Alma Mater Studiorum · Università di Bologna

DOTTORATO DI RICERCA IN MATEMATICA
CICLO XXVII

Settore Concorsuale di afferenza: 01/A2

Settore Scientifico disciplinare: MAT/03

HIGGS BUNDLES AND LOCAL SYSTEMS ON ELLIPTIC CURVES

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Esame finale 2016

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Introduction

Given a compact Riemann surface Σ, Hodge theory provides abelian topological invariants of the surface with extra structure, bridging the topological, smooth and holomorphic worlds. In particular these are embodied by the Betti, de Rham and Dolbeault cohomology groups. Furthermore

$$H^1_B(\Sigma, \mathbb{C}) = Hom(H_1(\Sigma, \mathbb{Z}), \mathbb{C}) = Hom(\pi_1(\Sigma), \mathbb{C})$$

so the cohomology groups are essentially related to representations of the fundamental group of Σ in the additive group of the complex numbers. Therefore, it is natural to ask which smooth and holomorphic objects realize representations of $\pi_1(\Sigma)$ in nonabelian groups, providing nonabelian topological invariants.
A partial answer is given by the Riemann-Hilbert correspondence: if we denote M_B the moduli space of representations of $\pi_1(\Sigma)$ in $GL_r(\mathbb{C})$, up to conjugacy, this correspondence gives an analytic isomorphism between M_B and M_{dR}, the moduli space of C^∞ vector bundles on Σ equipped with a flat connection.

The first result in the other direction is due to Narasimhan and Seshadri [23] and asserts that semistable vector bundles of degree zero on a complex nonsingular projective curve are precisely those associated to unitary representations. In [17] Hitchin introduces the notion of Higgs field on a holomorphic vector bundle, which encodes the non-unitary part of a $GL_r(\mathbb{C})$-representation, and in [28] Simpson proves the correspondence between $GL_r(\mathbb{C})$-representations of $\pi_1(\Sigma)$ and holomorphic vector bundles on Σ equipped with a Higgs field: the non-Abelian Hodge theorem states that, just as in the Narasimhan-Seshadri correspondence, M_B is naturally diffeomorphic to the moduli space of semistable Higgs bundles M_{Dol} parametrizing pairs (E, ϕ) consisting of a vector bundle E on Σ together with a Higgs field $\phi \in H^0(\Sigma, End(E) \otimes K_\Sigma)$, subject to a natural condition of stability.

The variety M_{Dol} has a rich geometry and in particular it is equipped with a projective map

$$h : M_{Dol} \to \mathbb{A}$$

the *Hitchin fibration*, where the target $\mathbb{A}$ is an affine space and the fibre of h over a general point $a \in \mathbb{A}$ is isomorphic to the Jacobian of a branched covering of Σ, the *spectral curve*.

While the algebraic varieties M_B and M_{Dol} are diffeomorphic, they are far from being biholomorphic: the former is affine and the latter is foliated by the fibers of the

Hitchin map, which are compact algebraic subvarieties. Moreover, the variety M_B does not depend on the complex structure of Σ, while M_{Dol} does.

In [5] it was proved that for a compact Riemann surface Σ of genus $g \geq 2$ the non-Abelian Hodge theory diffeomorphism between the twisted character variety M_B of representations into $GL_2(\mathbb{C})$ and the moduli space M_{Dol} of rank 2 degree 1 stable Higgs bundles on Σ identifies the weight filtration $\mathcal{W}$ on the cohomology groups of M_B with the perverse filtration $\mathcal{P}$ induced by the projective Hitchin map on the cohomology groups of M_{Dol}. This leads to the so-called $P = W$ conjecture, stating basically that this exchange of filtration should be true for other groups, such as $GL_n(\mathbb{C})$ for $n > 2$.

Though the results have been proved in genus $g \geq 2$, a similar exchange phenomenon can be observed in genus one. If C is an elliptic curve the moduli space of rank one and degree zero Higgs bundles on C is naturally isomorphic to $X := T^*C$, the total space of the cotangent bundle of C, and the corresponding character variety is the complex surface $Y := \mathbb{C}^* \times \mathbb{C}^*$. Thanks to [13], the punctual Hilbert scheme $X^{[n]}$ of X can be identified with the moduli space of stable *marked* Higgs bundles on C, i.e. triples (E, ϕ, v) where (E, ϕ) is a semistable Higgs bundle of rank n and degree 0 on C and $v \in E_o$ is a vector on the fiber of E over the fixed origin of C. The Hitchin fibration is the proper flat map

$$h_n : X^{[n]} \to \mathbb{C}^{(n)} \cong \mathbb{C}^n.$$

The main result of [4] establishes that there is a natural isomorphism of graded vector spaces

$$\varphi^{[n]} : H^*(X^{[n]}, \mathbb{Q}) \xrightarrow{\sim} H^*(Y^{[n]}, \mathbb{Q})$$

that exchanges the perverse Leray filtration on $X^{[n]}$ for the map h_n with the halved weight filtration on $Y^{[n]}$

$$\varphi^{[n]}(\mathcal{P}_{X^{[n]}}) = \mathcal{W}_{Y^{[n]}}.$$

The existence of the isomorphism $\varphi^{[n]}$ is explained by the following result.

Theorem. *Let C be a smooth projective curve of genus one.*

1. *The punctual Hilbert schemes $(T^*C)^{[n]}$ and $(\mathbb{C}^* \times \mathbb{C}^*)^{[n]}$ are diffeomorphic;*
2. *the isomorphism $\varphi^{[n]}$ is induced by a diffeomorphism*

$$\varphi : (T^*C)^{[n]} \xrightarrow{\simeq} (\mathbb{C}^* \times \mathbb{C}^*)^{[n]}.$$

It is important to note that this result does not follow from the version of the nonabelian Hodge theorem for parabolic Higgs bundles on punctured curves proved in [27], as $(\mathbb{C}^* \times \mathbb{C}^*)^{[n]}$ is not a character variety for the associated filtered local systems.

In this thesis we prove the theorem above and we give a complete description of the correspondence beetween the Hilbert scheme and the moduli space of marked Higgs bundles, giving an explicit description of Higgs bundles corresponding to subschemes of length $n \leq 3$. We also discuss a conjecture by Simpson on the compactification of

M_{Dol} and M_B and on the dual boundary complex of the character variety, proving a result (1.6.1) going in the direction of Simpson's conjecture.

The thesis is organized as follows: in the first chapter we first recall standards results about vector bundles on elliptic curves and focus on the the moduli spaces in rank one and their compactifications. We also recall basic facts about Hilbert scheme of points of smooth complex surfaces and explain the relation with the moduli space of Higgs bundles, giving a proof of the result above. We conclude the chapter with a conjecture of Simpson on the simple normal crossing compactification of the character variety.
The second chapter focus on Higgs bundles of higher rank on elliptic curves and their extensions: we give different descriptions, also using the so called factors of automorphy.
In the third chapter we give the details of the correspondence between the moduli space of marked Higgs bundles on C and the Hilbert scheme of points of T^*C, estabilished by a relative Fourier-Mukai functor, describing explicitly the Higgs bundles corresponding to subschemes of length $n \leq 3$.
In the last chapter we briefly introduce holomorphic connections and how they are related to Higgs bundles. Rank one flat connections on the elliptic curve C are parameterized by a surface $C^\natural$, biholomorphic to $\mathbb{C}^* \times \mathbb{C}^*$, whose Hilbert scheme $(C^\natural)^{[n]}$ parameterizes flat connections of rank n. In turn, as the zero-dimensional Hilbert schemes of biholomorphic surfaces are biholomorphic, we have that $(C^\natural)^{[n]}$ is biholomorphic, although not algebraically equivalent, to $(\mathbb{C}^* \times \mathbb{C}^*)^{[n]}$. The construction of a natural correspondence between Higgs bundles and bundles with flat connection, thus realizing a diffeomorphism between $(T^*C)^{[n]}$ and $(\mathbb{C}^* \times \mathbb{C}^*)^{[n]}$ will be pursued in future work.

Chapter 1

Nonabelian Hodge correspondence on elliptic curves

1.1 Vector bundles over elliptic curves

Let C be a smooth projective curve over $\mathbb{C}$ of genus one and let $o \in C$ be a distinguished point on it. We call the pair (C, o) an *elliptic curve*.
The Abel-Jacobi map

$$\begin{aligned} aj_n : \qquad C^{(n)} &\longrightarrow Pic^n(C) \\ D = \sum p_i &\longrightarrow \mathcal{O}(D) \end{aligned}$$

is surjective for any $n > 0$ and an isomorphism for $n = 1$. Furthermore, the distinguished point o gives an isomorphism

$$\begin{aligned} \psi_n : Pic^0(C) &\longrightarrow Pic^n(C) \\ L \quad &\longrightarrow L \otimes \mathcal{O}(o)^{\otimes n} \end{aligned}$$

for any $n > 0$ and in particular we can identify the curve and the dual variety

$$\begin{aligned} Aj = \psi_1^{-1} \circ aj_1 : C &\longrightarrow Pic^0(C) \\ p &\longrightarrow \mathcal{O}(p - o) \end{aligned}$$

We denote $(\hat{C}, o)$ the dual elliptic curve, where the distingueshed point corresponds to the trivial line bundle.

Vector bundles of degree zero and fixed rank over an elliptic curve have been classified in [1] by Atiyah. We briefly recall the principal results: we denote $I(n, 0)$ the set of indecomposable vector bundles on C of degree zero and rank n.

Theorem 1.1.1 ([1])**.** *There exists a vector bundle $F_n \in I(n, 0)$, unique up to isomorphism, such that $H^0(C, F_n) \neq 0$. Moreover $h^0(F_n) = 1$ and we have an exact sequence*

$$0 \to \mathcal{O} \to F_n \to F_{n-1} \to 0$$

where $F_1 = \mathcal{O}$ *and* $F_{n-1} \in I(n-1,0)$. *For any* $E \in I(n,0)$ *exists a unique* $L \in Pic^0(C)$ *such that* $E \cong L \otimes F_n$. *We have that* $det(E) \cong L^n$.

Proposition 1.1.2 ([1]). *The vector bundles* F_n *are selfdual for any* $n \in \mathbb{N}$ *and for any* $n \geq r \geq 1$

$$F_n \otimes F_r \cong F_{n-r+1} \oplus F_{n-r+3} \oplus \ldots \oplus F_{n-r+(2r-1)}$$

1.2 The correspondence in rank one

Let Σ be a compact Riemann surface of genus g over $\mathbb{C}$. We denote by K its canonical bundle. We introduce now the main object of this work: the definitions are given for vector bundles of arbitary rank on Σ but we focus our attention on the case of line bundles on an elliptic curve C.

Definition 1.2.1. A *Higgs bundle* of rank n and degree d over Σ is a pair (E, ϕ), where E is a holomorphic vector bundle on Σ and ϕ is a map

$$\phi : \ E \to E \otimes K$$

called the *Higgs field*.

Note that when $\Sigma = C$ is an elliptic curve, the canonical bundle is trivial and after choosing a trivialization dz of K_C, unique up to a multiplicative constant, the Higgs field turns out to be an endomorphism of the vector bundle, i.e. $\phi \in H^0(C, End\ E)$.

Definition 1.2.2. Two Higgs bundles (E, ϕ) and (F, θ) are *isomorphic* if there exists an isomorphism of holomorphic vector bundles $f : E \to F$ such that $\theta = (f \otimes id) \circ \phi \circ f^{-1}$, i.e. the diagram

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E \otimes K \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f\otimes id} \\ F & \xrightarrow{\theta} & F \otimes K \end{array}$$

commutes.

It is easy to see that the moduli space of Higgs bundles on Σ of degree zero and rank one, denoted $M_{Dol}(1,0)$, is always isomorphic to $T^*Pic^0(\Sigma)$: if L is a line bundle on Σ there is a canonical isomorphism

$$H^0(\Sigma, End\ L \otimes K) \cong H^0(\Sigma, K) = H^1(\Sigma, \mathcal{O})^* = T_o^* Pic^0(\Sigma)$$

so a Higgs field on L is just a one-form.
In the case of an elliptic curve the isomorphism

$$\hat{C} \times \mathbb{A}^1 \xrightarrow{\sim} M_{Dol}(1,0)$$

associates to a point $(L, \alpha) \in \hat{C} \times \mathbb{A}^1$ the Higgs bundle $(L, \alpha\ dz)$.

The following definition was suggested by Deligne and plays an important role in Simpson's theory.

Definition 1.2.3. Let E be an holomorphic vector bundle on Σ and $\lambda \in \mathbb{C}$. A λ-connection is a $\mathbb{C}$-linear operator $\nabla : E \to E \otimes K$ such that

$$\nabla(fs) = \lambda s \otimes (df) + f\nabla s$$

for any $f \in \mathcal{O}$ and $s \in E$.
For $\lambda = 1$ we simply speak about *holomorphic connection*.

We remark that a Higgs field ϕ on E is just a 0-connection, i.e. is $\mathcal{O}$-linear.
Moreover if $\mu \in \mathbb{C}$ and ∇ is a λ-connection then $\mu\nabla$ is a $\mu\lambda$-connection. In fact, this action sets up an equivalence between the category of 1-*connections* and that of λ-*connections* for every $\lambda \in \mathbb{C}^*$.

Nonabelian Hodge theory allows to relate the space M_{Dol} to the moduli space M_{dR} of holomorphic connections on the curve Σ.

Remark 1.2.4. If ∇ and ∇' are two λ-connections then

$$(\nabla - \nabla')(fs) = f(\nabla - \nabla')(s)$$

so their difference is a $\mathcal{O}$-linear operator. This means that given a vector bundle E on Σ, the space of λ-connections on it is an affine space for $H^0(\Sigma, End\ E \otimes K)$.

In particular in the rank one case we can give an easy description of the moduli space of rank one holomorphic connections on Σ, denoted $M_{dR}(1)$. First notice that $M_{dR}(1)$ has a natural group structure given by tensor product. It is an affine torsor $\Sigma^\natural$ on $Pic^0(\Sigma)$ with affine fiber $H^0(K)$. It defines an exact sequence of algebraic groups

$$0 \to \mathbb{C}^g \to \Sigma^\natural \xrightarrow{p} Pic^0(\Sigma) \to 0$$

where the projection p maps a couple (L, ∇) to the line bundle L.

By the Riemann-Hilbert correspondence ([6]) we know that M_{dR} is analytically isomorphic to the character variety M_B, i.e. the moduli space of representations of the fundamental group $\pi_1(\Sigma)$ of fixed rank modulo conjugation.
In particular in rank one, since $\mathbb{C}^*$ is abelian, the action is trivial and

$$M_B \cong (\mathbb{C}^*)^{2g}$$

One can consider (see [31]) the scheme $M_H \to \mathbb{A}^1$ whose fiber over a point $\lambda \in \mathbb{A}^1$ is the moduli space of rank one λ-connections on Σ. It is such that the fiber over $0 \in \mathbb{A}^1$

is $(M_H)_0 \cong M_{Dol}$ and $(M_H)_t \cong M_{dR}$ for every $t \in \mathbb{C}^*$. This construction can be done for arbitrary rank, but we will focus on the rank one case.

In the case of an elliptic curve C it is possible to find a compactification $\overline{M}_H$ of M_H such that $(\overline{M}_H)_0 \cong \overline{M}_{Dol}$ and $(\overline{M}_H)_\lambda \cong \overline{M}_{dR}$ for every $\lambda \neq 0$ and such that the divisors at infinity are the same.
Identify $\mathbb{A}^1$ with the space $Ext^1_C(\mathcal{O}, \mathcal{O})$ parametrizing extensions

$$0 \to \mathcal{O} \to E_t \to \mathcal{O} \to 0$$

We denote $M \to \mathbb{A}^1$ the universal family, i.e. $M_t = Tot(E_t)$ for every $t \in \mathbb{A}^1$; in particular $M_0 \cong \mathcal{O}^{\oplus 2}$ and $M_t \cong F_2$ for every $t \in \mathbb{C}^*$. M is a rank two vector bundle on $\mathbb{A}^1 \times C$ and so its total space is smooth.

Thus we can take as $\overline{M}_H$ the projective bundle $\mathbb{P}(M)$ over $\mathbb{A}^1 \times C$: the family $\overline{M}_H \to \mathbb{A}^1$ is thus such that over the zero

$$(\overline{M}_H)_0 = \mathbb{P}(\mathcal{O}^{\oplus 2}) = \mathbb{P}^1 \times C$$

that is a compactification $\overline{X}$ of the space $X = T^*C$ and the divisor at the infinity is a copy of C
Over any other closed point $t \in \mathbb{A}^1$ with $t \neq 0$ we have that $E_t \cong F_2$, thus

$$(\overline{M}_H)_t = \mathbb{P}(F_2)$$

Proposition 1.2.5. *The projective bundle $\mathbb{P}(F_2)$ is a compactification of $Y = \mathbb{C}^* \times \mathbb{C}^*$.*

Proof. First recall that Y is biholomorphic to $C^\natural$. When we compactify the affine bundle $C^\natural$ we get a $\mathbb{P}^1$-bundle $\mathbb{P}(V)$, where V is a rank two vector bundle. V must be indecomposable: if $V \cong L_1 \oplus L_2$, we can assume $L_1 \cong \mathcal{O}$ and so $\mathbb{P}(V)$ is the compactification of a line bundle L_2 but then it has two disjoint sections while $C^\natural$ has no compact subvarieties, since it is biholomorphic to an affine variety. $\square$

Here $\mathbb{P}(F_2)$ is a non trivial $\mathbb{P}^1$-bundle over C: the divisor at infinity C' is a section of this bundle, therefore a copy of C, but the bundle does not admit two disjoint sections. In fact the section is holomorphically rigid and has selfintersection 0 and the complement is Stein and biholomorphic to $\mathbb{C}^* \times \mathbb{C}^*$ but is not affine.

Proposition 1.2.6. *The normal sheaf of C' in $\mathbb{P}(F_2)$ is trivial, i.e. $N_{C'/\mathbb{P}(F_2)} = \mathcal{O}_C$.*

Proof. If we have an exact sequence

$$0 \to E \to F \to G \to 0$$

of vector bundles we always have the closed immersion $\mathbb{P}(E) \subset \mathbb{P}(F)$ and if E is a line bundle

$$N_{\mathbb{P}(E)/\mathbb{P}(F)} \cong G \otimes E^*$$

(cfr [26]). Applying this result to the exact sequence

$$0 \to \mathcal{O}_C \to F_2 \to \mathcal{O}_C \to 0$$

we get the statement of the proposition. $\square$

Proposition 1.2.7. *C' is rigid in $\mathbb{P}(F_2)$, i.e. C' is isolated in $Hilb(\mathbb{P}(F_2))$.*

Proof. If C' and C'' belong to the same component of $Hilb(\mathbb{P}(F_2))$, since $N_{C'/\mathbb{P}(F_2)} = \mathcal{O}_C$ we have that $C' = C''$ or $C' \cap C'' = \emptyset$. But since F_2 does not split we must have $C' = C''$. $\square$

In particular $\overline{X}$ and $\overline{Y}$ are compact deformation equivalent complex surfaces and so there is a diffeomophism $\phi : \overline{X} \to \overline{Y}$ (in fact real analytic).

So we have $\mathcal{X} := \mathbb{P}(M) \to \mathbb{A}^1$ a smooth family of projective surfaces and $\mathcal{C} \subset \mathcal{X} \to \mathbb{A}^1$ a family of projective curves. Denoting $\mathcal{Y} = \mathcal{X} \setminus \mathcal{C}$, we have $\mathcal{Y} \cong M_{Hod}$.

Proposition 1.2.8. *$\mathcal{C} \to \mathbb{A}^1$ is smooth.*

Then we can choose local analytic coordinates (z_1, z_2, t) around a point $p \in \mathcal{C}$ such that locally

$$\begin{aligned} \mathcal{X} \quad &\to \mathbb{A}^1 \\ (z_1, z_2, t) &\to t \end{aligned}$$

and $\mathcal{C}$ is locally given by $z_1 = 0$.

1.3 The Hilbert scheme of points on a smooth surface

In this chapter we will collect main definitions and basic facts about the Hilbert scheme of points on a surface, see [22] for an extensive treatment. Let Sch/S be the category of locally Noetherian schemes over a Noetherian scheme S and let X be a projective scheme over S, with a fixed projective embedding. For any fixed polynomial P the functor

$$\mathcal{H}ilb^P_{X/S} : Sch/S \to Sets$$

sends a locally Noetherian scheme S' over S to the set

$$\{Z \subset X \times_S S' \mid Z \textit{ closed}, \ Z \to S' \textit{ flat and } \chi(\mathcal{O}_{Z_s}(m)) = P(m) \textit{ for all } s \in S', \ m \geq 0\}$$

Namely, it is a functor which associates to a scheme S' a set of families of closed subschemes in X parameterized by S'. The crucial fact proved by Grothendieck is the following theorem.

Theorem 1.3.1. *The functor $\mathcal{H}ilb_X^P$ is representable by a projective scheme $Hilb_X^P$.*

Furthermore, if we have an open subscheme $Y \subset X$, we have the corresponding open subscheme $Hilb_Y^P \subset Hilb_X^P$ parameterizing subschemes in Y . In particular, $Hilb_Y^P$ is defined for a quasi-projective scheme Y .

Now let n be a positive integer. We will focus on the Hilbert scheme $Hilb_X^n$, denoted also $X^{[n]}$, which parametrizes the set of n-tuples of points in X. As the degree of the Hilbert polynomial of a variety Z is equal to the dimension of the variety, the closed points of $Hilb_X^n$ are 0-dimensional subvarieties Z with constant Hilbert polynomial $P \equiv n$, i.e. such that

$$dim\ H^0(Z, \mathcal{O}_Z) = n$$

For example, if we let Z be the union of n distinct closed points $p_1, \dots, p_n$ of X, the sheaf $\mathcal{O}_Z$ is the direct sum of the skyscraper sheaves over each point and thus satisfies this condition.

In general the Hilbert scheme is strictly related to the symmetric product: it always exists a map, called the *Hilbert Chow morphism*

$$\begin{aligned} H\ :\ & X^{[n]}_{red} \to X^{(n)} \\ & Z \to \sum_{x \in X} length(Z_x)[x] \end{aligned}$$

that associates to each subscheme its corresponding cycle.

Theorem 1.3.2 ([8],[22])**.** *Let S be a nonsingular connected surface over a field k. Then $Hilb_S^n$ is a nonsingular connected scheme of dimension 2n. If S is compact then $Hilb_S^n$ is also compact.*

Furthermore

Theorem 1.3.3 ([8])**.** *Let $X \to \Sigma$ be a smooth scheme of relative dimension 2 over a smooth curve Σ . Then $Hilb^n_{X/\Sigma}$ is a smooth variety of relative dimension 2n over Σ. If $X \to \Sigma$ is proper then $Hilb^n_{X/\Sigma} \to \Sigma$ is also proper.*

Now we can consider the relative Hilbert scheme of the family $\mathcal{X} \to \mathbb{A}^1$ constructed in the previous section: let $\mathcal{X}^{[n]} \to \mathbb{A}^1$ be the family such that $(\mathcal{X}^{[n]})_t = \mathcal{X}_t^{[n]}$.

Proposition 1.3.4. *$\mathcal{X}^{[n]}$ and $\mathcal{Y}^{[n]}$ are smooth and $\mathcal{X}^{[n]} \to \mathbb{A}^1$ is proper.*

Proof. Since $\mathcal{X}$ and $\mathcal{Y}$ are smooth of relative dimension 2 the results follow from the original results of Fogarty. □

Consider the pair (X, C) where X is a smooth surface and C is a smooth curve in X. To (X, C) one can associate a stratification

$$C^{(n)} = I_n \subset I_{n-1} \subset \dots I_1 \subset X^{[n]}$$

called the *incidence stratification*, where the closed stratum I_j denotes the locus of subschemes intersecting the curve C in length at least j and has codimension j in $X^{[n]}$. Note in particular that $I_1 \subset X^{[n]}$ is the divisor of subschemes intersecting C and so

$$X^{[n]} \setminus I_1 = (X \setminus C)^{[n]}$$

In general this stratification has complicated singularities except for the bottom stratum I_n that is smooth, but one can try to resolve the singularities of the incidence stratification by *stratified blow up*, i.e. blowing up I_n then blowing up the proper transform of I_{n-1} and so on. The stratified blow up indeed resolves the singularities of the incidence stratification.

Theorem 1.3.5 ([25]). *In the stratified blow up of the incidence stratification, the proper transform $\hat{I}_k$ of each closed stratum I_k is smooth and the total transform equals $\hat{I}_k + \hat{I}_{k+1} + \ldots + \hat{I}_n$ and has normal crossings.*

1.4 Hilbert schemes and Higgs bundles

Theorem 1.4.1 ([13]). *There is an isomorphism*

$$T^*C^{[n]} \cong \mathcal{M}$$

where $\mathcal{M}$ is the moduli space of rank n and degree 0 parabolic Higgs bundles on C. Its closed points parametrize triples (E, ϕ, v) where (E, ϕ) is a semistable Higgs bundle on C of rank n and degree 0 and $v \in E_o$ is such that there are not proper ϕ-invariant subsheaves $F \subset E$ with $\mu(F) \geq \mu(E)$ such that $v \in F_o$.

The punctual Hilbert scheme $(T^*C)^{[n]}$ is a 2n-dimensional nonsingular variety admitting a proper map

$$h_n : (T^*C)^{[n]} \to \mathbb{C}^{(n)} \cong \mathbb{C}^n$$

of relative dimension n, obtained composing the Hilbert Chow morphism

$$H : (T^*C)^{[n]} \to (T^*C)^{(n)}$$

with the symmetric power of the projection on the second factor

$$p : (T^*C)^{(n)} \to \mathbb{C}^{(n)}.$$

The map h_n is flat since the two spaces are nonsingular and the fibers have constant dimension. We will call this map the *Hitchin map* for $(T^*C)^{[n]}$ as it is a close analogue of the classical Hitchin map of [17].
Now let $Y = \mathbb{C}^* \times \mathbb{C}^*$ the corresponding character variety and consider its punctual Hilbert scheme $Y^{[n]}$.

Theorem 1.4.2 ([4]). *There is an isomorphism of graded vector spaces*

$$\varphi^{[n]} : H^*((T^*C)^{[n]}, \mathbb{Q}) \xrightarrow{\sim} H^*(Y^{[n]}, \mathbb{Q})$$

such that the perverse filtration $\mathcal{P}$ on $(T^*C)^{[n]}$ given by the Hitchin map corresponds to the weight filtration $\mathcal{W}$ on $Y^{[n]}$.

This exchange of filtration is conjectured to be typical of the Nonabelian Hodge theory and leads to give a modular interpretation to this cohomological isomorphism: it has been proved in [5] that for a compact Riemann surface of genus $g \geq 2$ the diffeomophism $M_{Dol} \cong M_B$ stemming from the Nonabelian Hodge theory induce an isomorphism on the rational cohomology groups

$$H^*(M_{Dol}, \mathbb{Q}) \xrightarrow{\sim} H^*(M_B, \mathbb{Q})$$

with the property that the perverse filtration on M_{Dol} is exchanged with the weight filtration on M_B.
This phenomenon can be explained by the following result.

Theorem 1.4.3. *Let C be a smooth projective curve of genus one.*

1. *The punctual Hilbert schemes $(T^*C)^{[n]}$ and $(\mathbb{C}^* \times \mathbb{C}^*)^{[n]}$ are diffeomorphic;*
2. *the isomorphism $\varphi^{[n]}$ is induced by a diffeomophism*

$$\varphi : (T^*C)^{[n]} \xrightarrow{\simeq} (\mathbb{C}^* \times \mathbb{C}^*)^{[n]}$$

Proof. First we remark that this does not follow from the fact that the family $\mathcal{Y}^{[n]} \to \mathbb{A}^1$ is smooth as this map is not proper so we can not apply Ehresmann's lemma.
We consider the relative Hilbert scheme $\pi : \mathcal{X}^{[n]} \to \mathbb{A}^1$ of the family $\mathcal{X}$. Since we can choose local coordinates (z_1, z_2, t) on $\mathcal{X}$ around a point $p \in \mathcal{C}$ such that t is a local coordinate on $\mathbb{A}^1$ and $\mathcal{C}$ is locally given by $z_1 = 0$, we can use 1.3.5 over $\mathbb{A}^1$ and we get a simple normal crossing compactification $\overline{\pi} : \overline{\mathcal{Y}^{[n]}} \to \mathbb{A}^1$ of $\mathcal{Y}^{[n]}$. Now we can apply a stratified version of Ehresmann's lemma.

Lemma 1.4.4 ([32]). *Let A be closed on X and let Y be a real smooth analytic space, $f : X \to Y$ a proper morphism on A and transvers to S, $y_0 \in Y$, X_0 and (A_0, S) the fibers over y_0. There exists an open neighborhood V of y_0 in Y and a stratified homeomorphism of $(A \cap f^{-1}(V), S \cap f^{-1}(V))$ on $(A_0 \times V, S_0 \times V)$ that preserves the stratifications and is compatible with projections on V.*

Denoting $\mathcal{D} = \overline{\mathcal{Y}^{[n]}} \smallsetminus \mathcal{Y}^{[n]}$ the simple normal crossing divisor it is possible to construct a stratified trivialization of $\overline{\pi}$, giving thus on the open smooth stratum a trivialization of $\mathcal{Y}^{[n]}$. Moreover denoting $\tilde{\pi} : \mathcal{D} \to \mathbb{A}^1$ the map induced by restriction

$$R\pi_! \mathbb{Q}_{\mathcal{Y}^{[n]}} = \; Cone(R\overline{\pi}_* \mathbb{Q}_{\overline{\mathcal{Y}^{[n]}}} \xrightarrow{f} R\tilde{\pi}_* \mathbb{Q}_{\mathcal{D}})$$

and since $\overline{\pi}$ and $\hat{\pi}$ are proper it follows that f is a map of local systems and thus $R\pi_! \mathbb{Q}_{\mathcal{Y}^{[n]}}$ is a local system. By duality also $R\pi_* \mathbb{Q}_{\mathcal{Y}^{[n]}}$ is a local system and thus the isomorphism in cohomology is the one induced by the stratified trivialization. ∎

1.5 Simpson's conjecture on the structure at infinity

Given a smooth projective curve Σ, Non-abelian Hodge Theory gives a homeomorphism beetween the moduli space of Higgs bundles on Σ of fixed rank n and degree zero, denoted M_{Dol}, and the moduli space of flat bundles on Σ of the same rank, denoted M_{dR}. The last one is biholomorphic, via the Riemann-Hilbert correspondence, to the character variety M_B, i.e. the moduli space of conjugacy classes of n-dimensional representations of the fundamental group $\pi_1(\Sigma)$.
It turns out that M_{Dol} and M_B are diffeomorphic, but they are far from being complex analytically equivalent: in fact M_B is affine, while M_{Dol} is equipped with the Hitchin fibration $h : M_{Dol} \to \mathbb{A}$, that is proper and whose general fibers are lagrangian abelian varieties.

However we can try to compare their structure "at infinity". Following [20] and [15], we can find a compactification $\overline{M}_{Dol}$ of M_{Dol} such that the Hitchin fibration extends to a map

$$\overline{h} : \overline{M}_{Dol} \to \hat{\mathbb{P}}^N$$

where $\hat{\mathbb{P}}^N$ is a weighted projective space. Let's call D' the divisor at infinity, N'_{Dol} a neighborhood of D' in $\overline{M}_{Dol}$ and $N_{Dol} = N'_{Dol} \smallsetminus D'$, i.e. a neighborhood of the infinity in M_{Dol}.

Definition 1.5.1. A divisor $D = \sum_I D_i$ of an algebraic variety X is a *normal crossing divisor (NCD)* if for all $p \in X$ a local equation for D is of the form $x_1 \cdot \ldots \cdot x_r$ with $x_i \in \mathcal{O}_{X,p}$ for some choice of local parameters.
The divisor D is a *simple normal crossing divisor (SNCD)* if is a NCD and all the D_i are smooth.

On the other hand we can take a simple normal crossing compactification $\overline{M}_B$ of the affine variety M_B. This means that we ask the divisor at infinity to be a simple normal crossing divisor $D = \sum_{i \in I} D_i$.

Definition 1.5.2. If $D = \sum_I D_i$ is a simple normal crossing divisor of an algebraic variety X the *dual complex* of D, denoted $\Delta(D)$, is a triangulated topological space such that the k-simplices correspond to the irreducible components of $D_J = \bigcap_{j \in J} D_j$ for $J \subset I$ with $|J| = k + 1$ and the inclusion of faces corresponds to inclusion of subvarieties.

The space $\Delta(D)$ is not always a simplicial complex but in general it is a regular CW complex. We can sequentially blow up the irreducible compontents of each D_J from smallest to largest to obtain a projective birational map $\phi : X' \to X$ that is an isomorphism on $X \smallsetminus D$ and such that $\hat{D} = \phi^{-1}(D)$ is a SNCD and all the $\hat{D}_J$ are empty or irreducible so that $\Delta(\hat{D})$ is a simplicial complex (corresponding to a baricentric subdivision of $\Delta(D)$).

Via the homeomorphism beetween M_{Dol} and M_B, the open set N_{Dol} is sent to an open set N_B such that $N'_B = N_B \cup D$ is an open neighborhood of the divisor at infinity.

Now the Hitchin map $\overline{h}$ induces by restriction a map : $N'_{Dol} \to U'$ where U' is a neighborhood of the divisor at infinity of $\hat{\mathbb{P}}^N$, so we get a continuous map defined up to isotopy

$$h : N_{Dol} \to \mathbb{S}^{2N-1}$$

On the other hand we can construct a map $N'_B \to \Delta(D)$ in the following way. We choose an open cover $\{U_i\}_{i \in I}$ such that each U_i is an open neighborhood of D_i. Then we choose a partition of unity $\{\psi_i\}$ subordinate to the open cover and consider the map $\psi = (\psi_1, \dots, \psi_{|I|}) : N'_B \to \mathbb{R}^{|I|}$. It is such that $Im(\psi) = \Delta(D)$ and by restriction it is defined a map

$$\psi : N_B \to \Delta(D)$$

The choice of the simple normal crossing compactification and that of the partition do not change the homotopy class of this map (see [24]), so ψ is well defined up to homotopy. Therefore we have a diagram

$$\begin{tikzcd} N_{Dol} \arrow[r, "\beta"] \arrow[d, "h"] & N_B \arrow[d, "\psi"] \\ \mathbb{S}^{2N-1} & \Delta(D) \end{tikzcd}$$

Conjecture 1 ([20])**.** *If M_{Dol} is a moduli space of Higgs bundles and M_B is the corresponding character variety via Non-abelian Hodge theory*

1. *$\Delta(D)$ is homotopy equivalent to $\mathbb{S}^{2N-1}$.*
2. *The maps h and $\psi \circ \beta$ are homotopy equivalent.*

As a simple case we can consider the rank one case. We know that $M_{Dol}(1,0) = T^*C$ and $\bar{M} = \mathbb{P}^1 \times C$, so the Hitchin map is the projection $h : \mathbb{P}^1 \times C \to \mathbb{P}^1$. Moreover $M_B = \mathbb{C}^* \times \mathbb{C}^*$ and we can choose a smooth toric compactification as $\mathbb{P}^1 \times \mathbb{P}^1$, so that $\Delta(D)$ is the one dimensional simplicial complex with four vertices homeomorphic to $\mathbb{S}^1$. In this case the diagram is

$$\begin{tikzcd} \mathbb{C}^* \times C \arrow[r, "\beta"] \arrow[d, "h"] & \mathbb{C}^* \times \mathbb{C}^* \setminus U(1) \times U(1) \arrow[d, "\phi"] \\ \mathbb{C}^* & \mathbb{S}^1 \end{tikzcd}$$

and β is the restriction of the diffeomophism

$$\beta : \mathbb{A}^1 \times C \to \mathbb{C}^* \times \mathbb{C}^*$$

given by non-abelian Hodge theory. In particular the zero section, i.e. Higgs bundles with zero Higgs fields, correspond to unitary representations

$$\beta : 0 \times C \to S^1 \times S^1$$

so that a unitary local system corresponds to the line bundle with the same transition functions. Moreover the diffeomorphism sends $\mathbb{A}^1$ isomorphically to $\mathbb{R}_+^* \times \mathbb{R}_+^*$ via

$$\phi \to (\ exp\ (-\int_{\gamma_1} \phi + \bar{\phi}\ ),\ exp\ (-\int_{\gamma_2} \phi + \bar{\phi}\ )\ ) \tag{1.1}$$

where γ_1 and γ_2 are generators of $\pi_1(C)$.
In our case $\phi = \alpha dz$ with $\alpha \in \mathbb{A}^1$ and we can suppose that an integer base of the lattice Λ associated to C is $(1, i)$. The map 1.1 thus turns to be

$$\alpha \to (\ exp(-2\ Re(\alpha)), exp(2\ Im(\alpha))\ )$$

Choosing real coordinates (θ_1, θ_2) for $C \cong S^1 \times S^1$ and $\alpha = x + iy$ for $\mathbb{A}^1$ the diffeomorphism is

$$\begin{gathered} \mathbb{C} \times C \to \mathbb{C}^* \times \mathbb{C}^* \\ ((x, y), (\theta_1, \theta_2)) \to (exp(-2x + i\theta_1), exp(2y + i\theta_2) \end{gathered}$$

Moreover on $N_B = \mathbb{C}^* \times \mathbb{C}^* \setminus U(1) \times U(1)$ we can choose the open cover

$$\begin{aligned} U_1^- &= \{(z_1, z_2) \in \mathbb{C}^* \times \mathbb{C}^* \mid \|z_1\| < 1\} & U_1^+ &= \{(z_1, z_2) \in \mathbb{C}^* \times \mathbb{C}^* \mid \|z_1\| > 1\} \\ U_2^- &= \{(z_1, z_2) \in \mathbb{C}^* \times \mathbb{C}^* \mid \|z_2\| < 1\} & U_2^+ &= \{(z_1, z_2) \in \mathbb{C}^* \times \mathbb{C}^* \mid \|z_2\| > 1\} \end{aligned}$$

The map $N_B \to \Delta(D)$ sends the intersections to 1-simplices and their complement to vertices.
A generator of $h^{-1}(S^1)$ can be parametrized by $(x, y) = (\cos\omega, \sin\omega)$ where $\omega \in [0, 2\pi]$ and one can verify that the maps h and $\phi \circ \beta$ are homotopy equivalent.

1.6 Dual boundary complex and weight filtration

For every quasiprojective smooth variety M of dimension d we can find a smooth compactification $\overline{M}$ such that $\partial M = \overline{M} \setminus M$ is a SNCD and the combinatoric of the boundary divisor contains informations about the weight filtration of M.
More precisely [24]

$$\widetilde{H}_{i-1}(\Delta\partial M, \mathbb{Q}) = Gr_{2d}^W H^{2d-i}(M).$$

Consider the complex affine surface $Y = \mathbb{C}^* \times \mathbb{C}^*$ and we denote by M the Hilbert scheme $Y^{[n]}$. We prove a result going in the direction of Simpson's conjecture by estabilishing that the rational cohomology of the dual boundary complex of M is that of the sphere.

Proposition 1.6.1.

$$\widetilde{H}_i(\Delta\partial M, \mathbb{Q}) = \begin{cases} \mathbb{Q} & for\ i = 2n - 1 \\ 0 & otherwise. \end{cases}$$

Proof. M is an affine variety of dimension $2n$ so $H^l(M) = 0$ for $l > 2n$ and

$$\widetilde{H}_k(\Delta\partial M, \mathbb{Q}) = 0 \qquad for\ k < 2n-1.$$

Moreover for $l < 2n$ $W_{2d-1}H^l(M) = H^l(M) = W_{2d}H^l(M)$ so that

$$\widetilde{H}_k(\Delta\partial M, \mathbb{Q}) = 0 \qquad for\ k > 2n-1.$$

The only non vanishing group for a smooth affine variety of dimension d is $\widetilde{H}_{d-1}$, so the dual boundary complex has in general the rational homology of a wedge of $(d-1)$-dimensional spheres.
In our case we can compute the dimension of

$$\widetilde{H}_{2n-1}(\Delta\partial M, \mathbb{Q}) = Gr^W_{4n}H^{2n}(M).$$

Denoting

$$H^*_\nu(S^{[n]}) = H^{*-2(n-l(\nu))}_\nu(S^{(\nu)})(l(\nu) - n)$$

where for any partion $\nu = (1^{\alpha_1}, 2^{\alpha_2}, \ldots, n^{\alpha_n})$ of n of length $l(\nu)$ we denote $M^{(\nu)} = \prod M^{(\alpha_i)}$ we have that

$$Gr^W_{4n}H^{2n}(M) = \bigoplus_{l(\nu)\leq n} H^{2n}_\nu(S^{[n]}) \Big/ \bigoplus_{l(\nu)\leq n-1} H^{2n}_\nu(S^{[n]}) = H^{2n}(S^{(n)})$$

and

$$H^{2n}(S^{(n)}) = H^{2n}(S^n)^{\sigma_n} = H^{2n}((\mathbb{S}^1 \times \mathbb{S}^1)^n)^{\sigma_n} = H^{2n}((\mathbb{S}^1 \times \mathbb{S}^1)^n) = \mathbb{Q}$$

because σ_n acts trivially on the fundamental class $[d\theta_{11} \wedge d\theta_{21} \wedge \ldots \wedge d\theta_{1n} \wedge d\theta_{2n}]$. The last equality follows also from the Macdonald formula

$$\sum_{n\geq 0} q^n P_t(S^{(n)}) = \frac{(1+qt)^2}{(1-q)(1-qt^2)} = 1 + \sum q^n \frac{(t+1)(t^{2n}-1)}{(t-1)}.$$

$\square$

Chapter 2

Higgs bundles over elliptic curves

In this chapter we will focus on Higgs bundles on a smooth projective curve of genus one. We denote by (C, o), or simply C, an elliptic curve with a marked point $o \in C$. We recall that in this case the canonical bundle K_C of the curve is trivial.

Definition 2.0.1. Given a vector bundle E on C the *slope* of E is defined by

$$\mu(E) = \frac{deg(E)}{rk(E)}$$

Definition 2.0.2. Given the Higgs bundle (E, ϕ), we say that a subbundle $F \subset E$ is *ϕ-invariant* if $\phi(F) \subseteq F \otimes K$. A Higgs bundle (E, ϕ) is *semistable* if the slope of any ϕ-invariant subbundle F satisfies

$$\mu(F) \leq \mu(E).$$

The Higgs bundle is *stable* if the above inequality is strict for every proper ϕ-invariant subbundle and *polystable* if it is semistable and isomorphic to a direct sum of stable Higgs bundles of the same slope.

If (E, ϕ) is a semistable Higgs bundle of slope μ, then it has a *Jordan-Hölder filtration* of ϕ-invariant subbundles

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_r = E$$

where the restriction of the Higgs field to every quotient E_i/E_{i-1} induces a stable Higgs bundle $(E_i/E_{i-1}, \phi_i)$ with slope μ. For every semistable Higgs bundle (E, ϕ) we define its *associated graded object*

$$gr(E, \phi) := \bigoplus_i (E_i/E_{i-1}, \phi_i).$$

The graded object $gr(E, \phi)$ associated to (E, ϕ) is well defined up to isomorphism, and we say that two semistable Higgs bundles (E, ϕ) and (F, θ) are *S-equivalent* if $gr(E, \phi) \cong gr(F, \theta)$.

So in the S-equivalence class of every semistable Higgs bundle there is a polystable object and it is clear from the definition that the two polystable objects are S-equivalent if and only if they are isomorphic.

Equivalently a Higgs bundle on an elliptic curve C can be described as a pair $(\mathcal{E}, \phi)$ where $\mathcal{E} \in Coh(C)$ is a locally free sheaf on C and $\phi \in H^0(C, \mathcal{E}nd(\mathcal{E}))$.
A morphism of Higgs sheaves $F : (\mathcal{E}, \phi) \to (\mathcal{F}, \theta)$ is a morphism of sheaves $f : \mathcal{E} \to \mathcal{F}$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\phi"] \arrow[d, "f"'] & \mathcal{E} \arrow[d, "f"'] \\ \mathcal{F} \arrow[r, "\theta"] & \mathcal{F} \end{tikzcd}$$

Definition 2.0.3. An extension of Higgs sheaves (or Higgs extension) is a short exact sequence

$$0 \to (\mathcal{E}_1, \phi_1) \to (\mathcal{E}, \phi) \to (\mathcal{E}_2, \phi_2) \to 0$$

A morphism between extensions of Higgs sheaves is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & (\mathcal{E}_1, \phi_1) \arrow[r] \arrow[d, "f_1"'] & (\mathcal{E}, \phi) \arrow[r] \arrow[d, "f"'] & (\mathcal{E}_2, \phi_2) \arrow[r] \arrow[d, "f_2"'] & 0 \\ 0 \arrow[r] & (\mathcal{E}'_1, \phi'_1) \arrow[r] & (\mathcal{E}', \phi') \arrow[r] & (\mathcal{E}'_2, \phi'_2) \arrow[r] & 0 \end{tikzcd}$$

and it is onto, resp. injective, resp an isomorphism if and only if the three vertical arrows are onto, resp. injective, resp isomorphisms.
If the morphism is injective then we call the extension in the first row a *subextension* of the extension in the second row.

Note that the data of a proper subextension is the same as a proper subsheaf $\mathcal{E} \subset \mathcal{E}'$ that is invariant under ϕ, i.e. a proper Higgs subbundle.

2.1 Spectral covers for elliptic curves

There are different approaches to define the extra structure on the holomorphic bundle E given by the Higgs field. One of these is to replace locally free coherent sheaves on the Riemann surface Σ by sheaves of pure dimension one on $T^*\Sigma$.

Proposition 2.1.1 (BNR correspondence)**.** *There is a natural equivalence between the groupoid of Higgs bundles $(\mathcal{E}, \phi)$ on Σ and the groupoid of quasi-coherent sheaves $\mathcal{F}$ on the cotangent bundle $T^*\Sigma$ such that $p_*\mathcal{F}$ is locally free, where $p : T^*\Sigma \to \Sigma$ is the natural projection.*

Proof. Denoting by Θ the tangent sheaf of Σ, the Higgs bundle $(\mathcal{E}, \phi)$ gives rise to a morphism
$$\Theta \otimes \mathcal{E} \to \mathcal{E}$$
which induces a morphism
$$Sym^{\bullet}\Theta \otimes \mathcal{E} \to \mathcal{E}$$
so that $\mathcal{E}$ is equipped with the structure of module over the algebra $Sym^{\bullet}\Theta = p_*\mathcal{O}_{T^*\Sigma}$. Since $p : T^*\Sigma \to \Sigma$ is an affine morphism this gives rise to a quasi-coherent sheaf $\mathcal{F}$ on $T^*\Sigma$ such that $p_*\mathcal{F} = \mathcal{E}$.
Vice versa, given a quasi-coherent sheaf $\mathcal{F}$ on $T^*\Sigma$, the push-forward $p_*\mathcal{F} = \mathcal{E}$ is endowed with the structure of module over $Sym^{\bullet}\Theta$ and in particular we have a map

$$\Theta \otimes \mathcal{E} \to \mathcal{E}$$

giving rise to a Higgs field on $\mathcal{E}$.

□

If $p_*\mathcal{F} = \mathcal{E}$ is a locally free sheaf of rank n, let S be the support of $\mathcal{F}$). The fiber of the projection $p : S \to C$ over a point $x \in C$ is a length n zero-dimensional subscheme of $T^*_x C$, hence $p : S \to C$ is a n-to-1 cover of C.
The curve S il called *spectral curve* as the fiber S_x over a point $x \in C$ represents the eigenvalues of $\phi_x : E_x \to E_x \otimes K_x$. For a detailed exposition we refer to [7].

Proposition 2.1.2. *Let C be an elliptic curve and C_n a spectral cover of C of order n, i.e. the projection $p : C_n \to C$ is finite of degree n. Then C_n is the disjoint union of multiple curves*
$$C_n = \coprod \mathcal{C}_i$$
where each C_i is a multiple curve of order n_i with $(\mathcal{C}_i)_{red} \cong C$ and $\sum n_i = n$.

Proof. For an elliptic curve C the canonical bundle is trivial and we have

$$T^*C = \mathbb{A}^1_C = \underline{Spec}(\mathcal{O}_C[y]).$$

The Hitchin map is
$$h : \mathcal{M} \to \mathbb{A} = \bigoplus_{i=1}^{n} H^0(C, K^{\otimes i})$$
and in this case the Hitchin base is

$$\mathbb{A} = \bigoplus_{i=1}^{n} H^0(C, K^{\otimes i}) = \bigoplus_{i=1}^{n} H^0(C, \mathcal{O}) \cong \mathbb{C}^n$$

If $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$, the spectral curve $C_a \subset \mathbb{A}^1_C$ is given by

$$C_a \ : \ y^n + a_1 y^{n-1} + \ldots + a_n = (y - \alpha_1) \cdot \ldots \cdot (y - \alpha_n) = 0$$

and we can identify

$$\mathbb{C}^n \to \mathbb{C}^{(n)}$$
$$a = (a_1, \dots, a_n) \to \alpha = \sum \alpha_i$$

When $n > 1$ the spectral curve C_a is connected only if it is non reduced. Furthermore

$$\mathbb{A}_{red} = \mathbb{C}^{(n)} \setminus \Delta$$

where Δ is the generalized diagonal. In this case C_a is the disjoint union of n copies of C.
Moreover

$$\mathbb{A} \setminus \mathbb{A}_{red} = \Delta = \coprod_{\lambda \vdash n} \Delta^\lambda$$

where elements in Δ^λ are of the form $\sum \lambda_i \alpha_i$.
So if $\alpha \in \Delta^\lambda$ then $C_\alpha = \coprod C^{\lambda_i}$, where

$$\mathbb{A}^1_C \supset C^{\lambda_i} \ : \ (y - \alpha_i)^{\lambda_i} = 0$$

is a multiple elliptic curve. $\square$

Since the Higgs bundle corresponding to a non connected spectral curve is just the direct sum of the Higgs bundle corresponding to each connected component, without loss of generality we can restrict to the case $\lambda = n$ and we can fix $\alpha = n \cdot 0$. In this case the spectral curve

$$C^n \ : \ y^n = 0$$

is irreducbile and we denote η its generic point.
Recall that $\mathbb{A}^1_C = \underline{Spec}(\mathcal{O}_C[y])$ and

$$C^n = \underline{Spec}(\mathcal{O}_C[y] \diagup y^n)$$

so that the nilradical $N = (y)$ is generated by one element. Note that N is exactly the ideal $\mathcal{I}_C$ of C in C^n and we have the exact sequence

$$0 \to N \to \mathcal{O}_{C^n} \to \mathcal{O}_C \to 0 \tag{2.1}$$

Now we will focus on the case $n = 2$. In this case the multiple curve is called a *ribbon*.

Proposition 2.1.3. *When $n = 2$, the nilradical N defines a line bundle on C that is trivial.*

Proof. In this case $\mathcal{I}_C^2 = 0$ so

$$\mathcal{I}_C = \mathcal{I}_C \diagup \mathcal{I}_C^2 = \mathcal{I}_C \otimes \mathcal{O}_C = \mathcal{O}(-C) \otimes \mathcal{O}_C = \mathcal{O}(-C)_{|C}$$

is a line bundle on C of degree $-C \cdot C = deg(T^*C) = 0$, where we denote $C \cdot C$ the *self-intersection number* of C (see [14]).
We can compute the degree also using the exact sequence (2.1): since

$$deg(N) = \chi(N) - \chi(\mathcal{O}_C)$$

we have

$$deg(N) + \chi(\mathcal{O}_C) = \chi(N) = \chi(\mathcal{O}_{C^2}) - \chi(\mathcal{O}_C)$$
$$deg(N) = \chi(\mathcal{O}_{C^2}) - 2\chi(\mathcal{O}_C) = (1 - g_2) - 2(1 - g) = 2g - g_2 - 1$$

where g is the genus of $C = C^2_{red}$ and g_2 is the arithmetic genus of the spectral curve C^2. In our case $g_2 = g = 1$ so the degree is zero and y defines a global section of N, which is thus the trivial line bundle.

□

Definition 2.1.4. Given a coherent sheaf $\mathcal{F} \in Coh(C^n)$ we define the *dimension* of $\mathcal{F}$ as $d(\mathcal{F}) = dim(supp(\mathcal{F}))$.
A sheaf $\mathcal{F}$ is *pure* if $d(\mathcal{F}) = d(\mathcal{G})$ for every non-zero subsheaf $\mathcal{G} \subset \mathcal{F}$.
An element $f \in \mathcal{O}_{C^n}(U)$ is a *non zero divisor on* $\mathcal{F}$ if the multiplication $f : \mathcal{F}(U) \to \mathcal{F}(U)$ is injective.
A sheaf $\mathcal{F}$ is *torsion free* if every non zero divisor on $\mathcal{O}_{C^n}$ is a non zero divisor on $\mathcal{F}$.
A sheaf $\mathcal{F}$ has rank 1 if $\mathcal{F}_\eta \cong \mathcal{O}_{C^n,\eta}$.
We call *generalized line bundle* a torsion free coherent sheaves of rank one.

Lemma 2.1.5. *If $\mathcal{F} \in Coh(C^n)$ and $d(\mathcal{F}) = 1$ then $\mathcal{F}$ is torsion free if and only if is pure.*

Proof. The sheaf $\mathcal{F}$ is not torsion free if and only if exists $m \in \mathcal{F}(U)$ such that $ann(m) \nsubseteq N_{|U}$. This is equivalent to the fact that exists $m \in \mathcal{F}(U)$ with finite support, so that $\mathcal{F}$ is not pure. □

Lemma 2.1.6. *([3]) If $\mathcal{F} \in Coh(C^2)$ is a pure sheaf of generic length 2 (i.e. $\mathcal{F}_\eta$ is a $\mathcal{O}_{C^2,\eta}$ module of rank 2), then the kernel of $\phi : \mathcal{O}_{C^2} \to End(\mathcal{F})$ is (0) or (N).*

Such a sheaf gives rise to a rank 2 Higgs bundles on C and the map ϕ encodes the $\mathcal{O}_{T^*C}$ module structure of $\mathcal{F}$ and therefore the Higgs field.

2.2 Differential geometric description and Higgs cohomology

Denoting the underlying smooth bundle of the holomorphic bundle E by E, we can describe the holomorphic structure on E by an integrable partial connection, i.e. by a $\mathbb{C}$-linear map

$$\bar{\partial}_E : A^0(\mathrm{E}) \to A^{0,1}(\mathrm{E})$$

which satisfies the $\bar{\partial}$-Leibniz formula and the integrability condition $(\bar{\partial}_E)^2 = 0$.
A Higgs bundle (E, ϕ) can thus be specified by a triple $(\mathrm{E}, \bar{\partial}_E, \phi)$ where $\phi \in A^{1,0}(\mathrm{End\ E})$ is such that $\bar{\partial}_E(\phi) = 0$ and $\phi \wedge \phi = 0$, where the last condition is always satisfied on a curve.

With this approach, if we consider an extension of Higgs bundles

$$0 \to (E_1, \phi_1) \to (E, \phi) \to (E_2, \phi_2) \to 0$$

we can fix a smooth splitting $\mathrm{E} = \mathrm{E}_1 \oplus \mathrm{E}_2$, so that the sub-Higgs bundle in the extension is described by the triple $(\mathrm{E}_1, \bar{\partial}_1, \phi_1)$, and the quotient Higgs bundle by $(\mathrm{E}_2, \bar{\partial}_2, \phi_2)$. The Higgs extension is then specified by the triple $(\mathrm{E}, \bar{\partial}_E, \phi)$ and with respect to the chosen frame $\bar{\partial}_E$ will be of the form

$$\begin{pmatrix} \bar{\partial}_1 & a \\ 0 & \bar{\partial}_2 \end{pmatrix}$$

where a is a holomorphic section of $A^{0,1}(\mathrm{Hom}(\mathrm{E}_2, \mathrm{E}_1))$, and the Higgs field ϕ is of the form

$$\begin{pmatrix} \phi_1 & b \\ 0 & \phi_2 \end{pmatrix}$$

where b is a section of $A^{1,0}(\mathrm{Hom}(\mathrm{E}_2, \mathrm{E}_1))$.

Definition 2.2.1. Given an Higgs bundle (E, ϕ) on C the *holomorphic Dolbeault complex* is the complex

$$E \xrightarrow{\phi\wedge} E \otimes \Omega^1 \xrightarrow{\phi\wedge} E \otimes \Omega^2 \dots$$

and its hypercohomology is called the *Dolbeault cohomology of* (E, ϕ)

$$H^i_{Dol}(E, \phi) := \mathbb{H}^i(E \xrightarrow{\phi\wedge} E \otimes \Omega^1 \dots)$$

Given the Higgs bundle $(\mathrm{E}, \bar{\partial}, \phi)$, we can define an operator $D'' = \bar{\partial} + \phi$

$$D'': \ A^0(\mathrm{E}) \to A^{0,1}(\mathrm{E}) \oplus A^{1,0}(\mathrm{E})$$

that satisfies $(D'')^2 = 0$ and the Leibniz rule

$$D''(fe) = \bar{\partial}(f)e + fD''(e)$$

for any $f \in A^0(E)$. It turns out that the complex

$$A^0(\mathrm{E}) \xrightarrow{D''} A^1(\mathrm{E}) \xrightarrow{D''} A^2(\mathrm{E}) \dots$$

is a fine resolution of the holomorphic Dolbeault complex so its cohomology is the *Dolbeault cohomology* of the Higgs field.

In the case $(E, \phi) = (\mathcal{O}, 0)$ we have that $D'' = \bar{\partial}$ and the hypercohomology of the Dolbeault complex is just the usual Dolbeault cohomology of C, i.e.

$$H^i_{Dol}((\mathcal{O}, 0)) = H^i_{Dol}(C)$$

We can also describe successive extensions of such Higgs bundles.

Proposition 2.2.2. *If we denote* $\mathrm{Ext}^1_{Dol}((\mathcal{O},0),(\mathcal{O},0))$ *the group of isomorphism classes of extensions of* $(\mathcal{O},0)$ *by* $(\mathcal{O},0)$ *in the category of Higgs bundles over* C*, then there are group isomorphisms*

$$\mathrm{Ext}^1_{Dol}((\mathcal{O},0),(\mathcal{O},0)) \cong H^1_{Dol}(C) = H^1(C,\mathcal{O}) \oplus H^0(C,\Omega^1)$$

If $s \in H^1_{Dol}(C)$ then we can write $s = \alpha + \beta$, where $\alpha \in H^1(C,\mathcal{O})$ represents an extension E of the bundles

$$0 \to \mathcal{O} \to E \to \mathcal{O} \to 0$$

and $\beta \in H^0(C,\Omega^1)$ is a holomorphic differential form on C; so in a local splitting the Higgs field on E will have the form

$$\phi = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$$

and this β does not depend on the choice of the splitting.
In particular if $\beta = 0$, so that $s \in H^1(C,\mathcal{O})$, then the Higgs extension is of the form $(E,0)$, with zero Higgs field.

Remark 2.2.3. In the genus one case, if (C,o) is an elliptic curve, we can identify the space $X = T^*\hat{C}$ with the moduli space of Higgs bundles of rank one and degree zero on C, and the Higgs bundle $(\mathcal{O},0)$ corresponds to the point $p = (o,0) \in X$. In this way the space $H^1_{Dol}(X) = H^1(X,\mathcal{O}) \oplus H^0(X,\Omega^1)$ can be identified with T_pX, the tangent space at p of X. In fact

$$X = T^*\hat{C} = H^0(C,\mathcal{O}) \times H^1(C,\mathcal{O})/H_1(C,\mathbb{Z})$$

and so

$$T_pX \cong H^1(C,\mathcal{O}) \oplus H^0(C,\mathcal{O})$$

2.3 Moduli space of Higgs bundles on elliptic curves

The triviality of the canonical line bundle simplifies the study of the semistability of Higgs bundles over an elliptic curve C.

Proposition 2.3.1. *([9]) A Higgs bundle* (E,ϕ) *on* C *is semistable if and only if* E *is semistable. If* $\gcd(n,d) = 1$*, then* (E,ϕ) *is stable if and only if* E *is stable.*

Now we look at semistable Higgs bundles of degree zero on C.

Theorem 2.3.2. *([9]) There are no stable Higgs bundles on* C *of rank* $n > 1$ *and degree zero and*

$$M_{Dol}(1,0) \cong T^*\hat{C}.$$

Moreover if (E,ϕ) is a polystable Higgs bundle of rank n and degree 0, then

$$(E,\phi)=\bigoplus_{i=1}^{n}(L_i,\phi_i)=gr(E,\phi)$$

where (L_i,ϕ_i) are stable Higgs bundles of rank one and degree 0 so that

$$M_{Dol}(n,0)\cong T^*C^{(n)}$$

Remark 2.3.3. It is clear that if E is polystable it is not true in general that (E,ϕ) is polystable. Consider, for instance, (E,ϕ) such that $E\cong\mathcal{O}\oplus\mathcal{O}$ and fix a holomorphic splitting (s_1,s_2). Then an endomorphism ϕ of E can be expressed by a 2×2 matrix A. If A is non-diagonalizable (E,ϕ) is an indecomposable Higgs bundle, i.e. we can not express (E,ϕ) as a direct sum of stable Higgs bundles.

In particular the singular set $Sing(M_{Dol}(n,0))$ coincides with the set S of points represented by polystable Higgs bundles for which at least two of the direct summands are isomorphic, corresponding to the generalized diagonal of $T^*C^{(n)}$. In particular, if $n\geq 2$, the set S has codimension 2. This can be proven also by looking at the infinitesimal deformation space T of (E,ϕ):

$$H^0(C,End\ E)\xrightarrow{a}H^0(C,End\ E)\to T\to H^1(C,End\ E)\xrightarrow{a^*}H^1(C,End\ E)$$

Here $a(f)=[f,\phi]$ so that $f\in ker(a)$ if and only if $f\in End((E,\phi))$.
If (E,ϕ) has no isomorphic direct summands f must be diagonal and so $dim(coker(a))=n$ and for duality $dim(ker(a^*))=n$ so that $dim(T)=2n$ and every small deformation of such a polystable object is again polystable.

The Hilbert scheme is a resolution of singularities of the symmetric product and in terms of moduli space we can see the Hilbert Chow map

$$\begin{aligned} H:\ T^*C^{[n]}&\to T^*C^{(n)}\\ (E,\phi,v)&\to(E,\phi)\end{aligned}$$

as the map that fogets the datum of the cyclic vector.

2.4 Extensions of Higgs bundles

Let (L_1,ϕ_1) and (L_2,ϕ_2) be two Higgs bundles on C of degree 0 and rank one.

Proposition 2.4.1. *There are nontrivial extensions*

$$0\to(L_1,\phi_1)\to(E,\phi)\to(L_2,\phi_2)\to 0 \tag{2.2}$$

if and only if $(L_1,\phi_1)\cong(L_2,\phi_2)$, i.e. they correspond to the same element in T^*C.
In this case the extensions are parametrized by the two dimensional vector space $V=H^0(\mathcal{O})\oplus H^1(\mathcal{O})$.

Proof. The extension fitting in the sequence 2.2 are controlled by the first hypercohomology of the complex

$$C^\bullet: \; L_2^* L_1 \to L_2^* L_1$$
$$f \to \phi_1 f - f\phi_2$$

and we have a long exact sequence

$$0 \to \mathbb{H}^0(C^\bullet) \to H^0(L_2^*L_1) \xrightarrow{\psi} H^0(L_2^*L_1) \to \mathbb{H}^1(C^\bullet) \to H^1(L_2^*L_1) \xrightarrow{\psi^*} H^1(L_2^*L_1) \to \mathbb{H}^2(C^\bullet) \to 0$$

By Riemann-Roch we have that $h^0(L_2^*L_1) = h^1(L_2^*L_1)$ and we denote this integer by k. If $L_1 \ncong L_2$ then $k = 0$ and so all the groups vanish. In particular there are no nontrivial extensions of (L_1, ϕ_1) by (L_2, ϕ_2) if the line bundles are not isomorphic.
We may thus assume $L_1 \cong L_2$ so that $k = 1$ and the sequence is

$$0 \to \mathbb{H}^0(C^\bullet) \to H^0(\mathcal{O}) \xrightarrow{\psi} H^0(\mathcal{O}) \to \mathbb{H}^1(C^\bullet) \to H^1(\mathcal{O}) \to H^1(\mathcal{O}) \to \mathbb{H}^2(C^\bullet) \to 0$$

with

$$\psi: \; H^0(\mathcal{O}) \to H^0(\mathcal{O})$$
$$f \to \phi_1 f - f\phi_2$$

where $f, \phi_1, \phi_2 \in \mathbb{C}$. If $\phi_1 \neq \phi_2$ then ψ is an isomorphism and $\mathbb{H}^0 = \mathbb{H}^1 = \mathbb{H}^2 = 0$.
In particular there are no nontrivial extensions of (L, ϕ_1) by (L, ϕ_2) when $\phi_1 \neq \phi_2$. Therefore the only interesting case is when $L_1 \cong L_2$ and $\phi_1 = \phi_2$. In this case $\psi = 0$ and

$$\mathbb{H}^0 = H^0(\mathcal{O}) \qquad \mathbb{H}^2 = H^1(\mathcal{O}) \qquad \mathbb{H}^1 \cong H^0(\mathcal{O}) \oplus H^1(\mathcal{O})$$

□

We can apply the same technique in higher rank.

Proposition 2.4.2. *Let (L, ϕ_1) and (M, ϕ_2) be two Higgs bundles on C of degree 0 with $r(L) = 1$ and $r(M) = n$ and suppose that M is indecomposable. Then there are nontrivial extensions*

$$0 \to (L, \phi_1) \to (E, \phi) \to (M, \phi_2) \to 0$$

if and only if (M, ϕ_2) is isomorphic to a successive extension of (L, ϕ_1) with itself .
In this case the extensions are parametrized by the two dimensional vector space $H^0(F_n) \oplus H^1(F_n)$.

Proof. The extensions fitting in the above sequence are controlled by the first hypercohomology of the complex

$$C^\bullet: \; M^*L \to M^*L$$
$$f \to \phi_1 f - f\phi_2$$

and we have a long exact sequence

$$0 \to \mathbb{H}^0(C^\bullet) \to H^0(M^*L) \xrightarrow{\psi} H^0(M^*L) \to \mathbb{H}^1(C^\bullet) \to H^1(M^*L) \xrightarrow{\psi^*} H^1(M^*L) \to \mathbb{H}^2(C^\bullet) \to 0$$

By Riemann-Roch we still have that $h^0(M^*L) = h^1(M^*L) = k$.
Let us first restrict to the case in which M is indecomposable. Then $M \cong F_n \otimes P$ for a unique $P \in Pic^0(C)$, such that $det(M) = P^{\otimes n}$.
If $P \ncong L$ then

$$k = H^0(M^*L) = H^0(F_n P^* L) = 0$$

and so all the hypercohomology groups vanish.
Thus we can restrict to the case $M \cong F_n \otimes L$ and in this case

$$k = H^0(M^*L) = H^0(F_n) = 1$$

and we still have that

$$\begin{aligned} \psi: \ & H^0(F_n) \to H^0(F_n) \\ & \qquad\qquad f \to \phi_1 f - f\phi_2 \end{aligned}$$

can be zero or an isomorphism. We recall that here $f \in H^0(F_n) \cong Hom(F_n L, L)$, while $\phi_2 \in End(F_n L) \cong End(F_n)$ and $\phi_1 \in End(L) \cong H^0(\mathcal{O}) \cong \mathbb{C}$. So we have

$$\begin{tikzcd} F_n \otimes L \arrow[r, "f"] \arrow[d, "\phi_2"] & L \arrow[d, "\phi_1"] \\ F_n \otimes L \arrow[r, "f"] & L \end{tikzcd}$$

but $f = id_L \otimes f'$ for some $f' \in Hom(F_n, \mathcal{O})$, $\phi_1 = \phi_1' id_L$ and $\phi_2 = \phi_2' \otimes id_L$ for some $\phi_2' \in End(F_n)$ so we can reduce to the case $L \cong \mathcal{O}$ and in this case

$$\begin{tikzcd} F_n \arrow[r, "f"] \arrow[d, "\phi_2"] & \mathcal{O} \arrow[d, "\phi_1"] \\ F_n \arrow[r, "f"] & \mathcal{O} \end{tikzcd}$$

In this case $\phi_2 = \lambda id_{F_n} + N$ with N a nilpotent endomorphism. If $\lambda \neq \phi_1$ then $(\phi_2 - \phi_1) \in Aut(F_n)$ and so $f(\phi_2 - \phi_1) = 0$ if and only if $f = 0$ and so ψ is an isomorphism and all the hypercohomology groups vanishes.
If $\lambda = \phi_1$ then ψ is zero and

$$\mathbb{H}^0 = H^0(F_n) = \mathbb{C} \qquad \mathbb{H}^2 = H^1(F_n) = \mathbb{C} \qquad \mathbb{H}^1 \cong H^0(F_n) \oplus H^1(F_n) \cong \mathbb{C}^2$$

$\square$

If M is decomposable suppose that M is of rank 2 and that we have a splitting $M = M_1 \oplus M_2$. Then

$$H^0(M^*L) = H^0({M_1}^* \oplus {M_2}^*L) = H^0({M_1}^*L \oplus {M_2}^*L) = H^0({M_1}^*L) \oplus H^0({M_2}^*L)$$

The interesting case is when $M_1 \cong M_2 \cong L$ and we have the exact sequence

$$0 \to \mathbb{H}^0(C^\bullet) \to H^0(\mathcal{O})^{\oplus 2} \xrightarrow{\psi} H^0(\mathcal{O})^{\oplus 2} \to \mathbb{H}^1(C^\bullet) \to H^1(\mathcal{O})^{\oplus 2} \xrightarrow{\psi^*} H^1(\mathcal{O})^{\oplus 2} \to \mathbb{H}^2(C^\bullet) \to 0$$

where

$$\begin{aligned} \psi: \ H^0(\mathcal{O})^{\oplus 2} &\to H^0(\mathcal{O})^{\oplus 2} \\ f &\to f(\phi_2 - \phi_1) \end{aligned}$$

with $f \in H^0(EndL)^{\oplus 2}$. If the Higgs bundle (M, ϕ_2) is not decomposable the Higgs field is of the form

$$\phi_2 = \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda \end{pmatrix}$$

Again if $\lambda \neq \phi_1$ then $(\phi_2 - \phi_1) \in Aut(L \oplus L)$ so ψ is an isomorphism and the hypercohomology groups all vanish.
When $\lambda = \phi$ then $(\phi_2 - \phi_1)$ is nilpotent and $Ker(\psi)$ is 1-dimensional so

$$\mathbb{H}^0 \cong H^0(\mathcal{O}) \cong \mathbb{C} \qquad \mathbb{H}^2 \cong H^1(\mathcal{O}) \cong \mathbb{C} \qquad \mathbb{H}^1 \cong H^0(\mathcal{O}) \oplus H^1(\mathcal{O}) \cong \mathbb{C}^2.$$

Hence we have proven the following proposition.

Proposition 2.4.3. *Let (L, ϕ_1) and (M, ϕ_2) be two indecomposable Higgs bundles on C of degree 0 with $r(L) = 1$ and $r(M) = 2$ and suppose that $M = M_1 \oplus M_2$ is a decomposable vector bundle. Then there are nontrivial extensions*

$$0 \to (L, \phi_1) \to (E, \phi) \to (M, \phi_2) \to 0$$

if and only if (M, ϕ_2) is isomorphic to an extension of (L, ϕ_1) with itself .
In this case the extensions (E, ϕ) are parametrized by the two dimensional vector space $H^0(\mathcal{O}) \oplus H^1(\mathcal{O})$.

2.5 Factors of automorphy on $\mathbb{C}$

We can give an explicit description of the vector bundles over an elliptic curve C and their endomorphisms using the so called *factors of automorphy*. We will use the results of ([19]) and use them to describe the endomorphisms of the vector bundles.
Let us call $\pi : \mathbb{C} \to C$ the universal cover of the elliptic curve and G its fundamental group. Then for any vector bundle E on C of fixed rank r and degree zero, π^*E is trivial and is equipped with an action of G, called a *factor of automorphy*.

Definition 2.5.1. A holomorphic function

$$f : G \times \mathbb{C} \to GL_r(\mathbb{C})$$

is called an r-dimensional factor of automorphy if

$$f(gh, z) = f(g, h \cdot z)f(h, z).$$

for any $g, h \in G$ and for any $z \in \mathbb{C}$.
We say that f is *analytically equivalent* to f', and we write $f \sim f'$, if there exists a holomorphic function $q : \mathbb{C} \to GL_r(\mathbb{C})$ such that

$$q(g \cdot z)f(g, z) = f'(g, z)q(z)$$

for any $g \in G$ and $z \in \mathbb{C}$.
A factor of automorphy f is said to be *flat* if it is constant on $\mathbb{C}$.

Remark 2.5.2. In particular, constant factors of automorphy are just n-dimensional representations of G. If two factors f and f' are flat we say that they are *flatly equivalent* if they are analytically equivalent and the map q realizing the equivalence is constant on $\mathbb{C}$, i.e. the two representations are isomorphic.
Note that even if two representations are not conjugated they can be analytically equivalent. Take for instance $G = \mathbb{Z}^2 \subset \mathbb{C}$, where $e_1 = (1, 0) = 1$ and $e_2 = (0, 1) = i$ and consider

$$\begin{aligned} f :\ & G \to \mathbb{C}^* \\ & e_1 \to 1 \\ & e_2 \to 1 \end{aligned} \qquad\qquad \begin{aligned} f' : G & \to \mathbb{C}^* \\ e_1 & \to 1 \\ e_2 & \to \alpha \end{aligned}$$

where $\alpha \in \mathbb{C}^*$ with $\alpha \neq 1$. In this case f and f' are obviously not conjugated because $\mathbb{C}^*$ is abelian so the two factors are not flatly equivalent, but if $\alpha = e^{2\pi n}$ with $n \in \mathbb{Z}$ they are analytically equivalent. In fact we are looking for a holomorphic function $q :\ \mathbb{C} \to \mathbb{C}^*$ such that

$$\begin{aligned} q(z + 1) &= q(z) \\ q(z + i) &= \alpha q(z) \end{aligned}$$

and we can take $q(z) = e^{-2\pi i n z}$.

If f is a factor of automorphy we can define an action of G on the trivial vector bundle of rank r on $\mathbb{C}$ by

$$g \cdot (z, v) = (g \cdot z, f(g, z)v)$$

and $E(f) = \mathbb{C} \times \mathbb{C}^r / G$ is a well defined holomorphic vector bundle on C.
Moreover there is a bijection between the set of equivalence classes of r-dimensional factors of automorphy and the set of isomorphism classes of vector bundles of rank n on C.

With this notation we can see a section of $E(f)$ as a so called *f-theta function*, i.e. a holomorphic function $s:\mathbb{C}\to\mathbb{C}^r$ such that

$$s(g\cdot z)=f(g,z)s(z).$$

Proposition 2.5.3. *Let*

$$f(g,z)=\begin{pmatrix} f_1(g,z) & \hat{f}(g,z) \\ 0 & f_2(g,z) \end{pmatrix}$$

be an r+s-dimensional factor of automorphy. Then f_1 and f_2 are factors of automorphy and f determines an extension of vector bundles

$$0\to E(f_1)\to E(f)\to E(f_2)\to 0.$$

Remark 2.5.4. Note that $\hat{f}(g,z)$ is not a factor of automorphy, but a holomorphic function $\hat{f}:\ G\times\mathbb{C}\to M_{r,s}(\mathbb{C})$ satisfying

$$\hat{f}(gh,z)=\hat{f}(g,h\cdot z)+\hat{f}(h,z)$$

Example 2.5.5. Every factor of automorphy of the form

$$f(g,z)=\begin{pmatrix} 1 & \hat{f}(g,z) \\ 0 & 1 \end{pmatrix}$$

gives rise to a vecor bundle $E(f)$ that is an extension

$$0\to\mathcal{O}\to E\to\mathcal{O}\to 0$$

The extension is trivial, i.e. $E\cong\mathcal{O}^{\oplus 2}$, if and only if $f\sim Id$, i.e. exists an holomorphic function $q:\mathbb{C}\to\mathbb{C}$ such that

$$\hat{f}(g,z)=q(g\cdot z)-q(z)$$

for every $g\in G$.

Proposition 2.5.6. *Let f_1 be an r-dimensional factor and f_2 an s-dimensional factor, then $f_1\otimes f_2$ is an rs-dimensional factor and $E(f_1\otimes f_2)\cong E(f_1)\otimes E(f_2)$.*

Now take $G=\mathbb{Z}\oplus\mathbb{Z}\tau$ where $\tau\in\mathbb{C}$ with $Im\ \tau>0$ and take $p=x+\tau y\in\mathbb{C}$. The function $\theta_p:\ \mathbb{C}\to\mathbb{C}$

$$\theta_p(z)=\sum_{k\in\mathbb{Z}}exp(\pi i(k+y)^2\tau)exp(2\pi i(k+y)(z+x))$$

is such that for every $g=m+n\tau\in G$

$$\theta_p(z+m+n\tau)=e_p(g,z)\theta_p(z)$$

where

$$e_p(g,z) = exp(2\pi i y g - \pi i n^2\tau - 2\pi i n(z+p))$$

is such that

$$e_p(g+h,z) = e_p(g,h+z)e_p(h,z)$$

so it is a 1-dimensional factor of automorphy, defining a line bundle $L(e_p)$ such that θ_p is a section of $L(e_p)$.

Proposition 2.5.7 ([19]). *$L(e_p)$ has degree 1 and $L(e_p) = \mathcal{O}([p+\frac{1}{2}+\frac{\tau}{2}])$.*

Proof. θ_p is a section of the line bundle and it has only a simple zero in the fundamental domain at the point $p' = [p+\frac{1}{2}+\frac{\tau}{2}]$. □

Example 2.5.8. We identify C with its dual $\hat{C}$ by the Abel Jacobi map

$$\begin{aligned} Aj:\ & C \to \hat{C} = Pic^0 C \\ & p' \to \mathcal{O}(p'-o) = \mathcal{O}(p') \otimes \mathcal{O}(o)^{-1} \end{aligned}$$

So we can associate to each $p = x + \tau y$ on the complex plane the line bundle with factor of automorphy $e_p \cdot e_o^{-1}$, i.e.

$$\begin{aligned} e_p \cdot e_o^{-1}(m+n\tau, z) =& exp[(2\pi i y\gamma - \pi i n^2\tau - 2\pi i n(z+p)) - (-\pi i n^2\tau - 2\pi i n z)] = \\ =& exp[2\pi i(y\gamma - np)] = exp[2\pi i(ym - nx)] \end{aligned} \tag{2.3}$$

If we have $p \in \mathbb{C}$ and we want to write $p = x + \tau y$ we use that

$$\frac{\bar{p}\tau - p\bar{\tau}}{\tau - \bar{\tau}} = \frac{(x+\bar{\tau}y)\tau - (x+\tau y)\bar{\tau}}{\tau - \bar{\tau}} = \frac{x(\tau - \bar{\tau})}{\tau - \bar{\tau}} = x$$

and

$$\frac{p - \bar{p}}{\tau - \bar{\tau}} = \frac{(x+\tau y) - (x+\bar{\tau}y)}{\tau - \bar{\tau}} = \frac{y(\tau - \bar{\tau})}{\tau - \bar{\tau}} = y$$

so for every $p \in \mathbb{C}$ the line bundle $\mathcal{O}(p'-o) \in \hat{C}$ has factor of automorphy

$$f(m+n\tau, z) = exp[\frac{2\pi i}{\tau - \bar{\tau}}(m(p-\bar{p}) + n(p\bar{\tau} - \bar{p}\tau))] \tag{2.4}$$

2.6 Factors of automorphy on $\mathbb{C}^*$

Since $\mathbb{C}^*$ as a complex variety is Stein of dimension one, it follows that every holomorphic vector bundle on $\mathbb{C}^*$ is trivial. This leads to an easier description. Denoting $u = exp(2\pi i z) \in \mathbb{C}^*$ and $q = exp(2\pi i\tau)$ we define for every $m, n \in \mathbb{Z}$

$$(m+n\tau)\cdot u = exp(2\pi i(m+n\tau))\ u = q^n u$$

This defines an action of $\mathbb{Z}$ on $\mathbb{C}^*$ of the form $n \cdot u = q^n u$ so that the elliptic curve C is biholomorphic to $\mathbb{C}^* / q^{\mathbb{Z}}$.

Pulling back a vector bundle E on C by the covering $\pi : \mathbb{C}^* \to C$ we still have that the bundle $\pi^* E$ is trivial and so we can describe E by a $\mathbb{C}^*$-factor of automorphy

$$\begin{aligned} A : \mathbb{Z} \times \mathbb{C}^* &\to GL_r(\mathbb{C}) \\ (n, u) &\to A(n, u) \end{aligned}$$

We can recover a factor of automorphy of $G = \mathbb{Z} \oplus \mathbb{Z}\tau$ on $\mathbb{C}$ by

$$f(m + n\tau, z) = A(n, u)$$

and the factor that we get is such that the first summand $\mathbb{Z} \subset G$ acts trivially on $\mathbb{C}$.

Proposition 2.6.1. *There is a one-to-one correspondence beetween $\mathbb{C}$-factors of automorphy $f : G \times \mathbb{C} \to GL_r(\mathbb{C})$ such that*

$$f(m + n\tau, z) = f(n\tau, z)$$

and $\mathbb{C}^*$-factors of automorphy $A : G \times \mathbb{C}^* \to GL_r(\mathbb{C})$ with the same property

$$A(m + n\tau, u) = A(n\tau, u)$$

Since A is a factor of autormophy $A(2\tau, u) = A(\tau, qu)A(\tau, u)$ and more in general

$$A(n, u) = A(q^{n-1}u) \dots A(qu)A(u)$$

so that we can define a $\mathbb{C}^*$-factor of automorphy simply giving an homolorphic function

$$A(\tau, u) = A : \mathbb{C}^* \to GL_r\mathbb{C}.$$

The functions corresponding to the line bundles above are

$$\begin{aligned} \mathcal{O}(o) : \;\; & A(u) = q^{-\frac{1}{2}}u^{-1} = exp(-\pi i\tau - 2\pi iz) \\ \mathcal{O}(p') : \;\; & A(u) = q^{-\frac{1}{2}}u^{-1}exp(-2\pi ip) \\ \mathcal{O}(p' - o) : \;\; & A(u) = exp(-2\pi ip) \end{aligned}$$

where $p \in \mathbb{C}$ and $p' \in C$ is the corresponding point on the elliptic curve.
In particular for $\mathcal{O}(p' - o)$ we recover the constant factor

$$\begin{aligned} f(1, z) &= 1 \\ f(\tau, z) &= exp(-2\pi ip) \end{aligned}$$

and this factor is equivalent (but obviously not conjugated) to the factor we have found in (2.3)

$$\begin{aligned} f'(1, z) &= exp(2\pi iy) \\ f'(\tau, z) &= exp(-2\pi ix) \end{aligned}$$

It means that there exists an holomorphic function $q : \mathbb{C} \to \mathbb{C}^*$ such that

$$q(z+1) = exp(2\pi i y)q(z)$$
$$q(z+\tau) = exp(2\pi i y\tau)q(z)$$

and we can choose $q(z) = exp(2\pi i y z)$.

Given two functions $A, B : \mathbb{C}^* \to GL_r(\mathbb{C})$ we say that they are *analytically equivalent* if exists an holomorphic function $l : \mathbb{C}^* \to GL_r(\mathbb{C})$ such that

$$A(u)l(u) = l(qu)B(u)$$

and in this case they define isomorphic vector bundles.
In particular A defines the trivial vector bundle if and only if $A(u) = l(qu)l(u)^{-1}$.

Proposition 2.6.2. *([19]) The constant $\mathbb{C}^*$-factor of automorphy*

$$f = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

corresponds to the indecomposable vector bundle F_n of rank n and degree 0.

Using this result we can give another proof of the following fact.

Proposition 2.6.3. *For every $n > 0$, $H^0(F_n) \cong \mathbb{C}$.*

Proof. A section of F_n corresponds to a holomorphic function

$$s : \mathbb{C}^* \to \mathbb{C}^n$$
$$u \to (s_1(u), \dots, s_n(u))$$

such that $s(qu) = f \cdot s(u)$. In this case $s_n(qu) = s_n(u)$ so $s_n(u) = s_n$ must be constant. Furthermore $s_{n-1}(qu) = s_{n-1}(u) + s_n$ so $s_n = 0$. The same argument show that $s_i = 0$ for every $i = 2, \dots, n$ and $s_1(z) = s_1$ is constant. $\square$

In particular $h^0(End\ F_n) = n$ in fact

$$H^0(End\ F_n) = H^0(F_n^* \otimes F_n) = H^0(F_n \otimes F_n) = H^0(F_{2n-1}) \oplus H^0(F_{2n-3}) \oplus \dots \oplus H^0(\mathcal{O})$$

With this notation we can describe the automorphisms of the bundle F_n. In fact when the factor of automorphy f is constant an element $a \in End(E(f))$ corresponds to a holomorphic map

$$s : \mathbb{C}^* \to M_2(\mathbb{C})$$

such that

$$s(qz) = fs(z)f^{-1}$$

and using the same techniques we can prove the following result.

Proposition 2.6.4. *An element $s \in End(F_n)$ corresponds to a constant map $s : \mathbb{C}^* \to GL_n(\mathbb{C})$ of the form*

$$s = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_n \\ 0 & \alpha_1 & \dots & \alpha_{n-1} \\ & & \ddots & \vdots \\ & & & \alpha_1 \end{pmatrix}$$

with $\alpha_i \in \mathbb{C}$.

Proof. In fact an element of $End\ F_n$ is a map

$$\phi : \mathbb{C}^* \to M_n\mathbb{C} \tag{2.5}$$

$$z \to \phi(z) = \phi_{ij}(z) \tag{2.6}$$

such that $\phi(qz) = f\phi(z)f^{-1}$. Using the fact that in this case

$$f_{il} = \begin{cases} 1 & l = i, i+1 \\ 0 & otherwise \end{cases} \qquad f^{-1}_{kj} = \begin{cases} 0 & k > j \\ (-1)^{k+j} & k \leq j \end{cases}$$

the matrix must satisfy

$$\phi_{ij}(qz) = \sum_k \sum_l f_{il} \cdot \phi_{lk}(z) \cdot f^{-1}_{kj} = \sum_{k \leq j} \sum_{l=i,i+1} (-1)^{i+j} \phi_{lk}(z)$$

In particular $\phi_{n1}(qz) = \phi_{n1}(z)$ so it is constant; moreover $\phi_{n2}(qz) = \phi_{n2}(z) - \phi_{n1}$ and using Laurent series one can verify that ϕ_{n1} must be zero. In fact setting $\phi_{n2}(z) = \sum_{\infty}^{+\infty} a_k z^k$ we get

$$\sum a_k q^k z^k = \sum a_k z^k - \phi_{n1}$$

so that $a_k = 0$ for $k \neq 0$ - and so ϕ_{n2} is constant - and $\phi_{n1} = 0$.
Iterating this procedure one can find the matrix above. $\square$

Chapter 3

The correspondence and the Fourier Mukai transform

In this chapter we will give details of the correspondence between the moduli space of marked Higgs bundles on C and the Hilbert scheme of points of T^*C, defined by means of a relative Fourier-Mukai transform; this will allow us to describe explicitly the Higgs bundles corresponding to subschemes of length $n \leq 3$.

Theorem 3.0.1 ([13]). *There is an isomorphism*

$$T^*C^{[n]} \cong \mathcal{M}$$

where $\mathcal{M}$ is the moduli space of rank n and degree 0 marked Higgs bundles on C, whose closed points parametrize triples (E, ϕ, v) where (E, ϕ) is a semistable Higgs bundle on C of rank n and degree 0 and $v \in E_o$ is such that there are not proper ϕ-invariant subsheaves $F \subset E$ with $\mu(F) \geq \mu(E)$ with $v \in F_o$.

Proof. We denote $\mathcal{P}$ the normalized Poincaré sheaf on $C \times \hat{C}$, such that $\mathcal{P}_{|C \times [L]} \cong L$ for every $L \in \hat{C}$, and Φ the Fourier Mukai transform with kernel $\mathcal{P}$

$$\Phi : \ D^b(C) \to D^b(\hat{C})$$

By the base change $b : \mathbb{A}^1 \to Spec\, \mathbb{C}$ we get the morphisms $f : T^*C \to C$, $g : T^*\hat{C} \to \hat{C}$ and $l : T^*C \times_{\mathbb{A}^1} T^*\hat{C} \to C \times \hat{C}$ and denoting π_1 and π_2 the projections of the fiber product on the two factors

$$\begin{tikzcd}
 & T^*C \times_{\mathbb{A}^1} T^*\hat{C} \arrow[dl, "\pi_1"'] \arrow[d, "l"] \arrow[dr, "\pi_2"] & \\
T^*C \arrow[d, "f"'] & C \times \hat{C} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & T^*\hat{C} \arrow[d, "g"] \\
C & & \hat{C}
\end{tikzcd}$$

the relative Fourier Mukai transform

$$\begin{aligned}\Phi_{\mathbb{A}^1} : D^b(T^*C) &\to D^b(T^*\hat{C})\\ \mathcal{E}^\bullet &\to R^\bullet \pi_{2*}(L\pi_1^*\mathcal{E}^\bullet \otimes l^*\mathcal{P})\end{aligned}$$

is still an equivalence of categories, with kernel $l^*\mathcal{P} \in D^b(T^*C \times_{\mathbb{A}^1} T^*\hat{C})$.

In this case, since f and g are flat morphisms, the pullback is exact and we have that for every $\mathcal{F} \in D^b(C)$

$$\Phi_{\mathbb{A}^1}(f^*\mathcal{F}) \cong g^*(\Phi(\mathcal{F}))$$

in the category $D^b(T^*\hat{C})$; so taking $\mathcal{F} = \mathcal{O}_C$ we get

$$\Phi_{\mathbb{A}^1}(\mathcal{O}_{T^*C}) \cong g^*(\mathcal{O}_o[-1]) \cong \mathcal{O}_{\hat{X}_0}[-1]$$

where $\hat{X}_0 = g^{-1}(0)$.

The structure sheaf $\mathcal{O}_Z$ of a length n subscheme $Z \subset T^*C$ is a torsion sheaf and $\Phi_{\mathbb{A}^1}(\mathcal{O}_Z)$ turns out to be a coherent sheaf on $T^*\hat{C}$, corresponding to a rank n Higgs bundle on C.
In fact denoting $\mathcal{F} = \pi_1^*\mathcal{O}_Z \otimes l^*\mathcal{P}$

$$\mathcal{M} = \Phi_{\mathbb{A}^1}(\mathcal{O}_Z) = R^\bullet\pi_{2*}(\mathcal{F})$$

but $\pi_2 :$ *supp* $\mathcal{F} \to \hat{X}$ is finite so $R^i\pi_{2*}(\mathcal{F})$ vanishes for $i \neq 0$ and $\mathcal{M}$ is actually a sheaf.

If we look at the subscheme structure of Z, given by a surjection $s : \mathcal{O}_{T^*C} \to \mathcal{O}_Z$, the relative Mukai functor endows the Higgs bundle $\Phi_{\mathbb{A}^1}(\mathcal{O}_Z)$ with a parabolic structure. The Higgs bundle $\Phi_{\mathbb{A}^1}(\mathcal{O}_Z)$ is semistable and the parabolic structure gives us a new stability condition, such that the marked Higgs bundle is stable.

The surjection $s : \mathcal{O}_X \to \mathcal{O}_Z$ encodes the datum of the embedding $Z \subset X$. To keep trace of this we must consider the image by the Fourier Mukai functor of the exact sequence

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_X \xrightarrow{s} \mathcal{O}_Z \to 0$$

and we get the exact triangle in $D^b(\hat{X})$

$$I \to \mathcal{O}_{\hat{X}_0}[-1] \to \mathcal{M} \to I[1]$$

The datum of the surjection s is thus translated by the relative Mukai functor into an element in

$$\begin{aligned}\mathrm{Hom}_{D^b(X)}(\mathcal{O}_{\hat{X}_0}[-1], \mathcal{M}) &\cong \mathrm{Hom}_{D^b(X)}(g^*\mathcal{O}_0, \mathcal{M}[1]) \cong \\ &\cong \mathrm{Hom}_{D^b(\hat{C})}(\mathcal{O}_0, M[1]) \cong \mathrm{Ext}^1_{\hat{C}}(\mathcal{O}_0, M)\end{aligned} \tag{3.1}$$

that can be thought as a vector $v \in M_0$, the fiber of M over 0, where $M = g_*\mathcal{M}$.

In fact $\mathcal{O}_0$ and M are both coherent sheaves on the elliptic curve $\hat{C}$. By Serre duality if $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves on the elliptic curve C, then we have a functorial isomorphism

$$\mathrm{Ext}^1(\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{F}, \mathcal{G})^*$$

so in our case

$$\mathrm{Ext}^1(\mathcal{O}_0, M) \cong \mathrm{Hom}(M, \mathcal{O}_0)^* \cong M_0$$

The stability condition for the marked Higgs bundle is the following: the Higgs bundle (M, ϕ) has no proper Higgs subbundle (N, ϕ) of degree zero such that $v \in N_0$.

In fact assuming that such proper subbundle (N, ϕ) of rank $k < n$ exists, its transform $\mathcal{F}$ would give rise to a commutative diagram

$$\begin{tikzcd}
\mathcal{O}_X \arrow[rr, two heads] \arrow[dr, two heads] & & \mathcal{O}_Z \\
& \mathcal{F} \arrow[ur] &
\end{tikzcd}$$

thus obtaining a contradiction. Similarly if the map s is not surjective, its image $\mathcal{F}$ would give rise to a non-trivial Higgs subbundle of (M, ϕ) containing v.

□

In fact if we denote $\pi = g \circ \pi_2$ and $\mathcal{G} = \pi_1^*\mathcal{O}_Z \otimes l^*\mathcal{P}$ we have that $M = \pi_*\mathcal{G}$ and $\pi :$ *supp* $\mathcal{G} \to \hat{C}$ is finite and flat, moreover each fiber over a point $t \in \hat{C}$ is isomorphic to Z and in particular $\mathcal{G}_o \cong \mathcal{O}_Z$, so by proper base change we have

$$M_0 \cong H^0(Z, \mathcal{G}_o) \cong H^0(Z, \mathcal{O}_Z)$$

and the vector $v \in M_0$ is the image of the identity element. For each point $t \in \hat{C}$ we have $\mathcal{G}_t \cong f^*L_{t|Z}$, so that $M_t \cong H^0(Z, f^*L_{t|Z})$, according with the construction in [12].

As seen in the proof of theorem 3.0.1 for every $\mathcal{F} \in D^b(C)$, we can compute $\Phi_{\mathbb{A}^1}(f^*\mathcal{F})$ using the fact that

$$\Phi_{\mathbb{A}^1}(f^*\mathcal{F}) \cong g^*(\Phi(\mathcal{F}))$$

. Unfortunately we can not use this formula to compute the image $\Phi_{\mathbb{A}^1}(\mathcal{O}_Z)$ where $Z \subset T^*C$ is a 0-dimensional subscheme, because we can not get $\mathcal{O}_Z = f^*\mathcal{F}$ for any $\mathcal{F} \in Coh(C)$.
However we can try to relate some properties of the subscheme Z with information on the corresponding Higgs bundle $\Phi^1_{\mathbb{A}}(\mathcal{O}_Z)$.

Lemma 3.0.2. *The Fourier Mukai transform commutes with the higher direct image functors, i.e.*

$$\Phi(Rf_*\mathcal{E}) \cong Rg_*\Phi^1_{\mathbb{A}}(\mathcal{E})$$

Proof. By the two commuting diagrams

$$\begin{tikzcd} T^*C \arrow[d, "f"] & T^*C \times_{\mathbb{A}^1} T^*\hat{C} \arrow[l, "\pi_1"'] \arrow[d, "l"] \\ C & C \times \hat{C} \arrow[l, "p_1"'] \end{tikzcd} \qquad \begin{tikzcd} T^*C \times_{\mathbb{A}^1} T^*\hat{C} \arrow[r, "\pi_2"] \arrow[d, "l"] & T^*\hat{C} \arrow[d, "g"] \\ C \times \hat{C} \arrow[r, "p_2"] & \hat{C} \end{tikzcd}$$

we have by flat base change (since p_i and π_i are flat) that

$$p_1^* Rf_*(\mathcal{E}) \cong Rl_* \pi_1^*(\mathcal{E})$$

and since $(g \circ \pi_2) = (p_2 \circ l)$ we have

$$Rg_* R\pi_{2*} \cong Rp_{2*} Rl_*$$

so that

$$\begin{aligned} Rg_* \Phi^1_{\mathbb{A}}(\mathcal{E}) = Rg_* R\pi_{2*}(\pi_1^*\mathcal{E} \otimes l^*\mathcal{P}) &= \\ = Rp_{2*} Rl_*(\pi_1^*\mathcal{E} \otimes l^*\mathcal{P}) &= \\ = Rp_{2*}(Rl_*\pi_1^*\mathcal{E} \otimes \mathcal{P}) &= \\ = Rp_{2*}(p_1^* Rf_*(\mathcal{E}) \otimes \mathcal{P}) &= \\ = \Phi(Rf_*(\mathcal{E}))& \end{aligned}$$

where we used that by the projection formula

$$Rl_*(\mathcal{F} \otimes l^*\mathcal{G}) \cong Rl_*\mathcal{F} \otimes \mathcal{G}$$

for every locally free $\mathcal{G}$ and quasi coherent $\mathcal{F}$. □

Using the last result we can describe more explicitly the Higgs bundle that we get from a subscheme or at least its underlying holomorphic vector bundle.

Proposition 3.0.3. *([16]) Let $S(r, 0)$ be the set of all isomorphism classes of semistable bundles of rank r and degree 0 on C. There is an isomorphism between $S(r, 0)$ and the set T_r of torsion sheaves of length r on C.*

$$\Phi : T_r \to S(r, 0)$$

We recall that for a coherent sheaf on a curve, being a torion sheaf is equivalent to having support on a finite number of closed points. The bijection between torsion sheaves and semistable bundles also allows the identification of indecomposable objects on both sides. Explicitly, torsion sheaves of the form $\mathbb{C}[x]/x^r$ give rise to indecomposable bundles and vice versa.

Composing this proposition with the previous lemma we get the following result.

Proposition 3.0.4. *Let $Z \subset T^*C$ be a 0-dimensional subscheme of length n and let (E, ϕ) the corresponding Higgs bundle. Then E is indecomposable if and only if $p_*\mathcal{O}_Z$ is an indecomposable torsion sheaf on C, i.e. of the form $\mathcal{O}_p[x]/x^n$.*

3.1 Rank one

Given a point $p \in T^*C$ we denote $\mathcal{O}_p$ the skyscraper sheaf with stalk $\mathbb{C}$ over p. If the point p parametrizes the rank one Higgs bundle (L, ϕ), the sheaf $\mathcal{O}_p$ is sent to the line bundle on the spectral curve S of ϕ, corresponding to (L, ϕ) via the BNR correspondence (see proposition 2.1.1).

Lemma 3.1.1. *Let us assume that $p : X \to B$ and $q : Y \to B$ are flat morphisms. If $\mathcal{E}^\bullet \in D^b(X)$, by denoting by j_t the immersions of both fibers $X_t = p^{-1}(t)$ and $Y_t = q^{-1}(t)$ over a closed point $t \in B$ into $X \times_B Y$, one has*

i) $Lj_t^*\Phi(\mathcal{E}^\bullet) \cong \Phi_t(Lj_t^*\mathcal{E}^\bullet)$ for every $\mathcal{E}^\bullet \in D^b(X)$;

ii) $j_{t*}\Phi_t(\mathcal{F}^\bullet) \cong \Phi(j_{t*}\mathcal{F}^\bullet)$ for every $\mathcal{F}^\bullet \in D^b(X_t)$.

In our case, denoting $X = T^*C$ and $\pi : X \to \mathbb{A}^1$, if $p \in X$ is a closed point such that $\pi(p) = t \in \mathbb{A}^1$ then $p \in X_t$ and $\mathcal{O}_p = j_{t*}\mathcal{O}_p$ so by (ii) $\Phi(\mathcal{O}_p)$ is just the line bundle $L_{f(p)}$ supported on X_t.
Moreover if $\mathcal{O}_Z$ is the structure sheaf of a 0-dimensional sheaf of length n supported on p, then X_t is the only reduced spectral curve passing through Z and $j_t^*\mathcal{O}_Z$ is an $\mathcal{O}_{X_t}$-module of rank n supported on $f(p)$ and $\Phi_t(j_t^*\mathcal{O}_Z)$ is the rank n vector bundle underlying the Higgs bundle.

3.2 Higher rank

We can use the previous lemma to describe the Higgs bundle corresponding to "horizontal" subschemes. We can fix local coordinates (z, y) on T^*C, where z is a coordinate on the curve and y on the fiber and suppose that the subscheme is supported on the origin $(o, 0) \in T^*C$.

Proposition 3.2.1. *A subscheme $Z \subset T^*C$ supported on the origin whose ideal is of the form $\mathcal{I}_Z = (z^n, y)$ corresponds to the Higgs bundle $(F_n, 0)$.*

Proof. A subscheme of this type is cointained in the spectral curve given by the zero section of T^*C and by the lemma above it correspond to the vector bundle $\Phi(\mathcal{O}_Z)$ supported on the same spectral curve. We have seen that since Z is an indecomposable subcheme of length n of C concentrated on o then $\Phi(\mathcal{O}_Z) = F_n$. It is supported on the zero section so the Higgs field vanishes. ∎

Remark 3.2.2. The algebraic group T^*C acts on M_{Dol} by tensor product and this corresponds via Fourier-Mukai transform to a translation of the support.
Moreover an Higgs bundle of type $\Phi_{\mathbb{A}^1}(\mathcal{O}_Z)$ is indecomposable if and only if the scheme Z is supported on a single point and the decomposition of a Higgs bundle into indecomposable factors corresponds to the decomposition

$$Z = \amalg\, Z_i$$

of a scheme Z into its irreducible components, each Z_i being supported on a distinct point.

For these reasons it suffices to restrict to the case of a subscheme Z supported on $(0, o) \in T^*C$.

Example 3.2.3. In the generic case a length n subscheme Z is given by n pairwise distinct points $p_1, \ldots, p_n \in T^*C$. Each of these points parametrizes a rank one degree zero Higgs bundle $(E_1, \phi_1), \ldots, (E_n, \phi_n)$ on C. In this case the Higgs bundle is given by the direct sum $\bigoplus_i (E_i, \phi_i)$. The parabolic structure is constructed by choosing a generic line l in the fiber E_0 over the origin, not contained in any subspace given by direct sums of the line bundles E_i. The parabolic Higgs bundle thus obtained is stable in the sense that (E, ϕ) has no proper Higgs subbundle which contains the chosen line. The isomorphism class in parabolic Higgs bundles in this case turns out to be independent of this choice.

Example 3.2.4. If $Z \subset T^*C$ is an "horizontal" subscheme supported on (p, t) the corresesoponing Higgs bundle is $(F_n \otimes L_p, t)$, where L_p is the line bundle parametrized by the point p and with t we mean the diagonal Higgs field $t \cdot id$.

If we have the monomial ideal $\mathcal{I} = (x^3, x^2y, y^2)$ we can represent it by means of a Young diagram

$$\begin{array}{|c|c|c|l}
\hline
1 & x & x^2 & x^3 \\
\hline
y & xy & \multicolumn{1}{l}{x^2y} & \\
\cline{1-2}
\multicolumn{1}{c}{y^2} & & &
\end{array}$$

The corresponding Higgs bundle will have underlying vector bundle $F_3 \oplus F_2$ and in general

$$\begin{array}{|c|c|c|c|c|l}
\hline
1 & x & x^2 & \ldots & & x^{i_0} \\
\hline
y & \ldots & & \multicolumn{3}{l}{x^{i_1}y} \\
\cline{1-3}
\multicolumn{1}{c}{\vdots} & & & & & \\
\cline{1-2}
y^k & \ldots & \multicolumn{4}{l}{x^{i_k}y^k} \\
\cline{1-2}
\multicolumn{1}{c}{y^{k+1}} & & & & &
\end{array}$$

will have underlying bundle $F_{i_0} \oplus F_{i_1} \oplus \ldots \oplus F_{i_k}$, the rows of the diagram corresponding to the indecomposable factor of the bundle.

3.2.1 Rank 2

From now on we denote $X = T^*C$, $\hat{X} = T^*\hat{C}$ and $p \in T^*C$ the point corresponding to the Higgs bundle $(\mathcal{O}, 0)$. We denote $\mathcal{O}_p$ the skyscraper sheaf with stalk $\mathbb{C}$ over p. Let $Z \subset X$ be a 0-dimensional subscheme of length 2 with support p and $\mathcal{O}_Z$ its structure sheaf. In particular this means that $dim\ H^0(Z, \mathcal{O}_Z) = 2$. The structure sheaf of a 0-dimensional subscheme of length n is a torsion sheaf on X that is always S-equivalent to a sheaf of the form $\mathcal{O}_{p_1} \oplus \ldots \oplus \mathcal{O}_{p_n}$, so in our case $\mathcal{O}_Z$ fits into the short exact sequence

$$0 \to \mathcal{O}_p \to \mathcal{O}_Z \to \mathcal{O}_p \to 0$$

and we can see $\mathcal{O}_Z$ as a non trivial extension in $\mathrm{Ext}^1_X(\mathcal{O}_p, \mathcal{O}_p)$;

Under these isomorphisms, the Hilbert-Chow morphism

$$Hilb^n(X) \to Sym^n(X)$$

is obtained by sending the ideal sheaf $\mathcal{I}_Z$ to the S-equivalence class of $\mathcal{O}_Z$.

In this case, applying the relative Fourier Mukai tansform, $\Phi_{\mathbb{A}^1}(\mathcal{O}_p) = \mathcal{F}$ is the trivial line bundle supported on the zero section of T^*C.
In this case g is affine so g_* is an exact functor and denoting $\mathcal{M} = \Phi_{\mathbb{A}^1}(\mathcal{O}_Z)$ we get

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, "g_*"] & \mathcal{M} \arrow[r] \arrow[d, "g_*"] & \mathcal{F} \arrow[r] \arrow[d, "g_*"] & 0 \\ 0 \arrow[r] & \mathcal{O} \arrow[r] & M \arrow[r] & \mathcal{O} \arrow[r] & 0 \end{tikzcd}$$

Now we look at the rank two extensions fitting in the above exact sequence.

First type The first case is a pair (E, ϕ) where $E = \mathcal{O}^{\oplus 2}$ and $\phi = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$.
In this case α can not be 0, giving rise to a trivial extension of Higgs bundles, otherwise for any $v \in E_0$ there would be a proper subbundle $L \subset E$ with $\phi(L) \subset L$ and $v \in L_0$. Furthermore, for any $\alpha \in \mathbb{C}^*$ these Higgs bundles are isomorphic: acting by automorphisms of the bundle we can reduce the Higgs field to its normal Jordan form fixing $\alpha = 1$. The automorphisms of $\mathcal{O}^2$ fixing the form of the Higgs field are

$$s = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$$

with $\alpha \neq 0$. So we can choose the vector v to be $(0, 1)^T$, as every vector $(a, b)^T$ with $b \neq 0$ is equivalent to this one (b can not be zero otherwise we could find a destabilizing subbundle).

Second type The second case is a pair (E, ϕ) where $E = F_2$ and $\phi = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$ with $\alpha \neq 0$. In this case for any two different values of $\alpha \in \mathbb{C}^*$ these Higgs bundles are not isomorphic, because any automorphism of F_2 fixes the form of the Higgs field and we can choose the vector $v = (0, 1)^T$, that is equivalent to any vector $(a, b)^T$ with $b \neq 0$.

Third type The third case is the limit of the previous one, when α vanishes. Here $E = F_2$ and $\phi = 0$, and the choice of a vector $v = (a, b)^T$ with $b \neq 0$ makes the Higgs bundle stable. The isomorphism class is independent of this choice.

In terms of torsion free sheaves on the spectral curve, the first type corresponds to a sheaf supported on the nonreduced part of C^2, the others correspond to generalized line bundles on C^2, parametrized by $H^1(C, \mathcal{O}_C)$. In particular the third type corresponds to the trivial class.

The sheaf O_Z has the structure of module over the local ring $\hat{\mathcal{O}}_p = C[[z, y]]$ where z is a local coordinate on C and y is the coordinate along the fiber of the cotangent bundle.

For the length 2 case, the punctual Hilbert scheme H_2 is a $\mathbb{P}^1$: all the ideals in $\mathbb{C}[[z, y]]$ of colength 2 can be reduced to the form $I_a = (z^2, az + y)$ or $I_\infty = (z, y^2)$ where a is an affine parameter. We will denote the corresponding subschemes by Z_a and Z_∞. Recall that in this case

$$T_pX \cong H^1_{Dol}(C) = H^1(C, \mathcal{O}) \oplus H^0(C, \mathcal{O})$$

In this case $\mathbb{C}[[z, y]]/I$ is equipped with a structure of $\mathbb{C}[[z, y]]$-module, that we can describe in terms of linear algebra by two commuting endomorphisms.

Proposition 3.2.5. *Under the correspondence given by the relative Fourier Mukai transform:*

1. *the subcheme Z_0 corresponds to the Higgs bundle $(F_2, 0)$;*
2. *the subschemes Z_a correspond to the Higgs bundles $(F_2, A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix})$;*
3. *the subscheme Z_∞ corresponds to the Higgs bundle $(\mathcal{O}^{\oplus 2}, A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})$.*

Proof. The subscheme Z_0 is horizontal and we have already seen that it corresponds to $(F_2, 0)$. In this case in fact $I_0 = (z^2, y)$ and

$$\mathbb{C}[[z, y]]/(y) \longrightarrow \mathbb{C}[[z, y]]/(z^2, y)$$

gives an embedding of the corresponding subscheme Z into the fiber $X_0 = \pi^{-1}(0)$ so we can use Lemma 3.1.1 to compute its image via the relative Fourier Mukai transform and we find the irreducible bundle F_2 supported on the nonreduced part of the spectral curve.

The subscheme Z_∞ is the only subscheme such that $p_*\mathcal{O}_Z = \mathcal{O}_o^{\oplus 2}$ is decomposable. In fact the ring $\mathbb{C}[z, y]/(z, y^2)$ has the structure of a decomposable rank 2 $\mathbb{C}[z]$-module. Moreover the equivalence gives an isomorphism of complex vector spaces

$$T_pX \cong Ext^1(\mathcal{O}_p, \mathcal{O}_p) \cong Ext^1(\phi^1_{\mathbb{A}}(\mathcal{O}_p), \phi^1_{\mathbb{A}}(\mathcal{O}_p))$$

and this gives an insomorphism

$$\mathbb{P}(T_pX) \cong \mathbb{P}(Ext^1_{T^*C}(\mathcal{O}_C, \mathcal{O}_C))$$

Note that in particular

$$Ext^1_{T^*C}(\mathcal{O}_C, \mathcal{O}_C) \cong Ext^1_{C^2}(\mathcal{O}_C, \mathcal{O}_C)$$

$\square$

We can compare this cases with the result of Lemma 2.1.6: we recall that for the torsion free sheaf $\mathcal{M}$ supported on the spectral curve the kernel of $\phi : \mathcal{O}_X \to End(\mathcal{M})$ is (0) or (N):

i) $Ker = (0) \Leftrightarrow \mathcal{M}$ is a generalized line bundle;

ii) $Ker = N \Leftrightarrow \mathcal{M} = i_*\mathcal{E}$ where $i : C_{red} \to C^2$ and $\mathcal{E}$ is a rank 2 vector bundle on $C_{red} \cong C$.

In the second case the $\mathcal{O}_X$ module structure on $\mathcal{M}$ is trivial, i.e. it factors through the $\mathcal{O}_C$ structure. So the corresponding Higgs bundle is just $(\mathcal{E}, 0)$. In particular if $\mathcal{E}$ is decomposable there is no parabolic datum that makes the Higgs bundle stable, so the only stable Higgs bundle of this kind is $(F_2, 0)$.
In the first case the nilradical $N = (y)$ does not act trivially on $\mathcal{M}$ thus $y \in \mathcal{O}_X \subseteq End(\mathcal{M})$ gives a nonzero nilpotent element that encodes the structure of $\mathcal{O}_X$ module of $\mathcal{M}$. So the corrisponding Higgs bundle is $(g_*\mathcal{M}, y)$.

Moreover the kernel is trivial so $\mathcal{M}$ is actually a line bundle on C^2: we have

$$0 \to H^1(C, \mathcal{O}_C) \to Pic(C^2) \xrightarrow{\pi} Pic(C) \to 0$$

so the generalized line bundles $\mathcal{M}$ such that $\pi(\mathcal{M}) = \mathcal{O}_C$ are parametrized by $H^1(C, \mathcal{O}_C) \cong \mathbb{C}$.

3.2.2 Rank 3

For the the length 3 case, we have to distinguish two cases, depending on $dim\ T_pZ$, i.e. the embedding dimension of the subscheme $Z \subset X$.
We call a 0-dimensional subscheme Z *curvilinear* if $dim\ T_pZ = 1$ and we denote H_3^c the subset of H_3 of curvilinear objects. In genearal H_n^c is an open dense of H_n but for $n = 3$ we know that $H_3 \setminus H_3^c$ is just a point, whose ideal is $\mathfrak{m}^2 = (z^2, yz, y^2)$.
Furthermore every ideal in H_n^c can be reduced to the form $I_{b,a} = (z^3, bz^2 + az + y)$ or $I_{b,\infty} = (y^3, by^2 + z)$, where b and a are affine coordinates. According to the previous notation, we will call the corresponding subschemes $Z_{b,a}$ and $Z_{b,\infty}$. In fact H_3^c turns out to be a $\mathbb{A}^1$-bundle over $H_2 \cong \mathbb{P}^1$.
Note that with this notation we have $I_{b,a} + \mathfrak{m}^2 = I_a$ and $I_{b,\infty} + \mathfrak{m}^2 = I_\infty$, that corresponds

to the inclusions $Z_a \subset Z_{b,a}$ and $Z_\infty \subset Z_{b,\infty}$ for each $b \in \mathbb{A}^1$. We can translate these relations into a surjection of $\mathcal{O}_X$-modules $\pi : \mathcal{O}_{Z_{b,a}} \to \mathcal{O}_{Z_a}$ that fits into the exact sequence

$$0 \longrightarrow \mathcal{O}_p \longrightarrow \mathcal{O}_{Z_{b,a}} \xrightarrow{\pi} \mathcal{O}_{Z_a} \longrightarrow 0$$

In particular the surjections $s : \mathcal{O}_X \to \mathcal{O}_{Z_a}$ and $s' : \mathcal{O}_X \to \mathcal{O}_{Z_{b,a}}$ that encode the $\mathcal{O}_X$-module structures are such that $s = \pi \circ s'$.
Applying the Mukai functor as before and the push forward under the map $g : T^*\hat{C} \to C$ we get

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, "g_*"] & \mathcal{N} \arrow[r] \arrow[d, "g_*"] & \mathcal{M} \arrow[r] \arrow[d, "g_*"] & 0 \\ 0 \arrow[r] & \mathcal{O} \arrow[r] & N \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

where the first row is an exact sequence of torsion sheaves on $T^*\hat{C}$ giving rise via the BNR correspondence to an exact sequence of Higgs bundles, and the second row is the sequence of the underlying holomorphic vector bundles.

We now want to look at the rank 3 stable triples. We start with considering the vector bundle E. The graded pieces must all be trivial, so if E is indecomposable it must be F_3. If it is decomposable it can either be $F_2 \oplus \mathcal{O}$ or $\mathcal{O}^{\oplus 3}$. We start with the last case. If $E = \mathcal{O}^{\oplus 3}$ we can put the Higgs field in its Jordan normal form; it can not be diagonalizable because in that case no vector would make it stable.

First type The first case is a pair (E, ϕ) where $E = \mathcal{O}^{\oplus 3}$ and

$$\phi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

We can choose the vector $v = (0,0,1)^T$, every vector $(a,b,c)^T$ with $c \neq 0$ being equivalent to this one : for every other vector $(a,b,0)$ we can always find a proper rank 2 destabilizing subbundle containing that vector.
This is the only Higgs bundle, up to isomorphism, with trivial underlying bundle. Even in this case the choice of the vector is unique up to isomorphism. The Higgs field can not have two Jordan block

$$\phi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

otherwise for every vector $v \in E_0$ we could find a proper rank 2 subbundle cointaning v.
This Higgs bundle can be considered as a nontrivial extension of the Higgs bundle corresponding to Z_∞ and correspond to the subscheme $Z_{0,\infty}$. Now we consider Higgs bundles whose underlying bundle is indecomposable.

Second type When $E = F_3$ the Higgs field must be of the form.

$$\phi = \begin{pmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}$$

And these correspond exactly to the subschemes $Z_{b,a}$ with $a \in \mathbb{C}$. In particular in the case $a = b = 0$ the Higgs field is diagonal.

Third type Here we consider Higgs bundles whose underlying vector bundle is $\mathcal{O} \oplus F_2$. The Higgs field must be

$$\phi = \begin{pmatrix} 0 & 0 & c \\ d & 0 & f \\ 0 & 0 & 0 \end{pmatrix}$$

with $c \neq 0$. If $d = 0$, acting by an automorphism of the bundle (exacltly $a_{11} = 1$, $a_{22} = c$ and $a_{21} = -f$) one can put the Higgs field in the form

$$\phi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so we get the Higgs bundle corresponding to Z_m described before.
If $d \neq 0$, we call $b = cd \in \mathbb{C}^*$ and by an an automorphism of the bundle (exacltly $a_{11} = 1$, $a_{22} = c$ and $a_{21} = -f$ and $a_{13} = 0$) one can put the Higgs field in the form

$$\phi = \begin{pmatrix} \phi & 0 & 1 \\ b & \phi & 0 \\ 0 & 0 & \phi \end{pmatrix}$$

i.e. a nontrivial extension of Z_∞ parametrized by the parameter $b \in \mathbb{C}^*$: these are the Higgs bundle corresponding to the subschemes $Z_{b,\infty}$.

Recall the notation: we denote H_3^c the subset of H_3 of curvilinear objects. $H_3 \setminus H_3^c$ is just a point Z_m, whose ideal is $\mathfrak{m}^2 = (z^2, yz, y^2)$.
Moreover every ideal in H_n^c can be reduced to the form $I_{b,a} = (z^3, bz^2 + az + y)$ or $I_{b,\infty} = (y^3, by^2 + z)$, where b and a are affine coordinates. According to the previous notation, we will call the corresponding subschemes $Z_{b,a}$ and $Z_{b,\infty}$. In fact H_3^c turns out to be a $\mathbb{A}^1$-bundle over $H_2 \cong \mathbb{P}^1$. We have $I_{b,a} + \mathfrak{m}^2 = I_a$ and $I_{b,\infty} + \mathfrak{m}^2 = I_\infty$, that correspond to the inclusions $Z_a \subset Z_{b,a}$ and $Z_\infty \subset Z_{b,\infty}$ for each $b \in \mathbb{A}^1$. We can translate these relations into surjection of $\mathcal{O}_X$-modules $\pi : \mathcal{O}_{Z_{b,a}} \to \mathcal{O}_{Z_a}$ that fits into the exact sequence

$$0 \longrightarrow \mathcal{O}_p \longrightarrow \mathcal{O}_{Z_{b,a}} \xrightarrow{\pi} \mathcal{O}_{Z_a} \longrightarrow 0$$

for $a \in \mathbb{C}$ and $a = \infty$.
The subscheme Z_m contains every length 2 subscheme, so we have surjections $\mathcal{O}_{Z_m} \to$

$\mathcal{O}_{Z_a}$ for every $a \in \mathbb{C}$ and $a = \infty$. The corresponding Higgs bundle has underlying vector bundle $\mathcal{O} \oplus F_2$ and its factor of automorphy is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

We have that $h^0(End(\mathcal{O} \oplus F_2)) = 5$ and these are exactly the matrices commuting with the factor:

$$\begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{22} \end{pmatrix}$$

If we consider the Higgs field

$$\begin{pmatrix} \phi & 0 & 1 \\ 0 & \phi & 0 \\ 0 & 0 & \phi \end{pmatrix}$$

we get a non trivial extension of the Higgs bundle corresponding to Z_0. Note that acting by conjugation with the automorphism with $a_{11} = a_{22} = 1$, $a_{21} = a$, the form of the Higgs field becomes

$$\begin{pmatrix} \phi & 0 & 1 \\ 0 & \phi & a \\ 0 & 0 & \phi \end{pmatrix}$$

so this Higgs bundle is also a nontrivial extension of the Higgs bundle corresponding to Z_a for any $a \in \mathbb{C}$.
Moreover $\mathcal{O} \subset F_2$ is invariant for the Higgs field and if we quotient by this Higgs subbundle we can recover it as a nontrivial extension of the Higgs bundle corresponding to Z_∞, i.e. $(\mathcal{O}^{\oplus 2}, \phi_A)$ where ϕ_A is not diagonalizable.
Note that in this special case the underlying vector bundle is decomposable and the Higgs field is also decomposable but we have an indecomposable Higgs bundle.

Proposition 3.2.6. *Under the correspondence given by the relative Fourier Mukai transform:*

1. *the subcheme $Z_{0,\infty}$ corresponds to the Higgs bundle $(\mathcal{O}^{\oplus 3}, \phi)$ where*

$$\phi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

2. *the subschemes $Z_{b,a}$ correspond to the Higgs bundles $(F_3, \phi_{b,a})$ where*

$$\phi_{b,a} = \begin{pmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}$$

3. the subschemes $Z_{b,\infty}$ *correspond to the Higgs bundles* $(\mathcal{O} \oplus F_2, \phi_b)$ *where*

$$\phi_b = \begin{pmatrix} 0 & 0 & 1 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

4. the subscheme Z_m *corresponds to the Higgs bundle* $(\mathcal{O} \oplus F_2, \phi)$ *where*

$$\phi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

3.3 Filtration

Recall that $X = T^*C$ is naturally equipped with an affine map $p : X \to C$ and if we denote z the coordinate on C, then any 0-dimensional subscheme of X is equipped with a natural filtration. It suffices as usual to consider the case of a subscheme supported on a point q of the fiber over the origin $o \in C$. We denote Z_k the subscheme of C whose ideal is (z^k) and X_k the fiber of p over Z_k. In particular each length n subscheme $Z \subset X$ supported on a point q over the origin is such that $Z \subset X_n$ and it is equipped with a "vertical" filtration

$$Z^{(1)} = Z \cap X_1 \ \subseteq \ Z^{(2)} = Z \cap X_2 \ \subseteq \ldots \subseteq \ Z^{(n)} = Z \cap X_n = Z$$

For instance for length 3 subschemes we get

$\mathcal{I}$	$\mathcal{I}_2 = \mathcal{I} + (z^2)$	$\mathcal{I}_3 = \mathcal{I} + (z)$
$(z^3, y + az + bz^2)$	$(z^2, y + az)$	(z, y)
$(y^3, z + by^2)$ $b \neq 0$	$(y^3, z + by^2)$	(z, y^2)
(y^3, z)	(y^3, z)	(z, y^3)
(z^2, zy, y^2)	(z^2, zy, y^2)	(z, y^2)

Chapter 4

Flat connections

Definition 4.0.1. A (linear) *connection* on a holomorphic vector bundle E over a holomorphic manifold M is a first order differential operator

$$\nabla : A^k(M,E) \to A^{k+1}(M,E)$$

such that

$$\nabla(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^q \alpha \wedge \nabla\beta$$

for all $\alpha \in A^q(M,\mathbb{C})$ and $\beta \in A^p(M,E)$.

In a local trivialization $\alpha(x) = \sum \sigma_j(x) \otimes e_j^\alpha(x)$ where $\sigma_j \in A^q(U_\alpha, \mathbb{C})$ so

$$\nabla\alpha = \sum d\sigma_j \otimes e_j + (-1)^q \sigma_j \wedge \nabla e_j$$

so it is enough to know ∇e_j, i.e. how ∇ acts on a base for local sections (0-forms) of E. In particular $\nabla e_j \in A^1(U_\alpha, E)$ so we can write locally

$$\nabla e_j = \sum \omega_{ij} \otimes e_i$$

where $\omega_{ij} \in A^1(U_\alpha, \mathbb{C})$, so that

$$\nabla\alpha = \sum_j d\sigma_j \otimes e_j + \sum_k (-1)^q \sigma_k \wedge \omega_{jk} \otimes e_j = \sum_j (d\sigma_j + \sum_k \omega_{jk} \wedge \sigma_k) \otimes e_j$$

and we have in local coordinates

$$\nabla\alpha = d\sigma + \Omega \wedge \sigma$$

If we change the trivialization from θ to θ' and $g : U_\alpha \to GL_r\mathbb{C}$, we have $\sigma' = g(\sigma)$ and

$$\nabla\alpha =_{\theta'} d\sigma' + \Omega' \wedge \sigma' =_\theta g^{-1}(d\sigma' + \Omega' \wedge \sigma') = g^{-1}(d(g\sigma) + \Omega' \wedge g\sigma) = d\sigma + (g^{-1}\Omega' g + g^{-1}dg) \wedge \sigma$$

so we get

$$\Omega = g^{-1}\Omega' g + g^{-1}dg$$

a so called *Gauge transformation.* Note that dg is a matrix of 1-forms with components (dg_{ij}).
If g is locally constant in particular d is globally defined and since $dg = 0$ we have that Ω is well defined globally as a 1-form with values endomorphisms.

A connection of type $(1,0)$ on a complex vector bundle E is a connection

$$\nabla' : A^{p,q}(X,E) \to A^{p+1,q}(X,E)$$

such that

$$\nabla'(\omega \wedge \alpha) = \partial\omega \wedge \alpha + (-1)^{deg(\omega)}\omega \wedge \nabla'\alpha$$

and similarly we define a connection ∇'' of type $(0,1)$. In local coordinates

$$\nabla' s = d\sigma + \Omega' \wedge \sigma$$
$$\nabla'' s = d\sigma + \Omega'' \wedge \sigma$$

where Ω' is a matrix of $(1,0)$ forms and Ω'' is a matrix of $(0,1)$ forms.

Every linear connection decompose uniquely as $\nabla = \nabla' + \nabla''$.
Locally $s = \sum \sigma^\alpha(z) \otimes e_j(z)$ with $\sigma^\alpha = g_{\alpha\beta}\sigma^\beta$ so

$$\bar\partial\sigma^\alpha = \bar\partial g_{\alpha\beta} \wedge \sigma^\beta + g_{\alpha\beta} \wedge \bar\partial\sigma^\beta = g_{\alpha\beta} \wedge \bar\partial\sigma^\beta$$

and since transition functions are holomorphic, it is globally defined.

In particular $\Omega^0(E) = ker(\bar\partial_E : A^0(E) \to A^{0,1}(E) \subset A^1(E))$ is called the set of *holomorphic sections* of E.

Let L be a complex line bundle on X. A trivialization τ induces an identification

$$A^k(X,\mathbb{C}) \to A^k(X,L)$$
$$\eta \to \eta\tau$$

and a connection $D_\tau : A^0(X,E) \to A^1(X,E)$ by the rule

$$D_\tau(f\tau) := df \wedge \tau$$

where $f \in A^0(X)$. This is the unique connection for which τ is parallel. With respect to this connection, an arbitrary connection D has the form $\nabla = D_\tau + \eta$ where $\eta \in A^1(X)$ and η acts by exterior multiplication. In particular the 1-form η is given by

$$\nabla(\tau) = \eta \wedge \tau$$

Let τ be a trivialization and let D_τ be the corresponding connection. If $g \in Aut(L)$ corresponds to a map $g : X \to \mathbb{C}^*$ then the action on a connection $D_\tau + \eta$ is given by:

$$g \cdot (D_\tau + \eta) := D_\tau + \eta + g^{-1}dg$$

In particular this action is independent of τ. Furthermore the $Aut(L)$-action preserves curvature.

4.1 Holomorphic connections

Let X be a complex holomorphic manifold of (complex) dimension n, E a holomorphic vector bundle of rank r on X and ∇ a connection on E compatible with the holomorphic structure of E. In this case we can see the action of ∇ on the holomorphic sections of E

$$\nabla : \ \Omega^0(E) \to A^{1,0}(E)$$

and locally

$$\nabla(f \cdot s) = \partial f \otimes s + f \otimes \nabla s$$

for any local holomorphic function f and local holomorphic section s. In general, even if ∇ is compatible with the holomorphic structure, the image of a holomorphic section is not necessarly holomorphic.

Definition 4.1.1. A holomorphic connection D on a holomorphic vector bundle E on X is a $\mathbb{C}$ linear map

$$D : \ \Omega^0(E) \to \Omega^0_X \otimes \Omega^0(E)$$

such that for any local holomorphic function f and local holomorphic section s

$$\nabla(f \cdot s) = \partial f \otimes s + f \otimes \nabla s.$$

Locally a holomorphic connection is of the form

$$Ds = \partial + A$$

where A is a matrix of *holomorphic* 1-forms. This shows that D induces a $\mathbb{C}$-linear map

$$D : \ A^0(E) \to A^{1,0}$$

and looks like the $(1,0)$ part of an ordinary connection. In fact $\nabla = \bar{\partial}_E + D$ defines an ordinary connection on E.

Proposition 4.1.2 ([21])**.** *There is a group isomorphism*

$$H^1_{DR}(C) = H^1_{DR}(C, (\mathbb{C}, d)) \cong Ext^1_{DR}((\mathbb{C}, d), (\mathbb{C}, d))$$

Proof. If (V, ∇) is an extension of $(\mathbb{C}, d)$ by itself in a suitable frame the connection matrix of ∇ will be of the form

$$A = \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix}$$

where $\omega \in A^1(C)$. Since the connection is flat, ω must be a closed 1-form. Even in this case acting by a gauge element (or equivalently choosing another frame compatible with the extension), the action on the matrix is

$$g \cdot A = g^{-1}Ag + g^{-1}dg.$$

To be compatible with the extension g must be of the form

$$A = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$$

so ω is replaced with $\omega + d\beta$. Hence the extension is independent of the choice of the representative $[\omega] \in H^1(C)$. □

4.2 Connections and factor of automorphy

If we have a constant $\mathbb{C}^*$-factor of automorphy it defines up to analytic equivalence a unique isomorphism class of vector bundles. However, when we choose a $\mathbb{C}^*$-factor in the equivalence class, the vector bundle comes equipped with a flat connection.
For instance we can choose for the bundle F_2 the $\mathbb{C}^*$-factor of automorphy

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

so that we consider on F_2 the connection ∇^0 whose flat section are the image of the constant sections of the trivial bundle on $\mathbb{C}$ via the quotient map. In particular flat sections are the functions $f : \mathbb{C} \to \mathbb{C}^2$ such that

$$\begin{cases} f(z+1) = f(z) \\ f(z+\tau) = A^{-1}f(z). \end{cases}$$

Moreover, we know that once we fix a connection ∇^0 on a bundle E, all the holomorphic connections are given by $\nabla = \nabla^0 + \phi$ where ϕ is an endomorphism of the bundle

$$\phi \in H^0(C, EndE)$$

4.3 Filtration

Recall that that $Y = C^\natural$ is naturally equipped with an affine map $p : Y \to C$ (see section 1.2) and denoting by z the coordinate on C, any subscheme, as in the case of T^*C, is equipped with a natural "vertical" filtration. We consider as usual the case of the fiber over the origin $o \in C$. We denote Z_n the subscheme of C whose ideal is (z^n)

and Y_n the fiber of p over Z_n. In particular each length n subscheme $Z \subset X$ supported on a point q over the origin is again equipped with a "vertical" filtration

$$Z^{(1)} = Z \cap Y_1 \ \subseteq \ Z^{(2)} = Z \cap Y_2 \ \subseteq \ldots \subseteq \ Z^{(n)} = Z \cap Y_n$$

Thus the flat bundle (E, ∇) parametrized by a subscheme Z comes equipped with a filtration

$$(E_1, \nabla) \subseteq (E_2, \nabla) \subseteq \ldots \subseteq (E_n, \nabla) = (E, \nabla)$$

This filtration is "vertical" in the sense that the graded objects obtained by the filtration are always of type $(\mathcal{O}^{\oplus k}, \nabla)$.

We remark that on T^*C we have used the notion of "horizontal" schemes because the coordinate z is well defined up to a multiplicative constant, while $C^\natural$ is not parallelizable and thus the only well defined notion is that of "vertical" subschemes.

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