithm for a particular class of functions

In this section we proceed to the design of a novel GSS algorithm, based on conjugate directions, that solves the optimization problem

$$\min_{x\in\mathbb{R}^n} f(x), \tag{4.17}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is the objective function. In particular, the novelty introduced is that the proposed algorithm exactly asymptotically solves problem (4.17) under the following assumptions.

(A0) f is continuously differentiable, globally lower bounded and it is not assumed to be known, but sampled measurements of it are supposed to be available every $\tau^\star > 0$, with $\tau^\star$ a tunable parameter;

(A1) the set $\{x \in \mathbb{R}^n : \nabla f(x) = 0\}$ of critical points of f is such that every local minimum is also a global minimum (i.e. all local minima share the same objective function value), every local maximum is an isolated point and f is analytic at every local maximum, and there are no saddle points;

(A2) the sublevel sets of f, namely the sets $\mathcal{L}_f(c) := \{x \in \mathbb{R}^n : f(x) \leq c\}$, are compact for all $c \in \mathbb{R}$.

The set of global minima will be denoted as $\mathcal{A}^\star := \{x^\star \in \mathbb{R}^n : f(x^\star) \leq f(x)\ \forall x \in \mathbb{R}^n\}$.

Assumptions (A0) and (A2) are standard for Direct Search methods, see Coope and Price (1999), Kolda et al. (2003) and Lucidi and Sciandrone (2002). Assumption (A0) can be relaxed by considering f to be locally Lipschitz, as shown in Kolda et al. (2003) and Popovic and Teel (2004), which requires the use of generalized gradients for analysis.

The reason for the particular structure of the set of critical points assumed in (A1) stems from the fact that our goal is to prove and guarantee convergence to the set of minima. While the assumptions on the value of the local minima is considered to simplify the structure of the problem, without the other assumptions on local maxima and saddle points, the proposed algorithm only guarantees convergence to the set of critical points. In particular, the assumption on analyticity at the maxima gives the possibility of escaping local maxima by adopting a non-analytic function of the step size as function imposing the sufficient decrease condition.

The proposed algorithm, shown in Fig. (4.3)-(4.4), is a GSS algorithm and belongs to the class of Algorithm 2 in Lucidi and Sciandrone (2002). It improves the results in Mayhew et al. (2007) by guaranteeing, under the less restrictive assumptions (A0) and (A1), asymptotic convergence to the set of minima. The main differences are highlighted in red. In particular, referring to the Algorithm in Fig. 4.3-4.4:

1) A different step size Δ_i is associated to each direction d_i in order to guarantee more freedom of exploration. As such, when a new direction is computed (lines 19-23) also a new step size is associated to the new direction (line 18);

2) In case no improvement is made along a direction (lines 8-10), the corresponding step size is reduced. This is the key step guaranteeing asymptotic convergence to the minima of the cost function;

3) The newly computed direction is "accepted" only if it keeps the directions in $\mathcal{G}$ linearly independent (lines 19-23), otherwise the previous set of directions is retained.

Remark 4.3. The step size associated to the newly computed direction is chosen as the maximum step size associated to the other directions, but any function bounded by the minimum and maximum of the step sizes would do. This is needed in order to guarantee that the step sizes are asymptotically reduced to zero. △

Remark 4.4. The reduction of the step size when no improvement is found, stems from the framework of GSS algorithms and Theorem 4.2. △

Remark 4.5. As pointed out in Zangwill (1967), 3), or similar steps guaranteeing linear independence of the directions, is necessary, also for convex quadratic functions, in order to guarantee convergence of the algorithm. △

The line minimization procedure explores a direction d_j from a starting point x_{kj} and returns the distance α_j traveled from x_{kj} to the found minimum of f along d_j. The main differences in the line minimization procedure are the following:

Algorithm 2 : Proposed RSP

Data: A set $\mathcal{G}_0 := \{d_{00}, ..., d_{0(n-1)}\}$ of linearly independent directions in $\mathbb{R}^n$, the set of initial step-sizes $\Delta_{0,n-1} := \{\Delta_{00}, ..., \Delta_{0(n-1)}\}$, each corresponding to a direction in $\mathcal{G}$, $\Phi > 0$ a global step size, $0 < \lambda_s < 1 < \lambda_t$, $\theta \in (0,1)$, $\gamma \geq 1$, $\mu \in (0, 1/\lambda_t)$, $\delta_{\text{det}} > 0$, and the initial position $x_\circ \in \mathbb{R}^n$.

Initialization: Let α be such that $x_\circ + \alpha d_{0(n-1)}$ is the minimizer of (4.17) resulting from a line minimization with step-size $\Delta_{0(n-1)}$, along the direction $d_{0(n-1)}$ from $x_\circ$.

$x_{00} \longleftarrow x_\circ + \alpha d_{0(n-1)}$
for $k \in \mathbb{N}$ **do**
    for $j \in \{0, 1, ..., n-1\}$ **do**
        Compute a line minimization with step size Δ_{kj} along d_{kj} from x_{kj} to obtain α_{kj}
        $x_{k(j+1)} \longleftarrow x_{kj} + \alpha_{kj} d_{kj}$
        if $\alpha_{kj} = 0$ **then**
            if $\theta \Delta_{kj} \leq \lambda_s \Phi_k$ **then**
                $\Delta_{kj} \longleftarrow \theta \Delta_{kj}$
            end
        end
    end
    if $\alpha_{kj} = 0 \ \forall j = 0, 1, ..., n-1$ **then**
        $\Phi_{k+1} = \mu \Phi_k$
        for $j \in \{0, 1, ..., n-1\}$ **do**
            if $\Delta_{kj} > \mu \Phi_k$ **then**
                $\Delta_{kj} = \mu \Phi_k$
            end
        end
    end
    $z \longleftarrow x_{k0} + \sum_{j=0}^{n-1} \alpha_{kj} d_{kj}$
    for $j \in \{0, 1, ..., n-2\}$ **do**
        $d_{(k+1)j} \longleftarrow d_{k(j+1)}$
        $\Delta_{(k+1)j} \longleftarrow \Delta_{k(j+1)}$
    end
    $\Delta_{(k+1)(n-1)} \longleftarrow \max_{j \in \{0,1,...,n-2\}} \Delta_{(k+1)j}$
    if $|\det(\text{col}(d^T_{(k+1)0}, d^T_{(k+1)1}, ..., d^T_{(k+1)(n-2)}, (z - x_{k0})^T))| \geq \delta_{det}$ **then**
        $d_{(k+1)(n-1)} \longleftarrow z - x_{k0} \quad (= \sum_{j=0}^{n-1} \alpha_{kj} d_{kj})$
    else
        $d_{(k+1)(n-1)} = d_{k0}$
    end
    Compute a line minimization with step size $\Delta_{(k+1)(n-1)}$ along $d_{(k+1)(n-1)}$ from z to obtain $\alpha_{(k+1)(n-1)}$
    $x_{(k+1)0} \longleftarrow z + \alpha_{(k+1)(n-1)} d_{(k+1)(n-1)}$
end

Figure 4.3: New RSP algorithm.

Line minimization procedure

Initialization: $i = 0$, $x_{kj0} = x_{kj}$, $\Delta_{kj0} = \Delta_{kj}$
while $f(x_{kji} + \Delta_{kji} d_{kj}) \leq f(x_{kji}) - \rho(\Delta_{kji})$ **do**
    $x_{kj(i+1)} \longleftarrow x_{kji} + \Delta_{kji} d_{kj}$
    if $\gamma \Delta_{kji} \leq \lambda_t \Phi_k$ **then**
        $\Delta_{kj(i+1)} \longleftarrow \gamma \Delta_{kji}$
    else
        $\Delta_{kj(i+1)} \longleftarrow \lambda_t \Phi_k$
    end
    $i \longleftarrow i + 1$
end
if $i = 0$ **then**
    while $f(x_{kji} - \Delta_{kji} d_{kj}) \leq f(x_{kji}) - \rho(\Delta_{kji})$ **do**
        $x_{kj(i+1)} \longleftarrow x_{kji} - \Delta_{kji} d_{kj}$
        if $\gamma \Delta_{kji} \leq \lambda_t \Phi_k$ **then**
            $\Delta_{kj(i+1)} \longleftarrow \gamma \Delta_{kji}$
        else
            $\Delta_{kj(i+1)} \longleftarrow \lambda_t \Phi_k$
        end
        $i \longleftarrow i + 1$
    end
end
$\alpha_{kj} \longleftarrow i \Delta_{kji}$, $x_{kj} \longleftarrow x_{kji}$, $\Delta_{kj} \longleftarrow \Delta_{kji}$

Figure 4.4: Line minimization procedure.

1) Newly explored points are accepted only if a *sufficient decrease* condition is satisified (lines 2 and 8);

2) When a new iteration is accepted, the step size is, possibly, increased (lines Fig. 4.3-4.4 and 10);

3) A different step size is associated to each direction and, also, a global step size Φ is considered in order to bound the different step sizes.

Remark 4.6. The sufficient decrease condition guarantees that the Armijo condition, needed for the algorithm convergence, is satisfied and also guarantees a margin of robustness to measurement noise, as we will see in Chapter 6. For the sufficient decrease condition, we adopt the function $\rho : \mathbb{R} \to \mathbb{R}$ defined as

$$\rho(\Delta) := \begin{cases} \Delta^{\frac{1}{\Delta}} & \Delta \leq e \\ \Delta + (e^{\frac{1}{e}} - e) & \Delta \geq e, \end{cases} \tag{4.18}$$

with e the Napier's constant. The function (4.18) is a strictly increasing function of Δ, that at $\Delta = 0$ is smooth (from the right) but non-analytic, and such that $\rho(\Delta) = o(\Delta^n)$ for $\Delta \to 0$ for all $n \in \mathbb{N}$, implying that, under assumption (A1), if $\bar{x} \in \mathbb{R}^n$ is a local maxima for f, there exists $\bar{\Delta} > 0$ such that for all $d \in \mathcal{G}$ and $\Delta \in (0, \bar{\Delta}]$, $f(\bar{x} + \Delta d) < f(\bar{x}) - \rho(\Delta)$. In particular, the following result is true.

Lemma 4.1. *For every $n \in \mathbb{N}_0$, the function $\rho : \mathbb{R} \to \mathbb{R}$ defined in* (4.18) *is $o(\Delta^n)$ for $\Delta \to 0$ and, given $\theta \in (0, 1)$ and $\Delta \in (0, 1)$, the series $\sum_{n=0}^{\infty} (\theta^n \Delta)^{\left(\frac{1}{\theta^n \Delta}\right)}$ is convergent.*

Proof. Throughout this proof we can consider, without loss of generality, $\rho(\Delta) = \Delta^{\frac{1}{\Delta}}$.

Let us show that $\Delta^{\frac{1}{\Delta}}$ is $o(\Delta)$. It comes directly from the definition of little-o notation, indeed

$$\lim_{\Delta \to 0} \frac{\Delta^{\frac{1}{\Delta}}}{\Delta} = \lim_{\Delta \to 0} \Delta^{\frac{1-\Delta}{\Delta}} = 0.$$

From the same reasoning it follows that for every $n > 0$, $\Delta^{\frac{1}{\Delta}}$ is $o(\Delta^n)$. Indeed notice that

$$\lim_{\Delta \to 0} \frac{\Delta^{\frac{1}{\Delta}}}{\Delta^n} = \lim_{\Delta \to 0} \Delta^{\frac{1-n\Delta}{\Delta}} = 0.$$

Regarding the series $\sum_{n=0}^{\infty}(\theta^n\Delta)^{\left(\frac{1}{\theta^n\Delta}\right)}$, we can rewrite it as

$$\sum_{n=0}^{\infty}(\theta^n\Delta)^{\left(\frac{1}{\theta^n\Delta}\right)} = \left(\sum_{n=0}^{\infty}\left(\left(\theta^{\frac{1}{\Delta}}\right)^n\right)^{\left(\frac{1}{\theta^n}\right)}\right)\left(\sum_{n=0}^{\infty}\left((\Delta)^{\left(\frac{1}{\Delta}\right)}\right)^{\left(\frac{1}{\theta^n}\right)}\right). \tag{4.19}$$

By assumption $\theta^{\frac{1}{\Delta}} \in (0,1)$.

Define as $\bar{\theta} \in \mathbb{R}_{>0}$ the smallest real number such that $\theta \leq \bar{\theta}$ and $1/\bar{\theta} \in \mathbb{N}$. Then, as the series $\sum_{n=0}^{\infty}\left(\left(\theta^{\frac{1}{\Delta}}\right)^n\right)^{\left(\frac{1}{\theta^n}\right)}$ is bounded by $\sum_{n=0}^{\infty}\left(\left(\bar{\theta}^{\frac{1}{\Delta}}\right)^n\right)^{\left(\frac{1}{\bar{\theta}^n}\right)}$, that is a subseries of $\sum_{n=0}^{\infty}\left(\bar{\theta}^{\frac{1}{\Delta}}\right)^n$, it will converge.

For the same reasoning, since $\sum_{n=0}^{\infty}\left((\Delta)^{\left(\frac{1}{\Delta}\right)}\right)^{\left(\frac{1}{\theta^n}\right)}$ can be bounded by a subseries of $\sum_{n=0}^{\infty}\left(\Delta^{\frac{1}{\Delta}}\right)^n$, and all the terms in (4.19) are positive, the whole series converges.
■

Notice that any other function with the same properties as (4.18) would also be an appropriate choice for ρ. △

Remark 4.7. The step size increase during the line minimization procedure helps in better exploiting the directions in which the cost function decreases. This step does not hinder convergence of the algorithm thanks to assumption (A2). △

We provide next the convergence results for the algorithm in Fig. 4.3-4.4.

Denote as $i^{\star}_{kj}$ the number of steps computed in the line minimization procedure at iteration k along direction d_j and as *blocked points* all the points x_{kj} such that

$$f(x_{kj} \pm \Delta_{kj}d_{kji}) > f(x_{kj}) - \rho(\Delta_{kji}). \qquad \forall j = 0, 1, ..., n-1$$

On the wake of the results of Garcia-Palomares and Rodriguez (2002), we can conclude the following Lemma 4.2 and Theorem 4.6.

Lemma 4.2. *The sequence of step sizes $\{\Delta_{kji}\}$ produced by the line minimization procedure in Fig. 4.4 is such that*

$$\lim_{k\to\infty}\lim_{i\to i^{\star}_{kj}} \Delta_{kji} = 0 \quad \forall j = 0, 1, ..., n-1.$$

Proof.

By construction, $\lambda_s\Phi_k \leq \Delta_{kji} \leq \lambda_t\Phi_k$ and Φ_k is a non-increasing sequence that reduces at blocked points. Hence, if blocked points occurred infinitely often, then we would have that, at blocked points, $\Phi_k \to 0$, and thus $\Delta_{kji} \to 0$.

By contradiction, if blocked points were not to occur infinitely often, then it means that there exists $w \in \mathbb{N}$ such that $\Phi_k = \Phi_w$ for all $k \geq w$. Thus, given $\lambda_s \Phi_w = \epsilon$, it follows that $\Delta_{kji} \geq \epsilon$ for all $k \geq w$. Hence, for all $k \geq w$, there exists $j \in \{0, 1, ..., n-1\}$ and $i^\star_{kj} > 0$ such that d_{kj} is such that

$$f(x_{kji} + \Delta_{kji} d_{kj}) \leq f(x_{kji}) - \rho(\Delta_{kji}) \leq f(x_{kji}) - \rho(\epsilon). \qquad \forall i \leq i^\star_{kj} \tag{4.20}$$

As such, $f(x_{kji})$ would decrease without a bound, contradicting (A2). Hence the claim is proved. ■

Theorem 4.6. *Every limit point x of the sequence of blocked points generated by the algorithm in Fig. 4.3-4.4 satisfies $\nabla f(x) = 0$.*

Proof. Denote as $\{\bar{x}_k\}$ the sequence of blocked points. Then

$$f(\bar{x}_k + \Delta_{kji} d_{kj}) - f(\bar{x}_k) > -\rho(\Delta_{kji}). \qquad \forall j$$

Notice that, by $\det(\mathrm{col}(d_0^T, d_1^T, ..., d_{n-1}^T)) > \epsilon$, compactness of the sublevel sets of f and the update rule for computing new directions, computing a new direction as the distance between two points explored points (and thus bounded by the diameter of the initial compact sublevel set), the norm of d_{kj}, for all $j = 0, 1, ..., n-1$ and $k \geq 0$, is upper bounded by $d_{max} := \max_{j=0,1,...,n}\{d_{0j}, \mathrm{diam}(\mathcal{L}_f(x_o))\}$, and lower bounded. The sequence $\{d_{kj}\}$ is thus bounded, and as such, considering any limit point $\bar{d}_j$, we can conclude that

$$\begin{aligned}
& f(\bar{x}_k + \Delta_{kj}\bar{d}_j) - f(\bar{x}_k) = \\
& f(\bar{x}_k + \Delta_{kj} d_{kj}) - f(\bar{x}_k) + f(\bar{x}_k + \Delta_{kj}\bar{d}_j) - f(\bar{x}_k + \Delta_{kj} d_{kj}) \\
& \geq -|\rho(\Delta_{kj})| + f(\bar{x}_k + \Delta_{kj}\bar{d}_j) - f(\bar{x}_k + \Delta_{kj} d_{kj}) \\
& \geq -|\rho(\Delta_{kj})| - |f(\bar{x}_k + \Delta_{kj}\bar{d}_j) - f(\bar{x}_k + \Delta_{kj} d_{kj})|,
\end{aligned}$$

thus, denoting as $\kappa > 0$ the Lipschitz constant of f at $\bar{x}_k + \Delta_{kj}\bar{d}_j$,

$$\frac{f(\bar{x}_k + \Delta_{kj}\bar{d}_j) - f(\bar{x}_k)}{\Delta_{kj}} \geq -\frac{|\rho(\Delta_{kj})|}{\Delta_{kj}} - \kappa\|\bar{d}_j - d_{kj}\|.$$

Then

$$\nabla f(\bar{x})^T \bar{d}_j = \lim_{\bar{x}_k \to x, \Delta_{kj} \to 0} \frac{f(\bar{x}_k + \Delta_{kj}\bar{d}_j) - f(\bar{x}_k)}{\Delta_{kj}} \geq$$
$$\geq \lim_{\Delta_{kj} \to 0} -\frac{\rho(\Delta_{kj})}{\Delta_{kj}} - \kappa\|\bar{d}_j - d_{kj}\| = 0.$$

Since this result is valid also for $-\bar{d}_j$, it follows that $\nabla f(x)^T \bar{d}_j = 0$. Moreover it is independent of j, and since $\{\bar{d}_0, \bar{d}_1, ..., \bar{d}_{n-1}\}$ span $\mathbb{R}^n$, we can conclude that $\nabla f(x) = 0$. ■

It is shown in Kolda et al. (2003) that Direct Search algorithm that continue the exploration along a direction as long as a (sufficient) decrease in the objective function is encountered, guarantee to achieve convergence to the minimum of subsequence of the blocked points. But as shown in Audet (2004), such a convergence result may also lead to converge to a set of infinite points, some of which may have gradient different from zero.

Some of the issues pointed out in Audet (2004) can be solved by exploring at each iteration all the directions belonging to the set $\mathcal{G}$, and continuing the exploration only along the directions where the most decrease in the cost function was found. This method, however, is not very practical for real application. Since, if we think about a vehicle that to take a step in a direction, it every time has to first visit all the directions belonging to $\mathcal{G}$, it is a really time consuming task.

As such, we next show how the structure of the cost function defined by assumptions (A0)-(A2) and the particular choice of the function ρ make it possible to conclude convergence of the whole sequence of iterates x_{kji} produced by the algorithm in Fig. 4.3-4.4, without resorting to explore all the directions in $\mathcal{G}$, but simply continuing the exploration along the first direction for which the sufficient decrease condition is satisfied. To the best of our knowledge, the following result is novel in the literature.

Theorem 4.7. *Consider the class of cost functions defined by (A0)-(A2), the sequence of iterate x_{kji} generated by the New RSP algorithm in Fig. 4.3-4.4 is such that*

$$\lim_{k \to \infty} \lim_{i \to i^\star_{kj}} \|x_{kji}\|_{\mathcal{A}^\star} = 0 \quad \forall j = 0, 1, ..., n-1. \tag{4.21}$$

Proof. We proceed by first showing that the every limit point of the sequence of blocked points is a minimum, to then prove that also the whole sequence of iterates converges to a set of minima.

By assumption (A1) and Theorem 4.6, we only need to show that every limit point of the sequence of blocked points is not a maximum. As we are assuming that every maximum is an isolated point, it follows that, considering a local maximum $\bar{x} \in \mathbb{R}^n$, there exists $\epsilon_m > 0$ such that $\forall x \neq \bar{x} \in \mathbb{R}^n$ such that $\|x - \bar{x}\| \leq \epsilon_m$, it follows that $f(x) < f(\bar{x})$.

Suppose there exists a subsequence of blocked points converging to $\bar{x}$. Denote it as $\{x_l\}$. Notice that since, for each $j = 0, 1, ..., n-1$, $\Delta_{kji} > 0$ and $\Delta_{kji} \to 0$ for $k \to \infty$, there exists $\bar{l} > 0$ such that $\forall l \geq \bar{l}$, $\|x_l - \bar{x}\| < \epsilon_m$.

If every point of the above sequence is such that $x_l \neq \bar{x}$, then, by the sufficient decrease condition and the definition of local maximum, it follows that $x_l \not\to \bar{x}$, since $f(\bar{x}) > f(x_l)$, thus contradicting that such a sequence exists.

So the only way for such a sequence to exist is if for some $\bar{\bar{l}} \geq \bar{l}$, $x_l = \bar{x}$ for all $l \geq \bar{\bar{l}}$.

As f is analytic at $\bar{x}$, there exists an even $m > 0$ such that the $m-th$ derivative of f with respect x is different from zero and, being $\bar{x}$ a maximum, its norm is lower than zero. Denote it as $f^m(x)$. Then, considering the Taylor expansion of $f(\bar{x}+\Delta d)$ around $\bar{x}$, and noticing that $\Delta^{\frac{1}{\Delta}}$ is $o(\Delta^m)$ and $\|d\|$ is lower bounded, there exists a $\bar{\Delta} \in (0,1)$ such that for all $\Delta \in (0, \Delta)$

$$f^m(x)\|d\|^m\Delta^m < -\rho(\Delta),$$

and thus there exists $\underline{l} > 0$ such that $x_{\underline{l}} \neq \bar{x}$ and $f(x_{\underline{l}}) < f(\bar{x})$.

Thus every limit point of the sequence of blocked points cannot be a maximum, hence they will all be minima.

To show that the whole sequence of iterates converges to the set of minima, notice that the subsequence of blocked points x_{b_k} of x_{kji} converges to $\mathcal{A}^\star$, by the above discussion. Suppose, by contradiction, that a subsequence x_{p_k} of x_{kji} does converge to a point $\bar{x} \notin \mathcal{A}^\star$.

By definition of converging sequence, given $\epsilon_p > 0$, there exists a $p^\star > 0$ such that, for all $p > p^\star$, $\|x_{p_k} - \bar{x}\| \leq \epsilon_p > 0$ and $\|\bar{x}\|_{\mathcal{A}^\star} > \epsilon_p$. Denote as $f_{\epsilon_p} := \min_{\{x:\|x-\bar{x}\|\leq\epsilon_p\}} f(x)$. Pick $\epsilon_b > 0$ such that $\max_{\{x:\|x\|_{\mathcal{A}^\star}\leq\epsilon_b\}} f(x) < f_{\epsilon_p}$ and notice that $\|\bar{x}\|_{\mathcal{A}^\star} > \epsilon_p + \epsilon_b$.

Then there exists a $b^\star > 0$ such that for all $b > b^\star$, $\|x_{b_k}\|_{\mathcal{A}^\star} \leq \epsilon_b$.

Pick $pb^\star = \max(p^\star, b^\star)$ and define as $\bar{b}^\star \geq pb^\star$ the smallest k such that $x_{\bar{b}^\star}$ is a blocked point and $\bar{p}^\star \geq pb^\star$ the smallest k such that $x_{\bar{p}^\star}$ belongs to the sequence x_{p_k}. Then, clearly, since $f(x_k)$ is a decreasing sequence (by the sufficient decrease condition), for $k \geq \bar{p}^\star$, no point in $\{x : \|x - \bar{x}\| \leq \epsilon_p\}$ can be selected, thus reaching a contradiction. ■

4.1.4 Examples

In this section we highlight with an example the features of the proposed RSP.

We will first consider a simple convex quadratic function and show asymptotic convergence of the algorithm. We consider the following cost function.

$$f(x) = x_1^2 + 5x_2^2, \tag{4.22}$$

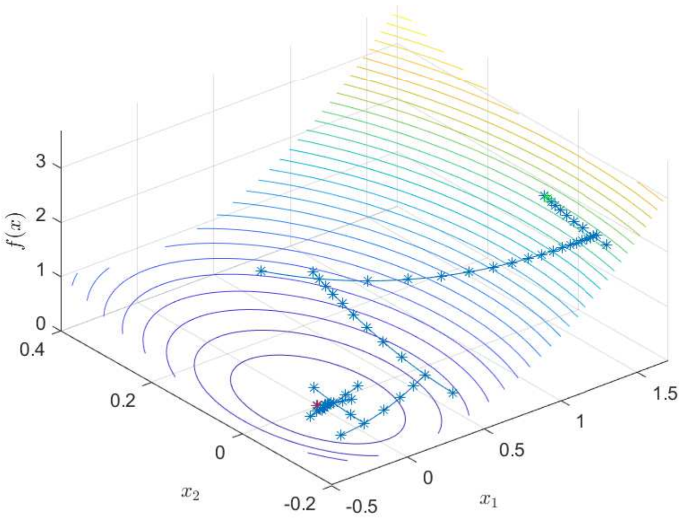
where $x = \text{col}(x_1, x_2) \in \mathbb{R}^2$. Results of a simulation of the proposed RSP in Matlab are shown in Fig. 4.5 for the initial conditions $x_\circ = \text{col}(1.5, 0)$, $\Phi_0 = 0.01$, $d_{01} = \text{col}(-\sin(\pi/8), \cos(\pi/8))$, $\Delta_{0j} = 0.01$, $j = 0, 1$, $\gamma = 1.2$, $\theta = 0.5$, $\delta_{\text{det}} = 0.001$, $\mu = 0.15$, $\lambda_s = 0.001$, and $\lambda_t = 5$.

From Fig. 4.5, it can be noticed as, at the same time, the step size converges to zero and the algorithm converges to the minimum. Notice, moreover, how the update of the set of directions with new (almost) conjugate directions speeds up the convergence as the exploration moves towards the minimum.

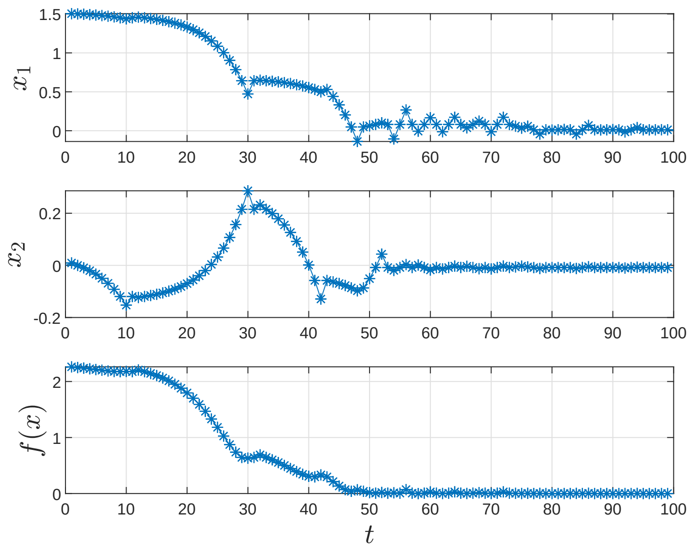
It comes as an intuition that, contrary to gradient descent methods which exploit only cost function information at the current iterate, Direct Search algorithms, which explore an arbitrary large neighborhood of the current iterate, can possibly escape undesired stationary points. In particular, we will show the ability of the proposed algorithm to escape maxima and, possibly, local minima. In this context, we apply the proposed RSP algorithm considering as objective function the "Drop Wave" function, whose expression is given next.

$$f(x) = -\frac{1 + \cos(12\sqrt{x_1^2 + x_2^2})}{0.5(x_1^2 + x_2^2) + 2}. \tag{4.23}$$

A section of the level sets of the Drop Wave function is shown in Fig. 4.6(a). It is a commonly used function for benchmarking ooptimization algorithms as it presents infinite global maximum points, infinite local minimum points, no saddle points, and



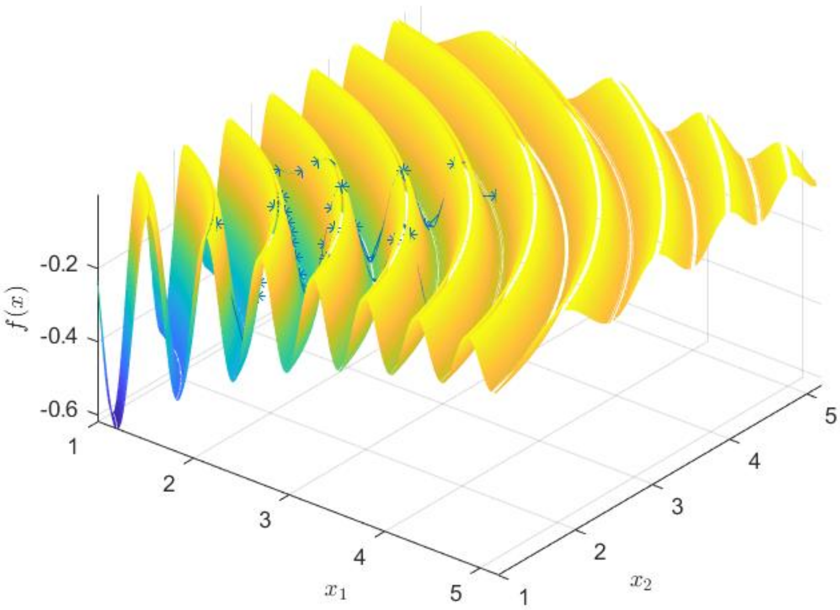
(a) Iterates of the proposed RSP algorithm versus the level sets of the quadratic convex fuction (4.22).



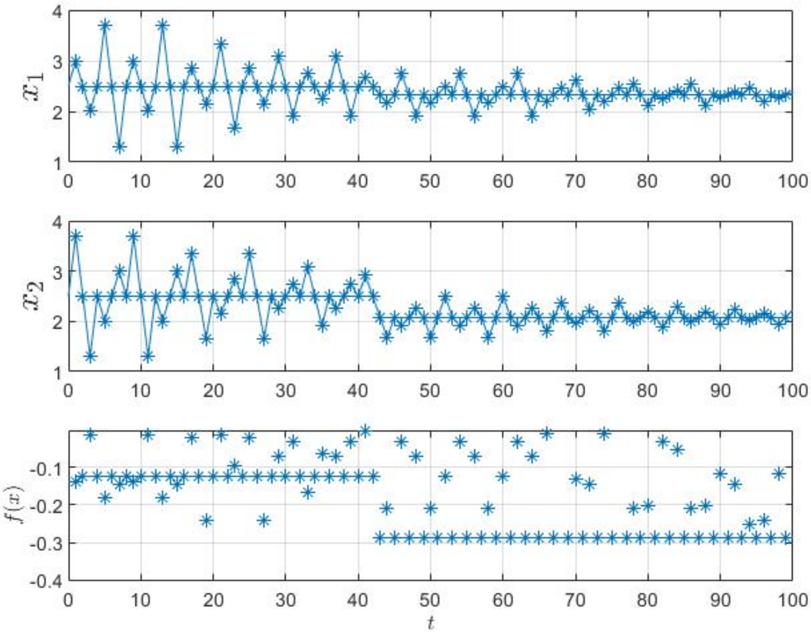
(b) Plot of the sequence explored points and their corresponding function value.

Figure 4.5: Plot of the trajectories sequence of iterates produced by the proposed RSP applied to the quadratic convex function (4.22). The dots '*' represent the position and function value of the points explored by the algorithms, while the "time" variable $t \in \mathbb{N}$ is a function $t : \mathbb{N}^3 \to \mathbb{N}$ given by $t(k, j, i) = \sum_{\bar{k}=0}^{k-1} \sum_{\bar{j}=0}^{n-1} i^{\star}_{\bar{k}\bar{j}} + \sum_{\bar{j}=0}^{j-1} i_{k\bar{j}} + i$, representing the sum of all the iterations computed by the line minimization procedure.

one global minimum. The simulation presented in Fig. 4.6 was obtained considering the following initial conditions for the new RSP algorithm, $x_\circ = \mathrm{col}(2.5, 2.5)$, $d_{01} = \mathrm{col}(-\sin(\pi/8), \cos(\pi/8))$, $\Phi_0 = 1.3$ $\Delta_{0j} = 1.3$, $j = 0, 1$, $\gamma = 1$, $\theta = 0.9$, $\delta_{\mathrm{det}} = 0.001$, $\mu = 0.7$, $\lambda_s = 0.9$, and $\lambda_t = 1.1$. We highlight as classic gradient descent algorithms would remain stuck at any of the stationary points of (4.23), while it is possible to notice as our algorithm not only escapes all the maxima, but also, if properly initialized, can move between level sets of local minima. In particular, we can notice this phenomenon happening between iteration 40 and 50, as evident from the shift in behavior of the function values in Fig. 4.6(b).



(a) Iterations of the proposed RSP algorithm versus the level sets of the Drop Wave fuction.



(b) Plot of the sequence explored points and their corresponding function value.

Figure 4.6: Plot of the sequence of iterates produced by the proposed RSP applied to the Drop Wave function. The dots '*' represent the position and function value of the points explored by the algorithms, while the "time" variable $t \in \mathbb{N}$ is a function $t : \mathbb{N}^3 \to \mathbb{N}$ given by $t(k,j,i) = \sum_{\bar{k}=0}^{k-1} \sum_{\bar{j}=0}^{n-1} i^{\star}_{\bar{k}\bar{j}} + \sum_{\bar{j}=0}^{j-1} i_{k\bar{j}} + i$, representing the sum of all the iterations computed by the line minimization procedure.

5

Hybrid Dynamical Systems

Hybrid dynamical systems aim at incorporating into a general framework both continuous time and discrete time dynamics, namely both differential and difference equations, or, more generally, inclusions. Early contributions to the study of hybrid systems can be found, between many, in Tavernini (1987), Lygeros (1996), and van der Schaft and Schumacher (1998). As the algorithms discussed in the previous section are inherently discrete, and as we aim at developing techniques for regulation and control of (continuous-time) dynamical systems in the case in which the error to be regulated is not measurable, but a proxy objective function is available in its stead, the framework of hybrid systems makes the perfect common ground to study and develop controllers for the interaction of these two dynamical systems.

The framework to which we will refer in this dissertation is the one proposed by Goebel, Sanfelice and Teel in Goebel et al. (2012).

In this chapter we will review the ingredients at the basis of hybrid systems, from the definition of a hybrid system, to solutions and stability concepts and theorems for hybrid systems. At last, as instrumental to prove the results in the next section, we also derive an extension to the hybrid setting of the sufficient

Lyapunov conditions proposed by Aeyels and Peuteman in Aeyels and Peuteman (1998).

5.1 Hybrid Systems

In the formalism of Goebel et al. (2012), a hybrid system $\mathcal{H}$ is defined as a 4-tuple (C, F, D, G), whose elements are called the *data* of the hybrid system. The data of a hybrid system are composed by two sets $C \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^n$, denoted respectively the *flow set* and the *jump set*, and two set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, denoted respectively the *flow map* and the *jump map*.

The flow map F and the jump map G encompass one the continuous dynamics of $\mathcal{H}$, or *flow*, the other the discrete dynamics of $\mathcal{H}$, or *jumps*. The possibly overlapping flow and jump set define the region of the state space in which it is admissible to flow or jump.

A general hybrid system $\mathcal{H}$ can be compactly represented in the following way

$$\mathcal{H} : \begin{cases} \dot{x} \ \in F(x) & x \in C \\ x^+ \in G(x) & x \in D. \end{cases} \tag{5.1}$$

As for continuous-time systems solutions are parameterized by a real time variable $t \in \mathbb{R}_{\geq 0}$, and for discrete-time are parametrized by a natural time variable $j \in \mathbb{N}$, solutions to (5.1) are specified by the couple $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$ on a subset of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ denoted *hybrid time domain.*

Definition 5.1 (Hybrid Time Domains)**.** *A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a compact hybrid time domain if*

$$E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq ... \leq t_J$. It is a hybrid time domain if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, ..., J\})$ is a compact hybrid time domain.

On any two points $(t, i), (s, j) \in E$, with E a hybrid time domain, we define the ordered relation $(t, i) \preceq (s, j)$ if $t + i \leq s + j$. We define in the same way the relations $\prec$, $=$, $\succ$, $\succeq$.

Definition 5.2 (Hybrid Arc)**.** *A function $\phi : E \to \mathbb{R}^n$ is a hybrid arc if E is a hybrid time domain and if for each $j \in \mathbb{N}$, the function $t \mapsto \phi(t,j)$ is locally absolutely continuous on the interval $I^j = \{t : (t,j) \in E\}$.*

Given a hybrid time domain E, denote as $\text{length}(E) = \sup_t E + \sup_j E$ the length of the hybrid time domain, where $\sup_t E = \sup\{t \in \mathbb{R}_{\geq 0} : \exists j \in \mathbb{N} \text{ such that } (t,j) \in E\}$ and $\sup_j E = \sup\{j \in \mathbb{N} : \exists t \in \mathbb{R}_{\geq 0} \text{ such that } (t,j) \in E\}$.

Hybrid arcs can be classified depending on the structure of their hybrid time domain. Given an hybrid arc ϕ, we will address to it as *complete* if $\text{length}(\text{dom}\,\phi) = \infty$ and *Zeno* if it is complete and $\sup_t \text{dom}\,\phi < \infty$.

We now have all the ingredients to define a solution to a hybrid system.

Definition 5.3 (Solution to a hybrid system)**.** *A hybrid arc ϕ is a solution to a hybrid system (C, F, D, G) if $\phi(0,0) \in \bar{C} \cup D$, and*

(S1) for all $j \in \mathbb{N}$ such that $I^j := \{t : (t,j) \in \text{dom}\,\phi\}$ has nonempty interior

$$\begin{aligned} &\phi \in C && \text{for all } t \in \text{int}\, I^j \\ &\dot{\phi} \in F(\phi(t,j)) && \text{for almost all } t \in I^j \end{aligned}$$

(S2) for all $(t,j) \in \text{dom}\,\phi$ such that $(t, j+1) \in \text{dom}\,\phi$,

$$\begin{aligned} &\phi(t,j) \in D, \\ &\phi(t,j+1) \in G(\phi(t,j)). \end{aligned}$$

A solution ϕ to $\mathcal{H}$ is *maximal* if there does not exist another solution ψ to $\mathcal{H}$ such that $\text{dom}\,\phi \subset \text{dom}\,\psi$ and $\phi(t,j) = \psi(t,j)$ for all $(t,j) \in \text{dom}\,\phi$.

Definition 5.4 (Pre-forward completeness)**.** *Given a set $S \subset \mathbb{R}^n_{\geq 0}$, a hybrid system $\mathcal{H}$ on $\mathbb{R}^n_{\geq 0}$ is* pre-forward complete *from S if every maximal solution to $\mathcal{H}$ from S is either bounded or complete.*

5.2 Nominally well-posed hybrid systems

Convergence of "well-behaved" sequences of hybrid arcs is often not satisfied in the point-wise, or uniform, sense, due to the discontinuous nature of hybrid arcs. As such, convergence of sequences of hybrid arcs, as for set-valued maps, is treated in the graphical sense.

Definition 5.5 (Graphical convergence of hybrid arcs)**.** *A sequence* $\{\phi_i\}_{i=1}^{\infty}$ *of hybrid arcs* $\phi_i : \operatorname{dom}\phi_i \to \mathbb{R}^n$ *converges graphically if the sequence of sets* $\{\operatorname{graph}\phi_i\}_{i=1}^{\infty}$*, where* $\operatorname{graph}\phi = \{(t,j,x) : (t,j) \in \operatorname{dom}\phi,\ x = \phi(t,j)\}$*, converges to the graph* $\operatorname{graph}\bar{\phi}$ *of a hybrid arc* $\bar{\phi}$*. In this case we write*

$$\bar{\phi} = \operatorname{gph-lim}_{i\to\infty}\phi_i$$

We define closeness of solutions to a hybrid system in the following way.

Definition 5.6 ((T,J,ε)-closeness of hybrid arcs)**.** *Given* $T \geq 0$*,* $J \geq 0$*. and* $\varepsilon > 0$*, two hybrid arcs* ϕ_1 *and* ϕ_2 *are* (T,J,ε)-close *if*

(a) for all $(t,j) \in \operatorname{dom}\phi_1$ *with* $t \leq T$*,* $j \leq J$*, there exists* s *such that* $(s,j) \in \operatorname{dom}\phi_2$*,* $|t-s| < \varepsilon$*, and* $|\phi_1(t,j) - \phi_2(t,j)| < \varepsilon$*;*

(b) for all $(t,j) \in \operatorname{dom}\phi_2$ *with* $t \leq T$*,* $j \leq J$*, there exists* s *such that* $(s,j) \in \operatorname{dom}\phi_1$*,* $|t-s| < \varepsilon$*, and* $|\phi_2(t,j) - \phi_1(t,j)| < \varepsilon$

Even in the graphical sense, however, sequential compactness of solutions to hybrid systems is in general not true. The particular class of hybrid systems for which sequences of solutions graphically converge to a solution are called *nominally well-posed* hybrid systems.

Definition 5.7 (Nominally well-posed hybrid systems)**.** *A hybrid system* $\mathcal{H}$ *is called nominally well-posed if the following property holds: for every graphically convergent sequence* $\{\phi_i\}_{i=1}^{\infty}$ *of solutions to* $\mathcal{H}$ *with* $\lim_{i\to\infty}\phi(0,0) = \xi$ *for some* $\xi \in \mathbb{R}^n$*,*

(a) if the sequence $\{\phi_i\}_{i=1}^{\infty}$ *is locally eventually bounded then the sequence* $\{\operatorname{length}(\phi_i)\}_{i=1}^{\infty}$ *is either convergent or properly divergent to* ∞ *and*

$$\phi = \operatorname{gph-lim}_{i\to\infty}\phi_i$$

is a solution to $\mathcal{H}$ *with* $\phi(0,0) = \xi$ *and* $\operatorname{length}(\phi) = \lim_{i\to\infty}\operatorname{length}(\phi_i)$*;*

(b) if the sequence $\{\phi_i\}_{i=1}^{\infty}$ *is not locally eventually bounded then there exists a number* $m \in (0,\infty)$ *for which there exist* $(t_i, j_i) \in \operatorname{dom}\phi_i$*,* $i = 1, 2, \ldots$*, such*

that $\lim_{i\to\infty} |\phi_i(t_i, j_i)| = \infty$ *and*

$$\phi = (\text{gph-lim}_{i\to\infty} \phi_i)|_{t+j<m}$$

is a maximal solution to $\mathcal{H}$ *with* $\text{length}(\phi) = m$ *and*

$$\lim_{t\to\sup_t \operatorname{dom}\phi} |\phi(t, sup_j \operatorname{dom}\phi)| = \infty.$$

As expressed in the next theorem, sufficient conditions for a hybrid system to be nominally well-posed are the following *hybrid basic conditions*

Definition 5.8 (Hybrid basic conditions)**.**

(1) C *and* D *are closed subsets of* $\mathbb{R}^n$*;*

(2) $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *is outer semicontinuous and locally bounded relative to* C*,* $C \subset \operatorname{dom} F$*, and* $F(x)$ *is convex for every* $x \in C$*;*

(3) $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *is outer semicontinuous and locally bounded relative to* D*, and* $D \subset \operatorname{dom} G$*.*

Theorem 5.1 (Hybrid basic conditions and nominal well-posedness)**.** *If the hybrid system* $\mathcal{H} = (C, F, D, G)$ *satisfies Assumption 5.8 then it is nominally well-posed.*

Denote as $T_C(x)$ the tangent cone to the set C at $x \in C$, namely all the $v \in \mathbb{R}^n$ such that exists a sequence $h_i \to 0^+$ and a sequence $v_i \to v$ such that for all $i \in \mathbb{N}$, $x + h_n v_n \in C$.

We report the following basic existence result for solutions to hybrid systems.

Theorem 5.2. *Let* $\mathcal{H} = (C, F, D, G)$ *satisfy the conditions in Definition 5.8. Take an arbitrary* $\xi \in C \cup D$*. If* $\xi \in D$ *or*

(VC) there exists a neighborhood U *of* ξ *such that for every* $x \in U \cap C$*,*

$$F(x) \cap T_C(x) \neq \emptyset, \tag{5.2}$$

then there exists a nontrivial solution ϕ *to* $\mathcal{H}$ *with* $\phi(0, 0) = \xi$*. If (VC) holds for every* $\xi \in C \setminus D$*, then there exists a nontrivial solution to* $\mathcal{H}$ *from every initial point in* $C \cup D$*, and every maximal solution* ϕ *to* $\mathcal{H}$ *satisfies exactly one of the following conditions:*

(a) ϕ *is complete;*

(b) $\operatorname{dom}\phi$ *is bounded and the interval* I^J*, where* $J = \sup_j \operatorname{dom}\phi$*, has a nonempty interior and* $t \mapsto \phi(t, J)$ *is a maximal solution to* $\dot{z} \in F(z)$*, in fact* $\lim_{t\to T} |\phi(t, J)| = \infty$*, where* $T = \sup_t \operatorname{dom}\phi$*;*

(c) $\phi(T, J) \notin C \cup D$*, where* $(T, J))\sup\operatorname{dom}\phi$*.*

Furthermore, if $G(D) \subset C \cup D$*, then* (c) *above does not occur.*

5.3 Stability Results

5.3.1 Stability Notions

In this section we recall the main and invariance definitions. We start by recalling the basic invariance definitions for a hybrid system $\mathcal{H}$ with respect to the set $\mathcal{A}$.

Definition 5.9. *Consider the hybrid system* $\mathcal{H}$*, the set* $\mathcal{A}$ *is*

- ***Weakly forward pre-invariant*** *if for every* $x_0 \in \mathcal{A}$ *there exists at least a maximal solution from* x_0 *such that* $x(t, j) \in \mathcal{A}$ *for all* $(t, j) \in \operatorname{dom} x$*.*

- ***Weakly backward pre-invariant*** *if for every* $x_0 \in \mathcal{A}$ *and every* $T > 0$*, there exists at least one maximal solution from* $\mathcal{A}$ *such that* $x(t_0, j_0) = x_0$ *for some* $(t_0, j_0) \in \operatorname{dom} x$ *fulfilling* $t_0 + j_0 \geq T$*, and* $x(t, j) \in \mathcal{A}$ *for all* $(t, j) \in \operatorname{dom} x$ *with* $(t, j) \preceq (t_0, j_0)$*.*

- ***Weakly pre-invariant*** *if it is both weakly forward pre-invariant and weakly backward pre-invariant.*

- ***Forward pre-invariant*** *if for every* $x_0 \in \mathcal{A}$*, all maximal solutions* x *to* $\mathcal{H}$ *from* x_0 *are such that* $x(t, j) \in \mathcal{A}$ *for all* $(t, j) \in \operatorname{dom} x$*.*

- ***Backward pre-invariant*** *if for every* $x_0 \in \mathcal{A}$ *and* $T > 0$*, every maximal solution* x *to* $\mathcal{H}$ *that satisfies* $x(t_0, j_0) = x_0$*, for some* $(t_0, j_0) \in \operatorname{dom} x$ *fulfilling* $t_0 + j_0 \geq T$*, is such that* $x(t, j) \in \mathcal{A}$ *for all* $(t, j) \preceq (t_0, j_0)$*,* $(t, j) \in \operatorname{dom} x$*.*

- ***Pre-invariant*** *if it is both forward pre-invariant and backward pre-invariant.*

Definition 5.10 (Attractivity notions)**.** *Given a hybrid system* $\mathcal{H}$ *and a subset* $X \subset C \cup D$*, the set* $\mathcal{A}$ *is said to be:*

- ***Pre-attractive*** *from X if every maximal solution x to $\mathcal{H}$ with initial condition in X is bounded and, if complete, $|x(t,j)|_{\mathcal{A}} \to 0$ for $t+j \to \infty$.*

- ***Attractive*** *from X if pre-attractive and pre-forward complete from X.*

- ***Uniformly pre-attractive*** *if for each $\epsilon > 0$ there exists $T > 0$ such that, for all maximal solutions x to $\mathcal{H}$ starting in X with $\mathrm{length}(\mathrm{dom}\, x) \geq T$, it holds that $|x(t,j)|_{\mathcal{A}} \leq \epsilon$ for all $(t,j) \in \mathrm{dom}\, x|_{\geq T}$.*

- ***Uniformly attractive*** *from X if uniformly pre-attractive and pre-forward complete from X.*

Definition 5.11 (Stability notions)**.** *Given a hybrid system $\mathcal{H}$, the set $\mathcal{A}$ is said to be:*

- ***Stable*** *if for every $\epsilon > 0$ there exists $\delta > 0$ such that $x(0,0) \in \mathcal{A} + \delta\mathbb{B}$ implies $|x(t,j)|_{\mathcal{A}} \leq \epsilon$ for all $(t,j) \in \mathrm{dom}\, x$.*

- ***Pre-asymptotically stable*** *from X if stable and pre-attractive from X.*

- ***Uniformly pre-asymptotically stable*** *from X if stable and uniformly pre-attractive from X.*

- ***Uniformly asymptotically stable*** *from X if stable and uniformly attractive from X.*

5.3.2 Lyapunov stability theorem

We report the basic Lyapunov stability theorem for hybrid systems, stating sufficient conditions for the asymptotic stability of a compact set, and refer the reader to Goebel et al. (2012), and the references therein, for further studies.

Definition 5.12 (Lyapunov function candidate)**.** *A function $V : \mathrm{dom}\, V \to \mathbb{R}$ is said to be a Lyapunov function candidate for the hybrid system $\mathcal{H} = (C, F, D, G)$ if the following hold:*

1. *$\bar{C} \cup D \cup G(D) \subset \mathrm{dom}\, V$;*

2. *V is continuously differentiable of an open set containing $\bar{C}$.*

Theorem 5.3 (Sufficient Lyapunov conditions)**.** *Let $\mathcal{H} = (C, F, D, G)$ be a hybrid system and let $\mathcal{A} \subset \mathbb{R}^n$ be closed. If V is a Lyapunov function candidate for $\mathcal{H}$ and there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and a continuous $\rho \in \mathcal{PD}$ such that*

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}) \quad \forall x \in C \cup D \cup G(D) \tag{5.3}$$

$$\langle \nabla V(x), f \rangle \leq -\rho(|x|_{\mathcal{A}}) \quad \forall x \in C, f \in F(x) \tag{5.4}$$

$$V(g) - V(x) \leq -\rho(|x|_{\mathcal{A}}) \quad \forall x \in D, g \in G(x) \tag{5.5}$$

then $\mathcal{A}$ is uniformly globally pre-asymptotically stable for $\mathcal{H}$

As it will be instrumental for a later analysis, we derive a hybrid extension of the relaxed Lyapunov sufficient conditions proposed by Aeyels and Peuteman in Aeyels and Peuteman (1998).

We first report a result by Goebel and Teel (2006a) regarding closeness of solutions.

Corollary 5.1. *Suppose that $\mathcal{H}$ is nominally well-posed and pre-forward complete at every $x \in \mathcal{A}$, for some compact set $\mathcal{A}$. For any $\varepsilon > 0$ and $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, there exists a $\delta > 0$ with the following property: for any maximal solution x_δ with initial condition in $\mathcal{A} + \delta\mathbb{B}$ there exists a solution x to $\mathcal{H}$ with $x(0,0) \in \mathcal{A}$ such that x_δ and x are $(T, J, \varepsilon) -$close.*

Theorem 5.4. *Consider a function $V : U \to \mathbb{R}$, with $U \subset \mathbb{R}^n$ an open neighborhood of the compact pre-invariant set $\mathcal{A}$ for $\mathcal{H}$, and assume that the hybrid system $\mathcal{H}$ is nominally well-posed and pre-forward complete from U. We assume the following conditions are satisfied.*

- *Condition 1: $V(\mathcal{A}) = 0$ and $\forall x \in U$: $\alpha(\|x\|_{\mathcal{A}}) \leq V(x) \leq \beta(\|x\|_{\mathcal{A}})$. The functions $\alpha(\cdot)$ and $\beta(\cdot)$ are class $\mathcal{K}_\infty$ functions.*

- *Condition 2: There exists $T > 0$ and there exists an open set $U' \subset U$ which contains $\mathcal{A}$ such that, for each hybrid arc $x : \operatorname{dom} x \to \mathbb{R}^n$, solution to $\mathcal{H}$, starting from U', there exists an increasing sequence of times $(t^\star_k, j^\star_k) \in \operatorname{dom} x$ $(k \in \mathbb{N})$ such that $(t^\star_{k+1} + j^\star_{k+1}) - (t^\star_k + j^\star_k) \leq T$ $(\forall k \in \mathbb{N})$, such that $\forall k \in \mathbb{N}$ and $\forall x(t^\star_k, j^\star_k) \in U' \setminus \{\mathcal{A}\}$:*

$$V(x(t^\star_{k+1}, j^\star_{k+1})) - V(x(t^\star_k, j^\star_k)) \leq 0. \tag{5.6}$$

Then the set $\mathcal{A}$ is uniformly stable.

Proof. Consider the set $\mathcal{A} + \epsilon\mathbb{B}$, with $\epsilon > 0$ small enough, such that $\mathcal{A} + \epsilon\mathbb{B} \subset U'$.

Since $\mathcal{A}$ is pre-invariant, any solution ϕ to $\mathcal{H}$ starting in $\mathcal{A}$ will remain into $\mathcal{A}$ for all $(t, j) \in \operatorname{dom}\phi$. Thus applying Corollary 5.1, given ϵ and T, there exists $\epsilon' > 0$ such that for all $x(t_0, j_0) \in \mathcal{A} + \epsilon\mathbb{B}$, $\|x(t, j)\|_{\mathcal{A}} \leq \epsilon'$ for all $(t, j) \in [(t_0, j_0), (t_0 + T', j_0 + J')] \subset \operatorname{dom} x$, with $T' + J' \leq T$.

Define $\delta' := \beta^{-1}(\alpha(\epsilon'))$ and consider the closed ball $\mathcal{A}+\delta'\mathbb{B}$. Apply now Corollary 5.1 again, to δ' and T, and get $\delta'' > 0$. For all $x(t_0, j_0) \in \mathcal{A}+\delta''\mathbb{B}$, $\|x(t, j)\|_{\mathcal{A}} \leq \delta'$ for all $(t, j) \in [(t_0, j_0), (t_0+T', j_0+J')] \subset \operatorname{dom} x$, with $T'+J' \leq T$. By Condition 2, there exists a $k_0 \in \mathbb{N}$ such that $(t^\star_{k_0}+j^\star_{k_0})-(t_0+j_0) \leq T$, implying that $\|x(t^\star_{k_0}, j^\star_{k_0})\|_{\mathcal{A}} \leq \delta'$.

For $x(t^\star_{k_0}, j^\star_{k_0}) \in \mathcal{A} + \delta'\mathbb{B}$, $\beta(\|x(t^\star_{k_0}, j^\star_{k_0})\|_{\mathcal{A}}) \leq \alpha(\epsilon')$, and thus $V(x(t^\star_{k_0}, j^\star_{k_0})) \leq \alpha(\epsilon')$.

Since, by Condition 2, $V(x(t^\star_{k_0+1}, j^\star_{k_0+1})) \leq V(x(t^\star_{k_0}, j^\star_{k_0}))$, $V(x(t^\star_{k_0+1}, j^\star_{k_0+1})) \leq \alpha(\epsilon')$. Since $\alpha(\|x(t^\star_{k_0+1}, j^\star_{k_0+1})\|_{\mathcal{A}}) \leq V(x(t^\star_{k_0+1}, j^\star_{k_0+1})) \leq \alpha(\epsilon')$, one obtains that $\|x(t^\star_{k_0+1}, j^\star_{k_0+1})\|_{\mathcal{A}} \leq \epsilon'$.

By the same argument, $\forall n \in \mathbb{N}$, $x(t^\star_{k_0+n}, j^\star_{k_0+n}) \in \mathcal{A} + \epsilon'\mathbb{B}$. It follows, from Corollary 5.1, that $\forall (t, j) \geq (t_0, j_0)$, $\|x(t, j)\|_{\mathcal{A}} \leq \epsilon$. Thus uniform stability is proved. ∎

6

Robust Hybrid Direct Search Controller

In the output regulation framework it is always assumed to have availability of the regulated error. The case in which, instead of the error variable, a "proxy" of the error is available under the form of a function to be minimized, of unknown expression but measurable, and whose minimum represents the reference signal, has yet to be studied in the literature. In particular, interesting is the case in which the minimum of the associated objective function is moving with time, possibly as a scalde solution of an exogenous dynamical system. One of the biggest challenges of this problem, from an output regulation point of view, is that the error dynamics are not locally observable at the minimum. This characteristic, however, possibly opens the path to a plethora of adaptive and optimization schemes, like the one proposed in this section, to, possibly, work in synergy with a regulator, based on the internal model principle, in order to solve this problem.

In this chapter, on an attempt to lay the ground for a future development of the just framed problem, aiming at a future interconnection with the results in

Part 1, we study the problem of steering a particular class of dynamical systems towards the minimum of an objective function, assumed to not be known but whose measurements are available at fixed intervals of time. To this end, we consider continuous-time dynamical systems that can be steered, by a known input, between any two points of the state space.

The problem at hand has been tackled in the literature with a variety of approaches, mostly related to source-seeking applications, from gradient descent methods applied to single vehicles Burian et al. (1996) and in the multi-agent framework Bachmayer and Leonard (2002), to stochastic approximation methods Azuma et al. (2012), to the extremum-seeking control technique Cochran and Krstic (2009).

This chapter is based on the preliminary work Mayhew et al. (2007) (see also Mayhew et al. (2008a) and Mayhew et al. (2008b)), where the source-seeking problem is solved by a hybrid controller based on the RSP algorithm. The choice to resort to a modified RSP algorithm stems from its simple form and ease of implementation.

In Mayhew et al. (2007), the classic RSP algorithm is implemented with discrete line minimizations with fixed step size. Practical asymptotic stability of the set of minimizers is shown for the 2-dimensional convex case, but due to the a dense exploration step introduced to guarantee the aforementioned stability result, no direct extension to the n dimensional scenario is possible. In Coope and Price (1999) an extension of the RSP was proposed in the general context of continuously differentiable functions. We adopt the algorithm proposed in Section 4.3 that, by using a decreasing step size asymptotically converging to zero, ensures asymptotic convergence to a stationary point.

Due to the inherent discrete dynamics of the algorithm, and the continuous dynamics of the dynamical system, on the wake of Mayhew et al. (2007), the controller is implemented as a hybrid controller, based on the hybrid systems framework developed in Goebel et al. (2012). In particular, the proposed hybrid controller addresses the optimization problem of an n-dimensional continuously differentiable function with a set of global minima, and possibly isolated local maxima, and it renders the set of minima almost globally asymptotically stable. We show, however, that asymptotic Direct Search methods based on line minimizations, as well as the algorithm developed in Section 4.3, are not robust to measurement noise. Thus we propose a robust algorithm, addressing n-dimensional objective functions (including the results of Mayhew et al. (2007) as a special case), highlighting that a trade-off between asymptotic convergence and robustness is inevitable. Moreover an explicit bound relating the step size to the supremum norm of the noise, acting on the objective function, is computed.

6.1 Hybrid Controller

In this section we design a hybrid controller $\mathcal{H}_c$ implementing the algorithm developed in Sectoin 4.3 to solve a minimization problem in $\mathbb{R}^n$ under the assumptions (A0)-(A2) in Section 4.1.3.

The reason for resorting to the hybrid systems' framework is to provide results regarding the stability and robustness of the proposed algorithm when applied to continuous-time dynamical systems.

We will consider the optimization problem constrained by the following dynamics

$$\dot{\xi} = \varphi(\xi, u) \qquad \xi = \mathrm{col}(x, \zeta) \in \mathbb{R}^{n+l}, u \in \mathbb{R}^m \tag{6.1}$$

with φ continuously differentiable in ξ and u. The state variables x represent the variables involved in the optimization problem, while ζ represent other possible states. Given $\tau^\star > 0$, we assume that for each x_0 and x_f in $\mathbb{R}^n$ there exists $t \mapsto u(t)$ such that the solution to $\dot{\xi} = \varphi(\xi, u(t))$ from $\xi_0 = (x_0, \cdot)$, reaches $\xi_f = (x_f, \cdot)$ after $\tau^\star$ seconds. We also assume that for each bounded input $\|u(t)\| \leq \bar{u} > 0$ for all $t \geq 0$, $\zeta(t)$ is bounded for all $t \geq 0$. The class of systems represented by (6.1), includes, for example, point-mass vehicles ($\xi = x$, with x representing the position) and Dubin's vehicles ($\xi = \mathrm{col}(x^T, \zeta)$, with x and ζ representing position and orientation).

The algorithm in Fig. 4.3-4.4 in Section 4.1.3 is implemented as a discrete time system, whose dynamics are set-valued in order to satisfy the *hybrid basic conditions* and have the closed-loop system $\mathcal{H}_{cl}$, given by the interconnection of $\mathcal{H}_c$ and (6.1), nominally well-posed.

6.1.1 State of $\mathcal{H}_c$

The full state of the controller x_c is composed by the state variables $\tau \in \mathbb{R}_{\geq 0}$, $d_j \in \mathbb{R}^n$, with $j = 0, 1, ..., n-1$, $\Delta_j \in \mathbb{R}_{\geq 0}$, with $j = 0, 1, ..., n-1$, $\Phi \in \mathbb{R}_{\geq 0}$, $\lambda \in \mathbb{R}$, $\alpha \in \mathbb{R}^n$, $\bar{\alpha} \in \mathbb{R}_{\geq 0}$, $p \in \{-1, 1\}$, $m \in \{0, 1\}$, $k \in \{0, 1, ..., n\}$, $q \in \{0, 1, 2\}$, $z \in \mathbb{R}$, $\Delta \in \mathbb{R}$, and $v \in \mathbb{R}^n$. In particular, we denote $x_c = \mathrm{col}(\tau, \Delta_0, ..., \Delta_{n-1}, d_0, ..., d_{n-1},$

$\Phi, \lambda, \alpha^T, \bar{\alpha}, p, m, k, q, z,\ \Delta, v^T)$ belonging to the domain $\mathcal{X}_c$ given by the Cartesian product of the domains of each state variable composing x_c, namely $\mathcal{X}_c := \mathbb{R}_{\geq 0} \times \mathbb{R}^n_{\geq 0} \times \mathbb{R}^{n\times n} \times \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}_{\geq 0} \times \{-1, 1\} \times \{0, 1\} \times \{0, ..., n\} \times \{0, 1, 2\} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$, where, with abuse of notation, we denoted the domain of the set of step sizes Δ_j as $\mathbb{R}^n_{\geq 0}$ and the domain of the set of directions d_j as $\mathbb{R}^{n\times n}$.

The state variable $\tau \in \mathbb{R}_{\geq 0}$ is a timer, that resets every $\tau^\star > 0$ seconds, and that regulates when new cost function evaluations are available.

Its hybrid dynamics are given by

$$\dot{\tau} = 1 \qquad (\xi, x_c) \in C := \{(\xi, x_c) \in \mathbb{R}^{n+l} \times \mathcal{X}_c : \tau \leq \tau^\star\}, \tag{6.2}$$

during flow, and

$$\dot{\tau} = 1 \qquad (\xi, x_c) \in C := \{(\xi, x_c) \in \mathbb{R}^{n+l} \times \mathcal{X}_c : \tau \geq \tau^\star\}, \tag{6.3}$$

during jumps.

The states $d_j \in \mathbb{R}^n$ and $\Delta_j \in \mathbb{R}_{\geq 0}$, with $j = 0, 1, ..., n-1$, represent, as in Fig. 4.3-4.4, the search directions and the step sizes corresponding to each direction. In the algorithm in Fig. 4.3-4.4, the variable α_j corresponds to the total distance traveled along each direction. The state $\Phi \in \mathbb{R}_{\geq 0}$ represents the global step size.

The state variable $\lambda \in \mathbb{R}$ which keeps track of the distance traveled along the currently explored direction, and the state variable $\alpha \in \mathbb{R}^n_{>0}$, which stores the total traveled vector from direction d_0, are related to the distance traveled along each direction, which is the variable α_j introduced in in Fig. 4.3-4.4. While the state $\bar{\alpha} \in \mathbb{R}_{\geq 0}$ is the total distance traveled during each cycle of directions exploration.

The proposed implementation of the new RSP explores each direction d_j first in the positive, and then in the negative sense. As such the positive or negative exploration along the current direction is determined by the state $p \in \{-1, 1\}$, and the variable $m \in \{0, 1\}$ indicates whether a switch in sense of exploration has already happened along the current direction.

To define in which operating point of the new RSP algorithm the controller is, the state variables $k \in \{0, 1, ..., n\}$ and $q \in \{0, 1, 2\}$ have been introduced. The variable k represents the state of the RSP, namely which direction is currently being explored. Notice that it has $n + 1$ components since the direction d_{n-1} is explored twice to be able to exploit the *Parallel Subspace Property*. The variable q, defining the state of the line minimization, assumes these values

- $q = 0$: the positive line minimization;
- $q = 1$: the negative line minimization;
- $q = 2$: the line minimization is completed.

The state variable $z \in \mathbb{R}$ is a memory state that keeps track of the best minimum value of f found satisfying the sufficient decrease condition.

Two more states have been added for ease of notation, namely $\Delta \in \mathbb{R}$ and $v \in \mathbb{R}^n$, that store the currently used step size and search direction.

6.1.2 Hybrid Controller Structure

The structure of $\mathcal{H}_c$ is given by

$$\mathcal{H}_c \ : \begin{cases} \dot{x}_c = F_c := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & (x, x_c) \in C \\ x_c^+ \in G_c(x_c, f(x)) := \begin{bmatrix} 0 \\ G_{c/\tau}(x_c, f(x)) \end{bmatrix} & (x, x_c) \in D \\ u = K(x, x_c, \tau^\star), \end{cases} \tag{6.4}$$

with sets $C,\ D$ defined before. The flow map F_c is a single-valued constant function with all components equal to zero except for the timer. The jump map $G_c : \mathcal{X}_c \times \mathbb{R} \to \mathcal{X}_c$ is a set-valued map, composed by the timer discrete dynamics and $G_{c/\tau} : \mathcal{X}_c/\mathbb{R} \times \mathbb{R} \to \mathcal{X}_c/\mathbb{R}$, which is defined next. The output of $\mathcal{H}_c$ is a function $K : \mathbb{R}^n \times \mathcal{X}_c \times \mathbb{R}_{>0} \to \mathbb{R}^m$ that steers the x-subsystem from $x(t_j, j)$ to $x(t_j + \tau^\star, j) = x(t_j, j) + p(t_j, j)\Delta(t_j, j)v(t_j, j)$, with $t_j = \inf_{t \in \mathbb{R}_{\geq 0}}(t, j) \in \operatorname{dom} x$, for all $j \in \mathbb{N}$.

The set-valued map $G_{c/\tau}$ is presented next. It is given by the composition of the maps $g_i(x_c, f(x_c))$ defined on the subsets D_i, $i = 1, 2, ..., 5$ of the jump set D, namely $G_{c/\tau}(x_c, f(x)) := g_i(x_c, f(x))$ for $(x_c, x) \in D_i$, where $D = \cup_{i=1}^5 D_i$. We omit the update law of the state variables that remain constant at jumps.

The sets D_i define the conditions under which the different operations of the algorithm proposed in Fig. 4.3-4.4, integrated in the functions g_i, take place.

In order to keep the presentation of the discrete dynamics of the controller compact, the conditions defined as "otherwise", represent the closure of the complement of the union of the subsets of the state space defined by the other conditions.

1) Continue a positive line search:

$D_1 = \{(\xi, x_c) \in \mathbb{R}^{n+l} \times \mathcal{X}_c : \; f(x) \leq z - \rho(\Delta), \, p = 1, \, q \in \{0,1\}, \, m = 0\}$

$g_1 : \; z^+ = f(x), \, q^+ = 1,$

$$\lambda^+ = \begin{cases} \lambda + \Delta_{n-1} p & \text{if } k = 0 \\ \lambda + \Delta_{k-1} p & \text{otherwise} \end{cases}$$

$$\Delta^+_{k-1} = \begin{cases} \gamma\Delta_{k-1} & \text{if } k = 1, 2, ..., n-1 \text{ and } \gamma\Delta_{k-1} \leq \lambda_t \Phi \\ \lambda_t \Phi & \text{if } k = 1, 2, ..., n-1 \text{ and } \gamma\Delta_{k-1} \geq \lambda_t \Phi \end{cases}$$

$$\Delta^+_{n-1} = \begin{cases} \gamma\Delta_{n-1} & \text{if } k = 0, n \text{ and } \gamma\Delta_{n-1} \leq \lambda_t \Phi \\ \lambda_t \Phi & \text{if } k = 0, n \text{ and } \gamma\Delta_{n-1} \geq \lambda_t \Phi \end{cases}$$

$$\Delta^+ = \begin{cases} \gamma\Delta_{k-1} & \text{if } k = 1, 2, ..., n-1 \text{ and } \gamma\Delta \leq \lambda_t \Phi \\ \gamma\Delta_{n-1} & \text{if } k = 0, n \text{ and } \gamma\Delta \leq \lambda_t \Phi \\ \lambda_t \Phi & \text{if} \gamma\Delta \geq \lambda_t \Phi \end{cases}$$

2) Correct overshoot:

$D_2 = \{(\xi, x_c) \in \mathbb{R}^{n+l} \times \mathcal{X}_c : f(x) \geq z - \rho(\Delta), \, q \in \{0,1\}, \, m = 0\}$

$g_2 : \; p^+ = -p, \, m^+ = 1, \, q^+ = q + 1,$

3) Starting negative line search:

$D_3 = \{(\xi, x_c) \in \mathbb{R}^{n+l} \times \mathcal{X}_c : m = 1, \, p = -1, \, q = 1\}$

$g_3 : \; z^+ = f(x), \, m^+ = 0, \, \lambda^+ = 0,$

4) Continue a negative line search:

$D_4 = \{(\xi, x_c) \in \mathbb{R}^{n+l} \times \mathcal{X}_c : f(x) \leq z - \rho(\Delta), \, p = -1, \, q = 1, \, m = 0\}$

$g_4 : \; z^+ = f(x),$

$$\lambda^+ = \begin{cases} \lambda + \Delta_{n-1} p & \text{if } k = 0 \\ \lambda + \Delta_{k-1} p & \text{otherwise} \end{cases}$$

$$\Delta^+_{k-1} = \begin{cases} \gamma\Delta_{k-1} & \text{if } k = 1, 2, ..., n-1 \text{ and } \gamma\Delta_{k-1} \leq \lambda_t \Phi \\ \lambda_t \Phi & \text{if } k = 1, 2, ..., n-1 \text{ and } \gamma\Delta_{k-1} \geq \lambda_t \Phi \end{cases}$$

$$\Delta^+_{n-1} = \begin{cases} \gamma\Delta_{n-1} & \text{if } k = 0, n \text{ and } \gamma\Delta_{n-1} \leq \lambda_t \Phi \\ \lambda_t \Phi & \text{if } k = 0, n \text{ and } \gamma\Delta_{n-1} \geq \lambda_t \Phi \end{cases}$$

$$\Delta^+ = \begin{cases} \gamma\Delta_{k-1} & \text{if } k = 1, 2, ..., n-1 \text{ and } \gamma\Delta \leq \lambda_t\Phi \\ \gamma\Delta_{n-1} & \text{if } k = 0, n \text{ and } \gamma\Delta \leq \lambda_t\Phi \\ \lambda_t\Phi & \text{if} \gamma\Delta \geq \lambda_t\Phi \end{cases}$$

5) Update direction and start positive line search:

$D_5 = \{(\xi, x_c) \in \mathbb{R}^{n+l} \times \mathcal{X}_c : q = 2\}$

$g_5 :$

$q^+ = 0,\ p^+ = 1,\ \lambda^+ = 0,\ m^+ = 0, z^+ = f(x)$

$$\alpha^+ = \begin{cases} \alpha + \lambda v & \text{if } k = 0, 1, ..., n-1 \\ 0 & \text{if } k = n \end{cases}$$

$$\bar{\alpha}^+ = \begin{cases} \bar{\alpha} + \|\lambda v\| & \text{if } k = 0, 1, ..., n-1 \\ 0 & \text{if } k = n \end{cases}$$

$k^+ = (k+1) \mod n+1$

$$\Delta^+ = \begin{cases} \Delta_{(k \mod n+1)} & \text{if } k = 0, 1, ..., n-1 \\ \max_{j \in \{0,1,...,n-2\}} \Delta_j & \text{if } k = n \end{cases}$$

$$\Phi^+ = \begin{cases} \mu\Phi & \text{if } k = n \text{ and } \bar{\alpha} + \|\lambda v\| \leq \min_{j \in \{0,1,...,n-1\}} \Delta_j \\ \Phi & \text{otherwise} \end{cases}$$

$$v^+ = \begin{cases} \phi(\alpha, \lambda v, M_{1,n-1}, d_0) & \text{if } k = n \\ d_k & \text{otherwise} \end{cases}$$

$$d_0^+ = \begin{cases} d_0 & \text{if } k = 0, ..., n-1 \\ d_1 & \text{if } k = n \end{cases}$$

$\vdots$

$$d_{n-2}^+ = \begin{cases} d_{n-2} & \text{if } k = 0, ..., n-1 \\ d_{n-1} & \text{if } k = n \end{cases}$$

$$d_{n-1}^+ = \begin{cases} d_{n-1} & \text{if } k = 0, 1, ..., n-1 \\ \phi(\alpha, \lambda v, M_{1,n-1}, d_0) & \text{if } k = n \end{cases}$$

$$\Delta_0^+ = \begin{cases} \theta\Delta_0 & \text{if } k = 1 \text{ and } |\lambda| \leq \frac{\Delta_0}{2} \text{ and } \theta\Delta_0 \geq \lambda_s\Phi \\ \lambda_s\Phi & \text{if } k = 1 \text{ and } |\lambda| \leq \frac{\Delta_0}{2} \text{ and } \theta\Delta_0 \leq \lambda_s\Phi \\ \Delta_0 & \text{if } k = 0, 2, ..., n-1 \text{ or} \\ & (k = 1 \text{ and } |\lambda| \geq \frac{\Delta_0}{2}) \\ \mu\lambda_t\Phi & \text{if } k = n \text{ and } \bar{\alpha} + \|\lambda v\| \leq \min_{j\in\{0,1,...,n-1\}} \Delta_j \text{ and } \Delta_1 \geq \mu\lambda_t\Phi \\ \Delta_1 & \text{otherwise} \end{cases}$$

$$\vdots$$

$$\Delta_{n-2}^+ = \begin{cases} \theta\Delta_{n-2} & \text{if } k = n-1 \text{ and } |\lambda| \leq \frac{\Delta_{n-2}}{2} \text{ and } \theta\Delta_{n-2} \geq \lambda_s\Phi \\ \lambda_s\Phi & \text{if } ((k = n-1 \text{ and } |\lambda| \leq \frac{\Delta_{n-2}}{2}) \text{ or} \\ & (k = n \text{ and } |\lambda| \leq \frac{\Delta_{n-1}}{2})) \text{ and } \theta\Delta_{n-2} \leq \lambda_s\Phi \\ \Delta_{n-2} & \text{if } k = 0, ..., n-2 \text{ or} \\ & (k = n-1 \text{ and } |\lambda| \geq \frac{\Delta_{n-2}}{2}) \\ \theta\Delta_{n-1} & \text{if } k = n \text{ and } |\lambda| \leq \frac{\Delta_{n-1}}{2} \text{ and } \Delta_{n-1} \geq \mu\lambda_t\Phi \\ \mu\Phi & \text{if } k = n \text{ and } \bar{\alpha} + \|\lambda v\| \leq \min_{j\in\{0,1,...,n-1\}} \Delta_j \text{ and } \Delta_{n-1} \geq \mu\lambda_t\Phi \\ \Delta_{n-1} & \text{otherwise} \end{cases}$$

$$\Delta_{n-1}^+ = \begin{cases} \theta\Delta_{n-1} & \text{if } k = 0 \text{ and } |\lambda| \leq \frac{\Delta_{n-1}}{2} \text{ and } \theta\Delta_{n-1} \geq \lambda_s\Phi \\ \lambda_s\Phi & \text{if } k = 0 \text{ and } |\lambda| \leq \frac{\Delta_{n-1}}{2} \text{ and } \theta\Delta_{n-2} \leq \lambda_s\Phi \\ \Delta_{n-1} & \text{if } k = 1, ..., n-1 \text{ or} \\ & (k = 0 \text{ and } |\lambda| \geq \frac{\Delta_{n-1}}{2}) \\ \mu\Phi & \text{if } k = n \text{ and } \bar{\alpha} + \|\lambda v\| \leq \min_{j\in\{0,1,...,n-1\}} \Delta_j \text{ and} \\ & \max_{j\in\{0,1,...,n-2\}} \Delta_j \geq \mu\lambda_t\Phi \\ \max_{j\in\{0,1,...,n-2\}} \Delta_j & \text{otherwise} \end{cases}$$

We highlight that the map $G_{c/\tau}$ is defined, and nonempty, on all D. This is ensured by the "otherwise" condition for some state variables and the fact that, for the remaining ones, the conditions determining the discrete dynamics involve only the state variable k, but are defined for all $k \in \{0, 1, ..., n\}$.

Even if the logic of the jump map $G_{c/\tau}$ resembles the one in the proposed RSP in Fig. 4.3-4.4, a couple of explanations are in order.

The computation of the new conjugate direction in g_5 is addressed by the function $\phi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times(n-1)} \times \mathbb{R}^n$ defined as

$$\phi(\alpha, \beta, M_{1,n-1}, d_0) = \begin{cases} \alpha+\beta & \det(\mathrm{col}(M_{1,n-1}^T, (\alpha+\beta)^T)^T) > \delta_{\det} \\ d_0 & \det(\mathrm{col}(M_{1,n-1}^T, (\alpha+\beta)^T)^T) < \delta_{\det} \\ \{d_0, \alpha+\beta\} & \text{otherwise,} \end{cases} \tag{6.5}$$

where $M_{1,n-1} := \mathrm{col}(d_1^T, ..., d_{n-1}^T)^T$. The conditions in ϕ check if the new direction $\alpha+\beta$, that is going to be computed exploiting the *Parallel Subspace Property*, is linearly independent from the last $n-1$ directions, namely if the determinant of the concatenation of $M_{1,n-1}$ and the new direction is bigger than a tunable parameter $\delta_{det} > 0$. In case this condition is not satisfied, the previous set of directions is retained.

The update rule of the states Δ_j, $j = 0, 1, ..., n-1$ also needs clarification. Let us consider Δ_{n-1}^+ since the same reasoning applies to the other state variables. The condition $|\lambda| < \Delta_{n-1}/2$ is a different way to express the condition $\lambda = 0$, while at the same time satisfying outer semicontinuity of the map g_5. Indeed $|\lambda| < \Delta_{n-1}/2$ is satisfied only for $\lambda = 0$, except perhaps at the initialization, since along direction d_{n-1}, Δ_{n-1} is the minimum displacement possible for λ.

When $k = n$, Δ_{n-1} is updated as the maximum of the other $n-1$ Δ_j state variables, but any function preserving the maximum value would do, since the final objective is to have $\Delta_j \to 0$, $j = 0, 1, ..., n$.

Notice that, in the current implementation, the step size is reduced and the timer limit is kept constant, and thus the speed of system (6.1) is reduced proportionally by reduction of the step size. In this way the distance traveled during the flow gets smaller and smaller, and the algorithm asymptotically converges to the set of minima.

6.2 Stability Analysis

Define the hybrid closed-loop $\mathcal{H}_{cl}$ as the interconnection of the dynamics (6.1) and the controller $\mathcal{H}_c$ developed in the previous section, namely

$$\mathcal{H}_{cl} \; : \; \begin{cases} \left.\begin{array}{l} \dot{\xi} = \varphi(\xi, K(x, x_c, \tau^\star)) \\ \dot{x}_c = F_c \end{array}\right\} & (\xi, x_c) \in C \\ \left.\begin{array}{l} \xi^+ = \xi \\ x_c^+ \in G_c(x_c, f(x)) \end{array}\right\} & (\xi, x_c) \in D \end{cases} \tag{6.6}$$

The flow and jump maps of the closed-loop system $\mathcal{H}_{cl}$ are thus defined as $F(\xi, x_c) := \mathrm{col}(\varphi(\xi, K(x, x_c, \tau^\star)), F_c)$ for all $(\xi, x_c) \in C$ and $G_c(\xi, x_c) := \mathrm{col}(\xi, G(x_c, f(x)))$ for all $(\xi, x_c) \in D$. We begin by showing nominal well-posedness of the closed loop system (6.6), a property of a class of hybrid systems instrumental to derive the next stability results.

Lemma 6.1. *Let assumptions (A0)-(A2) hold, and* $\tau^\star > 0$, $\delta_{det} > 0$, $0 < \lambda_s < 1 < \lambda_t$, $\mu \in (0, 1/\lambda_t)$, $\theta \in (0, 1)$ *and* $\gamma \in \mathbb{R}_{\geq 1}$. *Then the hybrid closed-loop system* $\mathcal{H}_{cl}$ *in* (6.6) *is nominally well-posed.*

Proof. In order to show nominal well-posedness of (6.6), we show first that (6.6) satisfies the *hybrid basic conditions* (Definition 5.8) and then invoke Theorem 5.1. It is straightforward that the sets C and D are closed.

Both F and K are continuous functions in C and thus outer semicontinuous and locally bounded. Moreover, being both single-valued, they are also convex for every $(x, x_c) \in C$.

The set-valued map $G(\cdot, f(\cdot))$ is composed by linear functions, apart for an instance of α^+ where the norm operator is present, which is continuous in the set of definition, and an instance of Δ^+_{n-1} where the *max* function is used, which is continuous as well. The map $G_{c\setminus\tau}$ is thus piecewise continuous. As all the inequalities defining the components of G_c on all the subspaces of D are not strict, at the points of discontinuity, G includes both the left and right limit. Hence G outer semicontinous by definition.

Since $G(\cdot, f(\cdot))$ is piecewise continuous, it is locally bounded by continuity.

The hybrid closed-loop system $\mathcal{H}_{cl}$ thus satisfies the hybrid basic conditions and is, thus, nominally well-posed. ■

Define the set of global minima of f as $\mathcal{A}^\star := \{x^\star \in \mathbb{R}^n : f(x^\star) \leq f(x)\ \forall x \in \mathbb{R}^n\}$ and, for a more compact notation, the set $\mathcal{A}_{dis} := \{-1, 1\} \times \{0, 1\} \times \{0, 1, ..., n\} \times \{0, 1, 2\}$. Then we consider the stabilization problem with respect to the sets $\mathcal{A} \subset \mathcal{A}_e \subset \mathbb{R}^{n+l} \times \mathcal{X}_c$, defined as

$$\begin{aligned} \mathcal{A} := \mathbb{R}^l \times \mathcal{A}^\star \quad & \times [0,1] \times \{0^n\} \times \mathbb{R}^{n\times n} \times \{0\} \times \{0\} \times \\ & \{0^n\} \times \{0\} \times \mathcal{A}_{dis} \times \{f(\mathcal{A}^\star)\} \times \{0\} \times \mathbb{R}^n, \end{aligned} \tag{6.7}$$

$$\begin{aligned} \mathcal{A}_e := \quad & \mathbb{R}^{n+l} \times [0,1] \times \{\{0^n\} \times \mathbb{R}^{n\times n} \times \{0\} \cup \mathbb{R}^n \times \\ & \{0^{n\times n}\} \times \mathbb{R}_{\geq 0}\} \times \{0\} \times \{0^n\} \times \{0\} \times \mathcal{A}_{dis} \\ & \times \mathbb{R} \times \{\{0\} \times \mathbb{R}^n \cup \mathbb{R} \times \{0^n\}\}. \end{aligned} \tag{6.8}$$

The set $\mathcal{A}$ represents the desired equilibrium set, namely the subset of $\mathbb{R}^{n+l} \times \mathcal{X}_c$ such that if $(\xi(0,0), x_c(0,0)) \in \mathcal{A}$, then $x(t,j) \in \mathcal{A}^\star$ for all $(t,j) \in \mathrm{dom}(\xi, x_c)$. Notice that invariance of $\mathcal{A}$ is guaranteed by all the step size variables being zero, so that $x(t_j + \tau^\star, j) = x(t_j, j)$. The equilibrium set $\mathcal{A}$ is given by $\zeta \in \mathbb{R}^l$, $x \in \mathcal{A}^\star$, $\Delta_j = 0$ for all $j = 0, 1, ..., n-1$, $d_j \in \mathbb{R}^n$ for all $j = 0, 1, ..., n-1$, $\Phi = 0$, $\lambda = 0$, $\alpha = \mathbf{0}$, $\bar{\alpha} = 0$, $z = f(\mathcal{A}^\star)$, and the remaining states belonging to their domain. In particular, notice how the auxiliary state ζ is irrelevant for the current control problem, as well as the directions d_j, since as long as the step sizes are zero, the displacement computed along each direction is zero, hence the state x remains constrained in the set $\mathcal{A}$ and the directions can, in principle, assume any value without hindering stability.

However, an initialization with $\Phi(0,0) = 0$ and/or $d_j(0,0) = 0$, even in the case in which the initial state $x(0,0)$ is not a minimum for f, i.e. $x(0,0) \notin \mathcal{A}^\star$, is possible. In such a case, no magnitude of the exploration steps along each direction d_j is zero, and as such the state variable x remains stuck in its initial position. We, thus, define a new set, $\mathcal{A}_e$, as the largest set of equilibrium points for $\mathcal{H}_{cl}$ for which no optimization step is performed due to an initialization with $\Phi = 0$ and/or $d_j = 0$ for all $j = 0, 1, ..., n-1$. It includes both the desired equilibrium set $\mathcal{A}$, both those, undesired, equilibrium states originated from a bad initialization. As such, it can be noticed, from (6.8), that the equilibrium set for the x state variable is $\mathbb{R}^n$, namely for a bad initialization, the optimization variable x can remain stuck everywhere in its domain.

Next we derive the stability properties of (6.6) with respect to the sets $\mathcal{A}$ and $\mathcal{A}_e$.

Theorem 6.1. *Let assumptions (A0)-(A2) hold, $\tau^\star > 0$, and the parameters of the algorithm in Fig. 4.3-4.4 satisfy $\delta_{det} > 0$, $0 < \lambda_s < 1 < \lambda_t$, $\mu \in (0, 1/\lambda_t)$, $\theta \in (0, 1)$ and $\gamma \geq 1$. Then, for the closed-loop system $\mathcal{H}_{cl}$, the set $\mathcal{A}$ in* (6.7) *is*

- *stable;*
- *almost globally attractive;*

hence it is almost globally asymptotically stable. Furthermore, the set $\mathcal{A}_e$ in (6.8) *is globally attractive for $\mathcal{H}_{cl}$.*

Proof. As we need the following result for the future discussions, we first show that all maximal solutions to $\mathcal{H}_{cl}$ are complete.

Lemma 6.2. *Let assumptions (A0)-(A2) hold, and $\tau^\star > 0$, $\delta_{det} > 0$, $0 < \lambda_s < 1 < \lambda_t$, $\mu \in (0, 1/\lambda_t)$, $\theta \in (0, 1)$ and $\gamma \in \mathbb{R}_{\geq 1}$. Then all maximal solutions to $\mathcal{H}_{cl}$ are complete.*

Proof. We prove completeness of maximal solutions to $\mathcal{H}_{cl}$ by invoking Proposition 6.10 in Goebel et al. (2012) on existence of solutions, and showing that no maximal solution jumps outside of $C \cup D$ or has finite escape time.

We first show that the viability condition in Proposition 6.10 in Goebel et al. (2012) holds for all $(\xi, x_c) \in C \setminus D$, namely that $F(\xi, x_c) \cap T_C(\xi, x_c) \neq \emptyset$, with $T_C : \mathbb{R}^{n+l} \times \mathcal{X}_c \to \mathbb{R}^{n+l} \times \mathcal{X}_c$ the Bouligand tangent cone of C at (ξ, x_c). Since $0 \in T_C(\xi, x_c)$ always, the viability condition is readily satisfied for all the state variables apart from ξ and τ, since F is zero in that case, and thus the intersection not empty. As the projection onto the ξ-subspace of $C \setminus D$ is equal to the domain of definition of ξ, i.e. $\mathbb{R}^{n+l}$, and thus the projection of $F(\xi, x_c)$ onto the ξ-subspace is never empty, the viability condition is satisfied also for the ξ state variable. Regarding the timer τ, define the projection of C and D onto the τ-subspace as $C_\tau := [0, \tau^\star]$ and $D_\tau := [\tau^\star, \infty)$. As the set $C_\tau \setminus D_\tau = [0, \tau^\star)$ is open to the right, we only need to check the viability condition at $\tau = 0$. Since at $\tau = 0$, $\dot{\tau} = 1$, the viability condition is satisfied also for τ.

Then, by Proposition 6.10 in Goebel et al. (2012), there exists a nontrivial solution from every initial condition in $\mathbb{R}^{n+l} \times \mathcal{X}_c$. Moreover, since $G(C \cup D) \subset C \cup D$, the solutions to $\mathcal{H}_{cl}$ or have finite time escape or are complete. Notice that for all solutions to $\mathcal{H}_{cl}$, $\zeta(t)$ does not have finite escape time by assumption. We show completeness by showing that the state variables (x, x_c), for all solutions to $\mathcal{H}_{cl}$, are bounded. Indeed, by condition (A2) and the update rule for the computation of new directions, (6.5), for all initial conditions $(\xi(0,0), x_c(0,0)) \in \mathbb{R}^{n+l} \times \mathcal{X}_c$, the state variables d_j, with $j = 0, 1, ..., n$, are upper bounded in norm by

$$d_{max} := \max_{j=0,1,...,n} \{\|d_j(0,0)\|, \operatorname{diam}(\mathcal{L}_f(\max\{f(x(0,0)), z(0,0)\}))\},$$

where, given $A \subset \mathbb{R}^n$, $\operatorname{diam}(A) := \sup_{x,y\in A} \|x - y\|$. Moreover, as the determinant of the matrix composed by the set of directions is lower bounded by $\delta_{\text{det}} > 0$, the directions d_j are also lower bounded in norm. Denote the lower bound as $d_{\min} \in \mathbb{R}$. Then Δ_j, $j = 0, 1, ..., n$, are upper bounded by

$$\Delta_{\max} := (1+\gamma) \max\left\{\max_{j=0,1,...,n} \Delta_j(0,0), \tfrac{\operatorname{diam}(\mathcal{L}_f(\max\{f(x(0,0)), z(0,0)\}))}{d_{\min}}, \lambda_s \Phi(0,0)\right\}.$$

Based on the same reasoning, $\Phi(t,j) \leq \Phi(0,0)$, $|\lambda(t,j)| \leq d_{\max}\Delta_{\max}$, $\|\alpha(t,j)\| \leq n d_{\max}\Delta_{\max}$, $\bar{\alpha}(t,j) \leq n d_{\max}\Delta_{\max}$, $z(t,j) \leq \max\{z(0,0), f(x(0,0))\}$, $\|x(t,j)\|_{\mathcal{A}^\star} \leq d^2_{\max}\Delta^2_{\max} d$ for all $(t,j) \in \operatorname{dom}(\xi, x_c)$. Hence any state variable of $\mathcal{H}_{cl}$ is bounded, thus all the maximal solutions to $\mathcal{H}_{cl}$ are complete. ■

In order to prove stability of $\mathcal{A}$, define the Lyapunov function $V(\xi, x_c) = z - f(\mathcal{A}^\star)$. We stress that, given assumption (A1), $f(\mathcal{A}^\star)$ is a scalar. Since V is $\mathcal{C}^1$, it is possible to bound the growth of V along any maximal solution ϕ to $\mathcal{H}_{cl}$ as

$$V(\phi(\bar{t},\bar{j})) - V(\phi(\underline{t},\underline{j})) \leq \int_{\underline{t}}^{\bar{t}} \frac{d}{dt} V(\phi(t, j(t)))dt + \sum_{j=\underline{j}+1}^{\bar{j}} [V(\phi(t(j), j)) - V(\phi(t(j), j-1))],$$

where

$$\frac{d}{dt} V(\phi(t, j(t))) = \dot{z}(t, j(t)) = 0, \tag{6.9}$$

$$V(\phi(t(j), j)) - V(\phi(t(j), j-1)) = \begin{cases} 0 & x_c \in D_{2,5} \\ z(t(j), j) - z(t(j), j-1) \leq 0 & x_c \in D_{1,3,4}, \end{cases} \tag{6.10}$$

where $t(j)$ and $j(t)$ denote respectively the least time t and the least index j such that $(t, j) \in \operatorname{dom} \phi$, $D_{2,5} := D_2 \cup D_5$ and $D_{1,3,4} := D_1 \cup D_3 \cup D_4$.

The above conditions follow directly from the definition of F_c and G_c. Indeed z changes only during jumps, and in that case, for $x \notin \mathcal{A}^\star$, it can decrease for $x_c \in D_{1,3,4}$ and remain unchanged for $x_c \in D_{2,5}$. However the Lyapunov function V is not strictly nonincreasing, nor positive definite, since there exist initial conditions for z and x such that $z(0,0) < f(x(0,0))$. However, after at most 3 timer-cycles, when D_3 is reached, z gets updated to $f(x)$, thus V becomes positive definite, and from that point onward, since, by the dynamics of z, z can only non-increase, we can show stability of $\mathcal{A}$, and apply a hybrid invariance principle to show attractivity of $\mathcal{A}_e$.

The above nonincreasing conditions on V are thus only valid for $t \geq 3\tau^\star$ and $j \geq 2$, where $(t, j) = (0, 0)$ initially. As we show next, this does not hinder the stability of the set $\mathcal{A}$ and convergence to the set $\mathcal{A}_e$ for the hybrid system $\mathcal{H}_{cl}$.

By the above discussion, compactness of $\mathcal{A}$, the conditions on V, and Theorem 7.6 in Sanfelice et al. (2007), for all $\varepsilon_1 > 0$ there exists a $\delta_1 > 0$ such that

$$\|(\xi(3\tau^\star, 2), x_c(3\tau^\star, 2))\|_{\mathcal{A}} < \delta_1 \implies \|(\xi(t, j), x_c(t, j))\|_{\mathcal{A}} < \varepsilon_1 \; \forall t + j \geq 3\tau^\star + 2 \tag{6.11}$$

Noticing that the set $\mathcal{A}$ is invariant for $\mathcal{H}_{cl}$, it follows, by Lemma 6.2 and Corollary 4.8 in Goebel and Teel (2006b), that

$$\forall \varepsilon_2 > 0, \forall T > 0, \exists \delta_2 > 0 : \|(\xi(0,0), x_c(0,0))\|_{\mathcal{A}} < \delta_2 \implies \|(\xi(t, j), x_c(t, j))\| < \varepsilon_2 \; \forall t + j < T \tag{6.12}$$

By choosing in (6.12) $T = 3\tau^\star + 2$ and $\varepsilon_2 = \delta_1$, we see that the definition of uniform stability is recovered with $\varepsilon = \varepsilon_1$ and $\delta = \delta_2$. Thus $\mathcal{A}$ is uniformly stable for $\mathcal{H}_{cl}$.

To show attractivity of $\mathcal{A}_e$ we invoke Theorem 4.7 in Sanfelice et al. (2007), setting $\mathcal{U} := \mathcal{R}_{\geq 3\tau^\star+2}(\mathbb{R}^{n+l} \times \mathcal{X}_c)$, namely the set of states that are reachable after $3\tau^\star + 2$ (see Definition 6.15 in Goebel et al. (2012)). Notice that $\mathcal{U}$ is forward invariant due to Lemma 6.2 and the definition of reachable set. By referring to the remark at the bottom of the proof of Theorem 4.7, we set $T = 3\tau^\star$ and $J = 2$, and defining u_C and u_D in the statement of Theorem 4.7 respectively as (6.9) and (6.10), for some $r \in V(\mathbb{R}^{n+l} \times \mathcal{X}_c)$, the trajectories of $\mathcal{H}_{cl}$ approach the largest weakly invariant subset of

$$V^{-1}(r) \cap \mathcal{U} \cap [u_C^{-1}(0) \cup (u_D^{-1}(0) \cap G(u_D^{-1}(0)))]. \tag{6.13}$$

The Lyapunov function V is constant along solutions to $\mathcal{H}_{cl}$ in D_2, D_5 and the set $\mathcal{A}_e$. By $m^+ = 1$ in g_2 and by $q^+ = 0$ in g_5 we can conclude that neither D_2 nor D_5 are (weakly) invariant. Indeed $\mathcal{A}_e$ is actually the largest (weakly) invariant set contained in $\mathcal{U}$ where V is constant along maximal solutions whose range is contained in $\mathcal{U}$. ■

Remark 6.1. From Theorem 6.1 and the structure of $\mathcal{A}$ and $\mathcal{A}_e$, it follows in particular that, for any initialization such that $\det(\mathrm{col}(d_0, d_1, ..., d_{n-1})) \neq 0$ and $\Phi \neq 0$, boundedness of the closed-loop trajectories and asymptotic convergence to the set $\mathcal{A}$ are guaranteed. △

Remark 6.2. Notice that, depending on the values of the constants δ_{det}, the quadratic termination property can be lost. Nonetheless, the asymptotic convergence property is preserved. △

6.3 Robustness Considerations

In this section we investigate the robustness of the proposed algorithm to noise acting on the cost function measurements. We start with a negative result showing that general Direct Search Algorithms based on line minimizations and asymptotic step size reduction are not robust to any bounded measurement noise.

Theorem 6.2. *Consider the class of Direct Search algorithms based on line minimizations and with asymptotic step size reduction, to which the algorithm in Fig. 4.3-4.4 belongs to, acting on a function $f : \mathbb{R}^n \to \mathbb{R}$ satisfying assumptions (A0) and (A2). Then, for any bound $\bar{n}_s > 0$, there exists a noise $n_s : \mathbb{R} \to \mathbb{R}$, with $|n_s(t)| \leq \bar{n}_s$ $\forall t \in \mathbb{R}$, such that, for noisy cost function measurements, namely $f(x(t)) + n_s(t)$, and all initial conditions apart from a set of measure zero, the sequence of iterate produced by such algorithms escapes any compact sub-level set of f.*

Proof. We will first show that, for any $\bar{n}_s > 0$, these class of algorithms can potentially remain stuck at every $x \in \mathbb{R}^n$. As such, by continuous differentiability of f, for every compact set $\mathcal{C} \subset \mathbb{R}^n$, there exists a maximum gradient norm $\nabla f_{\mathcal{C}}$.

Consider, without loss of generality, a unique step size variable $\Delta > 0$ and a single direction $d \in \mathbb{R}^n$. By the mean value theorem, it follows that, for all $x, y \in \mathcal{C}$, $|f(x) - f(w)| \leq \nabla f_{\mathcal{C}} \|x - w\|$, and, for $w = x - p\Delta d$, at iteration k in the algorithm

$$|f(x_k) - f(x_k - p_k \Delta_k d_k)| \leq \nabla f_{\mathcal{C}} \Delta_k \bar{d}, \tag{6.14}$$

where, by continuous differentiability of f, $\nabla f_{\mathcal{C}} < \infty$ for all compact $\mathcal{C} \subset \mathbb{R}^n$, and $\|d_k\| \leq \bar{d} > 0$.

Given a noise bound $\bar{n}_s > 0$, and remembering that $\Delta_k \to 0$ for $k \to \infty$, there exists a $k^\star > 0$ such that

$$\nabla f_{\mathcal{C}} \Delta_{k^\star} \bar{d} + \rho(\Delta_{k^\star}) < \nabla f_{\mathcal{C}} \Delta_{k^\star} \bar{d} \frac{1}{1-\theta} + \iota \Delta^\star_{k^\star} < \bar{n}_s, \tag{6.15}$$

with $\iota > 0$ and $\Delta^\star_{k^\star} > 0$ the value of the series $\sum_{n=0}^{\infty} (\theta^n \Delta_{k^\star})^{\left(\frac{1}{\theta^n \Delta_{k^\star}}\right)}$, proved to be convergent in Lemma 4.1. The term $1/(1-\theta)$ follows by noticing that given iteration k_1, after k_2 iterations of blocked points, then $\Delta_{k_1+k_2} = \theta^{k_2} \Delta_{k_1}$, and, as $k_2 \to \infty$, if we sum all the terms, we have a geometric series. We defined the bound in this way, since we build a noise function by iteratively summing previous noise values to produce the new one.

A noise signal defined to be $n_s(k) = 0$ for $k < k^\star$ and $n_s(k) = \nabla f_{\mathcal{C}} \Delta_k \bar{d} + \rho(\Delta_k) + n_s(k-1)$ for $k \geq k^\star$ will keep the algorithm stuck in $x = x_{k^\star}$ for all $k \geq k^\star$.

The reason is that the following relationship will always be satisfied

$$f(x_k + p_k \Delta_k d_k) + n_s(k) \geq f(x_k) + n_s(k-1) - \rho(\Delta_k),$$

where $f(x_k + p_k \Delta_k d_k) + n_s(k)$ is the cost function measurement obtained at iteration k and $f(x_k) + n_s(k-1)$ is the cost function measurement obtained at the previous iteration. Namely no improvement is ever found in any direction, since

$$n_s(k) = \nabla f_{\mathcal{C}} \Delta_k \bar{d} + \rho(\Delta_k) + n_s(k-1) \geq f(x_k) - f(x_k + p_k \Delta_k d_k) - \rho(\Delta_k) + n_s(k-1). \tag{6.16}$$

Now notice that at iterations where

$$f(x_k + p_k \Delta_k d_k) \geq f(x_k) - \rho(\Delta_k),$$

namely at iterations where no improvement would be found in case of no noise, the noise could act in order to mistakenly consider an improvement. Indeed in that case, with a noise of the form

$$n_s(k) = -\nabla f_{\mathcal{C}} \Delta_k \bar{d} - \rho(\Delta_k) + n_s(k-1), \tag{6.17}$$

for $k \geq k_1^\star \geq 0$ and $n_s(k_1^\star) = 0$, a wrong descent direction will be picked from everywhere in $\mathcal{C}$.

Alternating the noise values of (6.16) and (6.17), by considering $n_s(k-1) = 0$ when switching strategy, as long as $\Delta_k \leq \Delta_{k^\star}$, can steer the algorithm to every point in $\mathcal{C}$.

Consider now a compact set $\mathcal{C}_1 \supset \mathcal{C}$ and denote the maximum gradient norm of f on $\mathcal{C}_1$ as $\nabla f_{\mathcal{C}_1}$, where $\nabla f_{\mathcal{C}_1} \geq \nabla f_{\mathcal{C}}$.

Applying the noise (6.16) in $\mathcal{C}$, it is possible to notice that there exists a $k_1^\star \geq k^\star$ such that for $k = k_1^\star$ condition (6.15) is satisfied for $\nabla f_{\mathcal{C}_1}$.

Now, by switching between noise expressions (6.16) and (6.17), guaranteeing that $\Delta_k \leq \Delta_{k_1^\star}$, makes it possible to steer the sequence of iterate everywhere in $\mathcal{C}_1$ and in particular outside $\mathcal{C}$.

It is thus clear that repeating this procedure iteratively can make the sequence of iterate leave any compact sub-level set of f. ■

The above result shows that there is no robustness guarantee for the modified RSP algorithm, even if stability has been shown and convergence results are attainable for a proper choice of initial conditions.

This result translates to not robustness of the hybrid implementation of the proposed RSP algorithm. As such, we remark that it agrees with the robustness results of Chapter 7 of Goebel et al. (2012), where sufficient conditions for robustness of nominally well-posed hybrid systems are proposed. Indeed, it is proven in Goebel et al. (2012) that a sufficient condition for the robustness of a nominally well-posed hybrid system is pre-asymptotic stability of a compact set, while, in the current setting, only almost-global asymptotic stability of a compact set is shown. This fact opens up questions on the necessity, and possible relaxation, of the pre-asymptotic stability condition for robustness that will be investigated in future works.

Robustness to measurement noise for the hybrid closed-loop system $\mathcal{H}_{cl}$ is recovered by imposing a lower bound $\underline{\Phi} > 0$ on the global step size Φ, and modifying accordingly $G_{c\setminus\tau}$. In particular, in g_5, the discrete dynamics of Φ can be modified as follows.

$$\Phi^+ = \begin{cases} \mu\Phi & \text{if } k = n \text{ and } \bar{\alpha} + \|\lambda v\| \leq \min\limits_{j\in\{0,1,...,n-1\}} \Delta_j/2 \text{ and } \mu\Phi \geq \underline{\Phi} \\ \underline{\Phi} & \text{if } k = n \text{ and } \bar{\alpha} + \|\lambda v\| \leq \min\limits_{j\in\{0,1,...,n-1\}} \Delta_j/2 \text{ and } \mu\Phi \leq \underline{\Phi} \\ \Phi & \text{otherwise.} \end{cases} \tag{6.18}$$

Moreover, given $\delta_{\text{det}} > 0$, we restrict the domain of all the directions d_j to be such that $\det(\text{col}(d_0, d_1, ..., d_{n-1})) \geq \delta_{\text{det}}$. Without loss of generality, we will denote the desired equilibrium set within the restricted domain for the directions as $\mathcal{A}$.

Theorem 6.3. *Let assumptions (A0)-(A2) hold, $\underline{\Phi} > 0$, the parameters of the algorithm in Fig. 4.3-4.4 satisfy $0 < \lambda_s < 1 < \lambda_t$, $\delta_{det} > 0$, $\mu \in (0, 1/\lambda_t)$, $\theta \in (0, 1)$ and $\gamma \geq 1$, with the update of Φ modified such that $\Phi(t, j) \geq \underline{\Phi}$ for all $(t, j) \in \text{dom}\,\Phi$. Then the set $\mathcal{A}$ is semiglobally practically asymptotically stable on $\underline{\Phi} > 0$ for $\mathcal{H}_{cl}$.*

Proof. Let $\epsilon_1 > \epsilon_2 > 0$ and constants $0 < \lambda_s < 1 < \lambda_t$ be given.

Notice that, by the bounds on the state variables defined in the proof of Lemma 6.2, it is always possible to choose $\delta > 0$ to be the maximum radius of all the balls, one per state variable composing (x, x_c), such that the maximum of the bounds reported in the proof of Lemma 6.2 is upper bounded by ϵ_1. Namely pick δ such that for all initial conditions in $\delta\mathbb{B}(\mathcal{A})$,

$$\max_{\delta>0} \max_{(\xi(0,0),x_c(0,0))\in\delta\mathbb{B}(\mathcal{A})} \{d_{\max}, \Delta_{\max}, \Phi(0,0), d_{\max}\Delta_{\max}, nd_{\max}\Delta_{\max}, \\ \max\{z(0,0), f(x(0,0))\} - f(\mathcal{A}^\star), d_{\max}^2\Delta_{\max}^2\} < \epsilon_1.$$

Then pick $S_\Phi = (0, \delta]$ and notice that, by (6.18), for all $\underline{\Phi} \in S_\Phi$, $(\xi(0,0), x_c(0,0)) \in \delta\mathbb{B}(\mathcal{A}) \implies (\xi(t,j), x_c(t,j)) \in \epsilon_1\mathbb{B}(\mathcal{A})$ for all $(t,j) \in \text{dom}(\xi, x_c)$.

Denote as $\mathcal{L}_\epsilon$ the largest sublevel set of f subset of the closure of $\min\{\epsilon_2, \delta\}\mathbb{B}(\mathcal{A}^\star)$, and as f_ϵ the biggest value that f achieves in $\mathcal{L}_\epsilon$.

Pick $\mathcal{B} := \text{cl}\{x \in \mathbb{R}^n : x \in \epsilon_1\mathbb{B}(\mathcal{A}) \text{ and } x \notin \mathcal{L}_\epsilon\}$.

By assumptions (A0)-(A2) and the fact that the set of directions d_j, with $j = 0, 1, ..., n-1$, always span $\mathbb{R}^n$, it follows (from the fact that for any neighborhood small enough not containing any local maxima or local minima, the norm of the gradient of f is lower bounded away from zero) that for every compact set in $\mathbb{R}^n$ not containing a local minimum, there exists a $\bar{\Phi} > 0$, such that for all $\Phi \in (0, \bar{\Phi})$, there exists at least one direction, that, rescaled by $\lambda_s\Phi$, produces a sufficient decrease of f from every point in that compact set.

Since $\mathcal{B}$ is compact and does not contain a local minimum, it implies, by the above reasoning, that there exists $\bar{\Phi} > 0$, such that for all $\Phi_{bound} \in (0, \bar{\Phi})$ at least one direction is a descent direction for $\Phi = \Phi_{bound}$, hence, after at most n iterations, z decreases.

Define now the Lyapunov candidate function $V(\xi, x_c) = z - f_\epsilon$, and notice that it satisfies (6.9) and (6.10), after at most $3\tau^\star + 2$, on $\epsilon_1\mathbb{B}(\mathcal{A}) \setminus \min\{\epsilon_2, \delta\}\mathbb{B}(\mathcal{A})$. By Lemma 4.2 and picking $\underline{\Phi} \in (0, \bar{\Phi})$, it follows that there exist $(T, J) \in \mathrm{dom}(\xi, x_c)$ such that for all $(t, j) \in \mathrm{dom}(\xi, x_c)$ such that $t + j \geq T + J$, $\Phi = \underline{\Phi}$ and $\Delta_j = \lambda_s\underline{\Phi}$. This implies that as long as $x \in \mathcal{B}$, after at most every n iterations, z decreases, and thus, by applying Theorem 4.7 in Sanfelice et al. (2007), $(\xi, x_c) \to \epsilon_2\mathbb{B}(\mathcal{A})$. To conclude the proof, choose $\underline{\Phi} \in (0, \min\{\delta, \bar{\Phi}\})$. ■

The lower bound on Φ also guarantees an explicit bound on the allowable maximum noise that can be accepted without losing robustness.

Corollary 6.1. *For all parameters of the algorithm in Fig. 4.3-4.4 satisfying* $0 < \lambda_s < 1 < \lambda_t$, $\delta_{\text{det}} > 0$, $\mu \in (0, 1/\lambda_t)$, $\theta \in (0, 1)$, $\gamma \geq 1$, *and all measurement noise* $n_s : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ *added to* f, *with* $|n_s(t, j)| \leq \bar{n}_s$ *for all* $(t, j) \in \mathbb{R} \times \mathbb{N}$, *with* $\bar{n}_s > 0$, *pick* $\underline{\Phi}^\star > 0$ *such that*

$$\bar{n}_s = \frac{\rho(\lambda_s\underline{\Phi}^\star)}{2}. \tag{6.19}$$

Then the set $\mathcal{A}$ *is semiglobally practically asymptotically stable on* $\underline{\Phi} \geq \underline{\Phi}^\star$ *for* $\mathcal{H}_{cl}$, *with the update of* Φ *modified such that* $\Phi(t, j) \geq \underline{\Phi}$ *for all* $(t, j) \in \mathrm{dom}\,\Phi$.

Proof. Consider the Lyapunov function $V(x) = f(x) - f(\mathcal{A}^\star)$ for the case of lower bounded step size Φ, with lower bound $\underline{\Phi} > 0$, and the sequence of times $(t(j_k^\star), j_k^\star)$, where $t(j)$ is the biggest time t such that $(t, j) \in \mathrm{dom}\,x$, $(t(j_0^\star), j_0^\star)$ is such that $j_0^\star \geq j_0 + 3$ and, given $(t(j_k^\star), j_k^\star)$, $(t(j_{k+1}^\star), j_{k+1}^\star)$ is computed as $(t(j_k^\star + 1), j_k^\star + 1)$ if $(\xi(t(j_k^\star + 1), j_k^\star + 1), x_c(t(j_k^\star + 1), j_k^\star + 1)) \notin D_2$. We claim that such V satisfies Condition 1 and Condition 2 of Theorem 5.4.

Condition 1 Clearly $V(\mathcal{A}^\star) = 0$, moreover, since f is lower bounded, with lower bound $f(\mathcal{A}^\star)$ and continuous, V can be bounded by

$$\bar{\alpha}(\|x\|_{\mathcal{A}^\star}) \leq V(x) \leq \bar{\beta}(\|x\|_{\mathcal{A}^\star}),$$

where

$$\bar{\alpha}(\|x\|_{\mathcal{A}^\star}) := (-e^{-\|x\|_{\mathcal{A}^\star}} + 1) \inf_{\bar{x}\in\mathbb{R}^n:\|\bar{x}\|_{\mathcal{A}^\star}\geq\|x\|_{\mathcal{A}^\star}} (f(\bar{x}) - f(\mathcal{A}^\star))$$
$$\bar{\beta}(\|x\|_{\mathcal{A}^\star}) := \|x\|_{\mathcal{A}^\star} \max_{\bar{x}\in\mathbb{R}^n:\|\bar{x}\|_{\mathcal{A}^\star}\leq\|x\|_{\mathcal{A}^\star}} \|\nabla f(\bar{x})\|$$

Condition 2 We first show that $(t(j_k^\star) + j_k^\star) - (t(j_{k+1}^\star), j_{k+1}^\star) \leq T = 3T' + 3$, with $T' > 0$ the period of the timer.

Notice that D_2 is defined for $q \in \{0,1\}$, $\dot{q} = 0$ always, and for $(\xi, x_c) \in D_2$ $q^+ = q+1$. Since the jump rule is defined by the timer only, given $(\xi(t(j_k^\star), j_k^\star), x_c(t(j_k^\star), j_k^\star)) \in D \setminus D_2$, there exists $\bar{j} \in \{1,2,3\}$ such that $(\xi(t(j_k^\star+\bar{j}), j_k^\star+\bar{j}), x_c(t(j_k^\star+\bar{j}), j_k^\star+\bar{j})) \in D \setminus D_2$ again.

At $(t(j_k^\star), j_k^\star)$, $(\xi(t(j_k^\star), j_k^\star), x(t(j_k^\star), j_k^\star)) \in D$.

As a cycle in the algorithm results in the state $x(t(j), j)$ moving between D_i in the following order

$$D_5 \to D_1 \to ... \to D_1 \to D_2 \to D_3 \to D_4 \to ... \to D_4 \to D_5, \tag{6.20}$$

we can notice the following:
If $x(t(j_k^\star), j_k^\star) \in D_1$, then $y(t(j_k^\star), j_k^\star) = f(x(t(j_k^\star), j_k^\star)) + n_s(t(j_k^\star), j_k^\star) \leq z(t(j_k^\star), j_k^\star) - \rho(\Delta(t(j_k^\star), j_k^\star)) \leq f(x(t(j_k^\star-1), j_k^\star-1)) + \bar{n_s} - \rho(\Delta(t(j_k^\star), j_k^\star))$.
If $x(t(j_k^\star), j_k^\star) \in D_2$, then we do not consider the Lyapunov function there.
If $x(t(j_k^\star), j_k^\star) \in D_3$, then $f(x(t(j_k^\star), j_k^\star)) = f(x(t(j_k^\star-2), j_k^\star-2))$.
If $x(t(j_k^\star), j_k^\star) \in D_4$, then $y(t(j_k^\star), j_k^\star) = f(x(t(j_k^\star), j_k^\star)) + n_s(t(j_k^\star), j_k^\star) \leq z(t(j_k^\star), j_k^\star) - \rho(\Delta(t(j_k^\star), j_k^\star)) \leq f(x(t(j_k^\star-1), j_k^\star-1)) + \bar{n_s} - \rho(\Delta(t(j_k^\star), j_k^\star))$.
If $x(t(j_k^\star), j_k^\star) \in D_5$, then $f(x(t(j_k^\star), j_k^\star)) = f(x(t(j_k^\star-2), j_k^\star-2))$.

If we assume $\bar{n_s} = 0$, then we can notice that $V(x(t(j_{k+1}^\star), j_{k+1}^\star)) - V(x(t(j_k^\star), j_k^\star)) \leq 0$ for all hybrid arcs x, for all $k \in \mathbb{N}$.

$$V(x(t(j_{k+1}^\star), j_{k+1}^\star)) - V(x(t(j_k^\star), j_k^\star)) = \begin{cases} f(x(t(j_{k+1}^\star), j_{k+1}^\star)) - f(x(t(j_k^\star), j_k^\star) & \text{if } x(t(j_{k+1}^\star), j_k^\star) \in D_{1,4} \\ 0 & \text{if } x(t(j_{k+1}^\star), j_k^\star) \in D_{3,5} \end{cases} \tag{6.21}$$

In case we assume noise acting on the cost function measurement, $n_s(t,j) \leq \bar{n_s}$, then, for $x(t(j^\star_{k+1}), j^\star_k) \in D_{1,4}$,

$$V(x(t(j^\star_{k+1}), j^\star_{k+1})) - V(x(t(j^\star_k), j^\star_k)) = f(x(t(j^\star_{k+1}), j^\star_{k+1})) - f(x(t(j^\star_k), j^\star_k)) = \tag{6.22}$$

$$\delta_f(t(j^\star_{k+1}), j^\star_{k+1}). \tag{6.23}$$

Hence $\delta_f(t(j^\star_{k+1}), j^\star_{k+1})$ totally determines the sign of $V(x(t(j^\star_{k+1}), j^\star_{k+1})) - V(x(t(j^\star_k), j^\star_k))$.

Since

$$\begin{aligned}
&y(t(j^\star_{k+1}), j^\star_{k+1}) = f(x(t(j^\star_{k+1}), j^\star_{k+1})) + n_s(t(j^\star_{k+1}), j^\star_{k+1}) \leq z(t(j^\star_k), j^\star_k) - \Delta(t(j^\star_k), j^\star_k)^{\frac{1}{\Delta(t(j^\star_k), j^\star_k)}} \\
&= f(x(t(j^\star_k), j^\star_k)) + n_s(t(j^\star_k), j^\star_k) - \Delta(t(j^\star_k), j^\star_k)^{\frac{1}{\Delta(t(j^\star_k), j^\star_k)}} \implies \\
&f(x(t(j^\star_{k+1}), j^\star_{k+1})) \leq f(x(t(j^\star_k), j^\star_k)) + 2\bar{n}_s - \Delta(t(j^\star_k), j^\star_k)^{\frac{1}{\Delta(t(j^\star_k), j^\star_k)}} \implies \\
&f(x(t(j^\star_{k+1}), j^\star_{k+1})) - f(x(t(j^\star_k), j^\star_k)) = \delta_f(t(j^\star_{k+1}), j^\star_{k+1}) \leq 2\bar{n}_s - \rho(\Delta(t(j^\star_k), j^\star_k)).
\end{aligned}$$

If $2\bar{n}_s - \rho(\lambda_s \underline{\Phi}) \leq 0$, then $V(x(t(j^\star_{k+1}), j^\star_{k+1})) - V(x(t(j^\star_k), j^\star_k)) \leq 0$ for all $k \in \mathbb{N}$. Indeed, semiglobal practical stability is preserved for all $n_s : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$, with $n_s(t, j) \leq \bar{n}_s$ for all $(t, j) \in \mathbb{R} \times \mathbb{N}$ and such that

$$\bar{n}_s \leq \frac{\rho(\lambda_s \underline{\Phi})}{2} \tag{6.24}$$

■

Knowledge of a bound on the maximum norm of the measurement noise makes it possible to easily design robust hybrid controllers without resorting to further robust control techniques.

Remark 6.3. In Mayhew et al. (2007) an explicit characterization of the practical neighborhood of convergence to $\mathcal{A}$, as function of the step size, is provided. As the dense exploration procedure adopted in Mayhew et al. (2007) to guarantee such bounds cannot be extended to n-dimensional search spaces, a similar result cannot be achieved without further assumptions on f. Nonetheless, the norm of the gradient of f can be bounded at steady state by a function of $\underline{\Phi}$ and the equilibrium set of exploring directions (see Theorem 3.3 in Kolda et al. (2003)). △

Remark 6.4. The trade-off between practical global asymptotic stability and almost global asymptotic stability is, also, related to the lack of knowledge of $\mathcal{A}^\star$ or $f(\mathcal{A}^\star)$. By assuming, for example, knowledge of $f(\mathcal{A}^\star)$, the discrete dynamics of Φ can be extended with the addition of a term $\rho_f(|f(x) - f(\mathcal{A}^\star)|)$, where $\rho_f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $\rho_f(|f(x) - f(\mathcal{A}^\star)|) > 0$ for x such that $|f(x) - f(\mathcal{A}^\star)| > 0$. This term would prevent the algorithm to remain stuck at the initial position when Φ is initialized at zero and, thus, Theorem 6.1 could be extended to guarantee global asymptotic stability of the set of minimizers. △

6.4 Examples

In this section we show the results of different simulations of the proposed hybrid controller to the minimization of different objective functions.

Fig. 6.1 illustrates the level sets of the quadratic convex function

$$f(x) = x_1^2 + 5x_2^2, \tag{6.25}$$

where $x = \mathrm{col}(x_1, x_2)$. The trajectory of a point-mass vehicle, steered by the proposed hybrid controller in order to minimize (6.25), is superimposed to the level sets of (6.25), showing the value of $f(x)$ at each corresponding point of the trajectory. The control input was chosen as $K(x, x_c, \tau^\star) = p\Delta v/\tau^\star$. For this simulation, the initial values of the state variables of the hybrid closed loop were chosen as $x(0,0) = \mathrm{col}(1.5, 0)$, $\tau(0,0) = 0$, $\lambda(0,0) = 0$, $\alpha(0,0) = 0$, $z(0,0) = 0$, $p(0,0) = 1$, $q(0,0) = 0$, $m(0,0) = 0$, $k(0,0) = 0$, $\alpha(0,0) = 0$, $d_{00}(0,0) = \mathrm{col}(\cos(\pi/8), \sin(\pi/8))$, $d_{01} = \mathrm{col}(-\sin(\pi/8), \cos(\pi/8))$, $v(0,0) = d_{00}$ $\Delta_{0j}(0,0) = 0.01$, $j = 0, 1$, $\Delta(0,0) = \Delta_{00}$. The tunable parameters of the controller were defined as $\gamma = 1.2$, $\theta = 0.5$, $\delta_{\mathrm{det}} = 0.001$, $\mu = 0.15$, $\lambda_s = 0.001$, and $\lambda_t = 5$.

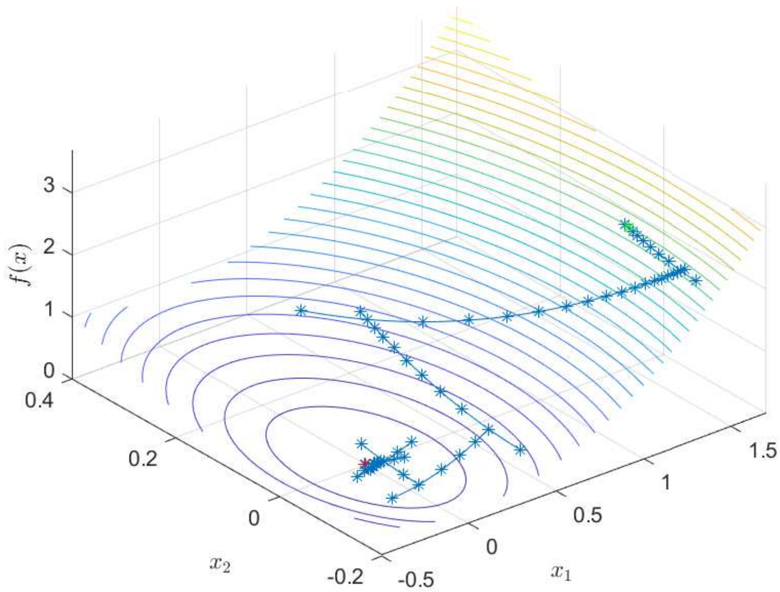
It can be noticed as in both Fig. 6.1(a) and Fig. 6.1(b), the distance to the minimizer tends asymptotically to zero as the step size converges to zero. We highlight how, at jump times, the state variable x, steered by the hybrid controller implementing the new RSP algorithm, coincides (up to a small numerical error given by the integration of the continuous time dynamics in the simulating environment) with the sequence of iterates produced by the new RSP algorithm for the same example in Section 4.1.4, under the assumption of "exact" controllability of the underlying continuous time dynamical system, i.e. the x subsystem.

The simulation reported in Fig. 6.2, instead, considered the nonconvex Rosenbrock function

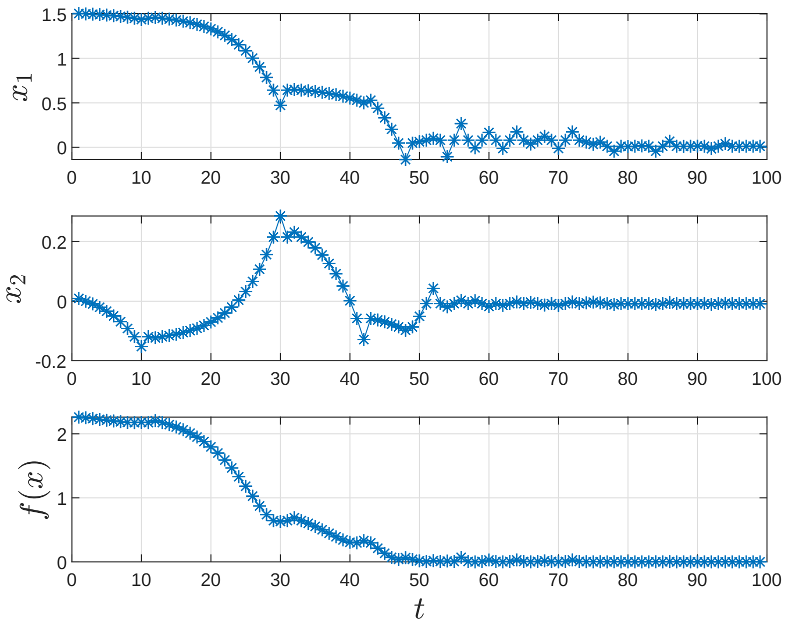
$$f(x) = (1 - x_1)^2 + 10(x_2 - x_1^2)^2, \tag{6.26}$$

and the Dubin's vehicle dynamics. The initial conditions and parameter values were kept the same of the previous simulation. In this case the minimizer is given by $x^\star = (1, 1)$ and, in spite of the nonconvex optimization problem, the trajectory of the state variable x is converging towards it, remarkably.

In Fig. 6.3 we show a comparison of the x-trajectories of $\mathcal{H}_{cl}$ for a point-mass vehicle in case measurement noise affecting (6.25) are considered. The initial conditions and parameter values were kept the same of the previous simulations apart from $\Phi(0, 0) = \Delta_j(0, 0) = \Delta(0, 0) = 1$. In Fig. 6.3(a)-6.3(b) no lower bound on Φ and no measurement noise is assumed, the x-trajectory indeed behaves similarly to the one in Fig. 6.1, converging asymptotically to the minimum $x^\star = (0, 0)$. In Fig. 6.3(c)-6.3(d) no lower bound on Φ is assumed, but a measurement noise $n_s(t, j)$, upper bounded by $\bar{n}_s = 0.04$ on f is considered. The measurement noise n_s is designed as proposed in the proof of Theorem 6.2. Notice that for $\Delta = 0.38$, $\rho(\Delta)/2 \simeq \bar{n}_s$. Indeed we highlight that when $\rho(\Delta_j) \leq \bar{n}_s$, for $j = 0, 1$, is satisfied, and it is for $\Delta_j \leq 0.38$ for all $j = 0, 1$, the effect of the noise tricks the hybrid controller into steering the x-subsystem away from the minimum. This behavior can be seen in the plot of $f(x)$ in Fig. 6.3(d) after about 20 seconds of simulation. In Fig. 6.3(e)-6.3(f) the same measurement noise is assumed, but, from (6.19), $\underline{\Phi} = 40$ is choosen, implying $\Delta_j(t, j) \geq 0.4$ for all $(t, j) \in \mathrm{dom}(\xi, x_c)$. As proven in Corollary 6.1, the imposed lower bound on Φ compensates the effects of the measurement noise, stabilizing the state x in a neighborhood of the minimum.



(a) x trajectory versus the level sets of a quadratic convex function



(b) $x(t)$ and $f(x(t))$

Figure 6.1: Plot of the trajectories of $x(t, j)$ and $f(x(t, j))$, where $f(x) = x_1^2 + 5x_2^2$. (a) Shows the vehicle path (blue with '*' where jump occurs) on the level sets of f. The initial point is indicated with a green '*' and the unique minimizer $(0, 0, 0)$ with a red '*'. (b) Shows the evolution of x and $f(x)$ as function of time.

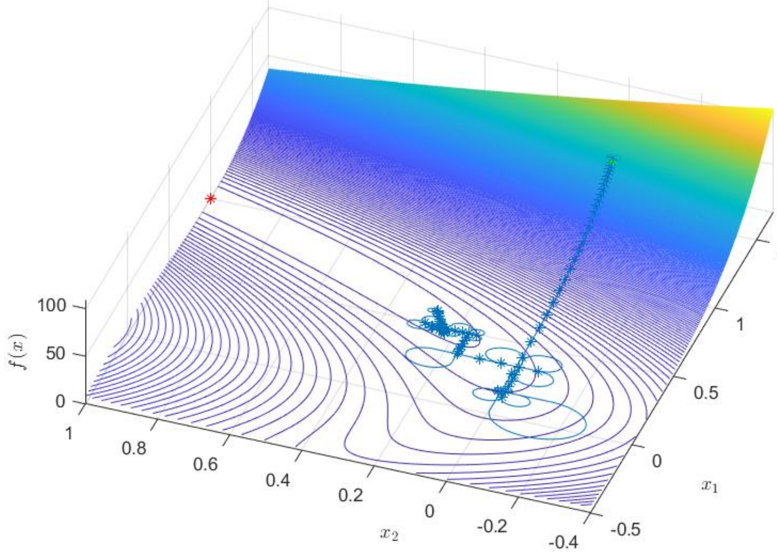
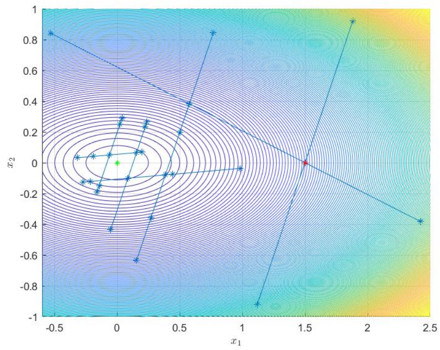
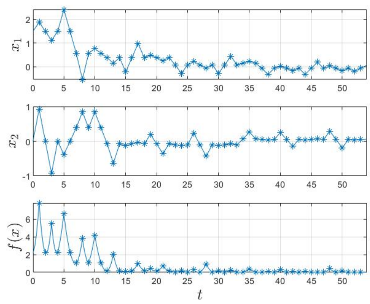


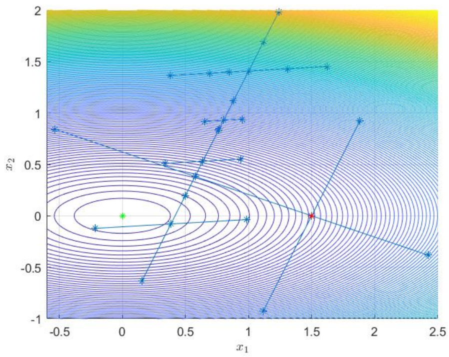
Figure 6.2: The Dubin's vehicle path on the level sets of the Rosenbrock function (6.26).



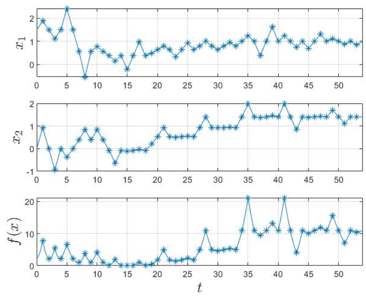
(a) x trajectory versus the level sets (6.25) assuming no lower bound on Φ and no measurement noise



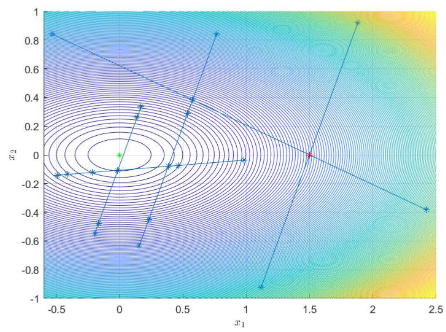
(b) $x(t)$ and $f(x(t))$



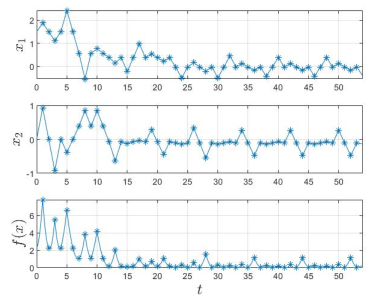
(c) x trajectory versus the level sets (6.25) assuming no lower bound on Φ but measurement noise added to f



(d) $x(t)$ and $f(x(t))$



(e) x trajectory versus the level sets (6.25) assuming a lower bound on Φ and measurement noise added to f



(f) $x(t)$ and $f(x(t))$

Figure 6.3: Comparison of the plots of the trajectories of $x(t, j)$ and $f(x(t, j))$, where $f(x) = x_1^2 + 5x_2^2$, under different assumptions on measurement noise and $\underline{\Phi}$. (a),(c) and (e) show the vehicle path (blue with '*' where jump occurs) on the level sets of f. The initial point is indicated with a red '*' and the unique minimizer $(0, 0, 0)$ with a green '*'. (b),(d) and (f) show the evolution of x and $f(x)$ as function of time.

Concluding Remarks

In this second part of the thesis we first introduced the class of derivative-free optimization algorithms denoted Direct Search algorithms, and their sub-class of Generating Set Search algorithms, together with the conjugacy property of vectors with respect to a matrix, adopted in Smith (1962) in order to reach the minimum of a convex quadratic function in a finite number of line minimizations. In this framework we developed an algorithm based on conjugate directions able to asymptotically reach the minimimum of a particular class of, possibly, non-convex function. The proposed algorithm computes a series of discrete line minimizations, based on a sufficient decrease condition, along a set of linearly independent directions spanning the search space, constantly updated with locally conjugate directions. When no further descent is achieved for the current value of the step size, the step size is reduced and the exploration starts again. We implemented the proposed optimization algorithm in a hybrid controller in order to study its stability and robustness properties when coupled with a continuous time dynamical system. Almost global asymptotic stability was shown for the proposed implementation with, however, a lack of robustness to noise acting on the objective function measurements. A robust solution is, thus, considered by imposing a lower bound on the admissible step sizes values. In particular an explicit expression for the relationship relating the minimum step size to the maximum norm of the measurement noise is computed.

Being intrinsically simple, one of the main advantages of the proposed hybrid controller is the ease of implementation. Indeed no complex operations are computed in the jump map G, and, apart from the state variable ξ, state of the continuous time dynamical system to be steered, and the timer, no other closed-loop state variable changes in continuous time, lending itself also for an easy digital implementation. Moreover, knowledge of the maximum bound on the noise will lead to a robust solution without any filtering of the measured objective function signal.

For the studied problem, the adopted hybrid system framework presented itself as a natural theory able to blend the inherent discrete dynamics of the proposed algorithm with a continuous time dynamical system executing the optimization algorithm. Other positive results of this idea can be found, for example, in the recently developed theory for hybrid extremum seeking control, see e.g. Poveda et al. (2018). In general, an extension of the proposed framework, based on a hybrid interconnection, with different optimization algorithms, is not straightforward as, in order to guarantee that the hybrid basic conditions are satisfied, particular attention has to be placed on the outer semicontinuity of functions adopted in each optimization algorithm. However, this will be the focus of future studies. A currently studied problem in this direction is the use of identification techniques to capture derivative information from function evaluations.

On the wake of these results, interesting research directions include further extensions to the hybrid system framework of optimization algorithms addressing more complex scenarios. In particular, we aim at further developing the proposed RSP algorithm to address constrained optimization problems. Applications, and techniques, regarding Direct Search algorithm framing this problem can, for example, be found in Section 8 in Kolda et al. (2003), but their hybrid implementation is not straight forward as, unless the geometry of the constraints is known exactly a priori, computing online new directions of exploration that belong to the tangent cone to the constraint set, while at the same time satisfying the hybrid basic conditions, is not trivial.

Further research directions under exploration are extensions to the non-smooth case, properly modifying and implementing the results in Popovic and Teel (2004), as well as to the multi-agent scenario, by noticing that the amount of line minimizations needed in order to apply the Parallel Subspace Property could be reduced by simultaneous parallel line minimizations computed by different agents collaborating to solve the same optimization problem.

On the other hand, the robustness analysis developed in the previous section fueled interest in further studying the necessity of the pre-asymptotic stability condition to guarantee robustness of hybrid systems, as well as providing conditions for the robustness (or lack of) for almost globally asymptotically stable hybrid dynamical systems. These topics will be material of future investigation.

Conclusions

This thesis work addressed two main topics, seemingly unrelated, but whose study has the same scope of paving the way towards robust solutions to the output regulation problem.

The first part dealt with the problem of adaptive linear output regulation when the exosystem is not assumed to be known. The proposed solution is based on the construction of an adaptive observer estimating the steady state variables of the internal model solving the regulator equations. The adaptive observer is built by the interconnection of an "extended" internal model unit and a continuous-time least-squares identifier. The achieved result, based on a small-gain argument, states that, under a persistence of excitation condition, if the identifier dynamics are "slow enough" with respect to the plant, stabilizer and internal model unit dynamics, then semiglobal asymptotic stability of the error zeroing subspace is proven. To be noticed is that, as no minimum phase assumptions were considered, the non-minimum phase case can be treated without any additional effort.

The second part of the thesis addressed the problem of steering the state of a continuous-dynamical system toward the minimum of an unknown, but sporadically measurable, objective function. The class of algorithms considered to solve this optimization problem are the Direct Search algorithms. In the framework of Direct Search algorithms, under particular assumptions on the structure of the, possibly non convex, objective function, a novel algorithm is developed, able to asymptotically converge to the set of minima. A hybrid controller is designed to implement the interconnection of the proposed algorithm with the the underlying dynamical system. The controller is shown to be almost globally asymptotically stable and, under a robustness analysis, shown to be, as all direct search methods based on asymptotic step size reduction, not robust to any bounded noise acting additively on the objective function measurements. An alternative design, encompassing the one proposed in Mayhew et al. (2007), is thus proposed and shown to be semiglobally practically asymptotically stable and robust to bounded measurement noise. Moreover, an expression relating the minimum step size and the noise bound is computed, making it possible to guarantee robustness of the proposed scheme without adding any processing of the measured signal. It is worth stressing that, under no additional assumptions, a trade-off between asymptotic stability and robustness is inevitable.

The studied topics should be framed in an attempt to seek more robust solutions to the output regulation problem, and not only. Indeed future work directions will be the study of the (linear) output regulation problem in a hybrid framework, considering the proposed new RSP algorithm as a possible identifier for the internal model parameters. Moreover, under the assumption of not measurability of the error, to solve the output regulation problem upon availability, sporadically, the measurements of an objective function whose minimum corresponds to the zero error steady state. In this regard, an attempt will be made into approaching the solution of this problem with the proposed hybrid direct search controller but from an output regulation point of view.

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