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REPETITIVE CONTROL SYSTEMS: STABILITY AND PERIODIC TRACKING BEYOND THE LINEAR CASE

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Esame finale anno 2019

Declaration of Authorship

I, Federico Califano, declare that this thesis titled, "Repetitive Control Systems: Stability and Periodic Tracking beyond the Linear Case" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed:

Date:

"Atoms with consciousness
matter with curiosity
Stands at the sea
wonders at wondering
I, a universe of atoms
An atom in the universe "

Richard P. Feynman

ALMA MATER STUDIORUM UNIVERSITY OF BOLOGNA

Abstract

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PhD in Biomedical, Electrical and Systems Engineering

Repetitive Control Systems: Stability and Periodic Tracking beyond the Linear Case

by Federico Califano

Periodic output regulation studies the problem of steering the output of a dynamical system along a periodic reference. This is a fundamental control problem which has a great interest from a practical point of view, since most industrial activities oriented to production are based on tasks with a cyclic nature.

Nevertheless this interest extends rapidly to a theoretical framework once the problem is formalized. Mathematical tools coming from different fields can be used to provide an insight to the output regulation problem in different ways.

An important control technique that is classically used to achieve periodic output regulation si called *Repetitive Control* (RC) and this thesis focuses on (but is not limited to) the development and the analysis with novel tools of RC schemes.

Periodic output regulation for nonlinear dynamical systems is a challenging topic. This thesis, besides of providing consistent and practically useful results in the linear case, introduces promising tools dealing with the nonlinear periodic output regulation problem, whose solution is presented for particular classes of systems.

The contribution of this research is mainly theoretical and relies on the use of mathematical tools like *infinite-dimensional port-Hamiltonian systems* and *autonomous discrete-time systems* to study stability and tracking properties in RC schemes and periodic regulation in general. Differently from the classical continuous-time formulation of RC, internal model arguments are not directly used is this work to study asymptotic tracking. In this way the linear case can be reinterpreted under a new light and novel strategies to consistently attack the nonlinear case are presented.

Furthermore an application-oriented chapter with experimental results is present which describes the possibility of implementing a discrete-time RC scheme involving trajectory generation and non-minimum phase systems.

Summary

This thesis is divided in 5 Chapters.

Chapter 1 introduces the reader to the Repetitive Control (RC) framework and gives an insight of applications that use this technique as well as the state of the art. Besides of the classical representation of RC, some key observations for the development of the topics in the following chapters are discussed within this introduction. References to the main related works are present as well as the detailed contribution of the thesis.

The main contribution of this research is present in Chapters 2, 3 and 4, that represent three different approaches to periodic output regulation. These three chapters can be read independently since they use different tools to treat the problem in different ways but are deeply related since they treat the same topic (especially chapter 2 and 3).

Chapter 2 introduces a novel way to tackle the main problems connected to continuous-time implementation of RC schemes. In particular *infinite-dimensional port-Hamiltonian systems* are introduced to study well-posedness and stability in the framework of *dissipative systems*. This approach leads to an original way to attack the nonlinear case.

Chapter 3 studies the periodic output regulation problem in a more abstract way in which RC will turn out to be a particular case. Here the analysis is based on system theory and in particular on *invariance analysis* of *autonomous discrete-time systems*.

Chapter 4 deals with a different way to apply RC. In fact it is about a discrete-time implementation of RC in connection to *B-spline trajectory generation* by means of dynamic filters. It is an application-oriented topic that aims at successfully achieving periodic tracking for linear non-minimum phase systems. Experimental results are presented to validate the proposed method.

Chapter 5 contains the conclusions and the future perspectives connected to this research.

Acknowledgements

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Moving outside the work area I deeply thank my family, especially my parents and by brothers Niccolò and Bubi. If I do not miss anything and live e regret-free life it is thank to my mother and her infinite power she was able to transmit me.

It is too important to me to thank my friends, which I consider like a second family, and which contributed in a fundamental way to my achievements. Thanks to whom has always been there even if distantly: Enrico, Paolo and the "*Chocka*" Andrea, Alessandro, Luca, Eric, Aldo, Nicole, Anna. Thanks to whom makes me feel welcome whatever I do: the guys from "*Franio Critto*" Francesco, Marco, Daniela, Omar. Thanks to whom gave me hospitality and friendship with infinite generosity during my period abroad in the Netherlands, the guys from "*Cena Pazza*" Luigi, Francesco, Kennet, Federico and Guido. Thanks to Chiara for turning me in a better person and thanks to all the other friends that I cannot manage to cite.

Finally since I am moving abroad for my next experience I have to thank again the friends that are shaping my daily life right now: Andrea, Riccardo and Daniela. To spend my time with all of you is a huge gift for me and it will hurt to leave, but what you gave me is far bigger than anything I may loose.

Thank you.

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List of Abbreviations

RC	**R**epetitive **C**ontrol
MRC	**M**odified **R**epetitive **C**ontrol
IMP	**I**nternal **M**odel **P**rinciple
ODE	**O**rdinary **D**ifferential **E**quation
PDE	**P**artial **D**ifferential **E**quation
PHS	**P**ort **H**amiltonian **S**ystem
dPHS	**d**istributed **P**ort **H**amiltonian **S**ystem
ILC	**I**terative **L**earning **C**ontrol
ZPETC	**Z**ero **P**hase **E**rror **T**racking **C**ontroller
CZPETC	**C**ontinuous **Z**ero **P**hase **E**rror **T**racking **C**ontroller

Dedicated to my family

Chapter 1

Repetitive Control: an Overview

1.1 History and Motivation

Repetitive Control (RC) is a control technique used to achieve *periodic output regulation*. The latter is the branch of control theory that studies the problem of steering an output of a dynamical system along a periodic reference with known and fixed time period.

More precisely RC is applied to dynamical systems in order to track arbitrary periodic signals of a fixed period (or equivalently reject unknown periodic disturbance signals of the same period). Thus RC systems can be seen as servosystems with a periodic exogenous signal. However there are two main reasons, described in the next two subsections, why this particular technique has been deserving since the early 80's a great amount of attention.

1.1.1 A simple controller for high precision tasks

The first one is related to the variety and large range of applications that has been successfully implemented with RC. Indeed it is not difficult to understand that a deep interest from an industrial point of view is present for RC systems. In industrial and production-oriented applications, control tasks are often of repetitive nature, and the increasingly high demand of quality and productivity has become a challenging practical problem. RC accomplishes perfectly these needs since the high precision that is achieved through this technique comes together with a quite simple implementation and little dependency on the physical parameters of the system to be controlled.

Historically RC was first developed in (Inoue, Iwai, and Nakano, 1981) where the motivation was to control proton synchrotron magnet power supply to a precise shape periodically, with a magnitude of order 10^3 V and a required precision of order 10^{-1} V. To achieving a tracking accuracy of a factor 10^{-4} robustly was an impossible task using techniques based on dynamic inversion of the system. The fortunate novel idea that was used to accomplish the task gave raise to the first SISO continuous-time RC scheme. The use of RC schemes in applications grew very fast in different fields[1] ranging from rejection of power supply interferences (Nakano and Hara, 1986"), accurate placement of the head of a disk-drive systems (Chew and Tomizuka, 1990), of optical disk-drives (Moon, Lee, and Chung, 1998), vibration suppression (Hillerstrom, 1996; Hattori, Ishida, and Hori, 2001) to control of robotic manipulators performing repetitive tasks (Omata, Hara, and Nakano, 1987; Tomizuka, Tsao, and Chew, 1988; Sadegh et al., 1990; Biagiotti, Moriello, and Melchiorri, 2015).

[1]The number of applications in which RC-based techniques are used is really big, here only some of the most historically important or to-the-author interesting examples are cited.

The main contribution of this thesis in terms of application is present in chapter 4 where a novel way to use RC at a trajectory generation layer in combination with a B-spline generator is presented. This control scheme is suitable to be used for linear non-minimum phase systems.

1.1.2 A theoretically challenging subject

The second reason why RC has been received a great amount of attention is related to the theoretical challenges that come together with its formal representation in its continuous-time description (Hara, Omata, and Nakano, 1985; Hara et al., 1988; Yamamoto and Hara, 1988; Yamamoto, 1993). As discussed more in detail in section 1.2, RC schemes in their initial design where based on the *internal model principle* (IMP) (Francis and Wonham, 1975) , which was formalized at that time only in the linear, finite-dimensional case. The main theoretical problem derives from the fact that an arbitrarily periodic signal has an infinite number of harmonics, and thus the exosystem that generates the periodic reference is in general infinite-dimensional. From a system theory perspective a continuous-time RC systems is indeed infinite-dimensional and for such systems most of the control theoretical main aspects, e.g. stabilizability, need to be re-investigate with finer mathematical tools, which where not necessarily present or used during the development of RC-related researches. Furthermore the extension to the case in which the systems to be controlled are non-linear is deeply involved because internal model based arguments can not be used to proof robust periodic tracking.

Chapter 2 is dedicated to the use of *distributed port-Hamiltonian systems* to study rigorously RC in its infinite-dimensional nature. With this new perspective, that does not rely on internal model arguments, the linear case can be reinterpreted under a new light and novel ways to attack the nonlinear case consistently are presented.
Another way to look at RC schemes is as *learning scheme*. In fact, since the aim of RC is to adjust cycle after cycle the control input in order to reduce iteratively the error signal, it can be interpreted as the system is *learning* from the previous cycles as periods go by. In this perspective there is a deep connection between RC and *iterative learning control* (ILC) (see Wang, Gao, and Doyle, 2009 for an excellent survey about this connection) with the subtle difference that the latter technique is not in general causal since the initial condition needs to be reset at the beginning of every iteration.

Chapter 3 exploits this learning paradigm to model a system that aims at achieving periodic regulation (RC will be a particular case) as an autonomous discrete-time system. System theory is used to derive consistent sufficient conditions for perfect tracking.

1.2 RC and Internal Model Principle

In this section the basics of RC are briefly summarized together with the classical interpretation that is behind its classical design. Most of the results stated in this section are present in (Hara et al., 1988), which collects all the main early results in the continuous-time RC framework and represents still now a cornerstone in the field.
The aim of RC systems is to solve the *periodic output regulation problem*, i.e. to achieve perfect tracking in a servosystem when the reference is a generic periodic signal with known time period τ. When such tracking problems are considered the *internal model principle* (IMP) (Francis and Wonham, 1975) plays a key role.

In the following, theorems are not stated and rigorous mathematical derivations are omitted since the aim is to introduce the reader to the RC framework in an intuitive way and summarize early results about RC that will be confirmed and developed in the next chapters using different mathematical tools, not involving directly the IMP. Nevertheless some non classical features and observations that will be used in the following chapters will be highlighted in this introduction.

1.2.1 RC interpreted with IMP

Classically the IMP states that a *necessary* condition for asymptotic perfect tracking of a servosystem is that the generator for the reference signal is included in the stable closed-loop system. Then, in the finite-dimensional case, *sufficiency* can be achieved through proper designs of stabilizing compensators.

As celebrated examples of this concept let us refer to the necessity of the model of the generator of some important reference signals to be present in the control loop. For a step signal as reference the corresponding model (which is the *Laplace transform*) $\frac{1}{s}$ must be present in the loop for perfect asymptotic regulation. The same reasoning can be repeated for a sinusoidal reference, say $sin(\omega t)$: the model that incapsulates the pair of generating poles of the reference, i.e. the transfer function $\frac{1}{s^2+\omega^2}$, must be present in the stable closed-loop system for perfect tracking.

In the RC setting the reference signal is a generic periodic signal $r(t)$ of period τ, i.e. $r(t+k\tau) = r(t)\ \ \forall k \in \mathbb{Z}$. This type of signal, contrarily of the case of the constant or sinusoidal reference, has in general an infinite number of harmonics.

In fact $r(t)$ can be characterized by its Fourier expansion

$$r(t) = \sum_{k=-\infty}^{+\infty} \hat{r}_k e^{j\omega k t}$$

where

$$\hat{r}_k = \frac{1}{T}\int_0^T r(t)e^{-j\omega_\tau k t}dt, \qquad k \in \mathbb{Z}$$

are the Fourier coefficients of the reference $r(\cdot)$ over one period and $\omega_\tau = \frac{2\pi}{\tau}$.

The IMP suggests then that an internal model based compensator, say $\mathcal{I}(s)$, must be present in the loop for perfect asymptotic tracking and should contain all poles that generate the periodic reference, i.e.

$$\mathcal{I}(s) = \frac{1}{s\prod_{k=1}^{\infty}(s^2+k^2\omega_\tau^2)}.$$

It is not surprising that this model contains an infinite number of poles since the exosystem that generates a generic periodical signal is indeed infinite-dimensional, and in particular a *neutral system* with infinite many poles on the imaginary axis, placed at multiple integers of the fundamental frequency ω_τ.

At this point a quite ironical fact that characterizes RC comes into the game: the minimal realization of such a compensator has a very simple structure considering its infinite-dimensional nature. In fact it is easy to see[2] that the poles of $\mathcal{I}(s)$ can be realized by a dynamical system with characteristic equation $e^{s\tau} - 1$. This suggests

[2] A formal derivation of this correspondence based on the identity $sinh(\pi s) = \pi s \prod_{k=1}^{\infty}(1 + (s^2/k^2))$ is present in (Yamamoto, 1993)

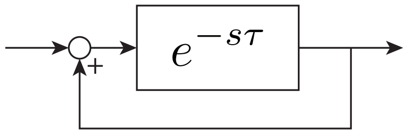


FIGURE 1.1: The repetitive compensator (Hara et al., 1988; Yamamoto, 1993)

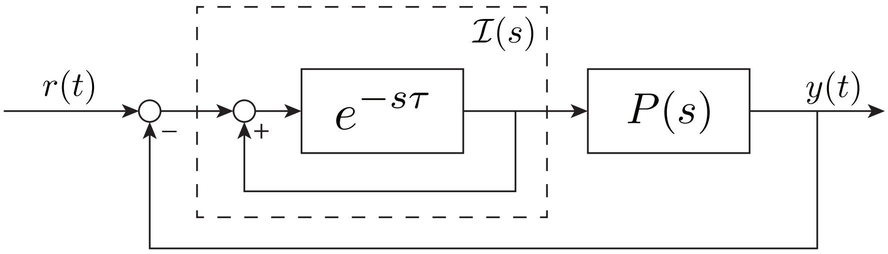


FIGURE 1.2: Continuous-time RC scheme.

that a valid internal model based compensator to be included in the loop is

$$\mathcal{I}(s) = \frac{1}{e^{s\tau} - 1}.$$

This dynamical system can be realized by a pure time delay of τ seconds surrounded by a positive feedback loop as shown in figure 1.1. Such a system is called *repetitive compensator*.

It is clear that such a system can generate any periodic signal of period τ autonomously, just by storing as initial condition in the delay the waveform of one period.

Now following the classical "internal model rule", a stabilizing compensator that achieves a stable closed-loop system would solve the problem of asymptotic perfect tracking for any periodic reference. Nevertheless this was proven in the finite-dimensional case and only in (Yamamoto, 1993) this nontrivial generalization in the RC case was carried out. In particular in the latter work it is proven that the presence in the control loop of the repetitive compensator is necessary for achieving perfect asymptotic tracking.

1.2.2 Stabilizability of linear RC

As consequence of the latter results, the problem of perfect tracking turns into a problem of stabilizability of the closed-loop system depicted in figure 1.2, where the controlled plant $P(s)$ is intended to merge from the factorization between a stabilizing compensator $R(s)$ and the uncontrolled plant $G(s)$, i.e. in the linear case $P(s) = R(s)G(s)$.

In particular the RC scheme design reduces to the choice of $R(s)$ such that the autonomous (with $r(t) = 0$) closed-loop system in figure 1.2 is exponentially stable.

From the stability analysis in e.g. (Hara et al., 1988) it turns out that not all plants $G(s)$ are admissible to be stabilized in a continuous-time RC scheme.

In particular only those plants which are not strictly proper, i.e. have relative degree 0, can be stabilized by means of a compensator $R(s)$.

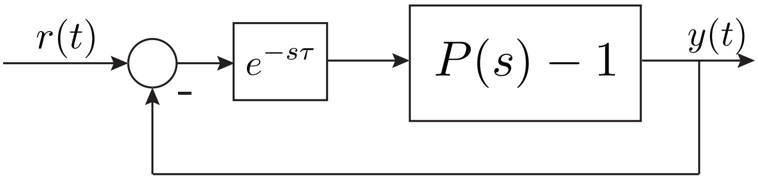


FIGURE 1.3: Equivalent scheme of RC for stability analysis.

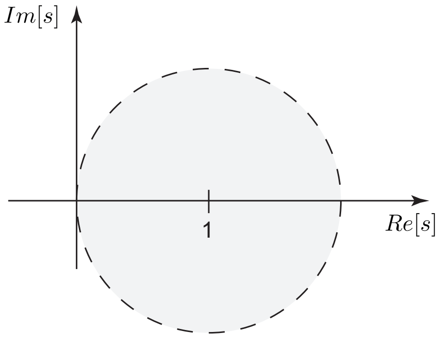


FIGURE 1.4: Stability region of RC for $P(j\omega)$.

A sufficient condition for stability in the linear case can be derived as follows[3] on the basis of classical Nyquist analysis.

The characteristic equation of the scheme is

$$1 + \frac{e^{-\tau s}}{1 - e^{-\tau s}} P(s) = 0.$$

After some algebraic manipulation, the latter can be rewritten as can be rewritten as

$$1 + e^{-\tau s}(P(s) - 1) = 0$$

which can be interpreted as the characteristic equation of the dynamic system shown in 1.3, in which the positive feedback loop of $\mathcal{I}(s)$ is no more present. By applying Nyquist criterion to the scheme of 1.3 it descends that the closed-loop system is exponentially stable if and only if the polar plot of the loop function $L(j\omega) = e^{-\tau j\omega}(P(j\omega) - 1)$ does not encircle or touch the critical point -1. This can be assured by imposing that

$$|P(j\omega) - 1| < 1, \;\; \forall \omega. \tag{1.1}$$

A graphic interpretation of this sufficient condition is shown in figure 1.4. The Nyquist plot of $P(j\omega)$ must lie entirely within the open unit disk centered in $(1, 0)$ in the complex plane.

This condition can be interpreted from a dynamic cancellation perspective: the more the control plant $P(s)$ is close to the identity operator (meaning that the stabilizing compensator is close to the inverse of the uncontrolled plant, i.e. $R(s) \simeq G^{-1}(s)$) the better the stability condition is satisfied. Intrinsic robustness of RC schemes allows imperfect cancellations up to margins characterized by the stability circle.

An observation representing the key to develop RC in the framework presented in chapter 2 is that the presented stability condition can be equivalently stated in

[3]In (Hara et al., 1988) the proof is slightly different but leads to the same sufficient condition.

a frequency free way. In particular dissipativity conditions on operators (and on dynamical systems in this case) having their roots in the *passivity framework* can be used to formulate in time-domain the same mathematical condition. This allows to extend consistently the analysis to nonlinear systems, for which frequency domain tools can not be used. In this case condition (4.9) can be equivalently formulated imposing the system $P(s)$ to be a α-output strict passive system, with $\alpha > \frac{1}{2}$ (see e.g. Lozano et al., 2000 for informations about dissipativity properties).

Even if this condition is only sufficient, intuitively it is very close to a necessary one since in a high-frequency range the phase shift caused by the pure delay can assume high values. The fact that for strictly proper systems $P(j\omega) \to 0$ when $\omega \to \infty$ suggests that these types of plants can not be stabilized.

Furthermore a specific proof that states that it is impossible to stabilize exponentially[4] RC schemes like in figure 1.2 is present in (Hara et al., 1988). This proof is based on the fact that in a neutral system in which a repetitive compensator is present the closed-loop poles will approach asymptotically the imaginary axis, which implies impossibility of exponentially stabilizing the system.

To summarize it is impossible to achieve perfect tracking in a continuous-time RC scheme for any periodic reference signal if the plant is strictly proper.

Intuitively the reason for this impossibility is quite clear: to track *any* periodic reference signal of fixed period using the same control scheme is requiring too much. In fact if the plant is strictly proper it will integrate the input at least once. Thus it is impossible to track periodic signals with arbitrarily high frequency modes (e.g. a square wave).

This represents of course a big limitation in the use of this technique and in the next section the two main approaches to overcome this problem in the linear case are presented.

1.3 Remedies to stabilizability problems for strictly proper plants in linear RC schemes

The main approaches used to solve the stabilizability problem for strictly proper controlled plants $P(s)$ aim at creating a trade-off between achieving exponential stability and capability of tracking periodic signals.

1.3.1 Modified RC

The approach that is discussed here leads to a control scheme called *modified RC* and merges from the addition of a low-pass filter $G_f(s)$ in series to the delay, as shown in figure 1.5. This leads to a modified repetitive compensator $\mathcal{I}_m(s)$. The main aspects of this scheme are summarized as follows:

- The modified repetitive compensator $\mathcal{I}_m(s)$ does not achieve anymore an internal model for any periodic signal. This happens because, depending on the bandwidth of the filter, the poles escape away from the imaginary axis on the complex left hand plane for high frequencies.

- For the same spectral reason that has been described before, it becomes now possible to exponentially stabilize the system even if the controlled plant $P(s)$

[4] Even if the system is linear, in infinite-dimensional systems exponential and asymptotic stability do not coincide. For the application of internal model arguments exponential stability is necessary.

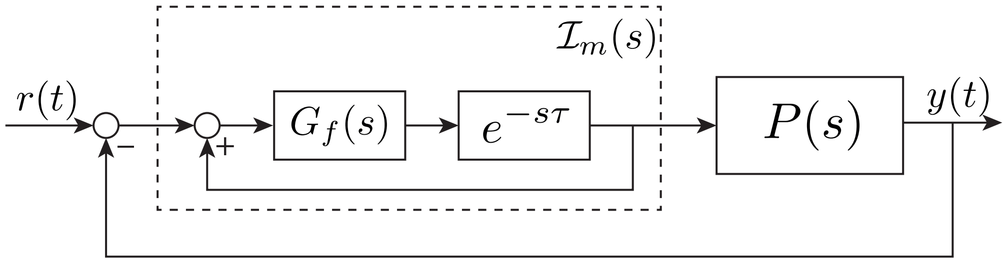


FIGURE 1.5: Continuous-time modified RC scheme.

is strictly proper. In particular the stability condition described by equation (4.9) in the *pure* RC case (i.e. if $G_f(s) = I$) becomes

$$|G_f(j\omega)(P(j\omega) - 1)| < 1, \ \ \forall \omega. \tag{1.2}$$

- As result this scheme achieves the possibility of exponentially stabilizing RC schemes with strictly proper plants through a minor modification of the original scheme. The price to pay is the impossibility of tracking arbitrarily high modes in periodic references.

In (Weiss, 1997) it is also highlighted that in continuous-time, without the use of modified RC, it is impossible achieving robustness with respect to arbitrarily small time delays at any point in the feedback loop.

From the previous considerations it seems necessary to consider the presence of the filter in the continuous-time case for periodic output regulation, and a section in chapter 2 is indeed dedicated to the stability analysis of modified RC scheme carried on in the time domain[5].

Nevertheless it is important to make one more comment which is not completely understood and that highlights somehow the limits of the modified RC scheme: adding a low-pass filter in the delay loop limits the class of controllers to the finite-dimensional case since the steady state control input is generated by the output of the low-pass filter, which is in fact a finite-dimensional system. Despite of this aspect the infinite-dimensional nature of the scheme is kept since the delay is still present. Thus one could ask why not to design directly a finite-dimensional controller, getting rid of the RC structure that contains the time-delay. This point of view was used in (Langari and Francis, 1994) and is not completely unrelated to the recent works involving periodic output regulation (Paunonen, 2017) in which RC is implicitly considered. The technique that the two latter references share to study the periodic output regulation at a finer level is called *lifting* and in chapter 3 the same tool is used to provide a result that is not explicitly based on the RC setting.

The latter consideration, together with the fact that the combined choice of the low-pass filter and the stabilizing compensator in the scheme of figure 1.5 is not an easy task and lacks of general systematic design procedures, represent the main difficulty in the use of modified RC.

On the other hand the very simple structure of the scheme was the key for the development and the fairly wide success of this technique.

[5]the classical stability analysis of modified RC (Hara et al., 1988), is based on frequency domain considerations and leads to the condition (4.9)

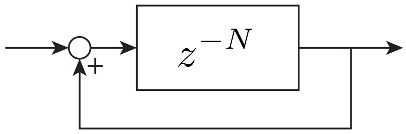


FIGURE 1.6: The digital repetitive compensator

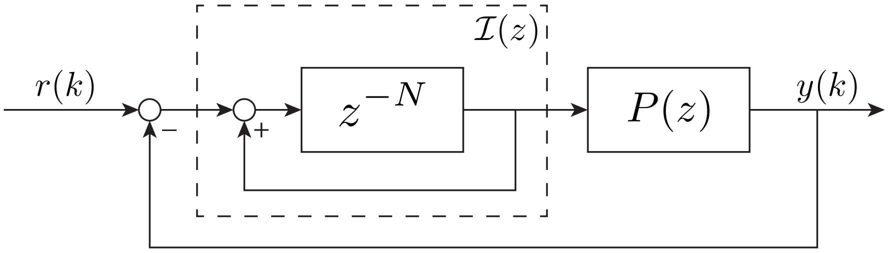


FIGURE 1.7: Discrete-time RC scheme.

1.3.2 Digital RC

The other main way that has been developed to overcome stabilizability problems of RC systems is to perform a discretization of the whole scheme and end up with a *digital RC scheme* (Tomizuka, Tsao, and Chew, 1988).

In this case a generic periodic reference signal does not contain an infinite number of harmonics since the maximum meaningful frequency is naturally upper bounded by the sampling frequency of the digital system.

Following the same idea of the continuous-time case, any periodic signal of period N[6] can be generated by an N-step delay with positive feedback loop, suggesting that the digital version of the repetitive compensator must be like in figure 3.3 where the complex function z^{-1} can be interpreted as the one-step delay operator.

This system contains N neutrally stable poles on the unit disc (instead of ∞ on the imaginary axis in the continuous-time case) whose linear combination can generate any periodic signal of period N in its discrete Fourier decomposition. This aspects confirms that the discrete-time repetitive compensator acts as internal model $\mathcal{I}(z)$ of a digital periodic reference.

Without entering too much into detail the reasons why stabilizability problems of digital RC schemes represented in figure 1.7 can be solved for strictly proper controlled plants $P(z)$ can be seen in the two following equivalent ways:

- The infinite-dimensional nature of the original scheme is lost because of discretization and stabilizability becomes possible also with strictly proper plants. In fact if delayed tracking is allowed (at it is always the case since the aim is to track signals periodically), the non null relative degree of the system does not represent a problem anymore.

- The frequency interval in which the polar plot has to be evaluated for stability is bounded (i.e. $< \infty$), and thus the problem of its convergence to the origin when $\omega \to \infty$ can be avoided in the digital regulator design phase. Mathematically speaking stability condition (4.9) becomes

$$|P(e^{j\omega}) - 1| < 1, \ \ \forall \omega \in [0, \pi/T]$$

[6]with this notation it is meant that the periodic signal has a period of N sampling times.

where T is the sampling period of the system. This condition can be satisfied also if $P(z)$ is strictly proper.

Even if the stabilizability problems seem to be solved using this approach, this solution is unfortunately not very satisfactory in applications that require a certain degree of smoothness: in fact the key observation is that considering the underlying digital system, tracking is achievable only at the sample instances. This may cause (even large) *intersample ripples* and a complete framework discussing this aspects and attenuation of high frequency components is still missing.

Even if this research focuses mainly on continuous-time RC, in chapter 4 an algorithm exploiting a digital RC scheme at a trajectory generation layer for non-minimum phase system is presented. In chapter 3 a novel interpretation of digital RC is carried out as particular case of the generic periodic output regulation problem.

1.4 Nonlinear RC and periodic solutions

All the techniques that have been described so far in this introductory chapter, which are used to described RC in most of the literature, are based on frequency-domain tools. Thus only the linear case can be considered.

In order to consider nonlinear RC, i.e. study the periodic output regulation problem if the system to be controlled is nonlinear, finer mathematical tools are needed.

What is meant by nonlinear RC scheme in this thesis is shown in figure 1.8, i.e. the same scheme of figure 1.2 with a generic nonlinear controlled plant Φ in lieu of $P(s)$.

Without considering techniques deeply related to RC like ILC[7], it can be stated that not much effort has been done in dealing with nonlinear RC schemes.

More specifically, in Omata, Hara, and Nakano, 1987 passivity theory is used to design a nonlinear RC scheme involving trajectory control for industrial manipulators. In Ghosh and Paden, 2000 nonlinear RC schemes are studied by approximating the compensator $\mathcal{I}(s)$ with a finite-dimensional system, and tracking is analyzed for a particular class of nonlinear systems. In Lin, Chung, and Hung, 1991, small-gain arguments are used to guarantee stability of linear RC schemes in which a sector nonlinearity at the input of a linear plant is present. In Owens, Li, and Banks, 2007 and similarly to the approach discussed in chapter 2, the stability analysis is performed by relying on a state-space representation of the plant, with the nonlinear case briefly addressed as perturbation of the linear one.

[7] As previously explained the techniques are related but not equivalent: since ILC has fixed initial iterative state, wider convergence conditions are allowed. On the contrary the design of RC historically performed in the frequency domain makes the nonlinear study more difficult.

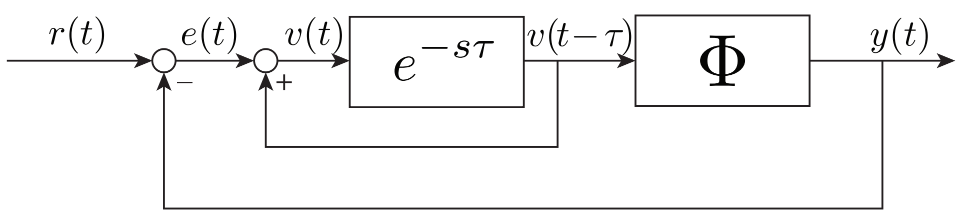


FIGURE 1.8: Continuous-time nonlinear RC scheme.

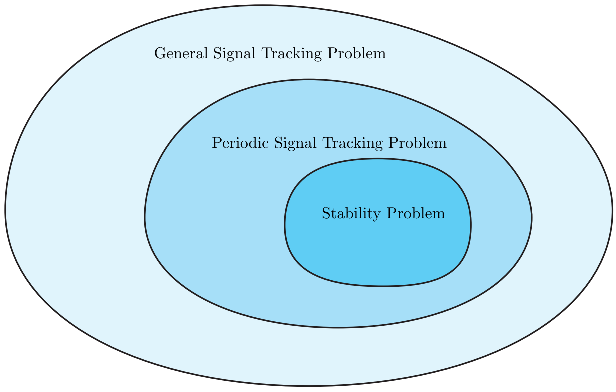


FIGURE 1.9: Venn diagram to illustrate qualitatively the inclusive properties of tracking problems and stability problem.

A nonlinear RC scheme presents both infinite-dimensional and nonlinear parts and many questions need to be re-investigated to study formally such a system. For example aspects like existence and regularity (namely *well-posedness*) of solutions[8] as well as stabilizability must be treated properly and the approach in chapter 2 allows such a rigorous analysis.

In the remaining part of this section the aim is to focus on one of the biggest issues that involves nonlinear RC schemes which is often misunderstood. In particular exponential stability for the closed-loop system is in general not sufficient for perfect tracking since the IMP does not hold in the nonlinear case. In fact the legitimacy of invoking the IMP in the linear infinite-dimensional case is due to (Yamamoto, 1993) but relies completely on the linear framework.

1.4.1 Periodic solutions and perfect tracking

The situation of tracking exogenous signals in servosystems is qualitatively depicted in figure 1.9, which shows that not every stabilization problem solves completely the tracking problem.

In the following an useful way that will be used in chapter 2 to determine perfect periodic tracking in a RC framework for a generic nonlinear controlled system Φ is described.

Refer to figure 1.8 and suppose the reference $r(t)$ is a generic τ-periodic signal. Asymptotic perfect periodic tracking means that $e(t) = v(t) - v(t-\tau) \to 0$ as $t \to \infty$. This is equivalent to ask that the signal $v(t)$ becomes τ-periodic as $t \to \infty$. Under this light a more intuitive criterion can be used to determine whether perfect tracking for periodic reference signals is achieved in a generic nonlinear RC scheme: if the state solutions are such that at steady state they end up to be periodic of period τ then the problem is solved.

This way of thinking can be interpreted as the IMP for periodic signals in the linear case: since exponentially stable linear systems produce asymptotically τ-periodic solutions if they are excited by a τ-periodic input, solving the stability problem of the closed-loop system implies also perfect tracking of the RC scheme.

[8]Such aspects have been often neglected in works related to nonlinear RC systems (e.g. in Owens, Li, and Banks, 2007) or strong assumptions about regularity of solutions have been done. This is not surprising since the mathematical tools available at this level are very recent.

Another interesting consideration involving this point of view is the expression of the error in the modified RC scheme of figure 1.5. In this case, assuming for simplicity $G_f(s) = \frac{1}{1+\bar{q}s}$, the expression of the error becomes $e(t) = v(t) - v(t - \tau) + \bar{q}\dot{v}(t)$. Now if the closed-loop is exponentially stable the signal $v(t)$ will be τ-periodic at steady state, implying $e(t) - \bar{q}\dot{v} \to 0$ as $t \to \infty$. This shows that the exact internal model is lost by adding a low-pass filter in the repetitive compensator since a periodic steady state error will be present depending on the design of the filter.

1.5 Contribution of the Thesis

Here the main contribution of this thesis is summarized and the tools that are used are highlighted.

1. In Chapter 2 a state-space approach using dissipativity tools and infinite dimensional port-Hamiltonian systems is performed in order to:

   - Analyse RC systems with a novel approach;
   - Analyse modified RC system with a novel approach;
   - Analyse a class of nonlinear RC systems leading to novel stability conditions.

   The analysis considers (nonlinear) RC schemes in their infinite-dimensional nature and no simplifying assumptions are done. Consequently aspects like well-posedness and regularity of solutions are consistently treated inside the framework.

2. In chapter 3 a state-space approach using invariance analysis of autonomous discrete-time systems is used in order to:

   - Describe periodic output regulation in a general way;
   - Design a controller that achieves robustly periodic output regulation in both continuous-time and discrete-dime;
   - Handle rigorously a nonlinear case, namely the case in which the system is a static nonlinearity.

   The analysis is performed in a framework that uses the *lifting* technique and is not directly based on the RC framework. It will be shown however that RC turns out to be a particular case merging from the selection of the simplest controller in this framework.

3. In Chapter 4 a digital scheme is presented that binds B-spline trajectory generation and digital RC. The scheme works on non-minimum phase linear systems and experimental results validating the theoretical method are presented.

Chapter 2

Analysis of Repetitive Control: the Port-Hamiltonian Approach

This chapter deals with a novel way to treat RC. Topics that come different fields of mathematics and system theory are consistently used in order to perform stability analysis of RC schemes.

The new point of view that is used differs from the classical frequency-based methods introduced in chapter 1 since it takes place in the framework of *dissipativity theory* applied to systems in which coupled *partial differential equations* (PDEs) and *ordinary differential equations* (ODEs) are present. The immersion of RC in this field will allow to derive the classical stability and tracking conditions in time domain since a state space approach will be used.

Furthermore, as it may happen at any level of human comprehension, a new perspective can push further the absolute knowledge of a subject. In fact the use of these tools and recent related results suggests ways to attack the stability analysis of RC schemes in its nonlinear version, besides of generating an alternative and useful derivation of the classical results for linear (modified) RC schemes.

2.0.1 Background on port-Hamiltonian Systems

Port-Hamiltonian systems (Maschke and van der Schaft, 1992) have been introduced to model lumped parameter physical systems in an unified manner (van der Schaft and Jeltsema, 2014) and their generalization to the infinite-dimensional scenario led to the definition of distributed port-Hamiltonian systems (van der Schaft and Maschke, 2002). These turned out to form an effective framework for describing distributed parameter physical systems as *boundary control systems* (BCS) (Fattorini, 1968), i.e. abstract systems whose dynamic is written in terms of a partial differential equation (PDE) with control and observation at the boundary of the spatial domain.

Recently general synthesis methodology of exponentially stabilizing control laws for the class of linear BCS in port-Hamiltonian form (Le Gorrec, Zwart, and Maschke, 2005; Jacob and Zwart, 2012) have been developed. In particular in (Le Gorrec, Zwart, and Maschke, 2005), *all* the admissible inputs defining a well-posed BCS are presented, together with a second similar parametrization that characterizes the (boundary) outputs. The distributed port-Hamiltonian system turns out to be dissipative van der Schaft, 2000, with its Hamiltonian as storage function, and quadratic supply rate.

In sections 2.1 and 2.2 these crucial tools that will be used are presented while in the rest of the chapter the actual RC analysis is performed.

2.1 Distributed port-Hamiltonian Systems

In the following the class of distributed port-Hamiltonian systems studied in (Le Gorrec, Zwart, and Maschke, 2005; Villegas et al., 2009; Jacob and Zwart, 2012; Ramírez et al., 2014) is considered. These systems belong namely to the class of *linear distributed port-Hamiltonian systems on a 1-D spatial domain* and are described by the following PDE

$$\frac{\partial x}{\partial t}(t,z) = P_1 \frac{\partial}{\partial z}\big(\mathcal{L}(z)x(t,z)\big) + (P_0 - G_0)\mathcal{L}(z)x(t,z). \tag{2.1}$$

Here x is the state variable and depends both on time t and on the spatial variable $z \in [a,\, b]$. The matrices P_1, P_0, G_0 and $\mathcal{L}(z)$ have the following properties[1]:

- $P_1 = P_1^{\mathrm{T}}$ and P_1^{-1} exists;
- $P_0 = -P_0^{\mathrm{T}}$;
- $G_0 = G_0^{\mathrm{T}}$;
- $\mathcal{L}(\cdot)$ is a bounded and Lipschitz continuous matrix-valued function such that $\mathcal{L}(z) = \mathcal{L}^{\mathrm{T}}(z) \geq \kappa I$, with $\kappa > 0$, for all $z \in [a,\, b]$.

For the sake of clearness, $(\mathcal{L}x)(t,z) := \mathcal{L}(z)x(t,z)$.
The state space is $X = L^2(a,b;\mathbb{R}^n)$, and is endowed with the inner product

$$\langle x_1 \mid x_2 \rangle_{\mathcal{L}} = \langle x_1 \mid \mathcal{L}x_2 \rangle$$

and norm $\|x\|^2_{\mathcal{L}} = \langle x \mid x \rangle_{\mathcal{L}}$, where $\langle \cdot \mid \cdot \rangle$ denotes the natural L^2-inner product[2]. The selection of this space for the state variable is motivated by the fact that $\|\cdot\|^2_{\mathcal{L}}$ is linked to the energy function of (2.1)[3]. Indeed for this class of systems it holds that the Hamiltonian function $H(x)$, which represents the energy of the system, equals half of the squared norm of the state, i.e.

$$H(x) = \frac{1}{2}\,\|x\|^2_{\mathcal{L}}\,. \tag{2.2}$$

As a consequence, X is also called the space of energy variables, and $\mathcal{L}x$ denotes the co-energy variables.

The following remark describes the presented distributed port-Hamiltonian system from an operator perspective, which will be very useful for further developments.

Remark 2.1.1. *The PDE (2.1) can be also written as $\dot{x} = \mathcal{J}x$, where $\mathcal{J}$ is the linear operator defined as $\mathcal{J}x := P_1\frac{\partial}{\partial z}(\mathcal{L}x) + (P_0 - G_0)\mathcal{L}x$, with domain $D(\mathcal{J}) = \{\mathcal{L}x \in H^1(a,b;\mathbb{R}^n)\}$. Here $H^1(a,b;\mathbb{R}^n)$ denotes the Sobolev space of order one.*

As consistent extension with respect to lumped PH systems, dPH systems have peculiar dissipative properties. For this purpose the PDE (2.1) has to be completed

[1]Refer to Jacob and Zwart, 2012 for a detailed derivation and discussion of these systems.

[2]As consequence $\|x\|_{\mathcal{L}} = \sqrt{\int_a^b (x^{\mathrm{T}}\mathcal{L}(z)x)dz}$

[3]In finite-dimensional physical systems this is always true since all the norms are equivalent in finite-dimensional complete metric spaces. This means that, e.g. in $\mathbb{R}^n$, the classical 2-norm is always linked to the energy of the system. On the contrary in infinite-dimensional systems norms are not equivalent and the concept of energy is different from the concept of squared norm.

by a set of boundary port variables. More precisely, the boundary port variables associated to (2.1) are the vectors f_∂, $e_\partial \in \mathbb{R}^n$ defined by

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix} \begin{pmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{pmatrix} \tag{2.3}$$

which means that they are a linear combination of the restriction of the co-energy variables on the boundary.

The definition of these variables is important because integration by parts shows that, considering the Hamiltonian defined in (2.2),

$$\dot{H}(x(t,\cdot)) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|x\|^2_{\mathcal{L}} = \langle \mathcal{J}x \mid x \rangle_{\mathcal{L}} \leq e^{\mathrm{T}}_\partial(t) f_\partial(t) \tag{2.4}$$

The problem of defining the boundary inputs and outputs for (2.1) to have a boundary control system on X in the sense of the semigroup theory (Curtain and Zwart, 1995) has been addressed in (Le Gorrec, Zwart, and Maschke, 2005). This important result is reported next.

Theorem 2.1.1. *Denote by W a $n \times 2n$ real matrix, then define $\mathcal{B} : H^1(a,b;\mathbb{R}^n) \to \mathbb{R}^n$ and the input $u(t)$ as*

$$u(t) = W \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} =: \mathcal{B}x(t) \tag{2.5}$$

If W has full rank and satisfies $W\Sigma W^{\mathrm{T}} \geq 0$, with

$$\Sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

then the system (2.1) with input (2.5) is a boundary control system on X. Furthermore, the operator $\bar{\mathcal{J}}x := P_1(\partial/\partial z)(\mathcal{L}x) + (P_0 - G_0)\mathcal{L}x$ with domain

$$D(\bar{\mathcal{J}}) = \left\{ \mathcal{L}x \in H^1(a,b;\mathbb{R}^n) \mid \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \in \operatorname{Ker} W \right\} = \left\{ \mathcal{L}x \in H^1(a,b;\mathbb{R}^n) \mid \mathcal{B}x = 0 \right\}$$

generates a contraction semigroup[4] *on X. Moreover, let $\tilde{W}$ be a full rank $n \times 2n$ matrix such that $\begin{pmatrix} W^{\mathrm{T}} & \tilde{W}^{\mathrm{T}} \end{pmatrix}$ is invertible and let P be given by*

$$P = \begin{pmatrix} W\Sigma W^{\mathrm{T}} & W\Sigma \tilde{W}^{\mathrm{T}} \\ \tilde{W}\Sigma W^{\mathrm{T}} & \tilde{W}\Sigma \tilde{W}^{\mathrm{T}} \end{pmatrix}^{-1}$$

Define the output as

$$y(t) = \tilde{W} \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} =: \mathcal{C}x(t) \tag{2.6}$$

with $\mathcal{C} : H^1(a,b;\mathbb{R}^n) \to \mathbb{R}^n$. Then, for $u \in C^2(0,\infty;\mathbb{R}^n)$ and $(\mathcal{L}x)(0) \in H^1(a,b;\mathbb{R}^n)$, the energy balance equation

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|x(t)\|^2_{\mathcal{L}} \leq \frac{1}{2} \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}^{\mathrm{T}} P \begin{pmatrix} u(t) \\ y(t) \end{pmatrix} \tag{2.7}$$

is satisfied.

[4] It is assumed that the reader is familiar with the concept of strongly continuous semigroup $T(t)$ on a Hilbert space, or C_0-semigroup. Every C_0-semigroup satisfies $\|T(t)\| \leq Me^{\omega t}$ for some real ω and M. A contraction semigroup is a subclass of C_0-semigroup such that $\|T(t)\| \leq 1$

This theorem is very important because it provides a way to chose the boundary input (2.5) properly, i.e. without loosing well-posedness of the system. Furthermore the following corollary can be stated.

Corollary 2.1.1. *The BCS of Theorem 2.1.1 is dissipative*[5] *with storage function* $H(x) = \frac{1}{2}\|x\|^2_{\mathcal{L}}$*, and supply rate*

$$s(u,y) =: \frac{1}{2}\begin{pmatrix} u \\ y \end{pmatrix}^{\mathrm{T}} \underbrace{\begin{pmatrix} U & S \\ S^{\mathrm{T}} & Y \end{pmatrix}}_{:=P_{W,\tilde{W}}} \begin{pmatrix} u \\ y \end{pmatrix}, \tag{2.8}$$

where $U = U^{\mathrm{T}}$*,* S*, and* $Y = Y^{\mathrm{T}}$ *are* $n \times n$ *real matrices*[6]*. This means that for all* $(\mathcal{L}x)(0) \in H^1(a,b;\mathbb{R}^n)$ *and* $u \in C^2(0,\infty;\mathbb{R}^n)$ *such that* $u(0) = \mathcal{B}x(0)$*, we can write that* $H(x(t)) - H(x(0)) \leq \int_0^t s(u(\tau), y(\tau))\,\mathrm{d}\tau$*.*

This result shows that dPH systems with boundary inputs chosen as suggested in Theorem 2.1.1 are always dissipative with respect to quadratic a supply rate[7]. This concept is the key element used in the following developments, which have been treated in the general case only very recently.

2.2 Exponential Stabilisation of Linear Boundary Control System

2.2.1 A general approach

In contrast with the generality of the result described in Theorem 2.1.1, most researches on stabilization techniques for distributed port-Hamiltonian systems (see e.g. Villegas et al., 2009; Schöberl and Siuka, 2013; Macchelli, 2013; Ramírez et al., 2014; Macchelli et al., 2017), are focused on a particular input-output mapping: the BCS has to be in *impedance form*, i.e. input and output are selected so that the system is passive[8], leading to the power balance

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{L}} \leq u^{\mathrm{T}}(t)y(t).$$

Then, control design relies on passivity theory, and the most common strategy is to add dissipation at the boundary, and / or to shape the energy function to shift the equilibrium, which is the natural extension to dPH systems of *energy shaping* and *damping injections* techniques developed for lumped PH systems.

[5]Here it assumed that the reader is familiar with the concept of dissipative systems, discussed in e.g. (van der Schaft, 2000)

[6]The (U, S, Y) matrices are the same of the (Q, S, R) ones used in classical literature on dissipative systems, (van der Schaft, 2000).

[7]In finite-dimensional port-Hamiltonian system the dissipativity property is specialized in *passivity*, i.e. it holds $\dot{H} \leq u^{\mathrm{T}}y$. Thus the corollary shows how dPH systems have wider dissipative properties.

[8]Referring to (2.8) this supply rate is obtained choosing boundary inputs and outputs such that $U = Y = 0$ and $S = I$. This passivity property is peculiar in lumped PH systems for Hamiltonian functions which are bounded from below.

Another important input-output mapping is the one that leads to a BCS in *scattering form*[9]. In this case the power balance is

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{L}} \leq \frac{1}{2}\|u(t)\|^2 - \frac{1}{2}\|y(t)\|^2 .$$

In the following a general approach, developed in Macchelli and Califano, 2018, is presented. The aim of this work is to extend the stability results to general dPH systems in the form (2.1) that present generic a dissipativity property (2.7), and not only the impedance passivity case.

At first, it is proved that the closed-loop system resulting from the feedback interconnection of a linear regulator and a BCS in port-Hamiltonian form is again a BCS in the sense of the semigroup theory Curtain and Zwart, 1995, Definition 3.3.2, in which the input is the reference signal. This happens if the controller is stable and dissipative with respect to a class of supply rates that is determined by the input-output mapping of the infinite-dimensional plant (subsection 2.2.2).

Moreover, the addition of dissipation makes the closed-loop storage function to decrease exponentially, thus implying exponential stability of the equilibrium. Thanks to these techniques, exponential stability is then proved for a large class of systems whose dynamic is described by coupled PDEs and ordinary differential equations (ODEs) (subsection 2.2.3). This result is an extension of Ramírez et al., 2014, where exponential stability was proved under the hypothesis that the regulator is a strictly input passive port-Hamiltonian system, and that the BCS is in impedance form. However, it is important to underline that such an extension relies on some technical lemmas presented in Ramírez et al., 2014, and generalized here to cope with a larger class of BCS in port-Hamiltonian form.

The potentialities of this approach are at first illustrated in case the BCS is in impedance or in scattering form, (van der Schaft, 2000, Chapter 4.4.3), and sufficient conditions on the the finite-dimensional controller to have exponential stability in closed-loop are provided (subsection 2.2.4).

After the presentation of these results, from section 2.3 till the end of the chapter, the theory will be applied to RC schemes, providing the novel analysis.

2.2.2 Well-posedness of systems with coupled PDEs and ODEs

In Sect. 2.2.3, the design of a control system for the PDE (2.1) that leads to an exponentially stable closed-loop system will be discussed. However, a preliminary problem is to understand if the linear system of coupled PDEs and ODEs associated to the closed-loop dynamics has a unique solution, and if it is a well-defined BCS. In this respect, let us consider the following control system

$$\begin{cases} \dot{x}_c(t) = A_c x_c(t) + B_c u_c(t) \\ y_c(t) = C_c x_c(t) + D_c u_c(t) \end{cases} \tag{2.9}$$

where $x_c \in \mathbb{R}^{n_c}$, $u_c, y_c \in \mathbb{R}^n$, and the matrices A_c, B_c, C_c and D_C have the appropriate dimensions. All the eigenvalues of A_c are non-positive, and the pair (A_c, B_c) is controllable. System (2.9) is in standard feedback interconnection with the BCS of

[9]Referring to (2.8) this supply rate is obtained choosing boundary inputs and outputs such that $U = -Y = I$ and $S = 0$.

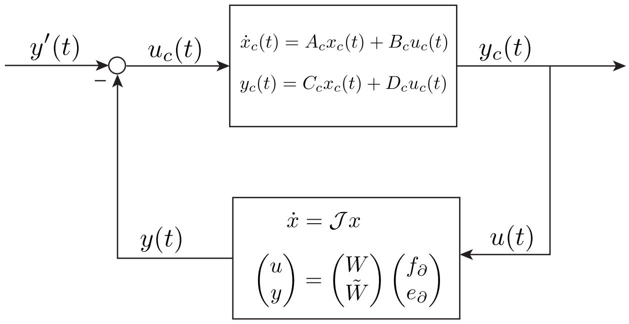


FIGURE 2.1: Standard feedback interconnection (2.10) of a finite dimensional controller (2.9) and a BCS defined in Theorem 2.1.1.

Theorem 2.1.1, i.e.

$$\begin{pmatrix} u_c(t) \\ y_c(t) \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} u(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} y'(t) \\ 0 \end{pmatrix}, \tag{2.10}$$

where $y' \in \mathbb{R}^n$ is the reference signal. The closed-loop system is shown in figure 2.1.

In order to proof that the closed loop system is again a well-posed BCS the main idea is characterize the controller by means of dissipativity propertied. In particular the major requirement is that there exists a symmetric, positive definite $n_c \times n_c$ real matrix Q_c such that (2.9) is dissipative with storage function

$$E_c(x_c) = \frac{1}{2} x_c^{\mathrm{T}} Q_c x_c \tag{2.11}$$

and supply rate

$$s_c(u_c, y_c) = \frac{1}{2} \begin{pmatrix} u_c \\ y_c \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} U_c & S_c \\ S_c^{\mathrm{T}} & Y_c \end{pmatrix} \begin{pmatrix} u_c \\ y_c \end{pmatrix} \tag{2.12}$$

with $U_c = U_c^{\mathrm{T}}$, and $Y_c = Y_c^{\mathrm{T}}$. This means that along all system trajectories for all $t \geq 0$, we have that

$$E_c(x_c(t)) - E_c(x_c(0)) \leq \int_0^t s_c\big(u_c(\tau), y_c(\tau)\big)\, \mathrm{d}\tau.$$

A necessary and sufficient condition for this property to hold is summarised in the next proposition, which is just a trivial extension of the Kalman-Yakubovich-Popov (KYP) lemma.

Proposition 2.2.1. *The linear system (2.9) is dissipative with storage function (2.11) in which $Q_c = Q_c^{\mathrm{T}} > 0$ is a real $n_c \times n_c$ matrix, and supply rate (2.12) if and only if:*

$$\mathcal{M}_c - \mathcal{N}_c \leq 0 \tag{2.13}$$

in which

$$\begin{aligned} \mathcal{M}_c &= \begin{pmatrix} Q_c A_c + A_c^{\mathrm{T}} Q_c & Q_c B_c \\ B_c^{\mathrm{T}} Q_c & 0 \end{pmatrix}, \\ \mathcal{N}_c &= \begin{pmatrix} C_c^{\mathrm{T}} Y_c C_c & C_c^{\mathrm{T}} Y_c D_c \\ D_c^{\mathrm{T}} Y_c C_c & U_c + D_c^{\mathrm{T}} Y_c D_c \end{pmatrix} + \begin{pmatrix} 0 & C_c^{\mathrm{T}} S_c \\ S_c^{\mathrm{T}} C_c & D_c^{\mathrm{T}} S_c + S_c^{\mathrm{T}} D_c \end{pmatrix}. \end{aligned} \tag{2.14}$$

The system resulting from the interconnection of (2.1) and (2.9) through the set of relations (2.10) can be written as

$$\begin{cases} \dot{\xi}(t) = \mathcal{J}_{cl}\xi(t) + B_{cl}y'(t) \\ D_c y'(t) = \mathcal{B}'\xi(t) := \begin{pmatrix} \mathcal{B} + D_c\mathcal{C} & -C_c \end{pmatrix} \xi(t) \end{cases} \tag{2.15}$$

where the operators $\mathcal{B}$ and $\mathcal{C}$ are defined in (2.5) and (2.6),

$$\xi = \begin{pmatrix} x \\ x_c \end{pmatrix} \in X_{cl} := X \times \mathbb{R}^{n_c} \tag{2.16}$$

is the state variable, $\mathcal{J}_{cl} : D(\mathcal{J}_{cl}) \subset X_{cl} \to X_{cl}$ and $B_{cl} : \mathbb{R}^n \to X_{cl}$ are the linear operators

$$\mathcal{J}_{cl}\xi := \begin{pmatrix} \mathcal{J} & 0 \\ -B_c\mathcal{C} & A_c \end{pmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix} \quad B_{cl}v := \begin{pmatrix} 0 \\ B_c v \end{pmatrix} \tag{2.17}$$

with $D(\mathcal{J}_{cl}) := D(\mathcal{J}) \times \mathbb{R}^{n_c}$, being $\mathcal{J}$ the operator introduced in Remark 2.1.1. Moreover, the state space X_{cl} is endowed with the inner product

$$\langle \xi_1 \mid \xi_2 \rangle_{X_{cl}} = \langle x_1 \mid x_2 \rangle_{\mathcal{L}} + x_{c,1}^{\mathrm{T}} Q_c x_{c,2}. \tag{2.18}$$

The following proposition provides sufficient conditions for well-posedness of the closed-loop system and is the main contribution of this subsection.

Proposition 2.2.2. *Let us consider the closed-loop system resulting from the feedback interconnection (2.10) of (2.1) and (2.9), which results in (2.15). If*

$$\begin{pmatrix} Y & -S^{\mathrm{T}} \\ -S & U \end{pmatrix} + \begin{pmatrix} U_c & S_c \\ S_c^{\mathrm{T}} & Y_c \end{pmatrix} \leq 0 \tag{2.19}$$

the operator $\bar{\mathcal{J}}_{cl}\xi := \begin{pmatrix} \mathcal{J} & 0 \\ -B_c\mathcal{C} & A_c \end{pmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix}$ *with domain*

$$D(\bar{\mathcal{J}}_{cl}) = \left\{ \begin{pmatrix} x \\ x_c \end{pmatrix} \in X_{cl} \mid x \in D(\mathcal{J}), \text{and } \mathcal{B}' \begin{pmatrix} x \\ x_c \end{pmatrix} = 0 \right\} \tag{2.20}$$

and $\mathcal{B}'$ *defined in (2.15) generates a contraction semigroup on* X_{cl}*. Moreover, (2.15) with* $\mathcal{J}_{cl}$ *and* B_{cl} *defined by (2.17) is a BCS on* X_{cl}*, with* $y' \in C^2(0,\infty;\mathbb{R}^n)$*.*

Proof. At first, note that the last condition in (2.20) can be equivalently written as

$$\begin{pmatrix} f_\partial^{\mathrm{T}} & e_\partial^{\mathrm{T}} & x_c^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}} \in \operatorname{Ker} W_{cl} \tag{2.21}$$

with $W_{cl} := \begin{pmatrix} W + D_c\tilde{W} & -C_c \end{pmatrix}$. Since $D(\bar{\mathcal{J}}_{cl})$ is dense in X_{cl}, and $\bar{\mathcal{J}}_{cl}$ is closed, according to the Lumer-Phillips theorem (Luo, Guo, and Morgul, 1999, Theorem 2.27), the first condition to check in order to prove that $\bar{\mathcal{J}}_{cl}$ generates a contraction semigroup on X_{cl} is that $\langle \bar{\mathcal{J}}_{cl}\xi \mid \xi\rangle_{X_{cl}} \leq 0$ for $\xi \in D(\bar{\mathcal{J}}_{cl})$. It is easy to see that

$$\begin{aligned}\langle \bar{\mathcal{J}}_{cl}\xi \mid \xi\rangle_{X_{cl}} \leq \frac{1}{2}\begin{pmatrix} f_\partial \\ e_\partial\end{pmatrix}^{\mathrm{T}} \begin{pmatrix} W \\ \tilde{W}\end{pmatrix}^{\mathrm{T}} \begin{pmatrix} U & S \\ S^{\mathrm{T}} & Y\end{pmatrix} \cdot \\ \cdot \begin{pmatrix} W \\ \tilde{W}\end{pmatrix} \begin{pmatrix} f_\partial \\ e_\partial\end{pmatrix} + \frac{1}{2} x_c^{\mathrm{T}}(Q_c A_c + A_c^{\mathrm{T}} Q_c) x_c - \\ - \frac{1}{2} x_c^{\mathrm{T}} Q_c B_c \tilde{W} \begin{pmatrix} f_\partial \\ e_\partial\end{pmatrix} - \frac{1}{2}\begin{pmatrix} f_\partial \\ e_\partial\end{pmatrix}^{\mathrm{T}} \tilde{W}^{\mathrm{T}} B_c^{\mathrm{T}} Q_c x_c \end{aligned} \tag{2.22}$$

where (2.7) and (2.8) have been taken into account. From (2.21), we have that $W\begin{pmatrix} f_\partial \\ e_\partial\end{pmatrix} = -D_c\tilde{W}\begin{pmatrix} f_\partial \\ e_\partial\end{pmatrix} + C_c x_c$, and because of (2.19), after some computations, we get

$$\begin{aligned}\begin{pmatrix} f_\partial \\ e_\partial\end{pmatrix}^{\mathrm{T}} \begin{pmatrix} W \\ \tilde{W}\end{pmatrix}^{\mathrm{T}} \begin{pmatrix} U & S \\ S^{\mathrm{T}} & Y\end{pmatrix} \begin{pmatrix} W \\ \tilde{W}\end{pmatrix} \begin{pmatrix} f_\partial \\ e_\partial\end{pmatrix} \leq -\begin{pmatrix} f_\partial \\ e_\partial\end{pmatrix}^{\mathrm{T}} \tilde{W}^{\mathrm{T}} \left(U_c + D_c^{\mathrm{T}} Y_c D_c + D_c^{\mathrm{T}} S_c + S_c^{\mathrm{T}} D_c\right) \cdot \\ \cdot \tilde{W}\begin{pmatrix} f_\partial \\ e_\partial\end{pmatrix} + x_c^{\mathrm{T}} C_c^{\mathrm{T}} (S_c + Y_c D_c)\tilde{W}\begin{pmatrix} f_\partial \\ e_\partial\end{pmatrix} + \begin{pmatrix} f_\partial \\ e_\partial\end{pmatrix}^{\mathrm{T}} \tilde{W}^{\mathrm{T}}\left(D_c^{\mathrm{T}} Y_c + S_c\right) C_c x_c - x_c^{\mathrm{T}} C_c^{\mathrm{T}} Y_c C_c x_c\end{aligned}$$

which implies that (2.22) can be rewritten as

$$\langle \bar{\mathcal{J}}_{cl}\xi \mid \xi\rangle_{X_{cl}} \leq \frac{1}{2}\begin{pmatrix} x_c \\ u_c\end{pmatrix}^{\mathrm{T}} \{\mathcal{M}_c - \mathcal{N}_c\} \begin{pmatrix} x_c \\ u_c\end{pmatrix}$$

because of (2.13), with $\mathcal{M}_c$ and $\mathcal{N}_c$ defined in (2.14). This proves that $\langle \bar{\mathcal{J}}_{cl}\xi \mid \xi\rangle_{X_{cl}} \leq 0$.

The second step consists in verifying that $(I - \bar{\mathcal{J}}_{cl})$ is surjective, i.e. that for all $(f, f_c) \in X \times \mathbb{R}^{n_c}$ there exists $(x, x_c) \in D(\bar{\mathcal{J}}_{cl})$ such that

$$\begin{pmatrix} f \\ f_c\end{pmatrix} = \begin{pmatrix} (I - \mathcal{J})x \\ B_c\mathcal{C}x + (I - A_c)x_c\end{pmatrix} \tag{2.23}$$

under the constraint (2.20) that now we rewrite as

$$(\mathcal{B} + D_c\mathcal{C})x = C_c x_c. \tag{2.24}$$

At first, note that $(I - A_c)$ is invertible because all the eigenvalues of A_c are non-positive, so from (2.23) we obtain that $x_c = (I - A_c)^{-1}(f_c - B_c\mathcal{C}x)$, which from (2.24) implies that $[\mathcal{B} + H_c(1)\mathcal{C}]\, x = C_c(I - A_c)^{-1} f_c$, where $H_c(1) = C_c(I - A_c)^{-1}B_c + D_c$. Now, let us assume that $\tilde{f}_c = C_c(I - A_c)^{-1} f_c$ and that $\tilde{x} = x - \tilde{B}\tilde{f}_c$, where $\tilde{B}$ is a linear operator such that $[\mathcal{B} + H_c(1)\mathcal{C}]\,\tilde{B} = I$. In this respect, the existence of $\tilde{B}$ follows from the proof of Theorem 2.1.1 (see Le Gorrec, Zwart, and Maschke, 2005 for more details), and from the fact that $W + H_c(1)\tilde{W}$ is full (row) rank. This latter property is verified because the matrix $\begin{pmatrix} W^{\mathrm{T}} & \tilde{W}^{\mathrm{T}}\end{pmatrix}$ is invertible. After these transformations, we can write that

$$\begin{aligned}(I - \mathcal{J})\tilde{x} &= f + (I - \mathcal{J})\tilde{B}\tilde{f}_c, \\ [\mathcal{B} + H_c(1)\mathcal{C}]\,\tilde{x} &= 0.\end{aligned} \tag{2.25}$$

The transfer matrix of the control system (2.9) is given by $H_c(s) = C_c(sI - A_c)^{-1}B_c +$

D_c, where $s \in \mathbb{C}$. In Willems, 1972, Remark 9, pg. 383 it is shown that (2.9) satisfies the conditions of Prop. 2.2.1, i.e. it is dissipative with storage function (2.11) and supply rate (2.12), if and only if

$$\begin{pmatrix} I \\ H_c(s) \end{pmatrix}^{\mathrm{H}} \begin{pmatrix} U_c & S_c \\ S_c^{\mathrm{T}} & Y_c \end{pmatrix} \begin{pmatrix} I \\ H_c(s) \end{pmatrix} \geq 0 \tag{2.26}$$

for all $s \in \mathbb{C}$ such that $\operatorname{Re} s \geq 0$. Here, $\cdot^{\mathrm{H}}$ denotes the conjugate transpose of a matrix. By taking $s = 1$ in (2.26), from (2.19) and with (2.12) in mind, we can write that

$$\begin{pmatrix} -H_c^{\mathrm{T}}(1) & I \end{pmatrix} P_{W,\tilde{W}} \begin{pmatrix} -H_c(1) \\ I \end{pmatrix} \leq 0. \tag{2.27}$$

The operator $\mathcal{J}$ applied to all the $\tilde{x}$ that satisfy the constraint expressed by the second relation in (2.25), i.e. $[\mathcal{B} + H_c(1)\mathcal{C}]\,\tilde{x} = 0$, generates a contraction semigroup if and only if $H_c(1)$ satisfies (2.27). This implies that $(I - \mathcal{J})$ has an inverse, and then $\tilde{x}$ can be computed from the first relation in (2.25) for all $(f, \tilde{f}_c) \in X \times \mathbb{R}^{n_c}$. Then, by definition we have $x = \tilde{x} + \tilde{B}\tilde{f}_c$, and finally $x_c = (I - A_c)^{-1}(f_c - B_c\mathcal{C}x)$, i.e. we have computed $(x, x_c) \in D(\bar{\mathcal{J}}_{cl})$ such that (2.23) holds. To prove that (2.15) is a boundary control system on X_{cl}, note that there exists a linear operator $B : \mathbb{R}^n \to X_{cl}$ such that for all $v \in \mathbb{R}^n$, we have that $Bv \in D(\mathcal{J}_{cl})$, and $\begin{pmatrix} \mathcal{B} + D_c\mathcal{C} & -C_c \end{pmatrix} Bv \equiv \mathcal{B}'Bv = v$. This fact follows from the proof of Theorem 2.1.1 in Le Gorrec, Zwart, and Maschke, 2005 since the matrix W_{cl} defined in (2.21) is full (row) rank. Similarly to Curtain and Zwart, 1995, Theorem 3.3.3, define $\tilde{\xi} = \xi - BD_cy'$, which from the second relation in (2.15) implies that $\mathcal{B}'\tilde{\xi} = \mathcal{B}'\xi - D_cy' = 0$. Moreover, from the first relation in (2.15), we have that $\dot{\tilde{\xi}} = \bar{\mathcal{J}}_{cl}\tilde{\xi} + \mathcal{J}_{cl}BD_cy' + B_{cl}y' - BD_c\dot{y}'$. Now, we have already proved that $\bar{\mathcal{J}}_{cl}$ generates a contraction semigroup on X_{cl}. Since B_{cl} and BD_c are bounded linear operators, and from Theorem 2.1.1 in Le Gorrec, Zwart, and Maschke, 2005 it is easy to see that also $\mathcal{J}_{cl}BD_c$ is bounded, from Curtain and Zwart, 1995, Theorem 3.1.3 we deduce that the previous differential equation has a unique classical solution provided that $y' \in C^2(0, \infty; \mathbb{R}^n)$. This proves that the closed-loop system (2.15) is a boundary control system in the sense of the semigroup theory, Curtain and Zwart, 1995, Definition 3.3.2. □

Remark 2.2.1. *If system (2.1) is in impedance form the control system (2.9) meets the condition of the previous proposition for example if it is passive, i.e. it is dissipative with respect to the supply rate* $s_c(u_c, y_c) = y_c^{\mathrm{T}}u_c$. *On the other hand, if (2.1) is in scattering form, the control system (2.9) can be selected such that it is dissipative with respect to the supply rate* $s_c(u_c, y_c) = \frac{1}{2}\gamma^2\|u_c\|^2 - \frac{1}{2}\|y_c\|^2$, *with* $|\gamma| \leq 1$. *In other words, (2.9) should have a* L^2*-gain lower than* γ, *with* $|\gamma| \leq 1$ *(van der Schaft, 2000).*

2.2.3 Exponential stability of systems with coupled PDEs and ODEs

This subsection aims at illustrating how to design the control system (2.9) that makes the closed-loop system exponentially stable. Let us consider the linear control system (2.9) in which A_c has all the eigenvalues with negative real part, and the pair (A_c, B_c) is controllable. Moreover, let us assume that there exists a storage function (2.11), with $Q_c = Q_c^{\mathrm{T}} > 0$, and a supply rate (2.12), with $U_c = U_c^{\mathrm{T}}$ and $Y_c = Y_c^{\mathrm{T}}$. The main requirement is that

$$\frac{1}{2}\left(\mathcal{M}_c - \mathcal{N}_c\right) \leq -\begin{pmatrix} -\delta_x(Q_cA_c + A_c^{\mathrm{T}}Q_c) & 0 \\ 0 & \delta_u I \end{pmatrix} \tag{2.28}$$

where δ_x and δ_u are two arbitrary small positive constants, and $\mathcal{M}_c$ and $\mathcal{N}_c$ are defined in (2.14). From a physical point of view, δ_x is related to the internal damping in the control system responsible for attenuating the lower frequencies in the plant dynamics. Differently, δ_u assures that the higher frequencies are damped.

Remark 2.2.2. *Since A_c is exponentially stable, for all $Q_c = Q_c^{\mathrm{T}} > 0$ there exists $P_c = P_c^{\mathrm{T}} \geq 0$ such that $Q_c A_c + A_c^{\mathrm{T}} Q_c = -P_c$. So, if (2.28) holds, then also (2.13) is satisfied, which means that (2.9) is dissipative with respect to the storage function $E_c(x_c) = \frac{1}{2} x_c^{\mathrm{T}} Q_c x_c$ and supply rate (2.12). Consequently, if the other conditions of Prop. 2.2.2 are met, the closed-loop system is well posed, and the state trajectories exist. Moreover, with Corollary 2.1.1 in mind, a natural candidate Lyapunov function for the stability analysis is the "closed-loop energy function"*

$$E_{cl}(x(t), x_c(t)) = \frac{1}{2} \|\xi(t)\|^2_{X_{cl}} \tag{2.29}$$

where $\xi \in X_{cl}$ is the state variable introduced in (2.16), and $\|\cdot\|_{X_{cl}}$ is the norm associated with the inner product $\langle \cdot \mid \cdot \rangle_{X_{cl}}$, defined in (2.18).

Since the control system is exponentially stable and, as discussed in Prop. 2.2.2, the autonomous closed-loop system generates a contraction C_0-semigroup, the following lemmas hold true.

Lemma 2.2.1 (Ramírez et al., 2014, Lemmas III.3 and III.4). *Let us consider the control system (2.9) in which A_c is exponentially stable, and denote by $E_c(x_c) = \frac{1}{2} x_c^{\mathrm{T}} Q_c x_c$ a storage function, with $Q_c = Q_c^{\mathrm{T}} > 0$. Then*

- *There exists positive constants χ_1, χ_2 and τ_0 such that for all $\tau > \tau_0$*

$$\int_0^\tau E_c \, \mathrm{d}t \leq -\chi_1 \int_0^\tau x_c^{\mathrm{T}} (Q_c A_c + A_c^{\mathrm{T}} Q_c) x_c \, \mathrm{d}t + \chi_2 \int_0^\tau \|u_c\|^2 \, \mathrm{d}t. \tag{2.30}$$

- *For every $\delta_1 > 0$, there exists $\delta_2 > 0$ such that for all $\tau > 0$*

$$\int_0^\tau \left[\delta_1 E_c + \|y_c\|^2 \right] \mathrm{d}t \leq \delta_2 \int_0^\tau \left[E_c + \|u_c\|^2 \right] \mathrm{d}t. \tag{2.31}$$

Lemma 2.2.2 (Ramírez et al., 2014, Lemma IV.I). *Let us consider the closed-loop system of Prop. 2.2.2 with $y'(t) = 0$. Then, the "energy function" (2.29) satisfies for τ large enough*

$$E_{cl}(\tau) \leq c(\tau) \int_0^\tau \|(\mathcal{L}x)(t, z_\partial)\|^2 \, \mathrm{d}t + \frac{2c(\tau)}{c_1} \int_0^\tau E_c(t) \, \mathrm{d}t, \tag{2.32}$$

where $z_\partial \in \{a, b\}$, $c(\tau)$ is a positive constant that only depends on τ, and c_1 is a positive constant.

The conditions for the exponential stability of the closed-loop system are presented in the next proposition and represent the main contribution of this subsection.

Proposition 2.2.3. *Under the same conditions of Prop. 2.2.2, assume that the control system (2.9) is such that A_c has all the eigenvalues with negative real part, the pair (A_c, B_c) is controllable, and (2.28) holds with $\delta_x > 0$ and $\delta_u > 0$ sufficiently small, but finite. Then, the closed-loop system (2.15) with $y'(t) = 0$ is exponentially stable.*

Proof. Let us consider the candidate Lyapunov function (2.29). As in the proof of Prop. 2.2.2, the variation of E_{cl} along system trajectories is

$$\dot{E}_{cl} \leq \frac{1}{2}\begin{pmatrix} x_c \\ u_c \end{pmatrix}^{\mathrm{T}} \{\mathcal{M}_c - \mathcal{N}_c\} \begin{pmatrix} x_c \\ u_c \end{pmatrix}.$$

Now, we follow the same steps in the proof of Theorem IV.2 in Ramírez et al., 2014. The previous inequality implies that

$$\begin{aligned} \dot{E}_{cl} \leq \delta_x x_c^{\mathrm{T}} \left(Q_c A_c + A_c^{\mathrm{T}} Q_c\right) x_c - \epsilon_1 \delta_u \left\|u_c\right\|^2 - \\ - \epsilon_1 \delta_u \left(\|u\|^2 + \|y\|^2\right) + \epsilon_2 \delta_u \left\|y_c\right\|^2 \end{aligned} \tag{2.33}$$

because of condition (2.28), with ϵ_1 and ϵ_2 positive and such that $\epsilon_1 + \epsilon_2 = 1$, and because of $y_c = u$ and $u_c = -y$. From (2.3), (2.5), and (2.6), it is easy to see that there exists $\epsilon > 0$ such that $\|u(t)\|^2 + \|y(t)\|^2 \geq \epsilon \left\|(\mathcal{L}x)(t, z_\partial)\right\|^2$, for $z_\partial \in \{a, b\}$. Then, (2.33) becomes $\dot{E}_{cl} \leq \delta_x x_c^{\mathrm{T}} \left(Q_c A_c + A_c^{\mathrm{T}} Q_c\right) x_c - \epsilon_1 \delta_u \|u_c\|^2 - \epsilon\epsilon_1 \delta_u \left\|(\mathcal{L}x)(z_\partial)\right\|^2 + \epsilon_2 \delta_u \|y_c\|^2$, relation that can be integrated on the interval $[0, \tau]$, with $\tau > 0$ sufficiently large, so that one obtains

$$\begin{aligned} E_{cl}(\tau) - E_{cl}(0) \leq \delta_x \int_0^\tau x_c^{\mathrm{T}} \left(Q_c A_c + A_c^{\mathrm{T}} Q_c\right) x_c \, \mathrm{d}t + \epsilon_2 \delta_u \int_0^\tau \|y_c\|^2 \, \mathrm{d}t - \\ - \delta_u \int_0^\tau \left[\epsilon_1 \|u_c\|^2 + \epsilon\epsilon_2 \left\|(\mathcal{L}x)(z_\partial)\right\|^2\right] \mathrm{d}t \end{aligned}$$

which from (2.32) in Lemma 2.2.2 implies that

$$\begin{aligned} \left(1 + \frac{\epsilon\epsilon_2\delta_u}{c(\tau)}\right) E_{cl}(\tau) - E_{cl}(0) \leq \delta_x \int_0^\tau x_c^{\mathrm{T}}(Q_c A_c + A_c^{\mathrm{T}} Q_c) x_c \, \mathrm{d}t - \\ - \epsilon_1 \delta_u \int_0^\tau \|u_c\|^2 \, \mathrm{d}t + \epsilon_2 \delta_u \int_0^\tau \left[\frac{2\epsilon}{c_1} E_c(t) + \|y_c\|^2\right] \mathrm{d}t. \end{aligned}$$

From (2.31), if $\delta_1 = \frac{2\epsilon}{c_1}$, we can write that

$$\begin{aligned} \left(1 + \frac{\epsilon\epsilon_2\delta_u}{c(\tau)}\right) E_{cl}(\tau) - E_{cl}(0) \leq \delta_x \int_0^\tau x_c^{\mathrm{T}}(Q_c A_c + A_c^{\mathrm{T}} Q_c) x_c \, \mathrm{d}t - \\ - (\epsilon_1 - \epsilon_2\delta_2)\delta_u \int_0^\tau \|u_c\|^2 \, \mathrm{d}t + \epsilon_2\delta_2\delta_u \int_0^\tau E_c \, \mathrm{d}t, \end{aligned}$$

while from (2.30) that

$$\begin{aligned} \left(1 + \frac{\epsilon\epsilon_2\delta_u}{c(\tau)}\right) E_{cl}(\tau) - E_{cl}(0) \leq (\delta_x - \epsilon_2\delta_2\delta_u\chi_1) \int_0^\tau x_c^{\mathrm{T}}(Q_c A_c + A_c^{\mathrm{T}} Q_c) x_c \, \mathrm{d}t - \\ - [\epsilon_1 - \epsilon_2\delta_2(1 + \chi_2)]\delta_u \int_0^\tau \|u_c\|^2 \, \mathrm{d}t. \end{aligned} \tag{2.34}$$

Since ϵ_2 can be arbitrarily small, the right side of (2.34) can be made lower than 0, which means that $\left(1 + \frac{\epsilon\epsilon_2\delta_u}{c(\tau)}\right) E_{cl}(\tau) \leq E_{cl}(0)$. Then, it is immediate that for some τ sufficiently large we have proved that $E_{cl}(\tau) < E_{cl}(0)$. This implies that the semigroup $T(t)$ generated by $\mathcal{J}_{cl}$ satisfies $\|T(\tau)\|_{X_{cl}} < 1$ for sufficiently large τ, i.e. the growth bound of the semigroup is strictly negative. This means that there exists positive scalars M and α such that $\|T(t)\|_{X_{cl}} \leq M e^{-\alpha t}$ for all $t \geq 0$, and this completes the proof. □

2.2.4 Stabilisation of distributed port-Hamiltonian systems in impedance and scattering form

As discussed in Remark 2.2.1, we start by assuming that (2.1) is in impedance form, i.e. dissipative with storage function given by the total energy $\frac{1}{2}\|x\|^2_{\mathcal{L}}$, and supply rate $s(u,y) = y^{\mathrm{T}}u$, where input $u(t)$ and output $y(t)$ are given in (2.5) and (2.6), respectively, with $W\Sigma W^{\mathrm{T}} = \tilde{W}\Sigma\tilde{W}^{\mathrm{T}} = 0$ and $\tilde{W}\Sigma W^{\mathrm{T}} = I$. From (2.28) in Prop. 2.2.3, (2.9) leads to an exponentially stable closed-loop system if there exists $Q_c = Q_c^{\mathrm{T}} > 0$, $\delta_x > 0$ and $\delta_u > 0$ sufficiently small such that

$$\begin{pmatrix} (1-2\delta_x)(Q_cA_c + A_c^{\mathrm{T}}Q_c) & Q_cB_c \\ B_c^{\mathrm{T}}Q_c & 0 \end{pmatrix} - \begin{pmatrix} 0 & C_c^{\mathrm{T}} \\ C_c & D_c + D_c^{\mathrm{T}} - 2\delta_u I \end{pmatrix} \leq 0. \tag{2.35}$$

This implies that $D_c + D_c^{\mathrm{T}} \geq 2\delta_u I > 0$, i.e. a positive feedthrough term must be present, and that

$$\dot{E}_c(x_c(t)) \leq y_c^{\mathrm{T}}(t)u_c(t) - \delta_u \|u_c(t)\|^2, \tag{2.36}$$

where $E_c = \frac{1}{2}x_c^{\mathrm{T}}Q_cx_c$ is the storage function of (2.9). This relation implies that the control system has to be input strictly passive (van der Schaft, 2000).

Corollary 2.2.1. *Under the same general conditions of Prop. 2.2.3, let us consider the BCS of Theorem 2.1.1 in impedance form, and denote the transfer matrix of (2.9) by $H_c(s) = C_c(sI - A_c)^{-1}B_c + D_c$. Then, the closed-loop system (2.15) is exponentially stable if the linear system with transfer matrix $H_c(s-\epsilon)$ is strictly input passive for some $\epsilon > 0$.*

Proof. Since $H_c(s-\epsilon)$ is strictly input passive, (2.36) holds and then it is easy to see that

$$\begin{pmatrix} Q_c(A_c + \epsilon I) + (A_c^{\mathrm{T}} + \epsilon I)Q_c & Q_cB_c \\ B_c^{\mathrm{T}}Q_c & 0 \end{pmatrix} - \begin{pmatrix} 0 & C_c^{\mathrm{T}} \\ C_c & D_c + D_c^{\mathrm{T}} - \delta_u I \end{pmatrix} \leq 0,$$

which is equivalent to (2.35) because $Q_c(A_c + \epsilon I) + (A_c^{\mathrm{T}} + \epsilon I)Q_c = Q_cA_c + A_c^{\mathrm{T}}Q_c + 2\epsilon Q_c$, and $-\delta_x(Q_cA_c + A_c^{\mathrm{T}}Q_c) \leq \epsilon Q_c$ for $\delta_x > 0$ since A_c is exponentially stable. □

Analogous considerations can be drawn if (2.1) is in scattering form, i.e. when input and output are selected so that the BCS is dissipative with storage function $\frac{1}{2}\|x\|^2_{\mathcal{L}}$ and supply rate $s(u,y) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\|y\|^2$. As reported in Remark 2.2.1, this is possible once W and $\tilde{W}$ in (2.5) and (2.6) are such that $W\Sigma W^{\mathrm{T}} = -\tilde{W}\Sigma\tilde{W}^{\mathrm{T}} = I$ and $\tilde{W}\Sigma W^{\mathrm{T}} = 0$. From (2.28) in Prop. 2.2.3, the linear regulator (2.9) exponentially stabilizes the boundary control system (2.1) if there exists $Q_c = Q_c^{\mathrm{T}} > 0$, and $\delta_x > 0$ and $\delta_u > 0$ sufficiently small such that

$$\begin{pmatrix} (1-2\delta_x)(Q_cA_c + A_c^{\mathrm{T}}Q_c) & Q_cB_c \\ B_c^{\mathrm{T}}Q_c & 0 \end{pmatrix} + \begin{pmatrix} C_c^{\mathrm{T}}C_c & C_c^{\mathrm{T}}D_c \\ D_c^{\mathrm{T}}C_c & D_c^{\mathrm{T}}D_c - (\gamma^2 - 2\delta_u)I \end{pmatrix} \leq 0 \tag{2.37}$$

for some γ such that $|\gamma| \leq 1$ (see also Remark 2.2.1). The LMI (2.37) implies that $D_c^{\mathrm{T}}D_c - (\gamma^2 - 2\delta_u)I \leq 0$, i.e. that $D_c^{\mathrm{T}}D_c - I < 0$, which means that the feedthrough gain has to be lower than 1 or, equivalently, that the following dissipation inequality with $|\gamma| < 1$ holds true:

$$\dot{E}_c(x_c(t)) \leq \frac{1}{2}\gamma^2 \|u_c(t)\|^2 - \frac{1}{2}\|y_c(t)\|^2. \tag{2.38}$$

Corollary 2.2.2. *Under the same general conditions of Prop. 2.2.3, let us consider the BCS of Theorem 2.1.1, which is assumed in scattering form. Moreover, let us denote by* $H_c(s) = C_c(sI - A_c)^{-1}B_c + D_c$ *the transfer matrix of the control system (2.9). Then, the closed-loop system (2.15) is exponentially stable if, for some* $\epsilon > 0$*, the linear system with transfer matrix* $H_c(s - \epsilon)$ *has* L^2*-gain* $\gamma < 1$ *or, equivalently, if (2.38) holds true.*

Proof. The result can be proved in the same way as Corollary 2.2.1. □

2.3 Stability Analysis of RC Systems: the port-Hamiltonian Approach

In this section a novel stability analysis will be applied on RC schemes. Since such schemes consist of coupled PDEs and ODEs, the idea is to rely on the previous results and on the properties of dissipative systems to perform an analysis in time-domain using a state-space approach, considering the infinite-dimensional nature of the system. As result a simpler and intuitive characterization of the controlled plants (that in Chapter 1 were referred to as $P(s)$) for which RC laws can be successfully applied is carried out.

The first step is to interpret the repetitive compensator of Figure 1.1 as a BCS in the sense of Theorem 2.1.1. Then the exponential stability result of Prop. 2.2.3 will be exploited to determine the conditions on the controlled plant to have a stable closed-loop system.

Proposition 2.3.1. *The repetitive compensator of Fig. 2.2 admits a port-Hamiltonian representation (2.1) with* $z \in [0, \tau]$, $P_1 = -I$, $P_0 = G_0 = 0$, *and* $\mathcal{L}(z) = I$. *If*

$$W = \sqrt{2}\begin{pmatrix} I & 0 \end{pmatrix} \quad \tilde{W} = \frac{\sqrt{2}}{2}\begin{pmatrix} -I & I \end{pmatrix} \tag{2.39}$$

the repetitive compensator is a BCS in the sense of Theorem 2.1.1 with input-output map defined as in (2.5) and (2.6), and the following energy-balance relation holds true:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|_2^2 = \frac{1}{2}\begin{pmatrix} u(t) \\ y(t) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} I & I \\ I & 0 \end{pmatrix} \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}. \tag{2.40}$$

Proof. The PDE (2.1) takes now the form

$$\frac{\partial x}{\partial t}(t,z) = -\frac{\partial x}{\partial z}(t,z) \tag{2.41}$$

and, since the spatial domain is $[0, \tau]$, it is easy to see that $x(t,\tau) = x(t-\tau, 0)$, which means that $x(t,\tau)$ is the delayed copy of $x(t,0)$, for all $t \geq \tau$. With Fig. 2.2 in mind, we can write that $y(t) = x(t,\tau)$ and that $u(t) + y(t) = x(t,0)$, which implies that $u(t) = x(t,0) - x(t,\tau)$. Since (2.41) is in the form (2.1), from (2.3) we have that

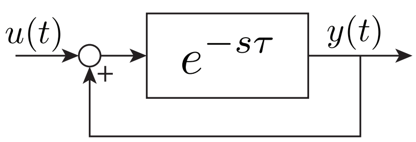


FIGURE 2.2: The repetitive compensator with I/O signals.

$f_\partial = \frac{1}{\sqrt{2}}[-x(\tau) + x(0)]$ and $e_\partial = \frac{1}{\sqrt{2}}[x(\tau) + x(0)]$, so $u(t)$ and $y(t)$ are obtained from (2.5) and (2.6) if the matrices W and $\tilde{W}$ are selected as in (2.39). Note that $W\Sigma W^{\mathrm{T}} = 0$, so (2.41) with input $u(t)$ is a BCS in the sense of Theorem 2.1.1. Finally, we have that $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|_2^2 = -\frac{1}{2}\|x(t,\tau)\|^2 + \frac{1}{2}\|x(t,0)\|^2$, which leads to (2.40) due to the definition of $u(t)$ and $y(t)$. □

The result reported in Prop. 2.3.1 allows to rely on the stability tools discussed in the previous section to determine under which conditions on the controlled plant the closed-loop system characterizing classical RC schemes is exponentially stable. In fact the RC scheme in Fig. 1.2 can be interpreted as merging from the standard feedback interconnection of the repetitive compensator as BCS and a finite-dimensional controlled plant described in the following.

Let us assume that the plant is the linear system

$$\begin{cases} \dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}\bar{u}(t) \\ \bar{y}(t) = \bar{C}\bar{x}(t) + \bar{D}\bar{u}(t) \end{cases} \tag{2.42}$$

where $\bar{x} \in \mathbb{R}^{\bar{n}}$, $\bar{u} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^n$ are the state variable, input and output, respectively. The matrices $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ have the appropriate dimensions. To perform the stability analysis of the repetitive control scheme, we note that for the closed-loop system the results of Prop. 2.2.2 are applicable. From (2.19) and (2.40) existence of solutions is guaranteed if in (2.42) $\bar{A}$ has no positive eigenvalues, the pair $(\bar{A}, \bar{B})$ is controllable, and if (2.42) is dissipative with respect to the storage function $\bar{E}(\bar{x}) = \frac{1}{2}\bar{x}^{\mathrm{T}}\bar{Q}\bar{x}$, with $\bar{Q} = \bar{Q}^{\mathrm{T}} > 0$, and supply rate

$$\bar{s}(\bar{u},\bar{y}) = \frac{1}{2}\begin{pmatrix}\bar{u}\\ \bar{y}\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}0 & I\\ I & -\sigma I\end{pmatrix}\begin{pmatrix}\bar{u}\\ \bar{y}\end{pmatrix}, \tag{2.43}$$

with $\sigma \geq 1$. Prop. 2.2.3 is, then, instrumental to characterize the class of linear systems (2.42) for which the repetitive control scheme is exponentially stable.

Proposition 2.3.2. *The repetitive control scheme of Fig. 1.2 is well-posed and exponentially stable if (2.42) is such that $\bar{A}$ is Hurwitz, the pair $(\bar{A}, \bar{B})$ is controllable, and*

$$\begin{pmatrix}\bar{Q}\bar{A} + \bar{A}^{\mathrm{T}}\bar{Q} & \bar{Q}\bar{B}\\ \bar{B}^{\mathrm{T}}\bar{Q} & 0\end{pmatrix} - \begin{pmatrix}-\sigma\bar{C}^{\mathrm{T}}\bar{C} & \bar{C}^{\mathrm{T}}(I-\sigma\bar{D})\\ (I-\sigma\bar{D}^{\mathrm{T}})\bar{C} & \bar{D}^{\mathrm{T}} + \bar{D} - \sigma\bar{D}^{\mathrm{T}}\bar{D}\end{pmatrix} \leq \\ \leq -2\begin{pmatrix}-\delta_x(\bar{Q}\bar{A} + \bar{A}^{\mathrm{T}}\bar{Q}) & 0\\ 0 & \delta_u I\end{pmatrix} \tag{2.44}$$

holds for a $\bar{Q} = \bar{Q}^{\mathrm{T}} > 0$, $\sigma \geq 1$, $\delta_x > 0$ and $\delta_u > 0$ sufficiently small.

Proof. The result follows from Prop. 2.2.3, in which the supply rate of the finite dimensional system is (2.43). □

The previous proposition is consistent with the classical stability conditions of repetitive control. In fact, a necessary condition for (2.44) to hold is that $\bar{D}^{\mathrm{T}} + \bar{D} - \sigma\bar{D}^{\mathrm{T}}\bar{D} \geq \delta_u I$, for all $\sigma \geq 1$, and $\delta_u > 0$ arbitrarily small. If for simplicity $\bar{D} = \gamma I$, we have that $\sigma\gamma^2 - 2\gamma < 0$, i.e. that $0 < \gamma < \frac{2}{\sigma}$. So, it is necessary that (2.42) is bi-proper, and that the feedthrough gain γ is positive and lower than 2, which corresponds to $\sigma = 1$. Moreover, from Prop. 2.2.2 and (2.43), since $\sigma \geq 1$, it can be deduced that the closed-loop system is described by a contraction C_0-semigroup, if (2.42) is ν-output strictly passive (van der Schaft, 2000), with $\nu \geq \frac{1}{2}$[10]. To have exponential stability, (2.44) forces the system (2.42) to have a non-null feed-through term and internal dissipation. Along the same line of Corollaries 2.2.1 and 2.2.2, it is easy to see that the following corollary of Prop. 2.3.2 holds true.

Corollary 2.3.1. *Under the same conditions of Prop. 2.3.2, let us denote by*

$$\bar{H}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D}$$

the transfer matrix of the plant (2.42). Then, the repetitive control scheme of Fig. 1.2 is exponentially stable if, for some $\epsilon > 0$, the linear system with transfer matrix $\bar{H}(s-\epsilon)$ is ν-output strictly passive, with $\nu > \frac{1}{2}$.

Remark 2.3.1. *It is worth to highlight that differently from the usual framework in which exponential stabilization of BCS is carried on, a rule inversion between controller and plant has been implicitly taken place in the latter analysis. In fact in RC schemes the infinite-dimensional BCS in port-Hamiltonian form (2.1) is indeed the controller, while the finite-dimensional plant (2.42), which is classically interpreted as the stabilizing controller for the BCS, represents the class of systems for which RC laws can be successfully applied.*

2.4 Extension to Modified RC Systems

In this section the proposed methodology is extended to the stability analysis of modified RC schemes (MRC), presented in subsection 1.3.1 and shown in Fig. 2.3

Once again the goal is to exploit different stabilization results developed for BCS in port-Hamiltonian form in the sense of Theorem 2.1.1, and analyse their relation with MRC schemes to provide alternative (with respect to (1.2)) and possibly simpler stability conditions in the time domain. The first step discussed below consists in the formulation of the pure delay $e^{-s\tau}$ as a BCS in port-Hamiltonian form.

Proposition 2.4.1. *The pure delay of τ units of time admits a distributed port-Hamiltonian representation (2.1) if*

$$P_1 = -I \qquad P_0 = G_0 = 0 \qquad \mathcal{L}(z) = I \tag{2.45}$$

with $z \in [0, \tau]$. Moreover, with Theorem 2.1.1 in mind, if the matrices that define the input and the output (2.5) and (2.6), respectively, are chosen as

$$W = \frac{\sqrt{2}}{2}\begin{pmatrix} I & I \end{pmatrix} \quad \tilde{W} = \frac{\sqrt{2}}{2}\begin{pmatrix} -I & I \end{pmatrix} \tag{2.46}$$

[10]Notice that in the LTI SISO case, this condition can be graphically interpreted as the stability region present in Fig. 1.4 deducted in frequency domain by e.g. Hara et al., 1988

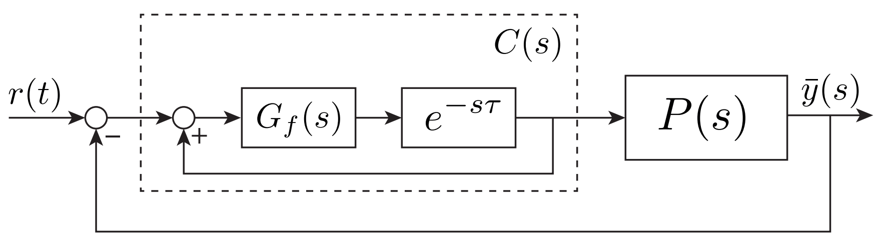


FIGURE 2.3: Modified Repetitive Control scheme.

the pure time delay is a BCS in the sense of the semigroup theory, and the following energy-balance relation

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|_2^2 = \frac{1}{2}u^{\mathrm{T}}(t)u(t) - \frac{1}{2}y^{\mathrm{T}}(t)y(t) \tag{2.47}$$

holds true.

Proof. The PDE (2.1) with the assumptions (2.45) is

$$\frac{\partial x}{\partial t}(t,z) = -\frac{\partial x}{\partial z}(t,z) \tag{2.48}$$

and, if the spatial domain is $[0, \tau]$, it is easy to see that $x(t,\tau) = x(t-\tau,0)$, which means that $x(t,\tau)$ is the delayed copy of $x(t,0)$, for all $t \geq \tau$. With the classical input / output description of a time delay block in mind, we can write that $y(t) = x(t,\tau)$ and that $u(t) = x(t,0)$. Since (2.48) is in the form (2.1), from (2.3) we have that

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -x(\tau)+x(0) \\ x(\tau)+x(0) \end{pmatrix}$$

so $u(t)$ and $y(t)$ are obtained from (2.5) and (2.6) if the matrices W and $\tilde{W}$ are selected as in (2.46). Note that $W\Sigma W^{\mathrm{T}} = I$, so (2.48) with input $u(t)$ is a boundary control system in the sense of Theorem 2.1.1. Furthermore it represents an exponentially stable BCS, and it is easy to check that its growth bound is $-\infty$ (Jacob and Zwart, 2012). Finally, we have that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|_2^2 = -\frac{1}{2}\|x(t,\tau)\|^2 + \frac{1}{2}\|x(t,0)\|^2 \tag{2.49}$$

which leads to the energy-balance relation (2.47) due to the definition of $u(t)$ and $y(t)$. □

2.4.1 Well-Posedness Analysis

Let us consider the MRC scheme reported in Fig. 2.3. Here, $G_f(s)$ is an exponentially stable low pass filter with minimal realization

$$\begin{cases} \dot{x}_f(t) = A_f x_f(t) + B_f u_f(t) \\ y_f(t) = C_f x_f(t) + D_f u_f(t) \end{cases} \tag{2.50}$$

in which $x_f \in \mathbb{R}^f$ is the state variable, and the matrices A_f, B_f, C_f and D_f have the appropriate dimensions. Differently, the dynamical system $P(s)$ is the controlled plant, which is supposed to be exponentially stable and with minimal realization

$$\begin{cases} \dot{x}_p(t) = A_p x_p(t) + B_p u_p(t) \\ y_p(t) = C_p x_p(t) + D_f u_p(t) \end{cases} \tag{2.51}$$

in which $x_p \in \mathbb{R}^p$ is the state variable, and the matrices A_p, B_p, C_p and D_p have the appropriate dimensions.

To frame the study of the properties of the MRC scheme within the standard theory of stabilization of BCS in port-Hamiltonian form, see e.g. (Villegas et al., 2009; Ramírez et al., 2014; Macchelli et al., 2017), an equivalent formulation of the block diagram of Fig. 2.3 is derived, and then shown in Fig. 2.4. With simple calculations, it is possible to check that $\bar{G}(s) = G_f(s)\,[P(s) - I]$, and for such system we assume

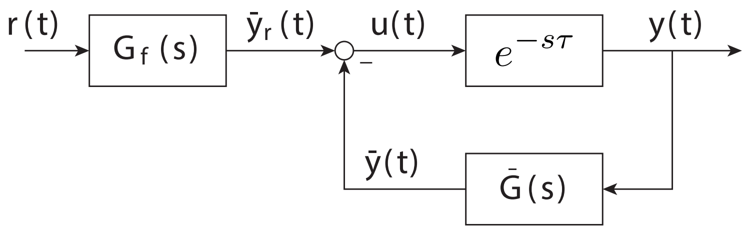


FIGURE 2.4: Equivalent formulation of the scheme of Fig. 2.3.

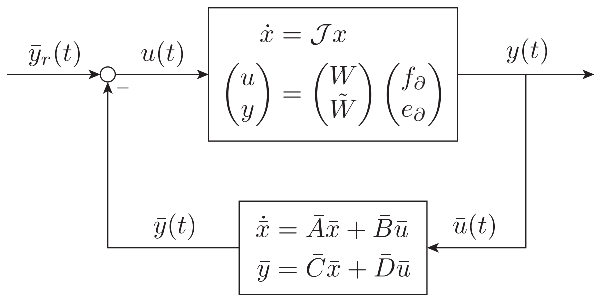


FIGURE 2.5: Port-Hamiltonian interpretation of MRC systems as boundary stabilization problem for the class of PDE (2.1).

that its minimal state-space realization is

$$\begin{cases} \dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}\bar{u}(t) \\ \bar{y}(t) = \bar{C}\bar{x}(t) + \bar{D}\bar{u}(t) \end{cases} \tag{2.52}$$

where $\bar{x} = (x_f, x_p) \in \mathbb{R}^{\bar{n}}$, with $\bar{n} = f + p$, and

$$\begin{aligned} \bar{A} &= \begin{pmatrix} A_f & 0 \\ B_p C_f & A_p \end{pmatrix} & \bar{B} &= \begin{pmatrix} B_f \\ B_p D_f \end{pmatrix} \\ \bar{C} &= \begin{pmatrix} (D_p - I)C_f & C_p \end{pmatrix} & \bar{D} &= (D_p - I)D_f \end{aligned} \tag{2.53}$$

Since $G_f(s)$ is an exponentially stable finite dimensional linear system, well-posedness and stability for the dynamical system of Fig. 2.4 can be studied by looking at the properties of the feedback loop that involves the pure delay and system $\bar{G}(s)$. For this system, we have that

$$u(t) = -\bar{y}(t) + \bar{y}_r(t) \qquad \bar{u}(t) = y(t) \tag{2.54}$$

and it is easy to see that it is in fact a particular case of the general situation depicted in Fig. 2.5, where a BCS in port-Hamiltonian form in interconnected to a finite dimensional system. In case of the MRC scheme, the BCS is the pure delay, while the finite dimensional system is $\bar{G}(S)$, which depends on both the filter and plant dynamics. Since from (2.47) in Prop. 2.4.1 we see that the pure delay is dissipative with respect to a specific supply rate that correspond to a BCS in the sense of Theorem 2.1.1 in scattering form, in the remaining part of the paper we restrict ourself to the case in which (2.1) is endowed with an input / output mapping defined by (2.5) and (2.6) with

$$W\Sigma W^{\mathrm{T}} = -\tilde{W}\Sigma\tilde{W}^{\mathrm{T}} = I \quad W\Sigma\tilde{W}^{\mathrm{T}} = 0 \tag{2.55}$$

Consequently, the energy balance relation (2.7) becomes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{L}} \leq \frac{1}{2}u^{\mathrm{T}}(t)u(t) - \frac{1}{2}y^{\mathrm{T}}(t)y(t)$$

which is in fact in the same for as (2.47).

The system of Fig. 2.5 can be compactly written as

$$\begin{cases} \dot{\zeta}(t) = \mathcal{J}_{cl}\zeta(t) \\ \bar{y}_r(t) = \begin{pmatrix} \mathcal{B}+\bar{D}\mathcal{C} & \bar{C} \end{pmatrix}\zeta(t) =: \mathcal{B}'\zeta(t) \end{cases} \tag{2.56}$$

where $\zeta = (x, \bar{x}) \in Z := X \times \mathbb{R}^{\bar{n}}$ is the state variable, and $\mathcal{J}_{cl} : D(\mathcal{J}_{cl}) \subset Z \to Z$ is the linear operator

$$\mathcal{J}_{cl}\zeta =: \begin{pmatrix} \mathcal{J} & 0 \\ \bar{B}\mathcal{C} & \bar{A} \end{pmatrix}\begin{pmatrix} x \\ \bar{x} \end{pmatrix} \tag{2.57}$$

with $\mathcal{J}$ introduced in Remark 2.1.1, and domain

$$D(\mathcal{J}_{cl}) = D(\mathcal{J}) \times \mathbb{R}^{\bar{n}} \tag{2.58}$$

Moreover, Z is endowed with the inner product

$$\langle \zeta_1 \mid \zeta_2 \rangle_Z = \langle x_1 \mid x_2 \rangle_{\mathcal{L}} + \bar{x}_1\bar{Q}\bar{x}_2$$

where

$$\bar{Q} = \begin{pmatrix} Q_f & 0 \\ 0 & Q_p \end{pmatrix} \tag{2.59}$$

with $Q_f = Q_f^{\mathrm{T}} > 0$ and $Q_p = Q_p^{\mathrm{T}} > 0$. Some fundamental properties associated to the system of coupled PDEs and ODEs that describes the closed-loop dynamics are presented in the next propositions.

Proposition 2.4.2. *Let us consider the closed-loop system (2.57) resulting from the interconnection (2.54) of the BCS of Theorem 2.1.1 in scattering form, i.e. input and output are defined as in (2.5) and (2.6) with W and $\tilde{W}$ so that (2.55) holds, and the linear system (2.52). If*

$$\begin{pmatrix} \bar{Q}\bar{A} + \bar{A}^{\mathrm{T}}\bar{Q} + \bar{C}^{\mathrm{T}}\bar{C} & \bar{Q}\bar{B} + \bar{C}^{\mathrm{T}}\bar{D} \\ \bar{B}^{\mathrm{T}}\bar{Q} + \bar{D}^{\mathrm{T}}\bar{C} & \bar{D}^{\mathrm{T}}\bar{D} - \gamma^2 I \end{pmatrix} \leq 0 \tag{2.60}$$

holds for some $\bar{Q} = \bar{Q}^{\mathrm{T}} > 0$ and $0 < \gamma \leq 1$, then (2.56) with $\mathcal{J}_{cl}$ defined in (2.57) and domain (2.58) is a BCS. Moreover, the operator $\bar{\mathcal{J}}_{cl}$ defined in the same way as in (2.57), but with domain

$$D(\bar{\mathcal{J}}_{cl}) = \left\{\zeta \in Z \mid x \in D(\mathcal{J}), \text{ and } \mathcal{B}'\zeta = 0\right\} \tag{2.61}$$

and with $\mathcal{B}'$ defined as in (2.56) generates a contraction semigroup.

Proof. The result follows applying Prop. 2.2.2 with $\begin{pmatrix} Y & -S^{\mathrm{T}} \\ -S & U \end{pmatrix} = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$ □

Remark 2.4.1. *It is easy to check that (2.60) holds if and only if the input/output map of (2.52) has finite L_2-gain, lower than γ, i.e. if and only if*

$$\dot{S}_{\bar{G}}(\bar{x}) \leq \frac{1}{2}\gamma^2\|\bar{u}\|^2 - \frac{1}{2}\|\bar{y}\|^2 \tag{2.62}$$

along system trajectories, and where $S_{\bar{G}}(\bar{x}) = \frac{1}{2}\bar{x}^T\bar{Q}\bar{x}$ *is a storage function for* $\bar{G}(s)$*. Moreover, because of the fact that* $G_f(s)$ *is an exponentially stable, finite dimensional, linear system, if the closed-loop system of Fig. 2.5 satisfies the requirements of Prop. 2.4.2, then such system is a BCS, and this implies that the complete system reported in Fig. 2.4 is a BCS. As a consequence, existence of trajectories when the reference period input is smooth enough is guaranteed also for the MRC scheme.*

2.4.2 Stability Analysis

In the previous section, well-posedness of the closed-loop system (2.56) reported in Fig. 2.5 when the BCS is in scattering form has been addressed in Prop. 2.4.2. In the next proposition, instead, the exponential stability of equilibria is investigated.

Proposition 2.4.3. *Under the same conditions of Prop. 2.4.2, assume that there exists* $\bar{Q} = \bar{Q}^{\mathrm{T}} > 0$, $\delta > 0$ *sufficiently small, and* $0 < \gamma < 1$ *such that*

$$\begin{pmatrix} (1-\delta)\left(\bar{Q}\bar{A} + \bar{A}^{\mathrm{T}}\bar{Q}\right) + \bar{C}^{\mathrm{T}}\bar{C} & \bar{Q}\bar{B} + \bar{C}^{\mathrm{T}}\bar{D} \\ \bar{B}^{\mathrm{T}}\bar{Q} + \bar{D}^{\mathrm{T}}\bar{C} & \bar{D}^{\mathrm{T}}\bar{D} - \gamma^2 I \end{pmatrix} \leq 0 \tag{2.63}$$

If $\bar{A}$ *is Hurwitz, then the closed-loop system (2.56) with* $\bar{y}_r(t) = 0$ *is exponentially stable.*

Proof. At first, if $\bar{A}$ is Hurwitz, for all $\bar{Q} = \bar{Q}^{\mathrm{T}} > 0$ there exists $\bar{P} = \bar{P}^{\mathrm{T}} > 0$ such that

$$\bar{Q}\bar{A} + \bar{A}^{\mathrm{T}}\bar{Q} = -\bar{P} \tag{2.64}$$

Now, a candidate Lyapunov function to study the stability of (2.56) is given by the sum of the Hamiltonian of (2.1) with the storage function of (2.52), i.e. by

$$E_{cl}(x(t), \bar{x}(t)) = \frac{1}{2}\|x(t)\|^2_{\mathcal{L}} + \frac{1}{2}\bar{x}^{\mathrm{T}}(t)\bar{Q}\bar{x}(t)$$

From (2.47), the variation of E_{cl} along system trajectories, whose existence is guaranteed by Prop. 2.4.2, is

$$\dot{E}_{cl} \leq \bar{x}^{\mathrm{T}}\bar{Q}(\bar{A}\bar{x} + \bar{B}\bar{u}) + \frac{1}{2}\|u\|^2 - \frac{1}{2}\|y\|^2$$

that can be rewritten as

$$\dot{E}_{cl} \leq \frac{1}{2}\begin{pmatrix} \bar{x} \\ \bar{u} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \bar{Q}\bar{A} + \bar{A}^{\mathrm{T}}\bar{Q} + \bar{C}^{\mathrm{T}}\bar{C} & \bar{Q}\bar{B} + \bar{C}^{\mathrm{T}}\bar{D} \\ \bar{B}^{\mathrm{T}}\bar{Q} + \bar{D}^{\mathrm{T}}\bar{C} & \bar{D}^{\mathrm{T}}\bar{D} - \gamma^2 I \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{u} \end{pmatrix}$$

since (2.54) holds true. Thanks to (2.63), the following upper-bound on the variation of the total energy is obtained:

$$\dot{E}_{cl} \leq \frac{1}{2}\begin{pmatrix} \bar{x} \\ \bar{u} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \delta\left(\bar{Q}\bar{A} + \bar{A}^{\mathrm{T}}\bar{Q}\right) & 0 \\ 0 & (\gamma^2 - 1)I \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{u} \end{pmatrix}$$

and this relation can be compactly rewritten as

$$\begin{aligned} \dot{E}_{cl} \leq \rho\bar{x}^{\mathrm{T}}\left(\bar{Q}\bar{A} + \bar{A}^{\mathrm{T}}\bar{Q}\right)\bar{x} - \sigma\|\bar{u}\|^2 \leq \rho\bar{x}^{\mathrm{T}}\left(\bar{Q}\bar{A} + \bar{A}^{\mathrm{T}}\bar{Q}\right)\bar{x} - \sigma\epsilon_1\|\bar{u}\|^2 - \\ - \sigma\epsilon_2\left(\|u\|^2 + \|y\|^2\right) + \sigma\epsilon_2\|\bar{y}\|^2 \end{aligned} \tag{2.65}$$

since from (2.54) we have that $\bar{u}(t) = y(t)$ and $u(t) = -\bar{y}(t)$ under the assumption that $\bar{y}_r(t) = 0$, and where $\epsilon_1 + \epsilon_2 = 1$, $\rho = \frac{\delta}{2} > 0$, and $\sigma = \frac{1}{2}(1-\gamma^2) > 0$, since $0 < \gamma < 1$. Note that (2.3), (2.5) and (2.6) define a linear and invertible relation between u and y, and the restriction of the co-energy variables at the boundary of the spatial domain, i.e. with $(\mathcal{L}x)(a)$ and $(\mathcal{L}x)(b)$. This means that there exists $\epsilon > 0$ such that $\|u(t)\|^2 + \|y(t)\|^2 \geq \epsilon \|(\mathcal{L}x)(t, z_\partial)\|^2$, for $z_\partial = a$ or $z_\partial = b$. With this in mind, relation (2.65) becomes

$$\dot{E}_{cl} \leq \rho \bar{x}^{\mathrm{T}} \left(\bar{Q}\bar{A} + \bar{A}^{\mathrm{T}}\bar{Q} \right) \bar{x} - \sigma\epsilon_1 \|\bar{u}\|^2 - \sigma\epsilon\epsilon_2 \|(\mathcal{L}x)(\cdot, z_\partial)\|^2 + \sigma\epsilon_2 \|\bar{y}\|^2 \quad (2.66)$$

Integrating (2.66) on the interval $[0, \kappa]$, with $\kappa > 0$ sufficiently large, one obtains

$$E_{cl}(\kappa) - E_{cl}(0) = \int_0^\kappa \rho \bar{x}^{\mathrm{T}} \left(\bar{Q}\bar{A} + \bar{A}^{\mathrm{T}}\bar{Q} \right) \bar{x} \,\mathrm{d}t - \\ - \int_0^\kappa \left[\sigma\epsilon_1 \|\bar{u}\|^2 + \sigma\epsilon\epsilon_2 \|(\mathcal{L}x)(t, z_\partial)\|^2 \right] \mathrm{d}t + \sigma\epsilon_2 \int_0^\kappa \|\bar{y}\|^2 \,\mathrm{d}t$$

Since A_p and A_f are Hurwitz, so is $\bar{A}$ because of (2.53). Consequently $\bar{Q}$ can always be selected so that (2.64) holds. Then, the remaining part of the proof follows exactly the same steps as in Prop. 2.2.3. To conclude, it is possible to show that the semigroup $T(t)$ generated by $\mathcal{J}_{cl}$ satisfies $\|T(\kappa)\|_Z < 1$ for sufficiently large κ, and this implies that the growth bound of the semigroup is strictly negative. As discussed e.g. in (Jacob and Zwart, 2012), this means that there exists positive scalars M and α such that $\|T(t)\|_Z \leq Me^{-\alpha t}$ for all $t \geq 0$. □

The class of systems for which the MRC scheme is exponentially stable is characterized in the next corollary.

Corollary 2.4.1. *The MRC scheme of Fig. 2.3 is exponentially stable if for some $\bar{Q} = \bar{Q}^T > 0$ defined in (2.59), $\delta > 0$, $0 < \gamma < 1$, the filter (2.50) and the controlled plant (2.51) are such that A_f and A_p are Hurwitz, and the following LMI holds true:*

$$\begin{pmatrix} \mathcal{A} & \mathcal{S} \\ \mathcal{S}^{\mathrm{T}} & \mathcal{D} \end{pmatrix} \leq 0 \quad (2.67)$$

where

$$\begin{aligned} \mathcal{A} &= (1-\delta) \begin{pmatrix} Q_f A_f + A_f^{\mathrm{T}} Q_f & C_f^{\mathrm{T}} B_p Q_p \\ Q_p B_p C_f & Q_p A_p + A_p^{\mathrm{T}} Q_p \end{pmatrix} + \begin{pmatrix} C_f^{\mathrm{T}} D_u^{\mathrm{T}} D_u C_f & C_f^{\mathrm{T}} D_u^{\mathrm{T}} C_p \\ C_p^{\mathrm{T}} D_u C_f & C_p^{\mathrm{T}} C_p \end{pmatrix} \\ \mathcal{S} &= \begin{pmatrix} Q_f B_f + C_f^{\mathrm{T}} D_u^{\mathrm{T}} D_u D_f \\ Q_p B_p D_f + C_p^{\mathrm{T}} D_u D_f \end{pmatrix} \\ \mathcal{D} &= D_f^{\mathrm{T}} D_u^{\mathrm{T}} D_u D_f - \gamma^2 I \end{aligned} \quad (2.68)$$

with $D_u = D_p - I$

Proof. It is easy to see that (2.67) can be obtained by substituting (2.53) in (2.63). □

In the next remarks, the general condition (2.67) is applied to two specific cases of interest. The first one is the standard RC scheme, while the second one deals with the situation in which both the filter and the controlled plant are strictly proper.

Remark 2.4.2. *In standard RC schemes, the filter is not present, i.e.* $A_f = 0$, $B_f = 0$, $C_f = 0$, *and* $D_f = I$ *in (2.50). In this case, condition (2.67) reduces to*

$$\begin{pmatrix} \mathcal{A}_{rc} & c\mathcal{S}_{rc} \\ \mathcal{S}_{rc}^{\mathrm{T}} & \mathcal{D}_{rc} \end{pmatrix} \leq 0 \tag{2.69}$$

where

$$\begin{aligned} \mathcal{A}_{rc} &= (1-\delta)\left(Q_p A_p + A_p^{\mathrm{T}} Q_p\right) + C_p^{\mathrm{T}} C_p \\ \mathcal{S}_{rc} &= Q_p B_p + C_p^{\mathrm{T}}(D_p - I) \\ \mathcal{D}_{rc} &= D_p^{\mathrm{T}} D_p - D_p^{\mathrm{T}} - D_p + (1-\gamma^2) I \end{aligned}$$

which can be fulfilled, for example, if (2.51) is α*-output strictly passive with* $\alpha > \frac{1}{2}$, *(Califano, Macchelli, and Melchiorri, 2017; Macchelli and Califano, 2018). In any case the feedthrough term* D_p *has to be different from zero, since* $\gamma < 1$ *and* $\mathcal{D}_{rc}$ *cannot be negative: this is a necessary condition for (2.69) to hold. This is consistent with classical stability results of RC, as discussed e.g. in (Hara et al., 1988; Biagiotti, Califano, and Melchiorri, 2016)). Differently, if the low-pass filter is present, it is clear from (2.68) that there is the chance to exponentially stabilize the system also if* $D_p = 0$. *In this case, in fact,* $\mathcal{D} = D_f^{\mathrm{T}} D_f - \gamma^2 I$ *can be made negative if* $|D_f| < 1$.

Remark 2.4.3. *An interesting case to investigate is when the filter and the plant are strictly proper systems: this is realistically what happens in continuous time applications. Then, we have that* $D_f = 0$ *and* $D_p = 0$, *i.e. that* $D_u = -I$. *In this case, (2.67) becomes*

$$\begin{pmatrix} \mathcal{A}_{sp} & \mathcal{S}_{sp} \\ \mathcal{S}_{sp}^{\mathrm{T}} & \mathcal{D}_{sp} \end{pmatrix} \leq 0 \tag{2.70}$$

where

$$\begin{aligned} \mathcal{A}_{sp} &= (1-\delta)\begin{pmatrix} Q_f A_f + A_f^{\mathrm{T}} Q_f & C_f^{\mathrm{T}} B_p Q_p \\ Q_p B_p C_f & Q_p A_p + A_p^{\mathrm{T}} Q_p \end{pmatrix} + \begin{pmatrix} C_f^{\mathrm{T}} C_f & -C_f^{\mathrm{T}} C_p \\ -C_p^{\mathrm{T}} C_f & C_p^{\mathrm{T}} C_p \end{pmatrix} \\ \mathcal{S}_{sp} &= \begin{pmatrix} Q_f B_f \\ 0 \end{pmatrix} \\ \mathcal{D}_{sp} &= -\gamma^2 I \end{aligned}$$

A necessary condition that depends on the filter only and is necessary to satisfy so that (2.70) holds true is the following:

$$\begin{pmatrix} Q_f A_f + A_f^{\mathrm{T}} Q_f + C_f^{\mathrm{T}} C_f & Q_f B_f \\ B_f^{\mathrm{T}} Q_f & -\gamma^2 I \end{pmatrix} \leq 0 \tag{2.71}$$

Such conditions implies that the filter must have L_2*-gain less than* γ, *being* $S_f(x_f) = \frac{1}{2} x_f^{\mathrm{T}} Q_f x_f$ *the storage function of the filter. The same relation describes the necessity of having a loop gain smaller than 1 in the delay subsystem. Indeed, it represents the intrinsic trade-off of continuous time repetitive control schemes: as discussed in (Weiss, 1997), it is impossible to track any periodic signal by relying on an exact internal model, and at the same time achieve exponential stability/robustness with respect to time delays at any point in the feedback loop.*

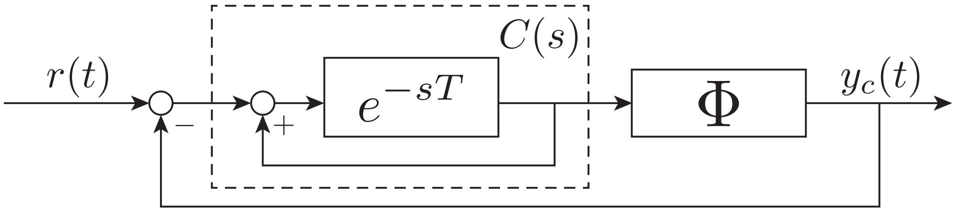


FIGURE 2.6: Continuous-time nonlinear RC scheme.

2.5 Extension to the Nonlinear Case

In this section the analysis is extended to the nonlinear case, i.e. to the RC scheme shown in Fig. 2.6. The proposed methodology, developed in time-domain using a state-space approach is thus amenable to be used in a nonlinear framework. With respect to the linear case some very recent results regarding stabilization of BCS through nonlinear boundary controllers developed in Ramírez, Zwart, and Le Gorrec, 2017 are consistently exploited.

Differently from what has been usually done in nonlinear RC, the proposed methodology does not rely on a finite-dimensional approximation of the repetitive compensator, nor on a linearisation of a nonlinear plant, or even on a nonlinear perturbation of a linear one. More specifically, in Ghosh and Paden, 2000 nonlinear RC schemes are studied by approximating the compensator $C(s)$ with a finite-dimensional system, and tracking is analysed for a particular class of nonlinear systems. In Lin, Chung, and Hung, 1991, small-gain arguments are used to guarantee stability of linear RC schemes in which a sector nonlinearity at the input of a linear plant is present. In Owens, Li, and Banks, 2007 and similarly to the approach discussed in this paper, the stability analysis is performed by relying on a state-space representation of the plant, with the nonlinear case briefly addressed as perturbation of the linear one. Note that well-posedness of the closed-loop system, i.e. existence and properties of solutions, is never addressed. When e.g. in Owens, Li, and Banks, 2007 the stability analysis in presence of nonlinear perturbations is carried on by differentiating a Lyapunov functional along solutions, it is implicitly assumed that such solutions in closed-loop exist, which is not correct in an infinite-dimensional framework.

2.5.1 Interconnection between a BCS in scattering form and a nonlinear system

2.5.2 Well-posedness analysis

To determine the class of nonlinear systems Φ for which the RC scheme of Fig. 2.6 is well-posed and stable, a preliminary step consists in studying the properties of the closed-loop system obtained from the feedback interconnection of a BCS in the sense of Theorem 2.1.1 and in scattering form (see e.g. Prop. 2.4.1), with a nonlinear system, denoted by Σ. The general framework is illustrated in Fig. 2.7. Note that

$$u(t) = -y_c(t) + r(t) \quad u_c(t) = y(t) \tag{2.72}$$

and that the closed-loop system is described by coupled PDEs and ODEs. With Ramírez et al., 2014; Ramírez, Zwart, and Le Gorrec, 2017; Califano, Macchelli, and

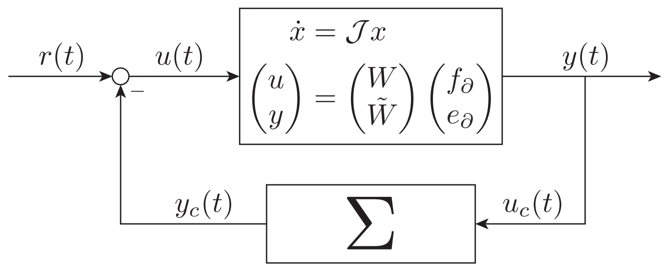


FIGURE 2.7: Feedback interconnection between a BCS in scattering form and the nonlinear system Σ.

Melchiorri, 2017 in mind, the problem of finding conditions on Σ to have a well-posed and exponentially stable system is in fact equivalent to the problem of finding the class of nonlinear control systems Σ able to stabilise the port-Hamiltonian system (2.1) in scattering form.

Let us consider the nonlinear system

$$\Sigma : \begin{cases} \dot{v}_1(t) = K_2 v_2(t) \\ \dot{v}_2(t) = -\dfrac{\partial P}{\partial v_1}(v_1(t)) - R(v_2(t))K_2 v_2(t) + B_c u_c(t) \\ y_c(t) = -D^{-\mathrm{T}} B_c^{\mathrm{T}} K_2 v_2(t) + D u_c(t) \end{cases} \tag{2.73}$$

where $v_1, v_2 \in \mathbb{R}^{n_c}$, $P : \mathbb{R}^{n_c} \to \mathbb{R}^+$ is a Fréchet differentiable function, B_c an $n_c \times n$ real matrix, $R(v_2)$ a locally Lipschitz-continuous matrix-valued function taking values in $\mathbb{R}^{n_c \times n_c}$, D a non-singular $n \times n$ matrix, $K_2 = K_2^{\mathrm{T}} > 0$ a $n_c \times n_c$ real matrix. Furthermore, $P(v_1) > P(0) = 0$ for all $v_1 \neq 0$, $P(v_1) \to \infty$ if $|v_1| \to \infty$, i.e. $P(v_1)$ is radially unbounded, and $\frac{\partial P}{\partial v_1}(v_1)$ is locally Lipschitz-continuous.

Proposition 2.5.1. *Assume that in (2.73)*

$$|D| \leq 1 \tag{2.74}$$

$$R(v_2) \geq \frac{1}{2} B_c D^{-1} D^{-\mathrm{T}} B_c^{\mathrm{T}}, \quad \forall v_2 \in \mathbb{R}^{n_c} \tag{2.75}$$

and define $\Theta(v_2) = K_2 \left[R(v_2) - \frac{1}{2} B_c D^{-1} D^{-\mathrm{T}} B_c^{\mathrm{T}} \right] K_2$, $\Delta = \frac{1}{2}\left(I - D^{\mathrm{T}} D\right)$, and $Q(v_1, v_2) = P(v_1) + \frac{1}{2} v_2^{\mathrm{T}} K_2 v_2$. Then, the variation of $Q(v_1, v_2)$, i.e. of the "energy" along the system trajectories is

$$\dot{Q} = -v_2^{\mathrm{T}} \Theta(v_2) v_2 - u_c^{\mathrm{T}} \Delta u_c + \frac{1}{2} u_c^{\mathrm{T}} u_c - \frac{1}{2} y_c^{\mathrm{T}} y_c, \tag{2.76}$$

where $\Theta(v_2) \geq 0$ and $\Delta \geq 0$ since (2.74) and (2.75) hold.

Proof. This result follows by computing $\dot{Q}$ along solutions of (2.73), which exist at least locally because the functions describing Σ are locally Lipschitz continuous. □

Remark 2.5.1. *Conditions (2.74) and (2.75) make the system (2.73) dissipative with storage function $Q(v)$ and supply rate $s(u_c, y_c) = \frac{1}{2} u_c^{\mathrm{T}} u_c - \frac{1}{2} y_c^{\mathrm{T}} y_c$, i.e. $\dot{Q} \leq s(u_c, y_c)$. In the same way as in Prop. 2.4.1, we can say that such system is in scattering form, which, from a dissipativity theory point of view, means that the input-output mapping of (2.73) has finite L_2-gain, lower than* 1.

For the closed-loop system sketched in Fig. 2.7 the storage function is

$$E_{tot}(\tilde{x}) = \frac{1}{2}\|x\|^2_{\mathcal{L}} + Q(v_1, v_2), \tag{2.77}$$

where $\tilde{x} = (x, v_1, v_2)$ is the state variable, and the dynamics is described by the following semi-linear differential equation

$$\begin{cases} \dot{\tilde{x}}(t) = \tilde{A}'\tilde{x}(t) + \tilde{B}f(\tilde{x}(t)) \\ r(t) = W_{cc}\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \\ v_2(t) \end{pmatrix} \end{cases} \tag{2.78}$$

with

$$\tilde{A}'\tilde{x} = \begin{pmatrix} \mathcal{J}x \\ K_2 v_2 \\ -v_1 + B_c\tilde{W}\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \end{pmatrix}, \qquad D(\tilde{A}') = D(\mathcal{J}) \times \mathbb{R}^{n_c}$$

$$f(\tilde{x}) = v_1 - \frac{\partial P}{\partial v_1}(v_1) - R(v_2)K_2 v_2,$$

$\tilde{B} = \begin{pmatrix} 0 & 0 & I \end{pmatrix}^{\mathrm{T}}$ and $W_{cc} = \begin{pmatrix} W + D\tilde{W} & -D^{-\mathrm{T}}B_c^{\mathrm{T}}K_2 \end{pmatrix}$.

When $r(t) = 0$, the linear operator is denoted by $\tilde{A}$, formally defined as $\tilde{A}'$, but with domain $D(\tilde{A}) = \{\mathcal{L}x \in H^1(a, b; \mathbb{R}^n),\ v_1, v_2 \in \mathbb{R}^{n_c} \mid (f_\partial, e_\partial, v_2) \in \operatorname{Ker} W_{cc}\}$. The new state space is $\tilde{X} = X \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_c}$, endowed with the with inner product $\langle \tilde{x}_1 \mid \tilde{x}_2 \rangle_{\tilde{X}} = \langle x_1 \mid \mathcal{L}x_2 \rangle + v_{1,1}^{\mathrm{T}} v_{1,2} + v_{2,1}^{\mathrm{T}} K_2 v_{2,2}$ and norm $\|\tilde{x}\|^2 = \langle \tilde{x} \mid \tilde{x} \rangle_{\tilde{X}}$. It is possible to prove that the linear operator $\tilde{A}$ is the generator of a contraction semigroup, and that its resolvent is compact, see Ramírez et al., 2014.

The first question that could arise is whether the closed-loop system (2.78) is well-posed, i.e. if at least local solutions exist for any initial condition. Due to the particular choice made for Σ in (2.73), it turns out that solutions exist globally.

Proposition 2.5.2. *Let us consider the nonlinear system (2.73), and assume that (2.74) and (2.75) are satisfied. Then, the autonomous closed-loop system (2.78) is well-posed and, in particular, for any initial condition, it possesses a unique global mild solution which is uniformly bounded.*

Proof. Since $\frac{\partial P}{\partial v_1}(v_1)$ and $R(v_2)$ are locally Lipschitz continuous, so is $f(\tilde{x})$ in (2.78); moreover, $\tilde{B}$ is a bounded linear operator. This implies, see e.g. Luo, Guo, and Morgul, 1999, that for any initial condition the closed-loop system possesses a unique mild solution on some time interval $[0, t_{max})$, which is classical if the initial condition belongs to the domain of $\tilde{A}$. Furthermore, if $t_{max} < +\infty$, then necessarily $\lim_{t \to t_{max}} \|\tilde{x}(t)\| = +\infty$. In fact, by taking the time derivative along classical solutions of the total energy (2.77), we obtain that

$$\dot{E}_{tot} \leq \frac{1}{2}u^{\mathrm{T}}u - \frac{1}{2}y^{\mathrm{T}}y - v_2^{\mathrm{T}}\Theta(v_2)v_2 - u_c^{\mathrm{T}}\Delta u_c + \frac{1}{2}u_c^{\mathrm{T}}u_c - \frac{1}{2}y_c^{\mathrm{T}}y_c,$$

where (2.76) has been taken into account, and it has been assumed that the BCS defined in Theorem 2.1.1 is in scattering form, i.e. the balance relation (2.47) in Prop. 2.4.1 holds. Thanks to (2.72) and after integration, this inequality becomes

$$E_{tot}(t) \leq E_{tot}(0) - \int_0^t \left[v_2^{\mathrm{T}}\Theta(v_2)v_2 + u_c^{\mathrm{T}}\Delta u_c \right] \mathrm{d}\tau \tag{2.79}$$

Since $\tilde{A}$ generates a contraction semigroup, its domain is dense in $\tilde{X}$ and solutions depends continuously on the initial conditions. This means that (2.79) is true for every initial condition. Due to the fact that $\Theta(v_2) \geq 0$ and $\Delta \geq 0$, the previous inequality shows that $E_{tot}(t)$ is uniformly bounded, and this means that so are $\|x\|_{\mathcal{L}}$, $P(v_1(t))$ and $\frac{1}{2}v_2^{\mathrm{T}}(t)K_2 v_2(t)$. Now boundedness of $|v_2(t)|$ is a consequence of the positive definiteness of K_2; boundedness of $|v_1(t)|$ follows from the assumption of radial unboundedness of $P(v_1(t))$, which in fact remains bounded and forces $|v_1(t)|$ to remain bounded as well. The conclusion is that $t_{max} = +\infty$, and thus global existence of solution is proved. □

2.5.3 Stability analysis with application to RC

The aim is now to study the exponential stability of the closed-loop system (2.78) so to characterize the class of nonlinear systems Φ that results into an exponentially stable RC scheme. The stability analysis is based on Ramírez, Zwart, and Le Gorrec, 2017, where exponential stability conditions for a similar closed-loop system have been addressed in case the BCS of Theorem 2.1.1 is passive, i.e. input and output are selected so that the balance relation (2.7) becomes $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x\|^2_{\mathcal{L}} \leq y^{\mathrm{T}}u$.

Proposition 2.5.3. *Let us consider the system (2.73), and assume that (2.75) holds. Furthermore, suppose that*

1. *There exist $\sigma_1, \sigma_2 > 0$ such that for all $v_1 \in \mathbb{R}^{n_c}$ we have that $v_1^{\mathrm{T}}\frac{\partial P}{\partial v_1}(v_1) \geq \sigma_1 P(v_1) \geq \sigma_2|v_1|^2$;*
2. *The input and output for the BCS of Theorem 2.1.1 are so that $|u(t)|^2 + |y(t)|^2 \geq \epsilon|(\mathcal{L}x)(b)|^2$ for some $\epsilon > 0$;*
3. *$n \geq n_c$, and B_c is full rank;*
4. *$|D| < 1$.*

Then, the closed-loop system (2.78) with $r(t) = 0$ is globally exponentially stable.

Proof. Note at first that (2.78) has the same structure of the closed-loop system described in Ramírez, Zwart, and Le Gorrec, 2017. Conditions 1) and 2) are in fact Assumptions 12 and 14 in Ramírez, Zwart, and Le Gorrec, 2017, respectively, and condition 3) combined with (2.75) makes Assumption 13 in Ramírez, Zwart, and Le Gorrec, 2017 automatically satisfied. Note that Assumption 15 in Ramírez, Zwart, and Le Gorrec, 2017 is replaced here by $\Delta > 0$, and this requirement is structurally true for the class of systems (2.73) provided that condition 4) holds. Thus, Theorem 20 in Ramírez, Zwart, and Le Gorrec, 2017 holds, and (2.78) is globally exponentially stable. □

Remark 2.5.2. *The major requirement to have exponential stability when the BCS of Theorem 2.1.1 is passive, see e.g. Ramírez et al., 2014; Ramírez, Zwart, and Le Gorrec, 2017, is the presence of a non-null feedthrough term in the controller, responsible for damping the high-frequency terms in the infinite-dimensional system. It this case, since systems in scattering form are considered, such requirement is transformed into a small gain condition, namely $|D| < 1$.*

Now the tools to apply the analysis to RC are present.

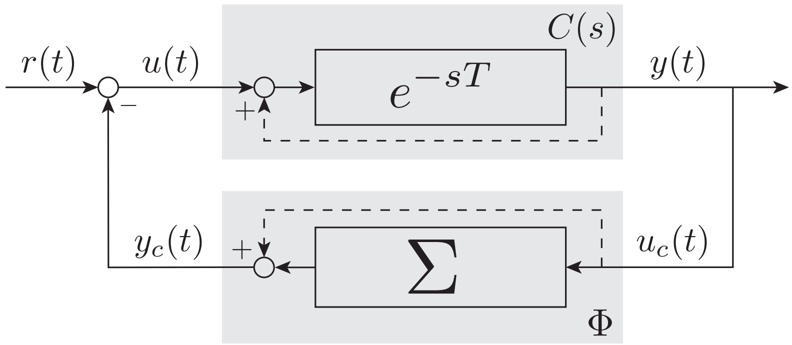


FIGURE 2.8: Equivalent representation of the RC scheme of Fig. 2.6.

Proposition 2.5.4. *Define the system Φ by adding a unitary contribution to the feedthrough term of Σ defined in (2.73):*

$$\Phi : \begin{cases} \dot{v}_1(t) = K_2 v_2(t) \\ \dot{v}_2(t) = -\dfrac{\partial P}{\partial v_1}(v_1(t)) - R(v_2(t))K_2 v_2(t) + B_c u_c(t) \\ y_c(t) = -D^{-\mathrm{T}} B_c^{\mathrm{T}} K_2 v_2(t) + (D+I)u_c(t) \end{cases} \tag{2.80}$$

Then, for the class of systems Φ, the nonlinear RC scheme of Fig. 2.6 is well-posed if (2.74) and (2.75) hold. Moreover, such closed-loop system is exponentially stable if conditions 1-4) in Prop. 2.5.3 are satisfied.

Proof. As discussed in Califano, Macchelli, and Melchiorri, 2017, the repetitive compensator $C(s)$ is a BCS in port-Hamiltonian form, but it is not in scattering form. To apply the results presented in the previous section, let us refer to the block diagram of Fig. 2.8 that is equivalent to the RC scheme of Fig. 2.7 once the positive feedback loops (dashed lines) around the delay e^{-sT} and system Σ are removed. Since as discussed in Prop. 2.4.1, the delay is a BCS in scattering form, the properties of the RC scheme can be studied by relying on the closed-loop system that is obtained from the feedback interconnection of Σ with the delay equation and, as a consequence, on the results presented in Propositions 2.5.2 and 2.5.3. □

Remark 2.5.3. *Thanks to Prop. 2.5.4, the nonlinear systems for which a RC scheme is well-posed and exponentially stable are characterized. This result is consistent with the stability conditions for RC schemes in the linear case. In fact, (2.80) must have a non-null feedthrough term to get an exponentially stable closed-loop system since $|D| < 1$. A finite-dimensional approximation of the repetitive compensator would relax this requirement, but in any case it would be impossible to track any periodic signal. So, it is convenient to keep the infinite-dimensional nature of the repetitive compensator, which does not pose any implementative issues, since it is just a time delay surrounded by a positive feedback loop. To be able to stabilize strictly proper plants, the solution consists in adding a low-pass filter in series to the delay. This leads to a modified RC scheme Hara et al., 1988.*

2.6 Asymptotic tracking in RC systems

2.6.1 Linear Case

Asymptotic tracking in the RC scheme of Fig. 1.2 is achieved if, for any periodic reference signal $r(t)$ of period T, the error $e(t) = r(t) - y(t)$ vanishes asymptotically,

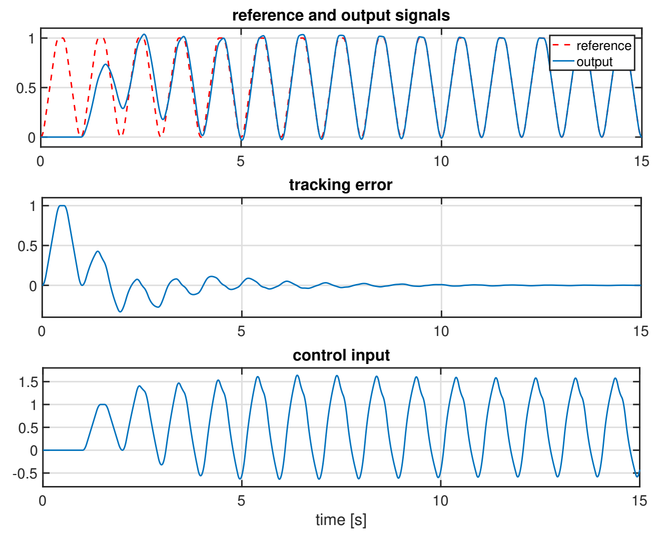


FIGURE 2.9: Example of perfect tracking tracking provided by exponentially stable RC scheme.

i.e. $\lim_{t\to\infty} e(t) = 0$. In the linear case, this property has been assumed to be satisfied once the closed-loop system is stable, Hara et al., 1988. In fact, according to internal model-based arguments that "classically" state that if the model of the exogenous signal generator is properly included in the loop, $C(s)$ in this case, and if the closed-loop system is exponentially stable, then asymptotic tracking of the exogenous signals is assured.

In subsection 1.4.1 the problem of tracking in RC schemes has been intuitively treated providing a new insight on the IMP when periodic signals are involved. Using that reasoning the following two facts in the linear case are clear without using internal model based arguments:

- In pure RC schemes exponential stability of the autonomous closed loop system implies also perfect tracking for any periodic signal. As simulative example in Fig. 2.9, a simulation results is reported in which the plant is

$$P(s) = \frac{1 + 0.1s}{1 + 0.35s}$$

  It is easy to see that with this choice the condition for exponential stability (2.44) is satisfied.

- In modified RC exponential stability of the autonomous closed loop system does not imply perfect tracking for any periodic signal, and in particular a periodic steady state error depending on the design of the filter and the harmonic content of the reference signal will be present. As simulative example In Fig. 2.10, the simulation results for a case in which both the plant and the filter are strictly proper is reported. In particular, we have that

$$G_f(s) = \frac{0.998}{(1 + 0.01s)} \quad P(s) = \frac{1}{(1 + 0.05s)}$$

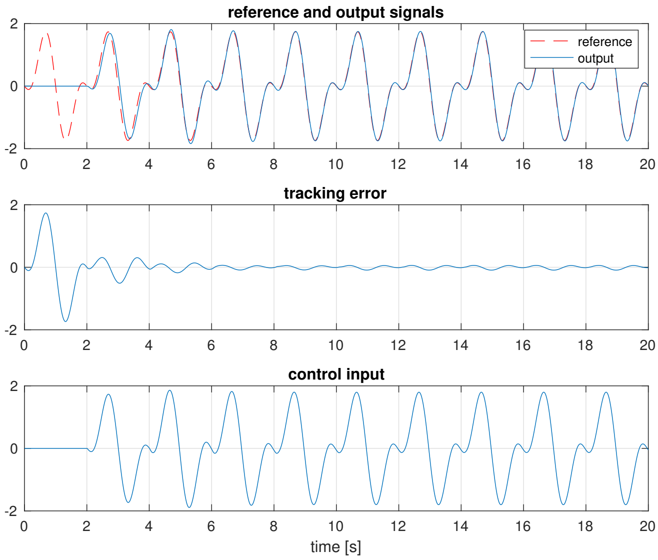


FIGURE 2.10: Example of tracking provided by exponentially stable MRC scheme.

Note that indeed a periodic steady-state error is present: the amplitude of the error depends on the reference signal to be tracked and on the bandwidth of the filter.

2.6.2 Nonlinear Case

The nonlinear case is more subtle since internal model-based arguments cannot be invoked to state that perfect tracking is assured once the closed-loop system is exponentially stable. In this section, asymptotic tracking of RC schemes applied to the class of nonlinear systems characterized in Prop. 2.5.4 is proved.

Definition 2.6.1. *A signal $s(t) : \mathbb{R} \to \mathbb{R}^n$ is asymptotically T-periodic if* $\lim_{t\to\infty} |s(t) - s(t+nT)| = 0$, $\forall n \in \mathbb{N}$.

Proposition 2.6.1. *Refer the RC scheme of Fig. 2.6, and consider the formulation of the delay as a BCS in port-Hamiltonian form of Prop. 2.4.1. Perfect tracking of periodic reference signals $r(t)$ of period T is achieved if and only if the state of the delay equation in the closed-loop system is such that for every $z \in [0,\ T]$, $x(t,z)$ is asymptotically T-periodic.*

Proof. The results follows once the RC scheme in Fig. 2.6 is considered. Asymptotic tracking is achieved if, at steady state, $e(t) = v(t) - v(t-T) = 0$, being $v(t)$ the input of the delay, and this is possible if and only if $v(t)$ is an asymptotic T-periodic signal, which implies that $x(t,z)$ is asymptotically T-periodic for almost every $z \in [0,\ T]$[11]. □

Now, the state of the delay system in Fig. 2.6 is asymptotically T-periodic in the sense of Prop. 2.6.1 if and only if the state of the delay system in Fig. 2.8, in which the (dashed) positive feedback loops are removed, is asymptotically T-periodic. Such closed-loop system can be described by (2.78). For this system, a local asymptotic tracking result is given, under the following assumption.

[11]The technical "almost every" condition is chosen because the aim is to zero the error in an L-2 sense, and not pointwise.

Assumption 2.6.1. *For system (2.78), there exist $R_0 \subset \mathbb{R}^n$, and $\widetilde{X}_0 \subset \widetilde{X}$ such that for all the initial conditions $\tilde{x}(0) \in \widetilde{X}_0$, and all the reference signals $r(t) \in R_0$ we have that*

$$\|\tilde{x}(t)\|_{\widetilde{X}} \leq \chi\left(\|\tilde{x}(0)\|_{\widetilde{X}}\right) + \gamma\left(|r|_\infty\right) \tag{2.81}$$

for all $t \in \mathbb{R}^+$, and for some class-$\mathcal{K}$ functions χ and γ.

Before stating the result that assures that local asymptotic tracking can be obtained in a nonlinear RC scheme under the conditions of Prop. 2.5.4, some preliminary results are presented. In what follows, for any signal $s(t)$, let $\Delta_n s(t) = s(t) - s(t+nT)$, with $T > 0$ and $n \in \mathbb{N}$.

Proposition 2.6.2. *Consider a system in the form*

$$\dot{\zeta}(t) = f(\zeta(t)) + bu(t) \tag{2.82}$$

with $\zeta \in \mathbb{R}^{n_\zeta}$, $u \in \mathbb{R}^{n_u}$, and $f : \mathbb{R}^{n_\zeta} \to \mathbb{R}^{n_\zeta}$ a smooth function. If (2.82) is 0-locally exponentially stable, then there exist ρ, $\varepsilon > 0$ any solution $\zeta(t)$ to (2.82) that fulfills $|\zeta(0)| \leq \rho$ and $|u|_\infty \leq \varepsilon$ also satisfies

$$\limsup_{t\to\infty} |\Delta_n \zeta(t)| \leq \alpha \limsup_{t\to\infty} |\Delta_n u(t)| \tag{2.83}$$

for some $\alpha > 0$. Thus, in particular, if $u(t)$ is asymptotically T-periodic, so is $\zeta(t)$.

Proof. A preliminary result is stated in the next lemma and its proof is based on standard ISS considerations.

Lemma 2.6.1. *Consider a system in the form (2.82), and assume that it is 0-locally exponentially stable. Then, for each $\mu \geq 0$, there exist $\bar{\zeta}$, $\bar{u} > 0$ such that all the solutions of (2.82) such that $|\zeta(0)| \leq \bar{\zeta}$ and $|u(t)| \leq \bar{u}$, for all $t \geq 0$, satisfies $|\zeta(t)| \leq \mu$ for all $t \geq 0$.*

We are now ready to present the proof of Prop. 2.6.2. For (2.82), define $\tilde{f}(\zeta) = f(\zeta) - A\zeta$, where $A = \frac{\partial f}{\partial \zeta}(0)$. Then, $\lim_{|\zeta|\to 0} \frac{|\tilde{f}(\zeta)|}{|\zeta|} = 0$. Denote by $P = P^{\mathrm{T}} > 0$ the unique solution to the Lyapunov equation $A^{\mathrm{T}}P + PA = -2I$, and define the function $V(\zeta) = \zeta^{\mathrm{T}}P\zeta$. Let $\tilde{f}'(\zeta) := \frac{\partial \tilde{f}}{\partial \zeta}(\zeta)$, since $\tilde{f}'(0) = 0$ and $\tilde{f}'(\cdot)$ is continuous, then $\lim_{\zeta\to 0} |\tilde{f}'(\zeta)| = 0$. Hence, for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $|\zeta| \leq \delta(\epsilon)$ implies $|\tilde{f}'(\zeta)| \leq \epsilon$. Moreover, since $\tilde{f}'(\zeta)$ is the (unique) linear map such that $\lim_{|h|\to 0} \frac{|\tilde{f}(\zeta+h) - \tilde{f}(\zeta) - \tilde{f}'(h)|}{|h|} = 0$, with $h \in \mathbb{R}^{n_\zeta}$, then there exists a function $\rho_\zeta : \mathbb{R}^{n_\zeta} \to \mathbb{R}^{n_\zeta}$ satisfying $\lim_{|h|\to 0} \frac{|\rho_\zeta(h)|}{|h|} = 0$ and such that $\tilde{f}(\zeta + h) - \tilde{f}(\zeta) = \tilde{f}'(\zeta)h + \rho_\zeta(h)$. This implies that, for all $\epsilon > 0$ there exists $\sigma(\epsilon) > 0$ such that $|\rho_\zeta(h)| \leq \epsilon|h|$ for all $|h| \leq \sigma(\epsilon)$. Now, let $\gamma(\epsilon) = \min\left\{\delta(\epsilon/2), \frac{1}{2}\sigma(\epsilon/2)\right\}$. Then, for all $\zeta_1, \zeta_2 \in \mathbb{R}^{n_\zeta}$ such that $|\zeta_1| \leq \gamma(\epsilon)$ and $|\zeta_2| \leq \gamma(\epsilon)$, we have that $|\zeta_2| \leq \delta(\epsilon/2)$, and $|\zeta_1 - \zeta_2| \leq \sigma(\epsilon/2)$, and consequently

$$|\tilde{f}(\zeta_1) - \tilde{f}(\zeta_2)| \leq |\tilde{f}'(\zeta_2)||\zeta_1 - \zeta_2| + |\rho_\zeta(\zeta_1 - \zeta_2)| \leq \epsilon|\zeta_1 - \zeta_2| \tag{2.84}$$

Given $\mu(\epsilon) = \frac{1}{2}\gamma(\epsilon)$, $\epsilon = \frac{1}{2|P|}$, and $\bar{\zeta}$ and $\bar{u}$ obtained from Lemma 2.6.1 with $\mu = \mu(\epsilon)$, let us consider any input $u(t)$ such that $|u|_\infty \leq \bar{u}$, and the corresponding solution $\zeta(t)$ so that $|\zeta(0)| \leq \bar{\zeta}$. For $n \in \mathbb{N}$, let $\zeta_n(t) = \zeta(t+nT)$ and $u_n(t) = u(t+nT)$. Then, $\dot{\zeta}(t+nT) = A\zeta(t+nT) + \tilde{f}(\zeta(t+nT)) + bu(t+nT)$, which implies that

$U_n(t) = V(\zeta(t) - \zeta_n(t))$ is such that

$$\dot{U}_n = 2(\zeta - \zeta_n)^{\mathrm{T}} P \big[A(\zeta - \zeta_n) + \tilde{f}(\zeta) - \tilde{f}(\zeta_n) + b(u - u_n) \big] \leq$$
$$\leq -2|\zeta - \zeta_n|^2 + 2|P||\zeta - \zeta_n||\tilde{f}(\zeta) - \tilde{f}(\zeta_n)| + 2|P||b||\zeta - \zeta_n||u - u_n|$$

In view of Lemma 2.6.1, it is possible to check that for all $t \geq 0$ we have that $|\zeta(t) - \zeta_n(t)| \leq 2\mu(\epsilon)$, and from (2.84) that $|\tilde{f}(\zeta(t)) - \tilde{f}(\zeta_n(t))| \leq \frac{1}{2|P|}|\zeta(t) - \zeta_n(t)|$, which implies that $\dot{U} \leq -|\zeta - \zeta_n|^2 + 4\mu(\epsilon)|P||b| \cdot |u - u_n|$. Such inequality implies (2.83), and if $|u(t) - u_n(t)| = |u(t) - u(t + nT)| \to 0$ then we have also that $|\zeta(t) - \zeta_n(t)| = |\zeta(t) - \zeta(t + nT)| \to 0$. The claim follows once $\rho = \bar{\zeta}$ and $\varepsilon = \bar{u}$. □

The next proposition summarizes the main contribution of this section and provides sufficient conditions for perfect asymptotic tracking for the studied class of nonlinear RC schemes.

Proposition 2.6.3. *Let us consider (2.78) under Assumption 2.6.1, and suppose that the conditions for exponential stability in Prop. 2.5.4 are satisfied. Then, there exist $\mathcal{R} \subset \mathbb{R}^n$, $\widetilde{\mathcal{X}} \subset \widetilde{X}$, and $\ell > 0$ such that, if D, B_c and K_2 in (2.73) fulfill $|D^{-\mathrm{T}} B_c^{\mathrm{T}} K_2| \leq \ell$, then for any solution $\tilde{x}(t)$ to (2.78) such that $\tilde{x}(0) \in \widetilde{\mathcal{X}}$ which corresponds to a T-periodic reference signal $r(t)$ such that $r(t) \in \mathcal{R}$ for all $t \geq 0$, the state evolution is asymptotically T-periodic. Thus, asymptotic perfect tracking of the corresponding RC scheme is achieved.*

Proof. Because of Prop. 2.6.1, it is sufficient to show the existence of $\mathcal{R}$, $\widetilde{\mathcal{X}}$, and ℓ such that the trajectories $\tilde{x}(t)$ of (2.78) are asymptotically T-periodic. Now, denote by $v = (v_1, v_2)$ the state of (2.73), system that is 0-locally exponentially stable, and then apply Prop. 2.6.2 with $(\zeta, u) \equiv (v, u_c)$. This yields the existence of ρ, ε, $\alpha > 0$ such that

$$\limsup_{t \to \infty} |\Delta_n v(t)| \leq \alpha \limsup_{t \to \infty} |\Delta_n u_c(t)| \tag{2.85}$$

for all the solutions that satisfy $|v(0)| \leq \rho$ and $|u_c(t)|_\infty \leq \varepsilon$, for all $t \geq 0$. Let us now consider the closed-loop system (2.78), and with $d = |D| < 1$ pick δ such that $d < \delta < 1$. Moreover, define $\ell = \frac{1}{\alpha}(\delta - d)$, and denote by R_0 and $\widetilde{X}_0$ the sets for which Assumption 2.6.1 holds. Let for convenience $L = |D^{-\mathrm{T}} B_c^{\mathrm{T}} K_2|$, and define the following sets

$$\widetilde{\mathcal{X}} = \left\{ (x, v) \in \widetilde{X}_0 \mid \|x\|_\infty \leq \varepsilon,\ |v| \leq \rho,\ \|(x, v)\|_{\widetilde{X}} \leq \chi^{-1} \left((1 - d) \frac{\varepsilon}{2L} \right) \right\}$$
$$\mathcal{R} = \left\{ r \in R_0 \mid |r| \leq (L\gamma + I)^{-1} \left[(1 - d) \frac{\varepsilon}{2L} \right] \right\}$$

where $L\gamma + I$ denotes the map $(L\gamma + I)(s) = L\gamma(s) + s$, with $s \in \mathbb{R}$ and positive. Let us select a T-periodic reference signal $r(t)$ such that $r(t) \in \mathcal{R}$, and an initial condition $\tilde{x}(0) \in \widetilde{X}$. This latter hypothesis implies that $|v(0)| \leq \rho$, and then that (2.85) holds, provided that $|u_c(t)| \leq \varepsilon$.

In this respect, with the formulation of the delay as a BCS given in Prop. 2.4.1 in mind, and with reference to Figures 2.7 and 2.8, suppose that $|x(t, T)| \leq \varepsilon$ for all $t \in [nT, (n + 1)T]$, and for some $n \in \mathbb{N}$. Then, for all $t \in [nT, (n + 1)T]$, $u_c(t) = x(t, T)$ fulfills $|u_c(t)| \leq \varepsilon$. Moreover, in view of (2.73) and (2.81), for all $t \in [nT, (n + 1)T]$ we have that

$$|y_c(t) - r(t)| \leq L|v(t)| + d|u_c(t)| + |r(t)| \leq L\chi \left(\|\tilde{x}(0)\|_{\widetilde{X}} \right) + L\gamma \left(|r|_\infty \right) + d\varepsilon + |r|_\infty$$
$$\leq (1 - d)\varepsilon + d\varepsilon = \varepsilon$$

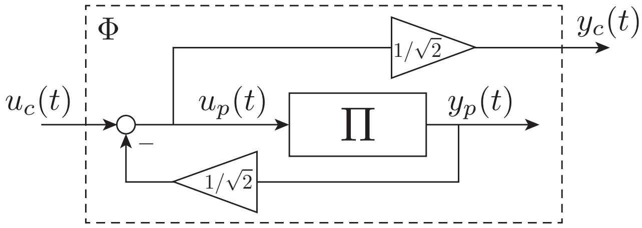


FIGURE 2.11: System (2.86) with the input / output Cayley transformation

Note that $x(t,0) = r(t) - y_c(t)$, which implies that $|x(t,0)| \leq \varepsilon$ for all $t \in [nT,(n+1)T]$, and then that $|x(t,T)| \leq \varepsilon$ for all $t \in [(n+1)T,(n+2)T]$. Since by definition of $\widetilde{\mathcal{X}}$ we have that $|x(t,T)| \leq \varepsilon$ for all $t \in [0,T]$, we claim by induction that $|x(t,T)| \leq \varepsilon$ for all $t \geq 0$. As a consequence, we have also that $|u_c(t)| = |x(t,T)| \leq \varepsilon$ for all $t \geq 0$, and then (2.85) holds.

Since $r(t)$ is T-periodic, then $|\Delta_n r(t)| = 0$ and, in view of (2.85), we have that

$$\begin{aligned}\limsup_{t\to\infty} |\Delta_n u_c(t)| &\leq \limsup_{t\to\infty} |\Delta_n y_c(t)| \leq L \limsup_{t\to\infty} |\Delta_n v(t)| + d \limsup_{t\to\infty} |\Delta_n u_c(t)| \\ &\leq (\alpha L + d) \limsup_{t\to\infty} |\Delta_n u_c(t)|\end{aligned}$$

If we suppose that $L \leq \ell$, the above relation yields

$$\limsup_{t\to\infty} |\Delta_n u_c(t)| \leq \delta \limsup_{t\to\infty} |\Delta_n u_c(t)|,$$

which implies that $\limsup_{t\to\infty} |\Delta_n u_c(t)| = 0$ since $d < \delta < 1$. This fact proves that $u_c(t)$ is asymptotically T-periodic, which in view of (2.85) implies that so are also $v(t)$ and, thus, $x(t,z)$ for all $z \in [0,T]$. Then, the asymptotic tracking for the RC scheme is an immediate consequence of Prop. 2.6.1. □

To illustrate the validity of the approach, in Fig. 2.12 it is shown how a RC scheme is able to let a system Φ in the form (2.80) to track a reference signal and reject an additive disturbance on the output, both periodic of period $T = 1$ s. The plant is obtained by starting from a 2^{nd}-order passive system Π that models e.g. a mechanical actuator with nonlinear damping:

$$\Pi : \begin{cases} \dot{v}_1(t) = v_2(t) \\ \dot{v}_2(t) = -v_1(t) - d\big(v_2(t)\big) + u_p(t) \\ y_p(t) = v_2(t) + S u_p(t) \end{cases} \tag{2.86}$$

Then, a Cayley transformation on the input / output mapping has been applied to get a system in scattering form and with the addition of an unitary feedthrough term (see Fig. 2.11). For reasonable values of the parameters, namely the nonlinear dissipation $d(v_2)$ and the feedthrough term S in (2.86), the transformed system fits in the class (2.80).

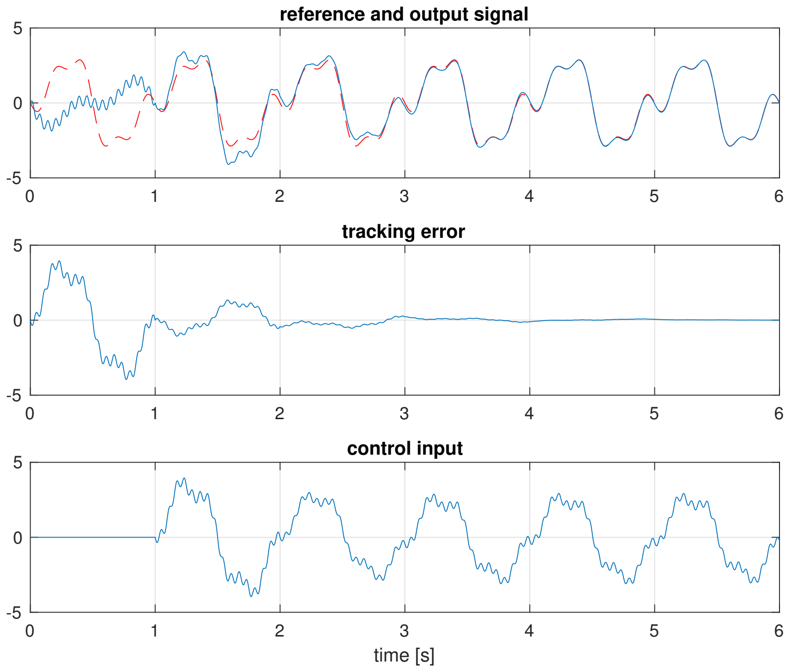


FIGURE 2.12: Asymptotic tracking and disturbance rejection of the RC system in which Φ is the plant depicted in Fig. 2.11. In (2.86), we have selected $S = 1.5$ and $d(v_2) = 5v_2^3$: with these choices the transformed system Φ fits in (2.80).

Chapter 3

Periodic Regulation and Invariance Analysis of Autonomous Systems

In this chapter an alternative approach to study the periodic output regulation problem is proposed. The aim is to provide sufficient conditions for perfect tracking without using internal-model based arguments. The idea is to represent a system fed by a periodic reference as an autonomous discrete-time system with infinite-dimensional input and output space. The main tool that is used is the *lifting technique* over a cycle of period τ of the continuous time systems, see Yamamoto, 1990; Bamieh and Pearson, 1992

3.1 Problem Statement and General Solution

Let us consider the following finite-dimensional LTI continuous-time system

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{3.1}$$

with initial condition $x(0) = x_0$. The system is defined on a finite-dimensional space $X = \mathbb{R}^n$, i.e. $x \in \mathbb{R}^n$. The input $u(\cdot)$ and output $y(\cdot)$ are supposed to take values in the same finite-dimensional space of dimension p, i.e. $u(\cdot), y(\cdot) \in \mathbb{R}^p$. The system matrices A, B, C and D have appropriate dimensions.

The output $y(t)$ has to track a periodic reference signal of period τ, described by the periodic repetition of $r \in U = L^2(0, \tau; \mathbb{R}^p)$. This periodic reference signal can be generated by the discrete-time exosystems

$$\Upsilon : \{ v^+ = v \tag{3.2}$$

where $v \in L^2(0, \tau; \mathbb{R}^p)$ and $v(0) = r$. Here the notation $(\cdot)^+$ referred to a discrete-time system means the signal at the next time step.

In order to tackle the periodic output regulation task a discrete-time controller in the following form is considered

$$\Gamma : \begin{cases} z^+ = \mathcal{I}z + \mathcal{H}e \\ u = \mathcal{K}z \end{cases} \tag{3.3}$$

where $z \in L^2(0, \tau; \mathbb{R}^p)$, $z(0) = z_0$, $e(\cdot) = v(\cdot) - y(\cdot) \in \mathbb{R}^p$ and

$$\mathcal{I}, \mathcal{H}, \mathcal{K} \in \mathcal{L}(L^2(0, \tau; \mathbb{R}^p), L^2(0, \tau; \mathbb{R}^p))$$

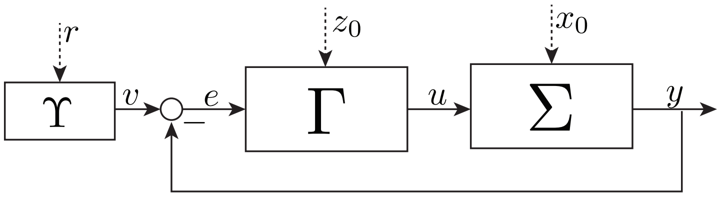


FIGURE 3.1: A graphical representation of the autonomous closed-loop system.

. The autonomous closed-loop system that merges from the interconnection is shown in Figure 3.1.

Using the *lifting technique* over the period τ we can represent the system Σ as the LTI discrete-time system

$$\Sigma_l : \begin{cases} \mathbf{x}^+ = \underline{A}\mathbf{x} + \underline{B}\mathbf{u} \\ \mathbf{y} = \underline{C}\mathbf{x} + \underline{D}\mathbf{u} \end{cases} \tag{3.4}$$

where $\mathbf{x}(k) = x(k\tau)$, $\mathbf{u}_k = u(k\tau + (\cdot))$ and $\mathbf{y}_k = y(k\tau + (\cdot))$. Notice that $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u}, \mathbf{y} \in U$ and $(\underline{A}, \underline{B}, \underline{C}, \underline{D})$ are such that $\underline{A} \in \mathbb{R}^{n\times n}$, $\underline{B} \in \mathcal{L}(L^2(0,\tau;\mathbb{R}^p), \mathbb{R}^n)$, $\underline{C} \in \mathcal{L}(\mathbb{R}^n, L^2(0,\tau;\mathbb{R}^p))$, $\underline{D} \in \mathcal{L}(L^2(0,\tau;\mathbb{R}^p), L^2(0,\tau;\mathbb{R}^p))$ defined as follows:

$$\begin{aligned} \underline{A} &= e^{A\tau} \\ \underline{B}\mathbf{u} &= \int_0^\tau e^{A(\tau-s)} B\mathbf{u}(s)ds \\ (\underline{C}\mathbf{x})(\cdot) &= Ce^{A(\cdot)}\mathbf{x} \\ (\underline{D}\mathbf{u})(\cdot) &= C\int_0^{(\cdot)} e^{A((\cdot)-s)} B\mathbf{u}(s)ds + D\mathbf{u}(\cdot) \end{aligned} \tag{3.5}$$

Remark 3.1.1. *Both the original system (3.1) and the lifted system (3.4) have a finite-dimensional state space, i.e. $x, \mathbf{x} \in \mathbb{R}^n$. The price to pay to deal with a discrete-time system is to end up with infinite-dimensional input and output spaces, making the system operators $\underline{B}, \underline{C}$ and $\underline{D}$ general linear operators, and not matrices.*

The lifted closed-loop system can be written as the following autonomous LTI discrete-time system

$$\begin{pmatrix} v^+ \\ \mathbf{x}^+ \\ z^+ \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & \underline{A} & \underline{B}\mathcal{K} \\ \mathcal{H} & -\mathcal{H}\underline{C} & (\mathcal{I} - \mathcal{H}\underline{D}\mathcal{K}) \end{pmatrix} \begin{pmatrix} v \\ \mathbf{x} \\ z \end{pmatrix} \tag{3.6}$$

where an element of the state space is $(v^{\mathrm{T}}\ \mathbf{x}^{\mathrm{T}}\ z^{\mathrm{T}})^{\mathrm{T}} \in U \times \mathbb{R}^n \times U$ which is an Hilbert space.

The aim is to regulate to zero the following error signal

$$e = \begin{pmatrix} I & -\underline{C} & -\underline{D}\mathcal{K} \end{pmatrix} \begin{pmatrix} v \\ \mathbf{x} \\ z \end{pmatrix} \tag{3.7}$$

which represents, as an L^2-element, the tracking error over a period.

The problem of achieving perfect periodic tracking can be equivalently reformulated by the following problem.

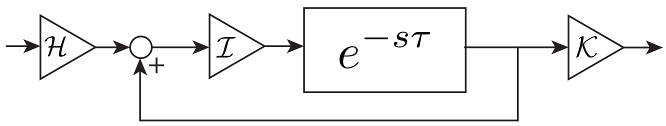


FIGURE 3.2: Continuous time representation of the controller (3.3). Note that for $\mathcal{I} = \mathcal{H} = \mathcal{K} = I$ the classical internal model based controller present in RC schemes merges.

Problem 3.1.1. *Given a τ-periodic reference signal as the periodic repetition of $r \in L^2(0,\tau;\mathbb{R}^p)$ assign $v(0) = r$. Choose the controller parameters $\mathcal{I}, \mathcal{H}, \mathcal{K}$ such that the system (3.6) admits a fixed point $\left(v(0)^{\mathrm{T}}\ \mathbf{x}_{ss}{}^{\mathrm{T}}\ z_{ss}{}^{\mathrm{T}}\right)^{\mathrm{T}} \in U \times \mathbb{R}^n \times U$ which is attractive $\forall z_0 \in U$ and $\forall x_0 \in \mathbb{R}^n$ and such that*

$$e_{ss} = \begin{pmatrix} I & -\underline{C} & -\underline{D}\mathcal{K} \end{pmatrix} \begin{pmatrix} v(0) \\ \mathbf{x}_{ss} \\ z_{ss} \end{pmatrix} = 0$$

Remark 3.1.2. *Notice that the controller (3.3) in the case in which $\mathcal{I} = \mathcal{H} = \mathcal{K} = I$ is equivalent to the continuous-time repetitive compensator, which is the infinite-dimensional controller present in repetitive control schemes Hara et al., 1988 (see figure 3.2). In fact with this choice of parameters, system (3.3) is the lifted representation of a pure time delay of τ seconds surrounded by a positive feedback loop and fed by the error signal.*

3.1.1 Invariance Analysis for a class of marginally stable linear systems

Here a mathematical result dealing with invariance analysis of autonomous linear systems is presented and then specialized to give sufficient conditions to solve problem 3.1.1.

Proposition 3.1.1. *Consider the discrete-time autonomous marginally stable linear system*

$$\begin{pmatrix} \mathbf{v}+ \\ \underline{\mathbf{x}}^+ \end{pmatrix} = \begin{pmatrix} I & 0 \\ \Theta & F \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ \underline{\mathbf{x}} \end{pmatrix} \tag{3.8}$$

with $\mathbf{v} \in L^2(0,\tau;\mathbb{R}^p)$ and initial condition $\mathbf{v}(0)$. Furthermore $\underline{\mathbf{x}} \in \underline{\mathbf{X}}$, being $\underline{\mathbf{X}}$ a Hilbert space and $\Theta \in \mathcal{L}(L^2(0,\tau;\mathbb{R}^p), \underline{\mathbf{X}})$ and $F \in \mathcal{L}(\underline{\mathbf{X}})$ bounded linear operators. If $\forall \lambda \in \sigma(F), |\lambda| < 1$, then the system converges to an invariant region of the state space for every initial condition $\underline{\mathbf{x}}(0)$.

Proof. Since the space $\mathcal{V}_1 = \mathrm{span}\begin{pmatrix} 0 \\ I \end{pmatrix}$ is invariant for (3.8), without loss of generality we can look at another invariant subspace in the form $\mathcal{V}_2 = \mathrm{span}\begin{pmatrix} I \\ \Pi \end{pmatrix}$ where $\Pi \in \mathcal{L}(L^2(0,\tau;\mathbb{R}^p), \underline{\mathbf{X}})$. Thus on the invariant the operator relation $\underline{\mathbf{x}}^+ = \underline{\mathbf{x}} = \Pi\mathbf{v} = \Theta\mathbf{v} + F\underline{\mathbf{x}} = (\Theta + F\Pi)\mathbf{v}$ must hold. This means that the equation $\Pi = \Theta + F\Pi$ must admit a unique solution Π. This is assured if $(I - F)^{-1} \in \mathcal{L}(\underline{\mathbf{X}})$. Furthermore, defining the vector $\underline{\tilde{\mathbf{x}}} = \Pi\mathbf{v} - \underline{\mathbf{x}}$, its dynamics can be written as

$$\underline{\tilde{\mathbf{x}}}^+ = \Pi\mathbf{v} - \underline{\mathbf{x}}^+ = (\Pi - \Theta)\mathbf{v} - F\mathbf{x} = F(\Pi\mathbf{v} - \underline{\mathbf{x}}) = F\underline{\tilde{\mathbf{x}}}$$

which shows that if the spectral radius of F is less than 1, i.e. $\forall \lambda \in \sigma(F), |\lambda| < 1$, the invariant is globally attractive. ◻

3.1.2 Perfect Tracking Conditions (general case)

Consider the system (3.6) with the error equation, i.e.

$$\begin{pmatrix} v^+ \\ \mathbf{x}^+ \\ z^+ \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & \underline{A} & \underline{B}\mathcal{K} \\ \mathcal{H} & -\mathcal{H}\underline{C} & (\mathcal{I} - \mathcal{H}\underline{D}\mathcal{K}) \end{pmatrix} \begin{pmatrix} v \\ \mathbf{x} \\ z \end{pmatrix}$$
$$e = \begin{pmatrix} I & -\underline{C} & -\underline{D}\mathcal{K} \end{pmatrix} \begin{pmatrix} v \\ \mathbf{x} \\ z \end{pmatrix} \tag{3.9}$$

The following theorem characterizes conditions for perfect tracking for periodic output regulation.

Theorem 3.1.1. *Problem 3.1.1 is solved for system (3.9) if the following hold:*

- $Ker\mathcal{H} = 0$
- $\mathcal{I} = I$
- $\forall \lambda \in \sigma \begin{pmatrix} \underline{A} & \underline{B}\mathcal{K} \\ -\mathcal{H}\underline{C} & (\mathcal{I} - \mathcal{H}\underline{D}\mathcal{K}) \end{pmatrix}, |\lambda| < 1$

Proof. Existence and global attractivity of the invariant region characterizing the fixed point of (3.6) is guaranteed by Proposition 3.1.1 with

$$F = \begin{pmatrix} \underline{A} & \underline{B}\mathcal{K} \\ -\mathcal{H}\underline{C} & (\mathcal{I} - \mathcal{H}\underline{D}\mathcal{K}) \end{pmatrix} \quad \Theta = \begin{pmatrix} 0 \\ \mathcal{H} \end{pmatrix}.$$

Furthermore if we look at the steady state map $\Pi = \begin{pmatrix} \Pi_x \\ \Pi_z \end{pmatrix}$ then

$$\Pi_z = \mathcal{H} - \mathcal{H}\underline{C}\Pi_x + (\mathcal{I} - \mathcal{H}\underline{D}\mathcal{K})\Pi_z = \mathcal{H}(I - \underline{C}\Pi_x - \underline{D}\mathcal{K}\Pi_z) + \mathcal{I}\Pi_z$$

Since at steady state the error e ranges on the space $\Pi_e = I - \underline{C}\Pi_x - \underline{D}\mathcal{K}\Pi_z$, the choice $\mathcal{I} = I$, together with the fact that $\mathrm{Ker}\mathcal{H} = 0$, forces $\Pi_e = 0$, i.e. perfect asymptotic tracking is achieved for any periodic reference of period τ. □

Remark 3.1.3. *On the invariant $I - \underline{C}\Pi_x - \underline{D}\mathcal{K}\Pi_z = 0$ and $\Pi_x = (I - \underline{A})^{-1}\underline{B}\mathcal{K}\Pi_z$. By substitution this implies*

$$I - \left[\underline{C}(I - \underline{A})^{-1}\underline{B} + \underline{D}\right]\mathcal{K}\Pi_z = 0 \tag{3.10}$$

which means that with the set of assumptions satisfied, the tracking error converging to zero implies that at steady state $\mathcal{K}\Pi_z$ is the right inverse of the transfer function of the lifted plant at the unitary frequency. By defining the operator $\underline{\mathbf{P}}(1) \in \mathcal{L}(L^2(0,\tau;\mathbb{R}^p), L^2(0,\tau;\mathbb{R}^p))$ as

$$\begin{aligned}\underline{\mathbf{P}}(1)\mathbf{u}(\cdot) &= \left[\underline{C}(I - \underline{A})^{-1}\underline{B} + \underline{D}\right]\mathbf{u}(\cdot) = \\ &= Ce^{A(\cdot)}(I - e^{A\tau})^{-1}\int_0^\tau e^{A(\tau-s)}B\mathbf{u}(s)ds + C\int_0^{(\cdot)} e^{A((\cdot)-s)}B\mathbf{u}(s)ds + D\mathbf{u}(\cdot)\end{aligned}$$

it can be concluded that $\underline{\mathbf{P}}(1)$ must be invertible to achieve perfect tracking. Now it is easy to see that the operator is invertible only if the original system (3.1) has a non null feedthrough term, i.e. $D \neq 0$. Since this operator is not invertible for $D = 0$ the general periodic output regulation problem can not be solved for strictly proper systems. This is true because $\underline{\mathbf{P}}(1)$

is not surjective if $D = 0$, since $\forall \mathbf{u} \in L^2(0, \tau; \mathbb{R}^p)$, $\underline{\mathbf{P}}(1)\mathbf{u}(\cdot) \in \mathcal{C}(0, \tau; \mathbb{R}^p)$, which is only continuously embedded in $L^2(0, \tau; \mathbb{R}^p)$.

Remark 3.1.4. *The above result does not involve internal model based considerations but can be post interpreted under this light. In fact the choice $\mathcal{I} = I$ makes the controller able to generate any periodic reference of period τ autonomously, by storing the base function to be repeated in the initial state of the controller (3.3). Furthermore the condition*

$$\forall \lambda \in \sigma \begin{pmatrix} \underline{A} & \underline{B\mathcal{K}} \\ -\mathcal{H}\underline{C} & (\mathcal{I} - \mathcal{H}\underline{D}\mathcal{K}) \end{pmatrix}, |\lambda| < 1 \tag{3.11}$$

is exactly the condition for exponential stability of the autonomous ($v = 0$) closed loop system.

The impossibility to track any periodic signal for strictly proper systems is somehow intuitive since the reference trajectory can present an unbounded harmonic content that will be smoothened to some extend without an algebraic relation between input and output. In the following perfect tracking conditions described in Theorem 3.1.1 will be specialized to specific subclasses of the reference signal by changing the state space of the exosystem (3.2).

3.2 Solution of Problem 3.1.1 in the Digital Case

In this section the the case in which the system Σ is embedded in a digital closed-loop system is considered. The periodic reference signal to be tracked is of period $\tau = qT$ where T is the sampling period of the digital system and $q \in \mathbb{N}$. The output of Σ is sampled and hold at the sampling frequency of the system in such a way to feed the discrete time controller with a piecewise constant error signal over one sampling period. The idea is to transform the state space of system (3.2) in such a way to end up with a generic state element of (3.6) $({v_a}^{\mathrm{T}}\ \mathbf{x}^{\mathrm{T}}\ {z_a}^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{qp} \times \mathbb{R}^n \times \mathbb{R}^{qp}$, i.e. $U = \mathbb{R}^{qp}$. This is possible because over one period the reference signal r and the state of the controller z is piecewise constant over a time period T, as shown in the following proposition.

Proposition 3.2.1. *Defining*

$$v_a(k) = (v_1(k)^{\mathrm{T}}\ v_2(k)^{\mathrm{T}}\ \cdots\ v_q(k)^{\mathrm{T}})^{\mathrm{T}} = (v(k\tau + T)^{\mathrm{T}}\ v(k\tau + 2T)^{\mathrm{T}}\ \cdots\ v(k\tau + qT)^{\mathrm{T}})^{\mathrm{T}}$$

and

$$z_a(k) = (z_1(k)^{\mathrm{T}}\ z_2(k)^{\mathrm{T}}\ \cdots\ z_q(k)^{\mathrm{T}})^{\mathrm{T}} = (z(k\tau + T)^{\mathrm{T}}\ z(k\tau + 2T)^{\mathrm{T}}\ \cdots\ z(k\tau + qT)^{\mathrm{T}})^{\mathrm{T}}$$

it yields $v_a(k), z_a(k) \in \mathbb{R}^{qp}$. Now system (3.6) can be expressed over the finite-dimensional state space $\mathbb{R}^{qp} \times \mathbb{R}^n \times \mathbb{R}^{qp}$ as follows

$$\begin{pmatrix} v_a^+ \\ \mathbf{x}^+ \\ z_a^+ \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & \underline{A} & \underline{B_a\mathcal{K}} \\ \mathcal{H} & -\mathcal{H}\underline{C_a} & (\mathcal{I} - \mathcal{H}\underline{D_a}\mathcal{K}) \end{pmatrix} \begin{pmatrix} v_a \\ \mathbf{x} \\ z_a \end{pmatrix} \tag{3.12}$$

where $\underline{B_a} \in \mathbb{R}^{n\times(qp)}$, $\underline{C_a} \in \mathbb{R}^{(qp)\times n}$, $\underline{D_a} \in \mathbb{R}^{(qp)\times(qp)}$ *defined as follows:*

$$\underline{B_a} = \left(\int_0^T e^{A(\tau-s)}Bds \quad \int_T^{2T} e^{A(\tau-s)}Bds \quad \cdots \quad \int_{((q-1)T)}^{\tau} e^{A(\tau-s)}Bds \right)$$

$$\underline{C_a} = \begin{pmatrix} Ce^{AT} \\ Ce^{A2T} \\ \cdots \\ Ce^{A\tau} \end{pmatrix}$$

$$\underline{D_a} = blockdiag\left\{ C\int_0^T e^{A(T-s)}Bds + D,\ C\int_0^{2T} e^{A(2T-s)}Bds + D, \cdots \right.$$
$$\left. \cdots, C\int_0^{\tau} e^{A(\tau-s)}Bds + D \right\}$$

where $blockdiag\{\cdot\}$ *returns the block-diagonal matrix of its arguments.*

Proof. The second row of (3.6) becomes

$$\mathbf{x}^+ = \underline{A}\mathbf{x} + \int_0^{\tau} e^{A(\tau-s)}B\mathcal{K}z(s)ds = \underline{A}\mathbf{x} + \sum_{i=1}^{q}\int_{(i-1)T}^{iT} e^{A(\tau-s)}\mathcal{K}z(s)ds =$$
$$= \underline{A}\mathbf{x} + \sum_{i=1}^{q}\left[\left(\int_{(i-1)T}^{iT} e^{A(\tau-s)}ds\right)\mathcal{K}z_i\right] = \underline{A}\mathbf{x} + \underline{B_a}\mathcal{K}z_a$$

The third row of (3.12) follows from the fact that $\forall i \in 1, ..., q$

$$z_i^+ = \mathcal{H}v_i - \mathcal{H}Ce^{AiT} + \mathcal{I}z_i - \mathcal{H}\left[C\int_0^{iT} e^{A(iT-s)}Bds + D\right]\mathcal{K}z_i$$

□

Notice that in (3.12) a slight abuse of notation is used since the operators

$$I, \mathcal{I}, \mathcal{H}, \mathcal{K} \in \mathbb{R}^{(qp)\times(qp)}$$

are expressed with the same name as in (3.6). This because it is clear from the context that now they represent simply gain matrices.

3.2.1 Invariance Analysis for a class of marginally stable linear systems (finite-dimensional case)

It is possible to specialize consistently the result of proposition 3.1.1 to deal with the finite-dimensional case.

Proposition 3.2.2. *Consider the discrete-time autonomous marginally stable linear system*

$$\begin{pmatrix} \mathbf{v}+ \\ \underline{\mathbf{x}}^+ \end{pmatrix} = \begin{pmatrix} I & 0 \\ \Theta & F \end{pmatrix}\begin{pmatrix} \mathbf{v} \\ \underline{\mathbf{x}} \end{pmatrix} \tag{3.13}$$

with $\mathbf{v}$ *and* $\underline{\mathbf{x}}$ *finite-dimensional vectors and* Θ *and* F *matrices of appropriate dimensions. Consider initial condition* $\mathbf{v}(0)$*. If* F *is Schur (i.e. all its eigenvalues have modulus less than* 1*), then the system converges to an invariant region of the state space for every initial condition* $\underline{\mathbf{x}}(0)$*.*

Proof. The proofs follows exactly like in proposition (3.1.1) with the steady state map Π that now is simply a matrix. As consequence the spectral condition turns into an eigenvalue condition. □

3.2.2 Perfect Tracking Conditions

Let us define an output $e_a \in \mathbb{R}^{qp}$ associated to system (3.12) which contains the tracking error at the sampling instances, i.e. the overall system becomes

$$\begin{aligned} \begin{pmatrix} v_a^+ \\ \mathbf{x}^+ \\ z_a^+ \end{pmatrix} &= \begin{pmatrix} I & 0 & 0 \\ 0 & \underline{A} & \underline{B_a}\mathcal{K} \\ \mathcal{H} & -\mathcal{H}\underline{C_a} & (\mathcal{I} - \mathcal{H}\underline{D_a}\mathcal{K}) \end{pmatrix} \begin{pmatrix} v_a \\ \mathbf{x} \\ z_a \end{pmatrix} \\ e_a &= \begin{pmatrix} I & -\underline{C_a} & -\underline{D_a}\mathcal{K} \end{pmatrix} \begin{pmatrix} v_a \\ \mathbf{x} \\ z_a \end{pmatrix} \end{aligned} \tag{3.14}$$

The following theorem characterizes conditions for perfect tracking for periodic output regulation in the digital case.

Theorem 3.2.1. *Problem 3.1.1 is solved for system (3.14), i.e. perfect asymptotic tracking is achieved for any periodic digital reference signal of period $\tau = qT$ if the following hold:*

- $Ker\mathcal{H} = 0$
- $\mathcal{I} = I$
- $\begin{pmatrix} \underline{A} & \underline{B_a}\mathcal{K} \\ -\mathcal{H}\underline{C_a} & (\mathcal{I} - \mathcal{H}\underline{D_a}\mathcal{K}) \end{pmatrix}$ *is Schur*

Proof. Existence and global attractivity of the invariant region characterizing the fixed point of (3.12) is guaranteed by Proposition 3.2.2 with

$$F = \begin{pmatrix} \underline{A} & \underline{B_a}\mathcal{K} \\ -\mathcal{H}\underline{C_a} & (\mathcal{I} - \mathcal{H}\underline{D_a}\mathcal{K}) \end{pmatrix} \qquad \Theta = \begin{pmatrix} 0 \\ \mathcal{H} \end{pmatrix}.$$

Furthermore if we look at the steady state matrix $\Pi = \begin{pmatrix} \Pi_x \\ \Pi_z \end{pmatrix}$ then

$$\Pi_z = \mathcal{H} - \mathcal{H}\underline{C_a}\Pi_x + (\mathcal{I} - \mathcal{H}\underline{D_a}\mathcal{K})\Pi_z = \mathcal{H}(I - \underline{C_a}\Pi_x - \underline{D_a}\mathcal{K}\Pi_z) + \mathcal{I}\Pi_z.$$

Since at steady state the error e_a ranges on the space $\Pi_e = I - \underline{C_a}\Pi_x - \underline{D_a}\mathcal{K}\Pi_z$, the choice $\mathcal{I} = I$, together with the fact that $\mathrm{Ker}\mathcal{H} = 0$, forces $\Pi_e = 0$, i.e. perfect asymptotic tracking is achieved at the sampling instances. □

Remark 3.2.1. *Similarly as before $I - \underline{C_a}\Pi_x - \underline{D_a}\mathcal{K}\Pi_z = 0$ with $\Pi_x = (I - \underline{A})^{-1}\underline{B_a}\mathcal{K}\Pi_z$ implies*

$$I - \left[\underline{C_a}(I - \underline{A})^{-1}\underline{B_a} + \underline{D_a}\right]\mathcal{K}\Pi_z = 0$$

which means that with the set of assumptions satisfied, the tracking error converging to zero implies that $\mathcal{K}\Pi_z$ is the right inverse of the transfer function of the lifted plant at the unitary frequency. This implies that the transfer matrix $\underline{\mathbf{P}_a}(1) \in \mathbb{R}^{qp \times qp}$ defined as $\underline{\mathbf{P}_a}(1) = \underline{C_a}(I - \underline{A})^{-1}\underline{B_a} + \underline{D_a}$ must be invertible if the set of assumptions are valid. Differently from the general case this is possible also if for the original system Σ the feedthrough term is null, i.e. $D = 0$. Thus it is possible to achieve digital periodic output regulation for strictly proper plants.

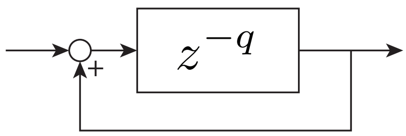


FIGURE 3.3: Discrete-time representation of the controller (3.3) for $\mathcal{I} = \mathcal{H} = \mathcal{K} = I$. It coincides with the digital version of the repetitive compensator present in RC schemes. Here z^{-1} represents the one-step delay operator.

Remark 3.2.2. *Similarly as in the continuous-time case, the result can be post-interpreted from an internal model perspective. In fact given the definition of z_a and v_a, with the choice $\mathcal{I} = I$ the controller achieves an internal model for any discrete time periodic signal of period $\tau = qT$. The simplest implementation of this controller (with $\mathcal{H} = \mathcal{K} = I$) is shown in Figure 3.3 and represents the controller of digital repetitive control schemes. The classical interpretation states that the tracking problem coincides then with a stabilization problem, represented by the matrix $\begin{pmatrix} \underline{A} & \underline{B_a}\mathcal{K} \\ -\mathcal{H}\underline{C_a} & (\mathcal{I} - \mathcal{H}\underline{D_a}\mathcal{K}) \end{pmatrix}$ to be Schur.*

3.3 Static nonlinear plant

Let us go beyond the linear case and consider as plant Σ a static nonlinearity $f(\cdot)$, i.e. pointwise $y = f(u)$. In order to consider the lifted nonlinear plant over a period τ we introduce the operator Λ_f by $(\Lambda_f(u))(\cdot) = f(u(\cdot))$ for $u \in D(\Lambda_f) = \{u \in L^2(0,\tau;\mathbb{R}^p)\}$. With this notation the lifted static nonlinearity can be expressed by the operator Λ_f which is in general not bounded (i.e. does not map necessarily $L^2(0,\tau;\mathbb{R}^p)$ in $L^2(0,\tau;\mathbb{R}^p)$). The dynamic equations for the closed loop systems become

$$\begin{pmatrix} v^+ \\ z^+ \end{pmatrix} = \begin{pmatrix} v \\ \mathcal{H}v + (\mathcal{I} - \mathcal{H}\Lambda_f\mathcal{K})z \end{pmatrix} \tag{3.15}$$

Proposition 3.3.1. *If the map $z \to (\mathcal{I} - \mathcal{H}\Lambda_f\mathcal{K})z$ is a contraction in $L^2(0,\tau;\mathbb{R}^p)$ then system (3.15) has a globally attractive fixed point. If in addition $\mathcal{I} = I$ and $\mathcal{H}$ is an injective map then the tracking error vanishes (in an L^2-sense) at steady state.*

Proof. Existence of a globally attractive fixed point for system (3.15) follows from the *Banach fixed point theorem* if the map $z \to \mathcal{H}v + (\mathcal{I} - \mathcal{H}\Lambda_f\mathcal{K})z$ is a contraction in $L^2(0,\tau;\mathbb{R}^p)$, and equivalently if the map $z \to (\mathcal{I} - \mathcal{H}\Lambda_f\mathcal{K})z$ is a contraction in $L^2(0,\tau;\mathbb{R}^p)$. Notice that the last requirement is exactly exponential stability of the autonomous ($v = 0$) closed-loop system. In this case and considering $\mathcal{I} = I$, the fixed point, i.e. the steady state solution for (3.15) is $(v, \bar{z})$, with $\bar{z} \in L^2(0,\tau;\mathbb{R}^p)$ such that $\mathcal{H}\Lambda_f\mathcal{K}\bar{z} = \mathcal{H}v$. The steady state error is $e = v - \Lambda_f\mathcal{K}\bar{z}$ which is equal to 0 if $\mathcal{H}$ is injective. □

Remark 3.3.1. *Well-posedness for the closed-loop nonlinear system is in general not assured for any static nonlinearity $f(\cdot)$ because the operator Λ_f could map L^2 functions outside the space. This is however implicitly solved by the requirement that the map $z \to (\mathcal{I} - \mathcal{H}\Lambda_f\mathcal{K})z$ is a contraction in $L^2(0,\tau;\mathbb{R}^p)$. In fact since $\mathcal{I}, \mathcal{K}$ and $\mathcal{H}$ are bounded operators, the functions $f(\cdot)$ which makes the operator Λ_f unbounded (i.e. functions $f(\cdot)$ that are not globally Lipschitz continuous) are automatically excluded.*

Remark 3.3.2. *In the case of pure repetitive control, i.e.* $\mathcal{I} = \mathcal{K} = \mathcal{H} = I$ *the condition on the static nonlinearity* $f(\cdot)$ *is to be globally Lipschitz continuous with Lipschitz constant* $L < 2$

Chapter 4

Repetitive B-Spline Trajectory Generation for Nonminimum Phase Systems

In this chapter a novel repetitive control scheme is presented and discussed, based on the so called B-spline filters. This type of dynamic filters are able to provide a B-spline trajectory if they are fed with the sequence of proper control points that define the trajectory itself. Therefore, they are ideal tools for generating online the reference signal with the prescribed level of smoothness for driving dynamic systems, e.g. with a feedforward compensator.

In particular, the so-called Continuous Zero Phase Error Tracking Controller (in brief CZPETC) can be used for tracking control of non-minimum phase systems but because of its open-loop nature cannot guarantee robustness with respect to modeling errors and exogenous disturbances. For this reason, CZPETC and trajectory generator have been embedded in a repetitive control scheme that allows to nullify interpolation errors even in non-ideal conditions, provided that the desired reference trajectory and the disturbances are periodic.

Asymptotic stability of the overall control scheme has been proven mathematically by the authors in Biagiotti, Califano, and Melchiorri, 2016. Here we extend those results since here an experimental set-up for a non-minimum phase system is presented. Different models of the same physical system are identified and the proposed model-based control scheme is implemented. Experimental results have been reported in order to show the validity of the proposed method and the importance of an accurate modeling step.

4.1 Motivation

Quite often, in industrial applications, the given tasks present a cyclic or repetitive nature; this means that, from a control perspective, the plant is required to track a periodic exogenous signal whose cycle time is supposed to be known in advance.

In a realistic scenario a complex reference signal is generated by means of tools such as Spline, Bezier, Nurbs curves and other similar functions (Biagiotti and Melchiorri, 2008). Therefore, it may be of interest to investigate how the use of this kind of curves, together with the cyclic nature of the task, can be exploited to improve the tracking accuracy.

Because of the cyclic nature of the task, a Repetitive Control (RC) approach represents a quite standard and effective solution to achieve asymptotic perfect tracking, being able to cancel tracking errors over repetitions by learning from previous iterations.

Here the modification of the control input is implemented at a level of trajectory generation. The idea of modifying a B-spline reference trajectory by applying a repetitive control scheme on the corresponding control points was proposed in Biagiotti, Moriello, and Melchiorri, 2015. Thanks to the possibility of generating B-spline trajectories by means of dynamic filters (Biagiotti and Melchiorri, 2010), the trajectory planner has been inserted in an external feedback control loop that modifies in real-time the control points of the B-spline curve so that the interpolation error at the desired via-points converges to zero.

This technique offers great advantages in real applications in terms of low complexity and ease of parameters tuning, without the need of a deep knowledge of the plant model. Moreover, it can be applied to all those systems, like servo-motors or robotic manipulators, that equipped with an (unmodifiable) off-the-shelf controller, only admit an external reference signal.

On the other hand, the proposed control scheme relies on the hypothesis that the plant $P(s)$, already controlled, is characterized by an acceptable tracking capability within a certain frequency range, i.e.

$$P(j\omega) \approx 1, \;\; \omega < \omega_m$$

where ω_m denotes the maximum frequency of the reference input. From a practical point of view this represents mainly a phase constraint since it is necessary that the controlled plant

$$|\angle P(j\omega)| < \frac{\pi}{2}, \;\; \omega < \omega_m. \tag{4.1}$$

Inspired by this necessity, a novel repetitive control scheme has been presented, in which the real-time modification of the control points defining the reference B-spline trajectory is combined with a feedforward compensation that, by widening the bandwidth of the plant, allows to remove the hypothesis on its dynamic behaviour. In particular, in order to deal also with non-minimum phase plants, a technique that generalizes the standard inversion-based feedforward control has been considered, i.e. the Zero Phase Error Tracking Controller (ZPETC) introduced in Tomizuka, 1987. Since this model-based controller is in general non-causal, tracking solutions for non-minimum phase systems require precognition of the reference signal to maintain bounded internal signals Rigney, Pao, and Lawrence, 2009. The most common way to overcome this difficulty is to design the controller in a discrete time framework, and anticipate the input signal to realize the necessary control action (see e.g. Adnan et al., 2012; Ismail et al., 2012; Tian et al., 2014; Ghazali et al., 2015). For example in Rodriguez, Pons, and Ceres, 2000 a repetitive control scheme was designed for ultrasonic motors with the use of a ZPET controller, exploiting the discrete time periodic reference trajectory to achieve precognition of the signal. It is worth to noticing that the combination of RC and ZPETC has been proposed by Tomizuka himself in Tomizuka, Tsao, and Chew, 1988, but again only in the discrete-time domain. Moreover, even if the RC scheme is defined plug-in since it does not require modification of the original feedback controller of the plant (Tomizuka, Tsao, and Chew, 1988; Kim and Tsao, 2002), it is based on the assumption that the control signal is accessible and therefore the RC contribution can be simply added to the original control. The solution that is proposed in the following is based on a continuous time version of the ZPET controller (called CZPETC) Park, Chang, and Lee, 2003, whose implementation has been made possible by the B-spline generator that, along with the reference trajectory profile, generates online its time derivatives. The major advantage of this approach in practical applications is the possibility to

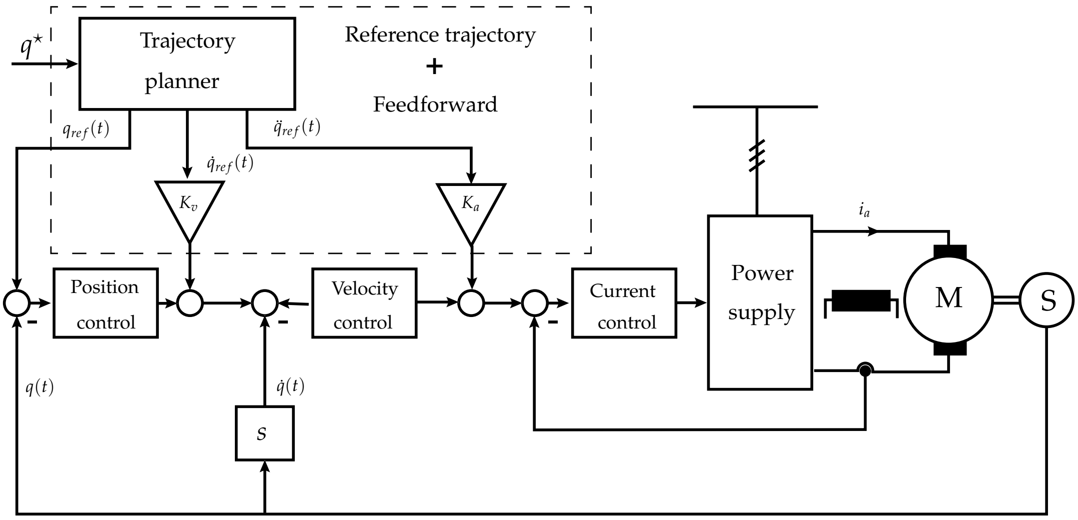


FIGURE 4.1: Standard cascade control structure with feedforward for electrical motor drives Leonhard, 2001.

design a true plug-in RC scheme for systems that, besides the reference trajectory, require a feedforward term. A notable example in the industrial field is represented by the control of electrical motors, that, in order to obtain high dynamics, is based on velocity/acceleration feedforward compensations besides the standard cascade control structure, see Fig. 4.1.

Because of the introduction of the CZPETC the control scheme strongly depends on the knowledge of the plant model. For this reason, it is of great interest to investigate how the accuracy of the model influences the performance of the overall algorithm.

Accordingly, the theoretical results reported in the conference paper Biagiotti, Califano, and Melchiorri, 2016 are extended in this chapter with a number of experimental results, that clearly show the improvement in the B-spline based RC due to the introduction of the CZPETC. For the sake of completeness, a non-minimum phase mechanical system obtained with variable stiffness actuators has been considered as benchmark.

In the following an overview of the main elements that are used in the final control scheme is given.

4.2 Overview of the Used Techniques

The proposed control scheme descends from the integration of three different different technologies, that are briefly summarized in this section.

4.2.1 The Continuous Zero Phase Error Tracking Controller

The CZPETC is a feedforward compensator representing an extension of the complete dynamic inversion for systems characterized by unstable zeros. A generic stable SISO LTI system $G(s)$ can be decomposed as

$$G(s) = \frac{N^s(s)N^u(s)}{D(s)}$$

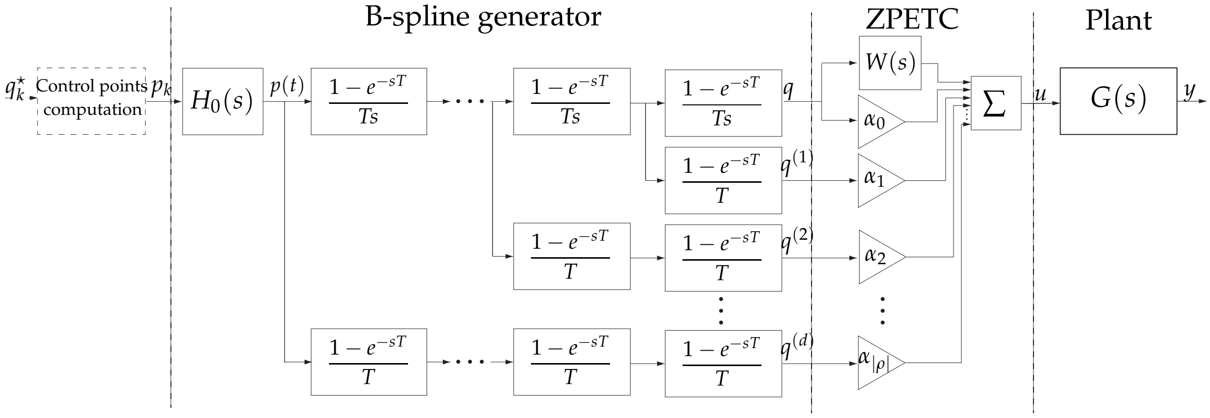


FIGURE 4.2: Block-scheme representation of feedforward control based on B-spline trajectory and CZPETC ($d = |\rho|$) for the plant $G(s)$.

where $D(s)$, $N^s(s)$ and $N^u(s)$ denote respectively the factors corresponding to the n poles, the m_s stable zeroes and m_u unstable zeroes of the plant. The CZPET controller assumes the form

$$R(s) = \frac{D(s)N^u(-s)}{N^s(s)[N^u(0)]^2}. \tag{4.2}$$

In this manner. if $m_u \neq 0$, the cascade $R(s)G(s)$ becomes

$$R(s)G(s) = \frac{N^u(-s)N^u(s)}{[N^u(0)]^2}. \tag{4.3}$$

This means that the residual dynamics of $R(s)G(s)$ is represented by pairs of zeros $\pm z_i$, $i = 1, \ldots, m_u$. Obviously, if $m_u = 0$, i.e. the system is minimum phase, $R(s)$ is a standard feedforward controller that completely cancels the dynamics of the plant ($R(s) = G^{-1}(s)$) and $R(s)G(s) = I$.
Since unstable zeros cannot be canceled or modified either with a feedforward regulator, for clear stability reasons, or with a feedback controller, the use of CZPETC represents a solution in those applications where phase shift is very critical, like e.g. (4.1). In fact, the function $R(s)G(s)$ in (4.3) is characterized by the following properties.

1. $|R(j\omega)G(j\omega)| \approx 1$ for $\omega < \omega^\star$, being $\omega^\star$ the break frequency corresponding to the smallest unstable zero. The asymptotic slope of the Bode plot for magnitude is $+2m_u \times 20$ db/decade.

2. $\angle R(j\omega)G(j\omega) = 0$ rad, $\forall\omega$, since, for any unstable zero, the cascade $R(s)G(s)$ contains also its opposite, and therefore their phase contributions compensate each other.

The major drawback of CZPETC, and in general of inversion-based feedforward approaches, concerns the practical implementation of the controller, which is generally non-causal. In fact, if γ is the (nonnegative) relative degree of the plant $G(s)$, the relative degree of $R(s)$ is $\rho = m_s - (n + m_u) = -\gamma - 2m_u$ which is strictly negative, unless $\gamma = m_u = 0$. A way to cope with this problem is based on the *precognition of the reference signal*. By diving numerator and denominator, $R(s)$ can be rewritten as

$$R(s) = \sum_{i=0}^{|\rho|} \alpha_i s^i + W(s)$$

where $W(s)$ is a strictly proper transfer function. Consequently, the control action of the ZPETC will result

$$U(s) = \sum_{i=0}^{|\rho|} \alpha_i s^i Y_r(s) + W(s)Y_r(s) \tag{4.4}$$

where $Y_r(s)$ denote the Laplace transform of the reference input, and in time domain

$$u(t) = \sum_{i=0}^{|\rho|} \alpha_i \frac{d^i y_r(t)}{dt^i} + \mathcal{L}^{-1}\{W(s)Y_r(s)\}. \tag{4.5}$$

Therefore, in order to compute the control signal $u(t)$ the knowledge of $y_r(t)$ and its first $|\rho|$ derivatives is necessary. Obviously, in order to guarantee the feasibility of $u(t)$, the $|\rho|$ derivatives must be limited and consequently $y_r(t) \in C^{|\rho|-1}$.

4.2.2 Set-point generation via B-spline filters and integration with the CZPETC

In many practical applications, smooth reference signals are defined using spline functions interpolating a set of desired via-points $q_k^\star$, $i = 0, \ldots, n-1$. In the following uniform B-spline curves are considered, i.e. splines in the so-called B-form and characterized by an equally-spaced distribution of the knots, and defined by

$$q_u(t) = \sum_{k=0}^{n-1} p_k \, B^d(t - kT), \qquad 0 \le t \le (n-1)T \tag{4.6}$$

where T is the constant distance between knots Biagiotti and Melchiorri, 2008. A novel way to generate spline curves was developed by the authors in (Biagiotti and Melchiorri, 2010), where it is proven that a uniform B-spline trajectory of degree d can be generated by means of a chain of d dynamic filters defined as

$$M(s) = \frac{1 - e^{-sT}}{Ts}$$

fed by the staircase signal $p(t)$ obtained by maintaining the value of each control point p_k defining the curve for the entire period $kT \le t < (k+1)T$, by means of a zero-order hold $H_0(s)$ applied to the sequence of impulses p_k of period T. The degree d of the spline and therefore the number of filters composing the B-spline generator determines the smoothness of the output trajectory. In fact the resulting spline is a function of class C^{d-1}. A fundamental property of the filter for B-spline generation, is the possibility to compute online the profiles of all the time derivatives of the trajectory up to the order d, as shown in Fig. 4.2. For this reason the implementation of (4.5) by means of the B-spline generator of Fig. 4.2 is straightforward provided that the order of the B-spline meets the condition

$$d \ge |\rho| = \gamma + 2m_u \tag{4.7}$$

In order to guarantee a good tracking of the B-spline function defined by the control points p_k it is necessary that the frequency spectrum of the trajectory is included in the bandwidth of $P(s)$. Since, the B-spline is obtained by applying a train of impulses to the cascade of filters

$$H_0(s)M^d(s), \tag{4.8}$$

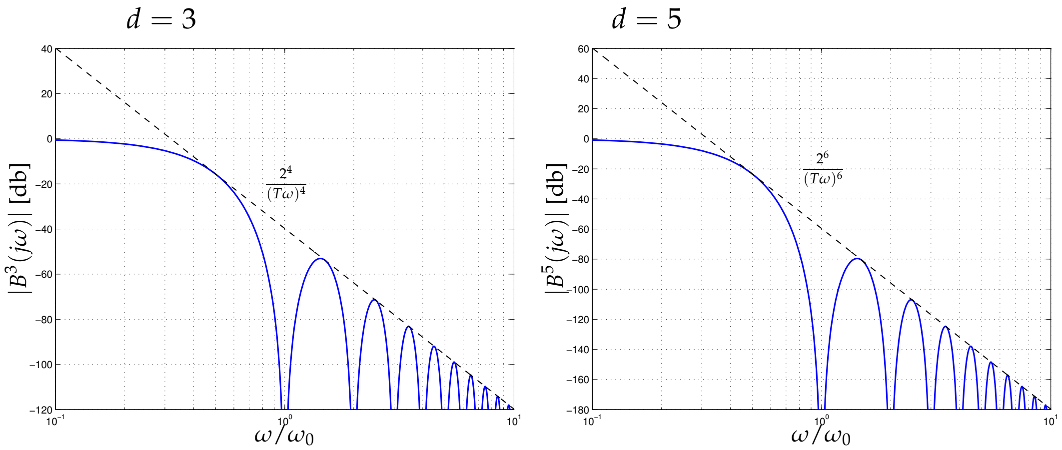


FIGURE 4.3: Frequency spectrum of the cubic ($d = 3$) and quintic ($d = 5$) B-spline filter as a function of a normalized frequency ω/ω_0 with $\omega_0 = \frac{2\pi}{T}$.

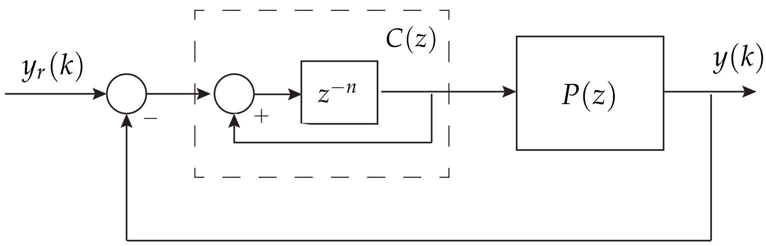


FIGURE 4.4: Discrete-time repetitive control.

its spectrum can be deduced by considering the frequency response of (4.8), i.e.

$$B^d(j\omega) = \left(\frac{\sin\left(\frac{\omega T}{2}\right)}{\frac{\omega T}{2}} \right)^{d+1} e^{-j\omega\frac{(d+1)}{2}T}.$$

The B-spline filter/signal is characterized by a pure delay of $\frac{(d+1)T}{2}$ seconds, while the magnitude decreases as $1/\omega^{d+1}$. By looking at the frequency spectra of cubic and quintic B-splines that are shown in Fig. 4.3, it can be seen that spectral components for $\omega \geq \omega_0 = \frac{2\pi}{T}$ can be neglected, in particular for higher values of d. This means that good tracking performances can be achieved if $\omega^\star > \omega_0$, where $\omega^\star$ denotes the break frequency of the smallest unstable zero of the CZPET controlled plant $P(s)$. Finally, is worth noticing that the slope of $|B^d(j\omega)|$ for $\omega \to \infty$ is $-(d+1) \times 20$ db/decade. As a consequence, being $d \geq 2m_u$ (see property 1 in subsection 4.2.1) the cascade of the trajectory generator and $P(s)$ will have a negative slope for high frequency values and will be always limited in magnitude.

4.2.3 Discrete-time Repetitive Control

Discrete-time repetitive control is a technique used to achieve output regulation for discrete-time periodic signals of known and fixed period, whose efficiency is based on the *internal model principle*. The basic scheme of RC in the discrete-time domain is shown in Fig. 4.4. It is composed by:

- a linear time invariant controlled plant $P(z)$;

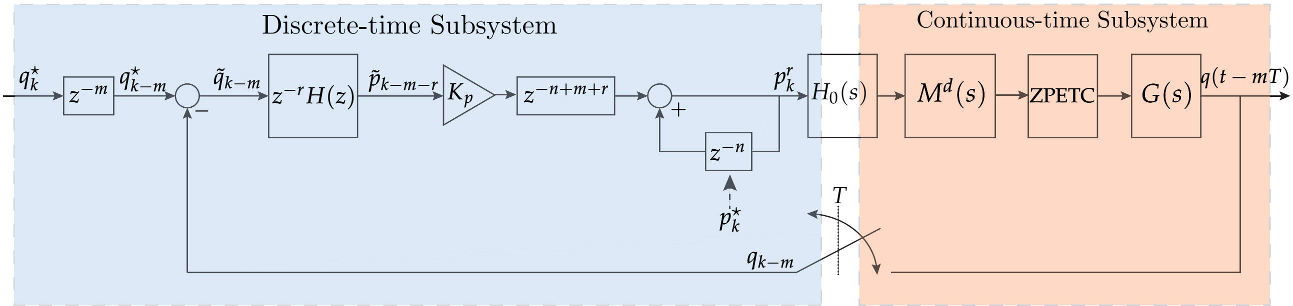


FIGURE 4.5: Discrete-time repetitive control scheme based on dynamic B-spline filter.

- an internal model based controller $C(z)$, that guarantees asymptotically zero tracking error of any digital periodic reference signal of period n, i.e. $y_r(k+nT) = y_r(k)$. Here T is the sampling time of the digital system[1].

The marginally stable system $C(z)$ contains the poles on the unit circle representing the modes whose linear combination is able to generate any digital periodic signal of period n. Thus, using internal-model based arguments, the main issue of this control approach consists in the choice of $P(z)$ that assures the stability of the closed-loop system. Stability condition for the RC scheme can be easily derived by means of e.g. classical Nyquist analysis and it results to be

$$|P(e^{j\omega T}) - 1| < 1, \ \ \forall \omega \in [0, \pi/T]. \tag{4.9}$$

Once this condition is satisfied, asymptotic perfect tracking for any reference signal of period n is achieved. Notice that, how explained in Chapter 1, differently with respect to continuous-time RC schemes also plants $P(z)$ that are not strictly proper are suitable to be stabilized in the closed-loop system. This follows because in the digital framework a periodic reference signal contains a finite number of harmonics, making the internal model based controller $C(z)$ finite-dimensional.

4.3 The B-spline CZPET Repetitive Controller

In order to eliminate the tracking error due to the open-loop structure of CZPETC, the overall system shown in Fig. 4.2, including a block for the computation of the control points from the desired via-points, has been embedded in the RC scheme, as shown in Fig. 4.5. Notice that the digital subsystem is characterized by the sampling time T, which is equal to the temporal distance among the desired via-points. In the following, a stability analysis of this scheme is performed. Perfect asymptotic tracking for periodic signals follows then by the properties of discrete-time RC.
The scheme is completely equivalent to the basic structure reported in Fig. 4.4, if one assumes that

$$P(z) = K_p \cdot H(z) \cdot z^m \cdot HMRG(z) \tag{4.10}$$

where

$$HMRG(z) = \mathcal{Z}\{H_0(s)M^d(s)R_{\text{ZPET}}(s)G(s)\}$$

[1] Notice that the same notation is used for the distance between knots in (4.6). This is consistent with the overall control scheme that is designed in this work.

denotes the z-transform of the continuous time system composed by B-spline generator, CZPET controller and plant, K_p is a proportional gain and, finally,

$$H(z) = \sum_{n=-r}^{r} h(n)\, z^{-n} \tag{4.11}$$

is a FIR filter that approximates the relationship between the via-points $q_k^\star$ and the control points p_k (details about this filter can be found in Biagiotti and Melchiorri, 2013 and Biagiotti, Moriello, and Melchiorri, 2015). Note that $H(z)$, characterized by $h(n) = h(-n)$, has a frequency response $H(e^{j\omega T})$ which is a positive real function of ω and whose argument is therefore null in the overall frequency range.
The RC scheme guarantees asymptotically a perfect interpolation of the via-points if $P(z)$ complies with (4.9). In particular, this condition can be met only if

$$\angle P(e^{j\omega T}) < \frac{\pi}{2}\text{rad}, \quad \omega \leq \frac{\pi}{T} \tag{4.12}$$

and this explains the role of the CZPET controller. The property 2 reported in Sec. 4.2.1, and the frequency response of the trajectory generator along with the zero-order hold guarantee

$$\angle H_0(j\omega) M^d(j\omega) R_{\text{ZPET}}(j\omega) G(j\omega) = -m\omega T.$$

Therefore, as already mentioned in Sec. 4.2.2 the continuous system is characterized by a pure delay of $m = \frac{d+1}{2}$ sample periods T caused by the trajectory generator, while the phase contribution of $R_{\text{ZPET}}(j\omega)G(j\omega)$ is null. The corresponding discrete-time system $HMRG(z)$ will have the same pure delay, and can be written as

$$HMRG(z) = z^{-m} L(z) \tag{4.13}$$

where $L(z)$ is a zero-phase filter. In Fig. 4.6 the typical frequency response of the system $HMRG(z)$ obtained by discretization, with sampling frequency $\omega_0 = 2\pi/T$, is shown and compared with the original system.
By substituting (4.13) in (4.10) the expression

$$P(z) = K_p \cdot H(z) \cdot L(z)$$

is obtained. Therefore, also $P(z)$ is a zero-phase filter. Moreover, since

$$0 < |H(e^{j\omega T})| \cdot |L(e^{j\omega T})| < \infty, \;\; \omega \leq \frac{\pi}{T}$$

by acting on K_p is always possible to impose $|P(e^{j\omega T})| \in]0, 2[$, i.e. inside the stability region. In many circumstances, e.g. when $\omega^\star > \omega_0$, $|H(e^{j\omega T})| \cdot |L(e^{j\omega T})| \approx 1$ in the overall frequency range and consequently $K_p = 1$. Note that if the plant is minimum phase and therefore its dynamics is fully cancelled by the CZPET controller $|H(e^{j\omega T})| \cdot |L(e^{j\omega T})| = 1$ in the overall frequency range.

Note that in (4.10) a time-anticipation z^m appears, but this is only due to analysis purposes and in the original scheme of Fig. 4.5 no anticipations are present. Moreover, the zero-phase filter $H(z)$, which is non-causal, in this scheme is delayed by r samples in order to make it feasible.

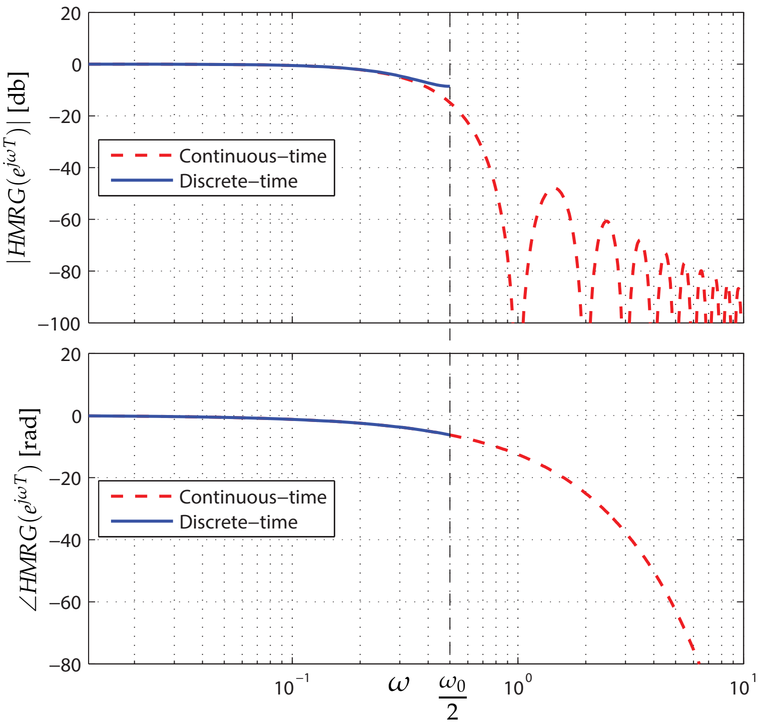


FIGURE 4.6: Frequency response of a discrete-time system $HMRG(z)$ obtained by sampling (with period T) the continuous-time system $H_0(s)M^d(s)R_{\text{ZPET}}(s)G(s)$ with $d = 3$ and $G(s)$ with an unstable zero.

4.4 Experimental evaluation

One of the goals of this work is to answer to a simple but fundamental question arising when a new control algorithm is proposed, i.e. "What modeling information is needed to design and tune the controller?" (Bernstein, 1999). For this reason, an extensive experimental activity, based on a nonminimum phase system, has been performed.

4.4.1 System description and modelling

As shown in Fig. 4.7, the experimental setup is basically a two-dof robotic manipulator with elastic joints built with QBMove - Maker Pro Variable Stiffness Actuators (VSA) by QBRobotics (*QBMove - maker pro datasheet*). These actuators implement the concept of variable stiffness servo-motors, i.e. motor units that allow the user to command both the position and the stiffness of the output shaft with external signals. For this reason, these actuators are very suitable for rapid prototyping robotic systems with variable stiffness joints (Catalano et al., 2011).
QBMove VSAs are provided with an easy to use Matlab/Simulink toolbox that can runs without particular restriction even on standard operating system and communicates with the actuators via USB. In the experiments reported in this section Matlab was running with a fixed step size $T_s = 2$ ms. For this reason, the B-spline trajectory generator (and the CZPETC) has been discretized as in Biagiotti, Moriello, and Melchiorri, 2015.
Obviously, the system of Fig. 4.7 can be modelled as a standard manipulator with

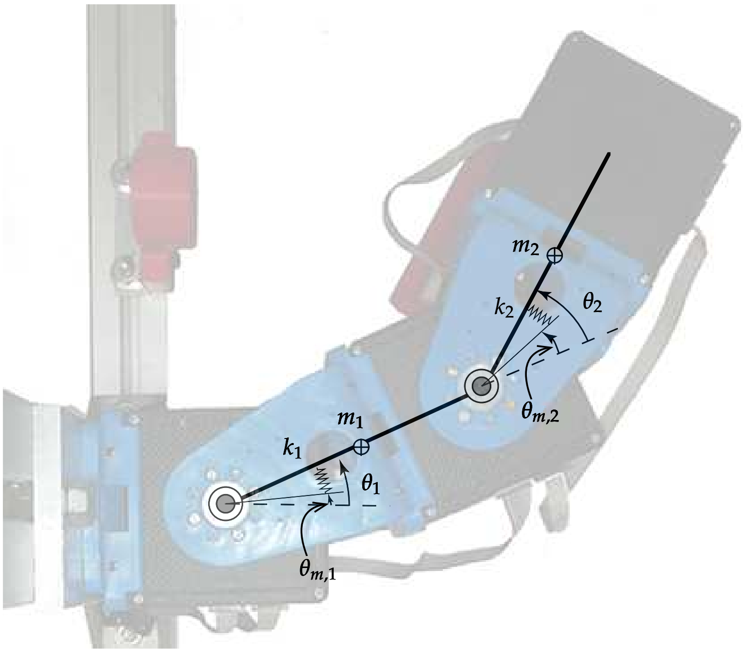


FIGURE 4.7: Two-link robotic arm with elastic joints based on QB-Move - Maker Pro VSAs.

the elastic joints commanded in position, i.e.

$$M(\theta)\ddot{\theta} + \mathbf{C}(\theta,\dot{\theta})\,\dot{\theta} + g(\theta) + K\,(\theta - \theta_m) + B\,(\dot{\theta} - \dot{\theta}_m) = 0$$

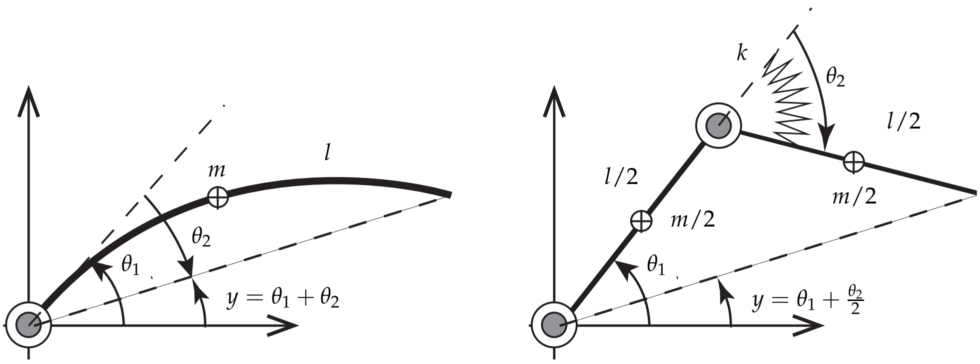
where $M(\theta)$, and $\mathbf{C}(\theta,\dot{\theta})$ are the inertia and centrifugal/Coriolis forces matrices, $g(\theta)$ represents the gravity term, $K = \mathrm{diag}\{k_i\}$, $B = \mathrm{diag}\{b_i\}$, $i = 1,2$ are the matrices of the transmission stiffness and viscous friction, θ and θ_m denote the vector of the joint positions at the link side and at the motor side, respectively. Because of the planar structure of the device, which is disposed in the horizontal plane, $g(\theta) = 0$. Moreover, it is imposed that the second joint behaves passively, by assuming that $\theta_{m,2} \equiv 0$. In this manner, the system emulates the dynamics of a planar flexible link in which the driven torque is $\tau = k_1(\theta_1 - \theta_{m,1})$, see Fig. 4.8(a).

As a matter of fact, the simplest lumped-parameters model of a flexible link, that takes into account only the first elastic mode, is the two-link system with an elastic hinge shown in Fig. 4.8(b). By assuming the tip angular position seen from the base as output variable, i.e. $y = \theta_1 + \frac{\theta_2}{2}$, it has proved in Luca, Lanari, and Ulivi, 1991 that the model of this system is non-minimum phase, characterized by order four and relative degree two. Therefore, by considering an additional pole, which takes into account the dynamics of the actuator at the base joint, it is reasonable to assume that the systems can be modeled by an LTI system with 5 poles and 2 zeroes.

Concerning the non-minimum phase nature of the system, its zero dynamics associated with any reasonable definition of an output is unstable regardless the accuracy of the model, whether it is chosen to be nonlinear or linear, infinite or finite-dimensional, with any number of elastic modes (Luca, Lanari, and Ulivi, 1991).

4.4.2 Experimental results

These manipulators present small and light actuators and are amenable to be composed in a modular way. The possibility of setting their stiffness via software allows



(a) (b)

FIGURE 4.8: One-link flexible arm (a) and equivalent model with the deformation concentrated in an elastic hinge (b).

to reproduce the system of Fig 4.8(b) with its non-minimum phase behaviour. In particular the experiments have been computed by setting a high stiffness value for the first actuator (13 Nm/rad) and a low stiffness value for the second one (0.5 Nm/rad). Since the manipulators are controlled in position, the input has been chosen to be the reference position of the first actuator θ_1^{ref} and the output the tip angular position $y = \theta_1 + \frac{\theta_2}{2}$. The reference of the second actuator θ_2^{ref} is kept constant to zero for the whole duration of the experiment, in order to reproduce the behaviour of a low-stiffness unactuated linear spring. The control scheme shown in Fig. 4.5 has been implemented in Matlab/Simulink. The ZPET controller has been designed on the basis of the the transfer function identified through the *identification toolbox* of Matlab by using the step response of the system as input data and other exciting dynamics that have been implemented as validation data. The performed identification confirmed the non-minimum phase nature of the system. In particular three different transfer function models have been identified for the controlled system which differ from the number of poles and zeroes that the user can select as input of the identification procedure. In particular

- $G_{5p2z}(s) = \frac{-19836(s-18.7)(s+5.6)}{(s+51.38)(s^2+1.96s+37.8)(s^2+10.3s+1083)}$
- $G_{3p2z}(s) = \frac{-17.6(s-6.9)(s+2.5)}{(s+16.75)(s^2+1.68s+17.8)}$
- $G_{2p1z}(s) = \frac{12.1(s+3.1)}{(s^2+1.81s+37.6)}$

are the transfer functions with respectively 5 poles and 2 zeroes, 3 poles and 2 zeroes and 2 poles and 1 zero.. Notice that the non-minimum phase nature of the system has been captured by the models $G_{5p2z}(s)$ and $G_{3p2z}(s)$ but not by $G_{2p1z}(s)$. This fact is helpful to highlight the effectiveness of the proposed method. In particular four sets of experiments have been performed: three with the model-based technique based on the ZPET compensator (using as models the three identified transfer functions) described in this paper, and one without ZPETC, reproducing basically the control scheme described in Biagiotti, Moriello, and Melchiorri, 2015. Every experiment starts with four open loop cycles (highlighted in grey in the plots) and then the loop is closed and repetitive control starts to act modifying the control points of the reference B-spline. The following considerations can be carried out:

Model \ T[s]	1	0.5	0.4	0.3	0.2
NO ZPETC	S	U	$\star$	$\star$	$\star$
$G_{2p1z}(s)$	S	S	U	$\star$	$\star$
$G_{3p2z}(s)$	S	S	S	S	U
$G_{5p2z}(s)$	S	S	S	S	U

TABLE 4.1: Stability Analysis of Experiments with $K_p = 1$

- The experiments performed without ZPETC, i.e. relying on the decentralized internal controller of the system only, show a stable behaviour for slow reference trajectories ($T = 1$ s) (Fig. 4.9(a)) because the internal controller manages to accomplish the RC stability condition for the range of frequencies involved in this case (e.g. the non-minimum phase dynamics are negligible at this speed). Nevertheless as soon as the reference trajectory becomes faster ($T = 0.5$ s) (Fig. 4.9(a)) the overall system becomes unstable, showing the limits of the model-free approach described in Biagiotti, Moriello, and Melchiorri, 2015 when non-rigid industrial manipulators are controlled.

- The experiments performed with the ZPETC show a better behaviour with respect to the model-free case that improves more if a better model is used for the ZPETC implementation. In this respect see Fig. 4.10,4.13 and 4.16 for the stable cases. The improvement with respect to the model-free approach is evident and Tab. 4.1 ('S' stands for stable and 'U' for unstable) confirms the effectiveness of the proposed method showing that the more the model is precise the better become the chances to stabilize the system. It is evident that the non-minimum phase models give better results aiming at stabilizing the control scheme, with $K_p = 1$, till the case $T = 0.3$ s. Fig. 4.12 and 4.15 together with Tab. 4.2 ('MS' stands for marginally stable, i.e. when empirically the error did neither converge nor diverge) show the stabilized cases when $T = 0.2$ s by reducing the gain K_p and highlight the fact that the model $G_{5p2z}(s)$ is more reliable for high frequencies than $G_{3p2z}(s)$.

- Fig. 4.11, 4.14 and 4.17 show a detail of the first and last cycle of one of the stable cases specified in the caption. It can be noticed that the interpolation error goes actually to zero through the modification of the reference B-spline trajectory that is performed by the RC scheme cycle after cycle. Furthermore a comment can be done for what concerns the intersample ripple of the error, which is considered to be an open question in discrete time RC schemes. It turns out that a better model (compare e.g. Fig. 4.11 and 4.17) improves the tracking performance of the system also in between the sampling instances since the continuous time ZPETC implementation aims at compensating the internal dynamics in a more reliable way.

Model \ K_p	1	0.5	0.1
$G_{3p2z}(s)$	U	U	MS
$G_{5p2z}(s)$	U	MS	S

TABLE 4.2: Stability Analysis of Experiments with $T = 0.2$

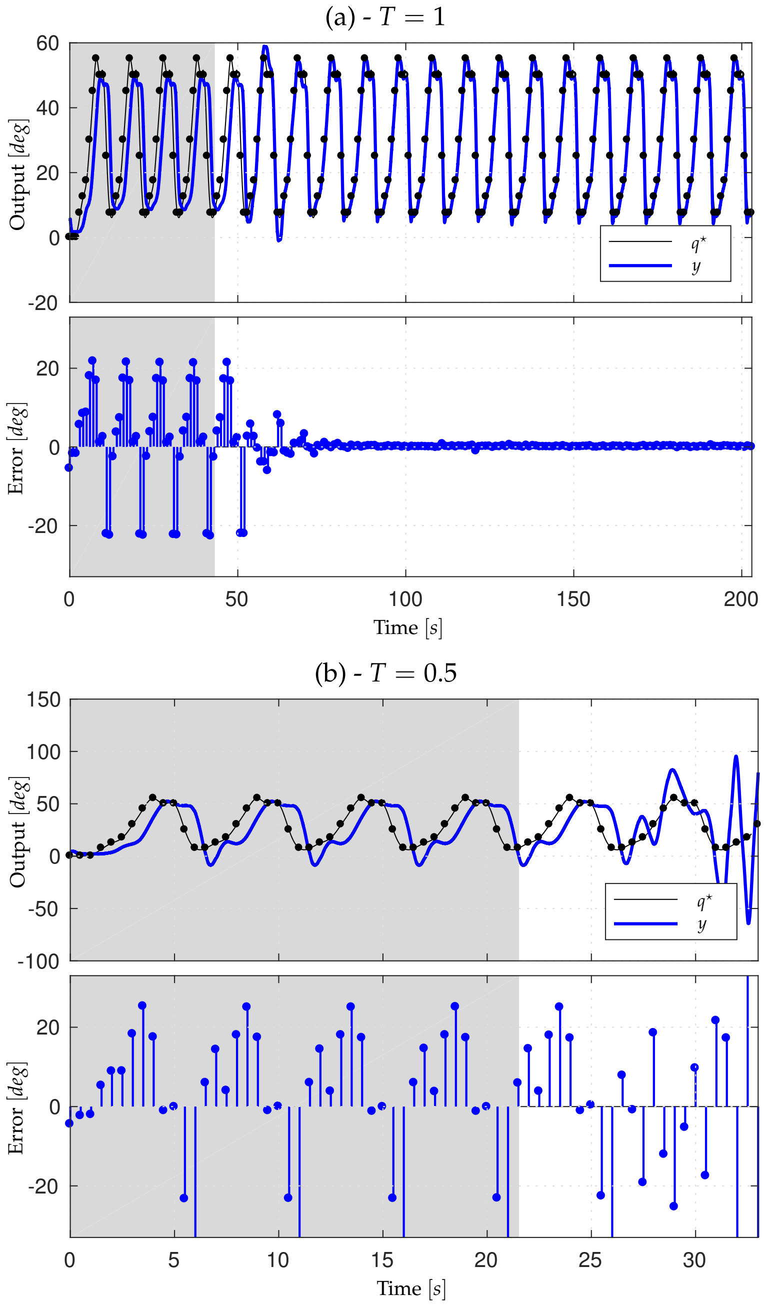


FIGURE 4.9: Output and tracking error without ZPETC, $Kp = 1$

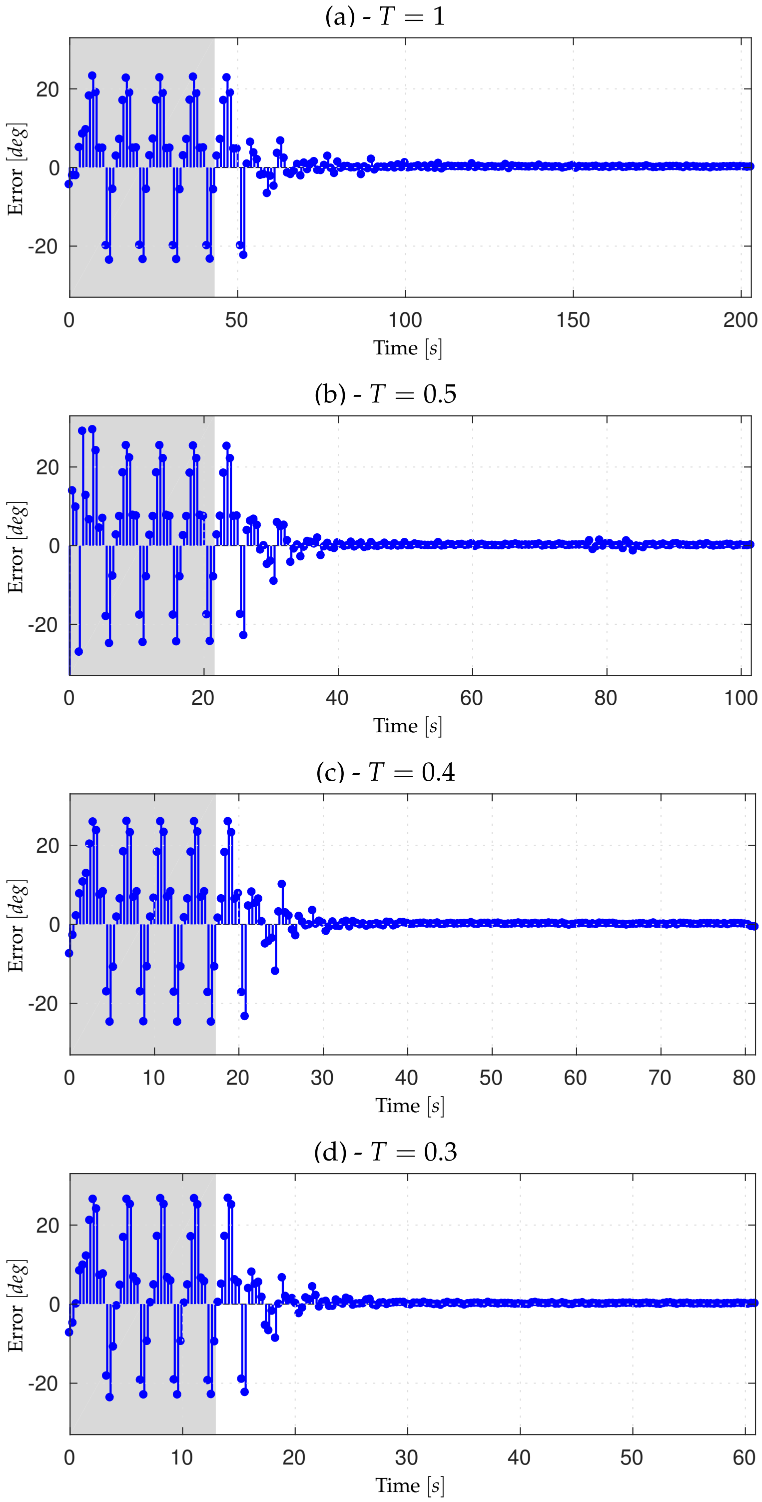


FIGURE 4.10: Tracking error with ZPETC implemented on the basis of the $G_{5p2z}(s)$ model, $Kp = 1$

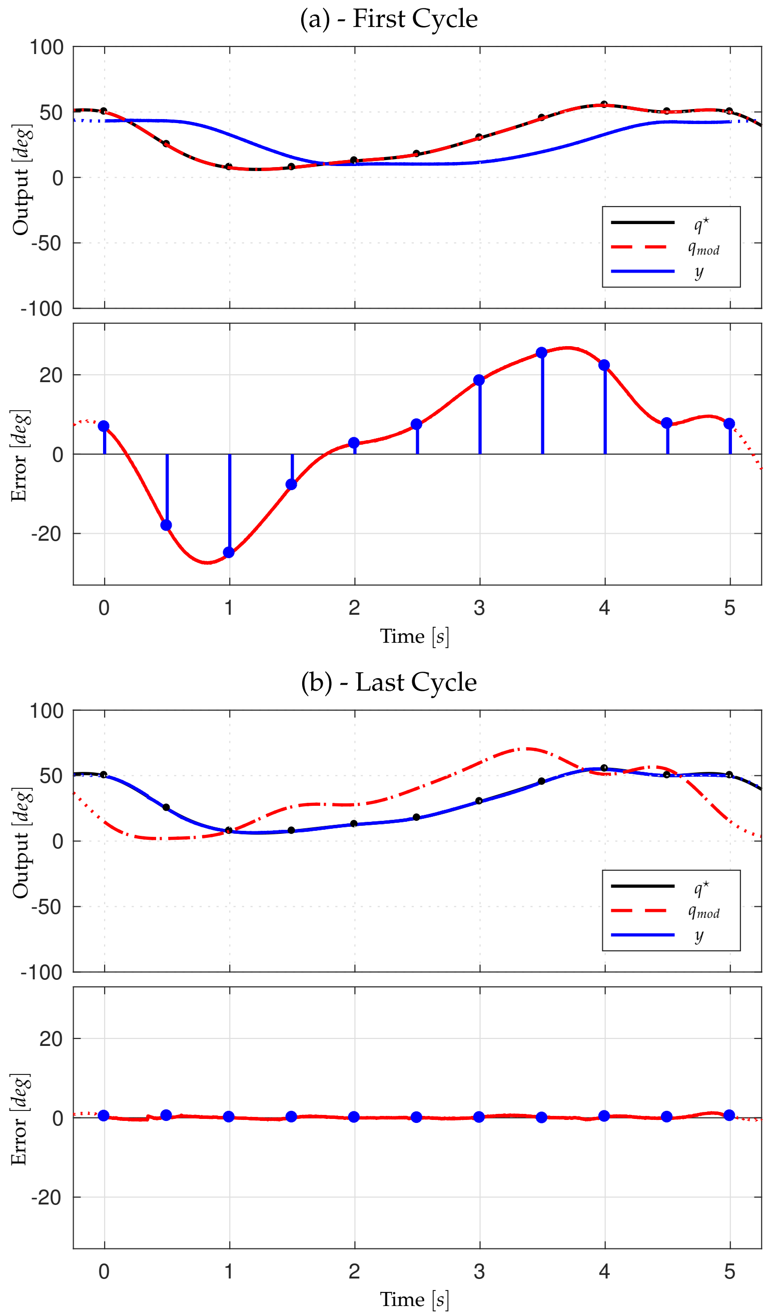


FIGURE 4.11: Time detail of the first and last cycle of experiment corresponding to Fig. 4.10(b)

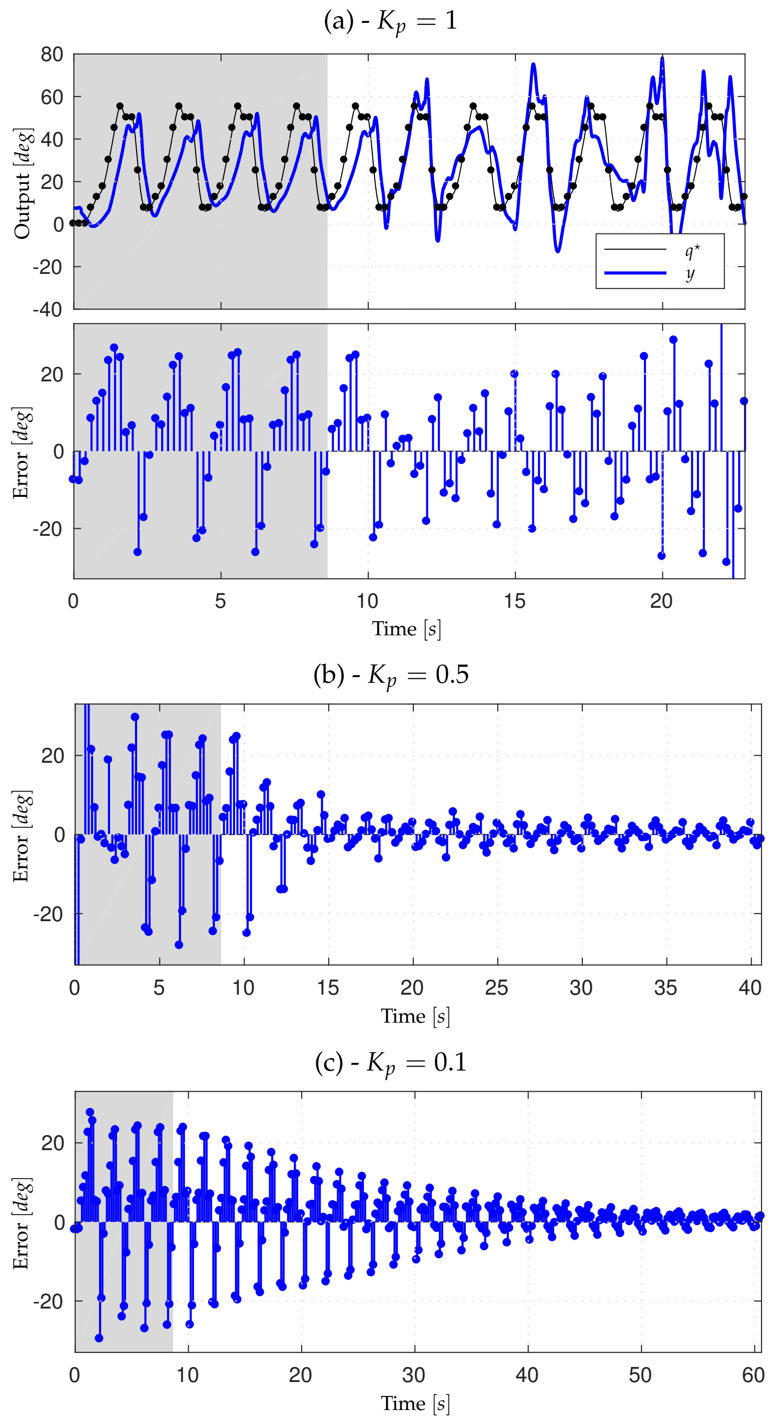


FIGURE 4.12: Output and tracking error with ZPETC implemented on the basis of the $G_{5p2z}(s)$ model, $T = 0.2$

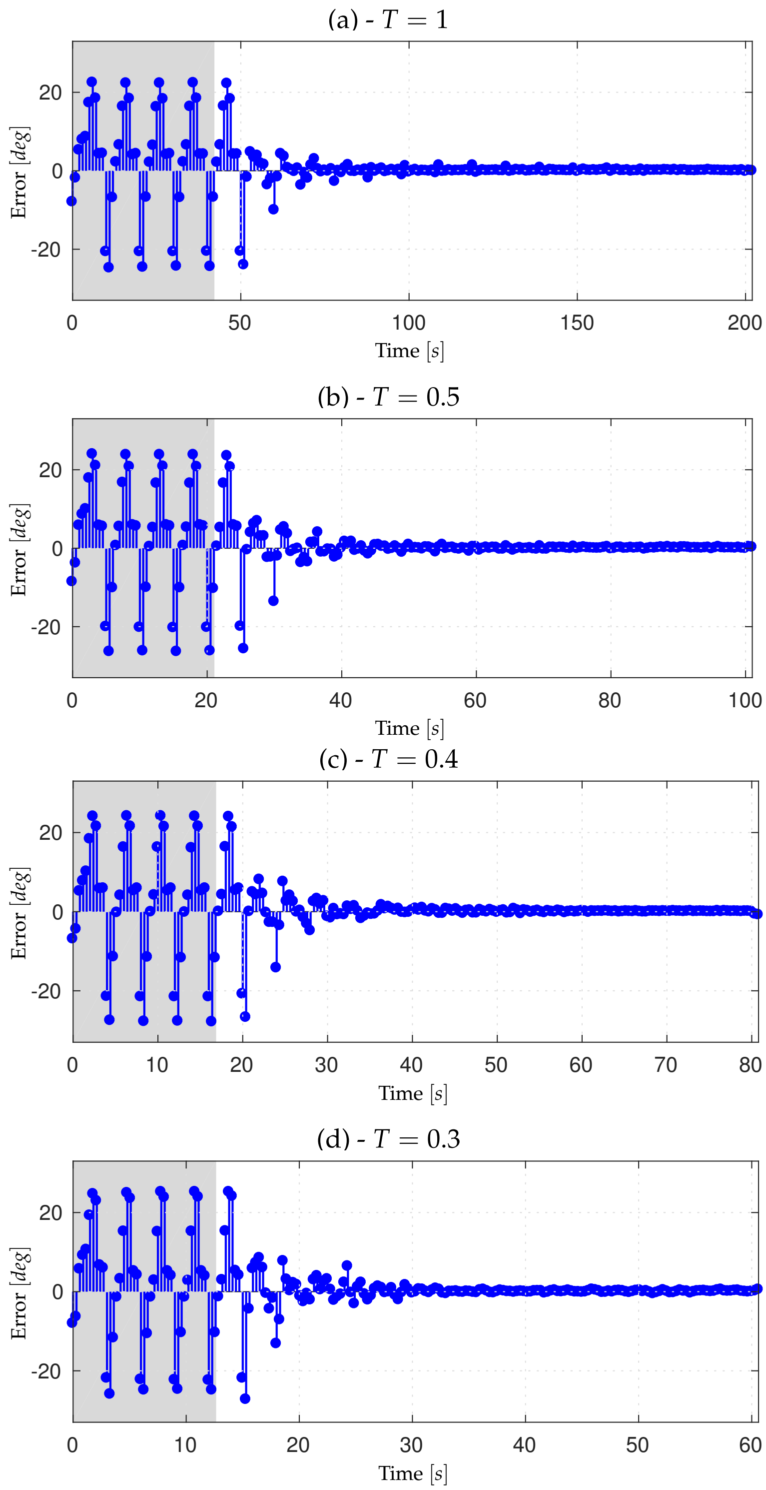


FIGURE 4.13: Tracking error with ZPETC implemented on the basis of the $G_{3p2z}(s)$ model, $Kp = 1$

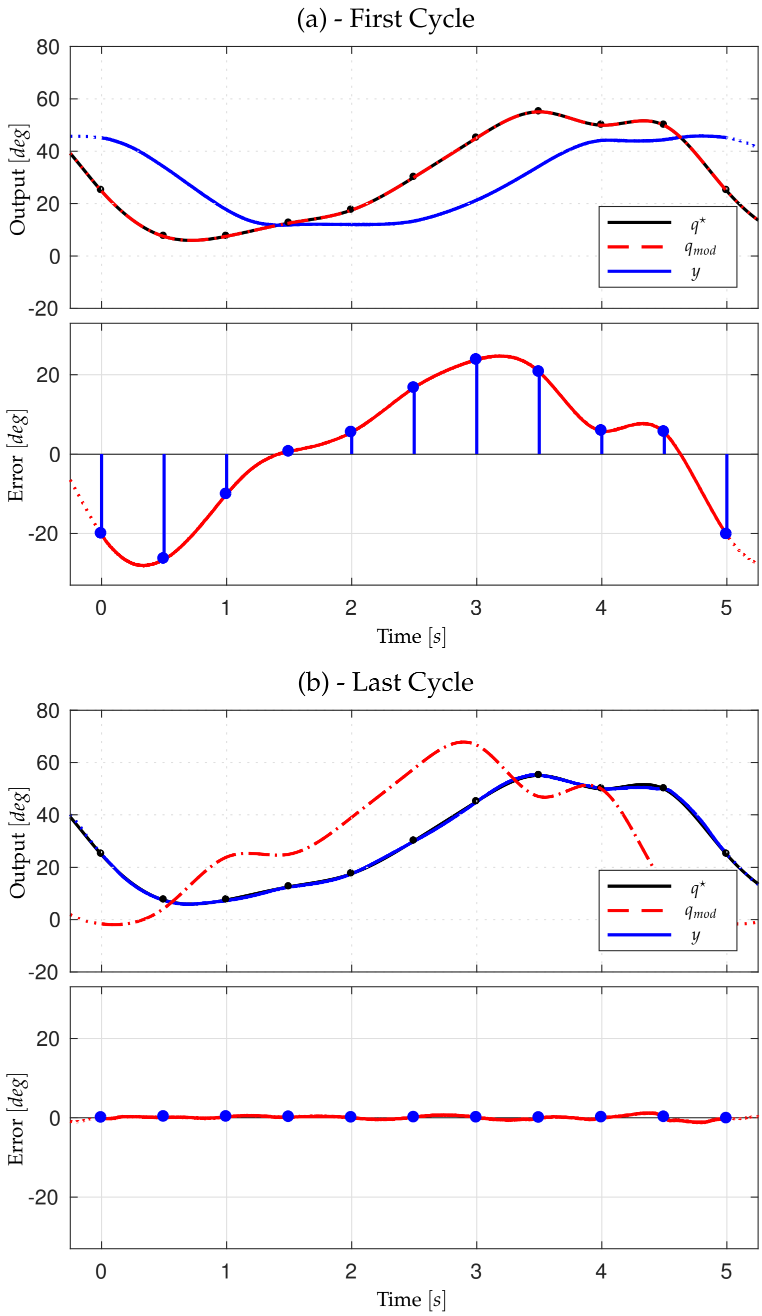


FIGURE 4.14: Time detail of the first and last cycle of experiment corresponding to Fig. 4.13(b)

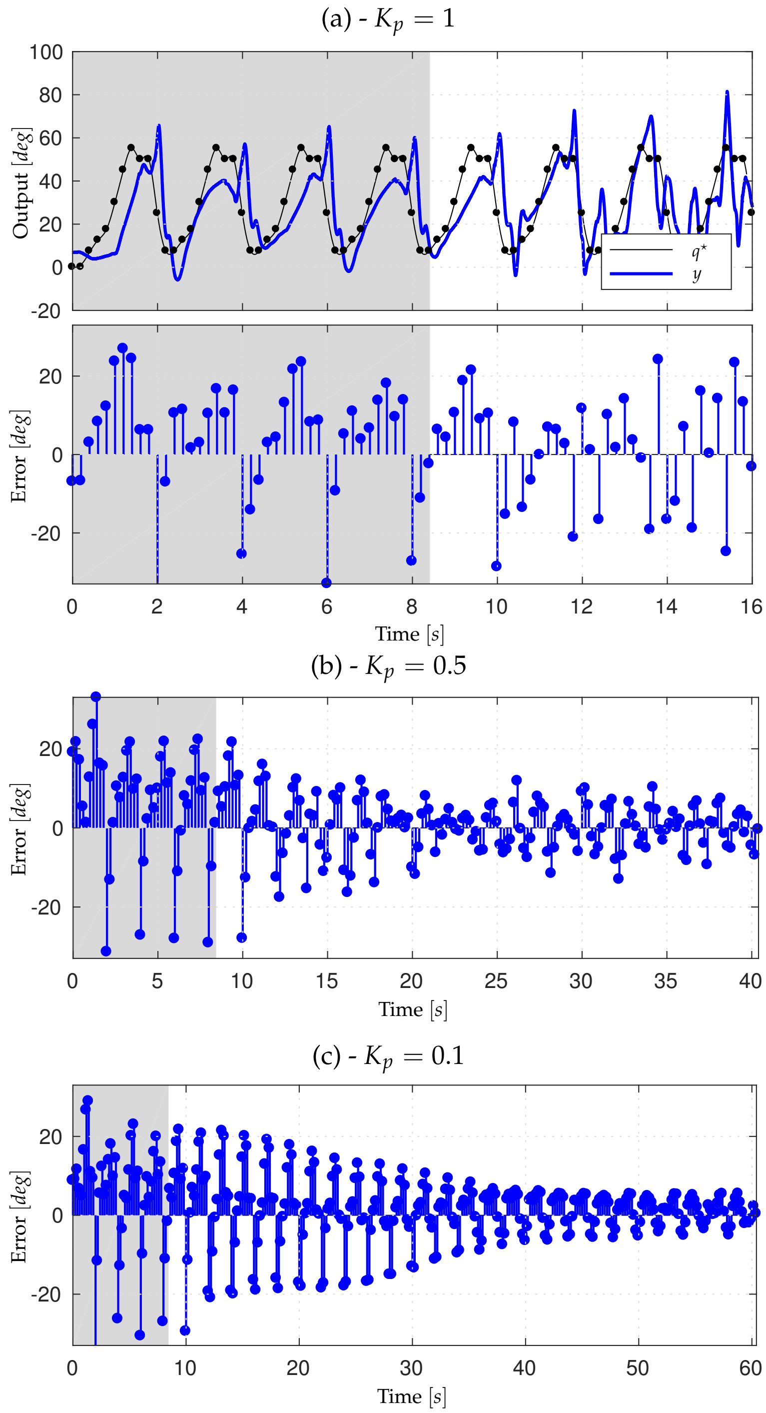


FIGURE 4.15: Output and tracking error with ZPETC implemented on the basis of the $G_{3p2z}(s)$ model, $T = 0.2$

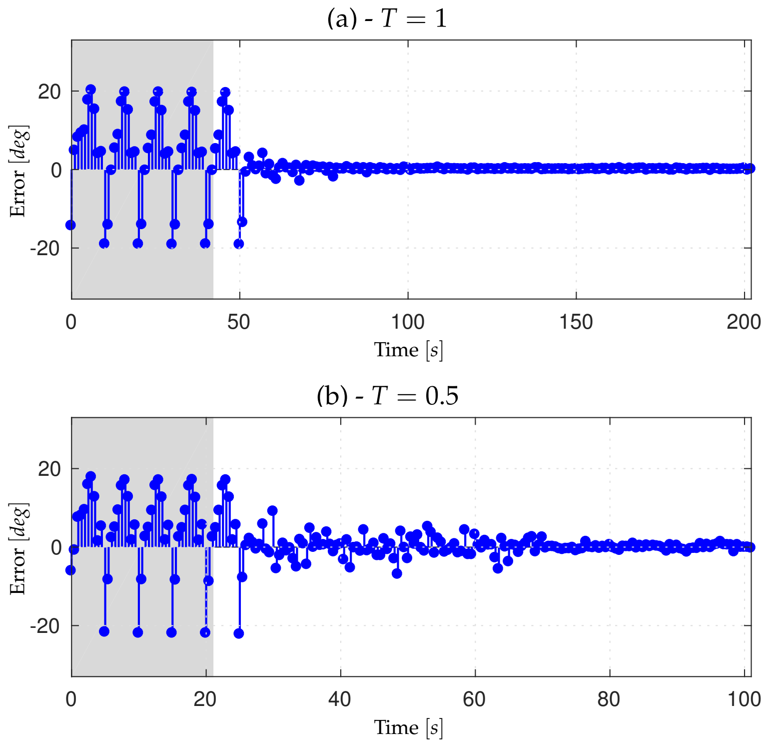


FIGURE 4.16: Tracking error with ZPETC implemented on the basis of the $G_{2p1z}(s)$ model, $Kp = 1$

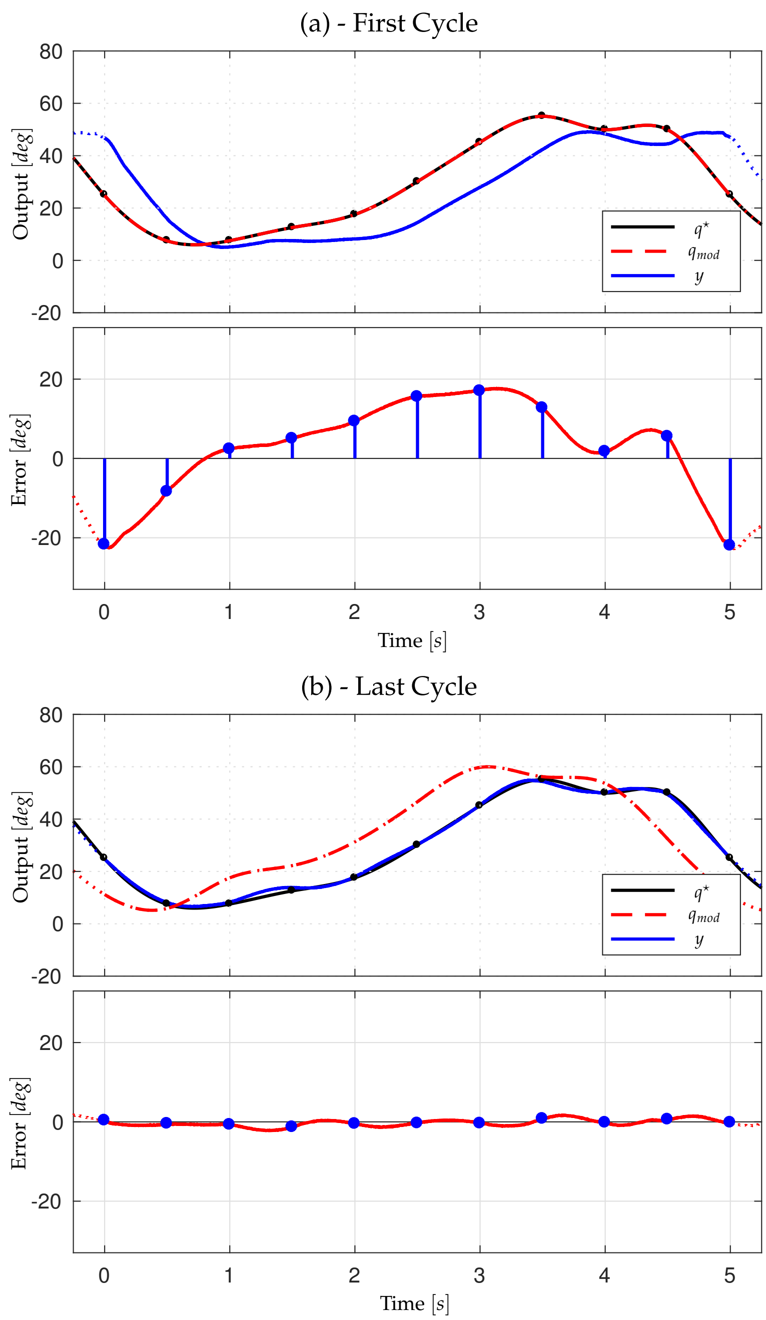


FIGURE 4.17: Time detail of the first and last cycle of experiment corresponding to Fig. 4.16(b)

Chapter 5

Conclusions and Future Work

5.1 Conclusions

In this thesis periodic output regulation has been treated in different fashions based on repetitive control (RC), a technique developed for achieving this issue in a general setting, i.e. for accomplishing the task if a generic periodic reference with known time period has to be tracked.

The novelty of the presented work consists in the mathematical tools that have been used to study aspects of the problem like well-posedness, stabilizability and tracking. In particular a time-domain approach has been adopted, getting rid of the classical frequency domain analysis that was performed in this framework. The used techniques range from infinite-dimensional port-Hamiltonian systems (Chapter 2) to invariance analysis of autonomous discrete-time systems (Chapter 3) and provide a new insight to study the problem in the linear case. Furthermore the proposed approaches represent starting points for attacking the nonlinear problem, that has also been solved in particular situations.

Since from a control theoretic point of view RC systems are in general infinite-dimensional and possibly nonlinear, finer mathematical tools with respect to the classical used ones are necessary to study existence and regularity of solutions (well-posedness) in RC schemes. This represents a crucial aspect to use Lyapunov-like arguments to study stability of such systems. This has been done in Chapter 2, where the port-Hamiltonian approach has also been extended to modified RC systems and to a particular class of nonlinear RC systems. Asymptotic tracking in this setting is proven in a way that does not rely on internal-model as principle to be invoked, but using constructive arguments which are amenable to be extended in the nonlinear case.

In Chapter 3 a more general way to study periodic output regulation is proposed, where the analysis exploits the lifting technique for LTI systems and for static nonlinearities. Continuous-time RC turns out to be a particular case of this analysis that is extended also to the digital framework.

Chapter 4 deals with a specific discrete-time RC schemes which combines different techniques to accomplish perfect tracking of possibly nonminimum phase. Experimental results performed on a QB-move manipulator are presented to show the validity of the proposed scheme, whose stability properties are proven mathematically.

5.2 Future Work

The outcomes of this works presented in this thesis open the possibility to several future developments. The port-Hamiltonian approach together with novel stabilization techniques for distributed parameter systems could lead to wider classes of nonlinear systems for which RC laws can be successfully applied. Furthermore it has been noticed that different parametrizations of the repetitive compensator that fit in the port-Hamiltonian form extend the classes of systems that result in a stable closed-loop RC scheme.

The main limitation of periodic output regulation in a general framework is the fact that using an infinite-dimensional controller it is impossible to stabilize the system unless the controlled plant is biproper. To face this problem MRC systems have been studied using a port-Hamiltonian approach and further results could be obtained by developing the theory in Chapter 3, considering e.g. finite-dimensional exosystems and consequently finite-dimensional controllers, i.e. limiting the class of the periodic references to be tracked. This could lead to the design of controllers that aim at achieving robustly output regulation for strictly proper plants in meaningful circumstances.

The nonlinear case is mainly unsolved and represents a big challenge for future research.

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