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Some Classes of Partial Differential Operators modelled on Sub-Laplacians

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To my family and to Annalisa,
the most beautiful surprise of my life.

Acknowledgments

At the exact moment when I started to write this thesis, I already knew that this part should have been the last and the most difficult to be written; the reason is that I am not very good at this kind of thing. However, I think that at the end of these nine years of study and research it is important to say “thank you” to all the people who made this path possible and so beautiful.

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Stefano Biagi

Introduction

As is clearly suggested by its title, all the results presented in this thesis concern linear partial differential operators whose main model is the following:

$$(1.0) \qquad L = -\sum_{j=1}^{m} X_j^{*\,\mu}\, X_j.$$

Here, $X_1,\ldots,X_m$ are smooth vector fields on $\mathbb{R}^N$ and, for every $j=1,\ldots,m$, $X_j^{*\,\mu}$ denotes the formal adjoint of X_j with respect to a Radon measure μ which assumed to be equivalent to the standard Lebesgue measure.

Among infinitely many other natural problems involving these operators, we are particularly interested in the following key issues, widely recognized as fundamental topics by the PDE community:

(1) the Strong Maximum Principle;

(2) existence and uniqueness of a "well-behaved" global fundamental solution;

(3) existence of a Lie group on $\mathbb{R}^N$ w.r.t. which $X_1,\ldots,X_m$ are left-invariant.

"Strong Maximum Principle", *"global fundamental solution"* and *"left-invariance on Lie groups"* are precisely the keywords of this thesis.

Now, if we denote by V the density of μ w.r.t. the Lebesgue measure (and if we assume that $V>0$ on $\mathbb{R}^N$), it is not difficult to recognize that any operator L as in (1.0) can be written in the following quasi-divergence form [1]

$$(1.1) \qquad \mathcal{L} := \frac{1}{V(x)}\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\Big(V(x)\,a_{i,j}(x)\,\frac{\partial}{\partial x_j}\Big), \qquad x\in\mathbb{R}^N.$$

As we shall describe in detail in Chpt. 4, such a class of PDOs is general enough to comprehend, along with our prototype operators (1.0), Hörmander operators, sub-Laplace operators on real Lie groups (e.g., on Carnot groups, [37]), as well as linear PDO intervening in the study of function theory of several complex variables, CR and Riemannian Geometry (see e.g., [74, 75, 96, 98, 123]).

To face problem (1) in the case of PDOs $\mathcal{L}$ of the form (1.1) and problems (2) and (3) in the case of PDOs L of the form (1.0), we adopt a unitary approach *which crucially relies on the study of the geometry of the integral curves of suitable vector fields associated with $\mathcal{L}$ (or with L) and of their composition.*

[1] Incidentally, this is the form of the operators studied by Feffermann and Phong since the early '80s (see, e.g., the fundamental papers [69, 70]).

We now turn to describe more closely how such a geometrical approach is exploited in order to study each one of the problems listed above.

(1) As concerns with the Strong Maximum Principle (SMP, for short) it is well-known that, in presence of the Hörmander rank condition, the SMP can be obtained by following the classical scheme:

Hörmander condition	$\Rightarrow$	Chow-Rashevsky connectivity theorem	$\Rightarrow$	Propagation of maxima	$\Rightarrow$	SMP.

Our aim is to prove the SMP for our operators $\mathcal{L}$ *without assuming* Hörmander's rank condition. In order to do this, we profitably exploit assumption (HY) plus a theorem of Control Theory due to Amano [7], based on the properties of the flows of the vector fields $A_1, \ldots, A_N$ associated with the quadratic form of $\mathcal{L}$ (see also [120]).

We can then schematize our approach as follows:

(HY) (plus (S)-to-(NTD))	$\Longrightarrow$	SMP without assuming Hörmander's rank condition.

(2) We now turn to discuss the problem of the existence (and of the uniqueness) of a *global* fundamental solution for our PDOs $\mathcal{L}$.

There is no doubt that this is a very difficult issue, even in the particular case of Hörmander operators. Indeed, to the best of our knowledge, in this general situation one can only prove the existence of *local fundamental solution* (see, e.g., the celebrated papers by Folland [72], Folland and Stein [75]) or of a parametrix (see Rothschild and Stein [123]). We thus limit ourselves to considering a further sub-class of the Hörmander operators, namely that of the *homogeneous Hörmander operators*: by this, we mean linear partial differential operators of the form

$$\mathcal{L} = \sum_{j=1}^{m} X_j^2,$$

where $\{X_1, \ldots, X_m\}$ is a Hörmander system on $\mathbb{R}^n$ and $X_1, \ldots, X_m$ are assumed to be homogeneous of degree 1 w.r.t. a family of non-isotropic dilations. A key tool for proving the existence of a *global* fundamental solution for these operators is the lifting method developed by Folland [72], where a fundamental rôle is played by the following map:

$$\pi : \mathrm{Lie}\{X_1, \ldots, X_m\} \to \mathbb{R}^n, \quad \pi(X) := \exp(X)(0).$$

The homogeneity assumption on the vector fields $X_1, \ldots, X_m$ implies that π is well-defined and that it can be turned into a global canonical projection. Note again the key role of the flows of the vector fields involved.

(3) Finally, we come to the problem of finding a Lie group on $\mathbb{R}^N$ with respect to which a PDO $\mathcal{L}$ of the form (1.1) is left-invariant.

Quite surprisingly, this problem seems to be exquisitely of algebraic/geometrical nature, in that it involves flows of vector fields and (in a crucial way) the Campbell-Baker-Hausdorff-Dynkin Theorem for ODEs.

> To be more precise, let us assume that $\mathcal{L}$ can be written as a sum of squares of m v.f.s $X_1, \dots, X_m$. Then, under suitable assumptions on
>
> $$\mathfrak{g} = \mathrm{Lie}\{X_1, \dots, X_m\},$$
>
> we can use the cited CBHD Theorem for ODEs to construct a local Lie group structure on $\mathbb{R}^N$ with Lie algebra (in the sense of local Lie groups) equal to $\mathfrak{g}$. To globalize this structure, we exploit a completeness result for the flows of time-dependent vector fields.

After this general picture of the ideas underlying this thesis, we now describe more closely the results we obtained for each one of the problems (1)-to-(3).

One of the main motivations for our interest in PDOs $\mathcal{L}$ of the form (1.1) is represented by the (well-known) fine properties of the sub-Laplace operators on Carnot groups, which take the cited form (1.1).

For this reason, the first natural question we answer is the following:

(Q): is it possible to find a set of independent necessary and sufficient conditions for $\mathcal{L}$ to be left-invariant on a suitable Lie group $\mathbb{G} = (\mathbb{R}^N, *)$?

In the particular case of operators $\mathcal{L}$ which can be written as a sum of squares of *real-analytic* vector fields, a complete answer to question **(Q)** is given by the following theorem, which is the main result of Chpt. 2.

Theorem A. *Let* $\mathfrak{g}$ *be a Lie algebra of real-analytic vector fields on* $\mathbb{R}^N$. *There exists a real-analytic Lie group* $\mathbb{G} = (\mathbb{R}^N, *)$ *with* $\mathrm{Lie}(\mathbb{G}) = \mathfrak{g}$ if and only if

(C): *every element of* $\mathfrak{g}$ *is a* complete *vector field;*

(H): $\mathfrak{g}$ *is a Hörmander system of vector fields;*

(ND): *the dimension of* $\mathfrak{g}$ *is equal to* N.

In a forthcoming study, we shall show that the hypothesis of real-analyticity of the vector fields can be replaced by their being *of class* C^∞ (in this case $\mathbb{G}$ will be a smooth Lie group); see Sec. 2.1.4.

Some examples, of relevance in the applied setting, to which our results apply are: the Kolmogorov-Fokker-Planck operators studied in [29, 32]; the degenerate Ornstein-Uhlenbeck operators in [33]; the homogeneous operators in [28, 31]. As regards Kolmogorov-Fokker-Plank operators, we also highlight the paper [91] by Helffer and Nier, where the authors show the relation between these operators and the Witten Laplacian, together with some notable applications.

The proof of Thm. A makes crucial use of the powerful tool provided by the *Campbell-Baker-Hausdorff-Dynkin* Theorem; for this reason, a deep study of the convergence of the Campbell-Baker-Hausdorff-Dynkin series (both from the algebraic and the analytic view-point) will be carried out in the Appendix. It is worth noting that the study of the convergence domain of the CBHD series has a long history, tracing back to Hausdorff [90].[2]

[2]Concerning this topic, we address the reader to the papers [25, 26, 27, 48, 49, 60, 63, 110, 111, 113, 114, 115, 118, 126, 128, 129, 133].

We now briefly describe the proof of Thm. A. First of all, by means of the cited Campbell-Baker-Hausdorff-Dynkin Theorem and by exploiting assumptions (C), (H) and (ND), we are able to show the existence of an open neighborhood $U \subseteq \mathbb{R}^N$ of 0 and of a real-analytic map

$$m : \mathbb{R}^N \times U \longrightarrow \mathbb{R}^N, \qquad (x, y) \mapsto m(x, y),$$

such that $(\mathbb{R}^N, m)$ is a local Lie group, with neutral element 0 and Lie algebra (in the sense of local Lie groups) equal to $\mathfrak{g}$. Then, we *globalize* this structure by using a completeness result for time-dependent vector fields.

In fact, for every fixed $x, y \in \mathbb{R}^N$ we prove that the function

$$]-\varepsilon, \varepsilon[\ \ni t \mapsto m(x, ty)$$

solves a Cauchy problem which possesses a maximal solution $\varphi_{x,y}$ defined on the whole of $\mathbb{R}$; hence, for every $x, y \in \mathbb{R}^N$ we define

$$x * y := \varphi_{x,y}(1).$$

By classical results of ODE Theory, the map $*$ is real-analytic on $\mathbb{R}^N \times \mathbb{R}^N$; furthermore, by Unique Continuation, it is not difficult to see that $*$ globalizes all the local group properties satisfied by m.

One of the most notable properties of sub-Laplace operators on Carnot groups is probably the existence of a global fundamental solution, which behaves like the fundamental solution of the classical Laplace operator on $\mathbb{R}^N$ (with $N \geq 3$). More precisely, if $\mathbb{G} = (\mathbb{R}^N, *, \delta_\lambda)$ is a Carnot group (with homogeneous dimension $Q > 2$) and if $\mathcal{L}_{\mathbb{G}}$ is a sub-Laplacian on $\mathbb{G}$, a deep result by Folland [72] ensures the existence of a homogeneous and continuous symmetric norm $d \in C^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{R})$ such that

$$\Gamma : \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y\} \longrightarrow \mathbb{R}, \qquad \Gamma(x, y) := d^{2-Q}\big(x^{-1} * y\big)$$

is the unique global fundamental solution for $\mathcal{L}_{\mathbb{G}}$ satisfying

$$\lim_{\|y\| \to \infty} \Gamma(x, y) \to 0, \quad \text{for every } x \in \mathbb{R}^N.$$

The existence of a global fundamental solution for $\mathcal{L}_{\mathbb{G}}$ brings along several consequences of great importance: for example, the availability of surface and solid mean value formulas for C^2 functions, the Strong and Weak Maximum Principles, some extensions of the classical Harnack Inequality and of the Hardy Inequality, and so on (see, e.g., [37]).

Motivated by these facts, we shall turn our attention to the problem of the existence (and of the uniqueness) of a global fundamental solution for our PDOs $\mathcal{L}$ of the form (1.1). As we said earlier, for a selected subclass of such operators, namely that of the *homogeneous Hörmander operators* on $\mathbb{R}^n$, we solve this problem by proving the following two theorems, which are the central results of Chpt. 3: Thm. B for the case of stationary (homogeneous Hörmander) operators $\mathcal{L}$, Thm. C for parabolic operators of the form $\mathcal{L} - \partial_t$ (with $\mathcal{L}$ as above).

In the following theorems, instead of the usual notation $\mathbb{R}^N$ we use $\mathbb{R}^n$, for a reason which will become apparent in a moment.

Theorem B. *Let $X = \{X_1, \dots, X_m\}$ be a set of linearly independent* [3] *smooth vector fields on $\mathbb{R}^n$ satisfying the following assumptions:*

(H1): $X_1, \dots, X_m$ *are homogeneous of degree 1 with respect to*

$$\delta_\lambda(x) = (\lambda^{\sigma_1} x_1, \dots, \lambda^{\sigma_n} x_n),$$

where $1 = \sigma_1 \leq \ldots \leq \sigma_n$;

(H2): $X_1, \dots, X_m$ *satisfy the Hörmander rank condition:*

$$\dim \big\{ X(x) : X \in \mathrm{Lie}\{X_1, \dots, X_m\} \big\} = n, \quad \text{for every } x \in \mathbb{R}^n.$$

We set $q := \sum_{j=1}^n \sigma_j$ and define $\mathcal{L} := \sum_{j=1}^m X_j^2$. If $q > 2$, there exists a unique global fundamental solution Γ for $\mathcal{L}$ satisfying the following properties:

(i) $\Gamma(x; y) = \Gamma(y; x)$ *for every $x, y \in \mathbb{R}^n$ with $x \neq y$;*

(ii) $\Gamma(x; \cdot) = \Gamma(\cdot; x)$ *is smooth and $\mathcal{L}$-harmonic on $\mathbb{R}^n \setminus \{x\}$;*

(iii) $\Gamma(x; \cdot) = \Gamma(\cdot; x)$ *vanishes at infinity (uniformly for x in compact sets);*

(iv) $\Gamma(x; \cdot) = \Gamma(\cdot; x)$ *is locally integrable on $\mathbb{R}^n$;*

(v) Γ *is locally integrable on $\mathbb{R}^n \times \mathbb{R}^n$ and C^∞ out of the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$.*

Now, the parabolic version of the above theorem.

Theorem C. *Let the assumptions and the notations in the above theorem apply. We consider the heat-type operator $\mathcal{H}$ associated with $\mathcal{L}$, that is,*

$$\mathcal{H} := \mathcal{L} - \partial_t = \sum_{j=1}^m X_j^2 - \partial_t, \quad \text{on } \mathbb{R}^{1+n} = \mathbb{R}_t \times \mathbb{R}^n_x.$$

Then, there exists a unique global fundamental solution Γ for $\mathcal{H}$ (usually referred to as a heat kernel for $\mathcal{H}$*) satisfying the following properties:*

(i) $\Gamma \geq 0$ *on its domain and, for every $(t, x), (s, y) \in \mathbb{R}^{1+n}$, we have*

$$\Gamma(t, x; s, y) = 0 \text{ if and only if } s \leq t.$$

(ii) *For every $(t, x) \neq (s, y) \in \mathbb{R}^{1+n}$, the function Γ depends on t and s only through the difference $s - t$: in fact, we have*

$$\Gamma(t, x; s, y) = \Gamma(0, x; s - t, y).$$

Moreover, Γ is symmetric w.r.t. the space variables $x, y \in \mathbb{R}^{1+n}$, that is,

$$\Gamma(t, x; s, y) = \Gamma(t, y; s, x).$$

[3] Here and throughout, we consider the set $\mathfrak{X}(\mathbb{R}^N)$ of the smooth vector fields on $\mathbb{R}^N$ as a *real vector space* and not as C^∞-module; therefore, the vector fields $X_1, \dots, X_m$ are linearly *dependent* if there exist $\lambda_1, \dots, \lambda_m \in \mathbb{R}$, not all vanishing, such that

$$\lambda_1 X_1 + \cdots + \lambda_m X_m = 0$$

as a first order linear PDO (i.e., all of its coefficient functions are identically equal to 0).

(iii) *For every* $\lambda > 0$ *and every* $(t,x) \neq (s,y) \in \mathbb{R}^{1+n}$*, we have*

$$\Gamma(\lambda^2 t, \delta_\lambda(x); \lambda^2 s, \delta_\lambda(y)) = \lambda^{-q}\,\Gamma(t,x;s,y).$$

(iv) Γ *is smooth out of the diagonal of* $\mathbb{R}^{1+n} \times \mathbb{R}^{1+n}$.

(v) *For every compact set* $K \subseteq \mathbb{R}^{1+n}$*, we have*

$$\lim_{\|\zeta\|\to\infty} \big(\sup_{z\in K} \Gamma(z;\zeta)\big) = \lim_{\|\zeta\|\to\infty} \big(\sup_{z\in K} \Gamma(\zeta;z)\big) = 0.$$

(vi) $\Gamma \in L^1_{\mathrm{loc}}(\mathbb{R}^{1+n} \times \mathbb{R}^{1+n})$ *and, for fixed every* $z \in \mathbb{R}^{1+n}$*, we have*

$$\Gamma(z;\cdot) \text{ and } \Gamma(\cdot;z) \in L^1_{\mathrm{loc}}(\mathbb{R}^{1+n}).$$

(vii) *For every fixed* $(t,x) \in \mathbb{R}^{1+n}$ *we have*

$$\int_{\mathbb{R}^n} \Gamma(t,x;s,y)\,\mathrm{d}y = 1, \quad \text{for every } s > t.$$

(viii) *For every fixed* $\varphi \in C_0^\infty(\mathbb{R}^{1+n}, \mathbb{R})$*, the function*

$$\Lambda_\varphi : \mathbb{R}^{1+n} \longrightarrow \mathbb{R}, \qquad \Lambda_\varphi(\zeta) := \int_{\mathbb{R}^{1+n}} \Gamma(z;\zeta)\,\varphi(z)\,\mathrm{d}z$$

is smooth, it vanishes at infinity and $\mathcal{H}(\Lambda_\varphi) = -\varphi$ *on* $\mathbb{R}^{1+n}$.

Furthermore, if we consider the function Γ^* *defined by*

$$\Gamma^*(t,x;s,y) := \Gamma(s,y;t,x), \quad \text{for every } (t,x) \neq (s,y) \in \mathbb{R}^{1+n},$$

then Γ^* *is a global fundamental solution for the adjoint operator* $\mathcal{H}^* = \mathcal{L} + \partial_t$*, satisfying the dual statements of* (i)-to-(viii).

The key ingredient for proving these results, which is at our disposal in the case of *homogeneous* Hörmander operators, is the notable lifting method for homogeneous vector fields proved by Folland in [73], plus an ad-hoc change of variables. Folland's approach is essentially a geometric re-interpretation of the lifting construction made by Rothschild and Stein [123], in the particular case when the vector fields involved are assumed to be homogeneous of degree 1 w.r.t. a family of non-isotropic dilations.

Under the latter assumption, Folland showed that the *local* lifting proved by Rothschild and Stein is actually *global*, and the vector fields can be directly related via a submersion π to left-invariant vector fields on a suitable higher-dimensional homogeneous Carnot group $\mathbb{G}$ on $\mathbb{R}^N$, with $N \geq n$. Taking into account this fact, we prove Thm.s B and C by using a *naive* saturation argument.

Let us take a closer look to the proof of Thm.s B and C. First of all, by using Folland's result and a suitable change of variables turning π into the canonical projection from $\mathbb{R}^N$ onto $\mathbb{R}^n$, we can prove the existence of a sub-Laplacian $\mathcal{L}_{\mathbb{G}}$ on a Carnot group $\mathbb{G} = (\mathbb{R}^N, \star, \delta_\lambda)$ which is a lifting of $\mathcal{L}$ on $\mathbb{R}^N$. We then use the following notation for the points of $\mathbb{R}^N$:

$$(x,\xi), \quad \text{with } x \in \mathbb{R}^n, \xi \in \mathbb{R}^p \text{ and } p = N - n.$$

If $\Gamma_{\mathbb{G}}(x,\xi;y,\eta)$ denotes the unique global fundamental solution for $\mathcal{L}_{\mathbb{G}}$ vanishing at infinity (with pole (x,ξ)), we show that the function

$$\mathbb{R}^p \ni \eta \mapsto \Gamma_{\mathbb{G}}(x,0;y,\eta)$$

is integrable on $\mathbb{R}^p$ for every $x \neq y \in \mathbb{R}^n$, this result being non-trivial. Setting

$$\Gamma(x;y) := \int_{\mathbb{R}^p} \Gamma_{\mathbb{G}}(x,0;y,\eta)\,\mathrm{d}\eta \qquad x \neq y \in \mathbb{R}^n,$$

we then prove that Γ defines a global fundamental solution for $\mathcal{L}$, further satisfying properties (i)-to-(v) in the statement of Thm. B.

As for Thm. C, we argue exactly in the same way: if $\Gamma_{\mathbb{G}}(z,\xi;\zeta,\eta)$ is the unique heat kernel for $\mathcal{H}_{\mathbb{G}} = \mathcal{L}_{\mathbb{G}} - \partial_t$ on $\mathbb{R}\times\mathbb{G}$ (with pole (z,ξ)), we prove that

$$\mathbb{R}^p \ni \eta \mapsto \Gamma_{\mathbb{G}}(z,0;\zeta,\eta)$$

is locally integrable on $\mathbb{R}^p$ for every fixed $z \neq \zeta \in \mathbb{R}^{1+n}$. Therefore, setting

$$\Gamma(z;\zeta) := \int_{\mathbb{R}^p} \Gamma_{\mathbb{G}}(z,0;\zeta,\eta)\,\mathrm{d}\eta \qquad z \neq \zeta \in \mathbb{R}^{1+n},$$

it turns out that Γ is a global fundamental solution for $\mathcal{H}$, further satisfying all the properties in the statement of the theorem.

In the literature, there are many examples of lifting involving meaningful PDOs: for instance, consider the case of the Grushin operator

$$\mathcal{G} = (\partial_{x_1})^2 + (x_1\,\partial_{x_2})^2$$

on $\mathbb{R}^2$, a lifting of which is given by the PDO

$$\widetilde{\mathcal{G}} = (\partial_{x_1})^2 + (\partial_{x_3} + x_1\,\partial_{x_2})^2 \quad \text{on } \mathbb{R}^3.$$

In turn, the latter is nothing but a copy (via a change of variable) of the well known Kohn-Laplacian on the first Heisenberg group. The idea of obtaining a fundamental solution for the Grushin operator $\mathcal{G}$ via a saturation argument applied to the (explicit!) fundamental solution of $\widetilde{\mathcal{G}}$ has already been exploited in the literature: see e.g., Bauer, Furutani, Iwasaki [16]; see also Calin, Chang, Furutani, Iwasaki [46, Sec. 10.3] for the Heat kernel; more generally, see Beals, Gaveau, Greiner, Kannai [19] for operators lifting to sub-Laplacians on 2-step Carnot groups. To the best of our knowledge, when the existence of a *global* fundamental solution Γ for a PDO is provided, it seems that in the vast majority of cases (though exceptions are available):

- PDOs with *polynomial coefficients* are considered;
- existence is a *by-product of an explicit (integral) formula* for Γ.

Note that the same happens in the present case, since our homogeneous operators necessarily have polynomial coefficients, and an integral representation for Γ (albeit not explicit) is furnished. Global fundamental solutions, without an explicit representation, are given for example, in Folland [72]; Nagel, Ricci, Stein [117]; Bonfiglioli and Lanconelli [32]; Bramanti, Brandolini, Lanconelli and Uguzzoni [40]. Existence results without an exact representation are also available, based on the so-called Levi parametrix method [107] (see also [40, 78, 95]); concerning Levi's parametrix method, we also highlight the recent paper by Bramanti, Brandolini, Manfredini, Pedroni [41], where a local Lifting technique and a local saturation argument are also applied. See also the paper [52] by Citti, Manfredini, where it is exploited a local Lifting technique involving hypoelliptic Hörmander operators and their local fundamental solutions.

Finally, as regards the use of a lifting argument to study some classes of PDOs, we address the reader to the notable paper [92] by Helffer and Nourrigat, where a lifting technique is profitably exploited in order to investigate the C^∞-hypoellipticity of PDOs modelled on Hörmander vector fields.

Once the existence of a global fundamental solution for our homogeneous Hörmander operators has been proved, we focus on the Strong Maximum Principle (SMP) and on Hardy's inequality for PDOs $\mathcal{L}$ of the form (1.1).

As is well-known, in the particular case of Hörmander operators, Bony proved in [39] the Strong Maximum Principle as a consequence of a Maximum Propagation argument, based on the Carathéodory-Chow-Rashevsky Theorem. Moreover, when a *strictly positive* global fundamental solution Γ exists, such a principle can be deduced from the mean value formulas related to Γ (see [1]).

As regards Harnack inequalities and Maximum Principles, during the 80's many important results on degenerate-elliptic operators under the divergence-form (1.1) were established; see e.g. [50, 64, 65, 66, 76, 77, 89, 96]. As for the assumptions made on the involved PDOs, in [96] a suitable *subellipticity* hypothesis is assumed, whereas in the other cited papers, operators of the form (1.1) are considered with very low regularity assumptions on the coefficients, but under the hypothesis that the degeneracy of $A(x)$ be controlled on both sides by some appropriate weights.

In the present thesis, we do not require $\mathcal{L}$ to be a Hörmander operator, our results holding true in the infinitely-degenerate case as well, nor do we make any assumption of subellipticity or weighted degeneracy. In obtaining our main results we are much indebted to the ideas in the pioneering paper by Bony, [39], where Hörmander operators are considered. The main novelty of our framework is that we have to renounce to the geometric information "of propagation-type", encoded in Hörmander's Rank Condition: indeed the latter implies a connectivity property (leading to the Strong Maximum Principle), as well as it implies hypoellipticity, due to Hörmander's theorem [94].

In our setting, the approach is somewhat reversed: hypoellipticity is the main assumption, and we need to derive from it some appropriate connectivity and propagation features, even in the absence of a maximal rank condition. This will be made possible by exploiting a Control Theory result by Amano [7] on hypoelliptic PDOs. Our result can be stated as follows (see Chpt. 4).

Theorem D. *Let $\mathcal{L}$ be a linear PDO of the form* (1.1) *and satisfying the following* structural assumptions*:* [4]

(S): *$\mathcal{L}$ has smooth coefficient functions $V, a_{i,j} \in C^\infty(\mathbb{R}^N, \mathbb{R})$ and V is* strictly positive *on the whole of $\mathbb{R}^N$;*

[4] The Strong Maximum Principle concerning $\mathcal{L}$-subharmonic functions, we could limit ourselves to only consider the case $V \equiv 1$, since the general case can be easily reduced to this particular one. The main reason why we preferred to deal with operators $\mathcal{L}$ of the general form (1.1) is the need to keep the rôle of the function V and of the matrix $A(x)$ distinct.

(DE): *$\mathcal{L}$ is* degenerate elliptic*, i.e.,* $A(x) = \big(a_{i,j}(x)\big) \geq 0$ *for every* $x \in \mathbb{R}^N$*;*

(NTD): *$\mathcal{L}$ is* non-totally degenerate at every point of $\mathbb{R}^N$ *(see* (4.1.2)*);*

(HY): *$\mathcal{L}$ is* C^∞-hypoelliptic in every open subset of $\mathbb{R}^N$*, that is, for every open set* $\Omega \subseteq \mathbb{R}^N$ *and every* $u \in \mathcal{D}'(\Omega)$ *we have*

$$\operatorname{sing\,supp}(u) = \operatorname{sing\,supp}(\mathcal{L}u).$$

If $\Omega \subseteq \mathbb{R}^N$ *is open and connected, the following facts hold true:*

(1) *Any function* $u \in C^2(\Omega, \mathbb{R})$ *satisfying* $\mathcal{L}u \geq 0$ *on* Ω *and attaining a maximum in* Ω *is constant throughout* Ω*.*

(2) *If* $c \in C^\infty(\mathbb{R}^N, \mathbb{R})$ *is nonnegative on* $\mathbb{R}^N$*, then any function* $u \in C^2(\Omega, \mathbb{R})$ *satisfying* $\mathcal{L}u - c\,u \geq 0$ *on* Ω *and attaining a nonnegative maximum in* Ω *is constant throughout* Ω*.*

Since we touched upon very classical facts together with a less known result by Amano, some clarification on our arguments may be welcome. Our proof of the Strong Maximum Principle in Thm. D follows, *à la Bony*, a rather classical scheme, in that it rests on a Hopf Lemma for $\mathcal{L}$. However, the passage from the Hopf Lemma to the SMP is, in general, non-trivial and the same is true in our framework. As anticipated, we are able to supply the lack of Hörmander's Rank Condition by using the notable control-theoretic property encoded in the hypoellipticity assumption (HY), proved by Amano in [7]: indeed, thanks to the hypothesis (NTD), we are entitled to use [7, Theorem 2] which states that (HY) ensures the *controllability* of the ODE system

$$\dot{\gamma} = \xi_0 A_0(\gamma) + \sum_{i=1}^N \xi_i A_i(\gamma), \qquad (\xi_0, \xi_1, \dots, \xi_N) \in \mathbb{R}^{1+N},$$

on every open and connected subset of $\mathbb{R}^N$. Here $A_1, \dots, A_N$ denote the vector fields associated with the rows of the principal matrix of $\mathcal{L}$, whereas A_0 is the drift vector field obtained by writing $\mathcal{L}$ in the form

$$\mathcal{L}u = \sum_{i=1}^N \frac{\partial}{\partial x_i}(A_i u) + A_0 u.$$

By definition of a controllable system, Amano's controllability result provides another geometric *connectivity property* (a substitute for Chow's Theorem): any couple of points can be joined by a continuous path which is piece-wise an integral curve of some vector field Y belonging to $\operatorname{span}_{\mathbb{R}}\{A_0, A_1, \dots, A_N\}$. Then, to complete the proof of Thm. D it suffices to show that there is a propagation of the maximum of any $\mathcal{L}$-subharmonic function u along all integral curves γ_Y

For example, operators of the general form (1.1) have been recently studied by Battaglia and Bonfiglioli [14] with the aim at obtaining invariant Harnack inequalities under a low regularity assumption on the coefficients. In this context, the rôles of the function V and of the matrix $A(x)$ are drastically different: on the one hand, V is a strictly positive L^1_{loc}-function in $\mathbb{R}^N$ which is aimed at being the density of a *doubling measure*; on the other hand, $A(x)$ is a matrix of measurable function which must satisfy some suitable X-elliptic. Therefore, it seems not convenient, for future purposes, to incorporate V into the matrix A.

of every vector field Y in $\mathrm{span}_{\mathbb{R}}\{A_0, A_1, \dots, A_N\}$. This, in turn, is done by exploiting a characterization by Bony [39] of the invariant sets for the C^1 vector fields in terms of (a suitable notion of) tangentiality. It is worth mentioning that this kind of operators (and the fine geometrical properties of the associated vector fields $A_1, \dots, A_N$) were first studied by Fefferman and Phong [70].

Finally, we turn our attention to a notable application of the existence of a fundamental solution: Hardy-type inequalities (see Sec. 4.2 for a list of related references). Following the techniques by Garofalo, Lin [82, 81] and by Garofalo, Lanconelli [80], we derive from the Hardy inequality a result of Unique Continuation for the solutions of the Schrödinger-type equation

$$-\mathcal{L}u + Vu = 0,$$

where $\mathcal{L}$ is a sub-Laplacian on a Carnot group $\mathbb{G}$, V is a continuous function on $\mathbb{G}$ (satisfying suitable estimates), and when u satisfies a (differential) growth condition. In the framework of Carnot groups and for $\mathcal{L}$-harmonic functions (i.e., when $V \equiv 0$), a thoroughly comprehensive analysis of Unique Continuation has been recently given by Garofalo and Rotz [83], by means of a new notion of Almgren's frequency function. In [83] it is also demonstrated that, without some control on the growth of u, the solutions of the above equation may fail to have a bounded frequency: see the example given in [83, Remark 7.5]. Thus, our assumption on the growth of u cannot be deleted without possibly losing the approach based on the boundedness of the frequency, which is the approach that we also follow (in line with [80, 81, 82, 83]) in proving Unique Continuation. See also Bahouri, [8], for the problems connected with perturbations of sum-of-squares and (the loss of) Unique Continuation.

To conclude the Introduction, let us briefly describe the structure of the thesis. There are four chapters plus an Appendix, whose contents are the following:

- Chpt. 1 is introductory and it is devoted to recalling the main notions and results concerning real Lie groups, with particular emphasis on homogeneous Carnot groups and on sub-Laplace operators on such groups.

- Chpt. 2 is subdivided into two sections: the first one is devoted to presenting the announced characterization of those Lie algebras $\mathfrak{g}$ of vector fields for which there exists a Lie group $\mathbb{G} = (\mathbb{R}^N, *)$ such that

  $$\mathrm{Lie}(\mathbb{G}) = \mathfrak{g};$$

  in the second section, we collect sufficient conditions (surely well-known) for a general linear second-order PDO (with smooth coefficients) to be re-written as a sum of squares of smooth vector fields.

- Chpt. 3 is totally devoted to proving the existence and the uniqueness of a well-behaved global fundamental solution for *homogeneous* Hörmander operators. Due to its relevance in our approach, we also present the lifting method for homogeneous vector fields introduced by Folland [73].

- Chpt. 4 is also subdivided into two sections: the first one is devoted to establishing the Strong Maximum Principle for *hypoelliptic* PDOs $\mathcal{L}$ of the form (1.1) and to presenting an application of such a principle to the

Harnack Inequality for $\mathcal{L}$; in the second section, we assume the existence of a global fundamental solution for $\mathcal{L}$ and, by making use of suitable mean value operators associated with $\mathcal{L}$, we prove a generalization of the classical Hardy Inequality. We also present, as an application of such an inequality, a Unique Continuation result for the solution of a class of Schrödinger-type equations on Carnot groups.

- Finally, Appendix A is devoted to briefly presenting a fine convergence result for the Campbell-Baker-Hausdorff-Dynkin in Banach-Lie algebras. Such a result is profitably exploited in Chpt. 2.

From the results contained in Sec. 2.1 of Chpt. 2 we obtained the recent paper [23]; from the results presented in Chpt. 3 we obtained the very recent paper [24]; from the results presented in Chpt. 4 we obtained the paper [15]. Finally, from the results contained in Appendix A we obtained the paper [22].

Contents

Chapter 1

Sub-Laplace operators on real Lie groups

The main purpose of this chapter is to recall some basic elements of Lie group Theory, in order the make this thesis as self-contained as possible. Since all the results we aim to present concern the analysis of partial differential equations and of partial differential operators on Euclidean spaces, we limit ourselves to only consider Lie groups on $\mathbb{R}^N$; in this sense, we closely follow the approach of Bonfiglioli, Lanconelli and Uguzzoni in [37], to which we refer the reader for a comprehensive exposition of this topic and for any further detail.

1.1 Generalities on Lie groups

In this first section, we briefly present the relevant definitions and properties concerning Lie groups on $\mathbb{R}^N$.

First of all we recall that, if $*$ is a group law on Euclidean space $\mathbb{R}^N$, then the couple $\mathbb{G} = (\mathbb{R}^N, *)$ is a **Lie group** (on $\mathbb{R}^N$) if the map

$$\mathbb{R}^N \times \mathbb{R}^N \ni (x, y) \mapsto x * y \in \mathbb{R}^N$$

is of class C^∞ on $\mathbb{R}^N \times \mathbb{R}^N$ (w.r.t. the usual differentiable structure). Unless otherwise specified, the (unique) neutral element of $\mathbb{G}$ will be denoted by e, while the inverse of $x \in \mathbb{R}^N$ will be denoted by x^{-1}; moreover, for every fixed $\alpha \in \mathbb{R}^N$ we also define

$$\tau_\alpha : \mathbb{R}^N \to \mathbb{R}^N, \quad \tau_\alpha(x) := \alpha * x;$$
$$\rho_\alpha : \mathbb{R}^N \to \mathbb{R}^N, \quad \rho_\alpha(x) := x * \alpha.$$

The maps τ_α and ρ_α are called, respectively, the **left-translation** and the **right-translation** by α (on $\mathbb{G}$). By exploiting the fact that $(\mathbb{R}^N, *)$ is a group, it is immediate to see that the following properties hold true:

- The maps τ_e and ρ_e coincide with the identity map on $\mathbb{R}^N$;
- For every $\alpha, \beta \in \mathbb{R}^N$, one has $\tau_\alpha \circ \rho_\beta = \rho_\beta \circ \tau_\alpha$;
- For every $\alpha, \beta \in \mathbb{R}^N$, one has $\tau_{\alpha*\beta} = \tau_\alpha \circ \tau_\beta$ and $\rho_{\alpha*\beta} = \rho_\beta \circ \rho_\alpha$;

- For every $\alpha \in \mathbb{R}^N$, the maps τ_α and $\tau_{\alpha^{-1}}$ are inverse to each others, whence τ_α is a smooth diffeomorphism of $\mathbb{R}^N$, and the same is true of ρ_α.

Let now $\mathbb{G} = (\mathbb{R}^N, *)$ be a Lie group on $\mathbb{R}^N$ and let X be a smooth vector field on $\mathbb{R}^N$ (meant as a first order PDO). We say that X is **left-invariant** on $\mathbb{G}$ if the following property is satisfied:

$$X\big(f \circ \tau_\alpha\big) = (Xf) \circ \tau_\alpha, \quad \text{for all } \alpha \in \mathbb{R}^N \text{ and every } f \in C^\infty(\mathbb{R}^N, \mathbb{R}). \tag{1.1.1}$$

We denote by $\mathfrak{g}$ the set of *all* left-invariant (smooth) vector fields on $\mathbb{G}$. A direct computation shows that $\mathfrak{g}$ is actually a *Lie sub-algebra* of the Lie algebra of vector fields on $\mathbb{R}^N$, that is,

$$\lambda X + \mu Y \in \mathfrak{g} \quad \text{for every } X, Y \in \mathfrak{g} \text{ and every } \lambda, \mu \in \mathbb{R}, \qquad \text{and}$$
$$[X, Y] = XY - YX \in \mathfrak{g}, \quad \text{for every } X, Y \in \mathfrak{g}.$$

For this reason, $\mathfrak{g}$ is called the **Lie algebra of** $\mathbb{G}$, and we shall also denote it by $\mathrm{Lie}(\mathbb{G})$. We now observe that, taking into account identity (1.1.1), it is very easy to see that a smooth vector field X is left-invariant on $\mathbb{G}$ *if and only*

$$XI(x) = \mathcal{J}_{\tau_x}(e) \cdot XI(e), \quad \text{for all } x \in \mathbb{R}^N, \tag{1.1.2}$$

where I denotes the identity map on $\mathbb{R}^N$ and $\mathcal{J}_{\tau_x}(e)$ denotes the Jacobian matrix of the map τ_x at the neutral element e of $\mathbb{G}$.

From this, one easily obtains the following simple but important characterization of the Lie algebra $\mathrm{Lie}(\mathbb{G})$ of $\mathbb{G}$, which shall be useful in the sequel (for a proof see, for example, [37, Proposition 1.2.7]).

Theorem 1.1.1. *Let $\mathbb{G} = (\mathbb{R}^N, *)$ be a Lie group on $\mathbb{R}^N$ and let $\mathrm{Lie}(\mathbb{G})$ be the Lie algebra of $\mathbb{G}$. Then the map*

$$\Lambda : \mathrm{Lie}(\mathbb{G}) \longrightarrow \mathbb{R}^N, \qquad \Lambda(X) := XI(e), \tag{1.1.3}$$

defines a linear isomorphism *of vector spaces. Hence, in particular,* $\mathrm{Lie}(\mathbb{G})$ *is a finite-dimensional real vector s pace and* $\dim(\mathrm{Lie}(\mathbb{G})) = N$.

Remark 1.1.2. Let $\mathbb{G} = (\mathbb{R}^N, *)$ be a Lie group on $\mathbb{R}^N$ and let X be a left-invariant vector field on $\mathbb{G}$. Let us assume that there exists a point $x_0 \in \mathbb{R}^N$ such that $XI(x_0) = 0$. Then $XI(x) = 0$ for every $x \in \mathbb{R}^N$.

Indeed, let $x \in \mathbb{R}^N$ be fixed and let $\alpha := x * (x_0)^{-1}$. By (1.1.2) we get

$$XI(x) = XI(\alpha * x_0) = \mathcal{J}_{\tau_\alpha}(x_0) \cdot XI(x_0) = 0,$$

and this proves that $X = 0$ in $\mathcal{X}(\mathbb{R}^N)$, as claimed.

By means of the map Λ in the statement of Thm. 1.1.1, it is possible to construct a distinguish basis for the Lie algebra $\mathrm{Lie}(\mathbb{G})$ of a Lie group $\mathbb{G}$.

Definition 1.1.3. Let $\mathbb{G} = (\mathbb{R}^N, *)$ be a Lie group on $\mathbb{R}^N$ and let $\mathrm{Lie}(\mathbb{G})$ be the Lie algebra of $\mathbb{G}$. Moreover, let Λ be the map defined in (1.1.1) and let $\mathcal{E} = \{e_1, \ldots, e_N\}$ be the canonical basis of $\mathbb{R}^N$. If we define

$$J_i := \Lambda^{-1}(e_i), \quad \text{for all } i = 1, \ldots, N, \tag{1.1.4}$$

then the vector fields $J_1, \ldots, J_N$ form a basis of $\mathrm{Lie}(\mathbb{G})$, which will be referred to as the **Jacobian basis** of $\mathrm{Lie}(\mathbb{G})$.

Remark 1.1.4. Let $\mathbb{G} = (\mathbb{R}^N, *)$ be a Lie group and let $\mathfrak{g}$ be the Lie algebra of $\mathbb{G}$. By Thm. 1.1.1, the set $\mathfrak{g}$ is a real vector space with dimension N; therefore, if $\{X_1, \ldots, X_N\}$ and $\{Y_1, \ldots, Y_N\}$ are two different basis of $\mathfrak{g}$ (as a vector space), there exists a non-singular $N \times N$ *constant* matrix $A = \big(a_{i,j}\big)_{i,j}$ such that

$$Y_j = \sum_{i=1}^{N} a_{i,j} X_i, \quad \text{for every } j = 1, \ldots, N.$$

Remark 1.1.5. Let $\mathbb{G} = (\mathbb{R}^N, *)$ be a Lie group and let $\{J_1, \ldots, J_N\}$ be the Jacobian basis of $\mathrm{Lie}(\mathbb{G})$. For every $1 \leq i \leq N$ and every $x \in \mathbb{R}^N$, one has

$$J_i I(x) \overset{(1.1.2)}{=} \mathcal{J}_{\tau_x}(e) \cdot J_i I(e) \overset{(1.1.3)}{=} \mathcal{J}_{\tau_x}(e) \cdot \Lambda(J_i) \overset{(1.1.4)}{=} \mathcal{J}_{\tau_x}(e) \cdot e_i;$$

hence, $J_i I(x)$ is nothing but the i-th column of the Jacobian matrix $\mathcal{J}_{\tau_x}(e)$.

Let now $\mathbb{G} = (\mathbb{R}^N, *)$ be a Lie group on $\mathbb{R}^N$ and let $X \in \mathrm{Lie}(\mathbb{G})$. Again by making use of identity (1.1.2), it is very easy to recognize that the maximal domains of all the integral curves of X do coincide; moreover, for every t in such a common maximal domain, the following identity holds true:

$$\alpha * \exp(tX)(\beta) = \exp(tX)(\alpha * \beta), \quad \text{for every } \alpha, \beta \in \mathbb{R}^N. \tag{1.1.5}$$

As a consequence of identity (1.1.5), we easily obtain the following remarkable result (for a proof see, e.g., [37, Lemma 1.2.23]).

Proposition 1.1.6. *Let* $\mathbb{G} = (\mathbb{R}^N, *)$ *be a Lie group on* $\mathbb{R}^N$. *Then any vector field belonging to* $\mathrm{Lie}(\mathbb{G})$ *is global.*

Thanks to Prop. 1.1.6, the following (central) definition is well-posed.

Definition 1.1.7 (Exponential Map of $\mathbb{G}$)**.** Let $\mathbb{G} = (\mathbb{R}^N, *)$ be a Lie group on $\mathbb{R}^N$ and let $\mathrm{Lie}(\mathbb{G})$ be the Lie algebra of $\mathbb{G}$. Then the function

$$\mathrm{Exp} : \mathrm{Lie}(\mathbb{G}) \longrightarrow \mathbb{R}^N, \qquad \mathrm{Exp}(X) := \exp(1 \cdot X)(e), \tag{1.1.6}$$

is well-defined and it is called the **Exponential Map** of $\mathbb{G}$.

We conclude this section with the following theorem, showing how the composition law of a Lie group $\mathbb{G}$ can be somehow "recovered" from the Exponential Map of $\mathbb{G}$; this result will play a fundamental role in Chpt. 2.

Theorem 1.1.8. *Let* $\mathbb{G} = (\mathbb{R}^N, *)$ *be a Lie group on* $\mathbb{R}^N$ *and let* $\mathrm{Lie}(\mathbb{G})$ *be the Lie algebra of* $\mathbb{G}$. *Moreover, let* $x \in \mathbb{R}^N$ *and let* $y \in \mathbb{R}^N$ *be such that* $y = \mathrm{Exp}(Y)$ *for a certain* $Y \in \mathrm{Lie}(\mathbb{G})$. *Then we have*

$$x * y = \exp(Y)(x). \tag{1.1.7}$$

Proof. This is a direct consequence of identity (1.1.5) and of the very definition of Exponential Map: in fact, we have

$$x * y = x * \mathrm{Exp}(Y) \overset{(1.1.7)}{=} x * \exp(Y)(e) \overset{(1.1.5)}{=} \exp(Y)(x).$$

This ends the proof. □

1.2 Homogeneous groups

In this second section we briefly introduce two selected classes of Lie groups, namely, the homogeneous groups and the Carnot groups. We shall recall the basic notions and properties concerning such groups, needed for a complete comprehension of Chpt. 3. Also in this case, we refer the Reader to the monograph [37] for a complete treatment of the argument.

Homogeneous groups

Let $\mathbb{G} = (\mathbb{R}^N, *)$ be a Lie group on $\mathbb{R}^N$. We say that $\mathbb{G}$ is a **homogeneous Lie group** (or simply a **homogeneous group**) if there exists a N-tuple $(\sigma_1, \ldots, \sigma_N)$ of positive real numbers satisfying the following properties:

1. $\sigma_1 = 1$ and $\sigma_i \leq \sigma_{i+1}$ for every $i = 1, \ldots, N$;
2. For every $\lambda > 0$, the dilation $\delta_\lambda : \mathbb{R}^N \to \mathbb{R}^N$ given by
$$\delta_\lambda(x) := (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_N} x_N),$$
is an automorphism of the group $\mathbb{G}$, that is,
$$\delta_\lambda(x * y) = \delta_\lambda(x) * \delta_\lambda(y), \quad \text{for every } x, y \in \mathbb{R}^N.$$

We shall denote by $\mathbb{G} = (\mathbb{R}^N, *, \delta_\lambda)$ the datum of a homogeneous Lie group, with composition law $*$ and family of dilations $\{\delta_\lambda\}_{\lambda>0}$, and we set
$$Q := \sum_{j=1}^{N} \sigma_j, \tag{1.2.1}$$
Such a number Q is called **homogeneous dimension of the group** $\mathbb{G}$.

We explicitly observe that, since the map δ_λ is an automorphism of $\mathbb{G}$ for every $\lambda > 0$, we necessarily have $\delta_\lambda(e) = e$ for every $\lambda > 0$, so that $e = 0$; this means that the neutral element of a homogeneous group is *always* 0. We also notice that the family of dilations $\{\delta_\lambda\}_{\lambda>0}$ forms a one-parameter group of automorphisms of $\mathbb{G}$ whose identity is $\delta_1 = I$, that is,
$$\delta_{\lambda\mu}(x) = \delta_\lambda\big(\delta_\mu(x)\big), \quad \text{for all } x \in \mathbb{R}^N \text{ and every } \lambda, \mu > 0.$$
In particular, the inverse of δ_λ is the dilation $\delta_{1/\lambda} = \delta_{\lambda^{-1}}$, since
$$\delta_\lambda \circ \delta_{1/\lambda} = \delta_{1/\lambda} \circ \delta_\lambda = \delta_1 = I.$$
In the theory of homogeneous Lie groups, a central role is played by homogeneous functions and homogeneous vector fields; we then quickly recall such notions. Let $\mathbb{G} = (\mathbb{R}^N, *, \delta_\lambda)$ be a fixed homogeneous Lie group on $\mathbb{R}^N$ and let $f : \mathbb{R}^N \to \mathbb{R}$. We say that f is **δ_λ-homogeneous** of degree $m \in \mathbb{R}$ if
$$f\big(\delta_\lambda(x)\big) = \lambda^m f(x), \quad \text{for all } x \in \mathbb{R}^N \text{ and for every } \lambda > 0. \tag{1.2.2}$$
Analogously, if P is a linear PDO on $\mathbb{R}^N$, we say that P is **δ_λ-homogeneous** of degree $m \in \mathbb{R}$ if, for every function $u \in C^\infty(\mathbb{R}^N, \mathbb{R})$, it satisfies
$$P\big(u(\delta_\lambda(x)\big) = \lambda^m \left(Pu\right)(\delta_\lambda(x)), \quad \text{for all } x \in \mathbb{R}^N \text{ and for every } \lambda > 0. \tag{1.2.3}$$

Finally, given a multi-index $(\alpha_1, \ldots, \alpha_N) \in \big(\mathbb{N} \cup \{0\}\big)^N$, we set

$$|\alpha|_{\mathbb{G}} := \langle \alpha, \sigma \rangle = \sum_{j=1}^{N} \alpha_j \sigma_j,$$

and we call it the **$\mathbb{G}$-length (or $\mathbb{G}$-height)** of the multi-index α.

Remark 1.2.1. Let $\mathbb{G} = (\mathbb{R}^N, *, \delta_\lambda)$ be a homogeneous Lie group. Then we have the following facts:

(i) The zero function and the identically vanishing linear PDO are δ_λ-homogeneous of *every* degree. Conversely, if a function f (or a linear PDO P) is δ_λ-homogeneous of two *different* degrees, then $f \equiv 0$ (or $P \equiv 0$ on $\mathbb{R}^N$).

(ii) For every $j = 1, \ldots, N$, the j-th projection $\pi_j(x) = x_j$ and the j-th partial derivative ∂/∂_{x_j} are δ_λ-homogeneous of degree σ_j. Moreover, if α is a fixed multi-index, the function $x \mapsto x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ and the linear PDO $\partial_{x_1}^{\alpha_1} \cdots \partial_{x_N}^{\alpha_N}$ are δ_λ- homogeneous of degree $|\alpha|_{\mathbb{G}}$.

(iii) If P is a linear δ_λ-homogeneous PDO of degree n and if $f \in C^\infty(\mathbb{R}, \mathbb{R})$ is δ_λ-homogeneous of degree $m \in \mathbb{R}$, then the function Pf is δ_λ-homogeneous of degree $m - n$, while the linear PDO fP is δ_λ-homogeneous of degree $n - m$.

We explicitly observe that, if $f \in C(\mathbb{R}^N, \mathbb{R})$ is δ_λ-homogeneous of degree m and if $f(x_0) \neq 0$ for a certain $x_0 \in \mathbb{R}^N$, then $m \geq 0$; analogously, if $g \in C(\mathbb{R}^N, \mathbb{R})$ is δ_λ-homogeneous of degree 0, then g is *constant* on $\mathbb{R}^N$. In the particular case of *smooth* δ_λ-homogeneous functions and *smooth* δ_λ-homogeneous vector fields, we have the following characterization (for a proof see, e.g., [37, Propositions 1.3.4 and 1.3.5 and Corollary 1.3.6]).

Theorem 1.2.2. *Let $\mathbb{G} = (\mathbb{R}^N, *, \delta_\lambda)$ be a homogeneous Lie group on $\mathbb{R}^N$. Then we have the following facts:*

(i) *A smooth function $f \in C^\infty(\mathbb{R}^N, \mathbb{R})$ is δ_λ-homogeneous of degree $m \in \mathbb{R}$* if and only if *f is a polynomial function of the form*

$$f(x) = \sum_{|\alpha|_{\mathbb{G}} = m} c_\alpha \, x^\alpha, \quad x \in \mathbb{R}^N,$$

where c_α are real constants. In particular, the function f only depends on the variables x_j such that $\sigma_j \leq m$, and $m \geq 0$.

(ii) *A smooth vector field X on $\mathbb{R}^N$ of the form*

$$X = \sum_{j=1}^{N} a_j \, \partial_{x_j},$$

is δ_λ-homogeneous of degree $n \in \mathbb{R}$ if and only if*, for every $j = 1, \ldots, N$, the function a_j is δ_λ-homogeneous of degree $\sigma_j - n$. Equivalently,*

$$\delta_\lambda\big(XI(x)\big) = \lambda^n \, XI\big(\delta_\lambda(x)\big), \quad \textit{for all } x \in \mathbb{R}^N \textit{ and every } \lambda > 0. \quad (1.2.4)$$

By gathering together statements (i) and (ii) in the preceding theorem, we can obtain a more explicit characterization of smooth δ_λ-homogeneous vector fields of *positive degree*.

Remark 1.2.3. Let $\mathbb{G} = (\mathbb{R}^N, *, \delta_\lambda)$ be a homogeneous Lie group on $\mathbb{R}^N$ and let X be a smooth vector filed of the form

$$X = \sum_{j=1}^{N} a_j(x)\,\partial_{x_j}.$$

Let us assume that X is δ_λ-homogeneous of *positive degree* $n > 0$. It then follows from statements (i) and (ii) of Theorem 1.2.2 that, for every $j = 1, \ldots, N$, the function a_j is δ_λ-homogeneous of degree $\sigma_j - n$, hence

$$a_j(x) = \sum_{|\alpha|_{\mathbb{G}} = \sigma_j - n} c^j_\alpha\, x^\alpha, \quad x \in \mathbb{R}^N,$$

where c^j_α are real constants. In particular, if a_j is not identically vanishing, then we must have $\sigma_j \geq n$, and a_j can only depends on those variables x_i such that $\sigma_i \leq \sigma_j - n$. As a consequence, since $n > 0$, we derive that a_j can only depends on $x_1, \ldots, x_{j-1}$ and that the v.f. X is actually a "pyramid-shaped" vector field of the form (we agree to let a_j be constant when $j = 1$)

$$X = \sum_{\substack{j=1 \\ \sigma_j \geq n}}^{N} a_j(x_1, \ldots, x_{j-1})\,\partial_{x_j}.$$

We conclude this first section by briefly describing the *structure* of a homogeneous Lie group on $\mathbb{R}^N$. To this end, we present two different theorems: the first one gives a somehow explicit expression of the *composition law* of a homogeneous Lie group, while the second one shows some interesting properties of the *Lie algebra* of such a group. For a proof of these theorems see, e.g., [37, Theorem 1.3.15 and Proposition 1.3.12].

Theorem 1.2.4 (Structure of a homogeneous group). *Let $\mathbb{G} = (\mathbb{R}^N, *, \delta_\lambda)$ be a homogeneous Lie group. Then the composition law $*$ of $\mathbb{G}$ has* polynomial component functions. *More precisely, for every $x, y \in \mathbb{R}^N$ one has*

$$\begin{aligned} (x * y)_1 &= x_1 + y_1, \qquad \textit{and} \\ (x * y)_j &= x_j + y_j + P_j(x, y), \quad \textit{for every } j = 2, \ldots, N, \end{aligned}$$

where, for every $j = 2, \ldots, N$, the function P_j satisfies the following properties:

(i) *P_j only depends on those variables x_i and y_i such that $\sigma_i < \sigma_j$;*

(ii) *P_j is actually a sum of mixed monimials in x and y;*

(iii) *$P_j\big(\delta_\lambda(x), \delta_\lambda(y)\big) = \lambda^{\sigma_j}\, P_j(x, y)$, for every $x, y \in \mathbb{R}^N$.*

In particular, we have $P_j \equiv 0$ for every $j \in \{2, \ldots, N\}$ such that $\sigma_j = 1$.

Theorem 1.2.5 (Lie algebra of homogeneous groups). *Let* $\mathbb{G} = (\mathbb{R}^N, *, \delta_\lambda)$ *be a homogeneous Lie group and let* $\mathrm{Lie}(\mathbb{G})$ *be the Lie algebra of* $\mathbb{G}$*. Then* $\mathrm{Lie}(\mathbb{G})$ *is* nilpotent *of step* $r \leq \sigma_N$*, that is, every commutators of vector fields in* $\mathrm{Lie}(\mathbb{G})$ *containing more than* σ_N *terms vanishes identically.*

Moreover, if $\{J_1, \ldots, J_N\}$ *is the Jacobian basis of* $\mathrm{Lie}(\mathbb{G})$ *(see Def. 1.1.3), then for every* $j = 1, \ldots, N$ *the vector field* J_i *is* δ_λ*-homogeneous of degree* σ_i*.*

By combining Thm.s 1.2.4 and 1.2.5, we obtain the following very explicit description for the *Jacobian basis* of the Lie algebra of a homogeneous group.

Corollary 1.2.6 (Jacobian basis of a homogeneous group). *Let* $\mathbb{G} = (\mathbb{R}^N, *, \delta_\lambda)$ *be a homogeneous Lie group on* $\mathbb{R}^N$ *and let* $\mathrm{Lie}(\mathbb{G})$ *be the Lie algebra of* $\mathbb{G}$*. Moreover, let* $\{J_1, \ldots, J_N\}$ *be the Jacobian basis of* $\mathrm{Lie}(\mathbb{G})$*. Then we have*

$$\begin{aligned} &J_i = \partial_{x_i} + \sum_{\substack{j=i+1 \\ \sigma_j > \sigma_i}}^{N} a_j^{(i)}(x)\,\partial_{x_j}, \quad \textit{for every } i = 1, \ldots, N-1, \qquad \textit{and} \\ &J_N = \partial_{x_N}, \end{aligned} \tag{1.2.5}$$

where $a_j^{(i)}$ *is a smooth* δ_λ*-homogeneous polynomial of degree* $\sigma_j - \sigma_i$

Proof. By Thm. 1.2.4, the matrix $\mathcal{J}_{\tau_x}(0)$ takes the following form

$$\mathcal{J}_{\tau_x}(0) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ a_2^{(1)}(x) & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_N^{(1)}(x) & \cdots & a_N^{(N-1)}(x) & 1 \end{pmatrix},$$

where, for every $i = 1, \ldots, N-1$ and every $j = 2, \ldots, N$, we have

$$a_j^{(i)}(x) = \frac{\partial P_j}{\partial y_i}(x, 0), \quad \text{for every } x \in \mathbb{R}^N.$$

Moreover, for every $j \in \{2, \ldots, N\}$, again from Thm. 1.2.4-(i) we infer that

$$a_j^{(i)} = \frac{\partial P_j}{\partial y_i}(\cdot, 0) \equiv 0, \quad \text{for every } i \in \{1, \ldots, N-1\} \text{ s.t. } \sigma_i \geq \sigma_j.$$

From this, recalling that the Jacobian basis of $\mathrm{Lie}(\mathbb{G})$ is given by the vector fields associated with the column vectors of $\mathcal{J}_{\tau_x}(0)$, we immediately derive that $J_1, \ldots, J_N$ are precisely of the form (1.2.5).

Finally, since we know from Thm. 1.2.5 that, for every $i = 1, \ldots, N-1$, the vector field J_i is δ_λ-homogeneous of degree σ_i, it follows from Thm. 1.2.2 that, for every $j = 2, \ldots, N$, the function $a_j^{(i)}$ is a δ_λ-homogeneous polynomial of degree $\sigma_j - \sigma_i$, and the proof is complete. □

Remark 1.2.7. Let $\mathbb{G} = (\mathbb{R}^N, *, \delta_\lambda)$ be a homogeneous Lie group and let $\alpha \in \mathbb{R}^N$. By exploiting Thm. 1.2.4, it is easy to see that the Jacobian matrices of the *translations* τ_α and ρ_α are both of the following *lower triangular form*

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ \star & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \star & \cdots & \star & 1 \end{pmatrix}; \tag{1.2.6}$$

therefore, the well-known Change of Variables formula implies that

$$\mathcal{H}^N\big(\tau_\alpha(E)\big) = \mathcal{H}^N\big(\rho_\alpha(E)\big) = \mathcal{H}^N(E), \quad \text{for every Borel set } E \subseteq \mathbb{R}^N, \tag{1.2.7}$$

that is, the standard Lebesgue measure on $\mathbb{R}^N$ is *invariant with respect to the left and the right translations on* $\mathbb{G}$ (any Radon measure on $\mathbb{G}$ with these properties is called a **bi-invariant Haar measure**).

We explicitly notice that the Lebesgue measure is also *Q-homogeneous* with respect to the family of dilations $\{\delta_\lambda\}_{\lambda>0}$: in fact, since for every $\lambda > 0$ one has

$$\mathcal{J}_{\delta_\lambda}(x) = \begin{pmatrix} \lambda^{\sigma_1} & 0 & \cdots & 0 \\ 0 & \lambda^{\sigma_2} & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda^{\sigma_N} \end{pmatrix}, \quad \text{for every } x \in \mathbb{R}^N,$$

again from the Change of Variables formula we obtain

$$\mathcal{H}^N\big(\delta_\lambda(E)\big) = \lambda^Q\,\mathcal{H}^N(E), \quad \text{for every Borel set } E \subseteq \mathbb{R}^N, \tag{1.2.8}$$

where $Q = \sum_{j=1}^N \sigma_j$ is the homogeneous dimension of $\mathbb{G}$ (see (1.2.1)).

Carnot groups

We conclude this second section by turning our attention to a particular subclass of homogeneous Lie groups, namely, the Carnot groups. Such a class of Lie groups is widely studied in the literature, and it represents the starting point for all the investigations carried out in this thesis.

Let $\mathbb{G} = (\mathbb{R}^N, *, \delta_\lambda)$ be a homogeneous Lie group on $\mathbb{R}^N$ (according to the definition recalled above). We say that $\mathbb{G}$ is a **homogeneous Carnot group** (or simply a **Carnot group**) if the vector fields of the Jacobian basis which are δ_λ-homogeneous of degree 1 form a set of Lie-generators of $\mathrm{Lie}(\mathbb{G})$.

The number m of such Jacobian vector fields is called the **number of generators** of the Carnot group $\mathbb{G}$.

Since Carnot groups are, in particular, homogeneous groups, all the results recalled so far do apply to such a class of groups; on the other hand, the structure of a Carnot group can be described in a very precise way, as the following theorem shows (for a proof see, e.g., [37, Section 1.4]).

Theorem 1.2.8. *Let* $\mathbb{G} = (\mathbb{R}^N, *, \delta_\lambda)$ *be a homogeneous Carnot group and let* $\mathrm{Lie}(\mathbb{G})$ *be the Lie algebra of* $\mathbb{G}$. *We define*

$$V_1 := \{X \in \mathrm{Lie}(\mathbb{G}) : X \text{ is } \delta_\lambda\text{-homogeneous of degree } 1\}.$$

Then the following facts hold true:

(i) *the exponents* $\sigma_1, \ldots, \sigma_N$ *are consecutive integers;*

(ii) $r = \sigma_N$ *is the step of nilpotency of* $\mathrm{Lie}(\mathbb{G})$*;*

(iii) $\mathrm{Lie}(\mathbb{G}) = V_1 \oplus \cdots \oplus V_r$, *where*

$$V_{i+1} = [V_1, V_i] \quad \text{for every } 1 \le i \le r-1, \qquad \text{and} \quad [V_1, V_r] = \{0\}.$$

Furthermore, if $1 \le i \le r$*, the vector space* V_i *is generated by the elements of the Jacobian basis which are* δ_λ*-homogeneous of degree* i*, and it coincides with the set of vector fields in* $\mathrm{Lie}(\mathbb{G})$ *which are* δ_λ*-homogeneous of degree* i*.*

It is worth noting that *not all homogeneous groups* are Carnot groups. For example, the classical Euclidean group $\mathbb{E} = (\mathbb{R}^2, +)$ is endowed with a structure of homogeneous group by the family of dilations $\{\delta_\lambda\}_{\lambda>0}$ given by

$$\delta_\lambda : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \qquad \delta_\lambda(x) = (\lambda x_1, \lambda^2 x_2).$$

However, $\mathbb{E} = (\mathbb{R}^2, +, \delta_\lambda)$ *is not* a Carnot group: in fact, the Jacobian basis of $\mathrm{Lie}(\mathbb{E})$ is given by $\mathfrak{J} = \{J_1 = \partial_{x_1}, J_2 = \partial_{x_2}\}$, and J_1 is the unique element of $\mathfrak{J}$ which is δ_λ-homogeneous of degree 1; since J_1 does not Lie-generates $\mathrm{Lie}(\mathbb{G})$, we conclude that $\mathbb{E}$ is not a Carnot group.

Let $\mathbb{G} = (\mathbb{R}^N, *, \delta_\lambda)$ be a homogeneous Carnot group, with m generators and nilpotent of step $r = \sigma_N$, and let $\mathfrak{g} = \mathrm{Lie}(\mathbb{G})$ be the Lie algebra of $\mathbb{G}$. Recalling that $\sigma_1, \dots, \sigma_N$ are consecutive integers between 1 and r, we can define a r-tuple $(N_1, \dots, N_r)$ of natural numbers in the following way:

$$N_i = \mathrm{card}\big\{j \in \{1, \dots, N\} : \sigma_j = i\big\}, \quad \text{for every } 1 \le i \le r$$

(note that, since $\mathbb{G}$ has m generators, we have $N_1 = m$). If we now denote a point $x \in \mathbb{R}^N$ by $x = (x^{(1)}, \dots, x^{(r)})$, with

$$x^{(i)} = (x_1^{(i)}, \dots, x_{N_i}^{(i)}) \in \mathbb{R}^{N_i}, \quad \text{for every } i = 1, \dots, r,$$

we have the split $\mathbb{R}^N = \mathbb{R}^{N_1} \times \cdots \mathbb{R}^{N_r}$ and we can write, for every $\lambda > 0$,

$$\delta_\lambda(x) = (\lambda x^{(1)}, \lambda^2 x^{(2)}, \dots, \lambda^r x^{(r)}), \quad \text{for every } x \in \mathbb{R}^N;$$

moreover, taking into account Thm. 1.2.4, for every $x, y \in \mathbb{R}^N$ we have

$$\begin{aligned} (x * y)^{(1)} &= x^{(1)} + y^{(1)}, \qquad \text{and} \\ (x * y)^{(j)} &= x^{(j)} + y^{(j)} + Q^{(j)}(x, y), \quad \text{for every } j = 2, \dots, r, \end{aligned}$$

where, for every $2 \le j \le r$, $Q^{(j)}$ is a $\mathbb{R}^{N_j}$-valued function such that

(i) $Q^{(j)}$ only depends on $x^{(1)}, \dots, x^{(j-1)}$ and $y^{(1)}, \dots, y^{(j-1)}$;

(ii) the components of $Q^{(j)}$ are sums of mixed monimials in x and y;

(iii) $Q^{(j)}\big(\delta_\lambda(x), \delta_\lambda(y)\big) = \lambda^j\, Q^{(j)}(x, y)$, for every $x, y \in \mathbb{R}^N$.

Finally, if we introduce the notation

$$J_1^{(1)}, \dots, J_{N_1}^{(1)}, \dots, J_1^{(r)}, \dots, J_{N_r}^{(r)},$$

for the Jacobian basis of $\mathrm{Lie}(\mathbb{G})$, we know from Thm. 1.2.5 that $J_i^{(p)}$ is δ_λ-homogeneous of degree p, and we derive form Cor. 1.2.6 that

$$J_i^{(p)} = \partial/\partial x_i^{(p)} + \sum_{h=p+1}^{r} \sum_{k=1}^{N_k} a_{jk}^{(ph)}(x)\, \partial/\partial x_k^{(h)},$$

where, the function $a_{jk}^{(ph)}$ is a δ_λ-homogeneous polynomial of degree $n_h - n_p$ (we agree to let the sum in the right-hand side be equal to 0 when $p = r$).

We conclude this section by briefly introducing a distinguished class of Carnot groups, namely, the *(prototype) groups of Heisenberg-type*.

Example 1.2.9 ((Prototype) H-type groups)**.** Let $m, n \in \mathbb{N}$ with $m \geq 2$ and let $B^{(1)}, \ldots, B^{(n)}$ be $m \times m$ matrices satisfying the following properties:

(1) $B^{(i)}$ is skew-symmetric and orthogonal for every $i = 1, \ldots, n$;

(2) $B^{(i)} B^{(j)} = -B^{(j)} B^{(i)}$ for every $i, j = 1, \ldots, n$ with $i \neq j$.

We then denote a generic point z in the product $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$ with $z = (x, t)$, where $x \in \mathbb{R}^m$ and $t \in \mathbb{R}^n$, and we define

$$\begin{aligned}&(x,t) * (\xi, \tau) := \\ &\quad \left(x + \xi, t_1 + \tau_1 + \frac{1}{2} \langle B^{(1)} x, \xi \rangle, \ldots, t_n + \tau_n + \frac{1}{2} \langle B^{(n)} x, \xi \rangle \right).\end{aligned} \tag{1.2.9}$$

It is very easy to recognize that $(\mathbb{R}^N, *)$ is a Lie group on $\mathbb{R}^N$, with neutral element 0 and where the inverse of an element $z = (x, t)$ is given by

$$z^{-1} = -z = (-x, t);$$

moreover, if we consider the family of dilations $\{\delta_\lambda\}_{\lambda > 0}$ given by

$$\delta_\lambda : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \qquad \delta_\lambda(z) = \delta_\lambda(x, t) = (\lambda x, \lambda^2 t), \tag{1.2.10}$$

it is not difficult to see that $\mathbb{H} = (\mathbb{R}^N, *, \delta_\lambda)$ is a homogeneous Lie group.

Now, a direct computation shows that, for every $z = (x, t) \in \mathbb{R}^N$, the Jacobian matrix of the left-translation τ_z at 0 takes the following block form:

$$\mathcal{J}_{\tau_z}(0) = \begin{pmatrix} & \mathrm{I}_m & & \mathbf{0}_{m \times n} \\ \frac{1}{2}\left(B^{(1)} x\right)_1 & \cdots & \frac{1}{2}\left(B^{(1)} x\right)_m & \\ \vdots & & \vdots & \mathrm{I}_n \\ \frac{1}{2}\left(B^{(n)} x\right)_1 & \cdots & \frac{1}{2}\left(B^{(n)} x\right)_m & \end{pmatrix};$$

therefore, according to Rem. 1.1.5, the Jacobian basis of Lie($\mathbb{G}$) is given by

$$\begin{aligned}&J_i^{(1)} = \partial/\partial x_i + \frac{1}{2} \sum_{j=1}^{n} (B^{(j)} x)_i \, \partial/\partial t_j, \quad \text{for every } i = 1, \ldots, m; \\ &J_j^{(2)} = \partial/\partial t_j, \quad \text{for every } j = 1, \ldots, n.\end{aligned} \tag{1.2.11}$$

Since conditions (1) and (2) imply the linear independence of the matrices $B^{(1)}, \ldots, B^{(n)}$, from (1.2.11) one can easily infer that $J_1^{(1)}, \ldots, J_m^{(1)}$, which are precisely those element of the Jacobian basis that are δ_λ-homogeneous of degree one, form a set of Lie-generators for Lie($\mathbb{H}$); therefore, according to the definition, $\mathbb{H}$ is a Carnot group, which is called a **(prototype) group of Heisenberg-type**, or simply a **(prototype) group of H-type**.

We point out that $\mathbb{H}$ has m generators and that Lie($\mathbb{H}$) is nilpotent of step 2; furthermore, we have Lie($\mathbb{H}$) $= V_1 \oplus V_2$, where

$$V_1 = \mathrm{span}(\{J_1^{(1)}, \ldots, J_m^{(1)}\}) \text{ and } V_2 = [V_1, V_1] = \mathrm{span}(\{J_1^{(2)}, \ldots, J_n^{(2)}\}).$$

1.3 Sub-Laplace operators on Carnot groups

The aim of this last section is to briefly introduce the sub-Laplace operators on Carnot groups: we shall recall the main definition and properties concerning such operators, and we will shortly describe the result by Folland on the existence of a (global) fundamental solution.

1.3.1 Main definitions and properties

Let $\mathbb{G} = (\mathbb{R}^N, *, \delta_\lambda)$ be a fixed homogeneous Carnot group on $\mathbb{R}^N$, with m generators and nilpotent of step r. As usual, we denote by $J_1, \ldots, J_N$ the N element of the Jacobian basis of $\mathrm{Lie}(\mathbb{G})$; accordingly, $J_1, \ldots, J_m$ are those elements of the basis which are δ_λ-homogeneous of degree 1. We also set

$$V_1 := \mathrm{span}(\{J_1, \ldots, J_m\}) = \{X \in \mathrm{Lie}(\mathbb{G})\colon\ X \text{ is } \delta_\lambda\text{-homogneous of degree } 1\}.$$

If $\mathcal{X} = \{X_1, \ldots, X_m\}$ is any (linear) basis of V_1, the second-order PDO

$$\mathcal{L} = \sum_{j=1}^{m} X_j^2$$

is called a **sub-Laplace operator** (or simply a **sub-Laplacian**) on $\mathbb{G}$. In particular, if we take $\mathcal{X} = \{J_1, \ldots, J_m\}$, the operator

$$\Delta_{\mathbb{G}} := \sum_{i=1}^{m} J_i^2,$$

is called the **canonical sub-Laplacian** on $\mathbb{G}$.

Example 1.3.1 (Sub-Laplacians on (prototype) H-groups)**.** Let $m, n \in \mathbb{N}$ with $m \geq 2$ and let $\mathbb{H} = (\mathbb{R}^N, *, \delta_\lambda)$ be (prototype) H-group on $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$ as in Exm. 1.2.9 (with group law $*$ given by (1.2.9) and dilations as in (1.2.10)).

By means of the expression of the Jacobian basis of $\mathrm{Lie}(\mathbb{H})$ obtained in (1.2.11) (and recalling that $B^{(1)}, \ldots, B^{(n)}$ are skew-symmetric and orthogonal), we can write explicitly the canonical sub-Laplacian on $\mathbb{H}$: we have

$$\begin{aligned}\Delta_{\mathbb{H}} &= \sum_{i=1}^{m} \partial_{x_i}^2 + \frac{1}{4}\,\|x\|^2 \sum_{i=1}^{n} \partial_{t_i}^2 + \sum_{i=1}^{m}\sum_{j=1}^{n} (B^{(j)}x)_i\, \partial_{x_i t_j} \\ &= \Delta_x + \frac{1}{4}\,\|x\|^2\, \Delta_t + \sum_{j=1}^{n} \langle B^{(j)}x, \nabla_x \rangle\, \partial_{t_j}.\end{aligned}$$

In particular, $\Delta_{\mathbb{H}}$ does not contain first order terms.

We now would like to list some simple yet important properties of any sub-Laplacian $\mathcal{L} = \sum_{j=1}^{m} X_j^2$ on $\mathbb{G}$, directly following from the properties of the vector fields $J_1, \ldots, J_m$. For a proof we refer, e.g., to [37, Section 1.5].

(P1) $\mathcal{L}$ is *invariant* w.r.t. the left-translations on $\mathbb{G}$, i.e., for every $\alpha \in \mathbb{R}^N$

$$\mathcal{L}\big(u \circ \tau_\alpha\big) = \big(\mathcal{L}u\big) \circ \tau_\alpha, \quad \text{for every } u \in C^\infty(\mathbb{R}^N, \mathbb{R}). \tag{1.3.1}$$

(P2) $\mathcal{L}_X$ is δ_λ-homogeneous of degree 2, i.e., for every $\lambda > 0$ one as

$$\mathcal{L}\big(u \circ \delta_\lambda\big) = \lambda^2 \,\big(\mathcal{L}u\big) \circ \delta_\lambda, \quad \text{for every } u \in C^\infty(\mathbb{R}^N, \mathbb{R}). \tag{1.3.2}$$

(P3) The operator $\mathcal{L}$ is of the following divergence form

$$\mathcal{L} = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{N} a_{i,j}(x) \, \frac{\partial}{\partial x_j} \right), \tag{1.3.3}$$

where the principal matrix $A(x) = (a_{i,j}(x))_{i,j}$ is given by the product

$$A(x) = S(x)\, S(x)^t, \quad \text{for every } x \in \mathbb{R}^N, \tag{1.3.4}$$

and $S(x)$ is the $N \times m$ matrix whose columns are the coefficient vectors of $X_1, \ldots, X_m$, that is,

$$S(x) = \big(X_1 I(x) \cdots X_m I(x)\big), \quad \text{for every } x \in \mathbb{R}^N. \tag{1.3.5}$$

As a consequence, $\mathcal{L}$ has (smooth) polynomial coefficients and it is (formally) self-adjoint on the space $L^2(\mathbb{R}^N)$, when restricted to smoothly and compactly supported functions, that is,

$$\int_{\mathbb{R}^N} \varphi \, \mathcal{L}\psi \, \mathrm{d}x = \int_{\mathbb{R}^N} \psi \, \mathcal{L}\varphi \, \mathrm{d}x, \quad \text{for every } \varphi, \psi \in C_0^\infty(\mathbb{R}^N, \mathbb{R}).$$

(P4) The principal $m \times m$ minor

$$A_{1,1}(x) = \begin{pmatrix} a_{1,1}(x) & \cdots & a_{1,m}(x) \\ \vdots & \ddots & \vdots \\ a_{m,1}(x) & \cdots & a_{m,m}(x) \end{pmatrix}, \quad x \in \mathbb{R}^N,$$

of the principal matrix $A(x)$ of $\mathcal{L}$ is *constant, symmetric and positive definite*. As a consequence, there exists $i \in \{1, \ldots, m\}$ such that

$$a_{ii} > 0. \tag{1.3.6}$$

(P5) $\mathcal{L}$ is C^∞-hypoelliptic on every open subset of $\mathbb{R}^N$, i.e., every distributional solution to the equation $\mathcal{L}u = f$ is of class C^∞ whenever f is of class C^∞.

Properties (P1)-to-(P5) listed above posses a large number of interesting consequences; we conclude this first part of the section by highlighting a couple of them, which will be important for us in the sequel.

For a proof of the following results we refer, e.g., to [37, Section 5.13]

Theorem 1.3.2 (Strong Maximum Principle for $\mathcal{L}$)**.** *Let $\mathbb{G} = (\mathbb{R}^N, *, \delta_\lambda)$ be a homogeneous Carnot group and let $\mathcal{L}$ be a sub-Laplacian on $\mathbb{G}$. Moreover, let $\Omega \subseteq \mathbb{R}^N$ be an open and connected set and let $u \in C^2(\Omega, \mathbb{R})$ be such that*

$$u \leq 0 \quad \text{and} \quad \mathcal{L}u \geq 0 \quad \text{in } \Omega.$$

If there exists a point $x_0 \in \Omega$ such that $u(x_0) = 0$, then $u \equiv 0$ throughout Ω.

Theorem 1.3.3 (Weak Maximum Principle for $\mathcal{L}$). *Let $\mathbb{G} = (\mathbb{R}^N, *, \delta_\lambda)$ be a homogeneous Carnot group and let $\mathcal{L}$ be a sub-Laplacian on $\mathbb{G}$. Moreover, let $\Omega \subseteq \mathbb{R}^N$ be an open and bounded set and let $u \in C^2(\Omega, \mathbb{R})$ be such that*

$$\begin{cases} \mathcal{L}u \geq 0 \text{ on } \Omega, \\ \limsup_{x \to \xi} u(x) \leq 0 \text{ for every } \xi \in \partial\Omega. \end{cases}$$

Then $u(x) \leq 0$ for every $x \in \Omega$.

Corollary 1.3.4. *Let $\mathbb{G} = (\mathbb{R}^N, *, \delta_\lambda)$ be a homogeneous Carnot group and let $\mathcal{L}$ be a sub-Laplacian on $\mathbb{G}$. If $u \in C^2(\mathbb{R}^N, \mathbb{R})$ is such that*

$$\mathcal{L}u = 0 \text{ on } \mathbb{R}^N \text{ and } \lim_{\|x\| \to \infty} u(x) = 0,$$

then $u(x) = 0$ for every $\mathbb{R}^N$.

1.3.2 Fundamental solution

As anticipated above, we conclude this section by briefly describing a deep result due to Folland, which will be of fundamental interest for us in Chpt. 3: roughly put, it ensures the existence (and the uniqueness) of a smooth global fundamental solution for any sub-Laplacian on a Carnot group.

To begin with, since there is no common agreement on the notion of what fundamental solutions are, we fix the relevant definitions.
In what follows we use the notation

$$D_x^\alpha = \Big(\frac{\partial}{\partial x}\Big)^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$

for higher order derivatives on $\mathbb{R}^N$. Here $\alpha = (\alpha_1, \ldots, \alpha_N) \in (\mathbb{N} \cup \{0\})^N$ and $|\alpha| = \alpha_1 + \cdots + \alpha_N$ is the length of α.

Definition 1.3.5 (Fundamental solution). On Euclidean space $\mathbb{R}^N$, we consider a generic linear partial differential operator of order $d \in \mathbb{N}$,

$$P = \sum_{|\alpha| \leq d} a_\alpha(x)\, D_x^\alpha,$$

with smooth real valued coefficients $a_\alpha(x)$ on $\mathbb{R}^N$. We say that a function

$$\Gamma : \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y\} \longrightarrow \mathbb{R},$$

is a **(global) fundamental solution for** P if it satisfies the following property:

(FS) For every $x \in \mathbb{R}^n$, the function $\Gamma(x; \cdot)$ is locally integrable on $\mathbb{R}^N$ and

$$\int_{\mathbb{R}^N} \Gamma(x; y)\, P^*\varphi(y)\, \mathrm{d}y = -\varphi(x) \qquad \text{for every } \varphi \in C_0^\infty(\mathbb{R}^N, \mathbb{R}), \quad (1.3.7)$$

where P^* denotes the usual formal adjoint of P [1].

[1] We point out that, since both the coefficient functions of the operator P and the test functions considered in identity (1.3.7) are assumed to be *real-valued*, the formal adjoint P^* of P actually coincide with the transpose P^T.

It is worth noting that, if P is a linear PDO as in Def. 1.3.5 and if Γ is a fundamental solution for P, identity (1.3.7) can be re-written as follows:

$$P\Gamma_x = -\mathrm{Dir}_x \quad \text{in } \mathcal{D}'(\mathbb{R}^N), \tag{1.3.8}$$

where Dir_x is the Dirac distribution supported at $\{x\}$.

Example 1.3.6 (The Laplace operator)**.** Probably, one of the best examples of fundamental solution is the one of the Laplace operator: if Δ is the classical Laplace operator on $\mathbb{R}^N$, with $N \geq 3$, then a (global) fundamental solution for Δ is given by the function

$$\Gamma(x, y) = \frac{1}{N\,\omega_N}\,\|x - y\|^{2-N}, \quad \text{with } x \neq y,$$

where ω_N denotes the (Lebesgue) measure of the Euclidean ball $B(0, 1)$.

Note that, together with property (FS) in Def. 1.3.5, the function Γ also satisfies the following additional properties:

(i) Γ is smooth and strictly positive out of the diagonal of $\mathbb{R}^N \times \mathbb{R}^N$;

(ii) $\Gamma(x, y) = \Gamma(y, x)$ for every $x, y \in \mathbb{R}^N$ with $x \neq y$;

(iii) $\Gamma \in L^1_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R}^N)$ and, for every fixed $y \in \mathbb{R}^N$, $\Gamma(\cdot; y) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$;

(iv) for every fixed $x \in \mathbb{R}^N$, the function $y \mapsto \Gamma(x; y)$ vanishes as $\|y\| \to \infty$;

(v) for every fixed $x \in \mathbb{R}^N$, the function $y \mapsto \Gamma(x; y)$ goes to ∞ as $y \to x$.

Before proceeding to the description of Folland's result, we give some remarks concerning the problem of the existence and the uniqueness of a global fundamental solution for a general linear PDO.

Remark 1.3.7. (a) The existence of a global fundamental solution for P is far from being obvious and it is, in general, a very delicate issue. In the particular case of C^∞-hypoelliptic linear PDOs P having a C^∞-hypoelliptic formal adjoint P^*, it is possible to prove the *local* existence of a fundamental solution on a suitable neighborhood of each point of $\mathbb{R}^n$ (see, e.g., [131]); moreover, in [39] Bony showed that any Hörmander operator admits a smooth fundamental solution on every bounded open set satisfying suitable regularity assumptions.

(b) Fundamental solutions are, in general, not unique since the addition of a P-harmonic function (that is, a smooth function h such that $Ph = 0$ in $\mathbb{R}^N$) to a fundamental solution produces another fundamental solution.

(c) Nonetheless, if P is a second order C^∞-hypoelliptic operator which fulfills the Weak Maximum Principle on every bounded open set of $\mathbb{R}^N$, then there exists at most one fundamental solution Γ for P such that

$$\lim_{\|y\|\to\infty} \Gamma(x, y) = 0, \quad \text{for every } x \in \mathbb{R}^N.$$

Indeed, if Γ_1, Γ_2 are two such functions, then (for every fixed $x \in \mathbb{R}^N$) the map $u_x := \Gamma_1(x, \cdot) - \Gamma_2(x, \cdot)$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and it is a solution of $Pu_x = 0$ in the sense of distributions on $\mathbb{R}^N$; the hypoellipticity of P ensures that u_x is (a.e. equal to) a smooth function on $\mathbb{R}^N$ which vanishes at infinity by the assumptions on Γ_1, Γ_2; from the Weak Maximum Principle for P it is readily obtained that $u_x \equiv 0$ (a.e.), that is, $\Gamma_1 \equiv \Gamma_2$ (a.e.). When continuity of $\Gamma(x, \cdot)$ is also requested, this gives $\Gamma_1 \equiv \Gamma_2$.

Now we have specified what we mean by fundamental solution of a general linear PDO, we can turn our attention to the particular case of the sub-Laplace operators on Carnot groups. In order to clearly state the notable result by Folland, we first recall the notion of homogeneous norm on a Carnot group.

Definition 1.3.8 (Homogeneous norm)**.** Let $\mathbb{G} = (\mathbb{R}^N, *, \delta_\lambda)$ be a homogeneous Carnot group on $\mathbb{R}^N$. We say that a *continuous* function

$$d : \mathbb{R}^N \longrightarrow [0, \infty[$$

is a **homogeneous norm on** $\mathbb{G}$ if it satisfies the following properties:

(i) $d(\delta_\lambda(x)) = \lambda d(x)$ for every $\lambda > 0$ and every $x \in \mathbb{R}^N$;

(ii) $d(x) > 0$ for every $x \in \mathbb{R}^N$ with $x \neq 0$.

Furthermore, we say that d is **symmetric** if

$$d(x^{-1}) = d(x), \quad \text{for every } x \in \mathbb{R}^N.$$

Let $\mathbb{G} = (\mathbb{R}^N, *, \delta_\lambda)$ be a homogeneous Carnot group and let $\sigma_1, \ldots, \sigma_N$ be the exponents in the family of dilations $\{\delta_\lambda\}_{\lambda>0}$ (recall that $\sigma_1, \ldots, \sigma_N$ are consecutive integers and that $\sigma_1 = 1$). Then, setting

$$\| \cdot \|_{\mathbb{G}} : \mathbb{R}^N \longrightarrow [0, \infty[, \qquad \|x\|_{\mathbb{G}} := \sum_{j=1}^{N} |x_j|^{1/\sigma_j}, \tag{1.3.9}$$

it is straightforward to recognize that $\| \cdot \|_{\mathbb{G}}$ is a homogeneous norm on $\mathbb{G}$, which is symmetric if $x^{-1} = -x$ for all $x \in \mathbb{G}$. Actually, it is very easy to prove that *all* the homogeneous norms on $\mathbb{G}$ are equivalent to $\| \cdot \|_{\mathbb{G}}$: more precisely, if d is a homogeneous norm on $\mathbb{G}$, there exists a constant $\mathbf{c} > 0$ such that

$$\mathbf{c}^{-1} \|x\|_{\mathbb{G}} \leq d(x) \leq \mathbf{c} \|x\|_{\mathbb{G}}, \qquad \text{for every } x \in \mathbb{R}^N. \tag{1.3.10}$$

With the notion of homogeneous norm at hand, we are finally in a position to state the announced theorem by Folland (the complete proof of this result can be found in [72, Theorem 2.1]; see also [37, Sections 5.1 and 5.3]).

Theorem 1.3.9 (Existence of the fundamental solution)**.** *Let $\mathbb{G} = (\mathbb{R}^N, *, \delta_\lambda)$ be a homogeneous Carnot group (with homogeneous dimension $Q > 2$) and let $\mathcal{L}$ be a sub-Laplacian on $\mathbb{G}$. It is then possible to find a homogeneous symmetric norm $d \in C^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{R})$ such that the function*

$$\Gamma : \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y\} \to \mathbb{R}, \quad \Gamma(x, y) := d^{2-Q}\big(x^{-1} * y\big), \tag{1.3.11}$$

is a fundamental solution for $\mathcal{L}$, further satisfying the following properties:

- Γ *is smooth and strictly positive out of the diagonal of* $\mathbb{R}^N \times \mathbb{R}^N$*;*
- Γ *is symmetric, that is,*

$$\Gamma(x, y) = \Gamma(y, x), \quad \textit{for every } x, y \in \mathbb{R}^N \textit{ with } x \neq y;$$

- Γ *is* δ_λ*-homogeneous of degree* $2-Q$*, that is, for every* $\lambda > 0$ *we have*

$$\Gamma\big(\delta_\lambda(x), \delta_\lambda(y)\big) = \lambda^{2-Q}\,\Gamma(x,y), \quad \textit{for every } x, y \in \mathbb{R}^N \textit{ with } x \neq y;$$

- *For every* $x \in \mathbb{R}^N$, $\Gamma(x,\cdot)$ *has a pole at* x *and it vanishes at infinity, i.e.,*

$$\lim_{y\to x} \Gamma(x,y) = \infty \qquad \textit{and} \qquad \lim_{\|y\|\to\infty} \Gamma(x,y) = 0.$$

Remark 1.3.10 (Uniqueness of the fundamental solution)**.** Let $\mathbb{G} = (\mathbb{R}^N, *, \delta_\lambda)$ be a homogeneous Carnot group on $\mathbb{R}^N$ and let $\mathcal{L}$ be a sub-Laplacian on $\mathbb{G}$.

Since the operator $\mathcal{L}$ is C^∞ hypoelliptic (see property **(P5)** on page 14) and since it satisfies the Weak Maximum Principle on every open and bounded subset of $\mathbb{R}^N$ (see Thm. 1.3.3), we infer from Rem. 1.3.7-(c) that the function Γ in Thm. 1.3.9 is the *unique* global fundamental solution for $\mathcal{L}$ such that

$$\lim_{\|y\|\to\infty} \Gamma(x,y) = 0, \quad \text{for every fixed } x \in \mathbb{R}^N.$$

Example 1.3.11 (The case of H-type groups)**.** Let $m, n \in \mathbb{N}$ with $m \geq 2$ and let $\mathbb{H} = (\mathbb{R}^N, *, \delta_\lambda)$ be (prototype) H-group on $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$ as in Exm. 1.2.9 (with group law $*$ given by (1.2.9) and dilations as in (1.2.10)). Denoting by (x,t) the points of $\mathbb{H}$, with $x \in \mathbb{R}^n$ and $t \in \mathbb{R}^m$, we know that the canonical sub-Laplacian $\Delta_{\mathbb{H}}$ on $\mathbb{H}$ takes the form (see Exm. 1.3.1)

$$\Delta_{\mathbb{H}} = \Delta_x + \frac{1}{4}\,\|x\|^2\,\Delta_t + \sum_{j=1}^{n} \langle B^{(j)}x, \nabla_x\rangle\,\partial_{t_j};$$

then, by a notable result by Kaplan [97], the (unique) global fundamental solution Γ of $\Delta_{\mathbb{H}}$ is *explicitly known*: if we set

$$d_{\mathbb{H}} : \mathbb{R}^N \longrightarrow [0,\infty[, \qquad d_{\mathbb{H}}(x,t) = \Big(\|x\|^4 + 16\,\|t\|^2\Big)^{1/4},$$

there exists a constant $c > 0$ such that, for every $x, y \in \mathbb{R}^N$ with $x \neq y$,

$$\Gamma(x,y) = c\, d_{\mathbb{H}}^{2-Q}\big(y^{-1} * x\big).$$

The constant c is somehow a "geometrical" constant, and it can be expressed as the integral of a suitable kernel, depending only on $d_{\mathbb{H}}$ (see [37, Theorem 5.5.6]).

Chapter 2

PDOs structured on complete vector fields

In this second chapter of the thesis we shall be concerned with linear partial differential operators (PDOs, in the sequel) of the following form

$$\mathcal{L} = \sum_{j=1}^{m} X_j^2 + X_0,$$

where $X_1, \dots, X_m$ and X_0 are smooth vector fields defined on $\mathbb{R}^N$. Obviously, a sufficient condition for $\mathcal{L}$ to be left-invariant w.r.t. some Lie group structure $\mathbb{G} = (\mathbb{R}^N, *)$ is that each $X_0, \dots, X_m$ belongs to the Lie algebra $\mathrm{Lie}(\mathbb{G})$ of $\mathbb{G}$, and the convenience to deal with left-invariant PDOs (both for the analysis of PDOs and PDEs) needs no further justifications.

Motivated by this fact, we shall dedicate Sec. 2.1.1 to the study of finite-dimensional Lie algebras of vector fields: more precisely, we shall provide *both necessary and sufficient conditions* for a Lie algebra $\mathfrak{g}$ to *coincide with* the Lie algebra of a Lie group $\mathbb{G}$ (on $\mathbb{R}^N$). In Sec. 2.2, instead, we shall consider second-order linear PDOs L of the general form

$$L = \sum_{i,j=1}^{N} a_{i,j}(x)\,\partial_{x_i x_j} + \sum_{j=1}^{N} b_j(x)\,\partial_{x_j},$$

(with smooth coefficients $a_{i,j}$ and b_j) and we shall present sufficient conditions allowing L to be re-written as a sum of squares of *smooth* vector fields.

2.1 Characterization of left-invariance

As anticipated, the main aim of this section is to provide an exhaustive answer to the following very natural question:

(Q) Given a Lie sub-algebra $\mathfrak{g}$ of the smooth vector fields on $\mathbb{R}^N$, is it possible to find a Lie group $\mathbb{G} = (\mathbb{R}^N, *)$ on $\mathbb{R}^N$ such that $\mathrm{Lie}(\mathbb{G}) = \mathfrak{g}$?

It is clear that, if we do not assume any hypothesis on $\mathfrak{g}$, the answer is negative: for example, if a vector field in $\mathfrak{g}$ is not global, then $\mathfrak{g}$ cannot be the

Lie algebra of any Lie group on $\mathbb{R}^N$ (see Prop. 1.1.6 on page 5). Taking into account the results recalled in Chpt. 1, it is not difficult to find some *necessary* conditions for (Q) to have a positive answer:

1. every $X \in \mathfrak{g}$ must be a global vector field, i.e., all of its integral curves must be globally defined on the real line;
2. $\mathfrak{g}$ must satisfy Hörmander's rank condition:
$$\dim\big(\{XI(x) \in \mathbb{R}^N \,:\, X \in \mathfrak{g}\}\big) = N, \quad \text{for every } x \in \mathbb{R}^N;$$
3. the dimension of $\mathfrak{g}$, as a linear subspace of $\mathcal{X}(\mathbb{R}^N)$, must be equal to N.

The main result of this chapter shows that, if all the vector fields in $\mathfrak{g}$ are assumed to be real analytic on $\mathbb{R}^N$, then the above conditions a re also a set of *independent and sufficient* conditions for $\mathfrak{g}$ to coincide with the Lie algebra of an analytic Lie group $\mathbb{G} = (\mathbb{R}^N, *)$ on $\mathbb{R}^N$. The tools we will use in order to prove this fact are the following:

- the Campbell-Baker-Hausdorff-Dynkin Theorem (for composition of flows of vector fields) in order to equip $\mathbb{R}^N$ with a local Lie-group structure;
- the use of a completeness result for time-dependent vector fields, in order to globalize this local Lie group (here, the hypothesis of real-analyticity of the vector fields in $\mathfrak{g}$ plays a crucial role).

As regards question (Q), we highlight the paper by Bonfiglioli and Lanconelli [32], where it is proved that, if $\mathfrak{g}$ is a Lie algebra of real-analytic vector fields satisfying the above (1)-to-(3) plus the assumption that *the local Lie group attached to* $\mathfrak{g}$ *can be globalized*, then it is possible to positively answer to (Q).

In the subsequent Sec. 2.1.3 we show that the latter Bonfiglioli and Lanconelli's globalization assumption is automatically guaranteed by the validity of (1)-to-(3); in this perspective, we give an improvement of Theorem 1.1 in [32].

2.1.1 Exponential map and Logarithmic map

The main goal of this section is to introduce, for a selected class of Lie algebras $\mathfrak{g} \subseteq \mathcal{X}(\mathbb{R}^N)$, the exponential map and the logarithmic map. Such maps will be fundamental to answer question (Q) posed above.

Definition 2.1.1. Let $\mathfrak{g} \subseteq \mathcal{X}(\mathbb{R}^N)$. We shall say that $\mathfrak{g}$ satisfies hypothesis

(C): if every $X \in \mathfrak{g}$ is a global vector field;

(H): if Hörmander's rank condition holds:
$$\dim\big(\{XI(x) \in \mathbb{R}^N \,:\, X \in \mathfrak{g}\}\big) = N \quad \text{for every } x \in \mathbb{R}^N; \tag{2.1.1}$$

(ND): if $\mathfrak{g}$ is N-dimensional, as a linear subspace of $\mathcal{X}(\mathbb{R}^N)$.

Remark 2.1.2. In order to distinguish the two dimensions appearing in conditions (H) and (ND) we observe that, for every linear subspace V of $\mathcal{X}(\mathbb{R}^N)$ and every $x \in \mathbb{R}^N$, one has

$$\dim\big(\{XI(x) \in \mathbb{R}^N \,:\, X \in V\}\big) \le \dim(V). \tag{2.1.2}$$

Indeed, setting $\Lambda_x : V \to \mathbb{R}^N,\ \Lambda(X) := XI(x)$, the map Λ_x is linear and

$$\Lambda_x(V) = \{XI(x) : X \in V\};$$

hence, $\dim\big(\Lambda_x(V)\big) \le \dim(V)$, as claimed.

We have already remarked that conditions (C), (H) and (ND) in Def. 2.1.1 are necessary for question (Q) to have a positive answer. We now highlight the independence of these conditions with the aid of the following examples.

Example 2.1.3 ((H)+(ND)$\not\Rightarrow$(C))**.** Let us consider, in $\mathcal{X}(\mathbb{R})$, the v.f.

$$X := (1 + x_1^2)\,\frac{\partial}{\partial x_1},$$

and let $\mathfrak{g} := \mathrm{Lie}\{X\}$. It is easy to recognize that $\mathfrak{g}$ satisfies conditions (H) and (ND) with $N = 1$, but it violates (C): indeed, the integral curve of X starting at 0 is the function $t \mapsto \tan t$, which is not defined on the whole of $\mathbb{R}$.

Example 2.1.4 ((C)+(ND)$\not\Rightarrow$(H))**.** Let us consider, in $\mathcal{X}(\mathbb{R})$, the v.f.

$$X := x_1\,\frac{\partial}{\partial x_1},$$

and let $\mathfrak{g} := \mathrm{Lie}\{X\}$. It is easy to recognize that $\mathfrak{g}$ satisfies conditions (C) and (ND) with $N = 1$. On the other hand, condition (H) does not hold, since (2.1.1) is not satisfied at $x = 0$.

Example 2.1.5 ((C)+(H)$\not\Rightarrow$(ND))**.** Let us consider, in $\mathcal{X}(\mathbb{R})$, the v.f.s

$$X := x_1\,\frac{\partial}{\partial x_1}, \quad Y := \frac{\partial}{\partial x_1},$$

and let $\mathfrak{g} := \mathrm{Lie}\{X, Y\}$. Since $[X, Y] = X$, condition (ND) does not hold: in fact, X and Y being linearly independent (remind that we are considering $\mathcal{X}(\mathbb{R})$ as a real vector space and not as a C^∞-module), we have

$$\mathfrak{g} = \mathrm{span}_{\mathbb{R}}\{X, Y\}, \text{ whence } \dim_{\mathbb{R}}(\mathfrak{g}) = 2.$$

On the other hand, $\mathfrak{g}$ satisfies conditions (C) and (H) with $N = 1$.

We remark that a Lie algebra $\mathfrak{g}$ can satisfy conditions (C) and (H) without being finite-dimensional (as a subspace of $\mathcal{X}(\mathbb{R}^N)$). This is the case, e.g., of the Lie algebra generated by

$$X := \frac{\partial}{\partial x_1}, \quad Y := \frac{1}{1 + x_1^2}\,\frac{\partial}{\partial x_2} \in \mathcal{X}(\mathbb{R}^2).$$

Thanks to Def. 2.1.1, we can state the main theorem of this chapter, which provides a complete answer to question (Q). The proof of this theorem is accomplished in Sec.s 2.1.2 and 2.1.3.

Theorem 2.1.6. *Let $\mathfrak{g} \subseteq \mathcal{X}(\mathbb{R}^N)$ be a Lie algebra of real-analytic vector fields satisfying conditions* (C), (H) *and* (ND) *of Def. 2.1.1.*

Then, there exists an analytic Lie group $\mathbb{G} = (\mathbb{R}^N, *)$ on $\mathbb{R}^N$, with neutral element 0*, such that* $\mathrm{Lie}(\mathbb{G}) = \mathfrak{g}$.

The first ingredient to prove Thm. 2.1.6 is the definition of exponentiation of a Lie algebra $\mathfrak{g} \subseteq \mathcal{X}(\mathbb{R}^N)$ satisfying condition (C). As we shall see in a moment, such a definition has a strong analogy with the definition of Exponential Map on a Lie group on $\mathbb{R}^N$.

Definition 2.1.7 (Exponentiation of $\mathfrak{g}$)**.** Let $\mathfrak{g} \subseteq \mathcal{X}(\mathbb{R}^N)$ be a Lie algebra satisfying condition (C). We set

$$\mathrm{Exp}_{\mathfrak{g}} : \mathfrak{g} \longrightarrow \mathbb{R}^N, \quad \mathrm{Exp}_{\mathfrak{g}}(X) := \gamma_{X,0}(1),$$

where $\gamma_{X,0}$ denotes the integral curve of X starting at 0. We shall often call this map the exponential map of $\mathfrak{g}$. We also denote $\mathrm{Exp}_{\mathfrak{g}}(X)$ by $\exp(X)(0)$.

We remark that assumption (C) on $\mathfrak{g}$ is essential for Def. 2.1.7 to make sense: indeed, if $X \in \mathfrak{g}$ is not complete, the integral curve of X (starting at 0) may not be defined for $t = 1$.

Remark 2.1.8. Let $\mathfrak{g} \subseteq \mathcal{X}(\mathbb{R}^N)$ satisfy condition (C), and let us suppose that there exists a Lie group $\mathbb{G} = (\mathbb{R}^N, *)$ with Lie algebra equal to $\mathfrak{g}$. If the neutral element of $\mathbb{G}$ is 0, then the exponential map $\mathrm{Exp}_{\mathfrak{g}}$ of $\mathfrak{g}$ is nothing but the Exponential Map of $\mathbb{G}$ (see Def. 1.1.7 on page 5).

Our next purpose is to investigate the regularity of the map $\mathrm{Exp}_{\mathfrak{g}}$: to this end, we need an additional structure on $\mathfrak{g}$ allowing us to talk about open sets and smooth functions. Hence, we assume that $\mathfrak{g}$ also satisfies condition (ND): if this is the case, the vector space $\mathfrak{g}$ can be endowed with a topological-differentiable structure by identifying it with $\mathbb{R}^N$ via the choice of a basis.

Lemma 2.1.9. *Let $\mathfrak{g} \subseteq \mathcal{X}(\mathbb{R}^N)$ satisfy conditions (ND) and (H). Then there exists a basis of $\mathfrak{g}$ (as a subspace of $\mathcal{X}(\mathbb{R}^N)$), say $\{J_1, \ldots, J_N\}$, such that*

$$\det(J_1 I(x) \cdots J_N I(x)) \neq 0 \quad \textit{for all } x \in \mathbb{R}^N, \tag{2.1.3}$$

$$(J_1 I(0) \cdots J_N I(0)) = \mathbb{I}_N, \tag{2.1.4}$$

where $\mathbb{I}_N$ is the $N \times N$ identity matrix.

Proof. First of all, since $\mathfrak{g}$ satisfies condition (ND), there exist $Z_1, \ldots, Z_N$ in $\mathfrak{g}$ s.t. $\{Z_1, \ldots, Z_N\}$ is a basis of $\mathfrak{g}$ (as subspace of $\mathcal{X}(\mathbb{R}^N)$). We claim that, for every $x \in \mathbb{R}^N$, the vectors $Z_1 I(x), \ldots, Z_N I(x)$ are linearly independent in $\mathbb{R}^N$.

Indeed, since $\mathfrak{g}$ also satisfies condition (H), there exist $W_1, \ldots, W_N \in \mathfrak{g}$ such that the vectors $W_1 I(x), \ldots, W_N I(x)$ are linearly independent (in $\mathbb{R}^N$); on the other hand, since $\mathcal{Z}$ is a basis of $\mathfrak{g}$, we have

$$\mathrm{span}_{\mathbb{R}}\{W_1 I(x), \ldots, W_N I(x)\} \subseteq \mathrm{span}_{\mathbb{R}}\{Z_1 I(x), \ldots, Z_N I(x)\},$$

and this shows that $Z_1 I(x), \ldots, Z_N I(x)$ are linearly independent, as claimed. We now perform a simple linear change of coordinates, in order to obtain from $\mathcal{Z}$ a basis satisfying conditions (2.1.3) and (2.1.4). We set

$$A = (a_{i,j})_{i,j=1,\ldots,N} := (Z_1 I(0) \cdots Z_N I(0))^{-1}, \tag{2.1.5}$$

and we define, for all $j = 1, \dots, N$,

$$J_j := \sum_{j=1}^{N} a_{i,j} Z_i. \tag{2.1.6}$$

Since A is invertible, $\mathcal{J} := \{J_1, \dots, J_N\}$ is a basis of $\mathfrak{g}$ (as a subspace of $\mathcal{X}(\mathbb{R}^N)$) and the vectors $J_1 I(x), \dots, J_N I(x)$ are linearly independent for all $x \in \mathbb{R}^N$; moreover, from the definition of $\mathcal{J}$ it follows that

$$\begin{aligned}(J_1 I(0) \cdots J_N I(0)) &\overset{(2.1.6)}{=} \Big(\sum_{j=1}^{N} a_{j,1} Z_j I(0) \cdots \sum_{j=1}^{N} a_{j,N} Z_j I(0) \Big) \\ &= (Z_1 I(0) \cdots Z_N I(0)) \cdot A \overset{(2.1.5)}{=} \mathbb{I}_N.\end{aligned}$$

This ends the proof. □

Remark 2.1.10. Let $\mathfrak{g} \subseteq \mathcal{X}(\mathbb{R}^N)$. The proof of Lem. 2.1.9 contains the following fact: if $\mathfrak{g}$ satisfies conditions (ND) and (H), then there exists a basis $\{J_1, \dots, J_N\}$ of $\mathfrak{g}$ (as a subspace of $\mathcal{X}(\mathbb{R}^N)$) such that

$$J_1 I(x), \dots, J_N I(x),$$

are linearly independent (in $\mathbb{R}^N$) for all $x \in \mathbb{R}^N$. On the other hand, if $\mathfrak{g}$ satisfies condition (ND) and if it is possible to find a basis $\{J_1, \dots, J_N\}$ of $\mathfrak{g}$ (as a subspace of $\mathcal{X}(\mathbb{R}^N)$) such that $J_1 I(x), \dots, J_N I(x)$ are linearly independent for all $x \in \mathbb{R}^N$, then it is easy to recognize that $\mathfrak{g}$ also satisfies condition (H).

We can then summarize these facts in the following way: if $\mathfrak{g} \subseteq \mathcal{X}(\mathbb{R}^N)$ satisfies condition (ND), then it fulfills condition (H) if and only if it fulfills condition (H'), where

(H'): there exists a basis $\{J_1, \dots, J_N\}$ of $\mathfrak{g}$ such that $J_1 I(x), \dots, J_N I(x)$ are linearly independent for every $x \in \mathbb{R}^N$.

Remark 2.1.11. Let $\mathfrak{g} \subseteq \mathcal{X}(\mathbb{R}^N)$ satisfy conditions (H) and (ND), and let us assume that $\mathbb{G} = (\mathbb{R}^N, *)$ is a Lie group on $\mathbb{R}^N$, with neutral element equal to 0, such that $\mathrm{Lie}(\mathbb{G})$ is equal to $\mathfrak{g}$. Then, a basis of $\mathfrak{g}$ as in lemma. 2.1.9 is unique, and it is nothing but that the Jacobian basis of $\mathrm{Lie}(\mathbb{G})$ (see Def. 1.1.3 on page 4, and recall that a left invariant vector fields X is completely determined by its value $XI(0)$ at the neutral element).

Proposition 2.1.12. *Let $\mathfrak{g} \subseteq \mathcal{X}(\mathbb{R}^N)$ satisfy conditions* (C), (H) *and* (ND).

Then $\mathrm{Exp}_\mathfrak{g}$ *is a smooth map on $\mathfrak{g}$ with non-singular differential at $X = 0 \in \mathfrak{g}$. If, in addition, every vector field in $\mathfrak{g}$ is real-analytic, then* $\mathrm{Exp}_\mathfrak{g}$ *is real analytic. Consequently, there exists an open and connected neighborhood $\mathcal{U}$ of* 0 *in $\mathfrak{g}$ such that* $(\mathrm{Exp}_\mathfrak{g})|_\mathcal{U}$ *is a diffeomorphism.*

Proof. We first prove the regularity of $\mathrm{Exp}_\mathfrak{g}$. To this end, let $\mathcal{J} = \{J_1, \dots, J_N\}$ be a basis of $\mathfrak{g}$ as in Lem. 2.1.9 and let $\pi : \mathbb{R}^N \to \mathfrak{g}$, $\pi(\xi) := \sum_{i=1}^N \xi_i J_i$. We set

$$E : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \quad E(\xi) := (\mathrm{Exp}_\mathfrak{g} \circ \pi)(\xi) = \mathrm{Exp}_\mathfrak{g}\Big(\sum_{k=1}^{N} \xi_k J_k \Big). \tag{2.1.7}$$

Then, by definition, $\mathrm{Exp}_{\mathfrak{g}}$ is smooth (resp. real-analytic) on $\mathfrak{g}$ if and only if E is smooth (resp. real-analytic) on $\mathbb{R}^N$. Now, the regularity of E follows from classical results of ODE Theory. Indeed, let us we define

$$f : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}^N, \quad f(x,\xi) := \sum_{k=1}^{N} \xi_k J_k I(x).$$

Obviously, f has the same regularity (w.r.t. both x and ξ) of $J_1, \dots, J_N$; moreover, $E(\xi)$ is nothing but $\gamma(1;\xi)$, where $\gamma(\cdot\,;\xi)$ is the unique maximal solution (which is defined on the whole of $\mathbb{R}$) of the parametric problem

$$\begin{cases} \dot{x} = f(x;\xi) \\ x(0) = 0. \end{cases}$$

Hence, we deduce from classical results ODE Theory that E has the same regularity (w.r.t. ξ) of $J_1, \dots, J_N$, as desired.

We now turn to show that the differential of $\mathrm{Exp}_{\mathfrak{g}}$ at $X = 0$ in non-singular. To this end, we consider once again the map E in (2.1.7) and we compute its Jacobian matrix at $\xi = 0$. From the Maclaurin expansion (with an integral remainder) of the map

$$t \mapsto \gamma(t;\xi) := \exp\left(t \sum_{k=1}^{N} \xi_k\, J_k\right)(0),$$

we obtain (here I stands for the identity map of $\mathbb{R}^N$)

$$\begin{aligned} E(\xi) = \gamma(1;\xi) &= \sum_{k=1}^{N} \xi_k\, J_k I(0) + \int_0^1 (1-s) \sum_{h,k=1}^{N} (\xi_h \xi_k\, J_h J_k I)(\gamma(s;\xi))\,\mathrm{d}s \\ &\overset{(2.1.4)}{=} \xi + \sum_{h,k=1}^{N} \xi_h\, \xi_k \int_0^1 (1-s)\,(J_h J_k I)(\gamma(s;\xi))\,\mathrm{d}s; \end{aligned} \qquad (2.1.8)$$

on the other hand, since $(t,\xi) \mapsto \gamma(t;\xi)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$, it is not difficult to recognize that

$$\sum_{h,k=1}^{N} \xi_h\, \xi_k \int_0^1 (1-s)\,(J_h J_k I)(\gamma(s;\xi))\,\mathrm{d}s = \mathcal{O}(\|\xi\|^2), \quad \text{as } \xi \to 0. \qquad (2.1.9)$$

Therefore, by gathering (2.1.8) and (2.1.9), we obtain

$$E(\xi) = \xi + \mathcal{O}(\|\xi\|^2), \qquad \text{as } \xi \to 0,$$

whence $\mathcal{J}_E(0) = \mathbb{I}_N$, which proves that the differential of $\mathrm{Exp}_{\mathfrak{g}}$ at 0 is a non-singular linear map. □

Remark 2.1.13. The proof of Prop. 2.1.12 contains the following fact: let $\mathfrak{g} \subseteq \mathcal{X}(\mathbb{R}^N)$ satisfy conditions (C), (H) and (ND) and let $\mathcal{J} := \{J_1, \dots, J_N\}$ be a basis of $\mathfrak{g}$ as in lemma. 2.1.9. If we set $\pi : \mathbb{R}^N \to \mathfrak{g}$, $\pi(\xi) := \sum_{k=1}^{N} \xi_k J_k$ and if we define $E := \mathrm{Exp}_{\mathfrak{g}} \circ \pi$, then E is a smooth map on $\mathbb{R}^N$ and

$$\mathcal{J}_E(0) = \mathbb{I}_N, \qquad (2.1.10)$$

where $\mathbb{I}_N$ is the identity $N \times N$ matrix.

Definition 2.1.14 (Logarithmic map on $\mathfrak{g}$)**.** Let $\mathfrak{g} \subseteq \mathcal{X}(\mathbb{R}^N)$ be a Lie algebra satisfying conditions (C), (H) and (ND), and let $\mathcal{U}$ be as in Prop.2.1.12.

We set $V := \mathrm{Exp}_{\mathfrak{g}}(\mathcal{U})$ and we denote by $\mathrm{Log}_{\mathfrak{g}} : V \longrightarrow \mathcal{U}$ the inverse map of $\mathrm{Exp}_{\mathfrak{g}} : \mathcal{U} \to V$. We call this map the logarithmic map of $\mathfrak{g}$ (relative to $\mathcal{U}$).

2.1.2 Construction of the local Lie group

In this section we shall show that, if $\mathfrak{g} \subseteq \mathcal{X}(\mathbb{R}^N)$ satisfies conditions (C), (H) and (ND), it is possible to endow $\mathbb{R}^N$ with a *local* Lie group structure in such a way that the vector fields in $\mathfrak{g}$ are left invariant. The results presented here are not new: indeed, we closely follow the approach in [32]; our new improvement will be given in the globalization of the local Lie group, in the next section.

To begin with, to keep the exposition clear, we fix once and for all the main notations used in the sequel:

- we denote by $\mathfrak{g}$ a fixed Lie algebra of real-analytic vector fields on $\mathbb{R}^N$ satisfying conditions (C), (H) and (ND) in Def. 2.1.1;
- we denote by Exp the exponential map $\mathrm{Exp}_{\mathfrak{g}}$ of $\mathfrak{g}$ and we let $\mathrm{Log} : V \to \mathcal{U}$ denote its local inverse (with $V := \mathrm{Exp}(\mathcal{U})$) as in Def. 2.1.14;
- we fix a basis $\mathcal{J} = \{J_1, \dots, J_N\}$ of $\mathfrak{g}$ as in lemma. 2.1.9 and we introduce the map $\pi : \mathbb{R}^N \longrightarrow \mathfrak{g}$ by setting $\pi(\xi) := \sum_{k=1}^N \xi_k J_k$.

With these notations, we set

$$m : \mathbb{R}^N \times V \longrightarrow \mathbb{R}^N, \qquad m(x,y) := \exp(\mathrm{Log}(y))(x). \tag{2.1.11}$$

As usual, if $X \in \mathcal{X}(\mathbb{R}^N)$ and if $x \in \mathbb{R}^N$, we denote by $\mathbb{R} \ni t \mapsto \exp(t\,X)(x)$ the maximal integral curve of X starting at x. By classical results of ODE Theory, we deduce that m is real-analytic on $\mathbb{R}^N \times V$.

Remark 2.1.15. Let us assume that there exists a Lie group $\mathbb{G} = (\mathbb{R}^N, *)$, with neutral element 0, and such that $\mathrm{Lie}(\mathbb{G}) = \mathfrak{g}$. As pointed out in Rem. 2.1.8, the exponential map Exp on $\mathfrak{g}$ coincides with the Exponential Map of $\mathbb{G}$; as a consequence, if $y \in V$ and if $Y = \mathrm{Log}(y) \in \mathfrak{g}$, Thm. 1.1.8 implies that

$$m(x,y) = \exp(Y)(x) = \gamma_{Y,x}(1) = x * y.$$

We want to show that m in (2.1.11) is locally associative near 0, and that 0 is a neutral element for m. To this end, we need the following result (for a proof see, e.g., [29] or [32]).

Theorem 2.1.16. *Let $\mathfrak{h}$ be a Lie algebra of real-analytic vector fields on $\mathbb{R}^N$ satisfying conditions* (C) *and* (ND)*, ant let $\|\cdot\|$ be a fixed norm on $\mathfrak{h}$. There exists a positive real number ε, depending on $\|\cdot\|$, such that the CBHD series*

$$Z(X,Y) := \sum_{h=1}^{\infty} Z_h(X,Y)$$

is totally convergent on $\mathcal{B}(0,\varepsilon) \times \mathcal{B}(0,\varepsilon)$, where $\mathcal{B}(0,\varepsilon) := \{V \in \mathfrak{h} : \|V\| < \varepsilon\}$. Furthermore, for every $X, Y \in \mathcal{B}(0,\varepsilon)$, we have the following ODE identity

$$\exp(Y)\big(\exp(X)(x)\big) = \exp\big(Z(X,Y)\big)(x), \quad \textit{for every } x \in \mathbb{R}^N. \tag{2.1.12}$$

As usual, we also use the notation $X \diamond Y := Z(X,Y)$.

In order to apply the remarkable identity (2.1.12) to our setting, we need to fix a norm on $\mathfrak{g}$; for simplicity, we consider the Euclidean norm obtained by identifying $\mathfrak{g}$ with $\mathbb{R}^N$ via the basis $\mathcal{J}$, that is,

$$\Big\| \sum_{k=1}^{N} \xi_k J_k \Big\|_{\mathcal{J}} := \sqrt{\xi_1^2 + \cdots + \xi_N^2}. \tag{2.1.13}$$

It is worth noting that, since $\mathfrak{g}$ is finite-dimensional (by assumption (ND)), all norms on $\mathfrak{g}$ are actually equivalent.

By means of Thm. 2.1.16, we are able to derive a powerful representation for the map m as in the next theorem.

Theorem 2.1.17. *Let $\varepsilon > 0$ be as in Thm. 2.1.16 and let us suppose (by possibly shrinking ε) that $\mathcal{B}(0,\varepsilon) \subseteq \mathcal{U}$. It is then possible to find an open and connected neighborhood $W \subseteq V$ of 0 such that the function*

$$\mathfrak{Z} : W \times W \longrightarrow \mathcal{B}(0,\varepsilon) \qquad \mathfrak{Z}(x,y) := \mathrm{Log}(x) \diamond \mathrm{Log}(y) \tag{2.1.14}$$

is well-defined and, for every $x, y \in W$, the following identity holds true

$$m(x,y) = \mathrm{Exp}(\mathfrak{Z}(x,y)). \tag{2.1.15}$$

Proof. Let $Z : \mathcal{B}(0,\varepsilon) \times \mathcal{B}(0,\varepsilon) \to \mathfrak{g}$, $Z(X,Y) := X \diamond Y$. Since, by Thm. 2.1.16 (and by the choice of ε), the CBHD series $X \diamond Y$ is totally convergent on the product $\mathcal{B}(0,\varepsilon) \times \mathcal{B}(0,\varepsilon)$, Z is well-defined and continuous on its domain; as a consequence, it is possible to find $0 < \varepsilon_1 < \varepsilon$ such that

$$Z(X,Y) \in \mathfrak{B}(0,\varepsilon), \quad \text{for all } X, Y \in \mathfrak{B}(0,\varepsilon_1). \tag{2.1.16}$$

Analogously, since Log is continuous on V and $\mathrm{Log}(0) = 0$, there exists $\delta > 0$ such that $B(0,\delta) \subseteq V$ and

$$\mathrm{Log}(x) \in \mathcal{B}(0,\varepsilon_1), \quad \text{for all } x \in B(0,\delta). \tag{2.1.17}$$

We then set $W := B(0,\delta)$ and we show that it satisfies all the properties in the statement of the theorem. To this end, let $x, y \in W$ be fixed.

By (2.1.17) we see that $\mathrm{Log}(x)$ and $\mathrm{Log}(y)$ belong to $\mathcal{B}(0,\varepsilon_1)$; therefore, recalling that $\varepsilon_1 < \varepsilon$, from (2.1.16) we infer that the series $\mathrm{Log}(x) \diamond \mathrm{Log}(y)$ is convergent, whence $\mathfrak{Z}$ is well-defined, and

$$\mathfrak{Z}(x,y) \in \mathcal{B}(0,\varepsilon), \quad \text{for every } x, y \in W.$$

As for identity (2.1.15) we observe that, since $W \subseteq V$, we have

$$\begin{aligned} m(x,y) &\overset{(2.1.11)}{=} \exp(\mathrm{Log}(y))(x) \quad \Big(x = \mathrm{Exp}(\mathrm{Log}(x))\Big) \\ &= \exp\big(\mathrm{Log}(y)\big)(\mathrm{Exp}(\mathrm{Log}(x))) \\ &= \exp\big(\mathrm{Log}(y)\big)(\exp(\mathrm{Log}(x))(0)). \end{aligned}$$

Thus, by gathering together (2.1.17) and (2.1.12), we conclude that

$$m(x,y) = \exp(\mathrm{Log}(x) \diamond \mathrm{Log}(y))(0) = \mathrm{Exp}(\mathfrak{Z}(x,y)),$$

which is exactly what we wanted to prove. □

Remark 2.1.18. Let W be the open (and connected) neighborhood of 0 constructed in the proof of Thm. 2.1.17. Then, for every $x \in W$, we have

$$\|\mathrm{Log}(x)\|_{\mathfrak{g}} < \varepsilon. \tag{2.1.18}$$

Indeed, if $x \in W$, we have $\mathrm{Log}(x) = \mathfrak{Z}(x, 0)$; thus, from Thm. 2.1.17 we infer that $\mathrm{Log}(x) \in \mathfrak{D}(0, \varepsilon)$, which is exactly (2.1.18). In particular, since $\mathcal{B}(0, \varepsilon)$ is symmetric, we have the following useful property

$$-\mathrm{Log}(x) \in \mathcal{B}(0, \varepsilon), \quad \text{for all } x \in W. \tag{2.1.19}$$

By exploiting Thm. 2.1.17, and in particular identity (2.1.15), we can provide a simple proof of the local associativity of m.

Theorem 2.1.19. *Let $W \subseteq V$ be as in Thm. 2.1.17. Then $m(a, b) \in V$ for every $a, b \in W$ and m is associative on W, that is,*

$$m(x, m(y, z)) = m(m(x, y), z) \qquad \text{for all } x, y, z \in W. \tag{2.1.20}$$

Furthermore, the point $0 \in \mathbb{R}^N$ provides a local neutral element for m, i.e.,

$$m(x, 0) = x, \quad \text{for all } x \in \mathbb{R}^N, \tag{2.1.21}$$

$$m(0, y) = y, \quad \text{for all } y \in V. \tag{2.1.22}$$

Proof. Let $a, b \in W$. Since $\mathfrak{Z}$ takes values in $\mathcal{B}(0, \varepsilon) \subseteq \mathcal{U}$, we have

$$m(a, b) \stackrel{(2.1.15)}{=} \mathrm{Exp}(\mathfrak{Z}(a, b)) \in \mathrm{Exp}(\mathcal{U}) = V. \tag{2.1.23}$$

We now turn to show identity (2.1.20). To this end, let $x, y, z \in W$. Firstly, by the above (2.1.23) (and since $W \subseteq V$), both sides of (2.1.20) are well-defined; moreover, by means of Thm. 2.1.17 we can write

$$\begin{aligned} \mathrm{Log}(m(y, z)) &\stackrel{(2.1.15)}{=} \mathrm{Log}(\mathrm{Exp}(\mathfrak{Z}(y, z))) \quad \big(\mathfrak{Z}(y, z) \in \mathcal{U}\big) \\ &= \mathrm{Log}(\mathrm{Exp}_{|\mathcal{U}}(\mathfrak{Z}(y, z))) = \mathfrak{Z}(y, z). \end{aligned} \tag{2.1.24}$$

As a consequence, the left-hand side of (2.1.20) can be rewritten as follows:

$$m(x, m(y, z)) = \exp(\mathrm{Log}(m(y, z)))(x) \stackrel{(2.1.24)}{=} \exp(\mathfrak{Z}(y, z))(x). \tag{2.1.25}$$

As for the right-hand side we observe that, by definition of m, we have

$$m(m(x, y), z) = \exp(\mathrm{Log}(z))(m(x, y)) = \exp\big(\mathrm{Log}(z)\big)(\exp(\mathrm{Log}(y)(x));$$

therefore, since $\mathrm{Log}(y), \mathrm{Log}(z) \in \mathcal{B}(0, \varepsilon)$ (by the choice of W, see identity (2.1.18)), we can apply identity (2.1.12), which gives

$$m(m(x, y), z) \stackrel{(2.1.12)}{=} \exp(\mathrm{Log}(y) \diamond \mathrm{Log}(z))(x) = \exp(\mathfrak{Z}(y, z))(x). \tag{2.1.26}$$

Finally, by comparing (2.1.25) and (2.1.25), we derive that

$$m(x, m(y, z)) = m(m(x, y), z),$$

which is (2.1.20). As for identity (2.1.21), it is a straightforward consequence of the definition of m: indeed, if $x \in \mathbb{R}^N$, we have (note that $0 \in V$ and $\mathrm{Log}(0) = 0$)

$$m(x,0) \stackrel{(2.1.11)}{=} \exp(\mathrm{Log}(0))(x) = \exp(0)(x) = x.$$

On the other hand, if $y \in V$, by definition of Exp we have

$$m(0,y) = \exp(\mathrm{Log}(y))(0) = \mathrm{Exp}(\mathrm{Log}(y)) = y,$$

and this is precisely the desired (2.1.22). This ends the proof. □

Definition 2.1.20. We set

$$\iota : V \longrightarrow \mathbb{R}^N, \qquad \iota(x) := \mathrm{Exp}(-\mathrm{Log}(x)). \tag{2.1.27}$$

As in the case of m, the real-analyticity of the vector fields in $\mathfrak{g}$ implies the real-analyticity of the map ι on its domain V.

Remark 2.1.21. Let us assume that there exists a Lie group $\mathbb{G} = (\mathbb{R}^N, *)$, with neutral element 0 and s.t. $\mathrm{Lie}(\mathbb{G}) = \mathfrak{g}$. Then, for every $x \in V$, we have

$$\iota(x) = x^{-1}.$$

In fact, since the map Exp is precisely the Exponential Map of $\mathbb{G}$, if $x \in V$ and if $X = \mathrm{Log}(x) \in \mathfrak{g}$, we then have

$$\begin{aligned} x * \iota(x) &= x * \mathrm{Exp}(-\mathrm{Log}(x)) = x * \exp(-X)(0) \stackrel{(1.1.5)}{=} \exp(-X)(x) \\ &= \exp(-X)(\mathrm{Exp}(X)) = \exp(-X)(\exp(X)(0)) \\ &= \exp((-X+X))(0) = 0, \end{aligned}$$

and this proves that $\iota(x) = x^{-1}$, as claimed.

We now prove that the map ι provides a local inverse for m.

Theorem 2.1.22. *Let $W \subseteq V$ be as in Thm. 2.1.17. Then the map ι in* (2.1.27) *provides a local inverse for m on W, that is,*

$$m(x, \iota(x)) = 0 \quad \text{for all } x \in W, \tag{2.1.28}$$

$$m(\iota(x), x) = 0, \quad \text{for all } x \in W. \tag{2.1.29}$$

Proof. Let $x \in V$ and let $X = \mathrm{Log}(x)$. By Rem. 2.1.18, $-X \in \mathcal{B}(0,\varepsilon)$, whence

$$\iota(x) = \mathrm{Exp}(-\mathrm{Log}(x)) = \mathrm{Exp}(-X) \in \mathrm{Exp}(\mathcal{U}) = V; \tag{2.1.30}$$

moreover, since $\mathcal{B}(0,\varepsilon) \subseteq \mathcal{U}$, we have

$$\mathrm{Log}(\iota(x)) = \mathrm{Log}\big(\mathrm{Exp}(-X)\big) = -X = -\mathrm{Log}(x). \tag{2.1.31}$$

From these identities we deduce that $m(x, \iota(x))$ is well-defined (since $\iota(x)$ belongs to V) and that

$$\begin{aligned} m(x,\iota(x)) &= \exp\big(\mathrm{Log}(\iota(x))\big)(x) \stackrel{(2.1.31)}{=} \exp(-X)(x) \\ &= \exp(-X)\big(\exp(X)(0)\big). \end{aligned}$$

As a consequence, since $X, -X \in \mathcal{B}(0,\varepsilon)$, we are entitled to apply the crucial identity (2.1.12), which gives

$$m(x,\iota(x)) \overset{(2.1.12)}{=} \exp(X \diamond (-X))(0). \tag{2.1.32}$$

The desired (2.1.28) now follows from (2.1.32), by noticing that

$$X \diamond (-X) = X + (-X) + \sum_{h=2}^{\infty} Z_h(X,-X) = -\sum_{h=2}^{\infty} Z_h(X,X) = 0.$$

As for identity (2.1.29) we observe, by definition of ι, we have

$$m(\iota(x),x) = \exp(\mathrm{Log}(x))(\iota(x)) = \exp\big(X\big)(\exp(-X)(0));$$

thus, by arguing as above, we conclude

$$m(\iota(x),x) \overset{(2.1.12)}{=} \exp((-X)\diamond X)(0) = \exp(0)(0) = 0.$$

This ends the proof. □

By gathering the results in Thm.s 2.1.19 and 2.1.22, we see that m actually defines a local Lie group structure on $\mathbb{R}^N$; we end this section by showing that the Lie algebra $\mathfrak{g}$ is deeply connected to this structure.

Theorem 2.1.23 (Local left-invariance of $\mathfrak{g}$)**.** *For every $X \in \mathfrak{g}$ it holds that*

$$XI(m(x,y)) = \frac{\partial m}{\partial y}(x,y)\cdot XI(y), \quad \textit{for every } (x,y)\in\mathbb{R}^N\times V. \tag{2.1.33}$$

Proof. We first prove that identity (2.1.33) holds for $y=0$, that is,

$$XI(x) = \frac{\partial m}{\partial y}(x,0)\cdot XI(0), \quad \text{for all } x\in\mathbb{R}^N. \tag{2.1.34}$$

To this end, let $x\in\mathbb{R}^N$ and let $\eta>0$ be such that $tX\in\mathcal{U}$ for all $t\in\mathbb{R}$ with $|t|<\eta$. For these values of t, we have $\mathrm{Exp}(tX)\in\mathrm{Exp}(\mathcal{U})=V$, whence

$$\exp(tX)(x) = \exp\big(\mathrm{Log}(\mathrm{Exp}(tX))\big)(x) \overset{(2.1.11)}{=} m\big(x,\mathrm{Exp}(tX)\big). \tag{2.1.35}$$

By taking the derivative w.r.t. t of both sides of identity (2.1.35) and evaluating at $t=0$, we get (since $t\mapsto\exp(tX)(x)$ is an integral curve of X)

$$\begin{aligned} XI(x) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big\{\exp(tX)(x))\big\} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big\{m\big(x,\mathrm{Exp}(tX)\big)\big\} \\ &= \frac{\partial m}{\partial y}(x,0)\cdot\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathrm{Exp}(tX) = \frac{\partial m}{\partial y}(x,0)\cdot XI(0), \end{aligned}$$

which is exactly the desired (2.1.34).

Let now W be an open and connected neighborhood of 0 as in Thm. 2.1.17. Since m is associative on W (as ensured by Thm. 2.1.19), we have

$$m(m(x,y),z) = m(x,m(y,z)), \quad \text{for all } x,y,z\in W;$$

thus, by differentiating w.r.t. z the above identity and evaluating at $z = 0$, we get (setting, to avoid ambiguities, $m = m(\alpha, \beta)$)

$$\frac{\partial m}{\partial \beta}(m(x,y),0) = \frac{\partial m}{\partial \beta}(x, m(y,0)) \cdot \frac{\partial m}{\partial \beta}(y,0), \quad \text{for all } x, y \in W. \tag{2.1.36}$$

From this, by multiplying both sides of (2.1.36) by the column vector $XI(0)$, we obtain (since $m(y,0) = y$)

$$\frac{\partial m}{\partial \beta}(m(x,y),0) \cdot XI(0) = \frac{\partial m}{\partial \beta}(x,y) \cdot \frac{\partial m}{\partial \beta}(y,0) \cdot XI(0),$$

which gives, by (2.1.34) (returning to the $m = m(x,y)$ notation),

$$\begin{aligned} XI(m(x,y)) &\overset{(2.1.34)}{=} \frac{\partial m}{\partial y}(m(x,y),0) \cdot XI(0) = \frac{\partial m}{\partial y}(x,y) \cdot \frac{\partial m}{\partial y}(y,0) \cdot XI(0) \\ &\overset{(2.1.34)}{=} \frac{\partial m}{\partial y}(x,y) \cdot XI(y), \quad \text{for all } x, y \in W. \end{aligned}$$

This is precisely the desired (2.1.33) for $x, y \in W$.

For the general case (that is, for $x \in \mathbb{R}^N$ and $y \in V$), we use the Unique Continuation Principle: since both sides of identity (2.1.33) are real-analytic in the couple (x,y) and since they coincide on the open set $W \times W$, they must be equal on the whole of $\mathbb{R}^N \times V$ (since V is connected), that is,

$$XI(m(x,y)) = \frac{\partial m}{\partial y}(x,y) \cdot XI(y), \quad \text{for all } (x,y) \in \mathbb{R}^N \times V.$$

This ends the proof. □

2.1.3 Local to global

The aim of this last section is to show that the local-group structure constructed in Sec. 2.1.2 can be (uniquely) continued to be global; this provides a complete answer to question (Q) and represents the main novelty with respect to the paper by Bonfiglioli and Lanconelli [32], where the prolongation of the local-group structure is assumed as an additional hypothesis.

In what follows, we take for fixed all the notations introduced so far.

To begin with, we prove that the map m can be analytically extended to the whole of $\mathbb{R}^N \times \mathbb{R}^N$. Our idea is the following: for every fixed $x, y \in \mathbb{R}^N$, let $\gamma_{x,y}$ be the curve defined by

$$\gamma_{x,y}(t) := m(x, t\,y).$$

Since m is defined on $\mathbb{R}^N \times V$, there exists a (possibly small) open neighborhood of $0 \in \mathbb{R}$ on which $\gamma_{x,y}$ is well-defined. We show that $\gamma_{x,y}$ satisfies a suitable Cauchy problem which possesses a (unique) global maximal solution, say $t \mapsto \varphi_{x,y}(t)$. Then it is natural to extend m as follows

$$x * y := \varphi_{x,y}(1).$$

Keeping in mind this idea, we start with establishing the following lemma.

Lemma 2.1.24. *Let $\mathfrak{J} = \{J_1, \ldots, J_N\}$ be a basis of $\mathfrak{g}$ as in lemma. 2.1.9. There exist N functions $a_1, \ldots, a_N \in C^\omega(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ such that*

$$y = \sum_{k=1}^N a_k(t,y) J_k I(ty), \quad \textit{for all } (t,y) \in \mathbb{R} \times \mathbb{R}^N. \tag{2.1.37}$$

Proof. For every $x \in \mathbb{R}^N$, we consider the matrix $J(x) := (J_1 I(x) \cdots J_N I(x))$ and we define ($J(x)$ being non-singular)

$$(a_1(t,y), \ldots, a_N(t,y)) := (J(t\,y))^{-1} \cdot y.$$

Obviously, $a_1, \ldots, a_N \in C^\omega(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ (since $J_1, \ldots, J_N$ are real-analytic); moreover, a direct computation shows that

$$\sum_{k=1}^N a_k(t,y) J_k I(ty) = J(ty) \cdot \big(\big(J(x)\big)^{-1} \cdot y \big) = y,$$

which is exactly the desired (2.1.37). This ends the proof. □

Remark 2.1.25. Let $a_1, \ldots, a_N$ be as in lemma. 2.1.24. We observe that, for fixed $t \in \mathbb{R}$ and $y \in \mathbb{R}^N$, the N-tuple $(a_1(t,y), \ldots, a_N(t,y)) \in \mathbb{R}^N$ is nothing but the solution x of the linear system

$$J(ty) \cdot x = y. \tag{2.1.38}$$

Since $J(ty)$ is non-singular for every choice of $(t,y) \in \mathbb{R} \times \mathbb{R}^N$, the system (2.1.38) has a unique solution, given by $(J(ty))^{-1} \cdot y$.

Theorem 2.1.26. *Let $x, y \in \mathbb{R}^N$ be fixed and let $I \subseteq \mathbb{R}$ be an open neighborhood of 0 such that $ty \in V$ for all $t \in I$. We set*

$$\gamma_{x,y} : I \longrightarrow \mathbb{R}^N, \qquad \gamma_{x,y}(t) := m(x, ty).$$

Then, for all $t \in I$, the function $\gamma_{x,y}$ is a solution of the following Cauchy problem (depending on the parameter y)

$$\begin{cases} \dot{z}(t) = \displaystyle\sum_{k=1}^N a_k(t,y) J_k I(z(t)) \\ z(0) = x, \end{cases} \tag{2.1.39}$$

where $a_1, \ldots, a_N$ are the functions given in lemma. 2.1.24.

Proof. Firstly, since m is real-analytic on $\mathbb{R}^N \times \mathbb{R}^N$, then $\gamma_{x,y} \in C^\omega(I, \mathbb{R}^N)$; moreover, by exploiting identity (2.1.33) in Thm. 2.1.23, we get

$$\begin{aligned} \dot{\gamma}_{x,y}(t) &= \frac{\partial m}{\partial y}(x, t\,y) \cdot y \overset{(2.1.37)}{=} \frac{\partial m}{\partial y}(x, t\,y) \cdot \big(\sum_{k=1}^N a_k(t,y) J_k I(t\,y) \big) \\ &= \sum_{k=1}^N a_k(t,y) \Big(\frac{\partial m}{\partial y}(x, t\,y) \cdot J_k I(t\,y) \Big) \overset{(2.1.33)}{=} \sum_{k=1}^N a_k(t\,y) J_k I(m(x, t\,y)) \\ &= \sum_{k=1}^N a_k(t\,y) J_k I(\gamma_{x,y}(t)), \end{aligned}$$

and this proves that $\gamma_{x,y}$ satisfies the ODE in (2.1.39). Finally, since 0 is a local neutral element for m, we have

$$\gamma_{x,y}(0) = m(x,0) \overset{(2.1.21)}{=} x,$$

and this shows that $\gamma_{x,y}$ solves the Cauchy problem (2.1.39). □

Our aim is now to prove that the Cauchy problem (2.1.39) admits a unique maximal solution which is actually defined on the whole of $\mathbb{R}$. To this end, we first establish the following result.

Theorem 2.1.27. *Let $X_1, \ldots, X_n \in \mathfrak{g}$ and let $\alpha_1, \ldots, \alpha_n \in C(\mathbb{R};\mathbb{R})$. Then, for every $\xi \in \mathbb{R}^N$, the maximal solution of the Cauchy problem*

$$\begin{cases} \dot{z}(t) = \sum_{k=1}^{n} \alpha_k(t) X_k I(z(t)), \\ z(0) = \xi, \end{cases} \tag{2.1.40}$$

is defined on the whole of $\mathbb{R}$.

Proof. Let $\varphi : \mathcal{D} \to \mathbb{R}^N$ be the unique maximal solution of (2.1.40) and let us assume, by contradiction, that $\mathcal{D} \neq \mathbb{R}$. To fix ideas, we suppose that

$$0 < T := \sup(\mathcal{D}) < \infty.$$

We then set $K := [0, T]$ and we choose $h > 0$ in such a way that the ball $B(0, h)$ is contained in V. Now, by exploiting classical results of ODE Theory, there exists $\varepsilon > 0$ such that the (unique) maximal solution u_s of the problem

$$\begin{cases} \dot{x} = \sum_{k=1}^{n} \alpha_k(t+s) X_k I(x), \\ x(0) = 0, \end{cases} \tag{2.1.41}$$

is defined at least on $[-\varepsilon, \varepsilon]$, uniformly for $s \in K$, and it satisfies

$$|u_s(t)| \leq h, \quad \text{for all } t \in [-\varepsilon, \varepsilon] \text{ and every } s \in K. \tag{2.1.42}$$

Let now $\tau \in]0, T[$ be such that $T - \tau < \varepsilon$ and let $x := \varphi(\tau)$ (note that x is well-defined, since $\tau \in]0, T[\subseteq \mathcal{D}$). We then define

$$\nu : [0, \varepsilon] \longrightarrow \mathbb{R}^N, \qquad \nu(t) := m(x, u_\tau(t)), \tag{2.1.43}$$

where u_τ is the maximal solution of the Cauchy problem (2.1.41) with $s = \tau$.

We observe that ν is well-defined and real-analytic on $[0, \varepsilon]$, since m belongs to $C^\omega(\mathbb{R}^N \times V, \mathbb{R}^N)$ and, by (2.1.42), we have

$$u_\tau(t) \in B(0, h) \subseteq V, \quad \text{for all } t \in [0, \varepsilon].$$

We claim that, on $[0, h]$, ν solves the following Cauchy problem

$$\begin{cases} \dot{z}(t) = \sum_{k=1}^{n} a_k(t+\tau) X_k I(z(t)), \\ z(0) = x. \end{cases} \tag{2.1.44}$$

Indeed, since $m(x, 0) = x$, we have

$$\nu(0) = m(x, u_\tau(0)) = m(x, 0) \overset{(2.1.21)}{=} x;$$

moreover, by Thm. 2.1.23, for every $0 \le t \le \varepsilon$ one has

$$\begin{aligned}
\dot\nu(t) &= \frac{\partial m}{\partial y}(x, u_\tau(t)) \cdot \dot u_\tau(t) \overset{(2.1.41)}{=} \frac{\partial m}{\partial y}(x, u_\tau(t)) \cdot \left(\sum_{k=1}^n a_k(t+\tau) X_k I(u_\tau(t)) \right) \\
&= \sum_{k=1}^n a_k(t+\tau) \left(\frac{\partial m}{\partial y}(x, u_\tau(t)) \cdot X_k I(u_\tau(t)) \right) \\
&\overset{(2.1.33)}{=} \sum_{k=1}^n a_k(t+\tau) X_k I(m(x, u_\tau(t))) \overset{(2.1.43)}{=} \sum_{k=1}^n a_k(t+\tau) X_k I(\nu(t)).
\end{aligned}$$

We then consider the gluing of φ and ν, that is, the map

$$\Phi : [0, \tau + h] \longrightarrow \mathbb{R}^N, \qquad \Phi(t) := \begin{cases} \varphi(t), & t \in [0, \tau] \\ \nu(t - \tau) & t \in \,]\tau, \tau + \varepsilon]. \end{cases}$$

By definition, $\Phi(0) = \varphi(0) \overset{(2.1.40)}{=} \xi$ and $\Psi \in C([0, \tau+\varepsilon], \mathbb{R}^N)$, since

$$\lim_{t \to \tau^-} \Phi(t) = \lim_{t \to \tau^-} \varphi(t) = \varphi(\tau) = x \overset{(2.1.44)}{=} \nu(0) = \lim_{t \to \tau^+} \nu(t - \tau) = \lim_{t \to \tau^+} \Phi(t);$$

moreover, it is not difficult to recognize that $\Phi \in C^1([0, \tau+\varepsilon], \mathbb{R}^N)$ and that Φ is a solution of (2.1.40). Indeed, for every $0 \le t < \tau$ we have

$$\dot\Phi(t) = \dot\varphi(t) = \sum_{j=1}^n a_j(t) X_j I(\varphi(t)) \overset{(2.1.40)}{=} \sum_{j=1}^n a_j(t) X_j I(\Phi(t)),$$

while, for every $\tau < t \le \varepsilon$, one has (since ν solves (2.1.44))

$$\dot\Phi(t) = \dot\nu(t - \tau) \overset{(2.1.44)}{=} \sum_{j=1}^n a_j(t - \tau + \tau)\, X_j(\nu(t - \tau)) = \sum_{j=1}^n a_j(t) X_j I(\Phi(t)).$$

Therefore, recalling that Φ is continuous on $[0, \tau + \varepsilon]$, we obtain

$$\lim_{t \to \tau^-} \dot\Phi(t) = \sum_{j=1}^n a_j(\tau)\, X_j(\Phi(\tau)) = \lim_{t \to \tau^+} \dot\Phi(t),$$

and this proves that $\Phi \in C^1([0, \tau+\varepsilon], \mathbb{R}^N)$ and it solves the problem (2.1.40).

As a consequence, since $\tau + \varepsilon > T$ (by the choice of τ), Φ turns out to be a prolongation of φ beyond $[0, T[$; this is clearly in contradiction with the maximality of φ, and the proof is complete. □

Remark 2.1.28. Let us assume that there exists a Lie group $\mathbb{G} = (\mathbb{R}^N, *)$ on $\mathbb{R}^N$ such that $\mathrm{Lie}(\mathbb{G}) = \mathfrak{g}$. In this case, the proof of the globality of a Cauchy problem of the form (2.1.40) can be accomplished by exploiting the existence of a group-inversion; however, we cannot follow this approach in order to prove Thm. 2.1.27, since we do not possess a global inversion yet.

From Thm. 2.1.27, we immediately deduce the following result.

Corollary 2.1.29. *Let $x, y \in \mathbb{R}^N$ and let $a_1, \dots, a_N$ be as in lemma. 2.1.24.*
Then the maximal solution $\varphi_{x,y}$ of the Cauchy problem (2.1.39) *(depending on the parameter y) is defined on the whole of $\mathbb{R}$.*

By means of Cor. 2.1.29, we are able the extend the map m.

Definition 2.1.30. Let $x, y \in \mathbb{R}^N$ and let $\varphi_{x,y}$ be the unique maximal solution of the Cauchy problem (2.1.39). We set

$$* : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}^N, \qquad x * y := \varphi_{x,y}(1). \tag{2.1.45}$$

As it is natural to expect, $*$ turns out to be a real-analytic extension of m.

Theorem 2.1.31. *The function $*$ defined in* (2.1.45) *is a real-analytic function on $\mathbb{R}^N \times \mathbb{R}^N$ which extends the function m, that is,*

$$x * y = m(x, y) = \exp(\mathrm{Log}(y))(x), \quad \textit{for all } (x, y) \in \mathbb{R}^N \times V. \tag{2.1.46}$$

Proof. We first prove the regularity of $*$. To this end, for every $x, y \in \mathbb{R}^N$, let $\varphi_{x,y}$ be the (unique) maximal solution of the parametric Cauchy problem

$$\begin{cases} \dot{z}(t) = \displaystyle\sum_{k=1}^{N} a_k(t, y) J_k I(z(t)) \\ z(0) = x. \end{cases} \tag{2.1.47}$$

Since the function defined on $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$

$$(t, z; y) \mapsto \sum_{k=1}^{N} a_k(t, y) J_k I(z)$$

is real-analytic w.r.t. t, z and y (since $a_1, \dots, a_N$ are real-analytic on $\mathbb{R}\times\mathbb{R}^N$), we infer that the map $(t; x, y) \mapsto \varphi_{x,y}(t)$ is real-analytic w.r.t. $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^N$; as a consequence, we deduce that

$$(x, y) \mapsto \varphi_{x,y}(1) = x * y$$

is real-analytic on $\mathbb{R}^N \times \mathbb{R}^N$, as desired.

To prove identity (2.1.46), we choose a real $r > 0$ such that $B(0, r) \subseteq V$ and we fix $x \in \mathbb{R}^N$ and $y \in B(0, r)$. Moreover, we choose an open interval I containing $[0, 1]$ (note that $B(0, r)$ is convex) and we define

$$\gamma_{x,y} : I \longrightarrow \mathbb{R}^N, \qquad \gamma_{x,y}(t) := m(x, ty).$$

By Thm. 2.1.26, $\gamma_{x,y}$ is solution of the Cauchy problem (2.1.47) for all $t \in I$; therefore, since $\varphi_{x,y}$ is the maximal solution of the same problem, we have

$$\gamma_{x,y}(t) = \varphi_{x,y}(t), \quad \text{for all } t \in I.$$

In particular, since $1 \in I$ (by the choice of I), we get

$$m(x, y) = \gamma_{x,y}(1) = \varphi_{x,y}(1) = x * y, \tag{2.1.48}$$

and this proves that $*$ coincides with m on $\mathbb{R}^N \times D(0, r)$.

To conclude the demonstration, we apply the Unique Continuation Principle: since both m and $*$ are real-analytic on $\mathbb{R}^N \times V$ and they coincide on the open set $\mathbb{R}^N \times B(0,r)$, we obtain (recall that V is connected)

$$m(x,y) = x * y, \quad \text{for every } (x,y) \in \mathbb{R}^N \times V,$$

which is precisely the desired (2.1.46). □

Remark 2.1.32. It is worth noting that, as a consequence of Thm. 2.1.31, the map $*$ defined in (2.1.45) is the unique analytic extension of m. Indeed, if $f \in C^\omega(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$ is another extension of m, we have

$$x * y = f(x,y) = m(x,y), \quad \text{for all } (x,y) \in \mathbb{R}^N \times V;$$

therefore, since both f and $*$ are real-analytic and they coincide on $\mathbb{R}^N \times V$, the Unique Continuation Principle ensures that

$$f(x,y) = x * y, \quad \text{for all } x, y \in \mathbb{R}^N.$$

As a consequence of the Unique Continuation Principle, the map $*$ inherits all the local properties of m proved in the previous section, turning them into global ones.

Theorem 2.1.33. *The map $*$ is globally associative on $\mathbb{R}^N$, that is*

$$x * (y * z) = (x * y) * z, \quad \text{for all } x, y, z \in \mathbb{R}^N. \tag{2.1.49}$$

Moreover, the point $0 \in \mathbb{R}^N$ is a global neutral element for $*$, that is,

$$x * 0 = 0 * x = x, \quad \text{for all } x \in \mathbb{R}^N. \tag{2.1.50}$$

Finally, the map ι in (2.1.27) *provides an inversion map for the $*$, that is,*

$$x * \iota(x) = \iota(x) * x = 0, \quad \text{for all } x \in V. \tag{2.1.51}$$

Proof. Let $W \subseteq V$ be as in Thm. 2.1.19 and let $x, y, z \in W$. Since, by Thm. 2.1.31, the map $*$ is an extension of m, we have (recall that both $m(x,y)$ and $m(y,z)$ belong to $W \subseteq V$)

$$\begin{aligned} x * (y * z) &\overset{(2.1.46)}{=} x * m(y,z) \overset{(2.1.46)}{=} m(x, m(y,z)); \\ (x * y) * z &\overset{(2.1.46)}{=} m(x * y, z) \overset{(2.1.46)}{=} m(m(x,y), z); \end{aligned}$$

therefore, since m is locally associative (as ensured by Thm. 2.1.19), we get

$$x * (y * z) = m(x, m(y,z)) \overset{(2.1.20)}{=} m(m(x,y), z) = (x * y) * z. \tag{2.1.52}$$

We can now apply the Unique Continuation Principle: since both sides of identity (2.1.52) are real-analytic w.r.t. x, y, z and since they coincide on the open set $W \times W \times W$, we necessarily have

$$x * (y * z) = (x * y) * z, \quad \text{for all } x, y, z \in \mathbb{R}^N.$$

This is precisely the desired (2.1.49).

Since $0 \in V$, by Thm. 2.1.31, we obtain

$$x * 0 \overset{(2.1.46)}{=} m(x,0) \overset{(2.1.21)}{=} x, \quad \text{for all } x \in \mathbb{R}^N.$$

On the other hand, if $x \in V$, again by Thm. 2.1.31 we get

$$0 * x \overset{(2.1.46)}{=} m(0,x) \overset{(2.1.22)}{=} x. \tag{2.1.53}$$

We can then apply once again the Unique Continuation Principle: since both sides of identity (2.1.53) are real-analytic on $\mathbb{R}^N$ and they coincide on the open set V, they must coincide on the whole of $\mathbb{R}^N$, that is,

$$0 * x = x, \quad \text{for all } x \in \mathbb{R}^N.$$

Finally, let $W \subseteq V$ be as in Thm. 2.1.22 and let $x \in W$. By Thm. 2.1.31, we have (recall that $\iota(x) \in V$, since $x \in W$)

$$\begin{aligned} x * \iota(x) &\overset{(2.1.46)}{=} m(x, \iota(x)), \\ \iota(x) * x &\overset{(2.1.46)}{=} m(\iota(x), x); \end{aligned}$$

hence, the map ι providing an inverse for m on W (by Thm. 2.1.22), we get

$$x * \iota(x) = m(x, \iota(x)) = m(\iota(x), x) = x * \iota(x) = 0, \quad \text{for all } x \in W. \tag{2.1.54}$$

We then apply the Unique Continuation Principle: since both maps

$$x \mapsto x * \iota(x) \qquad \text{and} \qquad x \mapsto \iota(x) * x$$

are real-analytic on V and since, by (2.1.54), they are equal to 0 on the open set W, they must identically vanish on the whole of V, that is,

$$x * \iota(x) = \iota(x) * x = 0, \quad \text{for all } x \in V.$$

This ends the proof. □

Now that we have globalized the local-group properties of m, we proceed by establishing a global version of Thm. 2.1.23. Before doing this, we give the following definition. To be noted that we do not yet know that the following maps are the left-/right-translations on a Lie group.

Definition 2.1.34. Let $x \in \mathbb{R}^N$ be fixed. We let

$$\tau_x : \mathbb{R}^N \longrightarrow \mathbb{R}^N \qquad \tau_x(y) := x * y, \tag{2.1.55}$$

$$\rho_x : \mathbb{R}^N \longrightarrow \mathbb{R}^N \qquad \rho_x(y) := y * x. \tag{2.1.56}$$

Theorem 2.1.35 (Global left-invariance of $\mathfrak{g}$)**.** *Every vector field X in $\mathfrak{g}$ is left-invariant w.r.t. $*$, that is, the following identity holds true*

$$XI(x * y) = \mathcal{J}_{\tau_x}(y) \cdot XI(y), \quad \textit{for all } x, y \in \mathbb{R}^N. \tag{2.1.57}$$

Proof. Let $X \in \mathfrak{g}$ be fixed. By Thm. 2.1.23 we know that

$$XI(m(x,y)) = \frac{\partial m}{\partial y}(x,y) \cdot XI(y), \quad \text{for all } (x,y) \in \mathbb{R}^N \times V;$$

from this, since $*$ coincides with m on the open set $\mathbb{R}^N \times V$, we get

$$\begin{aligned} XI(x*y) &\overset{(2.1.46)}{=} XI(m(x,y)) = \frac{\partial m}{\partial y}(x,y) \cdot XI(y) \\ &= \frac{\partial \tau_x}{\partial y}(y) \cdot XI(y) = \mathcal{J}_{\tau_x}(y) \cdot XI(y), \quad \text{for all } (x,y) \in \mathbb{R}^N \times V. \end{aligned} \tag{2.1.58}$$

We now apply the Unique Continuation Principle: since both sides of identity (2.1.58) are real-analytic on $\mathbb{R}^N \times \mathbb{R}^N$ (any vector field in $\mathfrak{g}$ being real-analytic) and since they coincide on the open set $\mathbb{R}^N \times V$, we must have

$$XI(x*y) = \mathcal{J}_{\tau_x}(y) \cdot XI(y), \quad \text{for all } x,y \in \mathbb{R}^N.$$

This is precisely the desired (2.1.57), and the proof is complete. □

Together with the map m and its analytic extension $*$, in order to prove our Thm. 2.1.6 we also need a global (analytic) extension of the inversion map ι, allowing us to define a group structure on $\mathbb{R}^N$.

Theorem 2.1.36. *For every $x \in \mathbb{R}^N$, the map τ_x in Def. 2.1.34 is a local diffeomorphism on $\mathbb{R}^N$ of class C^ω.*

Proof. We first prove that, if $x \in \mathbb{R}^N$ is fixed, then τ_x is an analytic diffeomorphism near the origin. To this end, we consider the following maps

$$\begin{aligned} &e_x : \mathbb{R}^N \longrightarrow \mathbb{R}^N \qquad && e_x(\xi) := \exp\left(\sum_{k=1}^N \xi_k J_k\right)(x), \\ &L : V \longrightarrow \mathbb{R}^N && L(x) := (\pi^{-1} \circ \mathrm{Log})(x). \end{aligned}$$

Since $*$ is a prolongation of m, by definition of τ_x we have

$$\tau_x(y) = x*y \overset{(2.1.46)}{=} m(x,y) \overset{(2.1.11)}{=} \exp\big(\mathrm{Log}(y)\big)(x) = e_x(L(y)); \tag{2.1.59}$$

therefore, to prove that τ_x is a C^ω-diffeomorphism at 0 we show that both L and e_x are real-analytic maps with non-singular Jacobian matrix at 0.

Now, the analyticity of L readily follows from that of Log; moreover, from Rem. 2.1.13 we deduce that $\mathcal{J}_L(0) = \mathbb{I}_N$. As for the map e_x, we first observe that, if $\gamma(\cdot\,;\xi)$ denotes the unique maximal solution of the Cauchy problemnce

$$\begin{cases} \dot z = \sum_{k=1}^N \xi_k J_k I(z) \\ z(0) = x, \end{cases}$$

then $e_x(\xi) = \gamma(1;\xi)$; therefore, $(t,\xi) \mapsto \gamma(t;\xi)$ is real-analytic on $\mathbb{R}^N \times \mathbb{R}^N$, and we deduce that $e_x \in C^\omega(\mathbb{R}^N, \mathbb{R}^N)$.

To compute the Jacobian matrix of e_x at 0, we write the Maclaurin expansion (with integral remainder) of $\gamma(\cdot\,;\xi)$ for $t = 1$:

$$\begin{aligned}\gamma(1;\xi) &= \gamma(0;\xi) + \dot\gamma(0;\xi) + \int_0^1 (1-s)\gamma''(s;\xi)\,\mathrm{d}s \\ &= x + \sum_{k=1}^N \xi_k J_k I(x) + \sum_{h,k=1}^N \xi_h\xi_k \int_0^1 (1-s)\,(J_h J_k I)(\gamma(s;\xi))\,\mathrm{d}s.\end{aligned} \tag{2.1.60}$$

From this, by arguing as in the proof of Prop. 2.1.12, we infer that

$$e_x(\xi) = \gamma(1;\xi) = x + \sum_{k=1}^N \xi_k J_k I(x) + \mathcal{O}(\|\xi\|^2), \quad \text{as } \xi \to 0,$$

and this proves that $\mathcal{J}_{e_x}(0) = (J_1 I(x) \cdots J_N I(x))$. Since $J_1 I(x), \ldots, J_N(x)$ are linearly independent in $\mathbb{R}^N$ (by the choice of the basis $\mathcal{J}$), we see that also $\mathcal{J}_{e_x}(0)$ is non-singular. By gathering together all these facts, the matrix

$$\mathcal{J}_{\tau_x}(0) \overset{(2.1.59)}{=} \mathcal{J}_{e_x}(0)\cdot\mathcal{J}_L(0) = \mathcal{J}_{e_x}(0) = \big(J_1 I(x)\cdots J_N I(x)\big)$$

is non-singular, whence τ_x is a C^ω-diffeomorphism near 0, as desired.

To conclude the demonstration of the theorem, we crucially exploit the associativity of $*$: since, for every $x, y, z \in \mathbb{R}^N$, we have

$$(x * y) * z = x * (y * z),$$

by differentiating both sides of the above identity w.r.t. z at $z = 0$, we get

$$\mathcal{J}_{\tau_{x*y}}(0) = \mathcal{J}_{\tau_x}(y * 0)\cdot\mathcal{J}_{\tau_y}(0) = \mathcal{J}_{\tau_x}(y)\cdot\mathcal{J}_{\tau_y}(0); \tag{2.1.61}$$

from this, since both matrices $\mathcal{J}_{\tau_{x*y}}(0)$ and $\mathcal{J}_{\tau_y}(0)$ are non-singular (as we have already proved), we infer that

$$\mathcal{J}_{\tau_x}(y) = \mathcal{J}_{\tau_{x*y}}(0)\cdot\big(\mathcal{J}_{\tau_y}(0)\big)^{-1},$$

is non-singular as well. By the Inverse Function Theorem, we then conclude that τ_x is a local diffeomorphism on $\mathbb{R}^N$, as desired. □

Since every local diffeomorphism (on $\mathbb{R}^N$) is an open map, Thm. 2.1.36 immediately gives the following non-trivial result.

Corollary 2.1.37. *For any $x \in \mathbb{R}^N$, the map τ_x in Def. 2.1.34 is an open map.*

By means of Thm. 2.1.36 and of Cor. 2.1.37, we can prove a simple topological result, which will allow us to extend the map ι to the whole of $\mathbb{R}^N$.

Proposition 2.1.38. *Let $W \subseteq V$ be as in Thm. 2.1.17. Then we have*

$$\mathbb{R}^N = \bigcup_{n=1}^\infty \Big\{ w_1 * \cdots * w_n : w_1, \ldots, w_n \in W \Big\}. \tag{2.1.62}$$

Proof. We denote by A the set in right-hand side of (2.1.62). To prove the proposition, we show that A is both open and closed (in $\mathbb{R}^N$).

A is open: We consider the family $\{A_n\}_{n\in\mathbb{N}}$ of subsets of $\mathbb{R}^N$ defined by

$$A_n := \big\{w_1 * \cdots * w_n : w_1, \dots, w_n \in W\big\}, \quad \text{for all } n \in \mathbb{N}. \tag{2.1.63}$$

By definition, $A = \bigcup_{n\geq 1} A_n$; moreover, it is easy to see that each A_n is open. Indeed, $A_1 = W$ is open and, if $n \geq 2$, we can write (since $*$ is associative)

$$A_n = \big\{x * w : x \in A_{n-1}, w \in W\big\} = \bigcup_{x\in A_{n-1}} \tau_x(W).$$

Since W is open and since any left-translation is an open map (by Cor. 2.1.37), we see that A_n is the union of open sets, whence it is open.

A is closed: Let $x_0 \in \overline{A}$ be fixed. Since ι is continuous (on V) and $\iota(0) = 0$, there exists an open neighborhood $U \subseteq W$ of 0 such that

$$\iota(U) \subseteq W; \tag{2.1.64}$$

moreover, since $\tau_{x_0}(U)$ is an open neighborhood of $x_0 = \tau_{x_0}(0)$ (τ_{x_0} being an open map), we must have $\tau_{x_0}(U) \cap A \neq \varnothing$. As a consequence, it is possible to find $w_1, \dots, w_n \in W$ and $u \in U$ such that

$$w_1 * \cdots * w_n = \tau_{x_0}(u) = x_0 * u.$$

From this, by the associativity of $*$, we get

$$\begin{aligned} (w_1 * \cdots * w_n) * \iota(u) &= (x_0 * u) * \iota(u) \overset{(2.1.49)}{=} x_0 * (u * \iota(u)) \\ &\overset{(2.1.51)}{=} x_0 * 0 = x_0; \end{aligned} \tag{2.1.65}$$

therefore, since $w_1, \dots, w_n, \iota(u) \in W$ (by (2.1.64)), we see that $x_0 \in A$.

Since obviously $A \neq \varnothing$, we conclude that $A = \mathbb{R}^N$, as desired. □

Remark 2.1.39. If $\mathbb{G} = (\mathbb{R}^N, *)$ is a Lie group with neutral element 0, the result contained in Prop. 2.1.38 is a straightforward consequence of the following more general fact: a connected topological group is generated (as a group) by any neighborhood of the identity.

From Prop. 2.1.38, we easily deduce the following crucial result.

Proposition 2.1.40. *For every $x \in \mathbb{R}^N$, there exists a unique $y_x \in \mathbb{R}^N$ s.t.*

$$x * y_x = y_x * x = 0. \tag{2.1.66}$$

Proof. Let $W \subseteq V$ be as in Thm. 2.1.17. By Prop. 2.1.38, it is possible to find $w_1, \dots, w_n \in W$, not necessarily unique, such that

$$x = w_1 * \cdots * w_n;$$

hence, we define $y := \iota(w_n) * \cdots * \iota(w_1)$. Since $W \subseteq V$, y is well-defined; moreover, by the associativity of $*$ we have

$$\begin{aligned} x * y &= (w_1 * \cdots * w_n) * (\iota(w_n) * \cdots * \iota(w_1)) \\ &\overset{(2.1.49)}{=} (w_1 * \cdots * w_{n-1}) * (w_n * \iota(w_n)) * (\iota(w_{n-1}) * \cdots * \iota(w_1)) \\ &\overset{(2.1.51)}{=} (w_1 * \cdots * w_{n-1}) * 0 * (\iota(w_{n-1}) * \cdots * \iota(w_1)) \\ &\overset{(2.1.50)}{=} (w_1 * \cdots * w_{n-1}) * (\iota(w_{n-1}) * \cdots * \iota(w_1)) = [...] = 0. \end{aligned}$$

Analogously, we have $y * x = 0$, whence y satisfies (2.1.66).

As for the uniqueness part, let $z \in \mathbb{R}^N$ be such that

$$z * x = x * z = 0.$$

By the associativity of $*$, we get

$$y \overset{(2.1.50)}{=} y * 0 \overset{(2.1.66)}{=} y * (x * z) \overset{(2.1.49)}{=} (y * x) * z \overset{(2.1.66)}{=} 0 * z \overset{(2.1.50)}{=} z,$$

whence y is the unique point in $\mathbb{R}^N$ satisfying (2.1.66). □

The result contained in Prop. 2.1.40 provides a very natural way to extend the map ι to the whole of $\mathbb{R}^N$.

Definition 2.1.41. For every $x \in \mathbb{R}^N$, let $y_x \in \mathbb{R}^N$ be the unique point satisfying identity (2.1.66) in Prop. 2.1.40. We define

$$\tilde{\iota} : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \quad \tilde{\iota}(x) := y_x. \tag{2.1.67}$$

As we did for $*$, we prove that $\tilde{\iota}$ is a real-analytic extension of ι.

Theorem 2.1.42. *The map $\tilde{\iota}$ defined in* (2.1.41) *is real-analytic on $\mathbb{R}^N$ and it extends the map ι, that is,*

$$\tilde{\iota}(x) = \iota(x), \quad \textit{for all } x \in V. \tag{2.1.68}$$

Proof. We first prove that $\tilde{\iota} \in C^\omega(\mathbb{R}^N, \mathbb{R}^N)$. To this end, let $x_0 \in \mathbb{R}^N$ and let $y_0 := \tilde{\iota}(x_0)$. Then, by definition, we have $x_0 * y_0 = 0$. We claim that the Jacobian matrix of $*$ at (x_0, y_0) has full rank. Indeed, recalling the definition of the maps τ_{x_0} and ρ_{x_0} (see Def. 2.1.34), we have

$$\mathcal{J}_*(x_0, y_0) = \Big(\mathcal{J}_{\rho_{y_0}}(x_0) \;\; \mathcal{J}_{\tau_{y_0}}(x_0)\Big); \tag{2.1.69}$$

therefore, since $\mathcal{J}_{\tau_{y_0}}(x_0)$ is non-singular (τ_{y_0} being a local diffeomorphism of $\mathbb{R}^N$, see Thm. 2.1.36), we deduce that

$$\mathrm{rank}\Big(\mathcal{J}_*(x_0, y_0)\Big) = N,$$

as claimed. By the Inverse Function Theorem (in the real-analytic setting), we can then find two open neighborhoods $U, U' \subseteq \mathbb{R}^N$ of 0 and a real-analytic function $f : U \to U'$ such that $f(x_0) = y_0 = \tilde{\iota}(x_0)$ and

$$\{(x, y) \in U \times U' : x * y = 0\} = \{(x, f(x)) : x \in U\}.$$

Now, by Prop. 2.1.40, for every $x \in \mathbb{R}^N$ there exists a unique point $y \in \mathbb{R}^N$ such that $x * y = 0$, which is precisely $\tilde{\iota}(x)$; therefore,

$$f(x) = \tilde{\iota}(x), \quad \text{for every } x \in U,$$

and this proves that $\tilde{\iota}$ is real-analytic on U. From the arbitrariness of x_0, we then conclude that $\tilde{\iota} \in C^\omega(\mathbb{R}^N, \mathbb{R}^N)$, as desired.

As for identity (2.1.68) we recall that, if $x \in V$, we have

$$x * \iota(x) = \iota(x) * x = 0;$$

hence, by Prop. 2.1.40 (and the very definition of $\tilde{\iota}$), we get $\iota(x) = \tilde{\iota}(x)$. □

Thanks to all the results proved in this section, we can provide a complete proof of Thm. 2.1.6 stated at the beginning of the chapter.

Proof (of Thm. 2.1.6). Let $\mathbb{G} := (\mathbb{R}^N, *)$, where $*$ is the map defined in (2.1.45). By Thm. 2.1.31, $*$ is real-analytic on $\mathbb{R}^N \times \mathbb{R}^N$; moreover, Thm.2.1.33 shows that $*$ is associative and that 0 provides a neutral element for $*$. Finally, if $\tilde{\iota}$ is the map defined in (2.1.67), by Thm. 2.1.42 we know that $\tilde{\iota} \in C^\omega(\mathbb{R}^N, \mathbb{R}^N)$ and that it provides an inversion map for $*$.

Summing up, $\mathbb{G}$ is a real-analytic group on $\mathbb{R}^N$ with neutral element 0. To conclude the demonstration of the theorem, we turn to show that $\mathrm{Lie}(\mathbb{G}) = \mathfrak{g}$. To this end we observe that, since Thm. 2.1.35 ensures that

$$XI(x * y) = \mathcal{J}_{\tau_x}(y) \cdot XI(y) \quad \text{for all } x, y \in \mathbb{R}^N \text{ and } X \in \mathfrak{g},$$

then $\mathfrak{g} \subseteq \mathrm{Lie}(\mathbb{G})$; from this, recalling that $\mathfrak{g}$ has dimension N (by assumption (ND)), we conclude that $\mathfrak{g} = \mathrm{Lie}(\mathbb{G})$, and the proof is complete. □

We easily obtain the following improvement of Thm. 2.1.6.

Theorem 2.1.43. *Let $\mathfrak{g}$ be a Lie algebra of real analytic vector fields on $\mathbb{R}^N$ satisfying conditions* (C), (H) *and* (ND) *in Def. 2.1.1. Then, for every x_0 in $\mathbb{R}^N$ there exists a unique real-analytic Lie group $\mathbb{G}_{x_0} = (\mathbb{R}^N, \star)$ with neutral element x_0 and Lie algebra equal to $\mathfrak{g}$. More precisely, we have*

$$x \star y = x * (x_0)^{-1} * y, \quad \textit{for all } x, y \in \mathbb{R}^N, \tag{2.1.70}$$

where $*$ is the map defined in (2.1.45) *and $(x_0)^{-1}$ is the inverse of x_0 w.r.t. $*$.*

As a consequence, the group $\mathbb{G} = (\mathbb{R}^N, *)$ is the unique (real-analytic) Lie group on $\mathbb{R}^N$ with neutral element 0 *and such that* $\mathrm{Lie}(\mathbb{G}) = \mathfrak{g}$.

Proof. Let $x_0 \in \mathbb{R}^N$ be fixed and let $\star$ be the map defined in (2.1.70). Moreover, let $\mathbb{G}_{x_0} := (\mathbb{R}^N, \star)$. Since $\mathbb{G} = (\mathbb{R}^N, *)$ is a Lie group with neutral element 0, it is very easy to recognize that $\mathbb{G}_{x_0}$ is also a (real-analytic) Lie group on $\mathbb{R}^N$, with neutral element x_0 (note that $\star$ is the *push-forward of $*$ via τ_{x_0}*).

To prove that $\mathrm{Lie}(\mathbb{G}_{x_0}) = \mathfrak{g}$ we first observe that, by the Chain Rule, we have (denoting by $\tau^\star_x$ the left-translation by x on $\mathbb{G}_{x_0}$)

$$\mathcal{J}_{\tau^\star_x}(x_0) = \mathcal{J}_{\tau_x}(0) \cdot \mathcal{J}_{\tau_{{x_0}^{-1}}}(x_0) = \mathcal{J}_{\tau_x}(0) \cdot \big(\mathcal{J}_{\tau_{x_0}}(0)\big)^{-1};$$

from this, recalling that the Jacobian basis of $\mathrm{Lie}(\mathbb{G}_{x_0})$ is given by the vector fields $J^\star_1, \ldots, J^\star_N$ associated with the columns of $\mathcal{J}_{\tau^\star_x}(x_0)$ (and, analogously, the

Jacobian basis of $\mathrm{Lie}(\mathbb{G})$ is given by the vector fields $J_1, \dots, J_N$ associated with the columns of $\mathcal{J}_{\tau_x}(0)$), we derive that

$$J_i^\star = \sum_{k=1}^{N} c_{k,i}\, J_k,$$

where $(c_{1,i}, \dots, c_{N,i})^T$ is the i-th column of $\mathcal{J}_{\tau_{x_0^{-1}}}(x_0)$. Finally, since $\mathrm{Lie}(\mathbb{G})$ and $\mathrm{Lie}(\mathbb{G}_{x_0})$ are both N-dimensional, we conclude that

$$\mathrm{Lie}(\mathbb{G}_{x_0}) = \mathrm{Lie}(\mathbb{G}) = \mathfrak{g}.$$

As for the uniqueness of the group $\mathbb{G}_{x_0}$, let us assume that there exists another (real-analytic) Lie group $\mathbb{F} = (\mathbb{R}^N, \circ)$, with neutral element x_0 and Lie algebra coinciding to $\mathfrak{g}$. Then, if we denote by $\mathrm{Exp}^\circ : \mathfrak{g} = \mathrm{Lie}(\mathbb{F}) \to \mathbb{R}^N$ the Exponential Map on $\mathbb{F}$ (and by $\mathrm{Exp}^\star$ the Exponential Map on $\mathbb{G}_{x_0}$), we have

$$\mathrm{Exp}^\circ(X) = \exp(1 \cdot X)(x_0) = \mathrm{Exp}^\star(X), \quad \text{for every } X \in \mathfrak{g}.$$

As a consequence, since $\mathrm{Exp}^\circ = \mathrm{Exp}^\star$ is local diffeomorphism of an open neighborhood $\mathcal{U}$ of $0 \in \mathfrak{g}$ onto an neighborhood U of $x_0 \in \mathbb{R}^N$, if $x \in \mathbb{R}^N$ and if $y = \mathrm{Exp}^\star(Y) = \mathrm{Exp}^\circ(Y) \in U$ (with $Y \in \mathcal{U}$), we obtain

$$x \star y = x \star \mathrm{Exp}^\star(Y) = \exp(Y)(x) = x \circ \mathrm{Exp}^\circ(Y) = x \circ y,$$

and this proves that $\star$ and $\circ$ do coincide on the open set $\mathbb{R}^N \times U$. Since both $\star$ and $\circ$ are real-analytic on $\mathbb{R}^N \times \mathbb{R}^N$, the Unique Continuation Principle ensures that they coincide on the whole of $\mathbb{R}^N \times \mathbb{R}^N$, and the proof is complete. □

2.1.4 The C^∞ case: a brief overview

The main aim of this section is to roughly describe how Thm. 2.1.6 can be proved also in the case of Lie algebras of *smooth* vector fields. More details will be given in a future planned thorough investigation (presently, in preparation).

To begin with, let us fix once and for all a Lie algebra $\mathfrak{g} \subseteq \mathcal{X}(\mathbb{R}^N)$ of smooth (but not necessarily real-analytic) vector fields on $\mathbb{R}^N$ satisfying the assumptions (C), (H) and (ND) introduced in Def. 2.1.1, that is,

(C): every $X \in \mathfrak{g}$ is a global vector field;

(H): the Hörmander rank condition holds for $\mathfrak{g}$;

(ND): $\mathfrak{g}$ is N-dimensional, as a linear subspace of $\mathcal{X}(\mathbb{R}^N)$.

We also choose a basis $\mathcal{J} = \{J_1, \dots, J_N\}$ of $\mathfrak{g}$ as in Lem. 2.1.9 and we set

$$\mathbf{J}(x) := \big(J_1 I(x) \cdots J_N I(x)\big), \qquad x \in \mathbb{R}^N. \tag{2.1.71}$$

Now, according to Prop. 2.1.12, the Exponential Map of $\mathfrak{g}$

$$\mathrm{Exp} : \mathfrak{g} \longrightarrow \mathbb{R}^N, \qquad \mathrm{Exp}(X) = \exp(X)(0)$$

is of class C^∞ on $\mathfrak{g}$; moreover, there exists an open and connected neighborhood $\mathcal{U} \subseteq \mathfrak{g}$ of 0 such that $\mathrm{Exp}_{|\mathcal{U}}$ is a smooth diffeomorphism, with inverse

$$\mathrm{Log} : V := \mathrm{Exp}(\mathcal{U}) \longrightarrow \mathcal{U}, \qquad \mathrm{Log}(x) = \big(\mathrm{Exp}_{|\mathcal{U}}\big)^{-1}(x).$$

Then, by proceeding exactly as in the real-analytic case, we define

$$m : \mathbb{R}^N \times V \longrightarrow \mathbb{R}^N, \qquad m(x,y) := \exp\big(\mathrm{Log}(y)\big)(x),$$

$$\iota : V \longrightarrow \mathbb{R}^N, \qquad \iota(x) := \mathrm{Exp}\big(-\mathrm{Log}(x)\big).$$

Obviously, Exp and Log being smooth on their domains of definition, we have

$$m \in C^\infty(\mathbb{R}^N \times V, \mathbb{R}^N) \qquad \text{and} \qquad \iota \in C^\infty(V, \mathbb{R}^N);$$

moreover, since Thm. 2.1.16 holds true also for (finite-dimensional) Lie algebras of *smooth* vector fields, by arguing *verbatim* as in Sec. 2.1.2 we can prove the following facts (see, respectively, Thm.s 2.1.19, 2.1.22 and 2.1.23):

(1) There exists an open and convex neighborhood $W \subseteq \mathbb{R}^N$ of 0 such that $m(a,b) \in V$ for every $a, b \in V$ and m is associative on W, that is,

$$m(x, m(y,z)) = m(m(x,y),z), \quad \forall\, x \in \mathbb{R}^N \text{ and } \forall\, y,z \in W. \tag{2.1.72}$$

Moreover, 0 is a neutral element for m, that is,

$$\begin{aligned} m(x,0) &= x, \quad \text{for all } x \in \mathbb{R}^N, \\ m(0,y) &= y, \quad \text{for all } y \in V. \end{aligned} \tag{2.1.73}$$

(2) If W is as in statement (1), we have $\iota(a) \in V$ for every $a \in W$ and ι provides a local inverse for m on W, that is,

$$m(x, \iota(x)) = m(\iota(x), x) = 0, \quad \text{for every } x \in W. \tag{2.1.74}$$

(3) Any vector field in $\mathfrak{g}$ is locally left-invariant w.r.t. m, that is,

$$XI(m(x,y)) = \frac{\partial m}{\partial y}(x,y) \cdot XI(y), \quad \text{for every } (x,y) \in \mathbb{R}^N \times W. \tag{2.1.75}$$

We explicitly observe that, in contrast to the real-analytic case, we *are not able to extend* (at this very point of the proof) identity (2.1.75) to the whole of $\mathbb{R}^N \times V$: indeed, due to the lack of analyiticity, we cannot exploit the Unique Continuation Principle (cf the proof of Thm. 2.1.23).

Summing up, also in the C^∞ case it is possible to construct a local Lie group $\mathbb{G}_{\mathrm{loc}} = (\mathbb{R}^N, m)$, with neutral element 0 and inverse given by ι, such that

$$\mathrm{Lie}(\mathbb{G}_{\mathrm{loc}}) = \mathfrak{g}, \qquad \text{in the sense of local Lie groups.}$$

We now turn to briefly describe how this local group can be globalized.

First of all, by following the same profitable idea explained in Sec. 2.1.3, we fix $(x,y) \in \mathbb{R}^N \times W$ and we consider the curve $\gamma_{x,y} : [0,1] \to \mathbb{R}^N$ defined by

$$\gamma(t) := m(x, ty), \quad 0 \le t \le 1.$$

Obviously, $\gamma \in C^\infty([0,1], \mathbb{R}^N)$; moreover, by exploiting identity (2.1.75) and by arguing as in the proof of Thm. 2.1.26, we see that $\gamma_{x,y}$ solves on $[0,1]$ the following Cauchy problem (recall that $m(x,0) = x$, see (2.1.73)):

$$\begin{cases} \dot z = \displaystyle\sum_{k=1}^N a_k(t,y) J_k I(z) \\ z(0) = x, \end{cases} \qquad \text{where} \begin{pmatrix} a_1(t,y) \\ \vdots \\ a_N(t,y) \end{pmatrix} = \big(\mathbf{J}(ty)\big)^{-1} \cdot y. \tag{2.1.76}$$

On the other hand, again by exploiting the crucial identity (2.1.75) (and by arguing as in the the proof of Thm. 2.1.27) we recognize that, for *every fixed* $x, y \in \mathbb{R}^N$, the Cauchy problem (2.1.76) possesses a unique maximal solution $\varphi_{x,y}$ which is defined on the whole of $\mathbb{R}$; therefore, we define

$$* : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}^N, \qquad x * y := \varphi_{x,y}(1). \tag{2.1.77}$$

We also introduce, for every fixed $x \in \mathbb{R}^N$, the notations τ_x and ρ_x for the left and the right translation by x associated with $*$, that is,

$$\begin{aligned} \tau_x &: \mathbb{R}^N \longrightarrow \mathbb{R}^N & \tau_x(y) &:= x * y, \\ \rho_x &: \mathbb{R}^N \longrightarrow \mathbb{R}^N & \rho_x(y) &:= y * x. \end{aligned}$$

Now, by classical results of ODE Theory we know that $*$ is of class C^∞ on $\mathbb{R}^N \times \mathbb{R}^N$; moreover, from the previous discussion we infer that

$$x * y = \varphi_{x,y}(1) = \gamma_{x,y}(1) = m(x,y), \quad \text{if } x \in \mathbb{R}^N \text{ and } y \in W.$$

In other words, $*$ is a *smooth prolongation* of m to the whole of $\mathbb{R}^N \times \mathbb{R}^N$. Hence, to complete the globalization of the local group $\mathbb{G}_{\mathrm{loc}}$, we have to show that $*$ inherits all the properties of m, turning them into *global* properties.

Unfortunately, due to the lack of analyticity, we have to renounce the powerful tool provided by the Unique Continuation Principle (which played a crucial rôle in the real-analytic case); *instead, we shall use the uniqueness of the maximal solution of a Cauchy problem.* The key steps are thus the following:

Step I: First of all we observe that, $\varphi_{x,y}$ being the unique maximal solution of the problem (2.1.76), for every $x, y \in \mathbb{R}^N$ and every $t \in \mathbb{R}$ we have

$$\tau_x(ty) = x * (ty) = \varphi_{x,ty}(1) = \varphi_{x,y}(t).$$

As a consequence, we obtain the following crucial identity:

$$\frac{\partial}{\partial y_i}\big(\varphi_{x,y}(t)\big) = \frac{\partial}{\partial y_i}\big(\tau_x(ty)\big) = t\,\mathcal{J}_{\tau_x}(ty) \cdot e_i, \tag{2.1.78}$$

holding true for every $i \in \{1, \ldots, N\}$, every $x, y \in \mathbb{R}^N$ and every $t \in \mathbb{R}$.

Step II: For every fixed $i \in \{1, \ldots, N\}$ and every $x, y \in \mathbb{R}^N$, we define

$$u_i : \mathbb{R} \longrightarrow \mathbb{R}^N, \qquad u_i(t) := t\,\mathbf{J}(x * ty) \cdot \big(\mathbf{J}(ty)\big)^{-1} \cdot e_i.$$

Obviously, $u_i \in C^\infty(\mathbb{R}, \mathbb{R}^N)$ and $u_i(0) = 0$; moreover, after some non-trivial computations we recognize that, for every $t \in \mathbb{R}$, one has

$$\dot{u}_i(t) = A(t) \cdot u_i(t) + b_i(t), \tag{2.1.79}$$

where we have used the notations

$$A(t) := \sum_{j=1}^N a_j(t,y) \cdot \mathcal{J}_{X_j I}(x * ty) \quad \text{and} \quad b_i(t) := \sum_{j=1}^N \frac{\partial a_j}{\partial y_i}(t,y)\, X_j I(x * ty).$$

Since the equation (2.1.79) is nothing but the variational ODE satisfied by the function $t \mapsto \partial/\partial_{y_i}(\varphi_{x,y}(t))$, from identity (2.1.78) we conclude that

$$\mathbf{J}(x*y)\cdot\big(\mathbf{J}(y)\big)^{-1}\cdot e_i = u_i(1) = \frac{\partial}{\partial y_i}\big(\varphi_{x,y}(1)\big) = \mathcal{J}_{\tau_x}(y)\cdot e_i. \qquad (2.1.80)$$

In particular, the matrix $\mathbf{J}(z)$ being non-singular for every $z \in \mathbb{R}^N$ (by Lem. 2.1.9), we infer that τ_x is a local diffeomorphism of $\mathbb{R}^N$ of class C^∞.

Step III: Thanks to identity (2.1.80) we can give an easy proof of the (global) associativity of $*$. Indeed, let $x, y, z \in \mathbb{R}^N$ be arbitrarily fixed and let $\gamma, \mu \in C^\infty(\mathbb{R}, \mathbb{R}^N)$ be the curves in $\mathbb{R}^N$ defined as follows:

$$\gamma(t) := x*(y*tz), \qquad \mu(t) := (x*y)*tz.$$

By crucially exploiting the cited (2.1.80), it is easy to see that γ and μ solves on $\mathbb{R}^N$ the *same Cauchy problem*, namely

$$\begin{cases} \dot{u} = \sum_{k=1}^{N} a_k(t,z) J_k I(z) \\ u(0) = x*y. \end{cases}$$

Thus, by uniqueness, we have $\gamma(t) = \mu(t)$ for every $t \in \mathbb{R}$, whence

$$x*(y*z) = \gamma(1) = \mu(1) = (x*y)*z.$$

This gives the global associativity of $*$.

Step IV: We now turn to show that 0 is a neutral element for $*$. To this end, let $x \in \mathbb{R}^N$ be fixed. Since, obviously, the constant function $\gamma(t) = x$ is a solution (on the whole of $\mathbb{R}$) of the Cauchy problem (see (2.1.76))

$$\begin{cases} \dot{z} = \sum_{k=1}^{N} a_k(t,0) J_k I(z) = 0, \\ u(0) = x, \end{cases}$$

we immediately infer that $x*0 = \varphi_{x,0}(1) = \gamma(1) = x$. On the other hand, a direct computation shows that the linear function $\mu(t) = tx$ is a solution (on the whole of $\mathbb{R}$) of the Cauchy problem

$$\begin{cases} \dot{z} = \sum_{k=1}^{N} a_k(t,x) J_k I(z), \\ z(0) = 0; \end{cases}$$

hence, by uniqueness, we conclude that $0*x = \varphi_{0,x}(1) = \mu(1) = x$.

Step V: Once Steps I-to-IV have been established, the existence of a global extension for the map ι can be proved as in the real-analytic case.

(i) First of all, we choose an open and connected neighborhood W_0 of 0 such that W_0 and $\iota(W_0)$ lie W; then, by arguing as in the proof of Prop. 2.1.62 (and recalling that the left-translation are smooth diffeomorphisms of $\mathbb{R}^N$, see Step II), we can write

$$\mathbb{R}^N = \bigcup_{n=1}^{\infty} \Big\{ w_1 * \cdots * w_n : w_1, \ldots, w_n \in W_0 \Big\}.$$

(ii) Let $x \in \mathbb{R}^N$ be arbitrarily fixed and, according with point (i), let $w_1, \ldots, w_n$ in W_0 be such that $x = w_1 * \cdots * w_n$. By crucially exploiting *all the properties* of $*$ established so far (and proceeding as in the proof of Prop. 2.1.40), we see that

$$y := \iota(w_n) * \cdots * \iota(w_1)$$

is the *unique* point of $\mathbb{R}^N$ such that $x * y = y * x = 0$.

(iii) Finally, let $\tilde{\iota} : \mathbb{R}^N \to \mathbb{R}^N$ be the function defined as follows: for every $x \in \mathbb{R}^N$, we set $y = \tilde{\iota}(x)$ to be the unique point of $\mathbb{R}^N$ such that

$$x * y = y * x = 0.$$

By the previous point (ii), $\tilde{\iota}$ is well-defined; moreover, by definition, we have $x * \tilde{\iota}(x) = \tilde{\iota}(x) * x = 0$. Thus, $\tilde{\iota}$ provides a global inverse for $*$ and, by uniqueness, it is an extension of ι, that is,

$$\iota(x) = \tilde{\iota}(x), \quad \text{for every } x \in W_0.$$

As for the regularity of $\tilde{\iota}$ we observe that, by definition, $\tilde{\iota}$ is implicitly defined (in a unique way) by the equation $x * y = 0$; moreover,

$$\mathrm{rk}\big(\mathcal{J}_*(x,y)\big) = \mathrm{rk}\Big(\big(\mathcal{J}_{\rho_y}(x) \mid \mathcal{J}_{\tau_x}(y)\big)\Big) = N, \quad \text{for every } x, y, \in \mathbb{R}^N.$$

Thus, the Implicit Function Theorem ensures that $\tilde{\iota} \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$.

By gathering together Steps I-to-V, we then recognize that $\mathbb{G} = (\mathbb{R}^N, *)$ is a Lie group on $\mathbb{R}^N$, with neutral element 0 and inversion map given by $\tilde{\iota}$; moreover, from identity (2.1.80) in Step II it follows that

$$J_i(x * y) = \mathcal{J}_{\tau_x}(y) \cdot J_i I(y),$$

for every $x, y \in \mathbb{R}^N$ and every $i \in \{1, \ldots, N\}$, whence $J_1, \ldots, J_N \in \mathrm{Lie}(\mathbb{G})$. As a consequence, we deduce that $\mathfrak{g} = \mathrm{span}\{J_1, \ldots, J_N\} \subseteq \mathrm{Lie}(\mathbb{G})$ and thus, both $\mathfrak{g}$ and $\mathrm{Lie}(\mathbb{G})$ being N-dimensional, we conclude that

$$\mathrm{Lie}(\mathbb{G}) = \mathfrak{g}.$$

This proves Thm. 2.1.6 also in the C^∞ case.

Remark 2.1.44. It is worth noting that, also in the C^∞ case, it is possible to prove a refinement of Thm. 2.1.6 analogous to Thm. 2.1.43: indeed, one can prove that *for every fixed* $x_0 \in \mathbb{R}^N$ there exists a *unique* Lie group $\mathbb{G} = (\mathbb{R}^N, \circ)$ with neutral element x_0 and $\mathrm{Lie}(\mathbb{G}) = \mathfrak{g}$. More precisely, we have

$$x \circ y = x * (x_0)^{-1} * y, \quad \text{for every } x, y \in \mathbb{R}^N,$$

where $*$ is the map defined in (2.1.77) and x_0^{-1} is the inverse of x_0 w.r.t. $*$.

2.2 Regularity of vector fields underlying second-oder PDOs

Let us consider, on an open set $\Omega \subseteq \mathbb{R}^N$, a generic linear PDO L of the form

$$L = \sum_{i,j=1}^{N} a_{i,j}(x)\,\partial_{x_i x_j} + \sum_{j=1}^{N} b_j(x)\,\partial_{x_j}, \quad x \in \Omega,$$

where $a_{i,j}, b_j \in C^\infty(\Omega, \mathbb{R})$ for every $i, j \in \{1, \ldots, N\}$. We also assume, without loss of generality, that the matrix $A(x) := (a_{i,j}(x))_{i,j}$ (usually referred to as the *principal matrix of* L) is symmetric and positive semi-definite for every $x \in \Omega$. As anticipated in the introduction of the chapter, the main aim of this section is to provide a simple sufficient condition allowing L to be re-written as a sum of squares of smooth vector fields (plus, possibly, a drift term).

First of all we observe that, if $\{X_0, \ldots, X_m\}$ is a set of smooth vector fields (on Ω) such that $L = \sum_{j=1}^m X_j^2 + X_0$, setting (for every $x \in \Omega$)

$$R(x) := \big(X_1 I(x) \cdots X_m I(x)\big),$$

we then have $A(x) = R(x) \cdot R(x)^T$; on the other hand, if $S(x) = (\sigma_{i,j}(x))_{i,j}$ is a $N \times m$ matrix of smooth functions $\sigma_{i,j} \in C^\infty(\Omega, \mathbb{R})$ such that

$$A(x) = S(x) \cdot S(x)^T, \quad \text{for every } x \in \Omega,$$

it is straightforward to see that $L = \sum_{j=1}^m Z_j^2 + Z_0$, where

$$Z_j = \sum_{i=1}^{N} \sigma_{i,j}(x)\,\partial_{x_i} \quad \text{and} \quad Z_0 = \sum_{k=1}^{N} \left(b_k(x) - \sum_{j=1}^{m} X_j(\sigma_{k,j}) \right) \partial_{x_k}.$$

Therefore, the operator L can be re-written as a sum of squares of smooth vector fields (plus, eventually, a drift) *if and only if* it is possible to find a *smooth matrix-valued* map $\Omega \ni x \mapsto S(x)$ such that

$$A(x) = S(x) \cdot S(x)^T, \quad \text{for every } x \in \Omega.$$

In the following section we shall prove, under a suitable *constant rank condition*, the existence of such a decomposition for $A(x)$. We point out that this result is not new: in fact, it can be found (although without a proof) in the introduction to the celebrated paper by Hörmander [94]. We also address the reader to the book by Oleĭnik and Radkevič [120] for an example of a linear PDO which cannot be re-written as a sums of squares of (smooth) vector fields.

2.2.1 The main result

We first introduce a couple of notations we shall use in the sequel. Given any $N \geq 1$, we denote by $\mathrm{Sym_N}(\mathbb{R})$ the real vector space of all $N \times N$ matrices with real entries; moreover, for every $1 \leq m \leq N$ we set

$$\mathrm{S}_N^+(m) := \{A \in \mathrm{Sym_N}(\mathbb{R}) : A \geq 0 \text{ and } \mathrm{rk}(A) = m\}. \tag{2.2.1}$$

We also denote by $\mathrm{M}_{N\times m}(\mathbb{R})$ the (real) vector space of the $N\times m$ matrices with real coefficients and we define

$$\Omega_N(m) := \{M \in \mathrm{M}_{N\times m}(\mathbb{R}) : \mathrm{rk}(M) = m\}. \tag{2.2.2}$$

We then have the following theorem, which is the main result of this section.

Theorem 2.2.1. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $A : \Omega \to \mathrm{S}_N^+(m)$ be a map of class C^k (with $k \in \mathbb{N}\cup\{\infty,\omega\}$). If Ω is contractible, it is possible to find a map $R : \Omega \to \Omega_N(m)$, with* the same regularity of that of A, *such that*

$$A(x) = R(x)\cdot R(x)^T \quad \textit{for all } x \in \Omega. \tag{2.2.3}$$

More precisely, the map R can defined as follows:

$$R(x) = \sqrt{A(x)}\cdot O(x), \quad \textit{for every } x \in \Omega, \tag{2.2.4}$$

where $O : \Omega \to \mathrm{M}_{N\times m}(\mathbb{R})$ is a map of class C^k on Ω and, for any $x \in \Omega$, the columns of $O(x)$ form an orthonormal basis of $\mathrm{Im}(A(x))$.

The key ingredient for proving Thm. 2.2.1 is notion of C^k-vector bundle on a manifold, which we now recall for the sake of completeness; we refer, e.g., to [93], [105], [109], and [134] for an exhaustive treatment of the argument.

Let $k \in \mathbb{N}\cup\{\infty,\omega\}$ and let M be a C^k-manifold. Moreover, let $m \in \mathbb{N}$ be fixed. A C^k-vector bundle of rank m over M is a triple $\mathcal{F} = (E, M, \pi)$, where E is a C^k-manifold and $\pi : E \to M$ is a map of class C^k such that:

(i) For every $x \in M$, the fiber $V(x) := \pi^{-1}(x) \subseteq E$ is endowed with the structure of a m-dimensional vector space;

(ii) For every $x \in M$, there exist an open neighborhood $U \subseteq M$ of x and a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^m$ of class C^k such that

- $\pi \circ \Phi = \Phi$ on U;
- for every $y \in U$, the restriction of Φ to $V(y)$ is a vector space isomorphism between $V(y)$ and $\mathbb{R}^m$.

If $\mathcal{F} = (E, M, \pi)$ is a C^k-vector bundle of rank m over M, a *(global) section* of $\mathcal{F}$ is map $\sigma : M \to E$ of class C^k such that $\pi\circ\sigma = \mathrm{id}$ on M, that is,

$$\sigma(x) \in V(x) = \pi^{-1}(x), \quad \text{for every } x \in M;$$

a *(global) frame* of $\mathcal{F}$ is an m-tuple $\{\sigma_1 \ldots, \sigma_m\}$ of sections of $\mathcal{F}$ such that, for any $x \in M$, the set $\mathcal{B}(x) := \{\sigma_1(x), \ldots, \sigma_m(x)\}$ is a basis of the fiber $V(x)$.

With the above preliminaries at hand, we can give the proof of Thm. 2.2.1.

Proof (of Thm. 2.2.1). For every $x \in \Omega$, let $V(x) := \mathrm{Im}(A(x))$. Since, by assumption, the dimension of $V(x)$ is constant and equal to m (and since the function $x \mapsto A(x)$ is of class C^k on Ω), the assignment

$$\Omega \ni x \mapsto V(x)$$

defines a C^k-vector bundle of rank m on Ω. More precisely, setting

$$E := \coprod_{x\in\Omega} V(x)$$

and denoting by π the canonical projection of E onto Ω, there exist a unique topology and a unique C^k-structure on E such that the triple $\mathcal{F}_A := (E, \Omega, \pi)$ is C^k-vector bundle of rank m on Ω (see, e.g., [106, Lemma 10.6]).

Now, since Ω is contractible, the bundle $\mathcal{F}_A$ is globally C^k-trivial (see, e.g., [93, Corollary 2.5]); therefore, it is possible to find a (global) frame $\{\sigma_1, \ldots, \sigma_m\}$ for $\mathcal{F}_A$ such that (see [106, Corollary 10.20])

$$\mathcal{B}(x) := \{\sigma_1(x), \ldots, \sigma_m(x)\}$$

is an *orthonormal* basis of $V(x)$ for every $x \in \Omega$. We then consider the function

$$R : \Omega \longrightarrow \mathrm{M}_{N\times m}(\mathbb{R}), \qquad R(x) := \sqrt{A(x)} \cdot O(x),$$

where $O : \Omega \to \Omega_N(m)$ is given by

$$O(x) := \big(\sigma_1(x) \cdots \sigma_m(x)\big), \quad x \in \Omega.$$

Obviously, $R(x)$ has rank equal to m for every $x \in \Omega$, since the same is true of $\sqrt{A(x)}$ and since $O(x)$ has full rank; moreover, since the sections $\sigma_1, \ldots, \sigma_m$ are of class C^k on Ω and since, by Cor. 2.2.7 in Sec. 2.2.2, the same is true of

$$x \mapsto \sqrt{A(x)},$$

the function R is of class C^k on Ω as well. Finally, the matrix $O(x) \cdot O(x)^T$ being the projection matrix onto $V(x)$ (recall that $\sigma_1(x), \ldots, \sigma_m(x)$ form an orthonormal basis of $V(x)$ for every $x \in \Omega$), we obtain

$$\begin{aligned} R(x) \cdot R(x)^T &= \big(\sqrt{A(x)} \cdot O(x)\big) \cdot \big(\sqrt{A(x)} \cdot O(x)\big)^T \\ &= \sqrt{A(x)} \cdot \big(O(x) \cdot O(x)^T \cdot \sqrt{A(x)}\big) = \sqrt{A(x)} \cdot \sqrt{A(x)} \\ &= A(x), \quad \text{for every } x \in \Omega. \end{aligned}$$

This is precisely the desired (2.2.3), and the proof is complete. □

From Thm. 2.2.1 we immediately derive the following result.

Corollary 2.2.2. *Let $A : \mathbb{R}^N \to \mathrm{S}_N^+(m)$ be a map of class C^k (for some $0 \le k \le \infty$). It is possible to find a map $R : \mathbb{R}^N \to \Omega_N(m)$, with* the same regularity of that of A, *such that*

$$A(x) = R(x) \cdot R(x)^T \quad \textit{for all } x \in \mathbb{R}^N. \tag{2.2.5}$$

Cor. 2.2.2 can be profitably used in order to obtain a simple result concerning the problem described at the beginning of the section: the possibility to re-write a general PDO as a sum of squares of vector fields (plus, eventually, a drift).

Theorem 2.2.3. *Let $\Omega \subseteq \mathbb{R}^N$ be a* contractible *open set and let $\mathcal{L}$ be a second-order linear operator on Ω of the general form*

$$\mathcal{L} := \sum_{i,j=1}^{N} a_{i,j}(x)\, \partial_{x_i x_j} + \sum_{j=1}^{N} b_j(x)\, \partial_{x_j} + c(x), \tag{2.2.6}$$

with (real and) smooth coefficients. We assume that, for every $x \in \Omega$, the principal matrix $\big(a_{i,j}(x)\big)_{i,j}$ of $\mathcal{L}$ is symmetric, positive semi-definite and that

$$\mathrm{rk}\big(A(x)\big) = m, \quad \textit{for every } x \in \Omega \qquad (\textit{with } 1 \leq m \leq N).$$

It is then possible to find $m+1$ smooth vector fields $X_0, \ldots, X_m$ on Ω s.t.

(i) $\dim\Big(\mathrm{span}_{\mathbb{R}}\big(\{X_1 I(x), \ldots, X_m I(x)\}\big)\Big) = m$ *for every $x \in \Omega$;*

(ii) $\mathcal{L} = \sum_{j=1}^{m} X_j^2 + X_0 + c$ *on Ω.*

Proof. Thanks to hypothesis on the matrix $(a_{i,j})_{i,j}$, we can define a smooth function $A : \Omega \to \mathrm{S}_N^+(m)$ by setting

$$A(x) := \big(a_{i,j}(x)\big)_{i,j}.$$

Then, by Thm. 2.2.1, there exists a smooth map $R : \Omega \to \Omega_N(m)$ such that

$$A(x) = R(x) \cdot R(x)^T, \quad \text{for every } x \in \Omega. \tag{2.2.7}$$

If $R(x) = (r_{i,j}(x))_{i,j}$, we define $m+1$ v.f.s $X_0, \ldots, X_m$ on Ω as follows:

- $X_j := \sum_{i=1}^{N} r_{i,j}(x) \partial_{x_i}$ for every $j = 1, \ldots, m$;
- $X_0 := \sum_{i=1}^{N} \Big(b_i(x) - \sum_{j=1}^{m} X_j(r_{i,j})(x)\Big)\, \partial_{x_i}$.

Since the map R and the coefficients of $\mathcal{L}$ are smooth on Ω, then $X_0, \ldots, X_m$ are smooth vector fields on $\mathbb{R}^N$; moreover, since $R(x) \in \Omega_N(m)$ for every x in Ω, condition (i) is fulfilled. Finally, from the very definition of X_0 and from identity (2.2.7) it easily follows that

$$\mathcal{L} = \sum_{j=1}^{m} X_j^2 + X_0 + c, \quad \text{on } \Omega.$$

This ends the proof. □

Remark 2.2.4. Let $\Omega \subseteq \mathbb{R}^N$ be a contractible open set and let $\mathcal{L}$ be a second-order linear PDO on Ω of the form (2.2.6) and such that

$$A(x) = \big(a_{i,j}(x)\big)_{i,j} \in \mathrm{S}_N^+(m), \quad \forall\, x \in \Omega \text{ and for some } m \in \{1, \ldots, N\}.$$

Since, by Thm. 2.2.1, it is possible to find a map $R : \Omega \to \Omega_N(m)$, with *the same regularity of* A and such that

$$A(x) = R(x) \cdot R(x)^T, \quad \text{for every } x \in \Omega,$$

we immediately obtain the following generalization of Thm. 2.2.3: *if the operator $\mathcal{L}$ is assumed to have (real) coefficients of class C^k on Ω (for some natural $k \geq 1$), then there exist m vector fields $X_1, \ldots, X_m$ of class C^k on Ω and a vector field X_0 of class C^{k-1} (on Ω) such that*

$$\mathcal{L} = \sum_{j=1}^{m} X_j^2 + X_0 + c, \quad \text{on } \mathbb{R}^N.$$

It is worth mentioning that Thm. 2.2.1 can be proved in a slightly different way, without using the regularity of $\sqrt{A(x)}$ but invoking the Homotopy Lifting Property for principal G-bundles. We are very grateful to one of the referees of the thesis for bringing to our attention the simpler proof presented here.

2.2.2 Appendix: Regularity of the square root

The aim of this brief appendix is to prove that the map

$$\sqrt{\cdot} : \mathrm{S}_N^+(m) \longrightarrow \mathrm{S}_N^+(m), \qquad A \mapsto \sqrt{A}$$

is a real-analytic diffeomorphism of $\mathrm{S}_N^+(m)$. It is well-known that $\mathrm{S}_N^+(m)$ is an *embedded* submanifold of $\mathrm{Sym_N}(\mathbb{R}) \equiv \mathbb{R}^{N(N+1)/2}$, with dimension

$$d_N(m) = Nm - \frac{m(m-1)}{2}; \tag{2.2.8}$$

furthermore, given a diagonal matrix $\Lambda \in \mathrm{S}_N^+(m)$, it is easy to describe the tangent space $T_\Lambda(\mathrm{S}_N^+(m))$ of $\mathrm{S}_N^+(m)$ at Λ: indeed, since the map

$$\Psi : \mathrm{Sym_N}(\mathbb{R}) \to \mathbb{R}^{N-d_N(m)}, \quad \Psi(A) := \Big(\det\big(a_{i,j}\big)_{\substack{i\in\{1,\ldots,m\}\cup\{p\}\\ j\in\{1,\ldots,m\}\cup\{q\}}}\Big)_{\substack{p,q=m+1,\ldots,N\\ p\leq q}}$$

can be used as a local defining function for $\mathrm{S}_N^+(m)$ in a suitable neighborhood of Λ, one has (due to the characterization of $T_\Lambda(\mathrm{S}_N^+(m))$ as $\ker(\mathrm{d}_\Lambda\Psi)$)

$$T_\Lambda(\mathrm{S}_N^+(m)) = \{H \in \mathrm{Sym_N}(\mathbb{R}) : h_{i,j} = 0 \text{ for all } i,j = m+1,\ldots,N\}. \tag{2.2.9}$$

We then have the following theorem.

Theorem 2.2.5. *For every $1 \leq m \leq N$, the map*

$$q : \mathrm{S}_N^+(m) \longrightarrow \mathrm{S}_N^+(m), \qquad q(M) := M^2, \tag{2.2.10}$$

is a real-analytic diffeomorphism from $\mathrm{S}_N^+(m)$ onto itself.

In order to prove Thm. 2.2.5, we first establish the following auxiliary lemma.

Lemma 2.2.6. *Let q be the map defined in (2.2.10) and let*

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m, 0, \ldots, 0) \in \mathrm{S}_N^+(m).$$

Then the differential of q at Λ is non-singular.

Proof. First of all we observe that, since Λ is positive semi-definite and its rank is equal to m, its non-zero eigenvalues $\lambda_1, \ldots, \lambda_m$ are (real and) strictly positive. We now compute the differential of the map q (as a map on $\mathrm{Sym_N}(\mathbb{R})$) at the point Λ. For every $t \in \mathbb{R}$ and for every $H \in \mathrm{Sym_N}(\mathbb{R})$, we have

$$q(\Lambda + tH) = (\Lambda + tH)^2 = (\Lambda + tH)(\Lambda + tH) = \Lambda^2 + t(\Lambda H + H\Lambda) + t^2 H^2,$$

and thus, for all $H \in \mathrm{Sym_N}(\mathbb{R})$, we have

$$\mathrm{d}_\Lambda q(H) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} q(\Lambda + tH) = \Lambda H + H\Lambda.$$

The differential of q at Λ, considered as a map from $T_\Lambda(\mathrm{S}_N^+(m))$ to $T_{\Lambda^2}(\mathrm{S}_N^+(m))$, is just the restriction of $\mathrm{d}_\Lambda q$ to $T_\Lambda(\mathrm{S}_N^+(m)) \subseteq T_\Lambda(\mathrm{Sym_N}(\mathbb{R})) \equiv \mathrm{Sym_N}(\mathbb{R})$.

To show that $\mathrm{d}_\Lambda q$ is an isomorphism between the two spaces $T_\Lambda(\mathrm{S}_N^+(m))$ and $T_{\Lambda^2}(\mathrm{S}_N^+(m))$ we prove that $\ker(d_\Lambda q) \cap T_\Lambda(\mathrm{S}_N^+(m)) = \{0\}$. To this end, let $H \in T_\Lambda(\mathrm{S}_N^+(m))$ be such that $\mathrm{d}_\Lambda q(H) = 0$. For every $j = 1, \ldots, N$ we then have

$$0 = \mathrm{d}_\Lambda q(H)\, e_j = (\Lambda H + H\Lambda)\, e_j = \Lambda \cdot (He_j) + H \cdot (\Lambda e_j).$$

If $1 \leq j \leq m$, from the above identity we then infer that He_j is an eigenvector for Λ where $(-\lambda_j)$ is the correspondent eigenvalue; since Λ is positive semi-definite, this implies that $He_j = 0$ for all $j = 1, \ldots, m$.

If, instead, $m + 1 \leq j \leq N$, then the above computation shows that He_j is an eigenvector for Λ with correspondent eigenvalue 0, whence the set $\{He_{m+1}, \ldots, He_N\}$ is included into the vector space spanned by $e_{m+1}, \ldots, e_N$. By exploiting the fact that H belongs to $T_\Lambda(\mathrm{S}_N^+(m))$ (and by using the expression of $T_\Lambda(\mathrm{S}_N^+(m))$ given in (2.2.9)), we immediately see that $He_j = 0$ for all $j = m + 1, \ldots, N$, and this proves that $H = 0$, as desired. □

We are now ready to give the proof of Thm. 2.2.5.

Proof (of Thm. 2.2.5). We first observe that the map q is real-analytic and bijective on $\mathrm{S}_N^+(m)$. Thanks to the Implicit Function Theorem, it then suffices to show that the differential of q (is non-singular at every point of $\mathrm{S}_N^+(m)$.

To this end, let $A \in \mathrm{S}_N^+(m)$ be arbitrarily fixed and let P be an orthogonal $N \times N$ matrix such that

$$P^T \cdot A \cdot P = \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m, 0, \ldots, 0),$$

for some (real and) positive $\lambda_1, \ldots, \lambda_m$. If we denote by L the linear map

$$L : \mathrm{Sym_N}(\mathbb{R}) \to \mathrm{Sym_N}(\mathbb{R}), \quad L(M) := P^T \cdot M \cdot P,$$

it is straightforward to see that L restricts to a smooth (actually, real-analytic) linear diffeomorphism from $\mathrm{S}_N^+(m)$ onto itself and that

$$q = (L^{-1} \circ q \circ L) \quad \text{on } \mathrm{S}_N^+(m).$$

By differentiating both sides of such an identity at A, we then obtain

$$\mathrm{d}_A q = L^{-1} \circ \mathrm{d}_{L(A)} q \circ L = L^{-1} \circ \mathrm{d}_\Lambda q \circ L, \quad \text{on} \quad T_A(\mathrm{S}_N^+(m)).$$

Since, by Lem. 2.2.6, the linear map $\mathrm{d}_\Lambda q$ is non-singular, we immediately infer that $\mathrm{d}_A q$ is non-singular as well, and the proof is complete. □

From Thm. 2.2.5 we immediately derive the following corollary.

Corollary 2.2.7. *Let $\Omega \subseteq \mathbb{R}^N$ be an open subset of $\mathbb{R}^N$ and let $A : \Omega \to \mathrm{S}_N^+(m)$ be a function of class C^k on Ω (for some $k \in \mathbb{N} \cup \{\infty, \omega\}$). If q is the map defined in* (2.2.10)*, then the function*

$$S : \Omega \to \mathrm{S}_N^+(m), \quad S(x) := \sqrt{A(x)} := q^{-1}\big(A(x)\big), \tag{2.2.11}$$

is of class C^k on Ω. Moreover, if A is locally-Hölder continuous with exponent $\alpha \in]0,1]$ on Ω, then the same is true of S.

Chapter 3

PDOs structured on homogeneous vector fields

Throughout this third chapter of the thesis, we shall be concerned with linear PDOs structured on *homogeneous* vector fields, that is, of the form

$$\mathcal{L} = \sum_{j=1}^{m} X_j^2,$$

where $X_1, \ldots, X_m$ are smooth vector fields on $\mathbb{R}^n$, homogeneous of degree 1 w.r.t. a suitable family of non-isotropic dilations $\{\delta_\lambda\}_{\lambda>0}$ on $\mathbb{R}^n$. Our main aim is to prove, for such operators $\mathcal{L}$ and for their parabolic counterpart $\mathcal{H} = \mathcal{L} - \partial_t$, the existence of a well-behaved global fundamental solution.

Roughly speaking, our argument consists of two steps:

(a) By means of a global Lifting method for homogeneous operators proved by Folland in [73], there exist a homogeneous Carnot group and a polynomial surjective map $\pi : \mathbb{G} \to \mathbb{R}^n$ such that the operator $\mathcal{L}$ is π-related to a sub-Laplacian $\mathcal{L}_{\mathbb{G}}$ on $\mathbb{G}$; we shall prove that it is always possible to perform a (global) change of variable on $\mathbb{G}$ such that the lifting map π becomes the projection of $\mathbb{G} \equiv \mathbb{R}^n \times \mathbb{R}^p$ onto $\mathbb{R}^n$.

(b) If $\Gamma_{\mathbb{G}}(x, \xi; y, \eta)$ $(x, y \in \mathbb{R}^n;\ \xi, \eta \in \mathbb{R}^p)$ is the fundamental solution of $\mathcal{L}_{\mathbb{G}}$, we shall show that $\Gamma_{\mathbb{G}}(x, 0; y, \eta)$ is always integrable with respect to the variables $\eta \in \mathbb{R}^p$, and its integral is a fundamental solution for $\mathcal{L}$.

    Analogously, if $\Gamma_{\mathbb{G}}(z, \xi; \zeta, \eta)$ $(z, \zeta \in \mathbb{R}^{1+n};\ \xi, \eta \in \mathbb{R}^p)$ is the fundamental solution of $\mathcal{H}_{\mathbb{G}} = \mathcal{L}_{\mathbb{G}} - \partial_t$, we shall exploit suitable (uniform) Gaussian e-stimates to prove that $\Gamma_{\mathbb{G}}(z, 0; \zeta, \eta)$ is always integrable w.r.t. the variables $\eta \in \mathbb{R}^p$, and its integral is a fundamental solution for $\mathcal{H} = \mathcal{L} - \partial_t$.

The main ingredient for step (b) is a general saturation argument for obtaining fundamental solutions, which will be presented in Sec. 3.1.

3.1 Saturation of fundamental solutions

As anticipated, the aim of this section is to present a general result concerning the possibility for obtaining (global) fundamental solutions via a saturation argument. Roughly put, it is about a generalization of a very well-know fact: if $\mathcal{H} = \Delta_x - \partial_t$ is the Heat operator on $\mathbb{R}^{N+1} = \mathbb{R}^N_x \times \mathbb{R}_t$ (with $N \geq 3$) and if

$$p_t(x,y) = \frac{1}{(4\pi t)^{N/2}} \exp\left(-\frac{\|x-y\|^2}{4t}\right), \quad x, y \in \mathbb{R}^N, t > 0$$

is the associated Heat kernel, then the "saturated function"

$$(x,y) \mapsto \int_0^\infty p_t(x,y)\,\mathrm{d}t$$

coincides with the fundamental solution for the Laplace operator Δ in $\mathbb{R}^N$.

In order to clearly describe this result, we begin with the following definition.

Definition 3.1.1 (Lifting of a linear PDO)**.** On Euclidean space $\mathbb{R}^N$, we consider a generic linear partial differential operator of order $d \in \mathbb{N}$,

$$P = \sum_{|\alpha| \leq d} a_\alpha(x)\, D_x^\alpha,$$

with smooth real valued coefficients $a_\alpha(x)$ on $\mathbb{R}^N$. We say that a linear PDO $\widetilde{P}$, defined on a higher-dimensional space $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^p$, is a **lifting of** P if the following conditions are fulfilled:

(a) $\widetilde{P}$ has smooth real coefficients, possibly depending on $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^p$;

(b) for every fixed $f \in C^\infty(\mathbb{R}^n)$, one has

$$\widetilde{P}(f \circ \pi)(x,\xi) = (Pf)(x), \quad \text{for every } (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^p, \tag{3.1.1}$$

where $\pi(x,\xi) = x$ is the canonical projection of $\mathbb{R}^n \times \mathbb{R}^p$ onto $\mathbb{R}^n$.

Remark 3.1.2. Let P be a linear PDO as in Def. 3.1.1 and let $\widetilde{P}$ be a linear PDO on a higher-dimensional space $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^p$ with smooth coefficients. It is immediate to recognize that (3.1.1) holds true if and only if

$$\widetilde{P} = P + R \quad \text{with} \qquad R = \sum_{\beta \neq 0} r_{\alpha,\beta}(x,\xi)\, D_x^\alpha D_\xi^\beta, \tag{3.1.2}$$

for (finitely many) real-valued coefficient functions $r_{\alpha,\beta} \in C^\infty(\mathbb{R}^N)$, possibly identically vanishing on $\mathbb{R}^N$. In other words, every summand of R operates, at least once necessarily, in the $\xi_1, \dots, \xi_p$ variables.

Remark 3.1.3. We explicitly observe that, if P is a linear PDO (of arbitrary order) on $\mathbb{R}^n$ with smooth coefficients, then a lifting for P *always exists*, and it is far from being *unique*: in fact, for every $p, k \in \mathbb{N}$, the operator

$$\widetilde{P}_{p,k} := P + \left(\sum_{j=1}^p \partial^2_{\xi_j}\right)^k = P + \left(\Delta_{\mathbb{R}^p}\right)^k, \quad \text{on } \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^p,$$

is a lifting for $\widetilde{P}$ on the Euclidean space $\mathbb{R}^n \times \mathbb{R}^p$, and $R = \widetilde{P} - P = \left(\Delta_{\mathbb{R}^p}\right)^k$.

Let now P be a linear PDO on $\mathbb{R}^n$ with smooth real coefficients, as in Def. 3.1.1. If $\widetilde{P}$ is a lifting of P and if $\widetilde{P}$ admits a fundamental solution $\widetilde{\Gamma}$, it is not at all obvious if the same holds true for P, nor if a fundamental solution for P may be obtained via a saturation argument. Technically, this is the case if the following heuristic argument can be made rigorous: by the very definition of fundamental solution for $\widetilde{P}$ we have

$$-\widetilde{\varphi}(x,\xi) = \int_{\mathbb{R}^n\times\mathbb{R}^p} \widetilde{\Gamma}(x,\xi;y,\eta)\,\widetilde{P}^*\widetilde{\varphi}(y,\eta)\,\mathrm{d}y\mathrm{d}\eta,$$

for every $\widetilde{\varphi} \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^p)$; if we take $\widetilde{\varphi}$ of the form $\varphi(x)\theta_j(\xi)$ (with φ in $C_0^\infty(\mathbb{R}^n)$ and θ_j in $C_0^\infty(\mathbb{R}^p)$) and we recall that $\widetilde{P} = P + R$, then the above equality gives, by choosing $\xi = 0$,

$$\begin{aligned}-\varphi(x)\,\theta_j(0) &= \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^p}\widetilde{\Gamma}(x,0;y,\eta)\,\theta_j(\eta)\,\mathrm{d}\eta\Big)P^*\varphi(y)\,\mathrm{d}y\\ &\qquad + \int_{\mathbb{R}^n\times\mathbb{R}^p}\widetilde{\Gamma}(x,0;y,\eta)\,R^*\big(\varphi(y)\theta_j(\eta)\big)\,\mathrm{d}y\mathrm{d}\eta\\ &=: \mathrm{I}_j + \mathrm{II}_j.\end{aligned} \tag{3.1.3}$$

We want to pass to the limit as $j \to \infty$ in such a way that (3.1.3) produces

$$-\varphi(x) = \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^p}\widetilde{\Gamma}(x,0;y,\eta)\,\mathrm{d}\eta\Big)P^*\varphi(y)\,\mathrm{d}y,$$

so that a fundamental solution for P is available by saturating the η variable in $\widetilde{\Gamma}(x,0;y,\eta)$. Our idea is to choose a sequence $\theta_j \in C_0^\infty(\mathbb{R}^p)$ such that the set

$$\{\eta \in \mathbb{R}^p : \theta_j(\eta) = 1\}$$

invades $\mathbb{R}^p$ as $j \to \infty$, and such that II_j in (3.1.3) goes to 0 as $j \to \infty$. This may be reasonably possible (together with some integrability assumptions on $\widetilde{\Gamma}$) provided some conditions are fulfilled by the remainder operator R:

- if one chooses $\theta_j(\eta) = \theta(\eta/j)$ for some $\theta \in C_0^\infty$ identically equal to 1 on a suitable neighborhood of the origin,
- if the operator R^* acts in the lifting variables, so that $R^*\big(\theta(\eta/j)\big)$ always gives out at least $1/j$,
- and if a dominated convergence argument can apply.

The above argument justifies the following definition of “saturable Lifting”; immediately after the technicalities, we show (see Remark 3.1.5) that a saturable Lifting is always available in meaningful cases.

Definition 3.1.4 (Saturable Lifting). Let P be a smooth linear PDO on $\mathbb{R}^n$, and $\widetilde{P} = P + R$ be a lifting of P on $\mathbb{R}^n \times \mathbb{R}^p$ as in (3.1.2). We say that $\widetilde{P}$ is a *saturable lifting* for P if the following conditions hold:

(S.1) Every summand of the formal adjoint R^* of R operates at least once in the ξ variables, i.e., R^* has the form

$$R^* = \sum_{\beta\neq 0} r^*_{\alpha,\beta}(x,\xi)\,D_x^\alpha D_\xi^\beta, \tag{3.1.4}$$

for (finitely many, possibly vanishing) smooth functions $r^*_{\alpha,\beta}(x,\xi)$.

(S.2) There exists a sequence $(\theta_j)_j$ in $C_0^\infty(\mathbb{R}^p, [0,1])$ such that[1]

$$\{\theta_j = 1\} \uparrow \mathbb{R}^p, \quad \text{as } j \uparrow \infty;$$

moreover, for every compact set $K \subset \mathbb{R}^n$ and for any coefficient function $r^*_{\alpha,\beta}$ of R^* as in (3.1.4), one can find constants $C_{\alpha,\beta}(K)$ such that

$$\Big| r^*_{\alpha,\beta}(x,\xi) \Big(\frac{\partial}{\partial \xi}\Big)^\beta \theta_j(\xi) \Big| \le C_{\alpha,\beta}(K), \tag{3.1.5}$$

for every $x \in K$, $\xi \in \mathbb{R}^p$ and for every $j \in \mathbb{N}$.

We next give some sufficient conditions for a lifting to be saturable. In what follows we always assume that P is a linear PDO on $\mathbb{R}^n$ with smooth coefficients, and that $\widetilde{P} = P + R$ is a lifting of P on $\mathbb{R}^n \times \mathbb{R}^p$, with R as in (3.1.2). The notation (x,ξ) for the points of $\mathbb{R}^n \times \mathbb{R}^p$ is always understood.

Remark 3.1.5. (a) *If the coefficients of R are independent of ξ, then $\widetilde{P}$ is a saturable lifting for P.* In fact, under this assumption, R takes the form

$$R = \sum_{\beta \ne 0} r_{\alpha,\beta}(x)\, D_x^\alpha D_\xi^\beta,$$

and thus its formal adjoint R^* acts on smooth functions ψ as follows:

$$\begin{aligned} R^*\psi &= \sum_{\beta \ne 0} (-1)^{|\alpha|+|\beta|}\, D_x^\alpha \Big(r_{\alpha,\beta}(x)\, D_\xi^\beta \psi(x,\xi) \Big) \\ &=: \sum_{\beta \ne 0} r^*_{\alpha,\beta}(x)\, D_x^\alpha D_\xi^\beta \psi(x,\xi). \end{aligned} \tag{3.1.6}$$

Thus condition (S.1) in Def. 3.1.4 is fulfilled. In order to verify (S.2) as well, we choose a function $\theta \in C_0^\infty(\mathbb{R}^p, [0,1])$ such that $\theta \equiv 1$ on the Euclidean ball centered at 0 and radius 1, and we set

$$\theta_j(\xi) := \theta\big(\xi/j\big), \quad \text{for every } \xi \in \mathbb{R}^p \text{ and every } j \in \mathbb{N}.$$

Clearly, $\{\theta_j = 1\} \uparrow \mathbb{R}^p$ as $j \uparrow \infty$; moreover, if $K \subseteq \mathbb{R}^n$ is compact, we have

$$\big| r^*_{\alpha,\beta}(x)\, D_\xi^\beta \theta_j(\xi) \big| \le (1/j)^{|\beta|} \max_K \big| r^*_{\alpha,\beta} \big| \max_{\mathbb{R}^p} |D^\beta \theta|,$$

and (3.1.5) follows.

(b) If, for every compact set $K \subseteq \mathbb{R}^n$, the coefficient functions of the operator R^* are bounded on $K \times \mathbb{R}^p$, then (S.2) of Definition 3.1.4 is satisfied. It suffices to take $\theta_j(\xi) = \theta(\xi/j)$ as in (a) above.

(c) If $\mathcal{L}$ is a smooth second-order operator on $\mathbb{R}^n$ and if we consider the associated Heat-type operator $\mathcal{H} = \mathcal{L} - \partial_t$ in $\mathbb{R}^n \times \mathbb{R}$, then we have the above

[1] By this we mean that, denoting by Ω_j the set $\{\xi \in \mathbb{R}^p : \theta_j(\xi) = 1\}$, one has

$$\bigcup_{j \in \mathbb{N}} \Omega_j = \mathbb{R}^p \quad \text{and} \quad \Omega_j \subset \Omega_{j+1} \quad \text{for any } j \in \mathbb{N}.$$

with $R = -\partial_t$. Since R has constant coefficients, we are in case (a) above and $\mathcal{H}$ is therefore a saturable Lifting of $\mathcal{L}$.

(d) As we shall prove in Section 3.2, *if $\mathcal{L}$ is a sum of squares of Hörmander vector fields which are δ_λ-homogeneous of degree 1 w.r.t. a suitable family of dilations δ_λ, then there exists a saturable lifting $\widetilde{\mathcal{L}}$ of $\mathcal{L}$*, which is actually a sub-Laplacian on a suitable Carnot group $\mathbb{G}$ on $\mathbb{R}^N$. This fact is non-trivial and it will be proved in Sec. 3.2 (see, precisely, Thm. 3.2.13).

We are now to prove the main result of this section.

Theorem 3.1.6. *Let P be a linear PDO on $\mathbb{R}^n$ with smooth coefficients and let $\widetilde{P}$ be a saturable ifting of P on $\mathbb{R}^n \times \mathbb{R}^p$, according to Def. 3.1.4.*

Let us assume that there exists a fundamental solution $\widetilde{\Gamma}$ for $\widetilde{P}$ on the whole of $\mathbb{R}^n \times \mathbb{R}^p$ (see Def. 1.3.5), further satisfying the following properties:

(i) *for every fixed $x, y \in \mathbb{R}^n$ with $x \neq y$, one has*

$$\eta \mapsto \widetilde{\Gamma}(x, 0; y, \eta) \quad \textit{belongs to} \quad L^1(\mathbb{R}^p); \tag{3.1.7}$$

(ii) *for every fixed $x \in \mathbb{R}^n$ and every compact set $K \subseteq \mathbb{R}^n$, one has*

$$(y, \eta) \mapsto \widetilde{\Gamma}(x, 0; y, \eta) \quad \textit{belongs to} \quad L^1(K \times \mathbb{R}^p); \tag{3.1.8}$$

Then the function $\Gamma : \{(x; y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\} \longrightarrow \mathbb{R}$ defined by

$$\Gamma(x; y) := \int_{\mathbb{R}^p} \widetilde{\Gamma}(x, 0; y, \eta)\, \mathrm{d}\eta, \tag{3.1.9}$$

is a global fundamental solution for P on $\mathbb{R}^n$.

Proof. First of all we observe that, thanks to (3.1.7), Γ is well-defined. In order to prove that Γ is a fundamental solution for P on $\mathbb{R}^n$, we have to prove the following fact: for every fixed $x \in \mathbb{R}^n$, one has $\Gamma(x; \cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and

$$P\Gamma(x; \cdot) = -\mathrm{Dir}_x, \quad \text{in } \mathcal{D}'(\mathbb{R}^n).$$

To this end, we fix a point $x \in \mathbb{R}^n$ and a function $\varphi \in C_0^\infty(\mathbb{R}^n)$. Moreover, the lifting $\widetilde{P}$ being saturable, it is possible to find a sequence of test functions θ_j as in Def. 3.1.4. Since the function $\widetilde{\Gamma}$ is a fundamental solution for $\widetilde{P}$ on $\mathbb{R}^n \times \mathbb{R}^p$, we have (for sufficiently large j's in such a way that $\theta_j(0) = 1$)

$$\int_{\mathbb{R}^n \times \mathbb{R}^p} \widetilde{\Gamma}(x, 0; y, \eta)\, \widetilde{P}^*\big(\varphi(y)\, \theta_j(\eta)\big)\, \mathrm{d}y\, \mathrm{d}\eta = -\varphi(x)\, \theta_j(0) = -\varphi(x);$$

thus, recalling that $\widetilde{P} = P + R$ (where R is a linear PDO operating in y and η),

$$\begin{aligned} -\varphi(x) = &\int_{\mathbb{R}^n \times \mathbb{R}^p} \widetilde{\Gamma}(x, 0; y, \eta)\, \theta_j(\eta)\, P^* \varphi(y)\, \mathrm{d}y \mathrm{d}\eta \\ &+ \int_{\mathbb{R}^n \times \mathbb{R}^p} \widetilde{\Gamma}(x, 0; y, \eta)\, R^*\big(\varphi(y)\, \theta_j(\eta)\big)\, \mathrm{d}y\, \mathrm{d}\eta =: \mathrm{I}_j + \mathrm{II}_j, \end{aligned} \tag{3.1.10}$$

with the obvious notation. Our aim is now to pass to the limit for $j \to \infty$ in (3.1.10). To this end we first notice that, if we denote by K the support of the function φ, then both integrals expressing I_j and II_j are actually performed over $K\times\mathbb{R}^p$. As for I_j, a simple application of the Lebesgue Dominated Convergence Theorem, made possible by (3.1.7), shows that

$$\lim_{j\to\infty}\mathrm{I}_j=\int_{\mathbb{R}^n}\Gamma(x;y)\,P^*\varphi(y)\,\mathrm{d}y. \tag{3.1.11}$$

We next turn to II_j. First we observe that, since the sets $\{\eta:\theta_j(\eta)=1\}$ increasingly invade $\mathbb{R}^p$, and since the operator R^* always differentiate w.r.t. η (see (S.1) in the definition of saturable lifting), we obtain that

$$\lim_{j\to\infty}R^*\big(\varphi(y)\,\theta_j(\eta)\big)=0,\quad \text{pointwise for } (y,\eta)\in K\times\mathbb{R}^p.$$

Moreover, by writing R^* as in (3.1.4), we get

$$\begin{aligned}\big|R^*\big(\varphi(y)\,\theta_j(\eta)\big)\big|&\le\sum_{\beta\neq 0}\big|r^*_{\alpha,\beta}(y,\eta)\big|\cdot\big|D^\alpha_y\varphi(y)\big|\cdot\big|D^\beta_\eta\theta_j(\eta)\big|\\&\le C(\varphi)\sum_{\beta\neq 0}\big|r^*_{\alpha,\beta}(y,\eta)\,D^\beta_\eta\theta_j(\eta)\big|.\end{aligned}$$

From this, by crucially exploiting property (3.1.5) of the sequence θ_j, we infer the existence of a positive constant $C=C(\varphi,K)>0$ such that

$$\big|\widetilde{\Gamma}\big(x,0;y,\eta\big)\,R^*\big(\varphi(y)\,\theta_j(\eta)\big)\big|\le C\,\big|\widetilde{\Gamma}\big(x,0;y,\eta\big)\big|,$$

uniformly for $(y,\eta)\in K\times\mathbb{R}^p$ and $j\in\mathbb{N}$. Therefore, due to property (3.1.8) of $\widetilde{\Gamma}$, we can apply once again a dominated convergence argument to infer that

$$\lim_{j\to\infty}\mathrm{II}_j=0. \tag{3.1.12}$$

Finally, by gathering together (3.1.11) and (3.1.12), we can pass to the limit for $j\to\infty$ in (3.1.10), obtaining

$$-\varphi(x)=\int_{R^n}\Gamma(x;y)\,P^*\varphi(y)\,\mathrm{d}y.$$

This ends the proof. □

Remark 3.1.7. Let P be a linear PDO on $\mathbb{R}^n$ with smooth real coefficients and let $\widetilde{P}$ be a saturable Lifting of P on $\mathbb{R}^n\times\mathbb{R}^p$.

It is worth noting that, if $\widetilde{\Gamma}$ is a fundamental solution for $\widetilde{P}$ on $\mathbb{R}^n\times\mathbb{R}^p$, then we have, for every fixed $x\in\mathbb{R}^n$ (see Def. 1.3.5-(i)),

$$(y,\eta)\mapsto\widetilde{\Gamma}\big(x,0;y,\eta\big)\in L^1_{\mathrm{loc}}(\mathbb{R}^n\times\mathbb{R}^p). \tag{3.1.13}$$

This means that the integrability assumption (3.1.8) in Thm. 3.1.6 is actually an integrability condition at infinity, which is equivalent to the following one:

(ii)' *for every $x\in\mathbb{R}^n$ and every compact set $K\subseteq\mathbb{R}^n$, there exists a compact set $K'\subseteq\mathbb{R}^p$ such that*

$$(y,\eta)\mapsto\widetilde{\Gamma}\big(x,0;y,\eta\big)\quad\textit{belongs to}\quad L^1\big(K\times(\mathbb{R}^p\setminus K')\big).$$

Example 3.1.8. Let P be a linear PDO on $\mathbb{R}^n$, with smooth coefficients and let $\mathcal{H}$ the *heat operator* related to P, that is,

$$\mathcal{H} = P - \partial_\xi, \quad \text{on } \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}.$$

As already pointed out in Rem. 3.1.5 - (c), the operator $\mathcal{H}$ is a *regular lifting* for P on the higher-dimensional space $\mathbb{R}^{n+1}$, so we can apply to this case the result contained in our Thm. 3.1.6: *if the operator $\mathcal{H}$ admits a fundamental solution $\widetilde{\Gamma} = \widetilde{\Gamma}\big((x,\xi);(y,\eta)\big)$ satisfying assumptions* (i) *and* (ii) *in the statement of the cited Thm. 3.1.6, then the function*

$$\Gamma(x;y) := \int_{\mathbb{R}} \widetilde{\Gamma}(x,0;y,\eta)\,\mathrm{d}\eta$$

is a fundamental solution for P on the whole of $\mathbb{R}^n$.

Let P be a linear PDO on $\mathbb{R}^n$ with smooth coefficients and let $\widetilde{P}$ be a saturable lifting for P admitting a (global) fundamental solution $\widetilde{\Gamma}$. It could happen that, together with properties (i) and (ii) in the statement of Thm. 3.1.6, the function $\widetilde{\Gamma}$ satisfies some additional properties: this is the case, e.g., if $\widetilde{P}$ is a sub-Laplacian on some Carnot group. We then conclude this section with a couple of results which give sufficient conditions on $\widetilde{\Gamma}$ in such a way that these additional properties are inherited by the Γ function in (3.1.9).

Proposition 3.1.9 (Continuity and limit at infinity)**.** *Let the notation and the hypotheses of Thm. 3.1.6 apply. Let us assume, in addition, that the fundamental solution $\widetilde{\Gamma}$ of $\widetilde{P}$ satisfies the following bound property:*

(B) *For every fixed $x \in \mathbb{R}^n$, there exist a compact set $K_x \subseteq \mathbb{R}^p$ and a nonnegative function $g_x \in L^1(\mathbb{R}^p \setminus K_x)$ such that*

$$\widetilde{\Gamma}\big(x,0;y,\eta\big) \le g_x(\eta), \quad \text{for every } y \in \mathbb{R}^n \text{ and every } \eta \in \mathbb{R}^p \setminus K_x. \tag{3.1.14}$$

Then the following facts hold true:

(a) *if, for every fixed $x \in \mathbb{R}^n$, the function $(y,\eta) \mapsto \widetilde{\Gamma}\big(x,0;y,\eta\big)$ is continuous away from $(x,0)$, then the function $y \mapsto \Gamma(x;y)$ is continuous on $\mathbb{R}^n \setminus \{x\}$;*

(b) *if, for every fixed $x \in \mathbb{R}^n$, the function $(y,\eta) \mapsto \widetilde{\Gamma}\big(x,0;y,\eta\big)$ vanishes at infinity, then the same is true of $y \mapsto \Gamma(x;y)$.*

Proof. (a) We fix a point $y_0 \in \mathbb{R}^n \setminus \{x\}$ and a real $\rho > 0$ such that the Euclidean ball $B_\rho(y_0)$ centered at y_0 and radius ρ is contained in $\mathbb{R}^n \setminus \{x\}$. Moreover, we choose a sequence $(y_j)_j$ in this ball converging to y_0 as $j \to \infty$. If $K_x \subseteq \mathbb{R}^p$ is as in assumption (B), for every $j \in \mathbb{N}$ we have

$$\Gamma(x;y_j) = \int_{K_x} \widetilde{\Gamma}\big(x,0;y_j,\eta\big)\,\mathrm{d}\eta + \int_{\mathbb{R}^p \setminus K_x} \widetilde{\Gamma}\big(x,0;y_j,\eta\big)\,\mathrm{d}\eta. \tag{3.1.15}$$

We pass to the limit as $j \to \infty$ in the right-hand side of (3.1.15). To this end we first observe that, under condition (a), we obviously have

$$\lim_{j\to\infty} \widetilde{\Gamma}(x,0;y_j,\eta) = \widetilde{\Gamma}(x,0;y_0,\eta), \quad \text{for every } \eta \in \mathbb{R}^p.$$

Moreover, since the set $K := \overline{B_\rho(y_0)} \times K_x$ is compact, there exists a positive real constant $M_x > 0$ such that

$$\widetilde{\Gamma}(x, 0; y_j, \eta) \le M_x, \quad \text{for every } j \in \mathbb{N} \text{ and every } \eta \in K_x.$$

By a dominated convergence argument, we then obtain

$$\lim_{j\to\infty} \int_{K_x} \widetilde{\Gamma}\big(x, 0; y_j, \eta\big)\, \mathrm{d}\eta = \int_{K_x} \widetilde{\Gamma}\big(x, 0; y_0, \eta\big)\, \mathrm{d}\eta. \tag{3.1.16}$$

As for the second integral in the right-hand side of identity (3.1.15), assumption (B) in the statement of the proposition is shaped in such a way that another dominated convergence argument can apply, so that

$$\lim_{j\to\infty} \int_{\mathbb{R}^p\setminus K_x} \widetilde{\Gamma}\big(x, 0; y_j, \eta\big)\, \mathrm{d}\eta = \int_{\mathbb{R}^p\setminus K_x} \widetilde{\Gamma}\big(x, 0; y_0, \eta\big)\, \mathrm{d}\eta. \tag{3.1.17}$$

By gathering together identities (3.1.16) and (3.1.17), we finally conclude that $\Gamma(x; y_j) \to \Gamma(x; y_0)$ as $j \to \infty$, whence the continuity of $\Gamma(x; \cdot)$ at y_0.

(b) Let $K_x \subseteq \mathbb{R}^N$ be as in assumption (B) and let $\{y_j\}_j \subseteq K_x$ be a sequence such that $\|y_j\| \to \infty$ as $j \to \infty$. For every $j \in \mathbb{N}$ we write

$$\Gamma(x; y_j) = \int_{K_x} \widetilde{\Gamma}(x, 0; y_j, \eta)\, \mathrm{d}\eta + \int_{\mathbb{R}^p\setminus K_x} \widetilde{\Gamma}(x, 0; y_j, \eta)\, \mathrm{d}\eta. \tag{3.1.18}$$

Since, by assumption, $\widetilde{\Gamma}(x, 0; \cdot)$ vanishes at infinity, we have

$$\lim_{j\to\infty} \widetilde{\Gamma}(x, 0; y_j, \eta) = 0, \quad \text{for every fixed } \eta \in \mathbb{R}^p;$$

as a consequence, it is possible to find a certain $j_0 \in \mathbb{N}$ such that

$$\widetilde{\Gamma}(x, 0; y_j, \eta) \le 1, \quad \text{for every } j \ge j_0 \text{ and every } \eta \in \mathbb{R}^p.$$

We are then entitled to apply the Lebesgue Dominated Convergence Theorem to the first integral in the right-hand side of (3.1.18), obtaining

$$\lim_{j\to\infty} \int_{K_x} \widetilde{\Gamma}(x, 0; y_j, \eta)\, \mathrm{d}\eta = 0. \tag{3.1.19}$$

As for the second integral, assumption (B) ensures that another dominated convergence argument can be applied, so that

$$\lim_{j\to\infty} \int_{\mathbb{R}^p\setminus K_x} \widetilde{\Gamma}(x, 0; y_j, \eta)\, \mathrm{d}\eta = 0. \tag{3.1.20}$$

By gathering together identities (3.1.19) and (3.1.20), we then get

$$\lim_{j\to\infty} \left(\int_{K_x} \widetilde{\Gamma}(x, 0; y_j, \eta)\, \mathrm{d}\eta + \int_{\mathbb{R}^p\setminus K_x} \widetilde{\Gamma}(x, 0; y_j, \eta)\, \mathrm{d}\eta \right) = 0.$$

This proves that $\Gamma(x; y_j) \to 0$ as $j \to \infty$, whence $\Gamma(x; \cdot)$ vanishes at infinity. $\square$

Proposition 3.1.10 (Pole of Γ). *Let the notation and the hypotheses of Theorem. 3.1.6 apply. Let us assume, in addition, that the fundamental solution $\widetilde{\Gamma}$ of $\widetilde{P}$ enjoys the following properties:*

(a) *$\widetilde{\Gamma}$ is nonnegative;*

(b) *for every $x \in \mathbb{R}^n$, the function $(y,\eta) \mapsto \widetilde{\Gamma}(x,0;y,\eta)$ is lower semi-continuous outside $(x,0)$;*

(c) *for every $x \in \mathbb{R}^n$, the function $\eta \mapsto \widetilde{\Gamma}\big(x,0;x,\eta\big)$ is not integrable on $\mathbb{R}^p$.*

Then the function $y \mapsto \Gamma(x;y)$ defined in (3.1.9) *has a pole at x, i.e.,*

$$\lim_{y\to x} \Gamma(x;y) = \infty.$$

Proof. Let $(y_j)_j$ be a sequence in $\mathbb{R}^n \setminus \{x\}$ converging to x as $j \to \infty$. Since, by our assumptions, the function $(y,\eta) \mapsto \widetilde{\Gamma}(x,0;y,\eta)$ is nonnegative and lower semi-continuous on $\mathbb{R}^n \times \mathbb{R}^p \setminus \{(x,0)\}$, from Fatou's lemma we obtain

$$\liminf_{j\to\infty} \Gamma(x;y_j) \geq \int_{\mathbb{R}^p} \liminf_{j\to\infty} \widetilde{\Gamma}\big(x,0;y_j,\eta\big)\,\mathrm{d}\eta \geq \int_{\mathbb{R}^p} \widetilde{\Gamma}\big(x,0;x,\eta\big)\,\mathrm{d}\eta = \infty,$$

thanks to hypothesis (c). This ends the proof. □

3.2 Lifting of homogeneous vector fields

With the general saturation argument described in Sec. 3.1 at hand, we now enter the "real core" of this chapter: as anticipated, in this section we shall prove that any homogeneous Hörmander operator admits a saturable lifting (in the sense of Def. 3.1.4), which is actually a sub-Laplacian on a Carnot group.

The main ingredient for establishing this fact is a general result on the lifting of homogeneous vector fields due to Folland [73]; for the sake of completeness, we now describe this notable result in all the details.

Compared with the contents of the cited paper [73], the contents of this section are essentially the same, modulo some changes of notation (due to the specific setting we are dealing with); [2] as we shall describe later on, we added a new feature to Folland's argument which will be fundamental for our aim: an ad hoc change of variables turning the map π into the canonical projection.

To begin with, let us fix a family $\{X_1,\ldots,X_m\}$ of linearly independent smooth vector fields on Euclidean space $\mathbb{R}^n$, satisfying the following properties:

(H1) $X_1,\ldots,X_m$ are δ_λ-homogeneous of degree 1 with respect to a family of non-isotropic dilations $\{\delta_\lambda\}_{\lambda>0}$ of the following type:

$$\delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n, \quad \delta_\lambda(x) = (\lambda^{\sigma_1}x_1,\ldots,\lambda^{\sigma_n}x_n),$$

where $\sigma_1 \leq \ldots \leq \sigma_n$ are positive integers and $\sigma_1 = 1$;

[2]The substantial difference between our setting and the one presented in [73] is that Folland considers finite-dimensional vector spaces V equipped with a suitable homogeneous structure, whereas we fix a basis in V and we write everything in coordinates, that is, we have $V \equiv \mathbb{R}^N$ with a family of non-isotropic dilations taking the usual form.

(H2) $X_1, \dots, X_m$ satisfy Hörmander's rank condition at 0, i.e.,

$$\dim \big\{X(0) : X \in \mathrm{Lie}\{X_1, \dots, X_m\}\big\} = n.$$

Remark 3.2.1. We explicitly observe that, by to Thm. 1.2.2 on page 7, the homogeneity assumption (H1) is equivalent to any of the following facts:

- If $X_j = \sum_{k=1}^n a_{k,j}(x)\,\partial_{x_k}$, the coefficient function $a_{k,j}$ is δ_λ-homogeneous of degree $\sigma_k - 1$.
- For every fixed $j \in \{1, \dots, m\}$, the following identity holds true

$$\delta_\lambda(X_j I(x)) = \lambda\, X I\big(\delta_\lambda(x)\big), \quad \text{for every } x \in \mathbb{R}^n \text{ and every } \lambda > 0. \quad (3.2.1)$$

In particular, the coefficient function $a_{k,j}$ is a polynomial and it is independent of $x_k, \dots, x_n$. This last fact ensures that the vector fields $X_1, \dots, X_m$ are complete.

Example 3.2.2. Let us consider the *Grushin vector fields* on $\mathbb{R}^2$

$$X_1 := \frac{\partial}{\partial x_1}, \qquad X_2 := x_1 \frac{\partial}{\partial x_2}.$$

Obviously, X_1, X_2 are linearly independent in the *real vector space* $\mathfrak{X}(\mathbb{R}^2)$ (although not *pointwise* linearly independent); moreover, since $[X_1, X_2] = \partial_{x_2}$, it is straightforward to recognize that

$$\big\{XI(0) : X \in \mathrm{Lie}\{X_1, X_2\}\big\} \supseteq \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} = \mathbb{R}^2,$$

whence X_1, X_2 satisfy assumption (H2) (that is, the Hörmander rank condition at 0). Finally, if we consider the family of dilations $\{\delta_\lambda\}_{\lambda>0}$ on $\mathbb{R}^2$ defined by

$$\delta_\lambda : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \qquad \delta_\lambda(x_1, x_2) = (\lambda x_1, \lambda^2 x_2),$$

we have $q = \sum_{i=1}^2 \sigma_i = 3$ and, for every $x \in \mathbb{R}^N$ and every $\lambda > 0$,

$$\delta_\lambda(X_1 I(x)) = \lambda X_1 I\big(\delta_\lambda(x)\big) \quad \text{and} \quad \delta_\lambda(X_2 I(x)) = \lambda X_2 I\big(\delta_\lambda(x)\big).$$

According to Rem. 3.2.1, this ensures that X_1, X_2 are both δ_λ-homogeneous of degree 1, whence they also fulfill assumption (H1).

We point out that *there cannot exists* any Lie group $\mathbb{G}$ on $\mathbb{R}^N$ with respect to which X_1, X_2 are left-invariant: to see this it suffices to notice that $X_2 I(0) = 0$ but X_2 is not the zero vector field (see Rem. 1.1.2 on page 4).

Our main goal is to prove the following theorem, by using Folland's results in [73] plus an ad hoc change of variable.

Theorem 3.2.3. *Let $N = \dim(\mathrm{Lie}\{X_1, \dots, X_m\})$. There exists a homogeneous Carnot group $\mathbb{G} = \big(\mathbb{R}^N, *, D_\lambda\big)$, with m generators and nilpotent of step r, and there exists a system $\{Z_1, \dots, Z_m\}$ of Lie-generators of* $\mathrm{Lie}(\mathbb{G})$*, such that*

$$Z_i \text{ is a lifting for } X_i \text{ on } \mathbb{R}^N, \quad \text{for every } i = 1, \dots, m.$$

The proof if this theorem is constructive, and it rests on the notable properties of the Campbell-Baker-Hausdorff operation. To begin with, let us denote by $\mathfrak{a}$ the Lie algebra generated by $X_1, \ldots, X_m$:

$$\mathfrak{a} := \mathrm{Lie}\{X_1, \ldots, X_m\}.$$

It follows from the homogeneity assumption (H1) that every commutator of $X_1, \ldots, X_m$ containing more than σ_n terms vanishes identically, hence $\mathfrak{a}$ is nilpotent of step $r \le \sigma_n$. Moreover, the rank condition (H2) ensures that r cannot be smaller than σ_n, so that $\mathfrak{a}$ is nilpotent of step equal to σ_n, which is therefore an integer which we also denote by r.

As a consequence, $\mathfrak{a}$ being finitely generated, its dimension (as a subspace of the linear space of the smooth vector fields on $\mathbb{R}^n$) is finite. We then set

$$N := \dim(\mathfrak{a}) \quad \text{and} \quad p := N - n,$$

and we assume from now on that $N > n$. Now, since $\mathfrak{a}$ is generated by $X_1, \ldots, X_m$ and since it is nilpotent of step r, we have

$$\mathfrak{a} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_r, \quad \text{with} \quad \begin{cases} \mathfrak{a}_1 := \mathrm{span}\{X_1, \ldots, X_m\}, \\ \mathfrak{a}_k := [\mathfrak{a}_1, \mathfrak{a}_{k-1}] \quad \text{for } 2 \le k \le r; \\ [\mathfrak{a}_1, \mathfrak{a}_r] = \{0\}. \end{cases} \tag{3.2.2}$$

In other words, the Lie algebra $\mathfrak{a}$ is stratified. In particular, a vector field $X \in \mathfrak{a}$ belongs to $\mathfrak{a}_k$ (with $1 \le k \le r$) if and only if X is δ_λ-homogeneous of degree k.

By means of the stratification (3.2.2), we can define a family $\{\Delta_\lambda\}_{\lambda>0}$ of dilations on $\mathfrak{a}$ in the following way:

$$\Delta_\lambda(X) = \sum_{k=1}^{r} \lambda^k V_k, \quad \text{where } X = \sum_{k=1}^{r} V_k \text{ and } V_k \in \mathfrak{a}_k \text{ for any } k = 1, \ldots, r. \tag{3.2.3}$$

Moreover, since $\mathfrak{a}$ is nilpotent, the Campbell-Baker-Hausdorff multiplication

$$X \diamond Y = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \cdots, \tag{3.2.4}$$

is actually a finite sum and it defines a group on $\mathfrak{a}$. We now transfer the operation $\diamond$ and the dilation $\{\Delta_\lambda\}_{\lambda>0}$ to a copy of $\mathfrak{a}$ by fixing a suitable coordinate system on (the finite-dimensional vector space) $\mathfrak{a}$.

To this end we first observe that, by means of (3.2.2) and of the rank condition (H2), we can complete $X_1, \ldots, X_m$ to form a basis

$$\mathcal{A} = \{X_1, \ldots, X_m, X_{m+1}, \ldots, X_N\}$$

of $\mathfrak{a}$ satisfying the following properties:

(P1) the set $\{X_1(0), \ldots, X_N(0)\}$ is a set of *generators* for the vector space $\mathbb{R}^n$;

(P2) the basis $\mathcal{A}$ is *adapted* to the stratification, that is,

$$\mathcal{A} = \{X_1^{(1)}, \ldots, X_{m_1}^{(1)}, \ldots, X_1^{(r)}, \ldots, X_{m_r}^{(r)}\},$$

where $m_1 = m$, $X_j^{(1)} = X_j$ for every $j = 1, \ldots, m$ and

$$m_k = \dim(\mathfrak{a}_k) \quad \text{and} \quad \mathfrak{a}_k = \mathrm{span}(\{X_1^{(k)}, \ldots, X_{m_k}^{(k)}\}),$$

for every $k = 2, \ldots, r$.

We then consider the linear isomorphism Φ associated with the basis $\mathcal{A}$, i.e.,

$$\Phi : \mathbb{R}^N \longrightarrow \mathfrak{a}, \qquad \Phi(a) = a \cdot X := \sum_{j=1}^{N} a_j \, X_j.$$

In the sequel we also set, for brevity, $a \cdot X := \sum_{j=1}^{N} a_j \, X_j$. Next we define an operation $*$ and a family of dilations $\{D_\lambda\}_{\lambda>0}$ on $\mathbb{R}^N$ by pushing $\diamond$ and D_λ:

$$a * b := \Phi^{-1}\big(\Phi(a) \diamond \Phi(b)\big), \quad \text{for every } a, b \in \mathbb{R}^N, \tag{3.2.5}$$

$$D_\lambda : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \quad D_\lambda(a) := \Phi^{-1}(\Delta_\lambda(\Phi(a))). \tag{3.2.6}$$

Remark 3.2.4. The following facts hold:

(a) For every $a, b \in \mathbb{R}^N$, the operations $*$ and $\diamond$ are related by the identity

$$(a * b) \cdot X = (a \cdot X) \diamond (b \cdot X). \tag{3.2.7}$$

(b) For every $\lambda > 0$ and every $a \in \mathbb{R}^N$, the dilations D_λ and Δ_λ are related by the notable identity

$$D_\lambda(a) \cdot X = \Delta_\lambda(a \cdot X). \tag{3.2.8}$$

As a consequence of identity (3.2.8), the dilation D_λ can be written as follows

$$D_\lambda(a) = (\lambda^{s_1} a_1, \ldots, \lambda^{s_N} a_N), \quad \text{for every } a \in \mathbb{R}^N,$$

where $1 = s_1 \leq \ldots \leq s_N$ are consecutive integers between 1 and r, and

$$\big(s_1, \ldots, s_N\big) = \big(\underbrace{1, \ldots, 1}_{m}, \underbrace{2, \ldots, 2}_{m_2}, \ldots, \underbrace{r, \ldots, r}_{m_r}\big). \tag{3.2.9}$$

With this notation, $X_1, \ldots, X_N$ are δ_λ-homogeneous of degrees $s_1, \ldots, s_N$ respectively, and one has

$$\Delta_\lambda(X_i) = \lambda^{s_i} \, X_i, \quad \text{for every } i = 1, \ldots, N.$$

As it is reasonable to expect, the following fact holds true (for a proof of this non-trivial result see, e.g., [37, Theorem 17.4.2]).

Theorem 3.2.5. *The triple $\mathbb{A} = (\mathbb{R}^N, *, D_\lambda)$ is a homogeneous Carnot group on $\mathbb{R}^N$, with m generators and nilpotent of step r. Furthermore, the Lie algebra* $\mathrm{Lie}(\mathbb{A})$ *of $\mathbb{A}$ is isomorphic to $\mathfrak{a}$.*

Example 3.2.6. Before proceeding, we illustrate the explicit construction of the group $\mathbb{A}$ in the case of the Grushin v.f.s. X_1, X_2 introduced in Exm. 3.2.2.

To begin with we observe that, since $X_3 := [X_1, X_2] = \partial_{x_2}$ and since X_3 commutes with both X_1 and X_2, we have

$$\mathfrak{a} := \mathrm{Lie}\{X_1, X_2\} = \mathrm{span}_{\mathbb{R}}\{X_1, X_2, X_3\} \quad \text{and} \quad N = \dim(\mathfrak{a}) = 3.$$

Moreover, $\mathfrak{a}$ is nilpotent of step $r = \sigma_2 = 2$ (note that $[X_i, X_3] = 0$ for every $i = 1, 2, 3$) and, according to (3.2.2), we can write

$$\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2, \quad \text{with} \quad \begin{cases} \mathfrak{a}_1 := \mathrm{span}\{X_1, X_2\}, \\ \mathfrak{a}_2 := [\mathfrak{a}_1, \mathfrak{a}_1] = \mathrm{span}\{X_3\}, \\ [\mathfrak{a}_1, \mathfrak{a}_2] = \{0\}. \end{cases}$$

We now consider the set $\mathcal{A} := \{X_1, X_2, X_3\} \subseteq \mathfrak{a}$ and we prove that it is a basis of $\mathfrak{a}$ satisfying properties (P1) and (P2) on page 65.

In fact, obviously, X_1, X_2 and X_3 are linearly independent in the vector space $\mathcal{X}(\mathbb{R}^2)$; moreover, $\mathcal{A}$ is adapted to the stratification $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$, since

$$\mathfrak{a}_1 = \mathrm{span}\{X_1, X_2\} \quad \text{and} \quad \mathfrak{a}_2 = \mathrm{span}\{X_3\}.$$

Finally, since $X_1 I(0) = e_1$, $X_2 I(0) = 0$ and $X_3 I(0) = e_2$ (where e_1 and e_2 denote the element of the canonical basis in $\mathbb{R}^2$), we deduce that

$$\{X_i I(0),\ i = 1, 2, 3\} \text{ is a system of generators of } \mathbb{R}^2.$$

If we thus introduce the linear isomorphism Φ associated with $\mathcal{A}$, that is,

$$\Phi : \mathbb{R}^3 \longrightarrow \mathfrak{a}, \qquad \Phi(a) = (a \cdot X) := \sum_{i=1}^{3} a_i\, X_i,$$

for every $a, b \in \mathbb{R}^3$ and every $\lambda > 0$ we can write (remind the definition of the Campbell-Baker-Hausdorff multiplication $\diamond$ and of the dilation Δ_λ):

$$\begin{aligned}
\Phi(a) \diamond \Phi(b) &= \left(\sum_{i=1}^{3} a_i\, X_i \right) \diamond \left(\sum_{i=1}^{3} b_i\, X_i \right) \\
&\quad \big(\text{by (3.2.4), since } \mathfrak{a} \text{ is nilpotent of step 2}\big) \\
&= \sum_{i=1}^{3} a_i\, X_i + \sum_{i=1}^{3} b_i\, X_i + \frac{1}{2} \left[\sum_{i=1}^{3} a_i\, X_i, \sum_{i=1}^{3} b_i\, X_i \right] \\
&\quad \big(\text{since } [X_1, X_2] = X_3 \text{ and } [X_1, X_3] = [X_2, X_3] = 0\big) \\
&= \sum_{i=1}^{2} (a_i + b_i)\, X_i + \left(a_3 + b_3 + \frac{1}{2}\, (a_1 b_2 - a_2 b_1) \right) X_3; \\
\Delta_\lambda\big(\Phi(a)\big) &= \Delta_\lambda \left(\sum_{i=1}^{3} a_i\, X_i \right) = \Delta_\lambda\big((a_1\, X_1 + a_2\, X_2) + (a_3\, X_3)\big) \\
&\quad \big(\text{by (3.2.3), since } (a_1\, X_1 + a_2\, X_2) \in \mathfrak{a}_1 \text{ and } (a_3\, X_3) \in \mathfrak{a}_2\big) \\
&= \lambda\big(a_1\, X_1 + a_2\, X_2\big) + \lambda^2\, a_3\, X_3.
\end{aligned}$$

Taking into account (3.2.5) and (3.2.6), we then obtain

$$\begin{aligned}
a * b = \Phi^{-1}\big(\Phi(a) \diamond \Phi(b)\big) &= \left(a_1 + b_1, a_2 + b_2, a_3 + b_3 + \frac{1}{2}\, (a_1 b_2 - a_2 b_1) \right); \\
D_\lambda(a) = \Phi^{-1}\big(\Delta_\lambda\big(\Phi(a)\big)\big) &= (\lambda a_1, \lambda a_2, \lambda^2 a_3).
\end{aligned} \tag{3.2.10}$$

It can be directly checked that $\mathbb{A} = (\mathbb{R}^3, *, D_\lambda)$ is a homogeneous Carnot group on $\mathbb{R}^3$, which is actually isomorphic to first Heisenberg group $\mathbb{H}^1$; furthermore, reminding that the Jacobian vector fields J_1, J_2, J_3 in $\mathrm{Lie}(\mathbb{A})$ are associated with the columns of $\mathcal{J}_{\tau_a}(0)$ (see Rem. 1.1.5 on page 5), we get

$$J_1 = \partial_{a_1} - \frac{1}{2}\, a_2\, \partial_{a_3}, \quad J_2 = \partial_{a_2} + \frac{1}{2}\, a_1\, \partial_{a_3}, \quad J_3 = \partial_{a_3}. \tag{3.2.11}$$

Since the structure constants of $\mathrm{Lie}(\mathbb{A})$ and of $\mathfrak{a}$ (with respect to $\{J_1, J_2, J_3\}$ and $\mathcal{A}$, respectively) are the same, we conclude that $\mathrm{Lie}(\mathbb{A}) \cong \mathfrak{a}$.

Following Folland [73], we now consider the crucial map

$$\pi : \mathbb{R}^N \longrightarrow \mathbb{R}^n, \quad \pi(a) := \exp(a \cdot X)(0) = \Psi_1^{a \cdot X}(0), \tag{3.2.12}$$

where, for every fixed vector field $V \in \mathfrak{a}$, we denote by $\Psi_t^V(0)$ the integral curve at time t of the vector field V starting from $0 \in \mathbb{R}^n$ at time 0. We also use the notation $\exp(tV)(0)$ for $\Psi_t^V(0)$. We explicitly observe that π is well-defined, since any vector field in $\mathfrak{a}$ is complete (see Remark 3.2.1).

The selected properties of π are given in the following result.

Theorem 3.2.7 (Folland, [73])**.** *Let π be the map defined in* (3.2.12)*. Then the following properties hold true:*

1. *For every fixed $\lambda > 0$, one has*

$$\pi\big(D_\lambda(a)\big) = \delta_\lambda\big(\pi(a)\big), \quad \textit{for every } a \in \mathbb{R}^N. \tag{3.2.13}$$

2. *π is a surjective polynomial map.*

3. *Let $J_1, \dots, J_N$ be the (unique) vector fields in* $\mathrm{Lie}(\mathbb{A})$ *coinciding at* $0 \in \mathbb{R}^N$ *with the coordinate partial derivatives; then, for every $j = i, \dots, N$,*

$$\mathrm{d}\pi(J_i)(a) = X_i(\pi(a)), \quad \textit{for every } a \in \mathbb{R}^N. \tag{3.2.14}$$

Proof. (1) For every $\lambda > 0$ and every $a \in \mathbb{R}^N$, one has

$$\pi(D_\lambda(a)) \overset{(3.2.12)}{=} \exp(\Phi(D_\lambda(a)))(0) \overset{(3.2.6)}{=} \exp(\Delta_\lambda(a \cdot X))(0),$$

while $\delta_\lambda(\pi(a)) = \delta_\lambda(\exp(a \cdot X)(0))$. We then consider, for every $t \in \mathbb{R}$, the following integral curves (recall that any v.f. in $\mathfrak{a}$ is complete):

$$\gamma(t) := \exp(t\,\Delta_\lambda(a \cdot X))(0) \quad \text{and} \quad \mu(t) := \delta_\lambda(\exp(t\,(a \cdot X))(0)).$$

One has $\gamma(0) = \mu(0) = 0$. Moreover, since X_j is δ_λ-homogeneous of degree s_j,

$$\begin{aligned}
\dot\mu(t) &= \delta_\lambda((a \cdot X)(\Psi_t^{a \cdot X}(0))) = \sum_{j=1}^N a_j \delta_\lambda(X_j(\Psi_t^{a \cdot X}(0))) \\
&\overset{(3.2.1)}{=} \sum_{j=1}^N \lambda^{s_j} a_j X_j(\delta_\lambda(\Psi_t^{a \cdot X}(0))) = \sum_{j=1}^N \lambda^{s_j} a_j X_j(\mu(t)) \\
&= (D_\lambda(a) \cdot X)(\mu(t)) \overset{(3.2.8)}{=} \Delta_\lambda((a \cdot X))(\mu(t)).
\end{aligned}$$

On the other hand, from the very definition of γ we get

$$\dot\gamma(t) = \Delta_\lambda((a \cdot X))(\gamma(t)),$$

and this shows that γ and μ solve the same Cauchy problem, whence they coincide; by taking $t = 1$ we get (3.2.13).

(2) Clearly $\pi \in C^\infty(\mathbb{R}^N, \mathbb{R}^n)$. Moreover, by Taylor's formula, we get

$$\pi(a) = (a \cdot X)(0) + \mathcal{O}(\|a\|^2), \quad \text{as } a \to 0.$$

This shows that the Jacobian matrix of π at $a = 0$ is given by the matrix

$$\mathcal{J}_\pi(0) = (X_1(0) \cdots X_N(0)), \tag{3.2.15}$$

and thus $\text{rank}(\mathcal{J}_\pi(0)) = n$. As a consequence, it is possible to find an open neighborhood W of $0 = \pi(0) \in \mathbb{R}^n$ such that $\pi : \mathbb{R}^N \to W$ is surjective. We claim that the homogeneity property (3.2.13) implies that π is also onto $\mathbb{R}^n$. Indeed, let $x \in \mathbb{R}^n$ be fixed and let $\lambda > 0$ be such that $y = \delta_\lambda(x) \in W$. Since π is onto W, there exists a point $a \in \mathbb{R}^N$ such that $\pi(a) = y$, and thus

$$\pi(D_{1/\lambda}(a)) \stackrel{(3.2.13)}{=} \delta_{1/\lambda}(\pi(a)) = \delta_{1/\lambda}(\delta_\lambda(x)) = x,$$

proving that π is surjective.

(3) Let $i \in \{1, \ldots, N\}$ be fixed and let e_i denote the i-th vector of the canonical basis of $\mathbb{R}^N$. By definition of J_i, for every $a \in \mathbb{R}^N$ we have

$$\begin{aligned} \mathrm{d}_a\pi(J_i(a)) = J_i(\pi)(a) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \pi(a * (t\, e_i)) \\ &\stackrel{(3.2.12)}{=} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\exp((a * (t\, e_i)) \cdot X)(0)) \\ &\stackrel{(3.2.7)}{=} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\exp((a \cdot X) \diamond ((t\, e_i) \cdot X))(0)) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\exp((a \cdot X) \diamond (t\, X_i))(0)). \end{aligned}$$

We now recall that, since $\mathfrak{a}$ is nilpotent, the Campbell-Baker-Hausdorff multiplication satisfies the remarkable formula (see also Thm. 2.1.16 on page 25)

$$\exp(W)(\exp(V)(x)) = \exp(V \diamond W)(x), \text{ for all } x \in \mathbb{R}^N \text{ and every } V, W \in \mathfrak{a};$$

therefore, by inserting this in the above computation, we obtain

$$\begin{aligned} \mathrm{d}_a\pi(J_i(a)) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\exp(t\, X_i)(\exp(a \cdot X)(0))) \\ &= X_i(\exp(a \cdot X)(0)) \stackrel{(3.2.12)}{=} X_i(\pi(a)). \end{aligned}$$

This is precisely the desired (3.2.14), and the proof is complete. □

Example 3.2.8. The aim of this example is to compute the explicit expression of the map π in the particular case of the Grushin vector fields X_1, X_2.

Keeping fixed all the notations introduced in Exm. 3.2.6, we choose a vector $a \in \mathbb{R}^3$ and we consider the following Cauchy problem (on $\mathbb{R}^2$)

$$\begin{cases} \dot\gamma = \sum_{i=1}^{3} a_i\, X_i I(\gamma), \\ \gamma(0) = 0 \end{cases} \iff \begin{cases} \dot\gamma_1 = a_1, \\ \dot\gamma_2 = a_2\, \gamma_1 + a_3, \\ \gamma(0) = 0. \end{cases}$$

Since $\dot\gamma_1 = a_1$ and $\gamma_1(0) = 0$, we obviously have $\gamma_1(t) = a_1 t$; moreover, by inserting this expression in the second equation of the problem, we get

$$\gamma_2(t) = \int_0^t (a_2\,\gamma_1(s) + a_3)\,\mathrm{d}s = a_3 t + \frac{a_1 a_2}{2} t^2, \quad \text{for every } t \in \mathbb{R}.$$

As a consequence, from the very definition of π we obtain

$$\pi(a) = \Psi_1^{a\cdot X}(0) = (\gamma_1(1), \gamma_2(1)) = \Big(a_1, a_3 + \frac{a_1 a_2}{2}\Big). \tag{3.2.16}$$

With this explicit expression of π at hand, we can check the validity of identity (3.2.14). In fact, taking into account the expression of the Jacobian basis $\{J_1, J_2, J_3\}$ of Lie($\mathbb{A}$) obtained in Exm. 3.2.6, for every $a \in \mathbb{R}^3$ we have

$$\mathcal{J}_\pi(a) \cdot J_1 I(a) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{a_2}{2} & \frac{a_1}{2} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -\frac{a_2}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = X_1 I(\pi(a));$$

$$\mathcal{J}_\pi(a) \cdot J_2 I(a) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{a_2}{2} & \frac{a_1}{2} & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ \frac{a_1}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ a_1 \end{pmatrix} = X_2 I(\pi(a));$$

$$\mathcal{J}_\pi(a) \cdot J_1 I(a) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{a_2}{2} & \frac{a_1}{2} & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = X_3 I(\pi(a)).$$

In order to construct a projection acting as a lifting for $X_1, \ldots, X_m$, we add a new feature to Folland's ideas: we find an appropriate change of coordinates of the group $\mathbb{A}$ constructed above which transforms the vector fields $J_1, \ldots, J_m$ on $\mathbb{A}$ into new vector fields $Z_1, \ldots, Z_m$ on $\mathbb{R}^N$ lifting $X_1, \ldots, X_m$ via the projection of $\mathbb{R}^N$ onto $\mathbb{R}^n$ (that is, $Z_1, \ldots, Z_m$ lift $X_1, \ldots, X_m$ in the sense of Def. 3.1.1). This change of variables is not contained in [73] and represents the main novelty of this section; moreover, it will play a crucial rôle in the sequel.

To begin with we observe that, since the vectors $X_1(0), \ldots, X_N(0)$ generate the whole of $\mathbb{R}^n$, we can find n indexes in $\{1, \ldots, r\}$

$$1 = i_1 < i_2 < \cdots < i_n,$$

such that the set $\mathcal{B} := \{X_{i_1}(0), \ldots, X_{i_n}(0)\}$ is a basis of $\mathbb{R}^n$. As a consequence, the vector fields $X_{i_1}, \ldots, X_{i_n}$ must be δ_λ-homogeneous of degree $\sigma_1, \ldots, \sigma_n$, respectively. We then set

$$\{j_1, \ldots, j_p\} := \{1, \ldots, r\} \setminus \{i_1, \ldots, i_n\} \qquad (p = N - n), \tag{3.2.17}$$

and we note that, from Hörmander's rank condition (H2), it follows that

$$j_p \le r - 1,$$

that is, all the vector fields in the basis $\mathcal{A}$ which are δ_λ-homogeneous of maximum degree $r = \sigma_n$ contribute to $\mathcal{B}$.

So far we have assumed that Hörmander's rank condition holds at 0 only; the last remark shows that it automatically holds at any point of $\mathbb{R}^n$.

Remark 3.2.9. With the above notation, we claim that

$$\dim\{X_{i_1}I(x),\dots,X_{i_n}I(x)\}=n,\quad \text{for every } x\in\mathbb{R}^n. \tag{3.2.18}$$

In order to see this, we consider the matrix-valued function $\mathbf{M}$ defined by

$$\mathbf{M}:\mathbb{R}^n\longrightarrow \mathrm{M}_n(\mathbb{R}),\quad \mathbf{M}(x):=\big(X_{i_1}I(x)\cdots X_{i_n}I(x)\big).$$

Since $\{X_{i_1}I(0),\dots,X_{i_n}I(0)\}$ is a basis of $\mathbb{R}^n$, the matrix $\mathbf{M}(0)$ is non-singular; therefore, it is possible to find a small open neighborhood $\mathcal{U}$ of 0 (in $\mathbb{R}^n$) such that $\det(\mathbf{M}(x))\neq 0$ for every $x\in\mathcal{U}$.

We now fix a point $x\in\mathbb{R}^n$ and we choose $\lambda>0$ such that $\delta_\lambda(x)\in\mathcal{U}$. Then, recalling that the vector fields $X_{i_1},\dots,X_{i_n}$ are δ_λ-homogeneous of degrees $\sigma_1,\dots,\sigma_n$ respectively, we have (see Rem. 3.2.1)

$$\begin{aligned}\mathbf{M}\big(\delta_\lambda(x)\big)&=\det\Big(X_{i_1}I\big(\delta_\lambda(x)\big)\cdots X_{i_n}I\big(\delta_\lambda(x)\big)\Big)\\&\overset{(3.2.1)}{=}\det\Big(\lambda^{-\sigma_1}\,\delta_\lambda\big(X_{i_1}I(x)\big)\cdots\lambda^{-\sigma_n}\,\delta_\lambda\big(X_{i_n}I(x)\big)\Big)\\&=\lambda^{-\sigma_1}\cdots\lambda^{-\sigma_n}\,\det\Big(\delta_\lambda\big(X_{i_1}I(x)\big)\cdots\delta_\lambda\big(X_{i_n}I(x)\big)\Big),\end{aligned}$$

and thus, since the point $\delta_\lambda(x)$ belongs to $\mathcal{U}$, we obtain

$$\det\Big(\delta_\lambda\big(X_{i_1}I(x)\big)\cdots\delta_\lambda\big(X_{i_n}I(x)\big)\Big)\neq 0.$$

This ensures that the vectors $\delta_\lambda\big(X_{i_1}I(x)\big),\dots,\delta_\lambda\big(X_{i_n}I(x)\big)$ form a basis of $\mathbb{R}^n$, whence the same is true of $X_{i_1}I(x),\dots,X_{i_n}I(x)$, since the map δ_λ is a (linear) isomorphism of $\mathbb{R}^n$. As a consequence, we see that $X_1,\dots,X_m$ satisfy Hörmander's rank condition not only at the origin 0 (see assumption (H2)), but at every point of $x\in\mathbb{R}^n$, that is,

$$\dim\big\{XI(x):X\in\mathrm{Lie}\{X_1,\dots,X_m\}\big\}=n,\quad \text{for every } x\in\mathbb{R}^n.$$

We are now ready to introduce our change of coordinates: we set, with reference to the above (3.2.17),

$$T:\mathbb{R}^N\longrightarrow\mathbb{R}^N,\quad T(a):=\big(\pi(a),a_{j_1},\dots,a_{j_p}\big). \tag{3.2.19}$$

We also define a new family $\{d_\lambda\}_{\lambda>0}$ of dilations on $\mathbb{R}^N$ by setting

$$d_\lambda(a):=(\lambda^{\sigma_1}a_1,\dots,\lambda^{\sigma_n}a_n,\lambda^{s_{j_1}}a_{n+1},\dots,\lambda^{s_{j_p}}a_N). \tag{3.2.20}$$

We then have the following crucial result.

Lemma 3.2.10. *The map T in* (3.2.19) *satisfies the following properties:*

(i) *For every fixed $\lambda>0$, one has*

$$T\big(D_\lambda(a)\big)=d_\lambda\big(T(a)\big),\quad \textit{for every } a\in\mathbb{R}^N; \tag{3.2.21}$$

(ii) *The map T is a C^∞-diffeomorphism of $\mathbb{R}^N$ onto itself.*

Proof. (i): For every $\lambda > 0$ and every $a \in \mathbb{R}^N$ we have

$$\begin{aligned} T\big(D_\lambda(a)\big) &\overset{(3.2.19)}{=} \Big(\pi\big(D_\lambda(a)\big), \big(D_\lambda(a)\big)_{j_1}, \dots, \big(D_\lambda(a)\big)_{j_p}\Big) \\ &= \Big(\delta_\lambda\big(\pi(a)\big), \lambda^{s_{j_1}} a_{j_1} \dots, \lambda^{s_{j_p}} a_{j_p}\Big) \\ &\overset{(3.2.20)}{=} d_\lambda\big(\pi(a), a_{j_1}, \dots, a_{j_p}\big) \\ &= d_\lambda\big(T(a)\big), \end{aligned}$$

which is precisely the desired identity (3.2.21).

(ii): Obviously $T \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$. Moreover,

$$\mathcal{J}_T(0) = \begin{pmatrix} \mathcal{J}_\pi(0) \\ e_{j_1} \\ \vdots \\ e_{j_p} \end{pmatrix} \overset{(3.2.15)}{=} \begin{pmatrix} X_1(0) \cdots X_N(0) \\ e_{j_1} \\ \vdots \\ e_{j_p} \end{pmatrix},$$

where $e_{j_1}, \dots, e_{j_p}$ denote some of the vectors (written as row $1 \times N$ vectors) of the canonical basis of $\mathbb{R}^N$. From this, by recalling that $X_{i_1}(0), \dots, X_{i_n}(0)$ form a basis of $\mathbb{R}^n$ and by (3.2.17), we derive that $\mathcal{J}_T(0)$ is invertible, so that there exist neighborhoods $\mathcal{U}, \mathcal{W}$ of 0 in $\mathbb{R}^N$ such that

$$T|_{\mathcal{U}} : \mathcal{U} \longrightarrow \mathcal{W}, \quad \text{is a } C^\infty\text{-diffeomorphism.}$$

We now claim that the homogeneity property (i) implies that the map T is actually a C^∞-diffeomorphism of $\mathbb{R}^N$ onto itself. To prove this claim, we first show that T is a bijection.

T is 1-1: Suppose that $a, b \in \mathbb{R}^N$ are such that $T(a) = T(b)$, and let $\lambda > 0$ be so small that $D_\lambda(a), D_\lambda(b) \in \mathcal{U}$. This gives

$$T(D_\lambda(a)) \overset{(3.2.21)}{=} d_\lambda(T(a)) = d_\lambda(T(b)) \overset{(3.2.21)}{=} T(D_\lambda(b)),$$

and thus, since $D_\lambda(a), D_\lambda(b) \in \mathcal{U}$ and $T|_{\mathcal{U}}$ is injective, we get $D_\lambda(a) = D_\lambda(b)$, hence $a = b$. This proves that T is injective.

T is onto: Let $u \in \mathbb{R}^N$ be fixed and let $\lambda > 0$ be such that $v = d_\lambda(u) \in \mathcal{W}$. Since $T|_{\mathcal{U}}$ is onto $\mathcal{W}$, it is possible to find a (unique) point $a \in \mathcal{U}$ such that $T(a) = d_\lambda(u)$, and thus

$$T(D_{1/\lambda}(a)) \overset{(3.2.21)}{=} d_{1/\lambda}(d_\lambda(u)) = u.$$

This proves that T is surjective.

In order to end the proof, we are left to show that the map T^{-1} (which is globally defined) is smooth. To this end we first notice that, from the homogeneity property (i) of T, we get

$$T^{-1}(d_\lambda(u)) = D_\lambda(T^{-1}(u)), \quad \text{for every } u \in \mathbb{R}^N. \tag{3.2.22}$$

Let now $u_0 \in \mathbb{R}^N$ and $\lambda > 0$ be such that $d_\lambda(u_0) \in \mathcal{W}$. The map d_λ being continuous, it is possible to find a positive $\rho > 0$ such that $d_\lambda(B(u_0, \rho)) \subseteq \mathcal{W}$; thus, for every $u \in B(u_0, \rho)$ we have

$$\begin{aligned} T^{-1}(u) &= T^{-1}(d_{1/\lambda}(d_\lambda(u))) \overset{(3.2.22)}{=} D_{1/\lambda}(T^{-1}(d_\lambda(u))) \\ &= (D_{1/\lambda} \circ (T^{-1})|_{\mathcal{W}} \circ d_\lambda)(u). \end{aligned}$$

This shows that T^{-1} coincides with the smooth function $D_{1/\lambda} \circ (T^{-1})\big|_{\mathcal{W}} \circ d_\lambda$ on the open ball $B(u_0, \rho)$, hence T^{-1} is smooth near u_0. The arbitrariness of the point u_0 completes the proof. □

Thanks to Lem. 3.2.10, we are entitled to use the change of variable T in order to define a new homogeneous Carnot group $\mathbb{G} = (\mathbb{R}^N, \star, D^\star_\lambda)$ starting from $\mathbb{A} = (\mathbb{R}^N, *, D_\lambda)$. We henceforth denote a generic point of $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^p$ by (x, ξ), with $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^p$, and we define

$$(x, \xi) \star (y, \eta) := T(T^{-1}(x, \xi) * T^{-1}(y, \eta)), \quad \forall\ (x, \xi), (y, \eta) \in \mathbb{R}^N; \tag{3.2.23}$$

$$D^\star_\lambda : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \quad D^\star_\lambda(x, \xi) := T(D_\lambda(T^{-1}(x, \xi))). \tag{3.2.24}$$

It is obvious that $\mathbb{G} = (\mathbb{R}^N, \star, D^\star_\lambda)$ is a homogeneous Carnot group on $\mathbb{R}^N$, with m generators and nilpotent of step r. Furthermore, the map T is a (smooth) isomorphism between $\mathbb{A}$ and $\mathbb{G}$, that is,

$$T(a) \star T(b) = T(a * b), \quad \text{for every } a, b \in \mathbb{R}^N.$$

We also have, for every $\lambda > 0$,

$$D^\star_\lambda(x, \xi) = T(D_\lambda(T^{-1}(x, \xi))) \overset{(3.2.22)}{=} T(T^{-1}(d_\lambda(x, \xi)) = d_\lambda(x, \xi).$$

There is therefore no reason to use the notation $D^\star_\lambda$ any longer, and we replace it by d_λ. In the new coordinates (x, ξ) it is useful to write

$$d_\lambda(x, \xi) = (\delta_\lambda(x), \delta^*_\lambda(\xi)),$$

where, for every $\lambda > 0$ and every $\xi \in \mathbb{R}^p$, we have

$$\delta^*_\lambda(\xi) = (\lambda^{\sigma^*_1}\xi_1, \ldots, \lambda^{\sigma^*_p}\xi_p), \quad \text{where } \sigma^*_i := s_{j_i} \text{ for any } i = 1, \ldots, p. \tag{3.2.25}$$

Now, since T is an isomorphism of Lie groups, it induces the Lie algebra isomorphism $\mathrm{d}T$ (see, e.g., [37, Section 2.1])

$$\mathrm{d}T : \mathrm{Lie}(\mathbb{A}) \longrightarrow \mathrm{Lie}(\mathbb{G}), \quad \mathrm{d}T(X)_{(x,\xi)} := \mathrm{d}T(X)_{T^{-1}(x,\xi)}. \tag{3.2.26}$$

We can then consider, in particular, the vector fields

$$Z_i := \mathrm{d}T(J_i), \quad \text{for every } i = 1, \ldots, N. \tag{3.2.27}$$

The map $\mathrm{d}T$ being an isomorphism of Lie algebras, we immediately infer that

- the set $\{Z_1, \ldots, Z_N\}$ is a basis of $\mathrm{Lie}(\mathbb{G})$;
- $\mathrm{Lie}(\mathbb{G}) = \mathrm{Lie}\{Z_1, \ldots, Z_m\}$.

We can finally prove the following result.

Theorem 3.2.11 (Lifting property). *Let* $Z_1, \dots, Z_N$ *be as in* (3.2.27). *Then*

(i) $Z_1, \dots, Z_N$ *are* d_λ*-homogeneous of degree* $s_1, \dots, s_N$ *respectively (see identity* (3.2.9) *in Rem. 3.2.4 for the definition of* $s_1, \dots, s_N$*);*

(ii) Z_i *is a lifting of* X_i*, that is,*

$$Z_i = X_i + R_i \quad (i = 1, \dots, N), \tag{3.2.28}$$

where R_i *is a vector field on* $\mathbb{R}^N$ *only operating in the* ξ *variables (with coefficients possibly depending on* (x, ξ)*). As a consequence, the sub-Laplacian* $\mathcal{L}_{\mathbb{G}} := \sum_{k=1}^m Z_k^2$ *on* $\mathbb{G}$ *is a lifting of the operator* $\mathcal{L} = \sum_{k=1}^m X_k^2$.

Thm. 3.2.11 proves Thm. 3.2.3 stated at the beginning of the section.

Proof. (i) We fix $i \in \{1, \dots, N\}$ and $\lambda > 0$. We recall that

$$J_i \text{ is } D_\lambda\text{-homogeneous of degree } s_i. \tag{3.2.29}$$

For every $(x, \xi) \in \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^p$, we have the following computation

$$\begin{aligned}
Z_i(d_\lambda(x,\xi)) &\overset{(3.2.27)}{=} \mathrm{d}T(J_i)(d_\lambda(x,\xi)) \overset{(3.2.26)}{=} J_i(T)(T^{-1}(d_\lambda(x,\xi))) \\
&\overset{(3.2.24)}{=} J_i(T)(D_\lambda(T^{-1}(x,\xi))) \overset{(3.2.29)}{=} \lambda^{-s_i}\, J_i(T \circ D_\lambda)(T^{-1}(x,\xi)) \\
&\overset{(3.2.21)}{=} \lambda^{-s_i}\, J_i(d_\lambda \circ T)(T^{-1}(x,\xi)) \\
&\overset{(3.2.26)}{=} \lambda^{-s_i}\, \mathrm{d}T(J_i)(d_\lambda)(x,\xi) = \lambda^{-s_i}\, d_\lambda(\mathrm{d}T(J_i)(x,\xi)) \\
&= d_\lambda(Z_i(x,\xi)),
\end{aligned}$$

and this proves that Z_i is d_λ-homogeneous of degree s_i, as claimed.

(ii) We fix $i \in \{1, \dots, N\}$ and $(x, \xi) \in \mathbb{R}^N$; we have

$$\begin{aligned}
Z_i I(x,\xi) &= \mathrm{d}T(J_i)(x,\xi) = J_i(T)(T^{-1}(x,\xi)) \\
&\overset{(3.2.19)}{=} \Big(J_i(\pi), J_i(a \mapsto a_{j_1}), \dots, J_i(a \mapsto a_{j_p})\Big)(T^{-1}(x,\xi)).
\end{aligned} \tag{3.2.30}$$

On the other hand, by (3.2.14) we infer

$$J_i(\pi)(T^{-1}(x,\xi)) = X_i(\pi(T^{-1}(x,\xi))). \tag{3.2.31}$$

Now, since $T(x, \xi) = (\pi(x), \xi)$, we derive that

$$\pi(T^{-1}(x,\xi)) = x; \tag{3.2.32}$$

therefore, by inserting (3.2.31) and (3.2.32) in (3.2.30), we infer

$$Z_i(x,\xi) = \Big(X_i(x,\xi), f_{i,1}(x,\xi), \dots, f_{i,p}(x,\xi)\Big),$$

where, for $k = 1, \dots, p$ we have used the notation

$$f_{i,k}(x,\xi) = J_i(a \mapsto a_{j_k})(T^{-1}(x,\xi)) = (J_i(T^{-1}(x,\xi)))_{j_k}.$$

This shows that Z_i can be written (as a vector field on $\mathbb{R}^N$) in the form

$$Z_i = X_i + R_i, \quad \text{with} \quad R_i = \sum_{k=1}^{p} f_{i,k}(x,\xi)\,\frac{\partial}{\partial \xi_k},$$

hence Z_i is a lifting for X_i and this ends the proof. □

Example 3.2.12. In this last example of the section we consider once again the Grushin vector fields X_1, X_2 introduced in Exm. 3.2.2 and we compute the explicit expression of the map T, of the group law $\star$ and of the dilation d_λ.

To begin with, keeping fixed all the notations introduced in the previous examples, we consider the basis $\mathcal{A} = \{X_1, X_2, X_3\}$ of $\mathfrak{a} = \mathrm{Lie}\{X_1, X_2\}$ and we construct the two sets of indexes defined in (3.2.17).

Since $X_1 I(0) = e_1$ and $X_3 I(0) = e_2$ (where e_1, e_2 denote the elements of the canonical basis of $\mathbb{R}^2$), we have $\{i_1, i_2\} = \{1, 3\}$ and

$$\{j_1\} = \{1,2,3\} \setminus \{i_1, i_2\} = \{2\};$$

therefore, according to the definition of T given in (3.2.19), we have

$$T(a) = (\pi(a), a_{j_1}) \overset{(3.2.16)}{=} \Big(a_1, a_3 + \frac{a_1 a_2}{2}, a_2\Big), \quad a \in \mathbb{R}^3. \tag{3.2.33}$$

With this expression of T at hand, we now write down the explicit expression of the composition $\star$ and of d_λ. In fact, a direct computations shows that

$$T^{-1}(x,\xi) = \left(x_1, \xi, x_2 - \frac{x_1\xi}{2}\right), \quad \text{for every } (x_1, x_2, \xi) \in \mathbb{R}^3;$$

as a consequence, by exploiting the expression of $*$ and of D_λ obtained in Exm. 3.2.6, for every $(x,\xi), (y,\eta) \in \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ and every $\lambda > 0$ we obtain

$$\begin{aligned}
(x,\xi) \star (y,\eta) &= T\big(T^{-1}(x,\xi) * T^{-1}(y,\eta)\big) \\
&= T\left(\left(x_1, \xi, x_2 - \frac{x_1\xi}{2}\right) * \left(y_1, \eta, y_2 - \frac{y_1\eta}{2}\right)\right) \\
&\overset{(3.2.10)}{=} T\left(x_1 + y_1, \xi + \eta, x_2 + y_2 - \frac{1}{2}(x_1\xi + y_1\eta) + \frac{1}{2}(x_1\eta - \xi y_1)\right) \\
&= (x_1 + y_1, x_2 + y_2 + x_1\eta, \xi + \eta); \\
d_\lambda(x,\xi) &= T\big(D_\lambda(T^{-1}(x,\xi))\big) = T\left(D_\lambda\left(x_1, \xi, x_2 - \frac{x_1\xi}{2}\right)\right) \\
&\overset{(3.2.10)}{=} T\left(\lambda x_1, \lambda\xi, \lambda^2\left(x_2 - \frac{x_1\xi}{2}\right)\right) \\
&= (\lambda x_1, \lambda^2\xi, \lambda x_2).
\end{aligned}$$

Finally, due to their relevance in our argument, we determine the explicit expression of the vector fields Z_1, Z_2, Z_3 introduced in (3.2.27). By exploiting the expression of J_1, J_2, J_3 given in (3.2.11), for every $(x,\xi) \in \mathbb{R}^3$ we have

$$Z_1 I(x,\xi) = \mathrm{d}T(J_1)I(x,\xi) = \mathcal{J}_T\big(T^{-1}(x,\xi)\big) \cdot J_1 I\big(T^{-1}(x,\xi)\big)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ \frac{\xi}{2} & \frac{x_1}{2} & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -\frac{\xi}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix};$$

$$Z_2 I(x,\xi) = \mathrm{d}T(J_2)I(x,\xi) = \mathcal{J}_T\big(T^{-1}(x,\xi)\big) \cdot J_2 I\big(T^{-1}(x,\xi)\big)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ \frac{\xi}{2} & \frac{x_1}{2} & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ \frac{x_1}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ x_1 \\ 1 \end{pmatrix};$$

$$Z_3 I(x,\xi) = \mathrm{d}T(J_3)I(x,\xi) = \mathcal{J}_T\big(T^{-1}(x,\xi)\big) \cdot J_3 I\big(T^{-1}(x,\xi)\big)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ \frac{\xi}{2} & \frac{x_1}{2} & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix};$$

as a consequence, we can write

$$Z_1 = \partial_{x_1}, \quad Z_2 = x_1\,\partial_{x_2} + \partial_\xi, \quad Z_3 = \partial_{x_2}.$$

In particular, we have $Z_1 = X_1, Z_3 = X_3$ and $Z_2 = X_2 + \partial_\xi$.

The next result motivates all the algebraic machinery developed so far.

Theorem 3.2.13. *The sub-Laplacian $\mathcal{L}_\mathbb{G} = \sum_{k=1}^m Z_k^2$ on the homogeneous Carnot group $\mathbb{G} = (\mathbb{R}^n \times \mathbb{R}^p, \star, d_\lambda)$ constructed in this section is a saturable lifting of the operator $\mathcal{L} = \sum_{k=1}^m X_k^2$, in the sense of Def. 3.1.4.*

Proof. With reference to the notation in Definition 3.1.4, we need to prove properties (S.1) and (S.2).

(S.1) Since $Z_1, \ldots, Z_m$ are d_λ-homogeneous of degree 1, the operator $\mathcal{L}_\mathbb{G}$ is (formally) self-adjoint on $L^2(\mathbb{R}^N)$. The same is true of $\mathcal{L}$, this time invoking the δ_λ-homogeneity of degree 1 of $X_1, \ldots, X_m$. Thus the formal adjoint R^* of

$$R = \mathcal{L}_\mathbb{G} - \mathcal{L}$$

coincides with R, whence ($\mathcal{L}_\mathbb{G}$ being a lifting for $\mathcal{L}$) it has the form (3.1.4).

(S.2) With reference to the dilations δ_λ^* introduced in (3.2.25), we consider the δ_λ^*-homogeneous map

$$N : \mathbb{R}^p \longrightarrow \mathbb{R}, \quad N(\xi) := \sum_{k=1}^p |\xi_k|^{1/\sigma_k^*}. \tag{3.2.34}$$

We now choose a smooth function $\theta \in C_0^\infty(\mathbb{R}^p, [0,1])$ such that

- $\mathrm{supp}(\theta) \subseteq \{\xi \in \mathbb{R}^p : N(\xi) < 2\}$;
- $\theta \equiv 1$ on $\{\xi \in \mathbb{R}^p : N(\xi) < 1\}$.

We then define a sequence θ_j in $C_0^\infty(\mathbb{R}^p)$ by setting, for every $j \in \mathbb{N}$,

$$\theta_j(\xi) := \theta(\delta^*_{2^{-j}}(\xi)), \quad (\text{with } \xi \in \mathbb{R}^p). \tag{3.2.35}$$

Obviously, any θ_j is valued in $[0,1]$; furthermore, since the function N is δ^*_λ-homogeneous of degree 1, we have

- $\mathrm{supp}(\theta_j) \subseteq \{\xi \in \mathbb{R}^p : N(\xi) < 2^{j+1}\}$;
- $\theta \equiv 1$ on $\{\xi \in \mathbb{R}^p : N(\xi) < 2^j\}$.

Consequently $\{\theta_j = 1\} \uparrow \mathbb{R}^p$ as $j \uparrow \infty$. In order to complete the verification of (S.2), let us fix a compact set $K \subseteq \mathbb{R}^n$ and let $r_{\alpha,\beta}$ be the coordinate coefficient function of the second-order PDO

$$R^* = R = \mathcal{L}_{\mathbb{G}} - \mathcal{L} = \sum_{k=1}^m (Z_k^2 - X_k^2) = \sum_{\alpha,\beta} r_{\alpha,\beta}(x,\xi)\, D_x^\alpha D_\xi^\beta.$$

The functions $r_{\alpha,\beta}$ are polynomials; a simple but tedious computation shows that any monomial decomposing $r_{\alpha,\beta}(x,\xi)$, has the following feature: as a function of ξ only it is δ^*_λ-homogeneous of degree not exceeding $|\beta|_* - 1$, where we have used the notation (see also (3.2.25))

$$|\beta|_* := \sum_{k=1}^p \beta_k\, \sigma_k^*, \quad \text{for every multi-index } \beta \in (\mathbb{N} \cup \{0\})^p.$$

With this notation, note that, for any $\xi \in \mathbb{R}^p$ and any multi-index β,

$$\big(\delta^*_\lambda(\xi)\big)^\beta = \lambda^{|\beta|_*}\, \xi^\beta. \tag{3.2.36}$$

We can write $r_{\alpha,\beta}$ in the following way

$$r_{\alpha,\beta}(x,\xi) = \sum_{|\gamma|_* \le |\beta|_* - 1} c_{\alpha,\beta,\gamma}(x)\, \xi^\gamma, \tag{3.2.37}$$

where $c_{\alpha,\beta,\gamma}(x)$ are polynomial functions only depending on x.

Now, for every multi-index γ with $|\gamma|_* \le |\beta|_* - 1$, every $(x,\xi) \in K \times \mathbb{R}^p$ and every $j \in \mathbb{N}$, we have the following estimate (we use the notation $\mathbf{1}_B$ for the characteristic function of a set B):

$$\Big| c_{\alpha,\beta,\gamma}(x)\, \xi^\gamma\, D_\xi^\beta \theta_j(\xi) \Big| \le \max_{x \in K} |c_{\alpha,\beta,\gamma}(x)| \cdot |\xi^\gamma| \cdot |D_\xi^\beta \theta_j(\xi)|$$

(recall that θ_j is constant outside the set $B_j := \{2^j \le N(\xi) \le 2^{j+1}\}$)

$$= \max_K |c_{\alpha,\beta,\gamma}| \cdot |\xi^\gamma| \cdot |D_\xi^\beta \theta_j(\xi)| \cdot \mathbf{1}_{B_j}(\xi)$$

$$\overset{(3.2.35)}{\le} \max_K |c_{\alpha,\beta,\gamma}| \cdot \sup_{\mathbb{R}^p} |D^\beta \theta| \cdot (2^{-j})^{|\beta|_*} \cdot |\xi^\gamma| \cdot \mathbf{1}_{B_j}(\xi)$$

(we denote by $\mathbf{c}_{\alpha,\beta,\gamma}$ a constant bounding the product of the first two factors, we write $\xi = \delta^*_{2^j} \circ \delta^*_{2^{-j}}(\xi)$ and we use (3.2.36))

$$\le \mathbf{c}_{\alpha,\beta,\gamma} \cdot (2^{-j})^{|\beta|_* - |\gamma|_*} \cdot \Big| (\delta^*_{2^{-j}}(\xi))^\gamma \Big| \cdot \mathbf{1}_{B_j}(\xi). \tag{3.2.38}$$

Observe that, if the point ξ belongs to the annulus $B_j = \{2^j \le N(\xi) \le 2^{j+1}\}$, then the point $\delta^*_{2^{-j}}(\xi)$ belongs to the compact set

$$B_1 = \{\xi \in \mathbb{R}^p : 1 \le N(\xi) \le 2\};$$

as a consequence, it is possible to find a positive constant $M_\gamma > 0$, only depending on γ but independent of $j \in \mathbb{N}$, such that

$$\left|\delta^*_{2^{-j}}(\xi)^\gamma\right| \cdot \mathbf{1}_{B_j}(\xi) \le M_\gamma, \quad \text{for every } \xi \in \mathbb{R}^p. \tag{3.2.39}$$

Since $|\beta|_* - |\gamma|_* \ge 1$, from (3.2.38) and (3.2.39), we then obtain

$$|c_{\alpha,\beta,\gamma}(x)\,\xi^\gamma\, D^\beta_\xi \theta_j(\xi)| \le \mathbf{c}_{\alpha,\beta,\gamma}\, M_\gamma, \quad \forall\, x \in K,\ \xi \in \mathbb{R}^p \text{ and } j \in \mathbb{N}. \tag{3.2.40}$$

We are now ready to conclude: by taking into account (3.2.37), for every for every $x \in K$, $\xi \in \mathbb{R}^p$ and $j \in \mathbb{N}$ we have

$$\begin{aligned} |r_{\alpha,\beta}(x,\xi) \cdot D^\beta_\xi \theta_j(\xi)| &\le \sum_{|\gamma|_* \le |\beta|_* - 1} |c_{\alpha,\beta,\gamma}(x)\,\xi^\gamma\, D^\beta_\xi \theta_j(\xi)| \\ &\overset{(3.2.40)}{\le} \sum_{|\gamma|_* \le |\beta|_* - 1} \mathbf{c}_{\alpha,\beta,\gamma}\, M_\gamma, \end{aligned}$$

and this completes the verification of property (S.2) of a saturable lifting. □

We conclude this section by turning our attention the homogeneous second-order linear PDO associated with $X_1, \ldots, X_m$, that is,

$$\mathcal{L} = \sum_{j=1}^m X_j^2.$$

By exploiting the δ_λ-homogeneity of $X_1, \ldots, X_m$ and Thm. 1.2.2 - (ii), it is easy to recognize that $\mathcal{L}$ can be written in the following divergence form

$$\mathcal{L} = \sum_{i=1}^n \partial_{x_i} \left(\sum_{j=1}^n a_{i,j}(x) \partial_{x_j} \right),$$

where the matrix $A(x) = \big(a_{i,j}(x)\big)_{i,j}$ is given by the product

$$A(x) = S(x) \cdot S(x)^T,$$

and $S(x)$ is the $n \times m$ matrix whose columns are the coefficient vectors of $X_1, \ldots, X_m$, that is,

$$S(x) = \big(X_1 I(x) \cdots X_m I(x)\big), \quad \text{for every } x \in \mathbb{R}^n.$$

Furthermore, $\mathcal{L}$ is *δ_λ-homogeneous of degree* 2: for every $\lambda > 0$, we have

$$\mathcal{L}\big(u \circ \delta_\lambda\big) = \lambda^2 \left(\mathcal{L}u\right) \circ \delta_\lambda, \quad \text{for every } u \in C^\infty(\mathbb{R}^n, \mathbb{R}).$$

Finally, since $X_1, \ldots, X_m$ satisfy Hörmander's rank condition at every point of $\mathbb{R}^n$ (see Rem. 3.2.9), we derive the following notable facts:

(a) The operator $\mathcal{L}$ is C^∞-hypoelliptic on every open subset of $\mathbb{R}^n$;

(b) The operator $\mathcal{L}$ satisfies the so-called Strong Maximum Principle on every *open and connected* subset Ω of $\mathbb{R}^n$: *any function* $u \in C^2(\Omega, \mathbb{R})$ *satisfying* $\mathcal{L}u \geq 0$ *on* Ω *and attaining a maximum in* Ω *is constant throughout* Ω;

(c) As a consequence of the Strong Maximum Principle, the operator $\mathcal{L}$ also satisfies the Weak Maximum Principle on every open subset U of $\mathbb{R}^n$:

$$\begin{cases} u \in C^2(U, \mathbb{R}); & \\ \mathcal{L}u \geq 0, & \text{on } U; \\ \limsup\limits_{x \to y} u(x) \leq 0 & \text{for every } y \in \partial U; \end{cases} \quad \Longrightarrow \quad u \leq 0 \quad \text{on } U.$$

Remark 3.2.14. As regards the lifting in the sub-elliptic contexts of Carnot groups, we briefly highlight the paper by Bonfiglioli and Uguzzoni [38].

Roughly put, in this paper the authors prove that any Carnot group $\mathbb{G}$ (nilpotent of step r and with m generators) can be lifted to the free Carnot group $\mathbb{F}_{m,r}$. Compared with the lifting by Folland presented in this section, the result by Bonfiglioli and Uguzzoni is essentially a lifting for vector fields generating the Lie algebra of a Carnot group; this, however, is not our case.

3.3 Fundamental solution for homogeneous Hörmander operators

Thanks to all the results proved in the previous sections, we are finally in a position to prove the main result of this chapter, namely the existence of a (global) fundamental solution for any homogeneous Hörmander operator.

For the reader's convenience (and to improve the readability of this section), we summarize in Thm. 3.3.1 below all the results we are going to prove.

Theorem 3.3.1. *Let* $X_1, \ldots, X_m$ *be a family of linearly independent smooth vector fields satisfying assumptions* (H1) *and* (H2) *in Sec. 3.2.*

Then the operator $\mathcal{L} := \sum_{j=1}^m X_j^2$ *admits a unique global fundamental solution* Γ *which satisfies the following (joint)* δ_λ*-homogeneity property:*

$$\Gamma(\delta_\lambda(x); \delta_\lambda(y)) = \lambda^{2-q}\,\Gamma(x; y), \quad \forall\ x, y \in \mathbb{R}^n \text{ with } x \neq y \text{ and } \lambda > 0.$$

Moreover, Γ *is continuous out of the diagonal of* $\mathbb{R}^n \times \mathbb{R}^n$*, and it is symmetric:*

$$\Gamma(x; y) = \Gamma(y; x) \qquad \text{for every } x, y \in \mathbb{R}^n \text{ with } x \neq y.$$

Finally, for every fixed $x \in \mathbb{R}^n$*, we have the following properties:*

(i) $\Gamma(x; \cdot) = \Gamma(\cdot; x)$ *is smooth and* $\mathcal{L}$*-harmonic on* $\mathbb{R}^n \setminus \{x\}$;

(ii) $\Gamma(x; \cdot) = \Gamma(\cdot; x)$ *vanishes at infinity (uniformly for* x *in compact sets);*

(iii) $\Gamma(x; \cdot) = \Gamma(\cdot; x)$ *is locally integrable on* $\mathbb{R}^n$*;*

(vi) Γ *is locally integrable on* $\mathbb{R}^n \times \mathbb{R}^n$ *and* C^∞ *out of the diagonal of* $\mathbb{R}^n \times \mathbb{R}^n$.

To give a complete proof of Thm. 3.3.1, we begin by fixing some notations.

Throughout this section, $\mathcal{L} = \sum_{j=1}^m X_j^2$ is a sum of squares of (linearly independent) vector fields satisfying assumptions (H1) and (H2) in the *incipit* of Section 3.2. Without further comments, we denote by $\mathbb{G} = (\mathbb{R}^N, \star, d_\lambda)$ the homogeneous Carnot group on $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^p$ constructed in the previous section, with the sub-Laplacian $\mathcal{L}_\mathbb{G} = \sum_{j=1}^m Z_j^2$ which lifts $\mathcal{L}$ through the projection of $\mathbb{R}^n \times \mathbb{R}^p$ onto $\mathbb{R}^n$. As usual, we denote a generic point of $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^p$ by (x, ξ), where $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^p$. We know that d_λ takes the form

$$d_\lambda(x,\xi) = (\delta_\lambda(x), \delta_\lambda^*(\xi)), \tag{3.3.1}$$

where δ_λ^* is the dilation on $\mathbb{R}^p$ introduced in (3.2.25). Three homogenous dimensions naturally arise:

- that of $(\mathbb{R}^n, \delta_\lambda)$, namely $q := \sum_{j=1}^n \sigma_j$;
- that of $(\mathbb{R}^p, \delta_\lambda^*)$, namely $q^* := \sum_{j=1}^n \sigma_j^*$;
- that of $(\mathbb{R}^N, d_\lambda)$, namely $Q = q + q^*$.

Let us now assume that the δ_λ-dimension of $\mathbb{R}^n$ is greater than 2:

$$q = \textstyle\sum_{j=1}^n \sigma_j > 2. \tag{3.3.2}$$

In the sequel, we consider the homogeneous norm on $\mathbb{G}$ (in the sense of Def. 1.3.8 on page 17) defined by

$$h(x,\xi) := \|(x,\xi)\|_\mathbb{G} = \sum_{j=1}^n |x_j|^{1/\sigma_j} + \sum_{k=1}^p |\xi_k|^{1/\sigma_k^*}. \tag{3.3.3}$$

Since $\mathcal{L}_\mathbb{G}$ is a sub-Laplacian on the Carnot group $\mathbb{G}$, we know from Thm. 1.3.9 that there exists a homogeneous norm $d \in C^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{R})$ such that

$$\Gamma_\mathbb{G}(x,\xi;y,\eta) := d^{2-Q}\big((x,\xi)^{-1} * (y,\eta)\big), \qquad \text{(with } (x,\xi) \neq (y,\eta)\text{)} \tag{3.3.4}$$

is the unique fundamental solution for $\mathcal{L}_\mathbb{G}$ satisfying the additional property

$$\lim_{\|(y,\eta)\| \to \infty} \Gamma_\mathbb{G}(x,\xi;y,\eta) = 0, \quad \text{for every fixed } (x,\xi) \in \mathbb{R}^N.$$

Moreover, by the equivalence of *all* the homogeneous norms on $\mathbb{G}$ (see identity (1.3.10) on page 17), there exists a (group) constant $\mathbf{c} > 0$ such that

$$\mathbf{c}^{-1}\, h^{2-Q}\big((x,\xi)^{-1} * (y,\eta)\big) \le \Gamma_\mathbb{G}(x,\xi;y,\eta) \le \mathbf{c}\, h^{2-Q}\big((x,\xi)^{-1} * (y,\eta)\big), \tag{3.3.5}$$

for every $(x,\xi), (y,\eta) \in \mathbb{R}^N$ with $(x,\xi) \neq (y,\eta)$. By means of this equivalence, we want to show that $\Gamma_\mathbb{G}$ satisfies the integrability assumptions Thm. 3.1.6 (plus the other good properties in Prop. 3.1.9). As a consequence, since we proved in Thm. 3.2.13 that $\mathcal{L}_\mathbb{G}$ is a saturable lifting of $\mathcal{L}$, then $\mathcal{L}$ admits a fundamental solution obtained by a saturation of $\Gamma_\mathbb{G}$.

Due to its central role in the saturation formula (3.1.9), we briefly study some properties of the "convolution-like" map

$$F : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^N, \quad F(x,y,\eta) := (x,0)^{-1} \star (y,\eta). \tag{3.3.6}$$

First of all we observe that, since the family $\{d_\lambda\}_{\lambda>0}$ forms a one-parameter group of automorphisms of $\mathbb{G}$, for every $x \in \mathbb{R}^n$ and every $(y,\eta) \in \mathbb{R}^N$ one has

$$\begin{aligned} F(\delta_\lambda(x), d_\lambda(y,\eta)) &= (\delta_\lambda(x),0)^{-1} \star d_\lambda(y,\eta) \\ &\overset{(3.3.1)}{=} (d_\lambda(x,0))^{-1} \star d_\lambda(y,\eta) = d_\lambda((x,0)^{-1} \star (y,\eta)) = d_\lambda(F(x,(y,\eta))); \end{aligned}$$

hence, if we consider the family of dilations $\{\widetilde{D}_\lambda\}_{\lambda>0}$ on $\mathbb{R}^n \times \mathbb{R}^N$ given by

$$\widetilde{D}_\lambda : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^n \times \mathbb{R}^N, \quad \widetilde{D}_\lambda(x,y,\eta) = (\delta_\lambda(x), d_\lambda(y,\eta)),$$

then the components of F, say

$$F_1, \dots, F_n, \quad F_{n+1}, \dots, F_N,$$

are $\widetilde{D}_\lambda$-homogeneous of degrees, respectively,

$$\sigma_1, \dots, \sigma_n, \quad \sigma_1^*, \dots, \sigma_p^*.$$

On the other hand, if we take $x = 0$, we get $F(0,(y,\eta)) = (y,\eta)$, whilst $F(x,(x,0)) = (0,0)$ (since the origin is the neutral element of $\mathbb{G}$). By all these facts, we deduce that the components of F are $\widetilde{D}_\lambda$-homogeneous polynomials, and that, for every $x \in \mathbb{R}^n$ and every $(y,\eta) \in \mathbb{R}^N$, they take the form

$$\begin{aligned} F_1(x,y,\eta)) &= y_1 - x_1, \\ F_i(x,y,\eta) &= y_i - x_i + p_i(x,y,\eta) \qquad (i = 2,\dots,n), \\ F_{n+k}(x,y,\eta) &= \eta_k + q_k(x,y,\eta), \qquad (k = 1,\dots,p), \end{aligned} \tag{3.3.7}$$

where, for every $i = 2,\dots,n$ and every $k = 1,\dots,p$, p_i and q_k are $\widetilde{D}_\lambda$-homogeneous polynomials of degrees σ_i and σ_k^*, respectively, and

- p_i only depends on those variables x_h, y_h and η_j such that $\sigma_h, \sigma_j^* < \sigma_i$;
- q_k only depends on those variables x_h, y_h and η_j such that $\sigma_h, \sigma_j^* < \sigma_k^*$;
- $p_i(0,y,\eta) = q_k(0,y,\eta) = 0$, for every $(y,\eta) \in \mathbb{R}^N$.

Remark 3.3.2. Let $x, y \in \mathbb{R}^n$ be fixed. Since the polynomial q_1 does not depend on $\eta_1,\dots,\eta_p$ and since, for every $k \in \{2,\dots,p\}$, the polynomial q_k only depends on $\eta_1,\dots,\eta_{k-1}$, we see that the map

$$\Psi_{x,y} : \mathbb{R}^p \longrightarrow \mathbb{R}^p, \quad \Psi_{x,y}(\eta) := \Big(F_{n+1}(x,y,\eta),\dots,F_N(x,y,\eta)\Big), \tag{3.3.8}$$

defines a C^∞-diffeomorphism of $\mathbb{R}^p$, with polynomial components. Hence, in particular, $\Psi_{x,y}$ is a proper map, which is equivalent to saying that

$$\lim_{\|\eta\|\to\infty} \Psi_{x,y}(\eta) = \infty. \tag{3.3.9}$$

Furthermore, by (3.3.7), we get

$$\det(\mathcal{J}_{\Psi_{x,y}}(\eta)) = 1, \quad \text{for every } \eta \in \mathbb{R}^p. \tag{3.3.10}$$

Summing up, from the estimate (3.3.5) we obtain (whenever $(y,\eta)\neq(x,0)$)

$$\mathbf{c}^{-1}\,K^{2-Q}(x,y,\eta)\leq \Gamma_{\mathbb{G}}(x,0;y,\eta)\leq \mathbf{c}\,K^{2-Q}(x,y,\eta), \tag{3.3.11}$$

where we have set

$$K(x,y,\eta):=h((x,0)^{-1}\star(y,\eta)),\quad \text{with } h \text{ as in (3.3.3)}. \tag{3.3.12}$$

Taking into account (3.3.3) and (3.3.7), a more explicit expression for K is

$$\begin{aligned} K(x,y,\eta) &= \sum_{i=1}^{n}\Big|F_i(x,y,\eta)\Big|^{1/\sigma_i}+\sum_{k=1}^{p}\Big|F_{n+k}(x,y,\eta)\Big|^{1/\sigma_k^*}\\ &= |y_1-x_1|+\sum_{i=2}^{n}\Big|y_i-x_i+p_i(x,y,\eta)\Big|^{1/\sigma_i}\\ &\quad +\sum_{k=1}^{p}\Big|\eta_k+q_k(x,y,\eta)\Big|^{1/\sigma_k^*}. \end{aligned} \tag{3.3.13}$$

Thanks to (3.3.11), we are now able to prove the following crucial result:

Theorem 3.3.3. *Suppose that* (3.3.2) *holds true. Then the fundamental solution* $\Gamma_{\mathbb{G}}$ *of* $\mathcal{L}_{\mathbb{G}}$ *satisfies assumptions* (i) *and* (ii) *in Thm. 3.1.6.*

Proof. First we prove condition (i). We need to show (3.1.7) when $\widetilde{\Gamma}$ is $\Gamma_{\mathbb{G}}$; due to (3.3.11), we need to prove that, for fixed $x\neq y$ in $\mathbb{R}^n$, we have

$$\eta\mapsto K^{2-Q}(x,y,\eta)\quad \text{belongs to}\quad L^1(\mathbb{R}^p). \tag{3.3.14}$$

We perform the change of variable $\eta=\Psi_{x,y}^{-1}(u)$ introduced in Rem. 3.3.2:

$$\begin{aligned} \int_{\mathbb{R}^p}K^{2-Q}(x,y,\eta)\,\mathrm{d}\eta &= \int_{\mathbb{R}^p}K^{2-Q}(x,y,\Psi_{x,y}^{-1}(u))\cdot|\det(\mathcal{J}_{\Psi_{x,y}^{-1}}(u))|\,\mathrm{d}u\\ &\overset{(3.3.10)}{=}\int_{\mathbb{R}^p}K^{2-Q}(x,y,\Psi_{x,y}^{-1}(u))\,\mathrm{d}u. \end{aligned}$$

We now observe that, since $x\neq y$, the function $u\mapsto K^{2-Q}(x,y,\Psi_{x,y}^{-1}(u))$ is continuous on $\mathbb{R}^p$, hence it is integrable on every compact subset of $\mathbb{R}^p$. In fact, $K(x,y,\Psi_{x,y}^{-1}(u))=0$ if and only if

$$(x,0)^{-1}\star(y,\Psi_{x,y}^{-1}(u))=0,$$

which necessarily implies $x=y$. Thus, if we consider the homogeneous norm N in (3.2.34), (3.3.14) will follow if we show that

$$\int_{\{N(u)\geq 1\}}K^{2-Q}(x,y,\Psi_{x,y}^{-1}(u))\,\mathrm{d}u<\infty.$$

By the expression of K given in (3.3.13) and the definition of $\Psi_{x,y}$, we infer

$$\begin{aligned} K(x,y,\Psi_{x,y}^{-1}(u)) &\overset{(3.3.13)}{\geq}\sum_{k=1}^{p}|F_{n+k}(x,y,\Psi_{x,y}^{-1}(u))|^{1/\sigma_k^*}\\ &\overset{(3.3.8)}{=}\sum_{k=1}^{p}|\Psi_{x,y}(\Psi_{x,y}^{-1}(u))|^{1/\sigma_k^*}=\sum_{k=1}^{p}|u_k|^{1/\sigma_k^*}=N(u). \end{aligned}$$

Therefore, we are left to show that

$$\int_{\{N(u)\geq 1\}} N^{2-Q}(u)\,\mathrm{d}u < \infty. \tag{3.3.15}$$

In proving (3.3.15), we use a typical argument on diadic annuli (modeled on the homogeneous norm N): setting, for $j \in \mathbb{N}$, $C_j := \{u \in \mathbb{R}^p : 2^{j-1} \leq N(u) < 2^j\}$, then (see (3.2.25) for the definition of δ^*_λ)

$$\begin{aligned}
\int_{\{N(u)\geq 1\}} N^{2-Q}(u)\,\mathrm{d}u &= \sum_{j=1}^{\infty} \int_{C_j} N^{2-Q}(u)\,\mathrm{d}u \quad (\text{change of variable } u = \delta^*_{2^j}(\eta)) \\
&= \sum_{j=1}^{\infty} (2^j)^{q^*} \int_{\delta^*_{2^j}(C_j)} N^{2-Q}(\delta^*_{2^j}(\eta))\,\mathrm{d}\eta \\
&= \left(\int_{\{1/2\leq N(\eta)\leq 1\}} N^{2-Q}(\eta)\,\mathrm{d}\eta \right) \sum_{j=1}^{\infty} (2^j)^{2-Q+q^*} < \infty,
\end{aligned}$$

since $2 - Q + q^* = 2 - q > 0$ by (3.3.2). This ends the proof of (i).

Finally we prove (ii) of Thm. 3.1.6. We need to prove (3.1.8) when $\widetilde{\Gamma}$ is $\Gamma_{\mathbb{G}}$. If $x \in \mathbb{R}^n$ is fixed and $K \subset \mathbb{R}^n$ is compact, we perform the change of variable $(u, v) = (y, \Psi_{x,y}(\eta))$ and we get (arguing as in part (i) to recognize that this substitution has Jacobian determinant identically 1)

$$\begin{aligned}
\int_{K\times\mathbb{R}^p} \Gamma_{\mathbb{G}}(x, 0; y, \eta)\,\mathrm{d}y\,\mathrm{d}\eta &= \int_{K\times\mathbb{R}^p} \Gamma_{\mathbb{G}}(x, 0; u, \Psi^{-1}_{x,u}(v))\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_{K\times\{N(v)\leq 1\}} \{\cdots\}\,\mathrm{d}u\,\mathrm{d}v + \int_{K\times\{N(v)>1\}} \{\cdots\}\,\mathrm{d}u\,\mathrm{d}v =: \mathrm{I} + \mathrm{II},
\end{aligned}$$

where N is as above. Clearly I is finite since we integrate a continuous function over a compact set. As for II, we use (3.3.11) and we have to prove the finiteness of the following integral:

$$\begin{aligned}
&\int_{K\times\{N(v)>1\}} K^{2-Q}(x, u, \Psi^{-1}_{x,u}(v))\,\mathrm{d}u\,\mathrm{d}v \\
&\overset{(3.3.13)}{\leq} \int_{K\times\{N(v)>1\}} \left(\sum_{k=1}^{p} |F_{n+k}(x, u, \Psi^{-1}_{x,u}(v))|^{1/\sigma^*_k} \right)^{2-Q} \mathrm{d}u\,\mathrm{d}v \\
&\overset{(3.3.8)}{=} \int_{K\times\{N(v)>1\}} \left(\sum_{k=1}^{p} |v_k|^{1/\sigma^*_k} \right)^{2-Q} \mathrm{d}u\,\mathrm{d}v = \mathbf{c} \int_{K\times\{N(v)>1\}} N^{2-Q}(v)\,\mathrm{d}u\,\mathrm{d}v.
\end{aligned}$$

The finiteness of the last integral follows by the same argument as in the previous part of the proof (and the fact that K is compact). This ends the proof. □

By gathering together Thm. 3.2.13 proved in Sec. 3.2 and Thm. 3.3.3, we can finally prove our existence result of a fundamental solution for $\mathcal{L}$.

Theorem 3.3.4 (Existence of a fundamental solution for $\mathcal{L}$). *Suppose that* (3.3.2) *holds true. Then the function*

$$\Gamma(x; y) = \int_{\mathbb{R}^p} \Gamma_{\mathbb{G}}\big(x, 0; y, \eta\big)\,\mathrm{d}\eta \quad (x \neq y)$$

is a fundamental solution for $\mathcal{L}$*. Moreover, if* h *is as in* (3.3.3)*, one has*

$$\mathbf{c}^{-1}\int_{\mathbb{R}^p} h^{2-Q}((x,0)^{-1}\star(y,\eta))\,\mathrm{d}\eta \leq \Gamma(x;y) \leq \mathbf{c}\int_{\mathbb{R}^p} h^{2-Q}((x,0)^{-1}\star(y,\eta))\,\mathrm{d}\eta,$$

holding true for every $x,y\in\mathbb{R}^n$ *with* $x\neq y$*. Here,* $\mathbf{c}>0$ *is a constant only depending on the homogeneous Carnot group* $\mathbb{G}$*,* Q *is the homogeneous dimension of* $\mathbb{G}$ *and* $\star$ *is the (polynomial) composition law of* $\mathbb{G}$*.*

Proof. By Thm. 3.2.13, we know that the sub-Laplacian $\mathcal{L}_{\mathbb{G}}$ is a saturable lifting for $\mathcal{L}$; moreover, Thm. 3.3.3 shows that the fundamental solution $\Gamma_{\mathbb{G}}$ of $\mathcal{L}_{\mathbb{G}}$ in (3.3.5) satisfies assumptions (i) and (ii) in Thm. 3.1.6.

Therefore, by applying the cited Thm. 3.1.6, we conclude that Γ is a (global) fundamental solution for $\mathcal{L}$. This ends the proof. □

The last part of this section is dedicated to establish some further notable properties of the fundamental solution Γ for $\mathcal{L}$ constructed in Thm. 3.3.4.

Proposition 3.3.5. *Let the assumption and the notation of Thm. 3.3.4 apply. Then the function* Γ *is (jointly)* δ_λ*-homogeneous of degree* $2-q$*, that is,*

$$\Gamma(\delta_\lambda(x);\delta_\lambda(y)) = \lambda^{2-q}\,\Gamma(x;y), \quad \forall\ x,y\in\mathbb{R}^n \text{ with } x\neq y \text{ and } \lambda>0. \tag{3.3.16}$$

Proof. Let $\lambda>0$ and let $x,y\in\mathbb{R}^n$ be distinct. We have

$$\Gamma(\delta_\lambda(x);\delta_\lambda(y)) = \int_{\mathbb{R}^p}\Gamma_{\mathbb{G}}(\delta_\lambda(x),0;\delta_\lambda(y),\eta)\,\mathrm{d}\eta \overset{(3.3.1)}{=} \int_{\mathbb{R}^p}\Gamma_{\mathbb{G}}(d_\lambda(x,0);\delta_\lambda(y),\eta)\,\mathrm{d}\eta.$$

By the substitution $\eta=\delta^*_\lambda(u)$, and thanks to the d_λ-homogeneity of degree $2-Q$ of $\Gamma_{\mathbb{G}}$ (see Thm. 1.3.9 on page 17), we obtain

$$\begin{aligned}\Gamma(\delta_\lambda(x);\delta_\lambda(y)) &= \lambda^{q^*}\int_{\mathbb{R}^p}\Gamma_{\mathbb{G}}(d_\lambda(x,0);d_\lambda(y,u))\,\mathrm{d}u\\ &= \lambda^{2-Q+q^*}\int_{\mathbb{R}^p}\Gamma_{\mathbb{G}}(x,0;y,u)\,\mathrm{d}u = \lambda^{2-q}\,\Gamma(x;y),\end{aligned}$$

since $Q=q+q^*$. This gives (3.3.16), and the proof is complete. □

Proposition 3.3.6. *Let the assumption and the notation of Thm. 3.3.4 apply. Then, for every fixed* $x\in\mathbb{R}^n$*, we have the following properties:*

(i) $\Gamma(x;\cdot)$ *is continuous on* $\mathbb{R}^n\setminus\{x\}$*;*

(ii) $\Gamma(x;\cdot)$ *vanishes at infinity, that is,* $\Gamma(x;y)\to 0$ *as* $y\to\infty$*.*

Proof. First of all, by performing (for every fixed $y\in\mathbb{R}^n$) the change of variables $u=\Psi_{x,y}(\eta)$ in Rem. 3.3.2, we can write

$$\begin{aligned}\Gamma(x;y) &= \int_{\mathbb{R}^p}\Gamma_{\mathbb{G}}(x,0;y,\Psi^{-1}_{x,y}(u))\cdot|\det(\mathcal{J}_{\Psi^{-1}_{x,y}}(u))|\,\mathrm{d}u\\ &\overset{(3.3.10)}{=} \int_{\mathbb{R}^p}\Gamma_{\mathbb{G}}(x,0;y,\Psi^{-1}_{x,y}(u))\,\mathrm{d}u\\ &= \int_{\{N(u)<1\}}\{\dots\}\,\mathrm{d}u + \int_{\{N(u)\geq 1\}}\{\dots\}\,\mathrm{d}u := \mathrm{I}+\mathrm{II},\end{aligned} \tag{3.3.17}$$

where N denotes the $\delta_\lambda{}^*$ - homogeneous norm in (3.2.34); moreover, by the properties of $\Gamma_{\mathbb{G}}$ (see Thm. 1.3.9 on page 17) and the continuity of the map $(y,\eta) \mapsto \Psi_{x,y}(\eta)$ (see Rem. 3.3.2), we have that

(a) $\mathbb{R}^N \ni (y,u) \mapsto \Gamma_{\mathbb{G}}(x,0;y,\Psi_{x,y}^{-1}(u))$ is continuous on $\mathbb{R}^N \setminus \{(x,0)\}$;

(b) $\Gamma_{\mathbb{G}}(x,0;y,\Psi_{x,y}^{-1}(u)) \to 0$ as $\|y\| \to \infty$, for every fixed $u \in \mathbb{R}^p$.

Therefore, to prove the proposition it suffices to show that a dominated convergence argument can be applied in the above (3.3.17).

(i) Let $y_0 \in \mathbb{R}^N$ and let $r > 0$ be such that $x \notin \overline{B}(y_0, r)$. Since the set

$$K := \overline{B}(y_0, r) \times \{N \le 1\} \subseteq \mathbb{R}^N$$

is compact and contained in $\mathbb{R}^N \setminus \{(x,0\}$, we deduce from property (a) that there exists a constant $C > 0$ such that

$$\Gamma_{\mathbb{G}}(x,0;y,\Psi_{x,y}^{-1}(u)) \le C, \quad \text{for every } (y,u) \in K;$$

therefore, we can apply a simple dominated convergence argument to pass to the limit for $y \to y_0$ in the integral I. As for integral II, we argue as in the proof of Thm. 3.3.3: by estimate (3.3.11), the expression of K given in (3.3.13) and the very definition $\Psi_{x,y}$, for every $(y,u) \in \mathbb{R}^n \times \{N(u) \ge 1\}$ we obtain

$$\Gamma(x,0;y,\Psi_{x,y}^{-1}(u)) \le \mathbf{c}\, K^{2-Q}(x,y,\Psi_{x,y}^{-1}(u)) = \mathbf{c}\, N^{2-Q}(u); \tag{3.3.18}$$

therefore, the function N being integrable on $\{N(u) \ge 1\}$ (as we have shows in the proof of Thm. 3.3.3), another dominated convergence argument allows us to pass to the limit for $y \to y_0$ also in this case.

(ii) By property (b), there exists a real $\rho > 0$ such that

$$\Gamma_{\mathbb{G}}(x,0;y,\Psi_{x,y}^{-1}(u)) \le 1, \quad \text{for every } (y,u) \in \mathbb{R}^N \text{ with } \|y\| \ge \rho;$$

therefore, a simple dominated converge argument ensures us the possibility for passing to the limit as $\|y\| \to \infty$ in the integral I. As for integral II, it suffices to observe that estimate (3.3.18) allows us to apply the Lebesgue Dominated Convergence Theorem to pass to the limit also for $\|y\| \to \infty$.

The proposition is thus completely proved. □

Corollary 3.3.7. *Let the assumption and the notation of Thm. 3.3.4 apply. Then, for every fixed $x \in \mathbb{R}^n$, we have $\Gamma(x;\cdot) \in C^\infty(\mathbb{R}^n \setminus \{x\}, \mathbb{R})$ and*

$$\mathcal{L}\Gamma(x;\cdot) = 0, \quad \text{on } \mathbb{R}^n \setminus \{x\}.$$

Proof. Since Γ is fundamental solution for $\mathcal{L}$, by definition we have (see identity (1.3.8) on page 16) $\mathcal{L}\Gamma(x;\cdot) = -\mathrm{Dir}_x$ in $\mathcal{D}'(\mathbb{R}^n)$; as a consequence, one has

$$\mathcal{L}\Gamma(x;\cdot) = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^n \setminus \{x\}).$$

Now, since $\mathcal{L}$ is C^∞-hypoelliptic on every open subset of $\mathbb{R}^n$ (see property (a) on page 79), it is possible to find a function $u \in C^\infty(\mathbb{R}^n \setminus \{x\}, \mathbb{R})$ such that

$$\mathcal{L}u = 0 \text{ on } \mathbb{R}^n \setminus \{x\} \text{ and } u \equiv \Gamma(x;\cdot) \text{ almost everywhere on } \mathbb{R}^n \setminus \{x\};$$

on the other hand, $\Gamma(x;\cdot)$ being continuous out of x (see Prop. 3.3.6 - (i)), we necessarily have $\Gamma(x;\cdot) \equiv u$ on $\mathbb{R}^n \setminus \{x\}$, and the proof is complete. □

Remark 3.3.8. Let the assumption and the notation of Thm. 3.3.4 apply. Then Γ is the unique fundamental solution for $\mathcal{L}$ such that, for every $x \in \mathbb{R}^n$,

$$\Gamma(x;\cdot) \in C(\mathbb{R}^n \setminus \{x\}, \mathbb{R}) \quad \text{and} \quad \lim_{\|y\|\to\infty} \Gamma(x;y) = 0.$$

This is a consequence of Rem. 1.3.7 - (c) on page 16, since the operator $\mathcal{L}$ is C^∞-hypoelliptic and it satisfies the Weak Maximum Principle on every open and bounded subset of $\mathbb{R}^n$ (see properties (a)-to-(c) listed on page 79).

Remark 3.3.9. Before proceeding, we would like to briefly comment the statements of Prop. 3.3.6 and of Cor. 3.3.7. First of all we observe that, since Γ is a fundamental solution of $\mathcal{L}$ and $\mathcal{L}$ is C^∞-hypoelliptic in $\mathbb{R}^n$, for every $x \in \mathbb{R}^n$ it is possible to find a function $u_x \in C^\infty(\mathbb{R}^n \setminus \{x\}, \mathbb{R})$ such that

$$\mathcal{L}u_x = 0 \text{ on } \mathbb{R}^n \setminus \{x\} \text{ and } u_x \equiv \Gamma(x;\cdot) \text{ almost everywhere on } \mathbb{R}^n \setminus \{x\};$$

as a consequence, there exists a (unique) smooth function in the equivalence class of $\Gamma(x;\cdot)$ in $L^1_{\text{loc}}(\mathbb{R}^n)$, which satisfies

$$\mathcal{L}u_x = -\text{Dir}_x, \quad \text{in } \mathcal{D}'(\mathbb{R}^n).$$

Our main issue, however, is that we need to know that this u_x is *everywhere* identical to the integral function defined in Thm. 3.3.4, not only out of a Lebesgue-neglibile set, which would unpleasantly depend on x. In fact, since we are interested in establishing some pointwise properties of Γ, it is important for us to know that, for every $x \in \mathbb{R}^N$, we do not need to modify $\Gamma(x;\cdot)$ in order to obtain a smooth function vanishing at infinity.

Having established Prop.s 3.3.5 and 3.3.6, our aim is to prove that the function Γ is actually symmetric, that is,

$$\Gamma(x;y) = \Gamma(y;x), \quad \text{for every } x, y \in \mathbb{R}^n \text{ with } x \neq y.$$

To this end, we need some preliminary results of independent interest.

Lemma 3.3.10. *Let $\Gamma_{\mathbb{G}}$ be the fundamental solution for $\mathcal{L}_{\mathbb{G}}$ introduced in* (3.3.4). *Then the following properties hold true:*

(i) *The map $(x,y,\eta) \mapsto \Gamma_{\mathbb{G}}(x,0;y,\eta)$ is locally integrable on $\mathbb{R}^n \times \mathbb{R}^N$;*

(ii) *For every $y \in \mathbb{R}^n$, the map $(x,\eta) \mapsto \Gamma_{\mathbb{G}}(x,0;y,\eta)$ belongs to $L^1_{\text{loc}}(\mathbb{R}^N)$.*

Proof. (i) Let $K_1 \subseteq \mathbb{R}^n$ and $K_2 \subseteq \mathbb{R}^N$ be compact sets. By Fubini's Theorem and by performing the change of variables $(y,\eta) = (x,0) \star (z,\zeta)$, one has

$$\int_{K_1 \times K_2} \Gamma_{\mathbb{G}}(x,0;y,\eta)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\eta \overset{(3.3.4)}{=} \int_{K_1} \left(\int_{\tau_x^{-1}(K_2)} d^{2-Q}(z,\zeta)\,\mathrm{d}z\,\mathrm{d}\zeta \right) \mathrm{d}x,$$

where τ_x denotes the left-translation by $(x,0)$ on the Carnot group $\mathbb{G}$. We now observe that, for every $x \in K_1$, the set $\tau_x^{-1}(K_2)$ is included in the compact set $H = \big(K_1 \times \{0\}\big)^{-1} \star K_2$; therefore, by recalling that $d^{2-Q} = \Gamma_{\mathbb{G}}(0;\cdot)$ is locally integrable on $\mathbb{R}^N$, we obtain property (i).

(ii) We fix a point $y \in \mathbb{R}^n$ and a compact set $K \subseteq \mathbb{R}^n$, and we set

$$C_y : \mathbb{R}^N \to \mathbb{R}^N, \quad C_y(x,\eta) := (x,0)^{-1} \star (y,\eta).$$

It can be easily deduced from identity (3.3.7) that C_y is a C^∞-diffeomorphism of $\mathbb{R}^N$ onto itself and that, for every $(x,\eta) \in \mathbb{R}^N$, one has

$$\left|\det\big(\mathcal{J}_{C_y}(x,\eta)\big)\right| = 1.$$

Therefore, by the change of variables $(x,\eta) = \mathrm{C}_y^{-1}(z,\zeta)$, we get

$$\int_K \Gamma_{\mathbb{G}}(x,0;y,\eta)\,\mathrm{d}x\,\mathrm{d}\eta = \int_{C_y^{-1}(K)} d^{2-Q}(z,\zeta)\,\mathrm{d}z\,\mathrm{d}\zeta.$$

Since $C_y^{-1}(K)$ is compact and $d^{2-Q} \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, we obtain property (ii). □

Proposition 3.3.11. *Let the assumption and the notation of Thm. 3.3.4 apply. Then the following properties hold true:*

(i) $\Gamma \in L^1_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R}^n)$;

(ii) *For every* $y \in \mathbb{R}^n$*, we have* $\Gamma(\cdot;y) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. (i) Let $K_1, K_2 \subseteq \mathbb{R}^n$ be compact sets and let

$$\Phi : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^n \times \mathbb{R}^N, \quad \Phi(x,y,\eta) := \big(x,y,\Psi_{x,y}(\eta)\big).$$

As pointed out in Rem. 3.3.2, $\Psi_{x,y}$ is a smooth diffeomorphism of $\mathbb{R}^p$ onto itself, and the map $(x,y,\eta) \mapsto \Psi_{x,y}(\eta)$ is smooth on $\mathbb{R}^n \times \mathbb{R}^N$; therefore, Φ defines a smooth diffeomorphism of $\mathbb{R}^n \times \mathbb{R}^N$ and,by (3.3.10),

$$\det\big(\mathcal{J}_\Phi(x,y,\eta)\big) = 1, \quad \text{for every } (x,y,\eta) \in \mathbb{R}^n \times \mathbb{R}^N.$$

From this, by applying Fubini's Theorem and by performing the change of variables $(x,y,\eta) = \Phi^{-1}(u,v,\nu)$, we get

$$\begin{aligned}
&\int_{K_1\times K_2} \Gamma(x;y)\,\mathrm{d}x\,\mathrm{d}y = \int_{K_1\times K_2\times\mathbb{R}^p} \Gamma_{\mathbb{G}}(u,0;v,\Psi^{-1}_{u,v}(\nu))\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\nu \\
&= \int_{K_1\times K_2\times\{N(\nu)<1\}} \{\ldots\}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\nu + \int_{K_1\times K_2\times\{N(\nu)\geq 1\}} \{\ldots\}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\nu =: \mathrm{I} + \mathrm{II},
\end{aligned}$$

where, as usual, we have set $N(\nu) = \sum_{k=1}^p |\nu_k|^{1/s_{j_k}}$.

Now, since the product $K_1 \times K_2 \times \{N(\nu) < 1\}$ is bounded in $\mathbb{R}^n \times \mathbb{R}^N$, it follows from Lem. 3.3.10 - (i) that I is finite. As for the integral II we notice that, by exploiting estimate (3.3.18) in the proof of Prop. 3.3.6, we have

$$\begin{aligned}
\mathrm{II} &\leq \mathbf{c} \int_{K_1\times K_2\times\{N(\nu)\geq 1\}} N^{2-Q}(\nu)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\nu \\
&= \mathbf{c}\cdot \mathrm{mis}(K_1\times K_2) \int_{\{N(\nu)\geq 1\}} N^{2-Q}(\nu)\,\mathrm{d}\nu;
\end{aligned}$$

therefore, the function N^{2-Q} being integrable on $\{N(\nu) \geq 1\}$ (see the proof of Thm. 3.3.3), we conclude that II is finite as well.

(ii) Let $y \in \mathbb{R}^n$ and let $K \subseteq \mathbb{R}^n$ be a compact set. We consider the map

$$\Phi_y : \mathbb{R}^N \to \mathbb{R}^N, \quad \Phi_y(x, \eta) := \big(x, \Psi_{x,y}(\eta)\big).$$

By arguing as above, we see that Φ_y defines a C^∞-diffeomorphism of $\mathbb{R}^N$ onto itself and that, for every $(x, \eta) \in \mathbb{R}^N$, one has

$$\det\big(\mathcal{J}_{\Phi_y}(x, \eta)\big) = 1;$$

therefore, by the change of coordinates $(x, \eta) = \Phi_y^{-1}(u, v)$, we get

$$\begin{aligned}\int_K \Gamma(x; y)\,\mathrm{d}x &= \int_{K\times\mathbb{R}^p} \Gamma_{\mathbb{G}}(u, 0; y, \Psi_{u,y}^{-1}(v))\,\mathrm{d}u\,\mathrm{d}v \\ &= \int_{K\times\{N(v)<1\}} \{\ldots\}\,\mathrm{d}u\,\mathrm{d}v + \int_{K\times\{N(v)\geq 1\}} \{\ldots\}\,\mathrm{d}u\,\mathrm{d}v =: \mathrm{I} + \mathrm{II}.\end{aligned}$$

Since the set $K \times \{N(v) < 1\} \subseteq \mathbb{R}^N$ is bounded, we deduce from Lem. 3.3.10 - (ii) that the integral I is finite; the finiteness of the integral II can be proved by exploiting estimate (3.3.18) and by arguing exactly as in (i). □

The next proposition provides a slight improvement of the results contained in Prop. 3.3.6, showing that Γ is well-behaved as a function of *both* x *and* y.

Proposition 3.3.12. *Let the assumption and the notation of Thm. 3.3.4 apply. Then the following facts hold true:*

(i) *Setting* $\mathcal{O} := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$, *then* $\Gamma \in C(\mathcal{O}, \mathbb{R})$;

(ii) *For every compact set* $K \subseteq \mathbb{R}^n$, *then*

$$\lim_{\|y\|\to\infty} \Gamma(x; y) = 0, \quad \text{uniformly } \textit{for } x \in K;$$

(iii) *For every fixed* $y \in \mathbb{R}^n$, *the function* $\Gamma(\cdot; y)$ *vanishes at infinity.*

Proof. (i) Let $(x_0, y_0) \in \mathcal{O}$ and let $r > 0$ be s.t. $\overline{B}(x_0, r) \cap \overline{B}(y_0, r) = \varnothing$. By performing the usual change of variables $u = \Psi_{x,y}(\eta)$, we have

$$\begin{aligned}\Gamma(x; y) &= \int_{\mathbb{R}^p} \Gamma_{\mathbb{G}}(x, 0; y, \Psi_{x,y}^{-1}(u))\,\mathrm{d}u \\ &= \int_{\{N(u)<1\}} \{\ldots\}\,\mathrm{d}u + \int_{\{N(u)\geq 1\}} \{\ldots\}\,\mathrm{d}u =: \mathrm{I} + \mathrm{II},\end{aligned}$$

where we have set $N(u) = \sum_{k=1}^p |u_k|^{1/s_{j_k}}$. Moreover, from Thm. 1.3.9 (on page 17) and the continuity of $(x, y, u) \mapsto \Psi_{x,y}^{-1}(u)$, we infer that

$$\Gamma_{\mathbb{G}}(x, 0; y, \Psi_{x,y}^{-1}(u)) \in C(\mathcal{O} \times \mathbb{R}^p, \mathbb{R}). \tag{3.3.19}$$

Therefore, to prove the continuity of Γ it suffices to show that a dominated convergence argument can be applied to I and II.

To this end we first observe that, since the product

$$K := \big(\overline{B}(x_0, r) \times \overline{B}(y_0, r)\big) \times \{N(u) \leq 1\} \subseteq \mathcal{O} \times \mathbb{R}^p$$

is compact, by (3.3.19) it is possible to find a constant $C > 0$ such that

$$\Gamma_{\mathbb{G}}(x, 0; y, \Psi_{x,y}^{-1}(u)) \leq C, \quad \text{for every } (x, y, u) \in K;$$

hence, a simple dominated convergence argument can be applied to pass to the limit as $(x, y) \to (x_0, y_0)$ in the integral I. As for the integral II, we argue as usual: by estimate (3.3.11), the expression of K given in (3.3.13) and the very definition $\Psi_{x,y}$, for every $(x, y) \in \mathcal{O}$ and every $u \in \{N(u) \geq 1\}$ we obtain

$$\Gamma(x, 0; y, \Psi_{x,y}^{-1}(u)) \leq \mathbf{c}\, K^{2-Q}(x, y, \Psi_{x,y}^{-1}(u)) = \mathbf{c}\, N^{2-Q}(u); \tag{3.3.20}$$

therefore, the function N^{2-Q} being integrable on the set $\{N(u) \geq 1\}$, we are entitled to apply the Lebesgue Dominated Convergence Theorem to pass to the limit for $(x, y) \to (x_0, y_0)$ also in this case.

(ii) Let $K \subseteq \mathbb{R}^n$ be a compact set. We consider the map

$$\mathrm{S} : (\mathbb{R}^n \setminus K) \times \mathbb{R}^p \longrightarrow \mathbb{R}, \qquad \mathrm{S}(y, u) := \sup_{x \in K} \Gamma_{\mathbb{G}}(x, 0; y, \Psi_{x,y}^{-1}(u)).$$

By (3.3.19), the function S is well-defined and continuous on $\mathbb{R}^n \setminus K$; moreover, by performing the usual change of variables $u = \Psi_{x,y}(\eta)$, we obtain

$$\begin{aligned}
\sup_{x \in K} \Gamma(x; y) &= \sup_{x \in K} \left(\int_{\mathbb{R}^p} \Gamma_{\mathbb{G}}(x, 0; y, \Psi_{x,y}^{-1}(u))\, \mathrm{d}u \right) \\
&\leq \sup_{x \in K} \left(\int_{\{N(u)<1\}} \{\ldots\}\, \mathrm{d}u \right) + \sup_{x \in K} \left(\int_{\{N(u)\geq 1\}} \{\ldots\}\, \mathrm{d}u \right) \\
&\leq \int_{\{N(u)<1\}} \mathrm{S}(y, u)\, \mathrm{d}u + \int_{\{N(u)\geq 1\}} \mathrm{S}(y, u)\, \mathrm{d}u =: \mathrm{I} + \mathrm{II}.
\end{aligned} \tag{3.3.21}$$

We now observe that, since $d^{2-Q} = \Gamma_{\mathbb{G}}(0, \cdot)$ vanishes at infinity (see Thm.1.3.9 on page 17) and $(x, 0)^{-1} \star (z, \zeta) \to \infty$ as $\|(z, \zeta)\| \to \infty$, *uniformly for* $x \in K$ (recall that the left-translations are diffeomorphisms), we have

$$\lim_{\|y\| \to \infty} \mathrm{S}(y, u) = 0, \quad \text{uniformly for } u \in \mathbb{R}^p; \tag{3.3.22}$$

therefore, to prove property (ii) it suffices to show that a dominated convergence argument can be applied in both integrals I and II.

As for I we notice that, by (3.3.22), there exists $\rho > 0$ such that

$$\mathrm{S}(y, u) \leq 1, \quad \text{for every } y \in \mathbb{R}^p \text{ with } \|y\| \geq \rho \text{ and every } u \in \mathbb{R}^p;$$

therefore, the Lebesgue Dominated Convergence Theorem allows us to pass to the limit for $\|y\| \to \infty$. As for integral II, we argue as in (i): by estimate (3.3.20) (holding true for every $(x, y) \in \mathcal{O}$ and every $u \in \{N(u) \geq 1\}$) we get

$$\mathrm{S}(y, u) \leq \mathbf{c}\, N^{2-Q}(u), \quad \text{for every } y \in \mathbb{R}^n \text{ and every } u \in \{N(u) \geq 1\};$$

thus, the function N^{2-Q} being integrable on $\{N(u) \geq 1\}$, a dominated convergence argument allows us to pass to the limit for $\|y\| \to \infty$ also in this case.

(iii) Let $y \in \mathbb{R}^n$ be fixed. By the usual change of coordinates $u = \Psi_{x,y}(\eta)$ in Rem. 3.3.2 and the expression of $\Gamma_{\mathbb{G}}$ given in (3.3.4), we can write

$$\begin{aligned}\Gamma(x;y) &= \int_{\mathbb{R}^p} \Gamma_{\mathbb{G}}(x,0;y,\Psi_{x,y}^{-1}(u))\,\mathrm{d}u \\ &= \int_{\mathbb{R}^p} d^{2-Q}\big((x,0)^{-1}\star(y,\Psi_{x,y}^{-1}(u)\big)\,\mathrm{d}u \\ &= \int_{\{N(u)<1\}}\{\ldots\}\,\mathrm{d}u + \int_{\{N(u)\geq 1\}}\{\ldots\}\,\mathrm{d}u =: \mathrm{I}+\mathrm{II};\end{aligned}$$

moreover, since both maps

$$C_y(x,\eta) = (x,0)^{-1}\star(y,\eta) \quad \text{and} \quad \Psi_y(x,\eta) = (x,\Psi_{x,y}(\eta)$$

are C^∞-diffeomorphisms of $\mathbb{R}^N$ (see the proofs of Lem. 3.3.10 and Prop. 3.3.11, respectively), we easily deduce that

$$\lim_{\|(x,u)\|\to\infty}(x,0)^{-1}\star(y,\Psi_{x,y}^{-1}(u)) = \lim_{\|(x,u)\|\to\infty}(C_y\circ\Psi_y^{-1})(x,u) = \infty.$$

From this, recalling that $d^{2-Q} = \Gamma_{\mathbb{G}}(0,\cdot)$ vanishes at infinity, we get

$$\lim_{\|x\|\to\infty} d^{2-Q}\big((x,0)^{-1}\star(y,\Psi_{x_j,y}^{-1}(u)\big) = 0, \quad \text{uniformly for } u\in\mathbb{R}^p. \tag{3.3.23}$$

To complete the demonstration of property (iii), we are then left to show that a dominated convergence argument can be applied to both integrals I and II.

As for I we observe that, by (3.3.23), there exists $\rho > 0$ such that

$$d^{2-Q}\big((x,0)^{-1}\star(y,\Psi_{x,y}^{-1}(u)\big) \leq 1$$

for every $x \in \mathbb{R}^n$ with $\|y\| \geq 1$ and every $u \in \mathbb{R}^p$; therefore, a simple dominated convergence argument allows us to pass to the limit for $\|x\| \to \infty$. As for integral II, we argue exactly as in (i): by estimate (3.3.20) we obtain

$$d^{2-Q}\big((x,0)^{-1}\star(y,\Psi_{x,y}^{-1}(u)\big) \leq \mathbf{c}\,N^{2-Q}(u), \quad \forall\, x\in\mathbb{R}^n \text{ and } u\in\{N(u)\geq 1\};$$

thus, the function N^{2-Q} being integrable on $\{N(u) \geq 1\}$, a dominated convergence argument allows us to pass to the limit for $\|x\| \to \infty$ also in this case. □

Thanks to all the results proved so far, we are now in a position to prove that Γ provides a right inverse for the operator $\mathcal{L}$ in the sense of distribution.

Theorem 3.3.13 (Γ right-inverts $\mathcal{L}$). *Let the assumption and the notation of Thm. 3.3.4 apply. For every fixed $\varphi \in C_0^\infty(\mathbb{R}^n,\mathbb{R})$, the function*

$$\Lambda_\varphi : \mathbb{R}^n \to \mathbb{R}, \quad \Lambda_\varphi(y) := \int_{\mathbb{R}^n}\Gamma(x;y)\,\varphi(x)\,\mathrm{d}x, \tag{3.3.24}$$

is well-defined and it satisfies the following properties:

(i) $\Lambda_\varphi \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$*;*

(ii) $\Lambda_\varphi \in C\big(\mathbb{R}^n \setminus \mathrm{supp}(\varphi),\mathbb{R}\big)$ *and it vanishes at infinity;*

(iii) $\mathcal{L}\big(\Lambda_\varphi\big) = -\varphi$ *in the sense of distributions on* $\mathbb{R}^n$.

Proof. By Prop. 3.3.11 - (ii) we know that, for every $y \in \mathbb{R}^n$, $\Gamma(\cdot; y) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$; thus Λ_φ is well-defined. We now turn to prove properties (i)-to-(iii).

(i) Let $K \subseteq \mathbb{R}^n$ be a compact set. Setting $K_0 := \mathrm{supp}(\varphi)$, one has

$$\int_K \Lambda_\varphi(y)\,\mathrm{d}y \le \sup_{\mathbb{R}^n} |\varphi| \int_{K\times K_0} \Gamma(x; y)\,\mathrm{d}x\,\mathrm{d}y;$$

therefore, the function Γ being locally integrable on $\mathbb{R}^n \times \mathbb{R}^n$ (as we know from Prop. 3.3.11 - (i)), we obtain property (i).

(ii) Let $y_0 \in \mathbb{R}^n \setminus K_0$ and let $r > 0$ be such that $\overline{B}(y_0, r) \subseteq \mathbb{R}^n \setminus K_0$. Since, by Prop. 3.3.12, Γ is continuous out of the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$, we have

$$\lim_{y\to y_0} \Gamma(x; y) = \Gamma(x; y_0), \quad \text{for every fixed } x \in K_0;$$

moreover, by the same reason, there exists a constant $C > 0$ such that

$$\Gamma(x; y) \le C, \quad \text{for every } (x, y) \in K_0 \times \overline{B}(y_0, r)$$

(note that $K_0 \times \overline{B}(y_0, r)$ is compact and $K_0 \cap \overline{B}(y_0, r) = \varnothing$). A simple dominated convergence argument thus ensures that $\Lambda_\varphi(y) \to \Lambda_\varphi(y_0)$ as $y \to y_0$, hence that $\Lambda_\varphi \in C(\mathbb{R}^n \setminus K_0, \mathbb{R})$. To prove that Λ_φ vanishes at infinity we observe that, for every y outside K_0, we have (by definition)

$$|\Lambda_\varphi(y)| \le \big(\sup_{x\in K_0} \Gamma(x; y)\big)\cdot \int_{K_0} |\varphi(x)|\,\mathrm{d}x;$$

thus, since we know from Prop. 3.3.12 that $\sup_{x\in K_0} \Gamma(x; y) \to 0$ as $\|y\| \to \infty$, we conclude that Λ_φ vanishes at infinity.

(iii) By Fubini's theorem, for every $\psi \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$ we have

$$\begin{aligned}\int_{\mathbb{R}^n} \Lambda_\varphi(y)\,\mathcal{L}\psi(y)\,\mathrm{d}y &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \Gamma(x; y)\,\varphi(x)\,\mathrm{d}x\right) \mathcal{L}\psi(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \varphi(x) \left(\int_{\mathbb{R}^n} \Gamma(x; y)\,\mathcal{L}\psi(y)\,\mathrm{d}y\right) \mathrm{d}x;\end{aligned}$$

therefore, by recalling that $\mathcal{L}\Gamma(x; \cdot) = -\mathrm{Dir}_x$ in $\mathcal{D}'(\mathbb{R}^n)$ (as it follows from very definition of fundamental solution, see (1.3.8) on page 16), we obtain

$$\int_{\mathbb{R}^n} \Lambda_\varphi(y)\,\mathcal{L}\psi(y)\,\mathrm{d}y = -\int_{\mathbb{R}^n} \varphi(x)\,\psi(x)\,\mathrm{d}x, \quad \text{for every } \psi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}),$$

This means that $\mathcal{L}(\Lambda_\varphi) = -\varphi$ in $\mathcal{D}'(\mathbb{R}^n)$, as desired. □

Corollary 3.3.14. *Let the assumption and the notation of Thm. 3.3.4 apply. For every fixed* $\varphi \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$, *one has*

$$\Lambda_{\mathcal{L}\varphi}(y) = \int_{\mathbb{R}^n} \Gamma(x; y)\,\mathcal{L}\varphi(x)\,\mathrm{d}x = -\varphi(y), \quad \textit{a.e. on } \mathbb{R}^n. \tag{3.3.25}$$

Proof. By Thm. 3.3.13, we have $\Lambda_{\mathcal{L}\varphi} \in L^1_{\rm loc}(\mathbb{R}^n)$ and $\mathcal{L}(\Lambda_{\mathcal{L}\varphi}) = -\mathcal{L}\varphi$ in the sense of distributions on $\mathbb{R}^n$; therefore, the operator $\mathcal{L}$ being C^∞-hypoelliptic on every open subset of $\mathbb{R}^n$, there exists $h \in C^\infty(\mathbb{R}^n, \mathbb{R})$ such that

- $h \equiv \Lambda_{\mathcal{L}\varphi}$ almost everywhere on $\mathbb{R}^n$;
- $\mathcal{L}h = -\mathcal{L}\varphi$ pointwise on $\mathbb{R}^n$.

Furthermore, since $\Lambda_{\mathcal{L}\varphi}$ is continuous outside $\mathrm{supp}(\varphi)$ (again by Thm. 3.3.13), we have $h = \Lambda_{\mathcal{L}_\varphi}$ on the open set $\mathbb{R}^n \setminus \mathrm{supp}(\varphi)$; as a consequence, the function $\Lambda_{\mathcal{L}_\varphi}$ vanishing at infinity, then the same is true of h.

We now consider the function $u := h + \varphi$. Obviously, $u \in C^\infty(\mathbb{R}^n, \mathbb{R})$ and

$$\mathcal{L}u = \mathcal{L}h + \mathcal{L}\varphi = -\mathcal{L}\varphi + \mathcal{L}\varphi = 0, \quad \text{on } \mathbb{R}^n; \tag{3.3.26}$$

moreover, since φ has compact support and h vanishes at infinity, we have

$$u(y) \to 0, \quad \text{as } \|y\| \to \infty. \tag{3.3.27}$$

By (3.3.26) and (3.3.27), we deduce from the Weak Maximum Principle for $\mathcal{L}$ (see property (c) on page 79) that u must vanish identically on $\mathbb{R}^n$, whence

$$h(y) = -\varphi(y), \quad \text{for every } y \in \mathbb{R}^n.$$

From this, by recalling that h coincides almost everywhere with Λ_φ, we obtain the desired identity (3.3.25). This ends the proof. □

We are finally ready to prove the announced symmetry of Γ.

Theorem 3.3.15 (Symmetry of Γ). *Let the assumption and the notation of Thm. 3.3.4 apply. Then the function Γ is symmetric, that is,*

$$\Gamma(x;y) = \Gamma(y;x), \quad \textit{for every } x, y \in \mathbb{R}^n \textit{ with } x \neq y. \tag{3.3.28}$$

Proof. For the sake of clarity, we split the proof into three different steps.

Step I: We first prove the existence of a measurable set $E \subseteq \mathbb{R}^n$, with vanishing Lebesgue measure, such that

$$\mathcal{L}\Gamma(\cdot;x) = -\mathrm{Dir}_x, \quad \text{for every } x \in \mathbb{R}^n \setminus E. \tag{3.3.29}$$

To this end we observe that, the space $C_0^\infty(\mathbb{R}^n, \mathbb{R})$ being separable (with the usual LF-topology), there exists a countable dense set $\mathcal{F} \subseteq C_0^\infty(\mathbb{R}^n, \mathbb{R})$; moreover, thanks to Cor. 3.3.14, for $\varphi \in \mathcal{F}$ it is possible to find a measurable set $E(\varphi)$, with vanishing Lebesgue measure, such that

$$\int_{\mathbb{R}^n} \Gamma(y;x)\, \mathcal{L}\varphi(y)\, \mathrm{d}y = -\varphi(x), \quad \text{for every } x \in \mathbb{R}^n \setminus E(\varphi).$$

We then set $E := \bigcup_{\varphi \in \mathcal{F}} E(\varphi)$. Since $\mathcal{F}$ is countable and $E(\varphi)$ has vanishing Lebesgue measure for every $\varphi \in \mathcal{F}$, we see that E has measure 0 as well; furthermore, for every $x \in \mathbb{R}^n \setminus E$, we have

$$\int_{\mathbb{R}^n} \Gamma(y;x)\, \mathcal{L}\varphi(y)\, \mathrm{d}y = -\varphi(x), \quad \text{for every } \varphi \in \mathcal{F}.$$

This proves that, for every $x \notin E$, the distribution $\mathcal{L}\Gamma(\cdot;x)$ coincides with $-\mathrm{Dir}_x$ on $\mathcal{F}$; the latter being dense, we immediately obtain the claimed (3.3.29).

STEP II: We now consider, for a fixed $x \notin E$, the function

$$u_x := \Gamma(x;\cdot) - \Gamma(\cdot;x).$$

Since both $\Gamma(x;\cdot)$ and $\Gamma(\cdot;x)$ are locally integrable on $\mathbb{R}^n$ (see Prop. 3.3.11 - (ii)), we see that $u_x \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$; moreover, thanks to identity (3.3.29), we have

$$\mathcal{L}\,u_x = \mathcal{L}\Gamma(x;\cdot) - \mathcal{L}\Gamma(\cdot;x) = -\mathrm{Dir}_x + \mathrm{Dir}_x = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^n).$$

As a consequence, the operator $\mathcal{L}$ being C^∞-hypoelliptic on every open subset of $\mathbb{R}^n$, there exists $h_x \in C^\infty(\mathbb{R}^n,\mathbb{R})$ such that

$$\mathcal{L}\,h_x = 0 \text{ on } \mathbb{R}^n \text{ and } h_x \equiv u_x \text{ almost everywhere on } \mathbb{R}^n.$$

In particular, as u_x is continuous on $\mathbb{R}^n \setminus \{x\}$ and it vanishes at infinity (since the same is true of both $\Gamma(x;\cdot)$ and $\Gamma(\cdot;x)$, see Prop. 3.3.12), we have

- $h_x(y) = u_x(y) = \Gamma(x;y) - \Gamma(y;x)$, for every $y \in \mathbb{R}^n \setminus \{x\}$;
- $h_x(y) \to 0$ as $\|y\| \to \infty$.

By gathering together all these facts, we deduce from the Weak Maximum Principle for $\mathcal{L}$ that h_x identically vanishes on $\mathbb{R}^n$, whence

$$\Gamma(x;y) = \Gamma(y;x), \quad \text{for every } x \notin E \text{ and every } y \in \mathbb{R}^n \setminus \{x\}. \tag{3.3.30}$$

STEP III: To complete the proof of the theorem, we show that identity (3.3.30) actually holds out of the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$. To this end, let $x, y \in \mathbb{R}^n$ with $x \neq y$ and let $r > 0$ be such that $y \notin B_r(x)$. Since $\mathbb{R}^n \setminus E$ is dense (as E has measure 0), there exists a sequence $\{x_j\}_j \subseteq (\mathbb{R}^n \setminus E) \cap B_r(x)$ converging to x as $j \to \infty$; hence, identity (3.3.30) implies that

$$\Gamma(x_j;y) = \Gamma(y;x_j), \quad \text{for every } j \in \mathbb{N}.$$

From this, as Γ is continuous out of the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$ (see Prop. 3.3.12), we deduce that $\Gamma(x;y) = \Gamma(y;x)$. This ends the proof. □

Corollary 3.3.16. *Let the assumption and the notation of Thm. 3.3.4 apply. Then the following properties hold true:*

(i) $\Gamma(\cdot;x) \in C^\infty(\mathbb{R}^n \setminus \{x\},\mathbb{R})$;

(ii) $\mathcal{L}\Gamma(\cdot;x) = 0$ *pointwise on* $\mathbb{R}^n \setminus \{x\}$.

Proof. This is an immediate consequence of Thm. 3.3.15 and of Cor. 3.3.7. □

We conclude this section with the following non-trivial result.

Theorem 3.3.17. *Let the assumption and the notation of Thm. 3.3.4 apply. Then the function* Γ *is smooth out of the diagonal of* $\mathbb{R}^n \times \mathbb{R}^n$.

Proof. We introduce the $2m$ vector fields $\widetilde{X}_1, \ldots, \widetilde{X}_m, \widetilde{Y}_1, \ldots, \widetilde{Y}_m$, operating on $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, defined in the following way:

$$\tilde{X}_j := \sum_{i=1}^{n} (X_j I)_i(x)\, \partial_{x_i}, \quad \tilde{Y}_j := \sum_{i=1}^{n} (X_j I)_i(y)\, \partial_{y_i} \quad (j = 1, \ldots, m).$$

We then set $\tilde{\mathcal{L}} := \sum_{j=1}^m (\tilde{X}_j^2 + \tilde{Y}_j^2)$. Obviously, $\tilde{\mathcal{L}}$ has smooth coefficients; moreover, since $[\tilde{X}_i, \tilde{Y}_j] = 0$ for every $i, j = 1, \ldots, m$, it is immediate to see that $\tilde{\mathcal{L}}$ is a Hörmander operator on the whole of $\mathbb{R}^n \times \mathbb{R}^n$, hence C^∞-hypoelliptic on the same space. Since, by Cor.s 3.3.7 and 3.3.16 we have

$$\tilde{\mathcal{L}}\Gamma(x;y) = \mathcal{L}\Gamma(\cdot;y) + \mathcal{L}\Gamma(x;\cdot) = 0, \quad \text{for every } x, y \in \mathbb{R}^n \text{ with } x \neq y,$$

and since Γ is continuous out of the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$ (see Prop. 3.3.12), we thus conclude that Γ is actually of class C^∞ on the same set. □

3.4 Some examples

This section is devoted to present some explicit examples of homogeneous Hörmander operators to which our theory applies.

Example 3.4.1 (Grushin operator on $\mathbb{R}^2$). Let us consider once again the Grushin vector fields X_1, X_2 introduced in Exm. 3.2.2, that is,

$$X_1 = \partial_{x_1}, \quad X_2 = x_1\, \partial_{x_2} \qquad \text{on } \mathbb{R}^2.$$

As already pointed out in Exm. 3.2.2, X_1, X_2 satisfy assumptions (H1) and (H2) of Sec. 3.2; in particular, they are homogeneous of degree 1 w.r.t. the dilations

$$\delta_\lambda(x_1, x_2) = (\lambda x_1, \lambda^2 x_2).$$

Taking into account all the explicit computations carried out in Exm. 3.2.12, we know that the relevant Carnot group is $\mathbb{G} = (\mathbb{R}^3, \star, d_\lambda)$ with

$$d_\lambda(x_1, x_2, \xi) = (\lambda x_1, \lambda^2 x_2, \lambda\, \xi), \quad Q = 4,$$

while the composition law is

$$(x_1, x_2, \xi) \star (y_1, y_2, \eta) := (x_1 + y_1, x_2 + y_2 + x_1\eta, \xi + \eta).$$

Furthermore, the vector fields Z_1, Z_2 lifting X_1 and X_2 are

$$Z_1 = \partial_{x_1}, \qquad Z_2 = x_1\, \partial_{x_2} + \partial_\xi. \tag{3.4.1}$$

The operator $\mathcal{L} = X_1^2 + X_2^2$ lifts to the sub-Laplacian

$$\mathcal{L}_\mathbb{G} = Z_1^2 + Z_2^2 = \partial_{x_1}^2 + (x_1\, \partial_{x_2} + \partial_\xi)^2.$$

The latter is (modulo a change of variable) the Kohn-Laplacian on the first Heisenberg group (remind that $\mathbb{A} \cong \mathbb{H}^1$, see Exm. 3.2.6), whence its fundamental solution with pole at the origin is given by the function

$$\Gamma_\mathbb{G}(x, \xi) = c\left((x_1^2 + \xi^2)^2 + 16\,(x_2 - 1/2\, x_1\xi)^2\right)^{-1/2}, \quad (x, \xi) \neq (0, 0),$$

where $c > 0$ is a suitable constant. According to Thm. 3.3.3, the function

$$\Gamma(x_1, x_2; y_1, y_2) = c \int_{\mathbb{R}} \frac{\mathrm{d}\eta}{\sqrt{((x_1 - y_1)^2 + \eta^2)^2 + 4\,(2\,x_2 - 2\,y_2 + \eta\,(x_1 + y_1))^2}}, \tag{3.4.2}$$

is the unique fundamental solution for the Grushin operator $\mathcal{L}$ vanishing at infinity. From (3.4.2) we also derive that, for every $x \in \mathbb{R}^2$, the function $\Gamma(x; \cdot)$ has a pole at x: in fact (see Proposition 3.1.10)

$$\liminf_{y \to x} \Gamma(x; y) \geq c \int_{\mathbb{R}} \frac{\mathrm{d}\eta}{\sqrt{\eta^4 + 16\, x_1^2 \eta^2}} = \infty.$$

Finally, the integral in (3.4.2) can be expressed in terms of Elliptic Functions. More precisely, we have

$$\Gamma(x; y) = \frac{c\sqrt{2}}{\sqrt[4]{(x_1^2 + y_1^2)^2 + 4\,(x_2 - y_2)^2}} \cdot \mathrm{K}\left(\frac{1}{2} + \frac{x_1 y_1}{\sqrt{(x_1^2 + y_1^2)^2 + 4\,(x_2 - y_2)^2}}\right),$$

where K denotes the complete elliptic integral of the first kind, that is,

$$\mathrm{K}(m) := \int_0^{\pi/2} (1 - m \sin^2(t))^{-1/2}\, \mathrm{d}t, \quad \text{for } -1 < m < 1.$$

This gives back a formula already obtained by Greiner [88] (see also the milestone works by Beals, Gaveau, Greiner [17, 18]; Beals, Gaveau, Greiner, Kannai [19]; Bauer, Furutani, Iwasaki [16]).

Example 3.4.2 (Another Grushin-type operator). Let us consider, on Euclidean space $\mathbb{R}^2$, the smooth vector fields

$$X_1 = \partial_{x_1}, \quad X_2 = x_1^2\, \partial_{x_2}.$$

Obviously, X_1, X_2 are linearly independent in the real vector space $\mathcal{X}(\mathbb{R}^2)$ and it is very easy to check that they are homogeneous w.r.t. the dilations

$$\delta_\lambda(x_1, x_2) = (\lambda x_1, \lambda^3 x_2);$$

moreover, since $X_3 := [X_1, X_2] = 2\,x_1\,\partial_{x_2}$ and $X_4 := [X_1, X_3] = 2\,\partial_{x_2}$, we see that X_1, X_2 satisfy the Hörmander rank condition at the origin. As a consequence, X_1, X_2 fulfill assumptions (H1) and (H2) of Sec. 3.2.

We now observe that, since X_2 commutes with all the X_js and since, by definition, $[X_1, X_2] = X_3$, $[X_1, X_3] = X_4$ and $[X_1, X_4] = [X_3, X_4] = 0$, we have

$$\mathfrak{a} := \mathrm{Lie}\{X_1, X_2\} = \mathrm{span}_{\mathbb{R}}\{X_1, X_2, X_3, X_4\} \quad \text{and} \quad N = \dim(\mathfrak{a}) = 4.$$

Moreover, $\mathfrak{a}$ is nilpotent of step $r = \sigma_2 = 3$ and, according to (3.2.2), one has

$$\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_3, \quad \text{with} \quad \begin{cases} \mathfrak{a}_1 := \mathrm{span}\{X_1, X_2\}, \\ \mathfrak{a}_2 := [\mathfrak{a}_1, \mathfrak{a}_1] = \mathrm{span}\{X_3\}, \\ \mathfrak{a}_3 := [\mathfrak{a}_1, \mathfrak{a}_2] = \mathrm{span}\{X_4\}, \\ [\mathfrak{a}_1, \mathfrak{a}_3] = \{0\}. \end{cases}$$

We now consider the set $\mathcal{A} := \{X_1, X_2, X_3, X_4\} \subseteq \mathfrak{a}$ and we prove that it is a basis of $\mathfrak{a}$ satisfying properties (P1) and (P2) on page 65.

In fact, obviously, X_1, X_2, X_3 and X_4 are linearly independent in the vector space $\mathcal{X}(\mathbb{R}^2)$; moreover, $\mathcal{A}$ is adapted to the stratification $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_3$, since

$$\mathfrak{a}_1 = \mathrm{span}\{X_1, X_2\}, \quad \mathfrak{a}_2 = \mathrm{span}\{X_3\} \quad \text{and} \quad \mathfrak{a}_3 = \mathrm{span}\{X_4\}.$$

Finally, since $X_1 I(0) = e_1$, $X_2 I(0) = X_3 I(0) = 0$ and $X_4 I(0) = 2e_2$ (where e_1 and e_2 denote the element of the canonical basis in $\mathbb{R}^2$), we deduce that

$$\{X_i I(0),\ i = 1, 2, 3, 4\} \text{ is a system of generators of } \mathbb{R}^2.$$

If we thus introduce the linear isomorphism Φ associated with $\mathcal{A}$, that is,

$$\Phi : \mathbb{R}^4 \longrightarrow \mathfrak{a}, \qquad \Phi(a) = (a \cdot X) := \sum_{i=1}^{4} a_i\, X_i,$$

for every $a, b \in \mathbb{R}^4$ and every $\lambda > 0$ we can write (remind the definition of the Campbell-Baker-Hausdorff multiplication $\diamond$ and of the dilation Δ_λ):

$$\begin{aligned}
\Phi(a) \diamond \Phi(b) &= \left(\sum_{i=1}^{4} a_i\, X_i\right) \diamond \left(\sum_{i=1}^{4} b_i\, X_i\right) \\
&\quad \big(\text{by (3.2.4), since } \mathfrak{a} \text{ is nilpotent of step 3}\big) \\
&= \sum_{i=1}^{4} a_i\, X_i + \sum_{i=1}^{4} b_i\, X_i + \frac{1}{2}\left[\sum_{i=1}^{4} a_i\, X_i, \sum_{i=1}^{4} b_i\, X_i\right] \\
&\quad + \frac{1}{12}\left[\sum_{i=1}^{4} (a_i - b_i)\, X_i, \left[\sum_{i=1}^{4} a_i\, X_i, \sum_{i=1}^{4} b_i\, X_i\right]\right] \\
&\quad \big(\text{by using the commutator identities between the } X_j\text{s}\big) \\
&= \sum_{i=1}^{2} (a_i + b_i)\, X_i + \left(a_3 + b_3 + \frac{1}{2}\,(a_1 b_2 - a_2 b_1)\right) X_3 \\
&\quad + \left(a_4 + b_4 + \frac{1}{2}\,(a_1 b_3 - a_3 b_1) + \frac{1}{12}\,(a_1 - b_1)\,(a_1 b_2 - a_2 b_1)\right) X_4;
\end{aligned}$$

$$\begin{aligned}
\Delta_\lambda\big(\Phi(a)\big) &= \Delta_\lambda\left(\sum_{i=1}^{4} a_i\, X_i\right) = \Delta_\lambda\big((a_1\, X_1 + a_2\, X_2) + (a_3\, X_3) + (a_4\, X_4)\big) \\
&\quad \big(\text{by (3.2.3), since } (a_1\, X_1 + a_2\, X_2) \in \mathfrak{a}_1, \\
&\quad\ (a_3\, X_3) \in \mathfrak{a}_2 \text{ and } (a_4\, X_4) \in \mathfrak{a}_3\big) \\
&= \lambda\big(a_1\, X_1 + a_2\, X_2\big) + \lambda^2\, a_3\, X_3 + \lambda^3\, a_4\, X_4.
\end{aligned}$$

Taking into account (3.2.5) and (3.2.6), we then obtain

$$\begin{aligned} a * b &= \Phi^{-1}\big(\Phi(a) \diamond \Phi(b)\big) \\ &= \bigg(a_1 + b_1, a_2 + b_2, a_3 + b_3 + \frac{1}{2}\,(a_1 b_2 - a_2 b_1), \\ &\qquad a_4 + b_4 + \frac{1}{2}\,(a_1 b_3 - a_3 b_1) + \frac{1}{12}\,(a_1 - b_1)\,(a_1 b_2 - a_2 b_1)\bigg); \end{aligned} \tag{3.4.3}$$

$$D_\lambda(a) = \Phi^{-1}\big(\Delta_\lambda\big(\Phi(a)\big)\big) = (\lambda a_1, \lambda a_2, \lambda^2 a_3, \lambda^3 a_4).$$

By Thm. 3.2.5, $\mathbb{A} = (\mathbb{R}^4, *, D_\lambda)$ is a Carnot group with Lie algebra isomorphic to $\mathfrak{a}$; in particular, the Jacobian vector fields of $\mathrm{Lie}(\mathbb{A})$ are given by

$$J_1 = \partial_{a_1} - \frac{1}{2}\,a_2\,\partial_{a_3} - \frac{1}{12}\,(6a_3 - a_1 a_2)\,\partial_{a_4}, \quad J_2 = \partial_{a_2} + \frac{1}{2}\,a_1\,\partial_{a_3} + \frac{1}{12}\,a_1^2\,\partial_{a_4},$$

$$J_3 = \partial_{a_3} + \frac{1}{2}\,a_1\,\partial_{a_4}, \quad J_4 = \partial_{a_4}.$$

We now turn to compute the explicit expression of the map π defined in (3.2.12). To this end, we fix $a \in \mathbb{R}^4$ and we consider the following Cauchy problem:

$$\begin{cases} \dot\gamma = \displaystyle\sum_{i=1}^4 a_i\, X_i I(\gamma), \\ \gamma(0) = 0 \end{cases} \iff \begin{cases} \dot\gamma_1 = a_1, \\ \dot\gamma_2 = a_2\,\gamma_1^2 + 2\,a_3\,\gamma_1 + 2\,a_4, \\ \gamma(0) = 0. \end{cases}$$

Since $\dot\gamma_1 = a_1$ and $\gamma_1(0) = 0$, we obviously have $\gamma_1(t) = a_1 t$; moreover, by inserting this expression in the second equation of the problem, we get

$$\gamma_2(t) = \int_0^t (a_2\,\gamma_1^2(s) + 2\,a_3\,\gamma_1(s) + 2\,a_4)\,\mathrm{d}s = 2\,a_4 t + a_1 a_3\,t^2 + \frac{a_1^2 a_2}{3}\,t^3.$$

As a consequence, from the very definition of π we obtain

$$\pi(a) = \Psi_1^{a\cdot X}(0) = (\gamma_1(1), \gamma_2(1)) = \left(a_1, 2\,a_4 + a_1 a_3 + \frac{a_1^2 a_2}{3}\right).$$

With this expression of π at hand, we proceed by writing down the explicit expression of the diffeomorphism T defined in (3.2.19). To this end, we first need to determine the two sets of indexes defined in (3.2.17).

Since $X_1 I(0) = e_1$ and $X_4 I(0) = 2e_2$, we have $\{i_1, i_2\} = \{1, 4\}$ and

$$\{j_1, j_2\} = \{1, 2, 3, 4\} \setminus \{i_1, i_2\} = \{2, 3\};$$

therefore, according to the definition of T given in (3.2.19), we have

$$T(a) = (\pi(a), a_{j_1}, a_{j_2}) = \left(a_1, 2\,a_4 + a_1 a_3 + \frac{a_1^2 a_2}{3}, a_2, a_3\right), \quad a \in \mathbb{R}^4. \tag{3.4.4}$$

Having established this expression of T, we can finally write down the expression of the group law $\star$, of the dilation d_λ and of the vector fields Z_1, Z_2 lifting X_1 and X_2. In fact, a direct computation shows that

$$T^{-1}(x, \xi) = \left(x_1, \xi_1, \xi_2, \frac{1}{6}\big(3x_2 - x_1^2 \xi_1 - 3x_1 \xi_2\big)\right), \quad (x_1, x_2, \xi_1, \xi_2) \in \mathbb{R}^4;$$

as a consequence, by exploiting the expression of $*$ and of D_λ written in (3.4.3), for every $(x,\xi),(y,\eta)\in\mathbb{R}^4=\mathbb{R}^2\times\mathbb{R}^2$ and every $\lambda>0$ we obtain

$$\begin{aligned}(x,\xi)\star(y,\eta) &= T\big(T^{-1}(x,\xi)*T^{-1}(y,\eta)\big)\\ &= T\Big(\Big(x_1,\xi_1,\xi_2,\frac{1}{6}\big(3x_2-x_1^2\xi_1-3x_1\xi_2\big)\Big)*\\ &\qquad *\Big(y_1,\eta_1,\eta_2,\frac{1}{6}\big(3y_2-y_1^2\eta_1-3y_1\eta_2\big)\Big)\Big)\\ &= \Big(x_1+y_1,x_2+y_2+x_1(x_1+y_1)\eta_1+2x_1\eta_2,\xi_1+\eta_1,\\ &\qquad \xi_2+\eta_2+1/2\,(x_1\eta_1-y_1\xi_1)\Big);\\ d_\lambda(x,\xi) &= T\big(D_\lambda(T^{-1}(x,\xi))\big)\\ &= T\Big(D_\lambda\Big(x_1,\xi_1,\xi_2,\frac{1}{6}\big(3x_2-x_1^2\xi_1-3x_1\xi_2\big)\Big)\Big)\\ &= (\lambda x_1,\lambda^3x_2,\lambda\,\xi_1,\lambda^2\xi_2).\end{aligned}$$

Furthermore, according to (3.2.27), for every $(x,\xi)\in\mathbb{R}^4$ we have

$$Z_1I(x,\xi)=\mathrm{d}T(J_1)I(x,\xi)=\mathcal{J}_T\big(T^{-1}(x,\xi)\big)\cdot J_1I\big(T^{-1}(x,\xi)\big)$$

$$=\begin{pmatrix}1&0&0&0\\ \xi_2+\frac{2\,x_1\xi_1}{3}&\frac{x_1^2}{3}&x_1&2\\ 0&1&0&0\\ 0&0&1&0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-\frac{\xi_1}{2}\\-\frac{1}{12}\big(6\,\xi_2-x_1\xi_1\big)\end{pmatrix}=\begin{pmatrix}1\\0\\0\\-\frac{\xi_1}{2}\end{pmatrix};$$

$$Z_2I(x,\xi)=\mathrm{d}T(J_2)I(x,\xi)=\mathcal{J}_T\big(T^{-1}(x,\xi)\big)\cdot J_2I\big(T^{-1}(x,\xi)\big)$$

$$=\begin{pmatrix}1&0&0&0\\ \xi_2+\frac{2\,x_1\xi_1}{3}&\frac{x_1^2}{3}&x_1&2\\ 0&1&0&0\\ 0&0&1&0\end{pmatrix}\cdot\begin{pmatrix}0\\1\\\frac{x_1}{2}\\\frac{x_1^2}{12}\end{pmatrix}=\begin{pmatrix}0\\x_1^2\\1\\\frac{x_1}{2}\end{pmatrix};$$

Summing up, the vector fields Z_1,Z_2 can be written as follows:

$$Z_1=\partial_{x_1}-\frac{\xi_1}{2}\,\partial_{\xi_2},\qquad Z_2=x_1^2\,\partial_{x_2}+\partial_{\xi_1}+\frac{x_1}{2}\,\partial_{\xi_2}.$$

Thanks to all this algebraic machinery, we can proceed by using Thm. 3.3.3 to find a global fundamental solution for the Grushin-type operator $\mathcal{L}=X_1^2+X_2^2$.

Indeed, since the sub-Laplacian $\mathcal{L}_{\mathbb{G}}=Z_1^2+Z_2^2$ lifts $\mathcal{L}$, the cited Thm. 3.3.3 ensures that, if $\Gamma_{\mathbb{G}}$ is the fundamental solution for $\mathcal{L}_{\mathbb{G}}$, the function

$$\begin{aligned}\Gamma(x;y)&=\int_{\mathbb{R}^2}\Gamma_{\mathbb{G}}((x,0)^{-1}\star(y,\eta))\,\mathrm{d}\eta\\ &=\int_{\mathbb{R}^2}\Gamma_{\mathbb{G}}\Big(y_1-x_1,y_2-x_2+x_1\eta_1(x_1-y_1)-2\,x_1\eta_2,\eta_1,\eta_2-\tfrac{1}{2}x_1\eta_1\Big)\,\mathrm{d}\eta_1\mathrm{d}\eta_2,\end{aligned}$$

is the unique fundamental solution for $\mathcal{L}$ vanishing at infinity.

Furthermore, from Thm. 3.3.3 we derive that $\Gamma(x;y)$ is bounded from above and from below (up to two structural constants) by

$$\int_{\mathbb{R}^2} K^{-5}(x,y,\eta)\,\mathrm{d}\eta_1\mathrm{d}\eta_2,$$

where the function K is

$$\begin{aligned} K(x,y,\eta) &= |y_1-x_1| + |y_2 - x_2 + x_1\eta_1(x_1-y_1) - 2\,x_1\eta_2|^{1/3} \\ &\quad + |\eta_1| + |\eta_2 - \tfrac{1}{2}x_1\eta_1|^{1/2}. \end{aligned}$$

In this case we are able to deduce that, for every fixed $x \in \mathbb{R}^2$, the function $\Gamma(x;\cdot)$ has a pole at x (see Prop. 3.1.10): indeed, for some constant $\mathbf{c} > 0$

$$\liminf_{y\to x} \Gamma(x,y) \geq \mathbf{c}^{-1} \int_{\mathbb{R}^2} \Big(|2x_1\eta_2|^{1/3} + |\eta_1| + |\eta_2 - 1/2x_1\eta_1|^{1/2}\Big)^{-5} \mathrm{d}\eta = \infty.$$

Example 3.4.3 (An Engel-type operator). Let us consider, on Euclidean space $\mathbb{R}^3$, the smooth vector fields

$$X_1 = \partial_{x_1}, \quad X_2 = x_1\,\partial_{x_2} + x_1^2\,\partial_{x_3}.$$

Obviously, X_1, X_2 are linearly independent in the real vector space $\mathcal{X}(\mathbb{R}^3)$ and it is very easy to check that they are homogeneous w.r.t. the dilations

$$\delta_\lambda(x_1,x_2,x_3) = (\lambda x_1, \lambda^2 x_2, \lambda^3 x_3);$$

moreover, since $X_3 := [X_1,X_2] = \partial_{x_2} + 2\,x_1\,\partial_{x_3}$ and $X_4 := [X_1,X_3] = 2\,\partial_{x_3}$, we see that X_1, X_2 satisfy the Hörmander rank condition at the origin. As a consequence, X_1, X_2 fulfill assumptions (H1) and (H2) of Sec. 3.2.

We now observe that, since X_2 commutes with all the X_js and since, by definition, $[X_1,X_2] = X_3$, $[X_1,X_3] = X_4$ and $[X_1,X_4] = [X_3,X_4] = 0$, we have

$$\mathfrak{a} := \mathrm{Lie}\{X_1,X_2\} = \mathrm{span}_{\mathbb{R}}\{X_1,X_2,X_3,X_4\} \quad \text{and} \quad N = \dim(\mathfrak{a}) = 4.$$

Moreover, $\mathfrak{a}$ is nilpotent of step $r = \sigma_3 = 3$ and, according to (3.2.2), one has

$$\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_3, \quad \text{with} \quad \begin{cases} \mathfrak{a}_1 := \mathrm{span}\{X_1,X_2\}, \\ \mathfrak{a}_2 := [\mathfrak{a}_1,\mathfrak{a}_1] = \mathrm{span}\{X_3\}, \\ \mathfrak{a}_3 := [\mathfrak{a}_1,\mathfrak{a}_2] = \mathrm{span}\{X_4\}, \\ [\mathfrak{a}_1,\mathfrak{a}_3] = \{0\}. \end{cases}$$

We now consider the set $\mathcal{A} := \{X_1,X_2,X_3,X_4\} \subseteq \mathfrak{a}$ and we prove that it is a basis of $\mathfrak{a}$ satisfying properties (P1) and (P2) on page 65.

In fact, obviously, X_1, X_2, X_3 and X_4 are linearly independent in the vector space $\mathcal{X}(\mathbb{R}^3)$; moreover, $\mathcal{A}$ is adapted to the stratification $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_3$, since

$$\mathfrak{a}_1 = \mathrm{span}\{X_1,X_2\}, \quad \mathfrak{a}_2 = \mathrm{span}\{X_3\} \quad \text{and} \quad \mathfrak{a}_3 = \mathrm{span}\{X_4\}.$$

Finally, since $X_1I(0) = e_1$, $X_2I(0) = 0$, $X_3I(0) = e_2$ and $X_4I(0) = 2\,e_3$ (where e_1, e_2 and e_3 denote the element of the canonical basis in $\mathbb{R}^3$), we deduce that

$$\{X_iI(0),\ i = 1,2,3,4\} \text{ is a system of generators of } \mathbb{R}^3.$$

If we thus introduce the linear isomorphism Φ associated with $\mathcal{A}$, that is,

$$\Phi : \mathbb{R}^4 \longrightarrow \mathfrak{a}, \qquad \Phi(a) = (a \cdot X) := \sum_{i=1}^{4} a_i\, X_i,$$

for every $a, b \in \mathbb{R}^4$ and every $\lambda > 0$ we can write (remind the definition of the Campbell-Baker-Hausdorff multiplication $\diamond$ and of the dilation Δ_λ):

$$\begin{aligned}
\Phi(a) \diamond \Phi(b) &= \Big(\sum_{i=1}^{4} a_i\, X_i\Big) \diamond \Big(\sum_{i=1}^{4} b_i\, X_i\Big) \\
&\quad \big(\text{by (3.2.4), since } \mathfrak{a} \text{ is nilpotent of step } 3\big) \\
&= \sum_{i=1}^{4} a_i\, X_i + \sum_{i=1}^{4} b_i\, X_i + \frac{1}{2}\Big[\sum_{i=1}^{4} a_i\, X_i, \sum_{i=1}^{4} b_i\, X_i\Big] \\
&\qquad + \frac{1}{12}\Big[\sum_{i=1}^{4} (a_i - b_i)\, X_i, \Big[\sum_{i=1}^{4} a_i\, X_i, \sum_{i=1}^{4} b_i\, X_i\Big]\Big] \\
&\quad \big(\text{by using the commutator identities between the } X_j\text{s}\big) \\
&= \sum_{i=1}^{2} (a_i + b_i)\, X_i + \Big(a_3 + b_3 + \frac{1}{2}\,(a_1 b_2 - a_2 b_1)\Big)\, X_3 \\
&\qquad + \Big(a_4 + b_4 + \frac{1}{2}\,(a_1 b_3 - a_3 b_1) + \frac{1}{12}\,(a_1 - b_1)\,(a_1 b_2 - a_2 b_1)\Big)\, X_4;
\end{aligned}$$

$$\begin{aligned}
\Delta_\lambda\big(\Phi(a)\big) &= \Delta_\lambda\Big(\sum_{i=1}^{4} a_i\, X_i\Big) = \Delta_\lambda\big((a_1\, X_1 + a_2\, X_2) + (a_3\, X_3) + (a_4\, X_4)\big) \\
&\quad \big(\text{by (3.2.3), since } (a_1\, X_1 + a_2\, X_2) \in \mathfrak{a}_1, \\
&\qquad (a_3\, X_3) \in \mathfrak{a}_2 \text{ and } (a_4\, X_4) \in \mathfrak{a}_3\big) \\
&= \lambda\big(a_1\, X_1 + a_2\, X_2\big) + \lambda^2\, a_3\, X_3 + \lambda^3\, a_4\, X_4.
\end{aligned}$$

Taking into account (3.2.5) and (3.2.6), we then obtain

$$\begin{aligned}
a * b &= \Phi^{-1}\big(\Phi(a) \diamond \Phi(b)\big) \\
&= \Big(a_1 + b_1, a_2 + b_2, a_3 + b_3 + \frac{1}{2}\,(a_1 b_2 - a_2 b_1), \\
&\qquad a_4 + b_4 + \frac{1}{2}\,(a_1 b_3 - a_3 b_1) + \frac{1}{12}\,(a_1 - b_1)\,(a_1 b_2 - a_2 b_1)\Big);
\end{aligned} \tag{3.4.5}$$

$$D_\lambda(a) = \Phi^{-1}\big(\Delta_\lambda\big(\Phi(a)\big)\big) = (\lambda a_1, \lambda a_2, \lambda^2 a_3, \lambda^3 a_4).$$

By Thm. 3.2.5, $\mathbb{A} = (\mathbb{R}^4, *, D_\lambda)$ is a Carnot group with Lie algebra isomorphic to $\mathfrak{a}$; in particular, the Jacobian vector fields of Lie($\mathbb{A}$) are given by

$$J_1 = \partial_{a_1} - \frac{1}{2}\, a_2\, \partial_{a_3} - \frac{1}{12}\,(6 a_3 - a_1 a_2)\, \partial_{a_4}, \quad J_2 = \partial_{a_2} + \frac{1}{2}\, a_1\, \partial_{a_3} + \frac{1}{12}\, a_1^2\, \partial_{a_4},$$

$$J_3 = \partial_{a_3} + \frac{1}{2}\, a_1\, \partial_{a_4}, \quad J_4 = \partial_{a_4}.$$

We explicitly notice that the group $\mathbb{A}$ just constructed coincides with the group (also denoted by $\mathbb{A}$) constructed in Exm. 3.4.2: this is a consequence of the fact that the Lie algebras involved have the same structure constants with respect to the chosen basis (in both cases denoted by $\mathcal{A}$).

We now turn to compute the explicit expression of the map π defined in (3.2.12). To this end, we fix $a \in \mathbb{R}^4$ and we consider the following Cauchy problem:

$$\begin{cases} \dot\gamma = \sum_{i=1}^4 a_i\, X_i I(\gamma), \\ \gamma(0) = 0 \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} \dot\gamma_1 = a_1, \\ \dot\gamma_2 = a_2\,\gamma_1 + a_3, \\ \dot\gamma_3 = a_2\,\gamma_1^2 + 2\,a_3\,\gamma_1 + a_4, \\ \gamma(0) = 0. \end{cases}$$

Since $\dot\gamma_1 = a_1$ and $\gamma_1(0) = 0$, we obviously have $\gamma_1(t) = a_1 t$; moreover, by inserting this expression in the last two equations of the problem we get

$$\gamma_2(t) = \int_0^t (a_2\,\gamma_1(s) + a_3)\,\mathrm{d}s = \frac{1}{2}\,a_1 a_2\,t^2 + a_3 t;$$
$$\gamma_3(t) = \int_0^t (a_2\,\gamma_1^2(s) + 2\,a_3\,\gamma_1(s) + a_4)\,\mathrm{d}s = \frac{1}{3}\,a_1^2 a_2\,t^3 + a_1 a_3\,t^2 + 2\,a_4\,t.$$

As a consequence, from the very definition of π we obtain

$$\begin{aligned} \pi(a) &= \Psi_1^{a\cdot X}(0) = \big(\gamma_1(1), \gamma_2(1), \gamma_3(1)\big) \\ &= \left(a_1, \frac{1}{2}\,a_1 a_2 + a_3, \frac{1}{3}\,a_1^2 a_2 + a_1 a_3 + 2\,a_4\right). \end{aligned}$$

With this expression of π at hand, we proceed by writing down the explicit expression of the diffeomorphism T defined in (3.2.19). To this end, we first need to determine the two sets of indexes defined in (3.2.17).

Since $X_1 I(0) = e_1$, $X_3 I(0) = e_2$ and $X_4 I(0) = 2\,e_3$, we have

$$\{i_1, i_2, i_3\} = \{1, 3, 4\} \text{ and } \{j_1\} = \{1, 2, 3, 4\} \setminus \{i_1, i_2, i_3\} = \{2\};$$

therefore, according to the definition of T given in (3.2.19), we have

$$\begin{aligned} T(a) &= (\pi(a), a_{j_1}) = (\pi(a), a_2) \\ &= \left(a_1, \frac{1}{2}\,a_1 a_2 + a_3, \frac{1}{3}\,a_1^2 a_2 + a_1 a_3 + 2\,a_4, a_2\right), \quad a \in \mathbb{R}^4. \end{aligned} \tag{3.4.6}$$

Having established this expression of T, we can finally write down the expression of the group law $\star$, of the dilation d_λ and of the vector fields Z_1, Z_2 lifting X_1 and X_2. In fact, a direct computation shows that, if $(x_1, x_2, x_3, \xi) \in \mathbb{R}^4$,

$$T^{-1}(x, \xi) = \left(x_1, \xi, x_3 - \frac{1}{2}\,x_1\xi, \frac{1}{12}\big(x_1^2\xi + 6\,x_3 - 6\,x_1 x_2\big)\right);$$

as a consequence, by exploiting the expression of $*$ and of D_λ written in (3.4.5), for every $(x,\xi),(y,\eta) \in \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ and every $\lambda > 0$ we obtain

$$\begin{aligned}(x,\xi) \star (y,\eta) &= T\big(T^{-1}(x,\xi) * T^{-1}(y,\eta)\big)\\ &= T\Bigg(\Big(x_1,\xi,x_3 - \frac{1}{2}\,x_1\xi, \frac{1}{12}\big(x_1^2\xi + 6\,x_3 - 6\,x_1x_2\big)\Big) *\\ &\qquad * \Big(y_1,\eta,y_3 - \frac{1}{2}\,y_1\eta, \frac{1}{12}\big(y_1^2\eta + 6\,y_3 - 6\,y_1y_2\big)\Big)\Bigg)\\ &= \Big(x_1+y_1, x_2+y_2+x_1\eta, x_3+y_3+2x_1y_2+x_1^2\eta, \xi+\eta\Big);\\ d_\lambda(x,\xi) &= T\big(D_\lambda(T^{-1}(x,\xi))\big)\\ &= T\Big(D_\lambda\Big(x_1,\xi,x_3-\frac{1}{2}\,x_1\xi,\frac{1}{12}\big(x_1^2\xi+6\,x_3-6\,x_1x_2\big)\Big)\Big)\\ &= (\lambda x_1, \lambda^2 x_2, \lambda^3 x_3, \lambda\xi).\end{aligned}$$

Furthermore, according to (3.2.27), for every $(x,\xi) \in \mathbb{R}^4$ we have

$$\begin{aligned}Z_1 I(x,\xi) &= \mathrm{d}T(J_1)I(x,\xi) = \mathcal{J}_T\big(T^{-1}(x,\xi)\big)\cdot J_1 I\big(T^{-1}(x,\xi)\big)\\ &= \begin{pmatrix} 1 & 0 & 0 & 0\\ \frac{\xi}{2} & \frac{x_1}{2} & 1 & 0\\ x_3 - \frac{x_1\xi}{6} & \frac{x_1^2}{3} & x_1 & 2\\ 0 & 1 & 0 & 0\end{pmatrix}\cdot\begin{pmatrix}1\\ 0\\ -\frac{\xi}{2}\\ -\frac{1}{12}\big(6\,x_3 - 4\,x_1\xi_1\big)\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix};\end{aligned}$$

$$\begin{aligned}Z_2 I(x,\xi) &= \mathrm{d}T(J_2)I(x,\xi) = \mathcal{J}_T\big(T^{-1}(x,\xi)\big)\cdot J_2 I\big(T^{-1}(x,\xi)\big)\\ &= \begin{pmatrix} 1 & 0 & 0 & 0\\ \frac{\xi}{2} & \frac{x_1}{2} & 1 & 0\\ x_3 - \frac{x_1\xi}{6} & \frac{x_1^2}{3} & x_1 & 2\\ 0 & 1 & 0 & 0\end{pmatrix}\cdot\begin{pmatrix}0\\ 1\\ \frac{x_1}{2}\\ \frac{x_1^2}{12}\end{pmatrix} = \begin{pmatrix}0\\x_1\\x_1^2\\1\end{pmatrix};\end{aligned}$$

Summing up, the group $\mathbb{G} = (\mathbb{R}^4, \star, d_\lambda)$ is isomorphic to the Engel group on $\mathbb{R}^4$ and the vector fields Z_1, Z_2 can be written as follows:

$$Z_1 = \partial_{x_1}, \qquad Z_2 = x_1\,\partial_{x_2} + x_1^2\,\partial_{x_3} + \partial_\xi.$$

Thanks to all this algebraic machinery, we can proceed by using Thm. 3.3.3 to find a global fundamental solution for the Engel-type operator $\mathcal{L} = X_1^2 + X_2^2$.

Indeed, since the sub-Laplacian $\mathcal{L}_\mathbb{G} = Z_1^2 + Z_2^2$ lifts $\mathcal{L}$, the cited Thm. 3.3.3 ensures that, if $\Gamma_\mathbb{G}$ is the fundamental solution for $\mathcal{L}_\mathbb{G}$, the function

$$\begin{aligned}&\Gamma(x_1,x_2,x_3;y_1,y_2,y_3)\\ &\quad = \int_\mathbb{R} \Gamma_\mathbb{G}\Big(y_1-x_1, y_2-x_2-x_1\eta, y_3-x_3+2x_1(x_2-y_2)+x_1^2\eta, \eta\Big)\,\mathrm{d}\eta\end{aligned}$$

is the unique fundamental solution for $\mathcal{L}$ vanishing at infinity.

Furthermore, from Thm. 3.3.3 we derive that $\Gamma(x;y)$ is bounded from above and from below (up to two structural constants) by

$$\int_\mathbb{R}\Big\{|y_1-x_1| + |y_2-x_2-x_1\eta|^{1/2} + \Big|y_3-x_3+2x_1(x_2-y_2)+x_1^2\eta\Big|^{1/3} + |\eta|\Big\}^{-5}\,\mathrm{d}\eta.$$

In this case we are able to deduce that, for every fixed $x \in \mathbb{R}^3$, the function $\Gamma(x;\cdot)$ has a pole at x (see Prop. 3.1.10): indeed, for some constant $\mathbf{c} > 0$

$$\liminf_{y\to x} \Gamma(x;y) \geq \mathbf{c}^{-1} \int_{\mathbb{R}} (|x_1\eta|^{1/2} + |x_1^2\eta|^{1/2} + |\eta|)^{-5}\, \mathrm{d}\eta = \infty.$$

We explicitly notice that, in each of the examples discussed in this section, the fundamental solution Γ of $\mathcal{L}$ obtained by saturating $\Gamma_{\mathbb{G}}$ satisfies the following additional property: *for every $x \in \mathbb{R}^N$, $\Gamma(x;\cdot)$ has a pole at x, i.e.,*

$$\lim_{y\to x} \Gamma(x;y) = \infty. \tag{3.4.7}$$

It is straightforward to check that this property guarantees that, for every fixed $x \in \mathbb{R}^N$, the family of superlevel sets of $\Gamma(x;\cdot)$, that is,

$$\left\{ y \in \mathbb{R}^N : \Gamma(x;y) > \frac{1}{r} \right\} \cup \{x\}, \qquad r > 0,$$

forms a basis of open neighborhoods of x which invades $\mathbb{R}^N$ (as $r \to \infty$) and which shrinks to x as $r \to 0$. Most importantly, it can be used to prove that the set of non-negative $\mathcal{L}$-superharmonic functions *separates the points of* $\mathbb{R}^N$.

On this account, property (3.4.7) plays an important rôle in developing a satisfactory Potential Theory for $\mathcal{L}$ (see, e.g., [1, 13, 34, 37]).

3.5 Fundamental solution for Heat Operators

The aim of this last section is to prove, by using the same techniques exploited in the previous sections, the existence of a "well-behaved" global fundamental solution for any "heat-type" operator $\mathcal{H}$ of the form

$$\mathcal{H} = \mathcal{L} - \partial_t, \quad \text{on } \mathbb{R}^{1+n} = \mathbb{R}_t \times \mathbb{R}^n_x,$$

where $\mathcal{L}$ is a homogeneous Hörmander operator on $\mathbb{R}^n$ (see Sec. 3.2). Although it could appear naïve, the idea of obtaining global fundamental solutions for heat-type operators via a saturation argument seems very natural in the Euclidean setting. Indeed, it is well-known that a global fundamental solution for the classical heat operator $\mathcal{H}_n = \Delta - \partial_t$ on $\mathbb{R}^{1+n}$ is given by the function

$$\Gamma_n(t,x) = \mathbf{1}_{]0,\infty[}(t) \cdot (4\,\pi\,t)^{-n/2} \cdot \exp\left(-\frac{\|x\|^2}{4\,t} \right);$$

thus, is we consider the heat operator on $\mathbb{R}^{1+n+p}$ and if we integrate its fundamental solution Γ_{n+p} with respect to the last p variables, we obtain

$$\begin{aligned} \int_{\mathbb{R}^p} \Gamma_{n+p}(t,x,\xi)\, d\xi &= \mathbf{1}_{]0,\infty[}(t)\, (4\pi\,t)^{-(n+p)/2} \exp\left(-\frac{\|x\|^2}{4\,t} \right) \times \\ &\quad \times \int_{\mathbb{R}^p} \exp\left(-\frac{\|\xi\|^2}{4\,t} \right) \mathrm{d}\xi \\ &= \mathbf{1}_{]0,\infty[}(t)\, (4\,\pi\,t)^{-n/2} \exp\left(-\frac{\|x\|^2}{4\,t} \right) \\ &= \Gamma_n(t,x). \end{aligned}$$

In other words, the fundamental solution Γ_n of $\mathcal{H}_n$ can be recovered by that of $\mathcal{H}_{n+p}$ by saturation. Motivated by this fact, we then try to extend such an approach to the more general setting of homogeneous Hörmander PDOs.

To begin with, let us fix a family $\{X_1, \ldots, X_m\}$ of linearly independent smooth vector fields on Euclidean space $\mathbb{R}^n$ satisfying the assumptions (H1) and (H2) already introduced in Sec. 3.2, that is,

(H1) $X_1, \ldots, X_m$ are homogeneous of degree 1 with respect to a family of non-isotropic dilations δ_λ of the following form

$$\delta_\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \qquad \delta_\lambda(x) = (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_n} x_n),$$

where $1 = \sigma_1 \le \ldots \le \sigma_n$ are positive integers;

(H2) $X_1, \ldots, X_m$ satisfy Hörmander's rank condition at 0.

Moreover, we set $\mathcal{L} := \sum_{j=1}^m X_j^2$. We then denote the point $z \in \mathbb{R}^{1+n}$ by $z = (t, x)$, where $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, and we consider the *heat operator* $\mathcal{H}$ associated with $\mathcal{L}$, that is, the linear PDO defined on $\mathbb{R}^{1+n}$ as follows:

$$\mathcal{H} := \mathcal{L} - \partial_t = \sum_{j=1}^m X_j^2 - \partial_t, \qquad \text{on } \mathbb{R}^{1+n}. \tag{3.5.1}$$

The following theorem summarizes all the results we are going to prove.

Theorem 3.5.1. *The operator $\mathcal{H}$ defined in (3.5.1) admits a (unique) global fundamental solution Γ which satisfies the following properties:*

(i) $\Gamma \ge 0$ *on its domain and, for every $(t, x), (s, y) \in \mathbb{R}^{1+n}$, we have*

$$\Gamma(t, x; s, y) = 0 \text{ if and only if } s \le t.$$

(ii) *For every $(t, x) \ne (s, y) \in \mathbb{R}^{1+n}$, the function Γ depends on t and s only through the difference $s - t$: in fact, we have*

$$\Gamma(t, x; s, y) = \Gamma(0, x; s - t, y).$$

Moreover, Γ is symmetric w.r.t. the space variables $x, y \in \mathbb{R}^{1+n}$, that is,

$$\Gamma(t, x; s, y) = \Gamma(t, y; s, x).$$

(iii) *For every $\lambda > 0$ and every $(t, x) \ne (s, y) \in \mathbb{R}^{1+n}$, we have*

$$\Gamma(\lambda^2 t, \delta_\lambda(x); \lambda^2 s, \delta_\lambda(y)) = \lambda^{-q}\, \Gamma(t, x; s, y).$$

(iv) Γ *is smooth out of the diagonal of* $\mathbb{R}^{1+n} \times \mathbb{R}^{1+n}$*;*

(v) *For every compact set $K \subseteq \mathbb{R}^{1+n}$, we have*

$$\lim_{\|\zeta\| \to \infty} \big(\sup_{z \in K} \Gamma(z; \zeta)\big) = \lim_{\|\zeta\| \to \infty} \big(\sup_{z \in K} \Gamma(\zeta; z)\big) = 0.$$

(vi) $\Gamma \in L^1_{\mathrm{loc}}(\mathbb{R}^{1+n} \times \mathbb{R}^{1+n})$ *and, for fixed every $z \in \mathbb{R}^{1+n}$, we have*

$$\Gamma(z; \cdot) \text{ and } \Gamma(\cdot; z) \in L^1_{\mathrm{loc}}(\mathbb{R}^{1+n}).$$

(vii) *For every fixed* $(t,x) \in \mathbb{R}^{1+n}$ *we have*

$$\int_{\mathbb{R}^n} \Gamma(t,x;s,y)\,\mathrm{d}y = 1, \quad \textit{for every } s > t.$$

(viii) *For every fixed* $\varphi \in C_0^\infty(\mathbb{R}^{1+n}, \mathbb{R})$, *the function*

$$\Lambda_\varphi : \mathbb{R}^{1+n} \longrightarrow \mathbb{R}, \qquad \Lambda_\varphi(\zeta) := \int_{\mathbb{R}^{1+n}} \Gamma(z;\zeta)\,\varphi(z)\,\mathrm{d}z,$$

is smooth, it vanishes at infinity and $\mathcal{H}(\Lambda_\varphi) = -\varphi$ *on* $\mathbb{R}^{1+n}$.

Furthermore, if we consider the function Γ^* *defined by*

$$\Gamma^*(t,x;s,y) := \Gamma(s,y;t,x), \quad \textit{for every } (t,x) \neq (s,y) \in \mathbb{R}^{1+n},$$

then Γ^* *is a global fundamental solution for the adjoint operator* $\mathcal{H}^* = \mathcal{L} + \partial_t$, *satisfying the dual statements of* (i)-to-(viii).

Remark 3.5.2. Before proceeding, we briefly highlight a couple of properties of the operator $\mathcal{H}$ which will be important in the sequel.

(a) $\mathcal{H}$ is C^∞-hypoelliptic on every open subset of $\mathbb{R}^N$: this is a consequence of the Hörmander Hypoellipticity Theorem, since $\{X_1, \dots, X_m, \partial_t\}$ is a Hörmander system on the whole of $\mathbb{R}^{1+n}$ (recall that $X_1, \dots, X_m$ satisfy the Hörmander rank condition at every point of $\mathbb{R}^n$, see Rem. 3.2.9).

(b) $\mathcal{H}$ satisfies the Weak Maximum Principle on every open and bounded subset of $\mathbb{R}^{1+n}$: this follows from the fact that the principal matrix $A(x)$ of $\mathcal{H}$ is given by $A(x) = S(x) \cdot S(x)^t$, where

$$S(x) = \big(X_1 I(x) \cdots X_m I(x)\big), \quad x \in \mathbb{R}^n.$$

Since $X_1, \dots, X_m$ are δ_λ-homogeneous of degree 1 and since they satisfy the Hörmander rank condition at 0, it is possible to find an index i in $\{1, \dots, N\}$ such that $a_{i,i}$ is constant and strictly positive.

Let now $\mathfrak{a} := \mathrm{Lie}\{X_1, \dots, X_m\}$ and let $N := \dim(\mathfrak{a})$. By Thm. 3.2.3, it is possible to find a homogeneous Carnot group $\mathbb{G} = (\mathbb{R}^N, \star, d_\lambda)$ on $\mathbb{R}^N$ (with m generators and nilpotent of step $r = \sigma_n$) and a system $\mathcal{Z} = \{Z_1, \dots, Z_m\}$ of Lie-generators of $\mathrm{Lie}(\mathbb{G})$ such that, for every $i = 1, \dots, m$,

$$Z_i \text{ is a lifting of } X_i \text{ on } \mathbb{R}^N.$$

As a consequence, if we denote by $\mathcal{L}_\mathbb{G}$ the sub-Laplacian on $\mathbb{G}$ associated with $\{Z_1, \dots, Z_m\}$, it is straightforward to recognize that the heat operator

$$\mathcal{H}_\mathbb{G} := \mathcal{L}_\mathbb{G} - \partial_t = \sum_{j=1}^m Z_j^2 - \partial_t, \quad \text{on } \mathbb{R}^{1+N} = \mathbb{R}_t \times \mathbb{R}^N_{(x,\xi)},$$

is a *lifting* of $\mathcal{H}$ on $\mathbb{R}^{1+N} = \mathbb{R}^{1+n} \times \mathbb{R}^p$. The following lemma shows that $\mathcal{H}_\mathbb{G}$ actually provides a *saturable lifting* for $\mathcal{H}$ (see Def. 3.1.4).

Lemma 3.5.3. *Let the above assumptions and notations apply. Then the operator $\mathcal{H}_{\mathbb{G}}$ is a saturable lifting of $\mathcal{H}$ on $\mathbb{R}^{1+N}$.*

Proof. First of all we observe that, by definition, we have

$$R := \mathcal{H}_{\mathbb{G}} - \mathcal{H} = \mathcal{L}_{\mathbb{G}} - \mathcal{L}, \quad \text{on } \mathbb{R}^{1+N};$$

thus, $\mathcal{L}_{\mathbb{G}}$ being a saturable lifting of $\mathcal{L}$ on $\mathbb{R}^N$ (as we know from Thm. 3.2.13), it is immediate to recognize that $\mathcal{H}_{\mathbb{G}}$ is a saturable lifting of $\mathcal{H}$. □

With Lem. 3.5.3 at hands, the path to the existence of a global fundamental solution for $\mathcal{H}$ is traced in Thm. 3.1.6, and it consists of two parts:

- Firstly, we need to prove that $\mathcal{H}_{\mathbb{G}}$ admits a fundamental solution $\Gamma_{\mathbb{G}}$;
- Secondly, we have to show that such a $\Gamma_{\mathbb{G}}$ satisfies the integrability assumptions (i) and (ii) in the statement of Thm. 3.1.6.

As for the existence of a global fundamental solution for $\mathcal{H}_{\mathbb{G}}$, we have the following fundamental result (for a proof see, e.g., [36, Theorem 2.1]).

Theorem 3.5.4 (Existence of a global fundamental solution for $\mathcal{H}_{\mathbb{G}}$)**.** *There exists a map $\gamma_{\mathbb{G}} \in C^\infty(\mathbb{R}^{1+N} \setminus \{0\}, \mathbb{R})$ such that the function*

$$\Gamma_{\mathbb{G}}(t,x,\xi;s,y,\eta) := \gamma_{\mathbb{G}}\big(s-t, (x,\xi)^{-1} \star (y,\eta)\big), \quad (t,x,\xi) \neq (s,y,\eta) \qquad (3.5.2)$$

is a global fundamental solution for $\mathcal{H}_{\mathbb{G}}$ (here, $\star$ is the composition law of the Carnot group $\mathbb{G}$). Moreover, $\gamma_{\mathbb{G}}$ satisfies the following additional properties:

(i) $\gamma_{\mathbb{G}}(t,x,\xi) \geq 0$ *for every* $(t,x,\xi) \in \mathbb{R}^{1+N} \setminus \{0\}$ *and*

$$\gamma_{\mathbb{G}}(t,x,\xi) = 0 \text{ if and only if } t \leq 0;$$

(ii) $\gamma_{\mathbb{G}}(t,x,\xi) = \gamma_{\mathbb{G}}(t,(x,\xi)^{-1})$ *for every* $(t,x,\xi) \in \mathbb{R}^{1+N} \setminus \{0\}$*;*

(iii) *For every $\lambda > 0$ and every $(t,x,\xi) \in \mathbb{R}^{1+N} \setminus \{0\}$, we have*

$$\gamma_{\mathbb{G}}\big(\lambda^2 t, D_\lambda(x,\xi)\big) = \lambda^{-Q}\,\gamma_{\mathbb{G}}(t,x,\xi),$$

where Q is the homogeneous dimension of the group $\mathbb{G}$;

(iv) *$\gamma_{\mathbb{G}}$ vanishes at infinity, that is, $\gamma_{\mathbb{G}}(t,x,\xi) \to 0$ as $\|(t,x,\xi)\| \to \infty$;*

(v) *For every $t > 0$, we have*

$$\int_{\mathbb{R}^N} \gamma_{\mathbb{G}}(t,x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi = 1.$$

Finally, if we consider the function $\Gamma^*_{\mathbb{G}}$ defined by

$$\Gamma^*_{\mathbb{G}}(t,x,\xi;s,y,\eta) := \Gamma_{\mathbb{G}}(s,y,\eta;t,x,\xi), \quad (t,x,\xi) \neq (s,y,\eta), \qquad (3.5.3)$$

then $\Gamma^*_{\mathbb{G}}$ is a global fundamental solution for the adjoint operator $\mathcal{H}^*_{\mathbb{G}} = \mathcal{L}_{\mathbb{G}} + \partial_t$.

Remark 3.5.5. It is worth noting that the function $\Gamma_{\mathbb{G}}$ defined in (3.5.2) is the *unique* global fundamental solution of $\mathcal{H}_{\mathbb{G}}$ s.t., for every $(t,x,\xi) \in \mathbb{R}^{1+N}$,

$$\Gamma_{\mathbb{G}}(t,x,\xi;s,y,\eta) \to 0 \qquad \text{as } \|(s,y,\eta)\| \to \infty.$$

This follows form Rem. 1.3.7 - (c), since $\mathcal{H}_{\mathbb{G}}$ is C^∞-hypoelliptic and it satisfies the Weak Maximum Principle on every open and bounded subset of $\mathbb{R}^{1+n}$.

The needed integrability properties of $\Gamma_{\mathbb{G}}$, instead, crucially rely on the following *uniform Gaussian estimates* (of $\gamma_{\mathbb{G}}$). For a proof of these profound estimates we refer, for example, to [36, Theorem 2.5].

Theorem 3.5.6 (Uniform Gaussian estimates of $\Gamma_{\mathbb{G}}$)**.** *Let the assumptions and the notations in Thm. 3.5.4 apply. Moreover, let $d \in C^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{R})$ be the unique homogeneous symmetric norm on $\mathbb{G}$ such that*

$$d^{2-Q}\big((x,\xi)^{-1} \star (y,\eta)\big), \qquad (x,\xi) \neq (y,\eta)$$

is the global fundamental solution of $\mathcal{L}_{\mathbb{G}}$ (see Thm. 1.3.9). It is then possible to find a constant $\mathbf{c} > 0$ s.t., for every $(x,\xi) \in \mathbb{R}^N$ and every $t > 0$, one has

$$\mathbf{c}^{-1}\, t^{-Q/2} \exp\left(-\frac{\mathbf{c}\, d^2(x,\xi)}{t} \right) \le \gamma_{\mathbb{G}}(t,x,\xi) \le \mathbf{c}\, t^{-Q/2} \exp\left(-\frac{d^2(x,\xi)}{\mathbf{c}\, t} \right). \tag{3.5.4}$$

Thanks to Thm. 3.5.6, we can now prove the following central result.

Theorem 3.5.7. *Let the assumptions and the notations of Thm. 3.5.4 apply. Then the global fundamental solution $\Gamma_{\mathbb{G}}$ of $\mathcal{H}_{\mathbb{G}}$ satisfies the integrability assumptions* (i) *and* (ii) *in the statement of Thm. 3.1.6.*

Proof. We first prove that $\Gamma_{\mathbb{G}}$ satisfies assumption (i). According with our Thm. 3.1.6 we have to show that, for fixed $(t,x) \neq (s,y) \in \mathbb{R}^{1+n}$, one has

$$\eta \mapsto \Gamma_{\mathbb{G}}(t,x,0;s,y,\eta) \in L^1(\mathbb{R}^p). \tag{3.5.5}$$

If $s \le t$, the above (3.5.5) is an immediate consequence of Thm. 3.5.4, since

$$\Gamma_{\mathbb{G}}(t,x,0;s,y,\eta) \overset{(3.5.2)}{=} \gamma_{\mathbb{G}}\big((s-t, (x,0)^{-1} \star (y,\eta)\big) = 0, \quad \text{for every } \eta \in \mathbb{R}^p.$$

We can then assume that $s > t$. In this case, by exploiting the Gaussian estimates of $\gamma_{\mathbb{G}}$ contained in Thm. 3.5.6 and by performing the usual change of variables $\eta = \Psi_{x,y}^{-1}(u)$ (see Rem. 3.3.2), we obtain

$$\begin{aligned} \int_{\mathbb{R}^p} \Gamma_{\mathbb{G}}(t,x,0;s,y,\eta)\,\mathrm{d}\eta \le \frac{\mathbf{c}}{(s-t)^{Q/2}} \times \\ \times \int_{\mathbb{R}^p} \exp\left(-\frac{d^2\big((x,0)^{-1} \star (y, \Psi_{x,y}^{-1}(u))\big)}{\mathbf{c}\,(s-t)} \right) \mathrm{d}u; \end{aligned}$$

on the other hand, since d is a homogeneous norm on $\mathbb{G}$, it is possible to find a universal constant $\alpha > 0$ such that (recall the definition of h given in (3.3.3))

$$\begin{aligned} d^2\big((x,0)^{-1} \star (y, \Psi_{x,y}^{-1}(u)\big) &\ge \alpha\, h\big((x,0)^{-1} \star (y, \Psi_{x,y}^{-1}(u)\big) \\ &\overset{(3.3.12)}{=} \alpha\, K(x,y,\Psi_{x,y}^{-1}(u)), \quad \text{for every } u \in \mathbb{R}^p. \end{aligned}$$

Thus, the very same computation carried out in the proof of Thm. 3.3.3 gives

$$d^2\big((x,0)^{-1}\star(y,\Psi_{x,y}^{-1}(u)\big)\geq \alpha\,N^2(u),\quad \text{for every } u\in\mathbb{R}^p.$$

where N is the homogeneous norm on $\mathbb{R}^p$ defined in (3.2.34). By gathering together all these facts, we see that (3.5.5) follows if we show that

$$u\mapsto \varphi(u):=\exp\left(-\frac{\alpha\,N^2(u)}{\mathbf{c}(s-t)}\right)\in L^1(\mathbb{R}^p). \tag{3.5.6}$$

Now, since $\varphi\in C(\mathbb{R}^p,\mathbb{R})$, we obviously have $\varphi\in L^1_{\mathrm{loc}}(\mathbb{R}^p)$; moreover, by using the classical inequality $\exp(z^2)\geq\beta_Q\,(1+z^2)^{-Q/2}$ (holding true for every $z\in\mathbb{R}$ and for a suitable constant $\beta_Q>0$ only depending on Q), we get

$$\varphi(u)\leq\frac{\beta_Q\,\big(\mathbf{c}\,(s-t)\big)^{Q/2}}{\big(\mathbf{c}(s-t)+\alpha\,N^2(u)\big)^{Q/2}}\leq\widetilde{\beta}\,N^{-Q}(u),\quad\text{for every } u\in\mathbb{R}^p\setminus\{0\}.$$

The function N^{-Q} being integrable on $\{N\geq 1\}$ (as one can recognize by arguing as in the proof of Thm. 3.3.3), we conclude that $\varphi\in L^1(\mathbb{R}^p)$, as desired.

To complete the demonstration of the theorem, we are left to prove that $\Gamma_{\mathbb{G}}$ also satisfies assumption (ii) in Thm. 3.1.6. We then have to show that, for every fixed $(t,x)\in\mathbb{R}^{1+n}$ and every compact set $K\subseteq\mathbb{R}^{1+n}$, one has

$$(s,y,\eta)\mapsto\Gamma_{\mathbb{G}}(t,x,0;s,y,\eta)\in L^1(K\times\mathbb{R}^p).$$

To this end, let $T>0$ be such that $K\subseteq[t-T,t+T]\times\mathbb{R}^p$. We have

$$\begin{aligned}
\int_{K\times\mathbb{R}^p}\Gamma_{\mathbb{G}}(t,x,0;s,y,\eta)\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}\eta &\leq\int_{t-T}^{t+T}\left(\int_{\mathbb{R}^N}\Gamma_{\mathbb{G}}(t,x,0;s,y,\eta)\,\mathrm{d}y\,\mathrm{d}\eta\right)\mathrm{d}s\\
&=\int_{t-T}^{t+T}\left(\int_{\mathbb{R}^N}\gamma_{\mathbb{G}}(s-t,(x,0)^{-1}\star(y,\eta))\,\mathrm{d}y\,\mathrm{d}\eta\right)\mathrm{d}s\\
&\quad\big(\text{by the change of variables } (y,\eta)=(x,0)\star(u,v)\big)\\
&=\int_{t-T}^{t+T}\left(\int_{\mathbb{R}^N}\gamma_{\mathbb{G}}(s-t,u,v)\,\mathrm{d}u\,\mathrm{d}v\right)\mathrm{d}s\\
&\quad\big(\text{by Thm. 3.5.4 - (i) and (v)}\big)\\
&=\int_{t}^{t+T}1\,\mathrm{d}s=T,
\end{aligned}$$

so that $(s,y,\eta)\mapsto\Gamma_{\mathbb{G}}(t,x,0;s,y,\eta)\in L^1(K\times\mathbb{R}^p)$, as desired. □

Remark 3.5.8. The proof of Thm. 3.5.7 contains the following remarkable fact: there exists an absolute constant $\mathbf{M}>0$ such that, for every $(t,x),(s,y)\in\mathbb{R}^{1+n}$ with $s>t$ and for every $u\in\mathbb{R}^p\setminus\{0\}$, one has

$$\Gamma_{\mathbb{G}}(t,x,0;s,y,\Psi_{x,y}^{-1}(u))=\gamma_{\mathbb{G}}\big(s-t,(x,0)^{-1}\star(y,\Psi_{x,y}^{-1}(u))\big)\leq\mathbf{M}\cdot N(u)^{-Q}.$$

On the other hand, since $\gamma_{\mathbb{G}}$ identically vanishes on $\big(]-\infty,0]\times\mathbb{R}^N\big)\setminus\{(0,0)\}$ (by Thm. 3.5.4 - (i)) and $\Psi_{x,x}(0)=0$, we conclude that

$$\gamma_{\mathbb{G}}\big(s-t,(x,0)^{-1}\star(y,\Psi_{x,y}^{-1}(u))\big)\leq\mathbf{M}\cdot N^{-Q}(u), \tag{3.5.7}$$

for every $(t,x)\in\mathbb{R}^{1+n}$ and every $(s,y,u)\in\mathbb{R}^{1+N}$ with $(t,x,0)\neq(s,y,u)$.

By gathering together Lem. 3.5.3, Thm. 3.5.7 and Thm. 3.1.6, we are finally in a position to prove the existence of a global fundamental solution for $\mathcal{H}$.

Theorem 3.5.9 (Existence of a fundamental solution for $\mathcal{H}$)**.** *Let the above assumptions and notations apply. Then the function*

$$\Gamma(t,x;s,y) := \int_{\mathbb{R}^p} \Gamma_{\mathbb{G}}(t,x,0;s,y,\eta)\,\mathrm{d}\eta, \qquad (t,x) \neq (s,y),$$

is a fundamental solution for $\mathcal{H}$. Moreover, if d is as in Thm. 3.5.6, one has

$$\begin{aligned}\mathbf{c}^{-1}\,(s-t)^{-Q/2} \int_{\mathbb{R}^p} \exp\bigg(-\frac{\mathbf{c}\,d^2((x,0)^{-1}\star(y,\eta))}{s-t}\bigg)\,\mathrm{d}\eta \;\leq\; \Gamma(t,x;s,y)\\ \leq \mathbf{c}\,(s-t)^{-Q/2} \int_{\mathbb{R}^p} \exp\bigg(-\frac{d^2((x,0)^{-1}\star(y,\eta))}{\mathbf{c}\,(s-t)}\bigg)\,\mathrm{d}\eta,\end{aligned}$$

holding true for every $(t,x),(s,y) \in \mathbb{R}^{1+n}$ with $s > t$. Here, $\mathbf{c} > 0$ is a constant only depending on the homogeneous Carnot group $\mathbb{G}$ and on the operator $\mathcal{L}$.

Proof. By Lem. 3.5.3, we know that the heat operator $\mathcal{H}_{\mathbb{G}} = \mathcal{L}_{\mathbb{G}} - \partial_t$ on $\mathbb{R}\times\mathbb{G}$ is a saturable lifting of $\mathcal{H}$; moreover, Thm. 3.5.7 ensures that the fundamental solution $\Gamma_{\mathbb{G}}$ of $\mathcal{H}_{\mathbb{G}}$ in (3.5.2) satisfies assumptions (i) and (ii) in Thm. 3.1.6.

Therefore, by the cited Thm. 3.1.6, we conclude that the function Γ is a global fundamental solution of $\mathcal{H}$, and the proof is complete. □

Remark 3.5.10. Let the assumptions and the notations of Thm. 3.5.9 apply. It is worth noting that, for every $(t,x) \neq (s,y) \in \mathbb{R}^{1+n}$, the function Γ depends on t and s only through the difference $s-t$: in fact, we have

$$\begin{aligned}\Gamma(t,x;s,y) &= \int_{\mathbb{R}^p} \Gamma_{\mathbb{G}}(t,x,0;s,y,\eta)\,\mathrm{d}\eta\\ &\quad\big(\text{by the definition of } \Gamma_{\mathbb{G}} \text{ in (3.5.2)}\big)\\ &= \int_{\mathbb{R}^p} \gamma_{\mathbb{G}}\big(s-t,(x,0)^{-1}\star(y,\eta)\big)\,\mathrm{d}\eta\\ &= \int_{\mathbb{R}^p} \Gamma_{\mathbb{G}}(0,x,0;s-t,y,\eta)\,\mathrm{d}\eta\\ &= \Gamma(0,x;y,s-t).\end{aligned} \tag{3.5.8}$$

As a consequence, for every $(t,x) \neq (s,y) \in \mathbb{R}^{1+n}$ we have

$$\Gamma(t,x;s,y) = \Gamma(-s,x;-t,y). \tag{3.5.9}$$

3.5.1 Further properties of Γ

As for Sec. 3.3, the last part of this section is devoted to establish some further properties of the fundamental solution Γ of $\mathcal{H}$ constructed in Thm. 3.5.9.

To begin with, we prove the following very simple lemma.

Lemma 3.5.11. *Let the assumptions and the notations of Thm. 3.5.9 apply. Then $\Gamma \geq 0$ on its domain and, for every $(t,x) \neq (s,y) \in \mathbb{R}^{1+n}$, we have*

$$\Gamma(t,x;s,y) = 0 \textit{ if and only if } s \leq t. \tag{3.5.10}$$

Proof. By Thm. 3.5.4 - (i), we know that $\gamma_{\mathbb{G}} \geq 0$ on $\mathbb{R}^{1+N} \setminus \{0\}$ and that, for every $(\tau, z) \in \mathbb{R}^{1+N}$, one has $\gamma_{\mathbb{G}}(\tau, z) = 0$ if and only if $\tau \leq 0$; as a consequence,

$$\begin{aligned}\Gamma(t,x;s,y) &= \int_{\mathbb{R}^p} \Gamma_{\mathbb{G}}(t,x,0;s,y,\eta)\,\mathrm{d}\eta \\ &\overset{(3.5.2)}{=} \int_{\mathbb{R}^p} \gamma_{\mathbb{G}}\big(s-t,(x,0)^{-1}\star(y,\eta)\big)\,\mathrm{d}\eta \geq 0, \quad \forall\ (t,x) \neq (s,y) \in \mathbb{R}^{1+n}\end{aligned}$$

and $\Gamma(t, x; s, y) = 0$ if and only if $s - t \leq 0$. This ends the proof. □

Since the operator $\mathcal{L}$ is δ_λ-homogeneous of degree 2, the fundamental solution Γ of $\mathcal{H}$ also satisfies the following homogeneity property.

Proposition 3.5.12. *Let the assumptions and the notations of Thm. 3.5.9 apply. Then, for every $\lambda > 0$ and every $(t, x) \neq (s, y) \in \mathbb{R}^{1+n}$ we have*

$$\Gamma(\lambda^2 t, \delta_\lambda(x); \lambda^2 s, \delta_\lambda(y)) = \lambda^{-q}\,\Gamma(t,x;s,y) \tag{3.5.11}$$

where $q = \sum_{j=1}^n \sigma_j$ is the sum of the exponents in the dilation δ_λ.

Proof. Let $\lambda > 0$ and let $(t, x) \neq (s, y) \in \mathbb{R}^{1+n}$. By definition, we have

$$\begin{aligned}\Gamma(\lambda^2 t, \delta_\lambda(x); \lambda^2 s, \delta_\lambda(y)) &= \int_{\mathbb{R}^p} \Gamma_{\mathbb{G}}(\lambda^2 t, \delta_\lambda(x), 0; \lambda^2 s, \delta_\lambda(y), \eta))\,\mathrm{d}\eta \\ &= \int_{\mathbb{R}^p} \gamma_{\mathbb{G}}\big(\lambda^2(s-t), (\delta_\lambda(x),0)^{-1}\star(\delta_\lambda(y),\eta)\big)\,\mathrm{d}\eta.\end{aligned}$$

On the other hand, the family of dilations $\{d_\lambda\}_{\lambda>0}$ of $\mathbb{G}$ taking the form

$$d_\lambda(x,\xi) = (\delta_\lambda(x), \delta^*_\lambda(\xi)), \quad \text{for every } (x,\xi) \in \mathbb{R}^N \text{ and every } \lambda > 0$$

(where δ^*_λ is the dilation on $\mathbb{R}^p$ introduced in (3.2.25)), we get

$$\Gamma(\lambda^2 t, \delta_\lambda(x); \lambda^2 s, \delta_\lambda(y)) = \int_{\mathbb{R}^p} \gamma_{\mathbb{G}}\big(\lambda^2(s-t), (d_\lambda(x,0))^{-1}\star(\delta_\lambda(y),\eta)\big)\,\mathrm{d}\eta.$$

From this, by performing the change of variables $\eta = {\delta_\lambda}^*(u)$ and by using the homogeneity property of $\gamma_{\mathbb{G}}$ in Thm. 3.5.4 - (iii), we obtain

$$\begin{aligned}\Gamma(\lambda^2 t, \delta_\lambda(x); \lambda^2 s, \delta_\lambda(y)) &= \lambda^{q^*} \int_{\mathbb{R}^p} \gamma_{\mathbb{G}}\big(\lambda^2(s-t), (d_\lambda(x,0))^{-1}\star d_\lambda(y,u)\big)\,\mathrm{d}u \\ &\quad \big(d_\lambda \text{ is automorphism of } \mathbb{G}\big) \\ &= \lambda^{q^*} \int_{\mathbb{R}^p} \gamma_{\mathbb{G}}\big(\lambda^2(s-t), d_\lambda((x,0)^{-1}\star(y,u))\big)\,\mathrm{d}u \\ &\quad \big(\text{by Thm. 3.5.4 - (iii)}\big) \\ &= \lambda^{-Q+q^*} \int_{\mathbb{R}^p} \gamma_{\mathbb{G}}(s-t,(x,0)^{-1}\star(y,u))\,\mathrm{d}u \\ &= \lambda^{-q}\,\Gamma(t,x;s,y),\end{aligned}$$

since $Q = q + q^*$ (see the beginning of Sec. 3.3). This is precisely the desired (3.5.11), and the proof is complete. □

Another interesting property of Γ is contained in the next proposition.

Proposition 3.5.13. *Let the assumptions and the notations of Thm. 3.5.9 apply. Then, for every fixed* $(t,x) \in \mathbb{R}^{1+n}$ *we have*

$$\int_{\mathbb{R}^n} \Gamma(t,x;s,y)\,\mathrm{d}y = 1, \quad \textit{for every } s > t. \tag{3.5.12}$$

Proof. Let $s > t$ be fixed. By Thm. 3.5.4 - (v) and the definition of Γ, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \Gamma(t,x;s,y)\,\mathrm{d}y &= \int_{\mathbb{R}^N} \Gamma_{\mathbb{G}}(t,x,0;s,y,\eta)\,\mathrm{d}y\,\mathrm{d}\eta \\
&= \int_{\mathbb{R}^N} \gamma_{\mathbb{G}}(s-t,(x,0)^{-1}\star(y,\eta))\,\mathrm{d}y\,\mathrm{d}\eta \\
&\quad \big(\text{by the change of variables } (y,\eta) = (x,0)\star(u,v)\big) \\
&= \int_{\mathbb{R}^N} \gamma_{\mathbb{G}}(s-t,u,v)\,\mathrm{d}u\mathrm{d}v \\
&\quad \big(\text{by Thm. 3.5.4 - (v), since } s > t\big) \\
&= 1.
\end{aligned}$$

This is precisely the desired (3.5.12), and the proof is complete. □

The following proposition, which is a sort of analogous of Prop. 3.3.12, concerns the regularity and the behavior at infinity of the function Γ.

Proposition 3.5.14. *Let the assumptions and the notations of Thm. 3.5.9 apply. Then the following facts hold true:*

(i) Γ *is continuous out of the diagonal of* $\mathbb{R}^{1+n} \times \mathbb{R}^{1+n}$*;*

(ii) *For every fixed compact set* $K \subseteq \mathbb{R}^{1+n}$*, we have*

$$\sup_{z\in K} \Gamma(z;\zeta) \to 0 \quad \textit{as } \|\zeta\| \to \infty. \tag{3.5.13}$$

(iii) *For every fixed* $\zeta = (s,y) \in \mathbb{R}^{1+n}$*, we have*

$$\Gamma(z;\zeta) \to 0 \quad \textit{as } \|z\| \to \infty. \tag{3.5.14}$$

Proof. (i) Let $z_0 = (t_0,x_0)$, $\zeta_0 = (s_0,y_0) \in \mathbb{R}^{1+n}$ be distinct and let $\rho > 0$ be such that $\overline{B}(z_0,\rho) \cap \overline{B}(\zeta_0,\rho) = \varnothing$. Moreover, let $\{z_n\}_{n\in\mathbb{N}} \subseteq \overline{B}(z_0,\rho)$ and $\{\zeta_n\}_{n\in\mathbb{N}} \subseteq \overline{B}(\zeta_0,\rho)$ be two sequences converging, respectively, to z_0 and ζ_0 as $n \to \infty$. We set $\mathcal{O} := \{(z,\zeta) \in \mathbb{R}^{1+n} \times \mathbb{R}^{1+n} : z \neq \zeta\}$ and we consider the function $\varphi : \mathcal{O} \times \mathbb{R}^p \to \mathbb{R}$ defined in the following way:

$$\varphi(z,\zeta,u) := \gamma_{\mathbb{G}}\big(s-t,(x,0)^{-1}\star(y,\Psi_{x,y}^{-1}(u))\big), \quad (z,\zeta) \in \mathcal{O}, u \in \mathbb{R}^p. \tag{3.5.15}$$

Since the map $(x,y,u) \mapsto \Psi_{x,y}^{-1}(u)$ is smooth on $\mathbb{R}^n \times \mathbb{R}^N$ and, by Thm. 3.5.4, $\gamma_{\mathbb{G}} \in C^\infty(\mathbb{R}^{1+N} \setminus \{0\}, \mathbb{R})$, it is readily seen that φ is continuous on $\mathcal{O} \times \mathbb{R}^p$;

moreover, by means of the map φ, for every $n \in \mathbb{N}$ we can write

$$\begin{aligned}\Gamma(z_n;\zeta_n) = \Gamma(t_n,x_n;s_n,y_n) &= \int_{\mathbb{R}^p} \gamma_{\mathbb{G}}(s_n - t_n,(x_n,0)^{-1}\star(y_n,\eta))\,\mathrm{d}\eta \\ &\quad \big(\text{by the change of variables } \eta = \Psi^{-1}_{x_n,y_n}(u)\big) \\ &= \int_{\mathbb{R}^p} \gamma_{\mathbb{G}}\big(s_n - t_n,(x_n,0)^{-1}\star(y_n,\Psi^{-1}_{x_n,y_n}(u))\big)\,\mathrm{d}u \\ &= \int_{\mathbb{R}^p} \varphi(z_n,\zeta_n,u)\,\mathrm{d}u.\end{aligned}$$

Our aim is now to pass to the limit as $n \to \infty$ in the above identity. To this end we first observe that, since $K := \overline{B}(z_0,\rho) \times \overline{B}(\zeta_0,\rho) \Subset \mathcal{O}$ and φ is continuous on the product $\mathcal{O} \times \mathbb{R}^p$, we obviously have

$$\lim_{n\to\infty} \varphi(z_n,\zeta_n,u) = \varphi(z_0,\zeta_0,u), \quad \text{for every } u \in \mathbb{R}^p;$$

furthermore, if N is the homogeneous norm on $\mathbb{R}^p$ defined in (3.2.34), it is possible to find a constant $M > 0$ such that

$$\varphi(z_n,\zeta_n,u) \le M, \quad \text{for every } n \in \mathbb{N} \text{ and every } u \in \{N \le 1\}.$$

Thus, a simple dominated convergence argument gives

$$\lim_{n\to\infty} \int_{\{N\le 1\}} \varphi(z_n,\zeta_n,u)\,\mathrm{d}u = \int_{\{N\le 1\}} \varphi(z_0,\zeta_0,u)\,\mathrm{d}u. \tag{3.5.16}$$

On the other hand, by exploiting estimate (3.5.7) in Rem. 3.5.7, we obtain

$$\varphi(z_n,\zeta_n,u) \le \mathbf{M} \cdot N^{-Q}(u), \quad \text{for every } n \in \mathbb{N} \text{ and every } u \in \{N > 1\};$$

therefore, the function N^{-Q} being integrable on $\{N > 1\}$ we are entitled to apply the Lebesgue Dominated Convergence Theorem, which gives

$$\lim_{n\to\infty} \int_{\{N> 1\}} \varphi(z_n,\zeta_n,u)\,\mathrm{d}u = \int_{\{N> 1\}} \varphi(z_0,\zeta_0,u)\,\mathrm{d}u. \tag{3.5.17}$$

By gathering together (3.5.16) and (3.5.17), we finally get

$$\begin{aligned}\lim_{n\to\infty} \Gamma(z_n;\zeta_n) &= \lim_{n\to\infty} \int_{\mathbb{R}^p} \varphi(z_n,\zeta_n,u)\,\mathrm{d}u \\ &\int_{\mathbb{R}^p} \varphi(z_0,\zeta_0,u)\,\mathrm{d}u \\ &= \int_{\mathbb{R}^p} \gamma_{\mathbb{G}}\big(s_0 - t_0,(x_0,0)^{-1}\star(y_0,\Psi^{-1}_{x_0,y_0}(u))\big)\,\mathrm{d}u \\ &\quad \big(\text{by the change of variables } u = \Psi_{x_0,y_0}(\eta)\big) \\ &= \int_{\mathbb{R}^p} \gamma_{\mathbb{G}}(s_0 - t_0,(x_0,0)^{-1}\star(y_0,\eta))\,\mathrm{d}\eta \\ &= \Gamma(z_0;\zeta_0),\end{aligned}$$

and this proves that Γ is continuous at (z_0,ζ_0), as desired.

(ii) Let $K \subseteq \mathbb{R}^{1+n}$ be a fixed compact set and let $\{\zeta_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^{1+n} \setminus K$ be such that $\|\zeta_n\| \to \infty$ as $n \to \infty$. We then consider the map Φ defined by

$$\Psi : \big(\mathbb{R}^{1+n} \setminus K\big) \times \mathbb{R}^p \longrightarrow \mathbb{R}, \qquad \Phi(\zeta, u) := \sup_{z\in K} \varphi(z, \zeta, u),$$

where φ is as in (3.5.15). Since, obviously, $\big(\mathbb{R}^{1+n} \setminus K\big) \subseteq \mathcal{O}$ and φ is continuous on $\mathcal{O} \times \mathbb{R}^p$, the function Φ is (well-defined and) continuous on its domain; moreover, by means of the map Φ, for every natural n we can write

$$\begin{aligned}\sup_{(t,x)\in K} \Gamma(t, x; s_n, y_n) &= \sup_{z\in K} \Gamma(z; \zeta_n) = \sup_{z\in K} \left(\int_{\mathbb{R}^p} \varphi(z, \zeta_n, u)\, \mathrm{d}u\right) \\ &\le \int_{\mathbb{R}^p} \sup_{z\in K} \varphi(z, \zeta_n, u)\, \mathrm{d}u \\ &= \int_{\mathbb{R}^p} \Phi(\zeta_n, u)\, \mathrm{d}u.\end{aligned}$$

Our aim is now to pass to the limit as $n \to \infty$ in the above identity. To this end we first notice that, since $\gamma_{\mathbb{G}}$ vanishes at infinity (by Thm. 3.5.4) and

$$\lim_{n\to\infty} \|(s_n - t, (x, 0)^{-1} \star (y_n, \Psi_{x,y_n}(u)))\| = \infty$$

uniformly for $z = (t, x) \in K$ and $u \in \mathbb{R}^p$ (as is easy to see), we have

$$\lim_{n\to\infty} \Phi(\zeta_n, u) = 0, \quad \text{uniformly for } u \in \mathbb{R}^p;$$

in particular, there exists a constant $M > 0$ such that

$$\Phi(\zeta_n, u) \le M, \quad \text{for every } n \in \mathbb{N} \text{ and every } u \in \mathbb{R}^p.$$

Thus, if N is as in (3.2.34), a simple dominated convergence argument gives

$$\lim_{n\to\infty} \int_{\{N\le 1\}} \Phi(\zeta_n, u)\, \mathrm{d}u = 0. \tag{3.5.18}$$

On the other hand, again by exploiting estimate (3.5.7) in Rem. 3.5.7, for every $n \in \mathbb{N}$ and every $u \in \{N > 1\}$ we have the following bound for Φ:

$$\sup_{z\in K} \varphi(z, \zeta_n, u) = \Phi(\zeta_n, u) \le \mathbf{M} \cdot N^{-Q}(u);$$

therefore, the function N^{-Q} being integrable at infinity, another application of the Lebesgue Dominated Convergence Theorem gives

$$\lim_{n\to\infty} \int_{\{N>1\}} \Phi(\zeta_n, u)\, \mathrm{d}u = 0. \tag{3.5.19}$$

By gathering together (3.5.18) and (3.5.19), we obtain

$$\limsup_{n\to\infty} \big(\sup_{z\in K} \Gamma(z; \zeta_n)\big) \le \lim_{n\to\infty} \int_{\mathbb{R}^p} \Phi(\zeta_n, u)\, \mathrm{d}u = 0,$$

which implies the desired (3.5.13), since $\Gamma \ge 0$ on its domain.

(iii) Let $\zeta = (s, y) \in \mathbb{R}^{1+n}$ be fixed and let $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^{1+n} \setminus \{\zeta\}$ be s.t. $\|z_n\| \to \infty$ as $n \to \infty$. We then consider the map θ defined as follows:

$$\theta(z, u) := \big(s - t, (x, 0)^{-1} \star (y, \Psi^{-1}_{x,y}(u)\big), \quad z = (t, x) \in \mathbb{R}^{1+n}, u \in \mathbb{R}^p.$$

Since the map $(x, y, u) \mapsto \Psi^{-1}_{x,y}(u)$ is smooth on $\mathbb{R}^n \times \mathbb{R}^N$, it is easy to see that θ is a smooth diffeomorphism of $\mathbb{R}^{1+N}$ onto itself; moreover, we can write

$$\begin{aligned}\Gamma(z_n; \zeta) = \Gamma(t_n, x_n; x, y) &= \int_{\mathbb{R}^p} \gamma_{\mathbb{G}}\big(s - t_n, (x_n, 0)^{-1} \star (y, \eta)\big)\,\mathrm{d}\eta \\ &\big(\text{by the change of variables } \eta = \Psi^{-1}_{x_n,y}(u)\big) \\ &= \int_{\mathbb{R}^p} (\gamma_{\mathbb{G}} \circ \theta)(z_n, u)\,\mathrm{d}u, \quad \text{for every } n \in \mathbb{N}.\end{aligned}$$

Our aim is now to pass to the limit as $n \to \infty$ in the above identity. To this end we first notice that, θ being a smooth diffeomorphism, we have

$$\|\theta(z, u)\| \to \infty, \quad \text{as } \|(z, u)\| \to \infty;$$

therefore, since $\gamma_{\mathbb{G}}$ vanishes at infinity (by Thm. 3.5.4 - (iv)), we get

$$\lim_{n\to\infty} \gamma_{\mathbb{G}}\big(\theta(z_n, u)\big) = 0, \quad \text{uniformly for } u \in \mathbb{R}^p.$$

As a consequence, there exists a constant $M > 0$ such that

$$\gamma_{\mathbb{G}}\big(\theta(z_n, u)\big) \le M, \quad \text{for every } n \in \mathbb{N} \text{ and every } u \in \mathbb{R}^p.$$

We now argue exactly as in the proof of statement (ii): since $\{N \le 1\}$ is compact, an obvious dominated convergence theorem gives

$$\lim_{n\to\infty} \int_{\{N\le 1\}} \gamma_{\mathbb{G}}\big(\theta(z_n, u)\big) = 0; \tag{3.5.20}$$

on the other hand, by exploiting once again estimate (3.5.7) in Rem. 3.5.7 and by recalling that N^{-Q} is integrable at infinity, we obtain

$$\lim_{n\to\infty} \int_{\{N>1\}} \gamma_{\mathbb{G}}\big(\theta(z_n, u)\big) = 0. \tag{3.5.21}$$

By gathering together (3.5.20) and (3.5.20) we finally conclude that $\Gamma(\cdot; \zeta)$ vanishes at infinity, and the proof is complete. □

Corollary 3.5.15. *Let the assumptions and the notations of Thm. 3.5.9 apply. Then, for every fixed $z = (t, x) \in \mathbb{R}^{1+n}$, one has*

(i) $\Gamma(z; \cdot) \in C^\infty(\mathbb{R}^{1+n} \setminus \{z\}, \mathbb{R})$;

(ii) $\mathcal{H}\Gamma(z; \zeta) = 0$ *for every* $\zeta = (s, y) \in \mathbb{R}^{1+n}$ *with* $\zeta \ne z$.

Proof. Since Γ is a global fundamental solution of $\mathcal{H}$, we have $\mathcal{H}\Gamma(z; \cdot) = -\mathrm{Dir}_z$ in $\mathcal{D}'(\mathbb{R}^{1+n})$ (see identity (1.3.8) on page 16); as a consequence, one has

$$\mathcal{H}\Gamma(z; \cdot) = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^{1+n} \setminus \{z\}).$$

From the C^∞-hypoellipticity of $\mathcal{H}$ (see Rem. 3.5.2 - (a)) and the continuity of $\Gamma(z; \cdot)$ out z (see Prop. 3.5.14 - (i)), we infer that $\Gamma(z; \cdot)$ is actually smooth out of z and that $\mathcal{H}\Gamma(z; \cdot) = 0$ on $\mathbb{R}^{1+n} \setminus \{z\}$. This ends the proof. □

Remark 3.5.16. The same remarks made about the stationary case apply now, *mutatis mutandis*, to the statements of Prop. 3.5.14 and of Cor. 3.5.15.

More precisely, the C^∞-hypoellipticity of $\mathcal{L}$ and the fact that Γ is a fundamental solution for $\mathcal{L}$ imply that, for every fixed $z \in \mathbb{R}^{1+N}$, there exists a smooth function $u_z \in C^\infty(\mathbb{R}^{1+N}, \mathbb{R})$ such that

$$\mathcal{H} u_z = 0 \text{ on } \mathbb{R}^{1+n} \setminus \{z\} \text{ and } u_z \equiv \Gamma(z; \cdot) \text{ a.e. on } \mathbb{R}^{1+n} \setminus \{z\}.$$

However, also in this case the point is that we need to know that u_z is everywhere identical to the integral function defined in Thm. 3.5.9, not only out of a set with vanishing Lebesgue measure (and depending on z).

Remark 3.5.17. Let the assumptions and the notations of Thm. 3.5.9 apply. Then Γ is the unique fundamental solution of $\mathcal{H}$ s.t., for every $z \in \mathbb{R}^{1+n}$,

$$\Gamma(z; \cdot) \in C(\mathbb{R}^{1+n} \setminus \{z\}, \mathbb{R}) \quad \text{and} \quad \Gamma(z; \zeta) \to 0 \text{ as } \|\zeta\| \to \infty.$$

This follows from Rem. 1.3.7 - (c), since the operator $\mathcal{H}$ is C^∞-hypoelliptic on every open subset of $\mathbb{R}^{1+n}$ and it satisfies the Weak Maximum Principle on every open and bounded subset of $\mathbb{R}^{1+n}$ (see Rem. 3.5.2).

Having established some interesting properties of Γ, we proceed in this section by proving that, as it happens for $\Gamma_{\mathbb{G}}$, the function

$$\Gamma^*(t, x; s, y) := \Gamma(s, y; t, x), \qquad (t, x) \neq (s, y)$$

provides a global fundamental solution for the adjoint operator $\mathcal{H}^* = \mathcal{L} + \partial_t$.

Theorem 3.5.18 (Fundamental Solution for $\mathcal{H}^*$). *Let the assumptions and the notations of Thm. 3.5.9 apply. Then the function*

$$\Gamma^*(t, x; s, y) := \Gamma(s, y; t, x), \qquad (t, x) \neq (s, y), \tag{3.5.22}$$

is a global fundamental solution for the adjoint operator $\mathcal{H}^* = \mathcal{L} + \partial_t$*. Moreover, if* d *is as in Thm. 3.5.6, for every* $(t, x), (s, y) \in \mathbb{R}^{1+n}$ *with* $t > s$ *one has*

$$\begin{aligned} \mathbf{c}^{-1} (t-s)^{-Q/2} \int_{\mathbb{R}^p} \exp\left(- \frac{\mathbf{c}\, d^2((y,0)^{-1} \star (x, \eta))}{t-s} \right) \mathrm{d}\eta \ &\leq \ \Gamma(s, y; t, x) \\ \leq \mathbf{c}\, (t-s)^{-Q/2} \int_{\mathbb{R}^p} \exp\left(- \frac{d^2((y,0)^{-1} \star (x, \eta))}{\mathbf{c}\,(t-s)} \right) \mathrm{d}\eta. & \end{aligned}$$

Remark 3.5.19. Let the assumptions and the notations of Thm. 3.5.18 apply. Since Γ is continuous out of the diagonal of $\mathbb{R}^{1+n} \times \mathbb{R}^{1+n}$ and $\Gamma(\cdot; z)$ vanishes at infinity for every $z = (t, x) \in \mathbb{R}^{1+n}$ (see Prop. 3.5.14), the function Γ^* defined in (3.5.22) is the unique fundamental solution for $\mathcal{H}^*$ such that

$$\Gamma^*(z; \cdot) \in C(\mathbb{R}^{1+n} \setminus \{z\}, \mathbb{R}) \quad \text{and} \quad \Gamma^*(z; \zeta) \to 0 \text{ as } \|\zeta\| \to \infty.$$

This follows once again from Rem. 1.3.7 - (c), since the operator $\mathcal{H}^*$ is C^∞-hypoelliptic on every open subset of $\mathbb{R}^{1+n}$ (by Hörmander's theorem) and it satisfies the Weak Maximum Principle on every open and bounded subset of $\mathbb{R}^{1+n}$ (the principal matrix of $\mathcal{H}^*$ being the same of $\mathcal{H}$).

The proof of Thm. 3.5.18 is not difficult, but it requires some preliminary results of independent interest. To begin with, we establish some further integrability properties of the fundamental solution Γ for $\mathcal{H}$.

Lemma 3.5.20. *Let the assumptions and the notations of Thm. 3.5.9 apply. Then the following facts hold true:*

(i) $\Gamma \in L^1_{\mathrm{loc}}(\mathbb{R}^{1+n} \times \mathbb{R}^{1+n})$;

(ii) *For every fixed* $\zeta = (s, y) \in \mathbb{R}^{1+n}$, *we have*

$$(t, x, \eta) \mapsto \Gamma_{\mathbb{G}}(t, x, 0; s, y, \eta) \in L^1_{\mathrm{loc}}(\mathbb{R}^{1+N}); \tag{3.5.23}$$

(iii) *For every fixed* $\zeta = (s, y) \in \mathbb{R}^{1+n}$, *we have* $\Gamma(\cdot; \zeta) \in L^1_{\mathrm{loc}}(\mathbb{R}^{1+n})$.

Proof. (i) Let $K_1, K_2 \subseteq \mathbb{R}^{1+n}$ be compact sets and let $T > 0$ such that

$$K_2 \subseteq [-T, T] \times \mathbb{R}^n.$$

By exploiting Fubini-Tonelli's theorem and Prop. 3.5.13, we obtain

$$\begin{aligned}
\int_{K_1 \times K_2} \Gamma(z; \zeta)\, \mathrm{d}z\, \mathrm{d}\zeta &\leq \int_{K_1} \left(\int_{-T}^{T} \left(\int_{\mathbb{R}^n} \Gamma(z; s, y)\, \mathrm{d}y \right) \mathrm{d}s \right) \mathrm{d}\zeta \\
&\leq \int_{K_1} \left(\int_{-T}^{T} 1\, \mathrm{d}s \right) \mathrm{d}\zeta \;\leq\; 2T \cdot \mathrm{meas}(K_1),
\end{aligned}$$

and this proves the integrability of Γ on $K_1 \times K_2 \subseteq \mathbb{R}^{1+n} \times \mathbb{R}^{1+n}$.

(ii) Let $\zeta = (s, y) \in \mathbb{R}^{1+n}$ be fixed and let $K \subseteq \mathbb{R}^{1+N}$ be a compact set. We then consider the map H_y defined as follows (see also Lem. 3.3.10):

$$H_y : \mathbb{R}^{1+N} \to \mathbb{R}^{1+N}, \quad H_y(t, x, \eta) := \big(s - t, (x, 0)^{-1} \star (y, \eta)\big).$$

By arguing as in the proof of Lem. 3.3.10 - (ii), it is easy to recognize that H_y defines a smooth diffeomorphism of $\mathbb{R}^{1+N}$ onto itself and that

$$\big|\det\big(\mathcal{J}_{H_y}(t, x, \eta)\big)\big| = 1.$$

Therefore, by performing the change of variables associated with H_y^{-1}, we get

$$\begin{aligned}
\int_K \Gamma_{\mathbb{G}}(t, x, 0; s, y, \eta)\, \mathrm{d}t\, \mathrm{d}x\, \eta &= \int_K \gamma_{\mathbb{G}}\big(s - t, (x, 0)^{-1} \star (y, \eta)\big)\, \mathrm{d}t\, \mathrm{d}x\, \eta \\
&= \int_{H_y^{-1}(K)} \gamma_{\mathbb{G}}(\tau, z)\, \mathrm{d}\tau\, \mathrm{d}z.
\end{aligned}$$

Since $\gamma_{\mathbb{G}} = \Gamma_{\mathbb{G}}(0; \cdot)$ is locally integrable on $\mathbb{R}^{1+N}$ and $H_y^{-1}(K)$ is compact, we immediately deduce the desired (3.5.23).

(iii) Let $K \subseteq \mathbb{R}^{1+n}$ be a fixed compact set. We define

$$T : \mathbb{R}^{1+N} \to \mathbb{R}^{1+N}, \quad T(t, x, u) := \big(t, x, \psi_{x,y}^{-1}(u)\big).$$

Since the map $(x, y, u) \mapsto \Psi_{x,y}^{-1}(u)$ is smooth and $\Psi_{x,y}^{-1}$ is a smooth diffeomorphism of $\mathbb{R}^p$ onto itself, it is readily seen that T defines a smooth diffeomorphism of $\mathbb{R}^{1+N}$; moreover, a direct computation shows that (see also Rem˙3.3.2)

$$\mathcal{J}_T(t, x, u) = 1, \quad \text{for every } (t, x, u) \in \mathbb{R}^{1+N}.$$

From this, by performing the change of variables associated with T, we obtain

$$\begin{aligned}\int_K \Gamma(z;\zeta)\,\mathrm{d}z &= \int_{K\times\mathbb{R}^p} \gamma_{\mathbb{G}}\big(s-t,(x,0)^{-1}\star(y,\eta)\big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\eta \\ &= \int_{K\times\mathbb{R}^p} \gamma_{\mathbb{G}}\big(s-t,(x,0)^{-1}\star(y,\Psi_{x,y}^{-1}(u))\big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}u \\ &= \int_{K\times\{N\le 1\}} \{\cdots\}\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}u + \int_{K\times\{N>1\}} \{\cdots\}\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\eta =: \mathrm{I}+\mathrm{II},\end{aligned}$$

where N denotes the homogeneous norm in $\mathbb{R}^p$ defined in (3.2.34). Now, since the product $K \times \{N \le 1\}$ is compact, we deduce from (3.5.23) that I is finite; on the other hand, by exploiting estimate (3.5.7) in Rem. 3.5.8, we get

$$\mathrm{II} \le \mathbf{M} \int_{K\times\{N>1\}} N^{-Q}(u)\,\mathrm{d}u \le \mathbf{M}\cdot\mathrm{meas}(K) \int_{\{N>1\}} N^{-Q}(u)\,\mathrm{d}u.$$

Since N^{-Q} is integrable at infinity we deduce that II is finite as well, and thus $\Gamma(\cdot;\zeta)$ is integrable on K. This ends the proof. □

Thanks to Lem. 3.5.20, we can now prove the following key result.

Proposition 3.5.21. *Let the assumptions and the notations of Thm. 3.5.9 apply. For every fixed* $\varphi \in C_0^\infty(\mathbb{R}^{1+n}, \mathbb{R})$*, the function*

$$\Lambda_\varphi : \mathbb{R}^{1+n} \longrightarrow \mathbb{R}, \qquad \Lambda_\varphi(\zeta) := \int_{\mathbb{R}^{1+n}} \Gamma(z;\zeta)\,\varphi(z)\,\mathrm{d}z, \tag{3.5.24}$$

is well-defined and it satisfies the following properties:

(i) $\Lambda_\varphi \in C^\infty(\mathbb{R}^{1+n}, \mathbb{R})$ *and* $\mathcal{H}(\Lambda_\varphi) = -\varphi$ *pointwise on* $\mathbb{R}^{1+n}$*;*

(ii) $\Lambda_\varphi(\zeta) \to 0$ *as* $\|\zeta\| \to \infty$.

Proof. By Lem. 3.5.20 - (iii), we know that $\Gamma(\cdot;\zeta) \in L^1_{\mathrm{loc}}(\mathbb{R}^{1+n})$ for every fixed $\zeta \in \mathbb{R}^{1+n}$; thus Λ_φ is well-defined. We now prove assertions (i) and (ii).

(i) We first show that the function Λ_φ is continuous on $\mathbb{R}^{1+n}$. To this end, let $\zeta_0 = (s_0, y_0) \in \mathbb{R}^{1+n}$ be fixed and let $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^{1+N}$ be a sequence converging to ζ_0 as $n \to \infty$. We then choose a real $T > 0$ such that

$$K_0 := \mathrm{supp}(\varphi) \subseteq [-T, T] \times \mathbb{R}^n,$$

and we consider the map $H_y : \mathbb{R}^{1+N} \to \mathbb{R}^{1+N}$ defined by (see Lem. 3.3.10):

$$H_y(t, x, \eta) := \big(s-t, C_y(x,\eta)\big) = \big(s-t,(x,0)^{-1}\star(y,\eta)\big).$$

As already pointed out in the proof of Lem. 3.5.20, H_y is a smooth diffeomorphism of $\mathbb{R}^{1+N}$ and, for every $(t, x, \eta) \in \mathbb{R}^{1+N}$ and every $y \in \mathbb{R}^n$, we have

$$\big|\det\big(\mathcal{J}_{H_y}(t,x,\eta)\big)\big| = 1.$$

Therefore, by means of such a map we can write, for every $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}
\Lambda_\varphi(\zeta_n) &= \int_{K_0} \Gamma(z;\zeta_n)\,\varphi(z)\,\mathrm{d}z \\
&\quad \big(\text{by Lem. 3.5.20 - (ii) and since } K_0 \subseteq [-T,T]\times\mathbb{R}^n\big) \\
&= \int_{[-T,T]\times\mathbb{R}^N} \Gamma_{\mathbb{G}}(t,x,0;s_n,y_n,\eta)\,\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\eta \\
&\quad \big(\text{by definition of } \Gamma_{\mathbb{G}}, \text{ see } (3.5.2)\big) \\
&= \int_{[-T,T]\times\mathbb{R}^N} \gamma_{\mathbb{G}}\big(s_n - t, (x,0)^{-1} \star (y_n,\eta)\big)\,\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\eta \\
&\quad \big(\text{by the change of variables } (t,x,\eta) = (s-\tau, C_y^{-1}(u,v))\big) \\
&= \int_{s_n-T}^{s_n+T}\int_{\mathbb{R}^N} \gamma_{\mathbb{G}}(\tau,u,v)\,\varphi\big(s_n-\tau, C_{y_n}^{-1}(u,v)\big)\,\mathrm{d}\tau\,\mathrm{d}u\,\mathrm{d}v.
\end{aligned}$$

We now aim to pass to the limit as $n \to \infty$ in the above identity. To this end we first notice that, $\{\zeta_n\}_{n\in\mathbb{N}}$ being bounded, there exists a real $T_0 > 0$ s.t.

$$[s_n - T, s_n + T] \subseteq [-T_0, T_0], \quad \text{for every } n \in \mathbb{N}\cup\{0\};$$

as a consequence, for every $n \in \mathbb{N}\cup\{0\}$ we can write

$$\Lambda_\varphi(\zeta_n) = \int_{[-T_0,T_0]\times\mathbb{R}^N} \gamma_{\mathbb{G}}(\tau,u,v)\,\varphi\big(s_n-\tau, C_{y_n}^{-1}(u,v)\big)\,\mathrm{d}\tau\,\mathrm{d}u\,\mathrm{d}v.$$

On the other hand, since the map $(y,u,v) \mapsto C_y^{-1}(u,v)$ is smooth on $\mathbb{R}^n \times \mathbb{R}^N$ (see identity (3.3.7)) and since, by assumptions, $\varphi \in C_0^\infty(\mathbb{R}^{1+n},\mathbb{R})$, one has

$$\lim_{n\to\infty} \varphi\big(s_n-\tau, C_{y_n}^{-1}(u,v)\big) = \varphi\big(s_0-\tau, C_{y_0}^{-1}(u,v)\big)$$

and there exists a real constant $M > 0$ such that

$$|\varphi\big(s_n-\tau, C_{y_n}^{-1}(u,v)\big)| \leq M, \quad \forall\, n\in\mathbb{N} \text{ and } \forall\,(\tau,u,v)\in\mathbb{R}^{1+N}.$$

Thus, $\gamma_{\mathbb{G}} = \Gamma_{\mathbb{G}}(0;\cdot)$ being integrable on $[-T_0,T_0]\times\mathbb{R}^N$ (as it follows from Thm. 3.5.4 - (v)), we are entitled to apply the Lebesgue Dominated Convergence Theorem, which shows that $\Lambda_\varphi(\zeta_n) \to \Lambda_\varphi(\zeta_0)$ as $n\to\infty$. Due to the arbitrariness of $\zeta_0 \in \mathbb{R}^{1+n}$, we conclude that $\Lambda_\varphi \in C^(\mathbb{R}^{1+n},\mathbb{R})$, as desired.

We now claim that $\mathcal{H}(\Lambda_\varphi) = -\varphi$ in $\mathcal{D}'(\mathbb{R}^{1+n})$. Indeed, if $\psi \in C_0^\infty(\mathbb{R}^{1+n},\mathbb{R})$ is fixed, by applying Fubini-Tonelli's theorem (and recalling that Γ is a global fundamental solution for $\mathcal{H}$), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^{1+n}} \Lambda_\varphi(\zeta)\,\mathcal{H}^*\psi(\zeta)\,\mathrm{d}\zeta &= \int_{\mathbb{R}^{1+n}}\left(\int_{\mathbb{R}^{1+n}} \Gamma(z;\zeta)\,\mathcal{H}^*\psi(\zeta)\,\mathrm{d}\zeta\right)\varphi(z)\,\mathrm{d}z \\
&\quad \big(\text{since } \mathcal{H}\Gamma(z;\cdot) = -\mathrm{Dir}_z\big) \\
&= -\int_{\mathbb{R}^{1+n}} \varphi(z)\,\psi(z)\,\mathrm{d}z,
\end{aligned}$$

which precisely mean that $\mathcal{H}\Lambda_\varphi = -\varphi$ in $\mathcal{D}'(\mathbb{R}^{1+n})$. From this, since the operator $\mathcal{H}$ is C^∞-hypoelliptic on every open subset of $\mathbb{R}^{1+n}$ and $\Lambda_\varphi \in C(\mathbb{R}^{1+n},\mathbb{R})$, we deduce that Λ_φ is actually smooth on the whole of $\mathbb{R}^{1+n}$ and that

$$\mathcal{H}(\Lambda_\varphi) = -\varphi \text{ point-wise on } \mathbb{R}^{1+n}.$$

(ii) By definition of Λ_φ, for every $\zeta \notin \mathrm{supp}(\varphi)$ we have

$$|\Lambda_\varphi(\zeta)| \leq \sup_{z \in K_0} \Gamma(z;\zeta) \cdot \int_{\mathbb{R}^{1+n}} |\varphi(z)|\, \mathrm{d}z;$$

thus, since we know from Prop. 3.5.14 that $\sup_{z\in K_0}\Gamma(z;\zeta) \to 0$ as $\|\zeta\| \to \infty$, we conclude that Λ_φ vanishes at infinity. This ends the proof. □

Corollary 3.5.22. *Let the assumptions and the notations of Prop. 3.5.21 apply. For every $\varphi \in C^\infty(\mathbb{R}^{1+n}, \mathbb{R})$ and every $\zeta \in \mathbb{R}^{1+n}$, we have*

$$\Lambda_{\mathcal{H}\varphi}(\zeta) = \int_{\mathbb{R}^{1+n}} \Gamma(z;\zeta)\, \mathcal{H}\varphi(z)\, \mathrm{d}z = -\varphi(\zeta). \tag{3.5.25}$$

Proof. We consider the function $u : \mathbb{R}^{1+n} \to \mathbb{R}$ defined as follows

$$u(\zeta) := \Lambda_{\mathcal{H}\varphi}(\zeta) + \varphi(\zeta), \quad \zeta = (s,y) \in \mathbb{R}^{1+n}.$$

From Prop. 3.5.21 - (i), we infer that $u \in C^\infty(\mathbb{R}^{1+n}, \mathbb{R})$ and

$$\mathcal{H}u = \mathcal{H}(\Lambda_{\mathcal{H}\varphi}) + \mathcal{H}\varphi = -\mathcal{H}\varphi + \mathcal{H}\varphi = 0, \quad \text{on } \mathbb{R}^{1+n};$$

moreover, since φ is compactly supported and $\Lambda_{\mathcal{H}\varphi}$ vanishes at infinity (see Prop. 3.5.21 - (ii)), one has $u(\zeta) \to 0$ as $\|\zeta\| \to \infty$. By summing up, u is a smooth $\mathcal{H}$-harmonic function on $\mathbb{R}^{1+n}$ vanishing at infinity; therefore, since $\mathcal{H}$ satisfies the Weak Maximum Principle on every open and bounded subset of $\mathbb{R}^{1+n}$, we have $u \equiv 0$ on $\mathbb{R}^{1+n}$. By the very definition of u, we then get

$$\Lambda_{\mathcal{H}\varphi} = -\varphi, \quad \text{on } \mathbb{R}^{1+n},$$

which is precisely the desired (3.5.25). This ends the proof. □

With Cor. 3.5.22 at hands, we are finally in a position to prove Thm. 3.5.18.

Proof (of Thm. 3.5.18). We recall that we have to prove the following fact:

$$\Gamma^*(z;\zeta) = \Gamma(\zeta;z) \text{ is a global fundamental solution for } \mathcal{H}^*.$$

Obviously, Γ^* is defined out of the diagonal of $\mathbb{R}^{1+n} \times \mathbb{R}^{1+n}$ and, by Lem. 3.5.20, $\Gamma^*(z;\cdot) = \Gamma(\cdot;z) \in L^1_{\mathrm{loc}}(\mathbb{R}^{1+n})$ for every fixed $z \in \mathbb{R}^{1+n}$; thus, according with Def. 1.3.5 on page 15, we are left to show that, for every $z \in \mathbb{R}^{1+n}$, one has

$$\mathcal{H}^*\Gamma^*(z;\cdot) = -\mathrm{Dir}_z, \quad \text{in } \mathcal{D}'(\mathbb{R}^{1+n}). \tag{3.5.26}$$

On the other hand, by Cor. 3.5.22, for every $\varphi \in C_0^\infty(\mathbb{R}^{1+n}, \mathbb{R})$ we have

$$\begin{aligned}(\mathcal{H}^*\Gamma^*(z;\cdot))(\varphi) &= \int_{\mathbb{R}^{1+n}} \Gamma^*(z;\zeta)\, \mathcal{H}\varphi(\zeta)\, \mathrm{d}\zeta = \int_{\mathbb{R}^{1+n}} \Gamma(\zeta;z)\, \mathcal{H}\varphi(\zeta)\, \mathrm{d}\zeta \\ &= \Lambda_{\mathcal{H}\varphi}(z) \overset{(3.5.25)}{=} -\varphi(z) = -\mathrm{Dir}_z(\varphi),\end{aligned}$$

which is precisely the needed (3.5.26). This ends the proof. □

Now we have established Thm. 3.5.18, we continue in this section by exploiting such a result to obtain some further properties of Γ and Γ^*.

To begin with, we prove the following *regularity theorems*.

Corollary 3.5.23. *Let the assumptions and the notations of Thm. 3.5.18 apply. Then, for every fixed $z = (t, x) \in \mathbb{R}^{1+n}$, one has*

(i) $\Gamma^*(z; \cdot) \in C^\infty(\mathbb{R}^{1+n} \setminus \{z\}, \mathbb{R})$;

(ii) $\mathcal{H}^*\Gamma^*(z; \zeta) = 0$ *for every* $\zeta = (s, y) \in \mathbb{R}^{1+n}$ *with* $\zeta \neq z$.

Proof. Since, by Thm. 3.5.18, Γ^* is a global fundamental solution for $\mathcal{H}^*$, we have $\mathcal{H}^*\Gamma^*(z; \cdot) = -\mathrm{Dir}_z$ in $\mathcal{D}'(\mathbb{R}^{1+n})$; in particular, one has

$$\mathcal{H}^*\Gamma^*(z; \cdot) = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^{1+n} \setminus \{z\}).$$

From this, since $\mathcal{H}^*$ is C^∞-hypoelliptic on every open subset of $\mathbb{R}^{1+n}$ and $\Gamma^*(z; \cdot)$ is continuous out of z (see Rem. 3.5.19), we infer that $\Gamma^*(z; \cdot)$ is smooth on $\mathbb{R}^{1+n} \setminus \{z\}$ and $\mathcal{H}^*\Gamma^*(z; \cdot) = 0$ point-wise on $\mathbb{R}^{1+n} \setminus \{z\}$. This ends the proof. □

Theorem 3.5.24 (Smoothness of Γ)**.** *Let Γ be the fundamental solution of $\mathcal{H}$ introduced in Thm. 3.5.9. Then Γ is smooth out of the diagonal of $\mathbb{R}^{1+n} \times \mathbb{R}^{1+n}$.*

Proof. As we did in the proof of Thm. 3.3.17, we consider the $2m$ vector fields $\tilde{X}_1, \ldots, \tilde{X}_m, \tilde{Y}_1, \ldots, \tilde{Y}_m$, operating on $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, defined as follows:

$$\tilde{X}_j := \sum_{i=1}^n (X_j I)_i(x)\, \partial_{x_i}, \quad \tilde{Y}_j := \sum_{i=1}^n (X_j I)_i(y)\, \partial_{y_i} \quad (j = 1, \ldots, m).$$

We then introduce the following linear PDO on $\mathbb{R}^{1+n} \times \mathbb{R}^{1+n}$:

$$\tilde{\mathcal{H}} := \sum_{j=1}^m \tilde{X}_j^2 + \partial_t + \sum_{j=1}^m \tilde{Y}_j^2 - \partial_s.$$

Obviously, $\tilde{\mathcal{H}}$ has smooth coefficients; moreover, since $[\tilde{X}_i, \tilde{Y}_j] = 0$ for every $i, j = \{1, \ldots, m\}$, it is readily seen that $\tilde{\mathcal{H}}$ is a Hörmander operator on the whole $\mathbb{R}^{1+n} \times \mathbb{R}^{1+n}$, hence C^∞-hypoelliptic on the same set. From this, since $\Gamma(z; \zeta)$ is continuous for $z \neq \zeta \in \mathbb{R}^{1+n}$ and, by Cor.s 3.5.15 and 3.5.23,

$$\tilde{\mathcal{H}}\Gamma(z; \zeta) = \mathcal{H}^*\Gamma(\cdot; \zeta) + \mathcal{H}\Gamma(z; \cdot) = \mathcal{H}^*\Gamma^*(\zeta; \cdot) + \mathcal{H}\Gamma(z; \cdot) = 0,$$

for every $z, \zeta \in \mathbb{R}^{1+n}$ with $z \neq \zeta$, we infer that Γ is actually smooth out of the diagonal of $\mathbb{R}^{1+n} \times \mathbb{R}^{1+n}$, as desired. This ends the proof. □

The next result shows that the fundamental solution Γ of $\mathcal{H}$ is actually symmetric with respect to the space variables $x, y \in \mathbb{R}^n$.

Theorem 3.5.25 (Spacial symmetry of Γ)**.** *Let Γ be the fundamental solution of $\mathcal{H}$ introduced in Thm. 3.5.9. Then, for every $(t, x) \neq (s, y) \in \mathbb{R}^{1+n}$ we have*

$$\Gamma(t, x; s, y) = \Gamma(t, y; s, x). \tag{3.5.27}$$

Proof. For the sake of clarity, we split the proof into three steps.

Step I: We first prove that the function G defined by

$$G(t, x; s, y) := \Gamma(t, y; s, x), \qquad (t, x) \neq (s, y), \tag{3.5.28}$$

is a global fundamental solution for $\mathcal{H}$. To this end we first notice that, obviously, G is defined out of the diagonal of $\mathbb{R}^{1+n} \times \mathbb{R}^{1+n}$; thus, according with Def. 1.3.5, we have to show that, for every fixed $z = (t, x) \in \mathbb{R}^{1+n}$, one has

(a) $G(z;\cdot) \in L^1_{\text{loc}}(\mathbb{R}^{1+n})$;

(b) $\mathcal{H}G(z;\cdot) = -\text{Dir}_z$ in $\mathcal{D}'(\mathbb{R}^{1+n})$.

As for assertion (a), let $K \subseteq \mathbb{R}^{1+n}$ be a compact set and let $T, r > 0$ be s.t.

$$K \subseteq [-T, T] \times \overline{B}(0, r) =: C(T, r).$$

Since $\Gamma \geq 0$ and $\Gamma(\cdot;\zeta) \in L^1_{\text{loc}}(\mathbb{R}^{1+n})$ for every $\zeta \in \mathbb{R}^{1+n}$, one then has

$$\begin{aligned}
\int_K G(t,x;s,y)\,\mathrm{d}s\,\mathrm{d}y &\leq \int_{C(T,r)} G(t,x;s,y)\,\mathrm{d}s\,\mathrm{d}y \\
&= \int_{C(T,r)} \Gamma(t,y;s,x)\,\mathrm{d}s\,dy \\
&\quad \big(\text{by identities (3.5.8) and (3.5.9) in Rem. 3.5.10}\big) \\
&= \int_{C(T,r)} \Gamma(t-s,y;0,x)\,\mathrm{d}s\,\mathrm{d}y \\
&\quad \big(\text{by the change of variables } (s,y) = (-\tau + t, y)\big) \\
&= \int_{t-T}^{t+T} \int_{\overline{B}(0,r)} \Gamma(\tau,y;0,x)\,\mathrm{d}\tau\,\mathrm{d}y < \infty.
\end{aligned}$$

We now turn to prove statement (b). To this end, let $\varphi \in C_0^\infty(\mathbb{R}^{1+n}, \mathbb{R})$ be fixed and let $\psi(s,y) := \varphi(-s,y)$. Since $\Gamma^*(w;\zeta) = \Gamma(\zeta;w)$ is a global fundamental solution for $\mathcal{H}^*$ (as we know from Thm. 3.5.18), we have

$$\begin{aligned}
-\varphi(t,x) = -\psi(-t,x) &= \int_{\mathbb{R}^{1+n}} \Gamma(s,y;-t,x)\,\mathcal{H}\psi(s,y)\,\mathrm{d}s\,\mathrm{d}y \\
&\overset{(3.5.8)}{=} \int_{\mathbb{R}^{1+n}} \Gamma(0,y;-t-s,x)\,\mathcal{H}\psi(s,y)\,\mathrm{d}s\,\mathrm{d}y \\
&\quad \big(\text{by the change of variables } (s,y) = (-\tau, y)\big) \\
&= \int_{\mathbb{R}^{1+n}} \Gamma(0,y;\tau-t,x)\,(\mathcal{H}\psi)(-\tau,y)\,\mathrm{d}\tau\,\mathrm{d}y \\
&\quad \big(\text{since } (\mathcal{H}\psi)(-\tau,y) = \mathcal{H}^*\varphi(\tau,y)\big) \\
&= \int_{\mathbb{R}^{1+n}} \Gamma(t,y;\tau,x)\,\mathcal{H}^*\varphi(\tau,y)\,\mathrm{d}\tau\,\mathrm{d}y \\
&= \int_{\mathbb{R}^{1+n}} G(t,x;\tau,y)\,\mathcal{H}^*\varphi(\tau,y)\,\mathrm{d}\tau\,\mathrm{d}y,
\end{aligned}$$

and this proves that $\mathcal{H}G(z;\cdot) = -\text{Dir}_z$ in $\mathcal{D}'(\mathbb{R}^{1+n})$, as desired.

STEP II: In this step we show that, for very $z = (t,x) \in \mathbb{R}^{1+n}$, one has

$$G(z;\cdot) \in C(\mathbb{R}^{1+n} \setminus \{z\}, \mathbb{R}) \;\text{ and }\; G(z;\zeta) \to 0 \text{ as } \|\zeta\| \to \infty.$$

On the one hand, the continuity of $G(z;\cdot)$ out of z is a direct consequence of the continuity of Γ out of the diagonal of $\mathbb{R}^{1+n} \times \mathbb{R}^{1+n}$; on the other hand, since $\Gamma(\cdot;\zeta)$ vanishes at infinity and, by Rem. 3.5.10, we have

$$G(t,x;s,y) = \Gamma(t,y;s,x) = \Gamma(t-s,y;0,x), \quad \text{for every } (s,y) \neq (t,x),$$

we immediately deduce that $G(z;\cdot)$ vanishes at infinity as well.

Step III: We are now ready to prove identity (3.5.28). Indeed, by Steps I and II, G is a fundamental solution for $\mathcal{H}$ such that, for every $z \in \mathbb{R}^{1+n}$,

$$G(z;\cdot) \in C(\mathbb{R}^{1+n} \setminus \{z\}, \mathbb{R}) \;\text{ and } G(z;\zeta) \to 0 \text{ as } \|\zeta\| \to \infty;$$

thus, by the uniqueness property of Γ in Rem. 3.5.17, we conclude that

$$\Gamma(t,x;s,y) = G(t,x;s,y) = \Gamma(t,x;s,y), \quad \text{for every } (t,x) \neq (s,y).$$

This ends the proof. □

Corollary 3.5.26. *Let Γ and Γ^* be the global fundamental solutions of $\mathcal{H}$ and $\mathcal{H}^*$, respectively. Then, for every $(t,x) \neq (s,y) \in \mathbb{R}^{1+n}$ we have*

$$\Gamma^*(t,x;s,y) = \Gamma^*(t,y;s,x) = \Gamma(s,x;t,y). \tag{3.5.29}$$

Proof. Let $(t,x) \neq (s,y) \in \mathbb{R}^{1+n}$ be fixed. By Thm. 3.5.25, we have

$$\Gamma^*(t,x;s,y) = \Gamma(s,y;t,x) \overset{(3.5.27)}{=} \Gamma(s,x;t,y) = \Gamma^*(t,y;s,x),$$

which is precisely the desired (3.5.29). This ends the proof. □

Thanks to Cor. 3.5.26, we can give an easy proof of the following theorem.

Theorem 3.5.27 (Properties of Γ^*)**.** *Let Γ^* be the global fundamental solution of the operator $\mathcal{H}^*$. Then the following facts hold true:*

(i) $\Gamma^* \geq 0$ *on its domain and, for every $(t,x),(s,y) \in \mathbb{R}^{1+n}$, we have*

$$\Gamma^*(t,x;s,y) = 0 \textit{ if and only if } s \geq t.$$

(ii) *For every $(t,x) \neq (s,y) \in \mathbb{R}^{1+n}$, the function Γ^* depends on t and s only through the difference $s-t$: in fact, we have*

$$\Gamma^*(t,x;s,y) = \Gamma^*(0,x;s-t,y).$$

(iii) *For every $\lambda > 0$ and every $(t,x) \neq (s,y) \in \mathbb{R}^{1+n}$, we have*

$$\Gamma^*(\lambda^2 t, \delta_\lambda(x); \lambda^2 s, \delta_\lambda(y)) = \lambda^{-q}\,\Gamma^*(t,x;s,y).$$

(iv) *For every fixed compact set $K \subseteq \mathbb{R}^{1+n}$, we have*

$$\sup_{z\in K} \Gamma^*(z;\zeta) \to 0 \quad \textit{as } \|\zeta\| \to \infty.$$

(v) *For every fixed $(t,x) \in \mathbb{R}^{1+n}$ we have*

$$\int_{\mathbb{R}^n} \Gamma^*(t,x;s,y)\,\mathrm{d}y = 1, \quad \textit{for every } s < t.$$

Proof. Statements (i)-to-(v) are straightforward consequence of the analogous properties of Γ established so far and of identity (3.5.29) in Cor. 3.5.26. □

Remark 3.5.28. Let Γ be the global fundamental solutions of $\mathcal{H}$ introduced in Thm. 3.5.9. By combining statement (v) of Thm. 3.5.27 with statement (ii) of Prop. 3.5.14 we recognize that, for every compact set $K \subseteq \mathbb{R}^{1+n}$,

$$\lim_{\|\zeta\|\to\infty}\big(\sup_{z\in K}\Gamma(z;\zeta)\big) = \lim_{\|\zeta\|\to\infty}\big(\sup_{z\in K}\Gamma(\zeta;z)\big) = 0.$$

Remark 3.5.29 ($\Gamma^*_{\mathbb{G}}$ lifts Γ^*)**.** Let Γ^* be the global fundamental solution of the operator $\mathcal{H}^*$. Then, for every $(t,x) \neq (s,y) \in \mathbb{R}^{1+n}$ we have

$$\Gamma^*(t,x;s,y) = \int_{\mathbb{R}^p} \Gamma^*_{\mathbb{G}}(t,x,0;s,y,\eta)\,\mathrm{d}\eta, \tag{3.5.30}$$

where $\Gamma^*_{\mathbb{G}}$ is the fundamental solution of the operator $\mathcal{H}^*_{\mathbb{G}} = \mathcal{L}_{\mathbb{G}} + \partial_t$ on $\mathbb{G}$. In fact, from Cor. 3.5.26 and the definition Γ in Thm. 3.5.9 we obtain

$$\begin{aligned}
\Gamma^*(t,x;s,y) &\overset{(3.5.29)}{=} \Gamma(s,x;t,y) = \int_{\mathbb{R}^p} \Gamma_{\mathbb{G}}(s,x,0;t,y,\eta)\,\mathrm{d}\eta \\
&= \int_{\mathbb{R}^p} \gamma_{\mathbb{G}}\big(t-s,(x,0)^{-1}\star(y,\eta)\big)\,\mathrm{d}\eta \\
&\quad \big(\text{by Thm. 3.5.4 - (ii)}\big) \\
&= \int_{\mathbb{R}^p} \gamma_{\mathbb{G}}\big(t-s,(y,\eta)^{-1}\star(x,0)\big)\,\mathrm{d}\eta \\
&= \int_{\mathbb{R}^p} \Gamma_{\mathbb{G}}(s,y,\eta;t,x,0)\,\mathrm{d}\eta \\
&\quad \big(\text{by definition of } \Gamma^*_{\mathbb{G}}, \text{ see (3.5.3)}\big) \\
&= \int_{\mathbb{R}^p} \Gamma^*_{\mathbb{G}}(t,x,0;s,y,\eta)\,\mathrm{d}\eta.
\end{aligned}$$

Remark 3.5.30. Let Γ^* be the global fundamental solution of the operator $\mathcal{H}^*$. Then Γ^* satisfies the following *dual statement* of Prop. 3.5.21: *for every fixed* $\varphi \in C_0^\infty(\mathbb{R}^{1+n},\mathbb{R})$*, the function* $\Lambda^*_\varphi : \mathbb{R}^{1+n} \to \mathbb{R}$ *defined by*

$$\Lambda^*_\varphi(\zeta) := \int_{\mathbb{R}^{1+n}} \Gamma^*(z,\zeta)\,\varphi(z)\,\mathrm{d}z, \qquad \zeta = (s,y) \in \mathbb{R}^{1+n},$$

is well-defined and it satisfies the following properties:

(i) $\Lambda^*_\varphi \in C^\infty(\mathbb{R}^{1+n},\mathbb{R})$ and $\mathcal{H}^*(\Lambda^*_\varphi) = -\varphi$ pointwise on $\mathbb{R}^{1+n}$;

(ii) $\Lambda^*_\varphi(\zeta) \to 0$ as $\|\zeta\| \to \infty$.

Indeed, since $\Gamma^*(z;\cdot)$ is locally integrable on $\mathbb{R}^{1+n}$, we see that Λ^*_φ is well-defined;

moreover, by identity (3.5.29), for every $\zeta = (s,y) \in \mathbb{R}^{1+n}$ we can write

$$\begin{aligned}
\Lambda^*_\varphi(\zeta) &= \int_{\mathbb{R}^{1+n}} \Gamma^*(t,x;s,y)\,\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x \\
&\overset{(3.5.29)}{=} \int_{\mathbb{R}^{1+n}} \Gamma(s,x;t,y)\,\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x \\
&\quad \big(\text{by identity (3.5.9) in Rem. 3.5.10}\big) \\
&= \int_{\mathbb{R}^{1+n}} \Gamma(-t,x;-s,y)\,\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x \\
&\quad \big(\text{by the change of variables } (t,x) = (-\tau,u)\big) \\
&= \int_{\mathbb{R}^{1+n}} \Gamma(\tau,u;-s,y)\,\varphi(-\tau,u)\,\mathrm{d}\tau\,\mathrm{d}u \\
&= \Lambda_\psi(-s,y),
\end{aligned} \tag{3.5.31}$$

where we have set $\psi(s,y) := \varphi(-s,y)$. From this, since Λ_ψ is smooth on $\mathbb{R}^{1+n}$ and it vanishes at infinity (see Prop. 3.5.21), we immediately infer that

$$\Lambda^*_\varphi \in C^\infty(\mathbb{R}^{1+n},\mathbb{R}) \;\text{ and }\; \Lambda^*_\varphi(\zeta) \to 0 \text{ as } \|\zeta\| \to \infty.$$

On the other hand, since we know from Prop. 3.5.21 - (ii) that $\mathcal{H}(\Lambda_\psi) = -\psi$ on $\mathbb{R}^{1+n}$, from the above (3.5.31) we also get, for every $\zeta = (s,y) \in \mathbb{R}^{1+n}$,

$$\begin{aligned}
\mathcal{H}^*(\Lambda^*_\varphi)(\zeta &= \mathcal{H}^*(\Lambda^*_\varphi)(s,y) = \mathcal{H}^*\big((\tau,u) \mapsto \Lambda_\psi(-\tau,u)\big)(s,y) \\
&= \mathcal{H}(\Lambda_\psi)(-s,y) = -\psi(-s,y) \\
&= -\varphi(s,y).
\end{aligned}$$

3.5.2 The Cauchy problem for $\mathcal{H}$

Now we have established several qualitative properties of the functions Γ and Γ^*, we finally conclude this section by briefly studying the existence and the uniqueness of (classical) solutions of the Cauchy problem for $\mathcal{H}$.

To begin with, we remind the following definition.

Definition 3.5.31. Let $\varphi \in C(\mathbb{R}^n,\mathbb{R})$ be fixed and let $\Omega :=]0,\infty[\, \times \mathbb{R}^n$. We say that a function $u : \Omega \to \mathbb{R}$ is a (classical) solution of the Cauchy problem

$$\begin{cases} \mathcal{H}u = 0, & \text{in } \Omega; \\ u(0,x) = \varphi(x), & \text{for every } x \in \mathbb{R}^n \end{cases} \tag{3.5.32}$$

if the following conditions are satisfied:

(i) $u \in C^2(\Omega,\mathbb{R})$ and $\mathcal{H}u(t,x) = 0$ for every $(t,x) \in \Omega$;

(ii) For every fixed $x \in \mathbb{R}^n$, we have

$$\lim_{t\to 0^+} u(t,x) = \varphi(x). \tag{3.5.33}$$

Remark 3.5.32. Let $\varphi \in C(\mathbb{R}^n, \mathbb{R})$ be fixed and let $u \in C^2(\Omega, \mathbb{R})$ be a classical solution of the Cauchy problem (3.5.32), according to Def. 3.5.31.

It is worth noting that, since the operator $\mathcal{H}$ is C^∞-hypoelliptic on every open subset of $\mathbb{R}^n$ and since, by definition, $\mathcal{H}u = 0$ point-wise in Ω, the function u is actually *smooth* on its domain. In other words, any classical solution of the Cauchy problem (3.5.32) is actually a smooth function.

We then have the following notable result.

Theorem 3.5.33. *Let* $\varphi \in C(\mathbb{R}^n, \mathbb{R})$ *be* bounded. *Then the function*

$$u : \Omega =]0, \infty[\times \mathbb{R}^n \longrightarrow \mathbb{R}, \qquad u(t,x) := \int_{\mathbb{R}^n} \Gamma(0, y; t, x)\, \varphi(y)\, \mathrm{d}y, \tag{3.5.34}$$

is a classical solution of the Cauchy problem (3.5.31), *further satisfying*

$$|u(t,x)| \leq \|\varphi\|_\infty, \quad \textit{for every } (t,x) \in \Omega. \tag{3.5.35}$$

Proof. First of all, by Thm. 3.5.1 - (ii) and (vii), the function u is well-defined and it satisfies the estimate (3.5.35): indeed, for every $(t,x) \in \Omega$ we have

$$\begin{aligned}\int_{\mathbb{R}^n} \Gamma(0,y;t,x)|\varphi(y)|\,\mathrm{d}y &\leq \|\varphi\|_\infty \int_{\mathbb{R}^n} \Gamma(0,y;t,x)\,\mathrm{d}y \\ &\quad \big(\text{by Thm. 3.5.1 - (ii) and (vii)}\big) \\ &= \|\varphi\|_\infty \int_{\mathbb{R}^n} \Gamma(0,x;t,y)\,\mathrm{d}y = \|\varphi\|_\infty.\end{aligned}$$

To complete the proof of the theorem, we then consider the following steps.

STEP I: In this step we prove that u is continuous on Ω. To this end, let $z_0 = (t_0, x_0) \in \Omega$ be arbitrarily fixed and let $\rho > 0$ be such that

$$K := [t_0 - r, t_0 + r] \times \overline{B}(x_0, r) \subseteq \Omega.$$

Moreover, let $\{z_n\}_{n\in\mathbb{N}} \subseteq K$ be a sequence converging to z_0 as $n \to \infty$. By arguing as in the proof of Lem. 3.5.20 - (ii), one can easily recognizes that

$$(y,\eta) \mapsto \Gamma_{\mathbb{G}}(0,y,0;t,x,\eta) \in L^1_{\mathrm{loc}}(\mathbb{R}^N), \quad \text{for every } (t,x) \in \mathbb{R}^{1+n};$$

therefore, by Fubini's theorem, for every $n \in \mathbb{N} \cup \{0\}$ we can write

$$\begin{aligned}u(z_n) &= u(t_n, x_n) = \int_{\mathbb{R}^n} \Gamma(0,y;t_n,x_n)\,\varphi(y)\,\mathrm{d}y \\ &\quad \big(\text{by definition of } \Gamma, \text{ see Thm. 3.5.9}\big) \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^p} \Gamma_{\mathbb{G}}(0,y,0;t_n,x_n,\eta)\,\mathrm{d}\eta \right) \varphi(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^N} \Gamma_{\mathbb{G}}(0,y,0;t_n,x_n,\eta)\,\varphi(y)\,\mathrm{d}y\,\mathrm{d}\eta \\ &\overset{(3.5.2)}{=} \int_{\mathbb{R}^N} \gamma_{\mathbb{G}}\big(t_n, (y,0)^{-1} \star (x_n,\eta)\big)\,\varphi(y)\,\mathrm{d}y\,\mathrm{d}\eta.\end{aligned}$$

We then consider, for fixed $x \in \mathbb{R}^n$, the map defined as follows:

$$C_x : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \qquad C_x(y,\eta) := (y,0)^{-1} \star (x,\eta).$$

As already pointed out in the proof of Lem. 3.5.20, C_x is a smooth diffeomorphism of $\mathbb{R}^N$ such that $|\det \mathcal{J}_{C_x}(y,\eta)| = 1$ for every $(y,\eta) \in \mathbb{R}^N$; therefore, by performing the change of variables associated with C_x^{-1} we obtain

$$u(z_n) = \int_{\mathbb{R}^N} \gamma_{\mathbb{G}}(t_n,u,v)\,\varphi\big(C_{x_n}^{-1}(u,v)\big)\,\mathrm{d}u\mathrm{d}v, \quad \text{for every } n \in \mathbb{N}\cup\{0\}. \tag{3.5.36}$$

Our aim is now pass to the limit as $n \to \infty$ in the above (3.5.36). To this end we first notice that, since $\varphi \in C(\mathbb{R}^n,\mathbb{R})$, $\gamma_{\mathbb{G}}$ is smooth on $\mathbb{R}^{1+N}\setminus\{0\}$ and

$$\{t_n\}_{n\in\mathbb{N}} \subseteq [t_0-r,t_0+r] \subseteq\,]0,\infty[,$$

we have (remind that $(x,u,v) \mapsto C_x^{-1}(u,v)$ is continuous on $\mathbb{R}^n \times \mathbb{R}^N$)

$$\lim_{n\to\infty} \gamma_{\mathbb{G}}(t_n,u,v)\,\varphi\big(C_{x_n}^{-1}(u,v)\big) = \gamma_{\mathbb{G}}(t_0,u,v)\,\varphi\big(C_{x_0}^{-1}(u,v)\big), \quad \forall\,(u,v)\in\mathbb{R}^N.$$

Moreover, by exploiting the Gaussian estimates for $\gamma_{\mathbb{G}}$ in Thm. 3.5.6 (and reminding that φ is bounded), for every $n \in \mathbb{N}$ and every $(u,v)\in\mathbb{R}^N$ we obtain

$$\begin{aligned}\big|\gamma_{\mathbb{G}}(t_n,u,v)\,\varphi\big(C_{x_n}^{-1}(u,v)\big)\big| &\le \|\varphi\|_\infty\,\gamma_{\mathbb{G}}(t_n,u,v)\\ &\le \mathbf{c}\,\|\varphi\|_\infty (t_n)^{-Q/2}\exp\left(-\frac{d^2(u,v)}{\mathbf{c}\,t_n}\right)\\ &\big(\text{since } \{t_n\}_{n\in\mathbb{N}} \subseteq [t_0-r,t_0+r]\big)\\ &\le \mathbf{c}\,(t_0+r)^{-Q/2}\,\|\varphi\|_\infty \exp\left(-\frac{d^2(u,v)}{\mathbf{c}\,(t_0-r)}\right).\end{aligned}$$

By gathering together all thee facts, we see that a dominated convergence argument can be applied in the identity (3.5.36) if we show that

$$\mathbb{R}^N \ni (u,v) \mapsto f(u,v) := \exp\left(-\frac{d^2(u,v)}{\mathbf{c}\,(t_0-r)}\right) \in L^1(\mathbb{R}^N).$$

Now, since d is a (continuous) homogeneous norm on $\mathbb{G}$, there exists a universal constant $\alpha > 0$ such that (cf the proof of Thm. 3.5.7),

$$\begin{aligned} d^2(u,v) \ge \alpha\,h^2(u,v) &\overset{(3.3.3)}{=} \alpha\left(\sum_{j=1}^n |u_j|^{1/\sigma_j} + \sum_{j=1}^p |v_j|^{1/\sigma_j^*}\right)^2\\ &\ge \alpha\left[\left(\sum_{j=1}^n |u_j|^{1/\sigma_j}\right)^2 + \left(\sum_{j=1}^p |v_j|^{1/\sigma_j^*}\right)^2\right], \quad \forall\,(u,v)\in\mathbb{R}^N;\end{aligned}$$

therefore, if we set $P(u) := \sum_{j=1}^n |u_j|^{1/\sigma_j}$ (with $u \in \mathbb{R}^n$), we get

$$f(u,v) \le \exp\left(-\frac{\alpha\,P^2(u)}{\mathbf{c}\,(t_0-r)}\right)\cdot\exp\left(-\frac{\alpha\,N^2(v)}{\mathbf{c}\,(t_0-r)}\right), \quad \text{for every } (u,v)\in\mathbb{R}^N,$$

where $N(v)$ is the homogeneous norm on $\mathbb{R}^p$ defined in (3.2.34).

From this, by arguing exactly as in the proof of Thm. 3.5.7 (note that P is δ_λ-homogeneous of degree 1), we infer that $f \in L^1(\mathbb{R}^N)$; hence, an application of the Lebesgue Dominated Convergence Theorem in the above (3.5.36) gives

$$\begin{aligned}\lim_{n\to\infty} u(z_n) &= \lim_{n\to\infty} \int_{\mathbb{R}^N} \gamma_{\mathbb{G}}(t_n, u, v)\, \varphi\big(C_{x_n}^{-1}(u,v)\big)\, \mathrm{d}u\mathrm{d}v \\ &= \int_{\mathbb{R}^N} \gamma_{\mathbb{G}}(t_0, u, v)\, \varphi\big(C_{x_0}^{-1}(u,v)\big)\, \mathrm{d}u\mathrm{d}v = u(z_0).\end{aligned}$$

Due to the arbitrariness of $z_0 \in \Omega$, we conclude that $u \in C(\Omega, \mathbb{R})$.

STEP II: We now turn to prove that $\mathcal{H}u = 0$ in $\mathcal{D}'(\Omega)$ (note that $u \in L^1_{\mathrm{loc}}(\Omega)$, as it is continuous on the same set). To this end we first observe that, if $K \subseteq \mathbb{R}^{1+n}$ is a fixed compact set, we have

$$(t,x,y) \ni K \times \mathbb{R}^n \mapsto \Gamma(0, y; t, x) \in L^1(K \times \mathbb{R}^n). \tag{3.5.37}$$

Indeed, by Thm. 3.5.1 - (ii) and (vii) we have

$$\begin{aligned}\int_{K\times\mathbb{R}^n} \Gamma(0,y;t,x)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y &= \int_K \left(\int_{\mathbb{R}^n} \Gamma(0,x;t,y)\,\mathrm{d}y\right) \mathrm{d}t\,\mathrm{d}x \\ &\leq \int_K 1\,\mathrm{d}t\,\mathrm{d}x = \mathrm{meas}(K) < \infty.\end{aligned}$$

Let now $\psi \in C_0^\infty(\Omega, \mathbb{R})$. By (3.5.37) and Fubini's theorem, we obtain

$$\begin{aligned}\int_{\mathbb{R}^{1+n}} u(\zeta)\, \mathcal{H}^*\psi(\zeta)\,\mathrm{d}\zeta &= \int_{\mathbb{R}^{1+n}} \left(\int_{\mathbb{R}^n} \Gamma(0,y;\zeta)\,\varphi(y)\,\mathrm{d}y\right) \mathcal{H}^*\psi(\zeta)\,\mathrm{d}\zeta \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^{1+n}} \Gamma(0,y;\zeta)\,\mathcal{H}^*\psi(\zeta)\,\mathrm{d}\zeta\right) \varphi(y)\,\mathrm{d}y \\ &\quad \big(\Gamma \text{ is a fundamental solution for } \mathcal{H}, \text{ see } (1.3.7)\big) \\ &= -\int_{\mathbb{R}^n} \psi(0,y)\,\varphi(y)\,\mathrm{d}y \\ &\quad \big(\text{since } \mathrm{supp}(\psi) \subseteq \Omega = \,]0,\infty[\, \times \mathbb{R}^n\big) \\ &= 0,\end{aligned}$$

and this proves that $\mathcal{H}u = 0$ in $\mathcal{D}'(\Omega)$, as desired. From this, since $\mathcal{H}$ is C^∞-hypoelliptic on every open subset of $\mathbb{R}^n$ and $u \in C(\Omega, \mathbb{R})$, we infer that

$$u \in C^\infty(\Omega, \mathbb{R}) \text{ and } \mathcal{H}u(t,x) = 0 \text{ for every } (t,x) \in \Omega.$$

STEP III: To conclude the demonstration of the theorem, we turn to prove that u satisfies condition (ii) in Def. 3.5.31. To this end, let $x \in \mathbb{R}^n$ be arbitrarily fixed and let $\{t_j\}_{n\in\mathbb{N}} \subseteq \,]0,1]$ be a sequence converging to 0 as $n \to \infty$.

By means of the diffeomorphism C_x already considered in Step I and of the

Gaussian estimates of $\gamma_{\mathbb{G}}$ in Thm. 3.5.6, for every natural n we can write

$$
\begin{aligned}
|u(t_n,x)-\varphi(x)| &= \left|\int_{\mathbb{R}^n}\Gamma(0,y;t_n,x)\,\varphi(y)\,\mathrm{d}y-\varphi(x)\right| \\
&\quad \big(\text{by Thm. 3.5.1 - (ii) and (vii)}\big) \\
&= \left|\int_{\mathbb{R}^n}\Gamma(0,y;t_n,x)\big(\varphi(y)-\varphi(x)\big)\,\mathrm{d}y\right| \\
&\le \int_{\mathbb{R}^n}\Gamma(0,y;t_n,x)|\varphi(y)-\varphi(x)|\,\mathrm{d}y \\
&\quad \big(\text{by definition of } \Gamma \text{ and by (3.5.2)}\big) \\
&= \int_{\mathbb{R}^N}\gamma_{\mathbb{G}}\big(t_n,(y,0)^{-1}\star(x,\eta)\big)\,|\varphi(y)-\varphi(x)|\,\mathrm{d}y\,\mathrm{d}\eta \\
&\quad \big(\text{by the change of variables } (y,\eta)=C_x^{-1}(u,v)\big) \\
&= \int_{\mathbb{R}^N}\gamma_{\mathbb{G}}(t_n,u,v)\,\big|\varphi\big(C_x^{-1}(u,v)\big)-\varphi(x)\big|\,\mathrm{d}u\,\mathrm{d}v \\
&\overset{(3.5.4)}{\le} \mathbf{c}\,(t_n)^{-Q/2}\int_{\mathbb{R}^N}\exp\left(-\frac{d^2(u,v)}{\mathbf{c}\,t_n}\right)\big|\varphi\big(C_x^{-1}(u,v)\big)-\varphi(x)\big|\,\mathrm{d}u\,\mathrm{d}v.
\end{aligned}
$$

On the other hand, since d is d_λ-homogeneous of degree 1, by performing the change of variables $(u,v)=d_{\sqrt{t_n}}(w,z)$ (for every fixed $n\in\mathbb{N}$), we obtain

$$
\begin{aligned}
&|u(t_n,x)-\varphi(x)|\le \\
&\qquad \mathbf{c}\int_{\mathbb{R}^N}\exp\left(-\frac{d^2(w,z)}{\mathbf{c}}\right)\big|\varphi\big((C_x^{-1}\circ d_{\sqrt{t_n}})(w,z)\big)-\varphi(x)\big|\,\mathrm{d}w\,\mathrm{d}z.
\end{aligned} \tag{3.5.38}
$$

We now claim that the rhs of the above (3.5.38) tends to 0 as $n\to\infty$. Indeed, since φ is continuous on $\mathbb{R}^n$ and $C_x(x,0)=(x,0)^{-1}\star(x,0)=(0,0)$, we have

$$
\lim_{n\to\infty}\varphi\big((C_x^{-1}\circ d_{\sqrt{t_n}})(w,z)\big)=\varphi\big(C_x^{-1}(0,0)\big)=\varphi(x);
$$

moreover, φ being bounded on the whole of $\mathbb{R}^n$, one has

$$
\begin{aligned}
&\exp\left(-\frac{d^2(w,z)}{\mathbf{c}}\right)\big|\varphi\big((C_x^{-1}\circ d_{\sqrt{t_n}})(w,z)\big)-\varphi(x)\big| \\
&\qquad\qquad \le 2\,\|\varphi\|_\infty\,\exp\left(-\frac{d^2(w,z)}{\mathbf{c}}\right),\quad \text{for every } (u,v)\in\mathbb{R}^N.
\end{aligned}
$$

Since the function $(u,v)\mapsto\exp(-d^2(u,v)/\mathbf{c})$ is integrable on $\mathbb{R}^N$ (see Step I), by applying the Lebesgue Dominated Convergence Theorem we obtain

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N}\exp\left(-\frac{d^2(w,z)}{\mathbf{c}}\right)\big|\varphi\big((C_x^{-1}\circ d_{\sqrt{t_n}})(w,z)\big)-\varphi(x)\big|\,\mathrm{d}w\,\mathrm{d}z=0.
$$

By gathering together this last identity and the above (3.5.38), we then get

$$
\lim_{n\to\infty}|u(t_n,x)-\varphi(x)|=0,
$$

and this shows that u satisfies (3.5.33), as desired. This ends the proof. □

We now turn our attention to the issue of the *uniqueness* of solutions of the Cauchy problem for $\mathcal{H}$. To this end, we prove the following proposition.

Proposition 3.5.34. *Let* $u \in C^2(\Omega, \mathbb{R})$ *be a* bounded *solution of the problem*

$$\begin{cases} \mathcal{H}u = 0, & \text{on } \Omega =]0,\infty[\times\mathbb{R}^n, \\ u(0,x) = 0, & \text{for every } x \in \mathbb{R}^n. \end{cases} \tag{3.5.39}$$

Then u *vanishes identically on* Ω.

Proof. We denote by $\pi : \mathbb{R}^N \to \mathbb{R}^n$ the standard canonical projection of $\mathbb{R}^N$ onto $\mathbb{R}^n$ and we consider the function v defined as follows:

$$v :]0,\infty[\times\mathbb{R}^N \longrightarrow \mathbb{R}, \qquad v(t,x,\xi) := (u \circ \pi)(t,x,\xi) = u(t,x).$$

Obviously, the function v is of class C^2 and bounded on its domain of definition (since the same is true of u); moreover, since u is a classical solution of the problem (3.5.39) and $\mathcal{H}_{\mathbb{G}} = \mathcal{L}_{\mathbb{G}} - \partial_t$ is a lifting of $\mathcal{H}$ on $\mathbb{R} \times \mathbb{G}$, we get

- $v(0,x,\xi) = u(t,x) = 0$ for every $(x,\xi) \in \mathbb{R}^N$;
- $\mathcal{H}_{\mathbb{G}} v = \mathcal{H}_{\mathbb{G}}(u \circ \pi) = \mathcal{H}u = 0$ point-wise on $]0,\infty[\times\mathbb{R}^N$.

Summing up, v is a bounded solution of the Cauchy problem

$$\begin{cases} \mathcal{H}_{\mathbb{G}} u = 0, & \text{on }]0,\infty[\times\mathbb{R}^N, \\ u(0,x) = 0, & \text{for every } x \in \mathbb{R}^N. \end{cases}$$

We are then entitled to apply [36, Theorem 2.1], which ensures that v vanishes identically on its domain; from this, we deduce that

$$u(t,x) = 0 \text{ for every } (t,x) \in]0,\infty[\times\mathbb{R}^n.$$

and the proof is complete. □

By combining Thm. 3.5.33 with Prop. 3.5.34, we obtain the following result.

Theorem 3.5.35. *Let* $\varphi \in C(\mathbb{R}^n, \mathbb{R})$ *be* bounded. *Then the Cauchy problem*

$$\text{(CP)} \qquad \begin{cases} \mathcal{H}u = 0, & \text{in } \Omega =]0,\infty[\times\mathbb{R}^n; \\ u(0,x) = \varphi(x), & \text{for every } x \in \mathbb{R}^n \end{cases} \tag{3.5.40}$$

admits a unique bounded solution $u \in C^\infty(\Omega, \mathbb{R})$*, which is given by*

$$u(t,x) = \int_{\mathbb{R}^n} \Gamma(0,y;t,x)\,\varphi(y)\,\mathrm{d}y, \quad \text{for every } (t,x) \in \Omega. \tag{3.5.41}$$

Proof. By Thm. 3.5.2, the function u defined in (3.5.41) is a bounded solution of the Cauchy problem (3.5.40) (see (3.5.35)) ; therefore, Prop. 3.5.34 ensures that u is actually the unique bounded solution of this problem. □

Chapter 4

Degenerate divergence-form PDOs

In this last chapter of the thesis we go beyond the sums of squares of vector fields considered so far, and we turn our attention to linear PDOs (possibly degenerate-elliptic) in quasi-divergence form

$$(\bigstar) \qquad \mathcal{L} := \frac{1}{V(x)} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(V(x)\, a_{i,j}(x)\, \frac{\partial}{\partial x_j}\Big), \qquad x \in \mathbb{R}^N,$$

where $V \in C^\infty(\mathbb{R}^N, \mathbb{R})$ is strictly positive, the matrix $A(x) := (a_{i,j}(x))_{i,j}$ is symmetric and positive semi-definite at every point $x \in \mathbb{R}^N$, and it has real-valued C^∞ entries. As is well-known, such a class of operators comprehends sub-Laplacians on Carnot groups, sums of squares of vector fields and differential operators arising from CR geometry and general Lie group theory; moreover, there exist linear PDOs of the quasi-divergence form ($\bigstar$) which are not Hörmander nor sub-elliptic (see Exm. 4.1.4 below). For these reasons, this class of linear PDOs has been extensively studied since the early 80's (see, e.g., the fundamental works by Fefferman and Phong [69, 70]).

In this context, our aim is twofold: on the one hand, by exploiting a Control Theory result on hypoellipticity to recover a meaningful geometric information on connectivity and maxima propagation, we shall establish for such operators the Strong Maximum Principle; on the other hand, by means of suitable geometrical objects properly introduced, we shall prove a Hardy-type inequality generalizing the classical Hardy inequality for the Laplace operator.

4.1 The Strong Maximum Principle

According with the *incipit* of the chapter, this first section is completely devoted to establish the Strong Maximum Principle (SMP, for short) for a suitable class of hypoelliptic PDOs of the (quasi-)divergence form ($\bigstar$). The key tool for proving this principle is a notable Control Theory result due to Amano, which we shall briefly present in Sec. 4.1.2; thanks to this result, we are able to demonstrate the Strong Maximum Principle for our PDOs by following the very same approach exploited by Bony in the case of Hörmander operators.

We now describe more closely how this first section is organized.

- In Sec. 4.1.1 we properly introduce the class of linear PDOs we aim to study, and we present several examples.
- In Sec. 4.1.2 we briefly describe a deep Control Theory result by Amano [6] on hypoelliptic PDOs (long forgotten in the PDE literature), relating the hypoellipticity of a PDO $\mathcal{L}$ with the controllability of a suitable family of vector fields naturally associated with $\mathcal{L}$: as anticipated, this will be a fundamental tool in order to prove the SMP.
- Sec. 4.1.3 is devoted to call up some elements of ODE Theory/Control Theory; in particular, we remind the notion of invariant set for a vector field and a related result by Bony [39].
- By means of all the result presented in the preceding sections, we give in Sec. 4.1.4 the proof of the Strong Maximum Principle.
- Finally, in Sec. 4.1.5 we briefly show how the Strong Maximum Principle can be profitably used in order to prove the solvability of the Dirichlet problem and a Harnack Inequality for $\mathcal{L}$.

The contents of Sec.s 4.1.3 and 4.1.4 are inserted in this thesis for the sake of completeness: in fact, by crucially exploiting the result by Amano presented in Sec. 4.1.2, the proof of the Strong Maximum Principle for our PDOs can be carried out by arguing exactly as in Bony [39].

4.1.1 Main assumptions and notations

Throughout this section, we shall be concerned with linear PDOs $\mathcal{L}$ satisfying the following properties, where the acronyms stand for

- **(DS)**: Divergence Structure;
- **(DE)**: Degenerate-Ellipticity;
- **(NTD)**: Non-Total-Degeneracy;
- **(HY)**: Hypo-Ellipticity;

Here we have the definitions:

(DS): $\mathcal{L}$ has the following (quasi-)*divergence structure*

$$\mathcal{L} := \frac{1}{V(x)} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(V(x)\, a_{i,j}(x)\, \frac{\partial}{\partial x_j}\Big), \tag{4.1.1}$$

where $a_{i,j} \in C^\infty(\mathbb{R}^N, \mathbb{R})$ for every $i, j \in \{1, \dots, N\}$ and V is real-valued, C^∞ and strictly positive on the whole of $\mathbb{R}^N$;

(DE): $\mathcal{L}$ is *degenerate-elliptic* on $\mathbb{R}^N$, that is, the principal matrix of $\mathcal{L}$

$$A(x) := (a_{i,j}(x))_{i,j}$$

is symmetric and positive semi-definite at every point $x \in \mathbb{R}^N$;

(NTD): $\mathcal{L}$ is *non-totally degenerate at every point of* $\mathbb{R}^N$, that is (recalling that $A(x)$ is symmetric and positive semi-definite),

$$\operatorname{trace}(A(x)) > 0, \quad \text{for every } x \in \mathbb{R}^N; \tag{4.1.2}$$

(HY): $\mathcal{L}$ is C^∞*-hypoelliptic in every open subset of* $\mathbb{R}^N$: following, e.g., Treves [131], this means that for every open set $\Omega \subseteq \mathbb{R}^N$, for every $u \in \mathcal{D}'(\Omega)$, for every open set $U \subseteq \Omega$ and every $f \in C^\infty(U, \mathbb{R})$, the equation $\mathcal{L}u = f$ in U implies that u is a function-type distribution associated with a C^∞ function (on U). Equivalently, we can say that

$$\text{sing supp}(u) = \text{sing supp}(\mathcal{L}\mathrm{u}),$$

for every open set $\Omega \subseteq \mathbb{R}^N$ and every $u \in \mathcal{D}'(\Omega)$.

A wide class of linear PDOs satisfying all the assumptions listed above, which also represents a main motivation for our investigation, consists of the sub-Laplacians on the Carnot groups:

Example 4.1.1. Let $\mathbb{G} = (\mathbb{R}^N, *, \delta_\lambda)$ be a homogeneous Carnot group and let $X = \{X_1, \ldots, X_m\}$ be a set of Lie-generators of $\operatorname{Lie}(\mathbb{G})$. Then the sub-Laplacian

$$\mathcal{L} = \sum_{j=1}^m X_j^2$$

satisfies all the assumptions listed above.

In fact, by the results recalled in Chap. 1 (see, precisely, properties (P1)-to-(P5) in Sec. 1.3), we know that $\mathcal{L}$ takes the quasi-divergence form (4.1.1) (with $V \equiv 1$) and it is semielliptic on $\mathbb{R}^N$, since $A(x) = S(x) \cdot S(x)^T$, where

$$S(x) = \big(X_1 I(x) \cdots X_m I(x)\big), \quad x \in \mathbb{R}^N;$$

from this, it also follows that $\mathcal{L}$ is non-totally degenerate at every $x \in \mathbb{R}^N$. As for assumption (HY), it is a consequence of Hörmander's Hypoellipticity Theorem, jointly with the fact that $X_1, \ldots, X_m$ Lie-generate $\operatorname{Lie}(\mathbb{G})$.

More generally, any sub-Laplacian on a Lie group $\mathbb{G} = (\mathbb{R}^N, *)$ (not necessarily homogeneous nor Carnot) satisfies all the other assumptions listed above.

Example 4.1.2. Let $\mathbb{G} = (\mathbb{R}^N, *)$ be a Lie group on $\mathbb{R}^N$, let $X = \{X_1, \ldots, X_m\}$ be a set of Lie generators for $\operatorname{Lie}(\mathbb{G})$ and let μ be a fixed Haar measure on $\mathbb{G}$[1].

Then the linear partial differential operator

$$\mathcal{L} := -\sum_{j=1}^m X_j^{*\,\mu} X_j$$

satisfies the assumptions (DS)-to-(HY) listed above (here, $X_j^{*\,\mu}$ denotes the formal adjoint of X_j with respect to the fixed Haar measure μ).

[1] We remind that a Radon measure $\mu : \mathcal{B}(\mathbb{R}^N) \to [0, \infty[$ is called a **Haar measure for** $\mathbb{G}$ if it is left-invariant, that is, for every fixed $x \in \mathbb{G}$ it holds that

$$\mu\big(\tau_x(E)\big) = \mu(E), \quad \text{for every } E \in \mathcal{B}(\mathbb{R}^N),$$

where τ_x denotes the left-translation by x on $\mathbb{G}$.

It is out of any doubt that the sum of squares $P := \sum_{j=1}^m X_j^2$ would be a noteworthy PDO to be studied. This is not however the PDO we study here, for the following reason: both $\mathcal{L}$ and P are left-invariant, but P is not necessarily self-adjoint neither with respect to Lebesgue measure (this depending on the divergence of the X_js), nor with respect to the more natural measure to be considered, namely the Haar measure of the group. Besides, we observe that self-adjointness implies the symmetry of Gamma, another pleasant feature.

To prove that $\mathcal{L}$ satisfies assumptions (DS)-to-(HY) we first remind that, μ being a Haar measure on $\mathbb{G}$, it is absolutely continuous with respect to the Lebesgue measure; more precisely, if e denotes the neutral element of $\mathbb{G}$, there exists a positive constant $\mathbf{c} > 0$ such that

$$\mu = \mathbf{c}\, V(x)\,\mathrm{d}x, \qquad \text{where } V(x) = \frac{1}{\det\big(\mathcal{J}_{\tau_x}(e)\big)}.$$

As a consequence, for every smooth vector fields $Z \in \mathcal{X}(\mathbb{R}^N)$ one has

$$Z^{*\,\mu} = -Z - \left(\mathrm{div}(ZI) + \frac{ZV}{V}\right). \tag{4.1.3}$$

In fact, if $\varphi, \psi \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$ are arbitrarily fixed, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} \psi\, Z^{*\,\mu}\varphi\,\mathrm{d}\mu &= \int_{\mathbb{R}^N} \varphi\, Z\psi\,\mathrm{d}\mu = \int_{\mathbb{R}^N} V\varphi\, Z\psi\,\mathrm{d}x \\
&\quad \Big(\text{by writing } Z = \sum_{i=1}^N a_i(x)\,\partial_{x_i}\Big) \\
&= \mathbf{c}\sum_{i=1}^N \int_{\mathbb{R}^N} (V a_i\,\varphi)\,\partial_{x_i}\psi\,\mathrm{d}x \\
&\quad \big(\text{by performing an integration by parts}\big) \\
&= -\mathbf{c}\sum_{i=1}^N \int_{\mathbb{R}^N} \psi\Big((a_i\,\partial_{x_i}V)\varphi + (\partial_{x_i}a_i)V\varphi + (a_i\,\partial_{x_i}\varphi)V\Big)\mathrm{d}x \\
&\quad \big(\text{since } \mu = \mathbf{c}\,V(x)\,\mathrm{d}x \text{ and } V(x) > 0 \text{ on } \mathbb{R}^N\big) \\
&= -\int_{\mathbb{R}^N} \psi\left(\left(\frac{ZV}{V} + \mathrm{div}(ZI)\right)\varphi + Z\varphi\right)\mathrm{d}\mu,
\end{aligned}$$

and this obviously implies the above formula (4.1.3) . We explicitly notice that $Z^{*\,\mu}$ does not depend on the constant $\mathbf{c}$, but only on the function V.

Having established identity (4.1.3), we can now proceed to show that $\mathcal{L}$ satisfies assumptions (DS)-to-(HY). In fact, by (4.1.3) and by writing

$$X_j = \sum_{h=1}^N \sigma_{h,j}(x)\,\partial_{x_h}, \quad \text{for every } j = 1, \dots, m,$$

a direct computation gives

$$\begin{aligned}
\mathcal{L} &= -\sum_{j=1}^{m} X_j^{*\,\mu} X_j \overset{(4.1.3)}{=} \sum_{j=1}^{m}\Big(X_j + \mathrm{div}(X_j I) + \frac{X_j V}{V}\Big) X_j \\
&= \sum_{j=1}^{m} X_j^2 + \frac{1}{V}\sum_{j=1}^{m}\big(V\cdot \mathrm{div}(X_j I) + X_j V\big) X_j \\
&= \sum_{h,k=1}^{N}\bigg(\sum_{j=1}^{m}\sigma_{h,j}(x)\,\sigma_{k,j}(x)\bigg)\partial_{x_h}\partial_{x_k} + \sum_{k=1}^{N}\bigg(\sum_{j=1}^{m}\sum_{h=1}^{N}\sigma_{h,j}(x)\,\partial_{x_h}\sigma_{k,j}(x)\bigg)\partial_{x_k} \\
&\qquad + \frac{1}{V}\sum_{k=1}^{N}\bigg(\sum_{j=1}^{m}\sum_{h=1}^{N}\sigma_{k,j}(x)\Big(V\,\partial_{x_h}\sigma_{h,j}(x) + \sigma_{h,j}(x)\,\partial_{x_h}V\Big)\bigg)\partial_{x_k} \\
&\quad \Big(\text{setting, for each } h,k\in\{1,\ldots,N\},\ \ a_{h,k}(x) := \sum_{j=1}^{m}\sigma_{h,j}(x)\,\sigma_{k,j}(x)\Big) \\
&= \sum_{h,k=1}^{N} a_{h,k}(x)\,\partial_{x_h}\partial_{x_k} + \frac{1}{V}\sum_{k=1}^{N}\bigg(\sum_{h=1}^{N}\Big(a_{h,k}(x)\,\partial_{x_h}V + V\,\partial_{x_h}a_{h,k}(x)\Big)\bigg)\partial_{x_k}.
\end{aligned}$$

Therefore, if we introduce the matrix $A(x) := \big(a_{h,k}(x)\big)_{h,k}$, we can write

$$\mathcal{L} = \frac{1}{V(x)}\sum_{h,k=1}^{N}\frac{\partial}{\partial x_h}\Big(V(x)\,a_{h,k}(x)\,\frac{\partial}{\partial x_k}\Big),$$

and this proves that $\mathcal{L}$ takes the form (4.1.1). In particular, we see that $\mathcal{L}$ has smooth coefficient functions. Moreover, since $X_1,\ldots,X_m$ satisfy Hörmander's rank condition (as they Lie-generate $\mathrm{Lie}(\mathbb{G})$) and since

$$A(x) = S(x)\cdot S(x)^T, \quad \text{where } S(x) := (X_1 I(x)\cdots X_m I(x)),$$

we deduce that $A(x)\geq 0$ and that $A(x)\neq 0$ for every $x\in\mathbb{R}^N$, that is, $\mathcal{L}$ satisfies assumptions (DE) and (NTD). Finally, $\mathcal{L}$ also satisfies assumption (HY): this is a consequence of the Hörmander Hypoellipticity Theorem and of the fact that $X_1,\ldots,X_m$ Lie-generate $\mathrm{Lie}(\mathbb{G})$ (see Exm. 4.1.1).

We explicitly notice that the operator $\mathcal{L}$ does not depend on the chosen Haar measure μ, but only on the function V naturally associated with the group $\mathbb{G}$; this is coherent with the fact that the formal adjoint $Z^{*,\mu}$ of a smooth vector field Z w.r.t. the measure μ only depends on the function V.

As an explicit example, let us consider the group $\mathbb{G} = (\mathbb{R}^2, *)$, where

$$(x_1,x_2)*(y_1,y_2) = (x_1 + y_1 e^{x_2}, x_2 + y_2).$$

A direct computation shows that, for every fixed $x\in\mathbb{R}^2$, one has (note that the neutral element e of the group $\mathbb{G}$ is 0)

$$\mathcal{J}_{\tau_x}(0) = \begin{pmatrix} e^{x_2} & 0 \\ 0 & 1 \end{pmatrix};$$

as a consequence, according to Rem. 1.1.5 on page 5, the relevant Jacobian vector fields J_1, J_2 of $\mathrm{Lie}(\mathbb{G})$ are given by

$$J_1 = e^{x_2}\,\partial_{x_1}, \qquad J_2 = \partial_{x_2},$$

and $\{J_1, J_2\}$ is a system of Lie-generator for $\mathrm{Lie}(\mathbb{G})$. Furthermore, we have

$$V(x) = \frac{1}{\det\big(\mathcal{J}_{\tau_x}(0)\big)} = e^{-x_2}, \qquad \text{for every } x \in \mathbb{R}^2.$$

Thus, if μ is an arbitrary fixed Haar measure on $\mathbb{G}$ (hence, $\mu = \mathbf{c}\,V(x)\,\mathrm{d}x$ for a suitable constant $\mathbf{c} > 0$), from the above general discussion we obtain

$$\begin{aligned}
\mathcal{L} &= -J_1^{*\,\mu}\,J_1 - J_2^{*\,\mu}\,J_2 = \frac{1}{V(x)}\sum_{h,k=2}^{N}\frac{\partial}{\partial x_h}\Big(V(x)\,a_{h,k}(x)\,\frac{\partial}{\partial x_k}\Big) \\
&\quad \big(\text{note that, in this case, } A(x) = \mathcal{J}_{\tau_x}(0)\cdot\mathcal{J}_{\tau_x}(0)^T\big) \\
&= e^{x_2}\,\mathrm{div}\left(\begin{pmatrix} e^{x_2} & 0 \\ 0 & e^{-x_2}\end{pmatrix}\cdot\begin{pmatrix}\partial_{x_1} \\ \partial_{x_2}\end{pmatrix}\right) \\
&= e^{x_2}\left(\frac{\partial}{\partial x_1}\left(e^{x_2}\,\frac{\partial}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(e^{-x_2}\,\frac{\partial}{\partial x_2}\right)\right) \\
&= e^{2x_2}\,\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{\partial}{\partial x_2}.
\end{aligned}$$

We point out that the operator $\mathcal{L}$ cannot be re-written as pure divergence-form operator; more precisely, there cannot exist a matrix-valued function $B(x)$ s.t.

$$\mathcal{L} = \mathrm{div}\Big(B(x)\cdot\nabla\Big) = \sum_{h,k=1}^{2}\frac{\partial}{\partial x_h}\left(b_{h,k}(x)\,\frac{\partial}{\partial x_k}\right).$$

Indeed, if such a matrix B existed, then it should necessarily coincide with the principal matrix $A(x)$ of $\mathcal{L}$, that is,

$$B(x) = A(x) = \begin{pmatrix} e^{2x_2} & 0 \\ 0 & 1\end{pmatrix}, \qquad \text{for every } x \in \mathbb{R}^2.$$

On the other hand, the PDO $\tilde{\mathcal{L}} = \mathrm{div}(A(x)\cdot\nabla)$ is different from $\mathcal{L}$: in fact,

$$\begin{aligned}
\tilde{\mathcal{L}} &= \mathrm{div}\left(\begin{pmatrix} e^{2x_2} & 0 \\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}\partial_{x_1} \\ \partial_{x_2}\end{pmatrix}\right) \\
&= \left(\frac{\partial}{\partial x_1}\left(e^{2x_2}\,\frac{\partial}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial}{\partial x_2}\right)\right) \\
&= e^{2x_1}\,\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \neq \mathcal{L}.
\end{aligned}$$

Another wide class of linear PDOs satisfying all the assumptions listed above is that of *homogeneous Hörmander operators.*

Example 4.1.3. Let $X = \{X_1, \ldots, X_m\}$ be a set linearly independent smooth vector fields on $\mathbb{R}^N$ satisfying the following assumptions (see Sec. 3.2):

(1) $X_1, \dots, X_m$ are homogeneous of degree 1 with respect to a suitable family $\{\delta_\lambda\}_{\lambda>0}$ of dilations on $\mathbb{R}^N$ of the form

$$\delta_\lambda(x) = (\lambda^{\sigma_1} x_1, \dots, \lambda^{\sigma_N} x_N),$$

where $1 = \sigma_1 \le \dots \le \sigma_N$ and $Q := \sum_{j=1}^N \sigma_j \ge 2$;

(2) $X_1, \dots, X_m$ satisfy Hörmander's condition at every point of $\mathbb{R}^N$.

Then the linear PDO

$$\mathcal{L} = \sum_{j=1}^m X_j^2,$$

satisfies all the assumptions listed above. In fact, a direct computation shows that $\mathcal{L}$ is of the form (4.1.1), with $V \equiv 1$ and with principal matrix

$$A(x) = S(x) \cdot S(x)^T,$$

where $S(x) = (X_1 I(x), \dots, X_m I(x))$; as a consequence, $\mathcal{L}$ is semielliptic and non-totally degenerate. Moreover, the hypoellipticity of $\mathcal{L}$ is a direct consequence of assumption (2) and of Hörmander's Hypoellipticity Theorem.

Finally, the following is an example of a class of PDOs satisfying all the assumptions listed above, but *not Hörmander nor sub-elliptic.*

Example 4.1.4. Let us consider the class of operators in $\mathbb{R}^2$ defined by

$$\mathcal{L}_a = \frac{\partial^2}{\partial x_1^2} + \Big(a(x_1) \frac{\partial}{\partial x_2} \Big)^2, \tag{4.1.4a}$$

where $a \in C^\infty(\mathbb{R}, \mathbb{R})$ is a smooth function satisfying the following properties:

- a is even and $a(x) = 0$ if and only if $x = 0$;
- a is nonnegative on $\mathbb{R}$;
- a is non-decreasing on $[0, \infty[$.

Then the operator $\mathcal{L}_a$ satisfies assumptions (DS) (being a sum of squares), (NTD) (obviously) and (HY), thanks to a result by Fediĭ, [68]; on the other hand, $\mathcal{L}_a$ does not satisfy Hörmander's Rank Condition at $x_1 = 0$ if all the derivatives of a vanish at 0, as for

$$a(x_1) = \begin{cases} 0 & \text{if } x_1 = 0; \\ \exp(-1/x_1^2) & \text{if } x_1 \neq 0. \end{cases}$$

Other examples of operators satisfying our assumptions (NTD) and (HY) but failing to be Hörmander operators can be found, e.g., in the following papers: Bell and Mohammed [20]; Christ [51, Section 1]; Kohn [99]; Kusuoka and Stroock [104, Theorem 8.41]; Morimoto [116]. Explicit examples are, for instance,

$$\frac{\partial^2}{\partial x_1^2} + \Big(\exp(-1/|x_1|) \frac{\partial}{\partial x_2} \Big)^2 + \Big(\exp(-1/|x_1|) \frac{\partial}{\partial x_3} \Big)^2 \quad \text{in } \mathbb{R}^3, \tag{4.1.4b}$$

$$\frac{\partial^2}{\partial x_1^2} + \Big(\exp(-1/\sqrt{|x_1|}) \frac{\partial}{\partial x_2} \Big)^2 + \frac{\partial^2}{\partial x_3^2} \quad \text{in } \mathbb{R}^3, \tag{4.1.4c}$$

$$\frac{\partial^2}{\partial x_2^2} + \Big(x_2 \frac{\partial}{\partial x_1} \Big)^2 + \frac{\partial^2}{\partial x_4^2} + \Big(\exp(-1/\sqrt[3]{|x_1|}) \frac{\partial}{\partial x_3} \Big)^2 \quad \text{in } \mathbb{R}^4. \tag{4.1.4d}$$

For the hypoellipticity of (4.1.4b) see [51]; for (4.1.4c) see [104]; for (4.1.4d) see [116]. Note that the smallest eigenvalue in all the above examples vanishes very quickly (like $\exp(-1/|x|^\alpha)$ for $x \to 0$, with positive α) and it cannot be bounded from below by any weight $w(x)$ with locally integrable reciprocal function.

4.1.2 Amano's Hypoellipticity result

As anticipated, the aim of this brief section is to present a profound result by Amano [6], which will be fundamental in order to prove the Strong Maximum Principle (see Sec. 4.1.4). To begin with, we give the following definitions.

Definition 4.1.5. Let Ω be an open set and let $\mathcal{F} \subseteq \mathcal{X}(\Omega)$ be a family of smooth vector fields on Ω. A function $\gamma : [0,T] \to \Omega$ is called an **integral curve of** $\mathcal{F}$ if it satisfies the following properties:

(i) $\gamma \in C([0,T],\Omega)$;

(ii) there exist a partition $0 = t_0 < t_1 < \cdots < t_p = T$ of $[0,T]$ and vector fields $X_1,\dots,X_p \in \mathcal{F}$ such that, for every $i = 1,\dots,p$, we have

- $\gamma|_{]t_{i-1},t_i[} \in C^1(]t_{i-1},t_i[,\Omega)$;
- $\dot\gamma(t) = X_i I(\gamma(t))$, for every $t \in]t_{i-1},t_i[$.

Remark 4.1.6. Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $X \in \mathcal{X}(\Omega)$. According with Def. 4.1.5, a function $\gamma : [0,T] \to \Omega$ is an integral curve of the family $\mathcal{F} = \{X\}$ if it is continuous and it is *piecewise* an integral curve of the vector field X.

Definition 4.1.7 (Controllable family)**.** Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $\mathcal{F}$ be a family of smooth vector fields on Ω. We say that $\mathcal{F}$ is **controllable** on Ω if, for every $x,y \in \Omega$, there exists an integral curve $\gamma : [0,T] \to \Omega$ of $\mathcal{F}$ s.t.

$$\gamma(0) = x \quad \text{and} \quad \gamma(T) = y.$$

Remark 4.1.8. Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $\mathcal{F} \subseteq \mathcal{X}(\Omega)$. From a geometric point of view, we see that $\mathcal{F}$ is controllable on Ω if any two points in Ω can be joined with a continuous curve which is piecewise an integral curve of some vector field in $\mathcal{F}$. Thus, for example, it is readily seen that the family

$$\mathcal{E} = \{\partial_{x_1},\dots,\partial_{x_N}\}$$

is controllable on the whole of $\mathbb{R}^N$.

Let now $\Omega \subseteq \mathbb{R}^N$ be a fixed open set, and let L be a second-order linear PDO on Ω of the following general form

$$L = \sum_{i,j=1}^N \alpha_{i,j}(x)\,\frac{\partial}{\partial x_i \partial x_j} + \sum_{i=1}^N \beta_i(x)\,\frac{\partial}{\partial x_i} + \gamma(x), \quad x \in \Omega.$$

We assume that the coefficients of L are real-valued and smooth on Ω, i.e.,

$$\alpha_{i,j}, \beta_i, \gamma \in C^\infty(\Omega,\mathbb{R}), \quad \text{for every } i,j \in \{1,\dots,N\},$$

and that the matrix $(\alpha_{i,j}(x))_{i,j}$ is symmetric and positive semi-definite for every $x \in \Omega$; furthermore, we assume that the second order terms and the first order ones of L never vanish simultaneously, that is,

$$\sum_{i,j=1}^{N} |\alpha_{i,j}(x)| + \sum_{i=1}^{N} |\beta_i(x)| \neq 0, \quad \text{for every } x \in \Omega.$$

We then introduce the following smooth vector fields (on Ω) associated with L, which we shall referred to as **L-canonical vector fields**:

$$X_i := \sum_{j=1}^{N} \alpha_{i,j}(x) \frac{\partial}{\partial x_j}, \quad \text{for every } i = 1, \ldots, N;$$
$$X_0 := \sum_{j=1}^{N} \left(\beta_j(x) - \sum_{i=1}^{N} \frac{\partial \alpha_{i,j}}{\partial x_i}(x) \right) \frac{\partial}{\partial x_j}.$$

By means of these vector fields, we can re-write L in the following way:

$$L = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{N} \alpha_{i,j}(x) \frac{\partial}{\partial x_j} \right) + X_0 + \gamma(x)$$
$$= \sum_{i=1}^{N} \frac{\partial}{\partial x_i} X_i + X_0 + \gamma(x), \quad \text{on } \Omega.$$

Moreover, they play a central role in the study of the hypoellipticity of L: in fact, as anticipated, Amano proved the subsequent result (for a demonstration of this profound theorem, we refer to [6, Theorems 1 and 2 and Remark 1]).

Theorem 4.1.9 (Amano [6])**.** *In the above assumptions and notations, if the operator L is C^∞-hypoelliptic on every open subset of Ω, then the family*

$$\mathcal{F}_L := \text{span}\{X_0, \ldots, X_N\}$$

is controllable on every open and connected subset of Ω. Conversely, if the family $\mathcal{F}_L$ is controllable on every open and connected subset of Ω, the set

$$\mathcal{C}_L := \big\{x \in \Omega : \dim\big\{\text{Lie}\{X_0, \ldots, X_N\}\big\}(x) < N\big\} \tag{4.1.5}$$

is closed in Ω and has no interior. As a consequence, $\Omega_1 := \Omega \setminus \mathcal{C}_L$ is an open dense subset of Ω, and L is C^∞-hypoelliptic on every open subset of Ω_1.

Remark 4.1.10. It is worth noting that, in the *real-analytic* case, the first part of Thm. 4.1.9 can be reversed, that is, the set $\mathcal{C}_L$ in (4.1.5) is actually empty.

To be more precise, let us assume that the coefficients of L are real-analytic on Ω and that the family $\mathcal{F}_L$ is controllable on every open and connected subset of Ω. Since the vector fields $X_0, \ldots, X_N$ are real-analytic on Ω (the same being true of the coefficients of L), it can be proved that the family $\{X_0, \ldots, X_N\}$ is a Hörmander system on Ω (see, e.g., [127]), whence

$$\mathcal{C}_L = \big\{x \in \Omega : \dim\big\{\text{Lie}\{X_0, \ldots, X_N\}\big\}(x) < N\big\} = \varnothing.$$

We are then entitled to apply [120, Theorem 2.8.2] by Oleĭnik and Radkevič: since $X_0, \dots, X_N$ satisfy Hörmander's rank condition on Ω, the operator

$$L = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} X_i + X_0,$$

is C^∞-hypoelliptic in every open subset of Ω, whence $\mathcal{C}_L = \varnothing$.

4.1.3 Invariant sets and the Nagumo-Bony Theorem

In this section we remind the notion of invariant set w.r.t. a vector field and a classical result by Bony [39], which characterizes such a notion in a very intuitive geometric way. Together with Amano's Thm. 4.1.9, this result will lead to a simple proof of the Strong Maximum Principle.

To begin with, we give the following important definition.

Definition 4.1.11. Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let F be a relatively closed subset of Ω. Moreover, let $y \in \Omega \cap \partial F$. We say that a vector $\nu \in \mathbb{R}^N \setminus \{0\}$ is **externally orthogonal to F at y** if

$$\overline{B}(y + \nu, |\nu|) \subseteq (\Omega \setminus F) \cup \{y\}. \tag{4.1.6}$$

If this is the case, we shall write $\nu \perp F$ at y. We also set

$$F^\star = \{y \in \Omega \cap \partial F : \text{there exists } \nu \text{ externally orthogonal to } F \text{ at } y\}.$$

Remark 4.1.12. Let the assumptions and the notations in Def. 4.1.11 apply. We explicitly observe that, if Ω is connected and if $F \neq \Omega$, then

$$F^\star \neq \varnothing.$$

In fact, since Ω is connected, we have $\Omega \cap \partial F \neq \varnothing$; we then choose a point $z \in \Omega \cap \partial F$, a real $R > 0$ such that $B(z, R) \subseteq \Omega$ and a point $x_0 \in B(z, R/2)$ not belonging to F. Since ∂F is closed, there exists $y \in \Omega \cap \partial F$ such that

$$\|y - x_0\| = \inf\{\|x_0 - x\| : x \in \partial F\}; \tag{4.1.7}$$

moreover, the vector $\nu := \frac{1}{2}\|y - x_0\|$ being externally orthogonal to F at y (as one can easily deduce from (4.1.7)), we conclude that $y \in F^\star$.

We then introduce the notion of invariant set w.r.t. a vector field.

Definition 4.1.13 (Invariant set w.r.t. a vector field)**.** Let $\Omega \subseteq \mathbb{R}^N$ be an open set, let X be a continuous v.f. on Ω and let $F \subseteq \Omega$ be a relatively closed set. We say that F is **positively X-invariant** (or **positively invariant w.r.t. X**) if, for every integral curve $\gamma : [0, T] \to \Omega$ of X such that $\gamma(0) \in F$, we have

$$\gamma(t) \in F, \quad \text{for every } 0 \leq t \leq T.$$

We say that F is **X-invariant** (or **invariant w.r.t. X**) if F is positively invariant with respect to both X and $-X$.

Remark 4.1.14. Let the assumption and the notations in Def. 4.1.13 apply. Obviously, the role of 0 is immaterial: more precisely, a simple re-parametrization argument shows that the set F is positively X-invariant if and only if, for every integral curve $\gamma : [a, b] \to \Omega$ of X such that $\gamma(a) \in F$, we have

$$\gamma(t) \in F, \quad \text{for every } t \in [a, b];$$

as a consequence, we see that F is invariant with respect to X if and only if, for every integral curve $\gamma : [a, b] \to \Omega$ of X s.t. $\gamma([a, b]) \cap F \neq \varnothing$, we have

$$\gamma(t) \in F, \quad \text{for every } t \in [a, b].$$

With the above Def.s 4.1.11 and 4.1.13 hand, we are now in a position to state the aforementioned theorem by Bony; for a proof of this result see, e.g., [39, Théorème 2.1] or [37, Section 5.13].

Theorem 4.1.15 (Bony [39])**.** *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, let X be C^1 vector field on Ω and let $F \subseteq \Omega$ be a relatively closed set.*

Then F is positively X-invariant if and only if

$$\langle XI(y), \nu \rangle \leq 0, \quad \text{for every } y \in F^\star \text{ and every } \nu \perp F \text{ at } y. \tag{4.1.8}$$

We end this section with the following simple yet crucial corollary.

Corollary 4.1.16. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, let X be C^1 vector field on Ω and let $F \subseteq \Omega$ be a relatively closed set. Then F is X-invariant* if and only if

$$\langle XI(y), \nu \rangle = 0, \quad \text{for every } y \in F^\star \text{ and every } \nu \perp F \text{ at } y. \tag{4.1.9}$$

Proof. By Def. 4.1.13, F is X-invariant if and only if F is positively invariant with respect to X and $-X$. By the Bony Thm. 4.1.15, this is equivalent to

$$\langle \pm XI(y), \nu \rangle \leq 0 \qquad \text{for every } y \in F^\star \text{ and every } \nu \perp F \text{ at } y,$$

which is possible in and only if (4.1.9) is satisfied. This ends the proof. □

4.1.4 The proof of the Strong Maximum Principle

Gathering all the results recalled in Sec.s 4.1.2 and 4.1.3, we are finally in a position to prove the announced Strong Maximum Principle for our PDOs $\mathcal{L}$ satisfying the structural assumptions in Sec. 4.1. As anticipated, the proof we are going to present is completely analogous to that given Bony in the case of Hörmander operators (see, precisely, [39, Corollaire 3.1]).

To begin with, we establish the following useful Hopf-type lemma.

Lemma 4.1.17 (Hopf-type lemma)**.** *Let $\mathcal{L}$ be a linear PDO satisfying the assumptions introduced in Sec. 4.1, and let $\Omega \subseteq \mathbb{R}^N$ be a connected open set.*

Then the following facts hold true:

(i) *Let $u \in C^2(\Omega, \mathbb{R})$ be such that $\mathcal{L}u \geq 0$ on Ω. If the set*

$$F(u) := \{x \in \Omega : u(x) = \max_\Omega u\} \tag{4.1.10}$$

is a proper subset of Ω (that is, $\varnothing \neq F \neq \Omega$), then

$$\langle A(y)\nu, \nu \rangle = 0 \quad \text{for every } y \in F(u)^\star \text{ and every } \nu \perp F(u) \text{ at } y. \tag{4.1.11}$$

(ii) *Let $c \in C^\infty(\mathbb{R}^N, \mathbb{R})$ be nonnegative on $\mathbb{R}^N$ and let $\mathcal{L}_c := \mathcal{L} - c$. Moreover, let $u \in C^2(\Omega, \mathbb{R})$ be such that $\mathcal{L}_c u \geq 0$ on Ω. If the set $F(u)$ in* (4.1.10) *is a proper subset of Ω and if $\max_\Omega u \geq 0$, then* (4.1.11) *holds true.*

Proof. (i) First of all we observe that, by assumptions, the function u attains the maximum in Ω; we then set $M := \max_\Omega u$ and, arguing by contradiction, we assume that there exist a point $y \in F(u)^\star$ and a vector $\nu \perp F(u)$ at y s.t.

$$\langle A(y)\nu, \nu\rangle > 0. \tag{4.1.12}$$

Then, by definition, we have $\overline{B}(y + \nu, \|\nu\|) \subseteq (\Omega \setminus F(u)) \cup \{y\}$ and

$$u(x) < M, \quad \text{for every } x \in \overline{B}(y + \nu, \|\nu\|) \setminus \{y\}. \tag{4.1.13}$$

We now consider the smooth function

$$w(x) := e^{-\lambda|x-(y+\nu)|^2} - e^{-\lambda|\nu|^2},$$

where $\lambda > 0$ is a constant which will be fixed later on. By definition, we have

- $w > 0$ on $B(y + \nu, \|\nu\|)$;
- $w = 0$ on $\partial B(y + \nu, \|\nu\|)$;
- $w < 0$ outside $B(y + \nu, \|\nu\|)$.

Moreover, a direct computation shows that

$$\mathcal{L}w(y) = 4\,\lambda^2 e^{-\lambda|\nu|^2}\left(\langle A(y)\nu, \nu\rangle + \mathcal{O}\left(\frac{1}{\lambda}\right)\right); \tag{4.1.14}$$

then, thanks to assumption (4.1.12), we can choose and fix $\lambda > 0$ in such a way that $\mathcal{L}w(y) > 0$. As a consequence, $\mathcal{L}w$ being continuous on $\mathbb{R}^N$, there exists $r > 0$ such that $V := B(y, r)$ is compactly contained in Ω and $\mathcal{L}w > 0$ on V. We now define, for $\varepsilon > 0$, a function $v_\varepsilon : \overline{V} \to \mathbb{R}$ by setting

$$v_\varepsilon(x) := u(x) + \varepsilon\, w(x).$$

Obviously, $v_\varepsilon \in C^2(V, \mathbb{R}) \cap C(\overline{V}, \mathbb{R})$; moreover, we claim that it is possible to choose $\varepsilon > 0$ in such a way that the maximum of v_ε on $\overline{V}$ is attained in V.

In fact, let us consider the splitting of ∂V given by the two sets

$$K_1 := \partial V \cap \overline{B(y + \nu, |\nu|)} \quad \text{and} \quad K_2 := \partial V \setminus K_1.$$

For every $x \in K_2$ (and every $\varepsilon > 0$), one has

$$v_\varepsilon(x) = u(x) + \varepsilon w(x) < u(x) \leq M;$$

on the other hand, for every $x \in K_1$ (and every $\varepsilon > 0$) we have

$$v_\varepsilon(x) \leq \max_{K_1} u + \varepsilon \max_{K_1} w.$$

Since $K_1 \subseteq \overline{B}(y+\nu, \|\nu\|)$ and $y \notin K_1$, we infer from (4.1.13) that $\max_{K_1} u < M$; as a consequence, it is possible to choose $\varepsilon > 0$ so small that

$$v_\varepsilon(x) < M, \quad \text{for every } x \in K_1.$$

By gathering together these facts we see that, for every $x \in \partial V$ and with the above choice of ε, we have (note that $y \in F(u) \cap \overline{V}$ and $w(y) = 0$)

$$v_\varepsilon(x) < M = u(y) = v_\varepsilon(y) \leq \max_{\overline{V}} v_\varepsilon,$$

and this proves the claim. We are now ready to conclude: from

$$\mathcal{L} v_\varepsilon = \mathcal{L} u + \varepsilon \mathcal{L} w \geq \varepsilon \mathcal{L} w > 0, \quad \text{on } V,$$

we infer that v_ε is a *strictly* $\mathcal{L}$-subharmonic function on V, that is, $\mathcal{L} v_\varepsilon > 0$ on V, admitting a maximum point on the open set V, say p_0; then we have (recall that $A(p_0) \geq 0$ and notice that $\nabla v_\varepsilon(p_0) = 0$ and $H(p_0) := (\partial_{i,j} v_\varepsilon(p_0))_{i,j} \leq 0$)

$$0 < \mathcal{L} v_\varepsilon(p_0) = \sum_{i,j} a_{i,j}(p_0) \partial_{i,j} v_\varepsilon(p_0) = \operatorname{trace}\big(A(p_0) \cdot H(p_0)\big) \leq 0, \qquad (4.1.15)$$

which is a contradiction.

(ii) We proceed exactly as in part (i), from which we also inherit all notations: we replace $\mathcal{L}$ with $\mathcal{L}_c$ and we notice that $w(y) = 0$, so that $\mathcal{L}_c w(y) = \mathcal{L} w(y)$ and (4.1.14) is left unchanged. Arguing as above (and using the same notations), we let again $p_0 \in V$ be such that $v_\varepsilon(p_0) = \max_{\overline{V}} v_\varepsilon$, which gives

$$v_\varepsilon(p_0) \geq v_\varepsilon(y) = u(y) = M.$$

Hence (4.1.15) becomes

$$0 < \mathcal{L}_c v_\varepsilon(p_0) = \operatorname{trace}\big(A(p_0) \cdot H(p_0)\big) - c(p_0)\, v_\varepsilon(p_0) \leq -c(p_0)\, M,$$

where in the last inequality we used the assumption $c \geq 0$ and the fact that $v_\varepsilon(p_0) \geq M$. By the assumption $M \geq 0$ (and again by the assumption on the sign of c), we have $-c(p_0)\, M \leq 0$, and we obtain another contradiction. $\square$

With the Hopf-type Lem. 4.1.17 at hand, we can finally state and prove the announced Strong Maximum Principle for our PDOs $\mathcal{L}$ as in Sec. 4.1.1.

Theorem 4.1.18 (Strong Maximum Principle for $\mathcal{L}$)**.** *Let $\mathcal{L}$ be a linear PDO satisfying the assumptions introduced in Sec. 4.1.1, and let $\Omega \subseteq \mathbb{R}^N$ be a connected open set. Then the following facts hold true:*

(i) *Any function $u \in C^2(\Omega, \mathbb{R})$ satisfying $\mathcal{L} u \geq 0$ on Ω and attaining a maximum in Ω is constant throughout Ω.*

(ii) *If $c \in C^\infty(\mathbb{R}^N, \mathbb{R})$ is nonnegative on $\mathbb{R}^N$ and if we set $\mathcal{L}_c := \mathcal{L} - c$, then any function $u \in C^2(\Omega, \mathbb{R})$ satisfying $\mathcal{L}_c u \geq 0$ on Ω and attaining a* nonnegative *maximum in Ω is constant throughout Ω.*

Proof. (i) For the sake of clarity, we split the proof into three steps.

Step I: Let $F(u)$ be the set introduced in the Hopf-type Lem. 4.1.17:

$$F(u) := \{x \in \Omega : u(x) = \max_{\Omega} u\}.$$

By assumptions, $F(u)$ is non-empty, say $\xi \in F(u)$; we thus prove that $F(u) = \Omega$. To this end, we first re-write the operator $\mathcal{L}$ in its canonical form

$$\begin{aligned}\mathcal{L} &= \frac{1}{V(x)} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(V(x)\, a_{i,j}(x)\, \frac{\partial}{\partial x_j}\Big) \\ &= \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{N} a_{i,j}(x)\, \frac{\partial}{\partial x_j}\right) + \frac{1}{V(x)} \sum_{i=1}^{N} \partial_{x_i} V(x) \left(\sum_{j=1}^{N} a_{i,j}(x)\, \frac{\partial}{\partial x_j}\right);\end{aligned}$$

from this, we see that the $\mathcal{L}$-canonical v.f.s are given by (see Sec. 4.1.2)

$$\begin{aligned}X_i &:= \sum_{j=1}^{N} a_{i,j}(x)\, \frac{\partial}{\partial x_j}, \quad \text{for every } i = 1, \ldots, N; \\ X_0 &:= \frac{1}{V(x)} \sum_{i=1}^{N} \partial_{x_i} V(x)\, X_i.\end{aligned} \tag{4.1.16}$$

Since $\mathcal{L}$ satisfies assumptions (NTD) and (HY) in Sec. 4.1.1, we are entitled to apply Amano's Thm. 4.1.9, which ensures that the family

$$\mathcal{F}_{\mathcal{L}} = \{X_0, \ldots, X_N\}$$

is controllable on every open and connected subset of $\mathbb{R}^N$. Thus, Ω being connected, any point of Ω can be joined to ξ by a continuous curve $\gamma : [0, T] \to \Omega$ which is piecewise an integral curve of a vector field belonging to $\mathcal{F}_{\mathcal{L}}$.

According with Def. 4.1.13, to prove the theorem it then suffices to show that $F(u)$ is *invariant with respect to any vector fields* belonging to $\mathcal{F}_{\mathcal{L}}$.

Step II: Let now $X \in \mathcal{F}_{\mathcal{L}}$ be fixed. By the Bony Thm. 4.1.15 (or, more precisely, by Cor. 4.1.16), we know that $F(u)$ is X-invariant if and only if

$$\langle XI(y), \nu\rangle = 0, \quad \text{for every } y \in F(u)^\star \text{ and every } \nu \perp F(u) \text{ at } y. \tag{4.1.17}$$

On the other hand, since X is a linear combination of $X_0, \ldots, X_N$, identity (4.1.17) follows if we show that, for every $i = 0, \ldots, N$, we have

$$\langle X_i I(y), \nu\rangle = 0, \quad \text{for every } y \in F(u)^\star \text{ and every } \nu \perp F(u) \text{ at } y.$$

Finally, since X_0 is a combination (with smooth coefficients) of $X_1, \ldots, X_N$ (hence, $X_0 I(x)$ is a linear combination of $X_1 I(x), \ldots, X_N I(x)$ for every $x \in \mathbb{R}^N$; see (4.1.16)), we can limit ourselves to prove that, for every $i = 1, \ldots, N$,

$$\langle X_i I(y), \nu\rangle = 0, \quad \text{for every } y \in F(u)^\star \text{ and every } \nu \perp F(u) \text{ at } y. \tag{4.1.18}$$

Step III: Let $i \in \{1, \ldots, N\}$ be fixed. Since, for every $x \in \mathbb{R}^N$, the vector $X_i I(x)$ is precisely the i-th column of the principal matrix $A(x)$ of $\mathcal{L}$, the Cauchy-Schwarz inequality provides a constant $\lambda(x) > 0$ such that

$$\langle X_i I(x), \nu\rangle^2 \le \lambda_i(x)\, \langle A(x)\nu, \nu\rangle \quad \text{for every } \nu \in \mathbb{R}^N. \tag{4.1.19}$$

From this, by exploiting identity (4.1.11) in the Hopf-type Lem. 4.1.17, we immediately obtain the desired (4.1.18). This completes the proof.

(ii) We consider once again the set $F(u) \neq \varnothing$ introduced above, and we prove that $F(u) = \Omega$. To this end, we proceed exactly as in part (i): by exploiting Amano's Thm. 4.1.9 and Bony's Thm. 4.1.15, we see that the needed identity $F(u) = \Omega$ follows if we show that, for every $i = 1, \dots, N$,

$$\langle X_i I(y), \nu \rangle = 0, \quad \text{for every } y \in F(u)^\star \text{ and every } \nu \perp F(u) \text{ at } y. \tag{4.1.20}$$

Moreover, by part (ii) of Lem. 4.1.17, we have at our disposal a Hopf-type lemma for operators of the form $\mathcal{L}_c$, and for functions u such that $\mathcal{L}_c u \geq 0$ and attaining a *nonnegative* maximum. Therefore, by combining identity (4.1.11) with the above (4.1.19), we obtain the desired (4.1.20). This ends the proof. □

Remark 4.1.19. A closer inspection of the proof of Thm. 4.1.18 shows that we have indeed demonstrated the following result as well.

Let $\mathcal{L}$ be a linear PDO satisfying assumptions (DS), (DE) *and* (NTD) *in Sec. 4.1.1, and let $c \in C^\infty(\mathbb{R}^N, \mathbb{R})$ be nonnegative on $\mathbb{R}^N$. Let us assume that the operator $\mathcal{L}_c := \mathcal{L} - c$ is hypoelliptic on every open subset of $\mathbb{R}^N$.*

If $\Omega \subseteq \mathbb{R}^N$ is a connected open set, any function $u \in C^2(\Omega, \mathbb{R})$ satisfying $\mathcal{L}_c u \geq 0$ on Ω and attaining a nonnegative maximum in Ω is constant on Ω.

In fact, let $F(u) = \{x \in \Omega : u(x) = \max_\Omega u\}$ and let $\xi \in F(u)$. Since the operator $\mathcal{L}_c$ is non-totally degenerate and C^∞-hypoelliptic on every open subset of $\mathbb{R}^N$, we infer from Amano's Thm. 4.1.9 that the vector space spanned by the $\mathcal{L}_c$-canonical vector fields is a controllable family on Ω. On the other hand, since the canonical vector fields of $\mathcal{L}_c = \mathcal{L} - c$ *are the same* as those of $\mathcal{L}$, we see once again that the identity $F(u) = \Omega$ follows if we show that $F(u)$ is invariant w.r.t. the $\mathcal{L}$-canonical vector fields $X_1, \dots, X_N$ introduced in (4.1.16). At this point, it suffices to argue as in the proof of Thm. 4.1.18 .

As it is well-know, the Strong Maximum Principle for a linear PDO easily implies the Weak one. More precisely, we have the following result.

Theorem 4.1.20 (Weak Maximum Principle for $\mathcal{L}$)**.** *Let $\mathcal{L}$ be a linear PDO satisfying the assumptions introduced in Sec. 4.1.1, and let $c \in C^\infty(\mathbb{R}^N, \mathbb{R})$ be nonnegative on $\mathbb{R}^N$ (the case $c \equiv 0$ is allowed).*

Setting $\mathcal{L}_c := \mathcal{L} - c$, then the operator $\mathcal{L}_c$ satisfies the Weak Maximum Principle *(*WMP*, for short) on every bounded open set $\Omega \subseteq \mathbb{R}^N$, that is:*

$$\begin{cases} u \in C^2(\Omega, \mathbb{R}) \\ \mathcal{L}_c u \geq 0 \text{ on } \Omega \\ \limsup\limits_{x \to x_0} u(x) \leq 0 \text{ for every } x_0 \in \partial\Omega \end{cases} \implies \quad u \leq 0 \text{ on } \Omega. \tag{4.1.21}$$

Proof. Let $\Omega \subseteq \mathbb{R}^N$ be bounded open set, and let $u \in C^2(\Omega, \mathbb{R})$ be as in the left-hand side of (4.1.21). Since Ω is bounded, there exists $x_0 \in \overline{\Omega}$ s.t.

$$\limsup_{x \to x_0} u(x) = \sup_\Omega u. \tag{4.1.22}$$

We then distinguish two cases.

• $x_0 \in \partial\Omega$. In this case, identity (4.1.22) and the above (4.1.21) give $\sup_\Omega u = \limsup_{x \to x_0} u(x) \leq 0$, whence $u(x) \leq 0$ for every $x \in \Omega$.

• $x_0 \in \Omega$. In this case, identity (4.1.22) implies that $u(x_0) = \max_\Omega u$, that is, x_0 is an interior maximum point of u. If $u(x_0) < 0$, we conclude as above that $\max_\Omega u = u(x_0) < 0$. If, instead, $u(x_0) \geq 0$, we consider the connected component $\Omega_0 \subseteq \Omega$ of Ω containing x_0. Thanks to part (ii) of the Strong Maximum Principle in Thm. 4.1.18, the existence of an interior maximum point of u on $\Omega \supseteq \Omega_0$ (and the fact that $u(x_0) \geq 0$) ensures that

$$u \equiv u(x_0), \quad \text{on } \Omega_0.$$

Thus, if we choose any $\xi_0 \in \partial\Omega_0 \subseteq \partial\Omega$, we obtain

$$\max_\Omega u = u(x_0) = \limsup_{\Omega_0 \ni x \to \xi_0} u(x) \leq \limsup_{\Omega \ni x \to \xi_0} u(x) \leq 0,$$

where the last inequality follows from the assumption in (4.1.21). □

Remark 4.1.21. By arguing as in the proof of Thm. 4.1.20 (and by exploiting Rem. 4.1.19 instead of Thm. 4.1.18 - (ii)) we also get the following result, where we alternatively replace the hypothesis of hypoellipticity of $\mathcal{L}$ by that of $\mathcal{L} - c$.

Let $\mathcal{L}$ be a linear PDO satisfying assumptions (DS), (DE) *and* (NTD) *in Sec. 4.1.1, and let $c \in C^\infty(\mathbb{R}^N, \mathbb{R})$ be nonnegative on $\mathbb{R}^N$. Let us assume that the operator $\mathcal{L}_c := \mathcal{L} - c$ is hypoelliptic on every open subset of $\mathbb{R}^N$.*

Then $\mathcal{L}_c$ satisfies the WMP *on every bounded open set $\Omega \subseteq \mathbb{R}^N$.*

4.1.5 Application to the Dirichlet problem and to Harnack's Inequality

We conclude this first part of the chapter by briefly describing how the Strong Maximum Principle can be profitably used for proving the solvability of the Dirichlet problem and the Harnack inequality for our PDOs $\mathcal{L}$ as in Sec. 4.1.1.

All the results we are going to present here can be found in the very recent paper [15]; for this reason, we prefer not to give any proofs of such results and we directly refer to the cited [15] for all the details.

The first result we aim to state concerns the solvability of the Dirichlet problem for our operators $\mathcal{L}$. Such a result can be proved in a standard way, by crucially exploiting the Strong Maximum Principle and a classical elliptic approximation argument (see [39, Section 5] for all the details).

Theorem 4.1.22 (Solvability of the Dirichlet problem for $\mathcal{L}$)**.** *Let $\mathcal{L}$ be a linear PDO satisfying assumptions* (DS), (DE) *and* (NTD) *in Sec. 4.1.1, and let $\varepsilon \geq 0$ be fixed (the case $\varepsilon = 0$ being admissible). We set $\mathcal{L}_\varepsilon := \mathcal{L} - \varepsilon$ and we assume that $\mathcal{L}_\varepsilon$ is C^∞-hypoelliptic on every open subset of $\mathbb{R}^N$.*

Then, there exists a basis $\mathcal{B}$ for the Euclidean topology of $\mathbb{R}^N$, independent of ε, made of open and connected sets Ω (with Lipschitz boundary) with the following properties: for every $f \in C(\overline{\Omega}, \mathbb{R})$ and for every $\varphi \in C(\partial\Omega, \mathbb{R})$, there exists one and only one solution $u \in C(\overline{\Omega}, \mathbb{R})$ of the Dirichlet problem

$$\begin{cases} \mathcal{L}_\varepsilon u = -f & \text{on } \Omega \quad \text{(in the sense of distributions)}, \\ u = \varphi & \text{on } \partial\Omega \quad \text{(pointwise)}. \end{cases} \tag{4.1.23}$$

Furthermore, if $f, \varphi \geq 0$ then $u \geq 0$ as well. Finally, if $f \in C^\infty(\Omega, \mathbb{R}) \cap C(\overline{\Omega}, \mathbb{R})$, then the same is true of u, and u is a classical solution of (4.1.23).

Remark 4.1.23. Let the assumptions and the notations in Thm. 4.1.22 apply. Before proceeding, we briefly describe (for the sake of completeness) how the basis $\mathcal{B}$ can be constructed. We closely follow the idea of Bony [39, Sec. 5].

For every fixed $x_0 \in \mathbb{R}^N$, the operator $\mathcal{L}_\varepsilon$ being nontotally degenerate (by assumption (NTD)), there exists a unitary vector $h_0 \in \mathbb{R}^N$ such that

$$\langle A(x_0)h_0, h_0\rangle > 0. \tag{4.1.24}$$

We then consider, for every $M, \delta > 0$, the neighborhood of x_0 defined as follows:

$$\Omega(x_0, M, \delta) := B(x_0 + M\,h_0, M + \delta) \cap B(x_0 - M\,h_0, M + \delta).$$

By exploiting (4.1.24), it is possible to find $\overline{M}_{x_0}, \overline{\delta}_{x_0} > 0$ such that, for any $\delta \le \overline{\delta}_{x_0}$ and any $M \ge \overline{M}_{x_0}$, the set $\Omega(x_0, M, \delta)$ satisfies the following property: *for every $y \in \Omega(x_0, M, \delta)$ there exists $\nu \in \mathbb{R}^N \setminus \{0\}$ such that*

$$\overline{B}(y + \nu, \|\nu\|) \cap \overline{\Omega}(x_0, M, \delta) = \{y\} \;\text{ and }\; \langle A(y)\,\nu, \nu\rangle > 0;$$

then, the basis $\mathcal{B}$ can be obtained as

$$\mathcal{B} := \big\{\Omega(x_0, \delta, M) : x_0 \in \mathbb{R}^N, \delta \le \overline{\delta}_{x_0} \text{ and } M \ge \overline{M}_{x_0}\big\}.$$

With the existence of the weak solution of the Dirichlet problem for $\mathcal{L}_\varepsilon$ on a bounded open set Ω, we can define the associated Green operator as usual.

Definition 4.1.24 (Green operator and Green measure)**.** Let the assumptions and the notations in Thm. 4.1.22 apply, and let $\Omega \in \mathcal{B}$.

We consider the operator (depending on $\mathcal{L}_\varepsilon$ and Ω; we avoid keeping track of the dependence on Ω in the notation)

$$G_\varepsilon : C(\overline{\Omega}, \mathbb{R}) \longrightarrow C(\overline{\Omega}, \mathbb{R}) \tag{4.1.25}$$

mapping $f \in C(\overline{\Omega}, \mathbb{R})$ into the function $G_\varepsilon(f)$ which is the unique distributional solution u in $C(\overline{\Omega}, \mathbb{R})$ of the Dirichlet problem

$$\begin{cases} \mathcal{L}_\varepsilon u = -f & \text{on } \Omega \quad \text{(in the sense of distributions)}, \\ u = 0 & \text{on } \partial\Omega \quad \text{(pointwise)}. \end{cases} \tag{4.1.26}$$

We call G_ε **the Green operator related to $\mathcal{L}_\varepsilon$ and to the open set Ω.**

By the Riesz Representation Theorem (which is applicable thanks to the monotonicity properties in Thm. 4.1.22 with respect to the function f), for every $x \in \overline{\Omega}$ there exists a (nonnegative) Radon measure $\lambda_{x,\varepsilon}$ on $\overline{\Omega}$ such that

$$G_\varepsilon(f)(x) = \int_{\overline{\Omega}} f(y)\,\mathrm{d}\lambda_{x,\varepsilon}(y), \quad \text{for every } f \in C(\overline{\Omega}, \mathbb{R}). \tag{4.1.27}$$

We call $\lambda_{x,\varepsilon}$ **the Green measure related to $\mathcal{L}_\varepsilon$ (to Ω and to x).**

Let now $\mathcal{L}$ be a linear PDO satisfying assumptions (DS) and (DE) in the above Sec. 4.1.1. We denote by ν the Radon measure on $\mathbb{R}^N$ with density V with respect to the standard Lebesgue measure, that is,

$$\nu(B) := \int_B V(x)\,\mathrm{d}x, \quad \text{for every Borel set } B \subseteq \mathbb{R}^N. \tag{4.1.28}$$

Thm. 4.1.25 below shows that the Green measure $\lambda_{x,\varepsilon}$ related to $\mathcal{L}_\varepsilon$ admits density with respect to the measure ν, which is extremely "well-behaved".

Theorem 4.1.25 (Green kernel for $\mathcal{L}_\varepsilon$). *Let the assumptions and the notations in Thm. 4.1.22 apply, and let $\Omega \in \mathcal{B}$.*

If G_ε and $\lambda_{x,\varepsilon}$ are the Green operator and the Green measure related to $\mathcal{L}_\varepsilon$ (Def. 4.1.24), there exists a function $k_\varepsilon : \Omega \times \Omega \to \mathbb{R}$, smooth and positive out of the diagonal of $\Omega \times \Omega$, such that the following representation holds true:

$$G_\varepsilon(f)(x) = \int_\Omega f(y)\, k_\varepsilon(x,y)\, \mathrm{d}\nu(y), \quad \textit{for every } x \in \Omega, \tag{4.1.29}$$

and for every $f \in C(\overline{\Omega}, \mathbb{R})$. We call k_ε the Green kernel related to $\mathcal{L}_\varepsilon$ *(and to the open set Ω).Furthermore, we have the following properties:*

(i) *Symmetry of the Green kernel:*

$$k_\varepsilon(x,y) = k_\varepsilon(y,x) \quad \textit{for every } x,y \in \Omega. \tag{4.1.30}$$

(ii) *For every fixed $x \in \Omega$, we have $\mathcal{L}_\varepsilon k_\varepsilon(x,\cdot) = 0$ on $\Omega \setminus \{x\}$; moreover, for every $\varphi \in C_0^\infty(\Omega, \mathbb{R})$ we have $G_\varepsilon(\mathcal{L}_\varepsilon \varphi) = -\varphi = \mathcal{L}_\varepsilon(G_\varepsilon(\varphi))$, that is*

$$-\varphi(x) = \int_\Omega \mathcal{L}_\varepsilon \varphi\, k_\varepsilon(x,\cdot)\, \mathrm{d}\nu = \mathcal{L}_\varepsilon\Big(\int_\Omega \varphi\, k_\varepsilon(x,\cdot)\, \mathrm{d}\nu \Big). \tag{4.1.31}$$

(iii) *For every fixed $x \in \Omega$, one has*

$$\lim_{y \to y_0} k_\varepsilon(x,y) = 0 \quad \textit{for any } y_0 \in \partial\Omega. \tag{4.1.32}$$

(iv) *For every fixed $x \in \Omega$, we have*

$$k_\varepsilon(x,\cdot) = k_\varepsilon(\cdot,x) \in L^1(\Omega) \quad \textit{and} \quad k_\varepsilon \in L^1(\Omega \times \Omega).$$

The existence of a very regular Green kernel for $\mathcal{L}_\varepsilon$ is a key fact for proving the announced Harnack Inequality for our PDOs $\mathcal{L}$: in fact, it brings along with a "Weak Harnack Inequality" for $\mathcal{L}$, from which we shall derive the classical one.

In order to proceed in this direction, we shall need a further assumption, very similar to (HY) (and, indeed, equivalent to it in many important cases):

(HY)$_\varepsilon$ *There exists $\varepsilon > 0$ such that the operator $\mathcal{L} - \varepsilon$ is C^∞-hypoelliptic in every open subset of $\mathbb{R}^N$.*

For operators $\mathcal{L}$ satisfying hypotheses (NTD), (HY) and (HY)$_\varepsilon$ we are able to prove the Harnack Inequality (see Thm. 4.1.30).

Remark 4.1.26. Hypothesis (HY)$_\varepsilon$ is implicit in hypothesis (HY) for notable classes of operators, whence our assumptions for the validity of the Harnack Inequality for $\mathcal{L}$ reduce to (NTD) and (HY) solely: namely, (HY) *implies* (HY)$_\varepsilon$ *in the following important cases:*

- for Hörmander operators, and, more generally, for second order *subelliptic* operators (in the usual sense of fulfilling a subelliptic estimate, see e.g., [96, 99]); indeed, any operator L in these classes of PDOs is hypoelliptic (see Hörmander [94], Kohn and Nirenberg [100]), and L still belongs to these classes after the addition of a smooth zero-order term;

- for operators with *real-analytic coefficients*. Indeed, in the C^ω case, one can apply known results by Oleĭnik and Radkevič ensuring that, for a general linear second-order PDO L with real-analytic coefficients, hypoellipticity is equivalent to the verification of Hörmander's Rank Condition for the L-canonical vector fields (see Sec. 4.1.2); this condition is clearly invariant under any change of the zero-order term of L, so that (HY) and $(\mathrm{HY})_\varepsilon$ are indeed equivalent.

The problem of establishing, in general, whether (HY) implies $(\mathrm{HY})_\varepsilon$ seems non-trivial. In this regard we recall that, for example, in the complex coefficient case the presence of a zero-order term (even a small ε) may drastically alter hypoellipticity (see for instance the example given by Stein in [125]); see also the very recent paper by Parmeggiani [122] for related topics.

We explicitly remark that the operators (4.1.4a)-to-(4.1.4d) in Exm. 4.1.4 are *not* subelliptic (nor C^ω), yet they satisfy hypotheses (NTD), (HY) and $(\mathrm{HY})_\varepsilon$. The lack of subellipticity is a consequence of the characterization of the subelliptic PDOs due to Fefferman and Phong [70, 69] (see also [99, Prop.1.3] or [96, Th.2.1 and Prop.2.1], jointly with the presence of a coefficient with a zero of infinite order in (4.1.4a)-to-(4.1.4d)). The second assertion concerning the verification of $(\mathrm{HY})_\varepsilon$ (the other hypotheses being already discussed) derives from the following result by Kohn, [99]: any operator of the form

$$L_1 + \lambda(x)\, L_2 \quad \text{in } \mathbb{R}^n_x \times \mathbb{R}^m_y$$

is hypoelliptic, where $\lambda \in C^\infty(\mathbb{R}_x)$, $\lambda \geq 0$ has a zero of infinite order at 0 (and no other zeroes of infinite order), and L_1 (operating in $x \in \mathbb{R}^n$) and L_2 (operating in $y \in \mathbb{R}^m$) are general second order PDOs with smooth coefficients and they are assumed to be subelliptic.

It is straightforward to recognize that by subtracting ε to any PDO in (4.1.4a)-to-(4.1.4d) we get an operator of the form $(L_1 - \varepsilon) + \lambda(x)\, L_2$, where λ has the required features, L_2 is uniformly elliptic (indeed, a classical Laplacian in all the examples), and $L_1 - \varepsilon$ is a uniformly elliptic operator (cases (4.1.4a)-to-(4.1.4c)) or it is a Hörmander operator (case (4.1.4d)).

The rôle of the perturbation $\mathcal{L} - \varepsilon$ of the operator $\mathcal{L}$ is clearly expressed by the following lemma, which is a simple consequence of the Weak Maximum Principle in Thm. 4.1.20. Such a result, plus some topological facts on hypoellipticity, is the key ingredient for the *Weak* Harnack Inequality related to $\mathcal{L}$.

Lemma 4.1.27. *Let $\mathcal{L}$ be a linear PDO satisfying the assumptions in Sec. 4.1.1 and assumption* $(\mathrm{HY})_\varepsilon$. *Moreover, let Ω be an open set in $\mathbb{R}^N$ as in the thesis of Thm. 4.1.22 and let Ω' be an open set containing $\overline{\Omega}$. Finally, we denote by k_ε the Green kernel related to $\mathcal{L}_\varepsilon$ and to the set Ω (as in Thm. 4.1.25).*

Then we have the estimate

$$u(x) \geq \varepsilon \int_\Omega u(y)\, k_\varepsilon(x,y)\, \mathrm{d}\nu(y), \quad \forall\, x \in \Omega, \tag{4.1.33}$$

holding true for every smooth nonnegative $\mathcal{L}$-harmonic function u in Ω'.

As anticipated, Lem. 4.1.27 gives the following Weak Harnack Inequality.

Theorem 4.1.28 (Weak Harnack inequality for derivatives)**.** *Let $\mathcal{L}$ be a linear PDO satisfying the assumptions in Sec. 4.1.1 and assumption* $(\mathrm{HY})_\varepsilon$.

Then, for every connected open set $O \subseteq \mathbb{R}^N$, every compact subset K of O, every $m \in \mathbb{N} \cup \{0\}$ and every $y_0 \in O$, it is possible to find a real constant $C(y_0) = C(\mathcal{L}, \varepsilon, O, K, m, y_0) > 0$ such that

$$\sum_{|\alpha| \le m} \sup_{x \in K} \left| \frac{\partial^\alpha u(x)}{\partial x^\alpha} \right| \le C(y_0)\, u(y_0), \tag{4.1.34}$$

for every nonnegative $\mathcal{L}$-harmonic function u in O.

In order to obtain, from Thm. 4.1.28, the classical version of the Harnack Inequality, we exploit the following result of Potential Theory. A proof of a more general abstract version of this useful result, in the framework of axiomatic harmonic spaces, can be found in the survey notes [42, pp.20–24] by Brelot, where this theorem is attributed to G. Mokobodzki.

Instead of appealing to an abstract Potential-Theoretic statement, we prefer to formulate the result under the following more specific form (where a harmonic sheaf related to a smooth PDO is considered).

Theorem 4.1.29. *Let L be a general second order linear PDO in $\mathbb{R}^N$ with smooth coefficients. Suppose the following conditions are satisfied.*

(Regularity) *There exists a basis $\mathcal{B}$ for the Euclidean topology of $\mathbb{R}^N$ (consisting of bounded open sets) such that, for every $\Omega \in \mathcal{B} \setminus \{\varnothing\}$ and for every $\varphi \in C(\partial\Omega, \mathbb{R})$, there exists a unique function $H^\Omega_\varphi \in C^2(\Omega) \cap C(\overline{\Omega})$ solving the Dirichlet problem (related to L)*

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

and satisfying $H^\Omega_\varphi \ge 0$ whenever $\varphi \ge 0$.

(Weak Harnack Inequality) *For every connected open set $O \subseteq \mathbb{R}^N$, every compact subset K of O and every $y_0 \in O$, it is possible to find a constant $C(y_0) = C(L, O, K, y_0) > 0$ such that*

$$\sup_K u \le C(y_0)\, u(y_0),$$

for every nonnegative L-harmonic function u in O.

Then, the following Strong Harnack Inequality *for L holds:* for every connected open set O and every compact subset K of O it is possible to find a real constant $M = M(L, O, K) \ge 1$ such that

$$\sup_K u \le M \inf_K u, \tag{4.1.35}$$

for every nonnegative L-harmonic function u in O.

By combining Thm. 4.1.28 with the above Thm. 4.1.29, we finally obtain the announced Harnack Inequality for our PDOs $\mathcal{L}$.

Theorem 4.1.30 (Harnack Inequality for $\mathcal{L}$). *Let $\mathcal{L}$ be a linear PDO satisfying the assumptions in Sec. 4.1.1 and assumption* $(\mathrm{HY})_\varepsilon$.

Then, for every connected open set $O \subseteq \mathbb{R}^N$ and every compact subset K of O, there exists a constant $M = M(\mathcal{L}, O, K) \geq 1$ such that

$$\sup_K u \leq M \inf_K u, \tag{4.1.36}$$

for every nonnegative $\mathcal{L}$-harmonic function u in O.

4.2 An Hardy-type inequality

In Euclidean space $\mathbb{R}^N$, with $N \geq 3$, it is very well known the following Hardy inequality, holding true for every function $u \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$:

$$\int_{\mathbb{R}^N} \frac{u^2(x)}{\|x\|^2}\,\mathrm{d}x \leq \left(\tfrac{2}{N-2}\right)^2 \int_{\mathbb{R}^N} \|\nabla u(x)\|^2\,\mathrm{d}x. \tag{4.2.1}$$

Obviously, this inequality is profoundly connected with the Euclidean setting under many respects: it involves the Euclidean norm $\|\cdot\|$ and the usual Euclidean gradient ∇, both being related -in their turn- to the classical Laplace operator Δ, since $\mathrm{div}(\nabla u) = \Delta u$ and since the (unique global) fundamental solution Γ of Δ is a constant multiple of $\|x\|^{2-N}$ (see Exm. 1.3.6 on page 16).

The research in the different variants, in the possible improvements and in new geometrical insight of the Hardy inequality (4.2.1) is still very active, as the following (partial) list of references show: [2, 3, 4, 5, 9, 11, 12, 21, 43, 44, 45, 47, 53, 57, 59, 71, 82, 85, 103, 108, 112, 121, 132]. See also the 2015 survey monograph [10]. Furthermore, in the last 15 years many remarkable contributions to Hardy-type inequalities have been provided in *subelliptic* contexts. This is especially true: in the setting of the Heisenberg group $\mathbb{H}^n$ [5, 55, 87, 119, 136]; for certain classes of linear and quasi-linear degenerate-elliptic operators [54, 56, 61, 62, 101, 137]; for Carnot groups [86, 102, 124]; for general Carnot-Carathéodory spaces [58]. In 1990, Garofalo and Lanconelli [80] first contributed to Hardy-type inequalities in the paramount prototype of subelliptic contexts: indeed, in [80] it is proved a Hardy-type inequality in the Heisenberg group $\mathbb{H}^n$, and it is derived from it an Uncertainty Principle as well. Moreover, the Hardy-type inequality is employed in obtaining a Unique Continuation result for the Schrödinger-type equation

$$-\Delta_{\mathbb{H}^n} u + Pu = 0,$$

where $\Delta_{\mathbb{H}^n}$ is the Kohn- Laplacian on $\mathbb{H}^n$ (and P is a suitable potential). In [80] it is followed (with all the novelties of a subelliptic context) the approach to Unique Continuation previously introduced by Garofalo and Lin in [81, 82] in the case of uniformly elliptic operators.

As it is proved by Garofalo in [79], when $\mathcal{L}$ admits a well-behaved positive and global fundamental solution Γ (this is the case, e.g., of homogeneous Hörmander operators), an Hardy-type inequality holds true, with the Euclidean gradient and the Euclidean distance replaced –roughly put– by the $\mathcal{L}$-gradient and by a substitute for a distance obtained by Γ over a normalizing kernel. Also, Lebesgue measure $\mathrm{d}x$ is replaced by the $\mathcal{L}$-weighted measure $V(x)\mathrm{d}x$.

We first derive a general L^2-Hardy inequality for C^2-functions (not necessarily compactly-supported) over the super-level sets $\Omega(x, r)$ of $\Gamma(x; \cdot)$: we frame this result as a consequence of the mean-value formulas naturally associated with the variational form of $\mathcal{L}$, these mean-value formulas having already showed to be very versatile in the study of the Potential Theory for $\mathcal{L}$, as in the recent investigations [1, 13, 34, 35] (where the same assumptions on $\mathcal{L}$ as in the present paper are made; in [34, 35] the case $V \equiv 1$ is considered).

For the sake of clarity, we now briefly describe how we aim to proceed.

- In Sec. 4.2.1 we describe the linear PDOs to which we aim to extend the Hardy inequality (4.2.1), and we fix some notations.
- Sec.s 4.2.2 and 4.2.3 are devoted to introduce the relevant "geometrical objects" needed for generalizing inequality (4.2.1).
- In Sec. 4.2.4 we prove some mean value formulas which generalize the very well known surface and solid ones of the Laplace operator.
- In Sec. 4.2.5 we present the generalization of the Hardy inequality obtained by Garofalo [79] to the PDOs $\mathcal{L}$ described in Sec. 4.2.1; the mean value formulas proved in Sec. 4.2.4 will be a fundamental ingredient.
- Finally, in Sec. 4.2.6 we describe how the Hardy-type inequality presented in Sec. 4.2.5 can be used to establish a result of (strong) Unique Continuation for the solutions of the Schrödinger-type equation

  $$-\mathcal{L}u + Pu = 0,$$

  where $\mathcal{L}$ is a sub-Laplacian on a Carnot group $\mathbb{G}$ and P is a continuous function on $\mathbb{G}$ (satisfying suitable estimates).

4.2.1 Main assumptions and notations

Throughout this section, we shall be concerned with linear PDOs $\mathcal{L}$ satisfying *all the properties* introduced in Sec. 4.1.1, that is,

- **(DS)**: Divergence Structure;
- **(DE)**: Degenerate-Ellipticity;
- **(NTD)**: Non-Total-Degeneracy;
- **(HY)**: Hypo-Ellipticity;

plus an important additional assumption which we now properly introduce. To this end, we first give the following important definition.

Definition 4.2.1 ($\mathcal{L}$-weighted measure). Let s be equal to $N-1$ or N. We denote by $\mu^s_{\mathcal{L}}$ the Borel measure on $\mathbb{R}^N$ with density V with respect to the usual s-dimensional Hausdorff measure H^s on $\mathbb{R}^N$, that is,

$$\mu^s_{\mathcal{L}}(A) := \int_A V(x)\,\mathrm{d}H^s(x), \quad \text{for every Borel set } A \subseteq \mathbb{R}^N. \tag{4.2.2}$$

Remark 4.2.2. Since V is continuous and strictly positive, a measurable function $f : \mathbb{R}^N \to \mathbb{R}$ is locally integrable on $\mathbb{R}^N$ with respect to H^s if and only if this holds true with respect to the measure $\mu^s_{\mathcal{L}}$.

In the particular case $s = N$, the quasi-divergence form (4.1.1) of $\mathcal{L}$ ensures that $\mathcal{L}$ is formally self-adjoint in the Hilbert space $L^2(\mathbb{R}^N, \mu^N_{\mathcal{L}})$, when restricted to the smooth and compactly supported functions, that is

$$\int_{\mathbb{R}^N} \varphi\, \mathcal{L}\psi \,\mathrm{d}\mu^N_{\mathcal{L}} = \int_{\mathbb{R}^N} \psi\, \mathcal{L}\varphi \,\mathrm{d}\mu^N_{\mathcal{L}}, \qquad \text{for all } \varphi, \psi \in C_0^\infty(\mathbb{R}^N). \tag{4.2.3}$$

With the definition of $\mathcal{L}$-weighted measure at hand, we now introduce the announced additional assumption we require on our PDOs $\mathcal{L}$ throughout this chapter (the acronym **(FS)** stands for Fundamental Solution):

(FS): $\mathcal{L}$ admits a *well-behaved global fundamental solution* with respect to the measure $\mu^N_{\mathcal{L}}$: by this, we mean that there exists a function

$$\Gamma : \mathcal{O} = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y\} \to \mathbb{R}$$

satisfying the following properties:

(a) For every $x \in \mathbb{R}^N$, we have $\Gamma(x; \cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} \Gamma(x; y)\, \mathcal{L}\varphi(y) \,\mathrm{d}\mu^N_{\mathcal{L}}(y) = -\varphi(x), \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^N, \mathbb{R}); \tag{4.2.4}$$

(b) for every $x \in \mathbb{R}^N$, $\Gamma(x; \cdot)$ has a pole at x and it vanishes at infinity:

$$\lim_{y\to x} \Gamma(x; y) = \infty \qquad \text{and} \qquad \lim_{\|y\|\to\infty} \Gamma(x; y) = 0; \tag{4.2.5}$$

(c) for every $x \in \mathbb{R}^N$, we have $\nabla(\Gamma(x; \cdot)) \neq 0$ on $\mathbb{R}^N \setminus \{x\}$;

(d) $\Gamma \in C^\infty(\mathcal{O}, \mathbb{R})$ and $\Gamma(x; y) > 0$ for every $x, y \in \mathcal{O}$;

(e) $\Gamma \in L^1_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R}^N)$.

For the sake of brevity, given $x \in \mathbb{R}^N$, in the sequel we set:

$$\Gamma_x : \mathbb{R}^N \setminus \{x\} \longrightarrow \mathbb{R}, \quad \Gamma_x(y) := \Gamma(x; y).$$

Remark 4.2.3. Before proceeding, it is appropriate to stop for a moment to compare the definition of fundamental solution w.r.t. $\mu^N_{\mathcal{L}}$ with the notion of fundamental solution introduced in Def. 1.3.5 on page 15.

To this end we observe that, since $\mathcal{L}$ is (formally) self-adjoint w.r.t. the measure $\mu^N_{\mathcal{L}}$ (see Rem. 4.2.2), we can write identity (4.2.4) in the following way

$$\int_{\mathbb{R}^N} \Gamma(x; y)\, \mathcal{L}^{*\mu}\varphi(y) \,\mathrm{d}\mu^N_{\mathcal{L}} = -\varphi(x), \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^N, \mathbb{R}), \tag{4.2.6}$$

where $\mathcal{L}^{*\mu}$ stands for the formal adjoint of $\mathcal{L}$ with respect to the measure $\mu^N_{\mathcal{L}}$ (note that, $\mathcal{L}$ being self-adjoint w.r.t. $\mu^N_{\mathcal{L}}$, we have $\mathcal{L}^{*\mu} = \mathcal{L}$). Written in this form, it is clear that identity (4.2.4) is totally analogous to identity (1.3.7) on page 15, if we replace the Lebesgue measure $\mathrm{d}x$ with the measure $\mu_{\mathcal{L}}^N$ and the classical adjoint $\mathcal{L}^*$ of $\mathcal{L}$ with the adjoint with respect to $\mu_{\mathcal{L}}^N$.

In the particular case when $V \equiv 1$ (that is, $\mathcal{L}$ is a pure divergence-form operator), identity (4.2.4) reduces to identity (1.3.7), and a fundamental solution w.r.t. $\mu_{\mathcal{L}}^N = H^N$ is a fundamental solution in the sense of Def. 1.3.5.

The main reason why, in the present chapter, we decided to deal with fundamental solutions w.r.t. the measure $\mu_{\mathcal{L}}^N$ comes from the Theory of sub-Laplace operators on real Lie groups. To be more precise, let $\mathbb{G} = (\mathbb{R}^N, *)$ be a Lie group (with neutral element e), let $\{X_1, \ldots, X_m\}$ be a system of Lie generators for Lie($\mathbb{G}$) and let μ be the Haar measure on $\mathbb{G}$ defined as follows:

$$\mu := V(x)\,\mathrm{d}x, \quad \text{where } V(x) = \frac{1}{\det\big(\mathcal{J}_{\tau_x(e)}\big)}.$$

As already discussed in Exm. 4.1.2, the linear PDO

$$\mathcal{L} := -\sum_{j=1}^{m} X_j^{*\,\mu}\, X_j$$

satisfies assumptions (DE)-to-(HY), and it represents the main prototype for our PDOs of the quasi-diverge form (4.1.1). Now, since $\mathcal{L}$ and μ are left-invariant on $\mathbb{G}$ and since $\mathcal{L}$ is self-adjoint with respect to μ, it is not difficult to prove that a (global) fundamental solution for $\mathcal{L}$ w.r.t. $\mu_{\mathcal{L}}^N = \mu$ (which is actually unique, see Remark (c) below) satisfies the following very natural properties:

$$\Gamma(x;y) = \Gamma(y;x) \quad \text{and} \quad \Gamma(x;y) = \Gamma_e\big(x^{-1} * y\big).$$

On the other hand, it is easy to check that these properties do not hold if Γ is a fundamental solution for $\mathcal{L}$ in the sense of Def. 1.3.5 on page 15.

It is thus clear that the notion of fundamental solution with respect to the measure $\mu_{\mathcal{L}}^N$ is the most natural to work with in the present context.

We now continue by highlighting, in the subsequent examples, some wide classes of linear PDOs satisfying all the assumptions listed above.

Example 4.2.4. Let $\mathbb{G} = (\mathbb{R}^N, *, \delta_\lambda)$ be a homogeneous Carnot group, with homogeneous dimension $Q > 2$, and let $\{X_1, \ldots, X_m\}$ be a set of Lie-generators of Lie($\mathbb{G}$). We already know from Exm. 4.1.1 that the sub-Laplacian

$$\mathcal{L} = \sum_{j=1}^{m} X_j^2$$

satisfies assumptions (DS)-to-(HY) (with $V \equiv 1$ on $\mathbb{R}^N$); moreover, by Folland's Thm. 1.3.9 on page 17, there exists a (unique) global fundamental solution for $\mathcal{L}$ with respect to the measure $\mu_{\mathcal{L}}^N = H^N$ satisfying properties (a)-to-(e).

Example 4.2.5. Let $X = \{X_1, \ldots, X_m\}$ be a set linearly independent smooth vector fields on $\mathbb{R}^N$ satisfying the following assumptions (see Sec. 3.2):

(1) $X_1, \ldots, X_m$ are homogeneous of degree 1 with respect to a suitable family $\{\delta_\lambda\}_{\lambda>0}$ of dilations on $\mathbb{R}^N$ of the form

$$\delta_\lambda(x) = (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_N} x_N),$$

where $1 = \sigma_1 \leq \ldots \leq \sigma_N$ and $Q := \sum_{j=1}^{N} \sigma_j \geq 2$;

(2) $X_1, \dots, X_m$ satisfy Hörmander's condition at every point of $\mathbb{R}^N$.

As already pointed out in Exm. 4.1.3, the homogeneous PDO

$$\mathcal{L} = \sum_{j=1}^{m} X_j^2$$

satisfies assumptions (DS)-to-(HY) (with $V \equiv 1$ on $\mathbb{R}^N$); moreover, thanks to all the results obtained in Chpt. 3 (see, for example, the summarizing Thm. 3.3.1 on page 79), we know that $\mathcal{L}$ admits a (unique) global fundamental solution Γ w.r.t. $\mu_{\mathcal{L}}^N = H^N$ which satisfies properties (a)-to-(c) and such that

$$\lim_{\|y\|\to\infty} \Gamma(x;y) = 0, \quad \text{for every fixed } x \in \mathbb{R}^N.$$

In many meaningful cases (as, for example, for the linear PDOs considered in Sec. 3.4), we are also able to prove that

$$\lim_{y\to x} \Gamma(x;y) = \infty, \quad \text{for every fixed } x \in \mathbb{R}^N.$$

We conclude this section with some remarks concerning assumption (FS).

(a) If $\mathcal{L}^*$ is the classical formal adjoint operator of $\mathcal{L}$ (in the usual Hilbert space $L^2(\mathbb{R}^N, \mathrm{d}x)$), we deduce from (4.2.3) that

$$\mathcal{L}^* u = V\mathcal{L}(u/V), \quad \text{for every } u \in C^\infty(\mathbb{R}^N, \mathbb{R});$$

as a consequence, we see that property (c) in our assumption (FS) is in fact equivalent to the following more familiar identity

$$\int_{\mathbb{R}^N} \Gamma(x;y)\, \mathcal{L}^*\varphi(y)\, \mathrm{d}y = -\frac{1}{V(x)}\, \varphi(x), \qquad \forall\, \varphi \in C_0^\infty(\mathbb{R}^N), \quad \forall\, x \in \mathbb{R}^N.$$

Hence $\mathcal{L}\Gamma_x = -\mathrm{Dir}_x/V(x)$ in the distribution sense and, since Γ_x is of class C^∞ on $\mathbb{R}^N \setminus \{x\}$ (as it follows from property (a)), we have

$$\mathcal{L}\Gamma_x(y) = 0, \quad \text{for every } y \in \mathbb{R}^N \setminus \{x\}. \tag{4.2.7}$$

(b) Assumption (e) on Γ is made only for technical purposes: we shall soon consider the level sets of Γ and we shall require that they be smooth manifolds; this is the reason why assumption (e) is made. It is worth noting that, if such an assumption is dropped, then Sard's Lemma ensures that almost every level set of Γ is a smooth manifold; hence our results may be restated in an obvious (but perhaps less effective) way.

(c) Since the operator $\mathcal{L}$ is C^∞-hypoelliptic on every open subset of $\mathbb{R}^N$ (by assumption (HY)) and it satisfies the Weak Maximum Principle (as a consequence of properties (DS)-to-(HY); see Thm. 4.1.20), we derive from Rem. 1.3.7 - (c) on page 16 that a global fundamental solution Γ for $\mathcal{L}$ with respect to the measure $\mu_{\mathcal{L}}^N$ (as in assumption (FS)) is actually unique.

4.2.2 Preliminaries on Γ-balls

The main aim of this section is to introduce the so-called Γ-balls, which represent the appropriate substitute for the Euclidean balls in our sub-elliptic context.

From now on, we denote by $\mathcal{L}$ a *fixed* linear PDO satisfying all the assumptions introduced in Sec. 4.2.1, and we denote by Γ its unique (global) fundamental solution with respect to $\mu_{\mathcal{L}}^N$ as in assumption (FS).

Definition 4.2.6 (Γ-ball)**.** Let $x \in \mathbb{R}^N$ be fixed and let $r > 0$. The set

$$\Omega(x,r) := \Big\{ y \in \mathbb{R}^N \setminus \{x\} : \Gamma(x;y) > 1/r \Big\} \cup \{x\} \tag{4.2.8}$$

will be called the **Γ-ball** (related to $\mathcal{L}$) of centre x and radius r.

Remark 4.2.7. Let $x \in \mathbb{R}^N$ be fixed. Since, by assumption (FS)-(a), we have $\Gamma_x > 0$ on $\mathbb{R}^N \setminus \{x\}$, we allow ourselves to consider the Γ-ball of centre x and "infinite radius": by an abuse of notation, we set

$$\Omega(x,\infty) := \Big\{ y \in \mathbb{R}^N \setminus \{x\} : \Gamma_x > 0 \Big\} \cup \{x\} = \mathbb{R}^N. \tag{4.2.9}$$

Example 4.2.8. Let us consider the classical Laplace operator $\mathcal{L} = \Delta$ on $\mathbb{R}^N$, with $N \geq 3$. Since the fundamental solution Γ of Δ is given by

$$\Gamma(x;y) = \frac{1}{N\,(N-2)\,\omega_N} \|x-y\|^{2-N}, \qquad x \neq y,$$

we easily see that, for every $x \in \mathbb{R}^N$ and every $r > 0$, we have

$$\Omega(x,r) = \{ y \in \mathbb{R}^N : \|y-x\| < \rho \}, \quad \text{with } \rho = \left(\frac{r}{N\,(N-2)\,\omega_N} \right)^{1/(N-2)}.$$

Remark 4.2.9. By crucially exploiting the properties of Γ contained in our assumption (FS), it is not difficult to see that, for every $x \in \mathbb{R}^N$:

- $\Omega(x,r)$ is a bounded open neighborhood of x and
$$\textstyle\bigcup_{r>0} \Omega(x,r) = \mathbb{R}^N, \qquad \bigcap_{r>0} \Omega(x,r) = \{x\}.$$
- the family $\{\Omega(x,r)\}_{r>0}$ is a basis of neighborhoods of x;
- for every compact set $K \subseteq \mathbb{R}^N$ and every $x \in \mathbb{R}^N$, it is possible to find $r = r(K,x) > 0$ such that $K \subseteq \Omega(x,r)$;
- the set $S(x,r) := \{ y \in \mathbb{R}^N \setminus \{x\} : \Gamma_x(y) = 1/r \}$ is a smooth submanifold of $\mathbb{R}^N$ of dimension $(N-1)$.

The following proposition shows that any Γ-ball is extremely well-behaved from a differentiable point of view.

Proposition 4.2.10. *For every fixed $x \in \mathbb{R}^N$ and every $r > 0$, the Γ-ball $\Omega(x,r)$ is an open subset of $\mathbb{R}^N$ with C^∞ boundary, coinciding with the interior of its closure. In particular, one has*

$$\partial\Omega(x,r) = \Big\{ y \in \mathbb{R}^N \setminus \{x\} : \Gamma_x(y) = 1/r \Big\}, \tag{4.2.10}$$

and the unit exterior normal to $\Omega(x,r)$ *is given by*

$$\nu^{\rm ext}_{x,r}(y) = -\frac{\nabla\Gamma_x(y)}{\|\nabla\Gamma_x(y)\|}, \quad \textit{for all } y \in \partial\Omega(x,r). \tag{4.2.11}$$

Proof. We first prove that (4.2.10) is fulfilled. To this end we observe that, since Γ_x is continuous, we obviously have $\partial\Omega(x,r) \subseteq S(x,r)$. To show the reverse inclusion, let $y \in S(x,r)$ be fixed and let ν be the (unit) vector given by

$$\nu := -\frac{\nabla\Gamma_x(y)}{\|\nabla\Gamma_x(y)\|} \tag{4.2.12}$$

(note that ν belongs to the normal space to $S(x,r)$ at y). If $\delta > 0$ is such that

$$\Big\{ y + t\nu : t \in \ (-\delta,\delta) \Big\} \ \subseteq \mathbb{R}^N \setminus \{x\},$$

by the Mean Value Theorem we can write (for a suitable $|\theta_t| < t$)

$$\begin{aligned} \Gamma_x(y+t\nu) &= \Gamma_x(y) + t\,\langle \nabla\Gamma_x(y+\theta_t\nu), \nu\rangle \\ &= 1/r + t\,\langle \nabla\Gamma_x(y+\theta_t\nu), \nu\rangle; \end{aligned} \tag{4.2.13}$$

from this, since $\theta_t \to 0$ as $t \to 0$, we get

$$\lim_{t\to 0} \langle \nabla\Gamma_x(y+\theta_t\nu), \nu\rangle = \langle \nabla\Gamma_x(y), \nu\rangle \overset{(4.2.12)}{=} -\|\nabla\Gamma_x(y)\| < 0. \tag{4.2.14}$$

By gathering together (4.2.13) and (4.2.14), we can then find $\delta_1 < \delta$ such that

(a) $\Gamma_x(y+t\nu) > 1/r$ for every $t \in (-\delta_1, 0)$,

(b) $\Gamma_x(y+t\nu) < 1/r$ for every $t \in (0, \delta_1)$;

hence, inequality (a) ensures that $y + t\nu \in \Omega(x,r)$ for every $-\delta_1 < t < 0$, and this proves that $y \in \partial\Omega(x,r)$, as desired.

With identity (4.2.10) at hand, it is easy to see that $\Omega(x,r)$ is regular for the Divergence Theorem: in fact, since $\partial\Omega(x,r) = S(x,r)$ is a smooth $(N-1)$-dimensional manifold, we have

$$\begin{aligned} \Omega(x,r) \ &\subseteq \ {\rm int}\big(\overline{\Omega(x,r)}\big) \subseteq \Omega(x,r) \cup {\rm int}\big(\partial\Omega(x,r)\big) \\ &\overset{(4.2.10)}{=} \Omega(x,r) \cup {\rm int}\big(S(x,r)\big) = \Omega(x,r), \end{aligned}$$

and this proves that $\Omega(x,r)$ is a regular open set of class C^∞.

Finally, let $y \in \partial\Omega(x,r) = S(x,r)$ and let $\nu^{\rm ext}_{x,r}(y)$ be the unit vector defined in (4.2.11). Obviously, $\nu^{\rm ext}_{x,r}(y)$ is orthogonal to $\partial\Omega(x,r)$ at y; moreover, inequalities (a) and (b) show that there exists $\delta_1 > 0$, depending on y, such that

$$y - t\nu^{\rm ext}_{x,r}(y) \in \Omega(x,r) \quad \text{and} \quad y + t\nu^{\rm ext}_{x,r}(y) \notin \overline{\Omega(x,r)}, \quad \text{for } 0 < t < \delta_1.$$

This demonstrates that the unit exterior normal to $\Omega(x,r)$ at y is precisely $\nu^{\rm ext}_{x,r}(y)$, and the proof is complete. □

Corollary 4.2.11. *For every fixed $x \in \mathbb{R}^N$ and every $0 < \rho < r$, the Γ-annulus*

$$\Omega(x, \rho, r) := \Big\{ y \in \mathbb{R}^N \setminus \{x\} : 1/r < \Gamma_x(y) < 1/\rho \Big\} \tag{4.2.15}$$

is a regular open set of class C^∞. In particular, we have

$$\partial\Omega(x, \rho, r) = \partial\Omega(x, r) \cup \partial\Omega(x, \rho),$$

and the unit exterior normal to $\Omega(x, \rho, r)$ is given by

$$\nu^{\mathrm{ext}}_{x,\rho,r}(y) = \begin{cases} -\dfrac{\nabla\Gamma_x(y)}{\|\nabla\Gamma_x(y)\|} =: \nu^{\mathrm{ext}}_{x,r}(y) & \text{if } y \in \partial\Omega(x, r), \\ +\dfrac{\nabla\Gamma_x(y)}{\|\nabla\Gamma_x(y)\|} =: -\nu^{\mathrm{ext}}_{x,\rho}(y) & \text{if } y \in \partial\Omega(x, \rho). \end{cases} \tag{4.2.16}$$

Another important property of the Γ-balls is expressed by the following lemma, which is a simple consequence of the Weak Maximum Principle for $\mathcal{L}$.

Lemma 4.2.12. *For every fixed $x \in \mathbb{R}^N$ and every $r > 0$, the open Γ-ball $\Omega(x, r)$ is a (path-wise) connected subset of $\mathbb{R}^N$.*

Proof. We assume, by contradiction, that $\Omega(x, r)$ is not connected. Hence, there exist two disjoint open sets $U_1, U_2 \subseteq \mathbb{R}^N$ such that

$$\Omega(x, r) = U_1 \cup U_2.$$

Only one of these sets, let U_2 say, contains x. Thus the function $u := \Gamma_x$ is smooth on $\mathbb{R}^N \setminus \{x\} \supseteq \overline{U_1}$ and it satisfies the following properties:

- $\mathcal{L}u = 0$ on U_1 (see identity (4.2.7));
- $u \equiv 1/r$ on $\partial U_1 \subseteq \partial\Omega(x, r)$ (see identity (4.2.10)).

Since U_1 is bounded (as it is a subset of $\Omega(x, r)$, see Rem. 4.2.9), the Weak Maximum Principle in Thm. 4.1.20 implies that $u \le 1/r$ on U_1. This is clearly a contradiction since, by definition, $u = \Gamma_x > 1/r$ on $U_1 \subseteq \Omega(x, r)$. □

We now establish some simple yet important properties of the measure $\mu^s_{\mathcal{L}}$.

To begin with, we observe that, by Rem. 4.2.9 and Prop. 4.2.10, for every $x \in \mathbb{R}^N$ and every $r > 0$ it holds that

- $\Omega(x, r)$ is $\mu^N_{\mathcal{L}}$ - measurable and its $\mu^N_{\mathcal{L}}$ - measure is *positive and finite*;
- $\partial\Omega(x, r)$ is $\mu^{N-1}_{\mathcal{L}}$ - measurable and its $\mu^{N-1}_{\mathcal{L}}$ - measure is *positive and finite.*

We also have the following simple lemma.

Lemma 4.2.13. *For every fixed $x \in \mathbb{R}^N$ we have*

$$\lim_{r \to 0^+} \frac{\mu^N_{\mathcal{L}}\big(\Omega(x, r)\big)}{r} = 0.$$

Proof. First of all we observe that, by definition, we have

$$\frac{\mu_{\mathcal{L}}^N\big(\Omega(x,r)\big)}{r} = \frac{1}{r}\int_{\Omega(x,r)} \mathrm{d}\mu_{\mathcal{L}}^N \leq \int_{\Omega(x,r)} \Gamma_x \,\mathrm{d}\mu_{\mathcal{L}}^N.$$

Therefore, since the function Γ_x is locally integrable on $\mathbb{R}^N$ with respect to the measure $\mu_{\mathcal{L}}^N$ (see Rem. 4.2.2) and since $\bigcap_{r>0}\Omega(x,r) = \{x\}$, we conclude that the above rhs vanishes as $r \to 0^+$, and the proof is complete. □

As for the integration of continuous functions on Γ-balls and on Γ-annuli (which are bounded subsets of $\mathbb{R}^N$), we have the following useful results.

Lemma 4.2.14. *Let $x \in \mathbb{R}^N$ and let $0 < \rho < r$. If f is a locally integrable function on the Γ-annulus $\Omega(x,\rho,r)$ and if $0 < \rho < a < b < r$, then we have*

$$\int_{1/b}^{1/a}\left(\int_{\partial\Omega(x,1/t)} \frac{f}{\|\nabla\Gamma_x\|}\,\mathrm{d}\mu_{\mathcal{L}}^{N-1}\right)\mathrm{d}t = \int_{\Omega(x,a,b)} f\,\mathrm{d}\mu_{\mathcal{L}}^N. \tag{4.2.17}$$

Proof. This is an immediate consequence of the notable Federer's Coarea Formula [67]: since Γ_x is smooth (hence, locally Lipschitz-continuous) out of x, and since $\Omega(x,a,b) \subseteq \mathbb{R}^N \setminus \{x\}$, we have (note that f is integrable on the bounded set $\Omega(x,a,b) \Subset \Omega(x,\rho,r)$ and recall the definition of $\mu_{\mathcal{L}}^s$)

$$\begin{aligned}\int_{\Omega(x,a,b)} f\,\mathrm{d}\mu_{\mathcal{L}}^N &= \int_{\Omega(x,a,b)} f\,V\,\mathrm{d}y = \int_{1/b}^{1/a}\left(\int_{\{\Gamma_x = t\}} \frac{f\,V}{\|\nabla\Gamma_x\|}\,\mathrm{d}H^{N-1}\right)\mathrm{d}t \\ &= \int_{1/b}^{1/a}\left(\int_{\partial\Omega(x,1/t)} \frac{f}{\|\nabla\Gamma_x\|}\,\mathrm{d}\mu_{\mathcal{L}}^{N-1}\right)\mathrm{d}t.\end{aligned}$$

This ends the proof. □

Proposition 4.2.15. *Let $\Omega := \Omega(x,r)$ be an open Γ-ball (also the case $r = \infty$ is allowed, see Rem. 4.2.7) and let $u \in C(\Omega \setminus \{x\}, \mathbb{R})$. Then the function*

$$m : (0,r) \longrightarrow \mathbb{R} \qquad m(\rho) := \int_{\partial\Omega(x,\rho)} u\,\mathrm{d}\mu_{\mathcal{L}}^{N-1}$$

is continuous on $(0,r)$. If, in addition, $u \in L^1_{\mathrm{loc}}(\Omega)$, then the function

$$M : (0,r) \longrightarrow \mathbb{R} \qquad M(\rho) := \int_{\Omega(x,\rho)} u\,\mathrm{d}\mu_{\mathcal{L}}^N$$

is of class C^1 on $(0,r)$ and

$$\frac{\mathrm{d}}{\mathrm{d}\rho}M(\rho) = \frac{1}{\rho^2}\int_{\partial\Omega(x,\rho)} \frac{u}{\|\nabla\Gamma_x\|}\,\mathrm{d}\mu_{\mathcal{L}}^{N-1}, \quad \textit{for } 0 < \rho < r. \tag{4.2.18}$$

Proof. Since $\partial\Omega(x,r)$ is a smooth manifold and the function Γ_x has no critical points in $\mathbb{R}^N \setminus \{x\}$ (see assumption (FS)-(e)), the continuity of m follows by standard arguments of Geometric Measure Theory.

As for the second assertion we observe that, if $u \in L^1_{\mathrm{loc}}(\Omega)$ and if $a \in (0,r)$, by Lem. 4.2.14 we can write, for every $a < \rho < r$,

$$\begin{aligned} M(\rho) &= M(a) + \int_{\Omega(x,a,\rho)} u \, \mathrm{d}\mu_{\mathcal{L}}^N \\ &\overset{(4.2.17)}{=} M(a) + \int_{1/\rho}^{1/a} \left(\int_{\partial\Omega(x,1/t)} \frac{u}{\|\nabla\Gamma_x\|} \, \mathrm{d}\mu_{\mathcal{L}}^{N-1} \right) \mathrm{d}t; \end{aligned}$$

therefore, since the integrand function in the far right-hand side of the above identity is continuous with respect to t (note that $u/\|\nabla\Gamma_x\|$ is continuous on $\Omega \setminus \{x\}$), the Fundamental Theorem of Calculus gives

$$\frac{\mathrm{d}}{\mathrm{d}\rho} M(\rho) = \frac{1}{\rho^2} \int_{\partial\Omega(x,\rho)} \frac{u}{\|\nabla\Gamma_x\|} \, \mathrm{d}\mu_{\mathcal{L}}^{N-1}, \quad \text{for } a < \rho < r.$$

By the arbitrariness of $a \in (0,r)$, we obtain the desired (4.2.18). □

Finally, we have the following useful Green-type formulas for $\mathcal{L}$.

Lemma 4.2.16 (Green's identities for $\mathcal{L}$). *Let $U \subseteq \mathbb{R}^N$ be an open set supporting the Divergence Theorem, and let $u, v \in C^2(\overline{U}, \mathbb{R})$. Then one has*

$$\int_U u \, \mathcal{L}v \, \mathrm{d}\mu_{\mathcal{L}}^N = \int_{\partial U} u \, \langle A\nabla v, \nu_U^{\mathrm{ext}} \rangle \, \mathrm{d}\mu_{\mathcal{L}}^{N-1} - \int_U \langle A\nabla u, \nabla v \rangle \, \mathrm{d}\mu_{\mathcal{L}}^N, \qquad (4.2.19)$$

$$\int_U \big(u \, \mathcal{L}v - v \, \mathcal{L}u \big) \, \mathrm{d}\mu_{\mathcal{L}}^N = \int_{\partial U} \Big(u \, \langle A\nabla v, \nu_U^{\mathrm{ext}} \rangle \; - v \, \langle A\nabla u, \nu_U^{\mathrm{ext}} \rangle \Big) \, \mathrm{d}\mu_{\mathcal{L}}^{N-1}, \qquad (4.2.20)$$

where ν_U^{ext} is the exterior normal on ∂U. We call formulas (4.2.19) *and* (4.2.20)*, respectively,* ***Green's first and second identities for $\mathcal{L}$.***

When $u \equiv 1$ one gets

$$\int_U \mathcal{L}v \, \mathrm{d}\mu_{\mathcal{L}}^N = \int_{\partial U} \langle A\nabla v, \nu_U^{\mathrm{ext}} \rangle \, \mathrm{d}\mu_{\mathcal{L}}^{N-1}. \qquad (4.2.21)$$

Proof. Identity (4.2.19) is an obvious consequence of the Divergence Theorem, taking into account the quasi-divergence form (4.1.1) of $\mathcal{L}$, and the very definition of the $\mathcal{L}$-weighted measures $\mu_{\mathcal{L}}^N$ and $\mu_{\mathcal{L}}^{N-1}$ (see Def. 4.2.1)

Identity (4.2.20) follows from (4.2.19) and from the symmetry of $A(x)$. □

4.2.3 Average operators

Now we have defined the Γ-balls related to $\mathcal{L}$, we introduce the $\mathcal{L}$-kernel and the $\mathcal{L}$-surface density. Such objects will be of fundamental importance in the Hardy-type inequality for $\mathcal{L}$ presented in Sec. 4.2.5.

Definition 4.2.17 ($\mathcal{L}$-gradient). Let $U \subseteq \mathbb{R}^N$ be a fixed open set and let $f : U \to \mathbb{R}$ be of class C^1. We introduce the function

$$\|\nabla_{\mathcal{L}} f\| : U \longrightarrow \mathbb{R}, \qquad \|\nabla_{\mathcal{L}} f\|(x) := \sqrt{\langle A(x)\nabla f(x), \nabla f(x) \rangle}. \qquad (4.2.22)$$

We say that $\nabla_{\mathcal{L}} f$ is the **gradient of f associated with $\mathcal{L}$.**

If $\mathcal{L} = \Delta$ is the classical Laplace operator on $\mathbb{R}^N$ (with $N \geq 3$), for every open set $U \subseteq \mathbb{R}^N$ and every $f \in C^1(U, \mathbb{R})$ we have (since $A(x) = \mathrm{Id}_N$)

$$\|\nabla_\Delta f(x)\| = \|\nabla f(x)\|.$$

We then give the following definition.

Definition 4.2.18 ($\mathcal{L}$-kernel and $\mathcal{L}$-surface density)**.** Let $x \in \mathbb{R}^N$ be arbitrarily fixed. The function

$$\psi_x^{\mathcal{L}} : \mathbb{R}^N \setminus \{x\} \longrightarrow \mathbb{R}, \qquad \psi_x^{\mathcal{L}}(y) := \|\nabla_{\mathcal{L}}\Gamma_x\|^2(y), \tag{4.2.23}$$

will be called the **$\mathcal{L}$-kernel**. Furthermore, we define

$$\mathcal{K}_x^{\mathcal{L}} : \mathbb{R}^N \setminus \{x\} \longrightarrow \mathbb{R}, \qquad \mathcal{K}_x^{\mathcal{L}}(y) := \frac{\psi_x^{\mathcal{L}}(y)}{\|\nabla\Gamma_x(y)\|}. \tag{4.2.24}$$

We shall call $\mathcal{K}_x^{\mathcal{L}}$ the **$\mathcal{L}$-surface density**.

Taking into account the definition of the $\mathcal{L}$-gradient, more explicitly we have

$$\begin{aligned} \psi_x^{\mathcal{L}}(y) &= \langle A(y)\nabla\Gamma_x(y), \nabla\Gamma_x(y)\rangle, \\ \mathcal{K}_x^{\mathcal{L}}(y) &= \frac{\langle A(y)\nabla\Gamma_x(y), \nabla\Gamma_x(y)\rangle}{\|\nabla\Gamma_x(y)\|}. \end{aligned} \tag{4.2.25}$$

Example 4.2.19. Let us consider once again the classical Laplace operator Δ on $\mathbb{R}^N$, with $N \geq 3$, and let $x \in \mathbb{R}^N$ be fixed. A direct computation, crucially based on the explicit expression of the global fundamental solution Γ of Δ (see Exm. 4.2.8), shows that, for every $y \in \mathbb{R}^N \setminus \{x\}$,

$$\psi_x^{\Delta}(y) = \|\nabla\Gamma_x(y)\|^2 = \left(\frac{1}{N\omega_N}\|x-y\|^{1-N}\right)^2.$$

From this, we deduce that the Δ-surface density is the function given by

$$\mathcal{K}_x^{\Delta}(y) = \frac{\psi^{\Delta}(y)}{\|\nabla\Gamma_x(y)\|} = \frac{1}{N\,\omega_N}\,\|y-x\|^{1-N}, \quad \text{for every } y \neq x.$$

Remark 4.2.20. Let $x \in \mathbb{R}^N$ be fixed. It is worth noting that, since Γ_x is smooth on $\mathbb{R}^N \setminus \{x\}$, then both $\psi_x^{\mathcal{L}}$ and $\mathcal{K}_x^{\mathcal{L}}$ are smooth on the same set. Furthermore, since $A(x)$ is positive semidefinite on $\mathbb{R}^N$, we have

$$\psi_x^{\mathcal{L}}(y),\ \mathcal{K}_x^{\mathcal{L}}(y) \geq 0 \quad \text{for every } y \in \mathbb{R}^N \setminus \{x\}.$$

In the next results we establish some integral identities satisfied by $\mathcal{K}_x^{\mathcal{L}}$.

Lemma 4.2.21. *Let $x \in \mathbb{R}^N$ be fixed. Then the integral function*

$$K(r) := \int_{\partial\Omega(x,r)} \mathcal{K}_x^{\mathcal{L}}(y)\,\mathrm{d}\mu_{\mathcal{L}}^{N-1}(y) \qquad (r > 0) \tag{4.2.26}$$

is constant on $(0,\infty)$. More precisely, $K \equiv 1$ on $(0,\infty)$.

Proof. We choose $r, \rho \in]0, \infty[$ with $\rho < r$. By applying identity (4.2.21) with U given by the Γ-annulus $\Omega(x, \rho, r)$ (with exterior normal denoted by $\nu^{\text{ext}}_{x,\rho,r}$) and with v set to be Γ_x, one has

$$\int_{\Omega(x,\rho,r)} \mathcal{L}\Gamma_x \, \mathrm{d}\mu^N_{\mathcal{L}} = \int_{\partial\Omega(x,r)\cup\partial\Omega(x,\rho)} \langle A\nabla\Gamma_x, \nu^{\text{ext}}_{x,\rho,r}\rangle \, \mathrm{d}\mu^{N-1}_{\mathcal{L}};$$

from this, since $\mathcal{L}\Gamma_x \equiv 0$ on $\mathbb{R}^N \setminus \{x\}$ (see (4.2.7)) and recalling (4.2.16), we get

$$\int_{\partial\Omega(x,r)} \langle A\nabla\Gamma_x, \nu^{\text{ext}}_{x,r}\rangle \, \mathrm{d}\mu^{N-1}_{\mathcal{L}} - \int_{\partial\Omega(x,\rho)} \langle A\nabla\Gamma_x, \nu^{\text{ext}}_{x,\rho}\rangle \, \mathrm{d}\mu^{N-1}_{\mathcal{L}} = 0, \tag{4.2.27}$$

where $\nu^{\text{ext}}_{x,r}$ is the exterior normal to $\partial\Omega(x, r)$. On the other hand, since we know from Prop. 4.2.10 that $\nu^{\text{ext}}_{x,r} = -\nabla\Gamma_x/\|\nabla\Gamma_x\|$ on $\partial\Omega(x, r)$, one has

$$\langle A(y)\nabla\Gamma_x(y), \nu^{\text{ext}}_{x,r}(y)\rangle = -\psi^{\mathcal{L}}_x/\|\Gamma_x(y)\| = -\mathcal{K}^{\mathcal{L}}_x(y) \quad \text{on } \partial\Omega(x, r); \tag{4.2.28}$$

therefore, the above (4.2.27) gives

$$\begin{aligned} 0 \;&=\; -\int_{\partial\Omega(x,r)} \mathcal{K}^{\mathcal{L}}_x(y) \, \mathrm{d}\mu^{N-1}_{\mathcal{L}}(y) + \int_{\partial\Omega(x,\rho)} \mathcal{K}^{\mathcal{L}}_x(y) \, \mathrm{d}\mu^{N-1}_{\mathcal{L}}(y) \\ &\overset{(4.2.26)}{=} -K(r) + K(\rho). \end{aligned}$$

As r and ρ are arbitrary, we infer that K is constant on $(0, \infty)$, say $K \equiv K_1$. We now turn to show that $K_1 = 1$. To this end, let $v \in C^\infty_0(\mathbb{R}^N, \mathbb{R})$ be such that $v(x) = 1$ and let $r > 0$ be such that (see Rem. 4.2.9)

$$\operatorname{supp}(v) \subseteq \Omega(x, r).$$

For any $0 < \rho < r$, we apply Green's second identity (4.2.20) with U given by $\Omega(x, \rho, r)$ and with $u = \Gamma_x$ and v as above: by recalling that $\mathcal{L}\Gamma_x \equiv 0$ outside x and that $v \equiv 0$ on $\partial\Omega(x, r)$, from (4.2.16) we get

$$\begin{aligned} \int_{\Omega(x,\rho,r)} \Gamma_x \, \mathcal{L}v \, \mathrm{d}\mu^N_{\mathcal{L}} = &- \int_{\partial\Omega(x,\rho)} \Gamma_x \, \langle A\nabla v, \nu^{\text{ext}}_{x,r}\rangle \, \mathrm{d}\mu^{N-1}_{\mathcal{L}} \\ &+ \int_{\partial\Omega(x,\rho)} v \, \langle A\nabla\Gamma_x, \nu^{\text{ext}}_{x,\rho}\rangle \, \mathrm{d}\mu^{N-1}_{\mathcal{L}}. \end{aligned} \tag{4.2.29}$$

We now aim to pass to the limit as $\rho \to 0^+$ in the above (4.2.29). As for the lhs we observe that, as Γ_x is locally integrable on $\mathbb{R}^N$,

$$\lim_{\rho\to 0^+} \int_{\Omega(x,\rho,r)} \Gamma_x \, \mathcal{L}v \, \mathrm{d}\mu^N_{\mathcal{L}} = \int_{\Omega(x,r)} \Gamma_x \, \mathcal{L}v \, \mathrm{d}\mu^N_{\mathcal{L}}. \tag{4.2.30}$$

Moreover, since $\Gamma_x \equiv 1/\rho$ on $\partial\Omega(x, \rho)$, by applying identity (4.2.21) to the first integral in the right-hand side of (4.2.29) we get

$$\begin{aligned} &\left| \int_{\partial\Omega(x,\rho)} \Gamma_x \, \langle A\nabla v, \nu^{\text{ext}}_{x,\rho}\rangle \, \mathrm{d}\mu^{N-1}_{\mathcal{L}} \right| \\ &= \frac{1}{\rho} \left| \int_{\Omega(x,\rho)} \mathcal{L}v \, \mathrm{d}\mu^N_{\mathcal{L}} \right| \le \sup |\mathcal{L}v| \cdot \frac{\mu^N_{\mathcal{L}}\big(\Omega(x, \rho)\big)}{\rho}; \end{aligned}$$

hence, thanks to Lem. 4.2.13, we obtain

$$\lim_{\rho\to 0^+}\int_{\partial\Omega(x,\rho)}\Gamma_x\,\langle A\nabla v,\nu^{\mathrm{ext}}_{x,\rho}\rangle\,\mathrm{d}\mu^{N-1}_{\mathcal{L}}=0. \tag{4.2.31}$$

As for the second integral in the rhs of (4.2.29) we observe that, by arguing as in (4.2.28) and by recalling that $K(\rho)=K_1$, we get

$$\begin{aligned}&\int_{\partial\Omega(x,\rho)}v\,\langle A\nabla\Gamma_x,\nu^{\mathrm{ext}}_{x,\rho}\rangle\,\mathrm{d}\mu^{N-1}_{\mathcal{L}}\\&\qquad=-\int_{\partial\Omega(x,\rho)}\big(v(y)-1\big)\,\mathcal{K}^{\mathcal{L}}_x(y)\,\mathrm{d}\mu^{N-1}_{\mathcal{L}}(y)-K_1=:-J(\rho)-K_1.\end{aligned}$$

On the other hand, since v is continuous on $\mathbb{R}^N$ and $v(x)=1$ (and again recalling that $K(\rho)\equiv K_1$), it is easy to recognize that that $\lim_{\rho\to 0^+}J(\rho)=0$; as a consequence we derive

$$\lim_{\rho\to 0^+}\int_{\partial\Omega(x,\rho)}v\,\langle A\nabla\Gamma_x,\nu^{\mathrm{ext}}_{x,\rho}\rangle\,\mathrm{d}\mu^{N-1}_{\mathcal{L}}=-K_1. \tag{4.2.32}$$

By gathering together identities (4.2.30), (4.2.31) and (4.2.32), we can pass to the limit in the above (4.2.29), obtaining

$$\int_{\Omega(x,r)}\Gamma_x\,\mathcal{L}v\,\mathrm{d}\mu^N_{\mathcal{L}}=-K_1. \tag{4.2.33}$$

We are now ready to conclude: since $v\in C_0^\infty(\mathbb{R}^N)$ is supported in $\Omega(x,r)$ and since Γ is a fundamental solution for $\mathcal{L}$ w.r.t. $\mu^N_{\mathcal{L}}$, we get (see identity (4.2.4))

$$-K_1\overset{(4.2.33)}{=}\int_{\Omega(x,r)}\Gamma_x\,\mathcal{L}v\,\mathrm{d}\mu^N_{\mathcal{L}}\overset{(4.2.4)}{=}-v(x)=-1,$$

and this gives out $K_1=1$, as desired. □

From Lem. 4.2.21, one straightforwardly obtains:

Corollary 4.2.22. *Let $x\in\mathbb{R}^N$ be fixed and let $U\subseteq\mathbb{R}^N$ be an open set containing x. If $u\in C(U,\mathbb{R})$, it holds that*

$$\lim_{\rho\to 0^+}\int_{\partial\Omega(x,\rho)}u(y)\,\mathcal{K}^{\mathcal{L}}_x(y)\,\mathrm{d}\mu^{N-1}_{\mathcal{L}}(y)=u(x). \tag{4.2.34}$$

Proof. Let $r>0$ be such that $\Omega(x,r)\subseteq U$. For every fixed $0<\rho<r$, it follows from Lem. 4.2.21 that

$$\begin{aligned}&\int_{\partial\Omega(x,\rho)}u(y)\,\mathcal{K}^{\mathcal{L}}_x(y)\,\mathrm{d}\mu^{N-1}_{\mathcal{L}}(y)\\&\qquad=\int_{\partial\Omega(x,\rho)}(u(y)-u(x))\,\mathcal{K}^{\mathcal{L}}_x(y)\,\mathrm{d}\mu^{N-1}_{\mathcal{L}}(y)+u(x);\end{aligned} \tag{4.2.35}$$

on the other hand, since v is continuous on U and by exploiting once again Lem. 4.2.21, we easily deduce that

$$\lim_{\rho\to 0^+}\int_{\partial\Omega(x,\rho)}(u(y)-u(x))\,\mathcal{K}^{\mathcal{L}}_x(y)\,\mathrm{d}\mu^{N-1}_{\mathcal{L}}(y)=0.$$

From this, by passing to the limit as $\rho\to 0^+$ in the above (4.2.35), we immediately obtain the desired (4.2.34). This ends the proof. □

We conclude this section by establishing another useful integrability property of the $\mathcal{L}$-kernel $\psi_x^{\mathcal{L}}$ (see Def. 4.2.18).

Lemma 4.2.23. *For every fixed $x \in \mathbb{R}^N$ and every real $\alpha > 1$, one has*

$$\int_{\Omega(x,r)} \frac{\psi_x^{\mathcal{L}}(y)}{\Gamma_x^\alpha(y)} \,\mathrm{d}\mu_{\mathcal{L}}^N(y) = \frac{r^{\alpha-1}}{\alpha-1}, \qquad \text{for all } r > 0. \tag{4.2.36}$$

Proof. Let $r > 0$ and let $0 < a < r$. Since the function $\psi_x^{\mathcal{L}}/\Gamma_x^\alpha$ is continuous on $\mathbb{R}^N \setminus \{x\}$, identity (4.2.17) in Lem. 4.2.14 gives

$$\int_{\Omega(x,a,r)} \frac{\psi_x^{\mathcal{L}}}{\Gamma_x^\alpha} \,\mathrm{d}\mu_{\mathcal{L}}^N = \int_{1/r}^{1/a} \frac{1}{t^\alpha} \left(\int_{\partial\Omega(x,1/t)} \frac{\psi_x^{\mathcal{L}}}{\|\nabla \Gamma_x\|} \,\mathrm{d}\mu_{\mathcal{L}}^{N-1} \right) \mathrm{d}t;$$

therefore, from Lem. 4.2.21 we derive that

$$\int_{\Omega(x,a,r)} \frac{\psi_x^{\mathcal{L}}}{\Gamma_x^\alpha} \,\mathrm{d}\mu_{\mathcal{L}}^N = \int_{1/r}^{1/a} \frac{1}{t^\alpha} \,\mathrm{d}t = \frac{a^{\alpha-1} - r^{\alpha-1}}{1-\alpha}.$$

Finally, passing to the limit as $a \downarrow 0^+$, the Monotone Convergence Theorem applied to the above identity produces the desired (4.2.36). □

4.2.4 Mean Value Formulas

Thanks to the integral properties of the $\mathcal{L}$-surface density $\psi_x^{\mathcal{L}}$ established in the previous section, we are in a position to prove the following Surface Mean Value formula for $\mathcal{L}$. As we shall see in a moment, such a formula generalizes to our setting the analogous one of the Laplace operator.

Theorem 4.2.24 (Surface Mean Value Formula for $\mathcal{L}$). *Let $U \subseteq \mathbb{R}^N$ be an open set and let $u \in C^2(U, \mathbb{R})$. For every $x \in U$ and every $r > 0$ such that $\overline{\Omega(x,r)} \subseteq U$, one has the integral identity*

$$\begin{aligned} u(x) = &\int_{\partial\Omega(x,r)} u(y)\, \mathcal{K}_x^{\mathcal{L}}(y) \,\mathrm{d}\mu_{\mathcal{L}}^{N-1}(y) \\ &- \int_{\Omega(x,r)} \Big(\Gamma_x(y) - \frac{1}{r}\Big) \mathcal{L}u(y) \,\mathrm{d}\mu_{\mathcal{L}}^N(y), \end{aligned} \tag{4.2.37}$$

which we shall refer to as the ***Surface Mean Value Formula for*** $\mathcal{L}$.

Proof. Let $x \in U$ and $r > 0$ be as in the statement above. For every $0 < \rho < r$, by applying Green's second identity (4.2.20) to the Γ-annulus $\Omega(x, \rho, r)$ and to the functions u and $v := \Gamma_x$, we get (by recalling (4.2.7), (4.2.16) and the fact that Γ_x is constant on the boundary of any Γ-ball)

$$\begin{aligned} -\int_{\Omega(x,\rho,r)} \Gamma_x \,\mathcal{L}u \,\mathrm{d}\mu_{\mathcal{L}}^N = &\int_{\partial\Omega(x,r)} u\, \langle A\nabla\Gamma_x, \nu_{x,r}^{\mathrm{ext}}\rangle \,\mathrm{d}\mu_{\mathcal{L}}^{N-1} \\ &- \int_{\partial\Omega(x,\rho)} u\, \langle A\nabla\Gamma_x, \nu_{x,\rho}^{\mathrm{ext}}\rangle \,\mathrm{d}\mu_{\mathcal{L}}^{N-1} \\ &- \frac{1}{r} \int_{\partial\Omega(x,r)} \langle A\nabla u, \nu_{x,r}^{\mathrm{ext}}\rangle \,\mathrm{d}\mu_{\mathcal{L}}^{N-1} \\ &+ \frac{1}{\rho} \int_{\partial\Omega(x,\rho)} \langle A\nabla u, \nu_{x,\rho}^{\mathrm{ext}}\rangle \,\mathrm{d}\mu_{\mathcal{L}}^{N-1}. \end{aligned}$$

By arguing as in (4.2.28) (in the proof of Lem. 4.2.21) and by exploiting (4.2.21) in Lem. 4.2.16, the above identity becomes

$$
\begin{aligned}
-\int_{\Omega(x,\rho,r)} \Gamma_x\, \mathcal{L}u \,\mathrm{d}\mu_{\mathcal{L}}^N = \int_{\partial\Omega(x,r)} & u(y)\, \mathcal{K}_x^{\mathcal{L}}(y)\,\mathrm{d}\mu_{\mathcal{L}}^{N-1}(y) \\
& - \frac{1}{r}\int_{\Omega(x,r)} \mathcal{L}u\,\mathrm{d}\mu_{\mathcal{L}}^N \\
& - \int_{\partial\Omega(x,\rho)} u(y)\; \mathcal{K}_x^{\mathcal{L}}(y)\,\mathrm{d}\mu_{\mathcal{L}}^{N-1}(y) \\
& + \frac{1}{\rho}\int_{\Omega(x,\rho)} \mathcal{L}u\,\mathrm{d}\mu_{\mathcal{L}}^N .
\end{aligned}
\tag{4.2.38}
$$

We now aim to pass to the limit as $\rho \to 0^+$ in (4.2.38). To this end we first observe that, since Γ_x is locally integrable in $\mathbb{R}^N$, we have

$$
\lim_{\rho\to 0^+}\int_{\Omega(x,\rho,r)} \Gamma_x\,\mathcal{L}u\,\mathrm{d}\mu_{\mathcal{L}}^N = \int_{\Omega(x,r)} \Gamma_x\,\mathcal{L}u\,\mathrm{d}\mu_{\mathcal{L}}^N; \tag{4.2.39}
$$

moreover, by Lem. 4.2.13 we get

$$
\lim_{\rho\to 0^+}\frac{1}{\rho}\int_{\Omega(x,\rho)} \mathcal{L}u\,\mathrm{d}\mu_{\mathcal{L}}^N = 0. \tag{4.2.40}
$$

Finally, Cor. 4.2.22 gives

$$
\lim_{\rho\to 0^+}\int_{\partial\Omega(x,\rho,r)} u(y)\; \mathcal{K}_x^{\mathcal{L}}(y)\,\mathrm{d}\mu_{\mathcal{L}}^{N-1}(y) = u(x), \tag{4.2.41}
$$

and thus, by gathering together identities (4.2.39), (4.2.40) and (4.2.41) and by letting $\rho \to 0^+$ in the above (4.2.38), we obtain the desired (4.2.37). □

Example 4.2.25. Let us consider, on Euclidean space $\mathbb{R}^N$ (with $N \geq 3$), the classical Laplace operator $\mathcal{L} = \Delta$. Moreover, let $U \subseteq \mathbb{R}^N$ be an open set and let $u \in C^2(U,\mathbb{R})$. If $x \in U$ and if $\rho > 0$ is such that $\overline{B(x,\rho)} \subseteq U$, we derive from Exm. 4.2.8 that $\overline{\Omega(x,r)} = \overline{B(x,\rho)} \subseteq U$, where

$$
\rho = \left(\frac{r}{N\,(N-2)\,\omega_N}\right)^{1/(N-2)}.
$$

Therefore, by applying the Surface Mean Value Formula (4.2.37) to the Γ-ball $\Omega(x,r)$ and to the function u, we get (note that, in this case, $\mu_{\mathcal{L}}^s = H^s$)

$$
\begin{aligned}
u(x) = & \int_{\partial B(x,\rho)} u(y)\,\mathcal{K}_x^{\mathcal{L}}(y)\,\mathrm{d}H^{N-1}(y) \\
& - \int_{B(x,\rho)} \Big(\Gamma_x(y) - \frac{1}{r}\Big)\,\Delta u(y)\,\mathrm{d}y.
\end{aligned}
$$

From this, by taking into account the explicit expression of Γ and of the Δ-surface density $\mathcal{K}_x^{\Delta}$ (see Exm. 4.2.19), we obtain

$$
\begin{aligned}
u(x) = & \frac{1}{N\,\omega_N\,\rho^{N-1}}\int_{\partial B(x,\rho)} u(y)\,\mathrm{d}H^{N-1}(y) \\
& - \int_{B(x,\rho)} \big(\Gamma(y-x) - \Gamma(\rho)\big)\,\Delta u(y)\,\mathrm{d}y,
\end{aligned}
$$

which is precisely the usual Surface Mean Value formula for Δ.

Due to the relevance of the Surface Mean Value Formula in Thm. 4.2.24, and inspired by the particular case of the Laplace operator in Exm. 4.2.25, we introduce the following operators.

Definition 4.2.26. Let $U \subseteq \mathbb{R}^N$ be an open set and let $u \in C(U, \mathbb{R})$. For every $x \in U$ and every $r > 0$ such that $\overline{\Omega(x,r)} \subseteq U$, we define

$$\begin{aligned} \mathcal{M}_r(u)(x) &:= \int_{\partial\Omega(x,r)} u(y)\, \mathcal{K}_x^{\mathcal{L}}(y)\, \mathrm{d}\mu_{\mathcal{L}}^{N-1}(y), \\ \mathcal{N}_r(u)(x) &:= \int_{\Omega(x,r)} u(y) \Big(\Gamma_x(y) - \frac{1}{r}\Big)\, \mathrm{d}\mu_{\mathcal{L}}^{N}(y). \end{aligned} \tag{4.2.42}$$

We shall refer to $\mathcal{M}_r$ as the **surface mean value operator (related to $\mathcal{L}$)**.

By means of the above operators $\mathcal{M}_r$ and $\mathcal{N}_r$, we can restate Thm. 4.2.24 as follows: if $U \subseteq \mathbb{R}^N$ is an open set and if $u \in C^2(U, \mathbb{R})$, for every $x \in U$ and every $r > 0$ such that the closure of $\Omega(x,r)$ is contained in U, we have

$$u(x) = \mathcal{M}_r(u)(x) - \mathcal{N}_r(\mathcal{L}u)(x). \tag{4.2.43}$$

Remark 4.2.27. Let $x \in \mathbb{R}^N$ be fixed and let $r > 0$. It worth noting that, by Lem. 4.2.21, we have the following remarkable property:

$$\mathcal{M}_r(1)(x) = 1.$$

The following lemma concerns the regularity of $r \mapsto \mathcal{M}_r, \mathcal{N}_r$.

Lemma 4.2.28. *Let $U \subseteq \mathbb{R}^N$ be an open set and let $u \in C(U, \mathbb{R})$. Moreover, let $x \in U$ and $r > 0$ be such that $\overline{\Omega(x,r)} \subseteq U$. Then we have*

(i) *the function $\rho \mapsto \mathcal{M}_\rho(u)(x)$ is continuous on $]0, r]$, and*

$$\lim_{\rho \to 0^+} \mathcal{M}_\rho(u)(x) = u(x); \tag{4.2.44}$$

(ii) *the function $\rho \mapsto \mathcal{N}_\rho(u)(x)$ is of class C^1 on the same interval, and*

$$\lim_{\rho \to 0^+} \mathcal{N}_\rho(u)(x) = 0. \tag{4.2.45}$$

If, in addition, $u \in C^2(U, \mathbb{R})$, for every $\rho \in (0, r]$ we have,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\rho} \mathcal{M}_\rho(u)(x) &= \frac{1}{\rho^2} \int_{\Omega(x,\rho)} \mathcal{L}u\, \mathrm{d}\mu_{\mathcal{L}}^N \\ &= -\frac{1}{\rho^2} \int_{\partial\Omega(x,\rho)} \frac{\langle A\nabla u, \nabla\Gamma_x \rangle}{\|\nabla\Gamma_x\|}\, \mathrm{d}\mu_{\mathcal{L}}^{N-1}. \end{aligned} \tag{4.2.46}$$

Proof. (i) Since, by assumption, u is continuous on $\overline{\Omega(x,r)} \subseteq U$ and since the $\mathcal{L}$-surface density $\mathcal{K}_x^{\mathcal{L}}$ is smooth on $\mathbb{R}^N \setminus \{x\}$, the regularity of the function $\rho \mapsto \mathcal{M}_\rho(u)(x)$ on $]0, r]$ directly follows from Prop. 4.2.15; moreover, the limit (4.2.44) is precisely (4.2.34) proved in Cor. 4.2.22.

(ii) The regularity of $\rho \mapsto \mathcal{N}_\rho(u)(x)$ again follows from Prop. 4.2.15, since u is continuous on $\overline{\Omega(x,r)}$ and $\Gamma_x \in C^\infty(\mathbb{R}^N \setminus \{x\}, \mathbb{R}) \cap L^1_{\mathrm{loc}}(\mathbb{R}^N)$. As for the limit (4.2.45) we observe that, since $\Gamma_x, u \in L^1_{\mathrm{loc}}(U)$, we have

$$\lim_{\rho \to 0^+} \int_{\Omega(x,\rho)} u(y)\, \Gamma_x(y)\, \mathrm{d}\mu_{\mathcal{L}}^N(y) = 0;$$

moreover, from the continuity of u on U and from Lem. 4.2.13 we get

$$\lim_{\rho\to 0^+} \frac{1}{\rho} \int_{\Omega(x,\rho)} u(y)\,\mathrm{d}\mu_{\mathcal{L}}^N(y) = 0.$$

By gathering together these identities, we obtain the desired (4.2.45).

To complete the demonstration, we turn to show (4.2.46). To this end we observe that, if $u \in C^2(U,\mathbb{R})$, by the Surface Mean Value Formula (4.2.37) we have, for every fixed $0 < \rho < r$,

$$\mathcal{M}_\rho(u)(x) = u(x) + \int_{\Omega(x,\rho)} \Gamma_x\,\mathcal{L}u\,\mathrm{d}\mu_{\mathcal{L}}^N - \frac{1}{\rho}\int_{\Omega(x,\rho)} \mathcal{L}u\,\mathrm{d}\mu_{\mathcal{L}}^N. \tag{4.2.47}$$

We then differentiate the last two summands in the above (4.2.47): by (4.2.18) (since $\Gamma_x, \mathcal{L}u \in C(U\setminus\{x\},\mathbb{R}) \cap L^1_{\mathrm{loc}}(U)$) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\int_{\Omega(x,\rho)} \Gamma_x\,\mathcal{L}u\,\mathrm{d}\mu_{\mathcal{L}}^N\right) &= \frac{1}{\rho^2}\int_{\partial\Omega(x,\rho)} \frac{\Gamma_x\,\mathcal{L}u}{\|\nabla\Gamma_x\|}\,\mathrm{d}\mu_{\mathcal{L}}^{N-1}\\ &= \frac{1}{\rho^3}\int_{\partial\Omega(x,\rho)} \frac{\mathcal{L}u}{\|\nabla\Gamma_x\|}\,\mathrm{d}\mu_{\mathcal{L}}^{N-1};\end{aligned}$$

moreover, again from (4.2.18) we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\int_{\Omega(x,\rho)} \mathcal{L}u\,\mathrm{d}\mu_{\mathcal{L}}^N\right) = \frac{1}{\rho^2}\int_{\partial\Omega(x,\rho)} \frac{\mathcal{L}u}{\|\nabla\Gamma_x\|}\,\mathrm{d}\mu_{\mathcal{L}}^{N-1},$$

Summing up, from (4.2.47) we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\rho}\mathcal{M}_\rho(u)(x) &= \frac{1}{\rho^3}\int_{\partial\Omega(x,\rho)} \frac{\mathcal{L}u}{\|\nabla\Gamma_x\|}\,\mathrm{d}\mu_{\mathcal{L}}^{N-1} + \frac{1}{\rho^2}\int_{\Omega(x,\rho)} \mathcal{L}u\,\mathrm{d}\mu_{\mathcal{L}}^N\\ &\qquad - \frac{1}{\rho}\,\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\int_{\Omega(x,\rho)} \mathcal{L}u\,\mathrm{d}\mu_{\mathcal{L}}^N\right)\\ &= \frac{1}{\rho^2}\int_{\Omega(x,\rho)} \mathcal{L}u\,\mathrm{d}\mu_{\mathcal{L}}^N.\end{aligned}$$

This is the first identity in (4.2.46). The second one is a consequence of (4.2.21) and of the explicit expression of the unit exterior normal $\nu_{x,r}^{\mathrm{ext}}$ on the boundary of $\Omega(x,r)$ (see Prop. 4.2.10). This ends the proof. □

Remark 4.2.29. Let the assumption and the notations in Lem. 4.2.28 apply. Since both functions $\rho \mapsto \mathcal{M}_\rho(u)(x)$ and $\rho \mapsto \mathcal{N}_\rho(u)(x)$ are continuous on $]0,r]$ and they have finite limit as $\rho \to 0^+$, there exists a real constant $c > 0$ s.t.

$$\big|\mathcal{M}_\rho(u)(x)\big| \le c \quad \text{and} \quad \big|\mathcal{N}_\rho(u)(x)\big| \le c, \quad \text{for every } 0 < \rho \le r.$$

As a consequence, both $\mathcal{M}_\rho(u)(x)$ and $\mathcal{N}_\rho(u)(x)$ belong to $L^\infty([0,r])$.

We close this section by deriving, from the Surface Mean Value Formula (4.2.37), a family of Solid Mean Value Formulas for the operator $\mathcal{L}$. As in the case of the Laplace operator Δ, we use a superposition argument.

To begin with, we choose a nonnegative L^1_{loc} function $\varphi : [0,\infty[\to \mathbb{R}$ and, for every fixed $\rho > 0$, we define

$$\mathbf{c}_\rho(\varphi) := \int_0^\rho \varphi(t)\,\mathrm{d}t.$$

We then consider an open set $U \subseteq \mathbb{R}^N$ and a function $u \in C^2(U,\mathbb{R})$. If $x \in U$ and if $r > 0$ is such that $\overline{\Omega(x,r)} \subseteq U$, from (4.2.37) we get

$$u(x) = \mathcal{M}_\rho(u)(x) - \mathcal{N}_\rho(\mathcal{L}u)(x), \quad \text{for every } 0 < \rho \le r;$$

therefore, by multiplying both sides of the above identity times $\varphi(\rho)$ and by integrating with respect to ρ on $[0,r]$, we obtain

$$\begin{aligned} u(x) &= \frac{1}{\mathbf{c}_r(\varphi)} \int_0^r \varphi(\rho)\,\mathcal{M}_\rho(u)(x)\,\mathrm{d}\rho - \frac{1}{\mathbf{c}_r(\varphi)} \int_0^r \varphi(\rho)\,\mathcal{N}_\rho(\mathcal{L}u)(x)\,\mathrm{d}\rho \\ &=: \mathrm{M}^\varphi_r(u)(x) - \mathrm{N}^\varphi_r(\mathcal{L}u)(x). \end{aligned} \tag{4.2.48}$$

We explicitly observe that $\mathrm{M}^\varphi_r(u)(x)$ and $\mathrm{N}^\varphi_r(\mathcal{L}u)(x)$ are well-defined, since $\mathcal{M}_\rho(u)(x)$ and $\mathcal{N}_\rho(\mathcal{L}u)(x)$ are bounded on $[0,r]$ (see Rem. 4.2.29) and φ is locally integrable in $[0,\infty[$; moreover, by means of Federer's Coarea Formula, we can rewrite $\mathrm{M}^\varphi_r(u)(x)$ as follows (recall that $\varphi \ge 0$ on $[0,\infty[$)

$$\begin{aligned} \mathrm{M}^\varphi_r(u)(x) &= \frac{1}{\mathbf{c}_r(\varphi)} \int_0^r \varphi(\rho) \left(\int_{\partial\Omega(x,\rho)} u(y)\, \frac{\psi^{\mathcal{L}}_x(y)}{\|\nabla\Gamma_x(y)\|}\,\mathrm{d}\mu^{N-1}_{\mathcal{L}}(y) \right) \mathrm{d}\rho \\ &\quad \big(\text{by performing the change of variable } \rho = 1/t\big) \\ &= \frac{1}{\mathbf{c}_r(\varphi)} \int_{1/r}^\infty \frac{\varphi(1/t)}{t^2} \left(\int_{\partial\Omega(x,1/t)} u(y)\, \frac{\psi^{\mathcal{L}}_x(y)}{\|\nabla\Gamma_x(y)\|}\,\mathrm{d}\mu^{N-1}_{\mathcal{L}}(y) \right) \mathrm{d}t \\ &= \frac{1}{\mathbf{c}_r(\varphi)} \int_{1/r}^\infty \left(u(y)\,\varphi\!\left(\frac{1}{\Gamma_x(y)}\right) \frac{\psi^{\mathcal{L}}_x(y)}{\Gamma_x(y)^2}\, \frac{1}{\|\nabla\Gamma_x(y)\|}\,\mathrm{d}\mu^{N-1}_{\mathcal{L}}(y) \right) \mathrm{d}t \\ &\quad \big(\text{by Federer's Coarea Formula}\big) \\ &= \frac{1}{\mathbf{c}_r(\varphi)} \int_{\Omega(x,r)} u(y)\,\varphi\!\left(\frac{1}{\Gamma_x(y)}\right) \frac{\psi^{\mathcal{L}}_x(y)}{\Gamma_x(y)^2}\,\mathrm{d}\mu^N_{\mathcal{L}}(y). \end{aligned}$$

Summing up, we have thus proved the following notable result.

Theorem 4.2.30. *Let $\varphi : [0,\infty[\to \mathbb{R}$ be a nonnegative L^1_{loc} function, and let*

$$\mathbf{c}_r(\varphi) = \int_0^r \varphi(t)\,\mathrm{d}t.$$

Moreover, let $U \subseteq \mathbb{R}^N$ be an open set and let $u \in C^2(U,\mathbb{R})$. For every $x \in U$ and every $r > 0$ such that $\overline{\Omega(x,r)} \subseteq U$, we have

$$\begin{aligned} u(x) &= \frac{1}{\mathbf{c}_r(\varphi)} \int_{\Omega(x,r)} u(y)\,\varphi\!\left(\frac{1}{\Gamma_x(y)}\right) \frac{\psi^{\mathcal{L}}_x(y)}{\Gamma_x(y)^2}\,\mathrm{d}\mu^N_{\mathcal{L}}(y) \\ &\quad - \frac{1}{\mathbf{c}_r(\varphi)} \int_0^r \varphi(\rho) \left(\int_{\Omega(x,\rho)} \mathcal{L}u(y) \left(\Gamma_x(y) - \tfrac{1}{\rho}\right) \mathrm{d}\mu^N_{\mathcal{L}}(y) \right) \mathrm{d}\rho. \end{aligned} \tag{4.2.49}$$

We shall call (4.2.49) *the φ-**Solid Mean Value Formula for** $\mathcal{L}$.*

In particular, if in Thm. 4.2.30 we choose

$$\varphi : [0, \infty[\longrightarrow \mathbb{R}, \qquad \varphi(\rho) := \rho^\alpha \quad (\text{with } \alpha > -1),$$

we obtain the following family of Solid Mean Value Formulas for $\mathcal{L}$.

Proposition 4.2.31 (α-Solid Mean Value Formulas for $\mathcal{L}$)**.** *Let $U \subseteq \mathbb{R}^N$ be an open set and let $u \in C^2(U, \mathbb{R})$. Moreover, let $\alpha > -1$ be fixed. For every $x \in U$ and every $r > 0$ such that $\overline{\Omega(x,r)} \subseteq U$, we have*

$$\begin{aligned} u(x) &= \frac{\alpha+1}{r^{\alpha+1}} \int_{\Omega(x,r)} u(y)\, \frac{\psi_x^{\mathcal{L}}(y)}{\Gamma_x(y)^{2+\alpha}}\, \mathrm{d}\mu_{\mathcal{L}}^N(y) \\ &\quad - \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^{\alpha+1} \left(\int_{\Omega(x,\rho)} \mathcal{L}u(y) \left(\Gamma_x(y) - \frac{1}{\rho} \right) \mathrm{d}\mu_{\mathcal{L}}^N(y) \right) \mathrm{d}\rho. \end{aligned} \tag{4.2.50}$$

We shall call (4.2.50) *the α-**Solid Mean Value Formula for** $\mathcal{L}$.*

Due to the relevance of the α-Solid Mean Value Formulas in Prop. 4.2.31, and in analogy with the surface mean value operators in Def. 4.2.26, we also define the following solid mean value operators.

Definition 4.2.32 (α-solid mean value operator for $\mathcal{L}$)**.** Let $U \subseteq \mathbb{R}^N$ be an open set and let $u \in C(U, \mathbb{R})$. Moreover, let $\alpha > -1$ be fixed. For every $x \in U$ and every $r > 0$ such that $\overline{\Omega(x,r)} \subseteq U$, we set

$$\begin{aligned} \mathrm{M}_r^\alpha(u)(x) &:= \frac{\alpha+1}{r^{\alpha+1}} \int_{\Omega(x,r)} u(y)\, \frac{\psi_x^{\mathcal{L}}(y)}{\Gamma_x(y)^{2+\alpha}}\, \mathrm{d}\mu_{\mathcal{L}}^N(y), \\ \mathrm{N}_r^\alpha(u)(x) &:= \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^{\alpha+1} \left(\int_{\Omega(x,\rho)} u(y) \left(\Gamma_x(y) - \frac{1}{\rho} \right) \mathrm{d}\mu_{\mathcal{L}}^N(y) \right) \mathrm{d}\rho. \end{aligned} \tag{4.2.51}$$

We shall refer to M_r^α as the **α-solid mean value operator (related to $\mathcal{L}$)**.

By means of the operators M^α and N^α, we can restate Prop. 4.2.31 as follows: if $U \subseteq \mathbb{R}^N$ is as open set, if $u \in C^2(U, \mathbb{R})$ and if $\alpha > -1$, for every $x \in U$ and every $r > 0$ such that $\overline{\Omega(x,r)} \subseteq U$ we have

$$u(x) = \mathrm{M}_r^\alpha(x)(u) - \mathrm{N}_r^\alpha(x)(\mathcal{L}u).$$

Remark 4.2.33. Let $U \subseteq \mathbb{R}^N$ be an open set and let $u \in C(U, \mathbb{R})$. Moreover, let $\alpha > -1$ be fixed. If $x \in U$ and if $r > 0$ is such that $\overline{\Omega(x,r)} \subseteq U$, we have

$$\mathrm{N}_\rho^\alpha(u)(x) = \frac{\alpha+1}{\rho^{\alpha+1}} \int_0^\rho t^{\alpha+1}\, \mathcal{N}_\rho(u)(x)\, \mathrm{d}t, \quad \text{for } 0 < \rho \le r;$$

therefore, since $\rho \mapsto \mathcal{N}_\rho(u)(x)$ is continuous on $]0, r]$ (see Lem. 4.2.28), we derive that $\rho \mapsto \mathrm{N}_\rho^\alpha(u)(x)$ is of class C^1 on the same interval.

Analogously, since we have (by Federer's Coarea Formula)

$$\mathrm{M}_\rho^\alpha(u)(x) = \frac{\alpha+1}{\rho^{\alpha+1}} \int_0^\rho t^{\alpha+1}\, \mathcal{M}_t(u)(x)\, \mathrm{d}t, \quad \text{for } 0 < \rho \le r;$$

again from Lem. 4.2.28 we deduce that $\rho \mapsto \mathrm{M}_\rho^\alpha(u)(x)$ is of class C^1 on $]0, r]$. Finally, if $u \equiv 1$ on $\mathbb{R}^N$, from Lem. 4.2.23 we derive that

$$\mathrm{M}_r^\alpha(1)(x) = \frac{\alpha+1}{r^{\alpha+1}} \int_{\Omega(x,r)} \frac{\psi_x^{\mathcal{L}}(y)}{\Gamma_x(y)^{2+\alpha}}\, \mathrm{d}\mu_{\mathcal{L}}^N(y) = 1, \quad \text{for every } \alpha > -1.$$

Remark 4.2.34. Let $U \subseteq \mathbb{R}^N$ be an open set and let $u \in C(U, \mathbb{R})$. Moreover, let $\alpha > -1$ be fixed. Since, by Rem. 4.2.33, $\mathrm{M}_r^\alpha(1)(x) = 1$ for every $x \in \mathbb{R}^N$ and every $r > 0$, it is very easy to recognize that

$$\lim_{r \to 0^+} \mathrm{M}_r^\alpha(u)(x) = u(x), \quad \text{for every } x \in U.$$

Remark 4.2.35. Let the notation and the assumption in Thm. 4.2.30 apply. It is worth noting that, by Lem. 4.2.23 (and the continuity of u), the integral

$$\int_{\Omega(x,r)} u(y)\, \frac{\psi_x^{\mathcal{L}}(y)}{\Gamma_x(y)^{2+\alpha}}\, \mathrm{d}\mu_{\mathcal{L}}^N(y)$$

is well-defined and finite precisely when $\alpha + 2 > 1$, that is, if $\alpha > -1$.

4.2.5 An L^2-Hardy-type inequality

Thanks to the Surface Mean Value Formula (4.2.24) and to Lem. 4.2.28 in Sec. 4.2.4, we can finally present the L^2-Hardy-type inequality for $\mathcal{L}$ obtained by Garofalo [79]. This inequality being a direct consequence of the results obtained so far, we give its proof for the sake of completeness.

Theorem 4.2.36 (Hardy-type inequalities for $\mathcal{L}$)**.** *For every $x \in \mathbb{R}^N$, every $r > 0$ and every $u \in C^2(\mathbb{R}^N, \mathbb{R})$, the following Hardy-type inequality holds true*

$$\int_{\Omega(x,r)} u^2\, \frac{\psi_x^{\mathcal{L}}}{\Gamma_x^2}\, \mathrm{d}\mu_{\mathcal{L}}^N \le 4 \left(\frac{r}{2}\, \mathcal{M}_r(u^2)(x) + \int_{\Omega(x,r)} \|\nabla_{\mathcal{L}} u\|^2\, \mathrm{d}\mu_{\mathcal{L}}^N \right). \tag{4.2.52}$$

More explicitly, due to the very definition of the operator $\mathcal{M}$, we have

$$\int_{\Omega(x,r)} u^2\, \frac{\psi_x^{\mathcal{L}}}{\Gamma_x^2}\, \mathrm{d}\mu_{\mathcal{L}}^N \le 4 \left(\frac{r}{2} \int_{\partial\Omega(x,r)} u^2\, \mathcal{K}_x^{\mathcal{L}}\, \mathrm{d}\mu_{\mathcal{L}}^{N-1} + \int_{\Omega(x,r)} \|\nabla_{\mathcal{L}} u\|^2\, \mathrm{d}\mu_{\mathcal{L}}^N \right).$$

Proof. For every fixed $0 < a < r$, we have

$$\begin{aligned}
\int_{\Omega(x,a,r)} \frac{u^2}{\Gamma_x^2}\, \psi_x^{\mathcal{L}}\, \mathrm{d}\mu_{\mathcal{L}}^N &\overset{(4.2.17)}{=} \int_{1/r}^{1/a} \left(\int_{\partial\Omega(x,1/t)} \frac{u^2}{\Gamma_x^2}\, \mathcal{K}_x^{\mathcal{L}}\, \mathrm{d}\mu_{\mathcal{L}}^{N-1} \right) \mathrm{d}t \\
&= \int_{1/r}^{1/a} \frac{1}{t^2} \left(\int_{\partial\Omega(x,1/t)} u^2\, \mathcal{K}_x^{\mathcal{L}}\, \mathrm{d}\mu_{\mathcal{L}}^{N-1} \right) \mathrm{d}t \\
&= \int_{1/r}^{1/a} \frac{1}{t^2}\, \mathcal{M}_{1/t}(u^2)(x)\, \mathrm{d}t,
\end{aligned}$$

where in the last equality we used the Def. 4.2.26 of the surface mean value operator $\mathcal{M}$. Thanks to Lem. 4.2.28 we can integrate by parts, obtaining

$$\begin{aligned}
\int_{\Omega(x,a,r)} \frac{u^2}{\Gamma_x^2}\, \psi_x^{\mathcal{L}}\, \mathrm{d}\mu_{\mathcal{L}}^N &= \left[-\frac{1}{t}\, \mathcal{M}_{1/t}(u^2)(x) \right]_{t=1/r}^{t=1/a} + \int_{1/r}^{1/a} \frac{1}{t}\, \frac{\mathrm{d}}{\mathrm{d}t} \Big(\mathcal{M}_{1/t}(u^2)(x) \Big)\, \mathrm{d}t \\
&= r\, \mathcal{M}_r(u^2)(x) - a\, \mathcal{M}_a(u^2)(x) - \int_{1/r}^{1/a} \frac{1}{t^3}\, \frac{\mathrm{d}}{\mathrm{d}\rho} \Big(\mathcal{M}_\rho(u^2)(x) \Big)\Big|_{\rho=1/t} \\
&\overset{(4.2.46)}{=} r\, \mathcal{M}_r(u^2)(x) - a\, \mathcal{M}_a(u^2)(x) \\
&\quad + \int_{1/r}^{1/a} \frac{1}{t} \left(\int_{\partial\Omega(x,1/t)} \frac{\langle A\nabla(u^2), \nabla\Gamma_x \rangle}{\|\nabla\Gamma_x\|}\, \mathrm{d}\mu_{\mathcal{L}}^{N-1} \right) \mathrm{d}t.
\end{aligned}$$

We now focus on the last summand: from $\nabla(u^2) = 2u\nabla u$ and from identity (4.2.17) in Lem. 4.2.14, this last summand is

$$
\begin{aligned}
2\int_{1/r}^{1/a} \frac{1}{t}\left(\int_{\partial\Omega(x,1/t)} u\,\frac{\langle A\nabla u, \nabla\Gamma_x\rangle}{\|\nabla\Gamma_x\|}\,\mathrm{d}\mu_{\mathcal{L}}^{N-1}\right)\mathrm{d}t \\
&= 2\int_{1/r}^{1/a}\left(\int_{\partial\Omega(x,1/t)} \frac{u}{\Gamma_x}\,\frac{\langle A\nabla u, \nabla\Gamma_x\rangle}{\|\nabla\Gamma_x\|}\,\mathrm{d}\mu_{\mathcal{L}}^{N-1}\right)\mathrm{d}t \\
&\overset{(4.2.17)}{=} 2\int_{\Omega(x,a,r)} \frac{u}{\Gamma_x}\,\langle A\nabla u, \nabla\Gamma_x\rangle\,\mathrm{d}\mu_{\mathcal{L}}^{N}.
\end{aligned}
$$

By gathering the above identities, we get

$$
\begin{aligned}
\int_{\Omega(x,a,r)} \frac{u^2}{\Gamma_x^2}\,\psi_x^{\mathcal{L}}\,\mathrm{d}\mu_{\mathcal{L}}^{N} &= r\,\mathcal{M}_r(u^2)(x) - a\,\mathcal{M}_a(u^2)(x) \\
&\quad + 2\int_{\Omega(x,a,r)} \frac{u}{\Gamma_x}\,\langle A\nabla u, \nabla\Gamma_x\rangle\,\mathrm{d}\mu_{\mathcal{L}}^{N}.
\end{aligned}
\tag{4.2.53}
$$

The next step is to give an estimate of the integral in the right-hand side of (4.2.53). Since A is positive semi-definite we have (we make use of the $\mathcal{L}$-energy notation introduced in Def. 4.2.17 and of (4.2.23))

$$
\begin{aligned}
\left|\frac{u}{\Gamma_x}\,\langle A\nabla u, \nabla\Gamma_x\rangle\right| &\le \frac{|u|}{\Gamma_x}\left(\langle A\nabla u, \nabla u\rangle\right)^{1/2}\cdot\left(\langle A\nabla\Gamma_x, \nabla\Gamma_x\rangle\right)^{1/2} \\
&= \frac{|u|}{\Gamma_x}\left(\|\nabla_{\mathcal{L}} u\|^2\right)^{1/2}\cdot\left(\psi_x^{\mathcal{L}}\right)^{1/2}.
\end{aligned}
$$

Thus, by Hölder's inequality with $p = q = 1/2$, we get

$$
\begin{aligned}
2\int_{\Omega(x,a,r)} \frac{u}{\Gamma_x}\,\langle A\nabla u, \nabla\Gamma_x\rangle\,\mathrm{d}\mu_{\mathcal{L}}^{N} &\le 2\int_{\Omega(x,a,r)} \frac{|u|}{\Gamma_x}\left(\|\nabla_{\mathcal{L}} u\|^2\right)^{1/2}\cdot\left(\psi_x^{\mathcal{L}}\right)^{1/2}\mathrm{d}\mu_{\mathcal{L}}^{N} \\
&\le 2\left(\int_{\Omega(x,a,r)} \|\nabla_{\mathcal{L}} u\|^2\,\mathrm{d}\mu_{\mathcal{L}}^{N}\right)^{1/2}\left(\int_{\Omega(x,a,r)} \frac{u^2}{\Gamma_x^2}\,\psi_x^{\mathcal{L}}\,\mathrm{d}\mu_{\mathcal{L}}^{N}\right)^{1/2} \\
&\quad \big(\text{by a Young's inequality } 2AB \le \varepsilon A^2 + B^2/\varepsilon \text{ with } \varepsilon = 2\big) \\
&\le 2\int_{\Omega(x,a,r)} \|\nabla_{\mathcal{L}} u\|^2\,\mathrm{d}\mu_{\mathcal{L}}^{N} + \frac{1}{2}\int_{\Omega(x,a,r)} \frac{u^2}{\Gamma_x^2}\,\psi_x^{\mathcal{L}}\,\mathrm{d}\mu_{\mathcal{L}}^{N}.
\end{aligned}
$$

By inserting this estimate in (4.2.53) and moving terms around, we obtain

$$
\begin{aligned}
\frac{1}{2}\int_{\Omega(x,a,r)} \frac{u^2}{\Gamma_x^2}\,\psi_x^{\mathcal{L}}\,\mathrm{d}\mu_{\mathcal{L}}^{N} &\le r\,\mathcal{M}_r(u^2)(x) - a\,\mathcal{M}_a(u^2)(x) \\
&\quad + 2\int_{\Omega(x,a,r)} \|\nabla_{\mathcal{L}} u\|^2\,\mathrm{d}\mu_{\mathcal{L}}^{N}.
\end{aligned}
\tag{4.2.54}
$$

In order to complete the proof, we now pass to the limit as $a \to 0^+$. In the left-hand side, one can use the non-negativity of the integrand, so that, in the limit, $\Omega(x,a,r)$ simply becomes $\Omega(x,r)$. The same happens for the last summand in the right-hand side. Finally, from (4.2.44) we get $\lim_{a\to 0^+} \mathcal{M}_a(u^2)(x) = u^2(x)$, so that $\lim_{a\to 0^+} a\,\mathcal{M}_a(u^2)(x) = 0$. Therefore, (4.2.54) becomes

$$\frac{1}{2}\int_{\Omega(x,r)} \frac{u^2}{\Gamma_x^2}\,\psi_x^{\mathcal{L}}\,\mathrm{d}\mu_{\mathcal{L}}^N \le r\,\mathcal{M}_r(u^2)(x) + 2\int_{\Omega(x,r)} \|\nabla_{\mathcal{L}} u\|^2\,\mathrm{d}\mu_{\mathcal{L}}^N,$$

which is exactly (4.2.52). This ends the proof. □

As a direct consequence of Thm. 4.2.36 we obtain the following result, which represents the “true” generalization of the Hardy inequality (4.2.1).

Corollary 4.2.37. *For every $x \in \mathbb{R}^N$ and every $u \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$, the following Hardy-type inequality holds true*

$$\int_{\mathbb{R}^N} u^2\,\frac{\psi_x^{\mathcal{L}}}{\Gamma_x^2}\,\mathrm{d}\mu_{\mathcal{L}}^N \le 4\int_{\mathbb{R}^N} \|\nabla_{\mathcal{L}} u\|^2\,\mathrm{d}\mu_{\mathcal{L}}^N. \tag{4.2.55}$$

Proof. Let $r > 0$ be such that $\mathrm{supp}(u) \subseteq \Omega(x,r)$. By applying the Hardy-type inequality (4.2.52) to u and $\Omega(x,r)$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{u^2}{\Gamma_x^2}\,\psi_x^{\mathcal{L}}\,\mathrm{d}\mu_{\mathcal{L}}^N &= \int_{\Omega(x,r)} \frac{u^2}{\Gamma_x^2}\,\psi_x^{\mathcal{L}}\,\mathrm{d}\mu_{\mathcal{L}}^N \\ &\le 4\left(\frac{r}{2}\int_{\partial\Omega(x,r)} u^2\,\mathcal{K}_x^{\mathcal{L}}\,\mathrm{d}\mu_{\mathcal{L}}^{N-1} + \int_{\Omega(x,r)} \|\nabla_{\mathcal{L}} u\|^2\,\mathrm{d}\mu_{\mathcal{L}}^N\right) \\ &\big(u \equiv 0 \text{ on } \partial\Omega(x,r)\big) \\ &= 4\int_{\Omega(x,r)} \|\nabla_{\mathcal{L}} u\|^2\,\mathrm{d}\mu_{\mathcal{L}}^N = 4\int_{\mathbb{R}^N} \|\nabla_{\mathcal{L}} u\|^2\,\mathrm{d}\mu_{\mathcal{L}}^N.\end{aligned}$$

This ends the proof. □

Example 4.2.38. Let us consider, on Euclidean space $\mathbb{R}^N$ (with $N \ge 3$), the classical Laplace operator $\mathcal{L} = \Delta$, and let $x \in \mathbb{R}^N$ be fixed.

By exploiting the explicit expression of Γ (see Exm. 4.2.8) and of ψ_x^Δ (see Exm. 4.2.19), inequality (4.2.55) takes the following form

$$(N-2)^2\int_{\mathbb{R}^N} \frac{u^2(y)}{\|y-x\|^2}\,\mathrm{d}y \le 4\int_{\mathbb{R}^N} \|\nabla u\|^2\,\mathrm{d}y, \quad \forall\, u \in C_0^\infty(\mathbb{R}^N, \mathbb{R}).$$

In particular, taking $x = 0$, we obtain the Hardy Inequality (4.2.1).

By means of the Hardy-type inequality (4.2.55), we can easily extend to $\mathcal{L}$ also the classical Heisenberg Uncertainty Principle of Quantum Mechanics.

Corollary 4.2.39 (Uncertainty Principle for $\mathcal{L}$)**.** *For every $x \in \mathbb{R}^N$, every $u \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$ and every $\alpha \in \mathbb{R}$, the following inequality holds true:*

$$\left(\int_{\mathbb{R}^N} u^2\,\frac{\psi_x^{\mathcal{L}}}{\Gamma_x^{2\alpha-2}}\,\mathrm{d}\mu_{\mathcal{L}}^N\right)^{1/2}\left(\int_{\mathbb{R}^N} \|\nabla_{\mathcal{L}} u\|^2\,\mathrm{d}\mu_{\mathcal{L}}^N\right)^{1/2} \ge \frac{1}{2}\int_{\mathbb{R}^N} u^2\,\frac{\psi_x^{\mathcal{L}}}{\Gamma_x^{\alpha}}\,\mathrm{d}\mu_{\mathcal{L}}^N. \tag{4.2.56}$$

Furthermore, if $\alpha > 3/2$ all the integrals are finite.

Proof. Let us consider the nonnegative functions u_1, u_2 defined on $\mathbb{R}^N \setminus \{x\}$ by

$$u_1 := \frac{|u|}{\Gamma_x}\sqrt{\psi_x^{\mathcal{L}}}, \qquad u_2 := \frac{|u|}{\Gamma_x^{\alpha-1}}, \sqrt{\psi_x^{\mathcal{L}}}.$$

Since $u_1 \cdot u_2 = u^2\, \psi_x^{\mathcal{L}}/\Gamma_x^{\alpha}$ on $\mathbb{R}^N \setminus \{x\}$, by applying Hölder's Inequality to the integral at the right-hand side of (4.2.56) we get

$$\int_{\mathbb{R}^N} u^2\, \frac{\psi_x^{\mathcal{L}}}{\Gamma_x^{\alpha}}\, \mathrm{d}\mu_{\mathcal{L}}^N \le \left(\int_{\mathbb{R}^N} u^2\, \frac{\psi_x^{\mathcal{L}}}{\Gamma_x^{2}}\, \mathrm{d}\mu_{\mathcal{L}}^N\right)^{1/2} \left(\int_{\mathbb{R}^N} u^2\, \frac{\psi_x^{\mathcal{L}}}{\Gamma_x^{2\alpha-2}}\, \mathrm{d}\mu_{\mathcal{L}}^N\right)^{1/2}.$$

By applying the Hardy-type Inequality (4.2.55) to the first factor in the above right-hand side, we obtain (4.2.56). Thanks to Lemma 4.2.23, a sufficient condition for all the integrals to be finite is $\alpha > 3/2$. □

4.2.6 Application to Unique Continuation

The aim of this last section is to show how the Hardy-type inequality (4.2.52) in the previous section can be profitably used in obtaining a Unique Continuation result for the solutions of the Schrödinger-type equation

$$-\mathcal{L}u + Pu = 0, \tag{4.2.57}$$

where $\mathcal{L}$ is a sub-Laplacian on a Carnot group $\mathbb{G}$, P is a potential satisfying suitable assumptions, and u fulfills some (differential) growth condition.

We mainly follow the approach by Garofalo and Lanconelli in [80], where a the Hardy-type inequality (4.2.52) is employed in obtaining a Unique Continuation result for the Schrödinger-type equation $-\Delta_{\mathbb{H}^n} u + Pu = 0$, where $\Delta_{\mathbb{H}^n}$ is the Kohn-Laplacian on the Heisenberg group $\mathbb{H}^n$ and V is a suitable potential.

Some preliminaries. Throughout the sequel, we denote by $\mathbb{G} = (\mathbb{R}^N, *, \delta_\lambda)$ a fixed homogeneous Carnot group on $\mathbb{R}^N$, with homogeneous dimension $Q > 2$. Moreover, we choose once and for all a system $\{X_1, \dots, X_m\}$ of Lie-generators for $\mathrm{Lie}(\mathbb{G})$ and we denote by $\mathcal{L}$ the associated sub-Laplacian on $\mathbb{G}$, that is,

$$\mathcal{L} = \sum_{j=1}^{m} X_j^2.$$

As already pointed out in Exm. 4.2.4, the operator $\mathcal{L}$ satisfies all the assumptions introduced in Sec. 4.2.1. More precisely, we know that

(i) $\mathcal{L}$ is in the divergence form (4.1.1) with $V \equiv 1$;

(ii) the principal matrix $A(x)$ of $\mathcal{L}$ is given by

$$A(x) = S(x) \cdot S(x)^T, \tag{4.2.58}$$

where $S(x) = (X_1 I(x) \cdots X_m I(x))$ for every $x \in \mathbb{R}^N$;

(iii) there exists a unique global fundamental solution Γ for $\mathcal{L}$ w.r.t. $\mu_{\mathcal{L}}^N$ satisfying properties (a)-to-(e) in assumption (FS), which is of the form

$$\Gamma(x; y) = d^{2-Q}(x^{-1} * y), \quad \text{for every } x, y \in \mathbb{R}^N \text{ with } x \ne y, \tag{4.2.59}$$

for a suitable homogeneous symmetric norm $d \in C^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{R})$ on $\mathbb{G}$.

As a direct consequence of (i), we derive that the $\mathcal{L}$-weighed measures $\mu_{\mathcal{L}}^{N-1}$ and $\mu_{\mathcal{L}}^{N}$ are simply the Hausdorff measures H^{N-1} and H^N (see Def. 4.2.1); moreover, identity (4.2.59) easily implies that, for every $x \in \mathbb{R}^N$ and every $r > 0$, one has (denoting by τ_x the left-translation by x on $\mathbb{G}$)

$$\Omega(x,r) = x * \delta_\lambda\big(\Omega(0,1)\big) = (\tau_x \circ \delta_\lambda)\big(\Omega(0,1)\big), \quad \text{with } \lambda = r^{1/(2-Q)}.$$

In particular, $H^N\big(\Omega(x,r)\big) = r^{Q/(2-Q)}\,\omega_Q$, where $\omega_Q = H^N\big(\Omega(0,1)\big)$.

Let now $U \subseteq \mathbb{R}^N$ be an open set and let $f \in C^1(U,\mathbb{R})$. By exploiting identity (4.2.58), we see that the function $\|\nabla_{\mathcal{L}} f\|$ introduced in Def. 4.2.17 satisfies

$$\|\nabla_{\mathcal{L}} f\|^2(x) = \textstyle\sum_{j=1}^m (X_j f(x))^2 \qquad (x \in U). \tag{4.2.60}$$

Therefore, if we define the **$\mathcal{L}$-horizontal gradient of** f as

$$\nabla_{\mathcal{L}} f(x) := (X_1 f(x), \ldots, X_m f(x)) = \nabla f(x) \cdot S(x)^T, \qquad x \in U,$$

then $\|\nabla_{\mathcal{L}} f\|^2$ turns out to be a genuine norm squared, that is,

$$\|\nabla_{\mathcal{L}} f\|^2 = \langle \nabla_{\mathcal{L}} f, \nabla_{\mathcal{L}} f\rangle.$$

By means of this fact, we can provide an easy proof of the subsequent result.

Lemma 4.2.40. *The following properties hold true:*

(i) $\psi_x^{\mathcal{L}}(y) = \psi_0^{\mathcal{L}}(x^{-1} * y)$, *for every* $x, y \in \mathbb{R}^N$ *with* $x \neq y$;

(ii) $\psi_0^{\mathcal{L}}\big(\delta_\lambda(x)\big) = \lambda^{2\,(1-Q)}\,\psi_0^{\mathcal{L}}(x)$ *for every* $x \in \mathbb{R}^N \setminus \{0\}$ *and every* $\lambda > 0$.

Proof. (i) Let $x \in \mathbb{R}^N$ be fixed. By recalling the very definition of $\psi_x^{\mathcal{L}}$ and by exploiting identities (4.2.59) and (4.2.60), we can write

$$\psi_x^{\mathcal{L}}(y) = \sum_{j=1}^m \big(X_j(y \mapsto \Gamma_0(x^{-1} * y)\big)^2, \quad \text{for every } y \in \mathbb{R}^N \setminus \{x\}.$$

From this, $X_1, \ldots, X_m$ being left-invariant on $\mathbb{G}$, we obtain

$$\psi_x^{\mathcal{L}}(y) = \sum_{j=1}^m \big((X_j\Gamma_0)(x^{-1} * y)\big)^2 = \psi_0^{\mathcal{L}}(x^{-1} * y), \quad \forall\, y \in \mathbb{R}^N \setminus \{x\},$$

which is exactly the desired property (i).

(ii) Let $\lambda > 0$ be fixed. By arguing as in (i), we have

$$\psi_0^{\mathcal{L}}\big(\delta_\lambda(x)\big) = \sum_{j=1}^m \big((X_j\Gamma_0)(\delta_\lambda(x))\big)^2, \quad \text{for every } x \in \mathbb{R}^N \setminus \{0\}.$$

From this, recalling that $X_1, \ldots, X_m$ are δ_λ-homogeneous of degree 1 and that $\Gamma_0 = d^{2-Q}$ is δ_λ-homogeneous of degree $2-Q$, we obtain

$$\psi_0^{\mathcal{L}}\big(\delta_\lambda(x)\big) = \sum_{j=1}^m \big(\lambda^{1-Q}\,(X_j\Gamma_0)(x)\big)^2 = \lambda^{2\,(1-Q)}\,\psi_0^{\mathcal{L}}(x), \quad \forall\, x \in \mathbb{R}^N \setminus \{0\},$$

and this proves that $\psi_0^{\mathcal{L}}$ is δ_λ-homogeneous of degree $2-2Q$. □

Remark 4.2.41. Let $U \subseteq \mathbb{R}^N$ be an open set and let $f, g \in C^1(U,\mathbb{R})$. By exploiting once again the above (4.2.58), we obtain the following useful identity

$$\langle \nabla_{\mathcal{L}} f(x), \nabla_{\mathcal{L}} g(x)\rangle = \langle A(x)\,\nabla f(x), \nabla g(x)\rangle, \quad \text{for every } x \in U.$$

Three distinguished vector fields After all these preliminaries, we can proceed towards the announced Unique Continuation result for (the solutions of) the Schrödinger-type equation (4.2.57). To this end, we first introduce three selected vector fields which will be of fundamental importance in the sequel.

Definition 4.2.42. We define the following three smooth vector fields:

$$\mathcal{X} := \sum_{j=1}^{m} X_j\Gamma_0 \cdot X_j = \langle \nabla_{\mathcal{L}}\Gamma_0(x), \nabla_{\mathcal{L}} \rangle \qquad \text{on } \mathbb{R}^N \setminus \{0\}; \tag{4.2.61}$$

$$\mathcal{Z} := \sum_{j=1}^{N} \sigma_j\, x_j\, \frac{\partial}{\partial x_j} \qquad \text{on } \mathbb{R}^N; \tag{4.2.62}$$

$$\mathcal{R} := \Gamma_0\, \mathcal{X} + \frac{1}{Q-2}\, \psi_0^{\mathcal{L}}\, \mathcal{Z} \qquad \text{on } \mathbb{R}^N \setminus \{0\}. \tag{4.2.63}$$

Here $(\sigma_1, \ldots, \sigma_N)$ denotes the N-tuple of the exponents defining the dilation δ_λ of $\mathbb{G}$. We say that $\mathcal{Z}$ is the **infinitesimal generator** of the dilations of $\mathbb{G}$, and (following [83]) we say that $\mathcal{R}$ is the **$\mathcal{L}$- discrepancy**.

Remark 4.2.43. Let $U \subseteq \mathbb{R}^N$ be an open set and let $u : U \to \mathbb{R}$ be of class C^1. For every fixed $x \in U$, a direct computation shows that

$$\mathcal{Z}u(x) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=1} u(\delta_\lambda(x)); \tag{4.2.64}$$

moreover, by Rem. 4.2.41, we have

$$\mathcal{X}u(x) = \langle A(x)\nabla u(x), \nabla\Gamma_0(x)\rangle. \tag{4.2.65}$$

As is expected, $\mathcal{Z}$ can be used in order to characterize the δ_λ-homogeneous functions on $\mathbb{G}$, as in the following Euler-type result.

Lemma 4.2.44. *The following facts hold true:*

(i) *A function $f \in C^1(\mathbb{R}^N \setminus \{0\}, \mathbb{R})$ is δ_λ-homogeneous of degree $m \in \mathbb{R}$ iff*

$$\mathcal{Z} f(x) = m\, f(x), \quad \textit{for every } x \in \mathbb{R}^N \setminus \{0\}. \tag{4.2.66}$$

(ii) *A C^1 vector field Y on $\mathbb{R}^N$ is δ_λ-homogeneous of degree $m \in \mathbb{R}$ iff*

$$[Y, \mathcal{Z}] = m\, Y. \tag{4.2.67}$$

Proof. (i) Let us assume that f is δ_λ-homogeneous of degree m, and let $x \neq 0$ be fixed. For every $\lambda > 0$, we have

$$f\big(\delta_\lambda(x)\big) = \lambda^m\, f(x);$$

therefore, by differentiating both sides of this identity w.r.t. λ and by taking $\lambda = 1$, we immediately obtain the desired (4.2.66).

Conversely, let us assume that f satisfies (4.2.66) and, for a fixed $x \neq 0$, let $g :]0, \infty[\to \mathbb{R}$ be the function defined as follows:

$$g(\lambda) := f(\delta_\lambda(x)) - \lambda^m\, f(x).$$

Obviously, $g \in C^1(]0,\infty[,\mathbb{R})$; moreover, we have the computation

$$g'(\lambda) = \lambda^{-1}\Big(\big(\mathcal{Z} f\big)\big(\delta_\lambda(x)\big) - m\,\lambda^m\, f(x)\Big)$$
$$\overset{(4.2.66)}{=} \frac{m}{\lambda}\Big(f\big(\delta_\lambda(x)\big) - \lambda^m\, f(x)\Big) = \frac{m}{\lambda}\, g(\lambda), \quad \text{for every } \lambda > 0.$$

This proves that g solves the linear ODE $y' = (m/\lambda)\, y$ on $]0,\infty[$; since, by definition, $g(1) = 0$, we conclude that $g \equiv 0$ on $]0,\infty[$, that is,

$$f(\delta_\lambda(x)) = \lambda^m\, f(x), \quad \text{for every } \lambda > 0.$$

Hence, f is δ_λ-homogeneous of degree m, as desired.

(ii) First of all, if $a_1,\dots,a_N \in C^1(\mathbb{R}^N,\mathbb{R})$ are the coefficient functions of Y, a direct computation based on the explicit expression of $\mathcal{Z}$ gives

$$[Y,\mathcal{Z}] = \sum_{j=1}^N \big(\sigma_j\, a_j(x) - \mathcal{Z}(a_j)(x)\big)\, \frac{\partial}{\partial x_j}. \tag{4.2.68}$$

Let us now assume that Y is δ_λ-homogeneous of degree m. Since, for every $j = 1,\dots,N$, the function a_j is δ_λ-homogeneous of degree $\sigma_j - m$ (see Thm. 1.2.2 on page 7), by combining identities (4.2.68) and (4.2.66) we get

$$[Y,Z] = \sum_{j=1}^N \big(\sigma_j a_j(x) - (\sigma_j - \alpha) a_j(x)\big)\, \frac{\partial}{\partial x_j} = \alpha \sum_{j=1}^N a_j(x)\, \frac{\partial}{\partial x_j} = m\,Y,$$

and this is precisely the desired (4.2.67).

Conversely, let us assume that Y satisfies identity (4.2.67). By equating the coefficient functions of $[Y,Z]$ and $m\,Y$, we obtain (see (4.2.68))

$$\sigma_j\, a_j - \mathcal{Z}(a_j) = m\, a_j, \quad \text{for every } j = 1,\dots,N.$$

From this and part (i) it then follows that any a_j is δ_λ-homogeneous of degree $\sigma_j - m$, hence (again by Thm. 1.2.2) Y is δ_λ-homogeneous of degree m. □

Remark 4.2.45. It is worth noting that, since the vector fields $X_1,\dots,X_m$ defining $\mathcal{L}$ are δ_λ-homogeneous of degree 1, identity (4.2.67) implies that

$$[X_j,\mathcal{Z}] = X_j, \text{ for every } i = 1,\dots,m, \quad \text{hence} \quad [\nabla_{\mathcal{L}},\mathcal{Z}] = \nabla_{\mathcal{L}}. \tag{4.2.69}$$

Remark 4.2.46. Let $f :]0,\infty[\to \mathbb{R}$ be of class C^1 and let $u : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ be the "radial function" defined by $u(x) := f(\Gamma_0(x))$.

By means of Lem. 4.2.44 in can be proved that $\mathcal{R}u \equiv 0$ on $\mathbb{R}^N \setminus \{0\}$: in fact, since Γ_0 is δ_λ-homogeneous of degree $2 - Q$, we have

$$\mathcal{R}u = f'(\Gamma_0)\Big(\Gamma_0 \sum_{j=1}^m (X_j\Gamma_0)^2 + \frac{1}{Q-2}\, \psi_0^{\mathcal{L}}\, \mathcal{Z}\Gamma_0\Big)$$
$$= f'(\Gamma_0)\Big(\Gamma_0\, \psi_0^{\mathcal{L}} + \frac{1}{Q-2} \psi_0^{\mathcal{L}}\, (2-Q)\, \Gamma_0\Big) = 0.$$

The vector fields $\mathcal{X}, \mathcal{Z}$ and $\mathcal{R}$ have distinguished properties in terms of their divergence and their action on the fundamental solution of $\mathcal{L}$.

Proposition 4.2.47. *According with Def. 4.2.42, we have:*

(1.a) $\operatorname{div}\big(\mathcal{X}I\big) = 0$ *on* $\mathbb{R}^N \setminus \{0\}$*;*

(1.b) $\mathcal{X}\Gamma_0 = \psi_0^{\mathcal{L}}$ *on* $\mathbb{R}^N \setminus \{0\}$*;*

(2.a) $\operatorname{div}(\mathcal{Z}I) = Q$ *on* $\mathbb{R}^N$*;*

(2.b) $\mathcal{Z}\,\Gamma_0 = (2-Q)\Gamma_0$ *on* $\mathbb{R}^N \setminus \{0\}$*;*

(3.a) $\operatorname{div}\big(\mathcal{R}I\big) = 0$ *on* $\mathbb{R}^N \setminus \{0\}$*;*

(3.b) $\mathcal{R}\Gamma_0 \equiv 0$ *on* $\mathbb{R}^N \setminus \{0\}$.

Proof. We prove each property separately.

(1.a) First of all we observe that, since $X_1, \ldots, X_m$ are δ_λ-homogeneous of degree 1, one has $\operatorname{div}(X_j) = 0$ for every $j = 1, \ldots, m$; moreover, Γ being a fundamental solution for $\mathcal{L}$, one has $\mathcal{L}\Gamma_0 \equiv 0$ on $\mathbb{R}^N \setminus \{0\}$ (see (4.2.7)).

As a consequence we have the following computation:

$$
\begin{aligned}
\operatorname{div}\big(\mathcal{X}I\big) &= \sum_{j=1}^m \operatorname{div}\Big(X_j\Gamma_0 \cdot X_jI\Big) \\
&= \sum_{j=1}^m \Big(\langle \nabla(X_j\Gamma_0), X_jI\rangle + X_j\Gamma_0 \operatorname{div}(X_jI)\Big) = \sum_{j=1}^m X_j^2\,\Gamma_0 \\
&= \mathcal{L}\Gamma_0 = 0, \qquad \text{on } \mathbb{R}^N \setminus \{0\}.
\end{aligned}
$$

(1.b) By the definition of $\mathcal{X}$ and that of $\psi_0^{\mathcal{L}} = \|\nabla_{\mathcal{L}}\,\Gamma_0\|^2$, we have

$$
\mathcal{X}\Gamma_0 = \langle \nabla_{\mathcal{L}}\,\Gamma_0, \nabla_{\mathcal{L}}\,\Gamma_0\rangle = \|\nabla_{\mathcal{L}}\Gamma_0\|^2 = \psi_0^{\mathcal{L}}, \quad \text{on } \mathbb{R}^N \setminus \{0\}.
$$

(2.a) By the very definition of $\mathcal{Z}$, we have

$$
\operatorname{div}(\mathcal{Z}I) = \sum_{j=1}^N \sigma_j = Q, \quad \text{on } \mathbb{R}^N.
$$

(2.b) It follows Lem. 4.2.44, since Γ_0 is δ_λ-homogeneous of degree $2 - Q$.

(3.a) First of all we observe that, since $\psi_0^{\mathcal{L}}$ is δ_λ-homogeneous of degree $2\,(1-Q)$ (see Lem. 4.2.40), Lem. 4.2.44 implies that

$$
\mathcal{Z}\,\psi_0^{\mathcal{L}} = 2\,(1-Q)\,\psi_0^{\mathcal{L}}, \quad \text{on } \mathbb{R}^N \setminus \{0\}. \tag{4.2.70}
$$

Moreover, on $\mathbb{R}^N \setminus \{0\}$ we have the following computation:

$$
\begin{aligned}
\operatorname{div}(\mathcal{R}I) &= \operatorname{div}\left(\Gamma_0\,\mathcal{X}I + \frac{1}{Q-2}\,\psi_0^{\mathcal{L}}\,\mathcal{Z}I\right) \\
&= \langle \nabla\Gamma_0, \mathcal{X}I\rangle + \Gamma_0 \operatorname{div}(\mathcal{X}I) + \tfrac{1}{Q-2}\,\langle \nabla\psi_0^{\mathcal{L}}, \mathcal{Z}I\rangle + \frac{1}{Q-2}\,\psi_0^{\mathcal{L}} \operatorname{div}(\mathcal{Z}I) \\
&= \mathcal{X}\Gamma_0 + \Gamma_0 \operatorname{div}(\mathcal{X}I) + \frac{1}{Q-2}\,\mathcal{Z}\psi_0^{\mathcal{L}} + \frac{1}{Q-2}\,\psi_0^{\mathcal{L}} \operatorname{div}(\mathcal{Z}I).
\end{aligned}
$$

By exploiting, respectively, (1.b), (1.a), (4.2.70) and (2.a) we then obtain

$$\operatorname{div}(\mathcal{R}I) = \psi_0^{\mathcal{L}} + 0 + \frac{2-2Q}{Q-2}\,\psi_0^{\mathcal{L}} + \frac{Q}{Q-2}\,\psi_0^{\mathcal{L}} = 0.$$

(3.b) Finally, by using (1.b) and (2.b), we get

$$\begin{aligned}\mathcal{R}\Gamma_0 &= (\Gamma_0\,\mathcal{X} + \frac{1}{Q-2}\,\psi_0^{\mathcal{L}}\,\mathcal{Z})\Gamma_0 = \Gamma_0\,\mathcal{X}\,\Gamma_0 + \frac{1}{Q-2}\,\psi_0^{\mathcal{L}}\,\mathcal{Z}\,\Gamma_0\\ &= \Gamma_0\,\psi_0^{\mathcal{L}} + \frac{2-Q}{Q-2}\,\psi_0^{\mathcal{L}}\,\Gamma_0 = 0.\end{aligned}$$

This ends the proof. □

Remark 4.2.48. The statement $\mathcal{R}\Gamma = 0$ in (3.b) of Prop. 4.2.47 is equivalent to saying that $\mathcal{R}I(x)$ is orthogonal to $\nabla\Gamma_0(x)$ at every point x in $\mathbb{R}^N \setminus \{0\}$.

Now, since the boundary of a Γ-ball $\Omega(0,r)$ centered at the origin is a level set of Γ (so that the normal space to $\partial\Omega(0,r)$ at any of its point x is generated by $\nabla\Gamma_0(x)$), we deduce that $\mathcal{R}I(x)$ is tangent to the sub-manifold $\partial\Omega(0,r)$.

By the above tangentiality property of $\mathcal{R}$ in Rem. 4.2.48, and since the divergence of $\mathcal{R}$ is null (see (3.a) in Prop. 4.2.47), we obtain the following result.

Corollary 4.2.49. *Let $r > 0$ and let $f : \Omega(0,r) \longrightarrow \mathbb{R}$ be of class C^1. Then*

$$\int_{\partial\Omega(0,\rho)} \frac{\mathcal{R}f}{\|\nabla\,\Gamma_0\|}\,\mathrm{d}H^{N-1} = 0 \quad \text{for every } 0 < \rho < r. \tag{4.2.71}$$

Proof. In the assumptions of the statement, we set

$$F(\rho) := \int_{\partial\Omega(0,\rho)} \frac{\mathcal{R}f}{\|\nabla\,\Gamma_0\|}\,\mathrm{d}H^{N-1}, \qquad \rho \in (0,r).$$

We also fix any pair of arbitrary $a, b \in \mathbb{R}$ such that $0 < a < b < r$. If we take $\rho = 1/t$ in both sides of the above identity, and if we integrate with respect to $t \in [1/b, 1/a]$, from Coarea Formula we obtain

$$\begin{aligned}\int_a^b \frac{F(s)}{s^2}\,\mathrm{d}s &= \int_{1/b}^{1/a} F(1/t)\,\mathrm{d}t = \int_{1/b}^{1/a}\left(\int_{\partial\Omega(0,1/t)} \frac{\mathcal{R}f}{\|\nabla\,\Gamma_0\|}\,\mathrm{d}H^{N-1}\right)\mathrm{d}t\\ &\overset{(4.2.17)}{=} \int_{\Omega(0,a,b)} \mathcal{R}f\,\mathrm{d}H^N.\end{aligned}$$

On the other hand, by using –respectively– the tangentiality of $\mathcal{R}$ on the boundary of the Γ-annulus $\Omega(0,a,b)$, the Divergence Theorem, and $\operatorname{div}(\mathcal{R}) \equiv 0$, we get (here $\nu_{0,\rho,r}^{\mathrm{ext}}$ is the exterior normal vector on $\partial\Omega(0,a,b)$, see (4.2.16)):

$$\begin{aligned}0 &= \int_{\partial\Omega(0,a,b)} f\,\langle\mathcal{R}, \nu_{0,\rho,r}^{\mathrm{ext}}\rangle\,\mathrm{d}H^{N-1} = \int_{\Omega(0,a,b)} \operatorname{div}\big(f\,\mathcal{R}I\big)\,\mathrm{d}H^N\\ &= \int_{\Omega(0,a,b)} f\operatorname{div}\big(\mathcal{R}I\big)\,\mathrm{d}H^N + \int_{\Omega(0,a,b)} \mathcal{R}f\,\mathrm{d}H^N = \int_{\Omega(0,a,b)} \mathcal{R}f\,\mathrm{d}H^N.\end{aligned}$$

As a consequence,

$$\int_a^b \frac{F(s)}{s^2}\,\mathrm{d}s = 0 \qquad \text{whenever } 0 < a < b < r.$$

From this, F being continuous on $]0,r[$ (see Prop. 4.2.15), we conclude that $F \equiv 0$ on $(0,r)$, which is what we intended to prove. □

Unique Continuation Now we have introduced and studied the vector fields $\mathcal{X}, \mathcal{Z}$ and $\mathcal{R}$, we are ready to enter the final part of this section, in which we state and prove the announced Unique Continuation result for the solutions of the Schrödinger-type equation (4.2.57). To begin with, we fix some notations.

First of all, given a nonnegative measurable function $f : (a, b) \to \mathbb{R}$ (where $-\infty \le a < b \le \infty$), we say that f is a **Dini function on** (a, b) if

$$\int_a^b \frac{f(z)}{z}\,\mathrm{d}z < \infty.$$

Moreover, we fix once and for all a real $R > 0$, and we consider a C^2 solution $u : \Omega(0, R) \to \mathbb{R}$ of the Schrödinger-type equation

$$-\mathcal{L}u + Pu = 0 \quad \text{on } \Omega(0, R), \tag{4.2.72}$$

where the potential $P : \Omega(0, R) \to \mathbb{R}$ satisfies the following assumption:

(P) P is continuous on $\Omega(0, R)$ and there exists a Dini function f on $(1/R, \infty)$, non-increasing and positive, such that P satisfies the estimate

$$|P(x)| \le f\big(\Gamma_0(x)\big)\,\frac{\psi_0^{\mathcal{L}}(x)}{\Gamma_0^2(x)}, \qquad \text{for almost every } x \in \Omega(0, R). \tag{4.2.73}$$

Remark 4.2.50. It is worth noting that the computations in this final part of the section can be adapted (as in [80]) to the more general case of weak solutions u in a suitable Γ^2-class (that is, $u \in L^2(\Omega(0, R))$ and $X_j u \in L^2(\Omega(0, R))$ for every $j = 1, \dots, m$) and singular potentials P (so that (4.2.73) can allow genuine singularities of P). We consider classical C^2 solutions u (and continuous potentials P) for the sake of the simplicity only.

Definition 4.2.51. Let $v \in C^1\big(\Omega(0, R), \mathbb{R}\big)$. The three functions

$$\begin{aligned}
H_v &:]0, R[\longrightarrow \mathbb{R}, \quad H_v(r) := \int_{\partial\Omega(0,r)} v^2\,\frac{\psi_0^{\mathcal{L}}}{\|\nabla\Gamma_0\|}\,\mathrm{d}H^{N-1},\\
D_v &:]0, R[\longrightarrow \mathbb{R}, \quad D_v(r) := \int_{\Omega(0,r)} \|\nabla_{\mathcal{L}} v\|^2\,\mathrm{d}H^N,\\
I_v &:]0, R[\longrightarrow \mathbb{R}, \quad I_v(r) := \int_{\Omega(0,r)} \Big(\|\nabla_{\mathcal{L}} v\|^2 + P\,v^2\Big)\,\mathrm{d}H^N
\end{aligned}$$

are called, respectively, the **$\mathcal{L}$-height of** v, the **$\mathcal{L}$-Dirichlet integral of** v, and the **$\mathcal{L}$-total energy of** v in $\Omega(0, R)$.

Remark 4.2.52. Let $v \in C^2(\Omega(0, R), \mathbb{R})$ and let $r \in (0, R)$. Due to the crucial use that we shall make of it in the sequel, we observe that the Hardy-type inequality (4.2.52) can be rewritten, with the above notations, as follows:

$$\int_{\Omega(0,r)} v^2\,\frac{\psi_0^{\mathcal{L}}}{\Gamma_0^2}\,\mathrm{d}H^N \le 4\left(\frac{r}{2}\,H_v(r) + D_v(r)\right),$$

As a consequence, we deduce that property (P) of P implies

$$\int_{\Omega(0,r)} |P|\,v^2\,\mathrm{d}H^N \le 4\,f(1/r)\left(\frac{r}{2}\,H_v(r) + D_v(r)\right). \tag{4.2.74}$$

Remark 4.2.53 (Regularity of H_v, D_v, I_v)**.** Let $v \in C^2(\Omega(0,R),\mathbb{R})$. We explicitly observe that, with our mean-value notation, the $\mathcal{L}$-height of v satisfies

$$H_v(r) = \mathcal{M}_r(v^2)(0), \quad \text{for every } 0 < r < R.$$

Thus, from Lem. 4.2.28 we infer that H_v is of class C^1 on $(0,R)$ and that

$$\lim_{r\to 0^+} H_v(r) = v^2(0).$$

Hence H_v is bounded on any interval $(0,\rho]$, with $0 < \rho < R$. Furthermore, from Lem. 4.2.15 we see that also D_v, I_v, are of class C^1 on $(0,R)$.

We now obtain some formulas for the first derivative of H_v, D_v and I_v.

Lemma 4.2.54. *Let $v \in C^2(\Omega(0,R),\mathbb{R})$. Then $H_v \in C^1((0,R),\mathbb{R})$ and*

$$H_v'(r) = \frac{2}{r(Q-2)} \int_{\partial\Omega(0,r)} v \cdot \mathcal{Z}v \, \frac{\psi_0^{\mathcal{L}}}{\|\nabla\Gamma_0\|} \, \mathrm{d}H^{N-1}, \quad r \in]0,R[. \tag{4.2.75}$$

If $u \in C^2(\Omega(0,R),\mathbb{R})$ is a solution of the Schrödinger-type equation (4.2.72)*, we have an alternative expression for I_u:*

$$I_u(r) = \frac{r}{Q-2} \int_{\partial\Omega(0,r)} u \cdot \mathcal{Z}u \, \frac{\psi_0^{\mathcal{L}}}{\|\nabla\Gamma_0\|} \, \mathrm{d}H^{N-1}, \quad r \in (0,R). \tag{4.2.76}$$

As a consequence

$$H_u'(r) = \frac{2}{r^2} I_u(r), \quad r \in (0,R). \tag{4.2.77}$$

Proof. As already observed in Rem. 4.2.53, one has $H_v(r) = \mathcal{M}_r(v^2)(0)$ for every $r \in (0,R)$ and $H_v \in C^1((0,R),\mathbb{R})$; moreover, from Lem. 4.2.28 we have

$$\begin{aligned} H_v'(r) &= \frac{\mathrm{d}}{\mathrm{d}r}\mathcal{M}_r(v^2)(0) = -\frac{1}{r^2}\int_{\partial\Omega(0,r)} \frac{\langle A\nabla v^2, \nabla\Gamma_0\rangle}{\|\nabla\Gamma_0\|} \, \mathrm{d}H^{N-1} \\ &\overset{(4.2.65)}{=} -\frac{1}{r^2}\int_{\partial\Omega(0,r)} \frac{\mathcal{X}(v^2)}{\|\nabla\Gamma_0\|} \, \mathrm{d}H^{N-1}. \end{aligned}$$

Since, by definition, $\mathcal{X} = (\mathcal{R} - \frac{1}{Q-2}\,\psi_0^{\mathcal{L}}\,\mathcal{Z})/\Gamma_0$ and $\Gamma \equiv 1/r$ on $\partial\Omega(0,r)$, we get

$$\begin{aligned} H_v'(r) &= -\frac{1}{r}\left(\int_{\partial\Omega(0,r)} \frac{\mathcal{Z}(v^2)}{2-Q}\,\frac{\psi_0^{\mathcal{L}}}{\|\nabla\Gamma_0\|} \, \mathrm{d}H^{N-1} + \int_{\partial\Omega(0,r)} \frac{\mathcal{R}(v^2)}{\|\nabla\Gamma_0\|} \, \mathrm{d}H^{N-1}\right) \\ &\overset{(4.2.71)}{=} \frac{1}{r(Q-2)}\int_{\partial\Omega(0,r)} \mathcal{Z}(v^2)\,\frac{\psi_0^{\mathcal{L}}}{\|\nabla\Gamma_0\|} \, \mathrm{d}H^{N-1}, \end{aligned}$$

and this gives (4.2.75). Let now $u \in C^2(\Omega(0,R),\mathbb{R})$ be a solution of the equation (4.2.72). A direct computation shows that

$$\mathcal{L}(u^2) = 2\Big(\|\nabla_{\mathcal{L}}u\|^2 + u\,\mathcal{L}u\Big) = 2\Big(\|\nabla_{\mathcal{L}}u\|^2 + P\,u^2\Big), \quad \text{on } \mathbb{R}^N;$$

hence, for every $0 < r < R$ we have

$$I_u(r) = \frac{1}{2}\int_{\Omega(0,r)} \mathcal{L}(u^2)\, \mathrm{d}H^N.$$

By using the Green identity (4.2.21) and identity (4.2.65) in Rem. 4.2.43, we obtain (recall the expression of the exterior normal on $\partial\Omega(0,r)$, see (4.2.11))

$$I_u(r) = -\frac{1}{2}\int_{\partial\Omega(0,r)} \frac{\langle A\nabla(v^2), \nabla\Gamma_0\rangle}{\|\nabla\Gamma_0\|}\,\mathrm{d}H^{N-1} = -\frac{1}{2}\int_{\partial\Omega(0,r)} \frac{\mathcal{X}(v^2)}{\|\nabla\Gamma_0\|}\,\mathrm{d}H^{N-1}.$$

From this, by arguing as above, one gets (4.2.76) and (4.2.77). □

The derivative of the $\mathcal{L}$-Dirichlet integral of a function plays a key role; hence we first give a general formula for it, for any C^2 function v which is not necessarily a solution of (4.2.72). This formula also highlights the role of the vector fields $\mathcal{Z}$ and $\mathcal{X}$. See also [84, Corollary 3.3].

Theorem 4.2.55. *Let $v \in C^2(\Omega(0,R),\mathbb{R})$. Then, for every $0 < r < R$, the following first-variation formula for D_v holds true:*

$$\begin{aligned} D_v'(r) = \frac{1}{r}\,D_v(r) &+ \frac{2}{r(2-Q)}\int_{\partial\Omega(0,r)} \frac{\mathcal{Z}v\cdot\mathcal{X}v}{\|\nabla\Gamma_0\|}\,\mathrm{d}H^{N-1}\,\mathrm{d}H^N \\ &+ \frac{2}{r(2-Q)}\int_{\Omega(0,r)} \mathcal{Z}v\cdot\mathcal{L}v. \end{aligned} \tag{4.2.78}$$

Proof. Let $v \in C^2(\Omega(0,R))$ and let $0<r<R$ be fixed. From (4.2.18) we have

$$D_v'(r) = \frac{\mathrm{d}}{\mathrm{d}r}\left(\int_{\Omega(0,r)} \|\nabla_{\mathcal{L}}v\|^2\,\mathrm{d}H^N\right) = \frac{1}{r^2}\int_{\partial\Omega(0,r)} \frac{\|\nabla_{\mathcal{L}}v\|^2}{\|\nabla\Gamma_0\|}\,\mathrm{d}H^{N-1} =: (\star).$$

Since $\Gamma \equiv 1/r$ on $\partial\Omega(0,r)$ and $\mathcal{Z}\Gamma_0 = (2-Q)\,\Gamma_0$ (see (2.b) in Prop. 4.2.47),

$$\begin{aligned} (\star) &= \frac{1}{r(2-Q)}\int_{\partial\Omega(0,r)} \frac{\|\nabla_{\mathcal{L}}v\|^2\,\mathcal{Z}\Gamma_0}{\|\nabla\Gamma_0\|}\,\mathrm{d}H^{N-1} \\ &= -\frac{1}{r(2-Q)}\int_{\partial\Omega(0,r)} \left\langle \|\nabla_{\mathcal{L}}v\|^2\mathcal{Z}, \frac{-\nabla\Gamma_0}{\|\nabla\Gamma_0\|}\right\rangle\mathrm{d}H^{N-1} \\ &\quad\text{(by the Divergence Theorem)} \\ &= -\frac{1}{r(2-Q)}\int_{\Omega(0,r)} \mathrm{div}\Big(\|\nabla_{\mathcal{L}}v\|^2\mathcal{Z}\Big)\,\mathrm{d}H^N =: (2\star). \end{aligned}$$

Recalling that $\mathrm{div}(\mathcal{Z}) = Q$ (see (2.a) in Prop. 4.2.47), we then get

$$\begin{aligned} (2\star) &= -\frac{1}{r(2-Q)}\int_{\Omega(0,r)} \Big(Q\,\|\nabla_{\mathcal{L}}v\|^2 + \mathcal{Z}\big(\|\nabla_{\mathcal{L}}v\|^2\big)\Big)\,\mathrm{d}H^N \\ &= \frac{Q}{r(Q-2)}D_v(r) + \frac{1}{r(Q-2)}\int_{\Omega(0,r)} \mathcal{Z}\big(\|\nabla_{\mathcal{L}}v\|^2\big)\,\mathrm{d}H^N =: (3\star). \end{aligned}$$

From the commutator identity in (4.2.69) we obtain

$$\begin{aligned} \mathcal{Z}\big(\|\nabla_{\mathcal{L}}v\|^2\big) &= 2\langle\nabla_{\mathcal{L}}v, \mathcal{Z}(\nabla_{\mathcal{L}}v)\rangle = 2\langle\nabla_{\mathcal{L}}v, \nabla_{\mathcal{L}}(\mathcal{Z}v) + [\mathcal{Z},\nabla_{\mathcal{L}}]v\rangle \\ &\overset{(4.2.69)}{=} 2\langle\nabla_{\mathcal{L}}v, \nabla_{\mathcal{L}}(\mathcal{Z}v)\rangle - 2\|\nabla_{\mathcal{L}}v\|^2. \end{aligned}$$

As a consequence, we have

$$(3\star) = \frac{1}{r}D_v(r) + \frac{2}{r(Q-2)}\int_{\Omega(0,r)} \langle\nabla_{\mathcal{L}}v, \nabla_{\mathcal{L}}(\mathcal{Z}v)\rangle\,\mathrm{d}H^N =: (4\star).$$

Finally, integrating by parts and recalling that, for every $i = 1, \dots, m$ one has $\mathrm{div}(X_i) = 0$, we get (here $\nu_{0,r}^{\mathrm{ext}}$ is the exterior normal on $\partial\Omega(0,r)$, see (4.2.11))

$$\begin{aligned}
(4\star) &= \frac{1}{r} D_v(r) + \frac{2}{r(Q-2)} \sum_{i=1}^m \int_{\Omega(0,r)} X_i v \cdot X_i(\mathcal{Z}v)\, dH^N \\
&= \frac{1}{r}\, D_v(r) + \frac{2}{r(Q-2)} \sum_{i=1}^m \int_{\partial\Omega(0,r)} \mathcal{Z}v \cdot X_i v\, \langle X_i I, \nu_{0,r}^{\mathrm{ext}} \rangle \,\mathrm{d}H^{N-1} \\
&\quad - \frac{2}{r(Q-2)} \sum_{i=1}^m \int_{\Omega(0,r)} \mathcal{Z}v \cdot X_i^2 v \,\mathrm{d}H^N \\
&= \frac{1}{r} D_v(r) + \frac{2}{r(2-Q)} \int_{\partial\Omega(0,r)} \frac{\mathcal{Z}v}{\|\nabla\Gamma_0\|} \left(\sum_{i=1}^m X_i\Gamma_0\, X_i v \right) \mathrm{d}H^{N-1} \\
&\quad + \frac{2}{r(2-Q)} \int_{\Omega(0,r)} \mathcal{L}v\, \mathcal{Z}v \,\mathrm{d}H^N.
\end{aligned}$$

This is precisely (4.2.78), if one recalls the definition (4.2.61) of $\mathcal{X}$. $\square$

By expressing $\mathcal{X}$ in terms of $\mathcal{R}$ and $\mathcal{Z}$ (see (4.2.63)), formula (4.2.78) gives, for a solution u of $\mathcal{L}u = Pu$, the following result.

Corollary 4.2.56. *Let $u \in C^2(\Omega(0,R), \mathbb{R})$ be a solution of (4.2.72) on $\Omega(0,R)$. Then, for every $0 < r < R$, the following first-variation formula holds true:*

$$\begin{aligned}
D_u'(r) &= \frac{1}{r}\, D_u(r) + \frac{2}{r(2-Q)} \int_{\Omega(0,r)} Pu\, \mathcal{Z}u \,\mathrm{d}H^N \\
&\quad + \frac{2}{2-Q} \int_{\partial\Omega(0,r)} \frac{\mathcal{R}u\mathcal{Z}u}{\|\nabla\Gamma_0\|} \,\mathrm{d}H^{N-1} + \frac{2}{(2-Q)^2} \int_{\partial\Omega(0,r)} \frac{\psi_0^{\mathcal{L}}\,(\mathcal{Z}u)^2}{\|\nabla\Gamma_0\|} \,\mathrm{d}H^{N-1}.
\end{aligned} \tag{4.2.79}$$

In the next proof we use for the first time the estimate (4.2.73) of P. Besides, we apply the Hardy-type inequality (4.2.52).

Proposition 4.2.57. *Let $u \in C^2(\Omega(0,R), \mathbb{R})$ be a solution of (4.2.72) with a potential P as in assumption (P). Then, there exists $\rho_0 \in (0,R)$, depending on the function f in (4.2.73), but independent of u, with the following properties:*

1. *if $r \in (0, \rho_0]$ is such that $H_u(r) = 0$, then also $I_u(r) = 0$ and $D_u(r) = 0$, so that $\nabla_{\mathcal{L}} u \equiv 0$ on $\Omega(0,r)$;*
2. *if $\|\nabla_{\mathcal{L}} u\|^2$ is not identically zero on every Γ-ball $\Omega(0,r)$, with $0 < r \leq \rho_0$, then $H_u(r) \neq 0$ for every $r \in (0, \rho_0]$.*

Proof. (1) First of all we observe that, since f is a non-increasing Dini function on $(1/R, \infty)$ (see assumption (P)), we have

$$\lim_{z\to\infty} f(z) = 0;$$

thus, it is possible to find $\rho_0 < R$ such that $f(1/\rho_0) < 1/8$.

Let now $r \in (0, \rho_0]$ be such that $H_u(r) = 0$. The Cauchy-Schwarz inequality applied to identity (4.2.76) in Lem. 4.2.54 gives out

$$|I_u(r)| \le \frac{r}{Q-2}\Big(H_u(r)\Big)^{1/2}\left(\int_{\partial\Omega(0,r)} (Zu)^2\,\frac{\psi_0^{\mathcal{L}}}{\|\nabla\Gamma\|}\,\mathrm{d}H^{N-1}\right)^{1/2} = 0.$$

Thus $I_u(r) = 0$. Since, by definition, $D_u(r) = I_u(r) - \int_{\Omega(0,r)} P\,u^2$, from the nullity of $I_u(r)$ and $H_u(r)$ we then get

$$\begin{aligned} D_u(r) &\le \int_{\Omega(0,r)} |P|\,u^2\,\mathrm{d}H^N \overset{(4.2.74)}{\le} 4\,f(1/r)\,\Big(\frac{r}{2}\,H_u(r) + D_u(r)\Big) \\ &= 4\,f(1/r)\,D_u(r). \end{aligned}$$

Note that we have applied here the result (4.2.74) of Rem. 4.2.52, depending on the Hardy inequality (4.2.52) and the estimate (P) of P.

Since f is non-increasing and $r \le \rho_0$, we have $f(1/r) \le f(1/\rho_0) < 1/8$, so that the above inequality gives $D_u(r) < D_u(r)/2$, which obviously implies $D_u(r) = 0$. We then obtain $\|\nabla_{\mathcal{L}} u\|^2 \equiv 0$ on $\Omega(0,r)$, as desired.

(2) It is a direct consequence of (1). □

From now on, we understand that $u \in C^2(\Omega(0,R), \mathbb{R})$ is a solution of the Schrödinger-type equation (4.2.72), with a potential P as in assumption (P). Following the approach of [80, 81, 82, 83] using Almgren's frequency function, we give the following crucial definition.

Definition 4.2.58. Let ρ_0 be as in Prop. 4.2.57. Suppose that $\|\nabla_{\mathcal{L}} u\|^2$ is not identically zero on every Γ-ball $\Omega(0,r)$, with $0 < r \le \rho_0$.

Then the following function is well defined

$$N_u : (0, \rho_0] \longrightarrow \mathbb{R} \qquad N_u(r) := \frac{I_u(r)}{r\,H_u(r)},$$

and it is called the **$\mathcal{L}$-frequency of** u. We also set

$$\Lambda(u) := \Big\{ r \in (0, \rho_0] \,:\, N_u(r) > \max\{1, N_u(\rho_0)\} \Big\}. \tag{4.2.80}$$

Remark 4.2.59. Let the assumptions and the notations in Def. 4.2.58 apply. Since both I_u and H_u are of class C^1 on $(0,R)$ (and since $H_u \ne 0$ on $(0,\rho_0]$), we have $N_u \in C^1((0,\rho_0], \mathbb{R})$. Moreover, a direct computation gives

$$N_u'(r) = \frac{1}{r^2}\left(\frac{r\,I_u'(r)}{H_u(r)} - \frac{I_u(r)}{H_u(r)} - r\,\frac{I_u(r)\,H_u'(r)}{H_u^2(r)}\right), \quad \text{on } (0,\rho_0]. \tag{4.2.81}$$

Finally, the continuity of N_u on $(0,\rho_0]$ ensures that $\Lambda(u)$ is a (relatively) open subset of $(0,\rho_0]$. Note that, a priori, $\Lambda(u)$ could be empty.

Remark 4.2.60. Let the assumptions and the notations in Def. 4.2.58 apply. By the very definition of $\Lambda(u) \subseteq (0,\rho_0]$, we have

$$I_u(r) > r\,\max\{1, N_u(\rho_0)\}\,H_u(r) \ge r\,H_u(r) > 0, \quad \text{for every } r \in \Lambda(u)$$

Hence, in particular, $I_u \ne 0$ on $\Lambda(u)$.

Remark 4.2.61. Let $u \in C^2(\Omega(0,R),\mathbb{R})$ be as in Def. 4.2.58. If u is δ_λ-homogeneous of degree $m \geq 0$, its $\mathcal{L}$-frequency N_u can be explicitly computed.

In fact, under this additional assumption, the expression of I_u given in (4.2.76) and Lem. 4.2.44 imply that, for every $0 < r < R$,

$$I_u(r) = -r\,\frac{m}{2-Q}\int_{\partial\Omega(0,r)} u^2\,\frac{\psi_0^{\mathcal{L}}}{\|\nabla\Gamma_0\|}\,\mathrm{d}H^{N-1} = \frac{r\,m}{Q-2}\,H_u(r).$$

Therefore, by the very definition of N_u we get

$$N_u(r) = \frac{m}{Q-2}, \quad \text{for every } 0 < r \leq \rho_0.$$

From now on, when N_u or $\Lambda(u)$ are involved, we tacitly assume that u satisfies the assumptions in Def. 4.2.58. Since, by Rem. 4.2.60, $I_u \neq 0$ on $\Lambda(u)$, the logarithmic derivatives in the following statement are well posed.

Proposition 4.2.62. *For every $r \in \Lambda(u)$ we have the following formulas for the logarithmic derivatives of I_u and N_u:*

$$\begin{aligned}\frac{I_u'(r)}{I_u(r)} &= \frac{1}{r} + \frac{1}{I_u(r)}\Bigg\{\frac{2}{r(2-Q)}\int_{\Omega(0,r)} Pu\,\mathcal{Z}u\,\mathrm{d}H^N - \frac{1}{r}\int_{\Omega(0,r)} Pu^2\,\mathrm{d}H^N \\ &\quad + \frac{2}{2-Q}\int_{\partial\Omega(0,r)}\left(\frac{(\mathcal{Z}u)^2}{2-Q}\,\psi_0^{\mathcal{L}} + \mathcal{Z}u\,\mathcal{R}u\right)\frac{\mathrm{d}H^{N-1}}{\|\nabla\Gamma_0\|} \\ &\quad + \frac{1}{r^2}\int_{\partial\Omega(0,r)}\frac{Pu^2}{\|\nabla\Gamma_0\|}\,\mathrm{d}H^{N-1}\Bigg\}.\end{aligned}$$

$$\begin{aligned}\frac{N_u'(r)}{N_u(r)} &= -\frac{2I_u(r)}{r^2\,H_u(r)} + \frac{1}{I_u(r)}\Bigg\{\frac{2}{r(2-Q)}\int_{\Omega(0,r)} \mathcal{Z}u\,Pu\,\mathrm{d}H^N \\ &\quad - \frac{1}{r}\int_{\Omega(0,r)} P\,u^2\,\mathrm{d}H^N \\ &\quad + \frac{2}{2-Q}\int_{\partial\Omega(0,r)}\left(\frac{(\mathcal{Z}u)^2}{2-Q}\,\psi_0^{\mathcal{L}} + \mathcal{Z}u\,\mathcal{R}u\right)\frac{\mathrm{d}H^{N-1}}{\|\nabla\Gamma_0\|} \\ &\quad + \frac{1}{r^2}\int_{\partial\Omega(0,r)}\frac{P\,u^2}{\|\nabla\Gamma_0\|}\,\mathrm{d}H^{N-1}\Bigg\}.\end{aligned}$$

Proof. Since, by definition, $I_u(r) = D_u(r) + \int_{\Omega(0,r)} Pu^2$ for every $r \in (0,R)$, identity (4.2.18) in Prop. 4.2.15 implies that

$$I_r'(u) = D_u'(r) + \frac{1}{r^2}\int_{\partial\Omega(0,r)}\frac{Pu^2}{\|\nabla\Gamma_0\|}\,\mathrm{d}H^{N-1}, \quad \text{for every } r \in (0,R).$$

Thus, the formula for I_u'/I_u easily follows from (4.2.79) and (4.2.18).

On the other hand, by identity (4.2.81) in Rem. 4.2.59 we have

$$\frac{N_u'(r)}{N_u(r)} = \frac{I_u'(r)}{I_u(r)} - \frac{1}{r} - \frac{2}{r}\,\frac{I_u(r)}{r\,H_u(r)}, \quad \text{for every } r \in \Lambda(u),$$

and thus the formula for N_u'/N_u follows from that of I_u'/I_u. □

From now on, we make the following growth assumption on the $\mathcal{L}$- discrepancy $\mathcal{R}u$ of our solution u (see Def. 4.2.42):

(D) there exists a Dini function g on $(1/R, \infty)$, non-increasing and positive, such that u satisfies the condition

$$|\mathcal{R}u(x)| \leq g\big(\Gamma(x)\big)\,\psi_0^{\mathcal{L}}\,|u(x)|, \quad \text{for almost every } x \in \Omega(0,R). \tag{4.2.82}$$

We then have the following keystone result.

Theorem 4.2.63. *Let $u \in C^2(\Omega(0,R),\mathbb{R})$ be a solution of* (4.2.72) *on $\Omega(0,R)$ satisfying condition* (D) *above, and such that $\|\nabla_{\mathcal{L}} u\|^2$ is not identically zero on every Γ-ball $\Omega(0,r)$, with $0 < r \leq \rho_0$. We also recall that the potential P fulfills assumption* (P)*, and ρ_0 is as in Prop. 4.2.57.*

Then there exists a real constant $M > 0$ (independent of u) such that

$$\frac{N_u'(r)}{N_u(r)} \geq -M\left(\frac{f(1/r)+g(1/r)}{r}\right), \quad \textit{for every } r \in \Lambda(u). \tag{4.2.83}$$

Here f and g are the Dini functions in the hypotheses (P) *and* (D)*, respectively.*

Proof. We omit the proof, since this can be obtained by arguing *verbatim* as in [80, pages 341–345], once the expression for the logarithmic derivative of $N_u(r)$ has been obtained (see Prop. 4.2.62), and by making use of the results obtained so far for I_u, N_u, D_u. We limit ourselves to remark that the Hardy-type inequality (4.2.52) has a key role in this arguments as well. □

It is well known that control from below of the logarithmic derivative of N_u as in Thm. 4.2.63 yields the boundedness of the frequency N_u and a doubling property for the mean-value of u^2, as in the following corollary.

Corollary 4.2.64. *Let $u \in C^2(\Omega(0,R),\mathbb{R})$ satisfy the hypothesis in the above Thm. 4.2.63, and let ρ_0 be as in Prop. 4.2.57. Then, there exist real constants $\alpha,\beta,\gamma > 0$ (depending on u) such that the following results hold:*

(Upper boundedness of N_u)

$$N_u(r) \leq \alpha, \quad \textit{for every } r \in (0,\rho_0]. \tag{4.2.84}$$

(Doubling properties)

$$H_u(2r) \leq \beta\, H_u(r), \quad \textit{for every } r \in (0,\rho_0/2]; \tag{4.2.85}$$

$$\int_{\Omega(0,2r)} \frac{u^2}{\Gamma_0^2}\,\psi_0^{\mathcal{L}}\,\mathrm{d}H^N \leq \gamma \int_{\Omega(0,r)} \frac{u^2}{\Gamma_0^2}\,\psi_0^{\mathcal{L}}\,\mathrm{d}H^N, \quad \forall\, r \in (0,\rho_0/2]. \tag{4.2.86}$$

Proof. Let $\Lambda(u)$ be as in (4.2.80). Outside $\Lambda(u)$, N_u is bounded from above by

$$\theta := \max\{1, N_u(\rho_0)\}.$$

We can thus restrict to find an upper bound for N_u on $\Lambda(u)$. If r is fixed in $\Lambda(u)$, we let (a,b) be the connected component of $\Lambda(u)$ containing r. Then (4.2.83) implies that (recall that f,g are positive Dini functions)

$$\log\left(\frac{N_u(b)}{N_u(r)}\right) = \int_r^b \frac{N_u'(t)}{N_u(t)}\,\mathrm{d}t \geq -M\int_{1/\rho_0}^{\infty}\left(\frac{f(z)+g(z)}{z}\right)\mathrm{d}z =: C.$$

Since $b \le \rho_0$ but $b \notin \Lambda(u)$, we have $0 < N_u(b) \le \max\{1, N_u(\rho_0)\}$; therefore the above inequality gives $N_u(r) \le \alpha$ for every $0 < r \le \rho_0$, where

$$\alpha = \alpha(u) = e^{-C} \max\{1, N_u(\rho_0)\} \ (\ge \theta).$$

This proves (4.2.84). Next, we have

$$\frac{H_u'(\rho)}{H_u(\rho)} \overset{(4.2.77)}{=} \frac{2\, I_u(\rho)}{\rho^2\, H_u(\rho)} = 2\,\frac{N_u(\rho)}{\rho}, \quad \text{for every } 0 < \rho \le \rho_0;$$

therefore, for every $0 < r \le \rho_0/2$, and by means of (4.2.84), we deduce that

$$\begin{aligned} \log\left(\frac{H_u(2r)}{H_u(r)}\right) &= \int_r^{2r} \frac{H_u'(\rho)}{H_u(\rho)}\,\mathrm{d}\rho = 2\int_r^{2r} \frac{N_u(\rho)}{\rho}\,\mathrm{d}\rho \\ &\le 2\,\alpha(u)\int_r^{2r} \frac{1}{\rho}\,\mathrm{d}\rho = 2\,\alpha(u)\,\log(2), \end{aligned}$$

which gives (4.2.85) with the choice $\beta = \beta(u) = \exp(2\,\alpha(u)\,\log(2))$. Finally, we consider the function $F : (0, R) \to \mathbb{R}$ defined by

$$F(r) := \int_{\Omega(0,r)} \frac{u^2}{\Gamma_0^2}\,\psi_0^{\mathcal{L}}\,\mathrm{d}H^N, \qquad r \in (0, R).$$

From Lem. 4.2.15, we have $F \in C^1((0, R), \mathbb{R})$ and

$$\begin{aligned} F'(r) &= \frac{1}{r^2}\int_{\partial\Omega(0,r)} \frac{u^2}{\Gamma_0^2}\,\frac{\psi_0^{\mathcal{L}}}{\|\nabla\Gamma_0\|}\,\mathrm{d}H^{N-1} \\ &= \int_{\Omega(0,r)} u^2\,\frac{\psi_0^{\mathcal{L}}}{\|\nabla\Gamma_0\|}\,\mathrm{d}H^{N-1} = H_u(r), \quad \text{for every } r \in (0, R). \end{aligned} \tag{4.2.87}$$

Thus, if $0 < a < r \le \rho_0/2$, from (4.2.85) we obtain

$$\begin{aligned} \frac{F(2r) - F(2a)}{2} &\overset{(4.2.87)}{=} \int_a^r H_u(2\rho)\,\mathrm{d}\rho \\ &\le \beta(u)\int_a^r H_u(\rho)\,\mathrm{d}\rho \overset{(4.2.87)}{=} \beta(u)(F(r) - F(a)). \end{aligned}$$

The function $u^2\,\psi_0^{\mathcal{L}}/\Gamma_0^2$ being locally integrable on $\Omega(0, R)$ (as it follows from Lem. 4.2.23), we infer that $F(a) \to 0$ as $a \to 0$; therefore, by passing to the limit as $a \to 0$ in the latter inequality, we obtain

$$F(2r) \le 2\,\beta(u)\,F(r), \quad \text{for every } 0 < r < \rho_0/2.$$

Taking into account the definition of F, we conclude that (4.2.86) holds with the choice $\gamma = \gamma(u) := 2\beta(u)$. This ends the proof. □

The boundedness of N_u and the doubling properties (4.2.85) and (4.2.86) in the above Cor. 4.2.64 are the final tools for the Unique Continuation. In order to clearly state such a result, we first give the following definition.

Definition 4.2.65. Let u be a bounded function defined on $\Omega(0,R)$. We say that u **vanishes to infinite order at** 0 if, for every fixed $\alpha \in \mathbb{N}$, one has:

$$\int_{\Omega(0,r)} u^2 \,\frac{\psi_0^{\mathcal{L}}}{\Gamma_0^2}\,\mathrm{d}H^N = \mathcal{O}\big(r^\alpha\big), \quad \text{as } r \to 0^+.$$

By this we mean, precisely, that for every fixed $\alpha \in \mathbb{N}$ it is possible to find $r_\alpha \in (0,R)$ and a real $\mathrm{C}_\alpha > 0$ (both depending on α) such that

$$\int_{\Omega(0,r)} u^2 \,\frac{\psi_0^{\mathcal{L}}}{\Gamma_0^2}\,\mathrm{d}H^N \le \mathrm{C}_\alpha\, r^\alpha, \quad \text{for every } 0 < r < r_\alpha. \tag{4.2.88}$$

We then have the following Unique Continuation property.

Theorem 4.2.66. *Let $u \in C^2(\Omega(0,R),\mathbb{R})$ be a solution of the equation*

$$-\mathcal{L}u + Pu = 0, \quad \text{on } \Omega(0,R).$$

We assume that the $\mathcal{L}$-discrepancy of u satisfies the growth estimate (D) *and that the potential P satisfies assumption* (P)*.*

If u vanishes to infinite order at 0, the $u \equiv 0$ in a neighborhood of the origin.

Proof. We begin by showing that $u(0) = 0$. To this end, we take $\alpha = 2$ in (4.2.88) and we let C_2, r_2 be as in (4.2.88) as well. For $0 < r < r_2$, we have

$$\begin{aligned} C_2\, r &\ge \frac{1}{r}\int_{\Omega(0,r)} u^2\,\frac{\psi_0^{\mathcal{L}}}{\Gamma_0^2}\,\mathrm{d}H^N = \frac{1}{r}\int_{\Omega(0,r)} (u^2 - u^2(0))\,\frac{\psi_0^{\mathcal{L}}}{\Gamma_0^2}\,\mathrm{d}H^N + u^2(0) \\ &=: J(r) + u^2(0). \end{aligned}$$

In the first equality we invoked (4.2.36) with $\alpha = 2$, which proves that

$$\int_{\Omega(0,r)} \frac{\psi_0^{\mathcal{L}}}{\Gamma_0^2}\,\mathrm{d}H^N = r.$$

We now let $r \to 0^+$ in the above inequality: since u is continuous at 0, one has $J(r) \to 0$ as $r \to 0^+$; thus, we get $0 \ge u^2(0)$, whence $u(0) = 0$, as desired.

Now, if ρ_0 is as in Prop. 4.2.57, only two cases can occur:

(1) there exists $\overline{r} \in (0,\rho_0]$ such that $\|\nabla_{\mathcal{L}} u\|^2 \equiv 0$ on $\Omega(0,\overline{r})$;

(2) $\|\nabla_{\mathcal{L}} u\|^2$ is not identically zero on every Γ-ball $\Omega(0,r)$, with $0 < r \le \rho_0$.

We show that (2) cannot occur and that under case (1) the theorem is proved.

In case (1), we have $\nabla_{\mathcal{L}} u \equiv 0$ on $U := \Omega(0,\overline{r})$, and we claim that u is constant on U. In fact, since $X_1,\ldots,X_m$ are Lie-generators for $\mathrm{Lie}(\mathbb{G})$, the nullity of the $\mathcal{L}$-horizontal gradient of u implies that

$$J_1 u(x) = \ldots = J_N u(x) = 0, \quad \text{for every } x \in U,$$

where $J_1,\ldots,J_N$ denotes, as usual, the elements of the Jacobian basis for $\mathrm{Lie}(\mathbb{G})$. As a consequence, we have (see Rem. 1.1.5 on page 5)

$$\nabla u(x)\cdot \mathcal{J}_{\tau_x}(0) = 0, \quad \text{for every } x \in U.$$

Since, for every $x \in U$, the matrix $\mathcal{J}_{\tau_x}(0)$ is non-singular (τ_x being a diffeomorphism), we have $\nabla u \equiv 0$ on U; from this, $U = \Omega(0, \overline{r})$ being connected (see Lem. 4.2.12), we infer that u is constant throughout U. Finally, since $u(0) = 0$ (see above), we conclude that $u \equiv 0$ on U and the theorem is proved.

We suppose, by contradiction, that we are in case (2) so that we are entitled to apply Cor. 4.2.64. We thus fix any $0 < r \leq \rho_0$ and we iterate k times the doubling inequality (4.2.86). This gives

$$\int_{\Omega(0,r)} u^2 \,\frac{\psi_0^{\mathcal{L}}}{\Gamma_0^2}\,\mathrm{d}H^N \leq \gamma^k \int_{\Omega(0,r/2^k)} u^2 \,\frac{\psi_0^{\mathcal{L}}}{\Gamma_0^2}\,\mathrm{d}H^N, \quad \text{for every } k \in \mathbb{N}. \qquad (4.2.89)$$

We fix some large integer α (to be chosen in a moment), in such a way that, if k is sufficiently large (namely $r/2^k < r_\alpha$), one has (see (4.2.88))

$$\gamma^k \int_{\Omega(0,r/2^k)} u^2 \,\frac{\psi_0^{\mathcal{L}}}{\Gamma_0^2}\,\mathrm{d}H^N \leq C_\alpha\, \gamma^k \left(\frac{r}{2^k}\right)^\alpha = C_\alpha\, r^\alpha \left(\frac{\gamma}{2^\alpha}\right)^k.$$

If we fix $\alpha \in \mathbb{N}$ such that $\alpha > \log_2(2\gamma)$, we have $\gamma/2^\alpha < 1/2$ and the above right-hand side vanishes as $k \to \infty$. This proves that the integral in the left-hand side of (4.2.89) is null, whence $u^2\psi_0^{\mathcal{L}} = 0$ on $\Omega(0,r) \setminus \{0\}$. This gives

$$H_u(\rho) = \int_{\partial\Omega(0,\rho)} u^2 \,\frac{\psi_0^{\mathcal{L}}}{\|\nabla\Gamma_0\|}\,\mathrm{d}H^{N-1} = 0 \quad \text{for all } \rho \in (0,r).$$

Since $r \leq \rho_0$ is arbitrary, we get $H_u(r) = 0$ for every $r \in (0, \rho_0]$. From (1) in Prop. 4.2.57 we deduce that $\|\nabla_{\mathcal{L}} u\|^2 \equiv 0$ on $\Omega(0, \rho_0)$ but this is clearly in contradiction with assumption (2). This ends the proof. □

Appendix A

Finer convergence domain for the Campbell-Hausdorff series

The aim of this Appendix is to briefly describe a convergence result for the Campbell-Baker-Hausdorff-Dynkin (CBHD, in the sequel) series

$$\sum_{n=1}^{\infty} Z_n(x,y) = x + y + \frac{1}{2}\,[x,y] + \frac{1}{12}[x,[x,y]] - \frac{1}{12}[y,[x,y]] + \cdots$$

in infinite-dimensional Banach-Lie algebras L. In the existing literature, this problem is solved when $L = \mathrm{Lie}(G)$ is the Lie algebra of a finite-dimensional Lie group G (see Blanes, Cases [25]) or of an infinite-dimensional Banach-Lie group G (see Mérigot [110]). Indeed, one can obtain a suitable ODE for the map

$$t \mapsto \gamma(t) = \sum_{n=1}^{\infty} Z_n(x, ty),$$

which follows from the well-behaved formulas for the differential of the Exponential Map of the Lie group G. The novelty of the approach we are going to present is to derive this ODE in any infinite-dimensional Banach-Lie algebra (not necessarily associated with a Lie group), as a consequence of an analogous abstract ODE firstly obtained in the most natural algebraic setting: that of the formal power series in two commuting indeterminates s, t over the free unital associative algebra generated by the non-commuting indeterminates x, y.

The plan of the chapter is the following:

- In Sec. A.1 we introduce the main definitions and notations concerning the abstract algebraic setting in which the CBHD series is studied.
- Sec. A.2 is devoted to establishing two (formal) PDEs for the series

$$Z(xs, yt) := \log\big(\exp(x\,s) \star \exp(y\,t)\big),$$

in the algebra of formal power series in the two commuting indeterminates s, t over the free unital associative algebra generated by the non-commuting indeterminates x, y. As a consequence of such PDEs, we are able to provide a simple a proof of the notable CBHD Theorem.

- In Sec. A.3 we exploit the PDEs obtained in Sec. A.2 (plus a simple argument of ODE Theory) in order to prove the announced convergence result for the CBHD series in any (infinite-dimensional) Banach-Lie algebra.
- Finally, as an application of the convergence result proved in Sec. A.3, we derive in Sec. A.4 some notable PDEs related to the CBHD series.

A.1 Algebraic background

As anticipated, this first section is aimed to fix the notation used in the sequel. First of all, we fix a field $\mathbb{K}$ of characteristic zero; all linear structures will be tacitly understood over $\mathbb{K}$. Moreover, for the sake of simplicity, $\mathbb{N}$ denotes the set of the nonnegative integers, so that $0 \in \mathbb{N}$.

By $(A, *)$ we shall always denote a fixed unital associative algebra, i.e., a vector space endowed with a bilinear associative operation $*$, possessing a neutral element, denoted by 1_A. Addition in A and the action of $\mathbb{K}$ on A are denoted, respectively, by $A \times A \ni (a, a') \mapsto a + a'$ and $\mathbb{K} \times A \ni (k, a) \mapsto k\, a$.

Definition A.1.1. A **formal power series** in the two commuting indeterminates s, t, with coefficients in A (or, simply, a power series in A) is a map

$$F : \mathbb{N} \times \mathbb{N} \longrightarrow A, \qquad F(i,j) := F_{i,j} \in A.$$

The set of all formal power series in A will be denoted by $A[\![s,t]\!]$. Moreover, given any $F \in A[\![s,t]\!]$, we shall write it in the following equivalent ways

$$F = \sum_{i,j \geq 0} F_{i,j}\, s^i t^j = \sum_{i,j=0}^{\infty} F_{i,j}\, s^i t^j.$$

and we shall refer to $F_{i,j}$ as **the coefficient of** F **of place** (i,j).

Remark A.1.2. We explicitly observe that any element $a \in A$ can be identified with the formal power series $F_a \in A[\![s,t]\!]$ defined by

$$F_a(0,0) = a \quad \text{and} \quad F_a(i,j) = 0 \quad \text{for every } i,j \geq 1.$$

As a consequence, we consider the algebra A as a subset of $A[\![s,t]\!]$ and we identify the power series F_a with the element a.

A distinguished subset of $A[\![s,t]\!]$ is that of the polynomials in the two commuting variables s and t, with coefficients in A.

Definition A.1.3. A power series $F \in A[\![s,t]\!]$ is called a **polynomial** (in s and t with coefficients in A) if there exists $n \in \mathbb{N}$ such that

$$F_{i,j} = 0 \quad \text{for every } i,j \in \mathbb{N} \text{ with } i + j > n.$$

The set of all polynomials in A will be denote by $A[s,t]$. Moreover, given any couple $(i,j) \in \mathbb{N}\times\mathbb{N}$, we shall denote by $s^i\, t^j$ the (unique) polynomial $F \in A[s,t]$ whose only non-vanishing value is

$$F(i,j) := 1_A.$$

Finally, given any polynomial $F \in A[s,t]$ we define

$$\deg(F) := \min\big\{n \in \mathbb{N} : F_{i,j} = 0 \text{ for every } i+j > n.\big\}.$$

and we call it the **jointly degree** of F.

Remark A.1.4. We explicitly observe that, by the very definition of $A[s,t]$, we have $A \subseteq A[s,t]$ (the elements of A are polynomials of degree 0). Moreover,

$$A[s,t] = \mathrm{span}_{\mathbb{K}}\big(\{s^i\, t^j : i,j \in \mathbb{N}\}\big).$$

As it is reasonable to expect, the set $A[\![s,t]\!]$ inherits from A the structure of unital associative algebra. More precisely, we have the following theorem.

Theorem A.1.5. *The set $A[\![s,t]\!]$ is endowed with a structure of unital associative algebra (over the field $\mathbb{K}$) by the following operations:*

$$A[\![s,t]\!] \times A[\![s,t]\!] \ni (F,G) \mapsto (F+G) := \sum_{i,j=0}^{\infty} (F_{i,j} + G_{i,j})\, s^i t^j;$$

$$\mathbb{K} \times A[\![s,t]\!] \ni (a,F) \mapsto (a\,F) := \sum_{i,j=0}^{\infty} (a\,F_{i,j})\, s^i t^j;$$

$$A[\![s,t]\!] \times A[\![s,t]\!] \ni (F,G) \mapsto (F*G) := \sum_{i,j=0}^{\infty} \Bigg(\sum_{\substack{h+h'=i\\ k+k'=j}} F_{h,k} * G_{h',k'} \Bigg) s^i t^j.$$

In particular, the neutral element with respect to $+$ is the polynomial $F = 0$ while the neutral element with respect to $\star$ is the polynomial $s^0\, t^0 = 1_A$.

Remark A.1.6. Some remarks on the algebraic structure of $A[\![s,t]\!]$ are in order.

(a) All the operations defined on $A[\![s,t]\!]$ are compatible with the immersion $A \hookrightarrow A[\![s,t]\!]$ mentioned in Rem. A.1.2; for this reason, we adopt for such operations the same notations used for the ones defined on A.

(b) The operation $*$ is the usual Cauchy product of formal power series, since

$$s^i\, t^j * s^h\, t^k = s^{i+h}\, t^{j+k}, \quad \text{for every } (i,j),(h,k) \in \mathbb{N} \times \mathbb{N}.$$

(c) A direct computation shows that $A[s,t]$ is closed with respect to all the operations defined on $A[\![s,t]\!]$; hence $A[s,t]$ is a sub-algebra of $A[\![s,t]\!]$.

We next show that $A[\![s,t]\!]$ is not only a unital associative algebra, but it can be naturally endowed with a metric structure turning it into a topological algebra, as well. To this end, we first give the following definition.

Definition A.1.7. Let $F \in A[\![s,t]\!]$ be fixed. We define

$$\mathrm{ord}(F) := \begin{cases} \infty, & \text{if } F = 0,\\ \min\big\{i+j :\ F_{i,j} \neq 0\big\}, & \text{if } F \neq 0, \end{cases}$$

and we call it the **order of** F.

By making use of the notion of order of a power series, and with the convention $2^{-\infty}=0$, we then introduce the following map.

Definition A.1.8. We define

$$d: A[\![s,t]\!]\times A[\![s,t]\!]\to[0,\infty),\qquad d(F,G):=2^{-\mathrm{ord}(F-G)}.$$

Thm. A.1.9 below summirizes the most relevant properties of d.

Theorem A.1.9. *Let d be the function introduce in the above Def. A.1.8. Then d is a metric on $A[\![s,t]\!]$, and the couple $(A[\![s,t]\!],d)$ is a* metric space.

Moreover, the following properties hold true:

(i) *d is translation-invariant, that is,*

$$d(F,G)=d(F+H,G+H),\quad \textit{for every } F,G,H\in A[\![s,t]\!].$$

(ii) *A basis of neighborhoods of $0\in A[\![s,t]\!]$ is given by the family of sets*

$$\big\{F\in A[\![s,t]\!]:\,\mathrm{ord}(F)\geq n\big\},\quad n\in\mathbb{N}.$$

As a consequence, $(A[\![s,t]\!],d)$ is first-countable.

(iii) *d is an ultra-metric, i.e., the triangle inequality holds in the stronger form*

$$d(F,G)\leq\max\{d(F,H),d(H,G)\},\quad \textit{for every } F,G,H\in A[\![s,t]\!].$$

(iv) *$(A[\![s,t]\!],d)$ is a* complete *metric space and, relative to the structure of associative algebra $(A[\![s,t]\!],*)$ introduced in Thm. A.1.5, it is a topological algebra (under the topology induced by the metric structure).*

(v) *$A[s,t]$ is dense in $A[\![s,t]\!]$; more precisely, for every $F\in A[\![s,t]\!]$ we have*

$$A[s,t]\ni\sum_{k=0}^{n}\left(\sum_{i+j=0}^{k}F_{i,j}\,s^it^j\right)\stackrel{d}{\to}F\quad \textit{as } n\to\infty.$$

Remark A.1.10. Some remarks are in order:

(a) Let $\{F^{(n)}\}_{n\in\mathbb{N}}$ be a sequence in $A[\![s,t]\!]$. As a consequence of properties (i)-to-(iv) above (in particular, of the ultra-metric condition), the series

$$\sum_{n\in N}F^{(n)}$$

is convergent in $(A[\![s,t]\!],d)$ if and only if

$$\lim_{n\to\infty}F^{(n)}=0;$$

in turn, by the definition of d, the latter condition is satisfied *iff*

$$\lim_{n\to\infty}\mathrm{ord}(F^{(n)})=\infty.$$

(b) Property (v) shows that the notation $F = \sum_{i,j=0}^{\infty} F_{i,j}\, s^i t^j$ is far from being a mere formal notation: F is indeed the sum of a convergent series in the complete metric space $(A[\![s,t]\!], d)$.

(c) From the invariance of d, we see that a linear map $\varphi : A[\![s,t]\!] \to A[\![s,t]\!]$ is continuous if and only if it is continuous in 0. A sufficient condition for the latter is the existence of $m(\varphi) \in \mathbb{N}$ such that (see (i) above)

$$\mathrm{ord}(\varphi(F)) \geq \mathrm{ord}(F) - m(\varphi), \quad \text{for every } F \in A[\![s,t]\!].$$

The main motivation for the introduction of $A[\![s,t]\!]$ is to introduce two *partial* differential operators ∂_s and ∂_t, which will play a crucial rôle in the sequel.

Definition A.1.11. We define the following endomorphisms of $A[\![s,t]\!]$:

$$\partial_s : A[\![s,t]\!] \to A[\![s,t]\!], \quad \partial_s\Big(\sum_{i,j=0}^{\infty} F_{i,j}\, s^i t^j\Big) := \sum_{i,j=0}^{\infty} (i+1)\, F_{i+1,j}\, s^i t^j;$$
$$\partial_t : A[\![s,t]\!] \to A[\![s,t]\!], \quad \partial_t\Big(\sum_{i,j=0}^{\infty} F_{i,j}\, s^i t^j\Big) := \sum_{i,j=0}^{\infty} (j+1)\, F_{i,j+1}\, s^i t^j.$$

We say that $\partial_s F$ and $\partial_t F$ are the partial derivatives of F w.r.t. s and w.r.t. t.

Remark A.1.12. Some remarks concerning the maps ∂_s and ∂_t are in order.

(a) It is easy to prove that both ∂_s and ∂_t are derivations of the associative algebra $(A[\![s,t]\!], *)$, i.e., they are linear and they satisfy Leibniz's rule:

$$\partial_s(F * G) = \partial_s(F) * G + F * \partial_s(G),$$
$$\partial_t(F * G) = \partial_t(F) * G + F * \partial_t(G).$$

(b) By Rem. A.1.10 - (c), both ∂_s and ∂_t are continuous maps on the metric space $(A[\![s,t]\!], d)$: in fact, for every $F \in A[\![s,t]\!]$ we have

$$\mathrm{ord}(\partial_s(F)) \geq \mathrm{ord}(F) - 1 \quad \text{and} \quad \mathrm{ord}(\partial_t(F)) \geq \mathrm{ord}(F) - 1.$$

(c) It is possible to give an explicit characterization of the kernel of (the linear maps) ∂_s and ∂_t. In fact, since $\mathbb{K}$ has characteristic zero, we see that

$$\ker(\partial_s) = \left\{ \sum_{i,j=0}^{\infty} F_{i,j} s^i t^j \in A[\![s,t]\!] : F_{i,j} = 0 \text{ for all } i \geq 1, j \geq 0 \right\}.$$

Roughly put, the kernel of ∂_s consists of those formal power series which are independent of s. In the same way, one can see that the kernel of ∂_t consists of those formal power series which do not depend on t.

We now introduce two important subsets of $A[\![s,t]\!]$, which will be relevant in defining the exp and log maps on $A[\![s,t]\!]$:

$$A[\![s,t]\!]_+ := \{F \in A[\![s,t]\!] : F_{0,0} = 0\}, \tag{A.1.1a}$$
$$1 + A[\![s,t]\!]_+ := \{F \in A[\![s,t]\!] : F_{0,0} = 1_A\}. \tag{A.1.1b}$$

Definition A.1.13. By taking into account the subsets of $A[\![s,t]\!]$ introduced in (A.1.1a) and (A.1.1b), we define the two following functions:

$$\begin{aligned} &\exp : A[\![s,t]\!]_+ \longrightarrow 1 + A[\![s,t]\!]_+, \quad \exp(F) := \sum_{n=0}^{\infty} \frac{1}{n!} F^n, \\ &\log : 1 + A[\![s,t]\!]_+ \longrightarrow A[\![s,t]\!]_+, \quad \log(F) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (F - 1_A)^n. \end{aligned} \tag{A.1.2}$$

We say that exp and log are the **exponential** and **logarithmic** maps of $A[\![s,t]\!]$.

Remark A.1.14. We explicitly observe that the definitions of exp and log are well-posed, since the series in (A.1.2) are convergent in $(A[\![s,t]\!], d)$.

In fact, due to Rem. A.1.10-(a), it suffices to notice that, for every F in $A[\![s,t]\!]_+$, the sequence F^n vanishes as $n \to \infty$ (for $\mathrm{ord}(G^n) \geq n\,\mathrm{ord}(G) \geq n$).

The following Thm. A.1.15 (whose elementary proof is skipped) contains the most important features of the maps exp and log just defined.

Theorem A.1.15. *The following facts hold true:*

(i) *The maps* exp *and* log *are inverse to each other, that is,*

$$\begin{aligned} \exp(\log(F)) = F \quad &\textit{for every } F \in 1 + A[\![s,t]\!]_+ \\ \log(\exp(G)) = G, \quad &\textit{for every } G \in A[\![s,t]\!]. \end{aligned}$$

(ii) *For every fixed* $F \in A[\![s,t]\!]_+$*, the element* $\exp(F)$ *is invertible in the unital algebra* $(A[\![s,t]\!], *)$*, with inverse element given by* $\exp(-F)$*:*

$$\exp(F) * \exp(-F) = 1_A = \exp(-F) * \exp(F). \tag{A.1.3}$$

(iii) *For every fixed* $a \in A$*, we have* $a\,s, a\,t \in A[\![s,t]\!]$ *and*

$$\begin{aligned} &\frac{\partial}{\partial s} \exp(a\,s) = a * \exp(a\,s) = \exp(a\,s) * a, \\ &\frac{\partial}{\partial s} \exp(a\,t) = 0. \end{aligned} \tag{A.1.4}$$

Analogous formulas hold true for $\frac{\partial}{\partial t}$.

Remark A.1.16. As already said, the proof of Thm. A.1.15 is elementary. We limit ourselves to point out that statement (i) is a simple consequence of the following identities, holding true in every field of characteristic 0:

$$\sum_{n=1}^{k} \sum_{\substack{i_1,\dots,i_n \geq 1 \\ i_1+\dots+i_n = k}} \frac{(-1)^{n+1}}{n \cdot i_1! \dots i_n!} = \sum_{n=1}^{k} \sum_{\substack{i_1,\dots,i_n \geq 1 \\ i_1+\dots+i_n = k}} \frac{(-1)^{i_1+\dots+i_n+n}}{n! \cdot i_1 \dots i_n} = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k > 1. \end{cases} \tag{A.1.5}$$

In their turn, these identities follow from

$$\exp(\log x) = x \text{ and } \log(\exp y) = y \quad (\text{valid for } x > 0 \text{ and } y \in \mathbb{R}),$$

by inserting the series expansions of exp and log, and by recalling that any field of characteristic 0 possesses a sub-field isomorphic to $\mathbb{Q}$.

A notable generalization of (A.1.4) is given by the following Thm. A.1.18. First we need a notation used throughout. Given a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ and given an element $g \in \mathfrak{g}$, we set, as customary,

$$\operatorname{ad} g : \mathfrak{g} \longrightarrow \mathfrak{g}, \qquad (\operatorname{ad} g)(g') := [g, g'], \quad \forall\, g' \in \mathfrak{g}.$$

Remark A.1.17. If $(A, *)$ is an associative algebra, when we make any reference to A as a Lie algebra, we tacitly mean that it is equipped with the Lie bracket associated with $*$, namely

$$[a, a']_* := a * a' - a' * a, \quad \forall\, a, a' \in A.$$

For example, if $F \in A[\![s,t]\!]$, by $\operatorname{ad} F$ we mean the map

$$(\operatorname{ad} F) : A[\![s,t]\!] \longrightarrow A[\![s,t]\!], \qquad (\operatorname{ad} F)(G) = F * G - G * F,$$

where $*$ is as in Thm. A.1.5.

Theorem A.1.18. *Let $F \in A[\![s,t]\!]_+$. Then, the following identities hold:*

$$\partial_s \exp(F) = \left(\sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\operatorname{ad} F)^n (\partial_s F) \right) * \exp(F), \tag{A.1.6a}$$

$$\partial_s \exp(F) = \exp(F) * \left(\sum_{n=0}^{\infty} \frac{1}{(n+1)!} (-\operatorname{ad} F)^n (\partial_s F) \right). \tag{A.1.6b}$$

Analogous identities hold true for the partial derivative with respect to t.

Remark A.1.19. Before sketching its proof, two remarks concerning the content of Thm. A.1.18 are in order:

(1) Let $F \in A[\![s,t]\!]_+$ be fixed. By Rem. A.1.10 - (a) the series in (A.1.6a) and (A.1.6b) are convergent since, for any $G \in A[\![s,t]\!]$, one has

$$\operatorname{ord}((\operatorname{ad} F)^n(G)) \geq n \operatorname{ord}(F) + \operatorname{ord}(G) \geq n \to \infty, \quad \text{as } n \to \infty.$$

We shall compactly rewrite (A.1.6a) and (A.1.6b) as

$$\partial_s \exp(F) = \frac{e^{\operatorname{ad} F} - 1}{\operatorname{ad} F} \Big(\partial_s F \Big) * \exp(F), \tag{A.1.7a}$$

$$\partial_s \exp(F) = \exp(F) * \frac{1 - e^{-\operatorname{ad} F}}{\operatorname{ad} F} \Big(\partial_s F \Big). \tag{A.1.7b}$$

(2) The above notation is not only formal: for any $F \in A[\![s,t]\!]_+$, the map

$$A[\![s,t]\!] \ni G \mapsto \frac{e^{\operatorname{ad} F} - 1}{\operatorname{ad} F}(G) := \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\operatorname{ad} F)^n(G) \tag{A.1.8}$$

is a well-posed linear map of the vector space $A[\![s,t]\!]$ into itself (see point (1) above). Moreover, this map is invertible and its inverse is given by

$$A[\![s,t]\!] \ni G \mapsto \frac{\operatorname{ad} F}{e^{\operatorname{ad} F} - 1}(G) := \sum_{n=0}^{\infty} \frac{B_n}{n!} (\operatorname{ad} F)^n(G), \tag{A.1.9}$$

where $\{B_n\}_{n\in\mathbb{N}}$ is the sequence of the Bernoulli numbers, defined by the generating holomorphic function

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}\,z^n,\quad |z|<2\pi.$$

As it is well-known, the B_n are rational numbers (hence meaningful in $\mathbb{K}$) since they satisfy the recursion formula

$$B_0=1,\qquad B_n=-n!\sum_{k=0}^{n-1}\frac{B_k}{k!\,(n+1-k)!}\quad (n\geq 1),\tag{A.1.10}$$

as it follows by expanding in a Cauchy product the identity

$$1=\frac{e^z-1}{z}\cdot\frac{z}{e^z-1}=\left(\sum_{n=0}^{\infty}\frac{z^n}{(n+1)!}\right)\cdot\left(\sum_{n=0}^{\infty}\frac{B_n}{n!}\,z^n\right).$$

Proof (of Thm. A.1.18). We start from a result of non-commutative algebra, whose proof follows by induction (see [30, Lemma 4.21]): for every fixed F in $A[\![s,t]\!]$ and every natural $n\geq 1$ one has

$$\begin{aligned}\partial_s(F^n)&=\sum_{k=0}^{n-1}\tbinom{n}{k+1}\,(\mathrm{ad}\,F)^k\Big(\partial_sF\Big)*F^{n-k-1}\\&=\sum_{k=0}^{n-1}\tbinom{n}{k+1}\,F^{n-k-1}*(-\mathrm{ad}\,F)^k\Big(\partial_sF\Big).\end{aligned}$$

Now, Thm. A.1.18 can be proved arguing exactly as in the proof of [30, Theorem 4.22], since ∂_s is a continuous derivation of $A[\![s,t]\!]$ (see Rem. A.1.12 - (b)).

Indeed, by passing ∂_s under the series sign, we get

$$\begin{aligned}\partial_s\exp(F)&=\sum_{n=1}^{\infty}\frac{1}{n!}\,\partial_s(F^n)=\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{k=0}^{n-1}\tbinom{n}{k+1}\,(\mathrm{ad}\,F)^k\Big(\partial_sF\Big)*F^{n-k-1}\\&=\sum_{k=0}^{\infty}\left(\frac{1}{(k+1)!}\,(\mathrm{ad}\,F)^k\Big(\partial_sF\Big)*\sum_{n=k+1}^{\infty}\frac{F^{n-k-1}}{(n-k-1)!}\right)\\&=\frac{e^{\mathrm{ad}\,F}-1}{\mathrm{ad}\,F}\Big(\partial_sF\Big)*\exp(F).\end{aligned}$$

Identity (A.1.6b) can be proved analogously. □

To end the section of algebraic preliminaries, we make our choice of the unital associative algebra $A=\mathscr{T}(x,y)$, defined as follows.

Let $\{x,y\}$ be a set of cardinality two, and let us denote by $\mathscr{T}(x,y)$ the (free) associative $\mathbb{K}$-algebra of the polynomials in the non-commuting indeterminates x,y. More precisely, if $V:=\mathbb{K}\langle x,y\rangle$ is the free $\mathbb{K}$-vector space generated by the set $\{x,y\}$ (i.e., the set of the formal $\mathbb{K}$-linear combinations of x and y), $\mathscr{T}(x,y)$ is simply the tensor algebra of V. The operation in $\mathscr{T}(x,y)$ is the usual (tensor) multiplication (denoted by juxtaposition), and its unit is $1_{\mathbb{K}}$ (the unit of $\mathbb{K}$); moreover (as already discussed) $\mathscr{T}(x,y)$ is turned into a Lie algebra with the Lie bracket associated with this associative multiplication (see Rem. A.1.17).

It is therefore well-posed the Lie-sub-algebra

$$\mathcal{L}(x,y) \subset \mathscr{T}(x,y),$$

which is the smallest Lie sub-algebra of $\mathscr{T}(x,y)$ containing x and y. We call $\mathcal{L}(x,y)$ the (free) Lie algebra generated by x and y, and any of its elements is said to be a Lie-polynomial in x,y.

Remark A.1.20. It is not difficult to recognize that both $\mathscr{T}(x,y)$ and $\mathcal{L}(x,y)$ possess the following *universal properties*:

(UPT) For every unital associative $\mathbb{K}$-algebra U and for every $a,b \in U$, there exists a unique morphism (of unital associative algebras)

$$\varphi_{a,b} : \mathscr{T}(x,y) \longrightarrow U$$

such that $\varphi_{a,b}(x) = a$ and $\varphi_{a,b}(y) = b$.

(UPL) For every Lie algebra (over $\mathbb{K}$) L and for every $a,b \in L$, there exists a unique morphism (of Lie algebras)

$$\varphi_{a,b} : \mathcal{L}(x,y) \longrightarrow U$$

such that $\varphi_{a,b}(x) = a$ and $\varphi_{a,b}(y) = b$.

A.2 Two formal PDEs in $A[\![s,t]\!]$ for the CBHD series

Throughout this section, A denotes the free unital associative algebra $\mathscr{T}(x,y)$ (over the field $\mathbb{K}$), as introduced in the previous section, from which we also inherit all other notation. Accordingly, $(A[\![s,t]\!],*)$ is the associated unital associative algebra of the formal power series in the commuting indeterminates s,t (equipped with the metric and associative structures in Sec. A.1).

We consider the notable element $Z(s,t)$ of $A[\![s,t]\!]$ defined as follows:

$$Z(s,t) := \log(\exp(x\,s) * \exp(y\,t)), \tag{A.2.1}$$

where exp and log are the maps on $A[\![s,t]\!]$ introduced in Def. A.1.13.

Remark A.2.1. We explicitly observe that $Z(s,t)$ is well posed, since

$$\exp(x\,s) * \exp(y\,t) \in 1 + A[\![s,t]\!]_+$$

(we denote by 1 the unit of $A[\![s,t]\!]$, coinciding with that of $\mathbb{K}$). Moreover, the element $Z(s,t) \in A[\![s,t]\!]$ is completely characterized by the following identity

$$\exp(Z(s,t)) = \exp(x\,s) * \exp(y\,t). \tag{A.2.2}$$

As an element of $A[\![s,t]\!]$, for any $i,j \geq 0$ there are uniquely defined elements $Z_{i,j}(x,y) \in \mathscr{T}(x,y)$ (occasionally denoted by $Z_{i,j}$) such that

$$Z(s,t) = \sum_{i,j\geq 0} Z_{i,j}(x,y)\, s^i t^j. \tag{A.2.3}$$

We shall refer to the above (A.2.3) as the **Campbell-Baker-Hausdorff-Dynkin** (CBHD, for short) **double series**. By unraveling the very definitions of exp and log, it is obvious that $Z_{0,0}(x,y) = 0$ and, for every $(i,j) \neq (0,0)$

$$Z_{i,j}(x,y) = \sum_{n=1}^{i+j} \frac{(-1)^{n+1}}{n} \sum_{\substack{(i_1,j_1),\dots,(i_n,j_n) \neq (0,0) \\ i_1+\dots+i_n = i \\ j_1+\dots+j_n = j}} \frac{x^{i_1} y^{j_1} \cdots x^{i_n} y^{j_n}}{i_1! j_1! \dots i_n! j_n!}. \tag{A.2.4}$$

We next compute $Z_{i,j}$ when one of i and j is null. These values of $Z_{i,j}$ will soon be used as "initial data" for two formal PDEs satisfied by $Z(s,t)$.

Lemma A.2.2. *In the above assumptions and notations, we have*

$$Z_{1,0}(x,y) = x, \qquad Z_{i,0}(x,y) = 0 \quad \forall\, i \neq 1; \tag{A.2.5a}$$

$$Z_{0,1}(x,y) = y, \qquad Z_{0,j}(x,y) = 0 \quad \forall\, j \neq 1. \tag{A.2.5b}$$

Proof. By exploiting identity (A.2.4), for every $k \geq 1$ one has

$$\begin{aligned} Z_{k,0}(x,y) &= \sum_{n=1}^{k} \frac{(-1)^{n+1}}{n} \sum_{\substack{i_1,\dots,i_n \neq 0 \\ i_1+\dots+i_n = k}} \frac{x^{i_1+\dots+i_n}}{i_1! \dots i_n!} \\ &= x^k \sum_{n=1}^{k} \frac{(-1)^{n+1}}{n} \sum_{\substack{i_1,\dots,i_n \neq 0 \\ i_1+\dots+i_n = k}} \frac{1}{i_1! \dots i_n!}. \end{aligned}$$

Therefore, (A.2.5a) (and analogously for (A.2.5b)) follows from (A.1.5). □

We are ready to prove the main result of this section.

Theorem A.2.3. *Let* $Z(s,t) \in A[\![s,t]\!]$ *be as in* (A.2.1) *and let* ∂_s, ∂_t *be the two derivations of the (associative and Lie) algebra* $A[\![s,t]\!]$ *introduced in Def. A.1.11.*

Then, if the B_n *denote the Bernoulli numbers (see* (A.1.10)*), we have*

$$\partial_s Z(s,t) = \sum_{n=0}^{\infty} \frac{B_n}{n!} \,(\mathrm{ad}\, Z(s,t))^n(x), \tag{A.2.6a}$$

$$\partial_t Z(s,t) = \sum_{n=0}^{\infty} \frac{B_n}{n!} \,(-\mathrm{ad}\, Z(s,t))^n(y). \tag{A.2.6b}$$

With the notations introduced for the (inverse to each other) automorphisms in (A.1.8) *and* (A.1.9)*, identities* (A.2.6a) *and* (A.2.6b) *can be rewritten as*

$$\partial_s Z = \frac{\mathrm{ad}\, Z}{e^{\mathrm{ad}\, Z} - 1}(x) \quad \textit{and} \quad \partial_t Z = \frac{\mathrm{ad}\, Z}{1 - e^{-\mathrm{ad}\, Z}}(y). \tag{A.2.7}$$

Proof. We prove (A.2.6a), since the proof of (A.2.6b) is analogous. We apply the derivation ∂_s on both sides of identity (A.2.2); on the left-hand side we apply formula (A.1.7a), while on the right-hand side we apply (A.1.4) (and the fact that ∂_s is a derivation, hence it satisfies Leibniz's rule). This gives

$$\frac{e^{\operatorname{ad} Z}-1}{\operatorname{ad} Z}\Big(\partial_s Z\Big) * \exp(Z) = x * \exp(x\,s) * \exp(y\,t) \overset{(A.2.2)}{=} x * \exp(Z).$$

By Thm. A.1.15 - (ii), the element $\exp(Z)$ is invertibile in $A[\![s,t]\!]$, with inverse given by $\exp(-Z)$; as a consequence, by multiplying both sides of the preceding identity by $\exp(-Z)$ we infer that

$$\frac{e^{\operatorname{ad} Z}-1}{\operatorname{ad} Z}\Big(\partial_s Z\Big) = x.$$

Since the endomorphism in (A.1.8) is invertible, its inverse being given by the map in (A.1.9), we immediately get (A.2.6a) from this last identity. □

Remark A.2.4. Since ∂_s and ∂_t are derivations of the associative algebra $(A[\![s,t]\!],*)$ (see Rem. A.1.12 - (a)), they are also derivations of the Lie algebra associated with $A[\![s,t]\!]$, i.e., for every $F,G\in A[\![s,t]\!]$ one has

$$\partial_s\big([F,G]_*\big) = [\partial_s F, G]_* + [F,\partial_s G]_*$$
$$\partial_t\big([F,G]_*\big) = [\partial_t F, G]_* + [F,\partial_t G]_*$$

where $[\cdot,\cdot]_*$ is the Lie bracket associated with $*$ (see Rem. A.1.17). Roughly speaking, this is why both equations in (A.2.7) (plus convenient initial data) can be profitably solved providing a solution

$$Z=\sum_{i,j} Z_{i,j}\, s^i t^j,$$

where $Z_{i,j}$ is a *Lie polynomial* in x,y (this is precisely the content of the Campbell-Baker-Hausdorff-Dynkin Theorem).

As it is reasonable to expect, identities (A.2.6a) and (A.2.6b) in Thm. A.2.3 boil down to a system of (recursive) identities involving the coefficients $Z_{i,j}(x,y)$. More precisely, we have the following result.

Corollary A.2.5. *For any given $i,j\geq 0$, let $Z_{i,j}:=Z_{i,j}(x,y)\in\mathscr{T}(x,y)$ be as in* (A.2.3) *(see also the explicit expression in* (A.2.4)*).*

Then, together with the initial conditions (proved in Lem. A.2.2)

$$Z_{0,0}=0, \qquad Z_{1,0}=x, \qquad Z_{i,0}=0 \quad \text{for every } i\geq 2, \tag{A.2.8a}$$
$$Z_{0,0}=0, \qquad Z_{0,1}=y, \qquad Z_{0,j}=0 \quad \text{for every } j\geq 2, \tag{A.2.8b}$$

we have the following recursive identities, for every $i,j\geq 0$ s.t. $(i,j)\neq(0,0)$:

$$(i+1)\,Z_{i+1,j} = \sum_{\substack{1\leq n\leq i+j\\ (i_1,j_1),\dots,(i_n,j_n)\neq(0,0)\\ i_1+\cdots+i_n=i\\ j_1+\cdots+j_n=j}} K_n\,[Z_{i_1,j_1},\cdots[Z_{i_n,j_n},x]\cdots], \tag{A.2.9a}$$

$$(j+1)\,Z_{i,j+1} = \sum_{\substack{1\leq n\leq i+j\\ (i_1,j_1),\dots,(i_n,j_n)\neq(0,0)\\ i_1+\cdots+i_n=i\\ j_1+\cdots+j_n=j}} (-1)^n K_n\,[Z_{i_1,j_1},\cdots[Z_{i_n,j_n},y]\cdots], \tag{A.2.9b}$$

Here we have set $K_n := B_n/n!$, where $\{B_n\}_n$ is the sequence of the Bernoulli numbers in (A.1.10)*. Furthermore, $[\cdot,\cdot]$ denotes the Lie bracket associated with the associative algebra $\mathscr{T}(x,y)$ (see also Rem. A.1.17).*

Proof. We only prove (A.2.9a) since the proof of (A.2.9b) is analogous. With the notation $K_n := B_n/n!$ and $Z := Z(s,t)$, identity (A.2.6a) is equivalent to

$$\partial_s Z = \sum_{n=0}^{\infty} K_n \,(\operatorname{ad} Z)^n(x).$$

If we insert the decomposition $Z = \sum_{i,j} Z_{i,j}\, s^i t^j$ and we use the definition of ∂_s, then the above identity gives (recall that, by definition, $Z_{0,0} = 0$)

$$\begin{aligned}
&\sum_{i,j=0}^{\infty} (i+1) Z_{i+1,j}\, s^i t^j \\
&= x + \sum_{n=1}^{\infty} K_n \sum_{(i_1,j_1),\dots,(i_n,j_n)\neq(0,0)} [Z_{i_1,j_1},\cdots[Z_{i_n,j_n},x]\cdots]\, s^{i_1+\cdots+i_n} t^{j_1+\cdots+j_n} \\
&= x + \sum_{(i,j)\neq(0,0)} \Bigg(\sum_{\substack{(i_1,j_1),\dots,(i_n,j_n)\neq(0,0)\\ i_1+\cdots+i_n=i\\ j_1+\cdots+j_n=j}} K_n\, [Z_{i_1,j_1},\cdots[Z_{i_n,j_n},x]\cdots] \Bigg) s^i t^j.
\end{aligned}$$

Since, in the inner sum, $i_1+j_1,\dots,i_n+j_n \geq 1$ and $i_1+j_1+\cdots+i_n+j_n = i+j$, we infer that $1 \leq n \leq i+j$. By equating the coefficients of $s^i t^j$ (for every fixed $i,j \geq 0$) we then obtain (A.2.9a). This ends the proof. □

Remark A.2.6. From Cor. A.2.5 we obtain a proof of the Campbell-Baker-Hausdorff-Dynkin (CBHD, for short) Theorem as follows.

1. Identities (A.2.9a) are not sufficient to determine all of the coefficients $Z_{i,j}$; one needs to add the information contained in (A.2.8b). Analogously, (A.2.9b) and (A.2.8a) determine all of the $Z_{i,j}$. Alternatively, one could use (A.2.9a), (A.2.9b) and $Z_{0,0} = 0$ to determine all of the terms $Z_{i,j}$.

   More precisely, if we think of Z as an infinite matrix $(Z_{i,j})_{i,j\geq 0}$, identity (A.2.9a) allows to determine an entry $Z_{i,j}$, provided that one knows all the entries in the (finite) sub-matrix with rows strictly less than i and columns less than or equal to j. Thus, in order to obtain all the entries of Z from (A.2.9a), one needs to know the entries in the first infinite row; these are given by (A.2.8b) (see also Lem. A.2.2). Analogous remarks hold for (A.2.9b), by reversing the rôles of columns and rows.

2. Since all the elements $Z_{0,j}$ belong to $\mathcal{L}(x,y)$ (see (A.2.8b)) and since the right-hand side of (A.2.9a) only involves Lie-bracketing, by the results in (1) above, we can prove by induction that

   $$Z_{i,j}(x,y) \in \mathcal{L}(x,y), \quad \text{for every } i,j \geq 0. \tag{A.2.10}$$

   Then, by using the Dynkin-Specht-Wever Lemma (see e.g., [30, Lemma 3.26]), from (A.2.10) and (A.2.4) we obtain the following (Dynkin's) Lie-representation of $Z_{i,j}(x,y)$, holding true for $(i,j) \neq (0,0)$

$$Z_{i,j}(x,y) = \sum_{n=1}^{i+j} \frac{(-1)^{n+1}}{n(i+j)} \times \\ \times \sum_{\substack{(i_1,j_1),\dots,(i_n,j_n) \neq (0,0) \\ i_1+\dots+i_n = i \\ j_1+\dots+j_n = j}} \frac{(\operatorname{ad} x)^{i_1} (\operatorname{ad} y)^{j_1} \cdots (\operatorname{ad} x)^{i_n} (\operatorname{ad} y)^{j_n - 1}(y)}{i_1! j_1! \dots i_n! j_n!}. \tag{A.2.11}$$

A.3 Convergence domain of the CBHD series in Banach-Lie algebras

The present section is the real core of this chapter and it is totally devoted to state and prove the announced convergence result for the Campbell-Baker-Hausdorff-Dynkin double series introduced in Sec. A.2.

To begin with, we need to introduce some preliminary definitions.

Definition A.3.1. Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie algebra over $\mathbb{K}$. Given $a, b \in \mathfrak{g}$, if $\varphi_{a,b} : \mathcal{L}(x,y) \to \mathfrak{g}$ is the Lie-algebra morphism in property (UPL) of the free Lie algebra $\mathcal{L}(x,y)$ (see Rem. A.1.20), we define

$$Z_{i,j}(a,b) := \varphi_{a,b}(Z_{i,j}(x,y)), \quad \text{for every } i,j \geq 0. \tag{A.3.1}$$

Here, as in (A.2.3), $Z_{i,j}(x,y)$ is the coefficient of place (i,j) in the expansion of $Z(s,t) = \log(\exp(x\,s) * \exp(y\,t))$ in $A[\![s,t]\!]$ (with $A = \mathscr{T}(x,y)$).

Remark A.3.2. We explicitly observe that, thanks to Rem. A.2.6, the preceding Def. A.3.1 is well-posed: in fact, identity (A.2.10) ensures that $Z_{i,j}(x,y)$ actually belongs to $\mathcal{L}(x,y)$ for every $i,j \geq 1$. Moreover, for any $(i,j) \in \mathbb{N} \times \mathbb{N}$, the map $(a,b) \mapsto Z_{i,j}(a,b)$ defines, unambiguously, a function from $\mathfrak{g} \times \mathfrak{g}$ to $\mathfrak{g}$.

Remark A.3.3. By using the explicit Lie representation of $Z_{i,j}(x,y)$ given in (A.2.11), one can define $Z_{i,j}(a,b)$ in the following alternative way:

$$Z_{i,j}(a,b) = \sum_{n=1}^{i+j} \frac{(-1)^{n+1}}{n(i+j)} \times \\ \times \sum_{\substack{(i_1,j_1),\dots,(i_n,j_n) \neq (0,0) \\ i_1+\dots+i_n = i \\ j_1+\dots+j_n = j}} \frac{(\operatorname{ad} a)^{i_1} (\operatorname{ad} b)^{j_1} \cdots (\operatorname{ad} a)^{i_n} (\operatorname{ad} b)^{j_n - 1}(b)}{i_1! j_1! \cdots i_n! j_n!}, \tag{A.3.2}$$

for any $(i,j) \neq (0,0)$, and $Z_{0,0}(a,b) = 0$. Obviously, in the above formula the adjoint map ad is related to the Lie algebra $\mathfrak{g}$.

From the Universal Property (UPL) of the free Lie algebra $\mathcal{L}(x,y)$ presented in Rem. A.1.20, one easily derives the following crucial result.

Corollary A.3.4. *Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie algebra over the field $\mathbb{K}$ and let $a, b \in \mathfrak{g}$ be fixed. Then, for every $m \in \mathbb{N} \setminus \{1\}$ and every $(i, j) \neq (0, 0)$, we have*

$$
\begin{aligned}
&Z_{1,0}(a,b) = a, \quad Z_{m,0}(a,b) = 0, \qquad Z_{0,1}(a,b) = b, \quad Z_{0,m}(a,b) = 0,\\
&Z_{i+1,j}(a,b) = \frac{1}{i+1} \sum_{\substack{1 \le n \le i+j \\ (i_1,j_1),\dots,(i_n,j_n) \neq (0,0) \\ i_1+\cdots+i_n = i \\ j_1+\cdots+j_n = j}} K_n \, [Z_{i_1,j_1}(a,b), \cdots [Z_{i_n,j_n}(a,b), a] \cdots],\\
&Z_{i,j+1}(a,b) = \frac{1}{j+1} \sum_{\substack{1 \le n \le i+j \\ (i_1,j_1),\dots,(i_n,j_n) \neq (0,0) \\ i_1+\cdots+i_n = i \\ j_1+\cdots+j_n = j}} (-1)^n K_n \, [Z_{i_1,j_1}(a,b), \cdots [Z_{i_n,j_n}(a,b), b] \cdots],
\end{aligned}
\tag{A.3.3}
$$

Here, as usual, we have set $K_n := B_n/n!$, where $\{B_n\}_{n\in\mathbb{N}}$ is the sequence of the Bernoulli numbers (see Rem. A.1.19 - (2)).

Proof. These formulas follow from identities (A.2.8a) through (A.2.9b), by the very definition (A.3.1) of $Z_{i,j}(a, b)$, and since the map $\varphi_{a,b}$ is a Lie-algebra-morphism (see the Universal Property (UPL) in Rem. A.1.20). □

Next, we consider the Lie algebras we are interested in for the rest of the chapter: *Banach-Lie algebras*. Here is the definition.

Definition A.3.5. Let $(L, [\cdot,\cdot])$ be a (possibly infinite-dimensional) Lie algebra over $\mathbb{R}$ or $\mathbb{C}$, and let $\|\cdot\| : L \to [0, \infty)$ be a norm on L. We say that L is a **Banach-Lie algebra** if the following conditions are satisfied:

(i) $(L, \|\cdot\|)$ is a Banach space;

(ii) the map $[\cdot,\cdot] : L \times L \to L$ is continuous (w.r.t. the product topology).

If L is a Banach-Lie algebra and if $a, b \in L$, the series in L defined by

$$
\sum_{n=1}^{\infty} \Big(\sum_{i+j=0}^{n} Z_{i,j}(a,b) \Big)
$$

is called the **homogeneous CBHD series related to** (a, b).

Remark A.3.6. Let $(L, [\cdot,\cdot])$ be a (real or complex) Lie algebra and let $\|\cdot\|$ be a norm on L. Since the bracket is bilinear, the continuity assumption (ii) in Def. A.3.5 is equivalent to the existence of a constant $M > 0$ such that

$$
\|[g, g']\| \le M \, \|g\| \, \|g'\|, \quad \text{for every } g, g' \in L.
$$

By replacing $\|\cdot\|$ with the equivalent norm $M\|\cdot\|$, we can suppose (and we shall do it henceforth) that $\|\cdot\|$ is *Lie-sub-multiplicative*, that is,

$$
\|[g, g']\| \le \|g\| \, \|g'\|, \quad \text{for every } g, g' \in L. \tag{A.3.4}
$$

Example A.3.7. Let $(A, *)$ be an associative algebra over $\mathbb{R}$ or $\mathbb{C}$ and let $\|\cdot\|$ be a norm on A. Let us assume that $(A, \|\cdot\|)$ is a Banach space and that the product $*$ is continuous, that is, there exists $M > 0$ such that

$$\|a * a'\| \leq M\,\|a\|\,\|a'\|, \quad \text{for every } a, a' \in A.$$

Since, for the Lie bracket $[\cdot,\cdot]_*$ associated with $*$, one has

$$\|[a, a']_*\| = \|a * a' - a' * a\| \leq 2M\,\|a\|\,\|a'\|, \quad \text{for every } a, a' \in A,$$

we conclude that the triple $(A, [\cdot,\cdot]_*, \|\cdot\|)$ is a Banach-Lie algebra, a Lie-sub-multiplicative norm being provided by $2M\|\cdot\|$.

Given a Banach-Lie algebra $(L, [\cdot,\cdot], \|\cdot\|)$, our aim is to provide a subset of $L \times L$ on which this series is convergent in L. More precisely, we shall prove the convergence of the majorizing (numerical) series

$$\sum_{i,j \geq 0} \|Z_{i,j}(a, b)\|.$$

With the background algebraic identities in Corollary A.3.4 at hands, this will be reduced to the problem of estimating the maximal domain of the solution of a real ODE. The latter is investigated in the next result.

Lemma A.3.8. *Let β be a nonnegative real constant and let*

$$F : (-2\pi, 2\pi) \to \mathbb{R}, \quad F(t) := 2 + \frac{t}{2}\Big(1 - \cot\Big(\frac{t}{2}\Big)\Big) \tag{A.3.5}$$

(with the obvious convention $F(0) := 1$). Moreover, for any fixed $\alpha \in (-2\pi, 2\pi)$, let γ be the maximal solution of the (real) Cauchy problem

$$\begin{cases} \gamma' = \beta\,F(\gamma) \\ \gamma(0) = \alpha. \end{cases} \tag{A.3.6}$$

Then, if $\gamma^{(i)}(0)$ denotes the i-th derivative of γ at 0, for every $n \geq 0$ one has

$$\gamma^{(0)}(0) = \alpha, \quad \frac{\gamma^{(n+1)}(0)}{n!} = \beta \sum_{m=1}^{\infty} |K_m| \Bigg(\sum_{\substack{i_1,\dots,i_m \geq 0 \\ i_1+\cdots+i_m = n}} \frac{\gamma^{(i_1)}(0)\cdots\gamma^{(i_m)}(0)}{i_1!\cdots i_m!} \Bigg), \tag{A.3.7}$$

all these series being convergent. Here $K_m = B_m/m!$, where $\{B_m\}_{m\in\mathbb{N}}$ is the sequence of the Bernoulli numbers. As a consequence, if $\alpha \in [0, 2\pi)$ then all the derivatives of γ at 0 are nonnegative real numbers.

Proof. To begin with, if $D_{2\pi}$ is the complex disc with radius 2π centered at 0, and $F(z)$ is as in (A.3.5) (for $z \in D_{2\pi}$), we claim that

$$F(z) = \textstyle\sum_{n=0}^{\infty} |K_n|\, z^n \text{ for every } z \in D_{2\pi}. \tag{A.3.8}$$

In fact, by definition of K_n, for any $z \in \mathbb{C}$ with $|z| < 2\pi$ we have

$$\psi(z) := \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} K_n\, z^n.$$

By well-known properties of the Bernoulli numbers (see e.g., [135]), we have

$$K_0 = 1,\ K_1 = -1/2, \quad K_{2n+1} = 0,\ K_{2n} = (-1)^{n-1}\,|K_{2n}| \quad (n \geq 1).$$

This ensures that $\psi(iz) = 1 - \frac{iz}{2} - \sum_{n=1}^{\infty} |K_{2n}|\, z^{2n}$ (whenever $|z| < 2\pi$), hence

$$\begin{aligned}\sum_{n=0}^{\infty} |K_n|\, z^n &= 1 + \frac{z}{2} + \sum_{n=1}^{\infty} |K_{2n}|\, z^{2n} = 2 + \frac{z}{2} - \frac{iz}{2} - \psi(iz)\\ &= 2 + \frac{z}{2}\left(1 - \cot\left(\frac{z}{2}\right)\right) = F(z),\end{aligned}$$

as we claimed in (A.3.8). Now, (A.3.7) directly follows from (A.3.8) by inserting the Maclaurin expansions of F and of $\gamma(t)$ (which is real-analytic since it solves (A.3.6), with F analytic) and by the standard power-series *Ansatz*. The last assertion of the statement follows from (A.3.7) by an induction argument. □

From Lem. A.3.8 we obtain the following central result.

Theorem A.3.9 (Estimate of $Z_{i,j}$ in a Banach-Lie algebra). *Let L be a Banach-Lie algebra, equipped with a Lie-sub-multiplicative norm $\|\cdot\|$ (i.e.,* (A.3.4) *holds), and let $a, b \in L$ be such that $\|a\|, \|b\| < 2\pi$. Moreover, if F is as in* (A.3.5)*, let*

$$\gamma = \gamma_{\|a\|,\|b\|} \quad \textit{and} \quad \mu = \gamma_{\|b\|,\|a\|}$$

be the maximal solutions of the following (real) Cauchy problems

$$\begin{cases} \gamma' = \|b\|\, F(\gamma) \\ \gamma(0) = \|a\|, \end{cases} \qquad \begin{cases} \mu' = \|a\|\, F(\mu) \\ \mu(0) = \|b\|. \end{cases}$$

If $Z_{i,j}(a,b)$ is as in (A.3.1)*, one has the estimates*

$$\begin{aligned} &\sum_{i=0}^{\infty} \|Z_{i,j}(a,b)\| \leq \frac{\gamma^{(j)}(0)}{j!} \quad \textit{for every } j \geq 0;\\ &\sum_{j=0}^{\infty} \|Z_{i,j}(a,b)\| \leq \frac{\mu^{(i)}(0)}{i!} \quad \textit{for every } i \geq 0. \end{aligned} \tag{A.3.9}$$

Proof. We prove the first family of inequalities in (A.3.9), proceeding by induction on $j \in \mathbb{N}$. The proof of the second family is analogous and is omitted.

If $j = 0$, from the first group of identities in Cor. A.3.4 we infer that

$$\sum_{i=0}^{\infty} \|Z_{i,0}(a,b)\| = \|Z_{1,0}(a,b)\| = \|a\| = \gamma^{(0)}(0).$$

Let now $j \geq 0$ be fixed and let us assume that

$$\sum_{i=0}^{\infty} \|Z_{i,h}(a,b)\| \leq \frac{\gamma^{(h)}(0)}{h!} \quad \forall\ h = 0, \dots, j. \tag{A.3.10}$$

We shall prove that the same holds true for $h = j+1$. On the one hand, if $j=0$, from (A.3.3) (and the sub-multiplicative property (A.3.4)), we get

$$\sum_{i=0}^{\infty} \|Z_{i,1}(a,b)\| = \|Z_{0,1}(a,b)\| + \sum_{i=1}^{\infty} \|Z_{i,1}(a,b)\| \overset{(A.3.3)}{=} \|b\|$$
$$+ \sum_{i=1}^{\infty} \Bigg\| \sum_{\substack{1\le n\le i \\ (i_1,j_1),\dots,(i_n,j_n)\neq(0,0) \\ i_1+\cdots+i_n=i \\ j_1+\cdots+j_n=0}} (-1)^n K_n \left[Z_{i_1,j_1}(a,b),\cdots[Z_{i_n,j_n}(a,b),b]\cdots\right] \Bigg\|$$
$$\le \|b\| + \sum_{i=1}^{\infty} \sum_{\substack{1\le n\le i \\ i_1,\dots,i_n\neq 0 \\ i_1+\cdots+i_n=i}} |K_n|\, \|Z_{i_1,0}(a,b)\|\cdots\|Z_{i_n,0}(a,b)\|\,\|b\|$$
$$\overset{(A.3.3)}{=} \|b\| \left(1 + \sum_{i=1}^{\infty} |K_i|\,\|a\|^i\right) \overset{(A.3.8)}{=} \|b\|\,F(\|a\|)$$
$$= \|b\|\,F(\gamma(0)) = \gamma^{(1)}(0).$$

On the other hand, if $j \ge 1$, again by exploiting the last identity in (A.3.3), we obtain (see the induction hypothesis (A.3.10))

$$\sum_{i=0}^{\infty} \|Z_{i,j+1}(a,b)\| \le \sum_{i=0}^{\infty} \Bigg(\frac{1}{j+1} \times$$
$$\times \sum_{\substack{1\le n\le i+j \\ (i_1,j_1),\dots,(i_n,j_n)\neq(0,0) \\ i_1+\cdots+i_h=i \\ j_1+\cdots+j_h=j}} |K_n|\,\|Z_{i_1,j_1}(a,b)\|\cdots\|Z_{i_n,j_n}(a,b)\|\,\|b\| \Bigg)$$
$$\le \frac{\|b\|}{j+1} \sum_{\substack{n\ge1,\ i_1,\dots,i_n\ge 0 \\ j_1+\cdots+j_n=j}} |K_n|\,\|Z_{i_1,j_1}(a,b)\|\cdots\|Z_{i_n,j_n}(a,b)\|$$
$$= \frac{\|b\|}{j+1} \sum_{n=1}^{\infty} \sum_{j_1+\cdots+j_n=j} |K_n| \left(\sum_{i_1=0}^{\infty} \|Z_{i_1,j_1}(a,b)\|\right) \cdots \left(\sum_{i_h=0}^{\infty} \|Z_{i_1,j_1}(a,b)\|\right)$$
$$\overset{(A.3.10)}{\le} \frac{\|b\|}{j+1} \sum_{n=1}^{\infty} |K_n| \left(\sum_{\substack{j_1,\dots,j_n\ge 0 \\ j_1+\cdots+j_n=j}} \frac{\gamma^{(j_1)}(0)\cdots\gamma^{(j_n)}(0)}{j_1!\cdots j_n!} \right) = \frac{\gamma^{(j+1)}(0)}{(j+1)!}.$$

In the last identity we used in a crucial way (A.3.7). This ends the proof. □

We are finally in a position to state and prove the announced convergence result for the homogeneous CBHD series in any Banach-Lie algebra.

Theorem A.3.10. *Let L be a possibly infinite-dimensional) Banach-Lie algebra on $\mathbb{R}$ or $\mathbb{C}$, equipped with a Lie-sub-multiplicative norm $\|\cdot\|$. We set*

$$G : [0, 2\pi) \to \mathbb{R}, \quad G(r) := \int_r^{2\pi} \frac{1}{2 + \frac{u}{2}\left(1 - \cot\left(\frac{u}{2}\right)\right)}\,\mathrm{d}u, \tag{A.3.11}$$

and we denote by ipo(G) *the ipograph of* G*, that is,*

$$\mathrm{epi}(G) = \{(r,s) \in \mathbb{R}^2 : r \in [0, 2\pi),\, 0 \le s < G(r)\}.$$

We then define the set

$$\Delta := \Big\{(a,b) \in L \times L : (\|a\|, \|b\|) \in \mathrm{epi}(G) \text{ or } (\|b\|, \|a\|) \in \mathrm{epi}(G)\Big\}. \qquad \text{(A.3.12)}$$

Then, for every $(a,b) \in \Delta$*, the homogeneous CBHD series related to* (a,b) *is convergent in* L*. More precisely, we have the following bound*

$$\sum_{i,j=0}^{\infty} \|Z_{i,j}(a,b)\| < 2\pi \quad \text{for any } (a,b) \in \Delta. \qquad \text{(A.3.13)}$$

Finally, one has the improved estimate

$$\sum_{i,j=0}^{\infty} \|Z_{i,j}(a,b)\| < C(a,b), \quad \text{for any } (a,b) \in \Delta, \qquad \text{(A.3.14)}$$

where $C(a,b) := \min\{M(a,b), M(b,a)\}$*, and* $M = M(a,b)$ *in* $[0, 2\pi]$ *is implicitly defined (in a unique way) by the following integral equation*

$$\int_{\|a\|}^{M} \frac{1}{2 + \frac{u}{2}\big(1 - \cot(\frac{u}{2})\big)}\, \mathrm{d}u = \|b\|, \quad \text{(with } (a,b) \in \Delta\text{)}. \qquad \text{(A.3.15)}$$

Proof. First of all we observe that the function G is well-posed (and finite-valued), since the map F introduced in (A.3.5) has a positive infimum on the interval $(-2\pi, 2\pi)$. We then fix any (a,b) belonging to the half-set

$$\Delta_1 := \Big\{(a,b) \in L \times L : \|a\| < 2\pi,\ \|b\| < G(\|a\|)\Big\}, \qquad \text{(A.3.16)}$$

and we pass to prove that, for every $(a,b) \in \Delta_1$, the series

$$\sum_{i,j=0}^{\infty} \|Z_{i,j}(a,b)\|$$

is convergent. The case of the set analogous to (A.3.16), with $\|a\|$ and $\|b\|$ interchanged, can be treated similarly, and is therefore omitted.

We can suppose that $b \neq 0$, since $\sum_{i,j} \|Z_{i,j}(a,0)\| = \|a\|$. We denote by $\gamma = \gamma_{\|a\|,\|b\|}$ the maximal solution of the (real) Cauchy problem

$$\begin{cases} \gamma' = \|b\|\, F(\gamma), \\ \gamma(0) = \|a\|, \end{cases} \qquad \text{(A.3.17)}$$

which is defined on its maximal domain, say $\mathcal{D} = (c,d) \subseteq \mathbb{R}$ (and $0 \in \mathcal{D}$). From the general theory of separable ODEs, we know that

$$c = \frac{1}{\|b\|} \int_{-2\pi}^{\|a\|} \frac{1}{F(u)}\, \mathrm{d}u, \qquad d = \frac{1}{\|b\|} \int_{\|a\|}^{2\pi} \frac{1}{F(u)}\, \mathrm{d}u = \frac{G(\|a\|)}{\|b\|}. \qquad \text{(A.3.18)}$$

Moreover, since F is real analytic, γ is real analytic too, and its Maclaurin series has a positive radius of convergence, say ρ. We claim that

$$\rho \geq d. \tag{A.3.19}$$

Indeed, by contradiction, let us suppose that $\rho < d$. We fix, throughout the sequel, the notation for the complex disk of center w and radius r

$$D(w,r) := \{z \in \mathbb{C} : |z-w| < r\}.$$

The complex power series

$$A(z) := \sum_{n=0}^{\infty} \frac{\gamma^{(n)}(0)}{n!}\, z^n$$

has radius of convergence ρ, and by Lem. A.3.8, we know that it has real non-negative coefficients. From the classical Vivanti-Pringsheim Theorem (see e.g., [130, Theorem 7.21]), it follows that the point $z = \rho$ must be a singular point for A, that is, ρ does not belong to the disc of convergence of any power series deduced from A. [1] Since γ is real analytic on its maximal domain $\mathcal{D} = (c,d)$, and since $0 < \rho < d$ (by our assumption), the complex power series

$$B(z) := \sum_{n=0}^{\infty} \frac{\gamma^{(n)}(\rho)}{n!}\, (z-\rho)^n,$$

has a positive radius of convergence. Therefore, there exists $\delta > 0$ so small that $D(\rho,\delta)$ is contained in the disc of convergence of B, and such that

$$0 < \rho - \delta < \rho + \delta < d.$$

In particular, we have the following crucial identity

$$A(t) = \gamma(t) = B(t) \quad \text{for every real } t \in (\rho - \delta, \rho). \tag{A.3.20}$$

Since $A(z)$ and $B(z)$ are both holomorphic on $O := D(0,\rho) \cap D(\rho,\delta)$, we infer that they coincide on the whole of O. Let now

$$t_0 = \rho - \frac{\delta}{3}.$$

Since B is holomorphic on $D(\rho,\delta)$, since the latter set contains t_0, and since the distance of t_0 from $\partial D(\rho,\delta)$ is $2\delta/3$, the power series

$$C(z) := \sum_{n=0}^{\infty} \frac{B^{(n)}(t_0)}{n!}\, (z-t_0)^n,$$

which is deduced from B, has radius of convergence $\geq 2\delta/3$. In particular, ρ belongs to the disc of convergence of C. From (A.3.20) we see that

$$B^{(n)}(t_0) = A^{(n)}(t_0), \quad \text{for every } n \geq 0,$$

[1] We say that a power series B is deduced from a power series A, if B is the Taylor series of the function $z \mapsto A(z)$ about some point belonging to the disc of convergence of A.

so that C is also deduced from the power series A. Since t_0 also belongs to the disc of convergence of A, we have obtained the power series

$$C(z) = \sum_{n=0}^{\infty} \frac{A^{(n)}(t_0)}{n!} (z - t_0)^n,$$

deduced from A, with disc of convergence containing ρ. This is in contradiction with the fact that ρ is a singular point for A. Therefore, (A.3.19) is proved. From this, recalling that ρ is the radius of convergence of $A(z)$, we get

$$\gamma(t) = \sum_{n=0}^{\infty} \frac{\gamma^{(n)}(0)}{n!} t^n, \quad \text{for every } t \in [0, d). \tag{A.3.21}$$

We are ready to conclude the demonstration: from the first family of inequalities in (A.3.9) (see Thm. A.3.9), it follows that

$$\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \|Z_{i,j}(a,b)\| \le \sum_{j=0}^{\infty} \frac{\gamma^{(j)}(0)}{j!}, \tag{A.3.22}$$

and since (a, b) belongs to the set Δ_1 in (A.3.16), from (A.3.18) we infer that

$$d = \frac{G(\|a\|)}{\|b\|} > 1,$$

so that $t = 1$ is an admissible value in (A.3.21). This shows that the power series on the right-hand side of (A.3.22) is indeed convergent, namely to $\gamma(1)$. Since $\gamma(t)$ solves the Cauchy problem (A.3.17) for every $t \in (c, d)$, we have $\gamma((c, d)) \subseteq (-2\pi, 2\pi)$ so that $0 < \gamma(1) < 2\pi$. We have the estimate

$$\sum_{n=1}^{\infty} \left\| \sum_{i+j=0}^{n} Z_{i,j}(a,b) \right\| \le \sum_{i,j=0}^{\infty} \|Z_{i,j}(a,b)\| \le \gamma(1).$$

By taking into account that $\gamma(1) < 2\pi$, one obtains the estimate (A.3.13), which also gives the (absolute) convergence of the homogeneous CBHD series related to $(a, b) \in \Delta$. The improved estimate (A.3.14) follows from

$$\sum_{i,j=0}^{\infty} \|Z_{i,j}(a,b)\| \le \min\left\{\gamma_{\|a\|,\|b\|}(1), \gamma_{\|b\|,\|a\|}(1)\right\},$$

where we recall that $\gamma(t) = \gamma_{\|a\|,\|b\|}(t)$ solves

$$\begin{cases} \gamma' = \|b\|\, F(\gamma), \\ \gamma(0) = \|a\|. \end{cases}$$

From the basic theory of separable ODEs, one recognizes that $\gamma(1)$ is implicitly defined by the following integral equation

$$\int_{\|a\|}^{\gamma(1)} \frac{1}{2 + \frac{u}{2}\left(1 - \cot(\frac{u}{2})\right)}\, \mathrm{d}u = \|b\|,$$

and thus $\gamma(1) = M(a, b)$. This ends the proof. □

A.4 Application: Some differential equations related to the CBHD series

The aim of this last section is to prove, as a consequence of the results in the previous sections, the following Thm. A.4.2, concerning some differential equations associated with the CBHD series in a generic (possibly infinite-dimensional) Banach-Lie algebra. First we need some notation.

Let L be a Banach-Lie algebra, equipped with a Lie-sub-multiplicative norm $\|\cdot\|$ as in (A.3.4) (see Rem. A.3.6). We fix the following notation:

$$\Psi(z) := \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} K_n\, z^n \quad (z \in \mathbb{C}: \ |z| < 2\pi),$$

where $K_n := B_n/n!$ and $\{B_n\}_{n\in\mathbb{N}}$ is the sequence of the Bernoulli numbers in (A.1.10). (In the literature, $\Psi(-z)$ is usually referred to as Todd's function.)

Lemma A.4.1. *Let $g \in L$ be such that $\|g\| < 2\pi$. Then the map*

$$\Psi(\operatorname{ad} g) : L \longrightarrow L, \qquad \Psi(\operatorname{ad} g)(g') := \sum_{n=0}^{\infty} K_n\, (\operatorname{ad} g)^n(g'). \tag{A.4.1}$$

defines a continuous endomorphism of the normed space L.

Proof. The claimed properties of $\Psi(\operatorname{ad} g)$ are consequences of the following estimate of the operator norm $\left|\!\left|\!\left| \Psi(\operatorname{ad} g) \right|\!\right|\!\right|$ of $\Psi(\operatorname{ad} g)$ (together with the fact that, by definition, $(L, \|\cdot\|)$ is a Banach space):

$$\begin{aligned}\left|\!\left|\!\left| \Psi(\operatorname{ad} g) \right|\!\right|\!\right| &= \sup_{\|g'\|\leq 1} \|\Psi(\operatorname{ad} g)(g')\| \leq \sup_{\|g'\|\leq 1} \sum_{n=0}^{\infty} |K_n|\, \|(\operatorname{ad} g)^n(g')\| \\ &\text{(see (A.3.4))} \\ &\leq \sup_{\|g'\|\leq 1} \sum_{n=0}^{\infty} |K_n|\, \|g\|^n\, \|g'\| = \sum_{n=0}^{\infty} |K_n|\, \|g\|^n = F(\|g\|) < \infty.\end{aligned}$$

Here F is the map as introduced in (A.3.5) (see also (A.3.8)), which is finite-valued on $(-2\pi, 2\pi)$. This ends the proof. □

We are ready to state the following result.

Theorem A.4.2. *Let L be Banach-Lie algebra, equipped with a Lie-sub-multiplicative norm $\|\cdot\|$, and let $a, b \in L$ be such that the couple (a, b) belongs to the set Δ introduced in* (A.3.12). *We consider the function*

$$Z : [-1, 1] \times [-1, -1] \to L, \quad Z(s, t) := \sum_{n=1}^{\infty} \Big(\sum_{i+j=n} Z_{i,j}(s\, a, t\, b) \Big). \tag{A.4.2}$$

Then, with the notation in (A.4.1), *for every $s, t \in [-1, 1]$ we have*

$$\begin{cases} \dfrac{\partial}{\partial s} Z(s,t) = \Psi(\operatorname{ad} Z(s,t))(a) \\ Z(0,t) = t\, b, \end{cases} \qquad \begin{cases} \dfrac{\partial}{\partial t} Z(s,t) = \Psi(-\operatorname{ad} Z(s,t))(b) \\ Z(s,0) = s\, a. \end{cases} \tag{A.4.3}$$

Finally, setting $Z(t) := Z(t,t)$ *for every* $t \in [-1,1]$, *one has*

$$\begin{cases} Z'(t) = \Psi(\mathrm{ad}\, Z(t))(a) + \Psi(-\mathrm{ad}\, Z(t))(b), \\ Z(0) = 0, \end{cases} \qquad \textit{on } [-1,1]. \tag{A.4.4}$$

Proof. Let $(a,b) \in \Delta$ be fixed. First, we observe that the series of functions in (A.4.2) is normally convergent for every $(s,t) \in [-1,1] \times [-1,1]$: in fact, since

$$Z_{i,j}(s\,a, t\,b) = s^i t^j\, Z_{i,j}(a,b)$$

(as it trivially follows from identity (A.3.2)), one has

$$\begin{aligned} \sum_{i,j=0}^{\infty} \sup_{(s,t)\in[-1,1]^2} \|Z_{i,j}(s\,a, t\,b)\| &= \sum_{i,j=0}^{\infty} \sup_{(s,t)\in[-1,1]^2} |s^i t^j|\, \|Z_{i,j}(a,b)\| \\ &= \sum_{i,j=0}^{\infty} \|Z_{i,j}(a,b)\| < 2\pi. \end{aligned}$$

Here we used the bound (A.3.13) in Thm. A.3.10. Therefore, Def. (A.4.2) is well-given, the series being uniformly (hence pointwise and absolutely) convergent on $[-1,1] \times [-1,1]$. Furthermore, this also shows that $Z(s,t)$ belongs to the open disc of L with radius 2π and center 0; therefore $\Psi(\pm\mathrm{ad}\, Z(s,t))$ are well posed for every $s,t \in [-1,1]$ (recall that we assumed that $(a,b) \in \Delta$).

For any given $s,t \in [-1,1]$, the absolute convergence of the double series

$$\sum_{i,j=0}^{\infty} Z_{i,j}(a,b)\, s^i t^j$$

allows us to commute and associate summands as we please. As a consequence, fixed $t \in [-1,1]$, we can reorder the series defining $Z(s,t)$ as a power series in the real variable s (valued in the Banach space L): the radius of convergence of the resulting power series is greater than or equal to 1 (as it follows from the above argument), and differentiation term-by-term is therefore allowed. Hence, the first system in the above (A.4.3) follows from the very definition of $\Psi(\mathrm{ad}\, Z(s,t))$ (see (A.4.1)), from the continuity of the adjoint map ad on $L \times L$, and – in a crucial way – from the family of algebraic identities in (A.3.3) (see Cor. A.3.4). The second system in (A.4.3) can be proved analogously.

Finally, (A.4.4) directly follows from (A.4.3) and the Chain Rule. □

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