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THE EDUCATIONAL VALUE OF MAXWELL'S APPROACH TO ELECTROMAGNETISM, FROM THE FOUNDATIONS OF THE CONCEPT OF FIELD TO THE FORMULATION OF INTERDISCIPLINARY PROBLEMS

Elaborato finale

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Introduction

Usually, electrostatics is the first step in the teaching of the theory of electromagnetism, both at the secondary school and at university level. In teaching electrostatics, everyday notions, like electricity, current, discharge are elaborated and interpreted within a precise physical framework, based on the Newtonian concept of action-at-a-distance. Other physical entities, however, are introduced to students: the electric charge and the electric field.
Many researches investigated the quality of learning in electromagnetism, finding that the comprehension of many parts of the theory is hard for students, particularly the concept of field. Current teaching often fails to guide students to elaborate a sophisticated idea of the electromagnetic field and, behind the formalism, the permanence of models derived either from daily experience or Newtonian perspective is revealed. The result is that students fail to appreciate the deep transformation provided by Maxwell and his electromagnetism in modeling interactions.
Indeed, Newtonian mechanics is usually the unique framework that is taught at the secondary school level and the students usually cope with a unique model of interaction, based on the concept of action-at-a-distance. The Coulomb Force is, in fact, the first law taught in electrostatics and the electric field is very seldom introduced explicitly as another possible model of interaction. Altogether, students continue to "think in term of force", referring their reasoning to the Newtonian physical framework.
Furthermore, problem solving activities both at the secondary school level and at university usually do not train students to "think in terms of field". It results that electric and magnetic fields appear to be mainly new mathematical tools useful to find the Coulomb or the Lorentz forces.

The main difficulties in the comprehension of the field concept importance appear to be:

- electric field seems to be simply another way to describe Coulomb force, and it appears to serve only to quantify it;

- electric field is not conceived as a real physical object;
- electric field is usually represented in different ways (a vector, a series of vectors, a series of lines, etc.) and students are not able to manage consistently and consciously the different representations;
- unlike Coulomb Force, electric field has not a defined, proper mathematical form;
- unlike Lorentz Force, magnetic field has not a defined, proper mathematical form.

Maybe the most important difference between field and action-at-a-distance mechanics is that field is an extended object, represented by *space-time functions* and it quantifies *local interactions* occurring with at a finite velocity within the field itself, while Newtonian force quantifies global interactions, that suddenly and instantly occur between two bodies.
The difficulty to elaborate a robust, grounded, significant representation of the electromagnetic field causes a cascade effects in the learning of all of the aspects of electromagnetism that are to be taught in the following stages, included current and circuits.
After electrostatics and circuits, students usually begin to study magnetostatics, introduced with the same approach and the same formal representations, but using different images: unlike electric charges, the magnetic ones are always macroscopic objects. Magnetostatics appear to be something very different from Electrostatics, and students often fail to develop a correct, coherent idea of the electromagnetic field.
With electromagnetic induction, the most important conceptual revolution in teaching electromagnetism at school is encountered; in fact, with the Faraday-Neumann-Lenz rule, for the first time, two changing in time physical extended entities are related together. Usually, textbooks motivate this rule with the Lorentz force, bringing it back to action-at-a-distance framework.
Researchers in physics education recognized two main categories describing students' conceptual profile in dealing with electromagnetism: Coulombian-Newtonian (action-at-a-distance schema) and Maxwellian (field schema). The ontological shift from the Coulombian-Newtonian schema to the Maxwellian one is shown to be very challenging, but fruitful also to enable students to cope with problems that acquired them to reason about the energy-momentum conservation from a relativistic perspective.

On the basis of these remarks, I pointed out the following research problems that I addressed in my PhD work:

1. How do students and teachers cope with the exercises in electromagnetism? What is in general the role ascribed to mathematics in problem solving and in the understanding key concepts, like electromagnetic induction?

2. How was the concept of field historically introduced by Maxwell and what elements can be re-considered from the historical path in order to foster understanding of the concept of field?

In order to answer the first question, I designed and realized, in collaboration with the research group in physics education, two empirical studies with university students and secondary school teachers about problem solving. In the design of the studies, I followed a well-known theoretical framework, elaborated by the research group of the University of Maryland: the framework of the epistemic games.
In order to answer the second question, I reconstructed the historical path that led to the introduction of the concept of field and I analyzed the original memories of Maxwell from an educational point of view. This work was the more demanding, since the texts of Maxwell are not easy and, mainly, refer to a model of aether that sounds, nowadays, rather far and artificial.
Maxwell's aether was necessary to rationalize and quantify the theory of local electromagnetic interactions in order to work out their mathematics: Maxwell apply the mathematics of continuous bodies to aether and invent those differential operators that allowed fields to gain an autonomous identity with respect to forces.
Starting from an historical introduction of these differential operators, I analyse their meaning from an educational point of view. Then, I will derive Maxwell's equations from experimental observations and from a particular aether model.
The interplay between physics and mathematics plays an important role in this work. In fact, differences between the Newtonian framework and the Maxwellian one in terms of their equations emerge. Further, I will start from the aether model to suggest a different way to look at Maxwell's equations from an Educational point of view.
The historical analyses resulted in a teaching guide targeted to teachers and teacher educators, based on Maxwell's paper "On Physical Lines of Force", where, for the very first time, Maxwell's equations appeared. Its aim is to present electromagnetic field under a whole new light, capable to address

well-known problems in understanding the concept of field: what makes the interaction modeled by fields different from the interaction modeled by forces? What is the meaning behind the concept of field used in the solution of the problem of the interaction at a distance? What's the meaning behind the perspective that tells us a field is something real and not a mere mathematical tool? What mathematical tools are needed to describe the field properties?
This guide is also an opportunity to look back into an historical case and to appreciate the structural role of mathematics in physics.

The difficulties on learning/teaching electromagnetism at school, both at the secondary school and at the university level, as reported from the Research in educational physics, are pointed out in chapter 1. The Epistemic Games model is resumed in chapter 2, while the two empirical studies on problem solving and problem posing are described in chapter 3. Chapter 4 is dedicated to the interdisciplinary documents set up in order to enter the meaning of electromagnetic field.

Chapter 1

Review of Research on Teaching/Learning Classic Electromagnetism

Force and *field* are two fundamental concepts in physics. The former concept is introduced by Newton in the second middle of 17th century. The latter appears in the first middle of 19th within the pioneering work done by Michael Faraday, but is implemented in mathematical language only in the 1862, in the renowned Maxwell's memory "On Physical Lines of Force". Each of these concepts are the core of a scientific revolution. At school, the electric field is usually the first "field" encountered by students in physics.
The effectiveness of the traditional way of teaching electromagnetism has been analyzed by many authors and researches. Many of them conclude that the same perspective change experienced by physicists in late 19th century is not encountered by students.

In this chapter I report the main research results on students' difficulties in EM. The presentation of the results is organized, following the classic organization of high school physics courses:

1. electrostatics;
2. electric currents and circuits;
3. magnetostatics;
4. electromagnetic induction;
5. electromagnetic waves.

Within each section, I will discuss the results paying special attention to four aspects (my analytic lenses): models, languages, representations, (way of) reasoning. I chose these lenses since they are fundamental to characterize the change of perspective that the learning of EM would require: "from looking at interaction in terms of forces to looking at interaction in terms of field". In fact, EM as a physical theory is built on its own models, develops its proper language, brings peculiar and useful representations, implies the acquisition of peculiar ways of reasoning. From this analytic point of view, it will emerge the extent to which many difficulties, well-known in the physics education research, ground their origin in a Newtonian view and revail the difficulty to enter EM as a new perspective.

1.1 Electrostatics

1.1.1 Beginning Modeling Electrostatics

With electrostatics point-like charged particles and the concepts of field and potential are introduced. Electrostatics is usually the first chapter into the electromagnetic world, and student arrive at this meeting with a large common experience with electrical things . So, when students face the first lessons about electrostatics and are engaged to cope with rubbing bodies, amber and so on, it is very plausible that they have already many ideas but no coherence model to interpret what they are observing. It is so plausible that it is when teachers propose the first models and representations (point-like charges, field, potential, infinite metal plate,etc.) that students start to build up models or search for new forms of coherence (Ferguson and de Jong, 1987; Danusso and Dupré, 1991; Greca and Moreira, 1997). In a study on how students react to the first encountering with electrostatics, Furiò and colleagues pointed out two main reactions among the students (Furió et al., 2004): the "electrics" – the students who base their reasoning on the concept of charges imagined within bodies – and the "creationists" – the students who base their reasoning on charge creation (by rubbing, by induction, etc...). In (Furió et al., 2004, p. 300), are reported the following examples of "creationist or electrics explanations":

Interviewer: How do you think the rubbed body has been charged?

Student: Well, there were no charges before, but by rubbing, heat is created, and so charges appear in the straw due to the heat.

Interviewer: Did the plastic straw possess charges before being rubbed?

Student: No. The body was uncharged before. The charges are due to the rubbing.

The "electrics explanations" include that idea that plastic bar is charged because particles inside the bar are separated in positive and negative parts. This is because they do not consider the whole system, made by the plastic bar and the cloth used for rubbing. This problem is known in literature with the name of "functional fixation" (Viennot, 2001). Again, from (Furió et al., 2004, p. 302):

Interviewer: How do you think the rubbed small plastic straw has become charged?

Student: When rubbing, the small plastic straw is heated and then the charges appear; I think they are negative because they are electrons.

Interviewer: Did the small plastic straw have charges before being rubbed?

Student: No, it is the rubbing that charges the small plastic straw, by giving it energy.

Interviewer: You mean there are no charges before, and after rubbing the charges appear in the small straw.

Student: Yes, that's right.

Interviewer: Why do you think the metal bar has not become charged?

Student: I don't know; we may not have rubbed it enough. Yes, to make charges appear it is necessary to give a minimum amount of heat. Probably, it must be rubbed harder.

Similar conclusions can be found in other papers (Galili, 1995; Park et al., 2001).

1.1.2 Electric Field, Field Flux and the Gauss' Law

The concept of electric field is particularly problematic and the sources of difficulties are multiple: they can be related to the semantic and syntactic relation with the concept of Coulomb's force, to the relation with the concept of flux and with the superposition principle. In all the cases, the results are that students tent to conceive the field as a mere mathematic tool with no conceptual, ontological and epistemological identity with respect to the concept of force (Allain, 2001).
The relation with the Coulomb's force is problematic since it tends to lead the students to underestimate the practical and conceptual useless of the concept of field within the electrostatics framework taught at school (Nardi and Carvalho, 1990). In other words, many students see no reasons why they need the concept of field, but to calculate the Coulomb force on a charge and/or they are unwilling to use the field concept in their reasoning as something different from a force (Kesonen et al., 2011; Furió and Guisasola, 1998; Saarelainen et al., 2007).
There difficulties are a first signal that teaching fails to guide the students through the transition between Coulombian and Maxwellian physics, also because the emphasis given to the relation $\vec{E} = \vec{F}/q$ that overlaps the

concepts of electric field and the Coulomb force. Furiò and Guisasola wrote: «there is not enough differentiation between magnitudes F and E, as the students have not yet come to master the Maxwellian profile . It is easy to find an explanation for this confusion due to the fact that, even though $E = F/q$ has been defined, E has not gained enough epistemological status (Furió and Guisasola, 1998, p. 518).» Again, from (Furió and Guisasola, 1998, p. 520): «The high level of failure may have been due to a functional reduction in the students' way of reasoning. The concepts of electric force and electric field intensity are epistemologically bound, but students reasoned on the basis of the operative definition that establishes the proportionality between force and intensity ($E = F/q$) and transformed it into an equivalence. For instance, some student believes that the "electric action" of the electric field is transmitted instantaneously to the electric charges, like the Coulomb force:

Student: The force of the interaction does not depend on the time. The electric action happens at the very moment the phenomenon starts, as on putting a charge at a distance from another, the interaction appears instantaneously (Furió and Guisasola, 1998, p. 520)

Viennot and Rainson, referring to the same problem, argue that the traditional teaching reduces functionally E to F. Researchers ask to what extent the the formula $F = qE$ «suggests that no field can exist at a given point if there is no charge placed at this point»(Viennot and Rainson, 1992, p. 485). For instance, in commenting Figure 1.1

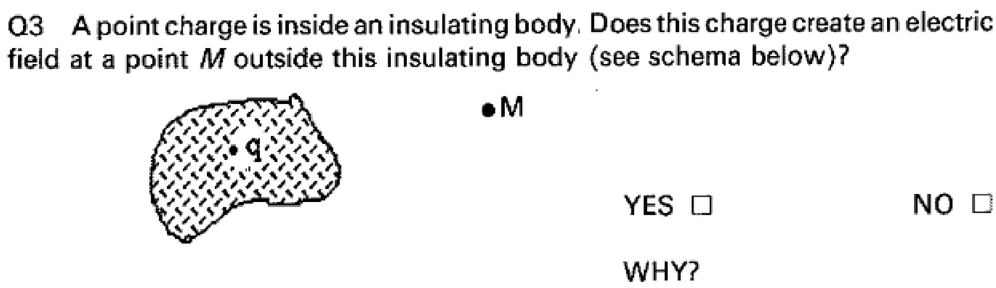

Figure 1.1: (Viennot and Rainson, 1992, p. 480)

students can say:

Student: It all depends on the charge at point M

Student: No, because the point is neutral from an electric point of view

Moreover, students see a «cause in the formula» (Rainson et al., 1994, p. 1027), interpreting the right hand-side of the formula as the cause of the left hand-side.

Different researchers found that the field-force overlap can be generated also by the way force and field are represented; they talk about «confusion by representation» between field and force (Arons, 1987; Törnkvist et al., 1993; Guisasola et al., 2004). In fact (Törnkvist et al., 1993, p. 338), «it is not a new discovery that students have shaky ideas about vectors as mathematical entities and show subsequent confusion between vectors representing different concepts.» For instance, answering to the uncomplicated question reported in Figure 1.2, some student drew curved vectors.

Interviewer: *Q 1-2*: Draw a force vector on the given charge in the given point in the given field.

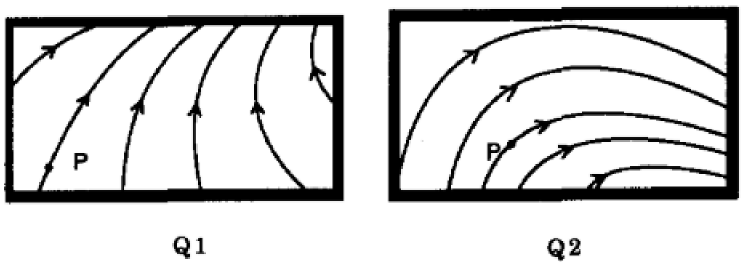

Figure 1.2: (Törnkvist et al., 1993, p. 337)

Rainson and colleagues show that many students do not reach an appropriate understanding of the electric field superposition principle. In particular, static and changing electric fields are perceived as two different concepts. Many students believe that any charge moving within an external electric field do not affect the field itself (Rainson et al., 1994).

As anticipated, further sources of difficulties come from the relation of the concept of field and the concept of flux (Albe et al., 2001; Rainson and Viennot, 1998). Students find it difficult to think that adding a charge outside a Gaussian surface does not modify the flux but it modifies internal electric field (Allen, 2001). Many times, thinking of Gauss law, students believe that not the flux, but the electric field depends only on the internal charges, especially when the configuration is misleading. In fact, Gauss law is often used too evaluate the electric field of a particularly symmetric configuration of charges. This aspect drives students to confuse the physical actor of the formula, usually exchanging the field and its flux (Viennot and Rainson, 1992; Rainson et al., 1994; Chandralekha, 2006).

Also in this case, some difficulties derive from the resources activated by the form $A = B$, that lead them to think that "things on the right are the cause of things on the left of the equal sign" (Camici et al., 2002; Rainson et al., 1994). For example, students can arrive to think that the presence of the electric field is associated only with the internal charges of the closed surface, on the basis of the formal relation $E = \sigma/\epsilon_0$ that is interpreted as "the surface density σ is the cause of the electric field". They hence did not appropriate the idea that : «All the universe's charges contribute to the electric field $\vec{E}$, not only surface charges» (Rainson et al., 1994, p. 1030). When the are asked "What does it happen when an external charge is added in the vicinity of the surface of a conductor?", typical answers are:

Student: The electric field becomes the sum between σ/ϵ_0 and the external one.

Some students believe that if the flux is zero, then the field is zero everywhere. In the following we report an exchange taken from (Guisasola et al., 2008, p. 1011):

Interviewer: Why do you say that the field on the Gaussian surface is zero?

Student: If the flux is zero, that means that there is no charge, does not it? Well, in Gauss's law, flux is proportional to charge, and if the charge is zero, this indicates that there is no field. In other words, if we use Gauss's law in this case, then if the flow is zero, the charge is zero and there is no field

Other researches pointed out that students often confuse the net electric field given by a charge distribution and the electric field generated by the single point charge, sometimes thinking that the whole electric field in a certain point depends only on the nearest charge. From (Chandralekha, 2006, p. 931):

Interviewer: [...] a point charge $+Q_1$ is at the center of an imaginary spherical surface and another point charge $+Q - 2$ is outside it. Point P is on the surface of the sphere. Let Φ_S be the net electric flux through the sphere and $\vec{E}_P$ be the electric field at point P on the sphere 1.3. Which one of the following statements is true?

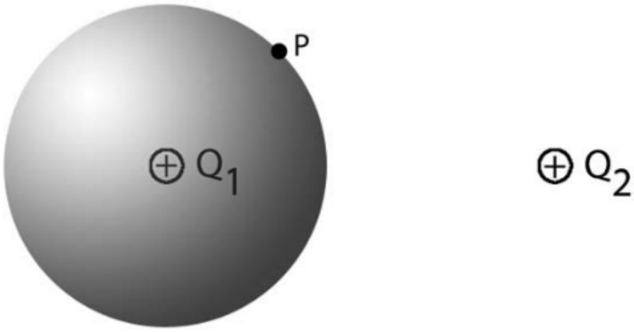

Figure 1.3: (Chandralekha, 2006, p. 931)

(A) Both charges $+Q_1$ and $+Q_2$ make nonzero contributions to Φ_S but only the charge $+Q_1$ makes a nonzero contribution to $\vec{E}_P$

(B) Both charges $+Q_1$ and $+Q_2$ make nonzero contributions to Φ_S but only the charge $+Q_2$ makes a nonzero contribution to $\vec{E}_P$

(C) Only the charge $+Q_1$ makes a nonzero contribution to Φ_S but both charges $+Q_1$ and $+Q_2$ make nonzero contributions to $\vec{E}_P$

(D) Charge $+Q_1$ makes no contribution to Φ_S or $\vec{E}_P$

(E) Charge $+Q_2$ makes no contribution to Phi_S or $\vec{E}_P$

Many students believe that total flux increases proportionally to the growth of the Gaussian surface; from (Allen, 2001, p. 45): «Since the students learn two mathematical representations for flux, $\Phi = \int_S \vec{E} \cdot d\vec{S}$ and $\Phi = Q_{\text{int}}/\epsilon_0$, it is possible that students tend to apply the first in situations where area varies in an effort to explicitly account for the variation in area and the second in cases where the area stays the same since then area does not seem to matter and does not appear in the equation.»
Some students, in addition, apply Gauss law to open surface (Chandralekha, 2006).

1.1.3 Potential and Voltage

«The concept of potential does not seem to have developed very far beyond the point at which Kirchhoff left it in the middle of the nineteenth century. There is now a cluster of at least four concepts which are closely related to each other and yet are thought of as somehow distinct, namely potential, emf, circuit voltage, and electrical potential energy. Engineers and teachers are quite confident that voltage is a genuine (and dangerous) physical property, while some theoretical physicists still suppose with Poisson and Green that potential is a mathematical artifact only. All of this suggests that a considerable effort is now required to distinguish, clarify, and formulate a coherent theory of potential» (Roche, 1989, p. 6).

from "Preface to a Treatise on Electricity and Magnetism" by J.C. Maxwell, 1881

Potential is typically introduced as a field corollary. For this reason, potential, like field, is perceived by students as an abstract and uselessness concept (Benseghir and Closset, 1993; Viennot, 2001; Guisasola and Montero, 2010). It is introduced in electrostatics, but becomes a primary actor during electric circuit lessons. Despite that, it remains a subordinate concept. In fact, students prefer to reason with charges and currents despite potential (Benseghir and Closset, 1993).

1.1.4 Electric Field Lines

The field lines were introduced by Faraday to represent electromagnetic field. Originally they were not a mathematical tool, but something real with matter properties, imaging them as. Since he had some difficulties to define them, he used the "graphic" representation before any property definition. In (Pocovi, 2007, p. 117):

«Whilst writing this paper I perceived that, in the late Series of these Researches [...] I have sometimes used the term lines of force so vaguely, as to leave the reader doubtful whether I intended it as a merely representative idea of the forces, or a description of the path along which the power was continuously exerted. [...] Wherever the expression line of force is taken to simply represent the disposition of the force, it shall have the fullness of that meaning; but that wherever it may seem to represent the idea of the physical mode of transmission of the force it expresses in that respect the opinion to which I incline at present. (Faraday, 1951, from "Experimental

Researches in Electricity")»

The quotation seems to show a certain discomfort about field lines, although it is a concept that he created. In the same memory he wrote that «physical lines of electric force» exist and they transport electric forces all through the space. Later, in a letter that Faraday wrote in 1855 to J. Tyndall, he seemed to change his mind: «*You are aware (and I hope others will remember) that I give the lines of force only as representations of the magnetic power, and do not profess to say to what physical idea they may hereafter point, or into what they will resolve themselves.* (From "Michael Faraday. A Biography" - P Williams – Chapon and Hall, London, 1965).
Also others physicists, like J.H. Poynting and W. Thompson, thought lines of force as real entities, with physical measurable quantities as longitudinal tension and lateral repulsion. Lorentz (1909) provided another explanation, believing lines of force are the representation of the latent forces (Roche, 1987). Even Maxwell, at the end of his life, said that: «[...] these lines must not be regarded as mere mathematical abstractions. They are the directions in which the medium is exerting a tension like that of a rope, or rather, like that of our own muscles» (Galili, 1995, p. 383).

In current teaching, the lines of force are introduced without deep reflections on their meaning and role and many students seem to attach them a "matter meaning", like in the very original meaning that physicists ascribed to them (Galili, 1995; Pocovi and Finley, 2002). In a very thorough research, Pocovi and Finley pointed out several nuances that can mirror students' attitude to attach "matter properties" to the lines of force (Pocovi, 2007): students can think that lines of force are gravity-sensitive, real paths followed by a charge, energy/charge-transporters, field-containers (tubes). The following examples are taken from (Pocovi, 2007)

1. Field lines have mass

   Interviewer: A point charge is located on the moon.

   Student A: [If a point charge is located on the moon, then] there would exist more lines of force coming out of the charge because there is no gravity. *[matter based concept: lines as gravity sensitive]*

   Student B: [If a point charge is located on the moon, then] the lines of force would have to be longer because there is no gravity. *[matter-based concept: lines as gravity sensitive]*

2. Field lines are possible charge path[1]

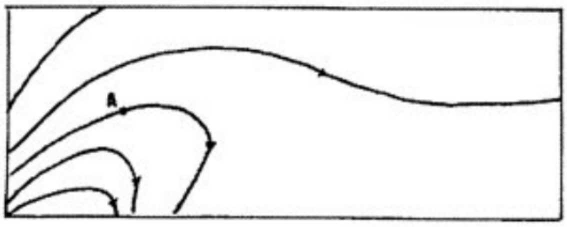

Figure 1.4: (Pocovi, 2007, p. 122)

Student A: [Figure 1.4] Lines of force are the real paths that a test charge would follow in a region where there exists an electric field. *[matter-based concept: lines as preestablished (prearranged, A/N.) path]*

3. Field lines are the force/field carrier

Student A: Lines of force affect the space where they are drawn transporting electricity where they are drawn. *[matter-based concept: lines transporting]*

Student B: Lines of force transport forces that push the charges. *[matter-based concept: lines transporting]*

Student C: Lines of force transmit charges. *[matter-based concept: lines transmitting]*

4. Field lines are the "interaction agent" of a charge

Student A: Lines of force are like very thin tubes located around a charge and cause the electric interaction. *[matter-based concept: lines as tubes]*

5. Field lines are the force container (Figure 1.5)

Student: There is no force acting on the charge at B because there is no line passing through it and the lines contain the field. *[matter-based concept: line as containing the field]*

[1] The same observations can be found in (Törnkvist et al., 1993) and (Galili, 1995)

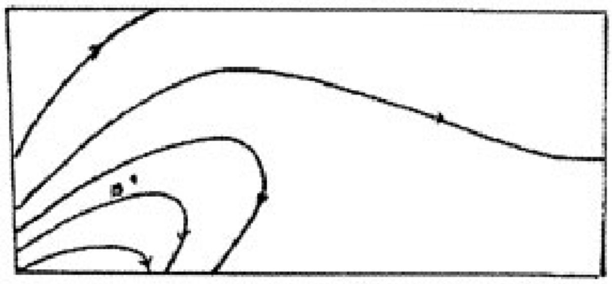

Figure 1.5: (Pocovi, 2007, p. 122)

6. Field lines self-interact (Figure 1.6)

Interviewer:	Figure 1.6 shows the lines of force that have been drawn for an infinite thin plane with a positive met charge $+Q$. If the charge of the plane is doubled, can you draw the lines for this new situation? Can you tell me why you draw ...?
Student A:	The lines' length will be doubled because the electric action of the plane has to be transported further. *[matter-based concept: lines as transporting]*
Student B:	The number of lines will remain the same but they will tend to curve themselves because they repel each other. *[matter-based concept: lines as repelling each other]*

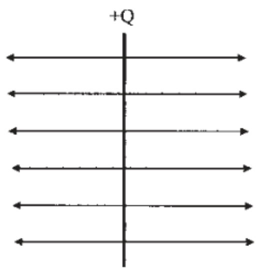

Figure 1.6: (Pocovi, 2007, p. 123)

7. Field lines are true force vectors which act on particles (Törnkvist et al., 1993; Maloney et al., 2001; Saglam and Millar, 2005; Thong and Gunstone, 2008; Smaill and Rowe, 2012).

Interviewer: A positive charge q is held at rest in a uniform magnetic field, and then released (Figure 1.7). You can ignore the effect of gravity on the charge

Figure 1.7: (Smaill and Rowe, 2012, p. 6)

How does the charge move after it is released?

- The charge moves to the right with constant velocity
- The charge moves to the right with constant acceleration
- The charge moves in a circle with constant speed
- The charge moves in a circle with increasing speed
- The charge stays at rest

Moreover, matter-based students think field lines as container of field/energy; they give to field lines an active role ("taking the charge from one place to another" (Pocovi, 2007, p. 125)). Other researches found that some students believe that field lines transport force vector in a rigid way, maintaining its length from source to target or that they create contact among interacting objects (Saarelainen et al., 2007).
Generally, many students represent the interaction between two objects with two models:

1. the "sending something" model – one body sends something like particles, light, force, etc., along a "path" to the other object;

2. the "fluid" model – something flows from one object to the other. These problems maybe reflects the difficulties to imagine the action at a distance (Loftus, 1996).

A further source of difficulty concerns the relation between the force vector's intensity and field lines' density. In general, the representation of fields' intensity is worse understood than that of fields' direction (Törnkvist et al., 1993). From (Chandralekha, 2006, p. 935):

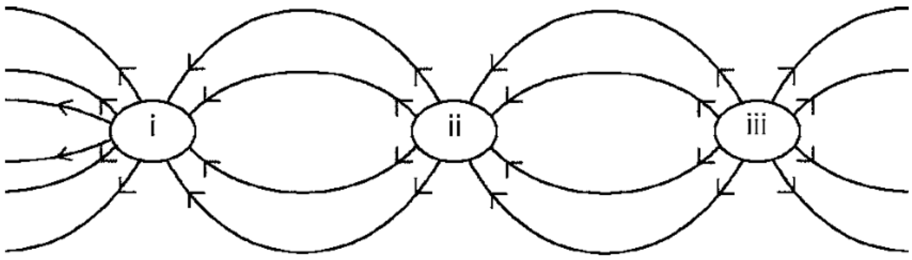

Figure 1.8: (Chandralekha, 2006, p. 935)

Interviewer: The diagram in Figure 1.8 shows the electric field lines in a region. Sadly, you do not know the field inside the three regions (i), (ii), and (iii). This cross-sectional drawing is qualitatively correct. Which region (or regions) carries a net charge of the greatest magnitude?

- (i) only
- (ii) only
- (iii) only
- (ii) and (iii) which have equal net charge
- (i), (ii), and (iii) which have equal net charge

«The most common distractor in the problem above was (1), which was chosen by 35% of the students (Chandralekha, 2006, p. 935).»
Many students draw force vector not on target charge but on the source. This common mistake can increase the difficulty in separating field lines (which begin from the source charge) from force vector (which start from the target charge) (Saarelainen et al., 2007).

In conclusion, we can comment that field lines are not usually understood as representation of the field function, that is, as the representations of the function properties (Törnkvist et al., 1993; Nguyen and Meltzer, 2003). This fact emerges again when two field lines meet each other in a space point (Figure 1.9): in that point no function exists, nevertheless students fail to find any inconsistency. A similar problem emerges when a field force line makes a loop or a kink: many students do not recognize that those configurations are impossible.

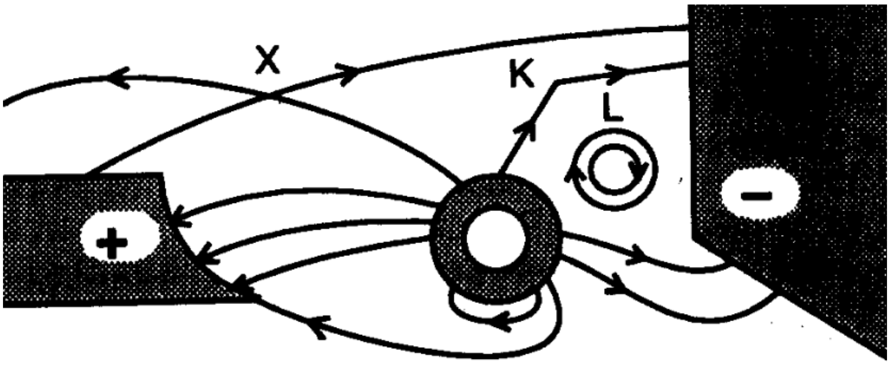

Figure 1.9: (Törnkvist et al., 1993, p. 336)

In (Törnkvist et al., 1993) the researchers report that students appear more confident in excluding loops and kinks than cross from possible field lines configuration. Other references found similar difficulties: (Martínez and Ley, 2014; Ferguson and de Jong, 1987; Greca and Moreira, 1997).

These studies show that apparently harmless representations in electromagnetism induce complex mental models. These results stimulate to design new methods to guide students to develop awareness in modeling representation forms, both through problems focused on representations and through educational research material.

1.1.5 Insulators, Conductors and the Electrostatic Equilibrium

Physics Education Research has pointed out many difficulties in learning properties and behaviors of insulating and conducting bodies. Students fail to acquire a coherent approach to the physics of macroscopic object, mainly because of their naive representations of charges and currents. Also in this case, the Newtonian approach to electromagnetism taught at school seems to emerge to be an obstacle.

One well-known difficulty for students is represented by thinking that charges can move in conducting bodies, so that a charged body can attract or repel any conductors, where attraction or repulsion depend on the quality of its charge; on the other hand, thinking that charges cannot move in insulating bodies, polarization effects are usually neglected (Park et al., 2001). For instance, many students think that insulators can block electric field, because charges do not move inside them. In Figure 1.10 we can see an example taken from the article of Furiò and colleagues (Furió et al., 2004, p. 305)

Interviewer: A sheet of charged plastic is placed near the end of a long wooden stick without touching it, as can be seen in the diagram. At the end of the stick there is a small ball of polyurethane. Explain whether it will be attracted or not to the ball

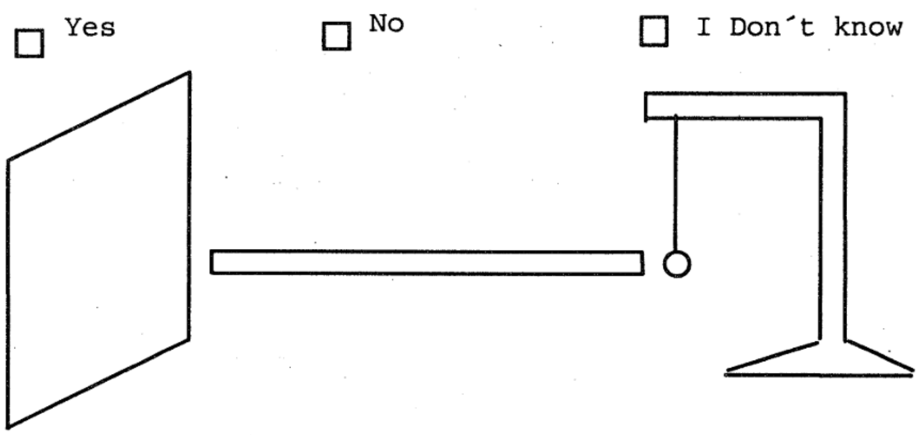

Figure 1.10: (Furió et al., 2004, p. 313)

Interviewer: What do you think will happen to the polyurethane ball?

Student A: We.., I think nothing, because it is very far from the charged plastic sheet. Besides, what there is in the middle is wood, which is an insulator.

Student B: Nothing will happen to it because the wooden stick is an insulator and does not conduct electricity.

The reasoning behind these answers is known as

field if mobility (Viennot and Rainson, 1992)

It is a version of another misunderstanding, that the presence of a force is sufficient for a charge to move. From (Viennot and Rainson, 1992, p. 483)

Student: The insulating property of the body prevents the field from penetrating it.

Guisasola and colleagues found that students generally do not consider the interaction between the charged bodies and its environment (Guisasola et al., 2002). Similar conclusions can be found in (Viennot and Rainson, 1992, 1999; Rainson et al., 1994; Chandralekha, 2006; Allen, 2001; Furió et al., 2004; Park et al., 2001).
Guruswamy et al. found that students make a real effort to imagine any transfer of charges among conductors charged by the same sign (Guruswamy et al., 1997). As an example, in Figure 1.13

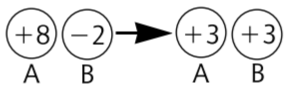

Figure 1.11: A

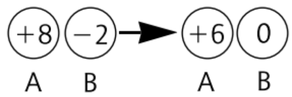

Figure 1.12: B

Figure 1.13: (Guruswamy et al., 1997, p. 94)

Many students (25%) choose the **B** hypothesis. They seem to have some difficulties to imagine and represent the concept of equilibrium. A large number of them think at the concept of electrostatic equilibrium paying their attention on the quantity of charge: two bodies are in equilibrium when they reach the same charge (and not when the ΔV between them is zero). The same problem can be found in fluid dynamics. For the principle of communicating vessels, fluid must reaches the same level (and not the same quantity, of course!) in each vessel[2]. From (Guisasola et al., 2002, p. 254):

Student: It will become charged until the charge in both bodies is the same. There will be a transfer of electrons from the most negatively charged to the other, until they become even *(second year of Engineering)*.

Learners usually do not consider forces among charges within the conductor (Guruswamy et al., 1997). Many work, moreover, conclude that students feel more comfortable when they talk in terms of charge than in terms of electric potential. We will discuss about this difficulty in section 1.2.

[2]Some authors believe that this metaphor can be a good start to separate charge and energy concepts

1.1.6 Newton Third Law in the Electrostatics Domain

We would like to point out, in passing, something interesting to you to think about. [...] Imagine two electrons with velocities at right angles, so that one will cross over the path of the other, but in front of it, so they don't collide. [...] We look at the force on q_1 due to q_2 and vice versa. On q_2 there is only the electric force from q_1, since q_1 makes no magnetic field along its line of motion. On q_1, however, there is again the electric force but, in addition, a magnetic force, since it is moving in a $\vec{B}$-field made by q_2. [...] The electric forces on q_1 and q_2 are equal and opposite. However, there is a sidewise (magnetic) force on q_1 and no sidewise force on q_2. Does action not equal reaction?

from "The Feynman Lectures on Physics" – Vol.2, Sec. 26-2 by R.P. Feynman

We will mention two further examples of momentum in the electromagnetic field. We pointed out in section 26-2 the failure of the law of action and reaction when two charged particles were moving in orthogonal trajectories. The forces on the two particles don't balance out, so the action and reaction are not equal: therefore the net momentum of the matter must be changing. It is not conserved. But the momentum in the field is also changing in such a situation. If you work out the amount of momentum given by the Pointing vector, it is not constant. However, the change of the particle momenta is just made up by the field momentum, so the total momentum of particles plus field is conserved.

from "The Feynman Lectures on Physics" – Vol.2, Sec. 27-6 by R.P. Feynman

Some researches notice that teaching fails to guide the students to apply correctly Newton's third law in the electromagnetism contest (Galili, 1995; Smaill and Rowe, 2012).
In fact, the application of Newton's third law in the electromagnetic framework is not trivial at all but, in case it is addressed, it stimulates very interesting reasonings and solutions. As pointed out by Feynman, the existence of a field is necessary for confirm Newton's third law for electromagnetic forces.

Again, Newtonian approach fails to explain electromagnetic phenomena, even including Newton own laws. The field approach would help students to understand and to visualize that part of the energy of the system can be transferred to the fields. We will see in Chapter 4 how a particular way of seeing interactions "in terms of field" can foster students' to imagine coherently electromagnetic interactions.

1.1.7 Modeling static electromagnetism

Furiò and colleagues organize students' ways of modeling electrostatic interactions in "four categories" (Furió et al., 2004, p. 307):

1. Creationist (few students): electricity appears in bodies when they are rubbed. Charges appear when dielectrics (plastic) are rubbed but not when metals are rubbed. Electrical induction phenomena are misunderstood.

2. Halo effect (few students): Charges bodies attract any other body that is nearby. Electricity is considered to be charges that create electric atmosphere.

3. Electric fluid (most students): Electricity is considered as a fluid that passes from one body to another; it passes into dialectrics through rubbing and into conductors through contact.

4. Newtonian (few secondary students, a minority of university students): Electricity is considered as a group of charges that acts at a distance. The electrical induction phenomena are explained as resulting from forces exerted by the charge of the charged body on the positive and negative, separated charges of the neutral body.

In their study, Furiò and colleagues see an analogy between these four categories and the historical development of modeling electricity, from electric effluvia to Coulombian theory. Following this research, students who do not reach the Newtonian category are not able to understand fully phenomena like polarization or induction.

However, as already seen in the previous sections, many times Newtonian approach do not lead to a successfully comprehension of electrostatics.To support this thesis, I report some interviews from (Furió and Guisasola, 1998), where two different conceptual profiles were compared: the *Coulombian* profile and the *Maxwellian* one. I summarized the main characteristics of these two profile from (Furió and Guisasola, 1998, p. 516):

Coulombian conceptual profile

- Charge is an intrinsic property of the matter, «it is situated in the matter itself.»

- A charge exerts "action-at-a-distance" on other charges through electric force, this force being analogues to the gravitational one.

- Action at a distance is exerted instantaneously; medium does not play any fundamental role.

Maxwellian conceptual profile

- «The electric interaction is no longer linked to its location in the material substratum, but extends to all the surrounding space.»

- «"Irradiation" of the electric interaction to the space requires the introduction of a new concept: the electric fiel [...] The importance of the idea of located charge diminishes, whereas that of the field extended to all the space gains in importance.»

- «It is impossible to interpret the electric relationships between charged bodies without consider- ing the medium in which the actions transmit [...]» Space geometry affects field's expansion, which has a finite velocity.

I report in Figure 1.14 an example from (Furió and Guisasola, 1998).

- Example of answer classified as being in the Coulombian category:

  Interviewer: Why is the sheet outside repelled, whereas the one inside remains vertical?

  Student: The paper outside has the charge on one side, whereas the one inside has it around. Then, all the forces exerted on the one inside nullify one another, and it remains vertical.

  Interviewer: But the paper is not place in the middle of the cylinder, don't you have to consider the distance when calculating the forces?

  Student: Well, yes, but in this case you see that it remains still, thus they nullify

  Interviewer: But is not that contradictory?

  Student: I don't know, the fact is that this is the way it is.

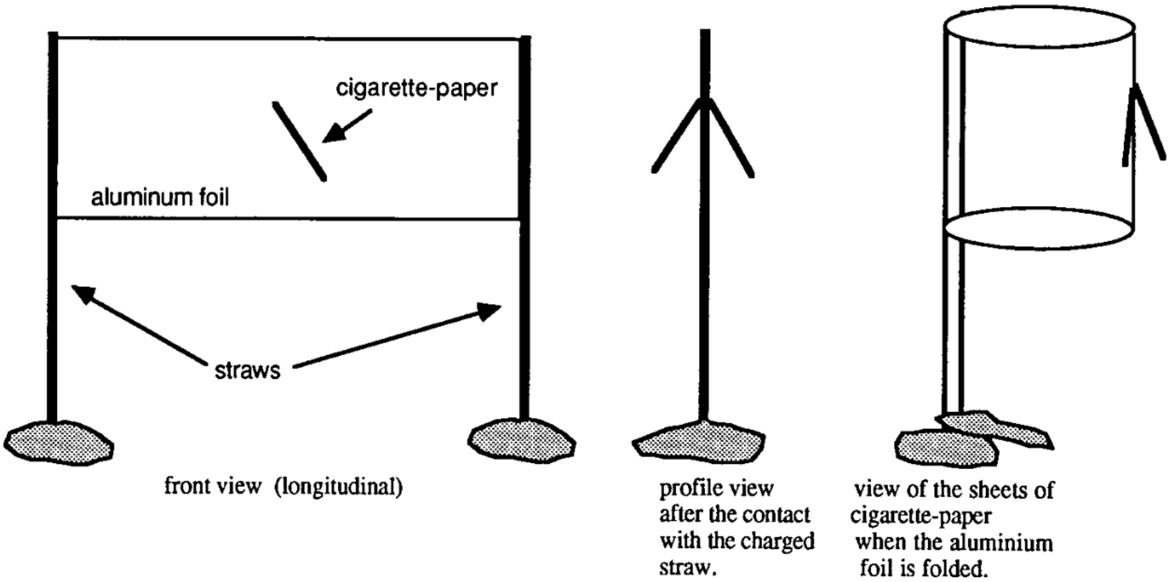

Figure 1.14: (Furió and Guisasola, 1998, p. 522)

- An example of answer classified as being in the Maxwellian category is:

Interviewer: Why is the sheet outside repelled, whereas the one inside remains vertical?

Student: Because inside the conductor there must not be field, well that... it is because inside... this is like a conductor, the sheet of metal, then on closing it you do as if it were a closed surface, then by Gauss; he says that inside a closed metallic surface the field is zero, then outside there is field but inside there is not, that is why the sheet of paper inside does not suffer any force and the one outside does.

In this example I can observe that the simple concept of force that "acts at a distance" does not give a complete vision of the system, 'cause students look only at charged objects. In this specific case, the distance appears to be the same between both piece of papers, but the effect is not the same. Students do not consider/see all the charges on the cylinder.

The ontological shift from the Coulombian schema to the Maxwellian one could be very engaging. At the same time, research has shown this shift is necessary to the complete comprehension of the whole physics of electrostatics.

1.2 Electric Circuits and Current

Students feel electricity (electric circuits and current) as an hard topic, also because many electromagnetic terms, especially in circuits analyses, belong to Western common language. The result is that students start their learning with an *undifferentiated electricity notion"* (Cohen et al., 1983; Osborne, 1983; Shipstone, 1985, 1988; Closset, 1989; McDermott and Shaffer, 1992a; Duit and Rhöneck, 1998; Borges and Gilbert, 1999; Engelhard and Beichner, 2004; Afra et al., 2009).
Researches of the group of Seattle and of Millar confirm that students do not show a coherent theoretical framework useful to solve a generic dc circuit (McDermott and Shaffer, 1992b; Millar and King Tom, 1993). Faced with an unfamiliar situation, they apply formulas, partial conceptual models and pieces of reasoning, apparently without any consistency. Usually their answers are based on intuition and personal experience, especially when maths cannot help them.

Typical expressions like "current consumption" suggest internal representations far from the scientific model (Danusso and Dupré, 1991). Current is the primary concept on which students base their analyses on electric circuits (Closset, 1983).

Research[3] has isolated four common conceptions about electric current (Figure 1.15):

1. unipolar (unidirectional flux without return) (Maichle, 1982; Shipstone, 1985). It emerges when the
    - circuit is open and the
    - current flows is imagined from the battery to the resistance.
2. bipolar ("clashing currents") (Osborne, 1983). It emerges when the
    - circuit is closed and the

[3] I cite papers on which the way of thinking appeared for the first time, as far as I know

- two types of current flows (positive and negative) are imagined to go from the battery to the resistance.

3. sequential model (Closset, 1983). It emerges when the
   - circuit is closed and the
   - battery is imagined to generate current, which is consumed through the wire becoming weaker at every resistance element ("attenuation model").
4. sharing model (Shipstone, 1985). It emerges when the
   - circuit is closed and the
   - current is imagined to be the same everywhere in the circuit *if* circuit's elements are identical; current is not intended to be conserved.
5. scientific model (Shipstone, 1985). It emerges when the
   - circuit is closed and the
   - battery generates efm; current is the same everywhere in the circuit, despite the circuit's elements.

Research has found that students easily substitutes unipolar conception with the bipolar one (Danusso and Dupré, 1991). At the high school students generally use the third conception of electric current, based on a local linear way of reasoning. Students focus on the «current destiny» (Danusso and Dupré, 1991).

Despite physics courses, scientific model is not always internalized: usually, facing with unknown or strange circuits, students return to the sequential model. In Figure 1.15 differents representations for currents are represented.

The great task to achieve in order to acquire a scientific conception is the necessity of the development of a systemic way of reasoning. In fact, typical circuits' physical variables are spatial global functions of time and Ohm's laws are systemic equations. To reach this goal, teachers should take time to clearly distinguish between current and voltage.

Usually, in the secondary school program, a deep link between electrostatics and electric circuits does not exist. Students have little and conceptually poor connections (Benseghir and Closset, 1996) between

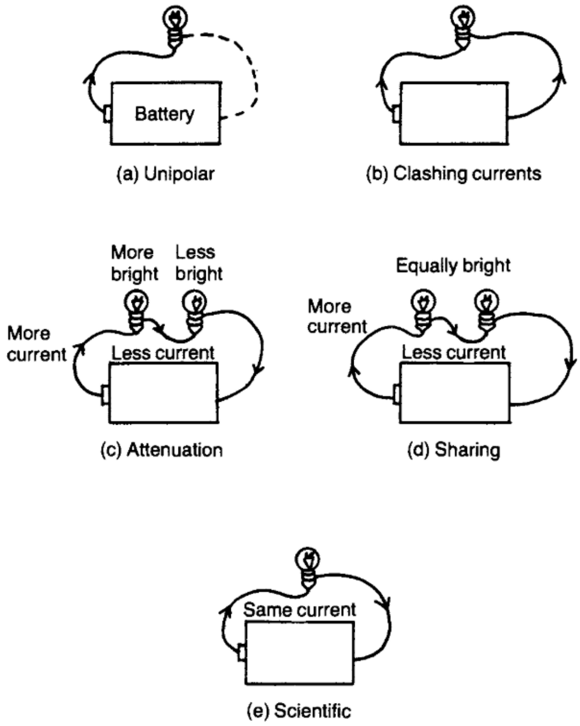

Figure 1.15: (Shipstone, 1985, p. 36)

potential and voltage, charges and current, conductors and circuits, field and circuit's energy[4]. Chabay and Sherwood deeply examined the correlation between microscopic and macroscopic models (Sherwood and Chabay, 1999; Guisasola, 2014). Eylon and Ganiel use the following words to describe the main conceptual problems that keep students away from the scientific conceptions about electric circuits (Eylon and Ganiel, 1990, p. 79):

At [secondary school], the mathematical tools for treating electric circuits are also available. Indeed, various studies (Osborne, 1981; Shipstone, 1988, 1985; Dupin and Johsua, 1989) have shown that students' general understanding does improve with age and instruction, and their mental models concerning current flow become more advanced: primitive models are abandoned in favor of more scientific ones. However, several studies (Haertel, 1982; Closset, 1983; Cohen et al., 1983) show that even after extensive instruction students do not grasp some of the very basic characteristics of an electric circuit. For example, students tend to be

[4]see (Varney and Fisher, 1980) for the historical motivations for this confusion

"current minded" rather than "voltage minded" (Cohen et al., 1983), thereby confusing cause and effect. Furthermore, the general idea that an electric circuit is an interactive system is not properly understood.

What will follow is a brief list of students' difficulties met studying DC electric circuits. Some repetition will be necessary because of the strong correlation among the elements of the list and because, at the basis of most of the difficulties there is the problem that *All the scientific concepts collapse under the global-undifferentiated notion of current/energy* (Psillos, 1998b). Usually, terms as "electrons", "charges", quantity of charge" and "process of energy transfer" are indifferently used to describe electric current (Mulhall et al., 2001; Licht, 1991).

1.2.1 Current Minded *vs* Voltage Minded Students

As anticipated in the quotation of Eylon and Ganiel, students prefer to reason about current – current minded students – instead of voltage – voltage minded students (Cohen et al., 1983; Psillos et al., 1988; Viennot and Rainson, 1992; Guisasola et al., 2002). As sentenced in (Psillos, 1998b, p. 1) «All the scientific concepts collapse under the global-undifferentiated notion of current/energy.» Usually, terms as "electrons", "charges", quantity of charge" and "process of energy transfer" are indifferently used to describe electric current (Mulhall et al., 2001; Licht, 1991). The exchange of current with voltage (or energy) is individuated in plenty of researches (Haertel, 1982; Shipstone, 1985; McDermott and Shaffer, 1992a; Stocklmayer and Treagust, 1996; Psillos, 1998a; Borges and Gilbert, 1999; Liégeois and Mullet, 2002; Engelhard and Beichner, 2004). This confusion can emerge in different forms. Students, for example, can believe that:

1. Current is generated by the battery; among others, (Licht, 1991; Stocklmayer and Treagust, 1996; Duit and Rhöneck, 1998; Sherwood and Chabay, 1999; Borges and Gilbert, 1999). From (Psillos, 1998b, p. 2):

   Interviewer: After all you have seen in this lesson up to now what do you think that volt indicates?

   Student: It is the quantity that a battery has.

   Interviewer: What quantity?

   Student: Current.

Interviewer: Do the others agree?

Student: Yes!

2. Current can assume different values inside the circuit; among others, (McDermott and Shaffer, 1992a; Smith and Van Kampen, 2011).

3. 3. current is not necessarily conserved (Shipstone et al., 1988; Eylon and Ganiel, 1990; Licht, 1991). Generally, pupils do not know the concept of "current conservation" (Duit and Rhöneck, 1998). Maybe this fact is due to the lack of differentiation between energy and current (Arons, 1987), maybe to a wrong model of current (Eylon and Ganiel, 1990).

4. Current is consumed through the wire; among others, (Haertel, 1982; Periago and Bohigas, 2005). Current is "used up" (McDermott and Shaffer, 1992a). It is noteworthy that only few students think at resistors as «voltage dividers» (Millar and King Tom, 1993, p. 339).

5. Current produces voltage or voltage is a property of current; among others, (Shipstone, 1988; Psillos, 1998b; Silva and Soares, 2007)

Students show typically a current minded attitude. For instance, when the students are asked to rank by brightness five identical bulbs in a ideal circuit (with ideal batteries - see Figure 1.16) and to explain their reasoning, a typical answer is

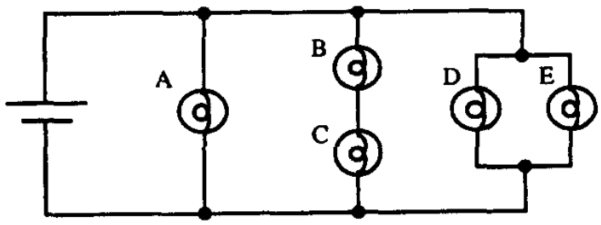

Figure 1.16: (McDermott and Shaffer, 1992a, p. 996)

Student: $A = B = C > D = E$. The current...is equally divided among the [three] paths. B and C are equal to A because the current travels through each bulb one at a time. Bulb D and Bulb E are less because the current splits between them (McDermott and Shaffer, 1992a, p. 996).»

1.2.2 Current in Circuits

In circuits, current is often described as "ordered charged particles flow through the conductor", as something that "flows out of" the negative plate of the battery, "passing through" resistances or "accumulating" on a capacitor's wall (Shipstone, 1985, 1988; Psillos, 1998b; Duit and Rhöneck, 1998). By these words, it seems that every single electron moves through the circuit after being produced by the battery (Mulhall et al., 2001). Many textbooks enforce this idea of current.

Generally, there is no uniform consensus in educational research on which type of representations is better to use for the description of electric current. Through interviews with secondary school and university students, Borges and Gilbert inquired their current models (Borges and Gilbert, 1999). They resumed these models in four categories: «electricity as flow, electricity as opposing currents, electricity as moving charges and electricity as a field phenomenon[5].»

- As a flow: poor differentiation among energy, charges, voltage, current. The current flows from the battery through the circuit. Students adopt a causal way of reasoning.

- As opposing currents: current is not clearly differentiated from energy; for that reason, current conservation is not considered. Students, sometimes, talk about protons and electrons. Again, battery is a current source and students adopt a causal way of reasoning.

- As moving charges: battery is the source of chemical energy, which is transferred to the electrical charges, which move through the circuit. Current is assumed to be conserved. Students adopt a causal way of reasoning.

- As field phenomenon: energy and current appear as different phenomenon; battery is the source of energy, while different current models are used. Current is conserved. Charges are moving through the circuit following potential differences, but the electric fields appear to be the very first actor of their movement. Circuit is perceived as a whole, and each perturbation can generate a new steady state.

[5] Note that here, as in the paper, electricity and current are synonymous. I maintain the original ambiguity to better adapt the text to students' vocabulary.

1.2.3 Voltage in Circuits

Voltage is directly linked to the electric potential. But if the latter is a mathematical space-time function, defined on a conservative electric field, the former is a global variable of time, usually referred to some particular points of a macroscopic and material circuit, related to a non-conservative electric field. emf is qualitatively different, being the (non-conservative) work made by the battery to produce voltage. Usually, teaching circuits, terms as "potential difference", "voltage", "emf" are used like synonyms, without any explications of their differences and their similarities.

Generally, their significance remains uncertain for students (Licht, 1991; McDermott and Shaffer, 1992a; Sherwood and Chabay, 1999; Mulhall et al., 2001). For instance, (Benseghir and Closset, 1996; Cohen et al., 1983) observe that only few university students have correctly learned the distinctions between potential difference and emf in a circuit. Students usually say that voltage is the strength of the current/battery (Shipstone, 1988).

1.2.4 Resistance in Circuits

The concept of resistance in contemporary textbooks is a synonymous of obstacle: the bigger is the obstacle, the greater is the resistance. This vision is enforced by the symbolic representation of the resistance while wire seems to have no resistance at all. Moreover, resistance, in the definition of many textbooks, appears as a universal property of the conductor, independent of the external environment or on other physical quantities. Resistance is also often confused with resistivity and their definitions are usually overlapping. In defining resistance, the geometric properties of the wire are fundamental, in order to define what resistance is and not what it seems to be like. In fact, augmenting volume means in the same time to augment atom nucleus, which contributes to resistance, and electrons, which contributes to current (Viard and Khantine-Langlois, 2001).

Students show many difficulties to separate the total resistance concept from the single resistances of the circuit (McDermott and Shaffer, 1992a). In fact, in order to evaluate the luminosity of a bulb, students calculate the total resistance and they use it to find the power emitted by the bulb itself; they do not realize that bulb luminosity is directly linked with its own resistance. Moreover, bulbs are not linear resistor, i.e. their resistance depends on current. For this reason a specific qualitative approach must be

developed (as done in (Smith and Van Kampen, 2011)).
Students often focus on the number of the elements in the circuits; thus, the more is the number of single resistances, the more is the total electrical resistance of the circuit (McDermott and Shaffer, 1992a; Viard and Khantine-Langlois, 2001).

1.2.5 Capacitance in Circuits

Although capacitors are often used in many practical application, textbooks spend less time to introduce them than they need. Usually, many simplifications are made by teachers and textbooks in order to save time: therefore capacitors seem to be plate, completely inductive (the same charge Q with opposite signs), parallels, very near, etc...Eventually, students believe capacitors to be something very different from what they really are. (Besson, 1995).

From a mathematical point of view, capacitance's formula[6] is a great simplification of a complex non linear problem: when two conductors are put near together, their capacitance is mutual dependent, differently from the isolated case. Capacitors' capacitance depends only on its geometry (Besson, 1995).
Some students believe that charged particles jump from one to the another plate of a capacitor; others think that voltage "flows" through them (Thacker et al., 1999). Students find hard to think of the space between plates as a store of energy (Guisasola et al., 2002). Nonetheless, extreme simplifications of the electric field inside the capacitor can lead to the violation of the energy conservation principle. For instance, looking at 1.17, if a charged particle starts between the two plates with zero initial velocity, it must has non zero velocity outside the capacitor, with no variation of the potential energy.

«A uniform electric field in a finite spatial region and anywhere else null is not conservative».
Another problem with infinite plates is the potential difference outside the capacitor: it results constant, not zero, as evaluated in the approximation for far distances (Besson, 1995).

Capacitance is usually intended by students as the amount of charge that a conductor can store. They do not think it as a property of a conductor's

[6] We define capacitance as the inverse ratio between potential difference and charge needed to keep the potential difference to zero.

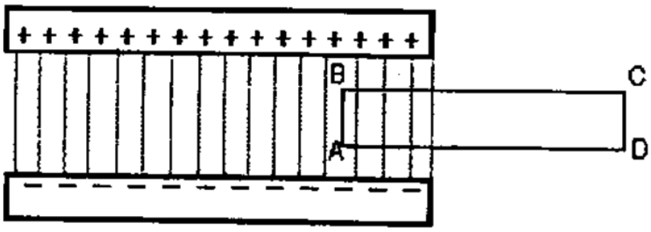

Figure 1.17: (Besson, 1995, p. 178)

system. Moreover, potential seems to be a secondary element of this system, and they focus attentions on charges. From the formula $C = Q/V$ they infer that more is the charge, more is the capacitance. For instance, in their study Guisasola and colleagues captured the following exchange between an interviewer and a university student:

«It is well known that a spherical cortex of radius R has a smaller electric capacitance than the system formed by the same cortex surrounded by another hollow sphere of radius $R' > R$ (Figure 1.18). Can you explain why?»

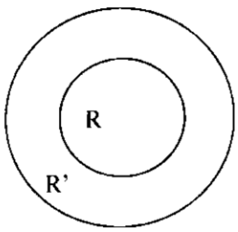

Figure 1.18: (Guisasola et al., 2002, p. 256)

Student: It is due to the fact that having a bigger radius, the sphere is now bigger, thus it can store more charge and capacitance $C = Q/V$ is bigger. *(1st year of Engineering)*

Their comment is: «The correct answer analyses that the process of induction that happens between both spheres results in a decrease of the difference of potential (Guisasola et al., 2002, p. 256).»

1.2.6 Linear View *vs* Systemic View

To analyze circuits with a systemic way of reasoning[7] is crucial to comprehend how they work (Besson, 2008). A local and causal/sequential way of reasoning seems however very resistant: this is one of the most important reason of failure in solving circuits.
Students visualize current as a particles flow that undertakes a travel from one point to another of the circuit, encounter different obstacles (like resistances, wire's splits, bulbs, etc...). They usually think that current diminishes through this travel, by overlapping the concepts of current and voltage. Also sudden modification of the system is usually interpreted locally and not a change on the whole system (Koumaras et al., 1997; Psillos, 1998b) among all.
In the following, I report examples from (Shipstone, 1988) which can be troublesome for students that are require to reason qualitatively (without the use of mathematics) and to guess voltage values at points indicated in Figure 1.21.

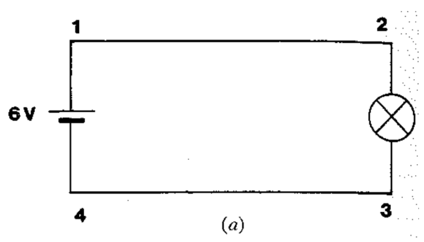

Figure 1.19: A

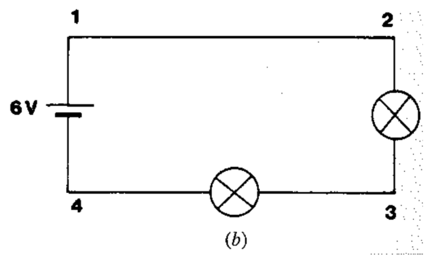

Figure 1.20: B

Figure 1.21: (Shipstone, 1988, p. 308)

A typical answer is that voltage is constant, whilst current decreases. Answering to question in Figure 1.22, students thought voltage, usually confused with current or perceived as undifferentiated from current, «divides into two equal parts at the junction before the bulbs.»

Some authors define two types of "wrong" approach (Cohen et al., 1983; Closset, 1983; Liégeois and Mullet, 2002): the *localist* approach and the *sequentialist* approach: «The localist approach is characterized by the fact that each part of the circuit tends to be treated separately. [...] The

[7](Stocklmayer and Treagust, 1996) observe that in certain cases engineers and students think differently: the former, for practical reasons, have developed a global, systemic view; the latter do not manifest the same needs and, consequently, they follow a localist approach.

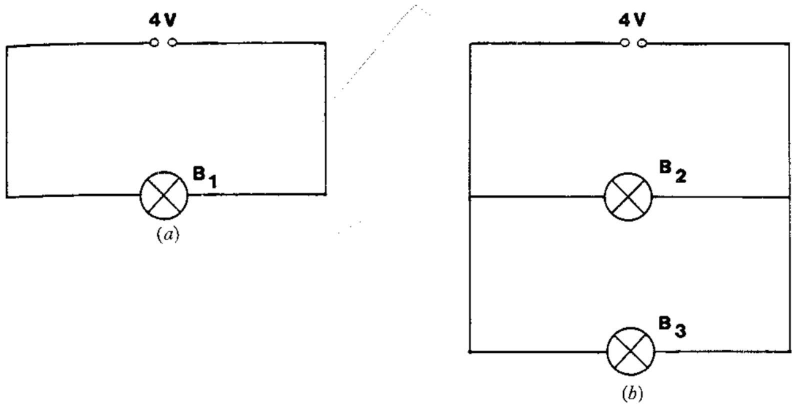

Figure 1.22: (Shipstone, 1988, p. 309)

sequentialist approach is characterized by the fact that some parts of the circuit tend to be considered before other parts (Liégeois and Mullet, 2002, p. 552).» If a resistence is added or modified, students tend to think that this

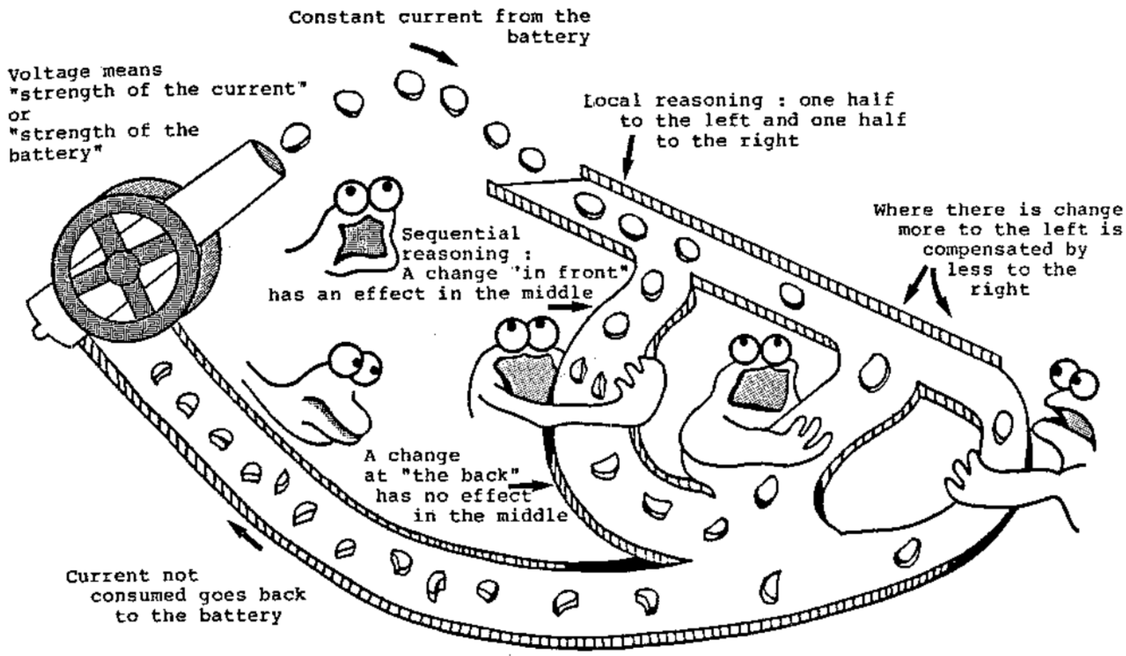

Figure 1.23: In (Shipstone, 1988, p. 315) a representation of both local and sequential reasoning

change does not have effects on the current until it returns to the modified

point (Duit and Rhöneck, 1998); instead, a single, local swing in the circuit produces a global change: only few students seem to deal with this systemic view (Millar and King Tom, 1993). Usually, students recognized this oddity; nonetheless they often continue to have a localist approach.
Adopting a sequential way of reasoning, students forget two key physical constraints: energy and current conservation.

I briefly reporte here that many Educational path have been proposed to induce a systemic view of circuits; many of them used the *hydraulic-fluid analogy*, without obtaining expected results (Haertel, 1993; Mosca and De Jong, 1993; Greenslade, 2003).

1.2.7 Ohm's Laws and Kirchhoff's Laws

Ohm's laws and Kirchhoff's laws are a clear example of a global, systemic mathematical formulation of equilibrium. They are not fundamental laws[8], but empirical. They can be applied in certain cases and they do not work when a change happens to the circuit's configuration. Ohm's law cannot be be applied in a localist approach (Jimenez and Fernandez, 1998; Psillos, 1998a; Periago and Bohigas, 2005), and others.
Ohm's law is misunderstood in different ways; this misunderstanding derives from a bad interpretation of its mathematical formulation. Research found that both younger and older secondary school students believe resistance directly proportional to both voltage and current, and not directly proportional to voltage and inversely proportional to current. «The resistance concept was thus very difficult to understand [...] For a majority of participants, irrespective of age and training, resistance was a direct function of both current and potential difference (Liégeois and Mullet, 2002, p. 561).»
Students show to think that «current and potential difference add their effects (Liégeois and Mullet, 2002, p. 562).»
Moreover, many students think that it does not exist voltage in empty space, not even between two capacitor's plates, because $V = i \cdot R$, and $i = 0$ (Cohen et al., 1983; Sherwood and Chabay, 1999).

(Reif, 1982, p. 1048) give a thorough introduction to Ohm's laws: «Consider any dissipative two-terminal system [...] A steady dc current i can flow through such a system only if the increase in the random internal

[8]Some researches found that Ohm's laws are believed to be more fundamental than Faraday's one (Bagno and Eylon, 1997)

energy of the system, caused by interactions of the moving charged particles with the other atomic particles in the system, is supplied by a compensating amount of work done on these charged particles. Thus the current i is zero if the work w done per unit charge is zero, while $i \neq 0$ if $w \neq 0$. If the current is not too large, the current i must then be simply proportional to w. Hence one can write $Ri = w$, where the proportionality constant R is called the "resistance" of the two-terminal system.»

In the same article, he describe what is intended for Generalized Ohm's laws:«It is important to note that w [...] consists generally of work done both by Coulomb forces and by non-Coulomb forces. The work per unit charge, done by the conservative Coulomb forces, can be expressed in terms of the electrostatic potential V and is simply equal to the potential drop $\widetilde{V} = \Delta V$ [...] The work per unit charge, done by all other non-Coulomb forces in charged particles moving inside the two terminal system [...] is (by definition) called the emf of the system. When both of these kinds of work are taken into account, the relation $Ri = w$ then yields the generalized form of Ohm's law for the current flowing [...]: $Ri = \widetilde{V} + emf$.
This general form of Ohm's law is applicable to any two-terminal system. In the special case of a two-terminal system with zero emf (i.e., a "resistor"), the generalized Ohm's law reduces to the traditional Ohm's law $Ri = \widetilde{V}$. In the special case of a two-terminal system with zero resistance (i.e., an "ideal battery"), the same law becomes $0 = \Delta V + emf$ and implies merely that the potential difference ΔV between the terminals is equal to the emf provided by chemical interactions in the battery.» See also (Smith and Van Kampen, 2011).

1.2.8 Microscopic and Macroscopic Approach

Many students can encounter serious difficulties in building a solid conceptual link between electrostatics and electrodynamics (especially circuits). They do not feel the need to link different models they have in mind. This produces a gap between the microscopic electrostatics' world and the macroscopic circuit's one. Students, due to their internal representations and maybe to electrostatic module taught them first, insist to interpret circuits' physics by a microscopic model; this model, moreover, is used by them to enforce their causal reasoning (Rosser, 1970; Closset, 1983; Preyer, 2000; Hirvonen, 2007; Muller, 2012).

Many textbooks do not examine in depth the relation between electrostatics and electrodynamics in circuits (Moreau et al., 1985). Many

mathematical formulas are taught only to solve particular problems, without the necessary explanation (Heald, 1984). Yet, the maths, the physics and the language, changes from electrostatics to circuits analyses: from the Newtonian, microscopic, linear model used in electrostatics, students face with circuits in a totally different manner, using systemic reasoning and dealing with macroscopic quantities. Moreover, from an almost completely theoretical approach, teachers begin to talk about "real-life" objects, like circuits, batteries and so on.
Stocklmayer and Treagust, after an analysis of physics textbooks from 1891 to 1991, observe few changes in the way electromagnetism is taught (Stocklmayer and Treagust, 1994). In order to link electrostatics with circuit's electrodynamics[9] and attempting to meet students' mental representations and ways of reasoning, many researchers built alternative approaches to the traditional teaching. Many of them have tried to build new electromagnetic curricola, aimed to make modeling coherent (Eylon and Ganiel, 1990; Chabay and Sherwood, 2015).

For example, the approaches that start from the microscopic point of view, circuit are introduced focusing the attention on the surface charges[10] and their distribution on the wire[11]. Their distribution, shaped by the battery potential difference (Figure 1.24), produces the (non-conservative) electric field[12][13][14] which causes the charges movement inside the wire (Sommerfeld, 1952; Jackson, 1996; Sherwood and Chabay, 1999). In this way, teachers can transport electrostatics within circuits analyses. Without electrostatics approach to circuits, students often fail to represent the electic field. For example, from (Sherwood and Chabay, 1999):

[9] (Haertel, 1987) was the first who tried to unify electrostatics with circuits analysis.

[10] Surface charges is the term with which a net amount of free charges on the surface of a conductor wire is indicated.

[11] (Rainson et al., 1994) found that almost nobody, among university's students sample, knows superficial charges role.

[12] Because the electric field on the conductor is very very small (1 from 200 Volt per meter), there is a small number of surface charges (few millions per centimeter if the cable is 1mm diameter): for this reason their electrostatic effects are difficult to detect. For quantitative and qualitative measures of this very small field, see (Jefimenko, 1962) and, respectively, (Muller, 2012; Jacobs et al., 2010).

[13] From (Haertel, 1987, p. 42): «Because of the enormous strength of the Coulomb interaction and the very high mobility of electrons in metals, it takes only a few electrons at the surface of the wire to push 10^{19} electrons around in a circle and to overcome the resistance of a metallic wire.»

[14] Although the electric field is discontinuous inside a charged sheet, the potential is not. This is a technical reason for prefer potential instead of electric field when a circuit is analyzing.

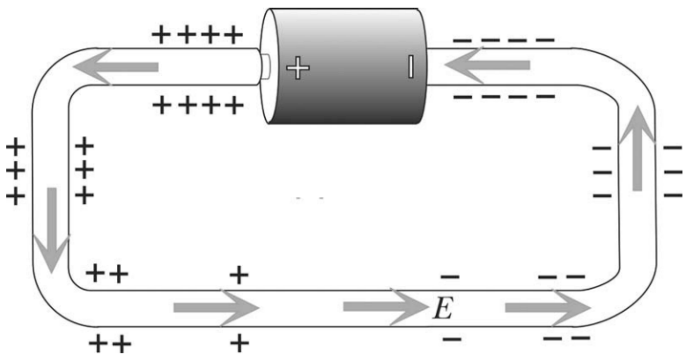

Figure 1.24: In (Chabay and Sherwood, 2006, p. 332)

«If the charges responsible for the electric field inside the bulb filament (Figure 1.25) are in and on the battery, shouldn't the bulb be much brighter when brought closer to the battery?» The surface charges' distribution on

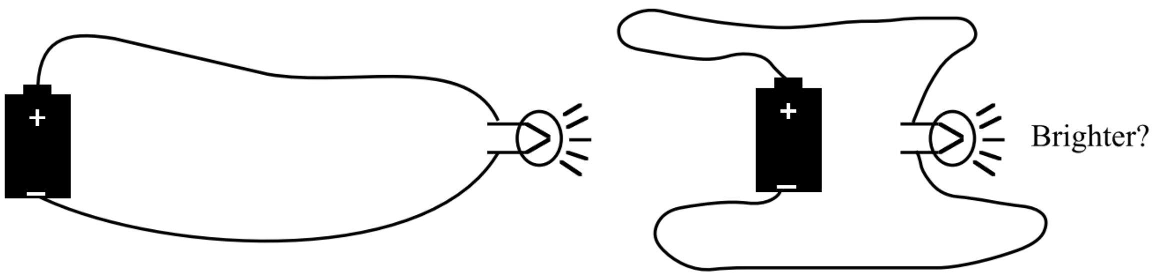

Figure 1.25: In (Sherwood and Chabay, 1999, p. 3)

circuit wires has three different roles (Jackson, 1996):

1. maintaining the potential around the circuit,
2. outside the wires, shaping the electric field,
3. inside the wires, generating an electric field which is parallel to the wires themselves, providing current confinement, direction and intensity.

In the microscopic case, battery is not the voltage source, but the electric field source, necessary to maintain uniform the superficial charges' distribution. The rate of change in a circuit could be considered small and not-interesting from a macroscopic view, but important in a microscopic approach (Muller, 2012).

The following table sums up the main differences between traditional approaches and microscopic approach, based on surface charges (Sherwood and Chabay, 1999):

Traditional treatment of circuits

- Little or no connection to electrostatics
- Solely in terms of potential and current
- Macroscopic only
- Steady state only
- Little sense of mechanism

New treatment of circuits

- Unified treatment of electrostatics and circuits
- Initially in terms of charge and field, followed up later by analyses in terms of potential and current
- Microscopic as well as macroscopic
- Transient polarization establishes the steady state
- Strong sense of mechanism

Further researches (Eylon and Ganiel, 1990; Thacker et al., 1999) state that a better understanding of macroscopic systems can be achieved developing a microscopic point of view, because of the proximity with learners knowledge. They infer that a microscopic analysis can be useless and too much complicated in solving problem, but that a clear microscopic point of view should help students developing a better understanding and a more complete framework about circuits. In confirmation of this approach, see (Kohlmyer et al., 2009).

On the contrary, (Duit and Rhöneck, 1998) state that it is necessary to develop a systemic view only, without passing through a particle description. Psillos attempt to build a course based on macroscopic approach is done, but "digressions" in the microscopic world have been necessary, in order to meet students' mental representations: «In our case, the conceptual part is based on the modeling of electrical phenomena at a macroscopic level including the concepts of voltage, current, energy, resistance, time. Simple use of microscopic entities (charged particles, electrons) is made only in response to students' questions regarding "what is flowing" (Psillos, 1998b, p. 3).» Other interesting attempts are made in order to switch on the systemic view using experiments with circuits, «batteries and bulbs» (James, 1978; McDermott et al., 1996).

Like in the case of thermodynamics, also this debate shows that each approach has its own language, models, typical forms of representation and of explanation. In Bologna we tend to advocate for multi-perspective, since we do believe that the comparison of different approaches has an impressive potential to engage the students, touch different interests and tastes and foster appropriation (Levrini et al., 2015).

1.2.9 Electric Circuits: Final Remarks

So far I listed many problematic situations that can be summarized in three points:

1. lack of consistent relations between electrostatics and electrodynamic circuits
2. lack of consistent relations between macroscopic and microscopic models
3. lack of systemic approach

As asserted in (Haertel, 1982; Dupin and Johsua, 1989), the huge number of possible approaches, models and representations causes too much confusion, which generates learning and teaching difficulties.
The lack of systemic approach development, moreover, hinders students to pursue qualitative reasoning (Eylon and Ganiel, 1990; Thacker et al., 1999; McDermott and Shaffer, 1992a; Cohen et al., 1983). Qualitative reasoning is proved to be harder than quantitative one and is fundamental to create appropriate connections between circuit schematic representations and real circuits (Gott, 1985; McDermott and Shaffer, 1992b). When «Students were asked to identify the corresponding standard circuit diagram for each of the sketches of a real circuit shown in Figure 1.26 **(a)**» Because of the «Lacking an adequate procedure for determining the types of connections between the bulbs, [they] often fail to recognize that the second circuit in Figure 1.26 is

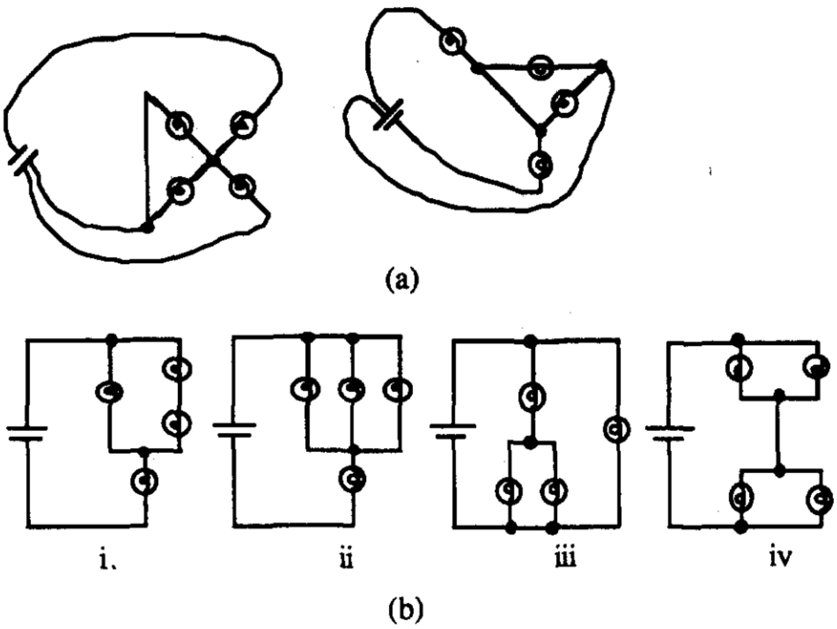

Figure 1.26: In (McDermott and Shaffer, 1992a, p. 999)

the correct diagram for both circuits in **(a)** (McDermott and Shaffer, 1992a, p. 999).»

Eylon and Ganiel stress how algebraic calculation cannot help to understand neither global nor local phenomena (Eylon and Ganiel, 1990; Millar and King Tom, 1993). Instead, teaching should be focused on functional relations among physical quantities, the causal explanations implied in the relations, on the construction of coherent frameworks that consistently shift from local - micro in (Eylon and Ganiel, 1990) - to global - macro in (Eylon and Ganiel, 1990) - models and vice versa.
In fact, students often prefer mathematics to qualitative reasoning. In Figure 1.27, this fact clearly emerges:

Interviewer: How does this (i.e., the configuration of the elements) explain the difference in currents?

Student: When the diode conducts, one has to consider the two resistors in parallel. The equivalent resistance is given by the equation $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$
If you compute it you find that resistance R is smaller than R_1

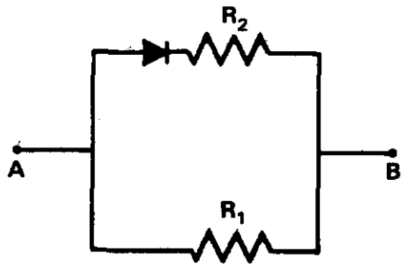

Figure 1.27: In (Cohen et al., 1983, p. 407)

Interviewer: Can you convince me without mathematical considerations that the current must be larger when the diode conducts?

Student: No. This is a mathematical fact[15]

Anyway, teaching circuits' analyzes remains a difficult task for teachers (Gunstone et al., 2009). As we have seen, many researchers point out that the local, Newtonian approach based on microscopic model often fail when quesions exit from the "confrot zone". Stocklmayer, in his paper, suggests: «The problem with the universal adaptation of the field model lies in its unfamiliarity. It is not within the "comfort zone" of many teachers, nor, indeed, many conventional physicist for whom the electron flow model has proved comprehensible and satisfactory [...] It will require the development of new resource materials, including textbooks and practical exercises, and extensive professional development for teachers (Stocklmayer, 2010, p. 1825).»

1.3 Magnetostatics

After electrostatics and circuits, students usually begin to study the magnetic properties of matter. They already know what magnets are, and they have built their own personal representations about magnetism.
Magnetic field and magnetic force are usually represented like the electric field and Coulomb force. The only significantly different aspect is the nature of charges: electric charges are point-like charges, while magnetic one are macroscopic objects.

[15] This qualitative problem could be solved easily thinking at the fluid analogy: more sections is equal to less resistance.

1.3.1 Confusion between Magnetostatics and Electrostatics

Many students confuse electric and magnetic charges: they say there is an excess or a lack of electricity on magnet pole. Students can think that magnetic charges are electric charges (Borges and Gilbert, 1999; Guisasola et al., 2004). Researchers argued the cause of this confusion can be found in the field lines representation (Maloney, 1985; Ambrose et al., 1999a; Maloney et al., 2001; Smaill and Rowe, 2012). Students often think that magnetic force is parallel to magnetic field lines, like in the electric case. As an example, from (Guisasola et al., 2004, p. 452):

Interviewer: Why do you think a magnet attracts iron material, as for instance a "paper clip"?

Student: I Think the clip, due to the magnetic field of the magnet, gets polarized and attracts it.

Interviewer: That is...

Student: Maybe, I didn't explain it correctly. The magnet has a magnetic field and it polarises the particles in the clip, it makes them move, and it attracts the charges, negative or positive depending on the pole that comes near it. And the clip moves and it comes closer to the magnet, that is, pulling force is created.

Also in (Borges and Gilbert, 1998) it is shown teachers answer the previous question in the same manner.

Many students think that electric charges at rest can be deviated by magnetic field (Allen, 2001). Moreover, they think magnetic force attracts "bodies", regardless physics nature of these bodies (Scaife and Heckler, 2010).
Also electrostatics knowledge is subjected to modifications due to magnetostatics module. For instance, students usually do not represent the electric field for a moving particle (Kesonen et al., 2011).
This misunderstanding is probably due to the usual representation of forces on a charged moving particles after the introduction of the Lorentz force. Furthermore, some students think that electric charges are responsible for electric field only, while currents produce only a magnetic field (Bagno and Eylon, 1997).

The right-hand rule is usually not-well comprehended. Besides, many students do not remember the non-commutativity of cross products (Scaife and Heckler, 2010) and often think that magnetic force is parallel to magnetic field lines, like in the electric case. Moreover, (Mosca, 1974) and (Onorato and Ambrosis, 2013) show students think Lorentz force can do work. These are examples of a more general difficulty, explains (Chandralekha, 2008, p. 1): «Some additional difficulties are due to the non-intuitive three dimensional nature of the relation between magnetic field, magnetic force and velocity of the charged particles or direction of current».

1.3.2 Ampère's Law and Field Lines Representation

Ampère's law shares with Gauss' law similar problems. In fact, students do not understand this law a fundamental law of magnetic interactions. So, they believe it is a tool for the evaluation of the magnetic field only. They do not appreciate the meaning of the circulation; indeed, they do not understand how circulation is independent from the chosen path[16].
For example, students do not separate the circulation concept from the field one. From (Guisasola et al., 2008, p. 1011):

Student: If we apply Ampere's law $0 = \oint \vec{B} \cdot d\vec{l} = B \oint dl \rightarrow B = 0$.

Students do not show to understand that the source of the magnetic field is electric current. Guisasola and colleagues found that some of them believes that the source of the field is the path chosen to evaluate the circulation (Guisasola et al., 2008). Probably they infer this information from the formulation of the law:

Student: According to Ampère's law, I applied the field circulation for that line [...] In the vertical segment and in the external part there is no circulation of $\vec{B}$. Therefore, *you would get field $\vec{B}$ from there, because we know the intensity that circulates through the loops*: $\mu_0 \sum i_{\text{internal}} = \oint \vec{B} \cdot d\vec{l} = B \oint dl = Bd$ (Guisasola et al., 2008, p. 1008).

Many students, like for electric field lines, think that magnetic field lines can attract or repel themselves. As for the electric field lines, researchers show that students give to field lines more reality than the necessary (Guisasola et al., 2004; Pocovi, 2007). For example:

[16] A similar analyses for the Gauss' law shows the same conceptual problems, that is the independence of the flux from the surface shape.

Student A: A magnet has two poles, N and S. Field lines generate in them, going out of N and into S. Such lines create a magnetic field around the magnet.

Student B: The magnet will create some field lines (which are the magnetic field), that will act on the clip attracting it (Guisasola et al., 2004, p. 452).

Allen showed that some students can't figure out in the correct manner how magnetic field believes outside a coil (Allen, 2001). Some researches notice that students can't apply correctly Newton's third law in the electromagnetism contest; in particular, when two current carrying wires attract themselves, students that the more is the current through the wire, the more is the force of attraction on the other wire (Galili, 1995; Smaill and Rowe, 2012).

1.3.3 Microscopic and Macroscopic

Also in magnetism, the two approaches are usually confused. Although magnetism is mainly presented as a macroscopic phenomenon, Lorentz force $\vec{F}_L = q\vec{v}\times\vec{B}$ plays a crucial role. It is typically introduced within the particle microscopic model, and it is usually transported into the macroscopic world in a naive way, in order to obtain $\vec{F}_L = i\vec{l} \times \vec{B}$. This formula is usually obtained from the equation $qAvdl = idl$. This equivalence contributes to the idea that current is a flow of charged particles. As already anticipated in the section on the circuits, some textbooks (Chabay and Sherwood, 2015) build a consistent microscopic model related to surface charges in order to bridge microscopic world with the macroscopic one.

Magnetism is a macroscopic phenomenon. At the microscopic level, magnetic charges became circular electric currents: magnetism and electricity become two aspects of the same effect. Magnetism «at a microscopic level is a property of all substances, although their macroscopic behavior may be very distinctive» (Erickson, 1994).

1.3.4 Models of Magnetism

Researchers found five different mental models of magnetism, which students have developed from early stages of learning to university courses (Erickson, 1994; Borges and Gilbert, 1998):

1. **Magnetism as pulling**: magnetism is the property of some bodies (called magnets) to attract other bodies – no poles are included in this view, neither other physical concepts like force or energy.

2. **Magnetism as a cloud**: magnetism is a "sphere of influence", a "force field" (in a very ingenuous sense) generated by a particularly ordered atoms disposition inside the magnet. From (Borges and Gilbert, 1998, p. 367):

   «Next, Patricia is shown a bar magnet and speaks about its uses and the origins of magnetism. Patricia explains that the magnet has the ability to attract metals. She asserts that the field is always present, but forces only exist when some object comes into the field.»

   The field is limited inside a three-dimensional region: outside the force/attraction is zero.

3. **Magnetism as electricity**: magnetic poles are region inside there's an excess or lack of electricity. There's no connection between this two different poles. In this model a distinction between poles takes place.

4. **Magnetism as electric polarization**: this is an evolution of the former model. Inside magnet, particles are polarized, giving a macroscopic electric field with two poles, one positive and one negative. The field is arranged like the true magnetic field, but it has the nature of an electric field.

5. **Magnetism as field**: particles inside a magnet are in regular motion and this generates the macroscopic magnetic field outside the magnet. In particular, from (Borges and Gilbert, 1998, p. 372):

   - «The view that magnetism is created by micro-currents circulating inside magnets, and also in ferromagnetic materials, which behave as small magnets. This is essentially the model adopted by Ampere and later on perfected by Weber. Most subjects equate the micro-currents with electrons moving round the atom, in closed orbits. [...] This is the view normally found in physics textbooks for secondary education.
   - The view that proposes the existence of small permanent magnetic dipoles within matter. These dipoles are not always related to electric currents.

- In a few cases the spin and orbital magnetic moments are distinguished. People holding this model can describe and account for the behavior of the magnet in a way consistent with mainstream science.»

1.4 Electromagnetic Induction

Usually, the electromagnetic induction is presented in the Faraday-Neumann-Lenz[17] (FNL) mathematical form:

$$emf = -\frac{d\Phi_S(\vec{B})}{dt} \tag{1.1}$$

An acceptable definition of this rule[18] can be found in (Romeni, 2012, p. 1039): «[the FNL law says that] the average induced emf in a circuit in the time Δt is equal to the opposite of the magnetic flux variation $\Delta\Phi_S(\vec{B})$ in the same time interval through any surface S having as border the circuit itself. »

Another textbook underlines that the FNL is in accord with experiments: «[...] every time a magnetic flux variation occurs (caused by a variation in the field intensity, in the surface or in the angle between field and the normal to the surface), it occurs an induced electromotive force, and so, an inducted current if the circuit is closed [...] The flux variation is equal to the electromotive force with the opposite sign (Amaldi, 2012, p. 958-959).»

This behavior was discovered experimentally by Michael Faraday in 1831. It is included in Maxwell's equations and represented a fundamental step towards the unification process between electricity and magnetism and in the construction of the field concept. For many reasons, electromagnetic induction is an hard topic for students at high school and university (Venturini and Albe, 2002).
Starting from the expression (1.1) I will analyze emerging didactic and epistemological difficulties, focusing on the mathematical formulation of the law.

[17]F.E. Neumann (1798-1895) from Koenigsberg proposed in *Allgemeine Gesctze der inducirten elektrischen Strome*-Annalen der Physik, 1846 the first quantitative formulation of the Faraday-Lenz law (for this called Faraday-Neumann-Lenz law). See also (Roche, 1987)

[18]As pointed out by many researches, FNL is a mathematical rule which defines quantitatively what happens in a circuit at rest in the case a magnetic flux variation occurs nearby (Munley, 2004)

1.4.1 What does FNL rule say?

FNL rule presents different theoretical challenges. Allen identifies three principal causes:

«Induction is comprised of multiple inter-related abstract quantities (non-linearity), that are inherently three-dimensional, and that are changing in time (Allen, 2001, p. 7).»

We analyze these challenges separating the equation in three parts.

The Right-Hand Side

Targeted researches point out difficulties related to the rate of change of a physical quantity. Students associate to a large flux a large rate of flux change and, if t increases, some show to believe that emf diminishes (Peters, 1984; Bagno and Eylon, 1997; Allen, 2001; Thong and Gunstone, 2008).

Often, students wrongly suppose that the induced magnetic field is in the opposite direction of the inducing field, instead of in the direction of the field change. For instance, many students fail in facing this type of question (Peters, 1984, p. 298):

«Consider a source of induced emf, possibly a long solenoid with steadily increasing current, giving a constant emf in any loop encircling the solenoid»

Another problem concerns the concept of magnetic flux. There is «lack of distinction between field, flux and flux variation (Allen, 2001, p. 353)». «Many students interpret that the magnetic field produces electromagnetic induction (Zuza et al., 2014, p. 2)». Also (Guisasola et al., 2013) underlined this difficulties. Students, who have already met this term in other physics topics, associate it to an undefined changing, mixing up flux with fluctuation (Allen, 2001).

Moreover, secondary school students show a persistent difficulty in interpreting the mathematical symbols and procedures they have just learned at school, especially derivatives (Chabay and Sherwood, 2006).

There is also a problem of stratification: the expression

$$\frac{d\Phi}{dt} \tag{1.2}$$

is built starting from the field, passing to the field flux and adding its rate of change. This stratification keeps students away from the comprehension of the expression (1.1) (Allen, 2001).

The Sign of =

Any teacher has experienced the following phenomenon, well-known in Educational research literature: students tend to think of the equal sign as a *procedure indicator*. Despite explanations of the FNL rule speaks about "equality" and not about "cause", the left-hand side of (1.1) – the emf - is felt as the effect of the right-hand side of (1.1) - the magnetic flux variation (Rainson et al., 1994; Camici et al., 2002). This way of reasoning leads to the following wrong interpretation: an inducted emf is *generated* by a variation in the magnetic flux.

It is necessary to insist on the word *induction* and to firmly state that induction does not mean *cause*, because many students continue to think of the FNL rule as the mathematical way to say that a current could be *produced* by a changing magnetic field or flux. In (Jefimenko, 2004, p. 294) the cause of electromagnetic induction is efficaciously described: «in time-variable systems electric and magnetic fields are always created simultaneously, because these fields have a common causative source: the changing electric current $\partial\vec{j}/\partial t$.» It is important to stress that the system is a time-variable one, in order to underline the very important difference in electromagnetic induction physics: time-variable quantities. From (Hill, 2010, p. 410):

«To establish causality, it is necessary to establish a time lag between the cause and the effect.»

The Left-Hand Side

A common difficulty emerges from the analysis of many specific tests submitted by researchers: «many students are not capable of recognizing electromagnetic induction when there is no induced current (Zuza et al., 2014, p. 2)». See also (Guisasola et al., 2013). It is the manifestation of the *no effect equal no cause* aspect of students' reasoning, appeared in many researches, (Rozier, 1989; Rainson et al., 1994; Viennot and Rainson, 1999). Reading again the explanations for (1.1), the word *circuit* is written explicitly, becoming a fundamental element for electromagnetic induction to occur.

1.4.2 What does FNL rule implicitly say?

FNL rule in (1.1) can be written in a more appropriate way as

$$\oint_{\partial S} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{S} \tag{1.3}$$

which apparently seems much more complicate. Indeed, on one hand, especially for a secondary school students without elementary knowledge on mathematical analysis, (1.3) is much more complicated than (1.1). But, on the other hand, (1.3) is much more detailed than (1.1): it is unambiguous that the contour of integration on the left-hand side is the border of the area of integration on the right-hand side; it is unambiguous that the element on the right is not the magnetic field; it emerges clearly the importance of the fields direction with respect to the direction of integration; furthermore, current is not a necessary element anymore (the word emf is usually linked to electric circuits). Thus, (1.3) is an important generalization of the FNL rule (1.1), because the integration path for the left-hand side can be considered any geometrical closed lines, a circuit or an imaginary line.

Anyway, referring at the expression in (1.3), I can address further problems revealed by research.
First, students encounter some difficulties in choosing the integration area: they usually choose the area enclosed by the circuit and they think that changing the area will change the flux and consequently the emf (Layton and Simon, 1998; Chabay and Sherwood, 2006; Zuza et al., 2014).
Second, students can find difficulties in representing fields (Saarelainen et al., 2007); in this case, the mathematical nature of field vectors does not emerge. Chandralekha moreover, discovered that many students see the field flux as a vector, because of the presence of a $\cos\theta$. citepChandralekha2006
Third, as pointed out in (Allen, 2001), students do not understand correctly the meaning of integration. In fact, many students pull fields out from the integral. For instance, from (Guisasola et al., 2008) if flux is zero, then

$$0 = \oint \vec{E} \cdot d\vec{l} = \vec{E} \cdot \oint d\vec{l} \rightarrow \vec{E} = 0 \tag{1.4}$$

1.4.3 What does FNL rule hide?

Two important problematic issues come out from this approach to induction through the FNL law: this law does not hold when different frames of reference are considered; its connection with the Lorentz force is not consistent.

It is known that Maxwell's electrodynamics—as usually understood at the present time—when applied to moving bodies, leads to asymmetries which *do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. For if the magnet is in motion and the conductor at rest, there arises in the neighborhood of the magnet an electric field with a certain definite energy, producing a current at the places where parts of the conductor are situated. But if the magnet is stationary and the conductor in motion, no electric field arises in the neighborhood of the magnet. In the conductor, however, we find an electromotive force, to which in itself there is no corresponding energy, but which gives rise—assuming equality of relative motion in the two cases discussed—to electric currents of the same path and intensity as those produced by the electric forces in the former case. Examples of this sort, together with the unsuccessful attempts to discover any motion of the earth relatively to the "light medium", suggest that the phenomena of electrodynamics as well as of mechanics possess no properties corresponding to the idea of absolute rest.*

from "On the Electrodynamics of moving bodies" by A Einstein - Annalen der Physik, 1905

In the beginning of his famous paper "On the Electrodynamics of moving bodies" Albert Einstein focused on the dynamics of relative motions.
What (1.1) and (1.3) do not say is what happens when the circuit is in motion. Before answering this, we briefly resume a long-lasting debate upon the question:

is the FNL just a rule or it is a fundamental law of physics?

Many authors claim that FNL is not a law of physics because there are exceptions (Barnett, 1912; Blondel, 1914; Nussbaum, 1972; Bartlett et al., 1977; Klein, 1981; Bradley, 1991; Guala-Valverde et al., 2002; Giuliani, 2002; Kelly, 2004; Hill, 2010; MacLeod, 2012; Zuza et al., 2014). These exceptions arise when circuits are in motion with respect to the magnetic flux. In this reference frame, expression (1.1) lead to infer that no emf is induced in the circuit; however, experiments show the opposite. To restore a correspondence between phenomena and theory it is necessary to introduce Lorentz's force and affirm that motion inside the circuit is induced by this force. It is impossible to derive Lorentz force from FNL expression (1.1). So, Lorentz force appears to be another law of physics, a distinct expression outside Maxwell's equations. From (Feynman, 2011, p. 17-3):

«We will now give some examples, due in part to Faraday, which show the importance of keeping clearly in mind the distinction between the two effects responsible for induced emf. Our examples involve situations to which the "flux rule" cannot be applied – either because there is no wire at all or because the path taken by induced currents moves about within an extended volume of conductor. We begin by making an important point: The part of the emf that comes from the $\vec{E}$-field does not depend on the existence of a physical wire (as does the $\vec{v} \times \vec{B}$ part.) The $\vec{E}$-field can exist in free space, and its line integral around any imaginary line fixed in space is the rate of change of the flux of $\vec{B}$ through that line. (Note that this is quite unlike the $\vec{E}$-field produced by static charges, for in that case the line integral of $\vec{E}$ around a closed loop is always zero.) [...] [The flux rule] must be applied to circuits in which the material of the circuit remains the same. When the material of the circuit is changing, we must return to the basic laws. The correct physics is always given by the two basic laws

$$\begin{aligned} \vec{F} &= q\left(\vec{E} + \vec{v} \times \vec{B}\right) \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \text{»} \end{aligned} \tag{1.5}$$

This theoretical problem is present within electromagnetism teaching (Galili and Kaplan, 1996). Lorentz force can be a useful tool in solving exercises, especially when FNL cannot be used[19].

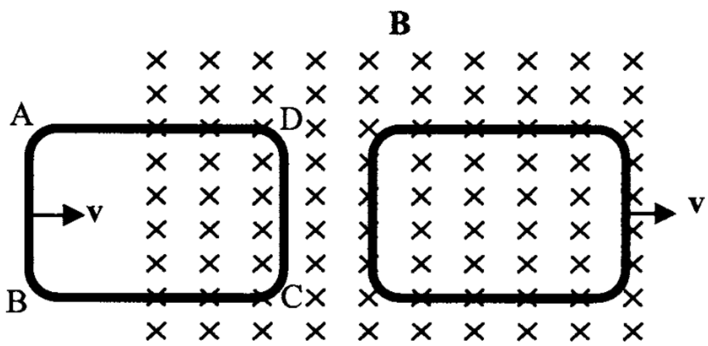

Figure 1.28: In (Galili et al., 2006, p. 341)

[19]«A number of students applied the Lorentz force on several of these questions [...] depending on which variable dominated the students' response (Allen, 2001, p. 355)»

For instance, in Figure 1.28 "motional emf" caused by Lorentz force is used to explain how charges behave in a circuits when magnetic flux is constant.

To have two ontologically different approaches to explain the same phenomenon represents an obstacle in the comprehension of electromagnetic induction.
«Most students do not understand the equivalence of the explanation based on a field model and on Lorentz's force for all induction phenomena (Zuza et al., 2014, p. 2).»

To overcome this problem, it need to take (1.3) and perform the total derivative $\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla$

$$\frac{d}{dt}\int_S \vec{B} \cdot d\vec{S} = \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} + \oint_{\partial S} \left(\vec{B} \times \vec{v}\right) \cdot d\vec{l} \quad [20] \qquad (1.6)$$

The last equation, combined with (1.3), gives the FNL rule for any path, at rest or in motion with velocity $\vec{v}$

$$\oint_{\partial S} \left[\vec{E} - \left(\vec{v} \times \vec{B}\right)\right] \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} \qquad (1.7)$$

In another reference frame the circuit is at rest. FNL (1.3) is always true, but here the electric field measures $\vec{E}'$

$$\oint_{\partial S} \vec{E}' \cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{S} \qquad (1.8)$$

Since the circuit is at rest, the last expression becomes

$$\oint_{\partial S} \vec{E}' \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} \qquad (1.9)$$

In a Galilean relativity,

$$\vec{E} = \frac{\vec{F}}{q} = \vec{E}' + \vec{v} \times \vec{B} \qquad (1.10)$$

This is exactly the Lorentz force, exerted on a charge q travels with velocity $\vec{v}$ with respect to the observer.

[20] $\frac{d\vec{B}}{dt} = \frac{\partial \vec{B}}{\partial t} + (\vec{v} \cdot \nabla)\vec{B} = \frac{\partial \vec{B}}{\partial t} + \nabla \times (\vec{B} \times \vec{v}) + \vec{v}(\nabla \cdot \vec{B})$

This approach makes Maxwell's equations clearer (Jackson, 2001). In fact, if $\vec{E}$ and $\vec{B}$ are measured in the same reference frame of the circuit at rest, using the Stokes theorem it is easy to find out from (1.3) that

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{1.11}$$

(1.7) is called *the general law of induction* (Scanlon et al., 1969; Nussbaum, 1972; Galili et al., 2006).

Electromagnetic induction is a fundamental step within the electromagnetic theory and an interesting beginning to move inside the Maxwellian paradigm. In fact, it contains new elements (macroscopic quantities, non-causal relations, time-variable quantities, three-dimensional interactions) hard to manage within the Newtonian approach. Lorentz force is an easy way to deal with charges-fields interactions. However, the research in educational physics reports that it must be included in the new theoretical framework, in order to prevent students to produce counterproductive representations and models.

1.5 Electromagnetic Waves

There are no many researches on students' comprehension of electromagnetic waves. Nevertheless, the work done by the Physics Education Group of the University of Washington can be considered - qualitatively and quantitatively – a fundamental study on this topic. It is a matter of fact that many students do not develop a basic wave model; they do not easily grasp concepts like wavelength, path length difference, and phase difference and they can not always explain correctly diffraction, interference and polarization phenomena (Ambrose et al., 1999b; Wosilait et al., 2001).

The typical representation of an electromagnetic plane wave is shown in Figure 1.29. Usually, this figure is drawn as much more similar as the mathematical expression of a plane electromagnetic wave

$$\vec{E}(\vec{x}, t) = E_0 \sin(kx + \omega t)\hat{z} \qquad \vec{B}(\vec{x}, t) = B_0 \sin(kx + \omega t)\hat{y} \tag{1.12}$$

«Experienced instructors know that the diagrammatic representation of a plane EM wave commonly used in introductory textbooks is often incomprehensible to students (Ambrose et al., 1999a, p.891).»

[21] Elaborated from: Izaak Neutelings (May 2018). Inspiration

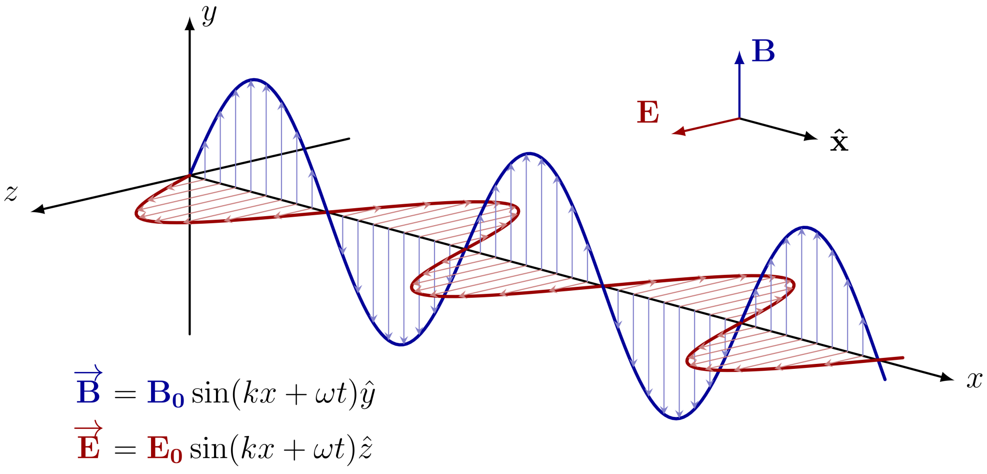

Figure 1.29: A plane electromagnetic wave[21]

Failure to interpret the typical representation of a plane electromagnetic wave

Students often learn from the representation shown in Figure 1.29 that electromagnetic wave exists only within the region shaped by the sinusoidal curve. They «attribute a spatial extent to the amplitude of the wave (Ambrose et al., 1999a, p.891).»

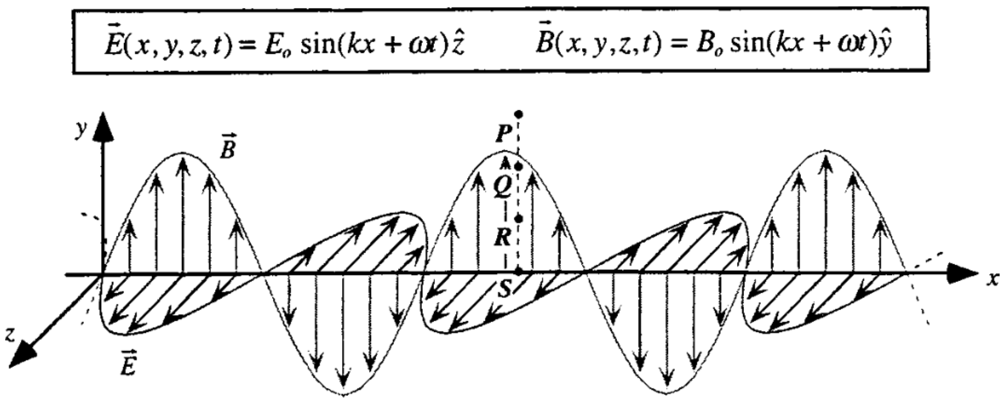

Figure 1.30: (Ambrose et al., 1999a, p. 892)

For instance, when students are asked to rank the points P, Q, R, and S in Figure 1.30 according to the magnitudes of the electric and magnetic fields at those points, only 10% out of 1275 students gave the correct answer. Moreover, when a plane wave like that in Figure 1.29 passes through a slit, many students believe it can be possible only if the wave is enough thinner with respect to the slit. From (Ambrose et al., 1999a, p. 893):

Student A: $P = 0$ because it is outside the boundary of the "reach" of the B field

Student B: $S = R = Q$, $P = 0$ [because] P lies off the wave where there is no field

Many errors seem to be related to the difficulty to distinguish the y and z coordinates from the $\hat{y}$ and $\hat{z}$ unit vectors. For example, from (Ambrose et al., 1999a, p. 893):

Student: $P > Q > R > S$, since y [referring to $\hat{y}$] corresponds to the strength of the magnetic field, and P is higher than Q, etc.

A "confusion by representations" (Törnkvist et al., 1993) is observed too (Ambrose et al., 1999a, p. 893):

Student: $Q = R = S$ because lines have the same spacing (the field is uniform below the curve). $P = 0$ because [there are] no field lines above the curve

Failure to interpret the electromagnetic wave as a field configuration which can interact with charges

Even though students show to know that electromagnetic wave is composed by the electric and magnetic fields, they often fail to recognize possible interactions among these fields and charges.
If Figure 1.30 represents a radio wave, students answer incorrectly if asked in which direction they would orient the antenna for best reception. Only about 10% of the students answer correctly; many students think that antenna has to be placed parallel to the direction of propagation. From (Ambrose et al., 1999a, p. 894):

Student: I would orient the antenna along the x-axis. This is because that's the direction of the wave, and it gets a maximum electrical and magnetic field (strong signal)

[22] From Dave3457, wikimedia commons

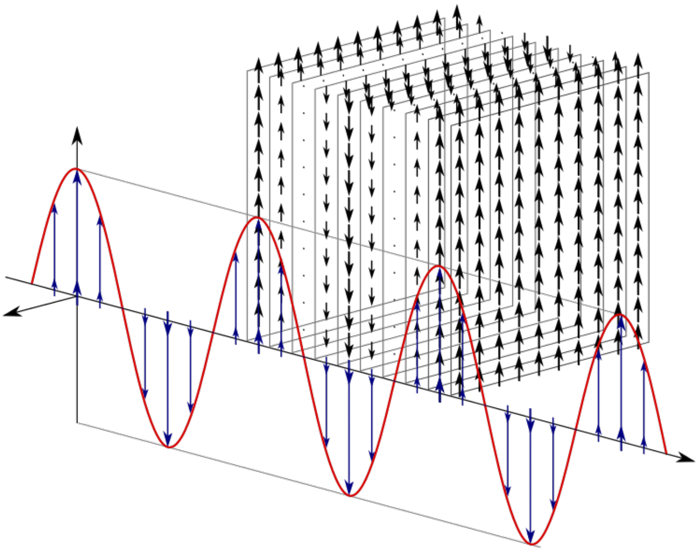

Figure 1.31: A plane electromagnetic wave[22]

In this case and in previous ones, students fail to apply the idea that an EM wave is a transverse wave.
Another common error is related to the difficulty to recognize the electric field as the only physical entity which can move almost fixed charges along its direction. For instance, (Ambrose et al., 1999a, p. 894):

Student: It seems that either the y-or z-axes would be good because [the antenna] would be perpendicular to the direction of propagation.

Failure to recognize the interdependency between the electric and the magnetic fields in an electromagnetic wave

"Several students treated the oscillating electric and magnetic fields in a light wave as independent entities. For example, a student correctly predicted that a polarizing filter placed in front of a single slit would decrease the intensity at the screen. He supported his answer, however, by saying that the polarizer consists of long molecular chains that form «very little [parallel] grooves. [The] only waves of light that are allowed to go through are the ones that are moving along that line, and the ones that are moving...perpendicular to that line will be canceled out.» When asked to consider the case in which the electric field of the incident light is parallel to the «little grooves», he stated that all of the electric field would be transmitted but none of the magnetic field (Ambrose et al., 1999a, p.894)." The same results were obtained in (Kesonen et al., 2011).

Failure to recognize the origin for the electromagnetic wave

(Kesonen et al., 2011) reports students are not able to say that accelerating particles are the origin for the electromagnetic wave[23]. They think the accelerating particles induce the magnetic field only. «This indicates that these students may have thought that an electric field is a stable property of a charge and that only the magnetic field can change (Kesonen et al., 2011, p. 531).»

[23] An exhaustive explanation of the electromagnetic production could be found in (Hecht, 2001)

Chapter 2

Mathematics-Physics Interplay and the Epistemic Games

The interplay between mathematics and physics in teaching is the topic of many important researches in physics education. Nevertheless, it is still considered a very problematic and open-ended question.
At the secondary school and the University, both subjects are usually taught separately; further, preservice teacher education programs often offer separated courses in Physics Education and in Mathematics Education.

In this chapter I will present the principal theoretical references I used to design the teaching/learning activities which I will describe in Chapters 3 and 4.

To frame the role of mathematics in physical modeling, I referred to Uhden, Karam, Pietrocola and Pospiech model. I will present this model in section 2.1.
In order to analyse the problem solving strategies carried out by university students and secondary school teachers (Chapter 3), I used the epistemic game theoretical framework elaborated by Tuminaro and Redish. This framework is presented in section 2.2.

In Chapter 4 I will show the specific manifestations of the interplay between mathematics and physics in the paradigm change "from force to field" worked by Faraday and Maxwell.

2.1 Big Eye and Little Eye Strategies within the Uhden Model

As I will show in Chapter 3, students and teachers have many different ideas on what is mathematics and what is physics, and they usually tend to separate mathematical terms from physical ones and the mathematical ways of reasoning from the physical ones.
In 2015 the *Science & Education* periodical published a special issue on this topic. Ricardo Karam, in the Introduction, wrote:

«In physics education, it is usual to find mathematics being seen as a mere tool to describe and calculate, whereas in mathematics education, physics is commonly viewed as a possible context for the application of mathematical concepts that were previously defined abstractly (Karam, 2015, p. 487)»

In this special issue, a series of historical case studies are presented, in order to enlarge and to problematize the interactions between physics and mathematics. For instance, Brush showed how mathematics has been «an instigator of Scientific Revolution» (Brush, 2015), while Kragh underlined «the creative power of physics (Kragh, 2015, p. 518)», and showed examples of how the formal structures shaped the ways of looking at physical phenomena.
The case studies reported in (Kragh, 2015), as those analyzed by (Tzanakis, 2016), stress to what extent mathematics has not been and is not a mere technical tool for physics, but it has been a main, fundamental actor in structuring the physical way of reasoning. The distinction between structural and technical role of mathematics in physics is the focal point of the approach and the model elaborated by Uhden, Karam, Pietrocola and Pospiech in 2012 (Uhden et al., 2012).
They wrote:

«If analysed more precisely, the role of mathematics in physics has multiple aspects: it serves as a tool (pragmatic perspective), it acts as a language (communicative function) and it provides a way of logical deductive reasoning (structural function) (Uhden et al., 2012, p. 486).»

These studies are based on the evidence that:

«The technical skills are associated with pure mathematical manipulations whereas the structural skills are related to the capacity of employing mathematical knowledge for structuring physical situations.

Similarly, [...] students should not only recognize that mathematics is a valuable tool for physics, but also that it can provide the underlying structure of a physical theory (Uhden et al., 2012, p. 493).»

The distinction between the structural role and the technical one arises in teaching when considering specific problems which impose to considerate mathematical modeling processes in physics. The same distinction do not arise facing with typical textbook exercises, as I will show in Chapter 3.

In Figure 2.1 the Uhden-model is represented. This picture highlights the distinction between technical skills, structural abilities and the role of mathematics in the process of modeling.

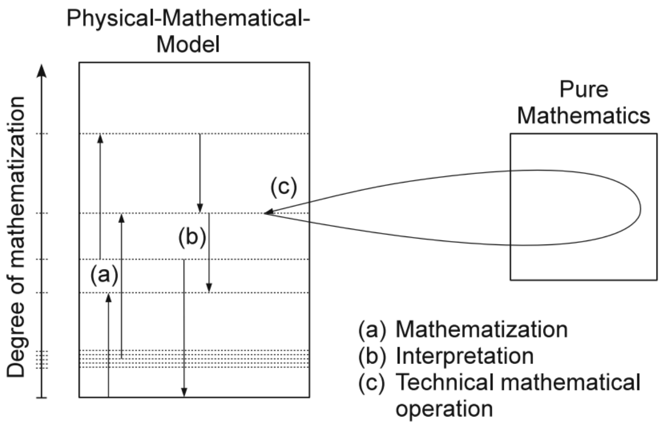

Figure 2.1: Schematic diagram of Uhden model (Uhden et al., 2012, p. 497).

In Figure 2.1, technical skills are represented in the loop at point (c). They do not have any substantial relation with physics contents: they are mare mathematical abilities, «related to the instrumental domain of algorithmic rules (e.g. isolating a variable, operating with fractions, differentiating/integrating a function and solving an equation), to the straightforward consult of a relation in a given list (e.g. differentiation rules, trigonometric identities and moments of inertia) or to the quotation of properties and theorems using arguments of authority (e.g. Pythagoras' or Stokes' theorem and the associative property) (Uhden et al., 2012, p. 498).»

Structural abilities correspond to processes called «mathematization» (a) and «interpretation» (b) and they represent the fundamental intertwining of mathematics and physics. Mathematization concerns the transformation process from a physical situation to a mathematical expression (at different levels), while interpretation «is related to the ability of "reading" equations, stating their meaning with the use of words and schemes, identifying special or limiting cases and making physical predictions from the formalism (Uhden et al., 2012, p. 498).»

Uhden and colleagues developed their model starting from the *modeling cycle* proposed by (Blum and Leib, 2005). This model was revised since it was based on a too clearcut distinction between mathematical model and physical one. Uhden and colleagues, instead, base their model on the claim that the physical-mathematical environment has to stress a fundamental interdisciplinary space where structural skills (mathematization and interpretation) are implied.
Within this mathematical-physical model different levels are present: the "zero" level, where qualitatively physics exists, represents the starting point, i.e. that level which must be reached to pass from the real world, through processes called respectively "idealization" and "validation". Passages among different mathematical internal growing levels are allowed by the structural skills employment (logical-deductive reasoning). When technical evaluations are needed, one go into the mere mathematics environment and, after the formal development, reasoning it supposed to re-enter the interdisciplinary space.

The Uhden model has been used by (Levrini et al., 2017; Branchetti et al., 2018) as a conceptual key to analyze original papers by Max Planck. Their objective was to infer the role of mathematics in the construction and the interpretation of the celebrated energy distribution law of the black body proposed by the German physicist, a milestone of modern physics.
The analyses of this historical case gave rise to two documents dedicated for teachers training courses. The first document was designed for the reconstruction of the Planck reasoning and the second one, a tutorial, for the analyses of the document. The application of the tutorial in three different contexts of preservice and in-service teacher education allowed the researchers to point out a widespread trend among teachers, especially if they have graduated in mathematics, called by researchers, «missing of the big eye»: teachers, as soon as they are asked to complete any mathematical passage made by Planck, they tended to develop very technical and detailed reasoning («little eye strategies») and to lose the entire sense of the modeling process. They noticed a trend to go into technical details and to get trapped in the pure mathematics square (Figure 2.1).
Starting from this evidence, they modified the tutorial, in order to foster the

acknowledgment, within the Planck reasoning, of “big eye strategies” and to foster the development of competences to consciously move back and forth from the detailed reasoning to the overall sense.
Examples of big eye strategies are (Branchetti et al., 2017, p. 16):

- Anticipation - choose a desired target and prefigure the result you would like to get through little eye strategies
- Analogy/Comparison - build a mapping between the faced problem and a problem formulated within another theory
- Placing the problem in a new theoretical background - framing the problem in a theory, in order to use its methods, principles, and results

The analyses of both historical cases and original papers, from the Uhden model point of view, show how “big eye strategies” are needed to emphasize the authentic scientific reasoning, the one that enhances the richness of the interplay between mathematics and physics and which underlines the structural role of mathematics.
However, at school often teaching focuses on the development of “little eye” technical competences. This induces strategies not helpful for problem solving. Both the model of Udhen and colleagues and the approach by Branchetti et al represented an important reference in the analysis of the historical papers of Maxwell and their educational reconstruction.

2.2 The Epistemic Game Model

In a 2007 famous paper, Tuminaro and Redish proposed to researchers an ontological classification for cognitive structures – the *vocabulary* – and a description for the relations among cognitive structures – the *grammar* – in order to describe the way students and experts solve physics problems and use mathematics in physical contexts (Tuminaro and Redish, 2007). This research is based on the cognitive model called “Resource Model”, built on results from neuroscience, cognitive and behavioral science. The model foresees the existence of different fundamental elements which are the base of every cognitive process (*resource*):

- knowledge base elements fixed in long-term memory (*knowledge element*);
- structure in which these elements are connected and associated (*knowledge element*);

- manner in which these structures are activated in different circumstances (*control structure*).

The "compilation" consists in combining between different "knowledge elements" within a "knowledge structure" to obtain a new knowledge element, that, depending on the context, can be a base element or a more complex one.
Learning consists of the modification of the network (*structure*) among different base *elements*. Principal resources (*resource*) are identified in the intuitive mathematical knowledge and in the «phenomenological primitives», i.e. in intuitive cognitive resources, that are intrinsic, irreducible and obtained from dealing with phenomena. From them, the «reasoning primitives» can be recognised, i.e. the everyday experience abstractions coming from generalizing different phenomenological primitives.

According to the "resource model" students are assumed to have a "resources" endowment and the question made by Tuminaro and Redish becomes: how are these resources organized and used by students to solve physics problems ?
Researchers propose to classify students strategies in six *control structures*, called "*epistemic games*". This concept has already been introduced by Collins and Ferguson, which defined it «general purpose strategies for analyzing phenomena in order to fill out a particular epistemic form. Epistemic form are target structures that guide inquiry (Collins and Ferguson, 1993, p. 25).»
This definition was then enriched and re-adapted in order to use this term also to students' behavior in problem solving context. Tuminaro and Redish define them as:

«a coherent activity that uses particular kinds of knowledge and processes associated with that knowledge to create knowledge or solve a problem (Tuminaro and Redish, 2007, p. 4).»

The term "epistemic" indicates that the activity implicates knowledge structure (*resources*) to build new knowledge; the term "game", indeed, refers to the fact that it is a recognizable and coherent activity, endowed, like every game, with *ontological components* (a common knowledge and representative forms) and *structural components* (a beginning and an end, moves and rules). An epistemic game has cognitive resources (both primitives and non-primitives, i.e. concepts, principles and equations) as ontological components and initial state and final state, permitted moving and rules as structural components. Further, it is a coherent activity because for a certain period of time (from few minutes to half an hour) students reason using a limited system of associated resources. However, this coherence does not imply awareness on problem solving: most of students do not choose consciously to play a particular epistemic game. In the following Table 2.1 we report schematically principal characteristic of an epistemic game.

Table 2.1: Principal epistemic game components

Ontological components	
Knowledge base	Set of cognitive resources used for a particular epistemic game
Epistemic form	Final representation that guides the research
Structural components	
Start and finish conditions	Conditions for the beginning and the end of a particular epistemic game, determined also by students expectations on the problem.
Moves	Activities which happen during the game; the different the context, the different the set of moves permitted

Tuminaro and Redish applied their model to analyze problem solving strategies used by university students and by experts. «The students in this study were enrolled in an introductory, algebra-based physics course (Tuminaro and Redish, 2007, p. 8).» From data analyzes they identified six different epistemic games. In the following, we enumerate them in descending order of complexity.

1. *Mapping Meaning to Mathematics*

It represents the most conceptually complex epistemic game. It begins with a conceptual comprehension of the physical situation described in the exercise text; then, it follows a quantitative evaluation. In Figure 2.2 a schematic diagram of principal moves is reported.

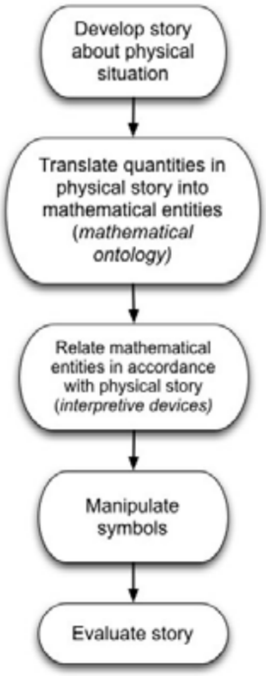

Figure 2.2: Schematic diagram of Mapping Meaning to Mathematics principal moves (Tuminaro and Redish, 2007, p. 6).

The knowledge base for this epistemic game is the whole set of physical and mathematical knowledge: physics fundamental principles, intuitive knowledge of the mathematics needed and intuitive knowledge of reasoning primitives (like "an action cause an effect"). The epistemic form is, generally, the series of mathematical expressions that solvers generate between the second and the third moves. The last move («Evaluate story») represents the moments in which solvers check their quantitative solution.

2. *Mapping Mathematics to Meaning*

The solver develops a conceptual story corresponding to a particular quantitative expression of a physical rule. Ontological components are the same as those of the previously described epistemic game. The difference is the starting point: here, a mathematical expression is the base from which the physical story begins. In Figure 2.3 a schematic diagram of principal moves is reported.

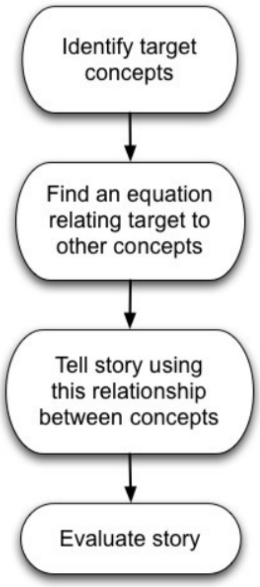

Figure 2.3: Schematic diagram of Mapping Mathematics to Meaning principal moves (Tuminaro and Redish, 2007, p. 6).

3. *Physical Mechanism Game*

The solver builds a coherent physical story, describing the situation read in the exercise text. It is based essentially on her/his intuition on the physical mechanism on which the phenomenon depends. In this epistemic game no explicit reference to a mathematical expression exists, the knowledge base used is only the intuitive one (*primitive*) without the intercession of the formal base. So, the epistemic form here is a mere description of the physical mechanism seen behind the phenomenon: there is a story, but it is impossible to find a real solution, because no thorough expression is used. In Figure 2.4 a schematic diagram of principal moves is reported.

4. *Pictorial Analyses*

The solver creates an external spatial representation to specify relations among various quantities (a free body diagram, a circuit diagram, etc.) The knowledge base comprehends the whole set of previously described resources plus resources of representative translation. The epistemic form is the schematic representation built by the solver. In Figure 2.5 a schematic diagram of principal moves is reported.

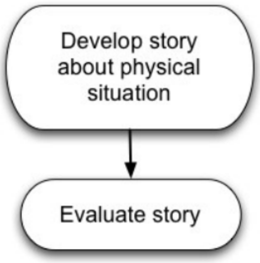

Figure 2.4: Schematic diagram of Physical Mechanism Game principal moves (Tuminaro and Redish, 2007, p. 7).

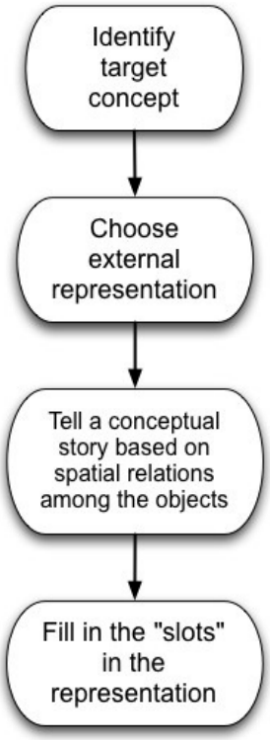

Figure 2.5: Schematic diagram of Pictorial Analyses principal moves (Tuminaro and Redish, 2007, p. 7).

5. *Recursive Plug-and-Chug*

The solver identifies unknown quantities (*target*) and she/he inserts them within some mathematical expressions related to them; the only purpose is to produce a numerical result without any conceptual comprehension of its physical implications. The nature of this epistemic game is recursive: if in the mathematical expression chosen there is another unknown quantity, the solver will look for another expression to evaluate the new unknown variable, until the desired result will come. The knowledge base is the intuitive syntactic comprehension (non conceptual) of physical symbols. Although the involved resources are very different among themselves, the epistemic form of this epistemic game is similar to that already seen for the "Mapping Meaning to Mathematics" and "Mapping Mathematics to Meaning". In Figure 2.6 a schematic diagram of principal moves is reported.

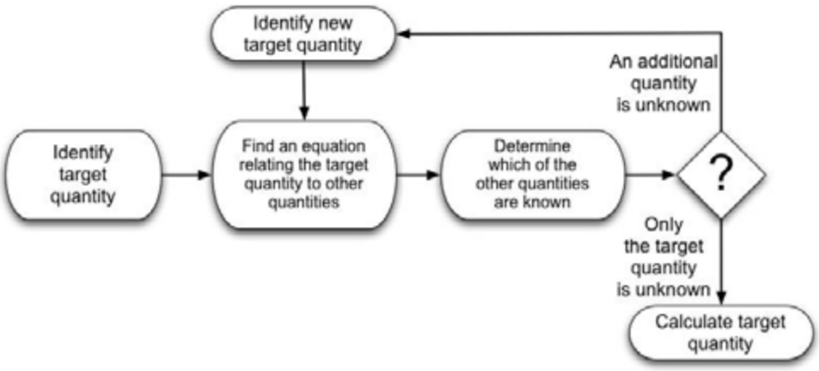

Figure 2.6: Schematic diagram of Recursive Plug-and-Chug principal moves (Tuminaro and Redish, 2007, p. 8).

6. *Transliteration to Mathematics*

The solver refers to examples already studied and solved to develop the solution to the new problem, adapting and translating quantities without developing a true conceptual comprehension. The knowledge base consists in resources associated to the equations syntactic structure. The epistemic form corresponds to the solution model. In Figure 2.7 a schematic diagram of principal moves is reported.

According to the type of the exercise and the solver's attitude for problem solving, the epistemic game activated will be more or less refined. Obviously, this is not an exhaustive list of all possible problem solving

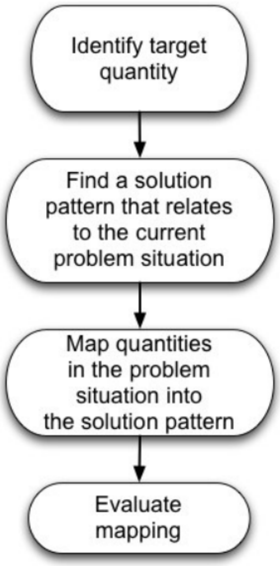

Figure 2.7: Schematic diagram of Recursive Plug-and-Chug principal moves (Tuminaro and Redish, 2007, p. 8).

strategies. A main result of Tuminaro and Redish research was to show that many students tend to activate the "Recursive Plug-and-Chug" and the "Transliteration to Mathematics" epistemic game. They are the epistemic games where mathematics plays a mere technical role (Uhden et al., 2012).

The epistemic game represents together with the Uhden modeling cycle, the principal theoretical reference I used to project teaching and learning activities which I described in the next chapter. I designed problem solving and problem posing activities in order to activate the more refined epistemic game, like "Mapping Meaning to Mathematics" and "Mapping Mathematic to Meaning". These games imply a proper knowledge base, representative form mastery and knowledge of rules and other structural components. However, they imply a proper attitude facing with the exercise: these epistemic-game request to use solver's own primitive resources.

Chapter 3

Epistemic Game: Developing Epistemic and Interdisciplinary Skills through Problem Solving and Problem Posing

We present two empirical studies designed to: i) acquire information on how teachers and university students deal with the relations between phenomenology, models, representation, mathematics in dealing with problem solving and, ii) measure the potential and the effects of specific problem solving and problem posing activities designed to develop awareness about these epistemic aspects. We used epistemic game to design and to implement these problem solving activities and to analyze it. They indeed have potential to develop epistemic and interdisciplinary skills.
In this chapter, we firstly present the two studies we carried out. The first one was carried out within the course of Physics Education, attended by physics, astrophysics and mathematics university students who intend to become secondary school teachers. The second one was carried out within a university course oriented to secondary school teachers of physics and mathematics. In particular we present the activities we designed, the context of their implementation and the results we obtained.
Data have been analyzed in order to inspect the relationship between exercise formulation and participants' way of reasoning and to get tips to build teaching materials aimed to promote epistemic skills (the awareness about the models, forms of representation, mathematical structures used in EM), as well as the conceptual change "from-forces-to-fields".

The first study has been realized within the course in Physics Education

of the Master degree in Physics at the University of Bologna. The course is attended by Physics, Astrophysics and Mathematics master students who are exploring the possibility to become secondary school teachers. 32 students include 15 females and 17 males, and 23 physics students, 5 astrophysics students and 4 mathematics students. They can be considered for the major part rather skill-equipped in physics problem solving because of their experience matured during their student career.
The second study has been realized within a training course oriented to secondary school teachers of physics and mathematics. 20 teachers include 12 females and 8 males. They can be considered skill-equipped in physics problem solving.
I will describe first the activities treated during the the course of Physics Education; the results obtained from data analyses have been used to develop the second course.

3.1 The First Study

3.1.1 The Activities

The activity was articulated in two lessons, as sketched in the time line in Figure 3.1: the first (1A) - from 15:00 to 17:00 of the 14th of May, 2018; the second (2A), divided in an initial discussion on homeworks (2A.1), a teamwork on an analytic grid (2A.2), a teamwork on exercise formulation, after a brief discussion on the analytic grid (2A.3) and a the final discussion – a presentation of their exercises (2A.4) - from 13:00 to 16:00 of the 16th of May, 2018.

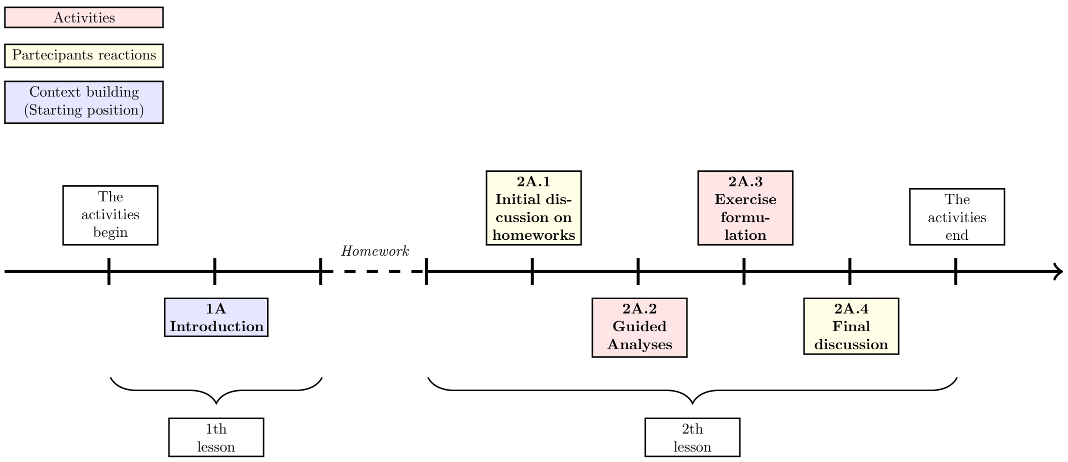

Figure 3.1: Activities timeline of the first empirical study

Introduction (1A)

Objectives:

- introducing the construct of epistemic game in a simple and practical way as a tool for reflecting on problem solving and on the interplay between mathematics a physics;
- to refresh the knowledge related to the exercises that will be considered in the activity (in our case electromagnetic induction) and align the students who can have different background;
- to present the main results in physics education research about the teaching/learning of the topic (in our case, the electromagnetic

induction) and to provide an example of comparative textbooks analysis to refresh the knowledge related to electromagnetic induction (a topic that many students had encountered a couple of years before the activities) and align the students who had a different background;

The introduction to the activities is comprised by two lectures, designed to create the playground for problem solving activities. One lecture aims to introduce the construct of epistemic game and the second to refresh the disciplinary theme that students were supposed to have already studied.
More specifically, the first lecture concerns an overview on epistemic game by Dr. Eleonora Barelli. She introduced the concept of epistemic game, mainly referring to the paper by (Tuminaro and Redish, 2007). Epistemic game are introduced as a theoretical framework elaborated within physics education research that would have played the role to provide the perspective and a common language to deal with the interplay between mathematics and physics in problem solving.
As for the second lecture, electromagnetic induction is refreshed by presenting how different textbooks address the topic. The books are compared and discussed on the basis of the main results achieved in physics education research. The focus of the lecture is the ontological shift "from-forces-to-fields"; in fact, electromagnetic induction is discussed as the quantitative equivalence between two specific field variations: the divergence of the magnetic field and the time-derivatives of the electric field. This equivalence does not describe a cause-effect relationship neither it is a local equivalence. This conceptual knot is presented to the classroom through a frontal lesson, that starts, as already mentioned, with an analytical comparison between two popular physics textbooks, (Amaldi, 2012) and (Romeni, 2012) - in the way they introduce electromagnetic induction. The comparison is carried out so as to make as clearest as possible their similarity and differences. The attention is focused on what particular models, representations, languages and ways of reasoning the two textbooks deal with.
In the lecture, the limits of "think in terms of forces" are discussed in some details. In fact, Lorentz force is usually described as the "ultimate" cause of the Faraday-Neumann effects. As already wrote in Chapter 1, this approach obstacles the shift from Newtonian to Maxwellian paradigm, because field seems to be causally generated by force.
Then, in the lecture, a problem solving situation of a moving spire passing through a uniform magnetic field is considered. Passing through an interdisciplinary reasoning on the interplay between mathematics and physics about the Faraday-Neumann-Lenz rule, we solved the exercise,

independently from any specific reference frame. In this excursus, the most important thing has been the passage from the classic formulation of the Faraday-Neumann-Lenz law

$$emf = -\frac{d\Phi_B}{dt} \tag{3.1}$$

to the more abstract one

$$\oint_{\partial S} \vec{E} \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} + \oint_{\partial S} \left(\vec{v} \times \vec{B}\right) \cdot d\vec{l} \tag{3.2}$$

Thanks to this mathematical abstraction, material stuff like circuits, currents, electromagnetic forces have been changed into more abstract entities such circulations, fields, potentials.
The whole activity requires about 2 hours.

The Initial Discussion (2A.1) and the Guided Analyzes (2A.2)

Objectives:

- to make students acquainted with the epistemic game classification;
- to enable students to use the epistemic game classification to analyze textbooks' exercises and their own resolution;
- to foster an epistemological discussion on the interplay between mathematics and physics in problem solving.

The activity consists of a guided analysis of a physics exercise on electromagnetic induction taken from the very popular secondary textbook (Amaldi, 2012). We have chosen a typical exercise, easy enough to introduce electromagnetic induction's exercises.
The students are asked:

- to analyze the resolution of the exercise, using epistemic game and, after that, to solve the exercise by themselves and to analyze their own resolution, by using epistemic game as meta-cognitive tool (they are asked to accomplish this part of the activity individually, as homework, before);
- to analyze the exercise following an analytic grid that we previously designed; they did it in teamwork.

The text of the exercise is the following:

A 20-turn coil has a cross-sectional area of $4\ cm^2$ and it is connected with a flashlight bulb; the circuit has no battery. If a magnet is repeatedly moving away and closer, the average magnetic field on the coil surface passes from zero to $9.4\ mT$. A boy moves the magnets near and far from the coil 2 times per second. What is the module of the emf induced in the circuit caused by this flux variation?[1]

The analytic grid that we designed to guide the teamwork discussion consists on an organized list of questions. In particular, the questions of the grid are organized in 5 parts:

1. Problem solving strategies – to activate and share reasonings to solve a typical textbook exercise, focusing on the exercise formulation.
2. Contents – to reflect about the physics of the situation, exploring similar scenarios through phenomenological exploration.
3. Representation and modeling – to think about the role of the pictures used to present the situation or to model possible solution strategies.
4. Mathematics-Physics interplay – to discuss about the role of the mathematics in the resolution of a physics exercise.
5. Critical considerations – a meta-reflection about the grid.

The activity requires one hour for homework and about one hour and half of teamwork. The initial discussion has been led by the research team, which let it be a free discussion on the homework and on students' comprehension of epistemic game. Group are named:

Group 1 Il mondo di Sofia

Group 2 Astronuplierra

Group 3 Cane che si morde la coda

Group 4 Il gruppo dei 6

Group 5 Il gruppo di Stefano

Group 6 4 mate e 1 fisi

[1] «Una bobina è composta da 20 spire, ognuna con un'area di $4\ cm^2$, ed è collegata a un circuito che contiene una lampadina (da torcia elettrica), ma nessun generatore. Avvicinando e allontanando una calamita, il campo magnetico medio sulla superficie della bobina passa dal valore zero al valore $9,4\ mT$. Un ragazzo sposta la calamita vicino e poi lontano dalla bobina 2 volte al secondo. Qual è il modulo della forza elettromotrice indotta nel circuito da tale variazione di flusso?»

Exercise Formulation (2A.3)

Objectives:

- to test students' confidence with epistemic game;
- to let students propose an interdisciplinary activity;
- to let students learn to formulate and to write the text of an *open problem*.

This was an activity of *problem posing*. It consists of asking the students to think (in groups) about the exercise formulation previously analyzed and to reformulate it in order to write an *open problem*, that is a problem which can induce "Mapping mathematics to meaning" or "Mapping meaning to mathematics" epistemic game and that it have no precise defined solution. This teamwork takes fifteen minutes. After this, they are asked to present and share the results of their exercise to the whole classroom, by motivating why the new formulation is expected to activate a specific epistemic game. This moment is particularly important for the whole study since it allows to test the confidence with epistemic game reached by the students.
The activity requires about an hour.

The Final Discussion (2A.4)

In the last part of the 2A, researchers and students take some time to wrap up the sense of the whole set of activities and discuss about epistemic game and about they role to activate epistemological reflections on the interplay between mathematics and physics. This discussion is an important moment from a research point of view, since it represents another source of data to check:

- if and to what extent epistemic game are understood by students, and
- if and how these activities can be adapted for secondary school teachers in mathematics and physics.

The activity requires about half an hour.

3.1.2 Data Collection and Methods to Analyze the Activity

Way of collecting data has been:

- audio recording of classroom open debates;
- audio recording of discussions in teamwork;
- students' written answers to the questions of the analytic grids;
- notes from researchers during the activities.

Each audio recording has been entirely transcribed.
Data have been analyzed through a qualitative, phenomenological approach, that is a bottom-up analysis from raw data to their organization and interpretation. Because of the specificity of the activities structure, I can not follow any specific path from an initial students' knowledge state to a final one. In fact, only the final activity could be seen as a comprehension test on their appropriation of the epistemic game description.

Three research questions have been chosen to inspect the collected data:

1. (RQ1) Did the students understand the construct of epistemic game and the specificities of the various epistemic game? In case, what difficulties did they met?

2. (RQ2) Did the activities induce a reflection on problem solving? More specifically, did they induce a reflection on the relationship between the exercise formulation and the epistemic game (the possible ways of resolution of the exercise) it can implicitly induce? If so, what kind of reflection?

3. (RQ3) Did the activities induce a reflection on the mathematics-physics interplay?

After a deep reading of the whole corpus, I identified the following data sources of important information to answer the three research questions (Table 3.1).

3.1.3 Results from the Analyses

Despite some elements of confusion among students, some of them have already demonstrated to have partially understood epistemic game since the initial discussion; in the following I will resume their ideas on epistemic game.

Table 3.1: Data sources

	Data Source	Brief Description
RQ1	A_2A.1	Audio recording of the initial debate 2A.1
	A_2A.2	Audio recording of the teamwork in 2A.2
	A_2A.4	Audio recording of the final debate 2A.4
RQ2	A_2A.1	Audio recording of the initial debate 2A.1
	A_2A.2	Audio recording of the teamwork in 2A.2
	A_2A.4	Audio recording of the final debate 2A.4
	A_IG MDThesis	Audio recording from (Giovannelli, 2017)
RQ3	A_2A.2	Audio recording of the teamwork in 2A.2
	W_2A.3	Written problem posing proposes in 2A.3

RQ1- Students' Comprehension of epistemic game

First, I looked for clues to help my self in measuring if and to what extent students understood epistemic game.
As already said, I found these clues in 2A.1 and 2A.3 especially. I collected evidences and signs useful to point out criticality but also to describe a fruitful process toward a significant comprehension of epistemic game by students.

The Criticality

After 2A.1, they could not distinguish clearly between epistemic game. For instance, a student said that (A_2A.1):

Student: *I did not find, in my approach, a clear distinction between one and the other*...I mean, maybe a bit 'a mix.

(Nel mio approccio *non ho trovato una netta distinzione tra l'utilizzo di uno o l'altro*...cioè, magari un po' un mescolarsi.)

They found the same difficulty distinguishing between "Transliteration to Mathematics" and "Mathematics to Meaning". For instance (A_2A.1):

Student: Well, sometimes it's not that one the formula to use, because it requests the right mathematical transliteration...I mean: *I understood physics behind and the mathematics is simply the language which I need to express it*, so, like in a speech, I choose the right words, I choose the right formula and methods in which I believe etc. etc., so, for me, *Transliteration to Mathematics it should not be snubbed.*

(`Ma molte volte non è quella la formula da usare perché richiede una giusta translitterazione matematica...cioè:` *`io ho compreso la fisica che sta dietro e la matematica è semplicemente il linguaggio che mi serve per esprimerla`*`, per cui, un po' come in un discorso, io scelgo le parole adatte, scelgo le formule adatte e metodi che mi paiono adatti eccetera eccetera, per cui secondo me` *`non va molto snobbato, diciamo, il Transliteration to Mathematics`*.)

Interviewer: [O. Levrini] Well, that is the pattern recognition. It should not be snubbed at all, similes recognition, patterns recognition, analogies recognition, that is an aspect...*the important thing is that recognition is not be done automatically, but consciously* [...]

(`Ma quello è il riconoscimento di pattern. Non va assolutamente snobbato, il riconoscimento di similitudine, il riconoscimento di pattern, il riconoscimento di analogie, questo è un aspetto...`*`l'importante è che non sia fatto in modo automatico ma consapevole`*) [...]

Student: *If I can't skip from reality to the model, what I do is Transliteration to Mathematics.*

(*`Se non riesco a passare dalla realtà al modello, quello che faccio è Transliteration to Mathematics.`*)

In the last extract, the student can not distinguish between Transliteration To Mathematics and Mathematics To Meaning. She seems to know the distinction between them, but her explanation is a bit inaccurate. In fact, as explained by Dr. Olivia Levrini, Transliteration to Mathematics provides a math model, but the use of this model is unintentional.

Students and researchers shared a **common difficulty analyzing text-books' resolutions with epistemic game** (A_2A.1):

Student: *It is very difficult, if you give me resolution written text, return to epistemic game to understand the reasoning behind.*

(*È molto difficile, se mi dai il testo scritto della risoluzione, risalire agli epistemic game, capire che ragionamento ci sta dietro.*)

Without a complete report of the way of reasoning behind the resolution, it is very troublesome to understand the epistemic game acted, because of the presence of many implicit (A_2A.1):

Interviewer: [O. Levrini] *There's a lot of implicit – it is a bad tool in analyzing already done exercises.*

(*C'è molto implicito - non è un buono strumento per l'analisi dei problemi già svolti.*)

Student: [they agree]

Interviewer: [E. Barelli] It was the same problem popped up before: either from any request formulation, nor from its epistemi realization form, I mean: *from [the written solution] is easy understand what resources are implied [within the problem resolution].*

(Era il problema che saltava fuori anche prima: *né dalla formulazione di qualunque richiesta, né dalla sua realizzazione sotto forma epistemica, cioè: dalla [risoluzione scritta] è facile capire quali risorse sono state impiegate [nella risoluzione del problema]*.)

Student: [they agree]

During the free discussion in 2A.1, many students shown to understand the order of the epistemic game with respect their complexity. However, someone thinks that, at the beginning of their Physics career, students reason with Transliteration to Mathematics; throughout their career they possibly acquired more refine epistemic game (A_2A.1):

Student: I mean: the order in which they were presented is the classification depending on the solver experience. *It's obvious that everybody, in the beginning, will approach the problem with the Transliteration.*

(`Nel senso...cioè: l'ordine con il quale sono presentati è l'ordine di esperienza anche della persona che fa esercizi.` *`È ovvio che tutti si approcceranno all'inizio con la Translitterazione.`*)

Audio recording of 2A.4 final discussion shows students better learned the significance of each single epistemic game with respect to 2A.1 initial discussion. For instance, students show to have acquired the significance of "Transliteration to Mathematics". In the following extract, they discuss about the effect of data in the exercise text proposed by the group "Il mondo di Sofia". This group re-formulate the exercise text dividing it in two parts: the first, where the physical situation is presented without data and without any question; the second, a table filled with data followed by the final question; their objective was to activate "Physical Mechanism" epistemic game. However, S1 replied that she could solve the problem looking at the final question only ("identify target quantity and find a solution pattern") and ignoring the initial description of the physical situation (A_2A.4 – Il mondo di Sofia).

Student A: "Imagine you have a coil linked to a circuit with, in addition to the coil itself, only a bulb. You decide to move a magnet near to and far from the coil, repeatedly." This is the text; further, we decided to give the data list. [...] And in the end we asked: "Evaluate the inducted efm module in the circuit."

(`''Si immagini di avere una bobina collegata ad un circuito contenente, oltre alla bobina stessa, solamente una lampadina. Si decide di muovere una calamita avvicinandola e allontanandola alla bobina ripetutamente.'' Questo è il testo, poi abbiamo deciso di dare a parte, come elenco, i dati. [...] E in fondo abbiamo chiesto: ''Si calcoli il modulo della forza elettromotrice indotta nel circuito''.`)

Interviewer: [E. Barelli] How do you list data?

(`Come sono elencati i dati?`)

Student A: [He reads data: 20 turns, etc.] [...]

Interviewer: [O. Levrini] And do you think this should develop a..."Develop story about physical situation, evaluate story,etc."?

(`E secondo voi questo dovrebbe svilupparvi un: ''Develop story about physical situation, evaluate story''?`)

Student A: We think "Yes", because you must, with this text, without data, you don't immediately begin to reason about data, from the mathematical point of view, but you must visualize the situation before, from a point of view of sketch.

(`Dal nostro punto di vista si, perché uno deve, avendo il testo così, senza i dati, non si mette a ragionare subito sui dati, dal punto di vista matematico, ma prima deve visualizzare la situazione, da un punto di vista di disegno.`)

Student B: *I don't know, maybe I would neither read the text, I have red the question, and data, without reading the above text. With data list in this way...I mean: you made me a favor, because you ordered them as I would have done [laughing]. I mean: I see data, don't I? Already written: this equals to that, that equals to this and a-a-ah, then the question. Easy!*

(*`Non lo so, io forse non avrei neanche letto il testo, avrei letto la domanda, letto i dati, senza neanche leggere il testo sopra. Con l'elenco dei dati così...cioè: mi hai fatto un favore, perché me li hai messi come li avrei messi io (risate). Cioè: io vedo i dati, no? Già scritti: questo uguale a quest, questo uguale a e-e-e-eh, poi la domanda. Eh!`*.)

Student C: *You don't read the text above at all.*

(*La parte sopra non la leggi proprio.*)

A student, at a certain point of the discussion, thought aloud on specific terms used by groups formulating the final question. She noted that "Mapping Meaning to Mathematics" doesn't start with identify target quantity, unlike the others (she forget the "Physical mechanism" epistemic game – we will return on this issue). She understood the capital importance of the words used formulating the final question and, indirectly, she understand the initial, fundamental difference between "Mapping Meaning to Mathematics" and the other epistemic game (A_2A.4).

Student: *I want to underline a think which I just noticed: "What happens? The light come on? How do you do? These are all three questions emerged in the first step, that -let's say- more qualitative [of the problem posing activity], those that in epistemic game recall the "Mapping Meaning to Mathematics" because it is the only one that doesn't start with: "Identify a target".* I mean, when the question is: "How does it value? Evaluate this..." I mean: the question, that one, you find...I identify what I need, I begin from the result that I want [...] to obtain from the problem and from there I start with all the other epistemic game. Indeed, the characteristic that [the text proposed] have in common[, those designed to activate], precisely, "Meaning to Mathematics" is to start from the question, that is not: "How much it value?" but it's "What happens? How do you do?"

(*Io volevo sottolineare una cosa che ho notato adesso: "Cosa succede? Si accende o no? Come fai?" Sono tutte e tre le domande che sono emerse nel primo step [dell'attività di problem posing], quello diciamo più qualitativo, che sono quelle che negli epistemic game si rifanno appunto al "Mapping meaning to mathematics" perché è l'unico che non parte come primo step con: "Identifica un target".* Cioè, quando la domanda è: "Quanto vale? Calcola questo..." Cioè: la domanda, quella lì, trovi...identifico cosa devo trovare, parto dal risultato che voglio [...] ottenere dal problema e da lì parto con tutti gli altri epistemic game. Invece la caratteristica che [i testi proposti] hanno tutti in comune[, quelli che hanno voluto attivare], appunto, ‘‘Meaning to mathematics’’ è partire da una domanda che non è: Quanto vale?’’ Ma è: ‘‘Cosa succede? Come fai?’’)

Most of the students understood “Transliteration to Mathematics” and “Mapping Meaning to Mathematics”. However, someone could not distinguish clearly between “Physical Mechanism” and “Mapping Meaning to Mathematics”. They didn’t noticed that the former problem solving has no mathematical form of reasoning, is grounded only on physical relations, while the latter is based on the math-physics interplay (A_2A.4 – Il mondo di Sofia).

Student: We tried to activate the “Physical Mechanism”

(Noi abbiamo provato ad attivare il ["Physical Mechanics"].)

Interviewer: [O. Levrini] There is an implicit, which we have never explained: in this epistemic game data are absents.

(C’è un implicito, che non abbiamo spiegato: in questo epistemic game (Physical mechanism) non ci sono i dati.)

Interviewer: [N. Vernazza] *You do not ask: “Describe what happens”. You ask a quantitative result, a numerical result derived from a formula.*

(*Voi non chiedete: ‘‘Descrivi cosa succede’’. Voi chiedete un dato quantitativo, un risultato numerico derivato da una formula.*)

Another example of this confusion could be found in the group “Astronuplierra” proposal (A_2A.4):

Student A: We do not change, in the sense...I mean: we putted two or three suggestions on, maybe, what to add, actually...For example: we would added a resistance, thus trying. Both to add misleading data [...] not asking, in this moment, the current intensity. So, you add R and aks: "How does the efm module value?" In this manner they already rose some questions. Further, only further, you ask: "*"What physical phenomenon doesn't happen anymore without the resistance?" Without asking how much the current intensity is, buy say: "If I remove R, what does it happen? What is that think, that physic phenomenon that's missing."* So, maybe, they begin to reason on the fact that before [with the resistance removed] I don't say there was nothing, [...] I mean: if I have the resistance I have both the fem and the current. With no resistance but with the circuit closed, I have efm withou current, I mean [...]

(`Noi non l'abbiamo cambiato, nel senso...cioè: abbiamo messo due o tre suggerimenti su, magari, cosa aggiungere, addirittura...Ad esempio: noi avremmo aggiunto un valore di resistenza, così. Sia per aggiungere dati che possano trarre in inganno [...] non chiedendo, in questo momento, l'intensità di corrente. Quindi, gli aggiungi R e gli fai la domanda: "Qual è il modulo della forza elettromotrice?" Così loro già si fanno qualche domanda. Poi, solo successivamente, chiedere:` *"Quale fenomeno fisico non avviene più se venisse tolta la resistenza? Senza chiedere qual è l'intensità di corrente, però dire: "Se io tolgo R, cosa accade? Qual è quella cosa, quel fenomeno fisico che manca?"* `Allora magari si mettono a ragionare sul fatto che prima [se non c'era la resistenza] non è che non c'era la fem, [...] cioè: se ho la resistenza ho sia fem che corrente. Se non ho la resistenza ma il mio circuito è chiuso ho fem senza corrente, cioè.`)
[...]

Student B: It occurs to understand the physical meaning of the problem and to try to restrict it in a formula. *However, our question were all at a phenomological level [...] Qualitatively questions change things...[when some part of the system varies qualitatively.]*

(`Qui c'è da capire il significato fisico del problema e cercare di comprimerlo in una formula.` *`Però le nostre domande erano tutte a livello fenomenologico.[...] Chiedergli qualitativamente che cosa cambia...[quando si varia qualitativamente una parte del sistema.]`*)

Almost nobody has shown to prefer "Pictorial Analyzes" epistemic game. Among the whole classroom, only two students declared, during 2A.1 activity, to use it. Probably, This is due to the fact that they have already seen the textbook drawn.
The first student, after having identified the target object in the problem formulation, chose an external representation (field lines) to visualize the interaction between the field and the coil. The second student represented the situation on time-space cartesian coordinate system (A_2A.2 – Gruppo Stefano).

Student B: No, nono, I don't say that: however, I start from this idea, I mean: *induction, in my mind, is linked to this figure.*

(`No, nono, non dico questo: parto però da quest'idea qui, cioè:` *`l'induzione per me è collegata a questo disegno.`*)

Interviewer: [G. Tasquier] You didn't add pictorial elements, did you. You were reasoning in a different way.

(`Tu non hai aggiunto elementi pittorici, quindi. Tu stavi ragionando in modo diverso.`)

Student B: What?

(`Cioè?`)

Interviewer: [G. Tasquier] Before, when we were talking about circuit elements, you didn't add them in an instrumental way or in a representative way, pictorial...right?

(`Prima, quando si parlava di elementi del circuito, non l'hai aggiunti in maniera strumentale o in maniera rappresentativa, pittorica...giusto?`)

Student B: On paper no, boh? Instead, thinking, yes...I mean, in the sens: *every time that I'm talking about induction, the first think that comes to my mind is this figure, and I try to refer myself to this figure, I mean: this is how it's shaped my platonic idea of induction.*

(`Sulla carta no, boh? Pensando, invece, si...cioè, nel senso:` *`tutte le volte che si parla di induzione la prima cosa che mi viene in mente è questo disegno e cerco di fare riferimento a questo disegno, cioè: l'idea platonica di induzione ha questa forma qui.`*)

Everyone else used the textbook representation, without adding any personal sketch. In fact, the sketch proposed by the textbook is drawn to simplify mathematics, adding further implicit informations to the exercise formulation. As emerged in (A_2A.3)

Interviewer: [O. Levrini] Did different attitudes among you emerge with respect to the problem?

(`Sono emersi tra di voi diversi atteggiamenti nei confronti del problema?`)

Student A: *Someone reasoned more "graphically", that is on the variation of the efm on the cartesian plane; someone else, approaching the problem from the request, the formula, and...he looks for the elements useful to complete the expression and to resolve.*

(*`Qualcuno ragionava in modo più "grafico", cioè su come varia in un grafico cartesiano la fem`*`; qualcuno invece che si approccia partendo da qual'era la richiesta, la formula, e...trova poi gli elementi che gli servono per completare la formula e risolve.`)

Interviewer: [O. Levrini] *Did other elements emerge? Why, for example, pictures are always [simplified]?*

(*`Altri elementi che sono emersi? Perché, ad esempio, i disegni sono sempre [semplificati]?`*)

Student B: *To simplify maths!*

(`Per semplificare la matematica!`)

Another important feature shared by almost all groups in 2A.4 was the difficulty in evaluating story. In fact, some exercises, after listed data, asked if the magnet movement lights on the bulb. Evaluating the situation, magnet can't light on the bulb because of the induced emf low intensity. Nevertheless, they answer the magnet light on the bulb, without any evaluation of the physical situation. In fact, in the exercise formulation the bulb voltage is not specified. For instance (A_2A.3 – Cane che si morde la coda):

Student: "A coil is composed by 20 turns, each one with a 4 centimeters square surface, and it is linked with a circuit containing a bulb which switches on a potential difference equal to one volt. Moving the magnet, the average magnetic field on the coil surface passes from 0 to nine [in 2 seconds]. Does the bulb turn on? What are the reasons for your answer? [...] The first thing we have said was: let's remove the word inducted electromotive force, let's insert at the beginning of the text "potential difference" *and at the end let's ask a phenomenological question: "Does this bulb turn on or does it does not?"*

(``Una bobina è composta da venti spire, ognuna con un'area di 4 centimetri al quadrato ed è collegata ad un circuito che contiene una lampadina che si accende con una differenza di potenziale di un volt. Muovendo una calamita, il campo magnetico medio sulla superficie della bobina passa dal valore 0 al valore nove [in 2 secondi]. La lampadina si accende? Motiva la risposta''. [...] La prima cosa che abbiamo detto è: eliminiamo la parola forza elettromotrice indotta, inseriamo all'inizio del testo ``differenza di potenziale'' *e alla fine facciamo la domanda fenomenologica sul cosa succede: ``Si accenderà questa lampadina o non si accenderà?''*)

The group "Gruppo dei 6" proposed a similar exercise. It is true, as they said, that "without the magnet movement bulb doesn't light on", but this condition is not sufficient to light on the bulb. Maths is used to resolve the exercise, not to evaluate the story (A_2A.4 – Gruppo dei 6).

Student: "A circuit is composed by a bulb and a coil, connected in series; there is a permanent magnet, too. How do yo light the bulb" We too chose [...] to remove data from the text. However, further, to add them, I mean: "If you have n turns [...], the section is given, the magnet generates a determined field, then [we ask to] evaluate the module of the electromotive force". But, I mean, we are obliged to give data...

(``Sono dati un circuito costituito da una lampadina e da una spira collegati in serie ed un magnete permanente. Come fai ad accendere la lampadina?'' Anche noi abbiamo scelto [...] di epurare inizialmente i dati numerici. Poi eventualmente aggiungerli, cioè: ``Se poi c'ho tot spire [...], la sezione è questa, il magnete genera un campo magnetico di un determinato tesla, allora poi dimmi qual è il modulo della forza elettromotrice'' Però, cioè, dobbiamo decidere anche noi di fare quest'operazione...)

Interviewer: [N. Vernazza] emphDo you give data?

(*Ma i dati li mettete oppure no?*)

Student: Meanwhile, they must say...how the bulb turns on...well: further yes, [we should give data]...

(Intanto devono dire se...come si accende...Bè: dopo si, dopo...)

Interviewer: [N. Vernazza] *Well: it's one thing to ask if the bulb turn on, but to answer data are needed.* [...]

(*No, perché allora: un conto se chiedete se si accende oppure no, però a quel punto avete bisogno dei dati.*) [...]

Interviewer: [L. Branchetti] emphTo say that [the bulb] will turn on, I need to motivate my answer...either one has a test circuit, or how he can to [answer?] He needs to pass through mathematics.

(*Per dire che poi [la lampadina] si accende devo motivare la risposta...o uno ha un circuito di prova oppure come fa a...deve passare dalla matematica.*)

Student: *Meanwhile, without any relative motion, nothing happens.*

(*Intanto se non c'è movimento relativo, non succede niente.*)

I didn't find any clue about students comprehension on "Recursive Plug-and-Chug". This epistemic game, known as the most basic one, is not probably a way of reasoning of these university students.
Another change has been happening during the activities in the students' understanding of epistemic game. During the 2A.1 initial discussion, they thought exercises are complicated if new (A2A.1):

Interviewer: [O. Levrini] What are, facing with the problem, things you find out complex? Facing with what kind of elements do you say: "This is a difficult problem to me"?

(Cosa sono quando vi trovate davanti a un problema, le cose che trovate complicate? Quali sono gli elementi che vi fanno dire: ''Questo è un problema difficile per me''?)

Student A: *A new problem.* I mean, in the sense...

(*Un problema nuovo.* Cioè, nel senso...)

Interviewer: [O. Levrini] So, when you can't bring back it to something [known]

(Quindi, quando non si riconduce a qualcosa di [già noto.])

Student A: Yes

(Si.)

Student B: Or emphwhen you must find out the phenomenon from something you have already studied, but which is not explicited.

(Oppure *se il fenomeno lo devi ricavare da qualcosa che hai già studiato, ma che non ce l'hai esplicito.*)

Student C: *In a purely quantitative manner, for me a problem was always complex when the resolution was not expressible with a pair of standard mathematical expression already seen in the textbook [...] An easy problem was for me: "OK, those are the quantities, this is the expression, end of discussion."*

(*In maniera puramente quantitativa, un problema io l'ho visto sempre difficile quando la risoluzione non era esprimibile matematicamente in un paio di formule standard già presenti sul libro.* [...] Un problema facile era per me: "Ok, le quantità son queste, la formula è questa, fine.")

Differently, in 2A.4, in order to activate the most complicated epistemic game, that is, in order to write down an hard exercise, they changed their mind: in fact, nobody proposed a new exercise but they modified exercise formulation. In this sense, a clear definition of "difficult exercise" is naively given by a student during the initial discussion (A_2A.1):

Student: Well, at the end, It seems to me that a problem is complex when it obliges you to think more than you were used to [laughing].

(Cioè, alla fine a me sembra che un problema sia difficile quando ti costringe a pensare più di quanto faresti normalmente (risate).)

The same exercise could activate different epistemic game depending on the student's attitude. For instance, a student – Student D –, speaking about an exercise, said that he could not solve it without to understand the physical situation ("the story"); another student – Student C – confirmed this attitude. Other, two students – Students A and B – answered that they would solved the same exercise using "Transliteration to Mathematics" epistemic game, so without evaluating the story (A_2A.4 – Il mondo di Sofia).

Student A: I don't know, maybe I would neither read the text, I have red the question, and data, without reading the above text. With data list in this way...I mean: you made me a favor, because you ordered them as I would have done [laughing]. I mean: I see data, don't I? Already written: this equals to that, that equals to this and a-a-ah, then the question. Easy!

(`Non lo so, io forse non avrei neanche letto il testo, avrei letto la domanda, letto i dati, senza neanche leggere il testo sopra. Con l'elenco dei dati così...cioè: mi hai fatto un favore, perché me li hai messi come li avrei messi io (risate). Cioè: io vedo i dati, no? Già scritti: questo uguale a quest, questo uguale a e-e-e-eh, poi la domanda. Eh!`)

Student B: *You don't read the text above at all.*

(*La parte sopra non la leggi proprio.*)

Student A: *I don't read that at all.*

(*Non la leggo proprio.*)

Student C: *But you need a situation.*

(*Però ti serve una situazione.*)

Student A: If you ask me: "Evaluate the electromotive force" Well! I have data, the resolutive formula, I write and solve. [...]

(`Se tu mi dici: "Calcola la forza elettromotrice" Va bene! Ho i dati sopra, la formula, scrivo e risolvo.`) [...]

Student D: Sincerely, as we have thought it, but evidently we were wrong, it was: since text is before data...I mean: data, expressed, yes, as a list, but...*I, on the contrary, if I should read data alone, I can't...I can't solve a problem, having data alone, because I can't...I don't understand the situation.*

(`Noi, detto sinceramente, come l'avevam pensato,` `che evidentemente abbiamo sbagliato, era più un: col fatto che sia prima il testo e i dati dati...cioè: e i dati, espressi, si, ad elenco, ma...`*`io, al contrario, se leggessi i dati e basta non mi farei...non saprei fare e risolvere un problema, avendo i dati solo così, perché non riesco...non capisco la situazione.`*)

Eventually, I report an interesting extract. A group reflect on how the exercise is submitted. They argued that **different epistemic game could be activated if the same exercise is written, on the PC or oral** (A_2A.4 – Il mondo di Sofia).

Student A: "A circuit is composed by a bulb and a coil, connected in series; there is a permanent magnet, too. How do yo light the bulb" We too chose [...] to remove data from the text. However, further, to add them, I mean: "If you have n turns [...], the section is given, the magnet generates a determined field, then [we ask to] evaluate the module of the electromotive force". But, I mean, we are obliged to give data...

(`"Sono dati un circuito costituito da una lampadina e da una spira collegati in serie ed un magnete permanente. Come fai ad accendere la lampadina?" Anche noi abbiamo scelto [...] di epurare inizialmente i dati numerici. Poi eventualmente aggiungerli, cioè: "Se poi c'ho tot spire [...], la sezione è questa, il magnete genera un campo magnetico di un determinato tesla, allora poi dimmi qual è il modulo della forza elettromotrice" Però, cioè, dobbiamo decidere anche noi di fare quest'operazione...`)

Interviewer: [N. Vernazza] emphDo you give data?

(`Ma i dati li mettete oppure no?`)

Student A: Meanwhile, they must say...how the bulb turns on...well: further yes, [we should give data]...

(`Intanto devono dire se...come si accende...Bè: dopo si, dopo...`)

Interviewer: [N. Vernazza] *So: it's one thing to ask if the bulb turn on, but to answer data are needed.* [...]

(No, perché allora: un conto se chiedete se si accende oppure no, però a quel punto avete bisogno dei dati. I dati li date oppure no?)

Student A: *If the question should be posed during an oral exam, we would give data later, I mean: I would ask the question before and then...sure, if data are written, they would read them. If the question should pose at the PC, one...thicks the question, then I give him data.*

(*Se fosse un'interrogazione li daremmo dopo, prima...cioè: farei la prima domanda e poi...certo è che se è scritto, dopo li leggono. Se fosse al pc, uno...se spunta la domanda poi dò i dati.*)

Student B: *Exactly, we changed the modality*

(*Esatto, abbiamo cambiato la modalità*)

A question arise: could an "open" exercise activate many more personal way of reasoning then a "closed" one? With "open" I mean an exercise in which the final question don't indicate the target quantity directly.

RQ2 - Relationship Between Exercise Formulation and Induced epistemic game: the Economy Principle

The research question focuses on the relationship existing between epistemic game and exercise formulation. Data from 2A.1, 2A.2 and 2A.4 show students have noticed that **way of reasoning is shortlisted by the exercise formulation**. For instance, a student argued that "to use" a particular epistemic game depends also on the exercise formulation (A_2A.1):

Student: I would say that instead to say "the more complex the way of reasoning the more complex the epistemic game activated", [it seems that] *the activation of determined epistemic game strongly depends from the problem, because...we are used to solve in a simple way simple problem, we have already done* *many problems, we have already seen them, so, maybe, for simple problems [we don't force ourself].* When we can't find the solution for a problem, then complex reasoning mechanisms get started, so we can activate other epistemic game...

`(Volevo dire che più che dire che un modo di ragionare complesso attiva epistemic game complessi, [sembra sia che]` ***`l'attivazione di determinati epistemic game dipende molto dal problema, perché...abbiamo l'abitudine di risolvere in maniera semplice problemi semplici, ne abbiamo già fatti molti, li abbiamo già visti, quindi magari per problemi semplici [non ci sforziamo].`*** `Quando un problema non riusciamo a trovare la soluzione, allora lì magari si mettono in moto meccanismi di ragionamento più complessi, quindi possiamo attivare altri epistemic game...)`

Interviewer: [O. Levrini] There is an economy principle [...]

`(C'è un principio di economia [...])`

Students said that they want to solve the exercise which they are facing with. They want to maximize their results with minimum effort. They look for the easiest way to solve the exercise, removing everything useless to reach their goal. This attitude could be synthesized in the expression **"economy principle". Following this principle, they use the simpler epistemic game they can to solve the exercise**. It is important to underline that students use a particular epistemic game depending on the text but also on their personal attitude (A_2A.1).

Student A: I mean, having already red the textbook solution, when I red the question "How would you do?" I felt a little trapped in a thing that, actually, I have never done in a different way, because it is the way of reasoning they have always taught to me, so...I would done it in the same way! [laughing] *Maybe...it is as he said: for simple problems, you reason for points as textbook, maybe if it would been some points which would have request a knowledge or some more complex reasonings, other epistemic game would came out.*

(Cioè, io, avendo letto prima la soluzione data dal libro, quando poi ho letto la domanda ''Tu come lo faresti?'' io mi sono sentita un po' ingabbiata in una cosa che in realtà non avrei fatto in maniera diversa, perché è il ragionamento che mi è stato sempre insegnato, quindi...l'avrei fatto così! (ride) *Forse...è come diceva lui: nel semplice, si ragiona per punti come ha fatto il libro, magari se ci fossero stati dei punti che avessero richiesto una conoscenza o comunque un ragionamento più intricato, sarebbero venuti fuori altri epistemic game.*)

Student B: I think the "Transliteration to Mathematics", actually, it is an economy principle, so, *why do I need to strain my mind when I [can solve easily the exercise?]* [rethoric question, author's note]

(Secondo me il ''Transliteration to Mathematics'', appunto, è un processo di economia, per cui, *perché devo ragionare tanto quando ho [la possibilità di risolvere l'esercizio facilmente?] [domanda retorica, nda]*)

3.1.4 The Economy Principle e Its Manifestations

The economy principle reveals itself in four manners, as found in 2A.1, 2A.2 and 2A.4.

1. The Cheapest Way to Solve an Exercise Is to Not Consider Useless Physical Circumstances

Facing with the exercise formulation, students didn't make any effort to imagine the physical situation, because they thought it is not helpful to solve the exercise. Although they know that the physical situation is quite complex[2], they know also that useless doubts (for the resolution of the

[2]

Question: Reading the textm what are the physical phenomena involved? (Leggendo il testo del problema, di quali fenomeni fisici si scrive?)

Student A: We must observe all phenomena, also the current transit through

exercise) could drive them away from their goal (to find the efm).
They are quite sure that knowing the whole situation don't help them at all, as it can be seen in the following extract (A_2A.2 – Il mondo di Sofia):

Question: What other phenomena you need to know to solve the problem?

(`Quali altri fenomeni è necessario conoscere per risolvere il problema?`)

Student A: *I think electromagnetism*, everything. Because...to understand what is the verse of the current, I think it's the least...isn't it?

(*`Penso l'elettromagnetismo`*`, qualsiasi. Perché tanto...capire in che verso va la corrente, credo che sia il minimo...o no?`)

Student B: *Actually., we need to know only the expression for the induction.*

(*`In realtà basta sapere la formula dell'induzione.`*)

Student C: *So: to solve that problem, it needs only the expression for the induction.*

(*`Allora: per risolvere questo problema basta veramente solo sapere la formula dell'induzione.`*)

Nonetheless, they know that this way of solving exercises does not improve their knowledge and their understanding of physics.

Question: You explicit, in words, the way of reasoning followed to solve the exercise.

(`Esplicita, solo a parole, il ragionamento condotto per la soluzione del problema.`)

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the circuit... (`Dobbiamo osservare tutti i fenomeni, anche il transito di corrente nel circuito...`)

Student B: All right. (`Infatti.`)

Student: I would solved it exactly like textbook, with all this formula...I mean, just, brutally, because, I think that this kind of problem...I mean, *this is the simplest and less elegant solution, maybe you don't understand nothing, but...you solve it.*

(`Io l'avrei risolto esattamente così, con tutta questa formula...cioè, proprio...brutalmente, perché, secondo me un problema del genere...cioè,` *`questa è la soluzione più semplice e meno elegante, magari puoi anche non aver capito niente, però...lo risolvi.`*)

2. The Cheapest Way to Solve the Exercise Is to Search a Formula in the Final Question or Which Contains Exercise Data

In the text exercise, students find that the final question contains a strong clue of the right formula (the FNL rule) to find in order to solve the exercise (A_2A.2 – Cdsmcm).

Question: What other phenomena you need to know to solve the problem?

(`Quali altri fenomeni è necessario conoscere per risolvere il problema?`)

Student: [When I'm facing with a physics problem] *basically I do in this way: I see the request, I write the resolutive formula, I write just the formula I need to answer to question, I look for data, if I miss one, I'm going to look for it.*

(`[Quando mi trovo a risolvere un problema di fisica]` *`Tendenzialmente faccio così: vedo qual è la richiesta, scrivo proprio la formula che mi serve per ottenere la richiesta, vedo se ho tutti i dati, se mi manca un dato vado a ricercarlo.`*)

Students are used to begin their reasoning from the final question (A_2A.2 – Cdsmcm):

Student A: *I start...from the question...of the problem*

(*`Io parto...dalla domanda...del problema...`*)

Student B: *Me too.*

(*Si, anch'io.*)

Moreover, they argued that the same formula is suggested by exercise data (A_2A.2 – Cane che si morde la coda):

Question: You explicit, with words, your way of reasoning.

(Esplicita, solo a parole, il ragionamento.)

Interviewer: [N. Vernazza] In your opinion, *why does the book put data [in the text so neatly]?*

(Secondo voi *perché il libro mette [così ordinatamente nel testo i] dati?*)

Student: *To recognize the resolutive formula...*

(*Per riconoscere che formula usare...*)

An interesting dialogue in (Giovannelli, 2017) can confirm our result (the same exercise is proposed to secondary school students of a scientific course) (A_IG Mdthesis):

Student A: [While the colleague is browsing the textbook] *Look for that exercise, maybe it is similar to the our. Let's compare texts and look for differences.*

([Mentre il compagno sfoglia il libro] *Guarda quell'esercizio, forse è simile al nostro. Confrontiamo i testi e guardiamo cosa c'è e cosa non c'è.*)

Student B: Mmh, here [on the book] there is the angle but here [in the assigned exercise] it isn't, on the contrary it is equal [...]

(Eh, qua [sul libro] c'è l'angolo ma qua [nell'esercizio assegnato] non c'è, sennò è uguale. [...])

Student A: *But, why is not the angle here?*

(*Ma perché qui allora non ci ha dato l'angolo?.*)

Student B: *You can see that it comes from a theoretical reasoning which we don't understand...*

(`Si vede che viene da un ragionamento teorico che noi non capiamo...`)

Student C: No, you will see we'll find out a formual with angle which we must invert to reach our goal!

(`Ma no, vedrai che ci sarà una formula con l'angolo che noi dobbiamo invertire per avere quello che ci manca!`)

Student B: Come on, let's search thoroughly within the formula!

(`Dai, rovista nelle formule!`)

Student A: In my opinion, it needs another formula to evaluate the flux...

(`Secondo me, serve un'altra formula per calcolare il flusso...`)

3. The Cheapest Way to Solve the Exercise Is to Recognize Some Familiar Elements in the Formulation which Could Recall Known Resolution Patterns

In many parts of the discussions, students affirmed that they have already seen similar exercises a number of times. So, they recognized to have simply reproduced resolution procedures previously acquired.
Moreover, at first sight, in the exercise formulation some elements appear useless, for instance the presence of a circuit or the presence of a bulb. Students instead see a reason for these "presences": they help students getting familiar with the physical situation and recognizing what arguments the exercise is dealing with (A_2A.2 – Oppurg).

Question: Is there useless elements?

(`Ci sono elementi inutili?`)

Student A: *To the calculations level, you're right. Either a bulb or an engine, it doesn't change anything*...I mean...The problem is that I always imagine with the eyes of a student, that is: *I* *need to see something that really...[...] the manifestation of something real.*

(*A livello di calcolo, hai ragione. Che ci sia una lampadina o un propulsore, tanto a te non cambia niente*...cioè...il problema è che io me lo immagino sempre con gli occhi dello studente, cioè: *io ho bisogno di vedere qualcosa che effettivamente...[...] la manifestazione di questa cosa indotta.*)

Student B: *But[...] it doesn't say: "The bulb turns on and off", I mean: it says "There is this bulb". Dot. I mean: it's awfull.*

(*Però [...] non ti dice: "La lampadina si spegne e si accende", cioè: ti dice: "C'è questa lampadina" punto. Cioè: è bruttissimo!*)

Student C: Perfect! Perfect. Student should know it, He must reach this goal alone, but you're right, arent'you?

(Perfetto! Perfetto. Lo dovrebbe saper lo studente, ci dovrebbe arrivar lo studente. Ma hai ragione, eh!?)

In fact, the resistivity of the bulb is an implicit which help students to familiarize with the situation (A_2A.2 – Il mondo di Sofia)

Question: Are there implicit details which are unwritten but that help you in imaginating the situation? If so, what?

(Ci sono dettagli impliciti che non sono scritti nel testo ma che hai immaginato per aiutarti nella rappresentazione? Se si, quali?)

Student: The bulb resistivity.

(La resistività della lampadina.)

A confirmation of this trend can be found in (Giovannelli, 2017) (A_IG Mdthesis):

Interviewer: [I. Giovannelli] Talking about the field, what do you need?

(`Cosa ti interessa del campo?`)

Student A: Its variation.

(`La variazione.`)

Interviewer: [I. Giovannelli] *Are you interested in the source?*

(*`Ti interessa la sorgente?`*)

Student A: *No.*

(*`No.`*)

Student B: *However, it helps to make the problem real.*

(*`Però aiuta a rendere un po' più concreto il problema!`*)

Interviewer: [I. Giovannelli] *So, you can figure out it better...*

(*`Quindi riesci a figurartelo meglio...`*)

Student B: *Exactly. If you have a magnet, it helps to better model the situation.*

(*`Esatto. Avere una calamita ti aiuta a modellizzare meglio.`*)

With "to model" student meant "to sketch out the resolution" of the exercise.

4. The Cheapest Way to Solve the Exercise Is to Have a Picture of the Physical Situation in Order to Simplify the Math Set Up of the Resolution

Most of the students declare to use or to think at pictures to represent the exercise (A_2A.1)

Student: [...] Talking about my way, when I'm facing with a physics problem, *the first thing I do is to make a figure, but I don't* *know either they let me do it a lot of time that it is the first thing's coming in my mind or I would done it independently from...*

`([...] per quanto mi riguarda, quando affronto un problema di fisica` ***`la prima cosa che mi viene da fare è fare il disegno`***`, però non so se è perché me l'hanno fatto fare talmente tante volte che è la prima cosa che mi viene in mente oppure l'avrei fatto indipendentemente da...)`

They say that pictures help them visualizing the situation, removing unnecessary elements (A_2A.4)

Student: [...] Having a similar text, without data, one doesn't start to reason on data immediately, for the mathematical point of view, *but he must visualize the situation, before.*

`([...] uno deve, avendo il testo così, senza i dati, non si mette a ragionare subito sui dati, dal punto di vista matematico,` ***`ma prima deve visualizzare la situazione, da un punto di vista di disegno.`***`)`

Usually, textbooks and teachers encourage this method in order to simplify the math modeling. Many students think that the "model" is the "picture" (A_2A.1 – Cdsmcm):

Student A: Maybe it happens only to me, but: *when it says: "moving near to and moving far away from a magnet" I don't think that magnet goes through turns. it is not obvious.* I can't think it passes nearby, or it moves near perpendicularly but...

`(Non so se accade solamente a me, ma:` ***`quando dice: "allontanando e avvicinando una calamita" io non penso che la calamita passi in mezzo alle spire. Non è scontato.`*** `Io posso pensare che gli passi accanto, o che si avvicini perpendicolarmente ma non...)`

Student B: Yes, it is not specified...

`(Si, non è specificato...)`

The exercise picture contains an additional simplification, probably with the objective to help students to reach their goal faster: the magnet enters perpendicularly into the coil. The only reason is to simplify the mathematics (A_2A.2 - Astronuplierra).

Question: Critical reflection.

`(Riflessione critica.)`

Student A: I mean, if I would had a closed circuit and the magnet inducing on that circuit, I would had to consider as effective area, I think, the circuit's one, and not that of the coil [...]

`(Cioè, se io avessi avuto un circuito chiuso e la calamita che invece induceva su quel circuito avrei dovuto considerare come area efficace, credo, quella del circuito, e non quella del solenoide.)` [...]

Student B: I don't understand your doubt...

`(Io non ho capito il tuo dubbio...)`

Student A: I mean, the fact is: you see, *the figure says clearly that magnets goes within [the coil perpendicularly.]*

(`Cioè, il fatto e che: vedi,` *`il disegno ti dice chiaramente che la calamita viene inserita [perpendicolarmente].`*)

Student B: Yes.

`(Si.)`

Student A: Now, if I insert the magnets, let's say, inside the circuit...

`(Se io la calamita, invece, fosse stata inserita qua dentro il circuito...)`

Student B: Ah, OK!

`(Ah, Ok!)`

Student A: Will Something change? [...]

(...sarebbe cambiato qualcosa? [...])

Interviewer: [O. Levrini] [Beyond these two cases] what does it happen with any else movement? [...] Did The same phenomenon happen?

([Al di là di questi due casi] Cosa succedeva con qualsiasi altro movimento? [...] C'era lo stesso il fenomeno?)

Student A: Yes, sure...

(Si, certo...)

Student C: Yes, sure...the flux exists, the variation...

(Si, certo...Il flusso c'è lo stesso, la variazione...)

Interviewer: [O. Levrini] The flux variation exists...What is the specificity to put it there?

(La variazione di flusso c'è lo stesso...qual è la specificità di averla messa lì?)

Student D: *To let us reason about...how to say...to unpack the problem, maybe, wuth a...I mean: to think to the circuit only as something for current passing through when efm exists, to let it pass 'till the bulb...*

(*Far ragionare...come dire...spacchettare il problema, forse, con un...cioè: pensare al circuito soltanto come qualcosa che fa passare la corrente quando si origina la fem, farla arrivare alla lampadina...*)

Interviewer: [O. Levrini] No, but when youtake the magnet...it moves...along, let's say, parallely ith respect to the center of the solenoid.

(No, ma il fatto di mettere la calamita...che si `muove...lungo, proprio, parallelamente al centro, parallelamente all'asse del solenoide...)`

Student D: So, the fact that within the flux the most important quantity is the scalar product between the type of field and the surface. *Maybe in this way too much reasoning on field geometries are stopped.*

`(Cioè, il fatto che nel flusso la grandezza più importante è il prodotto scalare tra tipo il campo e la superficie.` *`Forse in questo modo si evitano di fare troppi ragionamenti sulle geometrie del campo.`* `)` [...]

Interviewer: [O. Levrini] So, actually, this is a phenomenolgy done for what?

`(E quindi in realtà questo è una fenomenologia fatta per quale motivo?)`

Student C: Because students, maybe, I mean: they don't want to evaluate the cosine.

`(Perché gli studenti magari, cioè: non c'hanno voglia di calcolare il coseno...)`

Interviewer: [O. Levrini] [...] Special situation are creating, not 'cause of a physical motivation, [...] instead, to simplfy an evaluation [...] you want to avoid scalar product, don't you?

`(Si creano delle situazioni speciali, ma non per un motivo fisico, [...] ma proprio per semplificare un conto, [...] vuoi evitare un prodotto scalare, no?)`

Student C: I think you might [...] in the fourth or in fifth classroom they have already done a scalar product.

`(Solo che [...] in quarta o in quinta lo dovrebbero sapere, un prodotto scalare...)`

Student D: Yes, I did these thinks in the fifth school, but I meet it at the University.

(Io ho fatto queste cose in quinta e ho incontrato il prodotto scalare...all'università [...])

Student A: It would be simple to say: "Let's do a simplified exercise."

(*Basterebbe dirlo: "Facciamo il problema semplificato".*)

If not specified (by the text or by a picture) students are obliged to find a simplification in order to do their calculation, as demonstrated in (Giovannelli, 2017) (A_IG MDThesis):

Student A: Let's suppose the magnetic field would be perpendicular to the coil surface.

(*Supponiamo che il campo magnetico sia perpendicolare alla superficie della spira.*)

Student B: *...and that it would be uniform over the whole coil.*

(*...e che sia uniforme su tutta la spira.*)

Student A: *Yes, sure.*

(*Sì, esatto.*)

Student B: In each point of the coil it is constant.

(Costante in ogni punto della spira.)

Student A: Yes, but not over time, the field has the same value for each point.

(Sì, non costante nel tempo, ma il campo ha lo stesso valore in ogni punto.)

RQ3 - The Mathematics-Physics Interplay

The third research question focuses on students' personal feeling about the interplay between Mathematics and Physics. Mathematics is a fundamental aspect of Physics, as we have already discussed in Chapter 2: it is necessary to build models, to give sense to representations, to structure the way of

reasoning. Moreover, it is a language, the «universal language of Nature (G. Galilei)». Pure mathematics steps deal with formula manipulation and their resolution. Everything else, except the experiments, must be considered an interplay between Physics and Mathematics. Physics contains Mathematics and it is an essential element.
I analyze answers to questions 18 and 19 of the grid. I remind that the question 18 asks:

What are the terms that induce you a mathematical reasoning and what are those which induce you a physical reasoning?

It refers to the exercise formulation. The question 19 is:

In the resolution, where did you make a physical reasoning and where a mathematical reasoning?

Each question is quite challenging: in fact, many epistemological problems could arise defining what is pure mathematics and what is pure physics in the resolution of a physics exercise. While a pure mathematics reasoning is the manipulation of a math formula – and it could be done without thinking at physics – it is very difficult to identify what precisely is "pure physics", because physics always needs mathematics: for instance, numbers are fundamental in laboratory; models and representations are necessary in theoretical physics; each physical entities has a math correspondent symbol, and so on. Physics is a particular boundary for mathematics; physics determines a particular path from generalized math concepts to things happening in a laboratory. Without numbers, relationship, models, specific symbolic language, physics simply doesn't exist.
However, we need to define what could be considered a pure physical reasoning and what a math one to inspect students' answers. Pure math reasoning is to consider only quantitative relationships among math entities and to manipulate these relationships observing all rules of mathematics. Pure physical reasoning is to consider only qualitative relationships among physical entities. Any sort of manipulation which include quantitative or semi-quantitative reasoning could not be considered as pure physical. That is, solving exercises, pure physical reasoning is the path from a physical situation to a particular formula. There is another pure physical reasoning, which, for many practical reasons, almost nobody teaches at school: to derive new entities only from physical quantities considerations.
Data show **students do not consider Mathematics-Physics interplay: they try to divide everything in something physical or something**

math. However, there is no general consensus on what is Physics and what is Mathematics. As an example, I report a moment of "Il mondo di Sofia" discussion, in which a student explicitly speaks about "switch off" the Physicians and "switch on" the Mathematicians, as their reasoning would be separated in two different airlocks (A_2A.2 – Il mondo di Sofia)

Student: At the beginning we made a physical reasoning, then we turned off the physicist in us, we turned on the mathematicians, we said: "never mind physics, we need a number", but later, at the end, the physics returns to say: "the result is correct".

(`All'inizio abbiamo fatto un ragionamento fisico, poi dopo abbiamo spento il fisico che è in noi, abbiamo acceso il matematico, abbiamo detto ''chissenefrega della fisica, c'è da tirar fuori un numero'', però dopo alla fine ritorna il fisico e dice: ''è giusto il risultato''.`)

Other groups discuss about Physics and Mathematics as two separate airlocks too. This result can be found, a part of "Il mondo di Sofia", in the discussion of almost all groups. For the group "Astronuplierra" every term in the exercise formulation activates a physical reasoning (A_2A.2 - Astronuplierra)

Student A: I would say all physics.

(`Io direi tutto fisico.`)

Student B: Me too!

(`Anche io!`)

However, they seem to consider everything as being part of the physical dominion. For instance, they don't consider Algebra as a branch of Mathematics:

Student A: I would say, at the beginning, to understand what it happens, a physical reasoning. Then, you apply formula...

(`Io direi, all'inizio, per capire quello che succede, un ragionamento fisico. Poi, quando applichi le formule...`)

Table 3.2: What are the terms that induce you a mathematical reasoning and what are those which induce you a physical reasoning?

	Mathematics						Physics					
	G1	G2	G3	G4	G5	G6	G1	G2	G3	G4	G5	G6
«Module»	■		■			■		■				
"Turns number"	■		■	■				■				
«Area»	■		■	■				■				
«Average magnetic field»				■				■				
«Variation»					■			■				
"Numbers"						■		■				
"Magnets motion"							■	■	■	■		
«Flux variation»							■	■	■	■		
«Coil»								■			■	
«efm»								■		■	■	■
«Magnet»								■			■	

Student B: Pure mathematics no...unless we would consider the arithmetic as pure mathematics.

(`puramente matematico no...a meno che non vogliamo considerare l'aritmetica matematica pura.`)

Answers to the question 18 could be organized in the Table 3.2:

Answers are different but there is no superposition between mathematics and physics (a part Group2, for which "everything is physics").
Answers to the question 19 could be organized in the Table 3.3:

Answers are different and there is a superposition between what they consider "math reasoning" and what "physical reasoning". Group5 don't agree

Table 3.3: In the resolution, where did you make a physical reasoning and where a mathematical reasoning?

	Mathematics						Physics					
	G1	G2	G3	G4	G5	G6	G1	G2	G3	G4	G5	G6
To write the resolution formula	■		■		■	■		■		■		
To substitute data	■		■			■		■		■	■	
To manipulate mathematical expression	■				■			■	■			
To manipulate units of measurement			■					■				■
To link the situation to FNL rule							■	■	■	■		■

when exactly physics leave place to mathematics. For Student A "writing the formula" is a math reasoning, for Student B is a physical reasoning.

Student A: Well, I thought as you...mathematically the flux variation of the magnetic field over time. I mean, you see it as $d\Phi/dt$, I mean: that is mathematics. I mean, you link to that physical concept...

`(Ma, anch'io ho pensato come te...matematicamente variazione di flusso del campo magnetico rispetto al tempo. Cioè, tu la vedi come de phi su de t, cioè: quella è matematica. Cioè, tu associ a quel concetto fisico...)`

Student B: Here, in my opinion, there are no terms inducing a mathematical reasoning. At least as I understand it [...]

`(Qui, per me, non ci sono dei termini che inducono un ragionamento matematico. Almeno, non come io intendo [...])`

After that, Student B reaffirmed his opinion

Student B: Also because the formula "divergence of B equals 0" recalls in my min [...] closed field lines, I have in mind that monopoles don't exist...I mean: I feel it as physics, I don't see it as mathematics. [...]

`(Anche perché alla formula divergenza di B uguale zero mi viene in mente [...] delle linee di campo che sono chiuse, ho in mente che non ci sono i monopoli...cioè: leggo già fisica, non la vedo come matematica [...])`

Interviewer: [G. Tasquier] It's a relation that you see among [physical] concepts.

`(È una relazione che tu vedi tra concetti [fisici].)`

Student B: Yes. Yes. [...] The formula, it is physics, isn't it?

`( Si. Si. [...] È un senso fisico, proprio la formula, no?)`

During the discussion, some student changed his/her mind in relation to what is Mathematics and what is Physics. This another clear clue of the confusion on the interpretation of Mathematics-Physics interplay. For instance, some students in Group1 changed their mind on the nature of "flux variation", which changed its significance from "math property" to "physics phenomenon":

Student: Now we say [that the flux variation is a] "physical phenomenon", prior it wasn't a physical phenomenon! We have said that it was not a physical phenomenon! [...] Half an hour ago, while I was saying it was a physical phenomenon you said: "No, it is a property...a mathematical property."

```
(Adesso diciamo [che la variazione di flusso è
un] ‘‘fenomeno fisico’’, prima non era un fenomeno
fisico!  Prima avevamo detto che non era un
fenomeno fisico!  [...]  Mezz’ora fa dicevo che
era un fenomeno fisico e voi:  ‘‘Ma no, è una
proprietà...una proprietà matematica.’’)
```

3.2 The second study

3.2.1 The activities

The second empirical study has been carried out within an articulated course for in service teacher education. Here we refer, in particular, to an activity realized the second day of the course, the 23th of October, 2019, from 15:00 to 18:00 (1B); it was divided in four parts, as sketched in the time line in Figure 3.2. The only difference with the first proposal was the text of the exercise and the relative guided analyzes. The construct of epistemic game has been previously introduced to the audience by Dr. Eleonora Barelli.

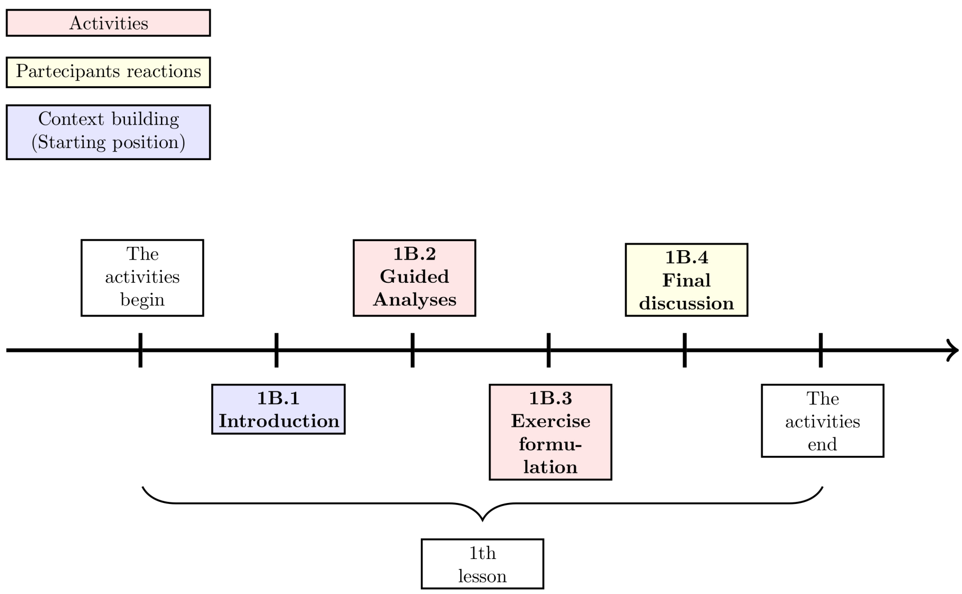


Figure 3.2: Activities timeline of the second empirical study

Introduction (1B.1)

Objectives:

- to refresh the knowledge related to the exercises that will be considered in the activity (in our case electromagnetic induction);
- to present the main results in physics education research about the teaching/learning of the topic (in our case, the electromagnetic induction) and to provide an example of comparative textbooks analysis.

Material was like that proposed to students in the first activity. See (1A).

The Guided Analyses 1B.2

Objectives:

- to enable teachers to use the epistemic game classification to analyze textbooks' exercises and their own resolution;
- to foster an epistemological discussion on the interplay between mathematics and physics in problem solving and on the types of models and representations involved in electromagnetism.

Like the first study, we proposed to teachers a guided analysis of a physics exercise on electromagnetic induction taken from the very popular secondary textbook (Romeni, 2012). The exercise is (like 2A.2) an entry-level exercise. Teacher are asked:

- to analyze the resolution of the exercise, using epistemic game and, after that, to solve the exercise by themselves and to analyze ones' own resolution, by using epistemic game as meta-cognitive tool (they are asked to accomplish this part of the activity individually, as homework, before);
- to analyze the exercise following an analytic grid that we previously designed; they did it in teamwork.

The text of the exercise is the following:

A coil is composed by 20 square turns, each one $l = 15$ cm side. The wire is very thin and curled unto itself. This coil is moved close to a large magnet ($L = 50$ cm), generating a $B = 0.12$ T magnetic field. The total resistance of the coil is $R = 5,0\ \Omega$; a 20 W bulb is linked to the coil, which is moving with a constant velocity of $v = 0,25\ m/s$. Find the electromotive force induced in the circuit [$fem_{max} = 0,45\ V$][3].

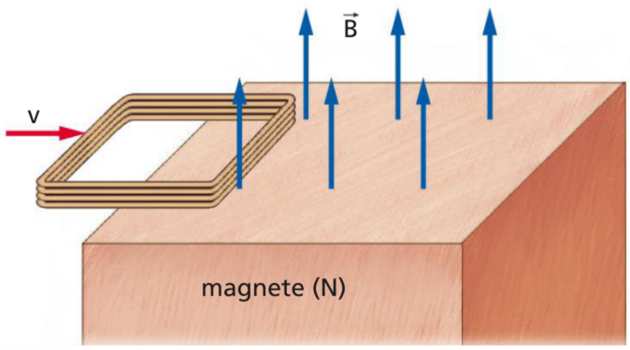


Figure 3.3: Exercise revisited from (Romeni, 2012).

The analytic grid that I designed to guide the teamwork discussion consists on an organized list of questions and it is printed in Appendix A. In particular, the questions of the grid are organized in 4 parts, differently from the first study grid (allegato). In this second study I focused more explicitly on the text of the problem and on the strategies on problem posing:

1. Problem solving strategies – to activate and share reasonings to solve a typical textbook exercise.
2. Text – to analyze the exercise formulation, its implicit elements and the way it possibly induces certain way of reasoning.
3. Contents – to reflect about the physics of the situation, exploring similar scenarios through phenomenological exploration.
4. Representation and modeling – to think about the role of the pictures used to present the situation or to model possible solution strategies.

[3] «Un avvolgimento è formato da 20 spire quadrate di lato $l = 15\ cm$ di filo molto sottile ed è chiuso su se stesso. Questo avvolgimento è fatto passare radente a un magnete largo $L = 50\ cm$ che genera un campo $B = 0.12\ T$. L'avvolgimento ha una resistenza complessiva di $R = 5,0\ \Omega$ ed è collegato ad una lampadina da 20 W. L'avvolgimento è spinto con velocità costante $v = 0,25\ m/s$. Determina la forza elettromotrice indotta nel circuito. [$fem_{\max} = 0,45\ V$]»

5. Mathematics-Physics interplay – to discussion about the role of the mathematics in the resolution of a physics exercise, about the representation of the situation, about the model used to solve the exercise.

Group are named:

Group 1 519

Group 2 520

Group 3 521

Group 4 522

Group 5 NOREC

The group NOREC preferred to not record the session.

Exercise Formulation (1B.3)

Objectives:

- to test teachers' confidence with epistemic game;
- to let teachers propose an interdisciplinary activity;
- to let teachers engage in formulating and writing the text of an *open problem.*

This was an activity of *problem posing.* It consists of asking the teachers to think (in groups) about the exercise formulation previously analyzed and to reformulate it in order to write an *open problem*, that is a problem which can induce "Mapping mathematics to meaning" or "Mapping meaning to mathematics" epistemic game and that it have no precise defined solution. Proposals were collected but no public lecture has been given.
The activity requires about a half an hour.

The Final Discussion (1B.4)

After the activity 1B.2, a collective discussion was promoted on the guided analyses. Teachers claimed to be very interested in questions about the interplay between mathematics and physics.

3.2.2 Data Collection and Methods to Analyze the Activity

Ways of collecting data have been:

- audio recording of group open debates;
- audio recording of discussions in teamwork;
- teachers' written answers to the questions of the analytic grids;
- notes from researchers during the activities.

Each audio recording has been entirely transcribed.
Data have been analyzed through a qualitative, phenomenological approach, that is a bottom-up analysis from raw data to their organization and interpretation.

Two research questions have been chosen to inspect the collected data:

1. (RQ1) Did the activities confirm the economy principle? Did they induce a reflection on problem solving?
2. (RQ2) Did the activities foster the debate on the mathematics-physics interplay?

After a deep reading of the whole corpus, I identified the following data sources of important information to answer the two research questions (Table 3.4.

3.2.3 Results from the Analyses

Teachers appear more aware than university students of the first study of usual students way of reasoning in problem solving. It seems they know students apply the economy principle.

Teacher: Can I say something? Best students don't loose their time in reasoning...they solve the exercise going right to the solution.

`(Posso dire una cosa? Quelli più bravi, gli studenti più bravi, non stanno a ragionare...loro risolvono e vanno dritti alla soluzione.)`
(A_1B.2-519).

Table 3.4: Data sources

	Data Source	Brief Description
RQ1	A_1B.1	Audio recording of the initial debate 1B.1
	A_1B.2	Audio recording of the teamwork in 1B.2
	A_1B.4	Audio recording of the final debate 1B.4
RQ2	A_1B.1	Audio recording of the initial debate 1B.1
	A_1B.2	Audio recording of the teamwork in 1B.2
	A_1B.4	Audio recording of the final debate 1B.4
	W_2B.3	Written problem posing proposes in 1B.3

Despite this, both from single written answers and from their written resolutions, I clearly found in three groups out of five that their resolutions belonged to "Transliteration to Mathematics".
Everybody agreed in saying that the exercise was simply and entry-level one. Some group noticed that the exercise is more abstract than real, as a result of the many simplifications.

Question: Have you evluated/discussed the result?

`(Avete valutato/discusso il risultato?)`

Teacher A: No

Teacher B: It isn't a real situation, so you can say: have the situation a physical meaning? The situation has been purged [...] and the number, maybe...you don't have all the elements to contextualise it within a physical situation...

`(Non è una situazione concreta, per cui ti viene da dire: la situazione ha un senso fisico? È una situazione così epurata [...] che il numero, forse...non hai gli elementi per contestualizzarlo in una situazione fisica...)`
(A_1B.2 – 521)

Only one group demonstrated (quantitatively) that the bulb does not turn on. The same group – group 522 – is the only one to write that data were useful only to «find a follow up.»

Even though the exercise was judged very simple, the grid allowed to point out a significant number of interesting physical and mathematical aspects from the situation described in the text. Structured discussions on the physical situation of the exercise emerged, both from the written answers and from the recorded discussions. Different critical aspects of the physics of the electromagnetic induction arose: side effects, self-induction, symmetric situations and variations to the text became arguments of rich debates[4]. For example, the group 519 discussed about the uniform linear motion of the coil, arguing that the velocity should be constant if there is a force equal to the magnetic one. In Question 3.4, "Something would change if the coil is substitued by a metal plate?", groups 520 and 522 answered that, in this case, plate should be decelerated by eddy currents.
Every group answered that nothing would change if the reference system were another one (the magnets in movement and the coil stationary.) Group 522 discussed if the form of the mathematical expressions would change in another reference system.

Teacher A: No, for the Einstein [...] relativity, no, but actually you wouldn't use the same equations.

(`No, per la relatività [...] di Einstein, no, però di fatto non useresti le stesse equazioni.`)

Teacher B: Exactly, the physical phenomenon behind it is...it is another one, isn't it?

(`Esatto, il fenomeno fisico che c'è dietro è...è un altro, no?`)

Teacher C: Actually, no, because there is the relative motion.

(`In realtà, no, perchè c'è il moto relativo.`)

Teacher B: But...I think you would write the equation in the same manner. Because it depends on the reference system that you chose [...]

(`Però...non so se scriveresti le equazioni nello`

[4]This result is confirmed in the first exploratory study and in (Giovannelli, 2017).

`stesso modo. Perchè dipende dal sistema di riferimento che scegli)` [...]

Teacher A: Luckily it is slow, so you can not consider the relativistic effects.

`(Per fortuna che va lento, per cui trascuri gli effetti relativistici.)`

It is possible to notice that debate evolves, touching some very important physical question.

Data show lively debates among teachers about what is "mathematics" and what is "physics". The group 520 said that flux «is both a physical and a mathematical concept». Group 519, on the contrary, believed that «everything is physics». Groups 521 and NOREC tried to separate mathematics from physics, often in a diametrically opposed manner. Group 522 is the only one which did not conceive a net distinction between mathematics and physics. From (A_1B.2 - 522)

Question: What are the terms in the text of the exercise that induced a physical reasoning? What are the terms in the text of the exercise that induced a mathematical reasoning?

`(Quali termini nella formulazione dell'esercizio hanno indotto un ragionamento fisico? Quali termini hanno indotto un ragionamento matematico?)`

Teacher: [written] It's not easy to separate the two aspects.

`(È difficile separare i due aspetti.)`

Teacher A: It's hard to divide something into mathematics and physics [...]

`(Separare matematica e fisica è molto complicato.)` [...]

Teacher B: It's hard to say where the physical reasoning ends and the mathematical one begins.

`(Si fa fatica a capire dove finisce il` `ragionamento fisico e dove inizia il ragionamento matematico.)`

Teacher A: I have studied physics, and I find hard to think the flux as a mathematical object, I see it as a physical object [...]

`(Io che ho fatto fisica, fatico a pensare al flusso come un oggetto matematico, lo vedo come un oggetto fisico.)` [...]

Teacher C: I have studied differential geometry, I see the flux as a mathematical object, a derivative [...]

`(Io che invece ho fatto geometria differenziale, vedo il flusso come un oggetto matematico, una derivata.)` [...]

Teacher A: You can't say that a model is pure mathematics or pure physics.

`(Non puoi dire che un modello è solo matematico o solo fisico.)`

Generally, they showed a numbers of different opinions on what is mathematics and what is physics. They all agreed interpreting the Figure A.1 as a mathematical model, useful to simplify mathematics. A teacher, for instance, said that «there was an hidden [scalar] product» "behind" the figure.

In the following, I show how teachers discussed to what extent the economy principle is stimulated by the formulation of the problem and how they are able to recognize its four manifestations.

1. The Cheapest Way to Solve an Exercise Is to Not Consider Useless Physical Circumstances

Almost every group began to solve the exercise without thinking at the useless physical circumstances for the resolution. Later, facing with questions in the analytic grid, they went deep into the physical situation.
For instance, in Figure 3.4, I report the resolution of the group NOREC.

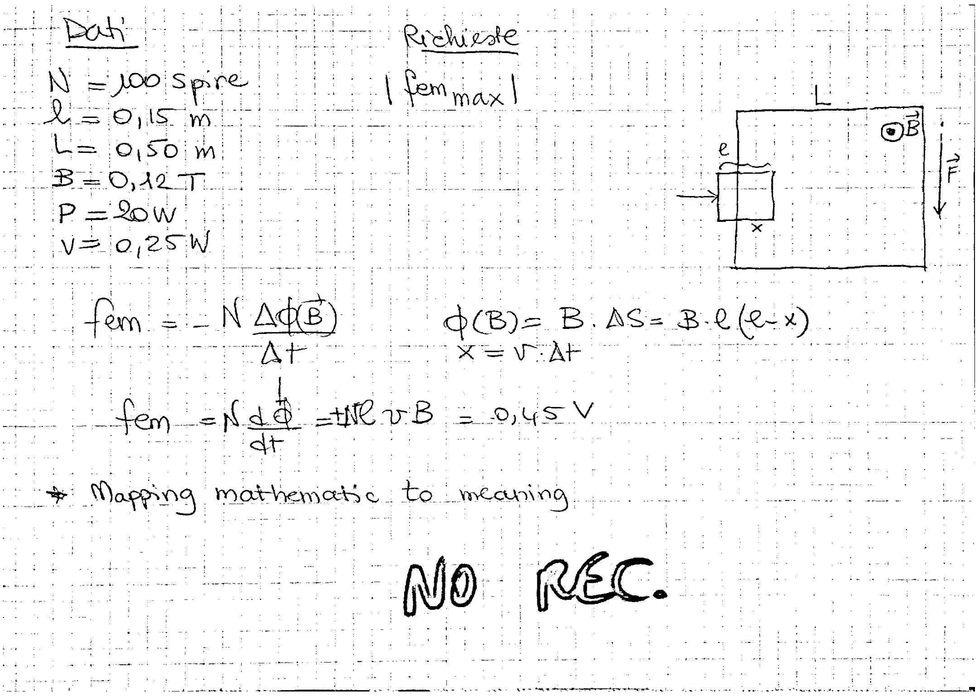


Figure 3.4: Resolution of the group NOREC

2. The Cheapest Way to Solve the Exercise Is to Search a Formula in the Final Question or Which Contains Exercise Data

Teachers of the group 519 began their resolution directly from the expression $emf = vBl$.

Question: Explicit the reasoning that guide you in solving the exercise, inferring the epistemic game used.

`(Esplicitate il ragionamento che vi ha condotto alla risoluzione dell'esercizio, deducendo quale/i EG avete applicato.)`

Teacher A: What was our reasoning? Nothing! [laughing] We recognized the context and we found the corresponding formula. Something more?

`(Che ragionamento abbiam fatto? Nessuno [risate] Abbiamo riconosciuto il contesto e abbiamo trovato la formula corrispondente. Qualcosa di più?)`

Teacher B: No
(A_1B.2 - 519)

The first thing group 520 did was «to remember the [FNL] law (A_1B.2 - 520).»
They observed data sequence helps in searching the resolutive expression, together with the final question.
Group 521 observed that the final question gives a strong hint for the exercise resolution, more than the data set.

3. The Cheapest Way to Solve the Exercise Is to Recognize Some Familiar Elements in the Formulation which Could Recall Known Resolution Patterns

The third manifestation means strictly that the solver applies Transliteration to mathematics EG. Many groups recognize that their resolutions (and the resolution of an "average" student) follow the Transliteration to mathematics structure. For instance:

Question: Explicit the reasoning that guide you in solving the exercise, inferring the epistemic game used.

(Esplicitate il ragionamento che vi ha condotto alla risoluzione dell'esercizio, deducendo quale/i EG avete applicato.)

Teacher: [written] Solution obtained though transliteration to mathematics.

(Soluzione ottenuta tramite transliteration to mathematics.)
(W_1B.2 – NOREC)

To the question "What are the words that induced you to remember similar exercises?" group 520 answered "coil, induced emf, turns, magnet, field, velocity" «there is a stream of words!». They wrote the text induces to reproduce resolution procedures already seen. From their discussion:

Question: What are the words that induced you to remember similar exercises?

(Quali parole vi inducono a ricordare esercizi simili?)

Teacher: They're words helpful to frame the problem just remembering others [...] Specific words makes you thinking at a specific type of exercise. You remember the expression...you have only to understand what you need. [referring to data, author's note]

(`Son parole che ti servono per inquadrare il problema proprio ricordandone altri [...] Determinate parole ti portano a pensare ad un determinato tipo di esercizio. La formula te la ricordi...devi solo vedere cosa ti serve.`)
[riferendosi ai dati, nda]
(A_1B.2 – 520)

To the question "Does the text induce to reproduce resolution pattern already done? Or does it induce to reproduce reasoning already made? the group 519 wrote on the grid "Yes"

4. The Cheapest Way to Solve the Exercise Is to Have a Picture of the Physical Situation in Order to Simplify the Math Set Up of the Resolution

Teachers of the group 519 specified that the figure simplified the mathematics of the problem.
The group 520 claimed that figure helps in simplifying mathematics. From their discussion:

Question: Are there in the picture additional hints with respect to the text?

(`Nel disegno ci sono indicazioni ulteriori rispetto al testo dell'esercizio?`)

Teacher: [written] Direction and verse of vectors and homogeneity, $\cos\alpha = 1$, relative dimension coil and magnet.

(`Direzione e verso dei vettori e uniformità,` $\cos\alpha = 1$`, dimensioni relative spira e magnete.`)

To the question "What are the words that induced you to remember similar exercises?" the group 519 answered "constant velocity" and "graphical representation («`il disegno che ci hanno dato`»)".

The group 521 answered “coil, field, is moved closed”; they wrote the text induces to reproduce resolution procedures already seen.
The group 522 answered “coil, magnetic field, moving, efm”. They claimed that useless elements can not help to identify the resolution.

The Mathematics-Physics Interplay

Answers to the question 18 could be organized in the Table 3.5:

Table 3.5: What are the terms that induce you a mathematical reasoning and what are those which induce you a physical reasoning?

	Mathematics					Physics				
	519	520	521	522	NR	519	520	521	522	NR
«Flux»	■			■		■			■	
«Flux variation»				■		■			■	
«Square»		■		■	■	■			■	
«Close»		■		■		■			■	
«Velocity»				■		■	■		■	■
«Turns»				■		■	■		■	■
«Turns number»				■	■	■			■	
«is moving with a constant v»			■	■		■			■	
«efm»			■	■		■			■	
«Magnet»				■		■			■	■
«Field»				■		■			■	■

3.3 Conclusions

From data collected from the two empirical studies described above, I deduced a main attitude which drives resolution strategies in problem solving. We called this attitude the "economy principle". This principle manifests itself in four practical ways of reasoning, enumerated in the previous sections. Each single exercise can be a way to reflect on the physics behind it, but also an incentive to improve mathematics-physics interdisciplinary skills; in principle, each single exercise can induce the most sophisticated epistemic game. However, since the aim of a resolution is to solve the exercise and not to understand the physical situation, most of the time students are looking for the *cheapest* way to obtain a numerical result with the least amount of effort.
The analytic grid proposed and discussed above is a great tool to induce a deep reasoning on four main aspects: the physical situation which arises from the text of the exercise, the episemic aspects involved in a disciplinary area (types of models, representations, language, forms of reasoning...), the interplay between physics and mathematics, the alternative ways to solve the same exercise.
Furthermore, it can be considered, from teachers point of view, a great tool to inspect students' attitude in problem solving and their level of comprehension.
Moreover, it is a possible tool to switch on the light over particular aspects of the model. For instance, (A_1B2.520):

Question: Have you overlooked the self-induction phenomenon? Why?

`(Avete trascurato il fenomeno dell'autoinduzione? Perché?)`

Teacher A: Yes...we didn't think about it!

`(Si...non ci abbiamo pensato!)`

Teacher B: It's true. I just thought about it right now, only thanks to this question. [laughing]

`(È vero. Ci ho pensato ora, solo perché ce lo hanno chiesto. [risate])`

Teacher C: Retrospectively, we saw the self-induction don't affect the efm.

```
(A posteriori, abbiamo valutato che
l'autoinduzione non incide sulla fem.)
```

An important aspect that arose from data analyses is the main importance that the exercise text has in problem solving. 3 out of 4 manifestations of the economy principle depends on the relation between the text and the "solvers". Activities 2A.3, 2A.4 and 1B.3, 1B.4 have been designed to give the opportunity to think about possible new ways to present the same physical situation of the exercise analyzed. Students and teachers have worked in teamwork on this problem posing interdisciplinary activity and then they have red their proposals in public.
These activities have been designed to broke individualist and automatic actions which cause the economy principle. The task now is to re-write the text in order to broke with the economy principle. Participants already know the solution of the exercise, so they can focus their attention on the text, recognizing terms, structures, implicit expressions which can be linked to the 4 manifestations of the economy principle. They are asked to understand the whole physical situation, in order to re-build the same system in a different manner, able to activate the most complicated epistemic game.
Thanks to these discussions, thinking at data analyses previously done, I propose a list of actions which can transform any exercise in physics in an *open problem*, that is a problem which can induce "Mapping mathematics to meaning" or "Mapping meaning to mathematics" epistemic game and that it have no precise defined solution.

1. To remove the figure.
2. To remove *building terms* – words which describe or activate a physical element.
3. To remove *evocative terms* – words which recall physical properties of a physical element.
4. To remove conceptual/mathematical simplification.
5. To present data outside the text (another sheet, a table, an Internet site, etc.)
6. To remove the target from the final question/To remove the final question at all.

These elements make up a shortcut to resolve the exercise. I removed all terms or words which could evocate some familiar pattern. I removed math simplification too. Furthermore, I removed exercise data and the final question. Finally, I invented a problematic situation.
For example, here after I will show how this rules work when applied to exercises proposed in the two empirical studies.

2A.3 – First study – Exercise re-formulation

A 20-turn coil has a cross-sectional area of 4 cm^2 and it is connected with a flashlight bulb; the circuit has no battery. If a magnet is repeatedly moving away and closer, the average magnetic field on the coil surface passes from zero to 9.4 mT. A boy moves the magnets near and far from the coil 2 times per second. What is the module of the emf induced in the circuit caused by this flux variation?

Building terms   Evocative terms   Simplifications   Target.

It becomes:

> You have to build a circuit to switch the bulb in Figure 3.5. You have a magnet and no battery.

Material	Resistivity
Aluminium	2.82×10^{-8}
Copper	1.72×10^{-8}
Gold	2.44×10^{-8}
Nichrome	150.0×10^{-8}

(a) Resistivities at 20 °C

Source	Magnetic field
Pulsar surface	$10^8 T$
Magnet neighborhood	$10^{-2} T$
Earth magnetic field	$10^{-5} T$

(b) Magnetic field

Figure 3.5: Flashbulb

1B.3 – Second study – Exercise re-formulation

The text of the second study exercise:

A coil is composed by 20 square turns, each one $l = 15\ cm$ side. The wire is very thin and curled unto itself. This coil is moved close to a large magnet ($L = 50\ cm$), generating a $B = 0.12\ T$ magnetic field. The total resistance of the coil is $R = 5,0\ \Omega$; a $20\ W$ bulb is linked to the coil, which is moving with a constant velocity of $v = 0,25\ m/s$. Find the electromotive force induced in the circuit [$fem_{\max} = 0,45\ V$]

Building terms   Evocative terms   Simplifications   Target.

It becomes:

For technical reasons you have to build a machine to produce electric current. You have a permanent magnet and a copper wire.
1) Draw a simply model of your machine.
2) Build your machine in order to generate a $220\ V$ tension.

Source	**Magnetic field**
Pulsar surface	$10^8 T$
Magnet neighborhood	$10^{-2} T$
Earth magnetic field	$10^{-5} T$

Table 3.6: Magnet

Chapter 4

The Interplay between Physics and Mathematics to Enter the Meaning of Electromagnetic Field

This chapter is dedicated to the presentation of the "electromagnetic field guide" elaborated during my PhD.
The guide is a document targeted to teachers and teacher educators. Its aim is to present electromagnetic field in a new light, able to address well-known problems in understanding the concept of field: what makes the interaction modeled by fields different from the interaction modeled by forces? What does it mean that the concept of field solves the problem of the interaction at a distance? What does it mean that a field is something real and not a mere mathematical tool? What mathematical tools are needed to describe field's properties? What physical meaning have divergence and curl? What do their names mean?

The guide is introduced by three preparatory documents. The first document is an historical introduction about the evolution of the aether concept, from Descartes to Maxwell. The second aims to pave the way to look at a generic field through the eyes of the mechanics of continuous and, from this point of view, to discuss the concept of pressure; the third proposes a nomenclature of differential operators, introduced by Maxwell, so as to recognize, behind the names, the conceptual meaning of mathematical tools. Finally, the guide is an educational presentation of the reasoning followed by Maxwell in his original paper "On Physical Lines of Force". In writing the documents and the guide I considered also the Maxwell's memory "On the Mathematical Classification of Physical Quantities". Both the original papers have been analyzed so as to reconstruct how Maxwell built his equations and how he discovered both the displacement and the electromagnetic waves. In the analysis, I paid special attention to the mathematics he invented, used and interpreted from a physical point of view.

Throughout the documents, the concept of aether plays a fundamental role. It represents not only the historical leading thread but also the essential element of the reasoning developed in the guide. The main thesis that this work intends to support is: the concept of aether is fundamental to understand Maxwell's equations since it can be an imaginative support to capture the physical meaning of the mathematical entities. Then, its overcoming, or better, the process of emancipation of physics from aether represents a fundamental intellectual tension with a great educational and cultural value: it marks the birth of 20th century physics and the birth of theoretical physics. Furthermore, its transformation into the concept of field, produced by Maxwell himself, is a very productive example, from an educational point of view, of the generative and structural role of mathematics in physics.

Teaching, both at secondary and university level, usually focuses on Maxwell's mathematics, and many understanding problems arise from the difficulties to manage it from a conceptual point of view. What physical meaning have divergence and curl? What do their names mean? My reconstruction aims to address questions like these since, as I will argue, their answers allow to enter the meaning of electromagnetic field.

In section 4.1, I historically contextualize Maxwell's memories and, after that, I present the overview of the guide on Maxwell's reasoning. Section 4.2 includes preparatory documents for the guide: in section 4.2.1, the history of the aether, from Descartes to Maxwell, and that of electromagnetism until Maxwell are resumed; section 4.2.2 is a brief digression about the pressure concept, in order to follow Maxwell argumentation; in sections 4.2.3,4.2.4 and 4.2.5 I follow Maxwell's paper "On the Mathematical Classification of Physical Quantities", where I illustrate Maxwell's invention of differential operator nomenclature. Section 4.3 reports the reconstruction of Maxwell's paper on "On Physical Lines of Force".

4.1 The Faraday Problem

When Maxwell began to write the first part of its memory "On Physical Lines of Force" the physical community knew that «if we strew iron fillings on paper near a magnet, [they will] form fibres, and these fibres will indicate the direction of the lines of force[1].»
The term *lines of force* was introduced by Michael Faraday in 1839 in its two volumes papers collection titled "Experimental Researches in electricity". Faraday was trying to represent what he saw when a magnet was near an iron fillings distribution (Figure 4.1). As time went by, he convinced himself – and the large majority of the English physical community – that those lines were something more than a simple representation: they could be something real.

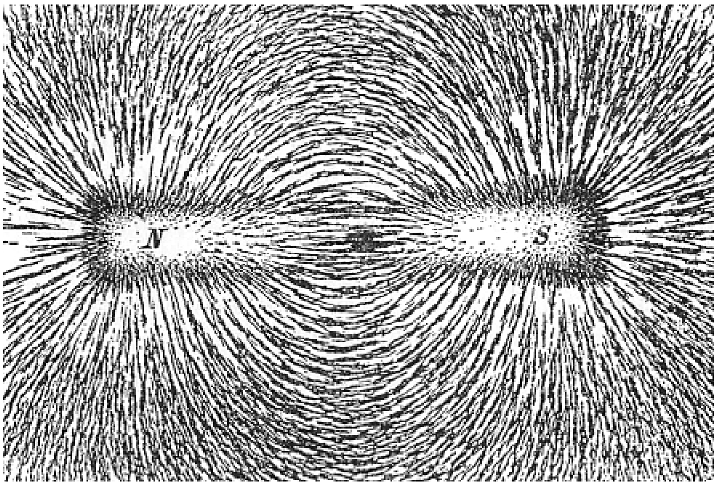


Figure 4.1: (Newton and Harvey, 1913)

This picture was used to present the idea of *field of forces* to English physicists. Faraday was the first to propose the idea of field, harnessed himself in the idea of *action-at-a-distance*. The idea was introduced and elaborated to solve two fundamental questions that were struggling the contemporary physicists: where does the force acting on each fiber come from? Why does the direction of interaction not follow a straight line?
The idea of lines of force was Faraday's answer. Forces, he argued, are carried by (or through) lines of forces, traveling from one pole of the magnet to the other one. Instead of forces, which are straight vector, lines of forces could be curved line. These remarks opened to a third, fundamental

[1]All citations are from "On Physical Lines of Force", unless otherwise stated

question: do forces travel with finite velocity?

I call these three questions *the Faraday problem*; they motivated a wide research but they did not find any reasonable answer by Faraday: he just had the intuition that the lines of force were the key to explain the magnetic interaction and that they could travel with a finite velocity. This made his work the most important seminal one in the creation of the field concept (Gooding, 2006), but it was Maxwell to solve the Faraday's problem. In fact, Maxwell, addressing the Faraday's problem, found the way for two crucial discoveries: the mathematical interpretation of induction and the electromagnetic waves.
Maxwell believed that mechanics could explain everything; furthermore, he thought that finding mechanical explanations of a phenomenon means having explained the phenomenon. Therefore, his aim was to inquire Faraday's hypothesis of the existence of electromagnetic lines of force by using a mechanical model.
To reach this goal, the Scottish physicist focused his attention on the space surrounding "magnetic charges" (Maxwell thought that, in analogy with the electric case, two magnetic poles existed), arguing that the whole space was filled by an "electromagnetic aether" which reacted to the presence of charges.
In order to reproduce the well-known effect of the magnetic interactions, Maxwell modeled the aether as an elastic solid, composed by vortexes which were supposed to rotate around a specific axis. Furthermore, Maxwell extended the model of aether to interpret induction and electric interactions, by adding the so-called "idle wheels", little particles whom movement inside the aether represents electric current.
With this apparently complicated system, Maxwell found 20 differential equations, nowadays celebrated (after some symbolic modifications) with the name of "Maxwell equations".

In the following I will underline three fundamental aspects of this system of equations, which can be considered the solution to "the Faraday problem":

1. Where does the force acting on each fiber come from?
    - Force is the manifestation of something more fundamental, i.e. the field of forces,
    - field is spatially extended
2. Why does the direction of interaction not follow a straight line?

    - Relations among physical quantities are local, that is, charges interact with the part of the field nearby; it is looking globally that it seems one particular charge acts on another not in straight line.

3. Do forces travel with finite velocity?

    - Yes, and this velocity is the velocity of the light.

Aether appears to be a necessary construction to answers "the Faraday problem". Aether is everywhere and, in Maxwell's view, it has energy because it is comprised of moving elements. Aether interacts locally with itself, and changing in the aether configuration are manifestation of charges; Aether, eventually, transmits informations with a finite velocity.

In the following I describe the itinerary followed in the guide based on Maxwell papers. My aim is to reach the following goals:

- to present "electromagnetic field" as a real physical object
    - the electromagnetic field has energy
- to differentiate "electromagnetic field" from Coulomb force
    - the electromagnetic field is not $E = F/q$
- to justify the introduction of the "electromagnetic field"
    - we need the concept of electromagnetic field to solve specific, fundamental problems, as well as to introduce the new physical object of electromagnetic waves, and the new physical framework of special relativity
- to imagine and to quantify "electromagnetic field"
    - the electromagnetic field is an extended body represented by space-time functions

The interplay between physics and mathematics will play an essential role to realize this program. Faraday couldn't go forward his problem because he wasn't able to rationalize his vision, that of a "tension" of the aether due to the presence of at least one charge. Maxwell rationalized and quantified this tension in the mathematical setting we will introduce later, finding not only all the laws of the electromagnetism already known, but also something new: the electromagnetic waves' equation. Furthermore, Maxwell found how to rationalize a new way of thinking at interactions.

Specifically, the mathematical way to answer to "the Faraday problem" is:

1. the electromagnetic field must be represented by space-time functions;
2. relations among electromagnetic quantities must be local – we need to find a system of partial differential equations;
3. actions don't act on straight lines – we need to know the form of differential operator of the fields.

4.2 Preparatory Documents for the Guide

As already mentioned, the guide has been accompanied by three preparatory documents, which can be used in different moments, either as introduction to the guide, or as insights, during the path. The documents, as they are, are not supposed to be used directly with secondary school students. They instead are thought to deepen the preparation of perspective teachers, during their university courses, or to enrich the preparation of in-service teachers.

The first document refers to the evolution of the aether concept throughout at least 3 Centuries. "The history of aether" begins with Descartes in 1600, it passes through Newton and Newtonian and it goes on with Faraday, Ampère, Weber and the modern physicists of nineteenth century. I preferred letting the main characters to talk to underline the meaning they give to physical concepts like action-at-a-distance, aether, mechanical explanation and so on. In parallel with this historical excursus, an epistemological reflection on the evolution of the physics is carried out, particularly focused on the role of the interplay between mathematics and physics in the scientific progress.
With this first document, I introduce the action-at-a-distance concept, from its first appearance as it was conceived by Newton to its last stage in the late 19th century. The complicated relationship among "aether" and "action-at-a-distance" broke down when Faraday introduced the idea of *electrotonic state*, a special state of tension of the space when charges are nearby. Faraday believed that this special state shown itself through lines of forces. So, the field of force became the rival of the action-at-a-distance.

At this point, two factions were formed: English physicists, who believed Faraday electrotonic state, and Continental physicists, who continued to use the action-at-a-distance approach. This in-depth introduction stop in 1862, with the publication of Maxwell's "On Physical Lines of Force", where the idea of electrotonic state became embedded in the Maxwellian aether.

The second document is a "little educational introduction to pressure". The aim of this document is to pave the way to understand Maxwell's view that treated aether as a continuous medium with mechanical properties. In particular, in order to solve Faraday's problem, he reasoned a lot about the concepts of pressure and tension in an homogeneous body.

The third document is based "On the Mathematical Classification of Physical Quantities" by James Clerk Maxwell (1870). The aim of these pages is to explain physically what it means each three-dimensional differential operator. I have tried to find a book or an Internet page where differential operators are explained physically, but this research has been nearly a bust. The only paper which has fulfilled my requirements was Maxwell's "On the Mathematical Classification of Physical Quantities".

4.2.1 The History of Aether

Descartes was born in 1596 in France and dead in 1650. He believed in the power of thought over the power of faith. The Universe, he said, must be a rational machine and Man, inquiring Universe with logic, can discover how it works. His rationalism had a revolutionary impact: following Descartes reasoning, many philosophers tried systematically, for the first time from many centuries, to think at the Universe as a machine, working with logic and rationality.
Descartes too spent many years in designing what he supposed the Universe was. In one of his most famous book, "Principia Philosophiae", written in 1644, he said:

«If for the mere fact that a body has length, width and depth we invariably expect it to be a substance and at the same time being the Nothingness by its very definition a fathomless lack of extension. Indeed the same thing ought to be postulated when speaking of the supposedly emptiness of space: since it has an extension it necessarily withholds substance *(trad. L Stefanini).*»

In the same book, he wrote that «Ex nihilo nihil fit (From nothing,

nothing grows)».
He imagined Universe as a plenum of massive vortexes whose movement explained all things happening in Nature (Figure (4.2)). He said: «Give me matter and movement and I'll build the Universe!».

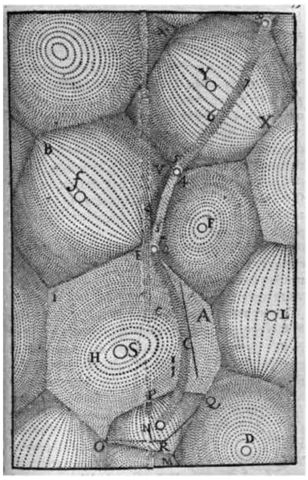


Figure 4.2: "The World" by R Descartes, 1664

It will be clear later in what sense Maxwell refers to Descartes: both of them rationalized aether (better, a kind of aether) by giving it matter and movement, in order to explain how Nature works, its inner mechanisms.
Descartes foresaw that mathematics could have an important role in rationalizing aether, but he believed that some deeper language should exist and he called it "universal Mathesis". At his time, mathematics was a subject used by engineers, plumbers, architects (John Wallis in (Heilbron, 1984)) , and it had to wait until the beginning of the 18th century to be restored and elevated as an independent subject. Furthermore, pioneering works of Newton and Galileo helped mathematics to became the Universal Language "spoken" by Nature, incorporating it into physics.
It is fundamental to remember that also physics it was not the same physics of today. It was the philosophy that inquired Nature's origin and manifestations. At the beginning of the 17th century, physics was not embedded in the mathematical logic:

«At the beginning of the 17th century the term "physics" used to indicate a qualitative and bookish science that included all kinds of natural bodies...it altogether ignored mathematics and the experimental method *(trad. L Stefanini)* (Heilbron, 1984, p. 15).»

For all this reasons, Descartes was one of the men who triggered the scientific revolution. (Whittaker, 1910, p. 3) writes:

«The grandeur of Descartes' plan, and the boldness of its execution, stimulated scientific thought to a degree unparalleled; and it was largely from its ruins that later philosophers constructed those more valid theories which have endured to our time. Descartes regarded the world as an immense machine.»

Unlike his method, Descartes results were easily proved to be wrong and its Universe disappears very soon. Bernard le Bovier de Fontanelle, a French author renowned for his scientific passion, wrote (Heilbron, 1984, p. 40): «Descartes is always to be admired but not always to be followed *(trad. L Stefanini).*»

Newton was born in 1643 and died in 1727. From (Whittaker, 1910, p. 1): «Until the seventeenth Century the only influence which is known to be capable of passing from star to star was that of light Newton added to this the force of gravity.»
In his celebrated masterpiece "Principia Mathematica Philosophiae Naturalis", published in 1687, he built a new framework, based on substantial empty and absolute space and time. Within this framework, he rationalized the action-at-a-distance concept, defining Force, Quantity of motion and the Universal Law for the Force of Gravity:

$$\vec{F} = \frac{d\vec{P}}{dt} \qquad \vec{F} = G\frac{M_1 M_2}{r^3}\vec{r} \tag{4.1}$$

Bodies change their state of motion if a force acts on them. The formalization of the force of Gravity opened a fundamental question immediately posed to Newton, destined to remain unanswered for centuries: How do bodies interact at a distance? Newton could not answer this question and the strategic line he preferred to choose was to focus on the effect of his laws and not on the cause, so as to avoid to "feign hypothesis": in this sense, he overtook Descartes in the path through the building of a Mathematical Philosophy of Nature (later called physics). His famous «hypotheses non fingo» must be red in its contest, to appreciate in depth the impact of Newton's refusal:

«I have not as yet been able to discover the cause for these properties of gravity from phenomena, and I do not feign hypothesis. For whatever is not deduced from the phenomena must be called a hypothesis; and hypothesis, whether metaphysical or physical, or based on occult quantities, or mechanical, have no place in experimental philosophy. In this philosophy particular proportions are inferred from the phenomena, and afterwards rendered general by induction[2].»

Newton overtook Descartes philosophy focusing his attentions on phenomena; he was interested on the effects of his philosophy, namely in the predictive power of his equations.
Newton's predictions worked very well and their success boosted the new approach to Nature: natural philosophers became physicists from the moment they move their attention from causes to effects (Williams, 1927). The path of building a Mathematical Philosophy of Nature (later called Physics) passed through the famous philosophical rule: "hypotheses non fingo".
Action-at-a-distance is characterized by three very important aspects:

1. $\vec{F}$ acts instantaneously
2. $\vec{F}$ acts in straight line
3. $\vec{F}$ exists between two bodies

Newton has never thought that these characteristics were the way the Nature works; he was simply not interested, at first, to inquire something which was impossible to verify, something which was too far from his "region of speculation".
However, his followers, the so called Newtonian physicists, were not as prudent as their leader. They elevated action-at-a-distance to the rank of law of Nature. This «rashness» (Aepinus (Heilbron, 1984, p. 76) brought aether on the back burner[3].
Voltaire, returning from London in 1727 wrote that «A Frenchman who arrives in London will find a big change in philosophy as well as in other thing. He had left the World packed with stuff and now finds it utterly empty *(trad. L Stefanini)* (Thompson, 1892).» Newton, although the fabric of cosmos was "made" of empty space and time, believed in the existence of some not so well defined aether. In a letter to Boyle he wrote: «All space is

[2] Wikipedia

[3] Newton, on the contrary, believed in the existence of some not so well defined aether. In a letter to Boyle he wrote: «All space is permeated by an elastic medium or aether, which is capable of propagating vibrations in the same way as the air propagates the vibrations of sound, but with far greater velocity (Whittaker, 1910, p. 17-18).»

permeated by an elastic medium or aether, which is capable of propagating vibrations in the same way as the air propagates the vibrations of sound, but with far greater velocity (Whittaker, 1910, p. 17-18)» Newtonian, instead, removed the idea of aether from physics horizons.
Newtonian positions were fought by other physicists, for example Leibniz and Huygens: they accuse directly Newton to have brought physics in the «old peripatetic obscurity» (Joseph Saurin, 1709 (Heilbron, 1984, p. 77). Euler brothers called action-at-a-distance a «mens deliria» (Heilbron, 1984, p. 111).
Fontenelle, in 1728, sentenced: «[Speaking of action-at-a-distance], idea banned by Cartesians [...] the caveat of not attributing any reality to it must not be neglected. In fact the risk of thinking to grasp its meaning is real *(trad. L Stefanini)* (Heilbron, 1984, p. 77).»

By the way, the eighteenth century was also characterized by technical improvement of the experiments: new materials, new methods, more attention to what didn't work and laid the foundation of the physics of the nineteenth century. Experimental results contributed to confirm Newton's physics and, indirectly, Newtonian vision of the Nature.
Nevertheless, in 1892 Thomson wrote «The Cartesian doctrine was widely adopted by mathematicians and philosophers in Continental Europe *(trad. L Stefanini).*»

In 1785, Charles Augustin de Coulomb (1736-1806) found that electric and magnetic charges (thin needles) followed the same law of gravitational charges:

$$\vec{F}_E = K\frac{Q_1Q_2}{r^3}\vec{r} \qquad\qquad \vec{F}_M = M\frac{P_1P_2}{r^3}\vec{r} \tag{4.2}$$

This discovery scarred a decisive goal in favor of Newtonian dynamics.

Thanks to the improvement of mathematics, especially with fluid dynamics and mechanics, aether brought back through the window, although with a different role from the one attached by Descartes.
At the beginning of 19th century, new fields of physics started to be inquired: thermodynamics, dynamics of continuous bodies, electricity and magnetism. Each of these subjects needed a special mathematics of continuum and different kinds of aether emerged in order to bring physics into a mathematical background. Aether became an instrument useful to apply mathematics to different situations. It was not important whether aether were real or not, but that it worked.

In the same period, many engineers and physicists reported the superiority of London artisans with respect to other Europeans colleagues. In 1782, Alessandro Volta wrote: «The machinery arrived from Paris are some mediocre pieces of kit...those we received from London instead to thoroughly meet the expectations of our Physics Department *(trad. L Stefanini)* (Heilbron, 1984, p. 123).»

On March twentieth, 1800, Volta presented his invention, called Pile, to the Royal Society. In the nineteenth century the second industrial revolution took place, and electricity and magnetism were the main characters of this process. «Leading country of the European industrialization process – especially Germany and United Kingdom – in those years underwent a powerful scientific and organizational boost in scientific research activities (De Marzo, 1978, p. 3).» The relationship between physics, technique and industry became even closer. The economic world demanded physicists to improve productive technique; the more their laboratories became up to date, the more their experiments were accurate. A virtuous cycle was established, and many physicists became acquainted with thermodynamics, engineering, mechanics, industrial machines and so on.
In this scenario, in April 1820, the Danish physicist Hans Christian Ørsted (1777-1851) discovered the magnetic power of current. He was looking for a magnetic connection between magnetism and electricity for years, but surprisingly the force exerted by the current on a magnet was "circular". In his own words: «From the preceding facts we may likewise infer that this conflict performs circles; for without this condition it seems impossible that the one part of the uniting wire, when placed below the magnetic pole, should drive it towards the east, and when placed above it towards the west; for it is the nature of a circle that the motions in opposite parts should have an opposite direction (Ørsted, 1820).»
What Ørsted found[4] was that some force does not act in straight line. At the time, this discovery wreaked havoc among physicists, because nobody knew how this new information could be putted in the action-at-a-distance framework.
Nonetheless, some physicists tried to lead back Ørsted effect in the action-at-a-distance framework. In the same year, André-Marie Ampère (1775-1836) discovered that two wires carrying current i_1 and i_2 can attract or repel with force directly proportional to their length l and inversely proportional to their distance d

[4]The Italian physicist Gian Domenico Romagnosi discovered the Ørsted effect before Ørsted and published his discovery in 1802 with the general indifference.

$$F = \mu \frac{i_1 i_2}{d} l \tag{4.3}$$

From the experiments, it is possible to observe that parallel currents attract, anti parallel currents repel.
At the same time, Jean-Baptiste Biot (1774-1862) and Félix Savart (1791-1841) found the mathematical law for the relation between a current i carried by a straight wire and the magnetic field induced

$$B = \frac{\mu i}{2\pi d} \tag{4.4}$$

μ is the nowadays magnetic permeability. Both these laws appeared to be Newtonian.
In 1825 Ampère proposed a law for the circulation of the magnetic field. Nowadays this law is called the Ampère law

$$\oint \vec{B} \cdot d\vec{l} = \mu I \tag{4.5}$$

By the way, this version of the Ampère law was formulated by Maxwell in On Physical Lines of Force. Despite that, Ampère was the first to propose that magnets are the manifestation of micro currents. In his "Thèorie Mathèmatique des Phénomènes Électro-dynamiques Uniquement Déduit de l'Expérience" of 1826 the French physicist speculated that the origin of magnetism was electric. He initially purposed magnets were constantly crossed by circling currents, which generates their macroscopic magnetic field. Fresnel criticized Ampère theory, saying that magnetic material are bad conductor, so electric current passing through them must heat them: but magnets are generally cold. But what about the magnets' atoms? Ampère, following a suggestion by Fresnel himself, found a solution to the Fresnel's problem, and this solution conceived the aether.
He speculated that aether was filled by an imponderable number of electric charges, normally in electric equilibrium. This equilibrium is kept until these charges enter into a magnet's atom. Electric charges, one negative and one positive, travel together with the same speed, in order to appear neutral. As soon as the neutral couple enter the spherical atom, one charge goes in one direction around the sphere and the other one travels on the opposite side; they meet at the opposite pole, continuing to travel together (Figure 4.3).

Aether was used many times to explain different phenomena. This instrumentalism caused a process of reification of aether, with the

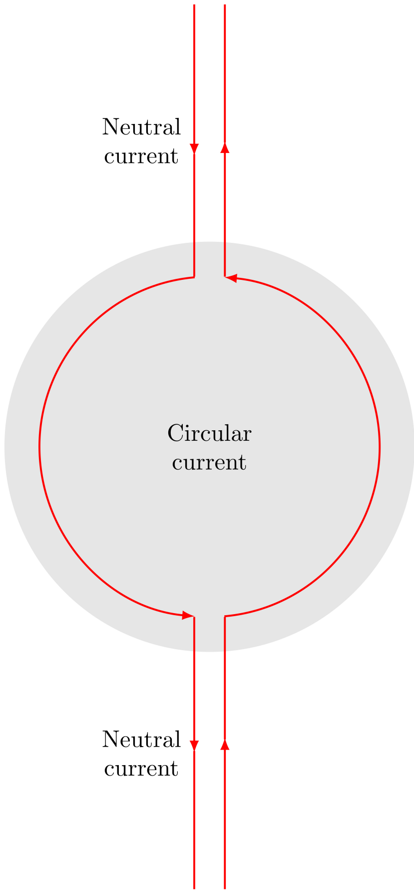


Figure 4.3: A sketch of Ampère's atom

«introduction of imponderable substances which transport forces linked to heat, light, fire, electricity, magnetism. At the end of the 18th century physicists distinguished between two electric fluids and two magnetic fluids, light corpuscles, phlogiston (Heilbron, 1984, p. 105).»
While the theory about electric and magnetic phenomena was going through a period of uncertainties, a great number of experiments were performed. All the greatest physicists all over the World were trying to found an explanation for electrical and magnetic phenomena and to put them into the action-at-a-distance framework. Many of them did not exclude that some features of the action-at-a-distance could be modified; Gauss, for instance, believed that the electrical force «is not instantaneous, but it propagates with time (as light) (Laugwitz, 1999).»
The first scholar who suggested a new way to look at interactions was

Michael Faraday (1791-1867) (Nersessian, 1985). In 1821 he created the first electrical engine. In 1831 he was the discoverer of magnetic induction[5]. In 1845 he discovered diamagnetic bodies and the so-called Faraday effect. He was well-renowned among physicists, which considered him one of the most brilliant experimentalist of the time.
Faraday did not know anything about mathematics. He was not involved in the mathematical discussion about Newtonian mechanics. He was free to suggest new representations of interactions (Gooding, 2006). So it happened that, in 1839 version of "Experimental Research in Electricity", for the first time he mentioned lines of force:

«By magnetic curves I mean the lines of magnetic force, [...] which would be depicted by iron filings, or those to which a very small magnetic needle would form a tangent. [...] Every line of force, therefore, at whatever distance it may be taken from the magnet, must be considered as a closed circuit, passing some part of its course through the magnet, and having an equal amount of force in the every part of its course (Wu and Yang, 2006, p. 3243).»

Faraday believed that experiments with iron filings (Figure 4.4[6]) demonstrated the existence of a special state of the "electric matter", which he called the electrotonic state. This is a state of tension, which pulls magnetic charges along special lines, the lines of force. As the years went by, Faraday declared that lines of force were real physical objects, with energy, forming together a field of force.

«I incline to the opinion that [the lines of magnetic force] have a physical existence correspondent to that of their analogue, the electric lines, and having that notion, am further carried on to consider whether they have a probable dynamic condition, analogous to the axis to which they consist in a state of tension round the electric axis, and may therefore be considered as static in their nature. Again and again the idea of an electrotonic state has been forced on my mind; such a state would coincide and become with that which would then constitute the physical lines of force (Wu and Yang, 2006, 3242).»

«It appears to me, that the outer forces at the poles can only have relation to each other by curved lines of force through the surrounding space; and I cannot conceive curved lines of force without the conditions of

[5]The first mathematical expression of this "law" appeared in (Neumann, 1846, p. 32)
[6]wellcomeimages.com

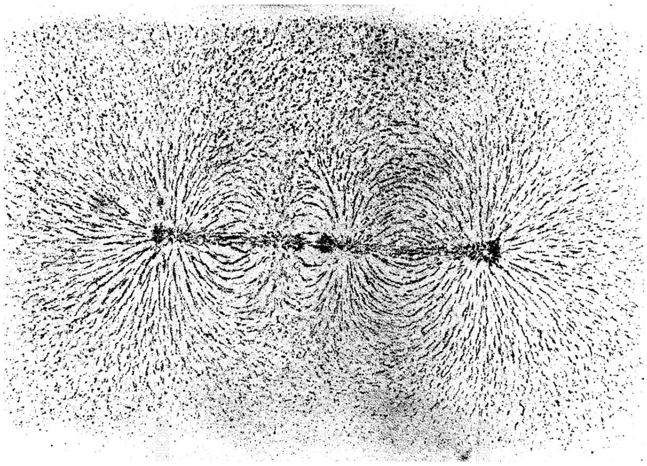


Figure 4.4: Results of Michael Faraday's iron filings experiments

a physical existence in that intermediate space (Faraday, 1852, p. 408).»

In the representation suggested by Faraday, interactions do not exists between two charges, but between the charge and the lines of forces in the surrounding space. The term *magnetic field* was firstly used by Faraday in 1845.

Faraday were not able to rationalize his vision of the magnetic interactions within a mathematical framework, and his idea did not find any followers in the Continent. Only a piece of English physicists community embraced the idea the lines of force and electrotonic state can be a new representation for interactions (Harman, 1982). Faraday was extremely humble and shy, in part because of his ignorance in mathematics. At the Royal Society, on November 24, 1831, he said:

«Whilst the wire is subject to either volta-electric of magneto-electric induction, it appears to be in a peculiar state; for it resists the formation of an electric current in it, whereas, if in its common condition, such a current would be produced; and when left uninfluenced it has the power of originating a current, a power which the wire does not possess under common circumstances. This electrical condition of matter has not hitherto been recognized, but it probably exerts a very important influence in many if not most of the phenomena produced by currents of electricity. For reasons which will be immediately apparent (paragraph 71), I have, after advising with several learned friends, ventured to designate it as the electrotonic state (Wu and Yang, 2006, p. 3241).»

«Am I not a bold man, ignorant as I am, to coin words? But I have consulted the scholars (Letter to R. Philips – 11/29/1831) (Wu and Yang, 2006, p. 3241).»

Maybe because he did not need mathematics, he did not conceive aether as something real. He said that «the aether doesn't exist. Masses, charged bodies and currents emanates lines of force in an empty space with which they interact (Faraday, 1852).»
The German physicist Hermann von Helmoltz (1821-1894), at the Chemical Society of London in 1881, during the Faraday Lectures, said:

«Now that the mathematical interpretation of Faraday's conceptions, regarding the nature of electric and magnetic forces has been given by J. C. Maxwell, we see how great a degree of exactness and precision was really hidden behind the words which to Faraday's contemporaries appear either vague or obscure; and it is in the highest degree astonishing to see what a large number of general theorems, the methodical deduction of which requires the highest powers of mathematical analysis, he found by a kind of intuition, with the security of instinct, without the help of a single mathematical formula (Wu and Yang, 2006, p. 3244).»

The physical world imagined by Faraday needed a mathematical support. Fortunately, England gave birth to two of the greatest mathematicians of the time.
One of them was William Thomson (1824-1907). In 1852 he wrote:

«During the 56 years from when Faraday for the first time hurt mathematical physicists with his closed lines of force, many workers and thinkers contributed to erect the plenum school of the nineteenth century (De Marzo, 1978).»

Thomson invented the method of analogy. Maxwell described this method in this way:

«By a physical analogy I mean that partial similarity between the laws of one science and those of another which makes each of them illustrate the other ("On Faraday's Lines of Force", 1865) (Maxwell, 1965a, p. 156).»

The laws of science are expressed in a mathematical form, so this method works at a pure mathematics level. The analogy, in other words, is a relation between relations and it has the form of a weak proportion:

$$A : B \sim C : D$$

Getting the sense of the analogy means understand to what extent the comparison works (Neri, 2011).
Thomson (aka Lord Kelvin) also derived the so-called Stokes theorem before Stokes himself (Thompson, 1851, p. 256), using it to evaluate that the divergence of the magnetic field is zero.
He was also interested in engineering. He had a primary role in the laying of the first Transatlantic Communications Cable (TCC) in 1858. This activity took him many years away from theoretical physics, although he continuously published many papers on electromagnetism.
At that time, both industrialists and physicists were interested in measuring the current's velocity in a cable. Thomson, like many others, worked on this problem, to improve the research on the electric impulses transmission through the TCC. In 1855, Wilhelm Eduard Weber (1804-1891) and Rudolf Kohlrausch (1809-1858) found that the ratio between the electrostatic unit and the electrodynamic unit was very similar to the velocity of light. In nowadays symbols

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \tag{4.6}$$

In 1857 Kirchhoff found that the "electric energy" travels inside cables at a velocity very close to the speed of light.

The "second" great mathematicians in England was a Scottish physicist, James Clerk Maxwell (1831-1879). Despite he died young, his career was extremely various. He substantially contributed to modern mathematics, all branches of physics, engineering, chemistry. He is considered the first modern theoreticians of the history of physics, but he became the first director of the Cambridge Cavendish Laboratory for his wide knowledge on experimental physics.
Maybe, the most impressive thing is the greatness of his humility. Very young, he decided to study electromagnetism. He found this work very hard. He remembered this period in the beginning of his 1855 paper "On Faraday's Lines of Force":

«The present state of electrical science seems peculiarly unfavourable to speculation. The laws of the distribution of electricity on the surface of conductors have been analytically deduced from experiment; some parts of the mathematical theory of magnetism are established, while in other parts the experimental data are wanting; the theory of the conduction of galvanism and that of the mutual attraction of conductors been reduced to mathematical formulae, but have not fallen into relation with the other parts of the science. No electrical theory can now be put forth, unless it shews the connexion not only between electricity at rest and current electricity, but between the attractions and inductive effects of electricity in both states. [...] the student must make himself familiar with a considerable body of most intricate mathematics, the mere retention of which in the memory materially interferes with further progress. [It is necessary to reduce] the results of previous investigation to a form in which the mind can grasp them, [...] a purely mathematical formula or a physical hypothesis. In the first case we entirely lose sight of the phenomena to be explained; [in the other case] we see the phenomena only through a medium, and are liable to that blindness to facts and rashness in assumption which a partial explanation encourages. [...] In order to obtain physical ideas without adopting a physical theory we must make ourselves familiar with the existence of physical analogies [and with a] partial similarity between the laws of one science and those of another (Maxwell, 1965a, p. 155-156).»

To help himself to reach «further progress», «before I began the study of electricity I resolved to read no mathematics on the subject till I had first read through Faraday's Experimental Researches in Electricity». He believed firmly in Faraday's intuition. In a letter to him dated 1857, 9th November Maxwell wrote:

«Now as far as I know you are the first person in whom the idea of bodies acting at a distance by throwing the surrounding medium into a state of constraints has arisen, as a principle to be actually believed in (Maxwell, 1990, p. 548).»

The first step of his work was the "geometrization" of lines of force. He used «Faraday's mathematical methods as well as his ideas». In his "On Faraday's Lines of Force" he wrote:

«The idea of the electro-tonic state, however, has not yet presented itself to my mind in such a form that its nature and properties may be clearly explained without reference to mere symbols, and therefore I propose in the following investigation to use symbols freely, and to take for granted the ordinary mathematical operations. By a careful study of the laws of elastic solids and of the motions of viscous fluids, I hope to discover a method of forming a mechanical conception of this electro-tonic state adapted to general reasoning (Maxwell, 1965a, p. 187-188).»

With this paper, Maxwell became acquainted with Faraday's world. He grasped the deep significance of the electrotonic state and lines of force. He felt ready to go beyond Faraday and to rationalize his concepts. He began to re-introduce aether to explain mechanically the manifestation of lines of force and the electrotonic state. He believed that an explanation can be only "mechanical", as he wrote:

«On the other hand, when a physical phenomenon can be completely described as a change in the configuration and motion of a material system, the dynamical explanation of that phenomenon is said to be complete. We cannot conceive any further explanation to be either necessary, desirable, or possible, for as soon as we know what is meant by the words configuration, motion, mass, and force, we see that the ideas which they represent are so elementary that they cannot be explained by means of anything else (Maxwell, 1875, p. 357).»

Aether will be the instrument used by the Scottish physicist to apply the law of mechanics in order to derive the law of electromagnetism. He will reach his goal, finding twenty equations which we call nowadays the "Maxwell's equations" (although they appeared written in a different form with respect to present time). After the accomplishment of the process of mathematization, physics was ready to re-interpret the results and to be aware that aether, at that point, was not longer needed.

4.2.2 A Summary on Pressure

The process of mathematization built by Maxwell is based on the mechanics of continuum and a special role to interpret the electrotonic state is played by the concept of *pressure*. Pressure, in Maxwell's paper, is related to a more complex mathematical structure than a scalar field and it is applied to continuous bodies. I report some notes on the concept of pressure in order to make Maxwell's argument easier to be followed.
To help me write this notes, I followed Besson's "Didattica della Fisica", but I have thought also at Maxwell's papers, in order to build an organic documents on the electromagnetic field.

Brief Educational Introduction to Pressure

Usually in teaching, pressure is introduced, for the sake of simplicity, as a scalar quantity

$$P = \frac{F_N}{S} \tag{4.7}$$

where F_N is the force perpendicular to the surface S.
Research in physics education found that many students all over the age are not able to accept pressure as a scalar quantity and, implicitly or explicitly, think that pressure is a vector. Picture and language used in the textbooks reinforce this idea (Figure 4.5).

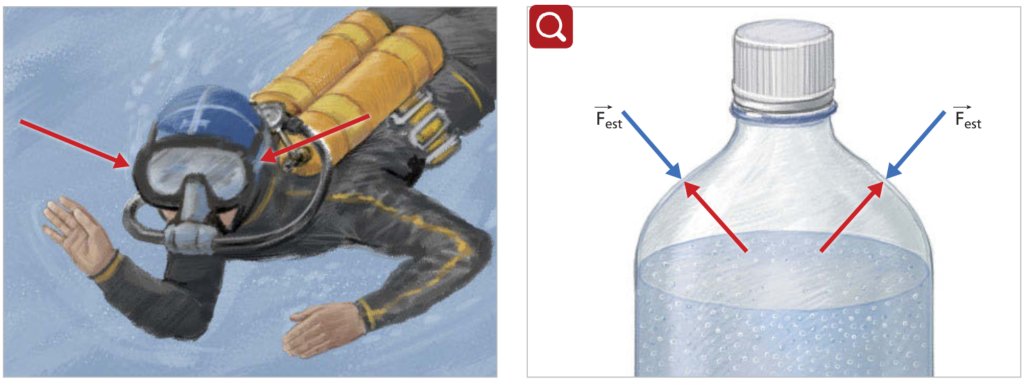


Figure 4.5: (Romeni, 2012, p. 267)

This idea has reasonable and acceptable intuitive roots and the problem is that the situations and problems where pressure is involved are different and multiform. In teaching, pressure is over-simplified to express all these meanings. The result is that students' intuitions do not find a form articulated enough to cover the span of contexts where pressure is required to be applied. What we usually call "pressure" it is only a very special case in the study of internal forces of a continuum.

Continuum Mechanics

In continuum mechanics, the fundamental element is not the point, but the infinitesimal volume element dV. Such an element (an infinitesimal cube) is characterized by a surface which separates the interior from the exterior.
The complementary of dV acts on dV with a force f on every points of its surfaces and, for the action/reaction principle, the interior acts on the

exterior with the a force equal and opposite on each point. Since each surface has infinite points, the superficial density (Besson, 2015, p. 123) of the force is considered,

$$f/dS \tag{4.8}$$

We call *stress* this superficial density and it is dimensionally a pressure, formally a vector. For now on, we will consider as positive the direction from the interior to the exterior of the volume.
Now, we would to give an idea of the Cauchy stress Theorem. Imagine a point of the fluid belonging to an arbitrary plane. It is possible evaluate the force perpendicular to this surface in order to find the stress on this point for that surface. It is possible to demonstrate (Cauchy's stress theorem) that this stress is always dependent on three stresses, each of them parallel to the cartesian axes.
This theorem can be explained visually: imagine the same point being the center of an infinitesimal cube; the parallel surfaces of this infinitesimal cube are near enough to consider them overlapped. So, to evaluate the stress, they can be considered as one. Since parallel surfaces contribute together to the total stress on the infinitesimal cube, it is possible to consider only three surfaces to evaluate the total stress on the infinitesimal volume.
On each surface, the stress could be decomposed into other the three, cartesian, direction, like in Figure 4.6.

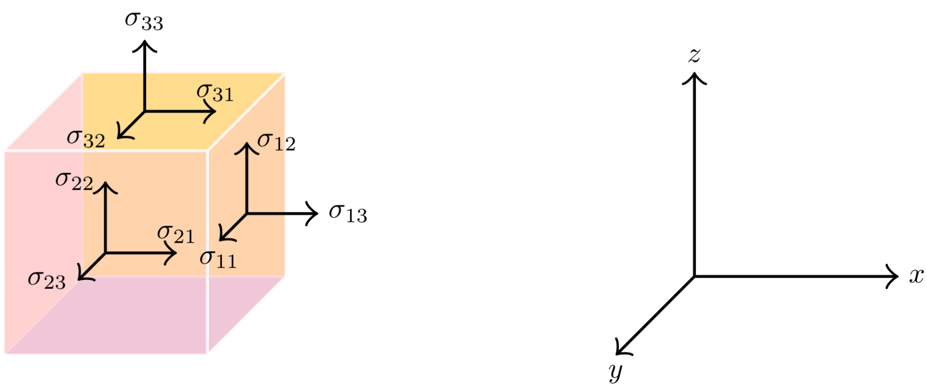


Figure 4.6: The stress tensor

Each surface could be associated with the perpendicular direction. We call dS_i the surface whom perpendicular vector is in the $i-$direction.

The result are nine projections – three per each surface. We call σ_{ij} the stress on the surface dS_i in the $j-$direction.
Since each stress acts on the same volume, we organize them in a matrix. We call it the *stress tensor*:

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} \tag{4.9}$$

Hydrostatic Equilibrium in a Fluid

For definition, a fluid in equilibrium has no superficial stresses ($\sigma_{ij} = 0$ if $i \neq j$); in this condition, it exists only principal stresses, those perpendicular to the dV surface.
The fluid is in hydrostatic equilibrium when no turbulence, no vortex, no whirlpool, no macroscopic movement is present. Principal stresses are equal, and the stress tensor reduces into a scalar

$$\sigma_{ij} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix} \tag{4.10}$$

The Pressure

What is pressure, in the general case?

- It can be seen as a tensor in the most complicate case (unequal forces in a continuum middle).
- It can be seen as a superficial density vector in other cases (for instance, when a stiletto heel pulls on a balloon or a ski on the snow).
- It is a scalar in the easiest case (a fluid in hydrostatic equilibrium).

Although the last case it is the easiest one, the pressure, when refers to superficial density of a force, maintains typical vector properties, for the reasons seen above. This sense of pressure will be fundamental to interpret Maxwell's paper.

4.2.3 On The Mathematical Classification Of Physical Quantities

At the beginning of this paper (Maxwell, 1965b, p. 257), Maxwell arguments about the evolution of a physical theory. Using his own words:

«The first part of the growth of a physical science consists in the discovery of a system of quantities on which its phenomena may be conceived to depend. The next stage is the discovery of the mathematical form of the relations between these quantities. After this, the science may be treated as a mathematical science, and the verification of the laws is effected by a theoretical investigation of the conditions under which certain quantities can be most accurately measured, followed by an experimental realisation of these conditions, and actual measurement of the quantities.»

The Scottish physicist describes what kind of evolution, in his opinion, represents the growth of a physical science. Briefly

1. at first, it is necessary to recognize a self-consistent system of quantities which describe the phenomena;
2. after that, it needs to discover the mathematical form of the relations between these quantities.

At this point, Maxwell says, the «physical science» turns to be a «mathematical science», whose results can be tested, both theoretically and experimentally, under certain conditions.
This abstraction can reveal similarities between different sciences. In fact, the mathematical form of a "specific science" can be similar to the mathematical form of another science. In this case, the method of analogy can link these two sciences (Larmor, 1937; Wigner, 1960; Turner, 1995; Nersessian, 2002; Neri, 2011; Bokulich, 2015). Two sciences could be different «in their physical nature, but agreeing in their mathematical form.»
For instance, in mechanics the fundamental law of motion could be expressed in the form

$$\vec{F} = \frac{d\vec{P}}{dt} \tag{4.11}$$

that is the same mathematical relation among electric field and vector potential (Bork, 1967)

$$\vec{E} = \frac{d\vec{A}}{dt} \tag{4.12}$$

The two physical sciences in the example differ in their physical nature; however, they are equal in their mathematical form – at least in this specif case. Maxwell gives another example, with a hint of irony:

«Thus, when Mossotti observed that certain quantities relating to electrostatic induction in dielectrics had been shewn by Faraday to be analogous to certain quantities relating to magnetic induction in iron and other bodies, he was enabled to make use of the mathematical investigation of Poisson relative to magnetic induction, merely translating it from the magnetic language into the electric, and from French into Italian».

Maxwell thought that mathematical classification of quantities could be helpful in learning physics. His "mathematical classification" is intended, in this paper, to be an insight into the meaning of differential operators. He wrote:

«I think that the progress of science, both in the way of discovery, and in the way of diffusion, would be greatly aided if more attention were paid in a direct way to the classification of quantities».

In order to classify mathematical quantities, I discuss the *nomenclature* of three differential operators, the nowadays well-known gradient, divergence and curl. These operators are introduced by Maxwell: he proposed a nomenclature to summarize in one word their meaning.

Differential Operator Nomenclature

Maxwell suggests to give a name to three differential operators, in order to evocate their meaning. He refers to stationary fields and, consistently, the differential operators represent spatial properties of field created by charges at rest.

1 - GRADIENT

He suggested *slope* for the operator

$$\nabla S = \left(\frac{\partial S}{\partial x} \; ; \; \frac{\partial S}{\partial y} \; ; \; \frac{\partial S}{\partial z} \right) \tag{4.13}$$

nowadays called *gradient*.
This operator is applied to a scalar field $S(\vec{x})$. The name evokes that the scalar field $S(\vec{x})$ increases along some direction, and it measures this growth.

Using Maxwell words, the fact that $\nabla S < 0$ indicates «the direction in which S decreases most rapidly, and measuring the rate of that decrease.»

2 - DIVERGENCE

He suggested *convergence* for the operator

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x} F_x + \frac{\partial}{\partial y} F_y + \frac{\partial}{\partial z} F_z \tag{4.14}$$

nowadays called *divergence*.

The divergence is applied to a vector field $\vec{F}(\vec{x})$. The name evokes that the intensity of the vector field $\vec{F}(\vec{x})$ increases through some point of the space if in that point $\nabla \cdot \vec{F} < 0$ (in nowadays convention). The divergence is the measure of this growth. In fact, in Maxwell's words, «if a closed surface [can] be described about any point, the surface integral of $\vec{F}$, which expresses the effect of the vector $\vec{F}$ considered as an inward flux through the surface, is equal to the volume integral of $\nabla \cdot \vec{F}$ throughout the enclosed space. [...] *that vector function [is] carrying its subject inwards towards a point* (Figure 4.7).»

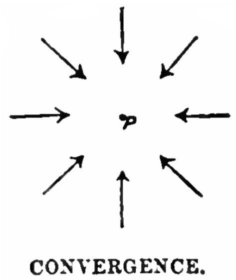


Figure 4.7: The local convergence of the field in the point P (Maxwell, 1965b, p. 257)

We are interested to the physical interpretation of $\nabla \cdot \vec{F}(\vec{x}) \neq 0$. Obviously, this is a local relation, and it means that a vector field with $\nabla \cdot \vec{F}(\vec{x}) \neq 0$ has a point of attraction/repulsion in $\vec{x}$: I will call these points *convergence/divergence points*. Field intensity grows/diminishes through that point in a particular way. The physical consequence is that there exist charges for the physical vector field $\vec{F}$.

Looking at Figure 4.7, it is possible to think that the field itself is moving, because the direction of "the arrows" can be associated with the direction of the field. Despite that, the field is not moving: only the substance carried

by the field is moving, with a rate depending on the field intensity and only if the net flux is not zero. If the physical vector field is the electromagnetic field, this "substance" is represented by the electric charges.
Note that Figure 4.7 is misleading for another reason: there exist vector fields with lines of force configuration like that in the picture Figure 4.8 and with no divergence. For instance the field

$$\vec{F} = \left(\frac{-x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right) \tag{4.15}$$

is $\nabla \cdot \vec{F}(\vec{x}) = 0 \quad \forall \ \vec{x}$.

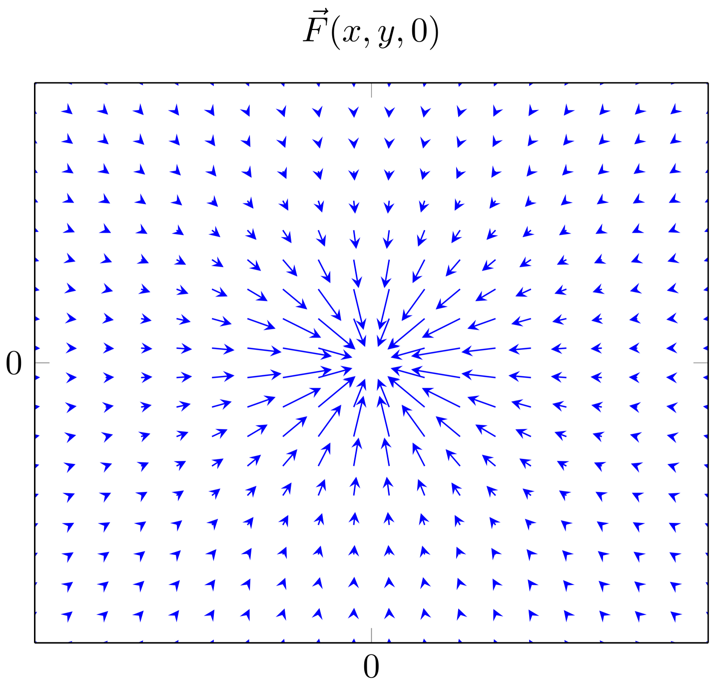


Figure 4.8: Despite the arrows are converging on the center of the figure, there is no convergence point in $(0; 0; 0)$ because the divergence (the local flux) is zero everywhere.

So, lines of force configuration with intersection points is not sufficiency for the divergence to be non-zero.
Since the divergence is a local operator, $\nabla \cdot \vec{V}(\vec{x})$ is a measure of the *local* flux in a point $\vec{x}$.
We can imagine this field configuration in two different ways:

1. in the “physical” way, convergence different from zero means the existence of charges;
2. in the “analogical” way, where the field is treated as a fluid, convergence different from zero means that the fluid experiences a perpetual infinitesimal expansion/contraction with respect to the divergence/convergence point $\vec{x}$; in this analogy, while the substance is moving through or far from the point, the velocities field remains uniform and it is represented by a static field.

3 - CURL

He suggested *curl* for the operator

$$\begin{aligned}\nabla \times \vec{F} = &(\partial_y F_z - \partial_z F_y)\,\hat{\imath} + \\ &+ (\partial_z F_x - \partial_x F_z)\,\hat{\jmath} + \\ &+ (\partial_x F_y - \partial_y F_x)\,\hat{k}\end{aligned} \tag{4.16}$$

nowadays it is the same[7].

The curl is applied to a vector field $\vec{F}(\vec{x})$. The name evokes that the vector field $\vec{F}(\vec{x})$ is associated with a circulation around some point of the space if $\nabla \times \vec{F} \neq 0$, and it measures the rate of this rotation. If a closed path can be described about any point, the line integral of $\vec{F}$, which expresses the circulation of the vector $\vec{F}$, is equal to the surface integral of $\nabla \times \vec{F}$ throughout the surrounded surface. In Maxwell's words, «It represents the direction and magnitude of the rotation of the subject matter carried by the vector $\vec{F}$ (Figure 4.9).»

We are interested to the physical interpretation of $\nabla \times \vec{F} \neq 0$. Obviously, this is a local relation. It means that a vector field with $\nabla \times \vec{F} \neq 0$ has a point of "circulation" in $\vec{x}$. I will call these points *circling/anticircling points*[8]. Field intensity grows/diminishes through that point in a particular way. The physical consequence is that there exist charges for the the physical vector field $\vec{F}$.

Looking at the Figure 4.9, it is possible to think that the field itself is rotating, because the of the direction of “the arrows” can be associated with

[7]At the Maxwell's time, the Scotland national sport was curling. In this sport, the act of curl the stone while throwing it's very important. *To curl* means «Move or cause to move in a spiral or curved course.» (from Oxford dictionary)

[8]“to draw a circle around something” . Cambridge Dictionary. « Circling the drain»

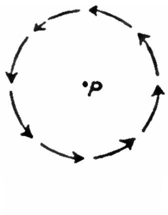


Figure 4.9: The local curl of the field in the point P (Maxwell, 1965b, p. 257)

the direction of the field. Despite that, the field is not rotating: only the substance carried by the field is rotating, with a rate depending on the field intensity and only if the net circulation is not zero. To make clear this distinction, Maxwell writes: «I have sought for a word which shall neither [...] connote motion [like Rotation, Whirl or Twirl], nor [...] indicate a helical or screw structure [like Twist].»
Note that Figure 4.9 picture is misleading for another important reason: there exist vector fields with lines of force configuration like that in the picture Figure 4.8 and with no divergence. For instance the field.

$$\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right) \tag{4.17}$$

is $\nabla \times \vec{F}(\vec{x}) = 0 \ \ \forall \ \vec{x}$.

So, lines of force configuration with whirlpool is not sufficiency for the curl to be non-zero.
Since the curl is a local operator, $\nabla \times \vec{V}(\vec{x})$ is a measure of the *local* circulation in a point $\vec{x}$.
Again, we could imagine this field configuration in two different ways:

1. in the "physical" way, curl different from zero means the existence of charges;

2. in the analogical way, where the field is treated as a fluid, curl different from zero means that the fluid experiences a perpetual infinitesimal rotation around the circling point $\vec{x}$; in this analogy, while the substance is moving around the point, the velocities field remains uniform and it is represented by a static field.

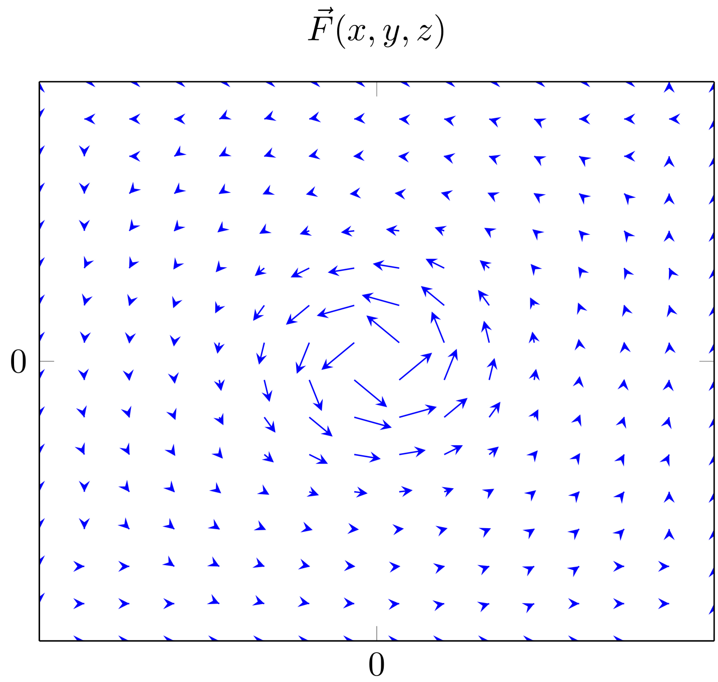


Figure 4.10: Despite the arrows are circulating around the center of the figure, there is no circling point in $(0; 0; 0)$ because the curl (the local circulation) is zero everywhere.

4.2.4 Divergence in in Educational Physics Context

I will show that a local field flux different from zero means that there exist *scalar charges* for the field. First, the Divergence theorem will be derived in order to show what *local flux* means and why divergence is its measure. Then, the divergence of the field will be locally compared with the density of the field's scalar charge.

Divergence Theorem.

$$\oint_{\partial V} \vec{F} \cdot d\vec{S} = \int_V \nabla \cdot \vec{F} \; dV \tag{4.18}$$

For any ball centered in $\vec{x}_P$ embracing only the charge Q, the Gauss' theorem holds

Gauss' Theorem.

$$\oint_{\partial V} \vec{F}(\vec{x}) \cdot d\vec{S} = Q = \int_V \rho(\vec{x}) \; dV \tag{4.19}$$

where Q is called the *scalar-charge* for the field $\vec{F}(\vec{x})$, $\rho(\vec{x})$ is the scalar-charge density and V is the volume of the ball.
If we consider a ball around $\vec{x}_P$, we can evaluate the net flow of the field through it. In the limiting case, the ball shrinks in the point $\vec{x}_P$ and we can evaluate the *local* net flow of the field in $\vec{x}_P$.
We will evaluate this case only for the x-direction, generalizing for the other two directions to obtain the Divergence Theorem expression (4.18). We are considering the case in Figure 4.11.

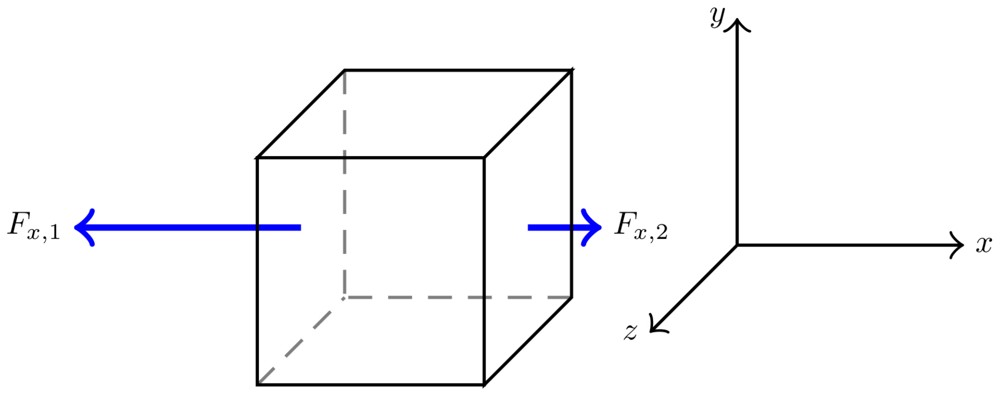


Figure 4.11: Expression (4.20)

First, we evaluate the net flow through the ball (we can consider the ball a little cube)

$$\begin{aligned}\text{Net flow in } x\text{-direction} &= -F_{x,1}\Delta y\Delta z + F_{x,2}\Delta y\Delta z = \\ &= (-F_{x,1} + F_{x,2})\,\Delta y\Delta z = \\ &= \left(\frac{F_{x,2} - F_{x,1}}{\Delta x}\right)\Delta x\Delta y\Delta z = \\ &= \left(\frac{F_{x,2} - F_{x,1}}{\Delta x}\right)\Delta V\end{aligned} \tag{4.20}$$

Then, the ball shrinks in the point $\vec{x}_P$. In this limiting case

$$\text{Net flow in } \vec{x_P} \text{ in } x\text{-direction} = \left(\frac{dF_x}{dx}\right) dV \tag{4.21}$$

Generalizing for the three coordinates,

$$\vec{F} \cdot d\vec{S} = \text{Net flow in } \vec{x_P} = \text{Net flow in } x + \text{Net flow in } y + \text{Net flow in } z \tag{4.22}$$

so

$$\vec{F} \cdot d\vec{S} = \left(\frac{dF_x}{dx} + \frac{dF_y}{dy} + \frac{dF_z}{dz} \right) dV \tag{4.23}$$

Now consider the net flow through the ball

$$\oint_{\partial V} \vec{F} \cdot d\vec{S} = \int_V \left(\frac{dF_x}{dx} + \frac{dF_y}{dy} + \frac{dF_z}{dz} \right) dV = \int_V \nabla \cdot \vec{F} \, dV \tag{4.24}$$

Taking Gauss theorem, it's easy to see that the local relation holds

$$\nabla \cdot \vec{F}(\vec{x}) = \rho(\vec{x}) \tag{4.25}$$

What does it means?

$$\oint_{\partial V} \vec{F} \cdot d\vec{S} = \int_V \nabla \cdot \vec{F} \, dV = Q \tag{4.26}$$

In words, from left to right, it means that the sum over the whole closed surface of the product between the field vector and the normal to the infinitesimal surface is equal to the sum over the whole volume of the field divergences in every points within the volume, which is equal to the value of the scalar-charge Q enclosed by the volume V.
So, the integration in (4.26) *counts* how many points in the volume V have scalar-charge density different from zero. As an example, the integration (4.26) on the volume V is shown in Figure 4.12: the whole set of cubes is V, while the colored ones are infinitesimal balls whose divergence is different from zero.

The colored part of the volume represents the scalar-charge Q.

So, the divergence is the measure of the local net flow of the field in any point $\vec{x}$ of the space and it accounts for scalar-chagre density.

4.2.5 Curl in an Eductional Physics Context

I will show that a local field circulation different from zero means that there exist *vector charges* for the field. First, the Stokes theorem will be derived

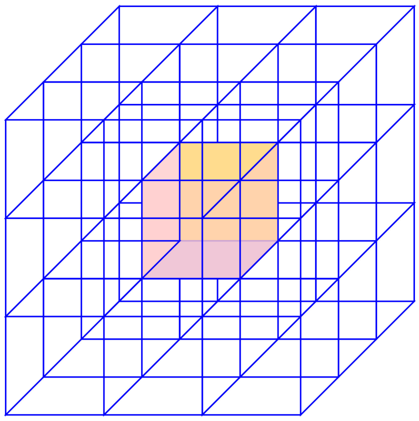


Figure 4.12: Colored cubes are the infinitesimal balls with non zero divergence

in order to show what *local circulation* means and why curl is its measure. Then, the curl of the field will be locally compared with the density of the field's vector charge.

Stokes' Theorem.

$$\oint_{\partial S} \vec{F} \cdot d\vec{l} = \int_S \nabla \times \vec{F} \cdot d\vec{S} \tag{4.27}$$

For any ball centered in $\vec{x}_P$ embracing only the charge i, the Ampère theorem reveals that

Ampère's Theorem.

$$\oint_{\partial S} \vec{F} \cdot d\vec{l} = i = \int_S \vec{j} \cdot d\vec{S} \tag{4.28}$$

where i is the module of $\vec{i}$, which we call the *vector-charge* for the field $\vec{F}(\vec{x})$, $\vec{j}(\vec{x})$ is the superficial vector-charge density and S is the surface of the ball.
If we consider a ball around $\vec{x}_P$, we can evaluate the field circulation on the ball around it. In the limiting case, the ball shrinks in the point $\vec{x}_P$ and we can evaluate the *local* field circulation in $\vec{x}_P$.
We will evaluate this case only for the a surface perpendicular to the z-direction, generalizing for the other two directions to obtain the Stokes Theorem expression (4.27). We are considering the case Figure 4.13. We

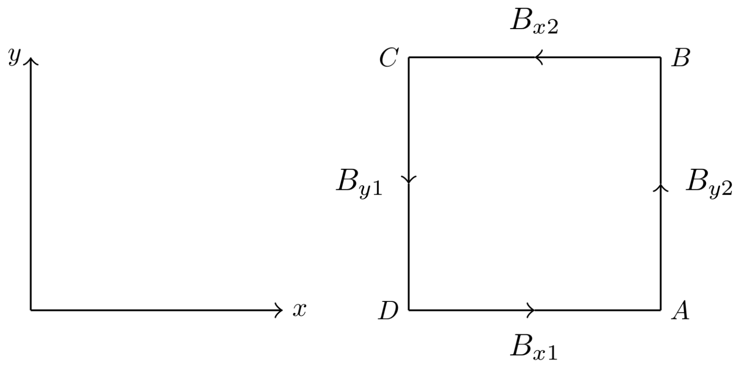


Figure 4.13: Expression (4.29)

call the z-circulation the circulation on the plane perpendicular to z.

First, we evaluate the circulation around any closed path on the ball.

$$\begin{aligned}\text{z-Circulation around } x_P =& F_{x,1}\Delta x + F_{y,2}\Delta y - F_{x,2}\Delta x - F_{y,1}\Delta y = \\ =& -\left(F_{x,2} - F_{x,1}\right)\Delta x + \left(F_{y,2} - F_{y,1}\right)\Delta y = \\ =& -\left(\frac{\Delta F_x}{\Delta y}\right)\Delta x \Delta y + \left(\frac{\Delta F_y}{\Delta x}\right)\Delta x \Delta y = \\ =& \left(\frac{\Delta F_y}{\Delta x} - \frac{\Delta F_x}{\Delta y}\right)\Delta S_z\end{aligned} \tag{4.29}$$

Where ΔS_z is the surface perpendicular to the z-direction. Then, the ball shrinks in the point $\vec{x_P}$. In this limiting case

$$z\text{-Circulation in } \vec{x_P} = \left(\frac{dF_y}{dx} - \frac{dF_x}{dy}\right) dS_z \tag{4.30}$$

Generalizing for the three coordinates,

$$\vec{F} \cdot d\vec{l} = \text{Circulation in } \vec{x_P} = x\text{-circulation} + y\text{-circulation} + z\text{-circulation} \tag{4.31}$$

so

$$\vec{F} \cdot d\vec{l} = \left(\frac{dF_z}{dy} - \frac{dF_y}{dz}\right) dS_x + \left(\frac{dF_x}{dz} - \frac{dF_z}{dx}\right) dS_y + \left(\frac{dF_y}{dx} - \frac{dF_x}{dy}\right) dS_z \tag{4.32}$$

Now consider the circulation around the ball

$$\oint_{\partial S} \vec{F} \cdot d\vec{l} = \int_S \left(\frac{dF_z}{dy} - \frac{dF_y}{dz} \right) dS_x + \left(\frac{dF_x}{dz} - \frac{dF_z}{dx} \right) dS_y + \left(\frac{dF_y}{dx} - \frac{dF_x}{dy} \right) dS_z =$$
$$= \int_S \nabla \times \vec{F} \cdot d\vec{S} \tag{4.33}$$

Taking Ampère theorem, it's easy to see that the local relation holds

$$\nabla \times \vec{F} = \vec{j} \tag{4.34}$$

What does it means?

$$\oint_{\partial S} \vec{F} \cdot d\vec{l} = \int_S \nabla \times \vec{F} \cdot d\vec{S} = i \tag{4.35}$$

In words, from left to right, it means that the sum over the closed path of the scalar product between the vector field and the infinitesimal path is equal to the sum over any enclosed surface of the field curl in every points of the surface, which is equal to the value of the vector-charge $i = |\vec{i}|$ passing through the surface S.
So, the integration in (4.35) counts how many points on the surface S have vector-charge density different from zero. As an example, the integration (4.35) on the surface S is shown in Figure 4.14: the whole set of circles fill S, while the red colored ones are the infinitesimal balls whose circulation is different from zero.

The red colored circles on the surface represents the vector-charge i (the current i is perpendicular to the surface and crosses it exactly where circles are red).

So, the curl is the measure of the local circulation of the field in any point $\vec{x}_P$ of the space and it accounts for vector-charge density.

4.2.6 Physical Meaning of Differential Operators

I resume four important consequences of the latter paragraph:

- $\nabla \cdot \vec{F}(\vec{x}) = A(\vec{x})$, with $A \neq 0$: there exist convergence/divergence points for the field $\vec{F}$.
- $\nabla \cdot \vec{F}(\vec{x}) = 0$: there are no convergence/divergence points for the field $\vec{F}$.

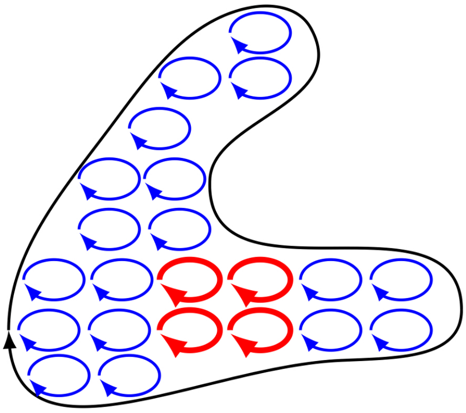


Figure 4.14: Red colored circles are the infinitesimal balls with non zero curl

- $\nabla \times \vec{F}(\vec{x}) = \vec{A}(\vec{x})$, with $\vec{A} \neq 0$: there exist circling/anticircling points for the field $\vec{F}$.
- $\nabla \times \vec{F}(\vec{x}) = \vec{0}$: there are no circling/anticircling points for the field $\vec{F}$

Now we are able to explain the meaning of the Maxwell's equations; in the guide, we will use results obtained in this section to derive Maxwell's equations from the Maxxwell's vortex model.

1. $\nabla \cdot \vec{D}(\vec{x}) = \rho(\vec{x})$: there exist scalar-charges for the electric field.
2. $\nabla \cdot \vec{B}(\vec{x}) = 0$: there are no scalar-charges for the magnetic field.
3. $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$: there exist vector-charges for the electric field.
4. $\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{j}$: there exist two types of vector-charges for the magnetic field.

Usually, with the word charge are called both convergence/divergence charges and circling/anticircling ones.

Further, thanks to this physical representation of differential operators, it is possible to derive that

$$\nabla \cdot \nabla \times \vec{F} = 0 \tag{4.36}$$

In fact, a circling point ($\nabla \times \vec{F} \neq o$) can't be at the same time a convergence point ($\nabla \cdot \nabla \times \vec{F} \neq 0$) or, in the physical view, a vector-charge can't be at the same time a scalar-charge.

4.3 The Guide

The document in this section discusses the electromagnetic field. It rises from a guided analyses of the Maxwell's paper "On Physical Lines of Force" (Maxwell, 1965b, p. 451).
The guide begins with the building of the aether model as imagined by Maxwell to model magnetic interactions. He believed that aether can represent the mechanical explanation of the magnetic interactions. Differently from Maxwell, I present the aether as a model useful to apply known mathematics and physics. My objective is to build a model that can support imagination in electromagnetism. The model of aether that I present is a sort of bridge (anchor model) aimed to support intuition in linking the formalism of Maxwell equations with the physics of the electromagnetism. The "aether model" will be named the *source system* [SOURCE]; the electromagnetic field will be called the *target system* [TARGET] so as to strengthen the role of aether: it is a source of knowledge that, analogically, will be used to interpreted the target phenomenon under investigation (electromagnetism). Thanks to this model, some "hidden" properties of the electromagnetic field arise. In particular, I will show how Ampère law and FNL rule and the electromagnetic waves follow the same *compensation principle*.
We supposed readers acquainted with the secondary school mathematics and physics. Sections 4.2.1 and 4.2.3 are required in order to deal with this guide.

I followed Maxwell's argumentation as much as I could and there are long sentences just reported and commented. However, some sections have been re-elaborated when the original text became too hard for a modern reader or too complex for the scope of present reconstruction. Luckily, Maxwell was a great writer, and his narrative was usually very clear and complete. Moreover, the text is very refined from an argumentative and methodological point of view. This allowed me to comment the text also for its epistemological value.

The first part of the guide follows the introduction of the paper. Faraday's electrotonic state and lines of force are described in details, with a metaphorical introduction to what it means "thinking an interaction in term

of fields" and "thinking an interaction in term of forces". Then, a model of aether is proposed, to explain mechanically the existence of lines of force.
Maxwell intended to explain the electromagnetism with a mechanical model; he thought mechanics was the foundation of all physics: each single branch of physics could be said "explained" only if its laws laid on a mechanical ground, he thought.
Many researches (f.i. (DiSessa, 1993)) argue that students reason on the same manner: they usually tend to feel satisfied if they have a mechanical vision of the physical system in exam. So, a coherent and complete mechanical model of the electromagnetic field might help students to grasp the concept of field.
The aether presented here is intended to be a model of the electromagnetic field. The aether is not supposed to exist; however, the mathematics derived from its mechanics will appear to be the same mathematics of the electromagnetic field.

Initially, we will use the known laws of magnetism to test the model. After that, the model is updated to include electric currents and, for this purpose, it is enriched with kind of idle wheels between vortexes. The induction phenomenon will rise from this upgrade. In the last part, following Maxwell, I will explain how vortexes and idle wheels interact. In this way we will show how Maxwell arrived to discover the displacement current and electromagnetic waves.
To build this guide I was inspired by (D'Agostino, 1956, 1968; Simpson, 1997; Branchetti et al., 2017). Crucial aspects of the construction of the Maxwell's equations are fleshed out, pointing out the "critical details" (Viennot et al., 2004) of his argumentation and of the electromagnetic field.

Note to the Reconstruction of Maxwell's Paper "On Physical Lines of Force"

Maxwell's argumentation is characterized by the development of an analogy between two systems.
The first system is an elastic solid body, called "aether". The presence of a magnets or of a current induces a particular partition of the aether, which divides into infinitesimal vortexes; the vortexes' rotation generates lines of tension, which, according to Faraday, attract or repel the magnetic bodies.
The second system is the electromagnetic field, which interacts with the magnetic charges; it is characterized by the mathematics of the so called "Maxwell equations".
This analogy would help reader

- to represent electromagnetic field and Maxwell's equations;
- to imagine electromagnetic field as a real object;
- to better understand the mathematical meaning of "local interactions" and "induction";
- to overcome usual misconceptions about electromagnetic waves;
- to appreciate Einstein's paradox at the beginning of his "On the Electrodynamics of Moving Bodies".

4.3.1 The Electrotonic State

On 1861-1862 Maxwell published On Physical Lines of Force with the deliberate intent to rationalize Faraday vision about lines of Force. He wrote:

«if we strew iron filings on paper near a magnet, each filing will be magnetized by [magnetic] induction, and the consecutive filings will unite by their opposite poles, so as to form fibres, and these fibres will indicate the direction of the lines of force.»

Faraday was interested on lines of force and on their physical nature. After decades of experiments, he concluded that these lines are *lines of tension*. On each line, he argued, a tension is exerted by poles, «like that of a rope.»
Faraday believed that lines of force were truly existing in nature, filling space between magnetic charges. The revolutionary impact of Faraday's vision could be resumed and stressed through the following reasoning, that aims to stress the difference between modeling an interaction in terms of forces and modeling an interaction in terms of lines of force.
Consider two point-like charges attracting each other.
I represent what it means "thinking in term of forces" in Figure 4.15, where two vector are drawn. They start from one of the charges and point through the other and, for the third law of Dynamics, their are of equal lenght.

Figure 4.15: Thinking in term of forces

In Figure 4.16, instead, the "thinking in term of field" way of reasoning is represented. Charges are connected via one "rope": a line of force. This line is a "line of tension", like that between two people playing tug-of-war. Since the rope links both charges together, it mediates attraction. In this case, the interaction travels along the rope with a finite velocity.

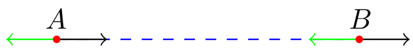


Figure 4.16: Thinking in term of fields

The rope in this example is exactly the line of force imagined by Faraday, that is, a line of tension.
A simple example for the existence of lines of tension between magnets is given looking at lines of force between two magnets. In the upper case of Figure 4.17 lines of tension pull magnets approaching them. In the other case, lines of tension keep them away.

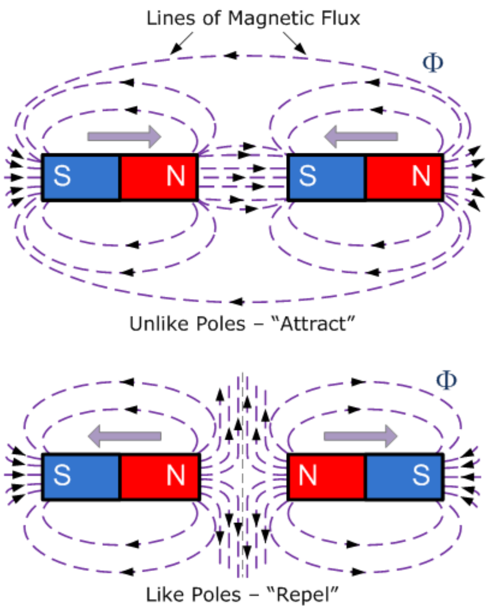


Figure 4.17: Lines of tension

Faraday called this state of tension “the electrotonic state” of the space.

Maxwell, for the reasons previously seen, wanted to go beyond action-at-a-distance model, and he was looking for a new model for interactions; indeed, he considered Faraday's lines of force the most promising model for magnetic interaction.
Despite that, he believed that each physical phenomenon must be explained with a proper mechanical model, and the lack of a mechanical view for lines of force left Maxwell dissatisfied. Instead, the aether just described appeared to be a good mechanical system to reproduce magnetic interactions and lines of force. Maxwell said:

«My object in this paper is to clear the way for speculation in this direction, by investigating the mechanical results of certain states of tension and motion in a medium, and comparing these with the observed phenomena of magnetism and electricity.»

4.3.2 The Aether: an Elastic, Solid, Anisotropic, Infinite, Continuous Body

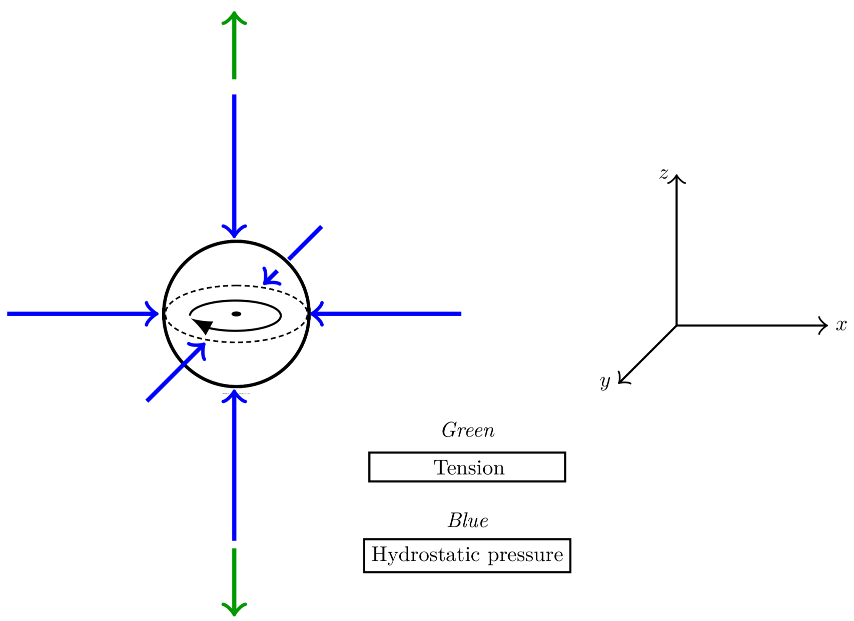


Figure 4.18: An infinitesimal portion of the continuous body. Tension means less pressure than the average

To explain mechanically the existence of this tension, let us enter the aether model: aether is an elastic, solid, anisotropic, infinite continuous body. The presence of magnetic objects induces lines of tensions (the analogical term of the lines of force) in the continuous body. Within the model, the lines of tension are direction along which the pressure is less than the average pressure of the solid. Less pressure than the average means tension (Figure 4.18).
In a fluid, the pressure anisotropy would led substance (part of the fluid itself) to move along the direction of lines of tension and the fluid would reach hydrostatic equilibrium by expanding along these lines, like in Figure 4.19.

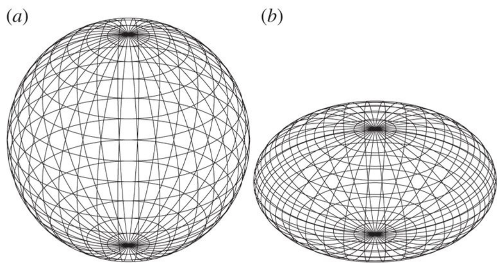


Figure 4.19: (Dritschel and Boatto, 2015)

The aether, as a solid, does not expand but it maintain this pressure anisotropy.
To explain mechanically how a pressure anisotropy is created in the ideal solid, Maxwell imagined this continuous body as filled completely by infinitesimal vortexes, rotating around the axis of symmetry, that is the axis along the lines of tension. In his words:

«What mechanical explanation can we give of this inequality of pressures in a [...] mobile medium? The explanation which most readily occurs to the mind is that the excess of pressure in the equatorial direction arises from the centrifugal force of vortexes or eddies in the medium having their axes in directions parallel to the lines of force. This explanation of the cause of the inequality of pressures at once suggests the means of representing the dipolar character of the line of force. Every vortex is essentially dipolar, the two extremities of its axis being distinguished by the direction of its revolution as observed from those points.»

In the sketch 4.20, a circular vortex is represented: with respect to the side of rotation, the vortex will be a north vortex (counterclockwise) or a south vortex (clockwise).

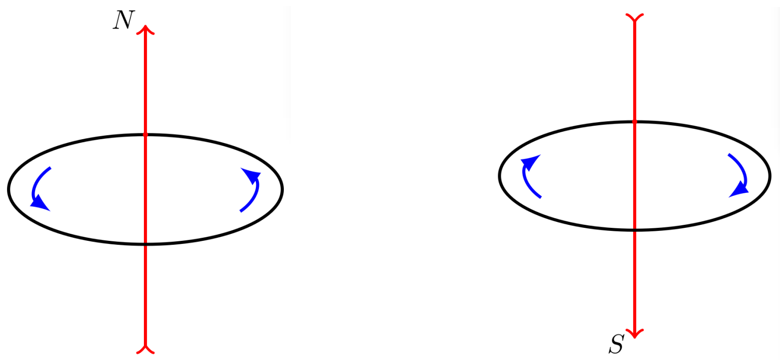


Figure 4.20: Counterclockwise vortex points through the north pole of the field, clockwise vortex through the south

In the direction of the axis of symmetry (the line of tension) the pressure must be lesser than the pressure on the equatorial plane. In fact, to create anisotropy along lines of force, vortexes are rotating around the axis of symmetry/line of tension: the centrifugal pressure plus the hydrostatic pressure on the equatorial plane will exceed the pressure along the axis. In this configuration, curved lines of forces are permitted. We underline here that this aether has two fundamental characteristics: it has mass, so it has energy; it is anisotropic, so it is in equilibrium.

«We shall suppose at present that all the vortices in any one part of the field are revolving in the same direction about axes nearly parallel, but that in passing from one part of the field to another, the direction of the axes, the velocity of rotation, and the density of the substance of the vortices are subject to change. We shall investigate the resultant mechanical effect upon an element of the medium, and from the mathematical expressions of this resultant we shall deduce the physical character of its different component parts».
Maxwell was able to derive a formal expression of the interaction between the field and a test magnetic charge. Such an expression is composed by three parts. Each of them can be interpreted in terms of properties of the magnetic field, already known qualitatively in the Faraday's model:

1. the density of the lines of force is proportional to the interaction intensity;

2. like poles repel, unlike poles attract;

3. lines of force can be curved.

Maxwell interpretes these results (which I have presented to teachers in the Second emprirical study) as the confirmation that the vortex model works.

In the following, we will inquire the properties acquired by the medium when static magnetic charges are present (magnets or currents). We will test the vortex model for magnetic interactions. Contemporary, the vortex model will help readers to give a physical meaning to the mathematics of Maxwell's equations.

4.3.3 Testing the Vortex Model: the Formal Description of Magnetic Interaction

This paragraph is dedicated to test the vortex model for magnetic interactions. At first, we will evaluate the relations between the tension of the lines of force and the vortexes' angular velocity. Then, I will demonstrate that also in the vortex model unlike poles attract and like poles repel with an intensity proportional to the lines of force density. Finally, the aether model will be used to interpret the force acting between a current carrying wire and a magnetic field.
Each step of the argumentation will be focused on the relations among vortexes in aether - charges will be only marginally considered. Analogously in TARGET space, the interactions occurring in magnetic field will be our main objects under investigation.
An important property of the magnetic field it will be highlighted: field always tends to conform, leveling any change. This property, which we called *compensation principle*, will be derived from the aether model. In this way, aether model will be overcome, in order to reach a new, more abstract, model for the electromagnetic field.

The Analogies Between the Source and the Target

Maxwell derived the tension t to be equal to

$$t = \frac{1}{4}\mu\omega^2 \tag{4.37}$$

where μ is the moment of inertia and ω the angular velocity. So, tension is directly proportional to the angular momentum $\mu\omega$.
Moreover, the equatorial pressure P_1 results

$$p_1 = \frac{1}{2}\mu\omega^2 + p_0 \tag{4.38}$$

where p_0 is the hydrostatic pressure of the aether.
In order to build the formal analogy between SOURCE and TARGET, Maxwell proposes the following comparisons:

SOURCE	TARGET
Angular velocity $\vec{\omega} = (\alpha, \beta, \gamma)$	Magnetic induction $\vec{H}$
Moment of inertia μ	Magnetic permeability μ
Angular momentum $\mu\vec{\omega}$	Magnetic Field $\vec{B}$

Moreover, because tension per unit volume has the dimension of an energy per unit of volume, in the TARGET the tension analogous will be the magnetic energy per unit of volume

SOURCE	TARGET
Angular velocity $\vec{\omega} = (\alpha, \beta, \gamma)$	Magnetic induction $\vec{H}$
Moment of inertia μ	Magnetic permeability μ
Angular momentum $\mu\vec{\omega}$	Magnetic Field $\vec{B}$
Tension $t = 1/4\mu\omega^2$	Magnetic energy density $u = B^2/(2\mu)$

The analogy can be tested in different ways.

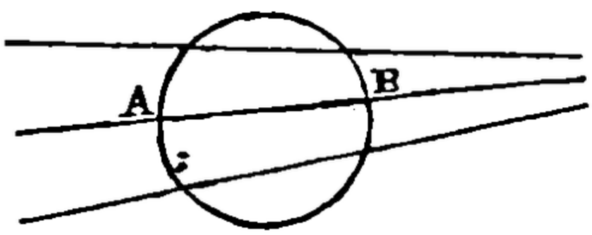


Figure 4.21: Tension is higher in B than in A

In the SOURCE, the intensity of the interactions depends on the intensity of the lines of force's tension. In the TARGET, analogously, the intensity of the interactions depends in the intensity of the magnetic energy, or on the module of the magnetic field.
In the SOURCE, it is possible to represent regions of growing tension with converging lines of force. Where the tension is higher, the angular momentum is higher too. In the TARGET, it is possible to represent regions of growing magnetic field with converging lines of force. Where the energy is higher, the magnetic field is higher too.
In the SOURCE, the intensity of the interaction depends also upon the constant μ, which is a measure of the mass distribution. In the TARGET, the intensity of the interaction depends also upon the magnetic permeability μ , which measures the magnetic response of the medium. So far, the analogy holds.
Maxwell explained the situation in Figure 4.21: the tension in A is less intense than the tension in B because the number of lines of force acting on A is lesser than that acting on B.

Like Poles Repel, Unlike Poles Attract

Following Maxwell, I will show in the following how vortex model can explain mechanically that unlike poles attract, while like poles repel.

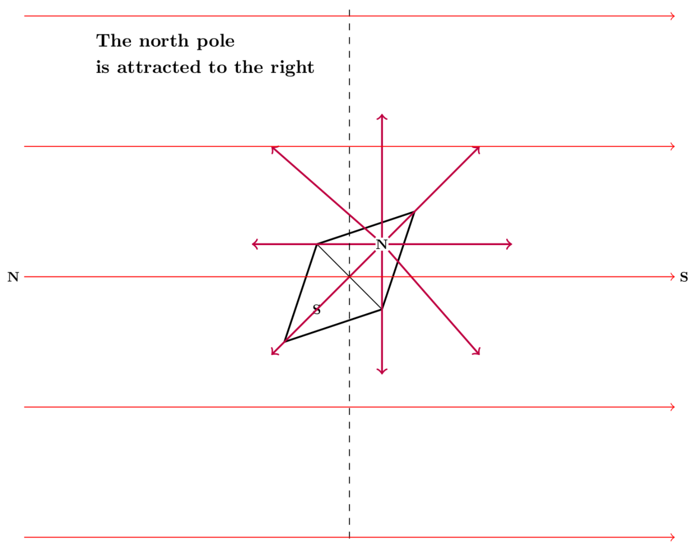


Figure 4.22: The external field acts on the magnet...

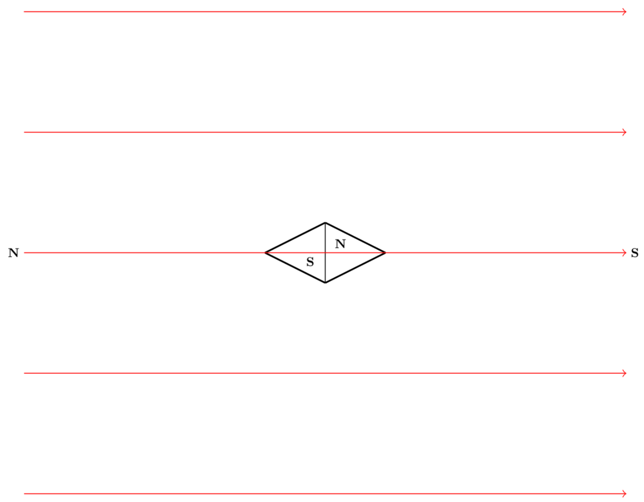


Figure 4.23: ...and the magnet aligns with the field.

In the TARGET, it is well-known that a magnet is aligned with the lines of force of an external magnetic field, like in Figure 4.22 We sketch only north pole's lines of force for simplicity. The magnet feel the external magnetic field. After a while, the magnets is aligned with the lines of force 4.23.

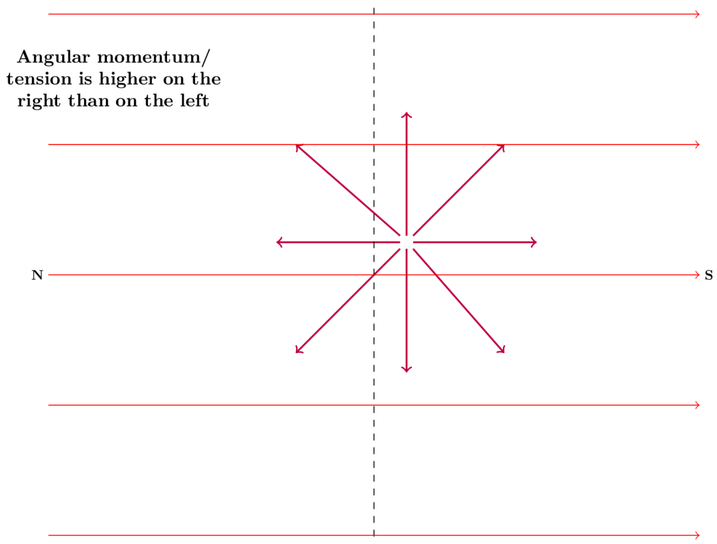


Figure 4.24

In the SOURCE, we better understand the fundamental characteristic of this magnetic interaction: magnet doesn't interact with the external field,

but it is its own field that interacts with the external one.
To understand this field interaction, we don't consider the magnet at all, but only its lines of force. When the lines of force configuration is like that in Figure 4.22, the angular momentum on the right is higher than that on the left. So, tension on the right is higher than that on the left. It means that in the aether there is a tension that pull on the right of the Figure 4.24

The Interactions Between Currents and Magnets

Ørsted discovery was immediately transposed in a mathematical form by Biot and Savart. In Nowadays symbols

$$B = \frac{\mu i}{2\pi d} \tag{4.39}$$

Ørsted himself found that the magnetic field induced by current is perpendicular to the direction of the current (Figure 4.25)

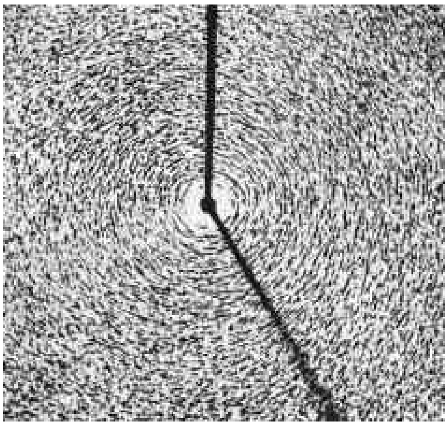


Figure 4.25: from the *Education Development Center*

Lines of force induced by a straight wire are circles on the perpendicular planes with respect to the direction of current. According with Biot-Savart law, their density diminishes with the distance from the wire (Figure 4.26)

If the wire is immersed in a uniform magnetic field (Figure 4.27) directed perpendicularly to the wire, a force will act on the wire.

From Figure 4.27 We note that the force is directed along the region with smaller magnetic field intensity, where lines of force subtract, to the left of this sheet of paper.
Ampère pursued Ørsted researches, finding that two current carrying wires attract or repel themselves according to the law

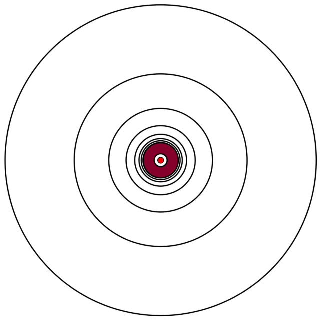


Figure 4.26: Lines of force around a current carrying wire

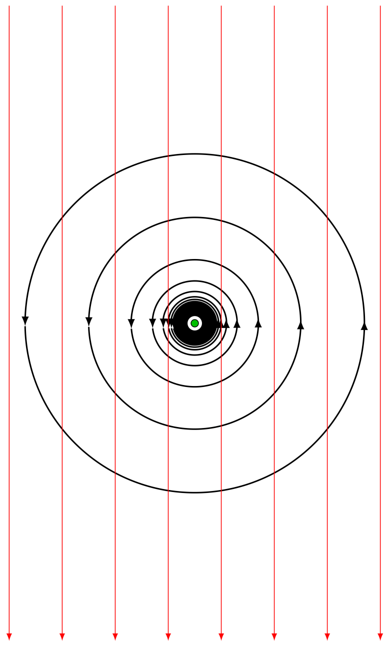


Figure 4.27: A current carrying wire – perpendicular to this sheet of paper – immersed in a uniform perpendicular magnetic field

$$F = \mu_0 \frac{i_1 i_2}{d} l \tag{4.40}$$

if we consider two current carrying wires nearby, as those in Figure 4.28 they will attract themselves if currents have the same verse.

The graphic representation of the magnetic field is that in Figures 4.29

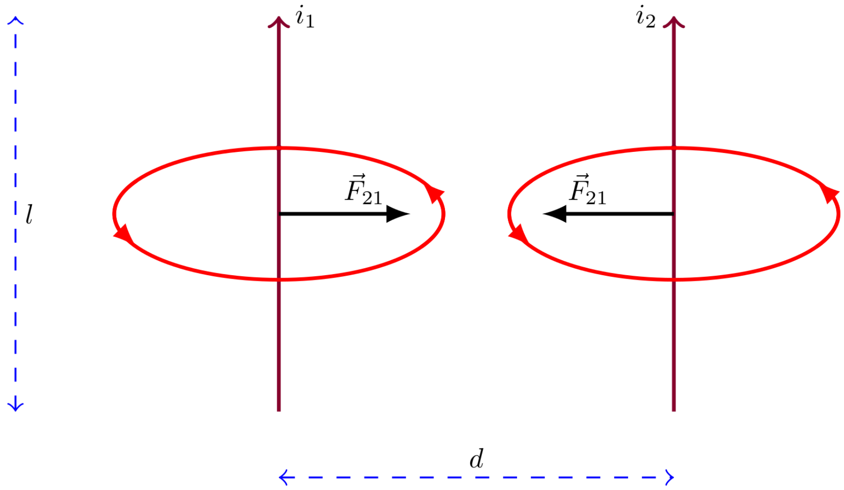


Figure 4.28: Two current carrying wires attracting themselves

and 4.30. Note that in 4.29 the magnetic field is intenser in the region between the current carrying wires than in the other regions. On the contrary, in 4.30 the magnetic field is intenser in the region outside the current carrying wires than in between. With the right-hand rule it is possible to test that the force is directed through regions with lower magnetic field.

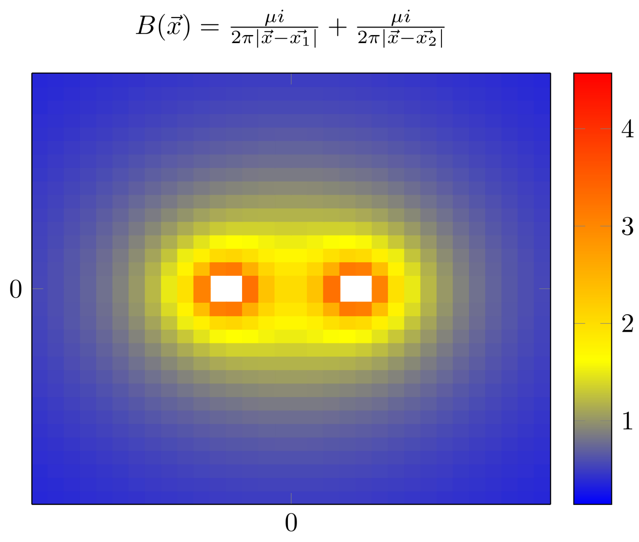


Figure 4.29: Two current carrying wires with the same current

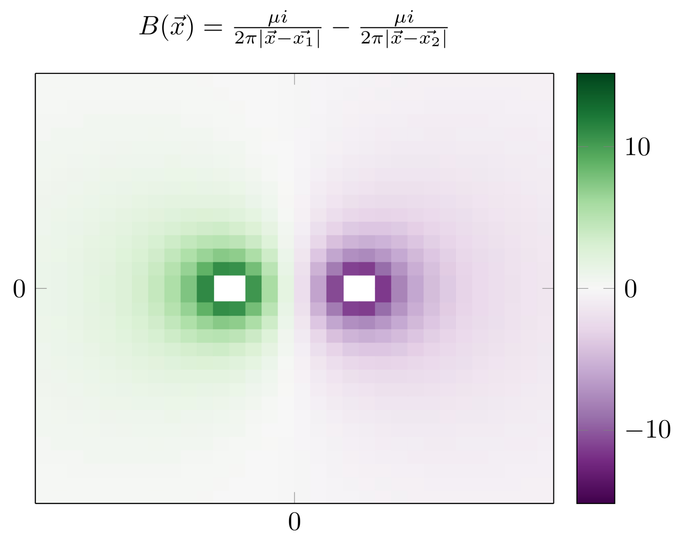


Figure 4.30: Two current carrying wires with opposite current

Again, we note that the force is directed along the region with smaller magnetic field intensity.
As already mentioned, this is a common feature of the interactions between currents and magnetic fields, which we called the *compensation principle*. Usually, in the electromagnetic context, students dealt with this property when they faced with the "Lenz" law. In the following, I will explain this feature for the Ampère law using the vortexes model.

In the TARGET, a current carrying wire induces a circling magnetic field on the plane perpendicular to the current direction. In Figure 4.31 I represented the vertical wire and the relative induced magnetic field.

In the SOURCE, we obtain the same configuration in this way

In this situation, equatorial pressure $p_1 = 1/2\mu\omega^2 + p_0$ between vortexes rotating in the same direction varies continuously, and the differences between equatorial pressures are infinitesimal. Moreover, the equatorial pressure between closed vortexes rotating in opposite direction is the same, because the angular velocity's module is the same. So, no net equatorial pressure arises from this configuration, and the system is in equilibrium.
In the same way, in the TARGET no force acts on the wire, which is in equilibrium too.

On the contrary, in the TARGET, if there is an external magnetic field

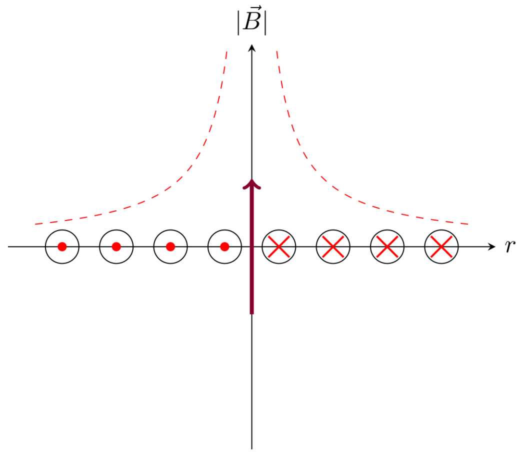


Figure 4.31: No External magnetic field

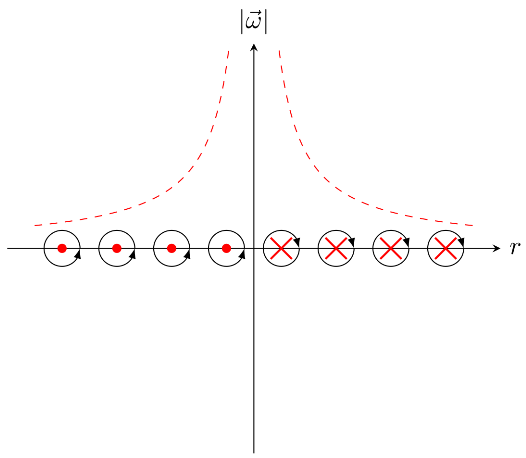


Figure 4.32: No discontinuities in the equatorial pressures

perpendicular to the wire and with the same direction of lines of force induced by the wire, the situation changes (Figure 4.33)

In the SOURCE, we obtain the same configuration in this way

In this situation, equatorial pressure $p_1 = 1/2\mu\omega^2 + p_0$ between vortexes rotating in the same direction varies continuously, and the differences between equatorial pressures are infinitesimal. Differently, the equatorial pressure between closed vortexes rotating in opposite direction is no longer the same. A net equatorial pressure arises from region of higher rotation to region of lower rotation. In both system a force from left to right acts on the center

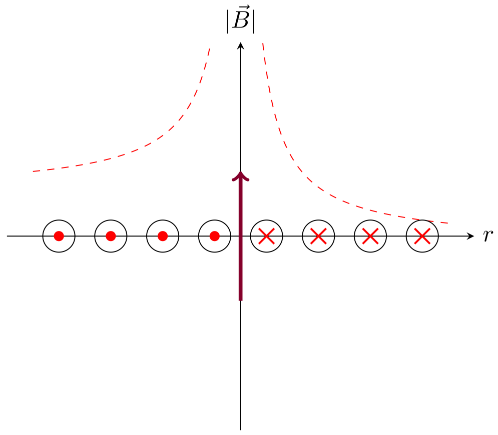


Figure 4.33: With External magnetic field

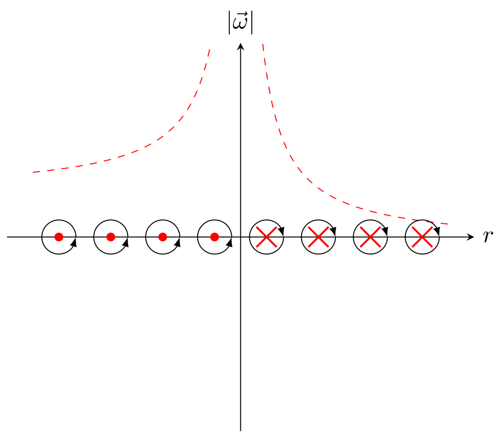


Figure 4.34: Discontinuities in the equatorial pressures

of the configuration.

4.3.4 Magnetostatics Equations

In order to find a mechanism for magnetic interactions, we need to derive the mathematical relations between charges and field. As already seen in the previous sections, differential operators provide all the information about the relations among charges and fields. In Newtonian paradigm, to know how charges interact in a specific physical framework, the laws of forces acting on them are required. In the Maxwellian paradigm, the attention shifts from charges to fields; all we need to know is the initial shape of the field and how it will change in time. Charges become the measure of the field behavior with respect to them. When the divergence and the curl of a three dimensional vector field are known, the theory can be considered satisfying.

Thus, we need to know the divergence and the curl of the vector field $\vec{B}$. So far, magnetic scalar-charges (also called magnetic monopoles) have been never observed and it is correct to write that (see section 4.2.4)

$$\nabla \cdot \vec{B} = 0 \tag{4.41}$$

On the contrary, it is possible to argue that magnetic vector charges exist. In fact, the lines of force configuration induced by a current carrying wire can be a clue for their existence. We can suppose that (see section 4.2.5)

$$\nabla \times \vec{B} = k\vec{j} \tag{4.42}$$

where $\vec{j}$ is the current surface density of the current i and k a proportionality constant. So

$$\int_S \nabla \times \vec{B} \cdot d\vec{S} = \int_S k\vec{j} \cdot d\vec{S} \tag{4.43}$$

For Stokes theorem (see section 4.2.3), this equation is equivalent to the following one

$$\oint_{\partial S} \vec{B} \cdot d\vec{l} = ki \tag{4.44}$$

The closed path can be chosen among infinite closed paths encircling the vector charge. If we choose a circular path laying on the plane perpendicular to the wire and centered on it, the magnetic field is always tangential to the path. In this case $\vec{B} \cdot d\vec{l} = Bdl$, with B constant on the path. Then, if d is the radius of the circumference,

$$\oint_{\partial S} \vec{B} \cdot d\vec{l} = B \oint_{\partial S} dl = 2\pi dB = ki \tag{4.45}$$

Calling $k = \mu_0$, we found the Biot-Savart law (see Section 4.2.1)

$$B = \frac{\mu_0 i}{2\pi d} \tag{4.46}$$

We can conclude that the surface current density is the vector charge for the magnetic field

$$\nabla \times \vec{B} = \mu_0 \vec{j} \tag{4.47}$$

We have seen that the aether model agrees with all properties of magnetism: like poles repel, unlike pole attract; the strength of interactions grows where lines of force converging; the Ampère law; the Gauss law for the magnetic field.
The value of the constant μ depends on the medium: in the empty space, we measure μ_0; measuring it in a solid body, its value changes, depending on the property of the body itself. In the SOURCE it is the mass density of the aether, in the TARGET it is the magnetic permeability of the vacuum. Thanks to this model, I hope that this constant appears as something real: it is not a constant of the vacuum, but a characteristic of the magnetic field in a medium.

I have shown how the Newtonian paradigm applied to magnetic charges interactions can be imaged in a different way. This way, named the Maxwellian paradigm, focuses on magnetic *fields* interactions. Fields "reacts" to changes in order to level them (*compensation principle*), following the mathematical expressions (4.41) and (4.47).
Now that the model was shown to work for magnetic interactions, it can be updated to include electric interactions. This is indeed the next step that Maxwell did in it paper.

4.3.5 The Theory of Aether Applied to Electric Currents

In the following, we report directly long quotations by Maxwell to introduce his model of electric currents. This model, in the SOURCE, allows to explain how vortexes are set in rotation. The consequence in the TARGET will be the electromagnetic induction.

«We have as yet given no answers to the questions, "How are these vortices set in rotation?" and "Why are they arranged according to the known laws of lines of force about magnets and currents?" These questions are certainly of higher order of difficulty than either of the former [...] We have, in fact, now come to inquire into the physical connexion of these vortices with electric currents, while we are still in doubt as to the nature of electricity, whether it is one substance, two substances, or not a substance at all, or in what way it differs from matter, and how it is connected with it.
We know that the lines of force are affected by electric currents, and we

know the distribution of those lines about a current; so that from the force we can determine the amount of the current. Assuming that our explanation of the lines of force by molecular vortices is correct, why does a particular distribution of vortices indicate an electric current? A satisfactory answer to this question would lead us a long way towards that of a very important one, "What is an electric current?"

I have found great difficulty in conceiving of the existence of vortices in a medium, side by side, revolving in the same direction about parallel axes (Figure 4.35). The contiguous portions of consecutive vortices must be moving in opposite directions; and it is difficult to understand how the motion of one part of the medium can coexist with, and even produce an opposite motion of part in contact with it.

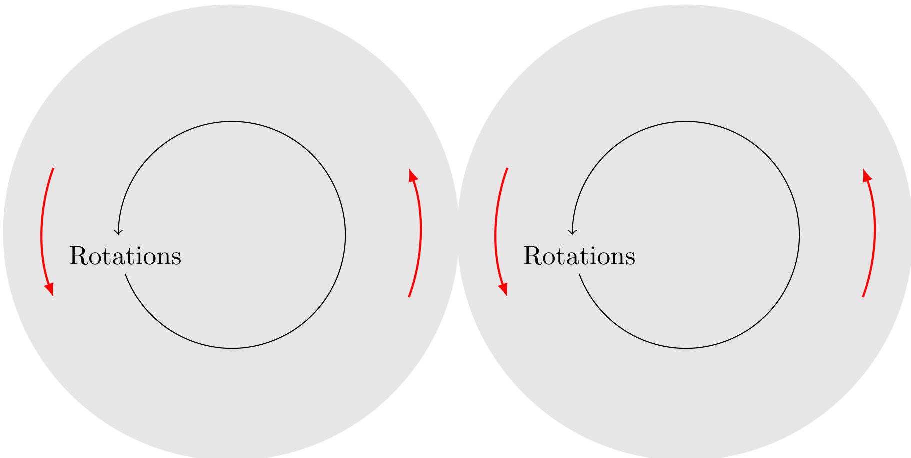


Figure 4.35: Two contiguous vortexes rotating in the same direction

The only conception which has at all aided me in conceiving of this kind of motion is that of the vortices being separated by a layer of particles, revolving each on its own axis in the opposite direction to that of the vortices, so that the contiguous surfaces of the particles and of the vortices have the same motion.

In mechanics, when two wheels are intended to revolve in the same direction, a wheel is placed between them so as to be in gear with both, and this wheel is called an "idle wheel" (Figure 4.37). The hypothesis about the vortices which I have to suggest is that a layer of particles, acting as idle wheels, is interposed between each vortex and the next, so that each vortex has a tendency to make the neighbouring vortices revolve in the same direction with itself.

In mechanics, the idle wheel is generally made to rotate about a fixed axle; but in epicyclic trains and other contrivances, as, for instance, in Siemens' governor for steam-engines, we find idle wheels whose centres are capable of motion.»

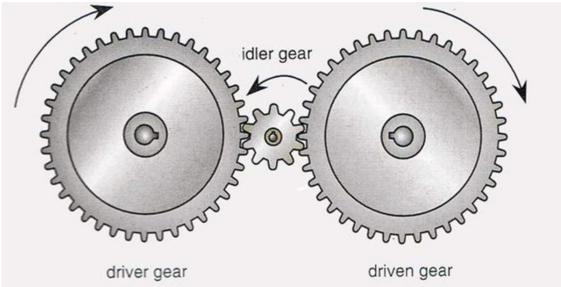


Figure 4.36: Siemens' idle wheels can translate other than rotate

I reported the beginning of the second part of "On Physical Lines of Force" almost entirely. In this part, Maxwell's way of reasoning and his technical background appears clearly. He believed that only mechanics can explain and explore a phenomenon. And he knew very well the technical innovations of his time.
So, idle wheels both rotate and translate between vortexes. Later, Maxwell represents the upgraded aether in Figure 4.38

Hexagonal[9] vortexes are separated by idle wheels, which can only rotate and translate, and they are incompressible. Plus and minus on vortexes indicate the verse of rotation: plus is counterclockwise, minus is clockwise.

«We may conceive that these particles are very small compared with the size of a vortex [...] The particles must be conceived to roll without sliding between the vortices which they separate, and not to touch each other, so that, as long as they remain within the same complete molecule, there is no loss of energy by resistance. When, however, there is a general transference of particles in one direction, they must pass from one molecule to another, and in doing so, many experience resistance, so as to waste electrical energy and generate heat.»

Aether does not experience resistance only if it is in vacuum. This fundamental characteristic denotes an exclusive nature of aether: it has mass, but this mass is of a different nature. Aether particles do not experience

[9]The shape of vortexes is not important

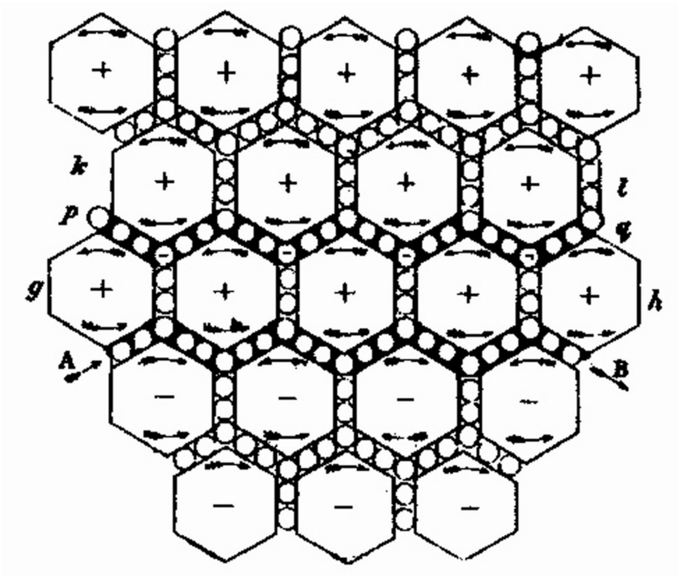


Figure 4.37: Transversal section of the aether

resistance among each others.
Electric current and vortexes start to move together, as Maxwell explains to justify Ampère law (Figure 4.38)

Maxwell evaluated the current density $\vec{j} = (p, q, r)$ in the SOURCE system, by considering that it is equivalent to the number of wheels per unit of time. The particle momentum is $m\vec{v}$ and the force acting on them is the tangential force $\vec{F}_t = m\vec{a}$.
Figure 4.38 indicates a possible solution for the interdependence between current and magnetic field. Maxwell said that «It appears therefore that, according to our hypothesis, an electric current is represented by the transference of the moveable particles interposed between the neighboring vortexes.»
At present, vortexes are rigid and they can not be deformed.

4.3.6 The Faraday-Neumann-Lenz Law

Maxwell proposes to update the analogy:

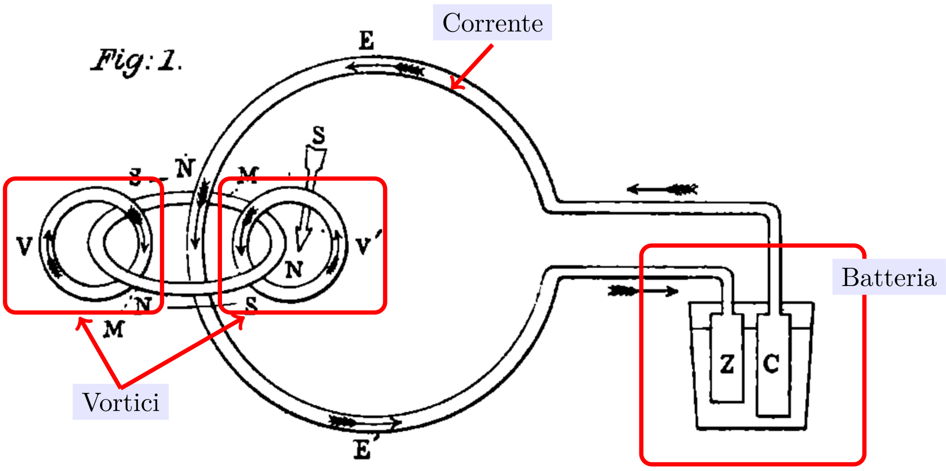


Figure 4.38: Currents induce magnetic field

SOURCE	TARGET
Angular velocity $\vec{\omega} = (\alpha, \beta, \gamma)$	Magnetic induction $\vec{H}$
Moment of inertia μ	Magnetic permeability μ
Angular momentum $\mu\vec{\omega}$	Magnetic Field $\vec{B}$
Number of idle wheels per unit of time	Current density $\vec{j}$
Tangential force $\vec{F}_t = (P, Q, R)$	Electric Field $\vec{E}$

Maxwell was able to derive from the vortex model that

$$-\nabla \times \vec{F}_t = \frac{d(\mu\vec{\omega})}{dt} \tag{4.48}$$

In the TARGET, the last expression (4.48) is analogous to

$$-\nabla \times \vec{E} = \frac{d\vec{B}}{dt} \tag{4.49}$$

This is the general law of induction, the third of the Maxwell's equations. Expression (4.49) means that a changing magnetic field is a vector-charge for the electric field.

Maxwell knew that electric charges exist, and they are positive or negative. So, for the electric field, the electric charge (a scalar-charge) is a convergence point. So, we can write that

$$\nabla \cdot \vec{E} = \varepsilon\rho \tag{4.50}$$

with ρ being the scalar-charge density for the electric charge Q and ε a constant.

In Figures 4.39-4.40, the mechanism which activates induction using the aether model is explained.

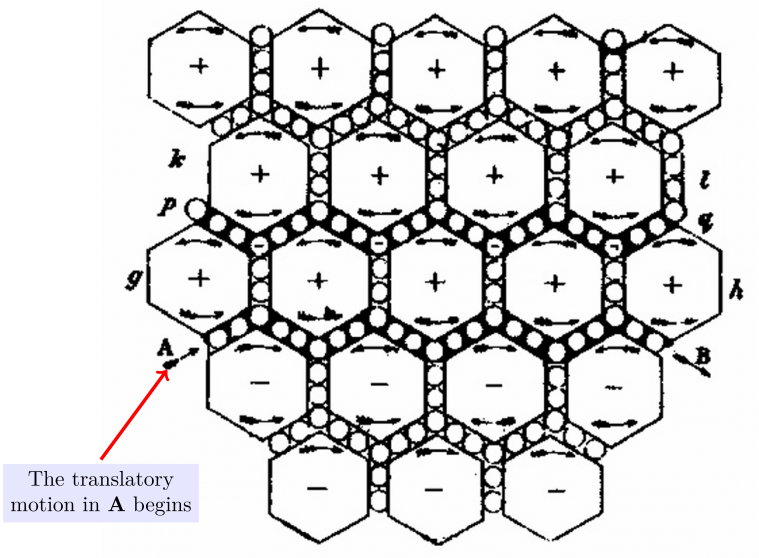
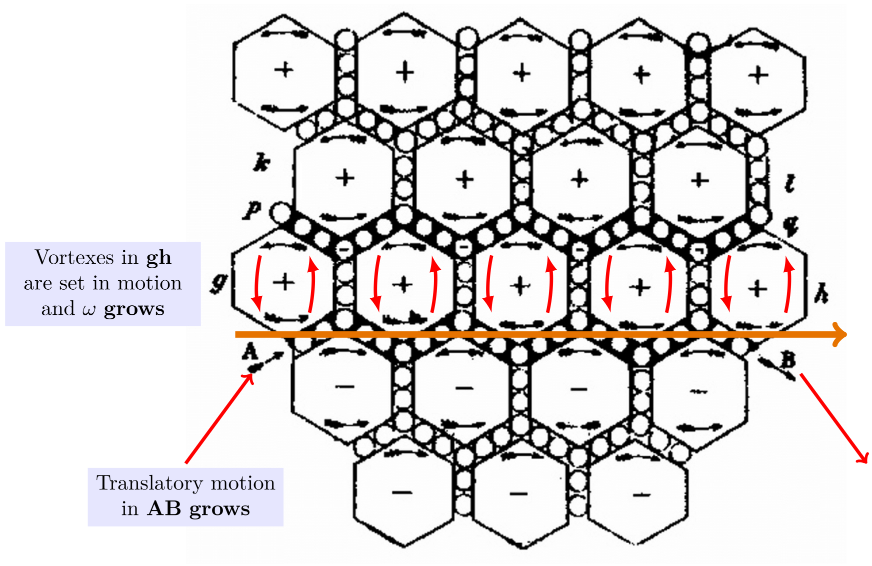
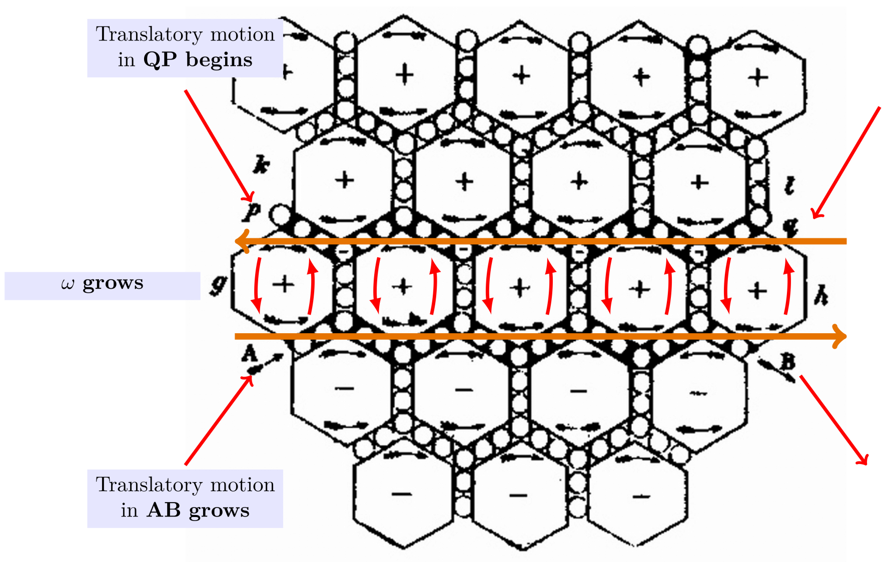
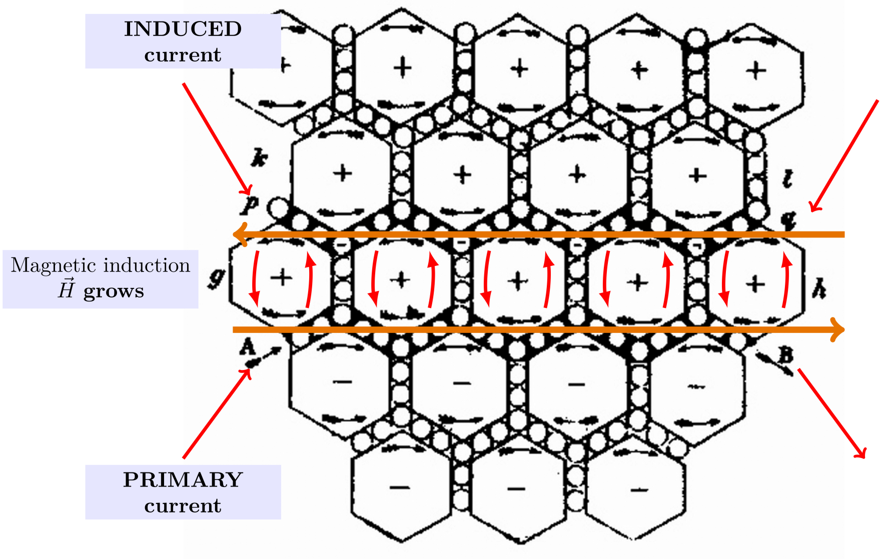
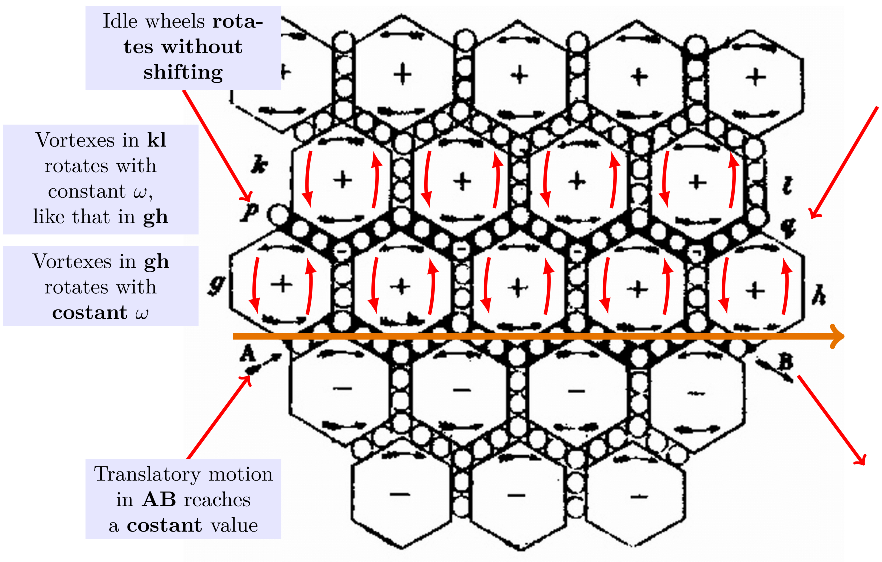


Figure 4.39: Induction begins


Vortexes in gh
are set in motion
and ω grows
Translatory motion
in AB grows
k
p
g
l
q
h
A
B


Translatory motion
in QP begins
ω grows
Translatory motion
in AB grows
k
p
g
l
q
h
A
B


INDUCED current
Magnetic induction $\vec{H}$ grows
PRIMARY current
k
p
l
q
g
h
A
B


Idle wheels rotates without shifting
Vortexes in kl rotates with constant ω, like that in gh
Vortexes in gh rotates with costant ω
Translatory motion in AB reaches a costant value
k
p
l
q
g
h
A
B

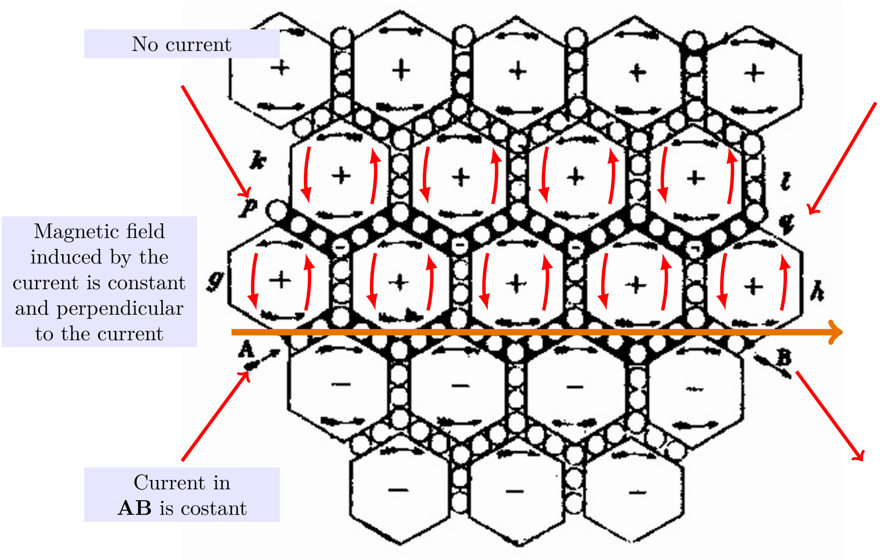


Figure 4.40: Induction ends

The Potential Vector

Idle wheels have a velocity and a mass, so it is possible to define their quantity of motion per unit of length $\vec{P} = (F, G, H)$. What is the TARGET analogous? To answer this question, we make the same observations done for the evaluation of the tangential force curl. We obtain

$$\begin{aligned} \left(\frac{dG}{dz} - \frac{dH}{dy}\right) &= \mu\alpha \\ \left(\frac{dH}{dx} - \frac{dF}{dz}\right) &= \mu\beta \\ \left(\frac{dF}{dy} - \frac{dG}{dx}\right) &= \mu\gamma \end{aligned} \tag{4.51}$$

Deriving with respect to time and considering the system (4.48), it is easy to conclude that

$$P = \frac{dF}{dt} \qquad Q = \frac{dG}{dt} \qquad R = \frac{dH}{dt} \tag{4.52}$$

The tangential force acting on the idle wheels is equal to the time derivative of their quantity of motion, that is, we have found the second law of mechanics.
In the TARGET, the latter equation can be written

$$\vec{E} = \frac{d\vec{A}}{dt} \tag{4.53}$$

and the upper system as

$$\nabla \times \vec{A} = -\vec{B} \tag{4.54}$$

where $\vec{A}$ is a new vector, which Maxwell defines the *electromagnetic impulse*. In fact, the vector $\vec{E}$ is the time derivative of $\vec{A}$, as the force $\vec{F}$ is the time derivative of the mechanical impulse $\vec{P}$. Nowadays, we call it the *vector potential* (with the opposite sign) and, as at the secondary school like at the university, it is not, let us say, the most important mathematical-physical entity of the electromagnetism. Maxwell, instead, believed that it was a fundamental concept for the understanding of the electromagnetic interactions: he thought that the vector potential was «that which Faraday has conjectured to exist, and has called the electrotonic state.»

We update the framework of the analogy with the potential vector:

SOURCE	TARGET
Angular velocity $\vec{\omega} = (\alpha, \beta, \gamma)$	Magnetic induction $\vec{H}$
Moment of inertia μ	Magnetic permeability μ
Angular momentum $\mu\vec{\omega}$	Magnetic Field $\vec{B}$
Number of idle wheels per unit of time	Current density $\vec{j}$
Tangential force $\vec{F_t} = (P, Q, R)$	Electric Field $\vec{E}$
Quantity of motion $\vec{P} = (F, G, H)$	Potential vector $\vec{A}$

In the case of the electromagnetic induction phenomenon, the Lenz law is the manifestation of the *compensation principle*. Many textbooks, while they are introducing the Lenz law, they speak about “opposition” Nevertheless, it is possible to find in the same text book a mention to a sort of “compensation”. For instance:

> **The Lenz law**
>
> [...]
>
> An induced current always flows in the direction which is *opposed* to the variation that caused it.
>
> [...]
>
> The induced current flows in order to oppose this variation and it generates [...] a field [...] which tend *to compensate* the [variation].
>
> (Walker, 2008, p. E166-167)

Textbooks’ introductions to Lenz law usually mention “current” and “circuit”, and the compensation principle is often presented in a qualitative way, sometimes speaking of energy conservation. I have already spoke about misunderstanding derived from this approach to electromagnetic induction. What I want to underline here is that the aether model presented in this thesis and the compensation principle described in Section 4.3.3 explains

also the electromagnetic induction in terms of field iteractions. Indeed, our approach explains that is the field that compensates variations occurring in the field itself, and the rise of an induced current is the manifestation of this principle when a closed circuit is in the neighborhood of these variations.
Usually students show to think that a flux variation generates current, without any reference to interactions on terms of field. Thinking the interactions in terms of fields can help them to see in the right way the electromagnetic induction and to enter the meaning of the electromagnetic field.

4.3.7 Energy transmission between vortexes and wheels

So far, Maxwell has obtained the mathematical formulation for all the electromagnetic phenomena known at his time. But he was unsatisfied; he wondered how energy was transmitted from vortexes to idle wheels. In his own words:

«I have not attempted to explain this tangential action, but it is necessary to suppose, in order to account for the transmission of rotation from the exterior to the interior parts of each [vortex], that the substance in the [vortexes] possesses elasticity [...]
According to our theory, the particles which form the partitions between the [vortexes] constitute the matter of electricity. The motion of these particles constitutes an electric current; the tangential force with which the particles are pressed by the matter of the [vortexes] is [the electric field] [...]
If we can now explain the condition of a body with respect to the surrounding medium when it is said to be "charged" with electricity, and account for the forces acting between electrified bodies, we shall have established a connexion between all the principal phenomena of electrical science [...]
Bodies which do not permit a current of electricity to flow through them are called insulators. But though electricity does not flow through them, electrical effects are propagated through them, and the amount of these effects differs according to the nature of the body; so that equally good insulators may act differently as dielectrics [...] A conducting body may be compared to a porous membrane which opposes more or less resistance to the passage of a fluid, while a dielectric is like an elastic membrane which may be impervious to the fluid, but transmits the pressure of the fluid on one side to that on the other.»
In the latter excerpt, Maxwell tried to resume how physicists conceive insulators and conductors. He made a metaphor: a conducting body is a porous membrane because it permits electricity to flow; insulators do not

permit electricity to flow, but they transmit the presence of a charge thanks to the polarization of their molecules. This polarization is explained by a mechanical analogy, that is, by comparing the insulators with an elastic body, which transmits only energy.
«[The electric field] acting on a dielectric produces a state of polarization of its parts [...] In a dielectric under induction, we may conceive that the electricity in each molecule is so displaced that one side is rendered positively, and the other negatively electrical, but that the electricity remains entirely connected with the molecule, and does not pass from one molecule to another.
The effect of this action on the whole dielectric mass is to produce a general displacement of the electricity in a certain direction. This displacement does not amount to a current, because when it has attained a certain value it remains constant, but it is the commencement of a current [...]»
Summarizing, Maxwell wanted to explain mechanically how tangential actions are transmitted from vortexes to idle wheels. Between two electrical charged bodies currents flows if a conductors is placed between them; otherwise, molecules of the insulating medium polarized. In the latter case, the "electricity" is confined inside molecules of the insulator.
In order to let the energy flow transmit between vortexes and idle sphere, Maxwell supposed the vortexes' substance to be elastic. While vortexes are deforming, they take off idle wheels from their equilibrium position.

Therefore, I will derive the vortex deformation force acting on a line of idle wheels. To reach the goal, I will apply Hook's law to evaluate the force acting on n idle wheels caused by the vortexes deformation. For Hook's law, the action of the vortex deformation on one idle wheels is, in the x-direction,

$$\Delta f_x = k\Delta x \tag{4.55}$$

where k is a constant factor characteristic of the vortex. The total force on n-idle wheels per unit of length is:

$$\Delta F_x = nk\Delta x \tag{4.56}$$

In the TARGET, the force acting on idle wheels is the electric field. The displacement of idle wheels is a displacement of electric particles, that is

$$n\Delta x = j_x\Delta t \tag{4.57}$$

If we use the constant $k = \varepsilon^{-1}$, we can write the analogous of Hook's law in the TARGET:

$$\Delta E_x = \varepsilon^{-1} j_x \Delta t \tag{4.58}$$

In this way we obtain

$$j_x = \varepsilon \frac{\Delta E_x}{\Delta t} \tag{4.59}$$

Maxwell called this density current the *displacement current*. This displacement is independent from the presence of a dielectric and it can happen also in the so-called "vacuum".

«These relations are independent of any theory about the internal mechanism of dielectrics [...] According to our hypothesis, the magnetic medium is divided into [vortexes], separated by partitions formed of a stratum of particles which play the part of electricity. When the electric particles are urged in any direction, they will, by their tangential action on the elastic substance of the [vortexes], distort each cell, and call into play an equal and opposite force arising from the elasticity of the [vortexes]. When the force is removed, the [vortexes] will recover their form, and the electricity will return to its former position.»

In the TARGET, the displacement of idle wheels is nowadays called *electric induction* $\vec{D}$, and

$$\vec{D} = \varepsilon \vec{E} \tag{4.60}$$

Maxwell concluded his analogy adding the last two terms

SOURCE	TARGET
Angular velocity $\vec{\omega} = (\alpha, \beta, \gamma)$	Magnetic induction $\vec{H}$
Moment of inertia μ	Magnetic permeability μ
Angular momentum $\mu\vec{\omega}$	Magnetic Field $\vec{B}$
Number of idle wheels per unit of time	Current density $\vec{j}$
Tangential force $\vec{F}_t = (P, Q, R)$	Electric Field $\vec{E}$
Quantity of motion $\vec{P} = (F, G, H)$	Potential vector $\vec{A}$
Constant of the vortexes' elasticity k	Electric permeability ε
Displacement of idle wheels $n\Delta x$	Electric induction $\vec{D}$

The complete electric current density vector is now

$$\vec{j} = \vec{j}_{\text{cond}} + \vec{j}_{\text{disp}} \tag{4.61}$$

where $\vec{j}_{\text{cond}}$ is the density of the conduction current and $\vec{j}_{\text{disp}}$ is the density of the displacement current, which nowadays is with the plus sign

$$\vec{j}_{\text{disp}} = \frac{d\vec{D}}{dt} \tag{4.62}$$

This term must be added to the Ampère law, obtaining

$$\nabla \times \vec{H} = \vec{j} = \vec{j}_{\text{cond}} + \frac{d\vec{D}}{dt} \tag{4.63}$$

This is the fourth of Maxwell's equations. With this equation we have derived the complete set of electromagnetic field equations from the Maxwell's model of aether and from the physical analyzes of differential operators made in section 4.2.3

$$\begin{aligned} \nabla \cdot \vec{D} &= \rho \\ \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{E} &= -\frac{d\vec{B}}{dt} \\ \nabla \times \vec{H} &= \vec{j} + \frac{d\vec{D}}{dt} \end{aligned} \tag{4.64}$$

Recalling the expression (4.36)

$$\nabla \cdot \nabla \times \vec{F} = 0 \tag{4.65}$$

So

$$\nabla \cdot \nabla \times \vec{H} = 0 \tag{4.66}$$

So

$$\nabla \cdot \vec{j} + \nabla \cdot \frac{d\vec{D}}{dt} = 0 \tag{4.67}$$

Using the first equation and exchanging the order of the derivation

$$\nabla \cdot \vec{j} + \frac{d\rho}{dt} = 0 \tag{4.68}$$

we have found the continuity equation for the electromagnetic charges, both scalar- and vector-charges. This equation is already contained in Maxwell's set: it means that electromagnetic charges are considered as a continuous body.
This equation can be red also in this way: current field has a scalar-charge, the derivative of the charge density. This means that the growth of the current in a point is induced by a growth of charge density in time in that point (see section 4.2.3).

4.3.8 The Electromagnetic Waves

The last question is: how fast do electromagnetic interactions travel? I will answer this question first with a description of what happens in the SOURCE. Then, by analogy, I will give the answer for the TARGET. Specifically, I will derive mathematically the expression for plane waves from Maxwell equations. I want to underline that the framework developed in this thesis can be used to find out the mathematical expression of plane electromagnetic waves, differently from the typical secondary school physics courses.
In the SOURCE, we have seen how the transversal motion of idle wheels deforms vortexes and that this motion is a «commencement» of a motion. So, motion of idle wheels ends after a short path. First, from 0 to $\Delta x/2$, they accelerate, then, from $\Delta x/2$ to Δx, they decelerate. Their motion ends after some finite Δt.
Vortexes begin to rotate when idle wheels begin to shift, and, at the same time, their shape changes. Again, this «commencement» of rotation experiences two phases: in the first $\Delta t/2$ time interval, their rotation accelerates, then, in the second $\Delta t/2$ time interval, their rotation decelerates, until the rotation ends exactly when idle wheels end to shift.
This motion is transmitted to the nearby idle wheels, which begins the same shift. On the other side, the first idle wheels (together with vortexes) begin to shift (rotate) in the opposite direction. If this motion does not experience resistance, it can travel through the aether forever, just like a perturbation wave.

In the TARGET, this means that a perturbation of the electromagnetic field behaves like a wave, called *electromagnetic wave*. This wave is a simultaneous variation of electric and magnetic field, one field being in phase with the other.
We will derive the mathematical expression of a plane wave from Maxwell's equation. This will be the mathematical demonstration that electromagnetic

waves exist and that their simpler form is double plane wave, with the electric field perpendicular to and in phase with the magnetic field.
I underline the most important aspect of the aether model in the derivation of the electromagnetic waves' equation: aether waves can't propagate in vacuum, they need a physical entity to move and this entity is the aether itself. An electromagnetic wave propagates in the vacuum, but it is a perturbation of the field itself. Again, the analogy can help to represent electromagnetism, useful

1. to imagine both propagation through the space and its oscillations in time,
2. to interpret how waves interact with charges,
3. to recognize the interdependency between the electric and the magnetic fields.

The Electromagnetic Wave Equations

In the TARGET, I will derive two quantitative relationships between the displacement current and the vector potential within an insulator.
To obtain the first relation, I recall that $\vec{E} = -d\vec{A}/dt$. In the three coordinates

$$P = -\frac{dF}{dt} \qquad Q = -\frac{dG}{dt} \qquad R = -\frac{dH}{dt} \tag{4.69}$$

From $\Delta E_x = \varepsilon^{-1} j_x \Delta t$ in the limit for infinitesimal variations

$$\vec{j}_{\text{disp}} = \varepsilon \frac{d\vec{E}}{dt} \tag{4.70}$$

so

$$\vec{j}_{\text{disp}} = -\varepsilon \frac{d^2\vec{A}}{dt^2} \tag{4.71}$$

To obtain the second relation, I recall the fourth of the Maxwell equations when $\vec{j}_{\text{cond}} = 0$

$$\vec{j}_{\text{disp}} = \nabla \times \vec{H} \tag{4.72}$$

In the three coordinate form, the last equation is (I will omit the subscript "disp" from now on)

$$
\begin{aligned}
j_x &= \left(\frac{d\gamma}{dy} - \frac{d\beta}{dz}\right) \\
j_y &= \left(\frac{d\alpha}{dz} - \frac{d\gamma}{dx}\right) \\
j_z &= \left(\frac{d\beta}{dx} - \frac{d\alpha}{dy}\right)
\end{aligned}
\tag{4.73}
$$

The three coordinates way to write the expression $\nabla \times \vec{A} = \vec{B}$, remembering that $\vec{B} = \mu(\alpha, \beta, \gamma)$, is

$$
\begin{aligned}
\alpha &= \frac{1}{\mu}\left(\frac{dG}{dz} - \frac{dH}{dy}\right) \\
\beta &= \frac{1}{\mu}\left(\frac{dH}{dx} - \frac{dF}{dz}\right) \\
\gamma &= \frac{1}{\mu}\left(\frac{dF}{dy} - \frac{dG}{dx}\right)
\end{aligned}
\tag{4.74}
$$

Substituting the second system (4.74) inside the first one (4.73), we obtain

$$
\begin{aligned}
j_x &= \frac{1}{\mu}\left(\frac{d^2G}{dxdy} - \frac{d^2F}{dy^2} - \frac{d^2F}{dz^2} + \frac{d^2H}{dxdz}\right) \\
j_y &= \frac{1}{\mu}\left(\frac{d^2H}{dydz} - \frac{d^2G}{dz^2} - \frac{d^2G}{dx^2} + \frac{d^2F}{dydx}\right) \\
j_z &= \frac{1}{\mu}\left(\frac{d^2F}{dzdx} - \frac{d^2H}{dx^2} - \frac{d^2H}{dy^2} + \frac{d^2G}{dzdy}\right)
\end{aligned}
\tag{4.75}
$$

With the following manipulations, I will find the expression of the plane waves for $\vec{A}$. To reach the goal, I have to compare the two ways to write the current density with respect to the vector potential, the expressions (4.71) and (4.75).

I will show how to manipulate the first line of the system, the other two lines being similar. I add and subtract d^2F/dx^2 to the first line (d^2G/dy^2 to the second line and d^2H/dz^2 to the third one), obtaining

$$
\left[-\frac{d^2F}{dx^2} - \frac{d^2G}{dy^2} - \frac{d^2H}{dz^2} + \frac{d}{dx}\left(\frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz}\right)\right] = -\mu\varepsilon\frac{d^2F}{dt^2}
\tag{4.76}
$$

The term $\left(\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}\right)=\nabla\cdot\vec{H}=0$.
I clonclude that

$$\left[\frac{d^2F}{dx^2}+\frac{d^2G}{dy^2}+\frac{d^2H}{dz^2}\right]=\mu\varepsilon\frac{d^2F}{dt^2} \tag{4.77}$$

Doing the same operations for the other two lines and writing down the obtained system in differential operators form, the final expression is

$$\nabla^2\vec{A}=\mu\varepsilon\frac{d^2\vec{A}}{dt^2} \tag{4.78}$$

A simple solution for label is

$$\vec{A}=(0,0,-A_0\cos{(kx+\omega t)}) \tag{4.79}$$

In this case, the magnetic field results

$$\vec{B}=\nabla\times\vec{A}=\frac{dA_z}{dx}\hat{\jmath} \tag{4.80}$$

$$\vec{B}=(0,kA_0\sin(kx+\omega t),0)=B_0\sin(kx+\omega t)\hat{y} \tag{4.81}$$

Analogously, the electric field is

$$\vec{E}=-\frac{d\vec{A}}{dt} \tag{4.82}$$

$$\vec{E}=(0,0,\omega A_0\sin(kx+\omega t))=E_0\sin(kx+\omega t)\hat{z} \tag{4.83}$$

where $\omega/k=c$.

Electric and magnetic field travel in phase with the same velocity (the speed of light), as drawn in Figure 4.41.

The velocity of light can be derived directly from the expression (4.78), knowing that in the usual plane wave equation the term multiplying the second derivative in time is the inverse of the square of a velocity

$$c=\frac{1}{\sqrt{\mu\varepsilon}} \tag{4.84}$$

At Maxwell's time, as already seen in section 4.2.1, this value has already known. If it is true that other physicists before Maxwell found the same result, they include it in a action-at-a-distance framework. This conceptual operation prevent them to include it in a general and complete theory of the electromagnetic interactions.

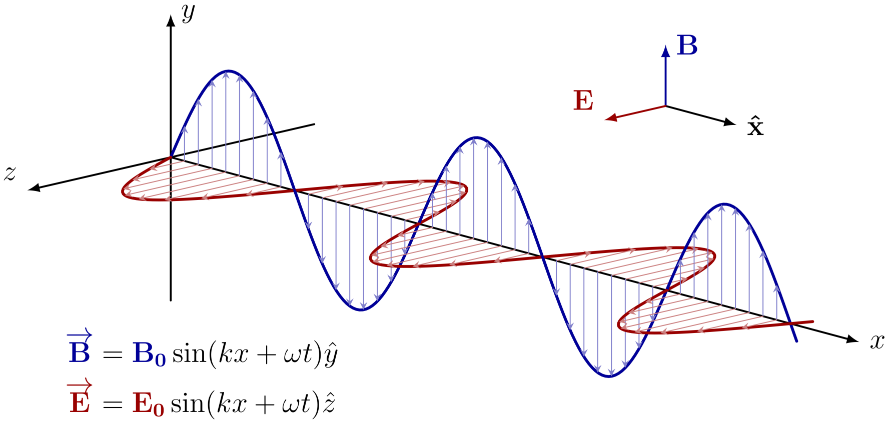


Figure 4.41: A plane electromagnetic wave

4.3.9 The Role of Aether for Maxwell

The guide ends with Maxwell own words. The following piece, extracted from the last part of Maxwell's paper, clears the approach of the Scottish physicist to aether. Summarizing, he stressed two important things: the electromagnetic aether of vortexes and idle wheels can be conceived a temporary model; despite that, the mathematics which describes the electromagnetic interactions, derived from this model, probably holds beyond the model itself.

«The conception of a particle having its motion connected with that of a vortex by perfect rolling contact may appear somewhat awkward. I do not bring it forward as a mode of connexion existing in nature, or even as that which I would willingly assent to as an electrical hypothesis. It is, however, a mode of connexion which is mechanically conceivable, and easily investigated, and it serves to bring out the actual mechanical connexions between the known electro-magnetic phenomena; so that I venture to say that any one who understands the provisional and temporary character of this hypothesis, will find himself rather helped than hindered by it in his search after the true interpretation of the phenomena. [...]
The facts of electro-magnetism are so complicated and various, that the explanation of any number of them by several different hypotheses must be interesting, not only to physicists, but to all who desire to understand how much evidence the explanation of phenomena lends to the credibility of a theory, or how far we ought to regard a coincidence in the mathematical

expression of two sets of phenomena as an indication that these phenomena are of the same kind.»

4.4 Beyond Maxwell

After that "On Physical Lines of Force" has been published in 1862, Maxwell wrote other two fundamental chapters of the electromagnetism history: "A Dynamical Theory of the Electromagnetic Field" (1865) and "A Treatise on Electricity and Magnetism" (1873). During this ten year period, he gradually abandoned his electromagnetic aether model, embracing the mathematical description of the electromagnetic field. However, he continued to believe in the material existence of aether until his death, happened in 1879. A number of physicists started from the results obtained by Maxwell to re-elaborate his theory. His works crossed the English Channel, expanding all over the Continent. Among others, two important physicists gave important theoretical contributions to the evolution of the electromagnetic theory and to the aether *dematerialization*: the English Oliver Heaviside (1850-1925) and the German Heinrich Rudolf Hertz (1857-1894). They raised Maxwell's equations to the rank of postulates: later, this approach was followed by many books, until nowadays, especially university textbooks. The actual form Maxwell's equations is due to Heaviside (Maxwell presented his equation as twenty equations in coordinates form and, later, using the quaternions formalism). Maxwell himself, in the "Treatise", used the Lagrangian approach in the derivation of his equations in order to reach the highest level of mathematical abstraction.
The history of aether ended only in 1905, when Albert Einstein (1879-1955) published his paper "On the Electrodynamic of moving bodies". Since that moment the aether has been erased from physics, except some isolated cases (Dirac, 1951).
It is significant that Einstein's paper begins with an electromagnetic paradox. This paradox, I believe, can be easily comprehended by contemporary students who possess an electromagnetic model based on locality and reality. This guide can be seen as an attempt to build a model of that kind.

4.4.1 The educational value of aether model

We have followed Maxwell in his construction of a model of aether which explains mechanically the electromagnetic interactions. Thanks to this model, we found the mathematical form for all type of electromagnetic interactions, i.e. the Maxwell's equations. In order to obtain them, the

action-at-a-distance model is abandoned in favor of the field model. What is the main characteristics of field that we can deduce from our approach?
Field interacts locally and has energy. Interactions happen within the field, and every interaction occurs when the field configuration changes. Reasoning within this new paradigm, students eventually put charges on the background, and they begin to focus on the field.
Fields must be intended as space-time functions which quantify specific anisotropies: the lines of force. Their configuration determines the shape of field, which is formally described by differential operators. For a three dimensional vector field it is sufficient to know local flux and local circuitation to determine its shape.
How do interactions appear in Maxwellian paradigm? When the local field's shape changes (the local flux or the local circuitation change) the field reacts to compensate changes occurred; information about these changes moves with a finite velocity. The field *destroys* and *creates* local flux and local circuitation continuously, apparently moving them from one place of the space to another one. As I have shown in section 4.3, Ampère law and Lenz law are two major examples of this property. Lorentz force can be another example of this universal principle. The electric field of two identical but opposite charges free to move tends to cancel off; in terms of force, the two charges are seen to attract each other to ideally form a neutral charge; in terms of field, the electric field changes so as to ideally destroy the two sources of local net flux.
In the Newtonian framework, local flux is scalar-charges and local circuitation vector-charges. In this framework, charges are acting on other charges through instantaneous forces, attracting or repelling themselves. In the Maxwellian framework, charges are physical objects which locally alter the space-time properties of the field.

The shift from Newtonian paradigm to Maxwellian paradigm is not only a change in the representation of interactions. It is also, maybe mainly, a change in the interplay between physics and mathematics. In fact, to complete this shift, it is necessary to learn new mathematical stuff and new tools for its interpretation. This thesis is supposed to suggest a new view about the electromagnetic field, so as to build a proper framework suitable to imagine electromagnetic interactions. The aim of this work is to go beyond the aether model I have presented in section 4.3, and to reach a deep understanding of Maxwell's equations and their context. I think that the interpretation of differential operators given in sections 4.2.3, 4.2.4, 4.2.5, and the attention given to field's shape in section 4.3 fit for the purpose to reach the way of reasoning called "thinking an interactions in terms of field". The following emancipation from aether and the construction of the differential operators will pave the way to the modern physical conception of the field as creation and annihilation operators.

Conclusions

After a large and systematic review of the studies on the teaching/learning electromagnetism, two main research issues have been selected and addressed in this research work.

The first one concerns the automatisms that typical exercises induce and prevent students to capture the real change produced by electromagnetism in the model of interaction.
The main aim of this study was to find a new approach to problem solving and posing, able to overcome hyper-simplification and trivializations that could prevent a meaningful understanding of EM and, in particular, of the topic of electromagnetic induction. In order to reach this aim I investigated how university students and secondary school teachers address electromagnetism exercises and, in particular, I investigated the role of representations, models, mathematics played in their problem solving strategies. Then, I designed an activity of problem analysis and problem posing to guide them to develop metacognitive and epistemological reflections.

The second study concerns a detailed conceptual, historical and epistemological analysis introduced by the concept of field by Maxwell. This analysis resulted in the production of a document where three main questions are addressed: What was the genesis of the concept of field? Which was Maxwell's contribution? What historical elements can be reconsidered today to promote the learning of the concept of field?

The main results I obtained are the following.

As for the first study, I found out a general tendency shown by university students and teachers to solve exercises: the tendency to find the result with the minimum effort. I called this attitude *economy principle*. My investigation led me to point out four different manifestations of this principle:

1. The Cheapest Way to Solve an Exercise Is to Not Consider Useless Physical Circumstances.

2. The Cheapest Way to Solve the Exercise Is to Search a Formula in the Final Question or Which Contains Exercise Data.

3. The Cheapest Way to Solve the Exercise Is to Recognize Some Familiar Elements in the Formulation which Could Recall Known Resolution Patterns.

4. The Cheapest Way to Solve the Exercise Is to Have a Picture of the Physical Situation in Order to Simplify the Math Set Up of the Resolution.

On the basis of these four manifestations, I designed both a guide to support a systematic and reflective analysis of the text of a problem, and a problem posing activity. These two activities can be applied to any physics exercise: as we have showed, they have the potential to enlarge the view not only on the single exercise and the physical situation described, but also on the entire physical system to which it refers. A fundamental role in the guide and in the problem poising analysis is given to the analyses of models, representations and mathematics in the resolution context. We worked on the electromagnetic induction, but, as we have already mentioned, these activities can be applied to all physics exercises.

As for the second study, I made an historical research to stress in what sense the field theory was built in the eighteenth century in order to overcome the Newtonian approach to interaction "in term of forces". The revolution was carrying on mainly by English physicists, and especially by M. Faraday and J. C. Maxwell. The experimental results of the former, together with its vision about electric and magnetic interactions, paved the way for Maxwell's theory, based on the existence of aether.
Aether was the topic of an historical reconstruction, from Descartes to Einstein, that introduced to Maxwell's world. Aether properties, by the method of analogy, were transferred by Maxwell to the new entity called field. In doing this, he invented new mathematical tools, the differential operators convergence (nowadays, the divergence) and curl. Like Maxwell, but in a more qualitative way, I started from empirical results to build up the electromagnetism theory in terms of field interactions. Loosely speaking, I described a model, abandoned for a long time, based on the existence of a

particular kind of aether, which is able to present the electromagnetic field as a real object, whose lines of force represent, mathematically, its properties of interaction.

This analysis of Maxwell papers provides the basis for a new possible way to re-think of the teaching of electromagnetism, aimed to support the formalism with a conceptual and epistemological reflection on the model of interaction introduced by the concept of field. In such a way, for example, the electric field would be no more introduced as a mere re-writing of Coulomb force, but it would acquire a new, coherent, real identity. Fields gain a mathematical form in Maxwell's equations. From our research emerges what we called *compensation principle*, as a new form of mechanism that could explain electromagnetic interaction. This principle expresses the tendency of the whole field to compensate local shape variations and it could represent the starting point to design an Educational proposal, aimed to teach electromagnetic interactions "in terms of field", highlighting the differences with electromagnetic interactions "in terms of force".

Appendices

Appendix A

Guided Analyses (Second Study)

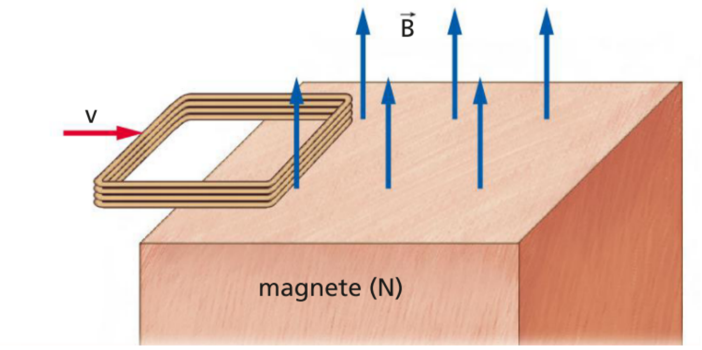


Figure A.1: Exercise revisited from (Romeni, 2012).

A coil is composed by 20 square turns, each one $l = 15\ cm$ side. The wire is very thin and curled unto itself. This coil is moved close to a large magnet ($L = 50\ cm$), generating a $B = 0.12\ T$ magnetic field. The total resistance of the coil is $R = 5,0\ \Omega$; a $20\ W$ bulb is linked to the coil, which is moving with a constant velocity of $v = 0,25\ m/s$. Find the electromotive force induced in the circuit [$fem_{\max} = 0,45\ V$]

Explicit the Reasoning Made to Solve the Exercise.

QUESTION	ANSWER
Explicit the reasoning made to solve the exercise (how do you set up the resolution, what kind of procedures do you follow to solve the exercise)	
What are all the physical phenomena present in the situation described in the exercise (list)?	
What physical phenomena do you have to know to solve the exercise (list)? What physical phenomena are useless to solve the exercise?	
Do you have validate/evaluate your results?	

Analyses of the resolution through an exploration of the exercise text.

QUESTION	ANSWER
What are the words that induced you to remember similar exercises?	
Does the text induce to reproduce procedures already done? Or does it induce to reproduce reasoning already done?	
The resolution way of reasoning is began from the situation analyses? Or first did you look for a resolutive expression?	
How do data and the final question influence your resolution?	
Are there in the exercise text useless elements for the resolution but which can help you to find the resolutive strategy?	
Are there implicit unwritten details which help you in some way? If yes, What are they?	

Analyses of the resolution through an exploration of the phenomenology.

QUESTION	ANSWER
Have you disregard side effects?	
Have you disregard self-induction? Why?	
If the magnet would be substituted with a current carrying coil, generating the same magnetic field, would you disregard self-induction too?	
If the coil would be substituted with a metal plate, would that make a difference? If the coil would be substituted with a wooden plate, would that make a difference?	
Are there in the exercise text useless elements for the resolution but which can help you to find the resolutive strategy?	

Analyses of the resolution through an exploration of the "mathematization".

QUESTION	ANSWER
Have you draw some picture of the exercise? If not, why do you not do it?	
Are there in your or in the textbook additional hints with respect to the text? Do this indications help the mathematics?	
Is the picture more abstract or more real? Please, explain.	
What are the terms that induce you a mathematical reasoning and what are those which induce you a physical reasoning?	
In the resolution, where did you make a physical reasoning and where a mathematical reasoning?	

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