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OUTPUT REGULATION FOR SYSTEMS WITH SYMMETRY

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"Physicists describe the two properties of physical laws that they do not depend on when or where you use them as symmetries of nature. By this usage physicists mean that nature treats every moment in time and every location in space identically symmetrically by ensuring that the same fundamental laws are in operation. Much in the same manner that they affect art and music, such symmetries are deeply satisfying; they highlight an order and coherence in the workings of nature. The elegance of rich, complex, and diverse phenomena emerging from a simple set of universal laws is at least part of what physicists mean when they invoke the term "beautiful"."
Greene (1999)

For Elettra and Ettore

Abstract

The problem of output regulation deals with asymptotic tracking/rejection of a prescribed reference trajectory/disturbance. The main feature of the output regulation is that references/disturbances to be tracked/rejected belong to a family of trajectories generated as solutions of an autonomous system typically referred to as exosystem. Tackling this problem in context of error feedback leads to solutions that embeds a copy of the exosystem properly updated by means of error measurements. The output regulation problem for linear systems has been fully characterized and solved in the mid seventies by Davison, Francis and Wonham and then has been generalized to the non-linear context by Isidori and Byrnes. It is worth noting, however, that most of the frameworks considered so far for output regulation deal with systems and exosytems defined on Euclidean real state space and not much efforts have been done to extend the results of output regulation to systems and exosystems whose states live in more general manifolds. The tools available for solutions of the output regulation problem can't be extented in a straightforward manner to non-linear systems whose states live in more general manifolds due to some restrictive structural assumption. The present thesis focuses on the problem of output regulation for *left invariant* systems defined on matrix Lie groups. In this framework we extend the idea of internal model-based control to systems defined on matrix Lie-groups taking advantages of the *symmetry* and *invariant structures* of the system considered. In particular we propose a general structure of the regulator for *left invariant* kinematic systems defined on general matrix Lie-group that solves the output regulation problem. Going further we study the output regulation problem for kinematics systems defined on the *special orthogonal group* and the *special Euclidean group*. We also show that the dynamics associated to the fully actuated system whose kinematic is defined on the *special orthogonal group* and the *special Euclidean group* can be handled taking advantages of *backstepping* techniques.

Sommario

Il problema della regolazione delle uscite si occupa dell'inseguimento asintotico (reiezione asintotica) di una traiettoria di riferimento. La caratteristica principale della regolazione delle uscite consiste nel considerare le traiettore da inseguire e i disturbi da reiettare appartenenti ad una famiglia di traiettorie generate come soluzioni di un sistema dinamico autonomo noto come esosistema. Nel contesto di retroazione delle uscite, il principio del modello interno porta a soluzioni in cui viene inserito nel loop di controllo una copia dell'esosistema opportunamente guidata da una funzione dell'errore di inseguimento. Il problema della regolazione delle uscite per sistemi lineari è stato ampiamente caratterizato e risolto nella metà degli anni settanta da Davison, Francis e Wonham. Il problema di regolazione delle uscite è stato successivamente esteso nel contesto non lineare da Isidori and Byrnes. Si noti tuttavia che nella maggioranza di lavori nel contesto di regolazione delle uscite vengono presi in considerazione sistemi ed esosistemi il cui spazio di stato è definito su uno spazio Euclideo e non vi è stato un grande sforzo nell'estensione del problema per sistemi il cui spazio di stato giace su varietà differenziali ed in generale su spazi non Euclidei. I classici strumenti per la soluzione del problema della regolazione non sono direttamente estendibili per sistemi non lineari il cui spazio di stato giace su varietà differenziabili a causa di alcune ipotesi restrittive. Questa tesi si focalizza sul problema della regolazione delle uscite per sistemi *sinistro-invarianti* definiti sui gruppi matriciali di Lie. In questo contesto, prendendo vantaggio delle *simmetrie* del sistema e delle sue *proprietà di invarianza*, verrà esteso il principio del modello interno per sistemi cinematici definiti sui gruppi di Lie. Inoltre, in questa tesi, verrà considerato il problema della regolazione delle uscite per sistemi definiti sul *gruppo speciale ortogonale* ed il *gruppo speciale Euclideo*. Verrà inoltre mostrato che il problema cinematico di regolazione può essere esteso dinamicamente per sistemi pienamente attuati utilizzando le classiche tecniche di *backstepping*.

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Finally, I would like to thank my family: my mother Rossana, my father Emidio, my brother Andrea and my sis-in-law Silvia for the continuous support and encouragement.

Last but not the least, I would like to thank my niece Elettra for making me rediscover the beauty of the simple things of life.

Contents

Notation

$\mathbb{R}$	set of real numbers
$\mathbb{R}^{n\times n}$	set of $n \times n$ square matrices
$\mathbb{R}_{\geq 0}$	set of non negative real numbers
$\mathbb{R}_{>0}$	set of real numbers larger than zero
$\mathbb{N}$	set of non negative integers
$\mathbb{N}_{>0}$	set of integers larger than zero
$\{A\}$	inertial frame
$\{B\}$	body-fixed frame
$\mathbf{G}$	general matrix Lie-group
$\mathfrak{g}$	Lie-algebra associated to the Lie-group $\mathbf{G}$
$\mathrm{SO}(n)$	special orthogonal group
$\mathfrak{so}(n)$	Lie-algebra associated to the special orthogonal group
R	Rotation matrix of $\{B\}$ relative to $\{A\}$
$\mathrm{SE}(n)$	special Euclidean group
$\mathfrak{se}(n)$	Lie-algebra associated to the special Euclidean group
$\mathrm{SL}(n)$	special linear group
$\mathfrak{sl}(n)$	Lie-algebra associated to the special linear group
$[\cdot,\cdot]$	Lie-bracket (matrix commutator)
mrp	matrix representation
vrp	vectorial representation
$\Omega_\times$	skew map
vex	inverse of the skew map
Ad_X	adjoint operator
$\in$	belongs to
$\subset$	subset

$\cup$	union
$\supset$	superset
$:=$	defined as
$\mapsto$	maps to
$\wedge$	vectorial product
$A^\top$	transpose
A^{-1}	inverse
$\|A\|$	Frobenius norm
$\mathbb{P}_a(A)$	anti-symmetric projection in square matrix space
$\mathbb{P}_s(A)$	symmetric projection in square matrix space
$\mathbb{P}_g(A)$	orthogonal projection of A onto $\mathfrak{g}$ with respect to the trace inner product
$A > 0$	positive definite matrix
$A \geq 0$	positive semi-definite matrix
$\det(A)$	determinant
$\mathrm{rank}(A)$	rank
$\mathrm{tr}\,(A)$	trace
$\lambda(A)$	eigenvalue of A
$\sigma(A)$	spectrum of A, the set of its eigenvalues
$0_{n\times m}$	matrix of dimension $n \times m$ whose entries are all zeros
I_n	an $n\times n$ identity matrix, also denoted with I when there is no need to emphasize the dimension
$\mathrm{diag}\,(a_1, \ldots, a_n)$	an $n \times n$ diagonal matrix with a_i as its i-th diagonal element
$\mathrm{col}\,(a_1, \ldots, a_n)$	column vector with elements $(a_1, \ldots, a_n)$
$\mathrm{vec}(A)$	column vector obtained by the concatenation of columns of the matrix A with elements $(a_{1,1}, ..., a_{n,1}, a_{1,2}, ..., a_{n,2}, ..., a_{1,n}, ..., a_{n,n})$
Hurwitz	matrix with all eigenvalues with strictly negative real part
$L_f h(x)$	Lie derivative of $h(x)$ along the vector field $f(x)$
$\square$	end of proof
$\|\|x\|\|$	Euclidean norm of x, with $x \in \mathbb{R}^n$

Introduction

SINCE the dawn of humanity, human being have been fascinated of the symmetries of the world surrounding them. Symmetry comes from the ancient Greek word $\sigma\upsilon\mu\mu\varepsilon\tau\rho\iota\alpha$, which literally means same ($\sigma\upsilon\mu$) measure ($\mu\varepsilon\tau\rho o\upsilon$). Over the centuries, symmetries have fascinated philosopher, architects, mathematicians, physicians, astronomers, musicians and "control people".

Broccoli, snowflakes, the Milky Way Galaxy, honeycomb, to name a few, exhibit many different types of symmetries.

It was Socrates who first attempted to express the concept of beauty and symmetry in mathematics "The straight line and the circle and the plane and solid figures formed from these by turning lashes and rulers and patterns of angle". The formal definition of symmetry and symmetric object was unknown at that time and the modern use of the word "symmetry" comes from Legendre (Hon and Goldstein (2005)).

An object, generally spiking, is said to be symmetric if after performing an action on it, it looks like the same. For example, a square look likes the same if rotated (around its center) of an angle multiple of $90°$. The circle instead looks like the same if you rotate it about its center for an arbitrary angle. The first kind of symmetry is called discrete while the second one is a continuous symmetry.

In mathematics, the set of all action you can perform on an object that after the action looks like the same form a group. In the case of continuous symmetries the group is known as Lie group.

Not only objects exhibit symmetries but also the laws of nature. In physics we say that a system is symmetric if some physical quantities remain unchanged under some action on them (such as a change of variables of the coordinates). For example the conservation laws of momentum and energy.

Underwater vehicles, aerial vehicles and terrestrial/ground vehicles naturally exhibit symmetries that lead to represent those systems on Lie Groups. It turns out that the kinematic law of motion of those kind of systems are invariant with respect to a change of coordinates.

In this work we are going to study the output regulation problem for systems defined on Matrix Lie group. The problem of output regulation has been intensively studied for both the linear and non-linear context, however the design tools available so far are not directly applicable on non Euclidean spaces. For the existing literature review on the output regulation problem the reader is referred to the Chapter 4 of the present thesis.

The output regulation problem on matrix Lie groups is motivated by a wide range of real world applications in robotics, aerospace and projective geometry. For example, the attitude control problem of a Low Earth Orbit (LEO) rigid satellite, whose configuration manifold is the *special orthogonal group* or the attitude control problem of a fully actuated *camera gimbal*. Also the control problem of an *omnidirectional* wheeled robots fits the framework of the present dissertation, indeed the configuration space of the wheeled robot is the *special Euclidean group* $\mathrm{SE}(2)$. The relevance of the proposed control problem is also relevant for the film industry. In particular the output tracking problem of *homographies*. Indeed is well known that the configuration manifold of the set of *homographies* is the *special linear group* $\mathrm{SL}(3)$.

The thesis is organized as follow. In Chapter 1 the reader can find a brief review of Manifolds, Lie Groups, Lie algebras and homogeneous spaces. In Chapter 2 we discuss about Matrix Lie group, focusing our attention on the two most important Lie group (for the mechanical point of view), that are the *special orthogonal group* and the *special Euclidean group*. In Chapter 3 we present some mathematical model of mechanical systems whose configuration space is a matrix Lie group. In Chapter 4 after discussing on the output regulation problem we present a novel regulator design for kinematic systems on matrix Lie groups with *invariant* relative error measurements. In Chapter 5 and 6 we study the output regulation problem for kinematic systems on the *special orthogonal group* and the *special Euclidean group*, respectively. In particular, exploiting the specific structure of the group considered we extend the local results obtained in Chapter 4 to almost global results. Going further a regulator design for dynamic fully actuated rigid body is presented.

"I am certain, absolutely certain that...these theories will be recognized as fundamental at some point in the future."

Sophus Lie

1

A brief Introduction to Lie Groups

Galois "inspired" the Norwegian mathematician Sophus Lie in the study of symmetry of differential equations. Galois theory provides a connection between the structure of groups and the structure of fields. The French mathematicians used this connection to describe how the roots of a given polynomial equation are related to one other. Taking advantages of the simpler structure of groups with respect to the structure of fields, Galois' study allowed to answer the question *"is it possible to find the roots of a* 5-th *order degree polynomial in terms of the polynomial's coefficients (with the usual operations* $+, -, \sqrt{}, \times$*)?"*. Analogously, roughly speaking, an highly non-linear object such as a Lie Group can be "characterized" by a simpler linear object known as Lie Algebra. Lie Groups nowadays constitute the foundation of many physic theory (see Hooft and Veltman, Gilmore (2008) and references therein) such as conformal field theory, string theory (*Virasoro algebra*) and general relativity, to name a few. Moreover, as we will discuss in detail later on (Chapter 3), they are of interest also from an engineering point of view since they describe for example the kinematic motion of satellites, unmanned aerial vehicles and mobile robotic systems.

In order to be self-contained and to prepare the reader for the next chapters, some basic facts about Lie groups are presented. It is not the purpose of this work to give a complete treatment of Lie groups (see books devoted to the subject as Hall (2003), Onishchik et al. (1993), Neeb and Hilgert (2011) for additional detail). The content of this chapter contains no novelty and can be found in many textbook as Arvanitogeorgos (2003), Marsden (1994), Varadarajan (1984), Spivak (1979). The present Chapter is based on the work of Arvanitogeorgos (2003) and Marsden (1994).

1.1 Manifolds

Intuitively, a manifold is a topological space $\mathbf{M}$ that locally looks like $\mathbb{R}^n$. In other words, each point of a n-dimensional manifold has a neighborhood that is homeomorfic to a Euclidean space of dimension n. More specifically one has:

Definition 1.1. *A **smooth n-dimensional manifold** is a Hausdorff topological space* $\mathbf{M}$ *with a family* $\{(\mathcal{U}_\alpha, \phi_\alpha)\}$ *with open sets* $\mathcal{U}_\alpha \subset M$ *and homeomorphism* $\phi_\alpha : \mathcal{U}_\alpha \mapsto \mathbb{R}^n$ *so that:*

1. $\mathbf{M}$ *is the union of all* $\mathcal{U}_\alpha$, $M = \cup_\alpha \mathcal{U}_\alpha$.
2. *Given* $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq 0$, *the coordinate transformation* $\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$ *is smooth.*
3. *The family* $\{(\mathcal{U}_\alpha, \phi_\alpha)\}$ *is maximal relative to 1 and 2.*

The pair $(\mathcal{U}_\alpha, \phi_\alpha)$ *is called chart and the family* $\{(\mathcal{U}_\alpha, \phi_\alpha)\}$ *is called atlas.*

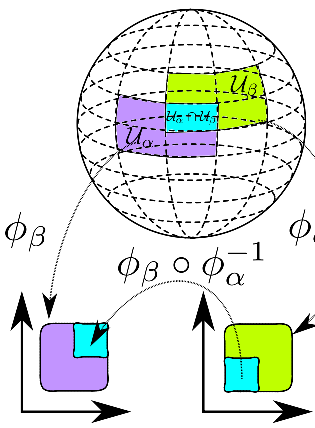

Figure 1.1: The sphere as smooth manifold.

Definition 1.2. *Two curves* $t \mapsto c(t)$ *and* $t \mapsto c'(t)$ *in* $\mathbf{M}$ *are equivalent at the point* $X \in \mathbf{M}$ *if*

1. $c(0) = c'(0) = X$
2. $\mathrm{d}t\,(\phi \circ c)\Big|_{t=0} = \mathrm{d}t(\phi \circ c')\Big|_{t=0}$

for some chart $(\mathcal{U}, \phi)$.

Definition 1.3. *A **tangent vector** v to a manifold $\mathbf{M}$ in a point $X \in \mathbf{M}$ is an equivalence class of curves at X.*

We denote by $\mathrm{T}_X\mathbf{M}$ the tangent space to the manifold $\mathbf{M}$ at the point $X \in \mathbf{M}$.

Definition 1.4. *The **tangent bundle** of the manifold $\mathbf{M}$, denoted by $\mathrm{T}\mathbf{M}$, is the disjoint union of the tangent spaces to $\mathbf{M}$ at the points $X \in \mathbf{M}$, that is*

$$\mathrm{T}\mathbf{M} = \bigcup_{X \in \mathbf{M}} \mathrm{T}_X\mathbf{M}.$$

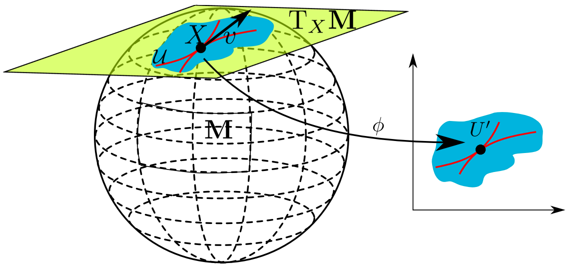

Figure 1.2: The tangent space of a sphere.

Definition 1.5. *A **vector field** F on a manifold $\mathbf{M}$ is a map $\mathrm{F} : \mathbf{M} \mapsto \mathrm{T}\mathbf{M}$ that assigns a vector $\mathrm{F}(X)$ at the point $X \in \mathbf{M}$.*

1.2 Lie Groups

Before introducing the formal definition of Lie-Groups we recall the notion of group.

Definition 1.6. *A **group** $(\mathbf{G}, \star)$ is a nonempty set $\mathbf{G}$ together with a binary operation $\star$ on $\mathbf{G}$ that satisfies the following group axioms.*

1. Closure*: for all $X, Y \in \mathbf{G}$ the element $X \star Y$ is also an element of $\mathbf{G}$.*
2. Associativity*: for all $X, Y, Z \in \mathbf{G}$, one has $X \star (Y \star Z) = (X \star Y) \star Z$.*
3. Identity element*: There exists an identity element $I_d \in \mathbf{G}$ such that, for all $X \in \mathbf{G}$, the following holds $I_d \star X = X \star I_d$.*
4. Inverse element*: For each $X \in \mathbf{G}$ there exists an inverse element $X^{-1} \in \mathbf{G}$ such that*

$$X \star X^{-1} = X^{-1} \star X = I_d.$$

Note that as consequence of the four axioms, the Identity element of the group is unique and each element of the group has a unique inverse element. A Lie Group, roughly speaking, is a smooth manifold and a group as well that satisfies the additional condition that the group operations are differentiable.

Definition 1.7. *A **Lie Group** $\mathbf{G}$ is a smooth manifold with:*

1. $\mathbf{G}$ *a group.*

2. *The group operation*

$$\star : \mathbf{G} \times \mathbf{G} \mapsto \mathbf{G}; \quad (X, Y) \mapsto XY$$

are smooth maps.

For example it is easy to verify that the Euclidean space is a smooth n-dimensional manifold covered by only one chart (since $\mathbb{R}^n$ looks like globally $\mathbb{R}^n$) with ϕ the identity map. The set $\mathbb{R}^n$ is also a abelian Group under the vector addition $+$

$$(X, Y) \mapsto X + Y, \quad X \mapsto -X.$$

As consequence $\mathbb{R}^n$ is a Lie Group.

Definition 1.8. *A **Lie subgroup** $\mathbf{H}$ of $\mathbf{G}$ is a submanifold of $\mathbf{G}$ which is also subgroup of the group $\mathbf{G}$.*

However is difficult to assert from the previous definition if a group $\mathbf{H}$ is a Lie subgroup or not. It is more easy, instead, to show that $\mathbf{H}$ is a subgroup of the Lie group $\mathbf{G}$, and the apply the following *Cartan's Theorem*.

Proposition 1.1. *(Cartan, see Onishchik et al. (1993))A subgroup $\mathbf{H}$ of a Lie group $\mathbf{G}$ is a Lie subgroup of $\mathbf{G}$ if it is closed under the topology of $\mathbf{G}$.*

We introduce the concept of left (right) maps and left (right) invariant vector fields, these notion are of extremely importance and will be used in the next section in order to define the Lie Algebra $\mathfrak{g}$ associated to the Lie Group $\mathbf{G}$.

Definition 1.9. *The map $\mathrm{L}_X : \mathbf{G} \mapsto \mathbf{G}$ is a **left translation** map if*

$$L_X(Y) = XY.$$

Definition 1.10. *The map $\mathrm{R}_X : \mathbf{G} \mapsto \mathbf{G}$ is a **right translation** map if*

$$R_X(Y) = YX.$$

It can be easily verified that the left translation map and the right translation map are smooth. Indeed, for $X, Y \in \mathbf{G}$ one has

$$\mathrm{L}_X(Y)^{-1} = X^{-1}Y = \mathrm{L}_{X^{-1}}(Y); \qquad \mathrm{R}_X(Y)^{-1} = YX^{-1} = \mathrm{R}_{X^{-1}}(Y).$$

Recalling that, given two manifolds $\mathbf{M}, \mathbf{N}$, a differentiable map $f : M \mapsto N$ is a diffeomorphism if it is bijective and its inverse is also differentiable, one concludes that the left and right translation maps are diffeomorphisms.

Definition 1.11. *For $X, Y, Z \in \mathbf{G}$, a map f is called* ***left invariant*** *if*

$$f(X, Y) = f(L_Z(X), L_Z(Y))$$

.

For example the map $E_L : G \times G \mapsto G$, $E_L(X, Y) := X^{-1}Y$ is a left invariant map since

$$E_L(L_Z(X), L_Z(Y)) = X^{-1}Z^{-1}ZY = X^{-1}Y.$$

This map will be used in Chapter 2 to define the natural error in the *special orthogonal group* and the *special euclidean group*.

Definition 1.12. *A vector field* F *is* ***left invariant*** *if for all $X, Y \in \mathbf{G}$ the following holds*

$$(T_Y \mathrm{L}_X)\mathrm{F}(Y) = \mathrm{F}(\mathrm{L}_X(Y)) = \mathrm{F}(XY).$$

Definition 1.13. *For $X, Y, Z \in \mathbf{G}$, a map f is called* ***right invariant*** *if*

$$f(X, Y) = f(R_Z(X), R_Z(Y))$$

.

Definition 1.14. *A vector field* F *is* ***right invariant*** *if for all $X, Y \in \mathbf{G}$ the following holds*

$$(T_Y \mathrm{R}_X)\mathrm{F}(Y) = \mathrm{F}(\mathrm{R}_X(Y)) = \mathrm{F}(YX).$$

Definition 1.15. *A vector field is called bi-invariant if it is both left and right invariant.*

1.3 Lie Algebras

Definition 1.16. *An n-dimensional vector space $\mathfrak{g}$ over a field* F *with a bilinear map,* ***Lie bracket*** *(or commutator) is a* ***Lie Algebra*** *if it is satisfies the following properties*

1. $[X, X] = 0$ *for all* $X \in \mathfrak{g}$.
2. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ *for all* $X, Y, Z \in \mathfrak{g}$ *(Jacobi identity).*

Definition 1.17. *A vector subspace $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a* ***Lie subalgebra*** *if* $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$.

As mentioned in section 1.2 left (right) invariant vector fields play an important role in the study of the geometry of a Lie group. Indeed consider a vector field F one has

$$\mathrm{F}(X) = \mathrm{F}(XI_d) = (T_{I_d}\mathrm{L}_X)\mathrm{F}(I_d), \quad \text{for all } X \in \mathbf{G},$$

this means that a left invariant vector field is completely characterized by its value at the identity element of the Lie group.

Proposition 1.2. *Let* $\mathbf{G}$ *be a Lie group. Then, the vector space of all left invariant vector fields on* $\mathbf{G}$ *is ismomorphic to* $T_{I_d}\mathbf{G}$.

Moreover it is possible to show that the Lie bracket of two left invariant vector fields is a left invariant vector field. Then, the tangent space of $\mathbf{G}$ at the identity, denoted by $T_{I_d}G$, is a Lie algebra.

Definition 1.18. *Let* G *be a Lie group and* $\chi_L(\mathbf{G})$ *the set of all left invariant vector fields on* $\mathbf{G}$*. The Lie algebra* $\mathfrak{g}$ *associated to the Lie group* $\mathbf{G}$ *is* $T_{I_d}\mathbf{G}$ *with the Lie bracket induced by its identification with* $\chi_L(\mathbf{G})$.

1.4 Group Actions and Homogeneous Spaces

Homogeneous spaces are symmetrical manifolds that do not necessarily possess a Lie group structure. For example the sphere is an homogeneous space but not a Lie group. One of the most interesting properties of an homogeneous manifold, in analogous manner of Lie groups, is that roughly speaking an homogeneous manifold looks locally the same at each point.

Definition 1.19. *Given a manifold* $\mathbf{M}$*, and a group* $\mathbf{G}$*, a* ***left group action*** *of* G *on* $\mathbf{M}$ *is a smooth map* $l : G \times M \mapsto M$*, such that*

1. *For all* $X, Y \in \mathbf{G}$ *and all* $m \in M$

$$l(X, l(Y, m)) = l(XY, m).$$

2. *For all* $m \in \mathbf{M}$

$$l(I_d, m) = m$$

where $I_d \in \mathbf{G}$ *is the identity element of the group.*

A left group action is called **linear left group action** if the map $l : G \times M \mapsto M$ is a linear map.

Definition 1.20. *Given a manifold* $\mathbf{M}$*, and a group* $\mathbf{G}$*, a* ***right group action*** *of* G *on* $\mathbf{M}$ *is a smooth map* $r : G \times M \mapsto M$*, such that*

1. *For all* $X, Y \in \mathbf{G}$ *and all* $m \in M$

$$r(X, r(Y, m)) = r(YX, m).$$

2. *For all* $m \in \mathbf{M}$

$$r(I_d, m) = m$$

where $I_d \in \mathbf{G}$ *is the identity element of the group.*

Definition 1.21. *The action* l *is* ***transitive*** *iff for any* $m, n \in \mathbf{M}$ *there exist* $X \in \mathbf{G}$ *such that*

$$l(X, m) = n.$$

Definition 1.22. *The action* r *is* ***transitive*** *iff for any* $m, n \in \mathbf{M}$ *there exist* $X \in \mathbf{G}$ *such that*

$$r(X, m) = n.$$

For example consider the group $\mathbf{G} \subset \mathbb{R}^{2\times 2}$ of 2×2 matrices

$$\left\{ X : \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \middle| \theta \in \mathbb{R} \right\},$$

the above matrix form a group under matrix multiplication, the inverse element is the transpose of the matrix and the identity element of the group is the identity matrix I_2. In Chapter 2 we will see that the set of this kind of matrices form a Lie group known as *special orthogonal group* $\mathrm{SO}(2)$. Let $\mathbf{M} = S^1$ be the unit circle, and $m \in S^1$ a unit norm column vector. We define a linear left action $\mathbf{G}$ on S^1 by matrix vector multiplication

$$l(X, m) = Xm.$$

Moreover is straightforward to verify that the left group action considered is a transitive action, indeed consider two elements of the unit circle $m = [a,\ b]^\top$ and $n = [c, d]^\top$ with $a^2 + b^2 = 1$ and $c^2 + d^2 = 1$. Thus

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix},$$

after basic algebraic manipulation one has

$$\begin{cases} \cos(\theta) = ac + bd \\ \sin(\theta) = bc - ad \end{cases}.$$

This implies that there exist a matrix X on the form

$$X = \begin{bmatrix} ac + bd & bc - ad \\ -(bc - ad) & ac + bd \end{bmatrix}$$

such that

$$l(X, m) = n.$$

We claim, then, that the unit circle S^1 is an homogeneous space; this claim is justified by the following definition.

Definition 1.23. *Let* $\mathbf{G}$ *a Lie group, a* $\mathbf{G}$-***homogeneous space*** *(or simply an* ***homogeneous space****) is a manifold with a transitive action of* $\mathbf{G}$.

It should be clear then, that a Lie group is an homogeneous space since $\mathbf{G} \times \mathbf{G}$ acts on $\mathbf{G}$ by left and right translations.

Definition 1.24. *The* ***Adjoint action*** *of* $\mathbf{G}$ *on* $\mathfrak{g}$ *is given by*

$$\mathrm{Ad} : \mathbf{G} \times \mathfrak{g} \mapsto \mathfrak{g}, \quad \mathrm{Ad}_X(\xi) := \mathrm{T}_{I_d}(R_{X^{-1}} \circ L_X)\xi$$

where $X \in \mathbf{G}$, $\xi \in \mathfrak{g}$.

1.5 Metric Space and Riemannian Metric

In this section we briefly introduce the notion of the length of a vector, the length of curve and the notion of the distance between two points on a manifold $\mathbf{M}$. The interested reader is referred to functional analysis books devoted to the argument as Alabiso and Weiss (2014), Lebedev et al. (2013) and differential geometry books as Spivak (1979) and Fecko (2006).

Definition 1.25. *Let* $\mathbf{G}$ *a group. A distance function* $d : \mathbf{G} \times \mathbf{G} \mapsto \mathbb{R}_+$ *is a* ***metric*** *on* $\mathbf{G}$ *if, for all* $X, Y, Z \in \mathbf{G}$*, the following hold*

1. *The map is symmetric* $d(X, Y) = d(Y, X)$.
2. $d(X, Y) = 0$, *iff* $X = Y$.
3. *Triangle inequality,* $d(X, Z) \leq d(X, Y) + d(Y, Z)$

Definition 1.26. *(see Alabiso and Weiss (2014)) A vector space V with $\|\cdot\|$ a norm on V, is called a **normed space** $(V, \|\cdot\|)$ if for all $X, Y \in V$ and $\alpha \in \mathbb{R}^n$ the following conditions hold*

- $\|X\| \geq 0$ *for* $X \neq 0$.
- $\|\alpha X\| = |\alpha|\,\|X\|$.
- $\|X + Y\| \leq \|X\| + \|Y\|$.

Definition 1.27. *(Alabiso and Weiss (2014)) If V is a normed space, then defining $d : V \times V \mapsto \mathbb{R}_+$ by*

$$d(x, y) = \|x - y\|$$

endows V with the structure of a metric space.

Definition 1.28. *A **Riemannian metric** on a smooth manifold $\mathbf{M}$ is a continuous collection of inner products $\left(\langle\cdot,\cdot\rangle_m\right)\big|_{m\in\mathbf{M}}$ in the tangent space $\mathrm{T}_m\mathbf{M}$ at each point $m \in \mathbf{M}$ such that for any smooth vector fields F_1 and F_2 , the map $m \mapsto \left\langle \mathrm{F}_1, \mathrm{F}_2 \right\rangle_m$ is smooth.*

Since a Lie group is a smooth manifold it is possible to endow the group $\mathbf{G}$ with a Riemannian metric. Moreover since a Lie group is also a group, we are interested in particular Riemannian metrics that takes into account the group structure of $\mathbf{G}$. It turns out that these particular metrics are those for which the left translations (or the right translations) are isometries.

Definition 1.29. *A metric $\langle\cdot,\cdot\rangle$ on a Lie group $\mathbf{G}$ is called **left invariant** (resp. **right invariant**) if*

$$\langle u, v\rangle_X = \langle (\mathrm{T}_X\mathrm{L}_Y)u(X), (\mathrm{T}_X\mathrm{L}_Y)v(X)\rangle_{\mathrm{L}_Y(X)}\,,$$

$$(\textit{resp. } \langle u, v\rangle_X = \langle (\mathrm{T}_X\mathrm{R}_Y)u(X), (\mathrm{T}_X\mathrm{R}_Y)v(X)\rangle_{\mathrm{R}_Y(X)}),$$

for all $X, Y \in \mathbf{G}$ and $u, v \in \mathrm{T}_X\mathbf{G}$.

"Manifolds are a bit like pornography: hard to define, but you know one when you see one."

S. Weinberger

2

Matrix Lie Group and their asociated Lie Algebras

Matrix Lie groups intuitively are Lie groups realized as group of $n \times n$ square matrices with real or complex entries. The realization theory plays major importance in some field such as quantum mechanics (see for instance Bés (2004) and Hall (2013)). However, in this work we do not use the representations theory and their classification (the interested reader is referred to Varadarajan (1984) and Hall (2003)). For the aim of the present work one should think of representation as a smart way to describe a point of a manifold $\mathbf{M}$ with matrices or vectors. The so called classical Lie groups are the following matrix Lie groups (over real numbers)

- The General Linear Group $\mathrm{GL}(n, \mathbb{R})$.
- The Special Linear Group $\mathrm{SL}(n, \mathbb{R})$.
- The Orthogonal Group $\mathrm{O}(n)$.
- The Special Orthogonal Group $\mathrm{SO}(n)$.
- The Euclidean Group $\mathrm{E}(n)$.
- The Special Euclidean Group $\mathrm{SE}(n)$.

- The Unitary Group $\mathrm{U}(n)$.
- The Special Unitary Group $\mathrm{SU}(n)$.

In this work we will focus our attention on the special orthogonal group $\mathrm{SO}(3)$ and the special Euclidean group $\mathrm{SE}(3)$ since they are associated with rigid body motion, it will be clear in the next sections. The material presented in this section is based in part on Murray et al. (1994) and in part on Siciliano et al. (2009).

2.1 General Linear Group

Definition 2.1. *The **general linear group** $\mathrm{GL}(n, \mathbb{R})$ of dimension n^2 is the set of $n \times n$ invertible matrices with real entries*

$$\mathrm{GL}(n) = \left\{ X \in \mathbb{R}^{n\times n} \middle| \det(X) \neq 0 \right\}$$

together with the operation of ordinary matrix multiplication.

Note that the set of $n \times n$ invertible matrices forms a group under the ordinary matrix multiplication, indeed the product between two invertible matrices is invertible

$$\det(AB) = \det(A)\det(B), \quad \text{for} \quad A, B \in \mathrm{GL}(n, \mathbb{R}).$$

As inverse of each element the matrix inverse ($\det(A) = \det(A)^{-1}$) and identity element of the group the identity matrix I_n.

Definition 2.2. *A **matrix Lie group** (with real entries) is a closed subgroup of* $\mathrm{GL}(n, \mathbb{R})$.

2.1.1 The Matrix Exponential and Matrix Logarithm

As we have seen in section 1.3 the tangent space of $\mathbf{G}$ at the identity element of the group is the Lie algebra $\mathfrak{g}$ associated to the Lie group $\mathbf{G}$. The matrix exponential plays an important role in the definition of a Lie algebra $\mathfrak{g}$ associated to a matrix Lie group $\mathbf{G}$.

Definition 2.3. *For $U \in \mathbb{R}^{n\times n}$, the **matrix exponential** $\exp(U)$ is defined by the series*

$$\exp(U) := \sum_{k=0}^{\infty} \frac{U^k}{k!}$$

Properties 2.1. *For $U, U' \in \mathbb{R}^{n\times n}$, the following hold*

1. $\exp(U)^0 = I_n$.
2. $\exp(U)$ *is always invertible.*

3. *if U and U' commute,* i.e. $UU' = U'U$*, then* $\exp(U + U') = \exp(U)\exp(U')$.

4. *For* $X \in \mathrm{GL}(n, R)$, $\exp(XUX^{-1}) = X\exp(U)X^{-1}$.

5. $\det(\exp(U)) = \exp(\mathrm{tr}(U))$.

Definition 2.4. *For $X \in \mathbb{R}^{n\times n}$, the* ***matrix logarithm*** $\log(U)$ *is the inverse map of the matrix exponential,* i.e. $\log(\exp(U)) = U$*, and is defined by the series*

$$\log(X) := \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(X - I_n)^k}{k!}.$$

2.1.2 The Lie Algebra Associated to a Matrix Lie group

Definition 2.5. *The Lie algebra $\mathfrak{g}$ of a matrix lie group $\mathbf{G}$ is the set*

$$\mathfrak{g} = \left\{U \in \mathbb{R}^{n\times n} \;\middle|\; \exp(tU) \in \mathbf{G}, \quad \textit{for all } t \in \mathbb{R}\right\}.$$

The Lie algebra $\mathfrak{g}$ is closed under the Lie bracket $[U, U'] = UU' - U'U$ for all $U, U' \in \mathbf{G}$.

From this definition it follows that the Lie algebra associated to the *general linear group* $\mathrm{GL}(n)$ denoted by $\mathfrak{gl}(n)$ is the set

$$\mathfrak{gl} = \left\{U \in \mathbb{R}^{n\times n} \;\middle|\; \exp(tU) \in \mathbf{G}, \quad \text{for all } t \in \mathbb{R}\right\}.$$

2.1.3 The Adjoint Action

For $X \in \mathbf{G}$ and $U \in \mathfrak{g}$, from definition 1.24 and using the matrix exponential one has

$$Ad_X(U) = X\frac{d}{dt}(\exp(tU))\Big|_{t=0} X^{-1} = XUX^{-1}.$$

In the present work we will make often use of the *Adjoint action* since, as will be more clear in the next sections, the *body fixed-frame* velocities and the velocities with respect to an *inertial frame* of a rigid body are related by the *Adjoint action*.

2.1.4 Right Invariant Systems

Proposition 2.1. *For $X \in \mathrm{GL}(n)$ and $U \in \mathfrak{gl}(n)$, a* ***right invariant system*** *on* $\mathrm{GL}(n)$ *is of the form*

$$\dot{X}(t) = U(t)X(t). \tag{2.1}$$

Proof. Consider a time-varying matrix $X(t) \in \mathrm{GL}(n)$, due to the fact that $det(X(t)) \neq 0$ one has

$$X(t)X^{-1}(t) = I_n$$

and differentiating with respect to time it yields

$$\begin{aligned} 0 &= \frac{d}{\mathrm{d}t}(X(t)X^{-1}(t)) \\ &= \dot{X}(t)X^{-1}(t) + X(t)\frac{d}{\mathrm{d}t}(X^{-1}(t)) \\ &= \dot{X}(t)X^{-1}(t) - \dot{X}(t)X^{-1}(t). \end{aligned}$$

Denoting $U = \dot{X}(t)X^{-1}(t)$, one has

$$\dot{X}(t) = U(t)X(t)$$

and note that U is an element of the Lie algebra $\mathfrak{gl}(n)$ associated to the general linear group $\mathrm{GL}(n)$. The tangent spaces $\mathrm{T}_X\mathrm{GL}(n)$, then, are identified with

$$\mathrm{T}_X\mathrm{GL}(n) := \left\{ UX \mid U \in \mathfrak{gl}(n) \right\} \subset \mathbb{R}^{n\times n}.$$

The map $\mathrm{T}_X R_Y : UX \mapsto UXY$ is given by right multiplication of the matrices in $\mathrm{T}_X GL(n)$ with a constant matrix Y. It is straightforward to verify that the system dynamics are *right invariant*, indeed

$$\frac{d}{\mathrm{d}t}(X(t)Y) = \dot{X}(t)Y = U(t)(X(t)Y)$$

and this concludes the proof. □

2.1.5 Left Invariant Systems

Proposition 2.2. *For $X \in \mathrm{GL}(n)$ and $U \in \mathfrak{gl}(n)$, a **left invariant system** on $\mathrm{GL}(n)$ is of the form*

$$\dot{X}(t) = X(t)U(t). \tag{2.2}$$

Proof. Consider a time-varying matrix $X(t) \in \mathrm{GL}(n)$, proceeding in a similar way of the right invariant case one has

$$X^{-1}(t)X(t) = I_n$$

and differentiating with respect to time one obtains

$$0 = X^{-1}(t)\dot{X}(t) - X^{-1}(t)\dot{X}(t).$$

Denoting $U = X^{-1}(t)\dot{X}(t)$, it yields

$$\dot{X}(t) = X(t)U(t)$$

and note that U is an element of the Lie algebra $\mathfrak{gl}(n)$ associated to the general linear group $\mathrm{GL}(n)$. The tangent spaces $\mathrm{T}_X\mathrm{GL}(n)$ are identified with

$$\mathrm{T}_X\mathrm{GL}(n) := \left\{XU \;\middle|\; U \in \mathfrak{gl}(n)\right\} \subset \mathbb{R}^{n\times n}.$$

The map $\mathrm{T}_X L_Y : XU \mapsto YXU$ is given by left multiplication of the matrices in $\mathrm{T}_X GL(n)$ with a constant matrix Y. The system dynamics are *left invariant*, indeed

$$\frac{d}{\mathrm{d}t}(YX(t)) = Y\dot{X}(t) = (YX(t))U(t)$$

and this concludes the proof. □

2.1.6 Matrix and Vectorial Representation of a Lie Algebra

Definition 2.6. *Let* $\mathbf{G}$ *a matrix Lie group and* $\mathfrak{g}$ *its associated Lie Algebra. The* ***matrix representation*** *is a mapping* $\mathrm{mrp} : \mathbb{R}^k \mapsto \mathfrak{g}$*, that maps a vector* $v \in \mathbb{R}^k$ *in an element of the algebra* $\mathfrak{g}$*, where* k *is the dimension of* $\mathbf{G}$*.*

Definition 2.7. *Let* $\mathbf{G}$ *a matrix Lie group and* $\mathfrak{g}$ *its associated Lie Algebra. The* ***vectorial representation*** *is a mapping* $\mathrm{vrp} : \mathfrak{g} \mapsto \mathbb{R}^k$*, that maps an element of the algebra* $\mathfrak{g}$ *in a vector* $v \in \mathbb{R}^k$*, where* k *is the dimension of* $\mathbf{G}$*.*

The vectorial operator is the inverse of the matrix operator, namely

$$\mathrm{vrp}(\mathrm{mrp}(v)) = v, \quad \text{for all } v \in \mathbb{R}^k.$$

2.1.7 Inner Product

Definition 2.8. *For any two matrices* $A, B \in \mathbb{R}^{n\times n}$ *the* ***Euclidean matrix inner product*** *is defined by*

$$\langle A, B\rangle = \mathrm{tr}(A^\top B).$$

The Euclidean inner product induces the Frobenius norm

$$\|A\|_F = \sqrt{\langle A, A\rangle},$$

and the Euclidean distance is the metric given by

$$d(A, B) = \|A - B\|_F, \quad \text{for all } A, B \in \mathbb{R}^{n\times n}.$$

Note that for any $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$

$$\langle A, B\rangle_F = \operatorname{tr}(A^\top B) = \operatorname{vec}(A)^\top \operatorname{vec}(B)$$

where $\operatorname{vec}(A) \in \mathbb{R}^{n^2}$ is the column vector obtained by the concatenation of columns of the matrix A as follows

$$\operatorname{vec}(A) = [a_{1,1}, ..., a_{n,1}, a_{1,2}, ..., a_{n,2}, ..., a_{1,n}, ..., a_{n,n}]^\top.$$

Definition 2.9. *Let* $\operatorname{vrp}(U) \in \mathbb{R}^k$ *and* $U \in \mathfrak{g}$ *an* $n \times n$ *matrix, with* $n \le k \le n^2$*. We call the matrix* $D \in \mathbb{R}^{n^2\times k}$ ***duplication matrix*** *if*

$$\operatorname{vec}(U) = D \operatorname{vrp}(U)\,.$$

The definition above states that the column vector obtained by the concatenation of columns of the matrix $U \in \mathfrak{g}$ is a linear combination of the vectorial representation $\operatorname{vrp}(U)$, for any $U \subset \mathfrak{gl}(n)$. As consequence, with $U \in \mathfrak{g}$ and $V \in \mathfrak{g}$ elements of the same Lie algebra one has

$$\operatorname{tr}\left(U^\top V\right) = \operatorname{vrp}^\top(U) Q_{\mathfrak{g}} \operatorname{vrp}(V)$$

where $Q_{\mathfrak{g}} = D^T D$ and D the *duplication matrix*. The matrix $Q_{\mathfrak{g}}$ will play an important role in Chapter 4 for the design of the regulator that solves the problem of *output regulation* for systems on matrix Lie groups.

We can endow $\mathbf{G}$ with a Riemannian metric and show that there is a one-to-one correspondence between left-invariant metrics on a Lie group $\mathbf{G}$, and inner products on the Lie algebra $\mathfrak{g}$. To this end, consider $U, V \in \mathfrak{g}$ two *left invariant vector fields*, one has

$$\langle U, V\rangle_X = \langle U(X), V(X)\rangle = \langle (\mathrm{T}_{I_d} L_X) U(I_d), (\mathrm{T}_{I_d} L_X) V(I_d)\rangle\,.$$

Moreover, if the metric considered is *left invariant* one obtains

$$\begin{aligned} \langle U, V\rangle_X &= \langle (\mathrm{T}_{I_d} L_X) U(I_d), (\mathrm{T}_{I_d} L_X) V(I_d)\rangle = \langle U(I_d), V(I_d)\rangle \\ &= \langle U, V\rangle_{I_d} \end{aligned}$$

for all $X \in \mathbf{G}$.

This means that an inner product on the Lie algebra can be extended to a Riemannian metric making use of left translation. As consequence also the matrix inner product on

$\mathfrak{gl}(n)$ can be extended to a Riemannian metric taking advantages of left or right translation, however this metric is not left nor right invariant. Indeed for $U,\ V \in \mathfrak{gl}(n)$ and $X \in \mathrm{GL}(n)$ one has,

$$\langle U(X), V(X)\rangle = \langle XU, XV\rangle = \mathrm{tr}(U^\top X^\top XV) \neq \langle U(I_d), V(I_d)\rangle \,.$$

Definition 2.10. *For all $A \in \mathbb{R}^{n\times n}$ the mapping $\mathbb{P}_{\mathfrak{g}}(A) : \mathbb{R}^{n\times n} \mapsto \mathfrak{g}$ is called the* ***orthogonal projection*** *of A onto $\mathfrak{g}$ with respect to the trace inner product if*

$$\langle U, A\rangle = \mathrm{tr}\left(U^\top A\right) = \mathrm{tr}\left(U^\top \mathbb{P}_{\mathfrak{g}}(A)\right) = \langle U, \mathbb{P}_{\mathfrak{g}}(A)\rangle$$

for any $U \in \mathfrak{g}$ and any $A \in \mathbb{R}^{n\times n}$.

Where the context is clear we will write $\mathbb{P}$ for $\mathbb{P}_{\mathfrak{g}}$. The orthogonal projection with respect to the trace inner product is significant in the *Observer design* for systems on Lie groups (see Mahony et al. (2012a) and Hua et al. (2011)), *Consensus and Synchronization* problems (see Sarlette et al. (2007) and references therein) and *Integral controls on Lie groups* (see Mahony et al. (2015) and Zhang et al. (2015)).

2.2 The Special Orthogonal Group

Definition 2.11. *The* ***special orthogonal group*** $\mathrm{SO}(n)$ *is the set of real $n \times n$ matrices with othonormal columns and determinant equal to* 1

$$\mathrm{SO}(n) := \left\{R \in \mathrm{GL}(n) \;\middle|\; RR^\top = I_n,\ \det(R) = 1\right\}$$

together with the operation of matrix multiplication. The ***special orthogonal group*** *is a $n(n-1)/2$ dimensional manifold.*

Note that the *special orthogonal group* is compact and connected. For $n = 3$ the *special orthogonal group* is also called the *rotation group* since, as we will see in the next section, it describes rigid body orientation.

2.2.1 Rotation Matrix and Rigid Body Attitude

Let $\{A\}$ and $\{B\}$ denote respectively an *inertial frame* and a *body-fixed* frame attached to the vehicle (see Figure 2.1). We denote by ${}^{\mathcal{A}}x_{\mathcal{B}},\ {}^{\mathcal{A}}y_{\mathcal{B}},\ {}^{\mathcal{A}}z_{\mathcal{B}},$ the coordinate of the unit vectors of $\{B\}$ with respect to the inertal frame $\{A\}$. In order to have a compact notation we can stack these unit vectors into a (3×3) matrix

$${}^{A}R_B = \begin{bmatrix} {}^{A}x_B & {}^{A}y_B & {}^{A}z_B \end{bmatrix}.$$

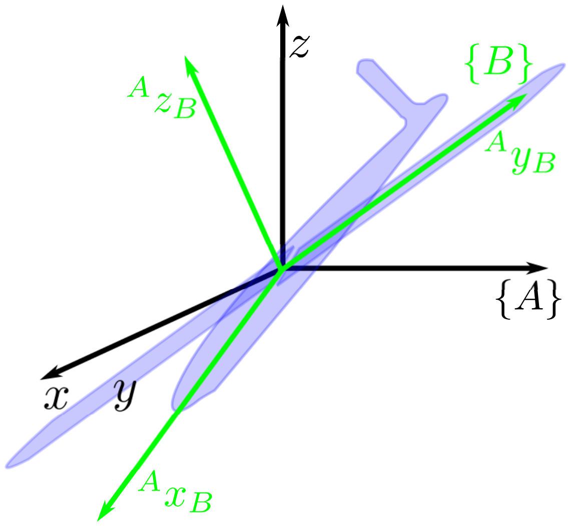

Figure 2.1: Rigid body Attitude

Note that since by construction these unit vector are the unit vectors of an orthonormal right handed frame one has

$$
\begin{aligned}
{}^A x_B^\top {}^A y_B &= 0, \qquad & {}^A x_B^\top {}^A z_B &= 0, \qquad & {}^A y_B^\top {}^A z_B &= 0, \\
{}^A x_B^\top {}^A x_B &= 1, \qquad & {}^A y_B^\top {}^A y_B &= 1, \qquad & {}^A z_B^\top {}^A z_B &= 1,
\end{aligned}
$$

and also

$$
{}^A x_B \wedge {}^A y_B = {}^A z_B, \qquad {}^A y_B \wedge {}^A z_B = {}^A x_B, \qquad {}^A z_B \wedge {}^A x_B = {}^A y_B.
$$

From the three properties above is straightforward to see that a rotation matrix is an element of SO(3) indeed the first and the second property yield to $R^\top R = I_3$. From the third one, recalling the relation between the mixed product and the determinant of a (3×3) matrix, it yields

$$
\det\left({}^A R_B\right) = {}^A x_B^\top \left({}^A y_B \wedge {}^A z_B\right) = {}^A x_B^\top {}^A x_B = 1.
$$

It follows, then, that the attitude of a rigid-body can be represented by a rotation matrix ${}^A R_B \in \mathrm{SO}(3)$ of the *body-fixed frame* $\{B\}$ relative to the *inertial frame* attached to the earth $\{A\}$. In order to have a more compact notation from now on, when the context is clear, we will drop the superscript A and the subscript B, *i.e.* $R \equiv {}^A R_B$.

A rotation matrix does not only defines the mutual orientation of two reference frames $\{A\}$ and $\{B\}$ but it also describes the *coordinate transformation* between the coordinates of a point expressed in two different frames. Indeed, for example, consider two

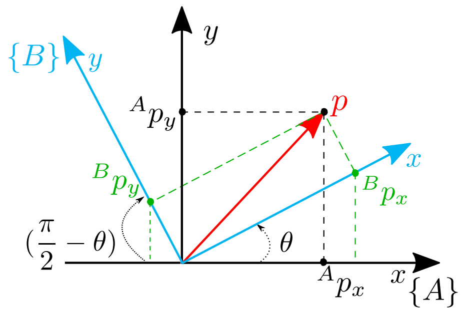

Figure 2.2: Representation of a point expressed in two different frames.

reference frame $\{A\}, \{B\}$ mutually rotated of an angle θ (see Figure 2.2). Denoting by $^{B}p_x$, $^{B}p_y$ the projections of the vector p along the x, y axis of the frame $\{B\}$ respectively, simple geometric calculations leads to

$$\begin{aligned} ^{A}p_x &= {}^{B}p_x \cos(\theta) - {}^{B}p_y \cos(\frac{\pi}{2} - \theta) = {}^{B}p_x \cos(\theta) - {}^{B}p_y \sin(\theta) \\ ^{A}p_y &= {}^{B}p_x \sin(\theta) + {}^{B}p_y \sin(\frac{\pi}{2} - \theta) = {}^{B}p_x \sin(\theta) + {}^{B}p_y \cos(\theta). \end{aligned}$$

The equation above can be written in the following compact form

$$\begin{bmatrix} ^{A}p_x \\ ^{A}p_y \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} ^{B}p_x \\ ^{B}p_y \end{bmatrix} = {}^{A}R_B \begin{bmatrix} ^{B}p_x \\ ^{B}p_y \end{bmatrix}.$$

Thus the rotation matrix $^{A}R_B$ defines the transformation of a vector from one frame to another

$$^{A}p = {}^{A}R_B \, {}^{B}p.$$

This property of a rotation matrix is of particular importance in order to link the angular velocity of a rigid object with respect to an inertial frame into a velocity in a body-fixed frame.

Note that since the set of rotation matrix forms a group under the usual matrix multiplication it follows that the product of two rotation matrix is still a rotation matrix and it represents *composition of successive rotations*. Indeed consider three frames $\{A\}, \{B\}, \{C\}$, if one knows the rotation matrix $^{\mathcal{B}}R_{\mathcal{C}}$ of the frame $\{C\}$ with respect to $\{B\}$ and the rotation matrix $^{\mathcal{A}}R_{\mathcal{B}}$ of the frame $\{B\}$ relative to $\{A\}$ then the mutual orientation of the reference frames $\{A\}$ and $\{C\}$ is given by

$$^{\mathcal{A}}R_{\mathcal{C}} = {}^{\mathcal{A}}R_{\mathcal{B}} \, {}^{\mathcal{B}}R_{\mathcal{C}}.$$

2.2.2 The Lie Algebra associated to the Special Orthogonal Group

Definition 2.12. *The Lie algebra associated to the special orthogonal group, denote by* $\mathfrak{so}(n)$*, is the set of* $n \times n$ *skew symmetric matrices*

$$\mathfrak{so}(n) := \left\{ U \in \mathfrak{gl}(n) \mid U + U^\top = 0 \right\}.$$

The Lie algebra $\mathfrak{so}(3)$ with the matrix commutator $[\cdot, \cdot]$ is isomorphic to $\mathbb{R}^3$ with the cross product. The map mrp, denoted in the particular case for systems posed on $\mathrm{SO}(3)$ as $(\cdot)_\times$, identifies $\mathrm{SO}(3)$ with $\mathbb{R}^3$. Indeed, let $\Omega_1, \Omega_2 \in \mathbb{R}^3$ then one has

$$[\Omega_{1\times}, \Omega_{2\times}] = (\Omega_1 \wedge \Omega_2)_\times.$$

The following identity is widely used in the present work.

Property 2.1. *Let* $v \in \mathbb{R}^3$ *and* $R \in \mathrm{SO}(3)$ *then*

$$(Rv)_\times = Rv_\times R^\top = \mathrm{Ad}_R\, v_\times.$$

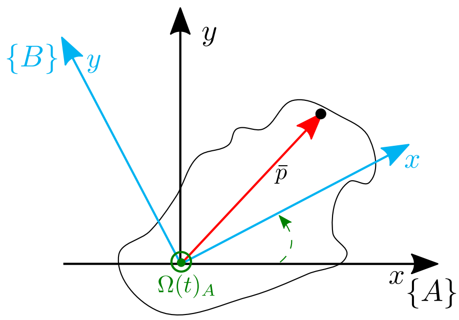

Figure 2.3: The angular velocity in the body-fixed frame and in the inertial frame.

Let $\{A\}$ and $\{B\}$ denote an inertial frame and a *body-fixed frame* and let $\Omega_A \in \mathbb{R}^3$ denotes the angular velocity of the *body-fixed frame* with respect to the *inertial frame* (see Figure 2.3), then the angular velocity in a body-fixed frame is

$$\Omega_B = R^\top \Omega_A \tag{2.3}$$

where R is the rotation matrix ${}^{\mathcal{A}}R_{\mathcal{B}}$ of the frame $\{B\}$ with respect to $\{A\}$. Using the isomorphism between $\mathfrak{so}(3)$ and $\mathbb{R}^3$ and recalling Property 2.1 it yields

$$(\Omega_B)_\times = (R^\top \Omega_A)_\times = R^\top \Omega_{A\times} R = \mathrm{Ad}_{R^\top}\, \Omega_{A\times}.$$

2.2.3 Right Invariant Systems on $\mathrm{SO}(n)$

Proposition 2.3. *For $R \in \mathrm{SO}(n)$ and $\Omega \in \mathfrak{so}(n)$, a **right invariant system** on $\mathrm{SO}(n)$ is of the form*

$$\dot{R}(t) = {}^{\circ}\Omega(t)R(t). \tag{2.4}$$

Proof. Consider a time-varying rotation matrix $R(t)$, due to the fact that $R(t)^{-1} = R(t)^{\top}$ one has

$$R(t)R^{\top}(t) = I_n$$

and differentiating with respect to time one obtains

$$\begin{aligned} 0 &= \frac{d}{\mathrm{d}t}(R(t)R^{\top}(t)) \\ &= \dot{R}(t)R^{\top}(t) + R(t)\dot{R}^{\top}(t). \end{aligned}$$

Denoting ${}^{\circ}\Omega = \dot{R}(t)R^{\top}(t)$, one has

$$\dot{R}(t) = {}^{\circ}\Omega(t)R(t)$$

it follows that

$${}^{\circ}\Omega + {}^{\circ}\Omega^{\top} = 0$$

as consequence ${}^{\circ}\Omega$ is a skew symmetric matrix. The tangent spaces $\mathrm{T}_R\mathrm{SO}(n)$, then, are identified with

$$\mathrm{T}_R\mathrm{SO}(n) := \left\{ {}^{\circ}\Omega R \;\middle|\; {}^{\circ}\Omega \in \mathfrak{so}(n) \right\} \subset \mathbb{R}^{n\times n}.$$

The map $\mathrm{T}_R R_{\bar{R}} :\mapsto {}^{\circ}\Omega R\bar{R}$ is given by right multiplication of the matrices in $\mathrm{T}_R\mathrm{SO}(n)$ with a constant matrix $\bar{R}$. From this one verifies that the vector field considered is *right invariant*, indeed

$$\frac{d}{\mathrm{d}t}(R(t)\bar{R}) = \dot{R}(t)\bar{R} = {}^{\circ}\Omega(t)(R(t)\bar{R})$$

and this concludes the proof. □

Note that a *right invariant* vector field on SO(3) has a clear physical interpretation, indeed it is well known from mechanics (see Figure 2.3) that

$$\dot{p}(t) = w \wedge R(t)\bar{p} \tag{2.5}$$

where $w(t) \in \mathbb{R}^3$ denotes the angular velocity of frame $\{B\}$ with respect to the frame

$\{A\}$. Recalling the $(\cdot)_\times$ map and defining ${}^\circ\Omega := w_\times$ it yields

$$\dot{p}(t) = \dot{R}(t)\bar{p} = {}^\circ\Omega(t)R(t)\bar{p}.$$

It follows that ${}^\circ\Omega(t)$ obtained in the derivation of the *right invariant* vector field, for the specific case of system on $\mathrm{SO}(3)$, is the angular velocity of the rigid body expressed in the inertial frame, we will refer to this velocity ${}^\circ\Omega$ as *right invariant angular velocity* or *spatial angular velocity*.

2.2.4 Left Invariant Systems on $\mathrm{SO}(n)$

Proposition 2.4. *For $R \in \mathrm{SO}(n)$ and $\Omega \in \mathfrak{so}(n)$, a **left invariant system** on $\mathrm{SO}(n)$ is of the form*

$$\dot{R}(t) = R(t)\Omega(t). \tag{2.6}$$

Proof.

Consider a time-varying rotation matrix $R(t) \in \mathrm{SO}(n)$, proceeding in a similar way of the right invariant case one has

$$R^\top(t)R(t) = I_n$$

and differentiating with respect to time one obtains

$$0 \quad = \quad \dot{R}^\top(t)R(t) + R^\top(t)\dot{R}(t).$$

Denoting $\Omega(t) = R^\top(t)\dot{R}(t)$, it yields

$$\dot{R}(t) = R(t)\Omega(t)$$

and note that $\Omega(t)^\top + \Omega(t) = 0$. The tangent spaces $\mathrm{T}_R\mathrm{SO}(n)$ are identified with

$$\mathrm{T}_R SO(n) := \left\{R\Omega \;\middle|\; \Omega \in \mathfrak{so}(n)\right\} \subset \mathbb{R}^{n\times n}.$$

The map $\mathrm{T}_R L_{\bar{R}} : R\Omega \mapsto \bar{R}R\Omega$ is given by left multiplication of the matrices in $\mathrm{T}_R\mathrm{SO}(n)$ with a constant matrix $\bar{R}$. It follows that the system dynamics are *left invariant*

$$\frac{d}{\mathrm{d}t}(\bar{R}R(t)) = \bar{R}\dot{R}(t) = (\bar{R}R(t))\Omega(t)$$

and this concludes the proof. □

Analogously to the right invariant case, the angular velocity Ω has clear physical interpretation. Indeed recalling that the angular velocity $w \in \mathbb{R}^3$ of the frame $\{B\}$ with respect to $\{A\}$ can be obtained from the angular velocity $w_B \in \mathbb{R}^3$ of the frame $\{A\}$ with respect to $\{B\}$ one has from (2.5)

$$\dot{p}(t) = w \wedge R(t)\bar{p} = Rw_B \wedge R(t)\bar{p}$$

and proceeding exactly as the right invariant case denoting $w_\times = {}^\circ\Omega$, $w_{B\times} = \Omega$

$$\dot{p}(t) = \dot{R}(t)\bar{p} = {}^\circ\Omega(t)R(t)\bar{p} = R(t)\Omega(t)R(t)R^\top\bar{p} = R(t)\Omega(t)\bar{p},$$

we will refer to $\Omega(t) \in \mathfrak{so}(3)$ as the *left invariant angular velocity* or *body angular velocity*.

2.2.5 Rodrigues' formula and Log map in $\mathrm{SO}(3)$

According to the classical Euler Theorem the orientation of a frame $\{B\}$ relative to a frame $\{A\}$ can be represented by means of a rotation about a fixed axis $\omega \in \mathbb{R}^3$ through an angle $\theta \in [0, 2\pi)$. From the definition of exponential map and of the Lie algebra associated to a matrix Lie group along with the Euler theorem one has

$$R = \exp(\omega_\times\theta) = \sum_{k=0}^{\infty}\frac{(\omega_\times\theta)^k}{k!}$$

for any $\theta \in [0, 2\pi)$ and any $\omega \in \mathbb{R}^3$ with $\|\omega\| = 1$. The equation above is an infinite series, the *Rodrigues' formula* is useful in order to express the matrix $R = \exp(\omega_\times\theta)$ in closed form

$$\exp(\omega_\times\theta) = I + \sin(\theta)\omega_\times + (1 - \cos(\theta))\omega_\times^2.$$

This method of representing a rotation about a fixed axis through an angle is known as *angle-axis representation*. The inverse map of the exponential map, namely the log map, is therefore given in closed form

$$\log(R) = \theta\omega_\times, \qquad \omega_\times = \frac{1}{2\sin(\theta)}(R - R^\top) \tag{2.7}$$

where (θ, ω) with $|\omega| = 1$ is the *angle-axis coordinates* of $R \in \mathrm{SO}(3)$.

2.2.6 Trace and Eigenvalues of a Rotation Matrix

Let $R \in \mathrm{SO}(3)$ be a rotation matrix, the eigenvalues of R can be determined from the roots of the characteristic polynomial

$$\det(\lambda I - R) = 0.$$

In order to do so, denoting the elements of the rotation matrix

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

yields to

$$\det(\lambda I - R) = -(\lambda - 1)(\lambda^2 + (r_{11} + r_{22} + r_{33} - 1)\lambda + 1 = 0.$$

It can be shown, using the *Rodrigues' formula* (see Murray et al. (1994)), that

$$\operatorname{tr}(R) = r_{11} + r_{22} + r_{33} = 1 + 2\cos(\theta) \tag{2.8}$$

where theta is the angle from *angle-axis representation*, hence the characteristic polynomial can be rewritten as

$$\det(\lambda I - R) = -(\lambda - 1)(\lambda^2 + 2\cos(\theta)\lambda + 1 = 0.$$

The eigenvalues of $R \in \mathrm{SO}(3)$ are therefore

$$\operatorname{eig}(R) = (1, \cos(\theta) + i\sin(\theta), \cos(\theta) - i\sin(\theta)). \tag{2.9}$$

Note that from (2.8) it follows

$$-1 \leq \operatorname{tr}(R) \leq 3. \tag{2.10}$$

2.2.7 Inner Products and Metrics on $\mathrm{SO}(n)$

The Euclidean distance in SO(3) is also known as *chordal distance* indeed for $R_A, R_B \in \mathrm{SO}(3)$ one has

$$\begin{aligned} d(R_A, R_B)_F &= \|R_A - R_B\| = \sqrt{\langle R_A - R_B, R_A - R_B \rangle} \\ &= \sqrt{\operatorname{tr}((R_A - R_B)^\top (R_A - R_B))} \\ &= \sqrt{2\operatorname{tr}(I_3 - \frac{1}{2}R_A^\top R_B - \frac{1}{2}R_B^\top R_A)} \\ &= \sqrt{2\operatorname{tr}(I_3 - R_A^\top R_B}). \end{aligned} \tag{2.11}$$

Denoting $\tilde{R} = R_A^\top R_B$, this new rotation matrix can be considered as coordinates of a new frame $\{C\}$, and recalling (2.8) one obtains

$$\frac{1}{\sqrt{2}} d(R_A, R_B)_F = \sqrt{\operatorname{tr}(I_3 - \tilde{R})} = \sqrt{2(1 - \cos(\theta))} = 2\sin(\frac{\theta}{2})$$

where θ is angle associated with the rotation from $\{B\}$ to the frame $\{C\}$.

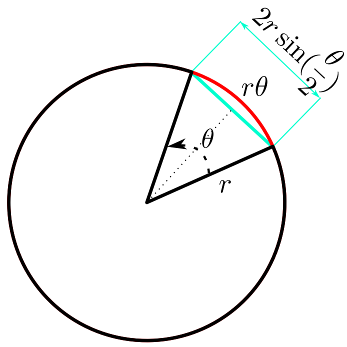

Figure 2.4: The chordal distance and the arc length of a circle.

Hence $\frac{1}{\sqrt{2}}d(R_A, R_B)_F$ represents the length of the chord between two points on a unit circle separated by an angle θ (see Figure 2.4). The Euclidean distance on SO(3) is both *left invariant*

$$d(L_{\bar{R}}(R_A), L_{\bar{R}}(R_B))_F^2 = \mathrm{tr}\left((R_A - R_B)^\top \bar{R}^\top \bar{R}(R_A - R_B)\right) = d(R_A, R_B)_F^2$$

and *right invariant*

$$\begin{aligned} d(R_{\bar{R}}(R_A), R_{\bar{R}}(R_B))_F^2 &= \mathrm{tr}\left(\bar{R}^\top (R_A - R_B)^\top (R_A - R_B)\bar{R}\right) \\ &= \mathrm{tr}\left((R_A - R_B)^\top (R_A - R_B)\bar{R}\bar{R}^\top\right) = d(R_A, R_B)_F^2 \end{aligned}$$

hence the metric considered is *bi-invariant*.

The *chordal distance* is widely used in the design of *attitude tracking regulators* (see Bertrand et al. (2009) and Bullo and Murray (1999)), *rigid body attitude synchronization* (see Sarlette et al. (2007), Nair and Leonard (2007)), *observer design* (see Mahony et al. (2012a) and Lageman et al. (2010b)) and *PI* control on SO(3) (see Maithripala and Berg (2014) and Mahony et al. (2015)). This metric is widely used due to the fact that it intrinsically captures the notion of error in SO(3) and it represents an *artificial potential energy* between the body-fixed frame and the reference frame. It is also important in the stability analysis and for the construction of a Lyapunov function candidate in SO(3).

We can endow SO(3) with a Riemannian metric, let $\Omega_1, \Omega_2 \in \mathfrak{so}(3)$ then

$$d(\Omega_1, \Omega_2)_R = \langle \Omega_1, \Omega_2 \rangle = \mathrm{tr}\,(\Omega_1^\top \Omega_2)$$

is a *bi-invariant* Riemannian metric. Indeed for $\bar{R} \in \mathrm{SO}(3)$ one has that the metric is *left invariant*

$$\langle L_{\bar{R}}(\Omega_1), L_{\bar{R}}(\Omega_2) \rangle = \mathrm{tr}\,(\Omega_1^\top \bar{R}^\top \bar{R}\Omega_2) = \langle \Omega_1, \Omega_2 \rangle$$

and *right invariant*

$$\langle R_{\bar{R}}(\Omega_1), R_{\bar{R}}(\Omega_2)\rangle = \operatorname{tr}(\bar{R}^\top \Omega_1^\top \Omega_2 \bar{R}) = \langle \Omega_1, \Omega_2 \rangle .$$

The Riemannian metric on SO(3) represents the length of the arc(angle) between two points on a unit circle, indeed taking advantage of the log map one has

$$d^2\left(\log(R_A^\top R_B), \log(R_A^\top R_B)\right)_R \quad = \quad \operatorname{tr}(\log(R_A^\top R_B)^\top \log(R_A^\top R_B))$$

with $R_A, R_B \in \mathrm{SO}(3)$. Denoting $\tilde{R} = R_A^\top R_B$ and recalling (2.7) it yields to

$$\begin{aligned} d^2\left(\log(\tilde{R}), \log(\tilde{R})\right)_R &= \operatorname{tr}(\omega_\times^\top \theta^\top \theta \omega_\times) \\ &= \operatorname{tr}(\theta^\top \theta \omega_\times^\top \omega_\times) = \theta^2 \operatorname{tr}(\omega_\times^\top \omega_\times) \end{aligned}$$

where (ω, θ) is the angle-axis representation of $\tilde{R}$. Using the fact that for any $\omega \in \mathbb{R}^3$

$$\|\omega\|^2 = \frac{1}{2}\operatorname{tr}(\omega_\times^\top \omega_\times) \tag{2.12}$$

we finally obtain

$$\frac{1}{2} d^2\left(\log(R_A^\top R_B), \log(R_A^\top R_B)\right)_R = \theta^2.$$

Note that since the Euclidean metric is the length of the chord $d(\cdot,\cdot)_F = 2\sin(\theta/2)$ and the Riemannian is the length of the arc $d(\cdot,\cdot)_R = \theta$, it follows that near the origin $\tilde{R} \approx I$ the two metrics are similar $d(\cdot,\cdot)_F \big|_I \approx d(\cdot,\cdot)_R \big|_i$.

2.2.8 Orthogonal Projection with respect to the Trace Inner Product

Proposition 2.5. *Let $A \in \mathbb{R}^{n\times n}$, then the* ***orthogonal projection*** *of A onto $\mathfrak{so}(n)$ with respect to the trace inner product is given by*

$$\mathbb{P}_{\mathfrak{so}(n)}(A) = \frac{1}{2}(A - A^\top). \tag{2.13}$$

Proof. For $A \in \mathbb{R}^{n\times n}$ and $\Omega \in \mathfrak{so}(n)$

$$\operatorname{tr}(\Omega^\top A) = \frac{1}{2}\operatorname{tr}\left(\Omega^\top \left((A - A^\top) + (A + A^\top)\right)\right)$$

where we have used the fact that a square matrix can be written as a sum of a symmetric matrix $\mathbb{P}_s(A) = \frac{1}{2}(A + A^\top)$ and skew-symmetric matrix $\mathbb{P}_a(A) = \frac{1}{2}(A - A^\top)$.

Note that for $B = B^\top \in \mathbb{R}^{n\times n}$and $C = -C^\top$ one has $\mathrm{tr}(B^\top C) = 0$, hence

$$\mathrm{tr}(\Omega^\top A) = \frac{1}{2}\,\mathrm{tr}\left(\Omega^\top \mathbb{P}_a(A)\right).$$

It follows that $\mathbb{P}_{\mathfrak{so}(n)}(A) = \mathbb{P}_a(A)$. □

If the matrix A considered is a rotation matrix $A \in \mathrm{SO}(3)$, then from (2.7) it turns out

$$\mathbb{P}_{\mathfrak{so}(3)}(A) = 2\sin(\theta)\omega_\times \tag{2.14}$$

where (θ, ω) are the *angle-axis coordinate* of the matrix A. The following identities are useful in the design of observers, PI and Internal model on SO(3) and are widely used in this work.

Properties 2.2. *For* $x, y \in \mathbb{R}^3$

1. $x^\top y = Q^{-1}_{\mathfrak{so}(3)} tr(x_\times^\top y_\times), \quad$ *where* $\quad Q^{-1}_{\mathfrak{so}(3)} = \frac{1}{2} I_3$
2. $x_\times y_\times = yx^\top - x^\top y I_3$
3. $yx^\top - xy^\top = (x \wedge y)_\times$

Properties 2.3. *For* $A = A^\top \in \mathbb{R}^{3\times 3}$ *and* $x \in \mathbb{R}^3$

1. $(Ax)_\times = \left(\frac{1}{2}\,\mathrm{tr}(A) I_3 - A\right) x_\times - x_\times \left(\frac{1}{2}\,\mathrm{tr}(A) I_3 - A\right)$
2. $\mathbb{P}(A x_\times) = \frac{1}{2}\left(\left(\mathrm{tr}(A) I_3 - A\right) x\right)_\times$

2.2.9 Kinematic Tracking on SO(3)

In this subsection we gather all the idea presented so far in order to design a regulator for a simple kinematic tracking problem for *left invariant* systems on SO(3). Consider a *left invariant* kinematic system on SO(3)

$$\dot{R} = R\Omega_\times$$

with $R \in \mathrm{SO}(3)$ the state and $\Omega_\times \in \mathfrak{so}(3)$ a velocity control input for the system. Consider a reference trajectory for the system, given in terms of a desired orientation $R_d(t)$ with respect to an inertial reference frame

$$\dot{R}_d = \Omega_{d\times} R_d$$

with $R_d \in \mathrm{SO}(3)$ and $\Omega_{d\times} \in \mathfrak{so}(3)$. The attitude kinematic tracking problem consists in designing a feedback control action $\Omega_\times$ along with a feed-forward terms, such that the orientation R of the body converges to R_d. Assume that the natural attitude error

$$E = R^\top R_d$$

and the reference velocity $\Omega_{d\times}$ are known. Consider as candidate Lyapunov function a modified version of the chordal distance in (2.4)

$$\mathcal{L} = \frac{1}{2} d^2(R, R_d) = \mathrm{tr}(K_p - EK_p).$$

with $K_p = \mathrm{diag}(k_1, k_2, k_3)$ with $k_3 > k_2 > k_1 > 0$. The reason why the positive gains are chosen to be distinct is technical and needed for the stability analysis. Differentiating $\mathcal{L}$ one obtains

$$\dot{\mathcal{L}} = -\,\mathrm{tr}(\dot{E}K_p).$$

By bearing in mind the definition of E, it turns out that the time derivative is given by

$$\begin{aligned}
\dot{E} &= \dot{R}^\top R_d + R^\top \dot{R}_d \\
&= \Omega_\times^\top R^\top R_d + R^\top \Omega_{d\times} R_d \\
&= -\Omega_\times R^\top R_d + R^\top \Omega_{d\times} R_d \\
&= -\Omega_\times R^\top R_d + R^\top \Omega_{d\times} R R^\top R_d \\
&= -\Omega_\times E + R^\top \Omega_{d\times} R E \\
&= -(\Omega_\times - R^\top \Omega_{d\times} R) E.
\end{aligned}$$

Note that the velocity $R^\top \Omega_{d\times} R = (R^\top \Omega_d)_\times$ is the desired *inertial velocity* expressed in the *body-fixed frame* (2.3). Introducing the expression of the time derivative of the error into the Lyapunov function it yields

$$\begin{aligned}
\dot{\mathcal{L}} &= \mathrm{tr}\left((\Omega_\times - R^\top \Omega_{d\times} R) E K_p\right) \\
&= \mathrm{tr}\left((\Omega - R^\top \Omega_d)_\times E K_P\right) \\
&= -\,\mathrm{tr}\left((\Omega_\times - R^\top \Omega_{d\times} R)^\top E K_p\right)
\end{aligned}$$

Recalling the orthogonal projection of EK_p onto $\mathfrak{so}(3)$ with respect to the trace inner product (Proposition 2.5) one has

$$\dot{\mathcal{L}} = -\,\mathrm{tr}\left(\left(\Omega_\times - R^\top \Omega_{d\times} R\right) \mathbb{P}(EK_p)\right)$$

and choosing as kinematic control input

$$\Omega_\times = R^\top \Omega_{d\times} R + k_p \mathbb{P}(EK_p)$$

one finally obtains

$$\dot{\mathcal{L}} = -\|\mathbb{P}(EK_P)\|^2.$$

Note that the control law obtained (Figure 2.5) is the superposition of a feed-forward term $R^\top \Omega_{d\times} R$ and a proportional feedback term $\mathbb{P}(EK_P)$. Lyapunov's direct method

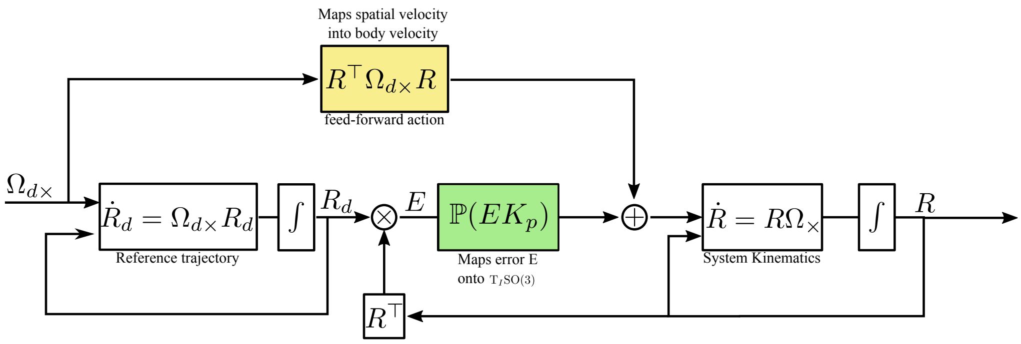

Figure 2.5: Block diagram of the proposed control law.

ensures that $\mathbb{P}(EK_P)$ converges asymptotically to zero. It follows that for $\dot{\mathcal{L}} = 0$ one has

$$E^\star K_p - K_p E^{\star\top} = 0$$

This implies that all eigenvalues of $E^\star$ are real, and bearing in mind the expression of the eigenvalues of an rotation matrix (2.9) it follows that $\theta = k\pi$ with $k = -1, 0, 1$ where θ is the angle of the *angle-axis representation*. For $\theta = 0$ one has $\mathrm{eig}(E^\star) = (1, 1, 1)$ and this implies $R = I$. For $\theta = \pm\pi$ one has $\mathrm{eig}(E^\star) = (1, -1, -1)$, hence $\mathrm{tr}(E^\star) = -1$. Therefore $\mathrm{tr}(E^\star) = -1$ also implies that $E^\star$ is a symmetric matrix, then

$$E^\star K_p - K_p E^\star = 0 \Rightarrow (k_i - k_j)E^\star_{ij} = 0, \quad \text{for all} \quad i, j$$

and since $k_3 > k_2 > k_1$ it follows that $E^\star_{ij} = 0$ for all $i \neq j$, hence $E^\star$ is a diagonal matrix. Thus there are four possible equilibria for the closed loop systems

$$\begin{aligned}
E_1^\star &= I \\
E_2^\star &= \mathrm{diag}(1, -1, -1) \\
E_3^\star &= \mathrm{diag}(-1, -1, 1) \\
E_4^\star &= \mathrm{diag}(-1, 1, -1).
\end{aligned}$$

Note that $E_1^\star$ is the desired equilibrium point while $E_i^\star$ for $i = 2, 3, 4$ are undesired equilibrium points. In order to prove that the set

$$\mathcal{S} := \{R, R_d \in \mathrm{SO}(3) \times \mathrm{SO}(3) \mid R = R_d\}$$

is almost globally asymptotically stable, we need to show that the desired equilibrium is stable while the others three equilibria are unstable. To this end it suffices to show that the eigenvalues of the linearized error system at $E = I$ have strictly negative real part. To this purpose consider the error dynamics for the closed-loop system

$$\begin{aligned}\dot{E} &= -\mathbb{P}(EK_p)E \\ &= -\frac{1}{2}\left(EK_p - K_pE^\top\right)E \\ &= \frac{1}{2}K_p - \frac{1}{2}EK_pE^\top.\end{aligned}$$

Consider a first order approximation of E around the equilibrium point $E_i^\star$

$$E = E_i^\star(I_3 + x_{i\times})$$

with $x_i \in \mathbb{R}^3$. The linearization of the closed-loop system dynamics is given by

$$\begin{aligned}\frac{d}{\mathrm{d}t}(E_i^\star(I_3 + x_{i\times})) &= \frac{1}{2}K_p - \frac{1}{2}E_i^\star(I + x_{i\times})K_pE_i^\star(I_3 + x_{i\times}) \\ &= \frac{1}{2}K_p - \frac{1}{2}E_i^\star K_pE_i^\star - \frac{1}{2}E_i^\star K_pE_i^\star x_{i\times} - \frac{1}{2}E_i^\star x_{i\times}K_pE_i^\star \\ &\quad -\frac{1}{2}E_i^\star x_{i\times}K_pE_i^\star x_{i\times}.\end{aligned}$$

Using the fact that $E_i^\star K_p$ commutes, i.e. $E_i^\star K_p = K_pE_i^\star$, and neglecting high order terms one obtains

$$E_i^\star \dot{x}_{i\times} = \frac{1}{2}K_p - \frac{1}{2}K_pE_i^\star E_i^\star - \frac{1}{2}E_i^\star K_pE_i^\star x_{i\times} - \frac{1}{2}E_i^\star x_{i\times}E_i^\star K_p.$$

Bearing in mind that $E_i^\star = E_i^{\star\top} = E_i^{\star-1}$ one has

$$\begin{aligned}\dot{x}_{i\times} &= -\frac{1}{2}K_pE_i^\star x_{i\times} - \frac{1}{2}x_{i\times}K_pE_i^\star \\ &= -\mathbb{P}(K_pE_i^\star x_{i\times}).\end{aligned}$$

Recalling the second item in Properties 2.3 it yields

$$\dot{x}_{i\times} = -\frac{1}{2}\left((\mathrm{tr}(K_pE_i^\star)I_3 - K_pE_i^\star)\,x_i\right)_\times$$

thus

$$\dot{x}_i = A_ix_i$$

where A_i is on the form $A_i = -\frac{1}{2}((\mathrm{tr}(K_pE_i^\star)I_3 - K_pE_i^\star)$, specifically

$$\begin{aligned}
A_1 &= \frac{1}{2}\,\mathrm{diag}(-k_2 - k_3, -k_1 - k_3, -k_1 - k_2),\\
A_2 &= \frac{1}{2}\,\mathrm{diag}(k_2 + k_3, -k_1 + k_3, -k_1 + k_2),\\
A_3 &= \frac{1}{2}\,\mathrm{diag}(-k_2 + k_3, k_1 - k_3, k_1 + k_2),\\
A_4 &= \frac{1}{2}\,\mathrm{diag}(k_2 - k_3, k_1 + k_3, k_1 - k_2).
\end{aligned}$$

It is straightforward to verify that the equilibrium $E = I$ is stable in the sense of Lyapunov, indeed all eigenvalues of the matrix A_1 have strictly negative real part. Note that for $i = 2, 3, 4$ the matrix A_i has at least one positive eigenvalue and not null eigenvalues, then from Lyapunov indirect method (Khalil (1996)) we could conclude that for $i = 2, 3, 4$ the system is unstable. However since in the next chapters we will deal with systems in which the indirect method is inconclusive we introduce an alternative method to prove the instability of the others three equilibria, namely the Chetaev's instability theorem (see Khalil (1996)). To this end consider the following cost function

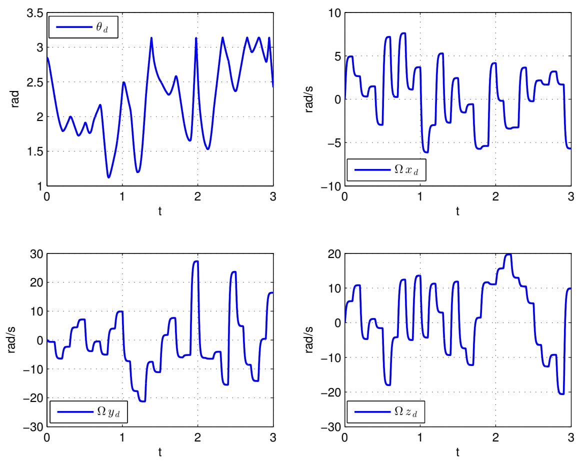

Figure 2.6: Simulations of the tracking algorithm. Reference trajectories $\theta_d, \Omega_{xd}, \Omega_{yd}, \Omega_{zd}$ are generated by a pink noise. θ_d represents the desired angle (angle-axis representation) and $\Omega_{xd}, \Omega_{yd}, \Omega_{zd}$ are the components of the reference velocity associated to the inertial frame.

$$\mathcal{I}_i = \frac{1}{2}x_i.'A_ix_i, \quad i = 2, 3, 4$$

and differentiating along the solutions of the linearized error dynamics one obtains

$$\dot{\mathcal{I}} = A_i^2 \, |x_i|^2 \, , \quad i = 2, 3, 4.$$

Since A_i^2 is positive definite it follows that the time derivative of the cost function $\mathcal{I}$ is always positive. For an arbitrarily small radius $r > 0$, define the set

$$\mathcal{U}_i = \{x_i \,|\, \mathcal{I}_i > 0, |x|_i < r\}, \quad i = 2, 3, 4$$

and note that the set $\mathcal{U}_i$ is non empty for each $i = 2, 3, 4$ since at least one eigenvalue of the matrix A_i is positive. As consequence a trajectory $x_i(t)$ inizialized near an equilibrium point $x_i^\star = 0$ will diverge from the compact set $\mathcal{U}_i$ since the derivative of the cost function is always positive. Moreover the trajectory $x_i(t)$ can not exit from the center of the ball since along the trajectory the level sets are $\mathcal{I}(x_i(t)) \geq \mathcal{I}(x_i(0))$. Thus trajectories arbitrary close to the origin must exit trough the ball $|x_i| = r$. Consequently the origin of the linearized system is unstable for $i = 2, 3, 4$. However note that there will be trajectories that converges to the unstable equilibria along the stable center manifold (Khalil (1996)). Anyhow such particular trajectories are of zero Lebesque measure.

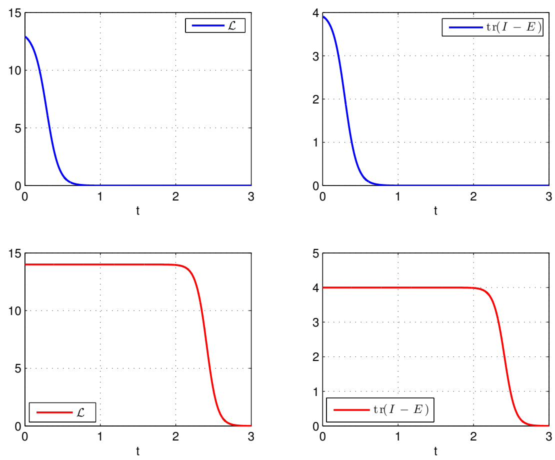

Figure 2.7: The time behavior of the Lyapunov function and the norm of the error. In red, the case in which the error $E(0)$ starts from an undesired equilibrium point. In blue, the case in which the error $E(0)$ starts from a generic point that differs from an equilibrium point.

Note that from Figure 2.7 one can see (from the red graph) that even if the error starts or converges to the unstable equilibria, this will not cause practical problems. Indeed, in this case, small integration errors of the MATLAB® solver used in the simulations moves out the trajectory from the unstable equilibrium. In a real world scenario, small disturbances such as the noise of the sensors or the quantization of the embedded controller will force the trajectory to exit from the undesired equilibrium point.

2.3 The Special Euclidean Group

Definition 2.13. *The **special Euclidean group*** $\mathrm{SE}(n)$ *is the set*

$$\mathrm{SE}(n) = \{(R, p) \mid R \in \mathrm{SO}(n),\ p \in \mathbb{R}^n\}$$

together with the group operation

$$(R_1, p_1)(R_2, p_2) \mapsto (R_1 R_2, R_1 p_2 + p_1).$$

The identity element is $(I_n, 0)$*, while the inverse element is given by*

$$(R, p)^{-1} = (R^\top, -R^\top p).$$

The *special Euclidean group* can be also defined as the set of mappings $f : R^n \mapsto R^n$ with

$$f_{(R,p)}(x) = Rx + p, \tag{2.15}$$

note that $f_{(R,p)}$ is a group action of $\mathrm{SE}(3)$ on $\mathbb{R}^3$. Moreover note that

$$f_{(R_1,p_1)} \circ f_{(R_2,p_2)} = f_{(R_1 R_2, R_1 p_2 + p_1)}$$

is the group operation in the definition of $\mathrm{SE}(3)$. Since the group operation and the inverse element involve a group action it turns out that $\mathrm{SE}(n)$ is the semi-direct product of $\mathrm{SO}(n)$ and $\mathbb{R}^n$

$$\mathrm{SE}(n) := \mathrm{SO}(n) \ltimes \mathbb{R}^n.$$

The dimension of $SE(n)$ as a manifold is $n(n+1)/2$.

2.3.1 Pose of a Rigid Body

Let $\{A\}$ denote an *inertial frame* (for example a frame attached to the earth) and $\{B\}$ a *body-fixed frame* attached to the rigid body. As we have already seen in the previous section, the attitude of a rigid body can be represented by a rotation matrix $R \in \mathrm{SO}(3)$ of the *body-fixed frame* $\{B\}$ with respect to the *inertial frame* $\{A\}$. The position of the *body-fixed frame*, expressed in the *inertial frame*, is denoted by a vector $p \in \mathbb{R}^3$ (see Figure 2.10). The pose (attitude and position) of the rigid-body is then represented by the tuple (R, p) with $R \in \mathrm{SO}(3)$ and $p \in \mathbb{R}^3$.

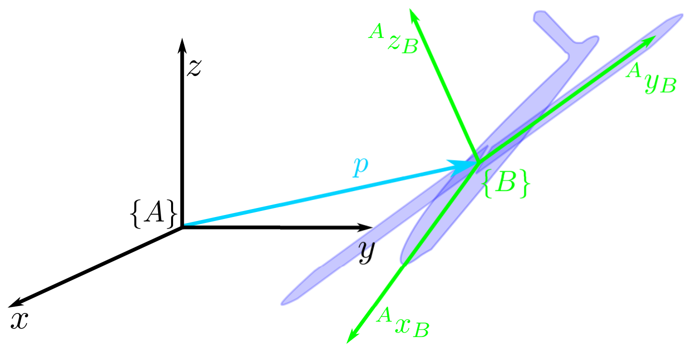

Figure 2.8: Pose of a rigid body.

Let $^A p_d$, $^B p_d$ the position of the point d with respect to the *inertial frame* and to the *body-fixed frame*, respectively. Assume that the coordinates of the point d with respect to the *inertial frame* are unknown, how we can obtain the coordinates of $^A p_d$ given p, $^B p_d$, R?

First of all we need to express the coordinates of the vector $^B p_d$ in the *inertial frame*. This can be done by the rotation matrix that defines the *transformation* of a vector from $\{B\}$ to $\{A\}$

$$^A p_{Bd} = R\,^B p_d.$$

Note that since in this case the origins of the two reference frames don not coincides, $^A p_{Bd}$ is not the position of the point d with respect to $\{A\}$. Then the position of the point d with respect to the frame $\{A\}$ is given by

$$^A p_d = {}^A p_{Bd} + p = R\,^B p_d + p.$$

Note that the new-found expression is exactly the *group action* in 2.15. Thus the pose of

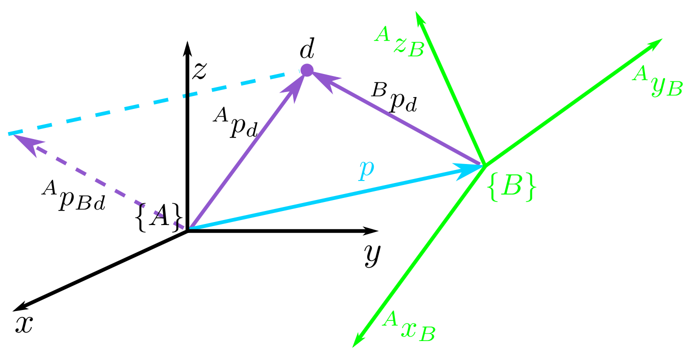

Figure 2.9: Position of a point d with respect to an inertial frame and a body-fixed frame.

a rigid body can be interpreted as an element of the *special Euclidean group* $\mathrm{SE}(3)$.

2.3.2 Representation of the Special Euclidean Group $SE(3)$

A convenient way to represent an element of SE(3) is the matrix representation of SE(3). Due to the fact that the group action in 2.15 is an *affine transformation* we can represent SE(3) with an augmented matrix $X \in \mathrm{SE}(3)$ i.e.

$$X = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}.$$

This representation is commonly known as *homogeneous* representation. Note that the group structure of SE(3), taking advantages of the matrix representation, is preserved under matrix multiplication. The identity element is I_4 while the inverse element is given by

$$X^{-1} = \begin{bmatrix} R^\top & -R^\top p \\ 0 & 1 \end{bmatrix}.$$

The group action then is obtained as "augmented" vector matrix multiplication

$$\begin{bmatrix} {}^A p_d \\ 1 \end{bmatrix} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}^B p_d \\ 1 \end{bmatrix}.$$

The "augmented" form of the vector is known as the *homogeneous* coordinates of the vector. From now on for any $p \in \mathbb{R}^4$ in homogeneous coordinates, the vector $\underline{p} \in \mathbb{R}^3$ denotes the first three components of the vector p, i.e. $p = \mathrm{col}(\underline{p}, 1)$.

2.3.3 The Lie Algebra associated to the Special Euclidean Group

Definition 2.14. *The Lie algebra associated to the special Euclidean group* SE(3)*, denote by* $\mathfrak{se}(3)$*, is the set of tuples*

$$\mathfrak{se}(3) := \left\{(\Omega, V) \,\middle|\, \Omega \in \mathfrak{so}(3), V \in \mathbb{R}^3\right\}.$$

Using the matrix representation we can define the Lie algebra associated to the special Euclidean group SE(3) as follow.

Definition 2.15. *The Lie algebra associated to the special Euclidean group, denote by* $\mathfrak{se}(3)$*, is the set of* 4×4 *matrices*

$$\mathfrak{se}(3) := \left\{U \in \mathbb{R}^{4\times 4} \,\middle|\, \exists \Omega, V \in \mathbb{R}^3 : U = \begin{bmatrix} \Omega_\times & V \\ 0 & 0 \end{bmatrix}\right\}.$$

2.3.4 The Vectorial and Matrix Representation of $\mathfrak{se}(3)$

Definition 2.16. *Let $\Omega, V \in \mathbb{R}^3$, the* ***vectorial representation*** $vrp : \mathfrak{se}(3) \mapsto \mathbb{R}^6$ *of*

$$U = \begin{bmatrix} \Omega_\times & V \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(3)$$

is given by

$$\mathrm{vrp}(U) = \begin{bmatrix} \Omega \\ V \end{bmatrix}.$$

Definition 2.17. *Let $\Omega, V \in \mathbb{R}^3$, the* ***matrix representation*** $mrp : \mathbb{R}^6 \mapsto \mathfrak{se}(3)$ *of*

$$T = \begin{bmatrix} \Omega \\ V \end{bmatrix}$$

is given by

$$\mathrm{mrp}(T) = \begin{bmatrix} \Omega_\times & V \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(3).$$

2.3.5 Right Invariant Systems on SE(3)

Proposition 2.6. *For $X \in \mathrm{SE}(3)$ and $U \in \mathfrak{se}(3)$, a* ***right invariant system*** *on* $\mathrm{SE}(3)$ *is of the form*

$$\dot{X}(t) = {}^\circ U(t)X(t). \tag{2.16}$$

Proof. Consider a time-varying matrix $X(t) \in \mathrm{SE}(n)$, differentiating with respect to time at the identity element of the group it yields

$$\begin{aligned} 0 &= \frac{d}{\mathrm{d}t}(X(t)X^{-1}(t)) \\ &= \dot{X}(t)X^{-1}(t) + X(t)\frac{d}{\mathrm{d}t}(X^{-1}(t)) \\ &= \dot{X}(t)X^{-1}(t) - \dot{X}(t)X^{-1}(t). \end{aligned}$$

Denoting ${}^\circ U := \dot{X}(t)X^{-1}(t)$, one has

$$\dot{X}(t) = {}^\circ U(t)X(t).$$

The tangent spaces $\mathrm{T}_X\mathrm{SE}(3)$, then are given by

$$\mathrm{T}_X\mathrm{SE}(3) := \left\{ {}^\circ UX \;\middle|\; {}^\circ U \in \mathfrak{se}(3) \right\} \subset \mathbb{R}^{4\times 4}.$$

It is straightforward to verify that system dynamics are *right invariant*, indeed

$$\frac{d}{\mathrm{d}t}(X(t)Y) = \dot{X}(t)Y = {}^{\circ}U(t)(X(t)Y)$$

with $Y \in \mathrm{SE}(3)$, and this concludes the proof. □

Bearing in mind the matrix representation of SE(3) one obtains

$$\begin{aligned} {}^{\circ}U = \dot{X}(t)X^{-1}(t) &= \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^{\top} & -R^{\top}p \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \dot{R}R^{\top} & -\dot{R}R^{\top}p + \dot{p} \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Recalling the angular velocity associated to a *right invariant* vector field one has

$${}^{\circ}U = \begin{bmatrix} {}^{\circ}\Omega_{\times} & -{}^{\circ}\Omega_{\times}p + v \\ 0 & 0 \end{bmatrix}$$

where $v := \dot{p}$ is the velocity of the origin of the *body-fixed frame* with respect to the *inertial frame*.

We can give a physical interpretation of the velocity ${}^{\circ}U$. To this end let $v \in \mathbb{R}^3$ the velocity of the origin of the body-fixed frame $\{B\}$ with respect to the inertial frame $\{A\}$ and ${}^{\circ}\Omega$ the angular velocity of the rigid-body with respect to the inertial frame (see Figure 2.10). It is well known from classical mechanics that the velocity v_d of a point d of the rigid body is given by

$$v_d = {}^{\circ}\Omega \wedge (p_d - p) + v$$

where p_d is the position of the point d with respect to the inertial frame. And in *homogeneous coordinates*

$$\begin{bmatrix} v_d \\ 0 \end{bmatrix} = \begin{bmatrix} {}^{\circ}\Omega_{\times} & -{}^{\circ}\Omega_{\times}p + v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_d \\ 1 \end{bmatrix} = {}^{\circ}U \begin{bmatrix} p_d \\ 1 \end{bmatrix}$$

it follows that the matrix ${}^{\circ}U$ is a mapping that relate the velocity of the origin of the *body-fixed frame* to the velocity of another point of the rigid body, both with respect to the *inertial frame*. Due to this fact we will call the vector $\mathrm{vrp}({}^{\circ}U) = \mathrm{col}({}^{\circ}\Omega, -{}^{\circ}\Omega_{\times}p + v)$ the spatial velocity of the rigid body.

2.3.6 Left Invariant Systems on SE(3)

Proposition 2.7. *For $X \in \mathrm{SE}(3)$ and $\Omega \in \mathfrak{se}(3)$, a* ***left invariant system*** *on* SE(3) *is of the form*

$$\dot{X}(t) = X(t)U(t). \tag{2.17}$$

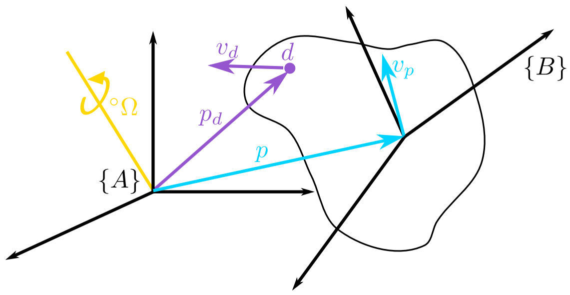

Figure 2.10: Velocity of a point of the rigid body.

Proof. Consider a time-varying matrix $X(t) \in \mathrm{SE}(3)$, proceeding in an analogous manner of the right invariant case, and differentiating at the identity element of the group one has

$$0 = X^{-1}(t)\dot{X}(t) - X^{-1}(t)\dot{X}(t).$$

Denoting $U = X^{-1}(t)\dot{X}(t)$, it yields

$$\dot{X}(t) = X(t)U(t)$$

Thus the tangent spaces $\mathrm{T}_X\mathrm{GL}(n)$ are identified with

$$\mathrm{T}_X\mathrm{GL}(n) := \left\{XU \mid U \in \mathfrak{gl}(n)\right\} \subset \mathbb{R}^{n\times n}.$$

The map $\mathrm{T}_X L_Y : XU \mapsto YXU$ is given by left multiplication of the matrices in $\mathrm{T}_X SE(3)$ with a constant matrix Y. The system dynamics are then *left invariant*

$$\frac{d}{\mathrm{d}t}(YX(t)) = Y\dot{X}(t) = (YX(t))U(t)$$

□

The physical interpretation of the velocity U is straightforward, indeed consider the *matrix representation* of U

$$\begin{aligned} U = X^{-1}(t)\dot{X}(t) &= \begin{bmatrix} R^\top & -R^\top p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} R^\top \dot{R} & R^\top \dot{p} \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Recalling the angular velocity associated to a *left invariant* vector field one has

$$U = \begin{bmatrix} \Omega_\times & R^\top \dot{p} \\ 0 & 0 \end{bmatrix}$$

and bearing in mind that the velocity of the origin of the body-fixed frame relative to the inertial frame is given by $V = R^\top v$, where v is the velocity of the origin of the body-fixed frame with respect to the inertial frame it yields

$$U = \begin{bmatrix} \Omega_\times & V \\ 0 & 0 \end{bmatrix}.$$

Thus the first three components of $\mathrm{vrp}(U) = \mathrm{col}(\Omega, V)$ represent the angular velocity of the inertial frame as seen from the body-fixed frame, while the second three components represent the velocity of the origin of the body coordinate frame relative to the spatial frame.

We have seen that ${}^\circ U$ represents inertial velocities while U represents body velocities, the two quantities are related by the adjoint action, indeed

$$\begin{aligned} {}^\circ U(t) &= \dot{X}(t)X^{-1}(t) = X(t)X^{-1}(t)\dot{X}(t)X^{-1}(t) \\ &= X(t)U(t)X^{-1}(t) = Ad_{X(t)}U(t). \end{aligned}$$

2.3.7 Metrics on SE(3)

For $X_A, X_B \in \mathrm{SE}(3)$ the *induced matrix norm* on $SE(3)$ is given by

$$\begin{aligned} d^2(X_A, X_B)_F &= \mathrm{tr}\left((X_A - X_B)^\top (X_A - X_B)\right) \\ &= \mathrm{tr}\left(\begin{bmatrix} (R_A - R_B)^\top & 0 \\ (p_A - p_B)^\top & 0 \end{bmatrix} \begin{bmatrix} (R_A - R_B) & (p_A - p_B) \\ 0 & 0 \end{bmatrix}\right) \\ &= \mathrm{tr}\left(\begin{bmatrix} (R_A - R_B)^\top (R_A - R_B) & (R_A - R_B)^\top (p_A - p_B) \\ (p_A - p_B)^\top (R_A - R_B) & (p_A - p_B)^\top (p_A - p_B) \end{bmatrix}\right) \\ &= \mathrm{tr}\left((R_A - R_B)^\top (R_A - R_B)\right) + (p_A - p_B)^\top (p_A - p_B) \\ &= d^2(R_A, R_B)_F + \|p_A - p_B\|^2. \end{aligned}$$

The metric $d(\cdot,\cdot)_F$ on SE(3) is left invariant, indeed considering a left translation $L_{\bar{X}}$

with $\bar{X} \in \mathrm{SE}(3)$ it yields

$$\begin{aligned} d^2(L_{\bar{X}}X_A, L_{\bar{X}}X_B)_F &= \mathrm{tr}\left((X_A - X_B)^\top \bar{X}^\top \bar{X}(X_A - X_B)\right) \\ &= \mathrm{tr}\left((X_A - X_B)^\top \begin{bmatrix} \bar{R}^\top & 0 \\ \bar{p} & 1 \end{bmatrix} \begin{bmatrix} \bar{R} & \bar{p} \\ 0 & 1 \end{bmatrix} (X_A - X_B)\right) \\ &= \mathrm{tr}\left((X_A - X_B)^\top \begin{bmatrix} I_3 & \bar{R}^\top \bar{p} \\ \bar{p}^\top \bar{R} & \bar{p}^\top \bar{p} \end{bmatrix} (X_A - X_B)\right) \\ &= \mathrm{tr}\left(\begin{bmatrix} (R_A - R_B)^\top & (R_A - R_B)^\top \bar{R}^\top \bar{p} \\ (p_A - p_B)^\top & (p_A - p_B)^\top \bar{R}^\top \bar{p} \end{bmatrix} \begin{bmatrix} (R_A - R_B) & (p_A - p_B) \\ 0 & 0 \end{bmatrix}\right) \\ &= d^2(X_A, X_B)_F. \end{aligned}$$

However it can be easily verified that the metric $d(\cdot,\cdot)_F$ is not right invariant. Often for the construction of a suitable Lyapunov function one needs a right invariant metric. This is for example the case of the design of *left observer* on Lie Groups (see Hua et al. (2015a) and Lageman et al. (2010a)). In the work of Lageman et al. (2010a), a systematic procedure for the construction of a right invariant metric for systems on Lie Groups is provided.

Proposition 2.8. *(Lageman et al. (2010a)) Let $\mathbf{G}$ a Lie group and let $f : \mathbf{G} \times \mathbf{G} \mapsto \mathbb{R}$ be a left invariant function. Then $\tilde{f} : \mathbf{G} \times \mathbf{G} \mapsto \mathbb{R}$ defined by*

$$\tilde{f}(X, Y) = f(X^{-1}, Y^{-1})$$

is a right invariant function.

Thus in our case the left invariant metric is given by

$$\begin{aligned} \tilde{d}^2(X_A, X_B)_F &= d^2(X_A^{-1}, X_B^{-1})_F \\ &= \mathrm{tr}\left(\begin{bmatrix} (R_A^\top - R_B^\top) & -R_A^\top p_A + R_B^\top p_B \\ 0 & 0 \end{bmatrix}^\top \begin{bmatrix} (R_A^\top - R_B^\top) & -R_A^\top p_A + R_B^\top p_B \\ 0 & 0 \end{bmatrix}\right) \\ &= \|R_A^\top - R_B\|_F^2 + \| - R_A^\top p_A + R_B^\top p_B\|^2. \end{aligned}$$

We can also endow SE(3) with a left invariant Riemannian metric, recalling the definition of tangent spaces for a *left invariant* vector field one has

$$\begin{aligned}
d^2(XU_1, XU_2)_R &= \langle XU_1, XU_2\rangle \\
&= \operatorname{tr}(U_1^\top X^\top X U_2) \\
&= \operatorname{tr}\left(\begin{bmatrix} -\Omega_{1\times} & 0 \\ V_1^\top & 0 \end{bmatrix}\begin{bmatrix} R^\top & 0 \\ p^\top & 1 \end{bmatrix}\begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \Omega_{2\times} & V_2 \\ 0 & 0 \end{bmatrix}\right) \\
&= -\operatorname{tr}(\Omega_{1\times}\Omega_{2\times}) + V_1^\top V_2 = \langle U_1, U_2\rangle = d^2(U_1, U_2)_R.
\end{aligned}$$

Or in analogous manner with a *right invariant* Riemannian metric, indeed by bearing in mind the definition of tangent spaces for a *right invariant* vector field one obtains

$$\begin{aligned}
d^2(^\circ U_1 X, {}^\circ U_2 X)_R &= \langle ^\circ U_1 X, {}^\circ U_2 X\rangle \\
&= \operatorname{tr}(X^\top {}^\circ U_1^\top {}^\circ U_2 X) \\
&= \operatorname{tr}\left(\begin{bmatrix} R^\top & 0 \\ p^\top & 1 \end{bmatrix}\begin{bmatrix} -^\circ\Omega_{1\times} & 0 \\ -p^\top {}^\circ\Omega_{1\times} + v_1^\top & 0 \end{bmatrix}\begin{bmatrix} ^\circ\Omega_{2\times} & -^\circ\Omega_{2\times} p + v_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} R & p \\ 0 & 0 \end{bmatrix}\right) \\
&= -\operatorname{tr}(^\circ\Omega_{1\times} {}^\circ\Omega_{2\times}) + v_1^\top v_2 \\
&= \langle ^\circ U_1, {}^\circ U_2\rangle = d^2(^\circ U_1, {}^\circ U_2)_R.
\end{aligned}$$

2.3.8 Orthogonal Projection with respect to the Trace Inner Product

Proposition 2.9. *Let $A \in \mathbb{R}^{4\times 4}$ a block matrix of the form*

$$A = \begin{bmatrix} A_1 & a_2 \\ a_3^\top & a_4 \end{bmatrix}$$

*with $A_1 \in \mathbb{R}^{3\times 3}$, $a_2, a_3 \in \mathbb{R}^3$ and $a_4 \in \mathbb{R}$. Then the **orthogonal projection** of A onto $\mathfrak{se}(3)$ with respect to the trace inner product is given by*

$$\mathbb{P}_{\mathfrak{se}(3)}(A) = \begin{bmatrix} \mathbb{P}_{\mathfrak{so}(3)}(A_1) & a_2 \\ 0 & 0 \end{bmatrix}. \tag{2.18}$$

"Spaghetti and Levi-Civita."

Albert Einstein

3

Models of Mechanical Systems whose configuration manifold is a Lie Group

The civil and commercial usage of Unmanned Aerial Vehicles (UAVs), Unmanned Ground Vehicles (UGV) and Unmanned Underwater Vehicles (UUV) has experienced an exponential growth in last past years. Unmanned vehicles have become very popular due to their ability to replace and help human beings (or cooperate with them) in dangerous environments. Nowadays unmanned vehicles are used in everyday life including aerial photography and filming, crop supervision, soil and field analyses, package delivery (Amazon and Google), infrastructure inspection (Marconi et al. (2012a)), seafloor mapping (Wynna et al. (2014)), research and rescue (Marconi et al. (2012b)), fire detection and monitoring (de Dios et al. (2006)).

A wide class of these robotic systems (UAV, UGV, UUV) share the fact that the kinematics laws of motion are *invariant* under a change of the configuration space. This invariace properties are known in physics as continuous symmetries. Such physical symmetries lead to structured state space representations on Lie groups. Indeed, as we have seen in the previous chapter the pose and the attitude of a rigid-body are represented in SE(3) and SO(3) respectively. In this chapter we are going to introduce some basic mechanical models for systems whose configuration space is a Lie Group, focusing our attention on fully actuated systems.

3.1 Unmanned Ground Vehicles

Roughly speaking, we can distinguish ground vehicles in base of the wheels mounted on them. Basically the wheels mounted on a vehicle can be *directional* wheels or *omni-directional* wheels. Conventional *directional* wheels, such as the wheels of a car, put constraints in the instantaneous velocity of the vehicle. Indeed, conventional wheels can't move in a direction parallel to the wheel axle. For example, for lateral parking, a car-like vehicle needs to perform complex maneuvers in order to park (see Figure 3.1).

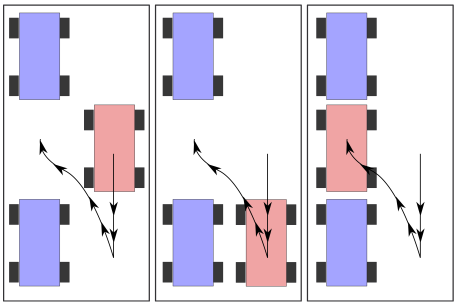

Figure 3.1: Lateral parking with conventional wheels.

This type of constraints are called *non-holonomic* constraints. For a deep treatment of wheeled robots the reader is referred to books devoted to the argument as Siciliano et al. (2009).

Vehicles with *omni-directional* wheels, instead, are able to instantaneously move the car in any direction regardless its current configuration (see Figure 3.2).

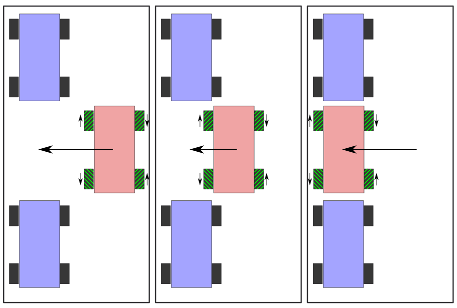

Figure 3.2: Lateral parking with Swedish wheels.

The *omni-directional* motion is obtained thanks to the particular design of *omni-directional* wheels. In *omni-directional* wheels, small rollers are located around the outer diameter of the wheel, mounted perpendicularly to the axle of the wheel. As consequence the wheel is free to move in the direction parallel to the rotation axis of the wheel (see Figure 3.4). The *omni-directional* wheel was first patented in 1919 by J. Grabowiecki. Another popular *omni-directional* wheel is the Swedish or Mecanum wheel. The wheel

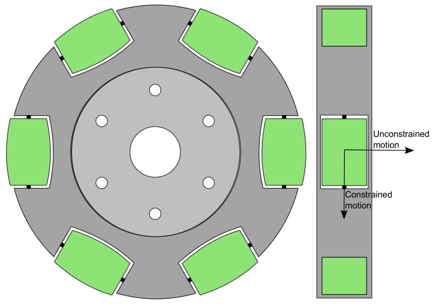

Figure 3.3: Side and front view of a roller wheel.

name come from the nationality of its inventor Bengt Ilon (1973) (Mecanum company). The first mobile robot with Mecanum wheels is Uranus (see Muir and Neuman (1987)). For an extensive analysis of the wheeled robots' kinematic the reader is referred to Muir and Neuman (1986) and Muir and Neuman (1987).

Figure 3.4: Uranus (Muir and Neuman (1987)).

Holonomic and non-*holonomic* vehicles, however, share the same configuration manifold. Indeed non-holonomic constraints do not restrict the configuration space. The configuration manifold of a planar rigid body (like vehicles) is the *special Euclidean group* SE(2)

$$\mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2.$$

In the next section we will briefly derive the kinematic model for a vehicle with three *omni-directional* wheels. We do not consider the dynamic model of the vehicle since it is strictly depended on the particular wheel used and because usually the commercial models have already a low level control for these wheels.

3.1.1 Ground Vehicle with Omni-directional Wheels

Figure 3.5 presents the classical configuration of a three *omni-directional* wheeled vehicle, each wheel is separated by an angle of $2\pi/3$ radiant and placed at distance d with respect to the center of the robot.

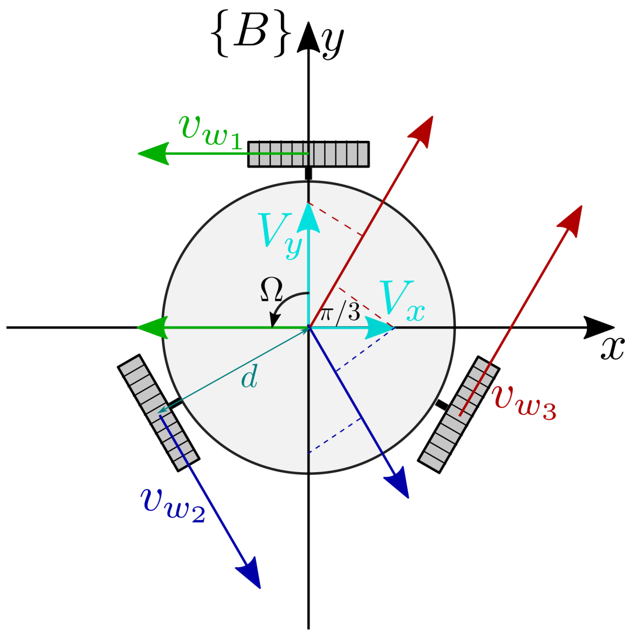

Figure 3.5: Ground vehicle with three roller wheels

We denote by v_{wi}, $i = 1, 2, 3$, the linear velocity of the *i-th* wheel, and by V_x, V_y, Ω the velocity along the x axis, the velocity along the y axis and the angular velocity, respectively. Linear velocities of the wheels with respect to the body-fixed frame $\{B\}$ velocities are given by

$$\begin{aligned}
v_{w1} &= -V_y + \Omega d \\
v_{w2} &= V_y \cos(\pi/3) - V_x \cos(\pi/6) + \Omega d \\
v_{w3} &= V_y \cos(\pi/3) + V_x \cos(\pi/6) + \Omega d.
\end{aligned}$$

And using a compact notation one gets

$$V_w = {}^{w}J_B \,\mathrm{vrp}(U)$$

where $V_w = \mathrm{col}(v_{w1}, v_{w2}, v_{w3})$, $\mathrm{vrp}(U) = \mathrm{col}(\Omega, V_x, V_y)$ and

$$^{w}J_B = \begin{bmatrix} 0 & -1 & d \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & d \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & d \end{bmatrix}.$$

The determinant of this matrix is given by

$$\det(^{w}J_B) = -\frac{3\sqrt{3}}{2}d$$

and bearing in mind that $d > 0$, it's possible to conclude that the matrix $^{B}J_w$ is always invertible. Thus, we can obtain the inverse transformation mapping the linear velocities of the wheels into the velocity of the body-fixed frame

$$\mathrm{vrp}(U) = {}^{B}J_w V_w$$

where

$$^{B}J_w = {}^{w}J_B^{-1} = \begin{bmatrix} 0 & -\sqrt{3}/3 & \sqrt{3}/3 \\ -2/3 & 1/3 & 1/3 \\ 1/(3d) & 1/(3d) & 1/(3d) \end{bmatrix}.$$

The kinematic model is

$$\begin{aligned} \dot{X} &= XU \\ \mathrm{vrp}(U) &= {}^{B}J_w V_w. \end{aligned} \tag{3.1}$$

with $X \in \mathrm{SE}(2)$ the state of the system.

3.2 Unmanned Aerial Vehicle

3.2.1 Rigid Satellite

The configuration manifold of a satellite is SO(3). The kinematics equation of motion are given by (2.6)

$$\dot{R} = R\Omega_\times$$

with R the rotation matrix of the *body-fixed frame* with respect to an *inertial frame* attached to the Earth and Ω the body angular velocities. Let J denotes the constant inertia matrix around the center of mass of the satellite (expressed in the body fixed frame).

The Newton-Euler equations of motion yield the following dynamic model

$$J\dot{\Omega} = -\Omega_\times J\Omega + \Gamma^{\text{ext}} + \Gamma \tag{3.2}$$

where $\Gamma \in \mathbb{R}^3$ is a torque control input and $\Gamma^{\text{ext}} \in \mathbb{R}^3$ is the resultant torque acting on the body. The control input Γ is usually actuated by thrusters or momentum wheels, however we do not consider the particular choice of the actuators in this work.

For satellites operating at high altitude the disturbance torque Γ^{ext} is negligible.

For low-Earth orbit satellites (LEO), orbits below below 10^3 to $5 * 10^3$ km, the gravitational field produces disturbances that are not negligible

$$\Gamma\text{grav} = 3\frac{\mu}{\rho^3}(R^\top \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top)_\times JR^\top \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top \tag{3.3}$$

where μ is the gravitation constant of the Earth and ρ is the radius of the orbit. Also the atmosphere produces not negligible disturbances, the atmosphere affects the satellite by generating aerodynamic drag lift. Drag depends on the ballistic coefficient and on the atmospheric density. For a deep treatment of space mission the reader is referred to Larson and Wertz (2005), for spacecraft control see books devoted to the argument Markley and Crassidis (2014).

3.2.2 Fully Actuated Multicopter

The configuration manifold of an aerial vehicle is SE(3). Rigid body's kinematic equations of motion are described by (2.17)

$$\dot{X} = XU$$

with $X \in \mathrm{SE}(3)$ and $U \in \mathfrak{se}(3)$. Let m denotes the mass of the body and let J denotes the constant inertia matrix around the center of mass (expressed in the body fixed frame). Newton-Euler equations of motion for a fully actuated aerial vehicle with 6DOF yield the following dynamic model

$$J\dot{\Omega} = -\Omega_\times J\Omega + \Gamma^{\text{ext}} + \Gamma \tag{3.4a}$$
$$m\dot{V} = -m\Omega_\times V + F^{\text{ext}} + F \tag{3.4b}$$

where $\Gamma \in \mathbb{R}^3$ and $F \in \mathbb{R}^3$ represent respectively the torques and forces control inputs. F^{ext} and Γ^{ext} represent the resultant of the external disturbances acting on the rigid body.

We can express the dynamic equation of motion in a compact form

$$\begin{bmatrix} J & 0 \\ 0 & m\,I_3 \end{bmatrix} \begin{bmatrix} \dot{\Omega} \\ \dot{V} \end{bmatrix} = \begin{bmatrix} -\Omega_\times J & 0 \\ 0 & -m\Omega_\times \end{bmatrix} \begin{bmatrix} \Omega \\ V \end{bmatrix} + \begin{bmatrix} \Gamma^{\text{ext}} \\ F^{\text{ext}} \end{bmatrix} + \begin{bmatrix} \Gamma \\ F \end{bmatrix}.$$

Denoting $\bar{F}^{\text{ext}} = \text{col}(\Gamma^{\text{ext}}, F^{\text{ext}})$ and $\bar{F} = \text{col}(\Gamma, F)$

$$\bar{M} = \begin{bmatrix} J & 0 \\ 0 & m\,I_3 \end{bmatrix}, \quad C(\Omega, V) = \begin{bmatrix} -\Omega_\times J & 0 \\ 0 & -m\Omega_\times \end{bmatrix}$$

and recalling the *vectorial representation* of $\mathfrak{se}(3)$ one obtains

$$\bar{M}\,\text{vrp}(\dot{U}) = C(\Omega, V)\,\text{vrp}(U) + \bar{F}^{\text{ext}} + \bar{F}.$$

For the sake of simplicity in the design of the control laws that we are going to present in the next chapters, we assume that the dynamics of the actuators are faster than vehicle dynamics. Moreover, for the sake of generality we are not going to consider the particular choice of the actuators. The structure presented is the basic structure of many of the nonlinear tracking control algorithms developed for fully-actuated vehicles, see Hua et al. (2015b), Naldi et al. (2008), Ryll et al. (2015) and Rajappa et al. (2015).

3.3 Unmanned Underwater Vehicle

UUV's can navigate vast distances and collect scientific data (seafloor mapping, temperature mapping, salinity) without any human control in very extreme environments. In this section we are going do present the kinematic and dynamic model of a 6DOF underwater vehicle. For a detailed analysis of marine vehicles (modeling and control) the reader is referred to Fossen (2002). The configuration manifold of a rigid body moving into an incompressible, irrotational and inviscid fluid is the group of rigid displacements SE(3). The kinematic equations of motion are described by (2.17)

$$\dot{X} = XU$$

with $X \in \text{SE}(3)$ and $U \in \mathfrak{se}(3)$. Let M and J denote the mass and the inertia of the body-fluid system. For the added mass effect (mass of the body-fluid system) the reader is referred to Fossen (2002). Kirchhoff equations of motion for a fully actuated immersed underwater vehicle with 6DOF read as

$$J\dot{\Omega} = -\Omega_\times J\Omega - V_\times MV + \Gamma^{\text{ext}} + \Gamma \tag{3.5a}$$

$$M\dot{V} = -\Omega_\times MV + F^{\text{ext}} + F \tag{3.5b}$$

where $\Gamma \in \mathbb{R}^3$ and $F \in \mathbb{R}^3$ represent respectively the torques and forces control inputs. F^{ext} and Γ^{ext} represent the resultant of the external disturbances acting on the vehicle. Kirchhoff dynamic equation of motion can be expressed in compact form

$$\begin{bmatrix} J & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} \dot{\Omega} \\ \dot{V} \end{bmatrix} = \begin{bmatrix} -\Omega_\times J & -V_\times M \\ 0 & -\Omega_\times M \end{bmatrix} \begin{bmatrix} \Omega \\ V \end{bmatrix} + \begin{bmatrix} \Gamma^{\text{ext}} \\ F^{\text{ext}} \end{bmatrix} + \begin{bmatrix} \Gamma \\ F \end{bmatrix}.$$

Denoting $\bar{F}^{\text{ext}} = \text{col}(\Gamma^{\text{ext}}, F^{\text{ext}})$, $\bar{F} = \text{col}(\Gamma, F)$

$$\bar{M} = \begin{bmatrix} J & 0 \\ 0 & M \end{bmatrix}, \quad C(\Omega, V) = \begin{bmatrix} -\Omega_\times J & -V_\times M \\ 0 & -\Omega_\times M \end{bmatrix}$$

one obtains

$$\bar{M}\,\text{vrp}(\dot{U}) = C(\Omega, V)\,\text{vrp}(U) + \bar{F}^{\text{ext}} + \bar{F}.$$

"My only wish would be to have 10 more lives to live on this planet. If that were possible, I'd spend one lifetime each in embryology, genetics, physics, astronomy and geology. The other lifetimes would be as a pianist, backwoodsman, tennis player, or writer for the "National Geographic"."

Joseph Murray

4

Output Regulation For Systems On Matrix Lie Groups

The output regulation problem is one of the central problems in control theory. This problem deals with asymptotic tracking of a reference trajectory or asymptotic rejection of external disturbances. A key characteristic in the context of internal model-based control is to model references to be tracked or disturbances to be rejected as belonging to the set of all possible solutions generated by an autonomous system typically referred to as exosystem. The framework can be considered as trade-off between scenarios in which the reference trajectory is completely known and the ones in which it is totally unknown. For linear MIMO (multiple-input multiple-output) systems, the output regulation problem was completely characterized and solved in the mid seventies by the pioneering works of Francis (1977), Francis and Wonham (1976) and Davison (1976), leading to the internal model principle. In this context, the regulator that solves the problem incorporates in the feedback path a suitably reduplicated model of the exosystem. The linear framework has been then extended to a quite general nonlinear context by Isidori and Byrnes (1990). After the seminal paper of Isidori and Byrnes (1990) there has been considerable interest in the theory, see among others Marconi et al. (2001), Marconi et al. (2006) and Huang and Lin (1994). A breakthrough in the nonlinear output regulation problem happened in Byrnes and Isidori (2004), in which has been recognized that the problem of output regulation can be cast as a problem of nonlinear observers design. In this new perspective, a large amount of paper has been published (see, among others, Priscoli et al. (2006), Bin et al. (2016), Marino and Tomei (2011) and for a deep treatment of observers and internal model Astolfi (2016)). A wide number of the existing works on output regulation deals with systems and exosystem defined on Euclidean real state space and there is only a small amount of papers that consider the output regulation problem on more general manifold. Only recently, some effort has been done to extend the internal model principle to systems defined on matrix Lie group. In particular, Schmidt et al. (2012) and Schmidt et al. (2014) consider the output regulation problem for *left invariant* systems and *left invariant* exosystems defined on the *special orthogonal group* $\mathrm{SO}(n)$ and the *special Euclidean group* $\mathrm{SE}(n)$. In both works full error information are assumed to be available. For a comprehensive treatment of the argument see Schmidt (2014). The main differences of the work of Schmidt (2014) with the present work is that we are going to consider the output regulation problem for *left invariant* systems and *right invariant* exosystems on Lie groups in which only partial relative measurement are supposed to be known, this fact will be clear in the next sections. Moreover in this work we are going to present a general internal model-based design for systems defined on matrix Lie groups.

4.1 Lie Output Regulation: An Illustrative Example

In what follows we consider one of the most simple abelian Lie group, that is $\mathbb{R}^n$. Due to some similarity with matrix Lie group, the design on $\mathbb{R}^n$ will give us some hint for a generic design. Consider the following linear system on $\mathbb{R}^n$

$$\dot{X} = U \tag{4.1a}$$

$X \in \mathbb{R}^n$ the state of the system and $U \in \mathbb{R}^n$ the control input. We consider an exosystem of the form

$$\begin{aligned} \dot{X}_d &= Cw \\ \dot{w} &= Sw \end{aligned} \tag{4.2}$$

where $X_d \in \mathbb{R}^n$, $w \in R^m$, $C \in \mathbb{R}^{n\times m}$ and $S \in \mathbb{R}^{m\times m}$ with $S = -S^\top$ and the pair (S, C) observable. The input w models exogenus signals that represent velocity references to be tracked. Note that assuming S skew-symmetric is equivalent to the classical assumption of neutral stability of the exosystem. Indeed consider an exosystem of the form

$$\dot{w} = Aw$$

if all eigenvalues of A have zero negative real part and multiplicity one in the minimal polynomial then the system is neutrally stable. Moreover is well known that a matrix A with all eigenvalues with zero real part and multiplicity one in the minimal polynomial can be always expressed, in suitable coordinates, as a skew symmetric matrix S. We also assume that all the trajectories of the exosystem are bounded backward and forward in time. Note that usually the assumption on the eigenvalues of the matrix S implies that the exosystem is bounded forward an backward in time. However in this particular formulation we have added an additional step of integration and due to this fact the neutral stability of the subsystem $\dot{w} = Sw$ does not implies the boundness of the state of the whole exosystem.

We assume that the matrices C and S are known and only relative position measurements are available, namely

$$e = X_d - X. \tag{4.3}$$

In this framework, the control problem is the design of a feedback control action U as a function of e, in such a way the error e converges with a large domain of attraction.

Note that considering the change of variable $Z = \begin{bmatrix} e & U \end{bmatrix}^\top$ the problem can be cast as in the classical linear Internal model framework. However we are going to proceed with a design based on the Lyapunov direct method since the linear design will be not applicable on the more general case of systems posed on matrix Lie group.

Proposition 4.1. *Consider the system* (4.1a) *along with exosystem* (4.2) *and let the controller be given by*

$$\begin{aligned} U &= k_p e + C\delta \\ \dot{\delta} &= S\delta + k_I C^\top e \end{aligned} \tag{4.4}$$

with k_p, k_I some positive gains. Then the set

$$\mathcal{S} = \{(X, \delta, (X_d, w)) \in \mathbb{R}^n \times \mathbb{R}^m \times (\mathbb{R}^n \times \mathbb{R}^m) : X = X_d,\ \delta = w\} \tag{4.5}$$

is globally asymptotically stable for the closed-loop system.

Proof. Consider the following Lyapunov function

$$\mathcal{L}(e, \tilde{w}) = \frac{1}{2}\|e\|^2 + \frac{1}{2k_I}\|\tilde{w}\|^2$$

where $\tilde{w} = w - \delta$, which is positive definite and $\mathcal{L}(0, 0) = 0$. The derivative of the position error with respect to time is given by

$$\dot{e} \;=\; \dot{X}_d - \dot{X} = Cw - U.$$

Taking the derivatives along the solution of (4.1a) and (4.2) one obtains

$$\begin{aligned}\dot{\mathcal{L}} &= e^\top \dot{e} + \frac{1}{k_I}\tilde{w}^\top \dot{\tilde{w}} \\ &= e^\top (Cw - U) + \frac{1}{k_I}\tilde{w}^\top (Sw - \dot{\delta})\end{aligned}$$

and substituting U and $\dot{\delta}$ from (4.4) one has

$$\begin{aligned}\dot{\mathcal{L}} &= -k_p e^\top e + e^\top C\tilde{w} + \frac{1}{k_I}\tilde{w}^\top S\tilde{w} - \tilde{w}^\top C^\top e \\ &= -k_p e^\top e.\end{aligned}$$

Substituting (4.4) into the derivative of the error for $\dot{\mathcal{L}} = 0$ one has

$$\begin{aligned}\dot{e} &= 0 = & C\tilde{w} \\ \dot{\tilde{w}} &= 0 = & S\tilde{w}\end{aligned}$$

and this along with the observably condition of the pair (S, C) in turn implies $\tilde{w} = 0$. From this using classical LaSalle arguments, it is possible to conclude that the set $\mathcal{S}$ is globally asymptotically stable.

□

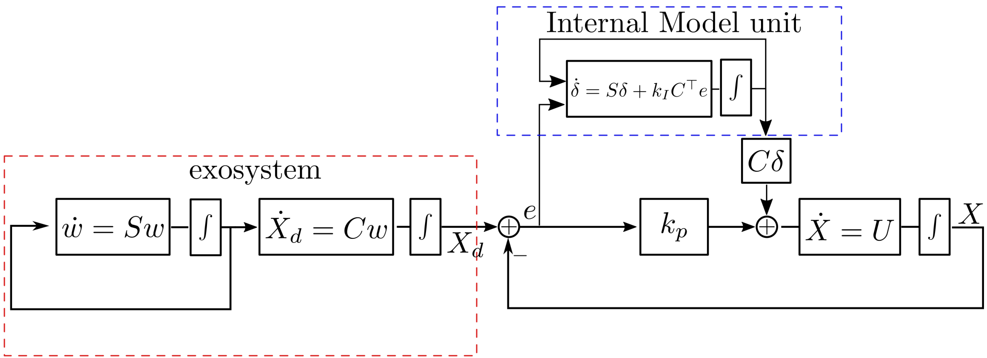

Figure 4.1: Block Diagram of the regulator in $\mathbb{R}^n$.

Note that, with the same spirit of the linear internal model principle, the control action U obtained is the superposition of a stabilizing unit and an internal model unit (see figure 4.1).

4.2 Kinematic Output Regulation For Systems on Matrix Lie Groups

As we have seen in the previous chapters many physical systems, such as aerial vehicles, mobile robotic vehicles and underwater vehicles can be described by geometric models with symmetries. Symmetric structures reflect the fact that the behavior of a symmetric system in one point is independent from the choice of a particular set of configuration coordinates. Preserving such a symmetry plays undoubtedly a key feature in the design of control and observer for mechanical systems with symmetries. Control of mechanical system has been intensively studied by Jurdjevic (1997) and Bullo and Lewis (2004). As shown by Byrnes and Isidori (2004) the output regulation problem is very close to the observer design problem. It is natural, then, to investigate and highlight the key principles in the design for invariant nonlinear observer. Aghannan and Rouchon (2003) first had pointed out the main role of *invariance* in the observer design. Recent works, based on the aforementioned paper, take advantage of the *left invariant* structure to define *invariant* error coordinates in order to build an *invariant* observer (see , among others, Bonnabel et al. (2008), Bonnabel et al. (2009), Hua et al. (2011), Trumpf et al. (2012), Lageman et al. (2009) and Lageman et al. (2010a)). As shown in Mahony et al. (2012b) and Khosravia et al. (2015) input measurements affected by bias lead to non-autonomus error dynamic, analogously the output regulation problem of this work will lead to an non-autonomous error system. Finally the last key concept to be highlighted is the construction of an *invariant cost function* on the output space. Properly chosen, these cost function give rise to non increasing Lyapunov function along the trajectories of the error system (see Khosravia et al. (2015), Mahony et al. (2013) and Maithripala and Berg (2014)). Part of the content of the present section has been accepted as brief paper for the Automatica journal (de Marco et al. (2016b)) and is under the revision process.

4.2.1 Problem formulation

In this section we consider a *left invariant* kinematic system of the form

$$\dot{X} = X(U + U_n), \quad X(0) \in \mathbf{G} \tag{4.6a}$$
$$U_n = \mathrm{mrp}\,(C_n w_n) \tag{4.6b}$$
$$\dot{w}_n = S_n w_n \tag{4.6c}$$

$X \in \mathbf{G}$ the state of the system , $U \in \mathfrak{g}$ the control input and where $U_n \in \mathfrak{g}$ represents velocity disturbances to be rejected. With $\mathrm{vrp}(U_n) \in \mathbb{R}^k$,$w_n \in \mathbb{R}^z$, $C_n \in \mathbb{R}^{k\times z}$, and $S_n \in \mathbb{R}^{z\times z}$ with $z \geq k$. We assume that $S_n = -S_n^\top$.

Reference trajectories to be tracked are generated by a right invariant system defined on the same Lie-Group $\mathbf{G}$ of the controlled system and driven by a linear oscillator defined on the Lie-algebra $\mathfrak{g}$ associated to the Lie group $\mathbf{G}$

$$\dot{X}_d = {}^\circ U_d X_d \tag{4.7a}$$
$${}^\circ U_d = \mathrm{mrp}(Cw) \tag{4.7b}$$
$$\dot{w} = Sw \tag{4.7c}$$

where $X_d \in \mathbf{G}$ and ${}^\circ U_d \in \mathfrak{g}$ are $n \times n$ matrices, $\mathrm{vrp}({}^\circ U_d) \in \mathbb{R}^\kappa$,$w \in \mathbb{R}^m$, $C \in \mathbb{R}^{\kappa\times m}$, and $S \in \mathbb{R}^{m\times m}$ with $m \geq \kappa$ and $S = -S^\top$. Note that the exosystem in (4.7) has a similar form of the exosystem presented in the illustrative example.

As we have seen in the previous chapters a natural choice to denote the state error as an element of the group $\mathbf{G}$ for systems defined on matrix Lie group is given by

$$E = X^{-1}X_d. \tag{4.8}$$

However we don not assume that the natural error is directly available for measurements since there aren't commercial or custom sensors capable of measuring that quantity. Instead, we assume that only partial relative geometrical information of the exosystem with respect to actual system is available for measurements. These measurements are assumed to be *invariant* and associated with a group action on a *homogeneous space* of the state space. In particular we consider a *linear left group* action (see Definition 1.19) of $\mathbf{G}$ on $\mathbb{R}^n$, $l(E, y) \mapsto Ey$ and reference vectors of the form

$$y_i = E\mathring{y}_i, \qquad i = 1, 2, ..., \nu$$

where $\mathring{y}_i$ are known constant reference vectors.

Note that the choice of a linear *left group action* is related to the fact that it is the natural action of SE(3) and SO(3) on R^4 and R^3, respectively. Moreover this choice simplifies the calculation in the derivation of the control law. However all results presented in this work would hold for a general *left group action*.

In this context we define an "error vector" $\mathring{e}_i$ by

$$\mathring{e}_i = \mathring{y}_i - E\mathring{y}_i.$$

If the goal is to maintain a certain relative "distance", for example in SE(3) in the formation flight problem, we can apply a constant "reference" $X_r \in \mathbf{G}$ to $\mathring{y}_i$ to generate constant reference vectors

$$y_i^r = X_r\mathring{y}_i.$$

Thus considering X_r one has

$$e_i = y_i^r - y_i = y_i^r - E_r y_i^r, \quad \text{where} \quad E_r = EX_r^{-1}\ .$$

The control problem considered is the design of a feedback control action U as a function of X and y_i, in such a way the error E_r converges to the identity element of the group and e_i converges to zero with a certain domain of attraction.

The control problem formulated above is solved under the assumption that there are at least a certain number of measurements and the exosystem state (X_d, U_d) is bounded, as formalized in the forthcoming assumptions.

Assumption 4.1. *There are sufficient independent measurements y_i with $i = 1, \ldots, \nu$ such that*

$$\ell(E) := \frac{1}{2}\sum_{i=1}^{\nu}\|e_i\|^2 = \frac{1}{2}\sum_{i=1}^{\nu}\|y_i^r - E_r y_i^r\|^2 \tag{4.9}$$

is locally positive definite in $E_r \in \mathbf{G}$ around the identity matrix $E_r = I$.

Assumption 4.2. *There exists a compact set $\mathcal{W}_d \subset \mathbf{G} \times \mathfrak{g}$ which is invariant for* (4.7).

The last assumption reflects the fact that in a real world scenario some trajectories will be forbidden even if they are generate by the neutrally stable subsystem $\dot{w} = Sw$, in an analogous way of the illustrative example. For example, on the *special Euclidean group* SE(3), due to this assumption, some trajectories such as constant linear trajectories or helical trajectories are forbidden since they will generate an unbounded exosystem state. Note that on the *special orthogonal group* SO(n) this assumption is automatically fulfilled since, as we have seen in Chapter 2, $SO(n)$ is a compact manifold.

Before moving to the next section we'd like to give a physical intuition of the measurements used in the present work. To this purpose consider two quadrotors as in figure 4.2. One of the quadrotor, the exosystem, is moving along a certain trajectory described by (4.7). The second vehicle, the controlled quadrotor, is measuring in its body-fixed frame its relative position with respect to the exosystem. In the example of figure 4.2 the measurements are taken by a stereo camera that observes some reference features on the exosystem quadrotor, represented in figure as optical markers. In this case a known vector $\mathring{y}_i$ is a reference feature on the exosystem quadrotor in its own frame of reference.

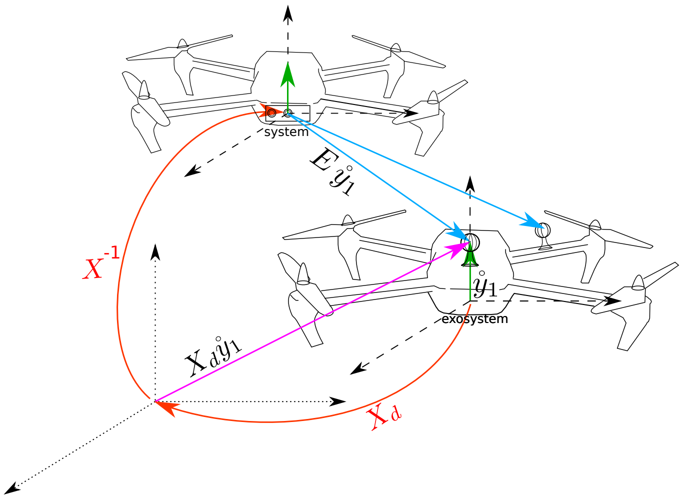

Figure 4.2: Reference vectors and error vectors in SE(3). The body fixed-frames are represented with dashed lines while the inertial reference frame is represented with dotted lines.(de Marco et al. (2016a))

The vector $X_d\mathring{y}_1$ represents the inertial coordinates of the point $\mathring{y}_1$, while $y_1 = E\mathring{y}_1$ are the coordinates of the marker $\mathring{y}_1$ in the body-fixed frame of the controlled quadrotor. Note that $y_1 = E\mathring{y}_1$ is exactly the relative measure of position taken by the onboard stereo camera.

The control goal then, is to steer the actual quadrotor in order to "follow" the exosystem quadrotor whit an orientation offset and position offset given by X_r.

4.2.2 Output regulation: Reference trajectory tracking

In this section we are going to present the structure of the regulator that solves the problem formulated in the previous section considering $U_n = 0$.

Theorem 4.1. *(de Marco et al. (2016b)) Consider system* (4.6a) *with* $U_n = 0$*, along with exosystem* (4.7)*. Let the controller be given by the control law*

$$U = Ad_{X^{-1}}\Delta - k_p \sum_{i=1}^{\nu} \mathbb{P}\left(e_i (y_i^r - e_i)^\top\right) \tag{4.10a}$$

$$\Delta = \mathrm{mrp}(C\delta) \tag{4.10b}$$

$$\dot{\delta} = S\delta + C^\top Q_{\mathfrak{g}}\, \mathrm{vrp}(\beta) \tag{4.10c}$$

$$\beta = -k_I \sum_{i=1}^{\nu} \mathbb{P}(X^{-\top} e_i (y_i^r - e_i)^\top X^\top) \tag{4.10d}$$

with k_p and k_I some positive gains. If Assumptions 4.1 and 4.2 hold then the compact set

$$\mathcal{S} = \{(X, \delta, (X_d, w)) \in \mathbf{G} \times \mathbb{R}^m \times \mathcal{W}_d \;\big|\; X^{-1}X_d = X_r,\ \delta = w\}$$

is locally asymptotically stable for the closed-loop system. Furthermore

$$(X, \delta, (X_d, w)) \in \mathcal{S} \Rightarrow e_i = 0 \quad \forall i = 1, \dots, \nu\,. \tag{4.11}$$

Proof. Condition (4.11) directly follows from the definition of the compact set $\mathcal{S}$ and from the definition of the error vectors e_i. We proceed by proving that the set $\mathcal{S}$ is locally asymptotically stable. Consider as candidate Lyapunov function the following function

$$\mathcal{L}(E, \tilde{w}) = \underbrace{\frac{1}{2}\sum_{i=1}^{\nu} \|e_i\|^2}_{\mathcal{L}_1} + \underbrace{\frac{1}{2k_I}\tilde{w}^\top \tilde{w}}_{\mathcal{L}_2} \tag{4.12}$$

where $\tilde{w} = w - \delta$. And note that by assumption (4.1) $\mathcal{L}$ is positive definite around $(E, \tilde{w}) = (I, 0)$ and $\mathcal{L}(I, 0) = 0$. Let's focus on the time derivative of the first term in the right hand side of the Lyapunov candidate

$$\dot{\mathcal{L}}_1 = \frac{1}{2}\sum_{i=1}^{\nu} \frac{d}{dt}\|y_i^r - E_r y_i^r\|^2 = -\sum_{i=1}^{\nu} e_i^\top \dot{E}_r y_i^r\,. \tag{4.13}$$

Recalling the expression of E_r in (4.8), it turns out that the derivatives along the solution of (4.6a) and (4.7) are given by

$$\begin{aligned}\dot{E}_r &= \frac{d}{dt}(X^{-1}X_dX_r^{-1}) \\ &= (X^{-1}(-\dot{X})X^{-1})X_dX_r^{-1} + X^{-1}\dot{X}_dX_r^{-1} \\ &= -X^{-1}XUX^{-1}X_dX_r^{-1} + X^{-1\circ}U_dX_dX_r^{-1} \\ &= -(UX^{-1} - X^{-1\circ}U_dXX^{-1})X_dX_r^{-1} \\ &= -(U - Ad_{X^{-1}}{}^\circ U_d)E_r.\end{aligned} \tag{4.14}$$

Substituting the time-derivative of E_r into $\dot{\mathcal{L}}_1$ one gets

$$\begin{aligned}\dot{\mathcal{L}}_1 &= \sum_{i=1}^{\nu} e_i^\top (U - Ad_{X^{-1}}{}^\circ U_d)E_r y_i^r \\ &= \sum_{i=1}^{\nu} e_i^\top (U - Ad_{X^{-1}}\tilde{\Delta} - Ad_{X^{-1}}\Delta)E_r y_i^r\end{aligned} \tag{4.15}$$

where $\tilde{\Delta}$ represents a velocity error in the Lie algebra $\mathfrak{g}$

$$\tilde{\Delta} = {}^\circ U_d - \Delta\,.$$

The i'th element of the equation above can be rewritten

$$\begin{aligned}e_i^\top \left(U - Ad_{X^{-1}}\tilde{\Delta} - Ad_{X^{-1}}\Delta\right) E_r y_i^r &= \operatorname{tr}\left(\left(U - Ad_{X^{-1}}\tilde{\Delta} - Ad_{X^{-1}}\Delta\right) E_r y_i^r e_i^\top\right) \\ &= \operatorname{tr}\left(\left(E_r y_i^r e_i^\top\right)^\top \left(U - Ad_{X^{-1}}\tilde{\Delta} - Ad_{X^{-1}}\Delta\right)^\top\right) \\ &= \operatorname{tr}\left(\left(U - Ad_{X^{-1}}\tilde{\Delta} - Ad_{X^{-1}}\Delta\right)^\top \left(E_r y_i^r e_i^\top\right)^\top\right).\end{aligned}$$

Introducing the projection $\mathbb{P}$ associated to the Lie algebra $\mathfrak{g}$ (see definition 2.10), one has

$$\begin{aligned}e_i^\top \left(U - Ad_{X^{-1}}\tilde{\Delta} - Ad_{X^{-1}}\Delta\right) E_r y_i^r = \operatorname{tr}&\left((U - Ad_{X^{-1}}\Delta)^\top \mathbb{P}\left(\left(E_r y_i^r e_i^\top\right)^\top\right)\right) \\ &- \operatorname{tr}\left(\tilde{\Delta}^\top \mathbb{P}\left(X^{-\top}\left(E_r y_i^r e_i^\top\right)^\top X^\top\right)\right).\end{aligned}$$

And substituting the above expression into $\dot{\mathcal{L}}_1$, it yields

$$\dot{\mathcal{L}}_1 = \sum_{i=1}^{\nu} \operatorname{tr}\left((U - Ad_{X^{-1}}\Delta)^\top \mathbb{P}\left(\left(E_r y_i^r e_i^\top\right)^\top\right)\right) - \sum_{i=1}^{\nu} \operatorname{tr}\left(\tilde{\Delta}^\top \mathbb{P}\left(X^{-\top}\left(E_r y_i^r e_i^\top\right)^\top X^\top\right)\right). \tag{4.16}$$

Now consider the time derivative of the second term of the Lyapunov candidate (4.12), one has

$$\dot{\mathcal{L}}_2 = \frac{1}{k_I}\tilde{w}^\top \dot{\tilde{w}} = \frac{1}{k_I}\tilde{w}^\top (Sw - \dot{\delta})$$

and, substituting $\dot{\delta}$ from (4.10c) into the above equation, one obtains

$$\frac{1}{k_I}\tilde{w}^\top (Sw - \dot{\delta}) = \frac{1}{k_I}\tilde{w}^\top S\tilde{w} - \frac{1}{k_I}\tilde{w}^\top C^\top Q_{\mathfrak{g}}\ \mathrm{vrp}(\beta)$$

where $Q_{\mathfrak{g}}$ is the duplication matrix. Recalling the fact that for a skew-symmetric matrix $B = -B^\top \in \mathbb{R}^{m\times m}$ and $x \in \mathbb{R}^m$

$$x^\top Bx = -x^\top Bx = 0$$

it yields

$$\begin{aligned}
\frac{1}{k_I}\tilde{w}^\top S\tilde{w} - \tfrac{1}{k_I}\tilde{w}^\top C^\top Q_{\mathfrak{g}}\ \mathrm{vrp}(\beta) &= -\frac{1}{k_I}(C\tilde{w})^\top Q_{\mathfrak{g}}\,\mathrm{vrp}(\beta) \\
&= -\frac{1}{k_I}\,\mathrm{vrp}^\top(\tilde{\Delta})Q_{\mathfrak{g}}\,\mathrm{vrp}(\beta) \\
&= -\frac{1}{k_I}tr(\tilde{\Delta}^\top\beta)
\end{aligned}$$

Bearing in mind the expression of $\dot{\mathcal{L}}_2$ and substituting β from (4.10d), one has

$$\dot{\mathcal{L}}_2 = \sum_{i=1}^{\nu} \mathrm{tr}\left(\tilde{\Delta}^\top \mathbb{P}\left(X^{-\top}\left(E_r y_i^r e_i^\top\right)^\top X^\top\right)\right). \tag{4.17}$$

Recalling that

$$\dot{\mathcal{L}} = \dot{\mathcal{L}}_1 + \dot{\mathcal{L}}_2$$

and substituting (4.16) and (4.17) into the equation above and introducing the expression of U (4.20a), it yields

$$\begin{aligned}
\dot{\mathcal{L}} &= \sum_{i=1}^{\nu} \mathrm{tr}\left((U - Ad_{X^{-1}}\Delta)^\top \mathbb{P}\left(\left(E_r y_i^r e_i^\top\right)^\top\right)\right) \\
&= -\,\mathrm{tr}\left(\left(\sum_{i=1}^{\nu} k_p \mathbb{P}\left(E_r y_i^r e_i^\top\right)^\top\right)\left(\sum_{i=1}^{\nu}\mathbb{P}\left(E_r y_i^r e_i^\top\right)^\top\right)^\top\right) \\
&= -k_p\left|\sum_{i=1}^{\nu}\mathbb{P}\left(\left(E_r y_i^r e_i^\top\right)^\top\right)\right|^2.
\end{aligned}$$

Since $\mathcal{L}$ is positive definite in the error state and since the exosystem state (X_d, w) lies in a compact set by assumption, it follows that the whole state is globally bounded and solutions exist for all time.

Let $\mathcal{I}$ be the largest invariant set for which the Lyapunov descend condition is zero. In what follows we are going to apply the LaSalle theorem showing that $\mathcal{S}$ is the largest invariant set in $\mathcal{I}$. To this end, let $\mathcal{A}_0$ denote the set

$$\mathcal{A}_0 = \{(X, \delta, (X_d, w)) \,:\, e_i = 0, \forall i = 1, \ldots, \nu\}\,.$$

It is easy to check that $\mathcal{A}_0$ is closed and $\mathcal{S} \subset \mathcal{A}_0$. Consider a set $\mathcal{A}_1 = \{\dot{\mathcal{L}} = 0\} - \mathcal{A}_0$ that contains the residual points in the state space for which $\dot{\mathcal{L}} = 0$, but which aren't in the set $\mathcal{A}_0$. It is straightforward to see that $\mathcal{A}_1$ is closed and disjoint from $\mathcal{A}_0$. And note that $\mathcal{I} \subset \mathcal{A}_0 \cap \mathcal{A}_1$. Let $\mathcal{I}_0 := \mathcal{I} \cap \mathcal{A}_0$ and note that $\mathcal{I}_0$ is invariant since $\mathcal{A}_0$ is disjoint from $\mathcal{A}_1$. Since $\mathcal{S} \subset \mathcal{I}_0$ is an invariant subset of $\mathcal{I}_0$ then $\mathcal{I}_0$ is not empty. Note that by assumption $\ell(E_r)$ is positive definite around $E_r = I$ and is identically zero on $\mathcal{I}_0$ hence $\dot{E}_r = 0$ on solution in $\mathcal{I}_0$, indeed substituting U of equation (4.10) into the time derivative of E_r one obtains

$$0 = \dot{E}_r = \left(Ad_{X^{-1}}\tilde{\Delta}\right)\,.$$

Hence $\tilde{\Delta} = 0$ on $\mathcal{I}_0$ and directly follows that $\tilde{w} = 0$ on $\mathcal{I}_0$. By construction $\mathcal{S} \subset \mathcal{I}_0$ since we have proved that $\mathcal{I}_0 \subset \mathcal{S}$ we get $\mathcal{I}_0 = \mathcal{S}$. Therefore the set $\mathcal{S}$ is locally asymptotically stable and this completes the proof. □

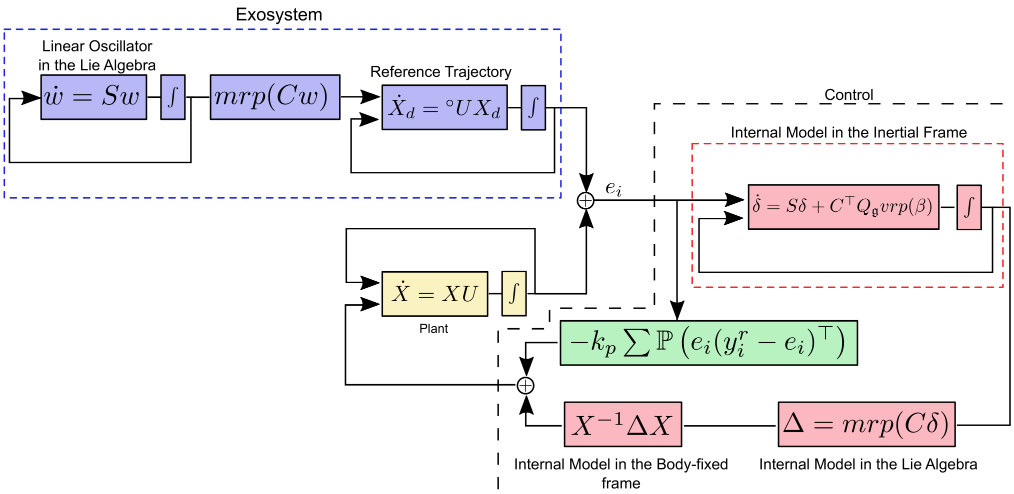

Figure 4.3: Block Diagram of the control law proposed in Theorem 4.1.

The regulator architecture, in the same spirit of the *linear Internal model*, contains a copy of the exosystem properly updated by means of error measurements. The control law proposed (see Figure 4.4) is composed by a stabilizing unit that maps the error e_i onto the tangent spaces $\mathrm{T}_I\mathbf{G}$ and an internal model unit that produces the control action in steady state.

Note that for the special case in which $^{\circ}\dot{U}_d = 0$, namely reference trajectories with constant velocities, the proposed control law (4.10) it's reduced to

$$U = Ad_{X^{-1}}\Delta - k_p \sum_{i=1}^{\nu} \mathbb{P}\left(e_i(y_i^r - e_i)^\top\right) \tag{4.18a}$$

$$\Delta = \mathrm{mrp}(\delta) \tag{4.18b}$$

$$\dot{\delta} = Q_{\mathfrak{g}}\,\mathrm{vrp}(\beta) \tag{4.18c}$$

$$\beta = -k_I \sum_{i=1}^{\nu} \mathbb{P}(X^{-\top} e_i (y_i^r - e_i)^\top X^\top) \tag{4.18d}$$

since for constant velocities one has that $S = 0$. Simple algebraic computation leads to

$$\begin{array}{rcl} U & = & Ad_{X^{-1}}\Delta - k_p \sum\limits_{i=1}^{\nu} \mathbb{P}\left(e_i(y_i^r - e_i)^\top\right) \\ \dot{\Delta} & = & -k_I \sum\limits_{i=1}^{\nu} \mathbb{P}(X^{-\top}(E_r y_i^r e_i^\top)^\top X^\top) \end{array} \tag{4.19}$$

that is exactly the *PI* control law for systems on matrix Lie group presented in Theorem 3.1 by Mahony et al. (2015).

4.2.3 Output regulation with disturbances rejection

Theorem 4.2. *Consider system* (4.6a)*, along with exosystem* (4.7)*. Let the controller be given by the control law*

$$
\begin{aligned}
U &= Ad_{X^{-1}}\Delta - \Delta_n - k_p \sum_{i=1}^{\nu} \mathbb{P}\left(e_i(y_i^r - e_i)^\top\right) && (4.20a)\\
\Delta &= \mathrm{mrp}(C\delta) && (4.20b)\\
\Delta_n &= \mathrm{mrp}(C_n\delta) && (4.20c)\\
\dot{\delta} &= S\delta + C^\top Q_{\mathfrak{g}}\,\mathrm{vrp}(\beta) && (4.20d)\\
\beta &= -k_I \sum_{i=1}^{\nu} \mathbb{P}(X^{-\top} e_i(y_i^r - e_i)^\top X^\top) && (4.20e)\\
\dot{\delta}_n &= S_n\delta_n + C_n^\top Q_{\mathfrak{g}}\,\mathrm{vrp}(\beta_n) && (4.20f)\\
\beta_n &= k_{nI} \sum_{i=1}^{\nu} \mathbb{P}\left(e_i(y_i^r - e_i)^\top\right) && (4.20g)
\end{aligned}
$$

with k_p, k_I and k_{nI} some positive gains. If Assumptions 4.1 and 4.2 hold then the compact set

$$\mathcal{S}_n = \{(X, w_n, \delta, \delta_n, (X_d, w)) \in \mathbf{G} \times \mathbb{R}^z \times \mathbb{R}^m \times \mathbb{R}^z \times \mathcal{W}_d \;\big|\; X^{-1}X_d = X_r,\ \delta = w,\ \delta_n = w_n\}$$

is locally asymptotically stable for the closed-loop system. Furthermore

$$(X, w_n, \delta, \delta_n, (X_d, w)) \in \mathcal{S}_n \Rightarrow e_i = 0 \quad \forall i = 1, \dots, \nu\,. \tag{4.21}$$

Proof. Consider as candidate Lyapunov function the following function

$$\mathcal{L}(E, \tilde{w}, \tilde{w}_n) = \underbrace{\frac{1}{2}\sum_{i=1}^{\nu} \|e_i\|^2}_{\mathcal{L}_1} + \underbrace{\frac{1}{2k_I}\tilde{w}^\top \tilde{w}}_{\mathcal{L}_2} + \underbrace{\frac{1}{2k_{nI}}\tilde{w}_n^\top \tilde{w}_n}_{\mathcal{L}_3} \tag{4.22}$$

where $\tilde{w} = w - \delta$ and $\tilde{w}_n = w_n - \delta_n$. It is straightforward to verify that the Lyapunov candidate is positive definite around $(E, \tilde{w}, \tilde{w}_n) = (I, 0, 0)$.

Proceeding in a similar way of the proof of theorem 4.1, the time derivative of the first term in the right hand side of the equation above is given by

$$\begin{aligned}\dot{\mathcal{L}}_1 \;=\; & \sum_{i=1}^{\nu} \operatorname{tr}\left((U + \Delta_n - Ad_{X^{-1}}\Delta)^\top \mathbb{P}\left(\left(E_r y_i^r e_i^\top\right)^\top \right)\right) \\ & -\sum_{i=1}^{\nu} \operatorname{tr}\left(\tilde{\Delta}^\top \mathbb{P}\left(X^{-\top}\left(E_r y_i^r e_i^\top\right)^\top X^\top\right)\right) + \sum_{i=1}^{\nu} \operatorname{tr}\left(\tilde{\Delta}_n^\top \mathbb{P}\left(\left(E_r y_i^r e_i^\top\right)^\top\right)\right).\end{aligned} \tag{4.23}$$

Note that $\mathcal{L}_2$ doesn't depend on the disturbances U_n, it follows that its time derivative is the same obtained in the proof of theorem 4.1. The time derivative of $\mathcal{L}_3$ is given by

$$\dot{\mathcal{L}}_3 = \frac{1}{k_{nI}} \tilde{w}_n^\top \dot{\tilde{w}}_n = \frac{1}{k_{nI}} \tilde{w}_n^\top (S_n w_n - \dot{\delta}_n)$$

and, substituting $\dot{\delta}_n$ from (4.20f) into the above equation, one obtains

$$\begin{aligned}\frac{1}{k_{nI}} \tilde{w}_n^\top (S_n w_n - \dot{\delta}_n) \;&=\; \frac{1}{k_I} \tilde{w}_n^\top S_n \tilde{w}_n - \frac{1}{k_I} \tilde{w}_n^\top C_n^\top Q_{\mathfrak{g}} \operatorname{vrp}(\beta_n) \\ &=\; -\frac{1}{k_{nI}} (C_n \tilde{w}_n)^\top Q_{\mathfrak{g}} \operatorname{vrp}(\beta_n) = -\frac{1}{k_{nI}} tr(\tilde{\Delta}_n^\top \beta_n).\end{aligned}$$

Introducing the expression of U, β and β_n from (4.20) into $\dot{\mathcal{L}}$ it yields

$$\dot{\mathcal{L}} = -k_p \left| \sum_{i=1}^{\nu} \mathbb{P}\left(\left(E_r y_i^r e_i^\top\right)^\top \right) \right|^2 .$$

From this, using similar LaSalle arguments exploited in Theorem 4.1, it is possible to conclude that the set S_n is locally asymptotically stable. And this concludes the proof. □

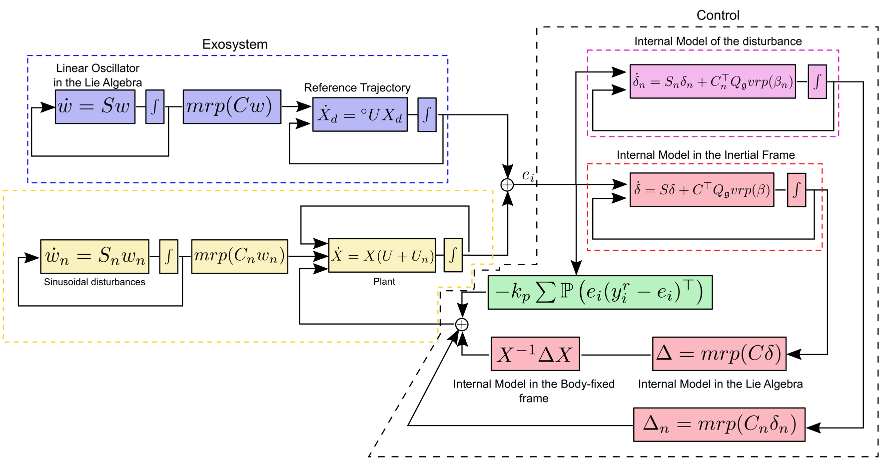

Figure 4.4: Block Diagram of the control law proposed in Theorem 4.2.

The control laws proposed in this chapter render the sets $\mathcal{S}, \mathcal{S}_n$ locally asymptotically stable. Exploiting the particular structure of the Lie group considered it is possible to extend the local properties of the control law (4.10) to almost global ones. In the next two chapters we are going to study the stability properties of the control architecture (4.10) for the particular case of systems posed on SO(3) and SE(3).

"Mathematics is a game played according to certain simple rules with meaningless marks on paper."

David Hilbert

5

Output Regulation for Systems on $\mathrm{SO}(3)$

In this chapter we provide a comprehensive stability analysis for the special case of system posed on the *special orthogonal group* $\mathrm{SO}(3)$, extending the local properties of the control law proposed in the previous chapter to almost global ones. Going further we present the particular case of a rigid body modeled as a dynamic system whose control input is a torque $\Gamma \in \mathbb{R}^3$ instead of a velocity input. In order to take into account also the dynamic of the system a backstepping procedure is developed. The content of this chapter is based on de Marco et al. (2016b).

Specializing the notation of the previous chapter and considering the Euler-Lagrange equations of motion of a rigid body derived in Chapter 3 yields to

$$\dot{R} = R\Omega_\times \tag{5.1a}$$

$$J\dot{\Omega} = -\Omega \wedge J\Omega + \Gamma^{\text{ext}} + \Gamma. \tag{5.1b}$$

Note that in this context the angular velocity Ω is a state component of the system. In this framework the exosystem is described by

$$\begin{aligned} \dot{R}_d &= {}^\circ\Omega_{d\times} R_d \\ {}^\circ\Omega_{d\times} &= (Cw)_\times \\ \dot{w} &= Sw \end{aligned} \tag{5.2}$$

where R_d represents the desired orientation ($R_d = X_d \in \mathrm{SO}(3)$) and ${}^\circ\Omega_d$ is the desired angular velocity expressed in the inertial reference frame (${}^\circ\Omega_{d\times} = {}^\circ U_d$, ${}^\circ\Omega_d = \mathrm{vrp}({}^\circ U_d)$).

In order to deal with the new control input Γ we first design the controller of the kinematic system using Ω as virtual input, then taking advantages of backstepping techniques we present a regulator design for fully actuated mechanical systems on SO(3).

Before moving into the next section, we specialize Assumption 4.1 in the specific case of systems posed on SO(3) as follow

Assumption 5.1. *There are at least two non collinear directions y_i^r available for measurements such that the symmetric matrix*

$$Y = \frac{k_p}{2}\sum_{i=1}^{\nu} y_i^r {y_i^r}^\top$$

has three distinct eigenvalues.

Note that under the assumption above, the natural error E could be algebraically reconstructed with algorithms such as the TRIAD (Black (1964)). Although this algorithm is simple, in presence of noise in the measurements is not guaranteed that the reconstructed matrix is an element of the *special orthogonal group*. There are more sophisticated algorithm to cope with noise in the measurements, however they introduce non negligible over-head for the computations. Thus, in the present chapter we are not going to reconstruct algebraically the error E.

5.1 Kinematic Output Regulation on $\mathrm{SO}(3)$

The control law (4.10), for the particular case of systems posed on SO(3), can be rewritten (denoting $R_e := E_r$, $\Omega := \Omega^c$) as

$$\Omega^c_\times = Ad_{R^\top}\Delta_\times + \frac{k_p}{2}\sum_{i=1}^{\nu}(e_i \wedge y_i^r)_\times \tag{5.3a}$$

$$\Delta_\times = (C\delta)_\times \tag{5.3b}$$

$$\dot{\delta} = S\delta + C^\top Q_{\mathfrak{so}(3)}\beta \tag{5.3c}$$

$$\beta = \frac{k_I}{2} R \sum_{i=1}^{\nu} (e_i \wedge y_i^r) \,. \tag{5.3d}$$

The forthcoming proposition extends the local properties of Theorem 4.1 to almost global ones.

Proposition 5.1. *(de Marco et al. (2016b)) Consider the system* (5.1a) *along with exosystem* (5.2) *and let the controller be given by* (5.3). *Let Assumption 5.1 holds. Then the set*

$$\mathcal{S} = \{(R, \delta, (R_d, w)) \in \mathrm{SO}(3) \times \mathbb{R}^m \times (\mathrm{SO}(3) \times \mathbb{R}^m) \;:\; R^\top R_d = X_r,\; \delta = w\} \tag{5.4}$$

is almost globally asymptotically and locally exponentially stable for the closed-loop system.

Note that almost global stability is the best we can get on SO(3) with a smooth control action due to the well-known topological obstructions (see S. P. Bath (2000)) on the special orthogonal group SO(3).

Proof. In order to prove that the set $\mathcal{S}$ is almost globally asymptotically and locally exponentially stable for the closed-loop system we proceed similarly to the kinematic tracking problem in Chapter 2 (subsection 2.2.9).

Starting from the fact that Theorem 4.1 ensures the local attractiveness of the set $\mathcal{S}$ we need to prove the following three facts:

1. The dynamic of the group error for the closed-loop system has only four isolated equilibrium points $(R_e, \Delta) = (R_{ej}^*, 0)$, $j = 1, \dots, 4$.

2. The equilibrium point $(R_{e1}^*, \Delta) = (I_3, 0)$ is locally exponentially stable.

3. The three equilibria with $(R_{ej}^*, \Delta) \neq (I_3, 0)$, $j = 2, \dots, 4$ are unstable.

We proceed by showing that there are only four isolated equilibria for the closed-loop system. To this purpose consider the error dynamic for the closed-loop system

$$\dot{R}_e = \left(Ad_{R^\top} \tilde{\Delta}_\times + \frac{k_p}{2} \sum_{i=1}^{\nu} (R_e y_i^r \wedge y_i^r)_\times \right) R_e \,.$$

As we have seen from Theorem 4.1, $\dot{R}_e = 0$ implies $\tilde{\Delta} = 0$, which in turn implies

$$\begin{aligned} 0 &= \frac{k_p}{2} \sum_{i=1}^{\nu} (R_e^* y_i^r \wedge y_i^r)_\times \\ &= R_e^* \frac{k_p}{2} \sum_{i=1}^{\nu} y_i^r {y_i^r}^\top - \frac{k_p}{2} \sum_{i=1}^{\nu} y_i^r {y_i^r}^\top {R_e^*}^\top = R_e^* Y - Y {R_e^*}^\top . \end{aligned} \tag{5.5}$$

Proceeding like in the proof of the tracking example in Chapter 2 one has that $R_e^* Y = Y R_e^{*\top}$ implies that R_e^* is a symmetric matrix. As consequence there are only four possible values of R_e^* that satisfy eq. (5.5), they are

$$\begin{cases} R_{e1}^* = I_3 \\ R_{e2}^* = u_1 u_1^\top - u_2 u_2^\top - u_3 u_3^\top \\ R_{e3}^* = -u_1 u_1^\top + u_2 u_2^\top - u_3 u_3^\top \\ R_{e4}^* = -u_1 u_1^\top - u_2 u_2^\top + u_3 u_3^\top \end{cases}$$

where u_1, u_2, u_3 are the eigenvectors of the matrix Y associated to the eigenvalues $\lambda_1, \lambda_2, \lambda_3$, with

$$0 \leq \lambda_1 < \lambda_2 < \lambda_3$$

and this concludes the proof of item 1.

We continue the analysis showing that the set $\mathcal{S}$ is locally exponentially stable (item 2).

The error dynamic for the closed-loop system can be rewritten as

$$\begin{aligned} \dot{R}_e &= \left(Ad_{R^\top} \tilde{\Delta}_\times + \frac{k_p}{2} \sum_{i=1}^{\nu} \left(R_e y_i^r \wedge y_i^r \right)_\times \right) R_e \\ &= \left(R_e R_d^\top \tilde{\Delta} + \frac{k_p}{2} \sum_{i=1}^{\nu} \left(R_e y_i^r \wedge y_i^r \right) \right)_\times R_e \end{aligned}$$

In order to simplify the algebra and without loss of generality consider $X_r = I_3$, and $C = I_3$. The dynamics of the velocity error are given by

$$\dot{\tilde{\Delta}}_\times = \left(S\tilde{\Delta} + k_I \sum_{i=1}^{\nu} R_d R_e^\top \left(R_e y_i^r \wedge y_i^r \right) \right)_\times . \tag{5.6}$$

and denoting

$$\begin{aligned} \underline{\tilde{\Delta}} &= \underline{R}\tilde{\Delta} \\ \underline{\dot{R}} &= -\underline{R}S \end{aligned} \tag{5.7}$$

where $\underline{R} \in \mathrm{SO}(3)$, one gets

$$\begin{aligned} \dot{R}_e &= \left(R_e R_d^\top \underline{R}^\top \underline{\tilde{\Delta}} + \frac{k_p}{2} \sum_{i=1}^{\nu} \left(R_e y_i^r \wedge y_i^r \right) \right)_\times R_e \\ \dot{\underline{\tilde{\Delta}}} &= k_I \sum_{i=1}^{\nu} \underline{R} R_d R_e^\top \left(R_e y_i^r \wedge y_i^r \right) . \end{aligned} \tag{5.8}$$

To prove the local exponential stability of the equilibrium $(R_{e1}^*, 0)$ of system (5.8), it suffices to prove that the origin of the linearized system is uniformly asymptotically stable. Thus, we proceed by linearizing system (5.8). To this purpose consider a first order approximation $R_e = I + x_\times$ and $\underline{\tilde{\Delta}} = \theta$, with $x, \theta \in \mathbb{R}^3$, of equation (5.8) around the equilibrium point $(R^*_{e1}, 0)$. Then the first order approximation of (5.8) is given by

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{k_p}{2}\sum_{i=1}^{\nu} y_{i\times} y_{i\times} & R_d^\top \underline{R}^\top \\ -k_I \underline{R} R_d \sum_{i=1}^{\nu} y_{i\times} y_{i\times}^\top & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix}. \tag{5.9}$$

Note that the linear system obtained is a time-varying system. From this, the proof of the exponential stability of the linear time-varying (LTV) system follows from a direct application of Theorem 1 in Loria and Panteley (2002), which establishes sufficient conditions for the uniform exponential stability of the origin of a linear time-varying system having the following standard form

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \mathcal{A}(t) & \mathcal{B}(t)^\top \\ -\mathcal{C}(t) & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix}. \tag{5.10}$$

Note that (5.9) is in standard form with

$$\begin{aligned} \mathcal{A}(t) &= \frac{k_p}{2}\sum_{i=1}^{\nu} y_{i\times} y_{i\times}, \qquad \mathcal{B}(t) = \underline{R} R_d, \\ \mathcal{C}(t) &= k_I \underline{R} R_d \sum_{i=1}^{\nu} y_{i\times} y_{i\times}^\top. \end{aligned}$$

Now we verify the three assumption of Theorem 1 in Loria and Panteley (2002). First, the first assumption of this theorem is satisfied since from Theorem 4.1 one has that $|\mathcal{B}(t)|$ and $\left|\frac{\partial \mathcal{B}(t)}{\partial t}\right|$ remain bounded for all time. The second assumption of this theorem is also satisfied since the symmetric matrices $\mathcal{P} = k_I \sum_{i=1}^{\nu} y_{i\times} y_{i\times}^\top$ and $\mathcal{Q} = k_I^{-1} k_p \mathcal{P}^2$ satisfying the conditions $\mathcal{P}\mathcal{B}^\top = \mathcal{C}^\top$ and $-\mathcal{Q} = \mathcal{A}^\top \mathcal{P} + \mathcal{P}\mathcal{A} + \dot{\mathcal{P}}$ are constant and positive definite. Finally, we need to prove that the term $\mathcal{B}$ is uniformly persistently exciting. It is straightforward to verify that $\mathcal{B}$ is uniformly persistently exciting, indeed for any positive number ε there exists $T > 0$ such that

$$\int_t^{t+T} \mathcal{B}(\tau)\mathcal{B}(\tau)^\top d\tau = \int_t^{t+T} \underline{R} R_d R_d^\top \underline{R}^\top d\tau = T I_3 > \varepsilon I_3$$

for all $t \geq 0$. Thus, all conditions of Theorem 1 in Loria and Panteley (2002) are satisfied, which in turn implies that the origin of (5.9) is uniformly exponentially stable and this concludes the proof of Item 2.

Let us prove Item 3, namely the instability of three equilibria with $(R^*_{ej}, \Delta) \neq (I_3, 0)$, $j = 2, \ldots, 4$. To this purpose we can proceed with similar arguments of the instability proof of the tracking example. The proof is based on a direct application of the Chetaev's Theorem. To this end consider the first order approximation of (5.8), $R_e = R^*_{ej}(I_3 + x_\times)$, $\Delta = \theta$ with $x,\ y \in \mathbb{R}^3$ around an equilibrium point $(R^*_{ej}, 0)$. Neglecting high order terms one has

$$
\begin{aligned}
\dot{x}_j &= \Upsilon_j x_j + R_d^\top \underline{R}^\top \theta \\
\dot{\theta}_j &= 2\frac{k_I}{k_p} \underline{R} R_d \Upsilon_j x_j
\end{aligned}
\tag{5.11}
$$

where $\Upsilon_j := \frac{k_p}{2}\sum_{i=1}^{\nu} R^*_{ej} y_{i\times} R^*_{ej} y_{i\times}$, with $j = 2, \ldots, 4$. Now, consider the continuously differentiable functions

$$
\mathcal{V}_j(x_j, \theta_j) = \frac{k_I}{2k_p} x_j^\top \Upsilon_j^\top x_j - \frac{1}{4}\left|\theta_j\right|^2, \quad j = 2, \ldots, 4
$$

and for each index $j = 2, \ldots, 4$ and for an arbitrary small radius $r > 0$ define

$$
\mathcal{U}_{j,r} := \{(x_j, \theta_j)^\top \mid \mathcal{V}_j(x_j, \theta_j) > 0, |x_j, \theta_j| < r\}.
$$

In order to ensure that the three equilibria with $(R^*_{ej}, \Delta) \neq (I_3, 0)$, $j = 2, \ldots, 4$ are unstable we need to prove that the set $\mathcal{U}_{j,r}$ is non-empty for each index $j = 2, \ldots, 4$ and show that the matrix Υ_j is not singular and at least one of its eigenvalues is positive. To this end consider the characteristic polynomial of the matrix Υ_j for each $j = 2, \ldots, 4$

$$
\det(\Upsilon_j - \bar{\lambda} I_3) \quad = \quad \det(Y R^*_{ej} - k_p \sum_{i=1}^{\nu} y_i^\top R^*_{ej} y_i I_3 - \bar{\lambda} I_3) \tag{5.12}
$$

and decomposing the symmetric matrices Y, R^*_{ej} as $Y = R_q \lambda_q R_q^\top$ and $R^*_{ej} = R_q \bar{R}_j R_q^\top$ with $\lambda_q = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ and

$$
\begin{aligned}
&\bar{R}_2 = \mathrm{diag}(1, -1, -1), \qquad \bar{R}_3 = \mathrm{diag}(-1, 1, -1) \\
&\bar{R}_4 = \mathrm{diag}(-1, -1, 1).
\end{aligned}
$$

one has

$$
\begin{array}{rcl}
\det(\Upsilon_j - \bar{\lambda} I_3) & = & \det(R_q \lambda_q R_q^\top R_q \bar{R}_j R_q^\top - \mathrm{tr}\left(\lambda_q \bar{R}_j\right) I_3 - \bar{\lambda} I_3) \\
& = & \det(\lambda_q \bar{R}_j - \mathrm{tr}\left(\lambda_q \bar{R}_j\right) I_3 - \bar{\lambda} I_3) \\
& = & \det(\lambda_q \bar{R}_j - \mathrm{tr}\left(\lambda_q \bar{R}_j\right) I_3 - \bar{\lambda} I_3).
\end{array}
\tag{5.13}
$$

Hence, the eigenvalues of Υ_j for $j = 2, 3, 4$ are

$$
\begin{array}{rcl}
\mathrm{eig}(\Upsilon_2) & = & [\lambda_2 + \lambda_3; \lambda_3 - \lambda_1; \lambda_2 - \lambda_1]^\top \\
\mathrm{eig}(\Upsilon_3) & = & [\lambda_3 - \lambda_2; \lambda_3 + \lambda_1; \lambda_1 - \lambda_2]^\top \\
\mathrm{eig}(\Upsilon_4) & = & [\lambda_2 - \lambda_3; \lambda_1 - \lambda_3; \lambda_1 + \lambda_2]^\top.
\end{array}
$$

From this, considering Assumption 5.1 it is possible to conclude that the matrix Υ_j is not singular and at least one of its eigenvalues is positive for each $j = 2, 3, 4$, which in turn implies that the set $\mathcal{U}_{j,r}$ is non empty.

It remains to show that the derivatives with respect to time of $\mathcal{V}_j(x_j, \theta_j)$ are always positive. Consider the time-derivative of $\mathcal{V}_j(x_j, \theta_j)$, one has

$$
\dot{\mathcal{V}}_j(x_j, \theta_j) = \frac{k_I}{k_p} x_j^\top \Upsilon_j^\top \Upsilon_j x_j.
$$

and due to the fact that the matrix Υ_j is not singular for each $j = 2, 3, 4$ one verifies that $\Upsilon_j^\top \Upsilon_j > 0$, with $j = 2, \ldots, 4$, which in turn implies that $\dot{\mathcal{V}}_j$ is always positive for each $(x, \theta) \in \mathcal{U}_{j,r}$. Since all conditions of the Chetaev's Theorem are satisfied it follows that the origin of (5.11) is unstable for $j = 2, 3, 4$ and this completes the proof. □

5.2 Dynamic Output Regulation for fully actuated systems on SO(3)

In order to take into account the dynamics of the system and consider the torques Γ as control input starting with the virtual input Ω^c in (5.3a), a backstepping procedure is developed. Define a new velocity error

$$
\tilde{\Omega} = \Omega - \Omega^c
$$

that is the velocity error between the real angular velocity of the rigid body and the virtual velocity Ω^c. By backstepping the new velocity error $\tilde{\Omega}$ it turns out that the following control law

$$\Gamma = \Omega_\times J\Omega - \Gamma^{\text{ext}} - J\Omega_\times R^\top \Delta + JR^\top \dot{\Delta} + 2\sum_{i=1}^{\nu} (e_i \wedge y_i^r) \tag{5.14a}$$
$$+ Jk_p \sum_{i=1}^{\nu} y_{i\times}^r (y_i^r - e_i)_\times (\tilde{\Omega} + \alpha) - k_D \tilde{\Omega}$$

$$\Omega_\times^c = Ad_{R^\top} \Delta_\times + \frac{k_p}{2} \sum_{i=1}^{\nu} (e_i \wedge y_i^r)_\times \tag{5.14b}$$

$$\Delta_\times = (C\delta)_\times \tag{5.14c}$$

$$\dot{\delta} = S\delta + C^\top Q_{\mathfrak{so}(3)} \beta \tag{5.14d}$$

$$\beta = \frac{k_I}{2} \left(\sum_{i=1}^{\nu} \left(k_p R (y_i^r - e_i)_\times y_{i\times}^r J^\top \tilde{\Omega} + R (e_i \wedge y_i^r) \right) \right) \tag{5.14e}$$

with $\alpha = k_p \sum_{i=1}^{\nu} 0.5 (e_i \wedge y_i^r)$ and k_p, k_I, k_D some positive arbitrary gain, solves the dynamic control problem as stated in the forthcoming proposition.

Proposition 5.2. *(de Marco et al. (2016b)) Consider system* (5.1a), (5.1b) *along with exosystem* (5.2) *and let the controller be given by* (5.14). *Let Assumption 5.1 holds.Then the set*

$$\mathcal{S}_{bs} = \{((R, R_d), (\delta, w), (\Omega_\times, {}^\circ\Omega_{d\times})) :\in \mathrm{SO}(3)^2 \times \mathbb{R}^{2m} \times \mathfrak{so}(3)^2 : R^\top R_d = X_r, \delta = w\,, \Omega = {}^\circ\Omega_d\}$$

is almost globally asymptotically stable and locally exponentially stable for the closed-loop system.

Proof. Consider the following Lyapunov candidate

$$\mathcal{L}_{bs}(E, \tilde{w}, \tilde{\Omega}) = \frac{1}{2} \sum_{i=1}^{\nu} \|e_i\|^2 + \frac{1}{2k_I} \tilde{w}^\top \tilde{w} + \frac{1}{2} \tilde{\Omega}^\top J \tilde{\Omega} \tag{5.15}$$

that under Assumption 5.1 is definite positive respect to the set $\mathcal{S}_{bs}$ and $\mathcal{L}_{bs}(I, 0, 0) = 0$. Differentiating the Lyapunov function with respect to time and bearing in mind the expression of Ω^c in (5.14b), one obtains

$$\begin{aligned}\dot{\mathcal{L}}_{bs} &= -\frac{k_p}{4}\left|\sum_{i=1}^{\nu}\left(R_e y_i^r \wedge y_i^r\right)_\times\right|^2 + \frac{1}{k_I}\tilde{w}^\top\left(Sw-\dot{\delta}\right)\\&+\frac{1}{2}tr\left(\tilde{\Omega}_\times^\top\sum_{i=1}^{\nu}\left(R_e y_i^r \wedge y_i^r\right)_\times\right)\\&-\frac{1}{2}tr\left(\tilde{\Delta}_\times^\top Ad_R\sum_{i=1}^{\nu}\left(R_e y_i^r \wedge y_i^r\right)_\times\right)\\&+\tilde{\Omega}^\top\left(-\Omega_\times J\Omega+\Gamma^{\text{ext}}+\Gamma-J\dot{\Omega}^c\right).\end{aligned}$$

And differentiating Ω^c along the solution of the closed-loop system it yields

$$\begin{aligned}\dot{\Omega}^c &= -\Omega_\times R^\top\Delta + R^\top\dot{\Delta} + \frac{k_p}{2}\sum_{i=1}^{\nu} y_{i\times}^r\left(R_e y_i^r\right)_\times\left(\tilde{\Omega}+\alpha\right)\\&-\frac{k_p}{2}\sum_{i=1}^{\nu} y_{i\times}^r\left(R_e y_i^r\right)_\times R^\top\tilde{\Delta}.\end{aligned}$$

Substituting the expression of $\dot{\Omega}^c$ in the Lyapunov function and recalling the fact that, for any two vectors A and B, $\text{tr}(A_\times^\top B_\times) = 2A^\top B$, it yields

$$\begin{aligned}\dot{\mathcal{L}}_{bs} = &-\frac{k_p}{4}\sum_{i=1}^{\nu}\left|\left(R_e y_i^r \wedge y_i^r\right)_\times\right|^2 + \frac{1}{k_I}\tilde{w}^\top\left(Sw-\dot{\delta}\right)\\&+\tilde{\Omega}^\top\left(2\sum_{i=1}^{\nu}\left(R_e y_i^r \wedge y_i^r\right) - \Omega_\times J\Omega + \Gamma^{\text{ext}} + \Gamma + J\Omega_\times R^\top\Delta - JR^\top\dot{\Delta} - Jk_p\sum_{i=1}^{\nu} y_{i\times}^r\left(R_e y_i^r\right)_\times\left(\tilde{\Omega}+\alpha\right)\right)\\&+\tilde{\Delta}^\top\left(k_p\sum_{i=1}^{\nu} R\left(R_e y_i^r\right)_\times y_{i\times}^r J^\top\tilde{\Omega} - R\sum_{i=1}^{\nu}\left(R_e y_i^r \wedge y_i^r\right)\right).\end{aligned}$$

Introducing the expression of Γ (5.14a) and $\dot{\delta}$ (5.14d) in the above expression, one has

$$\begin{aligned}\dot{\mathcal{L}}_{bs} = &-\frac{k_p}{4}\sum_{i=1}^{\nu}\left|\left(R_e y_i^r \wedge y_i^r\right)_\times\right|^2 - k_D\tilde{\Omega}^2\\&+\tilde{\Delta}^\top\left(k_p\sum_{i=1}^{\nu} R\left(R_e y_i^r\right)_\times y_{i\times}^r J^\top\tilde{\Omega} - R\sum_{i=1}^{\nu}\left(R_e y_i^r \wedge y_i^r\right) - \frac{2}{k_I}\beta\right).\end{aligned}$$

Finally substituting β from (5.14e) in the equation above one obtains

$$\dot{\mathcal{L}}_{bs} = -\frac{k_p}{4}\sum_{i=1}^{\nu}\left|\left(R_e y_i^r \wedge y_i^r\right)_\times\right|^2 - k_D\tilde{\Omega}^2. \tag{5.16}$$

It follows that the compact set S_{bs} is stable in the sense of Lyapunov and that $\tilde{\Omega}$ converges to zero. The proof can be completed using similar arguments to Theorem 4.1. □

5.3 Practical Output Regulation for systems on $\mathrm{SO}(3)$

In this section we briefly discuss on the robustness of control law (5.14) with respect to uncertainties in the parameters. We first show that even in the simpler case of Section 4.1, the backstepping procedure introduces non robust feed-forward terms. To this purpose consider system (4.1a) completed with the dynamic equation of motion

$$\dot{X} = U \tag{5.17a}$$

$$J\dot{U} = \Gamma \tag{5.17b}$$

$X \in \mathbb{R}^3$ and $U \in \mathbb{R}^3$ the state of the system, $\Gamma \in \mathbb{R}^3$ the control input and $J = J^\top > 0 \in \mathbb{R}^{3\times 3}$. We consider an exosystem of the form

$$\begin{array}{rcl} \dot{X}_d &=& Cw \\ \dot{w} &=& Sw \end{array} \tag{5.18}$$

where $X_d \in \mathbb{R}^3$, $w \in R^n$, $C \in \mathbb{R}^{3\times n}$ and $S \in \mathbb{R}^{n\times n}$ with $S = -S^\top$ and the pair (C, S) observable.

Define $\tilde{U}$ as

$$\tilde{U} = U - U^c.$$

Consider the following control law

$$\Gamma = (I + k_p^2 J - JCC^\top)e - (k_d I + k_p J + k_p JCC^\top J^\top)\tilde{U} - JCS\delta \tag{5.19a}$$

$$U^c = k_p e + C\delta \tag{5.19b}$$

$$\dot{\delta} = S\delta + k_I C^\top e - k_p C^\top J^\top \tilde{U} \tag{5.19c}$$

with k_D a positive arbitrary gain. By backstepping $\tilde{U}$ it turns out that control law (5.19) solves the stabilization problem as stated in the following proposition.

Proposition 5.3. *Consider system* (5.17a), (5.17b) *along with exosystem* (5.18) *and let the controller be given by* (5.19). *Then the set*

$$\mathcal{S}_1 = \{((X, X_d), (\delta, w), (U, U^c)) :\in (\mathbb{R}^3\times\mathbb{R}^3)\times\mathbb{R}^{2n}\times(\mathbb{R}^3\times\mathbb{R}^3) : X = X_d, \delta = w, U = U^c\}$$

is globally asymptotically stable for the closed-loop system.

Proof. Consider the following Lyapunov function

$$\mathcal{L}_{bs}(e, \tilde{w}, \tilde{U}) = \frac{1}{2}\|e\|^2 + \frac{1}{2k_I}\|\tilde{w}\|^2 + \frac{1}{2}\tilde{U}^\top J\tilde{U}$$

Differentiating $\mathcal{L}_{bs}$ along the solutions of the closed-loop system one has

$$\dot{\mathcal{L}}_{bs} \;=\; e^\top(Cw - \tilde{U} - U^c) + \frac{1}{k_I}\tilde{w}^\top(Sw - \dot{\delta}) + \tilde{U}^\top(\Gamma - J\dot{U}^c)$$

and bearing in mind the expression of U^c and substituting Γ, $\dot{\delta}$ from (5.20a), (5.20c) it yields

$$\dot{\mathcal{L}}_{bs} = -k_p e^\top e - k_d \tilde{U}^\top \tilde{U}.$$

From this using classical LaSalle arguments, it is possible to conclude that the set $\mathcal{S}_1$ is globally asymptotically stable. And this completes the proof. □

We show now that the control law obtained in (5.19) is not robust in the Francis and Wonham (1976) sense due to feed-forward terms used in the design of the control law. To this purpose we assume that the matrix J is uncertain and ranging over a given compact sect $\mathcal{P}$. We denote by J_0 its nominal value and with J its real one. Define

$$\tilde{J} = J - J_0$$

and consider the following Lyapunov function

$$\mathcal{L}_{bs}(e, \tilde{w}, \tilde{U}) = \frac{1}{2}\|e\|^2 + \frac{1}{2k_I}\|\tilde{w}\|^2 + \frac{1}{2}\tilde{U}^\top J \tilde{U}.$$

Differentiating $\mathcal{L}_{bs}$ along the solutions of the closed-loop system one has

$$\dot{\mathcal{L}}_{bs} \;=\; e^\top(Cw - \tilde{U} - U^c) + \frac{1}{k_I}\tilde{w}^\top(Sw - \dot{\delta}) + \tilde{U}^\top(\Gamma - J\dot{U}^c)$$

and choosing

$$\Gamma = (I + k_p^2 J_0 - J_0 CC^\top)e - (k_d I + k_p J_0 + k_p J_0 CC^\top J_0^\top)\tilde{U} - J_0 CS\delta \tag{5.20a}$$

$$U^c = k_p e + C\delta \tag{5.20b}$$

$$\dot{\delta} = S\delta + k_I C^\top e - k_p C^\top J_0^\top \tilde{U} \tag{5.20c}$$

one obtains

$$\dot{\mathcal{L}}_{bs} = -k_p e^\top e - k_d \tilde{U}^\top \tilde{U} + \tilde{U}^\top \tilde{J}(k_p I + CC^\top J_0)\tilde{U} + \tilde{U}^\top \tilde{J}(k_p^2 I - CC^\top)e - \tilde{U}\tilde{J}k_p\tilde{w} - \tilde{U}^\top \tilde{J}CS\delta.$$

Note that the last term in the right-hand side of the equation above depends on the state of the internal model, and this is one of the major issues related to the robustness (in the Wonham sense) of the proposed control law. Since the system considered is linear, it is straightforward to verify that the control law (5.20) with a suitable choice of k_p, k_d is practically robust with respect to parameter uncertainties.

Now we show that also for the SO(3) case the backstepping procedure prevents the robustness (in the Wonham sense) of control law (5.14). To this end consider the Lyapunov candidate function in (5.15)

$$\mathcal{L}_{bs}(E,\tilde{w},\tilde{\Omega}) = \frac{1}{2}\sum_{i=1}^{\nu}\|e_i\|^2 + \frac{1}{2k_I}\tilde{w}^\top\tilde{w} + \frac{1}{2}\tilde{\Omega}^\top J\tilde{\Omega}.$$

Differentiating $\mathcal{L}_{bs}$ along the solutions of the closed-loop system and recalling the expression of Γ, $\dot{\delta}$ and β in (5.14), it yields

$$\begin{aligned}
\dot{\mathcal{L}}_{bs} \;=\; & -\frac{k_p}{4}\left|\sum_{i=1}^{\nu}\left(R_e y_i^r \wedge y_i^r\right)_\times\right|^2 - k_D\tilde{\Omega}^2 \\
& +\tilde{\Omega}^\top\left(-\Omega_\times\tilde{J}\Omega + \tilde{J}\Omega_\times R^\top\Delta - \tilde{J}R^\top\dot{\Delta} - \tilde{J}k_p\sum_{i=1}^{\nu}y_{i\times}^r\left(y_i^r - e_i\right)_\times(\tilde{\Omega}+\alpha)\right) \\
& +\tilde{\Delta}^\top k_p\sum_{i=1}^{\nu}R\left(R_e y_i^r\right)_\times y_{i\times}^r\tilde{J}^\top\tilde{\Omega}.
\end{aligned}$$

That is exactly what we have found for the linear case. However, it is possible to verify that the control law (5.14) is robust respect to small variation in the inertia. Indeed local asymptotic stability implies local ISS, for suitable restriction on inputs and initial condition (see Lemma I.1 Sontag and Wang (1996)). This result is stated for system defined on the euclidean space, however it can be adapted to systems on manifold due its local nature.

It would be nice to prove an ISS property without restriction on inputs and initial condition for the closed-loop system obtained by backstepping, however it is well known that, due to topological obstruction (S. P. Bath (2000)), smooth continuous state feedback on smooth manifold (non homeomorfic to $\mathbb{R}^n$) will always lead to trajectories that do not converge to the origin. As consequence on non-Euclidean spaces global stabilization with a smooth vector field is not possible. Due to this fact the almost global stability concept was introduced. In this framework an equilibrium is said to be "Almost Globally Stable" if all trajectories converge asymptotically to the equilibrium point, except for a set of zero Lebesque measure. Recently Rantzer (2001) has recognized that the Lyapunov second method admits a dual method, based on density function, that naturally leads to almost global convergence results. Since global stability is a necessary condition for ISS, the new concept of Almost global Input-to-State Stability was introduced by Angeli (2004). In this work the combination of Lyapunov method with density function is suggested in order to provide almost ISS. Indeed almost ISS is obtained, as sketched by means of examples (Angeli (2004)), as the combination of local ISS (related to Lyapunov method, ultimate bound) with the concept of Weakly almost ISS (related to the density function method). The AISS property of the proposed control law is still an open problem and can be considered as future direction of work.

"Mathematics is the cheapest science. Unlike physics or chemistry, it does not require any expensive equipment. All one needs for mathematics is a pencil and paper."

George Pólya

6

Output Regulation for Systems on SE(3)

In this Chapter we are going to study the stability properties of the control architecture proposed in (4.1) for the special case of systems posed on SE(3). Due to the fact that stabilizing a point in SE(3) implies to stabilize a point on SO(3), and due to the topological obstructions discussed in the previous Chapter, the best one can gets with a smooth control action on SE(3) is almost global stability of the origin of the closed-loop system. Taking advantages of the specific structure of the *special Euclidean group* we extend the local results of (4.1) to almost global ones. Going further we also present a regulator design, based on backstepping techniques, for fully actuated dynamical mechanical systems whose kinematic space is defined on the *special Euclidean group* SE(3). As done in the previous Chapter we specialize Assumption 4.1 for the specific case of systems posed on SE(3).

Assumption 6.1. *There are at least three linearly independent measurement y_i ($i = 1, \dots, \nu$; $\nu \geq 3$) and the matrix*

$$Y := k_p \sum_{i=1}^{\nu} \underline{y}_i^r \underline{y}_i^{r\top} - \frac{k_p}{\nu} \sum_{i=1}^{\nu} \underline{y}_i^r \sum_{i=1}^{\nu} \underline{y}_i^{r\top}$$

has three distinct eigenvalues.

Where the vector $\underline{y} \in \mathbb{R}^3$ denotes the first three components of the vector y, *i.e.* $y = \begin{bmatrix}\underline{y} & 1\end{bmatrix}^\top$.

It is possible to verify that three is the minimum number of independent measurements such that the cost $\ell(E)$ in (4.9) is definite positive around the origin of the group SE(3). The second part of Assumption 6.1 on the eigenvalues of the matrix Y is technical and needed for the stability analysis.

The present chapter is based on de Marco et al. (2016a).

6.1 Kinematic Output Regulation on $\mathrm{SE}(3)$

By taking advantages of the group structure of the *special Euclidean group*, it is possible to extend the local results of the control law proposed in (4.1) to almost global results as stated in the following proposition.

Proposition 6.1. *(de Marco et al. (2016a)) Consider the system* (4.6a) *along with exosystem* (4.7) *and let the controller be given by* (4.10). *Let assumptions 4.2 and 6.1 hold. Then the compact set*

$$\mathcal{S} = \{(X, \delta, (X_d, w)) \in \mathrm{SE}(3) \times \mathbb{R}^m \times \mathcal{W}_d : X^{-1}X_d = X_r,\ \delta = w\}$$

is almost globally asymptotically stable and locally exponentially stable for the closed-loop system.

For the sake of analysis purpose, let us introduce an equivalent system to the dynamics of the group error E_r.

Recalling the dynamics of the error (4.14) for the closed loop system

$$\dot{E}_r = \left(Ad_{X^{-1}}\tilde{\Delta} + k_p \sum_{i=1}^{\nu} \mathbb{P}\left(e_i y_i^{r\top} E_r^\top\right)\right) E_r \tag{6.1}$$

and decomposing the error and the velocity error in

$$E_r = \begin{bmatrix} R_e & p_e \\ 0 & 1 \end{bmatrix}, \quad \tilde{\Delta} = \begin{bmatrix} \tilde{\Delta}^\Omega_\times & \tilde{\Delta}^v \\ 0 & 0 \end{bmatrix}$$

with $R_e \in \mathrm{SO}(3)$, $\tilde{\Omega}_\times \in \mathfrak{so}(3)$, $p_e \in \mathbb{R}^3$ and $\tilde{v} \in \mathbb{R}^3$ one obtains

$$
\begin{aligned}
\dot{E}_r &= \begin{bmatrix} R^\top & -R^\top p \\ 0^\top & 1 \end{bmatrix} \begin{bmatrix} \tilde{\Delta}^\Omega_\times & \tilde{\Delta}^v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} E_r + \mathbb{P}\left(\begin{bmatrix} I_3 - R_e & -p_e \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Xi & \mu \\ \mu^\top & k_p\nu \end{bmatrix} \begin{bmatrix} R_e^\top & 0 \\ p_e^\top & 1 \end{bmatrix} \right) E_r \\
&= \begin{bmatrix} R^\top \tilde{\Delta}^\Omega_\times R & R^\top \tilde{\Delta}^\Omega_\times p + R^\top \tilde{\Delta}^v \\ 0^\top & 0 \end{bmatrix} E_r + \begin{bmatrix} \mathbb{P}_a(\Xi R_e + \mu p_e^\top) & (I_3 - R_e)\mu - k_p \nu p_e \\ 0 & 0 \end{bmatrix} E_r \\
&= \begin{bmatrix} {}^B\tilde{\Delta}^\Omega_\times + \Omega_{e\times} & {}^B\tilde{\Delta}^v + v_e \\ 0 & 0 \end{bmatrix} E_r
\end{aligned}
$$

with

$$
\begin{aligned}
\mu &:= k_p \textstyle\sum_{i=1}^{\nu} \underline{y}_i \\
\Xi &:= k_p \textstyle\sum_{i=1}^{\nu} \underline{y}_i \underline{y}_i^\top
\end{aligned}
$$

and

$$
\begin{aligned}
{}^B\tilde{\Delta}^\Omega_\times &:= R^\top \tilde{\Delta}^\Omega_\times R \\
\Omega_{e\times} &:= 0.5 \left(\Xi R_e^\top - R_e \Xi + \mu p_e^\top - p_e \mu^\top \right) \\
{}^B\tilde{\Delta}^v &:= R^\top \tilde{\Delta}^\Omega_\times p + R^\top \tilde{\Delta}^v \\
v_e &:= (I_3 - R_e)\mu - k_p \nu p_e \,.
\end{aligned}
$$

System (6.1) is equivalent to the following system

$$
\begin{aligned}
\dot{R}_e &= ({}^B\tilde{\Delta}^\Omega_\times + \Omega_{e\times}) R_e \\
\dot{p}_e &= ({}^B\tilde{\Delta}^\Omega_\times + \Omega_{e\times}) p_e + {}^B\tilde{\Delta}^v + v_e \,.
\end{aligned}
\tag{6.2}
$$

As consequence of Theorem 4.1, $\tilde{\Delta}^\Omega_\times$, $\tilde{\Delta}^v$, $\Omega_{e\times}$ and v_e converge to zero which implies that the equilibrium points of (6.2) are characterized by

$$
p_e^* = (k_p \nu)^{-1} (I_3 - R_e^*) \mu \tag{6.3}
$$

$$
Y {R_e^*}^\top = R_e^* Y. \tag{6.4}
$$

Proof. The proof is similar to the SO(3) case, in what follows we proceed by step showing that:

1. System (6.2) has only four isolated equilibrium points $(R_e, p_e, \tilde{\Delta}^\omega, \tilde{\Delta}^v) = (R^*_{ej}, p^*_{ej}, 0, 0)$, $j = 1, \ldots, 4$. The trajectories of the error $(R_e(t), p_e(t), \tilde{\Delta}^\omega(t), \tilde{\Delta}^v(t))$ converge to one of these equilibria for any initial condition $(R_e(0), p_e(0), \tilde{\Delta}^\omega(0), \tilde{\Delta}^v(0))$.
2. The equilibrium point $(R_e, p_e, \tilde{\Delta}^\omega, \tilde{\Delta}^v) = (I_3, 0, 0, 0)$ is locally exponentially stable.

3. The equilibria $(R^*_{ej}, p^*_{ej}, 0, 0)$ with $(R^*_{ej}, p^*_{ej}) \neq (I_3, 0)$ are unstable.

We prove now that the error system has only four isolated equilibrium points. From (6.4) and from Assumption 6.1, using the same arguments of item 1 in Proposition 5.1, it is possible to show that there are only four possible equilibria for the attitude error R_e

$$\begin{cases} R^*_{e1} = I_3 \\ R^*_{e2} = u_1 u_1^\top - u_2 u_2^\top - u_3 u_3^\top \\ R^*_{e3} = -u_1 u_1^\top + u_2 u_2^\top - u_3 u_3^\top \\ R^*_{e4} = -u_1 u_1^\top - u_2 u_2^\top + u_3 u_3^\top \end{cases}$$

where u_1, u_2, u_3 are the eigenvectors of Y associated to the eigenvalues λ_1, λ_2, λ_3, with

$$0 \leq \lambda_1 < \lambda_2 < \lambda_3 \, .$$

From this, one verifies that the corresponding equilibrium points for the position error p^*_e in (6.3) are uniquely defined for $R_e = R^*_{ei}$. Thus, the convergence of the trajectories for the closed-loop systems to one of the four equilibria is a direct consequence of the Lyapunov analysis of Theorem 4.1, and this concludes the proof of the first item.

We proceed by showing that the set $\mathcal{S}$ is locally exponentially stable. To this end we first algebraically manipulate the error system in order to split the error dynamics in term of attitude and position errors. Then, following the same approach for the SO(3) case we linearize the error system around its origin. In order to simplify the algebra and without loss of generality consider $X_r = I_4$ and $C = I_6$. Using the following change of variable

$$\bar{p}_e = -R_e^\top p_e$$

one has

$$\begin{aligned} \dot{\bar{p}}_e &= -\dot{R}_e^\top p_e - R_e^\top \dot{p}_e \\ &= -R_e^\top {}^B\tilde{\Delta}^v - R_e^\top v_e \, . \end{aligned} \tag{6.5}$$

The linear velocity estimation error in the body-fixed frame is given by

$$\begin{aligned} {}^B\tilde{\Delta}^v &= R^\top \tilde{\Delta}^\Omega_\times p + R^\top \tilde{\Delta}^v \\ &= R_e R_d^\top \tilde{\Delta}^\Omega_\times p + R_e R_d^\top \tilde{\Delta}^v \\ &= -R_e R_d^\top \tilde{\Delta}^\Omega_\times R p_e + R_e R_d^\top \tilde{\Delta}^v + R_e R_d^\top \tilde{\Delta}^\Omega_\times p_d \\ &= R_e \left[R_d^\top \tilde{\Delta}^\Omega_\times R_d \bar{p}_e + R_d^\top \tilde{\Delta}^v + R_d^\top \tilde{\Delta}^\Omega_\times p_d \right] . \end{aligned} \tag{6.6}$$

Substituting the expression of the linear velocity estimation error in the body-fixed frame (6.6) into the time-derivative of the position error (6.5) it yields

$$\dot{\bar{p}}_e = \bar{p}_{e\times} R_d^\top \tilde{\Delta}^\Omega + R_d^\top p_{d\times} \tilde{\Delta}^\Omega - R_d^\top \tilde{\Delta}^v - k_p \sum_{i=1}^{\nu} (R_e^\top - I)\underline{y}_i - k_p \nu \bar{p}_e \,. \tag{6.7}$$

And bearing in mind the dynamic of the velocity estimation error for the closed loop system and recalling that $Q_{\mathfrak{se}(3)} = \mathrm{diag}(2,2,2,1,1,1)$ one obtains

$$\mathrm{vrp}\left(\dot{\tilde{\Delta}}\right) = S\,\mathrm{vrp}\left(\tilde{\Delta}\right) + \frac{k_I}{k_p}\begin{bmatrix} 2vex\left[(R\Omega_e)_\times + \mathbb{P}_a(Rv_e p^\top)\right] \\ Rv_e \end{bmatrix}. \tag{6.8}$$

Denoting

$$\begin{aligned} \mathrm{vrp}\left(\underline{\tilde{\Delta}}\right) &= \underline{R}\,\mathrm{vrp}\left(\tilde{\Delta}\right) \\ \dot{\underline{R}} &= -\underline{R}S \end{aligned} \tag{6.9}$$

where $\underline{R} \in \mathrm{SO}(6)$. From (6.8) one deduces

$$\begin{aligned} \mathrm{vrp}\left(\dot{\underline{\tilde{\Delta}}}\right) &= \frac{k_I}{k_p}\underline{R}\begin{bmatrix} 2vex\left[(R\Omega_e)_\times + \mathbb{P}_a(Rv_e p^\top)\right] \\ Rv_e \end{bmatrix} \\ &= \frac{k_I}{k_p}\underline{R}\begin{bmatrix} 2R_d R_e^\top \Omega_e + (p_d + R_d\bar{p}_e) \wedge R_d R_e^\top v_e \\ R_d R_e^\top v_e \end{bmatrix}. \end{aligned} \tag{6.10}$$

Then, the error dynamic for the closed loop system considering the change of variable $\mathrm{vrp}\left(\underline{\tilde{\Delta}}\right) = \underline{R}\,\mathrm{vrp}\left(\tilde{\Delta}\right)$ can be written as follow

$$\dot{R}_e = \left[\begin{bmatrix} R_e R_d^\top & 0 \end{bmatrix} \underline{R}^\top \mathrm{vrp}\left(\underline{\tilde{\Delta}}\right) + \frac{k_p}{2}\sum_{i=1}^{\nu}\left(R_e \underline{y}_i \wedge \underline{y}_i\right) + \frac{k_p}{2}\sum_{i=1}^{\nu}\left(\underline{y}_i \wedge R_e \bar{p}_e\right)\right]_\times R_e \tag{6.11}$$

$$\dot{\bar{p}}_e = \begin{bmatrix} \bar{p}_{e\times} R_d^\top + R_d^\top p_{d\times} & -R_d^\top \end{bmatrix} \underline{R}^\top \mathrm{vrp}\left(\underline{\tilde{\Delta}}\right) - k_p \sum_{i=1}^{\nu} (R_e^\top - I)\underline{y}_i - k_p \nu \bar{p}_e. \tag{6.12}$$

We are ready to linearize the system around the equilibrium point $(R_{ej}^*, \bar{p}_{ej}^*, 0)$. To this purpose consider the following first order approximation $R_e = R_{ej}^*(I_3 + x_{1\times})$, $\bar{p}_e = x_2 + \bar{p}_{ej}^*$ and $\mathrm{vrp}(\underline{\Delta}) = \theta$, with $x_1, x_2 \in \mathbb{R}^3$ and $\theta \in \mathbb{R}^6$. Denoting by $x = [x_1, x_2]^\top$

and neglecting high order terms it yields

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -k_p F A_j & B_j \underline{R}^\top \\ -k_I \underline{R} B_j^\top A_j & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix}. \tag{6.13}$$

where $F = \mathrm{diag}(0.5, 0.5, 0.5, 1, 1, 1)$ and

$$A_j = \begin{bmatrix} \sum_{i=1}^{\nu} (R_{ej}^* \underline{y}_i)_\times (\underline{y}_i - \bar{p}_{ej}^*)_\times^\top & -\sum_{i=1}^{\nu} (R_{ej}^* \underline{y}_i)_\times \\ \sum_{i=1}^{\nu} (R_{ej}^* \underline{y}_i)_\times & \nu I_3 \end{bmatrix},$$

$$B_j = \begin{bmatrix} R_d^\top & 0_{3\times 3} \\ p_{ej\times}^* R_d^\top + R_d^\top p_{d\times} & -R_d^\top \end{bmatrix}.$$

We proceed by proving the exponential stability of the origin of linear time-varying (LTV) system for $j = 1$, $(R_e, p_e, \tilde{\Delta}^\omega, \tilde{\Delta}^v) = (I_3, 0, 0, 0)$. The proof is analogous to the SO(3) case for a system having the standard form in (5.10).

The first assumption of Theorem 1 in Loria and Panteley (2002) is satisfied since by Assumption 4.2 $|\mathcal{B}|$ and $\left|\frac{\partial \mathcal{B}}{\partial t}\right|$ remain bounded for all time t. Assumption 2 of this Theorem is also satisfied since the matrix $\mathcal{P} = k_I A_1$ and $\mathcal{Q} = 2k_p k_I A_1 F A_1$ satisfy the required relations $\mathcal{P}\mathcal{B}^\top = \mathcal{C}^\top$ and $-\mathcal{Q} = \mathcal{A}^\top \mathcal{P} + \mathcal{P}\mathcal{A} + \dot{\mathcal{P}}$ and is easy to verify that are symmetric and positive definite. Indeed consider the Schur complement Sh of νI_3 in A_1 one obtains

$$\begin{aligned} Sh &= \sum_{i=1}^{\nu} \underline{y}_{i\times} \underline{y}_{i\times}^\top - \frac{1}{\nu} \sum_{i=1}^{\nu} \underline{y}_{i\times} \sum_{i=1}^{\nu} \underline{y}_{i\times}^\top \\ &= \frac{1}{\nu} \sum_{i=1}^{\nu} \sum_{\kappa<i}^{\nu} (\underline{y}_{i\times} - \underline{y}_{\kappa\times})(\underline{y}_{i\times} - \underline{y}_{\kappa\times})^\top. \end{aligned} \tag{6.14}$$

From the assumption on the measurements (Assumption 6.1) on verifies that Sh is positive definite, since νI_3 is positive definite it follows that the whole matrix A_1 is positive definite, which in turn implies that $\mathcal{P}$ is positive definite. Consider the $\mathcal{Q}$ matrix one has

$$z^\top \mathcal{Q} z = 2k_p k_I z^\top A_1 F A_1 z = 2k_p k_I (A_1 z)^\top F (A_1 z) > 0$$

for $\forall z \in \mathbb{R}^6$. Due to the fact that p_d is bounded and $B_1 \underline{R}^\top$ is not singular, one has that the term $\mathcal{B}(t)\mathcal{B}(t)^\top$ is positive definite. From this, one can verifies that the term $\mathcal{B}(t)$ is persistently exciting. It is seen that all condition of Loria and Panteley (2002) are fulfilled hence the set $\mathcal{S}$ is locally exponentially stable, and this concludes the proof of item 2.

In what follows we proceed by proving that the equilibria $(R^*_{ej}, \bar{p}^*_{ej}, 0)$ with $j = 2, \ldots, 4$ are unstable. To this purpose consider the smooth functions

$$\mathcal{V}_j(x,\theta) = -\frac{k_I}{2} x^\top A_j^\top x - \frac{1}{2}\theta^\top \theta, \quad \text{with} \quad j = 2, \ldots, 4.$$

For an arbitrarily small radius $r > 0$ define the set

$$\mathcal{U}_{j,r} := \{(x,\theta)^\top \mid \mathcal{V}_j(x,\theta) > 0, |x,\theta| < r\}, \quad j = 2, \ldots, 4.$$

We will show afterward that the set $\mathcal{U}_{j,r}$ is nonempty for each $j \in \{2,3,4\}$. The derivatives of $\mathcal{V}_j$ along the trajectories of the system are given by

$$\dot{\mathcal{V}}_j(x,\theta) = k_p k_I \, x^\top A_j^\top F A_j x.$$

The demonstration concludes showing that the matrix A_j is not singular $\forall j \in \{2,3,4\}$, at least one of its eigenvalues is negative and that the derivatives of the functions $\mathcal{V}_j$ are positive in $\mathcal{U}_{j,r}$ for all $j \in \{2,3,4\}$.

Consider the first block of the block matrix A_j one has

$$\begin{aligned}
\sum_{i=1}^{\nu} (R^*_{ej}\underline{y}_i)_\times (\underline{y}_i - \bar{p}^*_{ej})_\times^\top &= \sum_{i=1}^{\nu} R^*_{ej}\underline{y}_{i\times} R^*_{ej} \left(\underline{y}_{i\times} - \frac{1}{\nu}\sum_{i=1}^{\nu}\underline{y}_{i\times} + \frac{1}{\nu} R^*_{ej} \sum_{i=1}^{\nu} \underline{y}_{i\times} R^*_{ej} \right)^\top \\
&= -\frac{1}{\nu}\sum_{i=1}^{\nu}\sum_{\kappa<i}^{\nu} R^*_{ej}\left(\underline{y}_{i\times} - \underline{y}_{\kappa\times}\right) R^*_{ej} \left(\underline{y}_{i\times} - \underline{y}_{\kappa\times}\right)^\top - \frac{1}{\nu} R^*_{ej} \sum_{i=1}^{\nu}\underline{y}_{i\times} \sum_{i=1}^{\nu}\underline{y}_{i\times} R^*_{ej} \\
&= \frac{1}{k_p} \operatorname{tr}\left(Y R^*_{ej}\right) I_3 - \frac{1}{k_p} Y R^*_{ej} - \frac{1}{\nu} R^*_{ej} \sum_{i=1}^{\nu}\underline{y}_{i\times} \sum_{i=1}^{\nu}\underline{y}_{i\times} R^*_{ej}.
\end{aligned}$$

Let sh_j be the Schur complement of νI_3 in A_j, one verifies

$$\det(sh_j - \lambda I_3) = \det(\operatorname{tr}\left(Y R^*_{ej}\right) I_3 - Y R^*_{ej} - \lambda I_3)$$

as consequence one has

$$\operatorname{eig}(\Upsilon_2) = [-\lambda_2 - \lambda_3; \lambda_1 - \lambda_3; \lambda_1 - \lambda_2]^\top$$

$$\operatorname{eig}(\Upsilon_3) = [\lambda_2 - \lambda_3; -\lambda_1 - \lambda_3; \lambda_2 - \lambda_1]^\top$$

$$\operatorname{eig}(\Upsilon_4) = [\lambda_3 - \lambda_2; \lambda_3 - \lambda_1; -\lambda_1 - \lambda_2]^\top.$$

It is straightforward to verify that under Assumption 6.1 at least one of the eigenvalues of the Schur complement is negative, since νI_3 is positive definite one concludes that the matrix A_j is indefinite for $j \in \{2,3,4\}$. The matrix A_j for $j \in \{2,3,4\}$ is not singular, indeed one gets

$$\det(A_j) = \det(\nu I_3)\det(sh_j) \neq 0 \qquad \forall j \in \{2,3,4\}.$$

Since A_j has at least one negative eigenvalue and it is not singular it follows that the set $\mathcal{U}_{j,r}$ is nonempty for each $j \in \{2,3,4\}$ and the functions $\dot{\mathcal{V}}_j$ are positive in $\mathcal{U}_{j,r}$. It is seen that all condition of the Chateav's theorem are fulfilled hence the origin of system (6.13) for $j \in \{2,3,4\}$ is unstable, and this completes the proof. □

Note that in SE(3) Assumption 4.2, namely the boundness of the exosystem state, is needed to the proof. Indeed the forward and backward invariance of the exosystem ensures that solutions exist for all time and are instrumental for the LaSalle arguments. It follows that trajectories like a screw trajectory are forbidden since they will cause an unbounded exosystem state. Anyhow, it is possible to render the set compact by periodically re-inizialize the inertial frame to a new certain position. Note that since the control law is based on the relative error between the controlled system and the exosystem, such a re-inizialization will not cause discontinuities in the control action. A full treatment of this problem is behind the scope of the present work.

6.2 Dynamic Output Regulation for fully actuated systems on SE(3)

In this section we solve the output regulation problem for fully actuated system whose kinematic is described by a left invariant vector field on SE(3). In this context the system is described by the non linear differential equations in Chapter 3.2 (subsection 3.2.2 and section 3.3). We proceed exactly like the dynamic output regulation case for fully actuated system on SO(3). To this purpose denote the kinematic velocity input obtained in the previous section as $U := U^c$. Denote by

$$\tilde{U}^c := U - U^c = \begin{bmatrix} \Omega_\times & V \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \Omega^c_\times & V^c \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \tilde{\Omega}^c_\times & \tilde{V}^c \\ 0 & 0 \end{bmatrix}. \tag{6.15}$$

the error between the actual velocity U of the system and the virtual velocity U^c it should has. Consider the following control law

$$\begin{array}{rcl} \Gamma & = & -k_D^\Omega \tilde{\Omega}^c + \Gamma_{ff} + \Gamma_J \\ F & = & -k_D^v \tilde{V}^c + F_{ff} + F_m \end{array} \tag{6.16}$$

where Γ_{ff} and F_{ff} represent feed-forward terms

$$\begin{aligned}\Gamma_{ff} &= \Omega_\times J\Omega - \Gamma^{\text{ext}} + \sum_{i=1}^{\nu} \underline{e}_i \wedge \underline{y}_i^r \\ F_{ff} &= m\Omega_\times V - F^{\text{ext}} - \sum_{i=1}^{\nu} \underline{e}_i\end{aligned}$$

and Γ_J, F_m are terms that depend on the inertia tensor and on the mass of the vehicle

$$\begin{aligned}\Gamma_J &= 0.5J\left[k_p\sum_{i=1}^{\nu}\underline{y}_{i\times}^r(\alpha-\tilde{\Omega}^c)_\times(\underline{y}_i^r-\underline{e}_i)+k_p\sum_{i=1}^{\nu}\underline{y}_{i\times}^r\left(k_p\sum_{i=1}^{\nu}\underline{e}_i+\tilde{V}^c\right)-2\Omega_\times R^\top\Delta^\Omega+2R^\top\dot{\Delta}^\Omega\right] \\ F_m &= m\left[k_p(\alpha-\tilde{\Omega}^c)\sum_{i=1}^{\nu}(\underline{y}_i^r-\underline{e}_i)+k_p\nu(k_p\sum_{i=1}^{\nu}\underline{e}_i+\tilde{V}^c)+{}^B\dot{\Delta}^v\right].\end{aligned}$$

with $\alpha = -k_p\sum_{i=1}^{\nu} e_i^r \wedge y_i^r$ and k_D^Ω, k_D^v some positive gains. Along with the internal model

$$\begin{aligned}\Delta &= \text{mrp}(C\delta) \\ \dot{\delta} &= S\delta + C^\top Q\begin{bmatrix}\beta^\Omega+\beta^\Omega_{\Omega^c}+\beta^\Omega_{V^c} \\ \beta^V+\beta^V_{\Omega^c}+\beta^V_{V^c}\end{bmatrix}\end{aligned}$$

feed by means of the following terms

$$\begin{aligned}\beta^\Omega &= -k_I R\sum_{i=1}^{\nu}\underline{e}_i\wedge\underline{y}_i+\frac{1}{2}p\wedge R\sum_{i=1}^{\nu}\underline{e}_i \\ \beta^\Omega_{\Omega^c} &= \frac{k_I}{4}\left[k_pR\sum_{i=1}^{\nu}(\underline{y}_i^r-\underline{e}_i)_\times y_{i\times}-p_\times R\sum_{i=1}^{\nu}y_{i\times}\right]J^\top\tilde{\Omega}^c \\ \beta^\Omega_{V^c} &= m\left[k_pR\sum_{i=1}^{\nu}(\underline{e}_i-\underline{y}_i^r)+k_p\nu p_\times R\right]\tilde{V}^c \\ \beta^V &= k_IR\sum_{i=1}^{\nu}\underline{e}_i \\ \beta^V_{\Omega^c} &= \frac{k_Ik_p}{2}R\sum_{i=1}^{\nu}\underline{y}_{i\times}^rJ^\top\tilde{\Omega}^c \\ \beta^V_{V^c} &= -mk_p\nu R\tilde{V}.\end{aligned}$$

By backstepping U^c it turns out that the control law above solves the dynamic output regulation problem as stated in the following proposition.

Proposition 6.2. *(de Marco et al. (2016a)) Consider system* (4.6a), (3.4), *along with exosystem* (4.7) *and let the controller be given by* (6.16). *Let Assumption 4.2 and 6.1 hold. Then the* *set*

$$\mathcal{S} = \{((X, X_d), (\delta, w), (U, {}^{\circ}U_d)) :\in \mathrm{SE}(3)^2 \times \mathbb{R}^{2m} \times \mathfrak{se}(3)^2 : X^{-1}X_d = X_r, \delta = w\,, U = {}^{\circ}U_d\}$$

is almost globally asymptotically stable and locally exponentially stable for the closed-loop system.

The proof is omitted since it is similar to the backstepping procedure in SO(3) and it is very computational heavy without adding much more insights to the problem.

"If you can' t solve a problem, then there is an easier problem you can solve: find it."

George Pólya

7

Simulative Examples

THE present Chapter is dedicated to some examples in order to validate numerically the theory presented in the previous chapters. In particular we are going to simulate as illustrative example the output regulation problem for an *omnidirectional* wheeled robot and the attitude control problem for a fully actuated satellite.

7.1 Control of an Omnidirectional Wheeled Robot

In this section we consider the kinematic output regulation problem of an *omnidirectional* wheeled robot. We recall the kinematic model derived in (3.1)

$$\begin{aligned} \dot{X} &= X(U + U_n) \\ \mathrm{vrp}(U) &= {}^{B}J_w V_w. \\ U_n &= \mathrm{mrp}(C_n w_n) \\ \dot{w}_n &= S_n w_n \end{aligned} \tag{7.1}$$

where U_n is a velocity disturbance to be rejected. The matrices S_n and C_n are chosen to be

$$S_n = \begin{bmatrix} 0 & 71 & 0 & 0 & 0 & 0 \\ -71 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 100 & 0 & 0 \\ 0 & 0 & -100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 50 \\ 0 & 0 & 0 & 0 & -50 & 0 \end{bmatrix}, \qquad C_n \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Reference directions considered in the simulations are $y_1 = [1, 0, 1]^\top$ and $y_2 = [0, 0.5, 1]^\top$. It is straightforward to verify that two is the minimum number of independent measurements in SE(2) and the cross product between the two reference directions is not null. Moreover the matrix

$$Y := k_p \sum_{i=1}^{\nu} \underline{y}_i^r \underline{y}_i^{r\top} - \frac{k_p}{\nu} \sum_{i=1}^{\nu} \underline{y}_i^r \sum_{i=1}^{\nu} \underline{y}_i^{r\top} = \begin{bmatrix} 0 & -k_p/2 \\ -k_p/2 & 0 \end{bmatrix}$$

has two distinct eigenvalues, thus Assumption 6.1 for the particular case of systems posed on SE(2) is fulfilled. The initial state of the simulated system is chosen as $X(0) = I_3$ with initial zero velocity $U(0) = 0$. In this context the exosystem read as

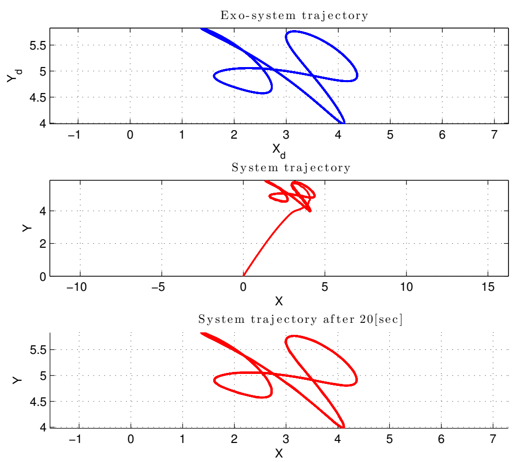

Figure 7.1: System and exosystem trajectories along the x and y axis

$$\dot{X}_d = {}^\circ U_d X_d \tag{7.2a}$$
$$ {}^\circ U_d = \mathrm{mrp}(Cw) \tag{7.2b}$$
$$\dot{w} = Sw \tag{7.2c}$$

with

$$S = \begin{bmatrix} 0 & 3 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & -2 & 0 \end{bmatrix}, \qquad C \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The simulated exosystem starts with $\pi/2[rad]$ as initial yaw and a relative distance from the inertial frame of $3[m]$ along the x axis and $5[m]$ along the y axis. In this framework

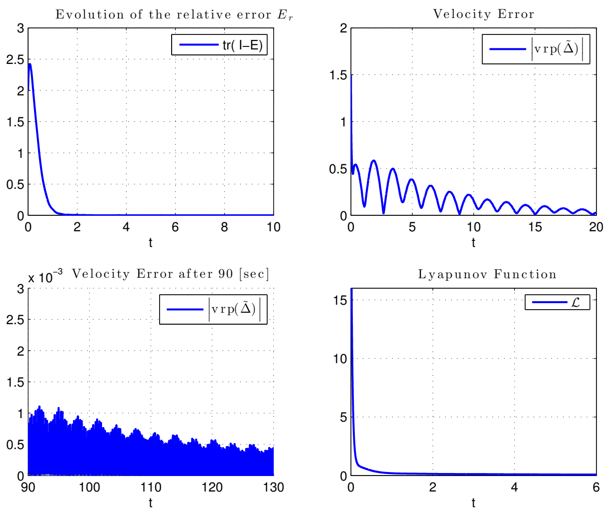

Figure 7.2: The time behavior of the norm of the closed-loop state components.

the control law that solves the output regulation problem is on the form

$$V_w = {}^B J_w^{-1}\, \mathrm{vrp}(U) \tag{7.3a}$$

$$U = Ad_{X^{-1}}\Delta - \Delta_n - k_p \sum_{i=1}^{2} \mathbb{P}_{\mathfrak{se}(2)}\left(e_i(y_i^r - e_i)^\top\right) \tag{7.3b}$$

$$\Delta = \mathrm{mrp}(C\delta) \tag{7.3c}$$

$$\Delta_n = \mathrm{mrp}(C_n\delta) \tag{7.3d}$$

$$\dot{\delta} = S\delta + C^\top Q_{\mathfrak{se}(2)}\, \mathrm{vrp}(\beta) \tag{7.3e}$$

$$\beta = -k_I \sum_{i=1}^{2} \mathbb{P}_{\mathfrak{se}(2)}(X^{-\top} e_i(y_i^r - e_i)^\top X^\top) \tag{7.3f}$$

$$\dot{\delta}_n = S_n\delta_n + C_n^\top Q_{\mathfrak{se}(2)}\, \mathrm{vrp}(\beta_n) \tag{7.3g}$$

$$\beta_n = k_{nI} \sum_{i=1}^{2} \mathbb{P}_{\mathfrak{se}(2)} \left(e_i (y_i^r - e_i)^\top \right). \tag{7.3h}$$

The controller gains are chosen as $k_p = 8$ and $k_I = k_{In} = 3$. Figure 7.2 shows the evolution of the $\mathrm{tr}(I - E)$, the norm of the estimation error $\left|\mathrm{vrp}(\tilde{\Delta})\right|$ an the time behavior of the Lyapunov function. Plot shows that in steady state the relative error converges to the identity element of the group (origin) and the velocity estimation Δ in the fixed frame converges to the desired velocity ${}^{\circ}U_d$.

7.2 Attitude Control of a rigid Satellite

In this section we consider the attitude control problem for a fully actuated satellite. The equations of motion of the rigid body where given in section 3.2 Eq. (3.2), while the control law is a direct application of the control law presented in Proposition 6.2. Reference direction are chosen to be $y_1 = [1, 0, 0]^\top$ and $y_2 = [0, 1.5, 0]^\top$, it is straightforward to verify that the two reference directions considered are not collinear moreover the matrix

$$Y = \frac{k_p}{2} \sum_{i=1}^{\nu} y_i^r y_i^{r\top} = \begin{bmatrix} k_p/2 & 0 & 0 \\ 0 & 9k_p/8 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

has three distinct eigenvalues, thus Assumption 5.1 is fulfilled. Initial states of the sys-

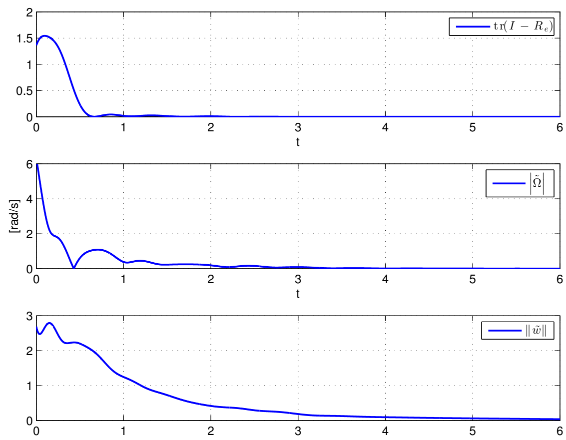

Figure 7.3: Evolution of the relative Error R_e, velocity error $\tilde{\Omega}$ and estimation error $\tilde{w}$ in SO(3). Case of perfect knowledge of the inertia matrix.

tem are chosen as $R(0) = I_3$ and initial zero angular velocity $\Omega(0) = 0$, while the control gains are $k_p = 4$, $k_d = 6$ and $k_I = 0.4$. The inertia tensor of the Satellite in the *body-fixed* *frame* is that of an non-axisymmetric rigid body $J = \mathrm{diag}(2, 1, 1.5)[Kg\, m^2]$.

The matrices of the linear oscillator in the Lie algebra of the exosystem are chosen to be

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & -5 & 0 \end{bmatrix}, \qquad C \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The initial yaw, pitch and roll of the desired trajectory are chosen to be $\pi/4[rad]$, $\pi/4[rad]$ and $\pi/3[rad]$, respectively. Figure 7.3 shows that in steady state the relative attitude error R_e converges to the identity element of the group, the angular velocity Ω of the satellite converges to the virtual velocity Ω^c and the norm of the estimation error $\tilde{w}$ converges to zero. We have run a second simulation considering a slight unknown variation in the

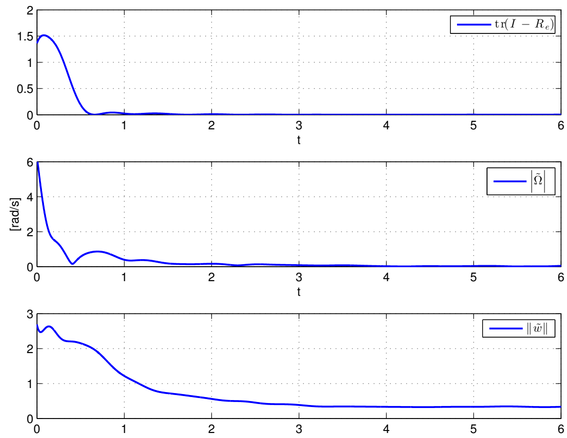

Figure 7.4: Evolution of the relative Error R_e, velocity error $\tilde{\Omega}$ and estimation error $\tilde{w}$ in SO(3). Case of imperfect knowledge of the inertia matrix.

inertia of the satellite. The inertia tensor implemented in the control law is $J_{\text{nom}} = \mathrm{diag}(2, 1, 1.5)[Kg\, m^2]$ while the real inertia of the system is $J_{\text{real}} = 0.85 J_{\text{nom}}$. Figure 7.4 and Figure 7.5 confirm, as seen in section 5.3, that only practical regulation can be achieved in presence of uncertainties in the dynamic parameters of the system.

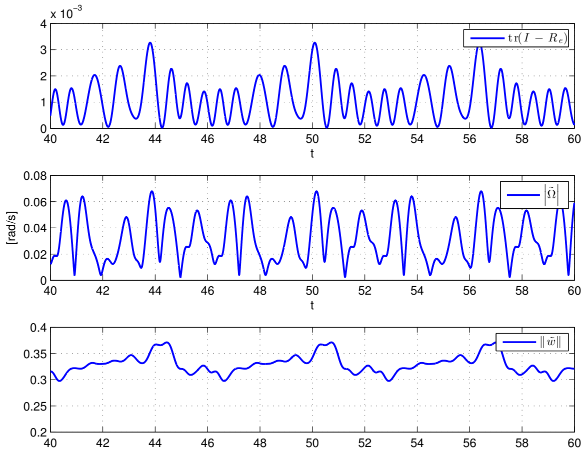

Figure 7.5: Evolution of the relative Error R_e, velocity error $\tilde{\Omega}$ and estimation error $\tilde{w}$ in SO(3) after 40 seconds of the simulation. Case of imperfect knowledge of the inertia matrix.

Conclusion

The output regulation problem for *left invariant* systems on matrix Lie group has been investigated. Taking advantages of the *symmetries* and *invariant* structures of the system considered, a novel *internal model* based design has been proposed. With the same spirit of the linear *internal model* principle, the proposed control architecture embeds a copy of the exosystem properly updated by means of partial *invariant* error measurements. Exploiting the particular structures of the *special orthogonal group* and the *special Euclidean group* the local properties of the control law presented in Chapter 4 has been extended to almost global ones in Chapter 5 and 6, respectively. For the particular case of systems posed on $\mathrm{SO}(3)$ and $\mathrm{SE}(3)$ the kinematic output regulation problem has been extended in Chapter 5 and 6, considering also the dynamics of the systems. The dynamic extension of the proposed control approach has been handled with classical *backstepping* techniques. In the same chapters the problem of robustness with respect to uncertainties in the system parameters has been considered.

Many problems investigated in this thesis are still open. First of all, it would be interesting to relax Assumption 4.2, since as pointed out a screw motion or a constant linear trajectory in $\mathrm{SE}(3)$ are forbidden. This problem can been addressed considering an hybrid analysis whit periodic resets of the coordinate of the *fixed frame* or considering $\mathrm{SE}(3)$ as the semi-direct product $\mathrm{SO}(3) \ltimes \mathbb{R}^3$. By considering $\mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$ it is possible to circumvent one of the major problematic in the inversion of the matrix X in the *adjoint action*.

Another open problem of major interest is the construction of a robust control law in the Wonham sense. Indeed as shown in Chapter 5 and 6 only practical regulation with restriction on initial state and input can be achieved with the presented control law. One of the major problem for the robustness of the control law proposed is related to the fact that a linear oscillator in the Lie algebra of a *right invariant* systems becomes a time varying oscillator in the body-fixed frame. Indeed the desired velocity in the body-fixed frame is $U = X_d^{-1} {}^\circ U X_d$. In order to know the frequency involved one would like to express the entries of the matrix X_d in closed form, however this can be done in very particular cases and it is strictly depended to the Lie group considered. Due to this fact classical *immersion assumptions* are really difficult to fulfill and even in very simple cases it turns out that one needs an infinite dimensional linear system to solve the problem. For example in the attitude control problem, in order to express each entry of the desired attitude R_d in closed form one could consider the linear time-varying system associated to (5.2). Then it is possible to express the state transition $\Upsilon(t, t_0)$ matrix of the equivalent linear time-varying system in closed form. To this purpose define

$$x := \mathrm{vec}(R_d)$$

where $\mathrm{vec}(R_d)$ is the column vector obtained by the concatenation of columns of the matrix R_d

$$\mathrm{vec}(R_d) = [r_{11}, r_{21}, r_{31}, r_{12}, r_{22}, r_{32}, r_{13}, r_{23}, r_{33}]^\top .$$

Exosystem (5.2), then, is equivalent to the following-time varying system

$$\dot{x} = A(t)x \tag{7.4}$$

where $A(t) := \mathrm{diag}({}^\circ\Omega_{d\times}, {}^\circ\Omega_{d\times}, {}^\circ\Omega_{d\times})$.

The *i-th* entry of the matrix ${}^\circ\Omega_{d\times}$, considering (5.2), is of the form

$${}^\circ\Omega_{di}(t) = a_{i0} + \sum_{k=1}^{n_i} (a_{ik} \sin(w_{ik}t) + b_{ik} \cos(w_{ik}t))$$

as consequence the time varying matrix $A(t)$ can be written as

$$A(t) = \sum_{i=1}^{3} \left[a_{i0} + \sum_{k=1}^{n_i} (a_{ik} \sin(w_{ik}t) + b_{ik} \cos(w_{ik}t)) \right] Q_i \tag{7.5}$$

where $Q_i = \mathrm{diag}(\bar{Q}_i, \bar{Q}_i, \bar{Q}_i)$, and

$$\bar{Q}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \bar{Q}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \bar{Q}_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The state transition matrix of a linear time-varying system can not, in general, be expressed in closed form. Closed form solutions are known only for a restricted class of systems, such as for example the commutative class. To the author's knowledge the class of systems (7.4), (7.5) in which

$$Q_iQ_j = (Q_jQ_i)^\top \quad \forall i,j = 1,2,3 \quad \text{with } i \neq j$$

has never been studied and a closed form solution is still unknown. Consider now the special case in which the exosystem is oscillating along one axis, for example along the x axis. The time varying matrix $A(t)$, then, can be written as

$$A(t) = \left[a_{10} + \sum_{k=1}^{n_1} (a_{1k} \sin(w_{1k}t) + b_{1k} \cos(w_{1k}t))\right] Q_1$$

and note that

$$A(t_1)A(t_2) = A(t_2)A(t_1).$$

The Peano-Baker series for the solution of the state transition matrix is given by

$$\begin{aligned}\Upsilon(t,t_0) &= I + \textstyle\int_{t_0}^t A(\tau)d\tau + \int_{t_0}^t A(\tau_1)\int_{t_0}^{\tau_1} A(\tau_2)d\tau_2 d\tau_1 \\ &\quad + \textstyle\int_{t_0}^t A(\tau_1)\int_{t_0}^{\tau_1} A(\tau_2)\int_{t_0}^{\tau_2} A(\tau_3)d\tau_3 d\tau_2 d\tau_1 + \ldots\end{aligned}$$

and since $A(t)\int_{t_0}^t A(\tau)d\tau = \int_{t_0}^t A(\tau)d\tau A(t)$ one has

$$\begin{aligned}\Upsilon(t,t_0) &= I + \textstyle\int_{t_0}^t A(\tau)d\tau + \int_{t_0}^t \left[\int_{t_0}^{\tau_1} A(\tau_2)d\tau_2 A(\tau_1)\right] d\tau_1 \\ &\quad + \textstyle\int_{t_0}^t \left[\int_{t_0}^{\tau_1}\left[\int_{t_0}^{\tau_2} A(\tau_3)d\tau_3 A(\tau_2)\right] d\tau_2 A(\tau_1)\right] d\tau_1 + \ldots \\ &= I + \textstyle\int_{t_0}^t A(\tau)d\tau + \dfrac{1}{2}\left[\int_{t_0}^t A(\tau)\right]^2 + \dfrac{1}{2}\dfrac{1}{3}\left[\int_{t_0}^t A(\tau)\right]^3 + \ldots + \dfrac{1}{k!}\left[\int_{t_0}^t A(\tau)\right]^k + \ldots \\ &= \exp\left[\textstyle\int_{t_0}^t A(\tau)d\tau\right] \\ &= \exp\left[a_{10}Q_1(t-t_0)\right]\textstyle\prod_{k=1}^{n_1}\exp\left[\int_{t_0}^t \left[a_{1k}\sin(w_{1k}\tau) + b_{1k}\cos(w_{1k}\tau)\right] d\tau Q_1\right].\end{aligned}$$

Denoting by $\lambda_k := \int_{t_0}^t \left[a_{1k}\sin(w_{1k}\tau) + b_{1k}\cos(w_{1k}\tau)\right] d\tau$ and considering the particular structure of the matrix Q_1 it yields

$$\Upsilon(t,t_0) = \exp\left[a_{10}Q_1(t-t_0)\right]\textstyle\prod_{k=1}^{n_1}\Theta_{\lambda_k}(t,t_0)$$

where $\Theta_{\lambda_k}(t,t_0) = \operatorname{diag}\left[\bar{\Theta}_{\lambda_k}(t,t_0), \bar{\Theta}_{\lambda_k}(t,t_0), \bar{\Theta}_{\lambda_k}(t,t_0)\right]$ and

$$\bar{\Theta}_{\lambda_k}(t,t_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\lambda_k) & -\sin(\lambda_k) \\ 0 & \sin(\lambda_k) & \cos(\lambda_k) \end{bmatrix}.$$

Consider now the case of a single linear oscillator in the Lie algebra $\mathfrak{so}(3)$

$$\dot{x}(t) = b_1 \cos(w_1 t) Q_1 x$$

one obtains

$$\Upsilon(t,0) \quad = \quad \Theta_{\lambda_1}(t,0)$$

where

$$\bar{\Theta}_{\lambda_1}(t,0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\frac{b_1}{w_1}\sin(w_1 t)) & -\sin(\frac{b_1}{w_1}\sin(w_1 t)) \\ 0 & \sin(\frac{b_1}{w_1}\sin(w_1 t)) & \cos(\frac{b_1}{w_1}\sin(w_1 t)) \end{bmatrix}.$$

Considering the Jacobi-Anger expansion of $\cos(\frac{b_1}{w_1}\sin(w_1 t))$ and $\sin(\frac{b_1}{w_1}\sin(w_1 t))$ one finally obtains

$$\bar{\Theta}_{\lambda_1}(t,0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & J_0(\frac{b_1}{w_1}) + 2\sum_{\kappa=1}^{\infty}(-1)^{\kappa} J_{2\kappa}(\frac{b_1}{w_1})\cos(2\kappa w_1 t) & -2\sum_{\kappa=1}^{\infty} J_{2\kappa-1}(\frac{b_1}{w_1})\sin[(2\kappa-1)w_1 t] \\ 0 & 2\sum_{\kappa=1}^{\infty} J_{2\kappa-1}(\frac{b_1}{w_1})\sin[(2\kappa-1)w_1 t] & J_0(\frac{b_k}{w_k}) + 2\sum_{\kappa=1}^{\infty}(-1)^{\kappa} J_{2\kappa}(\frac{b_1}{w_1})\cos(2\kappa w_1 t) \end{bmatrix}$$

where J_κ is the Bessel function of the first kind. This in turn implies that the *ij-th* element of the rotation matrix R_d is on the form

$$r_{ij} = a_{ij0} + \sum_{\kappa=1}^{\infty} {}^1a_{ij\kappa}\cos(2\kappa w_1 t) + \sum_{\kappa=1}^{\infty} {}^2a_{ij\kappa}\sin[(2\kappa-1)w_1 t]$$

in which $a_{ij0}, {}^1a_{ij\kappa}, {}^2a_{ij\kappa}$ are coefficients which depend on $R_d(0)$ and on the Bessel function of the first kind. From this, it should be clear that there not exists a finite dimensional observable linear system that solves the problem. However, in this particular case adding a certain number of oscillators it should be possible to make arbitrarily small the norm of the regulated output.

Anyhow the approach presented here seems to be a "brute force" solution, indeed with this approach we are not considering the particular structure of the system and we are not preserving the symmetries. A more fine solution should consider the Lagrangian-Hamiltonian structure of the system in order to solve the problem in a robust way.

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