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TWO APPLICATIONS OF THE DECOMPOSITION THEOREM TO MODULI SPACES

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Esame finale 2018

Ai miei nonni Rosa e Mirko,

che non hanno mai dubitato che questo giorno arrivasse

e che mi porto dentro ogni giorno.

Contents

Introduction

The decomposition theorem of Beilinson, Bernstein and Deligne is a powerful tool to investigate the topology of algebraic varieties and algebraic maps. Its statement emphasizes the central role played by a relatively new topological invariant, the intersection cohomology of an algebraic variety, or, more generally, of a local system defined on a locally closed nonsingular subset of an algebraic variety.

This invariant, introduced in the late 70's by Goresky and MacPherson to replace cohomology when the variety is singular so as to preserve Poincaré duality turns out to be a building block of the theory of perverse sheaves. Intersection cohomology is a complex of sheaves, and as such it lives in the derived category of constructible sheaves.

The decomposition theorem is a statement about the (derived) direct image of the intersection cohomology by an algebraic projective map. The decomposition theorem and more generally the theory of perverse sheaves have found many interesting applications, especially in representation theory (see [dCM2] for instance). Usually a lot of work is needed to apply it in concrete situations, to identify the various summands. This thesis proposes two applications of the decomposition theorem.

In the first, contained in chapter 2, we consider the moduli space of Higgs bundles of rank 2 and degree 0 over a curve of genus 2. The condition of degree 0 says that the moduli space is singular, while the choice for rank and genus are dictated by the fact that fairly explicit desingularization is known and turns out to be semismall: this is the case where the decomposition theorem has its simplest form. We stratify this space and its resolution. Applying the decomposition theorem, we are able to compute the weight polynomial of the intersection cohomology of this moduli space. This can be useful in view of investigating the so called $P = W$

conjecture for singular moduli spaces, where it is conceivable that the relevant filtrations to be compared will live on intersection cohomology groups.

The second result contained in this thesis is concerned with the general problem of determining the support of a map, and therefore in line with the "support theorem" by Ngô.

We consider families $\mathcal{C} \to B$ of integral curves with at worst planar singularities, and the relative "nested" Hilbert scheme $\mathcal{C}^{[m,m+1]}$. Along the lines of [MS1], and applying the technique of higher discriminants, recently developed by Migliorini and Shende, we prove that in this case, just as in the case of the relative Hilbert scheme and relative compactified Jacobian, there are no supports other than the whole base B of the family. Along the way we investigate smoothness properties of $\mathcal{C}^{[m,m+1]}$, which may be of interest on their own.

Chapter 1

The decomposition theorem

1.1 Preliminaries of Hodge theory

Complex algebraic varieties have provided an important motivation for the development of algebraic topology from its earliest days. On the other hand, algebraic varieties and algebraic maps enjoy many truly remarkable topological properties that are not shared by other classes of spaces and maps. These special features were first exploited by Lefschetz [L1] who claimed to have "planted the harpoon of algebraic topology into the body of the whale of algebraic geometry" ([L2], p.13), and they are almost completely summed up in the statement of the decomposition theorem and of its embellishments. The classical precursors to the decomposition theorem include the two theorems by Lefschetz, Hodge theorem, the Hodge-Riemann bilinear relations and Deligne's Theorem (1.1.1).

There are three known proofs of the decomposition theorem: the original proof by Beilinson, Bernstein, Deligne [BBD] and Gabber [G] is via arithmetic properties of the varieties over finite fields; the second one by Saito [Sa] uses the theory of mixed Hodge modules, while the last one by De Cataldo and Migliorini [dCM2] is via classical Hodge theory.

Standard references for what follows are [GH] and [V].

Let X be a nonsingular complex projective variety of dimension n embedded in some projective space $\mathbb{P}^N$, and let $D = H \cap X$ be the intersection of X with a generic hyperplane $H \subset \mathbb{P}^N$. First let us fix some notation: in the whole chapter, unless specified otherwise, we will consider cohomology with rational coefficients. Up to tensoring with $\mathbb{C}$, we know by De Rham theorem that there exist isomorphisms

$$H^i(X) \cong H^i_{sing}(X) \cong H^i_{dR}(X).$$

The *Lefschetz hyperplane theorem* states that the restriction map $H^i(X) \to H^i(D)$ is an isomorphism for $i < n-1$ and injective for $i = n-1$.
The cup product with the first Chern class η of the hyperplane bundle yields a mapping $\bigcup \eta :$ $H^i(X) \to H^{i+2}(X)$ which can be identified with the composition $H^i(X) \to H^i(D) \to H^{i+2}(X)$, the latter being a "Gysin" homomorphism.
The *Hard Lefschetz theorem* states that for all $0 \leq i \leq n$ the $i-$fold iteration of the cup product with η gives an isomorphism

$$\left(\bigcup \eta\right)^i : H^{n-i}(X) \to H^{n+i}(X).$$

The *Hodge decomposition* is a canonical decomposition

$$H^i(X, \mathbb{C}) \cong \bigoplus_{p+q=i} H^{p,q}(X).$$

The summands $H^{p,q}(X)$ can be thought as cohomology classes of differential (p,q) form (that is those with p dz's and q $d\bar{z}$'s).

This is the example we have to keep in mind when we define what a *rational pure Hodge structure* is.

Definition 1.1.1. Let H be a $\mathbb{Q}$-vector space. A pure Hodge structure of weight i on H is a direct sum decomposition

$$H_{\mathbb{C}} := H \otimes \mathbb{C} = \bigoplus_{p+q=i} H^{p,q}(X), \qquad H^{p,q} = \overline{H^{q,p}}.$$

This is equivalent to give a decreasing filtration $F^\bullet$, called the *Hodge filtration*, such that $F^p(H_{\mathbb{C}}) := \bigoplus_{p' \geq p} H^{p',q'}$. We may also define a *morphism of Hodge structures* as a linear map $f : H \to H'$ whose complexification (still denoted by f) is compatible with the Hodge filtration, i.e. $\mathrm{Im} f \cap F^p(H'_{\mathbb{C}}) = f(F^p(H'_{\mathbb{C}}))$.

For any fixed index $0 \leq i \leq n$ we can define a bilinear form $\mathcal{S}^H$ on $H^{n-i}(X)$ by

$$(a, b) \mapsto \mathcal{S}^H(a, b) := \int_X \eta^i \wedge a \wedge b = \deg([X] \cap \left(\eta^i \cup a \cup b\right)$$

where $[X]$ denotes the fundamental homology class of the naturally oriented X. The Hard Lefschetz theorem is equivalent to the nondegeneracy of the forms $\mathcal{S}^H$. The *Hodge-Riemann bilinear relations* ([dCM, 5.2.1]) establish their signature properties.

1.1.1 Families of nonsingular projective varieties

If $f : X \to Y$ is a $\mathcal{C}^\infty$ fibre bundle with nonsingular projective compact fibre F, let $R^j f_* \mathbb{Q}$ denote the local system on Y whose fibre at a point y is the $H^j(f^{-1}(y))$. We have the associated *Leray spectral sequence*

$$E_2^{i,j} = H^i(Y, R^j f_* \mathbb{Q}) \Rightarrow H^{i+j}(X) \tag{1.1}$$

and the the *monodromy representation*

$$\rho^i : \pi_1(Y, y_0) \to GL(H^i(F)). \tag{1.2}$$

Even when Y is simply connected, the Leray spectral sequence can be nontrivial, for example, the Hopf fibration $f : S^3 \to S^2$.

We define a *family of projective manifolds* to be a proper holomorphic submersion $f : X \to Y$ of nonsingular varieties that factors through some product $Y \times \mathbb{P}^N$ and for which the fibres are connected projective manifolds. By Ehresmann theorem, such a map is also a $\mathcal{C}^\infty$ fibre bundle. The following two results are due to Deligne [D1],[D2].

Theorem 1.1.1. *Suppose $f : X \to Y$ is a family of projective manifolds. Then*

(i) The Leray spectral sequence (1.1) degenerates at the E_2-page and induces an isomorphism

$$H^k(X) \cong \bigoplus_{i+j=k} H^i(Y, \underline{H}^j(F));$$

(ii) the monodromy representation (1.2) is semisimple, i.e. it is a direct sum of irreducible representations.

Item (i) gives a rather complete description of the cohomology of X. Part (ii) is remarkable because often the fundamental group of Y is infinite.

Remark 1. Even though theorem (1.1.1) is given in terms of cohomology groups, Deligne proved a stronger sheaf theoretic statement (see 1.2.1.)

Remark 2. For singular maps, the Leray spectral sequence can be very seldom degenerate. If $f : X \to Y$ is a resolution of singularities of a projective variety Y whose cohomology admits a mixed Hodge structure which is not pure, then the pullback f^* cannot be injective and this prohibits degeneration in view of the edge-sequence.

1.2 Singular varieties: mixed Hodge theory and intersection cohomology

The Lefschetz and Hodge theorem fail if X is singular. There are two somewhat complementary approaches to generalize these statements to singular projective varieties. They involve *mixed Hodge theory* [D2, D3] and *intersection cohomology* [GM, GM1].

Mixed Hodge theory

In mixed Hodge theory the topological invariant studied is the same as that investigated for nonsingular varieties, namely, the cohomology group of the variety, whereas the structure with which it is endowed changes. The (p, q) decomposition of classical Hodge theory is replaced by a more complicated structure. In particular, we have two filtrations: the *weight filtration* on the rational cohomology and the *Hodge filtration* on the complex one.

Definition 1.2.1. Let X be an algebraic variety. A mixed Hodge structure on the cohomology of X is the datum of:

(i) An increasing filtration $W_\bullet$, the weight filtration, on the rational cohomology groups $H^i(X, \mathbb{Q})$

$$\{0\} \subseteq W_0 \subseteq \ldots \subseteq W_{2i} = H^i(X, \mathbb{Q})$$

(ii) a decreasing filtration $F^\bullet$, the Hodge filtration,

$$H^i(X,\mathbb{C}) = F^0 \supseteq F^1 \supset \ldots \supset F^m \supset \{0\}$$

such that the filtrations induced by $F^\bullet$ on the graded pieces $\mathrm{Gr}_k^W := W_k/W_{k-1}$ endows them with a pure Hodge structure of weight k.

Example 1.1. Let X be a rational irreducible curve with one node (topologically this is a pinched torus). Then $H^1(X)$ has weight 0 and all the classes are of type $(0,0)$.

The definition of weights is due to Deligne [D2] and involves reduction to positive characteristic. However, the so called "Yoga of weights" tells us two fundamental restrictions on the weights:

1. if X is nonsingular, but possibly noncompact, then the weight filtration on $H^i(X)$ starts at W_i, that is $W_rH^i(X) = 0$ for any $r < i$;

2. If X is compact, but possibly singular, then the weight filtration on $H^i(X)$ ends at W_i, that is $W_rH^i(X) = H^i(X)$ for any $r \geq i$.

Intersection cohomology

In intersection cohomology, by contrast, is the topological invariant which is changed, whereas the (p,q)-decomposition turns out to have the same formal properties. We are going to describe intersection cohomology in the next section. For a beautiful introduction with also an historical point of view we refer to [Kl]. For now, let us just say that the intersection cohomology groups $IH^*(X)$ can be described using geometric "cycles" on the possibly singular varieties X and this gives a concrete way to compute simple examples. There is a natural isomorphism $H^i(X) \to IH^i(X)$ which is an isomorphism when X is nonsingular. Moreover these groups are finite dimensional, satisfy Mayer-Vietoris theorem and Künneth formula. Even though they are not homotopy invariant, they satisfy analogues of Poincaré duality and Hard Lefschetz theorem. The definition of intersection cohomology is very flexible as it allows for twisted coefficients: given a local system $\mathcal{L}$ on a locally closed nonsingular subvariety Y of X we can define the cohomology groups $IH(\overline{Y},\mathcal{L})$.

Example 1.2. Consider the nodal curve X of example (1.1): $H^1(X)$ has dimension 1, therefore it cannot admit a Hodge decomposition. If one considers the intersection cohomology group $IH^1(X)$, this turns out to be 0. Therefore Hodge decomposition is restored.

As an analogue of Deligne's theorem (1.1.1) we can now state the first version of the decomposition theorem.

Theorem 1.2.1 (**Decomposition theorem for intersection cohomology groups**). *Let $f : X \to Y$ a proper map of varieties. There exists finitely many pairs (Y_α, L_α) made of locally closed, nonsingular and irreducible algebraic subvarieties $Y_\alpha \subset Y$, semisimple local systems $\mathcal{L}_\alpha$ on Y_α and integer numbers d_α such that for every open set $U \subset Y$ there exists an isomorphism*

$$IH^i(f^{-1}(U)) \cong \bigoplus_\alpha IH^i(U \cap \overline{Y}_\alpha, \mathcal{L}_\alpha). \tag{1.3}$$

The pairs (Y_α, L_α) are essentially unique, independent of U and they will be described in the next sections. Setting $U = Y$ we get a formula for $IH^*(X)$ and therefore, if X is nonsingular, a formula for $H^*(X)$. If $X \to Y$ is a family of projective manifolds then the decomposition (1.3) coincides with the one of theorem (1.1.1). On the opposite side of the spectrum, when $f : X \to Y$ is a resolution of singularities of Y then the intersection cohomology groups of Y are direct summands of the cohomology groups $H^*(X)$.

When X is singular there is no direct sum decomposition for $H^*(X)$. Intersection cohomology turns out to be precisely the topological invariant apt to deal with singular varieties and maps. The rest of the chapter will be devoted to explain the notion of intersection cohomology groups and to describe the triples $(Y_\alpha, L_\alpha, d_\alpha)$ appearing in the decomposition (1.3).

1.3 Intersection complexes

Even though the statement of theorem (1.2.1) involves just intersection cohomology groups, there are not known proofs of such a decomposition, without first proving the statement at the level of complexes of sheaves. The language and theory of sheaves and homological algebra, specifically derived categories and perverse sheaves, plays an essential role in all the known proofs of the decomposition theorem, as well as in its numerous applications. We do not present them here, but the reader can find a detailed description in [I, dCM2]. We will just say that the intersection cohomology groups are defined as the hypercohomology of some complexes, called *intersection complexes*, that live in the derived category of constructible complexes. The intersection complexes are constructed from local systems defined on a locally closed subsets of an algebraic variety with a procedure called *intermediate extension* (see [BBD, 1.4.25,2.1.9, 2.1.11]).

Definition 1.3.1. Let X be an algebraic variety and let $Y \subset X$ be a locally closed subset contained in the regular part of X. Let $\mathcal{L}$ be a local system on Y. We define the *intersection complex* $IC_{\overline{Y}}(\mathcal{L})$ *associated with* $\mathcal{L}$ as a complex of sheaves on Y which extends the complex $\mathcal{L}[\dim Y]$ and is determined up to unique isomorphism in the derived category of constructible sheaves by the conditions

- $\mathcal{H}^j(IC_{\overline{Y}}(\mathcal{L})) = 0 \quad \text{for all } j < -\dim Y$,
- $\mathcal{H}^{-\dim Y}(IC_{\overline{Y}}(\mathcal{L}_{|U})) \cong \mathcal{L}$,
- $\dim \mathrm{Supp}\mathcal{H}^j(IC_{\overline{Y}}(\mathcal{L})) < -j$, for all $j > -\dim Y$,
- $\dim \mathrm{Supp}\mathcal{H}^j(\mathbb{D}IC_{\overline{Y}}(\mathcal{L})) < -j$, for all $j > -\dim Y$, where $\mathbb{D}IC_{\overline{Y}}\mathcal{L}$ denotes the *Verdier dual* of $IC_{\overline{Y}}\mathcal{L}$.

Remark 3. Let X be an algebraic variety with regular locus X_{reg}. In case $\mathcal{L} = \mathbb{Q}_{X_{reg}}$ then we just write IC_X for $IC_X(\mathcal{L})$ and we call it *intersection cohomology complex of* X. If X is nonsingular, then $IC_X \cong \mathbb{Q}_X[\dim X]$.

1.4 Intersection cohomology groups and decomposition theorem

Definition 1.4.1. Let X be an algebraic variety. We define the *intersection cohomology groups* of X as

$$IH^*(X) = H^{*-\dim X}(X, IC_X)$$

In general, given any local system $\mathcal{L}$ supported on a locally closed subset Y of X we define the cohomology groups of Y with coefficients in $\mathcal{L}$ as

$$IH^*(\overline{Y}, \mathcal{L}) = H^{*-\dim Y}(\overline{Y}, IC_{\overline{Y}}(\mathcal{L}))$$

Taking cohomology with compact support we obtain the *intersection cohomology groups with compact support* $IH_c^*(X)$ and $IH_c^*(\overline{Y}, \mathcal{L})$.

Remark 4. Here the shift is made so that for a nonsingular variety the intersection cohomology groups coincides with ordinary cohomology groups.

1.4.1 Hodge-Lefschetz package for $IH^*(X)$

By the properties of IC complexes we can deduce the following theorems for intersection cohomology groups (see [GM1] for further details).

Theorem 1.4.1 (Poincaré-Verdier duality). *Let X be an algebraic variety of dimension n. Then for any $0 \leq j \leq 2n$ there exists a nondegenerate bilinear form*

$$IH^j(X) \times IH_c^{2n-j}(X) \to \mathbb{Q}$$

Theorem 1.4.2 (Künneth formula). *Let X, Y be algebraic varieties. Then*

$$IH^k(X \times Y) = \bigoplus_{i+j=k} IH^i(X) \otimes IH^j(Y)$$

Theorem 1.4.3 (Lefschetz hyperplane theorem). *Let X be a projective variety of dimension n and D be a general hyperplane section. Then the restriction*

$$IH^i(X) \to IH^i(D)$$

is an isomorphism for $i < n-1$ and surjective for $i = n-1$.

1.4.2 The mixed Hodge structure on $IH^*(X)$

The proof of Hard-Lefschetz theorem for intersection cohomology appears in [BBD]. Therefore, at that point in time, intersection cohomology was known to fulfil the two Lefschetz theorems and Poincaré duality [GM, GM1]. The question concerning a possible Hodge structure in intersection cohomology, as well as Hodge-theoretic questions, was very natural at that juncture (see [BBD], p.165). The work of Saito [Sa1, Sa2] settled this issues completely with the use of mixed Hodge modules. We now summarize some of the mixed Hodge-theoretic properties of the intersection cohomology of complex quasi-projective varieties. For the proofs on projective varieties we refer to [dCM2, dCM3], whereas for the extension to quasi-projective varieties and proper maps we refer to [dC]. All these properties have been proved using classical Hodge theory(see [dCM1, Section 3.3]). The intersection cohomology groups carry natural mixed Hodge structures and so does intersection cohomology with compact support .

1. if f X is nonsingular, then the mixed Hodge structure coincides with the mixed Hodge structure on the cohomology;
2. if $f : X \to Y$ is a resolution of singularities of Y then the mixed Hodge structures on $IH^*(Y)$ and $IH^*_c(Y)$ are canonical sub-quotients of the mixed Hodge structures on respectively $H^*(X)$ and $H^*_c(X)$;
3. the intersection bilinear pairing in intersection cohomology is compatible with mixed Hodge structure, that is the resulting map $IH^{n-j}(X) \to (IH^{n+j}_c(X))^\vee(-n)$ is an isomorphism of mixed Hodge structures and the shift $(-n)$ increases the weights on (n,n);
4. the natural map $H^*(X) \to IH^*(X)$ is a map of mixed Hodge structures; if X is projective then its kernel is the subspace W_{*-1} of classes of weight $\leq * - 1$.

1.4.3 Decomposition theorem

We are now in a position to express the decomposition theorem in his sheaf theoretic statement.

Theorem 1.4.4 (Decomposition theorem and semisimplicity theorem). *Let $f : X \to Y$ be a proper map of complex algebraic varieties. There exists an isomorphism in the constructible derived category $D^b_c(Y)$:*

$$Rf_* IC_X \cong \bigoplus_{i \in \mathbb{Z}} {}^{\mathfrak{p}}\mathcal{H}^i(Rf_* IC_X)[-i].$$

Moreover the perverse cohomology sheaves ${}^{\mathfrak{p}}\mathcal{H}^i(Rf_*IC_X)$ *are semisimple, i.e. there exists a stratification of* $Y = \bigsqcup S_\beta$ *such that*

$$ {}^{\mathfrak{p}}\mathcal{H}^i(Rf_*IC_X) = \bigoplus_\beta IC_{\overline{S}_\beta}(\mathcal{L}_\beta). $$

Combining these two results we can express the decomposition theorem in its final form, i.e. the existence of a finite collection of pairs $(Y_\alpha, \mathcal{L}_\alpha)$ *such that*

$$ Rf_*IC_X \cong \bigoplus_\alpha IC_{\overline{Y}_\alpha}(\mathcal{L}_\alpha)[\dim X - \dim Y_\alpha] \tag{1.4} $$

Recalling that $IH^*(X) = H^{*-\dim X}(X, IC_X)$, the shifts in the formula are chosen so that they match with the ones of theorem (1.2.1), which is a consequence of this form.

Definition 1.4.2. We call *supports of* f the Y_α appearing in equation (1.4).

1.5 Semismall maps

In general, it is not easy to determine the supports Y_α and the local systems $\mathcal{L}_\alpha$. However Migliorini and De Cataldo [dCM1], following [BM], prove that for some proper maps, called *semismall maps*, the Decomposition theorem has a very explicit form and it is easy to describe both supports and local systems on them. Let us give some preliminary definitions.

Definition 1.5.1. Let $f : X \to Y$ be a map of algebraic varieties. A *stratification for* f is a decomposition of Y into finitely many locally closed nonsingular subsets Y_α such that $f^{-1}(Y_\alpha) \to Y_\alpha$ is a topologically trivial fibration. The subsets Y_α are called the *strata* of f.

Definition 1.5.2. Let $f : X \to Y$ be a proper map of algebraic varieties. We say that f is *semismall* if there exists a stratification $Y = \bigsqcup Y_\alpha$ such that for all α

$$ \dim Y_\alpha + 2d_\alpha \leq \dim X $$

where $d_\alpha := \dim f^{-1}(y_\alpha)$ for some $y_\alpha \in Y_\alpha$.

Remark 5. The condition on the dimensions of the preimages is equivalent to ask that the complex $f_*\mathbb{Q}_X[\dim X]$ satisfies both the support and co-support conditions, i.e. it is a perverse sheaf.

Definition 1.5.3. Keep the notation as above. We say that a stratum is *relevant* if

$$\dim Y_\alpha + 2d_\alpha = \dim X.$$

The decomposition theorem for semismall maps takes a particularly simple form: the only contributions come from the relevant strata Y_α and they consist of nontrivial summands $IC_{\overline{Y}_\alpha}(\mathcal{L}_\alpha)$, where the local systems $\mathcal{L}_\alpha$ turn out to have finite monodromy. Let Y_α be a relevant stratum, $y \in Y_\alpha$ and let $F_1, \ldots, F_l$ be the irreducible $(\dim Y_\alpha)-$dimensional components of the fibre $f^{-1}(y)$. The monodromy of the $F_i's$ defines a group homomorphism $\rho_\alpha : \pi_1(Y_\alpha) \to S_l$ from the fundamental group of Y_α to the group of permutations of the F^i's. The representation ρ_α defines a local system $\mathcal{L}_\alpha$ on Y_α. In this case the semisimplicity of the local system $\mathcal{L}_\alpha$ is an elementary consequence of the fact that the monodromy factors through a finite group, then by Maschke theorem it is a direct sum of irreducible representations. As a result, the local systems will be semisimple, that is it will be a direct sum of simple local systems. With this notation, the statement of the decomposition theorem for semismall maps is the following.

Theorem 1.5.1 (Decomposition theorem for semismall maps). *Let $f : X \to Y$ be a semismall map of algebraic varieties and let Λ_{rel} the set of relevant strata. For each $Y_\alpha \in \Lambda_{rel}$ let $\mathcal{L}_\alpha$ the corresponding local system with finite monodromy defined above. Then there exists a canonical isomorphism in $\mathcal{P}(Y)$*

$$IC_X \cong \bigoplus_{Y_\alpha \in \Lambda_{rel}} IC_{\overline{Y}_\alpha}(\mathcal{L}_\alpha)$$

1.6 Support type theorems

How can we deal with maps that are not semismall? We said that in general it is hard to find the supports of a map $f : X \to Y$. However, there exists a fairly general approach to the so called *support type theorems* like the decomposition theorem, which was developed by Migliorini and Shende in [MS2]. Such an approach relies on the fact that even though a stratum S might be necessary in the stratification of a map f, the change in the cohomology of the fibres of S can be predicted just by looking at the map on the strata containing S.

Therefore, Migliorini and Shende constructed a coarser stratification, the *stratification of higher* *discriminants*. Such a description refines the notion of discriminant: instead of looking at the inverse images of points one can consider the inverse images of discs $\mathbb{D}^r$ of varying dimension r. Clearly the bigger the disc is the more likely its inverse image will be nonsingular. Let us be more precise: suppose Y is nonsingular and let $Y = \bigsqcup Y_\alpha$. Take $y \in Y$ and let k be the dimension of the unique stratum containing y. Consider the codimension k slice, meeting the stratum only in y. Its inverse image will be a nonsingular codimension k subvariety of X. In case Y, we choose a local embedding $(Y, y) \subset (\mathbb{C}^n, 0)$ and we define a disc as the intersection of Y with a nonsingular germ of complete intersection T through y. The dimension of the disc is $\dim Y - \text{codim} T$.

Now we are ready for the definition of higher discriminant.

Definition 1.6.1. Keep the notation as above. We define the $i-th$ higher discriminant $\Delta^i(f)$ as

$$\Delta^i(f) := \{y \in Y \mid \text{there is no } (i-1) - \text{dimensional disc } \phi : \mathbb{D}^{i-1} \to Y,$$
$$\text{with } f^{-1}(\mathbb{D}^{i-1}) \text{ non singular , and } \text{codim}(\mathbb{D}^{i-1}, Y) = \text{codim}(f^{-1}(\mathbb{D}^{i-1}), X)\}$$

The higher discriminants $\Delta^i(f)$ are closed algebraic subsets, and $\Delta^{i+1}(f) \subset \Delta^i(f)$ by the openness of nonsingularity and the semicontinuity of the dimension of the fibres. Also we would like to remark that $\Delta^1(f)$ is nothing but the discriminant $\Delta(f)$ that is the locus of $y \in Y$ such that $f^{-1}(y)$ is singular.

One advantage of higher discriminants is that they are usually much easier to determine via differential method than the strata of a Whitney stratification. As we are supposing Y to be nonsingular, the implicit function theorem prescribes precise conditions under which the inverse image of a subvariety by a differentiable map is nonsingular: the tangent space of the subvariety must be transverse to the image of the differential. Hence, under this assumption we have the following

Proposition 1.6.1.

$$\Delta^i(f) := \{y \in Y \mid \textit{ for every linear subspace } I \subset T_yY, \textit{with } \dim I = i-1,$$
$$\textit{the composition } T_xX \xrightarrow{df} T_yY \to T_yY/I \textit{ is not surjective for some } x \in f^{-1}(y)\}$$

We may rephrase condition of proposition (1.6.1) saying that there is no $(i-1)$- dimensional subspace I transverse to f.

The following result shows the relevance of the theory of higher discriminants in determining the summands appearing in the decomposition theorem.

Theorem 1.6.2 ([MS2],Theorem B). *Let $f : X \to Y$ be a map of algebraic varieties. Then the set of i-codimensional supports of the map f is a subset of the set of i-codimensional irreducible components of $\Delta^i(f)$.*

This theorem restricts significantly the set of candidates for the supports. Furthermore, to check whether a component of a discriminant is relevant it is enough to check its generic point. We now describe a general strategy for proving support theorems.
We have seen that the stalks of intersection cohomology sheaves appearing in the decomposition theorem (as well as the intersection cohomology groups) are endowed with a mixed Hodge structure. Moreover the Saito proves that the isomorphism

$$H^k(f^{-1}(y)) = \mathcal{H}^k(Rf_*\mathbb{Q})_y \cong \bigoplus_\alpha \mathcal{H}^k(IC_{\overline{Y}_\alpha}(\mathcal{L}_\alpha))_y \tag{1.5}$$

provided by the decomposition theorem is actually an isomorphism of mixed Hodge structures. Whenever we have a mixed Hodge structure $H = \oplus H^i$ we can define the so called *weight polynomial* as

$$\mathfrak{w}(H)(t) := \sum (-1)^{i+j} t^i \dim \mathrm{Gr}_i^W H^j \quad \in \mathbb{Z}[t].$$

This polynomial has the additivity property, i.e. if $Z \subset X$ is a closed algebraic subvariety of X then

$$\mathfrak{w}(H^*(X))(t) = \mathfrak{w}(H^*(X \setminus Z))(t) + \mathfrak{w}(H^*(Z))(t).$$

As a result we have the following criterion

Proposition 1.6.3 ([M1], Prop. 8.4). *Let $f : X \to Y$ a proper map between algebraic varieties with Y nonsingular. Consider the stratification $Y = \bigsqcup Y_\alpha$ of (1.5) and take y in some stratum Y_α. If we call $I_\alpha := \{\beta \neq \alpha \mid Y_\alpha \subset \overline{Y}_\beta\}$ then the stratum Y_α is a support if and only if*

$$\mathfrak{w}(H^*(f^{-1}(y)) \neq \sum_{\beta \in I_\alpha} \mathfrak{w}\left(IC_{\overline{Y}_\beta}(\mathcal{L}_\beta)_y\right) \tag{1.6}$$

Although it is generally quite hard to determine weight polynomials, especially on the right hand side of (1.6), the criterion is nevertheless quite powerful. For example, in [MS1] and [MSV] this criterion is used to determine the supports for the relative Hilbert scheme and for the compactified jacobian family associated to a versal family of planar curves. We are using this criterion in chapter 3 to determine the supports for the relative nested Hilbert scheme associated to a versal family of planar curves.

Chapter 2

Intersection cohomology of the moduli space of Higgs bundles

2.1 Introduction

Let C be a smooth projective curve of genus $g \geq 2$. Its associated analytic space, which we still denote by C, is a Riemann surface and its fundamental group $\pi_1(C, x_0)$ is well known to be isomorphic to

$$\frac{\langle \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g \rangle}{\langle \prod [\alpha_i, \beta_i] \rangle},$$

the quotient of the free group on $2g$ generators modulo the normal subgroup generated by the product of the commutator $[\alpha_i, \beta_i] = \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1}$. A representation of $\pi_1(C, x_0)$ with values in $\mathrm{GL}(n, \mathbb{C})$ is uniquely determined by $2g$ matrices $A_1, \dots, A_g, B_1, \dots, B_g$ in $\mathrm{GL}(n, \mathbb{C})$ such that $\prod [A_i, B_i] = I_n$. We define the *Betti moduli space* $\mathcal{M}_B(n, 0)$ as the GIT quotient

$$\mathcal{M}_B(0, n) := \left\{ (A_1, \dots, A_g, B_1, \dots, B_g) \in \mathrm{GL}(n, \mathbb{C})^{\times 2g} \mid \prod [A_i, B_i] = I_n \right\} /\!/ \, \mathrm{GL}(n, \mathbb{C})$$

with $\mathrm{GL}(n, \mathbb{C})$ acting by conjugation. Doing the GIT quotient implies to eliminate points whose orbit is not closed, namely the points corresponding to representations which are not semisimple. $\mathcal{M}_B(0, n)$ is an affine variety, generally singular. Of course such a procedure can be done with any reductive algebraic Lie Group and we call the varieties obtained in this way *character varieties*. For the unitary group $U(n)$ the character variety can be constructed using a similar procedure; Narasimhan and Seshadri [NS] have shown that there exists a real analytic isomorphism between the character variety of unitary representations and the moduli space $\mathcal{N}(0,n)$ of *semistable* vector bundles on C of degree 0 and rank n. This variety, which has been the focus of several works in mathematics, parametrizes equivalence classes of *semistable* algebraic vector bundles V on C. Let us detail a bit the kind of equivalence relation.

Definition 2.1.1. Let V be an algebraic vector bundle on C.

$$\text{For any subbundle } W \subset V \text{ one has } \mu(W) := \frac{\deg W}{\operatorname{rank} W} \leq \frac{\deg V}{\operatorname{rank} V} =: \mu(V). \tag{2.1}$$

We call $\mu(V)$ the *slope* of V. A bundle is said to be *stable* if a strict inequality holds.

Also, we say that a vector bundle is polystable if it can be written as a direct sum of stable bundles. Whenever a bundle V is strictly semistable we can find subbundle W with least rank with the same slope as V: as a result the bundle V/W is a stable bundle with the same slope as V. Proceeding in this way we can construct a filtration, called the Jordan-Hölder filtration

$$0 = W_0 \subset W_1 \subset \ldots \subset W_k = V$$

such that W_i/W_{i-1} is a stable bundle with the same slope as V. Setting $Gr(V) := \oplus_i W_i/W_{i-1}$ this is a polystable bundle with the same slope as V. We say that V and V' are *S-equivalent* if $Gr(V') = Gr(V)$. Notice that S-equivalence is an equivalence relation and every class has a unique polystable representative up to isomorphism. Therefore we can think $\mathcal{N}(0,n)$ both as semistable bundles modulo S-equivalence and polystable bundles modulo isomorphism. The stable bundles form a smooth dense locus $\mathcal{N}^s(0,n)$, which corresponds to irreducible representations in the character variety. Moreover, if one wants to consider bundles of degree d, it suffices to replace the identity with $e^{\frac{2\pi i d}{n}}$ in the product of commutators which define the character variety. If one instead wants to consider bundles with trivial determinant then the representations in the character variety must take with values in $SU(n)$.

A natural question to ask is what happens when we consider representations in the whole $\mathrm{GL}(n,\mathbb{C})$, namely the Betti moduli space. Is there a corresponding geometrical object in terms of bundles over C? The answer has been given by Hitchin [H] and leads to the definition of *Higgs bundles*.

Definition 2.1.2. Let C be a smooth projective curve over $\mathbb{C}$. Let K_C denote the canonical bundle on C. A Higgs bundle is a pair (V, ϕ) where V is a holomorphic vector bundle on C and $\phi \in H^0(EndV \otimes K_C)$ is a holomorphic one form with coefficient in $EndV$, which we call Higgs field.

We say that $W \subseteq V$ is a Higgs subbundle if $\phi(W) \subset W$. As in the case of vector bundles we can define the notions of stability in the same way considering Higgs subbundles. We define $\mathcal{M}_{Dol}(d,n)$ to be the moduli space of equivalence classes of semistable Higgs bundles of rank n and degree d over C. Again if one wants to consider Higgs bundles with trivial determinant then the representation must take values in $SL(n, \mathbb{C})$.
$\mathcal{M}_{Dol}(0,n)$ is a quasi-projective normal irreducible variety, generally singular. The smooth locus is dense and parametrizes stable pairs. Observe that whenever d and n are coprime, every semistable pair is indeed stable, therefore the moduli space is smooth. If not, the singularities corresponds precisely to the strictly semistable pairs. Such a moduli space, comes equipped with a map to some affine space. Such a map is called the *Hitchin fibration* and maps a pair (V, Φ) to the characteristic polynomial of Φ.
The work of Corlette [Co] , Donaldson [Do], Hitchin [H] and Simpson [Si2] shows that there exists a real analytic isomorphism between the Dolbeault moduli space and the Betti one

$$\mathcal{M}_{Dol}(d,n) \cong \mathcal{M}_B(d,n). \tag{2.2}$$

The cohomology of these moduli spaces has been widely studied and computed in some particular cases. For the smooth case, Poincaré polynomials for $G = SL(2,\mathbb{C})$ character were computed by Hitchin in his seminal paper on Higgs bundles [H] and for $G = SL(3,\mathbb{C})$ by Gothen in [G]. More recently, the techniques involved in the computations by Gothen and Hitchin have been improved to compute the classes in the completion of the Grothendieck ring for these varieties in the $G = GL(4,\mathbb{C})$ case, and from their computations it is also possible to deduce the compactly supported Hodge polynomials [GH]. For the case of rank 2 and degree 1 Higgs bundles, which corresponds to the twisted character variety of $GL(2,\mathbb{C})$, De Cataldo, Hausel and Migliorini [dCHM] stated and proved the so called $P = W$ conjecture, which asserts that the *weight filtration* on the cohomology of the character variety corresponds in the

isomorphism in (2.2) to the *Perverse filtration* constructed from the Hitchin fibration. Furthermore, Hausel and Rodriguez-Villegas [HR] started the computation of the E-polynomials of $G-$character varieties focusing on $G = GL(n, \mathbb{C}), SL(n, \mathbb{C})$ and $PGL(n, \mathbb{C})$ using arithmetic methods inspired on the Weil conjectures. Following the methods of Hausel and Rodriguez-Villegas, Mereb [M] studied the case of $SL(n, \mathbb{C})$ giving an explicit formula for the E-polynomial in the case $G = SL(2, \mathbb{C})$. Also, Mellit in [Me] compute E-polynomials for nonsingular moduli spaces of Higgs bundles. The case we are interested in is the one of non twisted representations into SL$(2, \mathbb{C})$, which corresponds to Higgs bundles with rank 2 and degree 0 with trivial determinant. From now on we will denote this space simply by $\mathcal{M}_{Dol}$. Note that, as it is Lie algebra valued, the Higgs field in this case is traceless. First we describe the local structure of the singularities, using the fact that they are identical to those of the moduli space of rank 2 semistable sheaves on a K3 surface with a generic polarization studied by O' Grady in [OG], then following the idea of [KY] and [OG] we construct a desingularization of $\mathcal{M}_{Dol}$. After that we study the case of $g = 2$, for which there exists a desingularization $\tilde{\mathcal{M}}_{Dol}$ such that the map $\tilde{\mathcal{M}}_{Dol} \to \mathcal{M}_{Dol}$ is semismall thus we can apply an easier version decomposition theorem to compute the E-polynomial for the intersection cohomology of $\mathcal{M}_{Dol}$.
The results in this chapter are a first step in the direction of understanding the $P = W$ conjecture in the non coprime case, that is for singular moduli spaces of Higgs bundles. In fact in this case the theory behind the conjecture suggests that the natural invariant to look at on the Doulbeault side should be the intersection cohomology.

2.2 The structure of $\mathcal{M}_{Dol}$

Let us recall briefly the construction by Simpson of the moduli space $\mathcal{M}_{Dol}$.

- [Sim, Thm. 3.8] Fix a sufficiently large integer N and set $p := 2N + 2(1 - g)$. Simpson showed that there exist a quasi-projective scheme Q representing the moduli functor which parametrizes the isomorphism classes of triples (V, Φ, α) where (V, Φ) is a semistable Higgs pair with $detV \cong \mathcal{O}_X$, $tr(\Phi) = 0$ and $\alpha : \mathbb{C}^p \to H^0(C, V \otimes \mathcal{O}(N))$ is an isomorphism of vector spaces.

- [Sim, Thm. 4.10] Fix $x \in C$ and let $\tilde{Q}$ be the frame bundle at x of the universal bundle restricted to x. Then we have $SL(2,\mathbb{C}) \times GL(p,\mathbb{C})$ acting on $\tilde{Q}$. In fact $SL(2,\mathbb{C})$ acts as automorphisms of (V,Φ) while the action of $GL(p,\mathbb{C})$ acts on the α's. The action of $GL(p,\mathbb{C})$ on Q lifts to $\tilde{Q}$ and Simpson proves that such an action is free and every point in $\tilde{Q}$ is stable with respect to it, so we can define

$$\mathcal{R}_{Dol} = \tilde{Q}/\mathrm{GL}(p,\mathbb{C})$$

which represents the triples (V,Φ,β) where β is an isomorphism $V_x \to \mathbb{C}^2$.

- [Sim, Thm. 4.10] Every point in $\mathcal{R}_{Dol}$ is semistable with respect to the action of $\mathrm{SL}(2,\mathbb{C})$ and the closed orbits correspond to the polystable pairs (V,Φ,β) such that

$$(V,\Phi) = (L,\phi) \oplus (L^{-1},-\phi)$$

with $L \in Pic^0(C)$ and $\phi \in H^0(K_C)$.

Proposition 2.2.1. *[Sim, Thm. 4.10] The good quotient $\mathcal{R}_{Dol}//\,\mathrm{SL}(2,\mathbb{C})$ is $\mathcal{M}_{Dol}$.*

Thanks to proposition (2.2.2) it is possible to describe the singularities of $\mathcal{M}_{Dol}$ in terms of those on $\mathcal{R}_{Dol}$. Let G be a reductive group acting linearly on a complex projective scheme Y (here "linearly" means that the action lifts to $\mathcal{O}_Y(1)$), let W be a closed G-invariant subscheme and $\pi : \tilde{Y} \to Y$ be the blow-up of Y along W. Then G acts both on $\tilde{Y}$ and on the ample line bundle

$$D_l := \pi^*\mathcal{O}_Y(l) \otimes \mathcal{O}_{\tilde{Y}}(-E),$$

where l is an integer and E is the exceptional divisor of π. Thus the action is linearized and it makes sense to talk about stable and semistable points: we denote by $Y^{(s)s}$ the (semi)stable points with respect to $\mathcal{O}_Y(1)$ and $\tilde{Y}^{(s)s}(l)$ the (semi)stable points with respect to D_l.

Proposition 2.2.2. *[K, Prop. 3.1] Keep the notation as above. For$l \gg 0$ the loci $\tilde{Y}^{(s)s}(l)$ are independent of l and we have that*

$$\pi(\tilde{Y}^{ss}) \subset Y^{ss}$$

$$\pi(\tilde{Y}^{s}) \subset Y^{s}$$

In particular π induces a morphism

$$\bar{\pi} : \tilde{Y}//G \to Y//G$$

and for l sufficiently divisible such a morphism is identified with the blow-up along $W//G$.

Kirwan's proposition, roughly speaking, tells us that if we find a suitable desingularization of $\mathcal{R}_{Dol}$ and we quotient by the action of $\mathrm{SL}(2,\mathbb{C})$ we obtain something with at worst quotient singularities which has a birational map to $\mathcal{M}_{Dol}$. After another blow-up we can eliminate the singularities and find a desingularization of $\mathcal{M}_{Dol}$. As a consequence, it is of primary importance to understand the local structure of the singularities in $\mathcal{R}_{Dol}$. By a result of Simpson [Sim, Section 1], the criterion for GIT semistability of points in $\mathcal{R}_{Dol}$ coincides with the slope semistability of the corresponding Higgs bundles.
As a result, the singularities correspond to the strictly semistable bundles. If a Higgs bundle (V,Φ) is strictly semistable, then there exists a Φ-invariant line bundle L of degree 0. Call ϕ the restriction of Φ to $H^0(EndL \otimes K_C) \cong H^0(K_C)$. Then the singularities of $\mathcal{R}_{Dol}$ are of the following form:

- $\Omega_R^0 := \{(V,\Phi,\beta) \mid (V,\Phi) = (L,0) \oplus (L,0) \text{ with } L \cong L^{-1}\}$
- $\Omega_R' := \{(V,\Phi,\beta) \mid (V,\Phi) \text{ is a nontrivial extension of } (L,0) \text{ by itself}\}$
- $\Sigma_R^0 := \{(V,\Phi,\beta) \mid (V,\Phi) = (L,\phi) \oplus (L^{-1},-\phi) \text{ with } (L,\phi) \not\cong (L^{-1},-\phi)\}$
- $\Sigma_R' := \{(V,\Phi,\beta) \mid (V,\Phi) \text{ is a nontrivial extension of } (L,\phi) \text{ by } (L^{-1},-\phi)\}$

Since Ω_R' and Σ_R' are not polystable their orbits disappear when we quotient by the action of $\mathrm{SL}(2,\mathbb{C})$, thus we can avoid considering them. We call Ω_R and Σ_R the closures of respectively Ω_R^0 and Σ_R^0 in $\mathcal{R}_{Dol}$. By proposition (2.2.2), the singularities of $\mathcal{M}_{Dol}$ are the strictly semistable Higgs bundles

- $\Omega^0 := \{(V,\Phi) \mid (V,\Phi) = (L,0) \oplus (L,0) \text{ with } L \cong L^{-1}\}$
- $\Sigma^0 := \{(V,\Phi) \mid (V,\Phi) = (L,\phi) \oplus (L^{-1},-\phi) \text{ with } (L,\phi) \not\cong (L^{-1},-\phi)\}$.

As before, we call Ω and Σ their closures in $\mathcal{M}_{Dol}$. The loci Ω^0 and Σ^0 are the quotient of Ω_R^0 and Σ_R^0 with respect to the action of $\mathrm{SL}(2,\mathbb{C})$ modulo their stabilizers. The points in Ω^0 have $\mathrm{SL}(2,\mathbb{C})$ as stabilizer, so both Ω_R and Ω consists of 2^{2g} points corresponding to the roots of the trivial bundle $\mathcal{O}_C$. Observe that $\Sigma_0 \cong \left[(Pic^0(C) \times H^0(K_C)) \setminus (2^{2g} \text{ points })\right]/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts as the involution $(L,\phi) \mapsto (L^{-1},-\phi)$. Then Σ_R^0 is a $\mathbb{PSL}(2,\mathbb{C})$ bundle over Σ^0.

2.2.1 Strategy of the desingularization

Our strategy will be first to desingularize $\mathcal{R}_{Dol}$ and then quotient by the action of $\mathrm{SL}(2,\mathbb{C})$.

1) we first blow up $\mathcal{R}_{Dol}$ along the deepest singular locus Ω_R, set $\mathcal{P}_{Dol} := Bl_{\Omega_R}\mathcal{R}_{Dol}$ and call Σ_P the strict transform of the bigger singular locus;

2) we blow up again and set $\mathcal{S}_{dol} := Bl_{\Sigma_P}\mathcal{P}_{Dol}$;

3) If $g = 2$, $\hat{\mathcal{M}}_{Dol} := \mathcal{S}_{Dol}/\!/\,\mathrm{SL}(2,\mathbb{C})$ is smooth; if $g \geq 3$ it has at worst orbifold singularities and blowing up $\mathcal{S}_{Dol}$ along the locus of points whose stabilizer is larger than the centre $\mathbb{Z}_2$ of $\mathrm{SL}(2,\mathbb{C})$, we obtain $\mathcal{T}_{Dol}$ such that

$$\hat{\mathcal{M}}_{Dol} := \mathcal{T}_{Dol}/\!/\,\mathrm{SL}(2,\mathbb{C})$$

is a smooth variety obtained by blowing up $\mathcal{M}_{Dol}$ first along the points $(L,0) \oplus (L,0)$, secondly after the proper transform of orbit points of $(L,\phi) \oplus (L^{-1},-\phi)$ and third along a nonsingular subvariety lying in the proper transform of the exceptional divisor of the first blow-up.

2.3 Singularities of $\mathcal{M}_{Dol}$ and their normal cones

The rest of the chapter is devoted to the construction of the desingularization. The strategy for the desingularization is closely analogous to that in [OG]. The first thing to do is to describe the singular loci and their normal cones.

Let us give some preliminary results.

2.3.1 Normal cones and deformation of sheaves

If W is a subscheme of a scheme Z, we call $C_W Z$ the normal cone to W in Z. It is well known that the exceptional divisor of a blow up of Z along W is equal to $Proj(C_W Z)$, therefore we will have to determine both $C_\Omega \mathcal{R}_{Dol}$ and $C_\Sigma \bar{\mathcal{R}}_{Dol}$. The following theorem, known as Luna's étale slice theorem allows to see this problem in terms of deformation theory of sheaves. Thus we take a brief excursus on normal cones and their relations with deformation theory. For proofs and further details we refer to [OG].

Theorem 2.3.1. *[Luna's étale slice] Let G be a reductive group acting linearly on a quasi-projective scheme Y. Let $y_0 \in Y$ such that $\mathcal{O}(y_0)$ is closed in Y^{ss} (this implies $St(y_0)$ is reductive).Then there exists a slice normal to $\mathcal{O}(y_0)$, i.e. an affine subscheme $\mathcal{U} \hookrightarrow Y^{ss}$, containing y_0 and invariant under the action of $St(y_0)$, such that the following holds. The multiplication morphism*

$$G \times_{St(y_0)} \mathcal{U} \to Y^{ss}$$

has open image and is étale over its image. (Here $St(y_0)$ acts on $G \times \mathcal{V}$ by $h(g, y) := (gh^{-1}, hy)$). Moreover the morphism is G-equivariant with respect to the left multiplication on the first factor. The quotient map

$$\mathcal{U}//St(y_0) \to Y^{ss}//G$$

has open image and is étale over its image. If Y^{ss} is nonsingular at y_0, then $\mathcal{U}$ is also nonsingular at y_0.

Now if $W \subset Y^{ss}$ is a locally closed G-invariant subset containing y_0, we can describe the normal cone $C_W Z$ in terms of the normal slice. More precisely, we have the following corollary.

Corollary 2.3.2. *Let $\mathcal{W} := W \cap \mathcal{U}$. There exists a $St(y_0)$-equivariant isomorphism*

$$(C_W Y^{ss})_{y_0} \cong (C_{\mathcal{W}} \mathcal{U})_{y_0}$$

Now we go back to $\mathrm{SL}(2, \mathbb{C})$ acting $\mathcal{R}^{ss}_{Dol}$. The following result identifies the normal slice with a versal deformation space.

Proposition 2.3.3. *[OG, Prop. 1.2.3] Let* $v = (V, \Phi, \beta) \in \mathcal{R}^{ss}_{Dol}$ *a split extension (that is* v *has a closed orbit with respect to the action of* $SL(2, \mathbb{C})$*). Let* $\mathcal{U}$ *be a normal slice and* (U, v) *be the germ of* $\mathcal{U}$ *at* v*. Let* $\mathcal{V}$ *be the restriction to* $C \times (\mathcal{U}, v)$ *of the tautological quotient sheaf on* $C \times \mathcal{R}_{Dol}$*. The couple* $((\mathcal{U}, v), \mathcal{V})$ *is a versal deformation space of* (V, Φ, β)*.*

We also have some constraint on the dimension of the normal slice.

Proposition 2.3.4. *Keep the notation as above. Let* $v \in \mathcal{R}^{ss}_{Dol}$ *be a point with a closed orbit, and let* $\mathcal{U} \ni v$ *be a slice normal to the orbit* $O(v)$*. Then*

$$\dim_v \mathcal{U} \geq \dim Ext^1(V, V) - Ext^2(V, V)^0$$

where $Ext^i(V, V)$ *denotes extensions in the category of Higgs bundles and* $Ext^2(V, V)^0$ *are traceless extensions.*

The previous propositions permits to describe the normal cones to our singular loci as normal cones of other loci in the versal deformation space of semistable bundles. Let us provide tools which will turn out to be useful later.

Hessian cone

Let Y be a scheme and $B \subset Y$ a locally closed subscheme such that B is smooth and $T_b Y$ ha constant dimension for every $b \in B$. Therefore it makes sense to talk about a normal vector bundle $N_B Y$. Let $\mathcal{I}$ be the ideal sheaf of B in Y: the graded surjection

$$\bigoplus_{d=0}^{\infty} Sym^d(\mathcal{I}/\mathcal{I}^2) \to \bigoplus_{d=0}^{\infty} \mathcal{I}^d/\mathcal{I}^{d+1}$$

determines an embedding $i : C_B Y \hookrightarrow N_B Y$. We also observe that, as the map is an isomorphism in degree 1, the homogeneous ideal $\mathcal{I}(i(C_B Y))$ contains no linear terms. We define the *Hessian cone* to be the subscheme of $N_B Y$ whose corresponding homogeneous ideal is generated by the quadratic terms in $\mathcal{I}(i(C_B Y))$. Therefore we have a chain of cones

$$C_B Y \subset H_B Y \subset N_B Y.$$

Notice that for every $b \in B$

$$\mathbb{P}(H_b Y) \text{ is the cone over } \mathbb{P}(H_B Y)_b \text{ with vertex } \mathbb{P}(T_b B). \text{ (see [RU])} \tag{2.3}$$

Let $I_m := Spec(\mathbb{C}[t]/(t^{m+1}))$; thus tangent vectors to Y at b are identified with pointed maps $I_1 \to (Y, b)$.Then the reduced part of the hessian cone is

$$(H_bY)_{red} := \{f_1 : I_1 \to (Y, b) \mid \text{ there exists } f_2 : I_2 \to (Y, b) \text{ extending } f_1\}$$

Consider $\mathcal{E}$ to be a coherent sheaf over a projective scheme Y. If $(Def(\mathcal{E}, 0))$ is a parameter space for a versal deformation of $\mathcal{E}$ then one has that $T_0Def(\mathcal{E}) \cong Ext^1(\mathcal{E}, \mathcal{E})$. Consider the *Yoneda* cup product $Ext^1(\mathcal{E}, \mathcal{E}) \times Ext^1(\mathcal{E}, \mathcal{E}) \to Ext^2(\mathcal{E}, \mathcal{E})$ that maps a couple (e, f) in $e \cup f$. The Hessian cone is given by [OG, 1.3.5]

$$H_0(Def(\mathcal{E}))_{red} \cong \Upsilon_{\mathcal{E}}^{-1}(0)_{red} \tag{2.4}$$

where $\Upsilon_{\mathcal{E}} : Ext^1(\mathcal{E}, \mathcal{E}) \to Ext^2(\mathcal{E}, \mathcal{E})$ is the cup product of the extension class with itself. We call this map the *Yoneda square*.

In the following section we describe the local structure of the singularities and use the isomorphism in equation (2.4) to compute the normal cones along the singular loci.

2.3.2 Local structure of singularities

Let A^i denote the sheaf of $\mathcal{C}^\infty$ $i-$forms on C. For a polystable Higgs pair (V, ϕ) consider the complex

$$0 \longrightarrow End_0(V) \otimes A^0 \longrightarrow End_0(V) \otimes A^1 \longrightarrow End_0(V) \otimes A^2 \longrightarrow 0 \tag{1}$$

with differential $D'' = \bar{\partial} + [\phi, -]$. Splitting in (p, q) forms, we have that the cohomology of this complex is equal to the hypercohomology of the double complex

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & End_0(V) \otimes A^{1,0} & \xrightarrow{\bar{\partial}} & End_0(V) \otimes A^{1,1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow = & & \\
0 & \longrightarrow & End_0(V) \otimes A^0 & \xrightarrow{D''} & End_0(V) \otimes A^1 & \xrightarrow{D''} & End_0(V) \otimes A^2 & \longrightarrow & 0 \\
 & & \downarrow = & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & End_0(V) \otimes A^{0,0} & \xrightarrow{\bar{\partial}} & End_0(V) \otimes A^{0,1} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

This means that the cohomology groups T^i of (1) fit the long exact sequence (2)

$$0 \longrightarrow T^0 \longrightarrow H^0(End_0(V)) \xrightarrow{[\Phi,-]} H^0(End_0(V)\otimes K_C) \longrightarrow$$

$$T^1 \longrightarrow H^1(End_0(V)) \xrightarrow{[\Phi,-]} H^1(End_0(V)\otimes K_C) \longrightarrow T^2 \longrightarrow 0$$

Remark 6. Observe also that, by deformation theory for Higgs bundles, the T^i's parametrize the traceless extensions of Higgs bundles i.e. $T^i = Ext^i_0(V,V)$ in the category of Higgs sheaves. Moreover T^1 is precisely the Zariski tangent space to $\mathcal{M}_{Dol}$.

Thanks to sequence (2) we can now find the singularities of both $\mathcal{M}_{Dol}$ and $\mathcal{R}_{Dol}$. By [Sim, Lemma 10.7] one has that the dimension of the Zariski tangent space in a point $v = (V,\Phi,\beta)$ is equal to

$$\dim T_v\mathcal{R}_{Dol} = \dim T^1 + 3 - \dim T_0.$$

By Riemann-Roch theorem and (2) we have that

$$\dim T^1 = \chi(End_0(V)\otimes K_C) - \chi(End_0(V)) = 6g - 6 + 2\dim T^0.$$

As a result, we have a singular point (V,Φ,β) in $\mathcal{R}_{Dol}$ if and only if $\dim T^0 > 0$, that is there exists a section of $H^0(End_0(V))$ that commutes with the Higgs field.
If (V,Φ) is stable, no such section exists thus the singularities of $\mathcal{R}_{Dol}$ must be the strictly semistable orbits. Of course, as the condition does not depend from β, the same holds for the singularities of $\mathcal{M}_{Dol}$.

We can sum up the above remarks in the following proposition:

Proposition 2.3.5. *(i) The singularities of $\mathcal{R}_{Dol}$ are:*

- $\Omega^0_R := \{(V,\Phi,\beta) \mid (V,\Phi) = (L,0)\oplus(L,0)$ *with* $L \cong L^{-1}\}$
- $\Omega'_R := \{(V,\Phi,\beta) \mid (V,\Phi)$ *is a nontrivial extension of* $(L,0)$ *by itself*$\}$
- $\Sigma^0_R := \{(V,\Phi,\beta) \mid (V,\Phi) = (L,\phi)\oplus(L^{-1},-\phi)$ *with* $(L,\phi) \ncong (L^{-1},-\phi)\}$
- $\Sigma'_R := \{(V,\Phi,\beta) \mid (V,\Phi)$ *is a nontrivial extension of* (L,ϕ) *by* $(L^{-1},-\phi)\}$

(ii) The singularities of $\mathcal{M}_{Dol}$ are:

- $\Omega^0 := \{(V,\Phi) \mid (V,\Phi) = (L,0) \oplus (L,0)$ *with* $L \cong L^{-1}\}$
- $\Sigma^0_: = \{(V,\Phi) \mid (V,\Phi) = (L,\phi) \oplus (L^{-1},-\phi)$ *with* $(L,\phi) \ncong (L^{-1},-\phi)\}$

Remark 7. Let us remark that the singularities of $\mathcal{R}_{Dol}$ and $\mathcal{M}_{Dol}$ have a different origin. In fact, since the action of $GL(p)$ on $\tilde{Q}$ is free, the singularities of $\mathcal{R}_{Dol}$ are those of $\tilde{Q}$, whereas the singularities of $\mathcal{M}_{Dol}$ are coming form the singularities of $\mathcal{R}_{Dol}$ and the strictly semistable orbits for the action of $\mathrm{SL}(2,\mathbb{C})$.

Now that we know the singularities of $\mathcal{R}_{Dol}$, our aim is to describe their local structure, that is their normal cones. The following theorem by Simpson, describe the normal cone of the singular loci in terms of the extensions.

Theorem 2.3.6. *[Si2, Thm. 10.4] Consider* $\mathrm{SL}(2,\mathbb{C})$ *acting on* $\mathcal{R}_{Dol}$ *and suppose* (V,ϕ) *is a point in a closed orbit. Let* C *be the quadratic cone in* T^1 *defined by the map* $\eta \mapsto [\eta,\eta]$ *(where* $[,]$ *is the graded commutator) and* $\mathfrak{h}^\perp$ *be the perpendicular space to the image of* T^0 *in* $sl(2)$ *under the morphism* $H^0(End_0(V)) \to sl(2)$ *. Then the formal completion* $(\mathcal{R}_{Dol},(V,\phi))\hat{}$ *is isomorphic to the formal completion* $(C \times \mathfrak{h}^\perp, 0)\hat{}$.

Moreover this theorem hold also at the level of $\mathcal{M}_{Dol}$.

Proposition 2.3.7. *[Si2, Prop. 10.5] Let* $v = (V,\Phi)$ *be a point* $\mathcal{M}_{Dol}$ *and let* C *be the quadratic cone of* (V,Φ,β) *in the previous theorem. Then the formal completion of* $\mathcal{M}_{Dol}$ *at* v *is isomorphic to the formal completion of the good quotient* C/H *of the cone by the stabilizer of* (V,Φ,β).

Remark 8. We have seen in the introduction that there exists a real analytic isomorphism

$$\mathcal{M}_{Dol} \cong \mathcal{M}_B = \{\rho : \pi_1(C,c_0) \to SL(2,\mathbb{C})\}//\,\mathrm{SL}(2,\mathbb{C})\,.$$

This moduli space is constructed in the same way as $\mathcal{M}_{Dol}$, starting from a space $\mathcal{R}_B \cong Hom(\pi_1(C,c_0), SL(2,\mathbb{C}))$ which is still real analytic isomorphic to $\mathcal{R}_{Dol}$. The description of the singularities in theorem (2.3.6) is analogous to the one by Goldman and Millson in [GoM] for $\mathcal{R}_B$. They show that the singularities at a point in $\mathcal{R}_B$ are *quadratic*, that is the analytic germ of a point $\rho \in \mathcal{R}_B$ is equivalent to the germ of a quadratic cone at 0 in defined by a

bilinear map on the tangent space $T_\rho \mathcal{R}_B$.
Simpson's *isosingularity priciple* [Si2, Thm. 10.6] tells us that the formal completion of point in $\mathcal{R}_{Dol}$ and a formal completion to the corresponding point $\mathcal{R}_B$ are isomorphic, thus the singularities of $\mathcal{R}_{Dol}$ are quadratic as well.

Let us describe the spaces T^i and the graded commutator more explicitly: we consider our Higgs bundle (V, Φ) as an extension

$$0 \to (L_1, \phi_1) \to (V, \Phi) \to (L_2, \phi_2) \to 0.$$

The deformation theory of the above Higgs bundle is controlled by the hypercohomology of the complex

$$\begin{array}{rlcl} C^\bullet : & L_2^{-1} L_1 & \xrightarrow{\psi} & L_2^{-1} L_1 \otimes K_C \\ & f & \longmapsto & \phi_1 f - f \phi_2 \end{array}$$

and we have a long exact sequence

$$0 \longrightarrow Ext^0_H(L_1, L_2) \longrightarrow H^0(L_2^{-1} L_1) \xrightarrow{\psi} H^0(L_2^{-1} L_1 \otimes K_C) \longrightarrow$$

$$Ext^1_H(L_1, L_2) \longrightarrow H^1(L_2^{-1} L_1) \xrightarrow{\psi} H^1(L_2^{-1} L_1 \otimes K_C) \longrightarrow Ext^2_H(L_1, L_2) \longrightarrow 0$$

where $Ext^i_H(L_1, L_2) := \mathbb{H}^i(C^\bullet)$ are the extensions of (L_2, ϕ_2) with (L_1, ϕ_1) as Higgs sheaves. Observe that

$$T^i = \bigoplus_{i,j} Ext^i_H(L_i, L_j) \tag{2.5}$$

As we are considering bundles with trivial determinant and traceless endomorphisms L_2 will be the dual of $L_1 =: L$, $\phi_2 = -\phi_1 =: -\phi$, and we will not consider $Ext^i_H(L^{-1}, L^{-1})$ because they are just the opposites of elements in $Ext^i_H(L, L)$.

Yoneda Product

We want to consider the Yoneda product

$$\begin{array}{rccc} Yon: & Ext^1_H(V,V) \times Ext^1_H(V,V) & \to & Ext^2_H(V,V) \\ & (\alpha, \beta) & \mapsto & \alpha \cup \beta \end{array}$$

and the associated Yoneda square

$$\Upsilon : Ext^1_H(V,V) \to Ext^2_H(V,V), \quad \alpha \mapsto \alpha \cup \alpha.$$

Remark 9. If we think of elements in $Ext^1_H(V,V)$ locally as matrices of 1-forms in $sl(2)$ we have that such a product coincide with the graded commutator of Simpson's theorem.

If we use decomposition (2.5), we can write Yoneda square as

$$\begin{array}{rccc} \Upsilon : & Ext^1_H(L,L) \oplus Ext^1_H(L^{-1},L) \oplus Ext^1_H(L,L^{-1}) & \longrightarrow & Ext^2_H(L,L) \\ & (a,b,c) & \longmapsto & b \cup c \end{array}$$

Let

$$\begin{array}{rccc} \overline{\Upsilon} : & Ext^1_H(L^{-1},L) \oplus Ext^1_H(L,L^{-1}) & \longrightarrow & Ext^2_H(L,L) \\ & (a,b,c) & \longmapsto & b \cup c \end{array}$$

be the map induced by Υ on $Ext^1_H(V,V)/Ker\Upsilon$.

We now have all the tools to describe the normal cones of elements in the singular loci of $\mathcal{R}_{Dol}$. Their fibres will be the exceptional divisors of the blow-ups we described at the beginning of this section. We stress that, since the orbits of Γ^0 and Λ^0 are not closed they will disappear when performing the GIT quotient by the action of $\mathrm{SL}(2,\mathbb{C})$, therefore we do not compute their normal cones.

2.4 Construction of the desingularization $\hat{\mathcal{M}}_{Dol}$

For ease of the reader we present a short summary of the results in this section.

1) We compute the normal cones of the singularities of $\mathcal{R}_{Dol}$ and prove that

Proposition. *Σ^0_R is smooth and its normal cone $C_{\Sigma_R}\mathcal{R}_{Dol}$ is a locally trivial fibration over Σ^0_R with fibre the affine cone over a smooth quadric in $\mathbb{P}^{4g-5}$. More precisely we have that for a point $v = (V,\Phi,\beta)$ in Σ^0_R there is a canonical isomorphism*

$$(C_{\Sigma_R}\mathcal{R}_{Dol})_v \cong \{(b,c) \in Ext^1_H(L^{-1},L) \oplus Ext^1_H(L,L^{-1}) \mid b \cup c = 0\}$$

Proposition. *Ω_R^0 is a smooth closed subset of $\mathcal{R}_{Dol}$ and its normal cone $C_{\Omega_R}\mathcal{R}_{Dol}$ is a locally trivial fibration over Ω_R with fibre the affine cone over a reduced irreducible complete intersection of three quadrics in $\mathbb{P}^{6g-1}$. That is if $v = (V, \phi, \beta) \in \Omega_R$ with $(V, \Phi) = (L, 0) \oplus (L, 0)$ then*

$$(C_\Omega \mathcal{R}_{Dol})_v \cong \{f : sl(2) \to \Lambda^1 \mid f^*\omega = 0\} =: Hom^\omega(sl(2), \Lambda^1);$$

where $\Lambda^1 = Ext^1_H(L, L)$ and ω is the skew-symmetric bilinear form on Λ^1 induced by the Yoneda product on T^1.

2) We blow up $\mathcal{R}_{Dol}$ in Ω_R and set $\mathcal{P}_{Dol} := Bl_\Omega \mathcal{R}_{Dol} \xrightarrow{\hat{\pi}} \mathcal{R}_{Dol}$. We call Ω_P the exceptional divisor and Σ_P the strict transform of Σ_R under the blow-up. We describe the semistable points in both Ω_P and Σ_P and again we compute their normal cones in $\mathcal{P}_{Dol}$. More precisely we will show:

Proposition. *Let $[f]$ be an element of $Hom^\omega(sl(2), \Lambda^1)$. Then $[f]$ is semistable with respect to the action of $\mathrm{SL}(2, \mathbb{C})$ if and only if*

$$rkf \begin{cases} \geq 2 \ \textit{or} \\ = 1 \ \textit{and} \ kerf^\perp \ \textit{is non isotropic} , \end{cases}$$

where orthogonality and isotropy are with respect to the Killing form on $sl(2)$.

The semistable points in the strict transform Σ_P are described in the following proposition.

Proposition. *Consider the locus Σ_P^{ss} of semistable points in Σ_P. One has:*

(i) Σ_P^{ss} is smooth and reduced;

(ii) the intersection $\Sigma_P^{ss} \cap \Omega_P$ is smooth and reduced and in particular one has that if $v \in \Omega_R$ then

$$\pi_P^{-1}(v) \cap \Sigma_P^{ss} = \mathbb{P}Hom_1^{ss}(sl(2), \Lambda^1)$$

where $Hom_1^{ss}(sl(2), \Lambda^1)$ is the set of $f \in Hom^\omega(sl(2), \Lambda^1)$ which are semistable of rank ≤ 1 and has dimension $2g$;

(iii) $\Sigma_P^{ss} \setminus \Omega_P = \pi_P^{-1}(\Sigma_R^0)$;

(iv) the normal cone of Σ_P^{ss} *in* $\mathcal{P}_{Dol}$ *is a locally trivial bundle over* Σ_P^{ss} *with fibre the cone over a smooth quadric in* $\mathbb{P}^{4g-5}$.

3) Set $\pi_S : \mathcal{S}_{Dol} \to \mathcal{P}_{Dol}$ to be the blow-up of $\mathcal{P}_{Dol}$ along Σ_P. Put Ω_S the strict transform of Ω_P and Σ_S the exceptional divisor. By the previous propositions one has that for any $v \in \Omega_R$

$$(\pi_P \circ \pi_S)^{-1} = Bl_{\mathbb{P}Hom_1}\mathbb{P}Hom^{\omega}(sl(2), \Lambda^1) \Rightarrow \bigcup_{v\in\Omega_R} Bl_{\mathbb{P}Hom_1}\mathbb{P}Hom^{\omega}(sl(2), \Lambda^1) \subset \Omega_S.$$

Call Δ_S the closure of $\bigcup_{v\in\Omega_R} Bl_{\mathbb{P}Hom_1}\mathbb{P}Hom^{\omega}(sl(2), \Lambda^1)$ in Ω_P. By dimension counting we show that Δ_S is equal to the divisor Ω_S if and only if $g = 2$. We prove that:

Proposition. *(a)* Ω_S^{ss} *is smooth and all its points are stable;*

(b) Σ_S^{ss} *is smooth and all its points are stable;*

(c) $\mathcal{S}_{Dol}^{ss}$ *is smooth and all its points are stable;*

(d) Δ_S *is smooth.*

4) We blow up $\mathcal{S}_{Dol}$ along Δ_S and call the space so obtained $\mathcal{T}_{Dol}$. We call $\hat{\mathcal{M}}_{Dol} := \mathcal{T}_{Dol} /\!/ \mathrm{SL}(2, \mathbb{C})$ and we prove the following.

Proposition. $\hat{\mathcal{M}}_{Dol} \xrightarrow{\hat{\pi}} \mathcal{M}_{Dol}$ *is a desingularization of* $\mathcal{M}_{Dol}$.

2.4.1 Normal cones of the singularities in $\mathcal{R}_{Dol}$

In this section we compute the normal cones of the singular loci of $\mathcal{R}_{Dol}$.

Cones of elements in Σ_R^0

Consider

$$\Sigma_R^0 := \{(V, \Phi, \beta) \mid (V, \Phi) = (L, \phi) \oplus (L^{-1}, -\phi) \text{ with } (L, \phi) \ncong (L^{-1}, -\phi)\}.$$

We want to prove the following result.

Proposition 2.4.1. *Let Σ_R^0 be the set above. Then Σ_R^0 is nonsingular and the cone $C_{\Sigma_R}\mathcal{R}_{Dol}$ of Simpson's theorem is a locally trivial fibration over Σ_R^0 with fiber the affine cone over a nonsingular quadric in $\mathbb{P}^{4g-5}$. More precisely, for a point $v = (V, \Phi, \beta)$ in Σ_R^0 there is a canonical isomorphism*

$$(C_{\Sigma_R}\mathcal{R}_{Dol})_v \cong \{(b, c) \in Ext^1_H(L^{-1}, L) \oplus Ext^1_H(L, L^{-1}) \mid b \cup c = 0\}$$

Moreover the action of the stabilizer $\mathbb{C}^*$ of (V, Φ) on $C_{\Sigma_R}\mathcal{R}_{Dol}$ is given by

$$\lambda.(b, c) = (\lambda^{-2}b, \lambda^2 c)$$

The proof will proceed in several steps and lemmas. If want to use the strategy suggested by Simpson in theorem (2.3.6), we need to find the vector spaces T^i and find the quadratic cone in T^1 defined by the zero locus of the Yoneda square.

Lemma 2.4.2. *Let (V, Φ, β) be an element of Σ_R^0. Then the spaces $T^i = Ext^i_0(V, V)$ are*

$$\begin{aligned} T^0 &= Ext^0_H(L, L) \cong \mathbb{C} \\ T^1 &= Ext^1_H(L, L) \oplus Ext^1_H(L^{-1}, L) \oplus Ext^1_H(L, L^{-1}) \cong \mathbb{C}^{6g-4} \\ T^2 &= Ext^2_H(L, L) \cong \mathbb{C} \end{aligned}$$

Proof. Let us compute the $Ext^i_H(L_i, L_j)$ using (2). First we compute $Ext^i_H(L, L)$.
We have

$$0 \longrightarrow Ext^0_H(L, L) \longrightarrow H^0(\mathcal{O}) \xrightarrow{\psi} H^0(K_C) \longrightarrow$$

$$Ext^1_H(L, L) \longrightarrow H^1(\mathcal{O}) \xrightarrow{\psi} H^1(K_C) \longrightarrow Ext^2_H(L, L) \longrightarrow 0$$

where the map ψ sends an element $f \in H^0(\mathcal{O})$ in $f\phi - \phi f$. As ϕ is $\mathcal{C}^\infty$-linear, every $f \in H^0(\mathcal{O})$ commutes with the Higgs field ϕ we have that $Ext^0_H(L, L) \cong H^0(\mathcal{O}) \cong \mathbb{C}$. Moreover $Ext^0_H(L, L) \cong Ext^2_H(L, L)$ by Serre duality [1] and we have $Ext^1_H(L, L) \cong H^0(K_C) \oplus H^1(\mathcal{O})$. Thus

$$Ext^0_H(L, L) \cong \mathbb{C} \quad Ext^1_H(L, L) \cong \mathbb{C}^{2g} \quad Ext^0_H(L, L) \cong \mathbb{C}$$

[1] we mean Serre duality for Higgs bundles

Now we compute $Ext^i(L, L^{-1})$.

We have

$$0 \longrightarrow Ext^0_H(L, L^{-1}) \longrightarrow H^0(L^2) \xrightarrow{\psi} H^0(L^2 \otimes K_C) \longrightarrow$$

$$Ext^1_H(L, L) \longrightarrow H^1(L^2) \xrightarrow{\psi} H^1(L^2 \otimes K_C) \longrightarrow Ext^2_H(L, L^{-1}) \longrightarrow 0$$

We have to be careful in doing this computation. In fact even though (L, ϕ) and $(L^{-1}, -\phi)$ are not isomorphic as Higgs bundles, L and L^{-1} might be isomorphic as vector bundles. However we can see this does not change the nature of our description of the normal cone. Suppose first $L \ncong L^{-1}$: then L^2 is a nontrivial degree 0 line bundle thus it has no global sections and we can conclude that $Ext^0_H(L, L^{-1}) = Ext^2_H(L, L^{-1}) = 0$. Also, $Ext^1_H(L, L^{-1}) \cong H^0(L^2 \otimes K_C) \oplus H^1(L^2) \cong \mathbb{C}^{2g-2}$; if $L \cong L^{-1}$ we have that $H^0(L^2) \cong H^0(\mathcal{O}) \cong \mathbb{C}$, the map ψ sends f to $\phi f + f\phi$. Since there are no nonzero elements in $H^0(\mathcal{O})$ that commute with the Higgs fields, then $Ext^0(L, L^{-1})$ is still 0. By Serre duality we can conclude that $Ext^2_H(L, L^{-1})$ is 0 too and the alternate sum of the dimensions of vector spaces in the sequence tells us that $Ext^1_H(L, L^{-1}) \cong \mathbb{C}^{2g-2}$ in both cases. To sum up we have that

$$Ext^0_H(L, L^{-1}) = Ext^2_H(L, L^{-1}) = 0 \qquad Ext^1_H(L, L^{-1}) \cong \mathbb{C}^{2g-2}.$$

The factors $Ext^i_H(L^{-1}, L)$ are isomorphic to $Ext^0_H(L, L^{-1})$ as we have the involution $L \mapsto L^{-1}$. Summing up we have

$$\begin{aligned} T^0 &= Ext^0_H(L, L) \cong Ext^0_H(L^{-1}, L^{-1}) = \mathbb{C} \\ T^1 &= Ext^1_H(L, L) \oplus Ext^1_H(L^{-1}, L) \oplus Ext^1_H(L, L^{-1}) \cong \mathbb{C}^{6g-4} \\ T^2 &= Ext^2_H(L, L) \cong Ext^2_H(L^{-1}, L^{-1}) \cong \mathbb{C} \end{aligned}$$

□

Now we need to describe the Yoneda square.

Proposition 2.4.3. $\mathbb{P}(\overline{\Upsilon}^{-1}(0))$ *is a nonsingular quadric hypersurface in* $\mathbb{P}^{4g-5}$*. In particular, as* $g \geq 2$, $\mathbb{P}(\Upsilon^{-1}(0))$ *is a reduced irreducible quadric.*

Proof. By Serre duality, the Yoneda product

$$\begin{array}{ccccc} Ext^1_H(L, L^{-1}) & \times & Ext^1_H(L^{-1}, L) & \longrightarrow & Ext^2_H(L, L) \cong \mathbb{C} \\ b & & c & \mapsto & b \cup c \end{array}$$

is a perfect pairing. Hence

$$\mathbb{P}(\overline{\Psi}^{-1}(0)) \subset \mathbb{P}(Ext^1_H(L, L^{-1}) \times Ext^1_H(L^{-1}, L)) = \mathbb{CP}^{4g-5}$$

is a nonsingular quadric hypersurface. □

In order to prove proposition (2.4.1) we show that $(C_{\Sigma_R}\mathcal{R}_{Dol})v \cong \overline{\Upsilon}^{-1}(0)$ and that Σ_R^0 is smooth.

Let $\mathcal{U}$ be a slice normal to the closed $SL(2,\mathbb{C})$ orbit of v: by proposition (2.3.4), there is a natural isomorphism between $Def(\mathcal{U}, v) \cong Def(V, \Phi, \beta)$. In particular we have an embedding

$$C_v\mathcal{U} \subseteq Ext^1_H(V, V).$$

Proposition 2.4.4. *There are natural isomorphism of schemes*

$$C_v\mathcal{U} \cong H_v\mathcal{U} \cong \Upsilon^{-1}(0).$$

Proof. By the equality (2.4) and proposition (2.4.3)

$$\mathbb{P}(H_v\mathcal{U})_{red} \cong \mathbb{P}(\Upsilon^{-1}(0)).$$

As $\mathbb{P}(\Upsilon^{-1}(0))$ is a reduced irreducible quadric hypersurface and $\mathbb{P}(H_v\mathcal{U})$ is cut out by quadrics

$$\mathbb{P}(H_v\mathcal{U}) \cong \mathbb{P}(\Upsilon^{-1}(0)).$$

Consider the inclusion

$$C_v\mathcal{U} \subset H_v\mathcal{U} = \Upsilon^{-1}(0).$$

By what we said above, we have

$$\dim C_v\mathcal{U} = \dim \mathcal{U} \geq \dim Ext^1_H(V, V) - 1 = \dim \Upsilon^{-1}(0).$$

Since $\Upsilon^{-1}(0)$ is reduced irreducible, we must have $C_v\mathcal{U} = \Upsilon^{-1}(0)$. □

Lemma 2.4.5. *Let* $\mathcal{W} := \mathcal{U} \cap \Sigma_R^0$. *Then* $\mathcal{W}$ *is smooth at* v *and*

$$T_v \mathcal{W} \cong Ext_H^1(L, L).$$

Furthermore, up to shrink $\mathcal{U}$*, we can assume that*

$$\dim T_v \mathcal{U} = \dim T_{v'} \mathcal{U} \quad \forall v' \in \mathcal{U}.$$

Proof. Using the identification $(\mathcal{U}, v)$ with $Def(V, \Phi, \beta)$, we call $\mathcal{V}$ a first order deformation of (V, Φ, β) and $e = (a, b, c) \in Ext_H^1(V, V)$ its corresponding extension class. Then, by classical deformation theory, we have that e is tangent to $\mathcal{W}$ if and only if the following two exact sequences of Higgs bundles

$$0 \to L \to V \to L^{-1} \to 0$$

$$0 \to L^{-1} \to V \to L \to 0$$

lift to $\mathcal{V}$. By [OG2] this condition is equivalent to

$$b = c = 0$$

that is $e = (a, 0, 0) \in Ext_H^1(L, L)$. To prove smoothness, we observe that $\mathcal{W}$ parametrizes Higgs bundles $(L', \phi') \oplus (L'^{-1}, -\phi')$, where (L', ϕ') near (L, ϕ), this implies that the dimension of $\mathcal{W}$ at v is $\geq 2g$ (L lies in $Pic^0(C)$). On the other hand the right-hand of the equation has dimension $2g$, hence $\mathcal{W}$ is smooth at v. To prove the last statement it suffices to notice that $(\mathcal{U}, v)$ is a versal deformation. □

Now we are ready to start proving lemmas that will lead to the proof of proposition (2.4.1).

Lemma 2.4.6. *Keep the notation as above. Then* Σ_R^0 *is smooth.*

Proof. Let $v = (V, \Phi, \beta) \in \Sigma_R^0$ and $\mathcal{U}$ be a slice normal to the $SL(2, \mathbb{C})$-orbit $O(v)$ and $\mathcal{W} = \mathcal{U} \cap \Sigma_R^0$. By Luna's étale slice theorem there exists a neighbourhood of $v \in \Sigma_R^0$ isomorphic to a neighbourhood of $(1, v)$ in $SL(2, \mathbb{C}) \times_{St(v)} \mathcal{W}$. As $\mathcal{W}$ is smooth at v, $SL(2, \mathbb{C}) \times_{St(v)} \mathcal{W}$ is smooth at the point $(1, v)$ and v is a smooth point of Σ_R^0. □

Proof of proposition (2.4.1). We have already proved that Σ_R^0 is smooth in lemma (2.4.6). Now we just need to prove that the fiber of the normal cone is isomorphic to $\overline{\Upsilon}^{-1}(0)$.

We have seen that

$$(C_{\mathcal{W}}\mathcal{U})_v \cong (C_{\Sigma_R}\mathcal{R}_{Dol})_v,$$

therefore we must give an isomorphism

$$(C_{\mathcal{W}}\mathcal{U})_v \cong \overline{\Upsilon}^{-1}(0).$$

As $\mathcal{W}$ is smooth and $T_w\mathcal{U}$ has constant dimension for every $w \in \mathcal{W}$ then the normal bundle $N_w\mathcal{W}$ and we have the usual inclusions of cones

$$(C_{\mathcal{W}}\mathcal{U})_v \subset (H_{\mathcal{W}}\mathcal{U})_v \subset (N_{\mathcal{W}}\mathcal{U})_v.$$

By lemma (2.4.5), the fiber of the normal cone $N_{\mathcal{W}}\mathcal{U}$ is equal to $Ext^1_H(L, L^{-1}) \oplus Ext^1(L^{-1}, L)$. Now if we rewrite in (2.3) using the identifications of cones of the normal slice then up to projectivize we have

$$H_v\mathcal{U} \text{ is the cone over } (H_{\mathcal{W}}\mathcal{U})_v \text{ with vertex } T_v\mathcal{W},$$

since $T_v\mathcal{W} \cong Ext^1_H(L, L)$ and $H_v\mathcal{U} \cong \Upsilon^{-1}(0)$ then $(H_{\mathcal{W}}\mathcal{U})_v \cong \overline{\Upsilon}^{-1}(0)$. Arguing as in the previous proof we conclude that $(C_{\mathcal{W}}\mathcal{U})_v \cong (H_{\mathcal{W}}\mathcal{U})_v$. □

Finally we describe the action of the stabilizer.

Let $v = (V, \phi, \beta)$ be a point with a closed orbit in Σ_R and let $\mathcal{U}$ be a slice normal to the orbit $O(v)$ of v. As $St(v) = Aut(V)/\mathbb{C}^*$ the action of the stabilizer on $\mathcal{U}$. For any $g \in Aut(V)$ we define the differential

$$g_* : T_v\mathcal{U} \to T_v\mathcal{U}$$

of the corresponding to the action of g.

Lemma 2.4.7. *Keeping the notation as above, let*

$$e \in T_v\mathcal{U} \cong T_0Def(V) \cong Ext^1(V, V)$$

then $g_*(e) = g \cup e \cup g^{-1}$.

Proof. Suppose V' and W' be the first order deformations corresponding to e and $g_* e$ respectively. Consider the tautological quotient on $C \times \mathcal{R}_{Dol}$: the action of $Aut(V)$ restricts to $C \times \mathcal{U}$ compatibly with the action on $\mathcal{U}$. Then there exists an isomorphism $\alpha_g : W' \to V'$ fitting the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & tV & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & 0 \\ & & \uparrow g^{-1} & & \downarrow \alpha_g & & \downarrow g & & \\ 0 & \longrightarrow & tV & \longrightarrow & W' & \longrightarrow & V & \longrightarrow & 0 \end{array}$$

with whom it is possible to identify $g_*(e)$ with the deformation given by $g \cup e \cup g^{-1}$. □

If v is a point in Σ_0 then $Aut(V) \cong \mathbb{C}^*$. Write $V = L \oplus L^{-1}$; consider $g \in \mathbb{C}^*$ and $e \in Ext^1(V,V) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ with $a \in Ext^1(L,L), b \in Ext^1(L^{-1},L), c \in Ext^1(L,L^{-1})$. Then

$$g_*(e) = \begin{pmatrix} g^{-1} & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} = \begin{pmatrix} a & g^{-2}b \\ g^2 c & -a \end{pmatrix}$$

as stated in proposition (2.4.1).

Cone of elements in Ω_R

Let $v = (V, \Phi, \beta)$ be an element in Ω_R. Then we have

$$(V, \Phi) = (L, 0) \oplus (L, 0)$$

with $L \cong L^{-1}$. Then the bundle $End_0(V)$ is holomorphically trivial and we have that $H^0(End_0(V)) \cong sl(2)$ and we can think a generic element of this space as

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

with $a, b, c \in H^0(\mathcal{O})$. We now want to compute the T^i's and the quadratic cone defined by the graded commutator. In order to make the computation easier, we first notice that the second line of the long exact sequence is the Serre dual of the first one. Now we observe that T^0 are the elements in $sl(2)$ which commute with the Higgs field, which is 0, therefore

$$T^0 \cong T^2 \cong sl(2)$$

and the first map and the last map of the sequence are isomorphisms. To compute T^1 consider the central part of the sequence, which in this case is

$$0 \longrightarrow H^0(End_0(V) \otimes K_C) \longrightarrow T^1 \longrightarrow H^1(End_0(V)) \longrightarrow 0.$$

$H^0(End_0(V) \otimes K_C) = H^0(\mathcal{O}^{\oplus 3} \otimes K_C) \cong H^0(K_C) \otimes sl(2)$. Using Serre duality we have that $H^1(End_0(V)) \cong H^1(\mathcal{O}) \otimes sl(2)$, therefore we have that T^1 has dimension $6g$ and it is equal to

$$T^1 = (H^0(K_C) \oplus H^1(\mathcal{O})) \otimes sl(2) = Ext^1_H(L, L) \otimes sl(2)$$

Set

$$\Lambda^i = Ext^i_H(L, L)$$

and consider the composition of the Yoneda product on Λ^1 with the isomorphism $\Lambda^2 \cong \mathbb{C}$ given by the integration:

$$\Lambda^1 \times \Lambda^1 \to \Lambda^2 \cong \mathbb{C}.$$

This defines a skew-symmetric form which is non-degenerate bilinear form ω which is non-degenerate by Serre duality. Call

$$Hom^\omega(sl(2), \Lambda^1) := \{f : sl(2) \to \Lambda^1 \mid f^*\omega = 0\}.$$

We have a natural action of the automorphism group $\mathrm{SL}(2, \mathbb{C})$ of (V, Φ) given by the composition with the adjoint representation on $sl(2)$.

Remark 10. Let us remark that $Hom^\omega(sl(2), \Lambda^1)$ is precisely the set of those $f \in Hom(sl(2), \Lambda^1)$ whose image is an isotropic subspaces of Λ^1 with respect to the symplectic form ω on it.

Proposition 2.4.8. *Ω^0_R is a smooth closed subset of $\mathcal{R}_{Dol}$ and the normal cone is a locally trivial bundle over Ω^0_R and there exist a $\mathrm{SL}(2, \mathbb{C})$-equivariant isomorphism*

$$(C_{\Omega_R}\mathcal{R}_{Dol})_v \cong Hom^\omega(sl(2), \Lambda^1)$$

Proof. As we noticed in the previous paragraph, there is a natural isomorphism

$$T^1 = \Lambda^1 \otimes sl(2)$$

and the Yoneda product is the tensor product of the Yoneda product Υ on Λ^1 times the composition with bracket of $sl(2)$. Hence if $\Upsilon : \Lambda^1 \otimes sl(2) \to sl(2)$ is the Yoneda square,

$$\Upsilon(\sum_i \lambda_i \otimes m_i) = \sum_{i,j} \omega(\lambda_i, \lambda_j)[m_i, m_j]$$

Thanks to the self duality of $sl(2)$ as an algebra and to the identifications

$$sl(2) \otimes \Lambda^1 \cong Hom(sl(2), \Lambda^1) \qquad sl(2) \cong \bigwedge^2 sl(2)$$

we have a map

$$\begin{array}{rccc} \Upsilon : & Hom(sl(2), \Lambda^1) & \to & \bigwedge^2 sl(2) \\ & f & \mapsto & 2f^*\omega \end{array}$$

and $\Upsilon^{-1}(0) = Hom^{\omega}(sl(2), \Lambda^1)$.

To complete our proof we need to give an isomorphism for any $v \in \Omega^0_R$

$$C_v \mathcal{R}_{Dol} \cong \Upsilon^{-1}(0).$$

First we prove that this locus is reduced and we proceed as in the case of Σ^0_R. More precisely we have the following lemma:

Lemma 2.4.9. $\mathbb{P}(\Upsilon^{-1}(0))$ *a reduced irreducible complete intersection of three quadrics in* $\mathbb{P}^{6g-1}$.

Proof. We first observe that the quadrics that intersects are precisely those given by the isotropy conditions. In fact if $f \in Hom^{\omega}(sl(2), \Lambda^1)$ then $Im(f)$ is an isotropic subspace of Λ^1, therefore if $\{a_1, a_2, a_3\}$ is basis of $sl(2)$ then $\omega(f(a_i), f(a_j)) = 0$ for all $i, j = 1 \dots 3$ which gives us the three quadrics. Now we need to prove that their intersection is complete, irreducible and reduced. To do that we determine the critical locus of Υ. Consider the polarization of the quadratic form Υ

$$\tilde{\Upsilon}(\sum_i m_i \otimes \lambda_i, \sum_j n_j \otimes \mu_j) := \sum_{i,j} \omega(\lambda_i, \mu_j)[m_i, n_j] :$$

then the differential of Υ in a point $f = \sum_i m_i \otimes \lambda_i$ is given by

$$\begin{array}{rccc} d\Upsilon(f) : & sl(2) \otimes \Lambda^1 & \to & sl(2) \\ & \sum_j n_j \otimes \mu_j & \mapsto & \sum_{i,j} \omega(\lambda_i, \mu_j)[m_i, n_j] \end{array}$$

One can easily see that the rank of $d\Upsilon(f)$ depends just on rkf:

$$rk(d\Upsilon(f)) = \begin{cases} 3 \text{ if } rkf \geq 2 \\ 2 \text{ if } rkf = 1 \\ 0 \text{ if } f = 0 \end{cases}$$

Let $cr(\Upsilon)$ be the critical set of Υ: it is give by the $f \in Hom(sl(2), \Lambda^1)$ whose rank is ≤ 1. Then, as $g \geq 2$,

$$\dim \mathbb{P}(cr(\Upsilon)) = 2g + 1 < 6g - 4 = \dim \mathbb{P}(sl(2) \otimes \Lambda^1) - 3$$

the dimension of the critical set is strictly less than the dimension of $\Upsilon^{-1}(0)$, therefore the intersection of the three quadrics is reduced and complete. Now we need to prove irreducibility: from the above consideration we see that the dimension of the projectivization of the singular locus of $\Upsilon^{-1}(0)$ in $\Upsilon^{-1}(0)$ is strictly bigger than 1; on the other hand the above formula for the rank of the differential show that for every singular point p

$$\dim T_p \mathbb{P}\Upsilon^{-1}(0) = \dim \mathbb{P}\Upsilon^{-1}(0) + 1.$$

If $\mathbb{P}^{-1}(0)$ were reducible, as it is connected it should be the intersection of two irreducible components. However the above equality shows that the intersection of those components should be the intersection of two divisors in a smooth ambient space, hance it should have codimension 1 in $\mathbb{P}\Upsilon^{-1}(0)$, which contradicts what we said above. □

We are now ready to construct the isomorphism between $\Upsilon^{-1}(0)$ and the fibre of the normal cone $C_{\Omega_R}\mathcal{R}_{Dol}$. We first observe that since Ω_R consists of isolated points, then $(C_{\Omega_R}\mathcal{R}_{Dol})_v = C_v\mathcal{R}_{Dol}$. Proceeding as in the case of Σ and using the previous lemma we have that $\mathbb{P}(H_v\mathcal{R}_{Dol}) = \mathbb{P}(H_v\mathcal{R}_{Dol})_{red} = \mathbb{P}(\Upsilon^{-1}(0))$. Now consider the inclusion

$$C_v\mathcal{R}_{Dol} \subset H_v\mathcal{R}_{Dol} = \Upsilon^{-1}(0);$$

as

$$\dim C_v\mathcal{R}_{Dol} = \dim \mathcal{R}_{Dol} = 6g - 3 = \dim Ext^1_H(V, V) - 3 = \dim \Upsilon^{-1}(0)$$

then $C_v\mathcal{R}_{Dol}$ should be an irreducible component of $\Upsilon^{-1}(0)$, which is irreducible: thus $C_v\mathcal{R}_{Dol} = \Upsilon^{-1}(0)$. This completes the proof of proposition (2.4.8). □

2.4.2 The space $\mathcal{P}_{Dol}$, its singularities and normal cones

Call $\pi_P : \mathcal{P}_{Dol} \to \mathcal{R}_{Dol}$ the blow-up of $\mathcal{R}_{Dol}$ along Ω_R, and let Ω_P be its exceptional divisor. We have seen that this is isomorphic to $Hom^{\omega}(sl(2), \Lambda^1)$. As our aim is to compute the desingularization of the GIT quotient $\mathcal{M}_{Dol}$ of $\mathcal{R}_{Dol}$ by the action of $SL(2, \mathbb{C})$, we need to describe just the semistable points of $\hat{\Omega}$ because the other will disappear when we do the quotient.

Semistable points in Ω_P

Proposition 2.4.10. *Let $[f]$ be an element of $Hom^{\omega}(sl(2), \Lambda^1)$. Then $[f]$ is semistable with respect to the action of* $\mathrm{SL}(2, \mathbb{C})$ *if and only if*

$$rkf \begin{cases} \geq 2 \text{ or} \\ = 1 \text{ and } kerf^{\perp} \text{ is non isotropic ,} \end{cases}$$

where orthogonality and isotropy are with respect to the Killing form on $sl(2)$.

Proof. We observe that the action of $\mathrm{SL}(2, \mathbb{C})$ on Λ^1 is trivial, therefore we just consider the action on $Hom(sl(2), \Lambda^1) \cong sl(2) \otimes \Lambda^1 \cong sl(2)^{2g}$ with the adjoint representation applied simultaneously on every factor. We see that the torus $\mathbb{C}^*$ of $\mathrm{SL}(2, \mathbb{C})$ acts with weight 2 on E, -2 on F and 0 on H. If we apply the Hilbert-Mumford criterion we see that a point is not semistable if and only if it is either of type $(E, E, \dots, E)$ or $(F, F, \dots, F) \in sl(2)^{2g}$. To give this condition in a way which is invariant under conjugation, we ask precisely for the rank of f to be greater equal than 2 (which corresponds to the cases in which two different matrices (E, F, H) are present in the vector) or to be of dimension 1 with the orthogonal non isotropic (and this corresponds to the case $(H, H, \dots, H)$). □

Semistable points of Σ_P

Call Σ_P the strict transform of Σ_R under the blow-up. Again, we want to describe the locus Σ_P^{ss} of semistable points. We start by describing $\Sigma_P^{ss} \setminus \hat{\Omega}$: by proposition (2.2.2)

$$\Sigma_P^{ss} \setminus \Omega_P \subseteq \pi_P^{-1}(\Sigma_R^{ss} - \Omega_R) = \pi_P^{-1}(\Sigma_R^0 \coprod \Omega_R').$$

We want to prove the following result:

Proposition 2.4.11. *Keep the notation as above. Then*

$$\Sigma_P^{ss} \setminus \Omega_P = \pi_P^{-1}(\Sigma_R^0)$$

To prove the proposition, we need the following lemma [OG, Lem. 1.7.4].

Lemma 2.4.12. *Assume G is a reductive group acting linearly on a complex projective scheme Y and S be a closed G-invariant subscheme. Let $p : \tilde{Y} \to Y$ be the blow up of S. Let $\tilde{v} \in \tilde{Y}$ be a point such that $v := p(\tilde{v})$ is such that*

$$v \notin S, \quad \overline{O(v)} \cap S^{ss} \neq 0,$$

then $\tilde{v}$ is not semistable.

Now consider $w \in \mathcal{P}_{Dol}$ such that $\pi_P(w) = v \in \Omega'_R$. Then $\overline{O(v)} \cap \Omega_R \neq \emptyset$ hence by the above lemma w is not semistable. Hence $\pi_P^{-1}(\Omega'_R) \cap \mathcal{P}_{Dol}^s s = \emptyset$ and $\Sigma_P^{ss} \setminus \Omega_P \subseteq \pi_P^{-1}(\Sigma_R^0)$. We want to show the reverse inclusion, that is that every point in $\pi^{-1}(\Sigma_R^0)$ is semistable. Consider $w \in \pi_P^{-1}(\Sigma_R^0)$ and let $\pi_P(w) = v$. As $O(v)$ is closed in $\mathcal{R}_{Dol}^{ss}$ and disjoint from the $\mathrm{SL}(2, \mathbb{C})$-invariant closed subset Ω_R

Now consider the intersection $\Sigma_P^{ss} \cap \Omega_P$: again, by Kirwan's theorem, we can see that it contained in $\pi_P^{-1}(\Omega_R)$ which consists of 2^{2g} copies of $\mathbb{P}Hom^{\omega}(sl(2), \Lambda^1)$.

Lemma 2.4.13. *Let $v \in \Omega_R$. Then*

$$\pi_P^{-1}(v) \cap \Sigma_P^{ss} = \mathbb{P}Hom_1^{ss}(sl(2), \Lambda^1)$$

where $Hom_1^{ss}(sl(2), \Lambda^1)$ is the set of those $f \in Hom^{\omega}(sl(2), \Lambda^1)$ which are semistable and of rank ≤ 1 and has dimension $2g$.

Proof. If $w \in \Sigma_R^0$ then it has stabilizer $\mathbb{C}^*$. Thus $\dim St(\tilde{w}) \geq 1$ for any $\tilde{v} \in \Sigma_P$. In particular if

$$[f] \in \pi_P^{-1}(v) \cap \Sigma_P^{ss}$$

the stabilizer $St([f])$ has positive dimension. By the description given in the proof of proposition (2.4.10), we have that the stabilizer has positive dimension if and only if $\operatorname{rank} f = 1$ and this tells us that $\pi_P^{-1}(v) \cap \Sigma_P^{ss} \subset \mathbb{P}Hom_1^{ss}(sl(2), \Lambda^1)$.

Let's prove the other inclusion. Assume $[f] \in \mathbb{P}Hom_1^{ss}(sl(2), \Lambda^1)$. The isomorphisms $sl(2) \cong sl(2)^*$ allow to write

$$[f] = m \otimes \alpha \qquad m \in \mathrm{SL}(2, \mathbb{C}), \alpha \in \Lambda^1, \mathrm{Tr}(m^2) \neq 0.$$

As $\mathrm{Tr}(m^2) \neq 0$, m is diagonalizable and using a basis of eigenvectors we can write f as

$$f = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \qquad \lambda \in \Lambda^1, \tag{2.6}$$

Now we can deform the points in Ω_R on a curve, that is we can find a sheaf $\mathcal{L}$ on a smooth curve Γ such that for a given point $0 \in \Gamma$

$$\mathcal{L}_0 \cong \mathcal{L}_0^{-1} = (L, 0)$$

and $\mathcal{L}_p \ncong \mathcal{L}_p^{-1}$ for all $p \neq 0$. Call $\mathcal{K}$ and $\mathcal{K}^{-1}$ the Kodaira-Spencer map of $\mathcal{L}$ and $\mathcal{L}^{-1}$, then

$$\mathcal{K}(\partial/\partial t) = \lambda \qquad \mathcal{K}^{-1}(\partial/\partial t) = -\lambda, \qquad \partial/\partial t \in T_0\Gamma.$$

Set $\mathcal{V} = \mathcal{L} \oplus \mathcal{L}^{-1}$. If $\mathcal{U}$ is a slice normal to the orbit of v, then there exists a map $\psi : \Gamma \to \mathcal{U}, 0 \mapsto v$ such that $\mathcal{G}$ is the pullback of the quotient sheaf on $C \times \mathcal{U}$. By (2.6), the differential of ψ at 0 has image spanned by f. Also, since $\psi^{-1}(\Omega_R) = \{0\}$, there is a well defined lift $\tilde{\psi} : \Gamma \to \mathcal{P}_{Dol}$ such that $\tilde{\psi}(\Gamma) \subset \Sigma_P$. Thus $[f] = \tilde{\psi}(0) \in \Sigma_P^{ss} \cap \Omega_P$. □

The aim of this section is to prove the following proposition:

Proposition 2.4.14. *Keeping notation as above,*

(i) Σ_P^{ss} *is smooth;*

(ii) The intersection $\Sigma_P^{ss} \cap \Omega_P$ *is smooth and reduced;*

(iii) The normal cone of Σ_P^{ss} *in* $\mathcal{P}_{Dol}$ *is a locally trivial bundle over* Σ_P^{ss}*, with fibre the cone over a smooth quadric in* $\mathbb{P}^{4g-5}$.

We omit the proof of the first two points in the proposition, for which we refer to [OG, Prop. 1.7.10], and describe the normal cone. We observe that outside Ω_P, π_P is an isomorphism therefore the normal cone of $\Sigma_P^{ss} - \Omega_P$ is isomorphic to $C_{\Sigma_R}\mathcal{R}_{Dol}$, whose fibre is a smooth quadric in $\mathbb{P}^{4g-5}$.
Now let $w \in \Sigma_P^{ss} \cap \Omega_P$ and set $v := \pi_P(w)$, then w will be of the form $w = [f]$, where f is an element of $\mathbb{P}Hom_1^{ss}(sl(2), \Lambda^1)$. Since Ω_P and Σ_P^{ss} intersect transversely, then

$$(C_{\Sigma_P}\mathcal{P}_{Dol})_{[f]} \cong (C_{\Sigma_P \cap \Omega_P}\Omega_P)_{[f]};$$

also, since $\Omega_P^{ss} \to \Omega_R$ is a locally trivial fibration over 2^{2g} distinct points then

$$(C_{\Sigma_P \cap \Omega_P}\Omega_P)_{[f]} \cong (C_{\mathbb{P}Hom_1(sl(2),\Lambda^1)}\mathbb{P}Hom^{\omega}(sl(2), \Lambda^1))_{[f]}$$

If $[f] \in \mathbb{P}Hom_1(sl(2), \Lambda^1)$, Im$f$ is a one-dimensional isotropic subspace of Λ^1 with respect to the symplectic form ω defined in the previous section and it makes sense to consider the space $\mathrm{Imf}^{\perp_\omega}/\mathrm{Imf}$. We call ω_f the symplectic form induced by ω on $\mathrm{Imf}^{\perp_\omega}/\mathrm{Imf}$, which is a space of dimension $2g - 2$.

Lemma 2.4.15. *Keep the notation as above. Then*

$$(C_{\mathbb{P}Hom_1(sl(2),\Lambda^1)}\mathbb{P}Hom^{\omega}(sl(2), \Lambda^1))_{[f]} \cong Hom^{\omega_f}(\mathrm{Ker}f, \mathrm{Imf}^{\perp_\omega}/\mathrm{Imf})$$

Remark 11. Lemma (2.4.15) directly implies the proof of point (iii) in proposition (2.4.14): in fact $Hom(\mathrm{Ker}f, \mathrm{Imf}^{\perp_\omega}/\mathrm{Imf})$ is a vector space of dimension $4g-4$ and since ω_f is non-degenerate the isotropy condition given by ω_f on the images of basis of Ker$f$ defines a cone over a smooth projective quadric, which will live in $\mathbb{P}^{4g-5}$.

Proof of lemma 2.4.15. We first observe that

$$(C_{\mathbb{P}Hom_1(sl(2),\Lambda^1)}\mathbb{P}Hom^{0}(sl(2), \Lambda^1))_{[f]} \cong (C_{Hom_1(sl(2),\Lambda^1)}Hom^{\omega}(sl(2), \Lambda^1))_{[f]}$$

and we can work on the right-hand side. First we show that the Hessian cone to $Hom_1(sl(2), \Lambda^1)$ in $Hom^{\omega}(sl(2), \Lambda^1)$ is defined and that it is equal to the normal cone. We observe that $Hom_1(sl(2), \Lambda^1)$ is smooth. Also, $Hom^{\omega}(sl(2), \Lambda^1)$ is the zero set of Υ and $d\Upsilon^{-1}(0)$ has constant rank along $Hom_1(sl(2), \Lambda^1)$ therefore the tangent space to $Hom^{\omega}(sl(2), \Lambda^1)$ has constant rank along $Hom_1(sl(2), \Lambda^1)$. Now we want to compute the Hessian and normal cone: to do this we choose a basis $\{\lambda_1, \ldots, \lambda_{2g}\}$ of Λ^1 and $\{m_1, m_2, m_3\}$ of $sl(2)$ such that $f = \lambda_1 \otimes m_1$ and such that the matrix associated to ω is block diagonal with g blocks of order 2 of the form

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Using the formula for the differential in the proof of lemma 2.4.9 and noticing that $\omega(\lambda_1, \lambda_i) = 0$ whenever $i \neq 2$, we get that

$$d\Upsilon_0(\phi)(\sum_{i,j} Z_{ij}\lambda_i \otimes m_j) = Z_{22}[m_1, m_2] + Z_{23}[m_1, m_3],$$

hence

$$(THom^{\omega}(sl(2), \Lambda^1))_{\phi} = \left\{ \sum_{i,j} Z_{ij}\lambda_i \otimes m_j \mid Z_{22} = Z_{23} = 0 \right\}.$$

If we consider rank 1 applications, they have to be of the form $\sum_i Z_{ij}\lambda_i \otimes m_j$ for a fixed $j = 1, 2, 3$ and they annihilate the differential if and only if either j or $i = 1$. As a result,

$$(THom_1(sl(2), \Lambda^1))_{\phi} = \left\{ \sum_{i,j} Z_{ij}\lambda_i \otimes m_j \mid Z_{ij} = 0 \text{ if } i \geq 2, j \geq 2 \right\}.$$

Thus we have an isomorphism

$$(N_{Hom_1}Hom^{\omega}(sl(2), \Lambda^1))_{\phi} \cong \left\{ \sum_{i \geq 3, j \geq 2} Z_{ij}\lambda_i \otimes m_j \right\}. \tag{2.7}$$

Considering the natural isomorphism

$$(N_{Hom_1}Hom(sl(2), \Lambda^1))_{\phi} \cong Hom(\operatorname{Ker}\phi, \Lambda^1/\operatorname{Im}\phi)$$

given by writing the generators of the right hand side in terms of tensor products, we can view the normal bundle as the set of functions whose image is orthogonal to $\operatorname{Im}\phi$:

$$(N_{Hom_1}Hom^{\omega}(sl(2), \Lambda^1))_{\phi} \cong \{\alpha : \operatorname{Ker}\phi \to \Lambda^1/\operatorname{Im}\phi \mid Im\alpha \subset (\operatorname{Im}\phi^{\perp}/\operatorname{Im}\phi)\}.$$

Viewing $(N_{Hom_1}Hom^{\omega}(sl(2), \Lambda^1))_{\phi}$ as a deformation space and compute the Yoneda square as in equation (2.4) we get that the equation of the Hessian cone of $Hom_1(sl(2), \Lambda^1)$ in $Hom^{\omega}(sl(2), \Lambda^1))$ is

$$\sum_{2 \leq l \leq g} (Z_{2l-1,2}Z_{2l,3} - Z_{2l,2}Z_{2l-1,3}) = 0.$$

In particular the hypotheses of lemma (2.3.4) are satisfied, hence the normal cone is equal to the Hessian cone and we are done. □

Action of the stabilizers.

We want to describe the action of the $St(w)$ on $(C_{\Sigma_P}\mathcal{P}_{Dol})_w$ at a point $w \in \Sigma_P^{ss}$.
First we notice that if w is outside Ω_P, the action is the one described in proposition (2.3.4). In fact $(\Sigma_P^{ss}\Omega_P) = \pi_P^{-1}(\Sigma_R^0)$ and on this set π_P is an isomorphism. If instead $w \in \Omega_P \cap \Sigma_P^{ss}$ then by lemma (2.4.12) we can write $w = [f]$ for an element $[f] \in Hom_1^{ss}(sl(2), \Lambda^1)$. By the stability condition, $\operatorname{Ker} f$ must be non isotropic. We choose bases $\{\lambda_1, \dots, \lambda_{2g}\}$ of Λ^1 and $\{m_1, m_2, m_3\}$ of $sl(2)$ as in the previous section, adding the conditions

$$(m_1, m_2) = -\delta_{1i}$$
$$(m_j, m_j) = 0 \quad j = 2, 3$$
$$(m_2, m_3) = 1$$

and $m_1 \wedge m_2 \wedge m_3$ is the volume form (here we are exploiting again the isomorphism $sl(2) \cong \bigwedge^2 sl(2)$. We observe that an element $\theta \in \mathrm{SL}(2, \mathbb{C})$ stabilize $[f]$ if and only if it is an orthogonal transformation of $\operatorname{Ker} f$ with respect to the Killing form. The stabilizer $St(f)$ is generated by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

The action on the normal cone $C_{\Sigma_P}\mathcal{P}_{Dol}$ is given by multiplication of the above matrices with the m_j appearing in the expression of equation (2.7).

2.4.3 Semistable points of $\mathcal{S}_{Dol}$ and construction of the desingularization

Call $\pi_S : \mathcal{S}_{Dol} \to \mathcal{P}_{Dol}$ the blow-up of $\mathcal{P}_{Dol}$ along Σ_P. Let $\Omega_S \subset \mathcal{S}_{Dol}$ be the strict transform of Ω_P and $\Sigma_S \subset \mathcal{S}_{Dol}$ be the exceptional divisor (i.e. the inverse image Σ_P). Let $v = (V, \Phi, \beta) \in \omega_R$ and set $V = L \oplus L$. By lemma (2.4.12) and the second item of proposition (2.4.13),

$$(\pi_P \circ \pi_S)^{-1}(v) = Bl_{\mathbb{P}Hom_1}\mathbb{P}Hom^{\omega}(sl(2), \Lambda^1)$$

thus

$$\bigcup_{v \in \Omega_R} Bl_{\mathbb{P}Hom_1} \mathbb{P}Hom^{\omega}(sl(2), \Lambda^1) \subset \Omega_S.$$

Call Δ_S the closure of the left-hand side. Observe that $\mathcal{S}_{Dol}$ has dimension $6g - 3$ while $Bl_{\mathbb{P}Hom_1} \mathbb{P}Hom^{\omega}(sl(2), \Lambda^1)$ has dimension $4g$, thus

$$codim(\Delta_S, \mathcal{S}_{Dol}) = 2g - 3$$

As Ω_S is a divisor in $\mathcal{S}_{Dol}$ then $\Delta_S = \Omega_S$ if and only if $g = 2$.
Let now $\pi_T : \mathcal{T}_{Dol} \to \mathcal{S}_{Dol}$ be the blow up of $\mathcal{S}_{Dol}$ along Δ_S and denote by Ω_T and Σ_T the proper transforms of respectively Ω_S and Σ_S. We define

$$\hat{\mathcal{M}}_{Dol} := \mathcal{T}_{Dol} /\!/ \mathrm{SL}(2, \mathbb{C})$$

By proposition (2.2.2), there exists a map $\hat{\pi} : \hat{\mathcal{M}}_{Dol} \to \mathcal{M}_{Dol}$ which is induced by the equivariant map $\pi_P \circ \pi_S \circ \pi_T$. Set

$$\hat{\Omega} := \Omega_T /\!/ \mathrm{SL}(2, \mathbb{C}) \qquad \hat{\Sigma} := \Sigma_T /\!/ \mathrm{SL}(2, \mathbb{C})$$

We now prove, following the method by [OG], that $\hat{\mathcal{M}}_{Dol}$ is a desingularization of $\mathcal{M}_{Dol}$. In the next section, in the case of genus 2, we construct a desingularization $\tilde{\mathcal{M}}_{Dol}$ such that the map $\tilde{\pi} : \tilde{\mathcal{M}}_{Dol} \to \mathcal{M}_{Dol}$ is semismall.

Analysis of Ω_S

We have defined Ω_S as the strict transform of Ω_P under the map π_S.

Proposition 2.4.16. *The following holds:*

(i) Ω_S^{ss} *is smooth,*

(ii) $\Omega_S^{ss} = \Omega_S^{s}$.

To do that we need some preliminary lemmas.

Lemma 2.4.17. *Let* $v \in \Omega$. *Then the fibre* $(\pi_P \circ \pi_S)^{-1}(v)$, *which is equal to* $Bl_{\mathbb{P}Hom_1} \mathbb{P}Hom^{\omega}(sl(2), \Lambda^1)$, *is nonsingular.*

Proof. By lemma 2.4.15 the exceptional divisor is a locally trivial fibration over $\mathbb{P}Hom^{\omega}(sl(2), \Lambda^1)$ and the fibre over a point $[f]$ is

$$Hom^{\omega_f}(\mathrm{Ker} f, \mathrm{Imf}^{\perp_\omega}/\mathrm{Imf}),$$

that is a smooth quadric in $\mathbb{P}^{4g-5}$. As the base $\mathbb{P}Hom^{\omega}(sl(2), \Lambda^1)$ is smooth, so is the exceptional divisor. Thus the blow up is smooth along the exceptional divisor. Then the complement of the exceptional divisor is smooth by (2.4.3). □

Lemma 2.4.18. *All* $\mathrm{SL}(2, \mathbb{C})$ *semistable points of* $Bl_{\mathbb{P}Hom_1}\mathbb{P}Hom^{\omega}(sl(2), \Lambda^1)$ *are* $\mathrm{SL}(2, \mathbb{C})$ *stable. More explicitly:*

(i) Referring to the notation of (2.4.15), the semistable points in the exceptional divisor are given by

$$\left\{([f], [\alpha]) \mid [f] \in \mathbb{P}Hom_1^{ss}(sl(2), \Lambda^1), [\alpha] \in \mathbb{P}Hom^{\omega_f}(\ker f, \mathrm{Im}\, f^{\perp}/\,\mathrm{Im}\, f), \alpha(m_2) \neq 0 \neq \alpha(m_3)\right\}$$

Moreover, for $([f], [\alpha])$ *in the above set, the stabilizer*

$$St([f], [\alpha]) \cong \begin{cases} \mathbb{Z}_2 & \text{if } \mathrm{rank}\, \alpha = 2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } \mathrm{rank}\, \alpha = 1. \end{cases}$$

(ii) The semistable points which are not in the exceptional divisor are given by

$$\left\{[f] \in \mathbb{P}Hom^{ss}(sl(2), \Lambda^1) \mid \mathrm{rank}\, f = 3 \text{ or } \mathrm{rank}\, f = 2 \text{ and } \ker f \text{ non isotropic}\right\}$$

For $[f]$ *belonging to this set, stabilizer* $St([f])$ *is trivial if* $\mathrm{rank}\, f = 3$ *and equal to* $\mathbb{Z}_2$ *if* $\mathrm{rank}\, f = 2$.

Proof. By (2.2.2) the semistable points of the exceptional divisor are contained in the inverse image of $\mathbb{P}Hom_1^s s(sl(2), \Lambda^1)$. If we apply the Hilbert-Mumford criterion as in proposition (2.4.10), we are asking precisely for the images of E, F under the isomorphism of (2.4.10) not to vanish. Rephrasing this condition in an equivariant way we get item (i).Let's prove item (ii). Applying again the numerical criterion, we observe that all the points of the set are stable and by proposition (2.2.2) they remain so after the blow-up. We show that if $\mathrm{rank}\, f = 2$ and

Ker f is isotropic, then $[f]$ is not semistable. Choose $m \in sl(2)$ such that $m \in \operatorname{Ker} f^{\perp}$ and $m \notin \operatorname{Ker} f$. Then there exists a one parameter subgroup $\lambda : \mathbb{C}^* \to SL(2, \mathbb{C})$ such that

$$\lim_{t \to 0} \lambda(t).f = g$$

where $\operatorname{rank} g = 1$ and $\operatorname{Ker} g^{\perp} = m$. Thus $[g]$ should be in $\mathbb{P}Hom_1^s s(sl(2), \Lambda^1)$, which is the centre of the blow-up. However lemma (2.4.12) tells us that in this case $[f]$ cannot be semistable because it does not belong to centre of the blow-up but the closure of its orbit intersects the semistable points of it.

□

We are now ready to prove proposition (2.4.16).

Proof of proposition (2.4.16). By (2.2.2) we know that $(\pi_P \circ \pi_S)(\Omega_S^{ss}) \subset \Omega_R$. Let $v \in \Omega_R$, by lemma (2.4.17) the fibre $(\pi_P \circ \pi_S)^{-1}(v)$ is smooth. As semistability is an open condition, we get Ω_S^{ss}. The second item, follows directly from lemma (2.4.18). □

Analysis of Σ_S^{ss}

Proposition 2.4.19. *The following holds:*

(i) Σ_S^{ss} *is nonsingular,*

(ii) $\Sigma_S^{ss} = \Sigma_S^s$.

Proof. By (2.2.2), we have that $\Sigma_S^{ss} \subset \pi_S^{-1}(\Sigma_P^{ss}) = \mathbb{P}(C_{\Sigma_P^{ss}} \mathcal{P}_{Dol})$. Let $w \in \Sigma_P^{ss}$ and let $v = \pi_P(w)$. Then either $v \in \Sigma_P^0$ or $v \in \Omega_P^0$. In the latter case, the preimage has been described in the previous proposition. In the former case, we have that $\Sigma_S^{ss} \cap (\pi_P \circ \pi_S)^{-1}(v) = \mathbb{P}\{(b, c) \mid b \cup c = 0, b, c \neq 0\}$. Also, all semistable points are stable. The stabilizer of any point in the above set is $\mathbb{Z}_2$. Thus for every $w \in \Sigma_P^{ss}$, π_S^{-1} is a smooth quadric in $\mathbb{P}^{4g-5}$. By item (i) of (2.4.16) Σ_P^{ss} is smooth. Again, since stability is an open condition, we conclude that Σ_S^{ss} is smooth. The second item now follows from the previous claim. □

Analysis of $\mathcal{S}^{ss}_{Dol}$

By Kirwan's propositions we have that

$$\mathcal{S}^{ss}_{Dol} = \Sigma^{ss}_S \cup \Omega^{ss}_S \cup (\pi_S \circ \pi_P)^{-1}(\mathcal{R}^s_{Dol} \cup \Sigma'^0_R \cup \Omega'^0_R)$$

However, by (2.4.11) there are no semistable points in $(\pi_S \circ \pi_P)^{-1}(\Omega'^0_R)$ and if we apply lemma (2.4.12) to $Y = \mathcal{P}_{Dol}$, $\tilde{Y} = \mathcal{S}_{Dol}$ and $V = \Sigma_P$ we get that for any $w \in \pi_P^{-1}(\Sigma'^0_R)$ $O(w) \cap \Sigma^{ss}_P \neq 0$, therefore there are no semistable points in $(\pi_S \circ \pi_P)^{-1}(\Sigma'^0_R)$. Thus we conclude that

$$\mathcal{S}^{ss}_{Dol} = \Sigma^{ss}_S \cup \Omega^{ss}_S \cup (\pi_S \circ \pi_P)^{-1}(\mathcal{R}^s_{Dol}) \tag{2.8}$$

Proposition 2.4.20. *We have:*

(i) $\mathcal{S}^{ss}_{Dol}$ *is nonsingular,*

(ii) $\mathcal{S}^{ss}_{Dol} = \mathcal{S}^s_{Dol}$.

Proof. The first item follows from the fact that $(\pi_S \circ \pi_P)^{-1}(\mathcal{R}^s_{Dol})$ lies in the stable locus by lemma (2.2.2) , and we have just proved every pointΩ^{ss}_S and Σ^{ss}_S is indeed stable. To prove the second item we observe that $\mathcal{R}^s_{Dol}$ is smooth (this follows from the smoothness of the deformation space of any point $\mathcal{R}^s_{Dol}$). As $(\pi_S \circ \pi_P)$ is an isomorphism on the stable locus, then also $(\pi_S \circ \pi_P)^{-1}(\mathcal{R}^s_{Dol})$ is smooth. Now we conclude by noticing that both Ω^s_S and Σ^{ss}_S are nonsingular Cartier divisors, therefore $\mathcal{S}^s_{Dol}$ is smooth along them. □

Analysis of Δ^s_S

We defined Δ_S as the closure in $\mathcal{S}_{Dol}$ of the locus

$$\bigcup_{v \in \Omega_R} Bl_{\mathbb{P}Hom_1} \mathbb{P}Hom^{\omega}(sl(2), \Lambda^1).$$

Proposition 2.4.21. *Keep the notation as above. Then Δ_S is nonsingular.*

We want to see that Δ^s_S is nonsingular. As stability is an open condition, it suffices to prove that each one of the 2^{2g} fibres $Bl_{\mathbb{P}Hom_1}\mathbb{P}Hom^{\omega}(sl(2), \Lambda^1)$ is nonsingular.

We set

$$\mathrm{Gr}^{\omega}(k, \Lambda^1) := \{[A] \in \mathrm{Gr}(k, \Lambda^1) \mid \text{A is } \omega\text{-isotropic } \}$$
$$\tilde{\mathbb{P}}Hom^{\omega}_2(sl(2), \Lambda^1) := \{([K], [A], [f]) \in \mathbb{P}(sl(2)) \times \mathrm{Gr}(2, \Lambda^1) \times \mathbb{P}Hom^{\omega}_2(sl(2), \Lambda^1) \mid K \subset \mathrm{Ker}\, f, \mathrm{Im}\, f \subset A\},$$

and let $g : \tilde{\mathbb{P}}Hom_2^{\omega}(sl(2), \Lambda^1) \to \mathbb{P}Hom_2^{\omega}(sl(2), \Lambda^1)$ the projection onto the third factor.

Lemma 2.4.22. *There exists an $SL(2, \mathbb{C})$ equivariant isomorphism*

$$\tilde{g} : \tilde{\mathbb{P}}Hom_2^{\omega}(sl(2), \Lambda^1) \to Bl_{\mathbb{P}Hom_1}\mathbb{P}Hom_2^{\omega}(sl(2), \Lambda^1)$$

such that the map g corresponds to the blow down map.

As the $\tilde{\mathbb{P}}Hom_2^{\omega}(sl(2), \Lambda^1)$ is nonsingular by lemma (2.4.17), then lemma (2.4.22) implies that also $Bl_{\mathbb{P}Hom_1}\mathbb{P}Hom_2^{\omega}(sl(2), \Lambda^1)$ is, showing in this way that Δ_S^s is nonsingular.

Proof. By the Second Fundamental Theorem of Invariant Theory, the ideal I_{PHom_1} of $\mathbb{P}Hom_1$ is generated by 2×2 minors. Thus $g^* I_{PHom_1}$ is locally generated by the "determinant" of $\bar{f} : sl(2)/K \to A$, hence it is locally principal. By the universal property of the blow up, there exists a map $\tilde{g}$ as in the statement of the lemma. We now want to prove $\tilde{g}$ is an isomorphism. Choose bases of $sl(2)$ and Λ^1 and realize the blow up as the closure in $\mathcal{P}Hom_2^{\omega}(sl(2), \Lambda^1) \times \mathbb{P}^{4g-3}$ of

$$\{([f], \dots, [m_{IJ}(f)], \dots) \mid f \in Hom_2(sl(2), \Lambda^1), \operatorname{rank} f = 2, m_{IJ}(f) = (I \times J)\text{-minor}, \mid I \mid = \mid J \mid = 2\}$$

The map $\tilde{g}$ is given by

$$([K], [A], [f]) \mapsto ([f], \dots, [p_I(K) q_J(A)], \dots)$$

where $P_I(K)$ are the Plücker coordinates of $[K^{\perp}] \in Gr(2, sl(2)^*)$, and $q_J(A)$ are Plücker coordinates of $[A]$. This proves $\tilde{g}$ is an isomorphism and it is equivariant by construction. □

Smoothness of $\hat{\mathcal{M}}_{Dol}$

Proposition 2.4.23. *Let $\hat{\mathcal{M}}_{Dol} = \mathcal{T}_{Dol}//SL(2, \mathbb{C})$. Then $\hat{\pi} : \hat{\mathcal{M}}_{Dol} \to \mathcal{M}_{Dol}$ is a desingularization of $\mathcal{M}_{Dol}$.*

Proof. We are now ready to prove that $\hat{\pi} : \hat{\mathcal{M}}_{Dol} \to \mathcal{M}_{Dol}$ is a desingularization.
By the first item of (2.4.20) the semistable points of $\mathcal{S}_{Dol}$ are actually stable, hence

$$\mathcal{T}_{Dol}^{ss} = \pi_T^{-1}(\mathcal{S}_{Dol}^{ss}) = \pi_T^{-1}(\mathcal{S}_{Dol}^{s}) = Bl_{\Delta_S^s}\mathcal{S}_{Dol}^{s}.$$

As both Δ_S^s and $\mathcal{S}_{Dol}^s$ are nonsingular, so is the blow up $\mathcal{T}_{Dol}^s$. By (2.4.18), if $v \in \Omega_R$ then

$\Delta_S \cap \Sigma_S \cap (\pi_P \circ \pi_S)^{-1}(v) = \{([f], [\alpha]) \in \mathbb{P}Hom_1(sl(2), \Lambda^1) \times \in \mathbb{P}Hom^{\omega_f}(\operatorname{Ker} f, \operatorname{Im} f^{\perp} / \operatorname{Im} f) \mid \operatorname{rank} \alpha = 1\}$

By (2.4.19) and (2.4.18), for every point of $z \in \mathcal{T}_{Dol}^s$

$$\begin{cases} \{1\} & \text{if } z \notin \Sigma_T^s \cup \Delta_T^s \\ \mathbb{Z}_2 & \text{if } z \in (\Sigma_T^s \cup \Delta_T^s) \setminus (\Sigma_T^s \cap \Delta_T^s) \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } z \in \Sigma_T^s \cap \Delta_T^s \end{cases}$$

Since Σ_T^s and Δ_T^s are divisors, we conclude that $\hat{\mathcal{M}}_{Dol}$ is nonsingular. □

2.5 Construction of the semismall desingularization for $g = 2$

We now restrict ourselves to the case of genus 2. Starting from the desingularization $\hat{\mathcal{M}}_{Dol}$ of $\mathcal{M}_{Dol}$, we construct another desingularization $\tilde{\mathcal{M}}_{Dol}$, such that the map $\tilde{\pi} : \tilde{\mathcal{M}}_{Dol} \to \mathcal{M}_{Dol}$ is semismall. To do that we first describe the divisor $\hat{\Omega}$: its fibre over a point $v \in \Omega$ is isomorphic to the total space of the projective bundle $\mathbb{P}(S^2\mathcal{A})$ where $\mathcal{A}$ is the tautological $\mathbb{C}^2$ bundle over the symplectic Grassmannian $\mathrm{Gr}^{\omega}(2, \Lambda^1)$.

Thanks to Mori theory, we prove that if we do a contraction of $\hat{\mathcal{M}}_{Dol}$ over the $\mathbb{P}^2$-fibration $\mathbb{P}(S^2\mathcal{A}) \to \hat{\Omega} \to \mathrm{Gr}^{\omega}(2, \Lambda^1)$, we end up with a semismall desingularization $\tilde{\mathcal{M}}_{Dol}$ of $\mathcal{M}_{Dol}$.

2.5.1 Description of $\hat{\Omega}$

Let $\mathrm{Gr}^{\omega}(2, \Lambda^1)$ be symplectic Grassmannian over any point $v = (V, 0) \in \Omega$ and let $\mathcal{A}$ be the tautological $\mathbb{C}^2$ bundle over it. We will prove the following.

Proposition 2.5.1. *Keeping the notation as above, then for any $v \in \Omega$ the fibre of the exceptional divisor is isomorphic to the projective bundle $\mathbb{P}(S^2\mathcal{A})$*

$$\hat{\Omega}_v \cong \mathbb{P}(S^2\mathcal{A})$$

Given $v \in \Omega$ we define the classes $\hat{\epsilon}_v$ and $\hat{\gamma}_v$ in the cone of effective curves $NE_1(\hat{\Omega}_v)$ in the Neron-Severi cone $N_1(\hat{\Omega}_v)$(see [Ko] for further details). We let $\hat{\epsilon}_v$ be the class in $N_1(\hat{\Omega}_v)$

of a line in the fibre of $\mathbb{P}(S^2\mathcal{A}) \to \mathrm{Gr}^\omega(2, \Lambda^1)$. To define $\hat{\gamma}_v$ we notice that proposition 2.5.1 gives the isomorphism $\hat{\Omega}_v \cong \mathbb{P}(S^2\mathcal{A})$. Choose $[H] \in \mathbb{P}(\Lambda^1) = \mathbb{P}^3$ and $[q_l] \in \mathbb{P}(S^2H)$ and let $\{[A_t] \in \mathrm{Gr}^\omega(2, \Lambda^1)\}_{t\in\mathbb{P}^1}$ be a line through $[H]$ i.e. for every $t \in \mathbb{P}^1$ there exists an inclusion $i_t : H \hookrightarrow A_t$ and $[A_t/H] \in \mathbb{P}(H^\perp/H)$ varies in a line. We observe that $[i^t_* q_l]$ is a local section of $\mathbb{P}(S^2\mathcal{A})$, therefore we can set

$$\hat{\gamma}_v := \left[([A_t], [i^t_* q_l])\right]_{N_1(\hat{\Omega}_v)}$$

and obtained an element of $N_1(\hat{\Omega}_v)$ which is effective by definition. Letting $i_v : \hat{\Omega}_v \hookrightarrow \hat{\mathcal{M}}_{Dol}$ be the inclusion, we set

$$\hat{\epsilon} := i_v^* \hat{\epsilon}_v$$

$$\hat{\gamma} := i_v^* \hat{\gamma}_v$$

As the right-hand sides of the equalities do not depend on the point $v \in \Omega$, $\hat{\epsilon}$ and $\hat{\gamma}$ are well defined as elements in $NE_1(\hat{\mathcal{M}}_{Dol})$. We obtain the following result.

Proposition 2.5.2. *Keep notation as above. Then:*

(i) $R^+\hat{\epsilon}$ *is a* $K_{\hat{\mathcal{M}}_{Dol}}$*-negative extremal ray;*

(ii) let $\tilde{\mathcal{M}}_{Dol}$ *be the variety obtained by contracting* $R^+\hat{\epsilon}$*. Then* $\tilde{\mathcal{M}}_{Dol}$ *is a smooth quasi-projective desingularization of* $\mathcal{M}_{Dol}$*.*

(iii) The contraction of $R^+\hat{\epsilon}$ *is identified with the contraction of* $\hat{\mathcal{M}}_{Dol}$ *along the fibration* $\mathbb{P}(S^2\mathcal{A}) \to \hat{\Omega} \to Gr^\omega(2, \Lambda^1)$*.*

(iv) Call $\tilde{\pi}$ *the map obtained by* $\hat{\pi}$ *contracting its fibres over the points in* Ω*. Let* $\tilde{\Omega} := \tilde{\pi}^{-1}(\Omega)$ *and* $\tilde{\Sigma} := \tilde{\pi}^{-1}(\Sigma)$*. The fibre of* $\tilde{\pi}$ *over a point in* Ω *is isomorphic to the nonsingular quadric hypersurface* $Gr^\omega(2, \Lambda^1)$ *in* $\mathbb{P}^4$*.*

(v) The fibre of $\tilde{\pi}$ *over a point in* Σ^0 *is isomorphic to* $\mathbb{P}^1$*.*

By proposition (2.5.2) we can prove the main theorem of this section.

Theorem 2.5.3. *Consider* $\tilde{\pi} : \tilde{\mathcal{M}}_{Dol} \to \mathcal{M}_{Dol}$*. Then* $\tilde{\pi}$ *is semismall.*

Proof. We recall that a proper map $f : X \to Y$ of algebraic varieties is semismall if and only if, put $Y_k = \{y \in Y \mid \dim f^{-1}(y) = k\}$, then one has

$$\dim Y_k + k \leq \dim X - k. \tag{2.9}$$

First of all we notice that since $\tilde{\pi}$ is birational, then it is proper.
Set

$$\mathcal{M}_{Dol,k} := \left\{v \in \mathcal{M}_{Dol} \mid \dim \tilde{\pi}^{-1}(v) = k\right\}$$

We stratify $\mathcal{M}_{Dol}$ as

$$\mathcal{M}_{Dol} = \mathcal{M}^s_{Dol} \sqcup \Sigma^0 \sqcup \Omega,$$

where $\mathcal{M}^s_{Dol}$ denotes the smooth locus of $\mathcal{M}_{Dol}$.
Since C is a curve of genus 2, $\mathcal{M}_{Dol}$ is a quasi-projective variety of dimension 6. We have seen in section 2.2 that the singular locus

$$\Sigma^0 = \{(V, \Phi) \mid (V, \Phi) = (L, \phi) \oplus (L^{-1}, -\phi), \text{ with } (L, \phi) \ncong (L^{-1}, -\phi)\}$$

is given by $\left[(Pic^0(C) \times H^0(K_C)) \setminus (16 \text{ points })\right]/\mathbb{Z}_2$. $Pic^0(C)$ is a 2-dimensional torus, while $H^0(K_C) \cong \mathbb{C}^2$ therefore Σ^0 has dimension 4. The singular locus

$$\Omega = \{(V, \Phi) \mid (V, \Phi) = (L, 0) \oplus (L, 0) \text{ with } L \cong L^{-1}\}$$

parametrizing the fixed points of the involution $(L, \phi) \mapsto (L^{-1}, -\phi)$ consists just of 16 points, corresponding to the roots of the trivial bundle on C.

On $\mathcal{M}^s_{Dol}$, $\tilde{\pi}$ is an isomorphism and every point has just one pre-image, thus $\mathcal{M}^s_{Dol} = \mathcal{M}_{Dol,0}$. Thus it satisfies (2.9). Let now $v \in \Sigma^0$. By proposition (2.5.2, (iv)), $\tilde{\Sigma} \setminus \Omega = \tilde{\pi}^{-1}(\Sigma^0)$ is a $\mathbb{P}^1$-bundle over Σ^0. Then one has that Σ_0 correspond the stratum $\mathcal{M}_{Dol,1}$. Again it satisfies (2.9). Finally,by (2.5.2, (iv)), the fibre over each one of the 16 points of Ω is isomorphic to $Gr^{\omega}(2, \Lambda^1)$, which is a nonsingular hypersurface in $\mathbb{P}^4$. As a result it has dimension 3. This tells us that Ω is $\mathcal{M}_{Dol,3}$ and that it satisfies (2.9) as well. □

Remark 12. We observe that all the strata indeed satisfy the equality

$$\mathcal{M}_{Dol,k} + k = \dim \tilde{\mathcal{M}}_{Dol} - k,$$

that is they are *relevant strata* in the decomposition theorem for semismall maps (1.5.1).

We now prove proposition (2.5.1). We recall that for genus 2, $\mathcal{T}_{Dol} = \mathcal{S}_{Dol}$, hence $\hat{\mathcal{M}}_{Dol} = \mathcal{S}_{Dol} // \operatorname{SL}(2,\mathbb{C})$. We call $q : \mathcal{S}^s_{Dol} \to \hat{\mathcal{M}}_{Dol}$ the quotient map.

Proof of proposition 2.5.1. We have the isomorphism

$$\tilde{\mathbb{P}}Hom^\omega_2(sl(2),\Lambda^1)// \operatorname{SL}(2,\mathbb{C}) \cong Bl_{\mathbb{P}Hom_1}\mathbb{P}Hom^\omega_2(sl(2),\Lambda^1)// \operatorname{SL}(2,\mathbb{C})\,.$$

As $\operatorname{SL}(2,\mathbb{C})$ acts trivially on $\operatorname{Gr}^\omega(2,\Lambda^1)$ we get a map

$$h : \tilde{\mathbb{P}}Hom^\omega_2(sl(2),\Lambda^1)// \operatorname{SL}(2,\mathbb{C}) \to Gr^\omega(2,\Lambda^1), \quad ([K],[A],f) \mapsto [A]$$

As we are considering the case rank $f = 2$ the semistable points of $\tilde{\mathbb{P}}Hom^\omega_2(sl(2),\Lambda^1)$ are in the preimage of semistable points of ω_S, therefore by (2.4.18) we have

$$\tilde{\mathbb{P}}Hom^\omega_2(sl(2),\Lambda^1)^{ss} = \tilde{\mathbb{P}}Hom^\omega_2(sl(2),\Lambda^1)^{s} = \{([K],[A],f) \mid [K] \text{ is non isotropic}\},$$

hence the projection on the first factor $\tilde{\mathbb{P}}Hom^\omega_2(sl(2),\Lambda^1) \to \mathbb{P}(sl(2))$ maps the stable locus to the complement of the isotropic conic, i.e. $\mathbb{P}(sl(2))^{ss}$. The action of $\operatorname{SL}(2,\mathbb{C})$ by adjoint representation on $\mathbb{P}(sl(2))^{ss}$ is transitive, therefore

$$h^{-1}([A]) = \mathbb{P}Hom(K^\perp, A)//SO(K^\perp)$$

where $[K] \in \mathbb{P}(sl(2))^{ss}$ is any chosen point. Now observe that the map $\mathbb{P}Hom(K^\perp, A) \to \mathbb{P}(S^2A), \quad \alpha \mapsto \alpha \circ^t \alpha$ is the quotient map for the $SO(K^\perp)$ action. As a consequence we have $h^{-1}(A) \cong \mathbb{P}(S^2A)$ for any $A \in \operatorname{Gr}^\omega(2,\Lambda^1)$. □

To prove proposition (2.5.2) we will use Mori theory. Here we state and prove some technical lemmas.

Lemma 2.5.4.

$$\overline{NE}_1(\hat{\Omega}_v) = R^+\hat{\epsilon}_v \oplus R^+\hat{\gamma}_v$$

Proof. Consider the maps $g : \hat{\Omega}_v \to \operatorname{Gr}^\omega(2,\Lambda^1) \leftarrow \hat{\Omega}_v$, $h : \mathbb{P}Hom^\omega(sl(2),\Lambda^1)// \operatorname{SL}(2,\mathbb{C})$. One can easy verify that they are the contractions of $R^+\hat{\epsilon}_v$ and $R^+\hat{\gamma}_v$ respectively. Therefore they are extremal rays. Now, since g is a $\mathbb{P}^2$-fibration on $\operatorname{Gr}^\omega(2,\Lambda^1)$, which is a smooth quadric threefold, then $N_1(\hat{\Omega}_v)$ has rank 2 and the lemma is proved. □

Now, take $[A] \in \mathrm{Gr}^\omega(2, \Lambda^1)$. We want to prove that $\hat{\Omega}_{|\mathbb{P}(S^2A)} \cong \mathcal{O}_{\mathbb{P}(S^2A)}(-1)$.

Lemma 2.5.5.

$$q^*(\hat{\Omega}) \sim 2\Omega_S^s$$

where $\sim$ denotes numerical equivalence.

Proof. Since $q^{-1}\hat{\Omega} = \Omega_S^s$, all we have to do is determine the multiplicity of $q^*\hat{\Omega}$ at a generic point of Ω_S^s. Let $v \in \Omega_S^s \setminus \Sigma_S$, by (2.4.18) the stabilizer $St(v)$ is equal to $\mathbb{Z}_2$. Let now $\mathcal{U} \subset \mathcal{S}_{Dol}^s$ be a slice normal to $= O(v)$. By (2.3.1), $\mathcal{U}//\mathbb{Z}_2$ is isomorphic to some neighbourhood of $q(v)$ in $\hat{\mathcal{M}}_{Dol}$. Since the fixed locus of the action of $\mathbb{Z}_2$ is $\Omega_S \cap \mathcal{U}$, the claim is true on $\mathcal{U}$. □

Lemma 2.5.6. *Let $[K] \in \mathbb{P}(sl(2))^{ss}$. As K is non isotropic then there exists a straight line Θ in $\mathbb{P}Hom^\omega(sl(2), \Lambda^1)$. Then*

$$\Omega_S \cdot \Theta = -1$$

where $\cdot$ denotes the standard intersection form.

Proof. We have $\Omega_s \sim \pi_S^*\Omega_P$ and

$$\Omega_{P|\mathbb{P}Hom^\omega(sl(2),\Lambda^1)} \cong \mathcal{O}_{\mathbb{P}Hom^\omega(sl(2),\Lambda^1)}(-1).$$

Since the restriction of π_S to Θ is an isomorphism to a straight line in $\mathbb{P}Hom^\omega(sl(2), \Lambda^1)$, then the intersection must be -1. □

Now we can prove that $\hat{\Omega}_{|\mathbb{P}(S^2A)} \cong \mathcal{O}_{\mathbb{P}(S^2A)}(-1)$. Suppose $\hat{\Omega}_{|\mathbb{P}(S^2A)} \cong \mathcal{O}_{\mathbb{P}(S^2A)}(a)$. By (2.5.4) q maps Θ on-to-one onto a conic $\Gamma \subset \mathbb{P}(S^2A)$. Using the previous lemmas we get

$$2a = \Omega \cdot \Gamma = q^*\Omega \cdot \Theta = 2\Omega_S \cdot \Theta = -2$$

from which we conclude that $a = -1$.

2.5.2 Analysis of $\hat{\Sigma}$

Let $v \in \Sigma^0$. As before, we call $\hat{\Sigma}_v := \hat{\pi}^{-1}(v)$. Hence $\hat{\Sigma}_v \subset (\hat{\Sigma} \setminus \hat{\Omega})$.

Proposition 2.5.7. *Keep the notation as above. Then there exists an isomorphism $\hat{\Sigma}_v \cong \mathbb{P}^1$ and $\hat{\Sigma} \cdot \hat{\Sigma}_v = -2$.*

Proof. By (2.4.1), we have that

$$\hat{\Sigma}_v \cong \mathbb{P}\{(b,c) \in Ext^1(L^{-1}, L) \oplus Ext^1(L, L^{-1}) \mid b \cup c = 0\}//\mathbb{C}^*$$

and the action of $\mathbb{C}^*$ is the one described in (2.4.1). As we have already seen, this is a perfect pairing, therefore one gets that $\hat{\Sigma}_v \cong \mathbb{P}^1$.
Consider now the skew-symmetric isomorphism $\psi : Ext^1(L, L^{-1}) \to Ext^1(L^{-1}, L)$ given by the involution and let $\Theta := \{(b, c, \psi(c))\} \subset \mathbb{P}\{(b,c) \mid b \cup c = 0\}^s$. Then $q(\Theta) \cong \hat{\Sigma}_v$ and the map is an isomorphism. Thus

$$\hat{\Sigma} \cdot \hat{\Sigma}_v = q^* \hat{\Sigma} \cdot \Theta,$$

again arguing as in the proof of lemma (2.5.5) we see that $q^*\hat{\Sigma} \sim 2\Sigma_S^s$. Moreover, as Θ is a line in $\mathbb{P}\{(b,c) \mid b \cup c = 0\}$,then $\Sigma_S \cdot \Theta = -1$. Thus

$$\hat{\Sigma} \cdot \hat{\Sigma}_v = q^* \hat{\Sigma} \cdot \Theta = 2\Sigma_S^s \cdot \Theta = -2.$$

□

Let now $k_v : \hat{\Sigma}_v \hookrightarrow \hat{\mathcal{M}}_{Dol}$ be the inclusion. We need to prove the following result.

Lemma 2.5.8. *Keeping the notation as above,*

$$k_{v*}\overline{NE}_1(\hat{\Sigma}) = R^+\hat{\gamma}.$$

Proof. As $\hat{\Sigma} \cdot \hat{\Sigma}_v = -2$ then $k_{v*}\overline{NE}_1(\hat{\Sigma}) = R^+[\hat{\Sigma}_v]$. If we approach Ω from Σ, we see that $[\hat{\Sigma}_v]$ can be represented by a one cycle Γ on $\hat{\Omega}_v \cap \hat{\Sigma}$. The cycle Γ must be mapped to a single point by the map induced by π_S, and this implies that it must be a multiple of the cycle defining $\hat{\gamma}$. □

Finally, we are ready to prove the first item of proposition (2.5.2).

Proof of item (i) of (2.5.2). We start by proving the first item. Arguing as in the previous proofs we see that $K_{\hat{\mathcal{M}}_{Dol}} \sim 2\hat{\Omega}$. Given that $\hat{\Omega}_{|\mathbb{P}(S^2A)} \cong \mathcal{O}_{\mathbb{P}(S^2A)}$, we deduce that $K_{\hat{\mathcal{M}}_{Dol}} \cdot \hat{\epsilon} = -2$ i.e. $R^+\hat{\epsilon}$ is $K_{\hat{\mathcal{M}}_{Dol}}$-negative. We show that $\hat{\epsilon}$ and $\hat{\gamma}$ are linearly independent and that the image of the map $i_{v*} : \overline{NE}_1(\hat{\Omega}) \to \overline{NE}_1(\hat{\mathcal{M}}_{Dol})$ is injective with image $R^+\hat{\epsilon} \oplus R^+\hat{\gamma}$. This comes from

the fact that $\hat{\Omega} \cdot \hat{\epsilon} = i_v^*[\hat{\Omega}] \cdot \hat{\epsilon} = -1$, thus by (2.3.3) $\hat{\Omega} \cdot \hat{\gamma} = 0$. As a consequence $\hat{\epsilon}$ and $\hat{\gamma}$ define independent elements in $N_1(\hat{\mathcal{M}}_{Dol})$. Now, noticing that $R^+\hat{\epsilon} \oplus R^+\hat{\gamma} = \overline{NE}_1(\hat{\Omega})$, then the image of the inclusion must be generated by them.
Given the previous observations, the prove that $R^+\hat{\epsilon}$ is extremal is a consequence of the following lemma. □

Lemma 2.5.9. *Keeping the notation as above, $R^+\hat{\epsilon} \oplus R^+\hat{\gamma}$ is an extremal face of $\overline{NE}_1(\hat{\mathcal{M}}_{Dol})$.*

Proof. Suppose to have a positive linear combination of irreducible curves on $\hat{\mathcal{M}}_{Dol}$ $\sum_{\alpha \in I} t_\alpha \Gamma_\alpha \subset R^+\hat{\epsilon} \oplus R^+\hat{\gamma}$. We want to show that in this case any Γ_α lies in $\subset R^+\hat{\epsilon} \oplus R^+\hat{\gamma}$. As $\hat{\pi}^*\hat{\epsilon} = \hat{\pi}^*\hat{\gamma} = 0$, we get $\hat{\pi}_*\Gamma_\alpha$ is zero, therefore $\hat{\pi}(\Gamma_\alpha)$ is a point. We can then partition the set $I = I_\Omega \coprod I_\Sigma$ such that if $\alpha \in I_\Omega$ then $\Gamma_\alpha \subset \hat{\Omega}_v$ for some $v \in \Omega$; if $\alpha \in I_\Sigma$ then $\Gamma_\alpha \subset \hat{\Sigma}_w$ for some $w \in \Sigma^0$. If $\alpha \in I_\Omega$ the first item follows from $R^+\hat{\epsilon} \oplus R^+\hat{\gamma} = \overline{NE}_1(\hat{\Omega})$; if $\alpha \in I_\Sigma$ it follows from $k_{v*}\overline{NE}_1(\hat{\Sigma}) = R^+\hat{\gamma}$. To prove the second item, we use Mori theory. We know we have a fibration $\mathbb{P}^2 \to \hat{\Omega}_v \to \mathrm{Gr}^\omega(2, \Lambda^1)$, where the $\mathbb{P}^2$ fibre is isomorphic to $\mathbb{P}(S^2A)$ for any $A \in \mathrm{Gr}^\omega(2, \Lambda^1)$. If we show that the contraction of the extremal ray $R^+\hat{\epsilon}$ is identified with the contraction of $\hat{\mathcal{M}}_{Dol}$ along this fibration, then by standard Mori theory we have that $\tilde{\mathcal{M}}_{Dol}$ is smooth. Let Θ be a line in the fibre of the fibration, then $[\Theta] = \hat{\epsilon}$. What we need to show is that if $\Gamma \subset \mathcal{M}_{Dol}$ is an irreducible curve such that $[C] \in R^+\hat{\epsilon}$, then Γ belongs to the fibre. Notice that $\Gamma \cdot \hat{\Omega} < 0$, hence $\Gamma \subset \hat{\Omega}$. Furthermore, since $\hat{\pi}_*(\Gamma) = 0$, there exists a point $v \in \Omega$ such that $\Gamma \subset \hat{\Omega}_v$. As $R^+\hat{\epsilon} \oplus R^+\hat{\gamma} = \overline{NE}_1(\hat{\Omega})$, then $[\Gamma] \in R^+\hat{\epsilon}$, therefore Γ belongs to the fibre. □

Finally we prove the last three items of proposition (2.5.2), that is that $\tilde{\mathcal{M}}_{Dol}$ is nonsingular. The proof is a direct consequence of the following lemma. Recall that we have the $\mathbb{P}^2$-fibration

$$\mathbb{P}^2 \to \hat{\Omega} \to \mathrm{Gr}^\omega(2, \Lambda^1) \tag{2.10}$$

where the fibre over any point $[A]$ is $\mathbb{P}(S^2A)$.

Lemma 2.5.10. *The contraction of $R^+\hat{\epsilon}$ is identified with the contraction of $\hat{\mathcal{M}}_{Dol}$ along the fibration (2.10).*

Proof of (ii),(iii), (iv) in proposition (2.5.2) . Consider a line Θ in the fibre of (2.10): then $[\Theta] = \hat{\epsilon}$. Hence we must prove that if $\Gamma \subset \tilde{\mathcal{M}}_{Dol}$ is an irreducible curve such that $[\Gamma] \in R^+\hat{\epsilon}$,

then Γ belongs to a fibre of (2.10). We have seen that $\Gamma \cdot \hat{\Omega} < 0$, hence $\Gamma \subset \hat{\Omega}$. Furthermore, since $\hat{\pi}_*\Gamma = 0$ there exists a point $v \in \Omega$ such that $\Gamma \subset \hat{\Omega}_v$. Then $[\Gamma] \in R^+\hat{\epsilon}_v$, i.e. Γ belongs to a fibre of (2.10).
We observe that the $\mathbb{P}^2$ fibres of $\hat{\Omega}$ that have been contracted are contained in the fibres of $\hat{\pi}$. From the previous lemma we deduce straightforward that $\tilde{\Omega}_v \cong Gr^{\omega}(2, \Lambda^1)$ for every $v \in \Omega$. Now let $v \in \Sigma^0$. If we again define $\hat{\Sigma}_v := \hat{\pi}^{-1}(v)$, then $\hat{\Sigma}_v$ is contained in $(\hat{\Sigma} \setminus \hat{\Omega})$. However we observe that outside of $\hat{\Omega}$ nothing has changed, thus $\tilde{\Sigma}_v := \tilde{\pi}^{-1}(v) = \hat{\Sigma}_v$, which isomorphic to $\mathbb{P}^1$ by lemma (2.5.7). □

We are now ready to prove that the map $\tilde{\pi}$ is semismall.

2.6 Intersection cohomology of $\mathcal{M}_{Dol}$

In the previous section we constructed a semismall desingularization $\tilde{\mathcal{M}}_{Dol} \xrightarrow{\tilde{\pi}} \mathcal{M}_{Dol}$ of the moduli space $\mathcal{M}_{Dol}$ of Higgs bundles of rank 2, degree 0 and trivial determinant over a curve of genus 2.

We can thus apply the decomposition theorem for semismall maps (1.5.1) which we restate here for ease of the reader.

Theorem. *Let $f : X \to Y$ be a semismall map of algebraic varieties. Let Λ_{rel} the set of relevant strata, and for each $Y_\alpha \in \Lambda_{rel}$ let $\mathcal{L}_\alpha$ the corresponding local system with finite monodromy defined above. Then there exists a canonical isomorphism in $\mathcal{P}(Y)$*

$$f_*\mathbb{Q}_X[\dim X] \cong \bigoplus_{Y_\alpha \in \Lambda_{rel}} IC_{\overline{Y}_\alpha}(\mathcal{L}_\alpha)$$

As we have seen in chapter 1, in this case the only supports are the relevant strata, that is, the strata Y_k for which $\dim Y_k + k = \dim X - k$.
In the case of $\tilde{\pi} : \tilde{\mathcal{M}}_{Dol} \to \mathcal{M}_{Dol}$, we have seen in the proof of theorem (2.5.3) that all the strata satisfy the equality thus they are all relevant. In particular we showed

$$\mathcal{M}^s_{Dol} = \mathcal{M}_{Dol,0} \quad \Sigma^0 = \mathcal{M}_{Dol,1} \quad \Omega = \mathcal{M}_{Dol,3}.$$

We stratify $\tilde{\mathcal{M}}_{Dol}$ as follows

$$\hat{\mathcal{M}}_{Dol} = \tilde{\pi}^{-1}\mathcal{M}^s_{Dol} \sqcup (\tilde{\Sigma} \setminus \tilde{\Omega}) \sqcup \tilde{\Omega}.$$

By proposition (2.5.2)

1) $\tilde{\pi}$ is an isomorphism on the smooth locus of $\mathcal{M}_{Dol}$;

2) $\tilde{\Omega} := \tilde{\pi}^{-1}(\Omega)$ is the union of 16 copies of a nonsingular projective hypersurface $\mathrm{Gr}^{\omega}(2, \Lambda^1)$ in $\mathbb{P}^4$;

3) the fibre of $(\tilde{\Sigma} \setminus \tilde{\Omega}) = \pi^{-1}(\Sigma^0)$ over any point of Σ^0 is isomorphic to $\mathbb{P}^1$.

Applying the decomposition theorem for semismall maps we get that,

$$IC_{\tilde{\mathcal{M}}_{Dol}} = IC_{\mathcal{M}_{Dol}}(\mathcal{L}_{\mathcal{M}_{Dol}}) \oplus IC_{\Sigma}(\mathcal{L}_{\Sigma}) \oplus IC_{\Omega}(\mathcal{L}_{\Omega}) \tag{2.11}$$

We will use the above splitting to compute the *intersection E-polynomial* $IE(\mathcal{M}_{Dol})$ of $\mathcal{M}_{Dol}$.

Definition 2.6.1. The IE-polynomial of a variety X is defined as

$$IE(X)(u, v) = \sum_{h=0}^{2\dim X} (-1)^k \sum_{h,p,q} ih_c^{k,p,q} u^p v^q$$

where $ih_c^{k,p,q} = \dim \mathrm{Gr}^p_F Gr^W_{p+q} IH^k_c(X)$ and satisfies the following properties:

(i) if $Z \subset X$ then $IE(X) = IE(Z) + IE(X \setminus Z)$

(ii) $IE(X \times Y) = IE(X)IE(Y)$

If we consider ordinary cohomology groups instead of intersection cohomology we just call the polynomial obtained in this way the E-polynomial of X and we denote it by $E(X)$.
Let's go back to the splitting (2.11). Let us observe that we as the fibres of $\tilde{\pi}$ over both Ω and Σ^0 are irreducible, then the monodromy of the local system is trivial. Moreover since Ω and Σ^0 are nonsingular we have

$$IC_{\mathcal{M}_{Dol}}(\mathcal{L}_{\mathcal{M}_{Dol}})_{|\mathcal{M}^s_{Dol}} = \mathbb{Q}[6] \quad IC_{\Sigma}(\mathcal{L}_{\Sigma})_{|\Sigma^0} \cong \mathbb{Q}[4](-1) \quad IC_{\Omega}(\mathcal{L}_{\Omega})_{|pt} \cong \mathbb{Q}(-3)$$

where the shifts (-1) and (-3) correspond to the Hodge structures $\mathbb{Q}(-1)$ of respectively $\mathbb{P}^1$ and $Gr^{\omega}(2, \Lambda^1)$.

Taking hypercohomology with compact support in (2.11), we obtain the intersection cohomology groups and the splitting in the decomposition theorem becomes

$$H^*_c(\tilde{\mathcal{M}}_{Dol}) = IH^*_c(\mathcal{M}_{Dol}) \oplus H^{*-2}_c(\Sigma, IC_{\Sigma}(\mathcal{L}_{\Sigma})) \oplus H^{*-6}_c(\Omega, IC_{\Omega}(\mathcal{L}_{\Omega}))$$

The only contributions from the summands supported on Σ and Ω come from the highest cohomology groups of the fibres. Therefore, when we consider the cohomology with compact support to find the IE-polynomial of $\mathcal{M}_{Dol}$, we first sum the E-polynomials of each stratum and compute the E-polynomial of $\tilde{\mathcal{M}}_{Dol}$. After that, we subtract the contribution coming from the top cohomology of the fibres to get the IE-polynomial of $\mathcal{M}_{Dol}$. We will have that

Theorem 2.6.1 (Main Theorem).

$$IE(\mathcal{M}_{Dol})(u, v) = u^6v^6 + u^5v^5 + 15u^4v^4 + u^5v^3 + u^3v^5 + 15u^3v^3 + u^2v^4 + u^4v^2.$$

We observe that

$$E(\tilde{\mathcal{M}}_{Dol}) = E(\mathcal{M}^s_{Dol}) + E(\tilde{\Sigma} \setminus \tilde{\Omega}) + E(\tilde{\Omega}) \tag{2.12}$$

thus in the following sections we compute the E-polynomial of each each summand.

2.7 Cohomology of $\mathcal{M}^s_{Dol}$

The aim of this section is to compute the cohomology with compact support of the smooth part $\mathcal{M}^s_{Dol}$ of the moduli space $\mathcal{M}_{Dol}$, which parametrizes pairs (V, Φ) that are stable. We will show that

Theorem 2.7.1. *Let $\mathcal{M}^s_{Dol}$ be the locus of stable Higgs bundles. Then the E-polynomial of $\mathcal{M}^s_{Dol}$ is*

$$E(\mathcal{M}^s_{Dol}) = u^6v^6 + u^5v^5 + 16u^4v^4 + 11u^3v^3 - 17u^2v^2$$

It is well known that $\mathcal{M}^s_{Dol}$ contains the locus $\mathcal{N}^S$ of stable vector bundles as open dense subset, but there are several Higgs bundles whose underlying vector bundle is not stable. This is due to the fact that not all vector subbundles of V are Higgs subbundles: for example on may consider the bundle

$$V = K_C^{-1} \oplus K_C$$

where K_C denotes the canonical bundle on$X$. This vector bundle is not stable because the subbundle K_C has slope greater than the slope of V; however K_C is not a Higgs subbundle because to be Φ invariant $Hom(K_C, K_C^{-1}) \cong K_C^{-2}$ should have global sections, which is not not the case as it is of negative degree.

To compute the E-polynomial of $\mathcal{M}^s_{Dol}$ we will construct a suitable stratification, compute the E-polynomial of the strata and sum them up. We will sistematically apply the following well known result.

Proposition 2.7.2 (**Addivity property of compact support cohomology**). *Let Y be a quasi-projective variety. Let Z be a closed subset of Y and call U its complement. Then, given the inclusions $U \overset{j}{\hookrightarrow} Y \overset{i}{\hookleftarrow} Z$ there is a long exact sequence in cohomology*

$$\dots \longrightarrow H^i_c(U) \xrightarrow{j_!} H^i_c(Y) \xrightarrow{i^!} H^i_c(Z) \longrightarrow \dots$$

Therefore we will divide stable Higgs pairs in following three strata:

- pairs (V, Φ) with V stable vector bundle;
- pairs (V, Φ) with V strictly semistable vector bundle;
- pairs (V, Φ) with V unstable vector bundle.

2.7.1 The stable case

We want to parametrize all the stable Higgs bundles (V, Φ) where V is a stable vector bundle. Calling $\mathcal{S}$ the locus of stable vector bundles, the stable Higgs pairs (V, Φ) are parametrized by the cotangent bundle $T^*\mathcal{S}$. We will show the following:

Proposition 2.7.3. *Keep the notation as above. The $E-$polynomial of the locus $T^*\mathcal{S}$ of stable Higgs pairs (V, Φ) with V stable vector bundles is*

$$E(\mathcal{T}^*\mathcal{S})(u, v) = u^6v^6 - u^3v^5 - u^5v^3 - 3u^4v^4$$

Proof. Narasimhan and Ramanan [NR] proved that the locus of semistable vector bundles with trivial determinant modulo S-equivalence (equivalently polystable vector bundles up to isomorphism) on a nonsingular projective curve C of genus 2 is isomorphic to $\mathbb{CP}^3$. Considering polystable pairs, a vector bundle V is strictly semistable if and only if is of the form

$$V = L \oplus L^{-1}, \quad L \in Pic^0(C)$$

therefore strictly semistable vector bundles are parametrized by $\mathcal{J} := Pic^0(C)/\mathbb{Z}_2$ where $\mathbb{Z}_2$ is the involution $L \mapsto L^{-1}$. This is a compact Kummer variety with 16 singular points, which are precisely the fixed points of the involution, whose desingularization is a K3 surface obtained by blowing up $\mathcal{J}$ in the singular points. The locus of stable bundles is precisely the complement of $\mathcal{J}$ inside $\mathbb{P}^3$: our strategy will be to compute the compact support cohomology of this locus and using Poincaré duality to obtain the Betti numbers. First we need to compute the cohomology of $\mathcal{J}$: observe that this is given by the $\mathbb{Z}_2$ invariant part of the cohomology of $Pic^0(C)$, which is a 2-torus. The Betti numbers of $Pic^0(C)$ are

$$b_0 = 1 \quad b_1 = 4 \quad b_2 = 6 \quad b_3 = 4 \quad b_4 = 1$$

and the action of $\mathbb{Z}_2$ on the cohomology sends every generator γ of H^1 in $-\gamma$. Therefore the even cohomology groups are all $\mathbb{Z}_2$-invariant, while the odd ones are never; thus the Betti numbers of $\mathcal{J}$ are

$$b_0 = 1 \quad b_1 = 0 \quad b_2 = 6 \quad b_3 = 0 \quad b_4 = 1.$$

Alternatively, one can notice that the cohomology of $\mathcal{J}$ differs from the one of its desingularization just in the H^2 part, which has in addition the cohomology of the 16 exceptional divisors isomorphic to $\mathbb{P}^1$, and the Betti numbers of a K3 surface are

$$b_0 = 1 \quad b_1 = 0 \quad b_2 = 22 \quad b_3 = 0 \quad b_4 = 1.$$

Such a description is useful to compute the weights of the cohomology: we observe that the mixed Hodge structure on the cohomology of a K3 surface is pure and so is the cohomology of $\mathcal{J}$. In particular we have that $H^0(\mathcal{J})$ has weights (0,0), $H^2(\mathcal{J})$ splits in $4(1,1)+(2,0)+(0,2)$, and $H^4(\mathcal{J})$ has weights $(2,2)$. Consider now the inclusions $\mathcal{S} \overset{j}{\hookrightarrow} \mathbb{P}^3 \overset{i}{\hookleftarrow} \mathcal{J}$ as both $\mathbb{P}^3$ and

$\mathcal{J}$ are compact, we have the long exact sequence:

$$\dots \longrightarrow H^k_c(\mathcal{S}) \xrightarrow{j_!} H^k(\mathbb{P}^3) \xrightarrow{i^!} H^k(\mathcal{J}) \longrightarrow \dots$$

which splits in the following sequences

$$0 \longrightarrow H^0_c(\mathcal{S}) \longrightarrow \mathbb{C} \xrightarrow{i^!} \mathbb{C} \longrightarrow H^1_c(\mathcal{S}) \longrightarrow 0 \qquad (1)$$

$$0 \longrightarrow H^2_c(\mathcal{S}) \longrightarrow \mathbb{C} \xrightarrow{i^!} \mathbb{C}^6 \longrightarrow H^3_c(\mathcal{S}) \longrightarrow 0 \qquad (2)$$

$$0 \longrightarrow H^4_c(\mathcal{S}) \longrightarrow \mathbb{C} \xrightarrow{i^!} \mathbb{C} \longrightarrow H^5_c(\mathcal{S}) \longrightarrow 0 \qquad (3)$$

$$0 \longrightarrow H^6_c(\mathcal{S}) \longrightarrow \mathbb{C} \longrightarrow 0 \qquad \Rightarrow H^6_c(\mathcal{S}) \cong \mathbb{C} \qquad (4)$$

First we consider (1): the map $i^! = i^*$ is the restriction to a hyperplane sections, therefore it is an isomorphism by Lefschetz hyperplane theorem, thus $H^0_c(\mathcal{S}) = H^1_c(\mathcal{S}) = 0$.
Next we have (2): $i^!$ is the restriction of the fundamental class of $\mathbb{P}^1$ inside $\mathbb{P}^2$ which remains nonzero when we intersect it generically with $\mathcal{J}$, thus $i^!$ is an injection and we have $H^2_c(\mathcal{S}) = 0$ and $H^3_c(\mathcal{S}) = \mathbb{C}^5$. A similar argument shows that, in (3), $i^!$ is an isomorphism and that $H^4_c(\mathcal{S}) = H^5_c(\mathcal{S}) = 0$.
Using Poincaré duality one has that the Betti numbers are

$$b_0 = 1 \quad b_1 = 0 \quad b_2 = 0 \quad b_3 = 5 \quad b_4 = 0 \quad b_5 = 0 \quad b_6 = 0.$$

As $T^*\mathcal{S}$ is a vector bundle over $\mathcal{S}$, it inherits the cohomology of its base space, so the compact support cohomology groups of $\mathcal{S}$ are

$$\begin{aligned} H^9_c(\mathcal{S}) &= 5 \quad \text{with weights } (3,5) + (5,3) + 3(4,4) \\ H^{12}_c(\mathcal{S}) &= 1 \quad \text{with weights } (6,6) \\ H^i_c(\mathcal{S}) &= 0 \quad \text{otherwise.} \end{aligned}$$

As a result, the E-polynomial of the stable part is given by

$$E(\mathcal{T}^*\mathcal{S})(u,v) = u^6v^6 - u^3v^5 - u^5v^3 - 3u^4v^4$$

□

2.7.2 Strictly semistable case

We want to consider the pairs (V, Φ), where V is a strictly semistable vector bundle, and investigate when they become stable Higgs pairs. Again, we have to distinguish different cases:

(i) $V = L \oplus L^{-1}$ where $L \in Pic^0(C)$ and $L \ncong L^{-1}$;

(ii) V is a non trivial extension $0 \longrightarrow L \longrightarrow V \longrightarrow L^{-1} \longrightarrow 0$ with $L \ncong L^{-1}$;

(iii) $V = L \oplus L^{-1}$ where $L \in Pic^0(C)$ and $L \cong L^{-1}$;

(iv) V is a non trivial extension $0 \longrightarrow L \longrightarrow V \longrightarrow L^{-1} \longrightarrow 0$ with $L \cong L^{-1}$;

Type (i)

We call $\mathcal{S}_1$ the locus of stable Higgs bundles with underlying vector bundle of type (i). We will show that

Proposition 2.7.4. *The E-polynomial of the locus of stable Higgs bundles of type (i) is*

$$E(\mathcal{S}_1)(u,v) = u^5v^5 + u^3v^5 + u^5v^3 + 3u^4v^4 - 21u^3v^3 + 15u^2v^2$$

Proof. We have already seen that strictly semistable vector bundles are parametrized by $\mathcal{J} = Pic^0(C)/\mathbb{Z}_2$. We call $\mathcal{J}_0$ locus in $\mathcal{J}$ fixed by the involution and we set $\mathcal{J}^0 := \mathcal{J} - \mathcal{J}_0$ to be its complement. The locus of stable Higgs bundles with underlying vector bundle of type (i) will be a fibre bundle on $\mathcal{J}^0$. To compute the fibre we consider $V = L \oplus L^{-1}$ with $L \in Pic^0(C)$ such that $L \ncong L^{-1}$. We have that

$$H^0(End_0(V) \otimes K_C) = H^0(K_C) \oplus H^0(L^2K) \oplus H^0(L^{-2}K_C)$$

thus a Higgs field $\Phi \in H^0(End_0(V) \otimes K_C)$ will be of the form

$$\Phi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

with $a \in H^0(K_C)$, $b \in H^0(L^2K)$, $c \in H^0(L^{-2}K_C)$. A pair (V, Φ) is stable if and only if both L and L^{-1} are not preserved by Φ, that is $b, c \neq 0$. Now we need to understand when two different Higgs fields give rise to isomorphic Higgs bundles: since the automorphisms group of V is $(\mathbb{C}^* \times \mathbb{C}^*) \cap SL(2,\mathbb{C}) \cong \mathbb{C}^*$, two Higgs pairs (V,Φ_1) and (V,Φ_2) for $\Phi = (a_i, b_i, c_i)$ are isomorphic if and only if

$$\Phi_1 = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \phi_2 \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$$

that is $a_1 = a_2$, $b_1 = t^2 b_2$, $c_1 = t^{-2} c_2$. Therefore, the stable Higgs pairs (V,Φ) with fixed underline vector bundle V are parametrized by

$$H^0(K_C) \times \frac{(H^0(L^2K) - \{0\} \times H^0(L^{-2}K_C) - \{0\})}{\mathbb{C}^*} \cong \mathbb{C}^2 \times \mathbb{C}^*$$

(this is an actual quotient as all the points are semistable with respect to the action of $\mathbb{C}^*$). Letting V vary, we obtain a $\mathbb{C}^2 \times \mathbb{C}^*$ bundle $\mathcal{S}_1$ over $\mathcal{J}^0$ and we now compute the cohomology of its total space. Contracting the fibre to S^1 we can consider $\mathcal{S}_1$ as a sphere bundle over $\mathcal{J}^0$ and use the Gysin sequence to compute its cohomology. First, we need to find the cohomology of $\mathcal{J}^0$: to do that we proceed as before, computing compact support cohomology and applying Poincaré duality. Consider the two inclusions $\mathcal{J}^0 \xhookrightarrow{j} \mathcal{J} \xhookleftarrow{i} \mathcal{J}_0$ and the long exact sequence in cohomology

$$\ldots \longrightarrow H^k_c(\mathcal{J}^0) \xrightarrow{j_!} H^k(\mathcal{J}) \xrightarrow{i^!} H^k(\mathcal{J}_0) \longrightarrow \ldots$$

which splits in

$$0 \longrightarrow H^0_c(\mathcal{J}^0) \longrightarrow \mathbb{C} \xrightarrow{i^!} \mathbb{C}^{16} \longrightarrow H^1_c(\mathcal{J}^0) \longrightarrow 0 \qquad (1)$$

$$H^k_c(\mathcal{J}^0) \cong H^k_c(\mathcal{J}) \quad \forall k \geq 2 \qquad (2)$$

As $\mathcal{J}^0$ is not compact, $H^0_c(\mathcal{J}^0) = 0$ thus $H^1_c(\mathcal{J}^0) \cong \mathbb{C}^{15}$. By Poincaré duality and we have

$$H^0(\mathcal{J}^0) \cong \mathbb{C} \quad H^1(\mathcal{J}^0) = 0 \quad H^2(\mathcal{J}^0) = \mathbb{C}^6 \quad H^3(\mathcal{J}^0) = \mathbb{C}^{15} \quad H^4(\mathcal{J}^0) = 0$$

with the same weights as the cohomology of $\mathcal{J}$.

Applying the Gysin sequence

$$\ldots \to H^i(\mathcal{S}_1) \to H^{i-1}(\mathcal{J}^0) \to H^{i+1}(\mathcal{J}^0) \to \ldots$$

this splits in the following sequences

$$H^0(\mathcal{S}_1) \cong \mathbb{C} \quad H^3(\mathcal{S}_1) \cong \mathbb{C}^{21} \quad H^4(\mathcal{S}_1) \cong \mathbb{C}^{15} \qquad (2.13)$$

$$0 \to H^1(\mathcal{S}_1) \to \mathbb{C} \to \mathbb{C}^6 \to H^2(\mathcal{S}_1) \to 0 \tag{2.14}$$

$$H^i(\mathcal{S}_1) = 0 \quad \forall i \geq 5 \tag{2.15}$$

In (2.14) the map $\mathbb{C} \to \mathbb{C}^6$ is the product by the Euler class of a nontrivial bundle, which is nonzero, therefore $H^1(\mathcal{S}_1) = 0$ and $H^2(\mathcal{S}_1) = \mathbb{C}^5$. Recalling that in this case both the cup product with the Euler class and the pushforward increases weights of (1,1), we are able to compute weights of the cohomology. Therefore, applying Poincaré duality, the compact support cohomology groups of $\mathcal{S}_1$ are

$$\begin{aligned}
H^i_c(\mathcal{S}_1) &= 0 && \forall i = 0, \dots 5 \text{ and } i = 9\\
H^6_c(\mathcal{S}_1) &= \mathbb{C}^{15} && \text{with weight } (2,2)\\
H^7_c(\mathcal{S}_1) &= \mathbb{C}^{21} && \text{with weight } (3,3)\\
H^8_c(\mathcal{S}_1) &= \mathbb{C}^{5} && \text{with weight } 3(4,4)+(3,5)+(5,3)\\
H^{10}_c(\mathcal{S}_1) &= \mathbb{C} && \text{with weight } (5,5).
\end{aligned}$$

As a result, the E-polynomial of $\mathcal{S}_1$ is

$$E(\mathcal{S}_1)(u,v) = u^5v^5 + u^3v^5 + u^5v^3 + 3u^4v^4 - 21u^3v^3 + 15u^2v^2$$

□

Type (ii)

Now we want to compute the cohomology of the locus of stable pairs (V, Φ) where V is a nontrivial extension of L by L^{-1} with $L \ncong L^{-1}$.

Proposition 2.7.5. *Let V be a semistable vector bundle of type (ii). Then there is no Higgs field Φ such that the pair (V, Φ) is stable.*

Proof. Consider the universal line bundle $\mathcal{L} \to \mathcal{J}^0 \times C$ and let $p : \mathcal{J}^0 \times C \to \mathcal{J}^0$ be the projection onto the first factor. It is well known that non trivial extensions of $\mathcal{L}$ by $\mathcal{L}^{-1}$ are parametrized by $\mathbb{P}(R^1p_*\mathcal{L}^2)$: as $R^1p_*\mathcal{L}^2$ is a local system on $\mathcal{J}^0$ of rank one, we conclude that there exists a unique nontrivial extension up to isomorphism. Thus we can consider the universal extension bundle $\mathcal{V}$, which will be a bundle over $\mathcal{J}^0 \times C$ by the remark above. Such a bundle fits in the short exact sequence

$$0 \to \mathcal{L} \to \mathcal{V} \to \mathcal{L}^{-1} \to 0 \tag{2.16}$$

and parametrizes all the vector bundles V on C of type (ii). Now we have to take the Higgs field into account and ask for it not preserve the subbundle $\mathcal{L}$, which is the one that makes V strictly semistable. By an abuse notation, let us denote by K_C the pullback of the canonical bundle on C under the projection $\mathcal{J}^0 \times C \to C$: if we tensor the sequence (2.16) by K_C and apply the covariant functor $\mathcal{H}om(\mathcal{V}, -)$ restricted to traceless endomorphisms we obtain

$$0 \to Hom(\mathcal{V}, \mathcal{L} \otimes K_C) \to End_0(\mathcal{V}) \otimes K_C \to Hom(\mathcal{V}, \mathcal{L}^{-1} \otimes K_C) \to 0$$

If we pushforward to $\mathcal{J}^0$ we obtain the long exact sequence

$$0 \to p_*\mathcal{H}om(\mathcal{V}, \mathcal{L}K_C) \to p_*\mathcal{E}nd_0(\mathcal{V}) \otimes K_C \to p_*\mathcal{L}^{-2}K_C \to \tag{2.17}$$

$$\to R^1p_*\mathcal{H}om(\mathcal{V}, \mathcal{L}K_C) \to R^1p_*\mathcal{E}nd_0(\mathcal{V}) \otimes K_C \to R^1p_*\mathcal{L}^{-2}K_C \to 0 \tag{2.18}$$

We have that a Higgs pair $(\mathcal{V}, \Phi)$ is stable if and only if the Higgs field Φ it lies in the complement of the kernel of the map $p_*\mathcal{E}nd_0(\mathcal{V}) \otimes K_C \to p_*\mathcal{L}^{-2}$, that are precisely those Φ for which $\mathcal{L}$ is not invariant.

In order to prove the proposition, we show that the map $p_*\mathcal{E}nd_0(\mathcal{V}) \otimes K_C \to p_*\mathcal{L}^{-2}$ is 0. Starting again from (2.16) and applying the contravariant functor $p_*\mathcal{H}om(-, \mathcal{L}K_C)$, we end up with the long exact sequence

$$0 \to p_*\mathcal{L}K_C) \to p_*\mathcal{H}om(\mathcal{V}, \mathcal{L}K_C) \to p_*K_C \to \tag{2.19}$$

$$\to R^1p_*\mathcal{L}K_C) \to R^1p_*\mathcal{H}om(\mathcal{V}, \mathcal{L}K_C) \to R^1p_*K_C \to 0. \tag{2.20}$$

Consider the fibre of (2.19) on a point $L \in \mathcal{J}^0$. One has

$$H^1(L^2K_C) \to H^1(V^*LK_C) \to H^1(K_C) \to 0,$$

as $H^1(L^2K_C) = 0$ we have that $H^1(V^*LK_C) \cong H^1(K_C) \cong \mathbb{C}$, thus $R^1p_*\mathcal{H}om(\mathcal{V}, \mathcal{L}K_C)$ is a local system of rank 1 on $\mathcal{J}^0 \times C$. Now we can again consider (2.17) on the fibre over $L \in \mathcal{J}^0$ and obtain

$$0 \longrightarrow H^0(V^*LK_C) \longrightarrow H^0(\mathcal{E}nd_0(V) \otimes K_C) \longrightarrow H^0(L^{-2}K_C) \xrightarrow{ext} H^1(V^*LK_C) \longrightarrow$$

$$\longrightarrow H^1(\mathcal{E}nd_0(V) \otimes K_C) \longrightarrow H^1(L^{-2}K_C) \longrightarrow 0$$

As we have seen, $H^1(V^*LK_C) \cong H^1(K_C) \cong \mathbb{C}$ and $H^0(L^{-2}K_C) \cong \mathbb{C}$: the map "*ext*" is either 0 or an isomorphism. However, as V is a nontrivial extension, such a map has to be nonzero, thus it is an isomorphism. Therefore we have that the map

$$p_*\mathcal{E}nd_0(\mathcal{V}) \otimes K_C \to p_*\mathcal{L}^{-2}K_C$$

is zero.

□

Type (iii)

We now consider stable Higgs bundle with underlying vector bundle $V = L \oplus L^{-1}$ with $L \cong L^{-1} \in \mathcal{J}_0$.

Proposition 2.7.6. *Let $\mathcal{S}_3$ be the locus of stable Higgs bundles with underlying vector bundle $V = \mathcal{O} \oplus \mathcal{O}$. Then the locus of stable Higgs pairs of type (iii) is the union of 16 copies of $\mathcal{S}_3$ and its E-polynomial is*

$$E(16 \cdot \mathcal{S}_3)(u, v) = 16u^3v^3 - 16u^2v^2$$

Proof. Up to tensor by $L \in \mathcal{J}_0$ we may restrict to the case $L = \mathcal{O}$, so that V is just the trivial bundle $\mathcal{O} \oplus \mathcal{O}$. In this case $H^0(\mathcal{E}nd_0(V) \otimes K_C) \cong H^0(K_C) \otimes sl(2) \cong \mathbb{C}^2 \otimes sl(2)$ and the Higgs field is of the form

$$\Phi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \text{ with } a, b, c \in H^0(K_C)$$

The bundle is not stable if and only if Φ is conjugate to an upper triangular matrix of elements of $H^0(K_C)$. As the action of $SL(2,\mathbb{C})$ on $H^0(K_C)\otimes sl(2)$ is trivial on $H^0(K_C)$ we can consider it as the action of simultaneous conjugation on two matrices of $sl(2)$. Thus we are looking for the couples of matrices $(A,B)\in sl(2)\oplus sl(2)$ that are not simultaneously triangulable. This equivalent to say that the matrices have no common eigenspace. By a result of Shemesh [She] we have that two matrices $A,B\in sl(2)$ if and only if $Ker[A,B]\neq 0$, that is $det([A,B])=0$. If we write

$$A=\begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix} \qquad B=\begin{pmatrix} y_1 & y_2 \\ y_3 & -y_1 \end{pmatrix} \tag{2.21}$$

we have that

$$[A,B]=\begin{pmatrix} x_2y_3-y_2x_3 & 2(x_1y_2-x_2y_1) \\ 2(x_3y_1-x_1y_3) & -(x_2y_3-y_2x_3) \end{pmatrix}$$

and we can interpret the locus of simultaneously triangulable matrices $(A,B)\in sl(2)\oplus sl(2)$ as the locus

$Q:(x_2y_3-y_2x_3)^2+4(x_1y_2-x_2y_1)(x_3y_1-x_1y_3)=0$ in $\mathbb{C}^6$ with coordinates $(x_1,x_2,x_3,y_1,y_2,y_3)$.

Hence we have the following lemma

Lemma 2.7.7. *A Higgs bundle (V,Φ) of type (iii) is stable if and only if Φ lies in*

$$\mathcal{S}_3:=(\mathbb{C}^6-Q)//SL(2,\mathbb{C})$$

where the action of $SL(2,\mathbb{C})$ is the simultaneous conjugation on the matrices A and B as in (2.21).

Corollary 2.7.8. *The locus of stable Higgs bundles of type (iii) is isomorphic to 16 copies of $\mathcal{S}_3$, one for each point of $\mathcal{J}_0$.*

We start by looking at the quartic hypersurface Q in $\mathbb{C}^6$. If we set

$$\alpha = x_2y_3-y_2x_3$$

$$\beta = x_1y_2-x_2y_1$$

$$\gamma = x_3y_1-x_1y_3$$

then for every $(x_1, x_2, x_3, y_1, y_2, y_3) \in Q$, (α, β, γ) satisfy the equation

$$\alpha^2 + 4\beta\gamma = 0.$$

thus we have a map from our quartic Q to the cone $\mathcal{C} := \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid \alpha^2 + 4\beta\gamma = 0\}$

$$f : Q \to \mathcal{C}, \qquad (x_1, x_2, x_3, y_1, y_2, y_3) \mapsto (x_2y_3 - y_2x_3, x_1y_2 - x_2y_1, x_3y_1 - x_1y_3)$$

Now let us point out our strategy to compute the cohomology of $(\mathbb{C}^6 - Q)$:

1) thanks to the map f, we decompose Q as a disjoint union of the close set $Q_0 = f^{-1}(0)$ and its open complement $Q - Q_0 = f^{-1}(\mathcal{C} - \{0\})$;

2) we compute the cohomology with compact support of both Q_0 and $Q - Q_0$ and use the additivity property to compute the cohomology with compact support of Q;

3) again, as $\mathbb{C}^6 = Q \sqcup (\mathbb{C}^6 - Q)$, we use the additivity property of the cohomology with compact support to compute the cohomology of $\mathbb{C}^6 - Q$.

To compute the cohomology with compact support of our pieces, we first observe that α, β, γ are, up to multiplication, nothing but the minors of order 2 of the matrix

$$\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{pmatrix}. \tag{2.22}$$

Also,if we fix a point $(\alpha, \beta, \gamma) \in \mathcal{C}$ we notice that both (x_1, x_2, x_3) and (y_1, y_2, y_3) are orthogonal to $(\alpha, \frac{\gamma}{2}, \frac{\beta}{2})$, thus they satisfy the equations

$$2\alpha x_1 + \gamma x_2 + \beta x_3 = 0 \qquad 2\alpha y_1 + \gamma y_2 + \beta y_3 = 0$$

If $(\alpha, \beta, \gamma) \neq (0, 0, 0)$, let's say $\beta \neq 0$, then have that

$$x_3 = \frac{-2\alpha x_1 - \gamma x_2}{\beta} \qquad y_3 = \frac{-2\alpha y_1 - \gamma y_2}{\beta}$$

and when we substitute these values in (2.22) and compute the minors of order two we obtain three equations all identical to

$$x_1y_2 - x_2y_1 = \frac{\beta}{2}.$$

Therefore we conclude that the fibre of the map f in a point of $\mathcal{C} - \{0\}$ is a quadric in $\mathbb{C}^4$, which is isomorphic to $SL(2, \mathbb{C})$. Also, $\mathcal{C} - \{0\}$ is homotopy equivalent to $\mathbb{RP}^3$, thus it has fundamental group $\mathbb{Z}_2$ and the monodromy outside the origin is trivial as it equal to the one described in [FK, 3.1]. As a result, we can compute the cohomology with compact support of $Q - Q_0 = f^{-1}(\mathcal{C} - 0)$ via the Künneth formula. We have:

$$H^4_c(Q - Q_0) = \mathbb{C} \quad H^7_c(Q - Q_0) = \mathbb{C}^2 \quad H^{10}_c(Q - Q_0) = \mathbb{C} \quad H^i_c(Q - Q_0) = 0 \text{ otherwise}$$

Now, we need to compute the cohomology of Q_0: first observe that if α, β, γ are all zero, one has that the matrix (2.22) has rank ≤ 1 that is (y_1, y_2, y_3) is a multiple of (x_1, x_2, x_3). Thus points in Q_0 are parametrized by $(\mathbb{C}^3 - \{0\}) \times \mathbb{C} \sqcup \{0\} \times \mathbb{C}^3$. We observe that Q_0 has dimension 4 and the former is an open set in it, while the latter is closed. Therefore we can apply again the additivity property of compact support cohomology to find $H^i_c(Q_0)$. Observe that

$$\begin{array}{lll} H^3_c((\mathbb{C}^3 - \{0\}) \times \mathbb{C}) \cong \mathbb{C} & H^8_c((\mathbb{C}^3 - \{0\}) \times \mathbb{C}) \cong \mathbb{C} & H^i_c((\mathbb{C}^3 - \{0\}) \times \mathbb{C}) = 0 \text{ otherwise} \\ H^6_c(\{0\} \times \mathbb{C}^3) \cong \mathbb{C} & H^i_c(\{0\}) \times \mathbb{C}^3) = 0 \text{ otherwise} & \end{array},$$

hence

$$H^3_c(Q_0) \cong H^6_c(Q_0) \cong H^8_c(Q_0) \cong \mathbb{C}, \quad H^i_c(Q_0) = 0 \text{ otherwise}$$

Again we apply additivity of compact support cohomology to obtain the cohomology of Q:

$$\ldots \to H^i(Q - Q_0) \to H^i(Q) \to H^i(Q_0) \to H^{i+1}(Q - Q_0) \to \ldots$$

Now, $H^i_c(Q) = 0$ for any $i \geq 5$ since Q is affine and from the long exact sequence we conclude that $H^7_c(Q) \cong H^8_c(Q) \cong H^{10}_c(Q) \cong \mathbb{C}$ and $H^i_c(Q) = 0$ otherwise.

Finally, we compute the compact support cohomology of $\mathbb{C}^6 - Q$ and from the additivity property it is

$$H^8_c(\mathbb{C}^6 - Q) \cong H^9_c(\mathbb{C}^6 - Q) \cong H^{11}_c(\mathbb{C}^6 - Q) \cong H^{12}_c(\mathbb{C}^6 - Q) \cong \mathbb{C}, \quad H^i_c(\mathbb{C}^6 - Q) = 0$$

Now we notice that $SL(2, \mathbb{C})$ acts on $\mathbb{C}^6 - Q$ with a stabilizer which is at worst $\mathbb{Z}_2$, therefore we can compute the cohomology by considering $\mathbb{C}^6 - Q$ as a fibre bundle with fibre $SL(2, \mathbb{C})$ on $\mathcal{S}_3$

As $SL(2, \mathbb{C})$ has the same homotopy type as S^3 we can use the Gysin sequence

$$\ldots \to H^i(\mathbb{C}^6 - Q) \to H^{i-3}(\mathcal{S}_3) \to H^{i+1}(\mathcal{S}_3) \to \ldots$$

and we obtain

$$H^0(\mathcal{S}_3) \cong H^1(\mathcal{S}_3) \cong \mathbb{C}, \qquad H^2(\mathcal{S}_3) = 0 \tag{2.23}$$

$$0 \to H^3(\mathcal{S}_3) \to \mathbb{C} \to \mathbb{C} \to H^4(\mathcal{S}_3) \to \mathbb{C} \to \mathbb{C} \to H^5(\mathcal{S}_3) \to 0 \tag{2.24}$$

$$H^6(\mathcal{S}_3) = 0, \quad H^4(\mathcal{S}_3) \cong H^8(\mathcal{S}_3) \cong H^{12}(\mathcal{S}_3) \tag{2.25}$$

$$H^3(\mathcal{S}_3) \cong H^7(\mathcal{S}_3) \cong H^{11}(\mathcal{S}_3) \quad H^5(\mathcal{S}_3) \cong H^5(\mathcal{S}_3) \quad H^6(\mathcal{S}_3) \cong H^{10}(\mathcal{S}_3) = 0 \tag{2.26}$$

Since $\mathcal{S}_3$ is nonsingular connected but not compact, $H^{12}(\mathcal{S}_3) \cong H^0_c(\mathcal{S}_3) = 0$, thus $H^4(\mathcal{S}_3) \cong H^8(\mathcal{S}_3) = 0$. Therefore from (2.24) we deduce that $H^3(\mathcal{S}_3) \cong H^5(\mathcal{S}_3) = 0$, $H^7(\mathcal{S}_3) \cong H^{11}(\mathcal{S}_3) = 0$ and $H^9(\mathcal{S}_3) = 0$.

Therefore the E-polynomial of $\mathcal{S}_3$ is given by

$$E(\mathcal{S}_3)(u,v) = u^3v^3 - u^2v^2$$

□

Type (iv)

We now consider stable Higgs bundles of type (iv) and we prove the following result.

Proposition 2.7.9. *Let $\mathcal{S}_4$ be the locus of stable Higgs bundles whose underlying vector is a nontrivial extension of $\mathcal{O}$ by itself. Then the locus of stable Higgs bundles of type (iv) is the union of 16 copies of $\mathcal{S}_4$ and its E-polynomial is*

$$E(16 \cdot \mathcal{S}_4) = 16u^4v^4 + 16u^2v^2$$

Proof. As before, we can assume $L \cong \mathcal{O}$. Let V be a nontrivial extensions of $\mathcal{O}$ by itself: the isomorphism classes of such bundles are parametrized by

$$\mathbb{P}(Ext^1(\mathcal{O},\mathcal{O})) \cong \mathbb{P}^1. \tag{2.27}$$

Thus there exists a universal extension bundle on $\mathbb{P}^1 \times C$

$$0 \to \mathcal{O} \to \mathcal{V} \to \mathcal{O} \to 0.$$

Let $p : \mathbb{P}^1 \times C \to \mathbb{CP}^1$ be the projection : as in the type (ii) case we tensor the short exact sequence above by K_C, apply the covariant functor $Hom(\mathcal{V}, -)$ and pushforward to $\mathbb{P}^1$ and we end up with the long exact sequence (1)

$$0 \longrightarrow p_*Hom(\mathcal{V}, K_C) \longrightarrow p_*(End_0(\mathcal{V}) \otimes K_C) \longrightarrow p_*Hom(\mathcal{V}, K_C) \xrightarrow{ext} R^1p_*Hom(\mathcal{V}, K_C) \longrightarrow$$

$$\longrightarrow R^1p_*(End_0(\mathcal{V}) \otimes K_C) \longrightarrow R^1p_*Hom(\mathcal{V}, K_C) \longrightarrow 0$$

As before, stable Higgs bundles are precisely those with Higgs field in the complement of the kernel of the map

$$p_*(End_0(\mathcal{V}) \otimes K_C) \to p_*Hom(\mathcal{V}, K_C).$$

or, equivalently, the complement of the image of $p_*Hom(\mathcal{V}, K_C)$ in $p_*(End_0(\mathcal{V}) \otimes K_C)$. First we notice that $p_*Hom(\mathcal{V}, K_C) \cong p_*K_C$, which is a vector bundle of rank 2 and similarly we have that $R^1p_*Hom(\mathcal{V}, K_C) \cong R^1p_*K_C$. As the extension is nontrivial, we have that the map *ext* is nonzero and that its kernel has rank 1. Starting again from (2.27), we tensor with K_C, apply the contravariant functor $\mathcal{H}om(-, \mathcal{O})$ restricted to traceless endomorphisms and pushforward to $\mathbb{P}^1$ we obtain another long exact sequence (2)

$$0 \longrightarrow p_*K_C \longrightarrow p_*\mathcal{H}om(\mathcal{V}, K_C) \longrightarrow p_*K_C \xrightarrow{ext} R^1p_*K_C \longrightarrow \dots$$

We observe that since $R^1p_*K_C$ has rank 1 and the map *ext* is nonzero, the last map is surjective. Hence, the cokernel of $p_*\mathcal{H}om(\mathcal{V}, K_C) \to p_*K_C$ has rank 1 and consequently $p_*\mathcal{H}om(\mathcal{V}, K_C)$ has rank 3. Going back to the previous long exact sequence we conclude that $p_*\mathcal{E}nd_0(V) \otimes K_C$ is a vector bundle of rank 4, thus the locus of stable pairs is fibrewise the complement of a hyperplane.

Finally we need to see which Higgs fields define the isomorphic Higgs bundles: the group of automorphisms of a nontrivial extension of $\mathcal{O}$ by itself is the additive group $(\mathbb{C}, +) \subset SL(2, \mathbb{C})$, and an element $t \in \mathbb{C}$ acts on the Higgs field Φ by conjugation:

$$t.\Phi = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1a + tc & b - 2ta - t^2c \\ c & -a - tc \end{pmatrix}$$

Lemma 2.7.10. *$\mathcal{S}_4$ is a $\mathbb{C}^2$- bundle over a $\mathbb{C}^*$- bundle over $\mathbb{P}^1$.*

Proof. Let A be the kernel of the extension map in (1), minus the zero section: thus A is a $\mathbb{C}^*$-bundle over $\mathbb{P}^1$. We can think of $p_*(\mathcal{E}nd_0(\mathcal{V}) \otimes K_C) - p_*\mathcal{H}om(\mathcal{V}, K_C)$ as vector bundle of rank 3 over A. Similarly, the kernel of the extension map of (2) gives rise to a vector bundle $\mathcal{U}$ over A of rank 2 and the map

$$p_*\mathcal{H}om(\mathcal{V}) \to p_*(\mathcal{E}nd_0(\mathcal{V}) \otimes K_C)$$

lifts to a $\mathbb{C}$-equivariant map

$$[p_*(\mathcal{E}nd_0(\mathcal{V}) \otimes K_C) - p_*\mathcal{H}om(\mathcal{V}, K_C)] \to \mathcal{U}$$

of vector bundles over A whose kernel is of rank 2. Now we have to take automorphism into account: the action of $(\mathbb{C}, +)$ on $\mathcal{U}$ is linear $a \mapsto a + tc$, hence the quotient $\mathcal{U}/\mathbb{C}$ os actually A itself. As the map above is equivariant, we have that

$$[p_*(\mathcal{E}nd_0(\mathcal{V}) \otimes K_C) - p_*\mathcal{H}om(\mathcal{V}, K_C)]/\mathbb{C} \to \mathcal{U}/\mathbb{C} \cong A$$

is a vector bundle of rank 2 over A. □

Corollary 2.7.11. *The locus of stable Higgs bundles of type (iv) is isomorphic to 16 copies of $\mathcal{S}_4$, one for each point of $\mathcal{J}_0$.*

Thanks to lemma (2.7.10), we can now compute the Betti numbers of $\mathcal{S}_4$: first we notice that it is homotopy equivalent to a $\mathbb{C}^*$-bundle on $\mathbb{P}^1$. Using the Gysin sequence we have that the locus $\mathcal{S}_4$ of stable Higgs bundles of type (iv) has the following cohomology with compact support:

$$H^0(S_4) \cong H^0(\mathbb{P}^1) \cong \mathbb{C}$$

$$0 \to H^1(\mathcal{S}_4) \to \mathbb{C} \to \mathbb{C} \to H^2(\mathcal{S}_4) \to 0$$

$$H^3(S_4) \cong H^2(\mathbb{P}^1) \cong \mathbb{C}$$

$$H^i(\mathcal{S}_4) = 0 \text{ for all } i = 4 \dots 8.$$

As the central map of the second equation is the cup product with the Euler class of the bundle A, which is nontrivial, therefore it is nonzero and we have $H^1(\mathcal{S}_4) = H^2(\mathcal{S}_4) = 0$. Passing to compact support cohomology with Poincaré duality, the E-polynomial of $\mathcal{S}_4$ is

$$E(\mathcal{S}_4) = u^4v^4 + u^2v^2$$

□

2.7.3 Unstable case

Consider the locus $\mathcal{U}$ of stable Higgs bundles (V, Φ) where V is an unstable vector bundle with trivial determinant. Then there exists a line bundle L of degree $d > 0$ that fits an exact sequence

$$0 \longrightarrow L \longrightarrow V \longrightarrow L^{-1} \longrightarrow 0$$

If $d > 1$ then the bundle $L^{-2}K_C$ has no nonzero global section because it has negative degree, hence L is Φ-invariant for any Higgs field $\Phi \in H^0(End_0(V) \otimes K_C)$. The only case we have to check is $deg(L) = 1$. The line bundle $L^{-2}K_C$ has degree 0: it has global sections if and only if it is trivial, that is L is one of the 16 roots of the canonical bundle K_C. As a consequence, if there exists an unstable vector bundle V which is stable as a Higgs bundle, then it must be an extension of those bundles by their duals. We show the following

Proposition 2.7.12. *The locus $\mathcal{U}$ of stable Higgs bundles (V, Φ) with V unstable is isomorphic to $\mathbb{C}^3$. As a consequence its cohomology with compact support is given by*

$$H^6_c(\mathcal{U}) = \mathbb{C} \quad H^i_c(\mathcal{U}) = 0 \text{ otherwise.}$$

and the E-polynomial of $\mathcal{U}$ is $E(\mathcal{U}) = u^3v^3$.

Proof. **Trivial case**

If $V = L \oplus L^{-1}$ then

$$H^0(End_0(V) \otimes K_C) = H^0(K_C) \oplus H^0(L^2K_C) \oplus H^0(L^{-2}K_C) \cong \mathbb{C}^2 \oplus \mathbb{C}^3 \oplus \mathbb{C}$$

Thus the generic Higgs field will be of the form

$$\Phi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \text{ with } a \in H^0(K_C), b \in H^0(L^2K_C), c \in H^0(L^{-2}K_C).$$

Two Higgs fields define isomorphic Higgs bundles if and only if they are conjugate by an automorphism of the bundle, which will lie in $\mathbb{C}^* \times (H^0(K_C), +) \subset SL(2, \mathbb{C})$. The action of $\mathbb{C}^*$

on the Higgs field is precisely the one seen in the type (i) case. Therefore isomorphism classes of stable Higgs bundles are parametrized by the disjoint union of 16 copies of

$$H^0(K_C) \times \frac{(H^0(L^{-2}K_C) - \{0\}) \times H^0(L^2K_C))}{\mathbb{C}^*} \cong H^0(K_C) \times H^0(L^2K_C) \cong \mathbb{C}^5.$$

Then we have to consider the action of $(\mathbb{C}^2, +)$: if $\zeta \in H^0(K_C) = \mathbb{C}^2$ then it acts as

$$\begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} 1 & -\zeta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a - \zeta c & b + 2\zeta a - \zeta^2 c \\ c & -a + \zeta c \end{pmatrix}.$$

Such an action is linear and free on $a \in H^0(K_C)$ and whenever we fix $a - \zeta c$ then the value of $b + 2\zeta a - \zeta^2 c$ is fixed as well. Therefore the quotient of $H^0(K_C) \times H^0(L^2LK_C)$ by $(\mathbb{C}^2, +)$ is precisely $\mathbb{C}^3$.

Non trivial case

Non-trivial extensions of L by L^{-1} are parametrized by $\mathbb{P}(H^1(L^{-2})) = \mathbb{P}^2$ and fit the exact sequence

$$0 \to L \to V \to L^{-1} \to 0.$$

If we again tensor by K_C and apply the functor $Hom(V, -)$ restricted to traceless endomorphisms, when we take global sections we obtain

$$0 \to H^0(V^* \otimes LK_C) \to H^0(End_0(V) \otimes K_C) \to H^0(V^* \otimes L^{-1}K_C) \to H^1(V^* \otimes LK_C) \to \dots$$

Again, a Higgs bundle that has V as underlying vector bundle becomes stable if and only if its Higgs field lies in the complement of the kernel of $H^0(End_0(V) \otimes K_C) \to H^0(V^* \otimes L^{-1}K_C)$. First we notice that due to trace condition $Hom(V, L^{-1}K_XC = Hom(L, L^{-1}K_C) \cong \mathbb{C}$ and $H^1(L^{-2}K_C) \cong H^1(\mathcal{O}) \cong \mathbb{C}^2$. Applying the functor $\mathcal{H}om(, -LK_C)$ and taking global sections we have that the long exact sequence in cohomology splits in

$$0 \to Hom(L^{-1}, LK_C) \to Hom(V, LK_C) \to Hom(L, LK_C) \to 0 = H^1(L^2K_C)$$

$$0 \to H^1(V^* \otimes LK_C) \to H^1(K_C) \to 0.$$

From that we deduce that $H^1(V^*\otimes LK_C) \cong H^1(K_C) \cong \mathbb{C}$; also $Hom(L^{-1}, LK_C) \cong H^0(L^2K_C) \cong \mathbb{C}^3$ and $Hom(L, LK_C) \cong H^0(K_C) \cong \mathbb{C}^2$ thus $Hom(V, LK_C) \cong \mathbb{C}^5$. Coming back to the first long exact sequence one has

$$0 \to \mathbb{C}^5 \to H^0(\mathcal{E}nd_0(V) \otimes K_C) \to \mathbb{C} \to \mathbb{C} \to H^1(\mathcal{E}nd_0(V) \otimes K_C) \to \mathbb{C}^2 \to 0.$$

As the extension is nontrivial, one has that the map $\mathbb{C} \to \mathbb{C}$ is an isomorphism thus the map $H^0(End_0(V) \otimes K_C) \to \mathbb{C} \cong H^0(V^* \otimes L^{-1}K_C)$ is zero and therefore the destabilizing bundle is preserved by any Higgs field. We conclude that there are no non-trivial unstable extensions of L by its dual that give rise to a stable Higgs bundle.

□

2.8 Computation of the $IE(\mathcal{M}_{Dol})$

Now that we have computed the cohomology with compact support of all pieces we can sum them up to obtain the cohomology with compact support of $\mathcal{M}^s_{Dol}$. Let us do first a table to summarize the Betti numbers we have computed so far

	H^0_c	H^1_c	H^2_c	H^3_c	H^4_c	H^5_c	H^6_c	H^7_c	H^8_c	H^9_c	H^{10}_c	H^{11}_c	H^{12}_c
$\mathcal{S}_0$	0	0	0	0	0	0	0	0	0	5	0	0	1
$\mathcal{S}_1$	0	0	0	0	0	0	15	21	5	0	1	0	0
16$\times\mathcal{S}_3$	0	0	0	0	0	16	16	0	0	0	0	0	0
16$\times\mathcal{S}_4$	0	0	0	0	0	16	0	0	16	0	0	0	0
$\mathcal{U}$	0	0	0	0	0	0	16	0	0	0	0	0	0

If we sum up all the E-polynomials computed so far we conclude that the E-polynomial of $\mathcal{M}^s_{Dol}$ is

$$E(\mathcal{M}^s_{Dol}) = u^6v^6 + u^5v^5 + 16u^4v^4 + 11u^3v^3 - 17u^2v^2$$

2.8.1 Cohomology of $\tilde{\Sigma} \setminus \tilde{\Omega}$ and $\tilde{\Omega}$

Cohomology of $\tilde{\Omega}$

Lemma 2.8.1.

$$E(\tilde{\Omega})(u,v) = 16u^3v^3 + 16u^2v^2 + 16uv + 16$$

Proof. We recall that $\tilde{\Omega}$ consists of 16 copies of a nonsingular hypersurface $\mathrm{Gr}^{\omega}(2, \Lambda^1)$ in $\mathbb{P}^4$. Therefore its cohomology is given by

$$H^0(\tilde{\Omega}) = H^2(\tilde{\Omega}) = H^4(\tilde{\Omega}) = H^6(\tilde{\Omega}) = \mathbb{C}^{16}$$
$$H^1(\tilde{\Omega}) = H^3(\tilde{\Omega}) = H^5(\tilde{\Omega}) = 0,$$

thus the E-polynomial of $\tilde{\Omega}$ is

$$E(\tilde{\Omega})(u,v) = 16u^3v^3 + 16u^2v^2 + 16uv + 16$$

□

Cohomology of $\tilde{\Sigma} \setminus \tilde{\Omega}$

Lemma 2.8.2.

$$E(\tilde{\Sigma} \setminus \tilde{\Omega})(u,v) = u^5v^5 + 5u^4v^4 + u^5v^3 + u^3v^5 + 5u^3v^3 + u^2v^4 + u^4v^2 + u^2v^2 - 16uv - 16$$

We observe that $\tilde{\Sigma} \setminus \tilde{\Omega}$ is $\mathbb{P}^1$ bundle over Σ^0. Observe that $\Sigma^0 \cong (Pic^0(C) \times H^0(K_C)/\mathbb{Z}_2 \setminus \{16 \text{ points}\}$. First we notice that $\Sigma = (Pic^0(C) \times H^0(K_C)/\mathbb{Z}$ has the same cohomology $\mathcal{J}$ thus by Poincaré duality

$$\begin{aligned}
&H^4_c(\Sigma) \cong \mathbb{C} && \text{of weights } (2,2)\\
&H^2_c(\Sigma) \cong \mathbb{C}^6 && \text{of weights } 4(3,3) + (2,4) + (4,2)\\
&H^8_c(\Sigma) \cong \mathbb{C} && \text{of weights } (4,4)\\
&H^i_c(\Sigma) = 0 && \text{otherwise}
\end{aligned}$$

As $\Sigma_0 = \Sigma \setminus \{16 \text{ points}\}$, then it has the same cohomology groups as Σ except for $H^1_c(\Sigma^0) \cong \mathbb{C}^{16}$ of weight 0. By the properties of E-polynomials,

$$\begin{aligned}
E(\tilde{\Sigma} \setminus \tilde{\Omega})(u,v) = E(\mathbb{P}^1)E(\Sigma^0)(u,v) &= (uv+1)(u^4v^4 + u^2v^4 + u^4v^2 + 4u^3v^3 + u^2v^2 - 16)\\
&= u^5v^5 + 5u^4v^4 + u^5v^3 + u^3v^5 + 5u^3v^3 + u^2v^4 + u^4v^2 + u^2v^2 - 16uv - 16
\end{aligned}$$

As a result we have that

Theorem 2.8.3. *Let $\tilde{\mathcal{M}}_{Dol}$ the semismall desingularization of $\mathcal{M}_{Dol}$. The E-polynomial of $\tilde{\mathcal{M}}_{Dol}$ is*

$$E(\tilde{\mathcal{M}}_{Dol}) = u^6v^6 + 2u^5v^5 + 21u^4v^4 + u^5v^3 + u^3v^5 + 32u^3v^3 + u^2v^4 + u^4v^2.$$

By theorem (2.12), if we subtract the top cohomology of the fibres, we get that the E-polynomial for the intersection cohomology of $\mathcal{M}_{Dol}$ is

$$IE(\mathcal{M}_{Dol}) = u^6v^6 + u^5v^5 + 15u^4v^4 + u^5v^3 + u^3v^5 + 15u^3v^3 + u^2v^4 + u^4v^2.$$

Chapter 3

The cohomology of the nested Hilbert schemes of planar curves

3.1 Introduction

For the rest of this section curves are assumed to be complex, integral, complete and with locally planar singularities. We remind what locally planar singularities mean:

Definition 3.1.1. Let C be a complex curve. We say that C has *locally planar singularities* if for every $p \in C$ the completion $\hat{\mathcal{O}}_{C,p}$ of the local ring of C at p can be written as

$$\hat{\mathcal{O}}_{C,p} = \mathbb{C}[[x,y]]/(f_p)$$

for some reduced series $f_p \in \mathbb{C}[[x,y]]$.

Let C be a curve of arithmetic genus $p_a(C) := H^1(C, \mathcal{O}_C)$.
We consider the Hilbert scheme of points $C^{[m]}$, which parametrizes length m finite subschemes of C. More precisely the $m-$th Hilbert scheme of points of C is defined as

$$C^{[m]} := \{\text{zero dimensional closed subschemes Z} \subset C \mid \dim(\mathcal{O}_C/\mathcal{I}_Z) = m\}$$

where $\mathcal{I}_Z$ is the ideal sheaf of Z. Hilbert schemes have been introduced by Grothendieck in [Gr] and are now the focus of several works in mathematics. For a general introduction to

Hilbert schemes of points and their properties we refer to [Ko, R]. In [AIK] and [BGS], these varieties are proved to be nonsingular, complete, integral, m dimensional and locally complete intersections. Moreover there is a forgetful map $\rho : C^{[n]} \to C^{(n)}$ from the Hilbert scheme to the symmetric product of the curve that map any subscheme Z to his support. Such a map is an isomorphism of algebraic varieties when the curve C is nonsingular, while it is birational for singular curves.
We consider here the so called nested Hilbert scheme $C^{[m,m+1]}$ of length $m+1$ subschemes of C in which an ideal of colength 1 is fixed. More precisely we define $C^{[m,m+1]}$ as

$$\begin{aligned} C^{[m,m+1]} &:= \{(z', z) \mid z' \in C^{[m]}, z \in C^{[m+1]}, z' \subset z\} \\ &= \{(I, J) \text{ ideals of } \mathcal{O}_C \mid I \subset J \text{ and } \dim(\mathcal{O}_C/J) = m, \dim(\mathcal{O}_C/I) = m+1\} \end{aligned}$$

The theory nested Hilbert schemes of points on a curve have wide application, for example one may relate the topological invariants of these spaces to HOMFLY invariants for the link of the singularity of a curve [OS]. One can generalize the definition for $C^{[m',m+1]}$ in an obvious way for any $m' \leq m+1$, however the nested Hilbert schemes for $m' \neq m$ is always singular.

Also, we can consider the relative versions of $C^{[m]}$ and $C^{[m,m+1]}$(see [Ko] for details), that is if $\pi : \mathcal{C} \to B$ is a proper and flat family of curves we can define two families

$$\begin{aligned} \pi^{[m]} : &\quad \mathcal{C}^{[m]} \to B, \quad (\mathcal{C}^{[m]})_b = (\mathcal{C}_b)^{[m]} \\ \pi^{[m,m+1]} : &\quad \mathcal{C}^{[m,m+1]} \to B, \quad (\mathcal{C}^{[m,m+1]})_b = (\mathcal{C}_b)^{[m,m+1]} \end{aligned}$$

In [Sh], Shende proves that, under some assumptions on the basis, the total space of the relative Hilbert scheme $\mathcal{C}^{[m]}$ is smooth. As a result, the decomposition theorem applied to the map $\pi^{[m]}$ asserts that the complexes $R\pi_*^{[m]}\mathbb{C}$ decomposes in the derived category of constructible sheaves $D_c^b(B)$ as a direct sum of shifted intersection complexes associated to local systems on constructible subsets of the base.
Among them we find the intersection complex whose support is the whole base B. More precisely, if we denote by $\tilde{\pi} : \tilde{\mathcal{C}} \to \tilde{B}$ the restriction of the family to the smooth locus, then any fiber is a smooth curve and its Hilbert scheme coincides with the symmetric product; in particular the map $\tilde{\pi}^{[m]}$ is smooth. Hence the summand of $R\pi_*^{[m]}\mathbb{C}[m + \dim B]$ with support

equal to B is $\bigoplus IC_B(R^i\tilde{\pi}_*^{[m]}\mathbb{C})[-i]$ (the convention on the shift is the same as in theorem 1.2.1). Migliorini and Shende showed that this is in fact the only summand.

Theorem 3.1.1 ([MS1], Theorem 1). *Let $\mathcal{C} \to B$ be a proper and flat family of integral plane curves and let $\tilde{\pi} : \tilde{\mathcal{C}} \to \tilde{B}$ be its restriction to the smooth locus. If $\mathcal{C}^{[m]}$ is smooth then*

$$R\pi_*^{[m]}\mathbb{Q}[m + dimB] = \bigoplus IC_B(R^i\tilde{\pi}_*^{[m]}\mathbb{Q})[-i].$$

Here we prove that an analogous statement holds for the nested case.

Theorem 3.1.2. *Let $\mathcal{C} \to B$ be a proper and flat family of integral plane curves and let $\tilde{\pi} : \tilde{\mathcal{C}} \to \tilde{B}$ be its restriction to the smooth locus. If $\mathcal{C}^{[m,m+1]}$ is smooth then*

$$R\pi_*^{[m,m+1]}\mathbb{Q}[m + 1 + dimB] = \bigoplus IC_B(R^i\tilde{\pi}_*^{[m,m+1]}\mathbb{Q})[-i].$$

As a corollary, one may show that the *perverse filtration* on the cohomology groups of the nested Hilbert scheme does not depend on the map (cfr. [MS1, Prop. 24]).

The strategy for proving the theorem is the following. First we show that under some assumption on the basis, the relative nested Hilbert scheme $\mathcal{C}^{[m,m+1]} \to B$ is smooth. After that we prove, thanks to the theory of higher discriminants we introduced in Chapter 1, that the only candidates for the supports are the strata B_i of points whose fibre in the family is a curve of cogenus i. Using density of nodal curves in those strata, we verify the support criterion on weight polynomials for a generic nodal curve of cogenus δ.

3.2 Versal deformations of curves singularities

As we will systematically employ versal deformation of curve singularities (as analytic spaces), we recall here some known results. For further details we refer to [GLS].

Definition 3.2.1. Let (X, x) be the germ of a complex analytic space.

(i) A deformation $(i, \phi) : (X, x) \xrightarrow{i} (\mathcal{X}, x) \xrightarrow{\phi} (S, s)$ is a morphism ϕ of germs of complex analytic spaces, together with an injection i such that $X \cong i(X) = \mathcal{X}_x$.

(ii) A deformation $(i, \phi) : (X, x) \xrightarrow{i} (\mathcal{X}, x) \xrightarrow{\phi} (S, s)$ is called *complete* if, for any deformation $(j, \psi) : (X, x) \xrightarrow{j} (\mathcal{Y}, y) \xrightarrow{\psi} (T, t)$ of (X, x), there exists a morphism $\theta : (T, t) \to (S, s)$ such that (j, ψ) is isomorphic to the induced deformation $(\theta^* i, \theta^* \phi)$.

(iii) A deformation $(i, \phi) : (X, x) \xrightarrow{i} (\mathcal{X}, x) \xrightarrow{\phi} (S, s)$ is called *versal* if, for a given deformation (j, ψ) as above, the following holds: forn any closed embedding $k : (T', t) \to (T, t)$ of complex germs and any morphism $\theta' : (T', t) \to (S, s)$ there exists a morphism $\theta : (T, t) \to (S, s)$ satisfying

  (a) $\theta \circ k = \theta'$, and

  (b) $(j, \psi) = (\theta^* i, \theta^* \phi)$.

(iv) A deformation is *locally versal* if it induces versal deformations of all the singularities of X.

(v) A versal deformation is called *miniversal* if, with the notation of (iii), the Zariski tangent map $T(\theta) : T_{T,t} \to T_{(S,s)}$ is uniquely determined by (i, ϕ) and (j, ψ).

Consider a deformation $\mathcal{C} \to B$, such that the fibre C over the base point b_0 is a singular curve. The condition of being versal roughly says that any other deformation of C can be obtain (even though not uniquely) from $\mathcal{C} \to B$ by pullback. The condition of being locally versal can be interpreted in the following sense[FGVs]: if $\overline{\mathbb{V}}(C)$ is the product of the versal deformation spaces of the singularities of C, then there exists a tangent map $T_{b_0} B \to T_0 \overline{\mathbb{V}}(C)$ coming from the local-global spectral sequence for first order deformations of C. The deformation is locally versal whenever this map is surjective.

In the following section we will often use miniversal deformations since they can be described explicitly. More precisely let $(C, 0)$ be the germ at the origin of the zero locus of some $f \in \mathbb{C}[x, y]$ such that $f(0) = 0$. Fix $g_1 \dots g_t \in \mathbb{C}[x, y]$ whose images form a basis of the vector space $\mathbb{C}[x, y]/(f, \partial_x f, \partial_y f)$. Then consider $F : \mathbb{C}^t \times \mathbb{C}^2 \to \mathbb{C}^t \times \mathbb{C}$ given by $F(u_1, ..., u_t, x, y) = (u_1, \dots, u_t, (f + g_i u_i)(x, y))$. Taking the fibre over $\mathbb{C}^t \times 0$ gives a family of curves over $\mathbb{C}^t$; taking germs at the origin gives the miniversal deformation $(\mathcal{C}, 0) \to (\mathbb{C}^t, 0)$ of C. Moreover, if $g'_1, \dots, g'_s \in \mathbb{C}[x, y]$ are any functions and $(\mathcal{C}', 0) \to (\mathbb{C}^s, 0)$ the analogously formed deformation of C, then the tangent map $\mathbb{C}^s \to \mathbb{C}[x,y]/(f, \partial_x f, \partial_y f)$ is just induced by the quotient $\mathbb{C}[x,y] \to \mathbb{C}[x,y]/(f, \partial_x f, \partial_y f)$. As soon as this map is surjective, the family $(\mathcal{C}', 0) \to (\mathbb{C}^s, 0)$ is itself versal.

We would like to have a measure of "how singular" a curve is, for example we could look at how far a curve is from its normalization. Given a singular curve C and denoted its normalization by $\overline{C}$, we define the cogenus δ to be the difference between its arithmetic and geometric genera $\delta(C) := p_a(C) - p_a(\overline{C})$. For example, the cogenus of a curve with one node is precisely 1. The following theorem, show why the cogenus is a good candidate for our purpose. Moreover it will be the key result to reduce the proof of theorem (3.1.2) to the case of a family of nodal curves.

Theorem 3.2.1 ([T]). *Let $\mathcal{C} \to B$ be a family of curves. Then the cogenus is an upper semicontinuous function on B. Local versality is an open condition and in a locally versal family the locus of δ-nodal curves is dense in the locus of curves with cogenus at least δ. In particular, the locus of curves of cogenus δ in a locally versal family has codimension δ.*

As we are working with the cogenus we would like to have a result that allows us not to care about $p_a(\overline{C})$. In [L] Laumon showed that any curve singularity can be found on a rational curve. We will see that there exist an analogous result for families, that is given a family of curves $\mathcal{C} \to B$ then around a point $b_0 \in B$ one can find a different family of rational curves such that $\mathcal{C}'_{b_0} = \mathcal{C}_{b_0}$ and the two families induce the same deformations of the singularities of the central fiber. This is a consequence of the following proposition:

Proposition 3.2.2 ([FGVs]). *The map from the base of a versal deformation of an integral locally planar curve to the product of the versal deformations of its singularities is a smooth surjection.*

Corollary 3.2.3 ([MS1], Cor. 6). *Let $\pi : \mathcal{C} \to B$ be a family of curves. Fix $b_0 \in B$, and let $\overline{\mathcal{C}}_{b_0}$ be the normalization of $\mathcal{C}_{b_0}$.Then there exists a neighbourhood $b \in U \subseteq B$ and a family $\pi : \mathcal{C}' \to U$ such that $\mathcal{C}'_{b_0}$ is rational with the same singularities as $\mathcal{C}_{b_0}$, and $\mathcal{C}$ and $\mathcal{C}'$ induce the same deformations of these singularities on U. In particular, they have the same discriminant* *locus. Moreover, on U, we have an equality of local systems* $R^1\tilde{\pi}'_*\mathbb{C} \oplus H^1(\mathcal{C}_{b_0})$*, where* $H^1(\mathcal{C}_{b_0})$ *denotes the constant local system with this fiber.*

To make use of such a replacement we need to know that $\mathcal{C}'^{[m,m+1]}$ is smooth if $\mathcal{C}^{[m,m+1]}$ is. This follows from results on the smoothness of the nested Hilbert scheme which we are going to show. The results and their proof are closely analogous to [Sh, Prop. 17 and Thm.19], in which they are stated for $\mathcal{C}^{[m]}$.

3.3 Smoothness of the relative nested Hilbert scheme

Let $V \subset \mathbb{C}[x,y]$ be a finite dimensional smooth family of polynomials and consider the family of curves

$$\mathcal{C}_V := \{(f,p) \in V \times \mathbb{C}^2 \mid f(p) = 0\}.$$

If we consider the associated family of nested Hilbert scheme $\mathcal{C}_V^{[m,m+1]}$ then it is included in $V \times (\mathbb{C}^2)^{[m,m+1]}$. In [C], Cheah shows that the nested Hilbert scheme $(\mathbb{C}^2)^{[m,m+1]}$ is nonsingular for all m. Moreover she gives an explicit description of its tangent space: if (I,J) is a pair of ideals of $\mathbb{C}[x,y]$ with $I \subseteq J$ such that (I,J) defines a point in $(\mathbb{C}^2)^{[m,m+1]}$, then the tangent space $T_{(I,J)}(\mathbb{C}^2)^{[m,m+1]}$ is isomorphic to $Ker(\phi - \psi)$ where

$$\phi : Hom_{\mathbb{C}[x,y]}(I, \mathbb{C}[x,y]/I) \to Hom_{\mathbb{C}[x,y]}(I, \mathbb{C}[x,y]/J)$$
$$\psi : Hom_{\mathbb{C}[x,y]}(J, \mathbb{C}[x,y]/J) \to Hom_{\mathbb{C}[x,y]}(I, \mathbb{C}[x,y]/J)$$

are the obvious maps and

$$(\phi - \psi) : Hom_{\mathbb{C}[x,y]}(I, \mathbb{C}[x,y]/I) \oplus Hom_{\mathbb{C}[x,y]}(J, \mathbb{C}[x,y]/J) \to Hom_{\mathbb{C}[x,y]}(I, \mathbb{C}[x,y]/J)$$

is defined as $(\phi - \psi)(\eta_1, \eta_2) := \phi(\eta_1) - \psi(\eta_2)$.
Let us detail this isomorphism a little bit. The tangent space $T_J(\mathbb{C}^2)^{[m]}$ to the Hilbert scheme $(\mathbb{C}^2)^{[m]}$ in an ideal J is canonically isomorphic to $Hom_{\mathbb{C}[x,y]}(J, \mathbb{C}[x,y]/J)$ and the isomorphism is constructed in the following way. Given an element $\eta \in Hom_{\mathbb{C}[x,y]}(J, \mathbb{C}[x,y]/J)$ we choose a lifting $\tilde{\eta} : J \to \mathbb{C}[x,y]$ and such a lifting gives a tangent vector $J_{\epsilon,\eta} = J + \tilde{\eta}(J)$. The fact that η is a morphism of $\mathbb{C}[x,y]$-modules ensures that $J_{\epsilon,\eta}$ is indeed an ideal of $\mathbb{C}[x,y,\epsilon]/(\epsilon^2)$ and thus

that it defines a tangent vector.

Now we observe that

$$T_{(I,J)}(\mathbb{C}^2)^{[m,m+1]} \subset T_I(\mathbb{C}^2)^{[m+1]} \oplus T_J(\mathbb{C}^2)^{[m]} \cong Hom_{\mathbb{C}[x,y]}(I, \mathbb{C}[x,y]/I) \oplus Hom_{\mathbb{C}[x,y]}(J, \mathbb{C}[x,y]/J).$$

The last isomorphism sends a pair (η, ζ) in a couple of tangent vectors

$$(I_{\epsilon,\eta}, J_{\epsilon,\zeta}) \qquad \text{with } I_{\epsilon,\eta} = I + \tilde{\eta}(I), \quad J_{\epsilon,\zeta} = J + \tilde{\zeta}(J),$$

that do not satisfy the condition $I_{\epsilon,\eta} \subseteq J_{\epsilon,\zeta}$ a priori; this is ensured precisely by requiring that (η, ζ) lies in $Ker(\phi - \psi)$.

Choose a polynomial $f \in I \subset J$. If we write $(\tilde{I}, \tilde{J})$ for the image of the couple (I, J) in $\mathbb{C}[x,y]/(f)$ then we have an exact sequence of vector spaces

$$0 \to T_{f,(\tilde{I},\tilde{J})}\mathcal{C}_V^{[m,m+1]} \to T_f V \times T_{(I,J)}(\mathbb{C}^2)^{[m,m+1]} \to \mathbb{C}[x,y]/I, \tag{3.1}$$

where the last map is given by

$$(f + \epsilon g, (\eta, \zeta)) \mapsto \eta(f) - g \bmod I.$$

Even though ζ do not intervene explicitly in the last map, the condition $\eta(f) - g \equiv 0 \bmod I$ ensures that infinitesimally $f + \epsilon g$ is contained in $I_{\epsilon,\eta}$. Since $(\eta, \zeta) \in Ker(\phi - \psi)$, $I_{\epsilon,\eta} \subset J_{\epsilon,\zeta}$; thus $f + \epsilon g$ belongs to $J_{\epsilon,\zeta}$ as well.

Now, we observe that if f is reduced then all the fibers in a neighbourhood U of f are reduced and the relative nested Hilbert schemes $\mathcal{C}_U^{[m,m+1]}$ are reduced of pure dimension $\dim V + m + 1$. Also they are locally complete intersections [BGS]. Then $\mathcal{C}_V^{[m,m+1]}$ is smooth at a point $(f, (I, J))$ if the tangent space at this point has dimension $m + 1 + \dim V$.

Looking at dimensions of the vector spaces in (3.1), we notice that $\dim T_f V = \dim V$ as V is supposed to be smooth, $\dim T_{(I,J)}(\mathbb{C}^2)^{[m,m+1]} = 2m + 2$ by [C] and finally $\mathbb{C}[x,y]/I$ has dimension $m + 1$ by hypothesis: this tells us that $\dim T_{f,(\tilde{I},\tilde{J})}\mathcal{C}_V^{[m,m+1]} = \dim V + m + 1$ if and only if the last map in (3.1) is surjective. The easiest way to ensure this is to ask for surjectivity already in the case $\eta = \zeta = 0$, that is $T_f V \to \mathbb{C}[x,y]/I$ is surjective.

We are now ready to prove the smoothness of the relative nested Hilbert scheme.

Proposition 3.3.1. *Let $\mathcal{C} \to \mathbb{V}$ a family of versal deformations with base point $0 \in \mathbb{V}$. For sufficiently small representatives $\mathcal{C} \to \mathbb{V}$ the relative nested Hilbert scheme $\mathcal{C}_{\mathbb{V}}^{[m,m+1]}$ is smooth.*

Proof. Suppose f is the polynomial defining $\mathcal{C}_0$. Choose $\mathbb{V} \subset \mathbb{C}[x,y]$ containing f such that $\mathcal{C}_{\mathbb{V}} \to \mathbb{V}$ is a versal deformation of the singularity of $\mathcal{C}_0$ and $T_f\mathbb{V}$ contains all polynomials of degree $\leq m$. Then $T_f\mathbb{V}$ will be of dimension $\geq m+1$, thus for any I of colength $m+1$, $T_f\mathbb{V}$ will project surjectively onto $\mathbb{C}[x,y]/I$. By the considerations above, the dimensions counting in (3.1) implies that the relative nested Hilbert scheme $\mathcal{C}_{\mathbb{V}}^{[m,m+1]}$ is smooth. □

Remark 13. The smoothness of the relative nested Hilbert scheme over any versal deformation is equivalent to the smoothness over the miniversal deformations. In fact, if $\overline{\mathcal{C}} \to \overline{\mathbb{V}}$ is the miniversal deformations there are compatible isomorphisms $\mathbb{V} \cong \overline{\mathbb{V}} \times (\mathbb{C}^t, 0)$ and $\mathcal{C} \cong \overline{\mathcal{C}} \times (\mathbb{C}^t, 0)$ and hence also $\mathcal{C}^{[m,m+1]} \cong \overline{\mathcal{C}}^{[m,m+1]} \times (\mathbb{C}^t, 0)$

For a fixed pair of ideals (I, J) with I of colength $m+1$, if we choose the basis $\mathbb{V}$ to be $(m+1)$-dimensional then the relative nested Hilbert scheme $\mathcal{C}_{\mathbb{V}}^{[m,m+1]}$ is smooth by proposition (3.3.1). We would like to find a basis We will need the following lemma, which is stated and proved in [Sh].

Lemma 3.3.2. *Let $\mathcal{O}$ be the completion of the local ring of a point on a reduced curve, and let $\overline{\mathcal{O}}$ be a finite length quotient of $\mathcal{O}$. Let $W \subset \overline{\mathcal{O}}$ a generic k dimensional vector space. Then for I the image in $\overline{\mathcal{O}}$ of any ideal of colength $\leq k$, we have $W + I = \overline{\mathcal{O}}$.*

With this lemma, we are now ready to prove the main theorem of this section.

Theorem 3.3.3. *Let $(C, 0)$ be the analytic germ of a plane curve singularity and let $(\mathcal{C}, 0) \to (\mathbb{V}, 0)$ be an analytically versal deformation of $(C, 0)$. Then, for sufficiently small representatives $\mathcal{C} \to \mathbb{V}$ and a generic disc $0 \in \mathbb{D}^m \subset \mathbb{V}$, the space $\mathcal{C}_{\mathbb{D}^{m+1}}^{[h,h+1]}$ is smooth for $h \leq m+1$.*

Proof. As in proposition (3.3.1) it is enough to prove the theorem for any versal deformation $\mathcal{C} \to \mathbb{V}$. Let $(C, 0)$ be the analytic germ and let $f \in \mathbb{C}[x,y]$ be its equation. Choose $g_1, \ldots, g_s \in \mathbb{C}[x,y]$ such that their images in $\mathbb{C}[[x,y]]/(f, \partial_x f, \partial_y f) \cong \mathbb{C}^s$ form a basis. We have seen that the miniversal deformation $\mathcal{C} \to \mathbb{V} := \mathbb{C}^s$ has as fibres curves whose equation is of the form $f + \sum t_i g_i = 0$.

Let $0 \in \mathbb{D}^{m+1} \subset \mathbb{V}$ be a generic $(m+1)$-dimensional disc. Its tangent space W has dimension $m+1$ and lemma (3.3.2) ensures that $W \subset \mathbb{C}[[x,y]]/(f, \partial_x f, \partial_y f)$ is transverse to any ideal I of colength $h \leq m+1$. Thus for any $h \leq m+1$the final map of (3.1) is surjective, and $\mathcal{C}^{[h,h+1]}$ is smooth at points over $0 \in \mathbb{D}^{m+1}$ which correspond to subschemes supported at the singularity. Finally let $z \subset \mathcal{C}^{[h,h+1]}$ be any subscheme of length $h+1$; let z' be its component supported at the singularity, say of length h'. Then an analytic neighbourhood of z in $\mathcal{C}^{[h,h+1]}$ differs from an analytic neighbourhood of z' in $\mathcal{C}^{[h',h'+1]}$ by a smooth factor. □

Corollary 3.3.4. *Let $\mathcal{C} \to B$ be a family of integral locally planar curves, locally versal at $b_0 \in B$. Then for any generic, sufficiently small $b_0 \in \mathbb{D}^{m+1}$ the relative nested Hilbert scheme $\mathcal{C}^{[h,h+1]}$ is smooth for $h \leq m$.*

Proof. Such a situation is analytically locally smooth over that in theorem (3.3.3); a compactness argument yields smoothness uniformly over an open neighbourhood in the base. □

From the smoothness of the relative nested Hilbert scheme we can deduce an analogue result as the one in [MS1, $Thm.8$].

Corollary 3.3.5. *Let $\mathcal{C} \to B$ a family of curves and let $\mathbb{V}$ be the product of the versal deformations of curve singularities. Then given a point $b_0 \in B$,*

(i) the smoothness of $\mathcal{C}^{[m,m+1]}$ depends only on the image $\mathcal{T}$ of $T_{b_0}B$ in $T_0\mathbb{V}$;

(ii) if $\mathcal{C}^{[m,m+1]}$ is smooth along $\mathcal{C}_{b_0}^{[m,m+1]}$ then $\dim \mathcal{T} \geq min(\delta(\mathcal{C}_{b_0}), m+1)$;

(iii) if $\dim \mathcal{T} \geq m+1$ and $\mathcal{T}$ is general among such subspaces, then $\mathcal{C}^{[m,m+1]}$ is smooth $\mathcal{C}_{b_0}^{[m,m+1]}$;

(iv) $\mathcal{C}^{[m,m+1]}$ is smooth along $\mathcal{C}_{b_0}^{[m,m+1]}$ for all m if and only if $\mathcal{T}$ is transverse to the image of the equigeneric ideal. It suffices for $\mathcal{T}$ to be generic of dimension at least $\delta(\mathcal{C}_{b_0})$.

Proof. To prove (i) take a subscheme $z \in \mathcal{C}_{b_0}^{[m,m+1]}$ which decomposes as

$$z = (z_0, \ldots, z_k)$$

such that $z_0 \in \mathcal{C}_{b_0}^{[d_0,d_0+1]}$ is a subscheme supported at a point c_0 and $z_i \in \mathcal{C}_{b_0}^{[d_i]}$ are length d_i subschemes supported on points c_i.

Let $(\overline{\mathcal{C}}_i, c_i) \to (\mathbb{V}_i, 0)$ be the miniversal deformations of the singularities $(\mathcal{C}_{b_0}, c_i)$ and $(B, b_0) \to \prod(\mathbb{V}_i, 0)$ a map along which $\coprod(\overline{\mathcal{C}}_i, c_i) \to (B, b_0)$ pulls back. Then analytically locally, the germ $(\mathcal{C}^{[m,m+1]}, [z])$ pulls back from $(\mathcal{C}_0^{[d_0,d_0+1]}, [z_0]) \cdot \prod(\overline{\mathcal{C}}_i^{d_i}, [z_i])$ along the same map. We observe that the fibres of $(\overline{\mathcal{C}}_i^{d_i}, [z_i]) \to (\mathbb{V}_i, 0)$ are reduced of dimension d_i by [AIK] and the total space is nonsingular by [Sh, Prop. 17]. Moreover the same holds for $(\mathcal{C}_0^{[d_0,d_0+1]}, [z_0]) \to \mathbb{V}_0$ by proposition (3.3.1).
As the $\mathbb{V}_i$ were taken miniversal, the map $T_{b_0}B \to \prod T_0\mathbb{V}_i$ is uniquely defined and the smoothness of the pullback depends only on the image $\mathcal{T}$ of such a map. To check (ii) we might assume by (i) that the map $T_{b_0}B \to \prod T_0\mathbb{V}_i$ is an isomorphism and identify locally B with its image in some representatives $\overline{B}$ of $\prod(\mathbb{V}_i, 0)$. We can shrink $\overline{B}$ until it can be written as $B \times \mathbb{D}^k$ for some polydisc $\mathbb{D}^k$; as smoothness is an open condition we may shrink $\mathbb{D}^k$ further until $\mathcal{C}^{[m,m+1]}_{|B\times\epsilon}$ is smooth for all $\epsilon \in \mathbb{D}^k$. By [T], the locus of nodal curves with the same cogenus as $\mathcal{C}_{b_0}$ in $\prod \mathbb{V}_i$ is nonempty and of codimension $\delta(\mathcal{C}_{b_0})$; choose an ϵ such that $B \times \epsilon$ contains the point p corresponding to such a curve. If $m+1 \geq \delta$ the statement is trivial. If $m+1 \leq \delta$, we can find a point $z \in \mathcal{C}_p^{[m,m+1]}$, which is a subscheme supported at $m+1$ nodes. The Zariski tangent space $T_z\mathcal{C}_p^{[m,m+1]}$ has dimension $2m+2$, therefore $\mathcal{C}_p^{[m,m+1]}$ cannot be smoothed over a base of dimension less than $m+1$. For point (iii), we assume as above that B is embedded in $\overline{B} = \prod \mathbb{V}_i$. As the dimension of $\mathcal{T}$ is greater equal than $m+1$, then by lemma (3.3.2) it is transverse to any ideal of colength $\leq m+1$, therefore the relative nested Hilbert scheme is smooth. Finally, (iv) if $\mathcal{T}$ in $T_0\mathbb{V}$ is transverse to the equigeneric ideal then the map in (3.1) is surjective for any I and the relative nested Hilbert scheme is smooth. $\square$

3.4 Supports

In Chapter 1 we have defined the supports of a map to be the subvarieties $\overline{Y}_\alpha$ appearing in the Decomposition theorem with some associated non-zero local systems $\mathcal{L}_\alpha$. In the following section we want describe the supports of the map $\pi^{[m,m+1]} : \mathcal{C}^{[m,m+1]} \to B$. All the results we will state in this section holds also for Hilbert schemes and were proved,even though not stated, in [MS1]. In this section we are using the theory of higher discriminants we presented in the first chapter. We recall that whenever we have a map $f : X \to Y$ with Y nonsingular the higher discriminants are defined as

$$\Delta^i(f) := \{y \in Y \mid \text{ there is no } (i-1) - \text{dimensional subspace of } T_yY \text{trasnverse to } f\}$$

More precisely we want to construct a stratification of B such that the strata are precisely the higher discriminants of the map $\pi^{[m,m+1]} : \mathcal{C}^{[m,m+1]} \to B$. Let $b_0 \in B$ be the base point of B and suppose $\mathcal{C}_{b_0} = C$ is the curve with the highest cogenus, which we call δ. For any $i = 0 \ldots \delta$

$$B_i := \{b \in B \mid \delta(\mathcal{C}_b) = i\}$$

and we have that $B = \bigsqcup_i B_i$. As in the case of higher discriminants, we notice that B_0 is the nonsingular locus of the family. We want to show the following proposition:

Proposition 3.4.1. *Let $\pi : \mathcal{C} \to B$ be proper flat family of curves such that the relative nested Hilbert scheme $\pi^{[m,m+1]} : \mathcal{C}^{[m,m+1]} \to B$ is nonsingular for any m. Let δ be the highest cogenus we can find on a curve in the family. Then for any $i = 0 \ldots \delta$*

$$\Delta^i(\pi^{[m,m+1]}) = B_i.$$

Proof. Let $b \in B_i$. As the relative nested Hilbert scheme is nonsingular at b, then by items $(ii) - (iv)$ of corollary(3.3.5) then the image $\mathcal{T}$ of T_bB into the product of the first order deformations of the singularities C_b must be of dimension greater or equal than i. Therefore we have that $B_i \subseteq \Delta^i(\pi^{[m,m+1]})$. Conversely suppose $b \in \Delta^i(\pi^{[m,m+1]})$. If the cogenus of $\mathcal{C}$ were $< i$, then $\mathcal{T}$ would have dimension $< i$ contradicting item (ii) of corollary (3.3.5). □

As a consequence of theorem (1.6.2) if we have supports different from the smooth locus, then we will have to look for them in the i-codimensional irreducible components of the B_i's. We will prove theorem (3.1.2) by applying the criterion (1.6.3) on weight polynomials we stated in Chapter 1.

First we show the result for the Hilbert scheme in [MS1] with a direct computation, then we proceed to prove our theorem for the nested case. We recall that the criterion can be verified just on the generic points of the strata. By theorem (3.2.1) the generic points of the B_i are the nodal curves. Therefore we will prove theorem (3.1.2) for family of nodal curves.

3.5 Proof of theorem 3.1.2

Let $\pi : \mathcal{C} \to B$ a proper flat family of nodal curves, locally versal around a base point $b_0 \in B$. Suppose all the curves are rational: as we will see this is not a restrictive hypothesis. Call $\delta := \delta(\mathcal{C}_{b_0})$. Consider the nodes $\{x_1, \ldots, x_\delta\}$ of the central fiber $\mathcal{C}_{b_0}$. Shrinking B if necessary, we can assume the following facts:

1) The discriminant locus is normal crossing divisor $\Delta := \bigcup D_i$ with $i = 0, \ldots, \delta$, where D_i is the locus in which the $i-$th node x_i is preserved.

2) If $b \in B$ is such that $\mathcal{C}_b$ is nonsingular, then the vanishing cycles $\{\alpha_1, \ldots, \alpha_\delta\}$ associated with the nodes are disjoint.

As the curve $\mathcal{C}_b$ is irreducible, the cohomology classes in $H^1(\mathcal{C}_b)$ of these vanishing cycles are linearly independent, and can then be completed to a symplectic basis $\{\alpha_1, \beta_1, \ldots, \alpha_\delta \beta_\delta\}$. Let T_i be the generators of the (abelian) local fundamental group $\pi_1(B \setminus \Delta, b) \cong \mathbb{Z}^\delta$ where T_i corresponds to "going around D_i". Then the monodromy defining the local system $R^1\tilde{\pi}_*\mathbb{Q}$ on $B \setminus \Delta$ is given via the Picard-Lefschetz formula, and, in the symplectic basis above, the images of the generators of the fundamental group in $GL(H^1(\mathcal{C}_b)) = GL(2\delta, \mathbb{C})$ are given by block diagonal matrices consisting of one Jordan block of order 2 corresponding to a symplectic pair $\{\alpha_i, \beta_i\}$ and the identity elsewhere. Also, as the vanishing cycles are independent, we can consider $R^1\tilde{\pi}_*\mathbb{Q}$ as direct sum of δ modules V_i of rank 2 whose basis is $\{\alpha_i, \beta_i\}$. This makes much more easier to compute the invariants of any local system obtained by linear algebra operations from $R^1\tilde{\pi}_*\mathbb{Q}$. In our case we observe that, as $\mathcal{C}_b$ is nonsingular then

$$\mathcal{C}_b^{[m,m+1]} = \mathcal{C}_b^{(m,m+1)} = \mathcal{C}_b^{(m)} \times \mathcal{C}_b = \mathcal{C}_b^{[m]} \times \mathcal{C}_b.$$

By the MacDonald formula for the cohomology of the symmetric product we have

$$R^i\tilde{\pi}_*^{[m]}\mathbb{Q} = \bigoplus_{k=0}^{[\frac{i}{2}]} \bigwedge^{i-2k} R^1\tilde{\pi}_*\mathbb{Q}(-k) \cong R^{2m-i}\tilde{\pi}_*\mathbb{Q}(m-i) \tag{3.2}$$

where $(-k)$ denotes the weight shift of (k, k) in the mixed Hodge structure on the cohomology. Call the linear algebra operation above $\mathbb{S}^{i,m}$. Applying the Künneth formula and recalling that the cohomology of any curve $\mathcal{C}_b$ in the smooth locus has a pure Hodge structure given by

$$R^0\tilde{\pi}_*\mathbb{Q} = \mathbb{Q} \quad R^1\tilde{\pi}_*\mathbb{Q} \cong \mathbb{Q}^{2\delta} \quad R^2\tilde{\pi}_*\mathbb{Q} \cong \mathbb{Q}(-1)$$

we conclude that

$$(R^i\tilde{\pi}_*^{[m,m+1]}\mathbb{Q})_b = \Big((R^i\tilde{\pi}^{[m]}\mathbb{Q}) \oplus (R^{i-1}\tilde{\pi}_*^{[m]}\mathbb{Q} \otimes R^1\tilde{\pi}_*\mathbb{Q}) \oplus (R^{i-1}\tilde{\pi}^{[m]}\mathbb{Q}(-1))\Big)_b \tag{3.3}$$

Call $\mathbb{T}^{i,m}$ the linear algebra operation we apply to on $R^1\tilde{\pi}_*\mathbb{Q}$ to obtain $R^1\tilde{\pi}_*^{[m,m+1]}$:

$$\mathbb{T}^{i,m}(R^1\tilde{\pi}_*\mathbb{Q}) := \bigoplus_{j=0}^{2} \mathbb{S}^{i+j,m}(R^1\tilde{\pi}_*\mathbb{Q}) \otimes R^j\tilde{\pi}_*\mathbb{Q}$$

Then there exists natural isomorphisms

$$\left(\mathbb{S}^{i,m}H^1(\mathcal{C}_b)\right)^{\pi_1(B\setminus\Delta)} \cong \mathcal{H}^0\left(IC_B(R^i\tilde{\pi}_*^{[m]}\mathbb{Q})\right)_{b_0}$$

$$\left(\mathbb{T}^{i,m}H^1(\mathcal{C}_b)\right)^{\pi_1(B\setminus\Delta)} \cong \mathcal{H}^0\left(IC_B(R^i\tilde{\pi}_*^{[m,m+1]}\mathbb{Q})\right)_{b_0}$$

between the monodromy invariants on $\mathbb{S}^{i,m}H^1(\mathcal{C}_b)$(resp $\mathbb{S}^{i,m}H^1(\mathcal{C}_b)$) and the stalk at b_0 of the first non-vanishing cohomology sheaf of the intersection cohomology complex of $R^i\tilde{\pi}_*^{[m]}\mathbb{Q}$ (resp. $R^i\tilde{\pi}_*^{[m,m+1]}\mathbb{Q}$). The decomposition theorem implies that $H^*(\mathcal{C}_{b_0}^{[m]})$ and $H^*(\mathcal{C}_{b_0}^{[m,m+1]})$ contain respectively the Hodge structures

$$\mathbb{H}^m := \bigoplus_i \left(\mathbb{S}^{i,m}H^1(\mathcal{C}_b)\right)^{\pi_1(B\setminus\Delta)}$$

$$\mathbb{I}^m := \bigoplus_i \left(\mathbb{T}^{i,m}H^1(\mathcal{C}_b)\right)^{\pi_1(B\setminus\Delta)}$$

as a summand. We want to show that this is the unique summand by proving that that the weight polynomial of the cohomology of the nested Hilbert scheme of the $\mathcal{C}_{b_0}$ is equal to the weight polynomial of $\mathbb{H}^m$: this is a corollary of the criterion (1.6.3) in chapter 1.

Proposition 3.5.1. *[MS1, Prop. 15] Suppose $f : X \to Y$ is a proper map between nonsingular algebraic varieties. Let $\mathcal{F}$ be a summand of $Rf_*\mathbb{Q}[\dim X]$. If, for all $y \in Y$ we have that $\mathfrak{w}(\mathcal{F}_y[-\dim X]) = \mathfrak{w}(X_y)$, then $\mathcal{F} = Rf_*\mathbb{Q}[\dim X]$.*

Proposition 3.5.2. *Under the previous assumptions the following holds* *(i)* $\mathfrak{w}(\mathcal{C}_{b_0}^{[m]}) = \mathfrak{w}(\mathbb{H}^m)$

(ii) $\mathfrak{w}(\mathcal{C}_{b_0}^{[m,m+1]}) = \mathfrak{w}(\mathbb{I}^m)$

Remark 14. Suppose we have a family of curves of arithmetic genus p_a and let $\mathcal{C}_b$ be a curve in the smooth locus. Take a basis $\{\alpha_1, \beta_1, \ldots, \alpha_d, \beta_d, \omega_{\delta+1}, \eta_{\delta+1}, \ldots, \omega_{2r}, \eta_{2r}\}$ of $H^1(\mathcal{C}_b)$ where the first 2δ terms are the symplectic basis constructed from the vanishing cycles α_i. Then the monodromy acts as the identity on the others and the Hodge structures given by the monodromy invariants are extensions of Hodge structures $\mathbb{H}^m$ and $\mathbb{I}^m$ on a rational curve $\mathcal{C}'_b$ with the same singularities, whose existence is granted by corollary (3.2.3), tensorized with $\bigoplus_i \mathbb{S}^{i,m} H^1(\overline{C}_{b_0})$ or $\mathbb{T}^{i,m} H^1(\overline{C}_{b_0})$. Passing to weight polynomials, this is equivalent to multiply $\mathfrak{w}(\mathbb{H}^m(\mathcal{C}'_b))$ by the weight polynomial of $\overline{C}^{[m]}$. The same is true for $\mathbb{I}^m$. Summing over all m we get:

$$\sum_m \mathfrak{w}(\mathbb{H}^m(\mathcal{C}_b))q^m = \sum_m \mathfrak{w}(\mathbb{H}^m(\mathcal{C}'_b))q^m \cdot \sum_m \mathfrak{w}(\overline{C}^{[m]})q^m$$

$$\sum_m \mathfrak{w}(\mathbb{I}^m(\mathcal{C}_b)) = \sum_m \mathfrak{w}(\mathbb{I}^m(\mathcal{C}'_b)) \cdot \sum_m \mathfrak{w}(\overline{C}^{[m,m+1]})$$

On the other hand if $C = \mathcal{C}_{b_0}$ is the central singular fiber of the family, and we denote by $C_x^{[m]}$ and $C_x^{[m,m+1]}$ the (nested) Hilbert schemes supported at one node x, we have that splitting subschemes according to their support gives the following equality in the Grothendieck group of varieties:

$$\sum_m q^m \left[C^{[m]}\right] = \sum_m q^m \left[C_{reg}^{[m]}\right] \prod_{x_i} \sum q^m \left[C_{x_i}^{[m]}\right] =$$
$$\sum_m q^m \left[\overline{C}^{[m]}\right] \prod_{x_i} (1-q)^2 \sum q^m \left[C_{x_i}^{[m]}\right]$$

$$\sum_m q^m \left[C^{[m,m+1]}\right] = \sum_m q^m \left[C_{reg}^{[m,m+1]}\right] \prod_{x_i} \sum q^m \left[C_{x_i}^{[m,m+1]}\right] =$$
$$= \sum_m q^m \left[\overline{C}^{[m,m+1]}\right] \prod_{x_i} (1-q)^2 \sum q^m \left[C_{x_i}^{[m,m+1]}\right]$$

where the factor $(1-q)^2$ is given by the analytic local branches of C at the nodes.
As the normalization intervenes in both the formulas we can drop the arithmetic genus information in the proof and suppose it coincides with δ.

Remark 15. Even though we are supposing for simplicity that the family of curves is locally versal around b_0 , we may weaken our hypotheses by just asking that the family is regular around b_0 and that the locus of nodal curves is dense in every δ-stratum.

3.5.1 Hilbert scheme case

Let $\pi : \mathcal{C} \to B$ a locally versal deformation of a singular rational nodal curve $\mathcal{C}_{b_0} =: C$. As a warm up for the nested case, we will compute the weight polynomial of $C^{[m]}$ and the weight polynomial of the Hodge structure $\mathbb{H}^m$ given by the monodromy invariants and show they are equalm thus proving theorem [MS1, Theorem 1].

Computation of $\mathfrak{w}(C^{[m]})$

To compute $\mathfrak{w}(C^{[m]})$ we use power series to find a formula for the class of $C^{[m]}$ in the Grothendieck group. First we notice that

$$\sum_m q^m \left[C^{[m]}\right] = \sum_m q^m \left[C^{[m]}_{reg}\right] \prod_{x_i} \sum q^m \left[C^{[m]}_{x_i}\right] \tag{3.4}$$

As $C_{reg} = \mathbb{P}^1 \setminus 2\delta$ regular points $p_1, \dots p_{2\delta}$ then

$$\sum_m q^m \left[(\mathbb{P}^1)^{[m]}\right] = \sum_m q^m \left[C^{[m]}_{reg}\right] \prod_{p_i} \sum q^m \left[C^{[m]}_{p_i}\right]$$

Now observe that $(\mathbb{P}^1)^{[m]} = \mathbb{P}^m$; also as the p_i are regular points $\left[C^{[m]}_{p_i}\right] = 1$ for all m and we have:

$$\frac{1}{(1-q)(1-q\mathbb{L})} = \sum_m q^m \left[C^{[m]}_{reg}\right] \frac{1}{(1-q)^{2\delta}} \Rightarrow \sum_m q^m \left[C^{[m]}_{reg}\right] = \frac{(1-q)^{2\delta-1}}{(1-q\mathbb{L})}$$

where $\mathbb{L}$ denotes the weight polynomial of the affine line.
Now, in [R] Ran shows that $C^{[m]}_x$ consists of $m-1$ copies of $\mathbb{P}^1$ that intersects transversely. Thus

$$\prod_{x_i} \sum q^m \left[C^{[m]}_{x_i}\right] = \left(\sum q^m((m-1)\mathbb{L}+1)\right)^\delta = \frac{\left(1-q+q^2\mathbb{L}\right)^\delta}{(1-q)^{2\delta}}.$$

Substituting in equation (3.4), we get

$$\sum_m q^m \left[C^{[m]}\right] = \frac{\left(1-q+q^2\mathbb{L}\right)^\delta}{(1-q)(1-q\mathbb{L})}$$

The coefficient of q^m in the series is given by

$$\mathfrak{w}(C^{[m]}) = \sum_{s=0}^{m}(-1)^s \sum_{t=0}^{\delta}\binom{\delta}{t}\binom{t}{s-t}\mathbb{L}^{s-t}\cdot\sum_{l=0}^{m-s}\mathbb{L}^l = \sum_{s=0}^{m}(-1)^s\sum_{l=0}^{\delta}\binom{\delta}{t}\binom{t}{s-t}\mathbb{L}^{s-t}\cdot\frac{\mathbb{L}^{m-s+1}-1}{\mathbb{L}-1}. \tag{3.5}$$

Computation of $\mathfrak{w}(\mathbb{H}^m)$

Let b a point in the smooth locus. We now need to compute the invariants in the cohomology groups $H^i(\mathcal{C}_b)$ of the monodromy $\rho : \pi_1(B\setminus\Delta) \to H^1(\mathcal{C}_b)$. Also, we recall that all the vanishing cycles α_i have weight 0, while β_i have weight 2.
Considering the MacDonald formula to compute the cohomology of Hilbert scheme, we just need to understand the invariants of $\bigwedge^l H^1(\mathcal{C}_b)$ for any $l \geq 0$. As we observed before, $H^1(\mathcal{C}_b)$ can be viewed as a direct sum of 2-dimensional representations V_i on which a generator $T_j \in SL(2\delta,\mathbb{C})$ of the monodromy acts as the identity if $i \neq j$ and $T_i(\alpha_i) = \alpha_i$, $T_i(\beta_i) = \alpha_i + \beta_i$. Thus $H^1(\mathcal{C}_b) = \bigoplus_{i=1}^{\delta} V_i$ and we have

$$\bigwedge^l H^1(\mathcal{C}_b) = \bigoplus_{l_1+\ldots+l_\delta=l}\bigwedge^{l_1} V_1 \otimes \ldots \otimes \bigwedge^{l_\delta} V_\delta, \qquad 0 \leq l_i \leq 2. \tag{3.6}$$

Also, as $\dim V_i = 2$

$$\bigwedge^{l_i} V_i = \begin{cases} \mathbb{C} & \text{if } l_i = 0 \\ V_i & \text{if } l_i = 1 \\ \mathbb{C}(-1) & \text{if } l_i = 2 \end{cases}.$$

The only invariants of V_i are the α_i, of weight 0. In conclusion we have that for any $i = 0,\ldots,m$ we have

$$I(i,\delta) := \mathfrak{w}\left((H^i(\mathcal{C}_b^{[m]}))^{\pi_1(B\setminus\Delta)}\right) = (-1)^i\sum_{k=0}^{[\frac{i}{2}]}\mathbb{L}^k\sum_{j=0}^{[\frac{i-2k}{2}]}\binom{\delta}{j}\binom{\delta-j}{i-2k-2j}\mathbb{L}^j$$

where the index k is the one in MacDonald formula and j represents the number of second external power we take in (3.6).

Summing over m and taking the duality in (3.2) into account we get

$$\mathfrak{w}(\mathbb{H}^m) = \sum_{i=0}^{m-1}(-1)^i(1+\mathbb{L}^{m-i})\sum_{k=0}^{[\frac{i}{2}]}\mathbb{L}^k\sum_{j=0}^{[\frac{i-2k}{2}]}\binom{\delta}{j}\binom{\delta-j}{i-2k-2j}\mathbb{L}^j \tag{3.7}$$

$$+(-1)^m\sum_{k=0}^{[\frac{m}{2}]}\mathbb{L}^k\sum_{j=0}^{[\frac{m-2k}{2}]}\binom{\delta}{j}\binom{\delta-j}{m-2k-2j}\mathbb{L}^j \tag{3.8}$$

Proof of point (i) in Proposition 3.5.2. We start looking at $\mathfrak{w}(\mathbb{H}^m)$. First we notice that due to properties of binomial coefficient, the sum over j goes to δ while the sum in k can go to infinity. Also we have that $\binom{\delta}{j}\binom{\delta-j}{i-2k-2j} = \binom{\delta}{i-2k-j}\binom{i-2k-j}{j}$.

Setting $l = i-2k-j$ and applying the remarks above we get

$$\mathfrak{w}(\mathbb{H}^m) = \sum_{i=0}^{m-1}(-1)^i(1+\mathbb{L}^{m-i})\sum_{k=0}^{\infty}\mathbb{L}^k\sum_{l=0}^{\delta}\binom{\delta}{l}\binom{l}{i-2k-l}\mathbb{L}^{i-2k-l}+$$
$$+(-1)^m\sum_{k=0}^{\infty}\mathbb{L}^k\sum_{l=0}^{\delta}\binom{\delta}{l}\binom{l}{m-2k-l}\mathbb{L}^{m-2k-l}$$

Set $s = i-2k$ and split the sum in two parts with respect to the product with $(1+\mathbb{L}^{m-i})$.

$$\mathfrak{w}(\mathbb{H}^m) = \sum_{s=0}^{m}(-1)^s\sum_{k=0}^{\infty}\mathbb{L}^k\sum_{l=0}^{\delta}\binom{\delta}{l}\binom{l}{s-l}\mathbb{L}^{s-l}+$$
$$+\sum_{s=0}^{m}(-1)^s\mathbb{L}^{m-s}\sum_{k=0}^{\infty}\mathbb{L}^{-k}\sum_{l=0}^{\delta}\binom{\delta}{l}\binom{l}{s-l}\mathbb{L}^{s-l}$$

Taking out the sums in k and recalling that $\sum_{k=0}^{\infty}\mathbb{L}^k = \dfrac{1}{1-\mathbb{L}}$

$$\mathfrak{w}(\mathbb{H}^m) = \frac{1}{1-\mathbb{L}}\sum_{s=0}^{m}(-1)^s\sum_{l=0}^{\delta}\binom{\delta}{l}\binom{l}{s-l}\mathbb{L}^{s-l}+$$
$$-\frac{\mathbb{L}}{1-\mathbb{L}}\sum_{s=0}^{m}(-1)^s\mathbb{L}^{m-s}\sum_{k=0}^{\infty}\mathbb{L}^{-k}\sum_{l=0}^{\delta}\binom{\delta}{l}\binom{l}{s-l}\mathbb{L}^{s-l} =$$
$$= \frac{1}{1-\mathbb{L}}\sum_{s=0}^{m}(-1)^s(1-\mathbb{L}^{m-s+1})\sum_{l=0}^{\delta}\binom{\delta}{l}\binom{l}{s-l}\mathbb{L}^{s-l}$$

which is precisely $\mathfrak{w}(C^{[m]})$. □

3.5.2 Nested Hilbert scheme case

As above suppose $\pi : \mathcal{C} \to B$ is a locally versal deformation of a singular rational nodal curve $\mathcal{C}_{b_0} =: C$. We now want to show point (ii) of proposition (3.5.2), to conclude the proof of theorem (3.1.2). Again, we compute the weight polynomials $\mathfrak{w}(C^{[m,m+1]})$, $\mathfrak{w}(\mathbb{I}^m)$ and show that their are equal.

Computation of $\mathfrak{w}(C^{[m,m+1]})$

We start by stratifying $\mathcal{C}_{b_0}^{[m,m+1]}$. As the weight polynomial depends only on the class in the Grothendieck group, we can work there. Let $\mathcal{C}_{0,reg} := \mathcal{C}_0 \setminus \{x_1, \dots, x_\delta\}$. We can consider the colength 1 ideal of $\mathcal{C}_{b_0}^{[m,m+1]}$ as a copy of $\mathcal{C}_{b_0}^{[m]}$ to which we add a further point $p \in {\mathcal{C}_{b_0}}^{[m]}$. Whenever we add a regular point p the class does not change, while when the point is a node we need to be careful about the number of occurrences of the node in the colength one ideal. In [R], Ran shows that the nested Hilbert scheme $C_x^{[k,k+1]}$ supported on one node, is $2k-1$ copies of $\mathbb{P}^1$ alternating between those coming from $C_x^{[k]}$ and $C_x^{[k+1]}$. As a consequence $\left[C_x^{[k,k+1]}\right] = (2k-1)\mathbb{L}+1$.

We stratify $\mathcal{C}_0^{[m,m+1]}$ with respect to the number of times the nodes appear in $\left[\mathcal{C}_0^{[m]}\right]$:

$$\begin{aligned}\left[\mathcal{C}_{b_0}^{[m,m+1]}\right] &= \left[\mathcal{C}_0^{[m]} \times \mathcal{C}_{b_0,reg})\right] + \sum_{i=1}^{\delta}\sum_{k=0}^{m}\left[(\mathcal{C}_0 - x_i)^{[m-k]} \times C_{x_i}^{[k,k+1]}\right] = \\ &= \left[\mathcal{C}_{b_0}^{[m]} \times \mathcal{C}_{b_0,reg})\right] + \delta\sum_{k=0}^{m}\left[(\mathcal{C}_0 - x)^{[m-k]} \times C_x^{[k,k+1]}\right]\end{aligned}$$

We observe that for any $k \geq 0$ we can write $\left[C_x^{[k,k+1]}\right] = \left[C_x^{[k]}\right] + k\mathbb{L}$. Making a substitution in the above equation we get

$$\left[\mathcal{C}_{b_0}^{[m,m+1]}\right] = \left[\mathcal{C}_{b_0}^{[m]} \times \mathcal{C}_{b_0,reg})\right] + \delta\sum_{k=0}^{m}\left[(\mathcal{C}_{b_0} - x)^{[m-k]} \times C_x^{[k]}\right] + \delta\mathbb{L}\sum_{k=0}^{m} k\left[(\mathcal{C}_{b_0} - x)^{[m-k]}\right].$$

Since $\sum_{k=0}^{m}\left[(\mathcal{C}_{b_0} - x)^{[m-k]} \times C_x^{[k]}\right] = \left[{\mathcal{C}_{b_0}}^{[m]}\right]$, we have that the second term of the sum consists precisely of those δ copies of ${\mathcal{C}_{b_0}}^{[m]}$ which, added to the first term, give ${\mathcal{C}_{b_0}}^{[m]} \times \mathcal{C}_{b_0}$. Finally, we notice that $(\mathcal{C}_{b_0} - \times)$ can be considered as a curve $\tilde{\mathcal{C}}$ with $\delta-1$ nodes minus two regular points p, q. Then the class of its Hilbert scheme can be computed as $\left[\tilde{\mathcal{C}}^{[m]}\right] = \sum_{k=0}^{m}\left[(\mathcal{C}_{b_0} - x)^{[m-k]}\right] \times C_{p,q}^{[k]}$,

where $C_{p,q}^{[k]}$ is the Hilbert scheme with support $p \cup q$. As p and q are regular points, $\left[C_{p,q}^{[k]}\right]$ is just the number of length non ordered $k-$ple in p, q , which is equal to k.
In conclusion we can write

$$\left[\mathcal{C}_{b_0}^{[m,m+1]}\right] = {\mathcal{C}_{b_0}}^{[m]} \times \mathcal{C}_{b_0} + \delta \mathbb{L} \left[\tilde{\mathcal{C}}^{[m]}\right] \tag{3.9}$$

Computation of $\mathfrak{w}(\mathbb{I}^m)$

We remind that

$$H^i(\mathcal{C}_b^{[m,m+1]}) = H^i(\mathcal{C}_b^{[m]}) \oplus H^{i-1}(\mathcal{C}_b^{[m]}) \otimes H^1(\mathcal{C}_b) \oplus H^{i-2}(\mathcal{C}_b^{[m]})(-1).$$

We notice that, by applying the MacDonald formula to second term we get

$$H^{i-1}(\mathcal{C}_b^{[m]}) \otimes H^1(\mathcal{C}_b) = \bigoplus_{k=0}^{[\frac{i-1}{2}]} \bigwedge^{i-1-2k} H^1(\mathcal{C}_b) \otimes H^1(\mathcal{C}_b)(-k)$$

As a result we will have to find both the invariants of $\bigwedge^l H^1(\mathcal{C}_b)$ and those of $\bigwedge^l H^1(\mathcal{C}_b) \otimes H^1(\mathcal{C}_b)$. We have seen how to find the invariants of $\bigwedge^l H^1(\mathcal{C}_b)$ in the computation for the Hilbert scheme; when looking at the invariants of $\bigwedge^l H^1(\mathcal{C}_b) \otimes H^1(\mathcal{C}_b)$ we have to be more careful: there is more than just the invariant of $\bigwedge^l H^1(\mathcal{C}_b)$ times the invariant of $H^1(\mathcal{C}_b)$.
Let us be more precise: recall that $H^1(\mathcal{C}_b) = \bigoplus_{i=1}^{\delta} V_i$ and that we have

$$\bigwedge^l H^1(\mathcal{C}_b) = \bigoplus_{l_1+\ldots+l_\delta=l} \bigwedge^{l_1} V_1 \otimes \ldots \otimes \bigwedge^{l_\delta} V_\delta, \qquad 0 \leq l_i \leq 2. \tag{3.10}$$

Also, as $\dim V_i = 2$

$$\bigwedge^{l_i} V_i = \begin{cases} \mathbb{C} & \text{if } l_i = 0 \\ V_i & \text{if } l_i = 1 \\ \mathbb{C}(-1) & \text{if } l_i = 2 \end{cases}.$$

Thus

$$\bigwedge^l H^1(\mathcal{C}_b) \otimes H^1(\mathcal{C}_b) = (\bigoplus_{l_1+\ldots+l_\delta=l} \bigwedge^{l_1} V_1 \otimes \ldots \otimes \bigwedge^{l_\delta} V_\delta) \otimes (V_1 \oplus \ldots \oplus V_\delta). \tag{3.11}$$

By the considerations above, the monodromy invariants of summands of type $V_i \otimes V_j$ for $i \neq j$ are just an invariant of V_i tensor an invariant of V_j, while invariants of summands of type

$\bigwedge^2 V_i \otimes V_j$ are just the invariants of V_i with shifted weight.

The invariants which are not the tensor product of an invariant of $\bigwedge^l H^1(\mathcal{C}_b)$ times an invariant of $H^1(\mathcal{C}_b)$ come from the summands $V_i \otimes V_i = \bigwedge^2 V_i \otimes Sym^2(V_i)$. These summands provide additional invariants of weight 2, which are those of $\bigwedge^2 V_i$.

As equation (3.11) is symmetric in the V_i's it is sufficient to compute the invariants of

$$(\bigoplus_{l_1+\ldots+l_\delta=l} \bigwedge^{l_1} V_1 \otimes \ldots \otimes \bigwedge^{l_\delta} V_\delta) \otimes V_1$$

and multiply what we obtain by δ.

If $l_1 \neq 1$ then the formula we wrote for the Hilbert scheme still holds, while when $l_1 = 1$ we have a certain number of invariants of weight 2 to take into account.

$$\begin{aligned}\mathfrak{w}\left((H^i(\mathcal{C}_b^{[m]} \otimes H^1(\mathcal{C}_b)))^{\pi_1(B\setminus\Delta)}\right) = \delta \sum_{k=0}^{[\frac{i}{2}]} \mathbb{L}^k \sum_{j=0}^{[\frac{i-2k}{2}]} & \binom{\delta-1}{j-1}\binom{\delta-j}{i-2k-2j}\mathbb{L}^j + \\ & + (1+\mathbb{L})\binom{\delta-1}{j}\binom{\delta-1-j}{i-2k-2j-1}\mathbb{L}^j + \\ & + \binom{\delta-1}{j}\binom{\delta-1-j}{i-2k-2j}\mathbb{L}^j\end{aligned}$$

The first term in the sum represents the case in which $l_1 = 2$, the second one is the case of $l_1 = 1$ and the last one is $l_1 = 0$. As in the previous formula, the index k is the one in the MacDonald formula, while the index j represents the number of $l_i \neq l_1$ that are equal to 2.

Summing over i we get

$$\begin{aligned}\mathfrak{w}(\mathbb{I}^m) = &\sum_{i=0}^{m}(1+\mathbb{L}^{m+1-i})\sum_{k=0}^{[\frac{i}{2}]}\mathbb{L}^k\sum_{j=0}^{[\frac{i-2k}{2}]}\binom{\delta}{j}\binom{\delta-j}{i-2k-2j}\mathbb{L}^j+\\
&+\delta\sum_{k=0}^{[\frac{i-1}{2}]}\mathbb{L}^k\sum_{j=0}^{[\frac{i-1-2k}{2}]}\binom{\delta-1}{j-1}\binom{\delta-j}{i-1-2k-2j}\mathbb{L}^j+(1+\mathbb{L})\binom{\delta-1}{j}\binom{\delta-1-j}{i-1-2k-2j-1}\mathbb{L}^j+\\
&+\binom{\delta-1}{j}\binom{\delta-1-j}{i-1-2k-2j}\mathbb{L}^j+\mathbb{L}\sum_{k=0}^{[\frac{i-2}{2}]}\mathbb{L}^k\sum_{j=0}^{[\frac{i-2-2k}{2}]}\binom{\delta}{j}\binom{\delta-j}{i-2-2k-2j}\mathbb{L}^j+\\
&+(-1)^{m+1}\delta\sum_{k=0}^{[\frac{m}{2}]}\mathbb{L}^k\sum_{j=0}^{[\frac{m-2k}{2}]}\binom{d-1}{j-1}\binom{d-j}{m-2k-2j}\mathbb{L}^j+\\
&+(1+\mathbb{L})\binom{\delta-1}{j}\binom{\delta-1-j}{m-2k-2j-1}\mathbb{L}^j+\binom{\delta-1}{j}\binom{\delta-1-j}{m-2k-2j}\mathbb{L}^j+\\
&+2\mathbb{L}\sum_{k=0}^{[\frac{m-1}{2}]}\mathbb{L}^k\sum_{j=0}^{[\frac{m-1-2k}{2}]}\binom{\delta}{j}\binom{\delta-j}{m-1-2k-2j}\mathbb{L}^j.\end{aligned}$$

Looking at equation (3.9) we want to separate the invariants which are the tensor product of invariants of the Hilbert scheme and the invariants of the curve from those coming from the weight 2 part of the pieces $V_i \otimes V_i$, which we will prove to be precisely the invariants of the Hilbert scheme of the curves with $\delta - 1$ nodes. We want to show that the former are

$$\begin{aligned}A = &\sum_{i=0}^{m}(1+\mathbb{L}^{m+1-i})\sum_{k=0}^{[\frac{i}{2}]}\mathbb{L}^k\sum_{j=0}^{[\frac{i-2k}{2}]}\binom{\delta}{j}\binom{\delta-j}{i-2k-2j}\mathbb{L}^j+\\
&+\delta\sum_{k=0}^{[\frac{i-1}{2}]}\mathbb{L}^k\sum_{j=0}^{[\frac{i-1-2k}{2}]}\binom{\delta-1}{j-1}\binom{\delta-j}{i-1-2k-2j}\mathbb{L}^j+\binom{\delta-1}{j}\binom{\delta-1-j}{i-2k-2j-1}\mathbb{L}^j+\\
&+\binom{\delta-1}{j}\binom{\delta-1-j}{i-1-2k-2j}\mathbb{L}^j+\mathbb{L}\sum_{k=0}^{[\frac{i-2}{2}]}\mathbb{L}^k\sum_{j=0}^{[\frac{i-2-2k}{2}]}\binom{\delta}{j}\binom{\delta-j}{i-2-2k-2j}\mathbb{L}^j+\\
&+(-1)^{m+1}2\mathbb{L}\sum_{k=0}^{[\frac{m-1}{2}]}\mathbb{L}^k\sum_{j=0}^{[\frac{m-1-2k}{2}]}\binom{\delta}{j}\binom{\delta-j}{m-1-2k-2j}\mathbb{L}^j+\delta\sum_{k=0}^{[\frac{m}{2}]}\mathbb{L}^k\sum_{j=0}^{[\frac{m-2k}{2}]}\binom{\delta-1}{j-1}\binom{\delta-j}{m-2k-2j}\mathbb{L}^j+\\
&+\binom{\delta-1}{j}\binom{\delta-1-j}{m-1-2k-2j}\mathbb{L}^j+\binom{\delta-1}{j}\binom{\delta-1-j}{i-1-2k-2j}\mathbb{L}^j+\\
&+2\mathbb{L}\sum_{k=0}^{[\frac{m-1}{2}]}\mathbb{L}^k\sum_{j=0}^{[\frac{m-1-2k}{2}]}\binom{\delta}{j}\binom{\delta-j}{m-1-2k-2j}\mathbb{L}^j.\end{aligned}$$

while the latter are

$$B = \delta\mathbb{L}\sum_{i=0}^{m}(-1)^i(1+\mathbb{L}^{m+1-i})\sum_{k=0}^{[\frac{i-1}{2}]}\mathbb{L}^k\sum_{j=0}^{[\frac{i-1-2k}{2}]}\binom{\delta-1}{j}\binom{\delta-1-j}{i-2k-2j-1}\mathbb{L}^j+\binom{\delta-1}{j}\binom{\delta-1-j}{i-1-2k-2j}\mathbb{L}^j+$$

$$(-1)^{m+1}\delta\mathbb{L}\sum_{k=0}^{[\frac{m}{2}]}\mathbb{L}^k\sum_{j=0}^{[\frac{m-2k}{2}]}+\binom{\delta-1}{j}\binom{\delta-1-j}{m-2k-2j-1}\mathbb{L}^j.$$

Lemma 3.5.3.

$$A=\left[\mathcal{C}_{b_0}{}^{[m]}\times\mathcal{C}_{b_0}\right]=\mathfrak{w}(\mathbb{H}^m)(\mathbb{L}-\delta+1)$$

Proof. First we notice that, due to properties of binomial coefficients, the quantity

$$\sum_{k=0}^{[\frac{i-1}{2}]}\mathbb{L}^k\sum_{j=0}^{[\frac{i-1-2k}{2}]}\binom{\delta-1}{j-1}\binom{\delta-j}{i-1-2k-2j}\mathbb{L}^j+\binom{\delta-1}{j}\binom{\delta-1-j}{i-2k-2j-1}\mathbb{L}^j+\binom{\delta-1}{j}\binom{\delta-1-j}{i-1-2k-2j}\mathbb{L}^j$$

is equal to

$$\sum_{k=0}^{[\frac{i-1}{2}]}\mathbb{L}^k\sum_{j=0}^{[\frac{i-1-2k}{2}]}\binom{\delta}{j}\binom{\delta-j}{i-1-2k-2j}\mathbb{L}^j=I(i-1,\delta);$$

thus

$$A=\sum_{i=0}^{m}(-1)^i(1+\mathbb{L}^{m+1-i})I(i,\delta)+\delta I(i-1,\delta)+\mathbb{L}I(i-2,\delta)+(-1)^{m+1}\left(2\mathbb{L}I(m-1,\delta)+\delta I(m,\delta)\right).$$

Now, by setting $t=i-1$ we see that

$$\sum_{i=0}^{m}(-1)^i(1+\mathbb{L}^{m+1-i})\delta I(i-1,\delta)+\delta(-1)^{m+1}I(m,\delta)=-\delta\mathfrak{w}(\mathbb{H}^{[m]}).$$

Also,

$$\sum_{i=0}^{m}(-1)^i(1+\mathbb{L}^{m+1-i})I(i,\delta)+\mathbb{L}I(i-2,\delta)+(-1)^{m+1}2\mathbb{L}I(m-1,\delta)=$$

$$\sum_{i=0}^{m}(-1)^i(1+\mathbb{L}^{m+1-i})I(i,\delta)+\sum_{i=0}^{m}(-1)^i(\mathbb{L}+\mathbb{L}^{m+2-i})+(-1)^{m+1}2\mathbb{L}I(m-1,\delta),$$

setting $t=i-2$ this becomes

$$(1+\mathbb{L})\sum_{i=0}^{m-2}(-1)^i\mathbb{L}^{m-i})I(i,\delta)+\sum_{i=0}^{m-2}(-1)^i\mathbb{L}^{m-i}I(i,\delta)+$$

$$+(-1)^{m+1}2\mathbb{L}I(m-1,\delta)+(-1)^mI(m,d)(1+\mathbb{L})+(-1)^{m-1}(1+\mathbb{L}^2)=$$

$$=(1+\mathbb{L})\left(\sum_{i=0}^{m-1}(-1)^i\mathbb{L}^{m-i})I(i,\delta)+(-1)^mI(m,d)\right)=(1+\mathbb{L})\mathfrak{w}(\mathbb{H}^m).$$

□

Analogously, using properties of binomial coefficients and setting $t = i - 1$ we can prove

Lemma 3.5.4.

$$B = \delta \mathbb{L} \left[\tilde{\mathcal{C}}^{[m]} \right]$$

and this complete the proof of proposition (3.5.2) and theorem (3.1.2).

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