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SYNCHRONIZATION PROBLEMS IN NETWORKS OF NONLINEAR AGENTS

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Abstract

Over the last years, consensus and synchronization problems have been a popular topic in the systems and control community. This interest is motivated by the fact that, in several fields of application, a certain number of agents is interacting or has to cooperate to achieve a certain task. Robotic swarms, sensor networks, power networks, biological networks are only few outstanding examples where networks of agents displays behaviors which can be modeled and studied by means of consensus and synchronization techniques. The etymology of *consensus* and *synchronization* refers in fact to the property that systems reach some sort of agreement on a certain parameter or in their state evolution.

Two main aspects should be considered in dealing with networks problems. First, how to model the agents forming the network: for instance, one could consider linear or nonlinear systems, homogeneous or heterogeneous dynamics. Second, how to model the interaction and exchange of information: usually graph theory tools are exploited. With this respect several scenario can be considered: for instance, static topology, switching or time-varying topology, but also dynamical systems as links between the agents, rather than static couplings.

In this thesis we consider a general class of networked nonlinear systems in different operating frameworks and design control architecture to force the systems to reach synchronization and consensus on a target behavior. In particular, we consider the case of homogeneous and heterogeneous nonlinear agents with a static communication topology and design a static *high-gain*-based diffusive coupling and an *internal model*-based regulator respectively, to solve the problem of consensus. Then, we analyze the case of dynamical links and show under which conditions, synchronization for homogeneous nonlinear systems can be achieved. Depending on the structure of the dynamic links at

hand, static and dynamic regulators (based on the concept *extended state observers*) are proposed. Furthermore, we address the problem of disconnected topology and switching topology and derive under which conditions agents reach *cluster synchronization* and synchronization respectively. Last, we consider the problem of a sampled exchange of information between the agents and design a triggering rule locally at each agent such that the overall network reaches synchronization.

Acknowledgment

After such an important *chapter* of my life, there are definitely too many people I should thank. Even though I prefer to do this kind of thing *face-to-face*, I want to share a small part of this thesis with some of them.

First and foremost, I would like to thank my advisor Prof. Lorenzo Marconi, for the endless patience, dedication and motivation he always showed me. His support and guide has been fundamental for every result I have achieved during my PhD. I will always keep him as a model in my future career.

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A big thanks goes to my companions Jano, Francesco and Corrado, for the amazing support in my *secondary* life. Without you, all I have accomplished, musically and as a person, would not have been possible.

Last but not the least, my biggest and warmest thanks goes to my parents, Domenico and Luciana, to my brother Giovanni and to my sister Chiara. No words could explain how thankful I am to all of you for everything my life is and will be.

This thesis is dedicated to my hometown, Castelnovo ne' Monti and to all the mountains around the world I've been hiking in these years.

Contents

Notation

- We denote with $\mathbb{R}$ the set of real numbers. We denote $\mathbb{R}_{\geq 0} = [0, \infty)$ and $\mathbb{R}_{>0} = (0, \infty)$.
- For a real number $a \in \mathbb{R}$ we denote with $|a|$ its absolute value.
- We denote with $\mathbb{C}$ the set of complex numbers. We denote with i the imaginary unit, i.e. $\mathrm{i} = \sqrt{-1}$. Given a complex number $c \in \mathbb{C}$ we denote with $\Re(c)$ its real part, and with $\Im(c)$ its imaginary part, so that $c = \Re(c) + \mathrm{i}\Im(c)$.
- For a complex number $c \in \mathbb{C}$ we denote with $|c|$ its absolute value defined as $|c| = \sqrt{\Re(c)^2 + \Im(c)^2}$.
- For a vector $x \in \mathbb{R}^n$, we denote with $\|x\|$ its Euclidean norm, i.e. $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$.
- For a matrix $A \in \mathbb{R}^{n \times m}$ we denote with $\|A\|$ its norm defined as $\|A\| = \sup\{\|Ax\|; x \in \mathbb{R}^m \text{ with } \|x\| = 1\}$.
- We define $\mathrm{diag}\,(a_1, \ldots, a_n) = \begin{pmatrix} a_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n \end{pmatrix}$.
- We denote with $0_{n \times m}$ a matrix of dimension $n \times m$ whose entries are all zeros.
- We denote with I_n the identity matrix of dimension $n \times n$. When the dimension is not ambiguous, it can be also defined simply as I.

- A triplet S, B, C is said to be in prime form when

$$S = \begin{pmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ 0 & 0_{1\times(n-1)} \end{pmatrix}, \qquad B = \begin{pmatrix} 0_{(n-1)\times 1} \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0_{1\times(n-1)} \end{pmatrix}.$$

  Without misunderstanding we use the notation S in prime form, the pair S, B or the pair S, C in prime form.

- $A \otimes B$ denotes the Kronecker product of the two matrices $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{p\times q}$, namely

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1m}B \\ \vdots & & & \vdots \\ a_{n1}B & a_{n2}B & \dots & a_{nm}B \end{bmatrix}$$

  It has the following properties:

  1) $A \otimes (B + C) = A \otimes B + A \otimes C$
  2) $(A + B) \otimes C = A \otimes C + B \otimes C$
  3) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$
  4) if A, B, C, D are matrices of compatible size
     $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

[...] I'll tell you what hermits realize. If you go off into a far, far forest and get very quiet, you'll come to understand that you're connected with everything. [...]

Alan Watts

1 Introduction

When I first came into the quote above, I thought it could have been the best statement to start my thesis with. It summarizes the reasons why the scientific community is paying so much attention to the analysis and understanding of networked systems. In the everlasting process of discovery humanity is immersed into, we have realized that in order to understand the complexity of the world we live in, it is not possible to look at a single part. We have to consider each single aspect of life in its intrinsic interconnection with its surroundings and to understand the underlying structure that links us to each other. Friedrich Nietzsche used to say: *"Invisible threads are the strongest ties"*. Connection is everywhere, whether we look at the world around us from a material or a spiritual perspective.

In recent years multi-agent systems and networked systems have attracted a lot of interest from the control community, due to the challenging problems they convey and the many application areas which they cover. The increasing interest in practical frameworks concerning multi-agent systems is becoming a huge motivation to adapt *well know* theoretical results to understand and to control the behavior of networks.

In particular, we think about all the possible situations in which a certain number agents are required to collaborate and to achieve some sort of agreement in order to fulfill the assigned task. With this respect, synchronization and consensus are a typical mathematical representation of such a scenario, in which agents interact and communicate with each other and aims to find an *accordance* in their behavior. Robotic swarms, power networks, sensors networks are some outstanding examples where synchronization and consensus can play a fundamental role in achieving the desired goal. Furthermore, a mathematical analysis of networks finds a lot more fields of application, not merely engineering and sometimes even *fancy*. Usually network analysis tools are exploited to model complex systems dynamics, such as animal collective behavior, traffic monitoring, social networks, opinion dynamics, economic modeling and many others.

In the analysis of multi-agent systems, there are essentially two main components to be considered: the agents dynamics and the communication/interaction through which the agents are interconnected. The former is deeply interlaced with the particular framework of the analysis. The latter involves the representation of the exchange of information by means of graph theory and is a fundamental tool in the analysis of networked systems. It is indeed possible to consider consensus and synchronization problems from different perspective, for instance whether the dynamics of the agents is linear or nonlinear, whether the communication topology is static or dynamically changing, whether the interaction between the agents is static or dynamic.

The purpose of this thesis is to provide an extensive overview on the topics related to networks of nonlinear agents. The following summary shows the outline of this thesis and briefly recaps its main contributions.

Chapter 2: In this chapter we review basic concepts about graph theory and graph representations. Most of the concepts and results presented in this chapter are available in the literature: the reader is referred to Godsil and Royle (2004) for a theoretical perspective and to Wieland (2010) for a more engineering point of view. We focus our attention on:

- graph representation and basic definitions
- the use of matrix to represent graphs
- the possibility the represent complex graphs as a connection of smaller sub-graphs

Chapter 3: In this chapter, we focus our attention on the problem of synchronization and consensus for nonlinear systems networks. We present novel result based on Isidori et al. (2014), Isidori et al. (2013). In particular:

- we consider a network of homogeneous nonlinear agents and obtain sufficient conditions to achieve synchronization

- we consider a network of heterogeneous nonlinear agents and by means of internal model principle, we design a control architecture capable of inducing consensus in the network

Chapter 4: This chapter is devoted to the analysis of networks with dynamical links. Conventionally edges between nodes are considered as ideal *wires*, but in several applications such as power networks, links are dynamical systems themselves and modify the structure of the network deeply. The results presented in this chapter are partly inspired by Casadei et al. (2014a), but a substantial part is completely novel. In particular:

- we first consider the case of dynamical edges with algebraic connection between their input and their outputs: this scenario can be seen as a natural extension of the results in Chapter 3
- then, we consider the case of *purely* dynamic links connecting the agents in the network and propose a dynamic control architecture, including local observers at each node, to achieve of synchronization

Chapter 5: In this chapter we consider the problem of networks with disconnected topology. In particular:

- we study the behavior of nonlinear systems when disconnected topology occurs
- then, we define a control architecture capable of enforcing *clustered* consensus inside a disconnected network network
- last, we consider the case of switching topology and by means of hybrid tools, derive sufficient conditions under which agents achieve synchronization despite time intervals in which the topology is not connected

The results in this chapter are based on Casadei et al. (2014b) and Casadei et al. (2015): however, a major section is completely novel.

Chapter 6: In the last chapter we consider the problem of networks in which the exchange of information is sampled. We define an event triggering rule locally at each agent through which agents sample their neighbors information only when *necessary*. The results in this chapter are completely novel.

Supplementary material is provided in the appendices, aiming to make this thesis as *self-contained* as possible.

[...] so the movements of most of the people who live in cities have lost their connection with the earth; they hang, as it were, in the air, hover in all directions, and find no place where they can settle.

Rainer Maria Rilke

2

Graph Theory

In the study of synchronization and consensus problems, a network of dynamical systems is interconnected according to a specific *communication topology*. The communication topology describes the flow of information between the agents and thus determines the set of neighbors for each agents. Based on the exchange of information, we aim to define suitable control laws to achieve synchronization between the agent state or their outputs.

Communication can be interpreted from different perspectives: on a higher and more general level, communication represents the exchange of information between agents. However, it is also possible to consider communication from a more physical perspective, as the exchange of relative sensing or a *material* connection between the systems involved. This is especially the case when considering dynamical links: in this scenario, links are not considered as ideal connection between agents but rather introduces a dynamical effect to the whole network.

Nevertheless, the analysis of the network by means of graph theory is a fundamental step in establishing control architecture to achieve synchronization and consensus between agents. In this chapter, we introduce basic concepts about graph theory which will be used all over the thesis. For further details on the topic, the reader is referred to Godsil and Royle (2004).

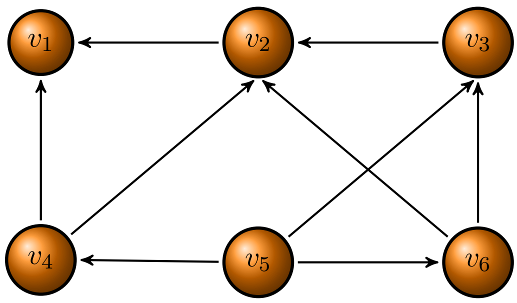

Figure 2.1: Example of a directed graph

2.1 Basic definitions about graphs

In general, graphs are composed of *nodes* (or *vertexes*) and *edges* (or *arcs*) connecting the nodes. The edges can be directed or undirected, weighted or unweighted. Directed edges implies that the flow of information is from an agent (the *tail* of the arc) to a receiver (the *head* of the arc): in contrast with oriented edges, which just have an arbitrary representative orientation, directed edges implies that the flow of information is not bidirectional. Weighted edges embeds a further information about the information exchange, in the sense that each edge has a *trust/relevance/importance* relatively to the other agents.

These different properties about edges generates different families of graphs:

- *unweighted graphs*: the edges are not directed, in the sense that every connection between agents has to intended as bidirectional and no weight is applied to the arcs;
- *wighted graphs*: weights are applied to the different edges;
- *unwighted digraphs*: the edges have a precise direction representing the flow of information from an agent to another;
- *weighted digrpahs*: the more general family of graphs, in which the edges are both weighted and directed

In this thesis, we deal with *weighted digrpahs* so all the results which are given about graphs can be extended to the other families of graphs without loss of generality. For the sake of generality, we will give basic definitions about graphs in the time varying framework: when we consider fixed graphs, the dependence on time will be omitted.

Definition 2.1. *A time varying weighted digrpahs is a triplet* $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}(t), A(t)\}$ *in which:*

- $\mathcal{V}$ *is a set of* N nodes $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$*, one for each of the* N *agents in the set.*

- $\mathcal{E}(t) \subset \mathcal{V} \times \mathcal{V}$ *is a set of* edges *that models the interconnection between nodes, according to the following convention:* (v_k, v_j) *belongs to* $\mathcal{E}(t)$ *if there is a flow of information from node* j *to node* k.
- *the flow of information from node* j *to node* k *is* weighted *by the* (k, j)*-th entry* $a_{kj}(t) \geq 0$ *of the adjacency matrix* $A(t) \in \mathbb{R}^{N \times N}$.

Furthermore, it satisfies the following properties

P1) there are no self-loops, i.e. that $(v_k, v_k) \notin \mathcal{E}$.

P2) the elements $a_{kj}(\cdot) : R \to R_+, k, j \in \mathbb{N}_N$ *of the adjacency matrix* A *are non-negative, piecewise continuous, and bounded functions of time.*

In the definition above, $v_k \in \mathcal{V}$ with $k \in \mathbb{N}_N$, represent one of the N nodes of the network. The edge $(v_k, v_j) \in \mathcal{E}(t)$ represents the connection and thus the flow of information from agent k (conventionally the *tail*) to agent j (conventionally the *head*).

The elements $a_{kj}(t)$ of the adjacency matrix $A(t)$ represents the weight associated to the edge (v_k, v_j). The case of *un-weighted* graphs can be derived from the definition, by imposing that $a_{kj}(t) = \{0, 1\}$, meaning that $a_{kj}(t) = 1$ when $(v_k, v_j) \in \mathcal{E}(t)$ and $a_{kj}(t) = 0$ if there is no connection from agent k to agent j. Property $P2$) is relevant only in the case of time varying graphs and basically requires the topology to have a finite number of discontinuity in a closed intervals of its domain and at the points of discontinuity the left and right limits exist.

It is worth noticing that the adjacency matrix $A(t)$ depends on the particular numbering of the vertexes. However its spectral properties are invariant on the particular numbering, thus the results given in this chapter do not depend on the chosen numbering.

The set of *neighbors* of node v_k is the set $\mathcal{N}_k(t) = \{v_j \in \mathcal{V} : a_{kj}(t) \neq 0\}$. A *path* from node v_j to node v_k is a sequence of r distinct nodes $\{v_{\ell_1}, \ldots, v_{\ell_r}\}$ with $v_{\ell_1} = v_k$ and $v_{\ell_r} = v_j$ such that $(v_i, v_{i+1}) \in \mathcal{E}$. Using the concept of path we can also define the set of *descendants* of node v_k at time t as

$$\mathcal{D}(v_k, \mathcal{G}(t)) = \{v_j \in \mathcal{V} : \exists \text{ a path from } v_k \text{ to } v_j \text{ at time } \ t\}$$

For instance, in the example of Figure 2.1, the neighbors of agent 4 are $\mathcal{N}_4 = \{1, 2\}$, while agent 1 has no neighbors. The path from node 5 to node 1 is the sequence of $(v_5, v_6) \to (v_6, v_3) \to (v_3, v_2) \to (v_2, v_1)$.

It comes without saying that, in order to achieve synchronization and consensus between a set of agents, it is necessary to assume that a certain flow of information is available to all agents. This consideration involves the *so called* connectedness of a graph.

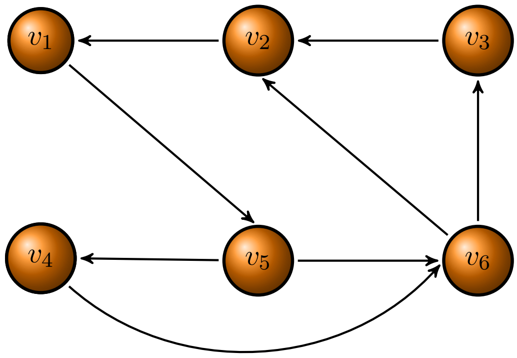

Figure 2.2: Example of a directed *strongly connected* graph

Depending on the particular point of view, it is possible to introduce different definitions of connectedness.

Definition 2.2. *A graph $\mathcal{G}(t)$ is said to be* connected *at time t, if there is a node v such that, for any other node $v_k \in \mathcal{V} \setminus \{v\}$, there is a path from v to v_k, and v is the centroid of the graph. In other words, there exists a node v such that all $v_k \in \mathcal{V} \setminus \{v\}$ are descendants of v, i.e. $v \cup \mathcal{D}(v, \mathcal{G}(t)) = \mathcal{V}$.*

A graph $\bar{\mathcal{G}}(t)$ obtained by removing all the edges which do not belong to any of the paths from node v_k to all the others is called *spanning tree*: the node v is the root of the spanning tree. So equivalently to Definition 2.2, we can say that a graph $\mathcal{G}(t)$ is said to be *connected* at time t, if there exists a spanning tree at time t.

A stronger notion of connectedness is now introduced: in particular we consider the case in which information can be exchanged between each pair of agents in the network.

Definition 2.3. *A graph $\mathcal{G}(t)$ is said to be* strongly connected *at time t, if all nodes $v_k \in \mathcal{V}$ are centroid of the graph. In other words, taken any pair of nodes v_i, v_j, $v_i \in \mathcal{D}(v_j, \mathcal{G}(t)) = \mathcal{V}$, $v_j \in \mathcal{D}(v_i, \mathcal{G}(t)) = \mathcal{V}$.*

The example of Figure 2.1 is a connected graph since there exists a path from vertex v_5 to all other vertexes. However, it is not strongly connected, since not all the nodes are centroids of the graph. Instead the graph depicted in Figure 2.2 is strongly connected, since from every node there is a path to all the others.

Besides instantaneous connection, it is often interesting to consider connection over a certain time interval. To this purpose, we introduce the concept of T-averaged adjacency matrix

$$A_T(t) = \frac{1}{T} \int_t^{t+T} A(\tau) d\tau$$

with $T \in \mathbb{R}_+$, and consequently derive the T-averaged graph, conventionally called *union graph*, as the triplet $\mathcal{G}_T(t) = \{\mathcal{V}, \mathcal{E}_T(t), A_T(t)\}$.

Figure 2.3: Example of a connected *union graph*

Definition 2.4. *A graph $\mathcal{G}(\cdot)$ is said to be* uniformly connected, *if there is a node $v \in \mathcal{V}$ and a finite time horizon $T \in \mathbb{R}_+$, such that v is a centroid of the union graph $\mathcal{G}_T(\cdot) = \{\mathcal{V}, \mathcal{E}_T(\cdot), A_T(\cdot)\}$.*

An example of *uniform connectivity* is presented in Figure 2.3: the union graph of the two topologies switching between each other over any time interval $T > 0$ is connected. Similarly to previous definitions, it would be possible to introduce the concept of *uniform strong connectedness*: this definition is a trivial extension of previous definitions, thus it is omitted.

2.2 Laplacian matrix and its properties

A fundamental tool in order to study connectivity and thus the possibility of inducing synchronization in a network of agents is the use of matrix to represent the graph. Given an adjacency matrix, describing the graph $\mathcal{G}(t)$ (modulo vertex permutation), it is possible to define another matrix, the *Laplacian* matrix, which turns out to be an *expressive* description of the connectivity properties of the graph.

Given the *degree* matrix $D(t) \in \mathbb{R}^{N \times N}$ as

$$D(t) = \mathrm{diag}(A(t)1_N)$$

the so-called *Laplacian* matrix $L(t)$ of the graph is defined as

$$L(t) = D(t) - A(t)$$

or, equivalently as

$$\ell_{kj}(t) = \begin{cases} -a_{kj}(t) & \text{for } k \neq j \\ \sum_{j=1}^{N} a_{kj}(t) & \text{for } k = j \end{cases}$$

By definition, the diagonal entries of L are non-negative, the off-diagonal entries are non-positive and the sum of all entries on each row is zero. A matrix with these properties is usually referred to as a *Metzler* matrix (see Luenberger (1979)).

Example 2.1. The Laplacian Matrix L associated to the graph depicted in Figure 2.2 is the 6×6 matrix

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

where, for the sake of simplicity, we dropped the dependence on the weight and considered all $a_{kj} = \{0, 1\}$ depending on whether the connection between two nodes is present or not. △

As a consequence, the all-ones N-vector $1_N = \text{col}(1, 1, \ldots, 1)$ is an eigenvector of $L(t)$, associated with the eigenvalue $\lambda_1 = 0$. In the synchronization framework, this eigenvector describes the consensus subspace, *i.e.* the subspace where all agents have reached an agreement on a certain variable. The corresponding eigenvalue $\lambda_1 = 0$ means that, once synchronization is achieved, the defined *coupling input* should be zero. Let the other (possibly nontrivial) $N-1$ eigenvalues of $L(t)$ be denoted as $\lambda_2(L(t)), \ldots, \lambda_N(L(t))$.

Lemma 2.1. *A graph $\mathcal{G}(t)$ is connected if and only if its Laplacian matrix $L(t)$ has only one trivial eigenvalue $\lambda_1 = 0$ and all other eigenvalues $\lambda_2(L(t)), \ldots, \lambda_N(L(t))$ have positive real parts.*

The proof of this result can be found, for instance, in Wieland (2010) (Corollary 2.14). In other words, by ordering the eigenvalues of $L(t)$ in ascending order, a graph $\mathcal{G}(t)$ is connected if and only if $\Re(\lambda_2(L(t))) > 0$. It is worth noticing that, even though it might be considered as the dual of *algebraic connectivity* for directed graphs, $\Re(\lambda_2(L(t))))$ does not have many properties that the former has. For instance, when adding and edge to an undirected graph, the *algebraic connectivity* increases, while in the case of directed graphs $\Re(\lambda_2(L(t)))$ might decrease.

By means of the Geršgorin Disk Theorem, it is possible to characterize the Laplacian matrix spectrum and bound its nontrivial eigenvalues. Define the maximum degree of $\mathcal{G}(t)$ as

$$d_{max}(t) = \max_{k \in \mathbb{N}_N} d_{kk}(t)$$

with d_{kk} the diagonal entris of $D(t)$. The spectrum of $L(t)$, denoted by $\sigma(L(t))$ is thus bounded as

$$\sigma(L(t)) \subset \{x \in \mathbb{C} : \; |x - d_{max}(t)| \leq d_{max}(t)\} \subset \mathbb{C}_+$$

This means that all the eigenvalues of $L(t)$ are contained inside a disk centered in $d_{max} + j0$, with radius d_{max} and thus in the right-half of the complex plane. The definition we provided depends explicitly on the time t: however, by recalling $P2$) in Definition 2.1, since the elements of $A(t)$ are bounded it is possible to derive a *worst case* bound, depending on the maximum value the elements of $A(t)$ can take.

2.3 Independent Connected components of a graph

When studying complex networks, *i.e.* when the number of nodes N is high, it can be helpful to separate the analysis of the graph into smaller groups of agents. This is often the case in the *model reduction* approach to network, where groups of agents subject to being connect, are considered as a unique system and then coupled to the other groups (see Ishizaki et al. (2014), Monshizadeh and van der Schaft (2014)). Even in dealing with disconnected topology, dividing the graph into sub-graphs subject to being connected can be really helpful (see Chapter 5.2), in order to conclude something on the behavior of the whole network.

For this reason in this section, we give basic definitions and tools to analyze the components of a graph. We start by recalling the definition of an *independent strongly connected component* of a di-graph $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}(t), A(t)\}$ presented in (Wieland, 2010).

Definition 2.5. *An independent connected component of a di-graph $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}(t), A(t)\}$ is the maximal subgraph $\tilde{\mathcal{G}}(t) = \{\tilde{\mathcal{V}}, \tilde{\mathcal{E}}(t), \tilde{A}(t)\}$ that is strongly connected and such that there is no edge in $\mathcal{E}$ with a tail outside $\tilde{\mathcal{V}}$ and the head in $\tilde{\mathcal{V}}$.*

In Figure 2.4, the sub-graphs $\tilde{\mathcal{G}} = \{v_1, v_2, v_3, v_4\}$ and $\tilde{\mathcal{G}} = \{v_5, v_6, v_7, v_8\}$ are *ISCC*, since the nodes are strongly connected and there is no edge with tail outside the *ISCC* and head inside it.

However, it is possible to give another and *milder* definition which asks the component to be *simply* connected.

Definition 2.6. *An independent connected component of a di-graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, A\}$ is the*

Figure 2.4: Example of a graph composed by two ISCC component

maximal subgraph $\tilde{\mathcal{G}} = \{\tilde{\mathcal{V}}, \tilde{\mathcal{E}}, \tilde{A}\}$ that is connected and such that there is no edge in $\mathcal{E}$ with a tail outside $\tilde{\mathcal{V}}$ and the head in $\tilde{\mathcal{V}}$.

In Figure 2.5, beside the *ISCC* described befor, the sub-graphs $\tilde{\mathcal{G}} = \{v_5, v_6, v_8\}$ is an *ICC*, since the nodes are connected (namely, there is a path from node v_8 to node v_6 and v_5) and there is no edge with tail outside the *ICC* and head inside it.

Let $c \geq 1$ be the number of independent connected components of the di-graph. We can partition the whole graph in $c + 1$ subgraphs: the first c subgraphs represent the independent connected components. The additional one is a "residual graph" (possibly empty) composed by "residual vertexes" that might have incoming edges from the independent connected components but without outgoing arcs toward independent connected components.

In the example in Figure 2.5, the sub-graph $\tilde{\mathcal{G}} = \{v_7\}$ is a residual component, since he has incoming edges from the two *ICC*s, $\tilde{\mathcal{G}}_1 = \{v_1, v_2, v_3, v_4\}$ and $\tilde{\mathcal{G}}_2 = \{v_5, v_6, v_8\}$.

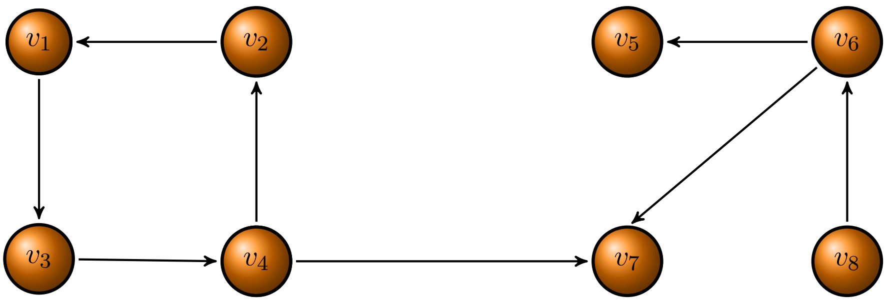

Figure 2.5: Example of a graph with *ICC*s and residual component

By relabeling the vertexes of the whole graph, so that the vertexes of independent connected components are consecutive and the residual subgraph vertexes are confined at the end, it turns out that the adjacency matrix of a non connected graph can always be written as

$$A^i = \begin{pmatrix} A_1^i & 0 & \cdots & 0 & 0 \\ 0 & A_2^i & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_c^i & 0 \\ \star & \star & \cdots & \star & A_{res}^i \end{pmatrix}$$

where the A_j^i, $j = 1, \ldots, c$, and A_{res}^i are the adjacency matrices of the subgraphs and the $\star$ denotes the weight of the incoming edges to the residual sub-graph. Similarly, the Laplacian matrix takes the form

$$L = \begin{pmatrix} L_1 & 0 & \cdots & 0 & 0 \\ 0 & L_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & L_c & 0 \\ & & \Gamma & & L_{res} \end{pmatrix}$$

where Γ denotes the matrix of incoming edges to the residual component.

We observe that the eigenvalues of the Laplacians L_1, ..., L_c are one zero and the rest are positive. It turns out that the eigenvalues of L_{res} are all positive and thus L_{res} is always invertible [1]. This fact is a necessary condition for the next lemma, in which N_1, ..., N_c, and N_{res} denote the number of vertexes in each of the c independent connected components.

Lemma 2.2. *Let $V \in \mathbb{R}^{N \times c}$ be the matrix defined as*

$$V = \begin{pmatrix} 1_{N_1 \times 1} & 0 & \ldots & 0 \\ 0 & 1_{N_2 \times 1} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1_{N_c \times 1} \\ \gamma_1 & \gamma_2 & \cdots & \gamma_c \end{pmatrix}$$

with $[\gamma_1, \gamma_2, \ldots, \gamma_c]$ defined as

$$[\gamma_1, \gamma_2, \ldots, \gamma_c] = -L_{res}^{-1}\Gamma \begin{pmatrix} 1_{N_1 \times 1} & 0 & \ldots & 0 \\ 0 & 1_{N_2 \times 1} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1_{N_c \times 1} \end{pmatrix}$$

[1] For the proof, see Wieland (2010) Appendix A.3.2, proof of theorem 2.13.

where $\begin{pmatrix} \Gamma & L_{res} \end{pmatrix}$ *denotes the matrix obtained by extracting the last* N_{res} *rows of* L*. Then the following holds:*

- $dim(Ker(L)) = c$;
- *the columns of* V *form a basis of* $Ker(L)$.

Proof. The c vectors columns of V are clearly linearly independent. They are in the null space of the matrix obtained by extracting the first $N_1 + N_2 + \ldots + N_c$ rows from the Laplacian L for all possible γ_i. They are thus in the null space of L if

$$\Gamma \begin{pmatrix} 1_{N_1\times 1} & 0 & \ldots & 0 \\ 0 & 1_{N_2\times 1} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1_{N_c\times 1} \end{pmatrix} + L_{res} \begin{pmatrix} \gamma_1 & \gamma_2 & \ldots & \gamma_c \end{pmatrix} = 0$$

from which the result follows using the definition of the ψ_i, $i = 1, \ldots, c$. □

With the previous result in hand, we define the transformation T

$$T = \begin{pmatrix} V & \mathcal{T} \end{pmatrix} \tag{2.1}$$

with $\mathcal{T} \in \mathbb{R}^{N-c\times N-c}$ an opportune matrix to be determined.

By applying such a transformation to L we obtain $\tilde{L} = T^{-1}LT$, which has the following structure

$$\tilde{L} = \begin{bmatrix} 0_{c\times c} & L_{12} & 0_{c\times N_{res}} \\ 0_{N-c-N_{res}\times 1} & L_{22} & 0_{N-c-N_{res}\times N_{res}} \\ 0_{N_{res}\times c} & \tilde{\Gamma} & \tilde{L}_{res} \end{bmatrix}$$

for some appropriately defined L_{12}, L_{22}, $\tilde{L}_{res}$, with all the eigenvalues of L_{22} and $\tilde{L}_{res}$ that are positive.

By opportunely relabeling the vertexes, it turns out that $L_{12} = \text{blkdiag}(L_{12_1}, \ldots, L_{12_c})$ and $L_{22} = \text{blkdiag}(L_{22_1}, \ldots, L_{22_c})$.

Remark 2.1. It is worth noticing that, under the condition that the spectrum of L_{res} and L_{22} are disjoint, namely

$$\sigma(L_{22}) \cap \sigma(L_{res}) = \varnothing$$

it is possible to define the transformation T in (2.1) in such a way that $\tilde{\Gamma} = 0$ (see Roth

(1952) for more details). If this is the case, the matrix $\tilde{L}$ would turn out to be

$$\tilde{L} = \begin{bmatrix} 0_{c\times c} & L_{12} & 0_{c\times N_{res}} \\ 0_{N-c-N_{res}\times 1} & L_{22} & 0_{N-c-N_{res}\times N_{res}} \\ 0_{N_{res}\times c} & 0_{N_{res}\times N-c-N_{res}} & \tilde{L}_{res} \end{bmatrix}$$

Nevertheless, the hypothesis that the spectrum of L_{res} and L_{22} are disjoint is often too restrictive. Hence, for the sake of generality, in the rest of the thesis we will consider the case in which $\tilde{\Gamma} \neq 0$. $\triangle$

[...] once the realization is accepted that even between the closest people infinite distances exist, a marvelous living side-by-side can grow up for them [...]

Rainer Maria Rilke

3

Synchronization of nonlinear systems

This chapter is devoted to the analysis of networks of homogeneous and heterogeneous nonlinear systems. From the seminal result about consensus in Moreau (2004) and Jadbabaie et al. (2003), which studied the case of integrators network, literature is now covering a huge variety of consensus and synchronization problems. In the linear framework Wieland (2010) presents an extensive survey both for graph theory analysis and control of homogeneous and heterogeneous networks. For the case of homogeneous linear dynamics, it is worth mentioning also Scardovi and Sepulchre (2009), Seo et al. (2009), while in the heterogeneous networks framework, other major reference can be found in Kim et al. (2011) and Wieland et al. (2011). For the nonlinear dynamics framework, extensive coverage of current results can be found in Stan and Sepulchre (2007), Arcak (2007), Qu et al. (2007), Hale (1997), both for the case of homogeneous and heterogeneous networks.

Recently, networks of heterogeneous nonlinear agents have attracted a lot of attention. Heterogeneous systems may differ in parameters, functions and dimension of the state. This indeed introduces complications in thinking about synchronization and consensus. Since the systems are potentially completely different, how is it possible to define an agreement? Even considering *only* an output agreement, what is the *behavior* the agents can agree on?

In order to solve this problem, in Wieland et al. (2011) the authors proposes an *internal model* strategy: once a common and desired behavior is defined, each agent embeds a copy of this behavior (generally referred to as exosystem) in its regulator and synchronizes its copy with the copies of the other agents. Simultaneously, an internal model regulator is designed in order to track the reference provided locally by the exosystems.

A different approach, proposed in Panteley and Loria (2015) (see also "A stability-theory perspective to synchronisation of heterogeneous networks", HDR), is based on averaging approach. The behavior of the interconnected systems is determined by two main components: on one hand the stability of an averaged dynamics, relative to an attractor called *emergent dynamic*. On the other hand, the synchronization of each system in the network relative to this emergent dynamics. With this approach, practical synchronization can be achieved.

In the first part of this chapter, we consider a network of homogeneous nonlinear agents and we show that by tuning opportunely a *gain* parameter, a *standard* diffusive coupling between the agents guarantees the achievement of synchronization. In the second part instead, we consider a network of heterogeneous nonlinear agents: by taking inspiration from the approach of Wieland et al. (2011), we show that, by exploiting an internal model strategy, the agents achieves consensus on a particular desired trajectory.

The content of this chapter have been presented in Isidori et al. (2014) and, partially, in Isidori et al. (2013).

3.1 Synchronization of Homogeneous Nonlinear Systems

In this section, in order to achieve synchronization within a network of nonlinear systems, the *synchronizing input* is defined locally at each agent, computing only the available information (*i.e.* the output) of the neighbors. In contrast with the conventional approaches proposed in literature, an *high gain* technique is proposed to deal with the nonlinearity of the systems.

3.1.1 Problem Formulation

We consider a network of N identical *nonlinear* systems, whose output can be modeled as an ordinary differential equation of order d

$$y^{*(d)} = \phi(y^*, y^{*(1)}, \dots, y^{*(d-1)}) \tag{3.1}$$

or in the equivalent state-space form of a d-dimensional system with output

$$\begin{array}{rcl} \dot{w} & = & s(w) \qquad w \in \mathbb{R}^d \\ y^* & = & c(w) \end{array} \tag{3.2}$$

in which

$$s(w) = Sw + B\phi(w)\,, \qquad c(w) = Cw \tag{3.3}$$

and (S, B, C) is a triplet of matrices in *prime* form, that is S is a shift matrix (all 1's on the upper diagonal and all 0's elsewhere), $B^T = (0 \cdots 0 \;\; 1)$ and $C = (1 \;\; 0 \cdots 0)$.

For further details on the systems at hand we refer the reader to Appendix A. This structure of nonlinear agents will recur often during the thesis thus a deep explanation of the systems properties has been given in the appendix-form as a possible reference along the whole thesis.

The communication between the agents is described by a fixed *weighted di-graph*, $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, A\}$.

Assumption 3.1. *The graph $\mathcal{G}$ is connected, namely there is a node from which a path to all other nodes exists. In other words*

$$\begin{array}{rcl} \lambda_1(L) & = & 0 \\ \Re(\lambda_i(L)) & > & 0 \qquad \forall\, i = 2, \dots, N \end{array}$$

The assumption that the graph is connected is reasonable considering that synchronization is expected to be achieved by the whole network: in contrast with other approaches (see for instance the passivity arguments introduced in Arcak (2007), where graphs are not directed), we do not ask the graph to be strongly connected. Each of the N agents is coupled to its neighbor through a control input which processes the output of the neighbors, namely

$$v_k = K \sum_{j=1}^{N} a_{kj}(y_j - y_k) \qquad \forall\, k = 1, \ldots, N \tag{3.4}$$

where a_{kj} are the elements of the adjacency matrix and K is a design parameter to be defined. Equivalently, using the Laplacian matrix, (3.4) reads as

$$v_k = K \sum_{j=1}^{N} \ell_{kj}\, y_j \qquad \forall\, k = 1, \ldots, N$$

Following the definition of the *diffusive coupling*, each agent in the network reads as

$$\begin{array}{rcll} \dot{w}_k & = & S w_k + B\phi(w_k) + K \sum_{j=1}^{N} \ell_{kj} C w_j & \qquad \forall\, k = 1, \ldots, N \\ y_k & = & C w_k & \end{array} \tag{3.5}$$

Since we are seeking nontrivial consensus trajectories, in what follows we will consider the case in which (3.2) possesses a nontrivial compact invariant set W. Moreover, we will assume that the function $\phi(\cdot)$ is globally Lipschitz. In presence of systems of the form (3.3) in which the $\phi(\cdot)$ is only locally Lipschitz, this assumption can always be enforced by properly modifying the function outside the compact set W by using appropriate extension theorems, such as Kirszbraun's Theorem (see, for instance, Theorem 2.10.43 in Federer (1996)). This two requirements are specified in the following assumption.

Assumption 3.2. *The function $\phi(w)$ in (3.3) is globally Lipschitz and there exists a compact set $W \subset \mathbb{R}^d$ invariant for (3.2) such that the system*

$$\dot{w} = S w + B\phi(w) + v$$

is input-to-state stable with respect to v relative to W, namely there exist a class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a class-$\mathcal{K}$ function $\gamma(\cdot)$ such that[1]

$$\|w(t,\bar{w})\|_W \leq \max\{\beta(\|\bar{w}\|_W, t),\ \gamma(\sup_{\tau\in[0,t)} \|v(\tau)\|)\}\,.$$

For details on ISS and its properties for interconnected systems, we refer the reader

[1] Here and in the following we denote by $\|w\|_W = \min_{x\in W} \|w - x\|$ the distance of w from W. Furthermore, $w(t,\bar{w})$ denotes the solution of (3.2) at time t with initial condition $\bar{w}$ at time $t = 0$.

to Appendix B and Isidori (1999).

Remark 3.1. Note that the existence of a compact set W satisfying the property above for $v = 0$ is guaranteed by the assumption that the trajectories of (3.2) are ultimately bounded (see, e.g., Hale and Isidori (2008)). Assumption 3.2, in addition, considers the effect of a perturbance v for which an ISS property is required. If system (3.2) does not have the property of ultimate boundedness and W is just a compact invariant set, the property in question could be achieved by properly modifying the function $s(\cdot)$ in (3.2) by adding appropriate dissipative terms outside the compact set W of interest. This is, for instance, the case of an harmonic oscillator with frequency $\Omega > 0$ in which $d = 2$, the function $\phi(w) = -\Omega^2 w_1$, and the set W is a set of the form $W = \{w \in \mathbb{R}^2 : \|\text{diag}(\Omega, 1)w\| \leq r\}$ with r an upper bound of the amplitude of the oscillations. In this case a possible modification of the function $s(\cdot)$ in (3.2) could be of the form

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\Omega^2 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} - \frac{f(\|\text{diag}(\Omega, 1)w\|)}{\|\text{diag}(\Omega, 1)w\|} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

where $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a smooth function such that $f(R) = 0$ for $R \leq r$, and $f(R)$ strictly increasing and radially unbounded for $R > r$. Simple computations show that the set W is globally asymptotically stable for the previous dynamics with the ISS property in Assumption 3.2 that is fulfilled. A similar technique could be used in the case of nonlinear oscillators, such as the Duffing oscillator. Consider a modification of the harmonic oscillator in polar coordinates of the form $\dot{R} = -f(R)$ and $\dot{\theta} = -\Omega$ where $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a smooth function such that $f(R) = 0$ for $R \leq r$, and strictly increasing and radially unbounded for $R > r$. Considering the cartesian coordinates $x_1 = \frac{R}{\Omega} \cos\theta$ and $x_2 = R \sin\theta$, of the form $\phi(w) = -\Omega^2 w_1 + \delta(w_1, w_2)$ with $\delta(\cdot, \cdot)$ a globally Lipschitz function such that $\delta(w_1, w_2) = 0$ if $\|w\| \leq r$ and $\delta(w_1, w_2) = -w_2$ otherwise. △

3.1.2 Main Result

To the purpose of inducing consensus in the network (3.4)-(3.5), we choose the vector K in (3.4) as

$$K = D_g K_0 \,, \tag{3.6}$$

where $D_g = \text{diag}(g, g^2, \ldots, g^d)$, with g a "gain" parameter and K_0 a vector to be designed.

Due to Assumption 3.1, it is known that the Laplacian matrix L has only one trivial eigenvalue and all remaining eigenvalues have positive real part. Hence there exists a

$\mu > 0$ such that

$$\text{Re}[\lambda_i(L)] \geq \mu \qquad i = 2, \dots, N\,. \tag{3.7}$$

With this in mind, let $T \in \mathbb{R}^{N\times N}$ be defined as

$$T = \begin{bmatrix} 1 & 0_{1\times(N-1)} \\ 1_{(N-1)\times 1} & I_{N-1} \end{bmatrix} \tag{3.8}$$

and note that

$$\tilde{L} = T^{-1}LT = \begin{bmatrix} 0 & L_{12} \\ 0_{(N-1)\times 1} & L_{22} \end{bmatrix}$$

in which the eigenvalues of L_{22} coincide with $\lambda_2(L), \dots, \lambda_N(L)$. Then, the following result holds.

Lemma 3.1. *Let P be the unique positive definite symmetric solution of the algebraic Riccati equation*

$$SP + PS^T - 2\mu PC^TCP + aI = 0 \tag{3.9}$$

with $a > 0$, S and C as in (3.3) and μ as in (3.7). Take K_0 as

$$K_0 = PC^T\,. \tag{3.10}$$

Then, the matrix [2]

$$[(I_{N-1} \otimes S) - (L_{22} \otimes K_0C)]$$

is Hurwitz. $\triangleleft$

The proof of this Lemma can be found in Seo et al. (2009) or in Wieland (2010). Using this, we can now proceed by introducing the main result of this Chapter.

Proposition 3.1. *Suppose Assumptions 3.1 and 3.2 hold. Consider the network of N coupled systems*

$$\dot{w}_k = Sw_k + B\phi(w_k) + D_gK_0 \sum_{j=1}^{N} \ell_{kj}Cw_j$$

with $k = 1, \dots, N$. Let K_0 be chosen as in (3.10). Then, there exists a number $g^\star > 0$ such that, for all $g \geq g^\star$, the compact invariant set

$$\mathbf{W} \;=\; \{(w_1, w_2, \dots, w_N) \in W \times W \times \cdots \times W \;:\; w_1 = w_2 = \cdots = w_N\} \tag{3.11}$$

[2] $A \otimes B$ denotes the Kronecker product of the two matrices A and B.

is globally asymptotically stable. ◁

Proof. In order to prove the result of Proposition 3.1, set $\mathbf{w} = \text{col}(w_1, \ldots, w_N)$ and accordingly rewrite the entire set of N controlled agents as

$$\dot{\mathbf{w}} = [(I_N \otimes S) - (L \otimes D_g K_0 C)]\, \mathbf{w} + (I_N \otimes B)\Phi(\mathbf{w})$$

where

$$\Phi(\mathbf{w}) = \text{col}(\phi(w_1), \ldots, \phi(w_N))\,.$$

Consider the change of variables $\tilde{\mathbf{w}} = (T^{-1} \otimes I_d)\mathbf{w}$, in which T is the matrix introduced in (3.8). The system in the new coordinates reads as

$$\begin{aligned} \dot{\tilde{\mathbf{w}}} &= (T^{-1} \otimes I_d)\, [(I_N \otimes S) - (L \otimes D_g K_0 C)]\, (T \otimes I_d)\tilde{\mathbf{w}} \\ &\qquad + (T^{-1} \otimes I_d)(I_N \otimes B)\Phi((T \otimes I_d)\tilde{\mathbf{w}}) \\ &= \Big[(I_N \otimes S) - (\tilde{L} \otimes D_g K_0 C)\Big]\, \tilde{\mathbf{w}} + (T^{-1} \otimes B)\Phi((T \otimes I_d)\tilde{\mathbf{w}})\,. \end{aligned}$$

By definition of (3.8), observe that

$$T^{-1} = \begin{bmatrix} 1 & 0_{1\times(N-1)} \\ -1_{(N-1)\times 1} & I_{N-1} \end{bmatrix}$$

and thus, $\tilde{\mathbf{w}}$ can be written as

$$\tilde{\mathbf{w}} = \text{col}(w_1, w_2 - w_1, \ldots, w_N - w_1)$$

Consequently, define $z_k = w_k - w_1$, for $k = 2, 3, \ldots, N$, and

$$\mathbf{z} = \text{col}(z_2, z_3, \ldots, z_N)\,,$$

yielding $\tilde{\mathbf{w}} = \text{col}(w_1, \mathbf{z})$. Then, it is readily seen that the system above exhibits a triangular structure of the form

$$\begin{aligned} \dot{w}_1 &= S w_1 + B\phi(w_1) - (L_{12} \otimes D_g K_0 C)\mathbf{z} \\ \dot{\mathbf{z}} &= [(I_{N-1} \otimes S) - (L_{22} \otimes D_g K_0 C)]\, \mathbf{z} + \Delta\Phi(w_1, \mathbf{z}) \end{aligned} \tag{3.12}$$

where

$$\Delta\Phi(w_1, \mathbf{z}) = (I_{N-1} \otimes B) \begin{pmatrix} \phi(w_1 + z_2) - \phi(w_1) \\ \phi(w_1 + z_3) - \phi(w_1) \\ \cdots \\ \phi(w_1 + z_N) - \phi(w_1) \end{pmatrix} .$$

Note that $\Delta\Phi(w_1, \mathbf{z})$ is globally Lipschitz in $\mathbf{z}$ uniformly in w_1 and $\Delta\Phi(w_1, 0) \equiv 0$ for all $w_1 \in \mathbb{R}^n$. Consider now the rescaled state variable

$$\zeta = (I_{N-1} \otimes D_g^{-1})\mathbf{z} .$$

and that by definition of S, B, C and D_g,

$$D_g^{-1} S D_g = gS, \quad D_g^{-1} B = \frac{1}{g^d} B, \quad C D_g = gC,$$

With simple calculations, it follows that the triangular system (3.12) is mapped into

$$\begin{aligned} \dot{w}_1 &= S w_1 + B\phi(w_1) - (L_{12} \otimes D_g K_0 C)(I_{N-1} \otimes D_g)\zeta \\ \dot{\zeta} &= g\left[(I_{N-1} \otimes S) - (L_{22} \otimes K_0 C)\right]\zeta + \\ &\qquad \frac{1}{g^d}\Delta\Phi(w_1, (I_{N-1} \otimes D_g)\zeta) . \end{aligned} \tag{3.13}$$

It is known from Lemma 3.1 that the proposed choice of K_0 guarantees that the matrix $[(I_{N-1} \otimes S) - (L_{22} \otimes K_0 C)]$ is Hurwitz. As a consequence, standard high-gain arguments lead to the conclusion that, if g is chosen sufficiently large, the equilibrium $\zeta = 0$ of the lower subsystem is colglobally exponentially stable, uniformly in w_1, and actually with a quadratic Lyapunov function that is independent of w_1. A detailed proof of the global exponential stability of $\zeta = 0$ is developed in Section 3.1.3, where we derive an explicit expression to determine g.

To conclude the proof of Proposition 3.1, consider that the ISS property in Assumption 2 guarantees that w_1 converges to the invariant set W. Since $w_k = w_1 + z_k$, $k = 2, \ldots, N$, and $z_k \to 0$ as $t \to \infty$, the result follows. □

Remark 3.2. Proposition 3.1 shows that, under Assumption 3.1-3.2, if g is large enough the set $\mathbf{W}$ is globally *asymptotically* stable. One may wonder under which extra conditions the set in question would also be *locally exponentially* stable. Standard arguments show that this is the case if, in Assumption 3.2, the function $\beta(r, t)$ is bounded – for small r – by a function of the form $\mathcal{M}e^{-\alpha t}r$ for some positive $\mathcal{M}$ and α and the function $\gamma(r)$ is locally Lipschitz at the origin. △

The following corollary, is a straightforward consequence.

Corollary 3.1. *Let the hypotheses of the previous Proposition hold and let K_0 be chosen as in (3.10). There is a number $g^\star > 0$ such that, if $g \geq g^\star$, the states of the N systems (3.5) reach consensus, i.e. for every $w_k(0) \in \mathbb{R}^d$, $k = 1, \ldots, N$, there is a function $w^\star : \mathbb{R} \to \mathbb{R}^d$ such that*

$$\lim_{t\to\infty} \|w_k(t) - w^\star(t)\| = 0 \qquad \textit{for all } k = 1, \ldots, N\,. \ \triangleleft$$

3.1.3 A bound for the minimal gain $g^\star$ in Proposition 3.1

The proof of Proposition 3.1 shows that, in the proposed setting, consensus is achieved if the gain parameter g exceeds a minimum value $g^\star$. One may wonder how the value $g^\star$ scales with number N of agents. In this respect, it can be shown that the number $g^\star$ is not influenced by the number N *per se*, but rather by parameters related to the Jordan form of the Laplacian matrix L. More precisely, let M be a matrix such that $ML_{22}M^{-1}$ is in Jordan form. Then, parameters which determine the value of $g^\star$ are: an upper bound on the dimension of the Jordan blocks of L_{22}, a lower bound on the real parts of the eigenvalues of L_{22} (which is positive since the graph is connected), lower and – respectively - upper bounds on the minimal and – respectively – maximal eigenvalue of the real part of the Hermitian matrix M^*M. So long as such bounds remain independent of N, so does the value $g^\star$.

Proposition 3.2. *The control gain parameter g in (3.6) does not depend on the number of agents in the network N. More precisely, it depends on the Lipschitz constant of the function $\phi(\cdot)$ in (3.3) and on spectral properties of the Laplacian matrix.*

Proof. Recall that the crucial step in the proof of Proposition 3.1 is to establish, via *high-gain* arguments, the global asymptotic stability of the equilibrium $\zeta = 0$ of the lower subsystem of (3.13), rewritten here for convenience

$$\dot{\zeta} = g\,[I_{N-1} \otimes S - L_{22} \otimes K_0 C]\,\zeta \ + \ \frac{1}{g^d}\Delta\Phi(w_1, (I_{N-1} \otimes D_g)\zeta)\,. \tag{3.14}$$

To investigate the influence of N on $g^\star$, we begin by finding an explicit (quadratic) Lyapunov function for the linear system

$$\dot{\zeta} = [(I_{N-1} \otimes S) - (L_{22} \otimes K_0 C)]\,\zeta\,, \tag{3.15}$$

which, for large g, determines the “dominant" part of (3.14).

Suppose, for the time being, that the Laplacian matrix L has a purely diagonal Jordan form: hence, also L_{22} has a purely diagonal Jordan form. Let M be a matrix that diagonalizes L_{22}, i.e. such that

$$ML_{22}M^{-1} = \mathrm{diag}(\lambda_2(L), \ldots, \lambda_N(L))$$

(note that M is in general a matrix of complex numbers) and consider the change of variables $\xi = (M \otimes I_d)\zeta$. This yields to rewrite (3.15) as

$$\begin{aligned}\dot{\xi} &= (M \otimes I_d)\left[I_N \otimes S - L_{22} \otimes K_0 C\right](M \otimes I_d)^{-1}\xi \\ &= (I_N \otimes S - ML_{22}M^{-1} \otimes K_0 C)\xi := A\xi\end{aligned}$$

where we have set

$$A = \mathrm{diag}(A_2, \ldots, A_N), \qquad \text{with} \qquad A_i = S - \lambda_i(L)K_0C\,.$$

Recall now the definition of P given in Lemma 3.1 and observe that, since $K_0 = PC^T$,

$$\begin{aligned}P^{-1}A_i + A_i^* P^{-1} &= P^{-1}(S - \lambda_i(L)PC^TC) + (S^T - \lambda_i^*(L)C^TC)P^{-1} \\ &= P^{-1}S + S^TP^{-1} - 2\Re[\lambda_i(L)]C^TC \\ &= 2(\mu - \Re[\lambda_i(L)])C^TC - aP^{-2}\,.\end{aligned}$$

Since $\mu \le \Re[\lambda_i(L)]$ and aP^{-2} is positive definite, we deduce the existence of a number $\bar{a} > 0$ (possibly depending on P) such that

$$(P^{-1}A_i + A_i^*P^{-1}) < -\bar{a}\, I_d$$

for $i = 2, \ldots, N$. As a consequence

$$(I_{N-1} \otimes P^{-1})A + A^*(I_{N-1} \otimes P^{-1}) < -\bar{a}\,(I_{N-1} \otimes I_d)\,.$$

Consider now, for the full system (3.14), the candidate Laypunov function

$$V(\zeta) = \zeta^T(M^* \otimes I_d)(I_{N-1} \otimes P^{-1})(M \otimes I_d)\zeta = \zeta^T(M^*M \otimes P^{-1})\zeta\,,$$

The matrix M^*M, which is Hermitian and positive definite, can be expressed as

$$M^*M = H_R + iH_I$$

in which $H_R = H_R^T$ is positive definite and $H_I = -H_I^T$. Hence, also the matrix

$M^*M \otimes P^{-1}$ is Hermitian positive definite and

$$M^*M \otimes P^{-1} = H_R \otimes P^{-1} + iH_I \otimes P^{-1}\,,$$

in which $H_R \otimes P^{-1} = (H_R \otimes P^{-1})^T$ is positive definite and $H_I \otimes P^{-1} = -(H_I \otimes P^{-1})^T$. As a consequence

$$V(\zeta) = \zeta^T(H_R \otimes P^{-1})\zeta\,.$$

Taking the derivative of $V(\zeta)$ along the trajectories of (3.14) yields

$$\dot{V} = g\,2\zeta^T(H_R \otimes P^{-1})(I_{N-1} \otimes S - L_{22} \otimes K_0C)\zeta + 2\zeta^T(H_R \otimes P^{-1})(I_{N-1} \otimes B)\delta\phi(\zeta)$$

in which

$$\delta\phi(\zeta) = \frac{1}{g^d}\,\mathrm{col}(\phi(w_1 + D_g\zeta_2) - \phi(w_1), \cdots, \phi(w_1 + D_g\zeta_N) - \phi(w_1))\,.$$

As long as the first term is concerned, observe that

$$\begin{aligned}
&2\,\zeta^T(H_R \otimes P^{-1})(I_{N-1} \otimes S - L_{22} \otimes K_0C)\zeta \\
&\quad = 2\,\Re[\zeta^T(M^*M \otimes P^{-1})(I_{N-1} \otimes S - L_{22} \otimes K_0C)\zeta] \\
&\quad = 2\,\Re[\zeta^T(M^* \otimes I_d)(I_{N-1} \otimes P^{-1})A(M \otimes I_d)\zeta] \\
&\quad = \zeta^T(M^* \otimes I_d)[(I_{N-1} \otimes P^{-1})A + A^*(I_{N-1} \otimes P^{-1})](M \otimes I_d)\zeta \\
&\quad \leq -\,\bar{a}\,\zeta^T(M^* \otimes I_d)(I_{N-1} \otimes I_d)(M \otimes I_d)\zeta \\
&\quad = -\,\bar{a}\,\zeta^T(M^*M \otimes I_d)\zeta = -\,\bar{a}\,\zeta^T(H_R \otimes I_d)\zeta\,.
\end{aligned}$$

This yields to

$$\dot{V} \leq -g\,\bar{a}\,\zeta^T(H_R \otimes I_d)\zeta + 2\zeta^T(H_R \otimes P^{-1}B)\delta\phi(\zeta)\,. \qquad (3.16)$$

Since H_R is positive definite, there exists a unitary matrix Q such that

$$H_R = Q^T\Lambda Q$$

where Λ is a diagonal matrix of positive real numbers. Setting

$$E = (Q \otimes I_d)$$

and observing that

$$H_R \otimes I_d = E^T(\Lambda \otimes I_d)E$$

we obtain

$$\dot{V} \leq -g\,\bar{a}\,\zeta^T E^T(\Lambda \otimes I_d)E\zeta + \zeta^T E^T(\Lambda \otimes I_d)E(I_{N-1} \otimes PB)\delta\phi(\zeta)$$

Setting $x = E\zeta$ the latter is rewritten as

$$\begin{aligned}\dot{V} &\leq -g\,\bar{a}\,x^T(\Lambda \otimes I_d)x + x^T(\Lambda \otimes I_d)(Q \otimes I_d)(I_{N-1} \otimes PB)\delta\phi(\zeta) \\ &= -g\,\bar{a}\,x^T(\Lambda \otimes I_d)x + x^T(\Lambda \otimes I_d)(Q \otimes PB)\delta\phi(\zeta)\end{aligned}$$

Let's now discuss the two terms separately. Let $\lambda_i(H_R)$, with $i = 1, \ldots, N-1$, be the eigenvalues of H_R and, in particular, let $\lambda_{\min}(H_R)$ and $\lambda_{\max}(H_R)$ denote the minimal and – respectively – maximal of such eigenvalues. Split x in $N-1$ vectors $x_1, \ldots, x_{N-1}$ each of dimension d and observe that

$$\begin{aligned}x^T(\Lambda \otimes I_d)x &= \sum_{i=1}^{N-1} \lambda_i(H_R)|x_i|^2 \\ &\geq \lambda_{\min}(H_R)\textstyle\sum_{i=1}^{N-1}|x_i|^2 \\ &= \lambda_{\min}(H_R)|x|^2\,.\end{aligned}$$

Hence

$$-g\,\bar{a}\,x^T(\Lambda \otimes I_d)x \leq -g\,\bar{a}\lambda_{\min}|x|^2\,.$$

For the second term, set $v = (Q \otimes PB)\delta\phi(\zeta)$, which we partition in $N-1$ vectors v_i, each of dimension d. Then, we have

$$\begin{aligned}x^T(\Lambda \otimes I_d)v &= \sum_{i=1}^{N-1} \lambda_i(H_R)x_i^T v_i \leq \sum_{i=1}^{N-1} \lambda_i(H_R)|x_i^T|\,|v_i| \\ &\leq \lambda_{\max}(H_R)\sum_{i=1}^{N-1}|x_i^T|\,|v_i| \leq \lambda_{\max}(H_R)|x|\,|v| \\ &\leq \lambda_{\max}(H_R)|x|\,|(Q \otimes PB)\delta\phi(\zeta)| \\ &\leq \lambda_{\max}(H_R)|x||(Q \otimes PB)|\,|\delta\phi(\zeta)|\ .\end{aligned}$$

Finally, observe that

$$\begin{aligned}|\delta\phi(\zeta)|^2 &\leq \frac{1}{g^{2d}}\sum_{i=2}^{N}|(\phi(w_1+D_g\zeta_i)-\phi(w_1)|^2\\ &\leq \frac{1}{g^{2d}}\sum_{i=2}^{N}\Upsilon^2|D_g\zeta_i|^2\\ &\leq \sum_{i=2}^{N}\Upsilon^2|\zeta_i|^2=\Upsilon^2|\zeta|^2\end{aligned}$$

where Υ is the Lipschitz constant of the function $\phi(\cdot)$. Therefore, since $\zeta = E^T x$ and E is a unitary matrix, we have

$$\begin{aligned}x^T(\Lambda\otimes I_d)v &\leq \lambda_{\max}(H_R)|x||(Q\otimes PB)||\delta\phi(\zeta)|\\ &\leq \lambda_{\max}(H_R)|(Q\otimes PB)|\Upsilon|x|^2\end{aligned}$$

It remains to estimate the norm of $(Q\otimes PB)$. Bearing in mind the fact that $|A| = \sqrt{\lambda_{\max}(A^TA)}$ we see that

$$\begin{aligned}|(Q\otimes PB)| &= \sqrt{\lambda_{\max}((Q\otimes PB)^T(Q\otimes PB))}\\ &= \sqrt{\lambda_{\max}((Q^T\otimes(PB)^T)(Q\otimes PB))}\\ &= \sqrt{\lambda_{\max}(I_n\otimes(PB)^TPB)}\\ &= \sqrt{\lambda_{\max}(B^TPPB)}\end{aligned}$$

Putting all these bounds together, it is seen that

$$\dot{V}\leq[-g\,\bar{a}\lambda_{\min}(H_R)+\sqrt{\lambda_{\max}(B^TPPB)}\,\Upsilon\,\lambda_{\max}(H_R)]|x|^2$$

from which it is concluded that $\dot{V}$ is bounded by a negative definite function so long as $g > g^\star$, with

$$g^\star = \sqrt{\lambda_{\max}(B^TPPB)}\,\frac{\Upsilon}{\bar{a}}\,\frac{\lambda_{\max}(H_R)}{\lambda_{\min}(H_R)}$$

In this formula, Υ is the Lipschitz constant of the function $\phi(\cdot)$, P and $\bar{a}$ depend solely on the value of μ, a lower bound on the real part of the eigenvalues of the Laplacian matrix L, while $\lambda_{\max}(H_R), \lambda_{\min}(H_R)$ are the maximal and minimal eigenvalues of the matrix H_R, the real part of the matrix M^*M, where M is a matrix that diagonalizes L_{22}. Thus, this estimate proves the claim in Propostion 3.2. A similar result can be shown also if the Jordan form of the matrix L_{22} is not purely diagonal, provided that *the dimension of the Jordan blocks remain bounded, so long as N varies*. A detailed proof uses arguments pretty similar to those presented above combined with standard ISS analysis.

□

3.1.4 Simulation Result

We now present simulation results about the proposed solution: we consider 5 Van Der Pol oscillators

$$\begin{array}{rcl} \dot{x}_{i_1} & = & x_{i_2} \\ \dot{x}_{i_2} & = & 2(1-x_{i_1}^2)x_{i_2} - x_{i_1} \end{array} \qquad y_i = x_{i_1} \tag{3.17}$$

The initial conditions of the agents are $w_1 = (1,1)^T, w_2 = (2,2)^T, w_3 = (3,3), w_4 = (5,10)$ and $w_5 = (10,-7)$. Furthermore, we choose K according to (3.10) with $g = 4$ and and $a = 1$. The graph is described by the Laplacian matrix

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Figure 3.1 shows the behavior of the Vand Der Pol oscillators achieving consensus: on the right column a zoom of the transient is shown. Figure 3.2 instead shows the phase plot of the oscillators during the synchronization.

We also show simulation results, considering 5 Lorentz oscillators connected on the same graph. The Lorentz oscillators are described by

$$\begin{array}{rcl} \dot{x}_{k_1} & = & \sigma(x_{k_2} - x_{k_1}) \\ \dot{x}_{k_2} & = & x_{k_1}(\rho - x_{k_3}) - x_{k_2} \\ \dot{x}_{k_3} & = & x_{k_1}x_{k_2} - \beta x_{k_3} \end{array} \qquad y_k = x_{k_1}\,. \tag{3.18}$$

for $k = 1, \ldots, 5$. The values of parameters (σ, ρ, β) are $\sigma = 10$, $\rho = 28$ and $\beta = 8/3$. System (3.18) can be embedded into the fourth order system

$$\begin{array}{rcl} \dot{w}_{k_1} & = & w_{k_2} \\ \dot{w}_{k_2} & = & w_{k_3} \\ \dot{w}_{k_3} & = & w_{k_4} \\ \dot{w}_{k_4} & = & \Phi(w_{k_1}, w_{k_2}, w_{k_3}, w_{k_4}) \end{array} \tag{3.19}$$

fitting into the structure of (3.2) and fulfilling the requested assumption.

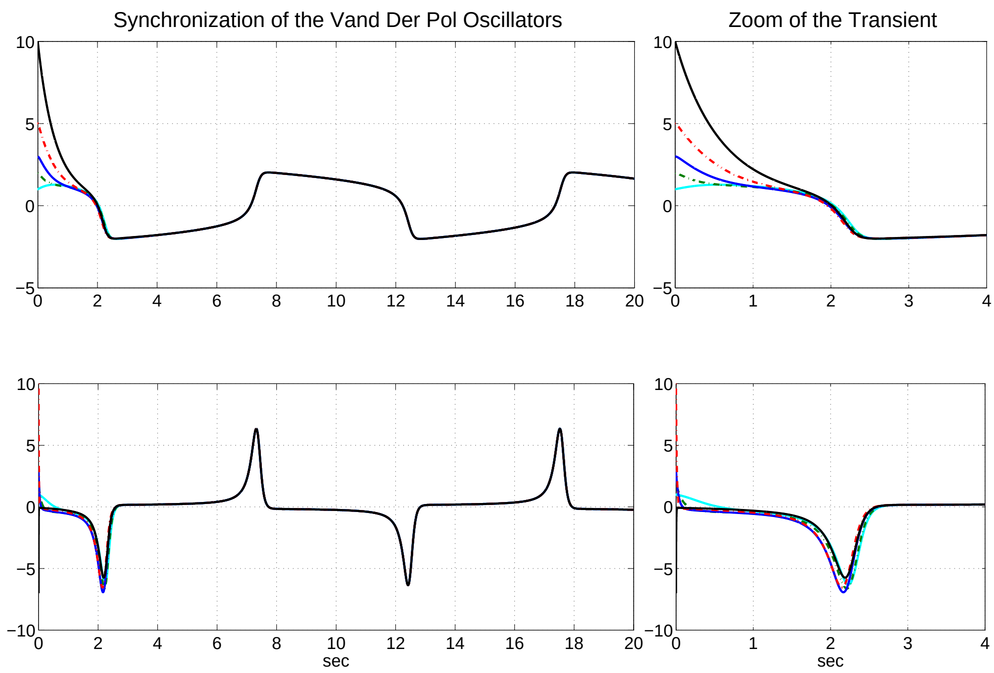

Figure 3.1: Synchronization of the two components of the Van Der Pol oscillators.

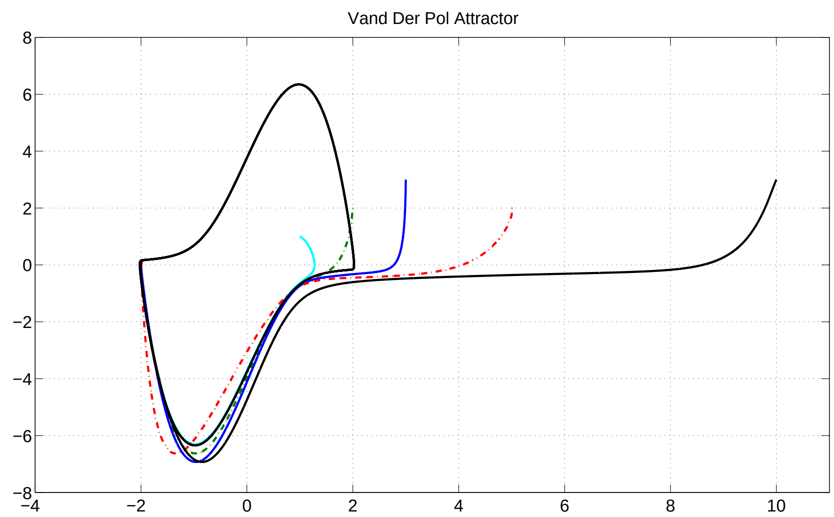

Figure 3.2: Phase plot of the synchronization of the two components of the Van Der Pol oscillators.

In new coordinates, the agents' initial conditions are $w_1 = (1.5; 1; 0; 0)$, $w_2 = (1; 5; 5; 5)$, $w_3 = (2; 10; 10; 10)$, $w_4 = (0.5; 7; 7; 7)$ and $w_5 = (0; 15; 15; 15)$. K is chosen according to (3.10) with the gain parameter $g = 50$ and $a = 1$.

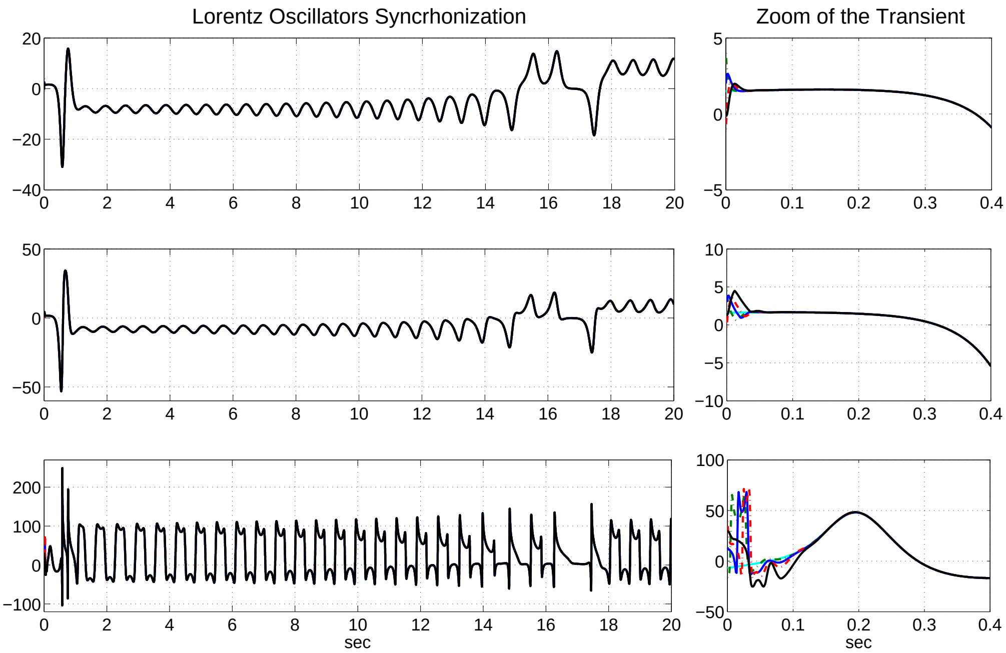

Figure 3.3: Synchronization of the three components of the Lorentz oscillators.

Figure 3.3 shows the three component of the Lorentz oscillators in the original coordinate during synchronization: the right column illustrates the zoom of the transient.

3.2 Synchronization of Hetherogeneous Nonlinear systems

In this section, we show how a network of heterogeneous agents can be controlled in such a way that their outputs asymptotically track the output of a prescribed nonlinear exosystem. Following a similar approach to the one of Wieland et al. (2011), the problem is solved in two steps. In the first step, the problem of achieving consensus among (identical) nonlinear reference generators is addressed. With respect to this first step, we refer to the results of Section 3.1. In the second step, the theory of nonlinear output regulation is applied in a decentralized control framework, to force the output of each agent of the network to robustly track the (synchronized) output of each local reference model.

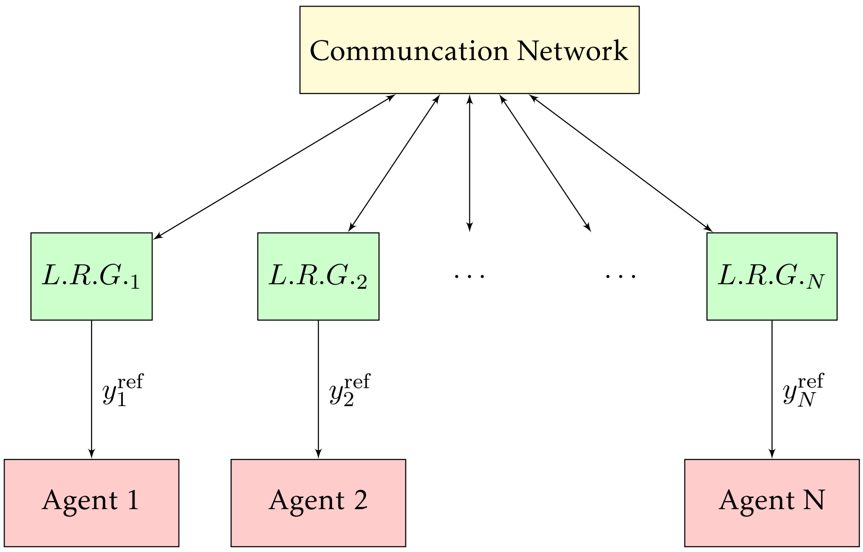

Figure 3.4: Two steps approach: on a first level the *Local Reference Generators* agree on a nontrivial trajectory. Their output becomes the reference for each agent.

The control paradigm is briefly illustrated in Figure 3.4. On the upper layer, a set of *local reference generators* achieves synchronization on a nontrivial trajectory by communicating their outputs through the network. They provide the reference to be tracked to the lower level, namely the controlled agents. Each agent will locally compute the reference and generate the control action to track such a reference.

3.2.1 Problem formulation

We consider the problem of inducing consensus between the outputs of N non-identical nonlinear systems, which exchange information through a communication graph $\mathcal{G}$. The control system is decentralized, i.e. there is no leader sending information to each individual system, but rather each system exchanges information only with a set of neighboring systems, being the information in question only the *relative* values of the respective controlled outputs. The N nonlinear agents are described by

$$\begin{aligned} \dot{x}_k &= f_k(x_k) + g_k(x_k)u_k \\ y_k &= h_k(x_k) \end{aligned} \qquad x_k \in \mathbb{R}^{n_k}\,, u_k, y_k \in \mathbb{R} \tag{3.20}$$

$k = 1, \ldots, N$, where u_k and y_k are the local control input and output, with the inputs u_k that must be designed in such a way that the outputs y_k of the N systems asymptotically reach consensus on a nontrivial common trajectory $y^\star(t)$. Each agent is controlled by a local output-feedback controller of the form

$$\begin{aligned} \dot{\varsigma}_k &= \Phi_k(\varsigma_k, y_k, \nu_k) \quad \varsigma_k \in \mathbb{R}^{\bar{n}_k},\ \nu_k \in \mathbb{R}^p \\ u_k &= \Gamma_k(\varsigma_k, y_k, \nu_k) \\ \vartheta_k &= \Theta_k(\varsigma_k, y_k) \qquad \vartheta_k \in \mathbb{R}^p \end{aligned} \tag{3.21}$$

in which ϑ_k and ν_k are outputs and inputs that characterize the exchange of relative information between individual (controlled) agents, which takes the form

$$\nu_k = \sum_{j=1}^{N} a_{kj}(\vartheta_j - \vartheta_k) \tag{3.22}$$

or equivalently, using the Laplacian notation

$$\nu_k = \sum_{j=1}^{N} \ell_{kj}\vartheta_j \tag{3.23}$$

In general terms, the control problem can be cast as follows. Let $X_k \in \mathbb{R}^{n_k}$, $k = 1, \ldots, N$, be fixed compact colsets of admissible conditions for (3.20). The problem is to find N local controllers of the form (3.21), exchanging information as in (3.22), and compact sets $\Sigma_k \in \mathbb{R}^{\bar{n}_k}$, $k = 1, \ldots, N$, of admissible initial conditions for all such controllers, so that the positive orbit of the set of all admissible initial conditions is bounded and output consensus is reached, i.e. for each admissible initial condition $(x_k(0), \varsigma_k(0)) \in X_k \times \Sigma_k$, $k = 1, \ldots, N$, there is a function $y^\star : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{t\to\infty} |y_k(t) - y^\star(t)| = 0 \qquad \text{for all } k = 1, \ldots, N\,,$$

uniformly in the initial conditions.

We expect that the consensus trajectory $y^\star(t)$ can be thought of as generated by a *nonlinear* autonomous system, which could be modeled as an ordinary differential equation of order d, as (3.1) or in the equivalent state-space form of a d-dimensional system (3.2)-(3.3).

Since we are seeking nontrivial consensus trajectories, following the same reasoning of Chapter 3.1, in what follows we will consider the case in which (3.2) possesses a nontrivial compact invariant set W. Moreover, we will assume that the function $\phi(\cdot)$ is globally Lipschitz. For further details on these assumptions see Section 3.1.1.

3.2.2 Structure of local controllers and communication protocol

Bearing in mind the possibility of modeling all solutions of (3.1) as outputs of the autonomous system (3.2)-(3.3), in what follows, we consider for the local controllers (3.21) a structure of the form

$$\begin{array}{rcl} \dot{w}_k & = & s(w_k) + K\nu_k \\ \dot{\eta}_k & = & \varphi_k(\eta_k, y_k - c(w_k)) \\ u_k & = & \gamma_k(\eta_k, y_k - c(w_k)) \\ \vartheta_k & = & c(w_k) \end{array} \tag{3.24}$$

in which $\nu_k = \sum_{j=1}^{N} a_{kj}(c(w_j) - c(w_k))$. It is readily seen that this structure consists of a set of N *local reference generators*

$$\begin{array}{rcl} \dot{w}_k & = & s(w_k) + K\nu_k \\ y_k^{\text{ref}} & = & c(w_k)\,, \end{array} \tag{3.25}$$

coupled via

$$\nu_k = \sum_{j=1}^{N} a_{kj}(y_j^{\text{ref}} - y_k^{\text{ref}})\,, \tag{3.26}$$

each one of which provides a reference y_k^{ref} to be tracked by a *local regulator*

$$\begin{array}{rcl} \dot{\eta}_k & = & \varphi_k(\eta_k, e_k) \\ u_k & = & \gamma_k(\eta_k, e_k) \end{array} \tag{3.27}$$

driven by the local tracking error

$$e_k = y_k - y_k^{\text{ref}}\,.$$

This control structure enables us to solve the problem in two stages. In the first stage, the colvector K of design parameters is chosen in such a way as to induce consensus among the N *local reference generators* (3.25). In the second stage, the local regulators are designed in such a way that each of the outputs y_k tracks its own reference y_k^{ref}. It goes without saying that in the second step we ought to be able to use – off the shelf – a large amount of existing results about the design of output regulators for nonlinear systems in the presence of exogenous signals generated by a nonlinear exosystem.

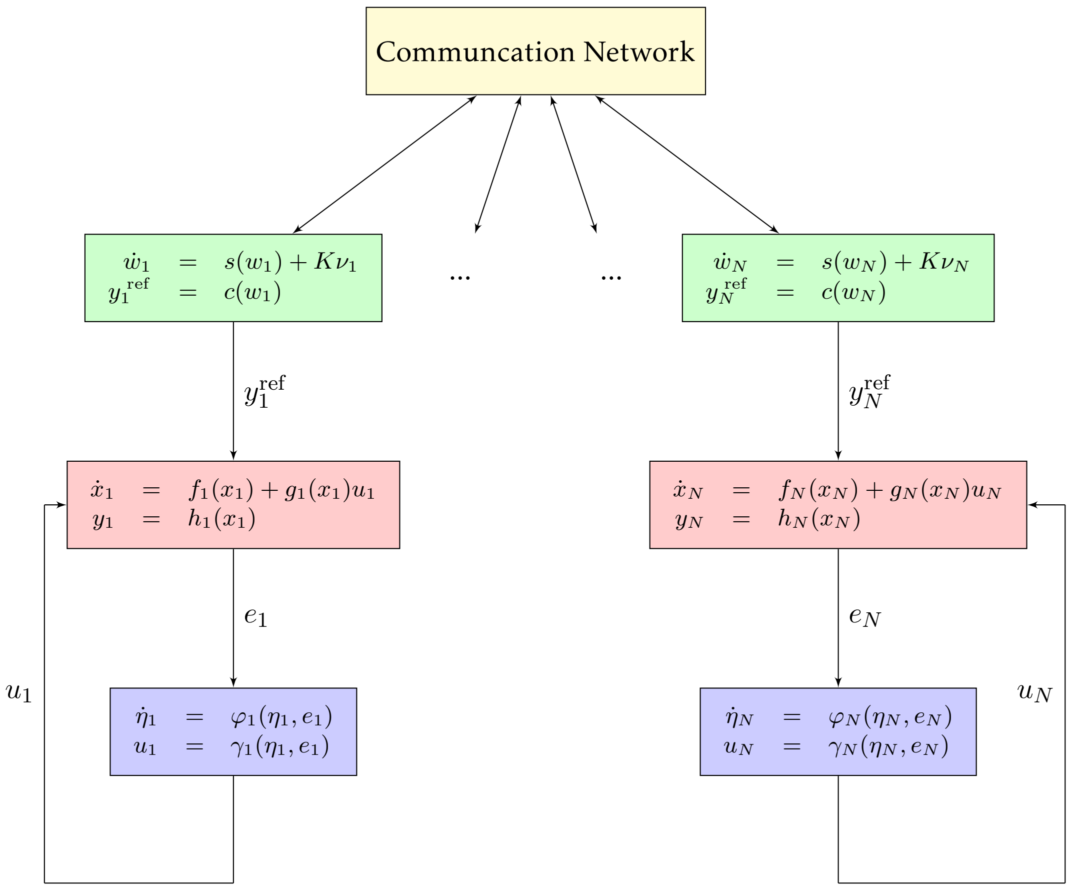

Figure 3.5: Control architecture: by exchanging information through the network, the *Local Reference Generators* achieve synchronization. They provide an output to be tracked locally by each agent. The regulators process their respective errors and generates the control action to track the reference.

3.2.3 Main Result

In this framework, given the results provided in Chapter 3.1 about the synchronization of the *local reference generators*, we proceed now with the second step of the design, i.e. we design local regulators for each agent. In what follows, we assume that the vector K_0 and the values of g have been fixed such that the conclusion of Proposition 3.1 holds, *i.e.* such that the synchronization set (3.11) is globally asymptotically stable for the network (3.5).

In what follows, we assume that each individual agent has a well defined relative degree r between input u_k and output y_k and possesses a globally defined normal form, see Isidori (1995). To streamline the exposition, we consider the special case in which $r = 1$. The case of higher relative degree only entails heavier *notation* complexity and no conceptual differences. Thus we assume that the individual agent (3.20) is modeled, with a mild abuse of notation, by equations of the form

$$\begin{aligned} \dot{z}_k &= f_k(z_k, y_k) \\ \dot{y}_k &= a_k(z_k, y_k) + b_k(z_k, y_k)u_k \,, \end{aligned} \tag{3.28}$$

where $z_k \in \mathbb{R}^{n_k-1}$ and where $b_k(z_k, y_k)$, which is the high-frequency gain of the k-th agent, is bounded away from zero. In particular, we assume that, for all $k = 1, \dots, N$, there exists $\bar{b}_k > 0$ such that $b_k(z_k, y_k) \geq \bar{b}_k$ for all $(z_k, y_k) \in \mathbb{R}^{n_k-1} \times \mathbb{R}$. Possible static or dynamic uncertainties affecting the controlled plant can be thought of as embedded in the z_k dynamics. For instance, in presence of *constant* parametric uncertainties μ_k affecting the functions $f_k(\cdot)$, $a_k(\cdot)$, $b_k(\cdot)$, the state variable z_k can be thought of as partitioned as $z_k = \mathrm{col}(z'_k, \mu_k)$ governed by the dynamics

$$\begin{aligned} \dot{z}'_k &= f'_k(\mu_k, z'_k, y_k) \\ \dot{\mu}_k &= 0 \,. \end{aligned}$$

More complex dynamic uncertainties can clearly be included in a similar fashion.

As anticipated, with this system we associate a local tracking error of the form

$$e_k = y_k - Cw_k \,.$$

The problem is to design a robust local regulator, driven by the regulation error e_k, to the purpose of steering e_k to zero. In this respect, it should be brought in mind that w_k is a "portion" of the state of the coupled system (3.20)-(3.21) and hence the entire dynamics of the latter should be taken into account in the analysis. To this end, recall the arguments used in the proof of Proposition 3.1 to obtain (3.13). Let T_k be a matrix in which all the elements on the diagonal and those on the k-th column are 1's, while all other elements are 0, and consider the change of variables $\tilde{\mathbf{w}} = (T_k^{-1} \otimes I_d)\mathbf{w}$. In the new coordinates, the entire set of the N networked local reference generators can be seen as a system modeled by equations of the form

$$\begin{aligned} \dot{w}_k &= Sw_k + B\phi(w_k) - (L_{k,12} \otimes D_g K_0 C)(I_{N-1} \otimes D_g)\zeta_k \\ \dot{\zeta}_k &= g\left[(I_{N-1} \otimes S) - (L_{k,22} \otimes K_0 C)\right]\zeta_k + \frac{1}{g^d}\Delta\Phi_k(w_k, (I_{N-1} \otimes D_g)\zeta_k) \,. \end{aligned}$$

in which

$$\zeta_k = \text{col}(w_1 - w_k, \ldots, w_{k-1} - w_k, w_{k+1} - w_k, \ldots, w_N - w_k)$$

and $L_{k,12}$, $L_{k,22}$, $\Delta\Phi_k(\cdot)$ are suitably defined. These can be written in compact form as

$$\begin{aligned} \dot{w}_k &= s(w_k) + \Upsilon_k \zeta_k \\ \dot{\zeta}_k &= \psi(\zeta_k, w_k) \end{aligned} \tag{3.29}$$

in which $\psi(0, w_k) = 0$. By assumption, the upper subsystem is input-to-state stable, with respect to the input ζ_k, to the set W. Moreover, as observed in the proof of Proposition 3.1, the equilibrium $\zeta_k = 0$ of the lower subsystem is globally exponentially stable. As a consequence, the set

$$\{(\zeta_k, w_k) \in \mathbb{R}^{(N-1)d} \times \mathbb{R}^d : \zeta_k = 0, w_k \in W\}$$

is a globally asymptotically stable compact invariant set of (3.29)Sontag (1995).

In view of this, we can represent the aggregate of (3.28) and of (3.29) as a standard exosystem-plant interconnection

$$\begin{aligned} \dot{\zeta}_k &= \psi(\zeta_k, w_k) \\ \dot{w}_k &= s(w_k) + \Upsilon_k \zeta_k \\ \dot{z}_k &= f_k(z_k, y_k) \\ \dot{y}_k &= a_k(z_k, y_k) + b_k(z_k, y_k) u_k \\ e_k &= y_k - C w_k \,. \end{aligned} \tag{3.30}$$

As usual, we change variables replacing y_k by e_k and obtain a system of the form

$$\begin{aligned} \dot{\zeta}_k &= \psi(\zeta_k, w_k) \\ \dot{w}_k &= s(w_k) + \Upsilon_k \zeta_k \\ \dot{z}_k &= f_k(z_k, C w_k + e_k) \\ \dot{e}_k &= q_k(w_k, z_k, \zeta_k, e_k) \; + \; b_k(z_k, C w_k + e_k) u_k \end{aligned} \tag{3.31}$$

where

$$q_k(w_k, z_k, \zeta_k, e_k) = a_k(z_k, C w_k + e_k) - C[s(w_k) + \Upsilon_k \zeta_k] \,.$$

This system is ready for the design (under appropriate hypotheses) of a local regulator of the form (3.27), which will now be written – with a mild abuse of notation – as

$$\begin{aligned} \dot{\eta}_k &= \varphi_k(\eta_k) + G_k v_k \quad && \eta_k \in \mathbb{R}^{m_k} \\ u_k &= \gamma_k(\eta_k) + v_k \\ v_k &= \kappa_k(e_k) && v_k \in \mathbb{R} \end{aligned} \tag{3.32}$$

according to the procedures suggested in Byrnes and Isidori (2004) or in Marconi et al. (2007). The basic assumption needed to make the design possible is that the zero dynamics of (3.31) namely, those of

$$\begin{aligned} \dot{\zeta}_k &= \psi(\zeta_k, w_k) \\ \dot{w}_k &= s(w_k) + \Upsilon_k \zeta_k \\ \dot{z}_k &= f_k(z_k, Cw_k) \end{aligned} \tag{3.33}$$

possess a compact invariant set which is asymptotically stable with a domain of attraction that contains the prescribed set of initial conditions. To make this assumption precise, let W_k be the set of admissible initial conditions of w_k, let S_k be the set of admissible initial conditions of ζ_k and Z_k the set of admissible initial conditions of z_k. Then, the standing assumption can be formulated as follows.

Assumption 3.3. *There exists a (possibly set-valued) map* $\pi_k : w_k \in W \mapsto \pi_k(w_k) \subset \mathbb{R}^{n_k-1}$ *such that the set*

$$\mathcal{A}_k \quad = \quad \{(\zeta_k, w_k, z_k) : \zeta_k = 0, w_k \in W, z_k \in \pi_k(w_k)\}$$

is an asymptotically stable invariant set for (3.33) with a domain of attraction containing $S_k \times W_k \times Z_k$.

We note that this assumption is the natural formulation, in the current framework of a networked system, of the (weak) minimum-phase assumption that one would assume in solving a problem of output regulation for the k-th agent if high-gain arguments were to be used for stabilization purposes.

We proceed now with the design of the functions $(\varphi_k(\cdot), \gamma_k(\cdot), G_k)$ in (3.32), whose key properties are captured in the following definition, taken from Marconi and Isidori (2007).

Definition: The triplet $(\varphi_k(\cdot), \gamma_k(\cdot), G_k)$ is said to have the asymptotic internal model property if there exists a C^1 map $\tau_k : \mathbb{R}^d \times \mathbb{R}^{n_k-1} \to \mathbb{R}^{m_k}$ such that the following holds:

i) for all $(w_k, z_k) \in \mathrm{gr}(\pi_k)$

$$\begin{aligned} \frac{\partial \tau_k}{\partial w_k} s(w_k) + \frac{\partial \tau_k}{\partial z_k} f_k(z_k, Cw_k) &= \varphi_k(\tau_k(w_k, z_k)) \\ -\frac{q_k(w_k, z_k, 0, 0)}{b_k(w_k, Cw_k)} &= \gamma_k(\tau_k(w_k, z_k)) \end{aligned}$$

ii) the set

$$\mathcal{S}_k = \{(\zeta_k, w_k, z_k, \eta_k) \,:\, \zeta_k = 0, (w_k, z_k) \in \mathrm{gr}(\pi_k), \eta_k = \tau_k(w_k, z_k)\}$$

is locally asymptotically stable for the system

$$\begin{aligned} \dot{\zeta}_k &= \psi(\zeta_k, w_k) \\ \dot{w}_k &= s(w_k) + \Upsilon_k \zeta_k \\ \dot{z}_k &= f_k(z_k, C w_k) \\ \dot{\eta}_k &= \varphi_k(\eta_k) - G_k\Big[\gamma_k(\eta_k) + \frac{q_k(w_k, z_k, \zeta_k, 0)}{b_k(w_k, C w_k)}\Big] \end{aligned}$$

with a domain of attraction containing $S_k \times W_k \times Z_k \times M_k$, where M_k is the compact set of initial conditions of (3.32).

If a triplet with the asymptotic internal model property can be designed then the problem of steering the regulation error e_k of the k-th agent to zero is solved as claimed by the following lemma proved in Marconi et al. (2007).

Lemma 3.2. *Let $S_k \subset \mathbb{R}^{(N-1)d}$, $W_k \subset \mathbb{R}^d$, $Z_k \subset \mathbb{R}^{n_k-1}$, $E_k \subset \mathbb{R}$ and $M_k \subset \mathbb{R}^{m_k}$ be compact sets of initial conditions for the closed-loop system (3.31), (3.32). Let the triplet $(\varphi_k(\cdot), \gamma_k(\cdot), G_k)$ be designed so that it has the asymptotic internal model property. Then there exists a continuous function $\kappa_k : \mathbb{R} \to \mathbb{R}$ such that the trajectories of the closed-loop system originating from $S_k \times W_k \times Z_k \times E_k \times M_k$ are bounded and $\lim_{t\to\infty} e_k(t) = 0$ uniformly in the initial conditions.*

A triplet having the asymptotic internal model property always exists as detailed in the next result coming from a slight adaptation of the results presented in Marconi et al. (2007).

Proposition 3.3. *Let $m_k \geq 2(d + n_k - 1) + 2$. Then there exists a $\lambda_k^* < 0$ and, for almost colall the possible choices of controllable pairs $(F_k, G_k) \in \mathbb{R}^{m_k \times m_k} \times \mathbb{R}^{m_k \times 1}$ such that colall eigenvalues of F_k have a real part which is less then or equal to λ_k^*, there exists a continuous $\gamma_k : \mathbb{R}^{m_k} \to \mathbb{R}$, such that the triplet $(\varphi_k(\cdot), \gamma_k(\cdot), G_k)$ with $\varphi_k(\eta_k) = F_k\eta_k + G_k\gamma_k(\eta_k)$ has the asymptotic internal model property.*

The previous results, although conceptually interesting, is not constructive in the design of the function $\gamma_k(\cdot)$. The reader is referred to Marconi and Praly (2008) for practical numerical design of the function.

As shown in Byrnes and Isidori (2004), it turns out that a constructive design procedure can be given if an extra assumption is invoked. In particular, assume that there exists a $m_k > 0$ and a locally Lipschitz function $\varrho_k : \mathbb{R}^{m_k} \to \mathbb{R}$ with the property that, for all $(w_k(0), z_k(0)) \in \mathrm{gr}(\pi_k)$, the solution $w_k(t), z_k(t)$ of

$$\begin{aligned} \dot{w}_k &= s(w_k) \\ \dot{z}_k &= f_k(z_k, Cw_k) \end{aligned}$$

is such that the function

$$\rho(t) = -\frac{q_k(w_k(t), z_k(t), 0, 0)}{b_k(w_k(t), Cw_k(t))}$$

satisfies

$$\rho^{(m_k)}(t) = \varrho_k(\rho(t), \rho^{(1)}(t), \ldots, \rho^{(m_k-1)}(t)) \quad \forall\, t \in \mathbb{R}\,.$$

If this assumption holds then the following result can be proved, by means of a slight adaptation of the results presented in Marconi and Isidori (2007).

Proposition 3.4. *Let $(A_k, B_k, C_k) \in \mathbb{R}^{m_k \times m_k} \times \mathbb{R}^{m_k \times 1} \times \mathbb{R}^{1 \times m_k}$ be a triplet of matrices in prime form. Furthermore, let $\bar{\varrho}_k : \mathbb{R}^{m_k} \to \mathbb{R}$ be a bounded locally Lipschitz function that agrees with $\varrho_k(\cdot)$ on $B_R = \{\xi \in \mathbb{R}^{m_k} : \|\xi\| \le R\}$, let $D_\ell = \mathrm{diag}(\ell, \ell^2, \ldots, \ell^{m_k})$ with ℓ a positive design parameter, and let $(c_0, \ldots, c_{m_k-1})$ be such that the polynomial $\lambda^{m_k} + c_0\lambda^{m_k-1} + \ldots + c_{m_k-1}$ is Hurwitz. Then there exist $R > 0$ and $\ell^* > 0$ such that for all $\ell \ge \ell^*$ the triplet $(\varphi_k(\cdot), \gamma_k(\cdot), G_k)$ defined as*

$$\varphi_k(\eta_k) = A_k\eta_k + B_k\bar{\varrho}_k(\eta_k)\,, \quad \gamma_k(\eta_k) = C_k\eta_k\,,$$

$G_k = D_\ell\, col(c_0, \ldots, c_{m_k-1})$ has the asymptotic internal model property.

Implicit in the proof of Lemma 3.2 above is the fact that the choice of the gain function $\kappa_k(\cdot)$ in the "local" controller of agent k depends on the choice of compact sets W_k, S_k, Z_k, E_k, M_k of admissible initial conditions of the various components of (3.31)–(3.32). In this respect, it should be observed that, while W_k, Z_k, E_k, M_k are sets associated with the k-th agent and its local controller, the set S_k depends by definition on *all* compact sets W_j, $j = 1, \ldots, N$, that is on the sets of admissible initial conditions of *all* "local reference generators". If all sets W_j are known *a priori*, and so is therefore S_k, then the function $\kappa_k(\cdot)$ is determined only by the choice of the sets W_k, Z_k, E_k, M_k of admissible initial conditions of the k-th agent and its local controller. However, if this is not the case, *i.e.* if at the time of the design of the "local" controller of agent k the sets S_k are not directly available, the previous design strategy must be enhanced, for instance in the following way.

The reason why the knowledge of the compact set S_k is needed for the choice of $\kappa_k(\cdot)$ resides in the high-gain arguments used in the proof of Lemma 3.2. In fact, $\kappa_k(\cdot)$

is expected to dominate the influence of $q_k(\cdot)$ in the dynamics of e_k, so as to make the set $\mathcal{A}_k \times \{0\}$ locally asymptotically stable for system (3.31). Since the function $q_k(\cdot)$ depends on ζ_k, directly through the term $C\Upsilon\zeta_k$ and "indirectly" through the influence of ζ_k on the dynamics of w_k, it turns out that the knowledge of S_k is crucial in enforcing the asymptotic properties of the closed-loop system. In order to avoid the necessity of knowing S_k, a possibility would be to define a reference signal for the output of the k-th agent, to the purpose of asymptotically recovering the "true" reference signal Cw_k but that, in the initial transient, is bounded along with its time derivative by a bound not dependent on the initial conditions of ζ_k.

By following this intuition, consider a controller of the form (3.25) and (3.32) but with the error e_k defined as

$$e_k = y_k - C\xi_k \tag{3.34}$$

with ξ_k generated by

$$\dot{\xi}_k = S\xi_k + B\phi(\xi_k) - D_h G\,(\,C\xi_k - \theta_{\rm s}(w_k)\,) \tag{3.35}$$

where $\theta_{\rm s} : \mathbb{R}^d \to \mathbb{R}$ is a smooth *bounded* function such that, colfor some $\bar{c} > 0$,

$$\|w_k\|_W \leq \bar{c} \qquad \Rightarrow \qquad \theta_{\rm s}(w_k) = Cw_k\,,$$

$G = \mathrm{col}(c_0, c_1, \ldots, c_{d-1})$ with the c_i's such that the polynomial $\lambda^d + c_o\lambda^{d-1} + \ldots + c_{d-1}$ is Hurwitz, and $D_h = \mathrm{diag}(h,\, h^2,\, \ldots, h^d)$ with h a positive design parameter. It turns out that it is possible to choose the design parameters of (3.35) in such a way that $C\xi_k(t)$ asymptotically converges to the local reference signal $Cw_k(t)$, uniformly with respect to the initial conditions of ζ_k. This is detailed in the next proposition, in which by $\Xi_k \subset \mathbb{R}^d$ we denote the compact set of initial conditions for ξ_k.

Proposition 3.5. *There exist a positive $h^\star$ and, for all $h \geq h^\star$, positive constants d_1 and d_2, such that for all $\xi_k(0) \in \Xi_k$, $\zeta_k(0) \in \mathbb{R}^{(N-1)d}$, $w_k(0) \in \mathbb{R}^d$, the trajectories of system (3.29) and (3.35) satisfy $\|\xi_k(t)\| \leq d_1$, $\|\dot{\xi}_k(t)\| \leq d_2$, and*

$$\lim_{t\to\infty}\,(\,\xi_k(t) - w_k(t)\,) = 0\,. \tag{3.36}$$

Proof. Let $\bar{\xi}_k = D_h^{-1}\xi_k$ and observe that $D_h^{-1}SD_h = hS$, $D_h^{-1}B = \frac{1}{h^d}B$, $CD_h = hC$. The rescaled dynamics read as

$$\dot{\bar{\xi}}_k = h(S - GC)\bar{\xi}_k + \frac{1}{h^d}B\phi(D_h\bar{\xi}_k) + G\theta_s(w_k)$$

in which $(S - GC)$ is a Hurwitz matrix. From this, the existence of the constants d_1 and d_2 follows from standard high-gain arguments by using boundedness of $\theta_s(\cdot)$ and the fact that ϕ is globally Lipschitz. To prove (3.36), note that, by Proposition 3.1, there exists a time $t^\star > 0$ such that $\theta_s(w_k(t)) = Cw_k(t)$ for all $t \geq t^\star$. By changing coordinates as $\xi_k \mapsto \tilde{\xi}_k = D_h^{-1}(\xi_k - w_k)$, system (3.35) for $t \geq t^\star$ is described by

$$\dot{\tilde{\xi}}_k = h(S - GC)\tilde{\xi}_k + \frac{1}{h^d} B\Delta\phi(\tilde{\xi}_k, w_k) + D_h^{-1}\Upsilon_k\zeta_k$$

in which $\Delta\phi(\tilde{\xi}_k, w_k) = \phi(D_h\tilde{\xi}_k + w_k) - \phi(w_k)$. The result then follows by the same high-gain arguments above, using the fact that ζ_k converges asymptotically to zero. □

Using the reference signal $C\xi_r$ instead of Cw_r in the definition of the error e_k guarantees that the overall closed loop system in the error coordinates is still written as in (3.31) but with $q_k(\cdot)$ defined as

$$\begin{aligned} q_k(w_k, z_k, \xi_k, e_k) &= a_k(z_k, C\xi_k + e_k) - C\dot{\xi}_k \\ &= a_k(z_k, C\xi_k + e_k) - C[s(\xi_k) + D_h G(C\xi_k - \theta_s(w_k))]\,. \end{aligned}$$

This, in turn, is a locally Lipschitz function with a bound on the Lipschitz constant that is uniform with respect to the initial condition of ζ_k. This fact makes it possible to continue the analysis as we did in the first part of the section and to claim a result similar to the one of Lemma 3.2, with a gain function $\kappa_k(\cdot)$ depending only on the sets of admissible initial conditions of ξ_k, z_k, e_k, η_k and not affected by the set of initial conditions of (w_k, ζ_k). Details are omitted, since the analysis follows, with simple adaptations, the one presented above.

3.2.4 Simulation Results

In this section, we present simulation results for a network of heterogeneous nonlinear agents. We consider the case of three uncertain heterogeneous systems described by

$$\begin{aligned} \dot{x}_{k1} &= x_{k2} \\ \dot{x}_{k2} &= a_k(\mu_k, x_{k1}, x_{k2}) + u_k \end{aligned} \qquad y_k = x_{k1} \tag{3.37}$$

where

$$a_k(\mu_k, x_{k1}, x_{k2}) = \begin{cases} -\mu_k\, x_{k1} & k = 1 \\ \mu_k\,(1 - x_{k1}^2)x_{k2} - x_{k1} & k = 2 \\ -\mu_k\, x_{k1} + x_{k1}^3 & k = 3 \end{cases}$$

with states $(x_{k1}, x_{k2}) \in \mathbb{R} \times \mathbb{R}$, control input $u_k \in \mathbb{R}$, output $y_k \in \mathbb{R}$, and uncertain parameter $\mu_k \in \mathbb{R}$, $k = 1, 2, 3$. It is readily seen that the three agents are respectively given by an uncertain harmonic ($k = 1$), Van der Pol ($k = 2$) and Duffing ($k = 3$) controlled oscillator. In the following we assume that the uncertainties μ_k are constant, i.e. they fulfill $\dot{\mu}_k = 0$, and their value ranges in a known compact set P_k, $k = 1, 2, 3$. Our goal is to control the three agents so that their outputs achieve a consensus on a trajectory $y^\star(t)$ generated by an exosystem of the form (3.2) and (3.3) with $d = 2$ and $\phi(w) = 2(1 - w_1^2)w_2 - w_1$. The reference output is thus generated by a Van der Pol oscillator. The controllers exchange information through a connected communication network described by the Laplacian matrix

$$L = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

By following the theory presented in Section 3.1, the three controllers are given by “local reference generators", which in this case take the form

$$\begin{pmatrix} \dot{w}_{k1} \\ \dot{w}_{k2} \end{pmatrix} = \begin{pmatrix} w_{k2} \\ 2(1 - w_{k1}^2)w_{k2} - w_{k1} \end{pmatrix} + K \sum_{j=1}^{3} a_{kj}(w_{j1} - w_{k1}) \tag{3.38}$$

with K designed according to Lemma 3.1 and Proposition 3.1, and “local regulators" that process a local regulation error $e_k = x_{k1} - w_{k1}$. As shown in Section 3.2.3, by changing coordinates in the appropriate way for each $k = 1, 2, 3$, the local reference generator (3.38) can be written as in (3.29) with $\Upsilon_k = (\Upsilon_{k1}\, \Upsilon_{k2})^T$ a vector of $\mathbb{R}^2$ suitably defined and with ζ_k an asymptotically vanishing state variable that depends on the state of all local reference generators.

In order to give the expressions of the local regulators some preliminary steps are necessary since the three heterogenous systems (3.37) have relative degree 2 from the input u_k to the output y_k and thus they are not in the class of systems considered in Section 3.2.3. To fit in the class of the relative-degree 1 systems considered above, consider the change of variables

$$\begin{aligned} x_{k1} &\mapsto e_{k1} = x_{k1} - w_{k1} \\ x_{k2} &\mapsto e_{k2} = x_{k2} - w_{k2} - \Upsilon_{k1}\zeta_k + e_{k1} \end{aligned}$$

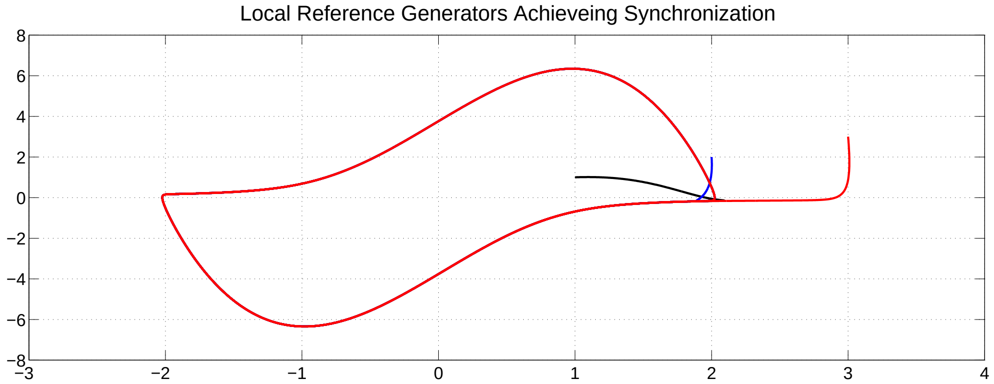

Figure 3.6: Synchronization of the 3 Local Reference Generators, defined as Van Der Pol oscillators.

by which the dynamics associated to the k-th agent are given by (3.29), $\dot{\mu}_k = 0$, and

$$\begin{aligned} \dot{e}_{k1} &= -e_{k1} + e_{k2} \\ \dot{e}_{k2} &= a_k(\mu_k, e_{k1} + w_{k1}, e_{k2} + w_{k2} + \Upsilon_{k1}\zeta_k + e_{k1}) - \\ &\qquad \phi(w_k) - \Upsilon_{k2}\zeta_k - \Upsilon_{k1}\psi(\zeta_k, w_k) + u_k \,. \end{aligned}$$

It is readily seen that this system is formally equivalent to (3.31) by replacing the state variables z_k and e_k in (3.31) respectively with (μ_k, e_{k1}) and e_{k_2} and with the function $q_k(w_k, z_k, \zeta_k, e_k)$, $f_k(z_k, w_k)$, and $b_k(z_k, w_k)$ in (3.31) taking the form

$$\begin{aligned} & q_k(w_k, (\mu_k, e_{k1}), \zeta_k, , e_{k2}) = \\ & \quad a_k(\mu_k, e_{k1} + w_{k1}, e_{k2} + w_{k2} + \Upsilon_{k1}\zeta_k + e_{k1}) - \phi(w_k) - \\ & \qquad \Upsilon_{k2}\zeta_k - \Upsilon_{k1}\psi(\zeta_k, w_k) \,. \end{aligned}$$

$f_k(z_k, w_k) = \begin{pmatrix} 0 & -e_{k1} + e_{k2} \end{pmatrix}^T$ and $b_k(z_k, w_k) = 1$. In particular, we observe that Assumption 3.2 is fulfilled with W given by the (compact) set consisting of all points lying on the limit cycle of the Van der Pol oscillator and in its interior, and

$$\pi_k(w_k) = \{(\mu_k, e_{k1}) \in P_k \times \mathbb{R} \;:\; e_{k1} = 0\} \,. \tag{3.39}$$

By following Lemma 3.2 and Proposition 3.3, we thus concentrate on an internal model-based regulator of the form

$$\begin{aligned} \dot{\eta}_k &= F_k\eta_k + G_k\gamma_k(\eta_k) + G_k v_k \\ u &= \gamma_k(\eta_k) + v_k \\ v_k &= \kappa_k(e_{k2}) \end{aligned} \tag{3.40}$$

in which $(F_k, G_k) \in \mathbb{R}^{m_k \times m_k} \times \mathbb{R}^{m_k}$ is a controllable pair with F_k Hurwitz, $\gamma_k(\cdot)$ is a suitably defined nonlinear function designed so that the previous system has the "asymptotic internal model property", and $\kappa_k(\cdot)$ is a suitably defined "high-gain" stabilizer. Since, by definition, $e_{k2} = \dot{e}_k + e_k$ with $e_k = x_{k1} - w_{k1}$ the "true" regulation error, a pure "error feedback" regulator can be always obtained from (3.40) by substituting e_{k2} with an estimate $\hat{e}_{k2}$ provided by a "dirty derivatives observer" of the form (see Esfandiari and Khalil (1992))

$$\begin{aligned} \dot{\hat{e}}_{k1} &= \hat{e}_{k2} + \ell c_0 (e_{k1} - \hat{e}_{k1}) \\ \dot{\hat{e}}_{k2} &= \ell^2 c_1 (e_{k1} - \hat{e}_{k1}) \end{aligned} \tag{3.41}$$

in which ℓ is a design parameter and the c_i's taken so that the polynomial $\lambda^2 + c_0\lambda + c_1$ is Hurwitz. The simulation results that follows have been thus obtained by considering local controllers of the form (3.38), (3.40) with e_{k2} substituted by $\hat{e}_{k2}$, and (3.41). The design parameters of this controller have been fixed in the following way.

The K in (3.38) has been fixed as $K = D_g K_0$ with $g = 10$ and K_0 as in Lemma 3.1 with $a = 1$. The triplet $(F_k, G_k, \gamma_k(\cdot))$ in (3.40) has been fixed so that it has the "asymptotic internal model property". Since F_k is Hurwitz, by bearing in mind the expression of $\pi_k(w_k)$ in (3.39), it follows that the requirements (i) and (ii) in the definition are fulfilled if $\gamma_k(\cdot)$ is designed so that

$$\gamma_k \circ \tau_k(w_k, \mu_k) = a_k(\mu_k, w_{k1}, w_{k2}) - \phi(w_k)$$

where $\tau_k(\cdot)$ is the solution of the PDE

$$\frac{\partial \tau(w_k, \mu_k)}{\partial w_k} s(w_k) = F_k \tau_k(w_k, \mu_k) + G_k(a_k(\mu_k, w_{k1}, w_{k2}) - \phi(w_k))$$

for all $(w_k, \mu_k) \in W \times P_k$. By following the results in Marconi et al. (2007), which are at the basis of Proposition 3.3, a $\gamma_k(\cdot)$ and a $\tau_k(\cdot)$ fulfilling the above relations can be always designed if the dimension m_k of (3.38) is taken $m_k \geq 6$. In the simulation we have thus chosen $m_k = 6$ with F_k and G_k in prime form with the eigenvalues of F_k taken as $\{-1, -2+j, -2-j, -3, -4+2j, -4-2j\}$ for all the agents. The expression of $\gamma_k(\cdot)$ has been then obtained by using the approximated numerical formula presented in Marconi and Praly (2008). The tuning of the internal model-based regulator (3.40) has been achieved by taking the stabilizer as $\kappa_k(s) = -\kappa s$ with κ that has been fixed, after a few numerical tests, to $\kappa = 15$. Finally, the dirty derivatives observer (3.41) has been tuned by taking $c_0 = c_1 = 1$ and $\ell = 20$.

The uncertainties and the initial conditions of the three agents are set as $\mu_1 = 2$,

$\mu_2 = 3$, $\mu_3 = 1$ and $x_1 = (5,5)$, $x_2 = (7,10)$, $x_3 = (1,1)$ and zero initial conditions for (3.40) and (3.41). We observe that the value of μ_2 has been fixed so that the Van der Pol dynamics of the system (3.37) with $k = 2$ is different from the Van der Pol oscillator that generates the reference output. The local reference generators trajectories are shown in Figure 3.6. The tracking results instead are shown in Figure 3.7 for each single agent and for both components. The *dot-dashed* red lines are the references generated locally, while the black lines represent the state components of the agents.

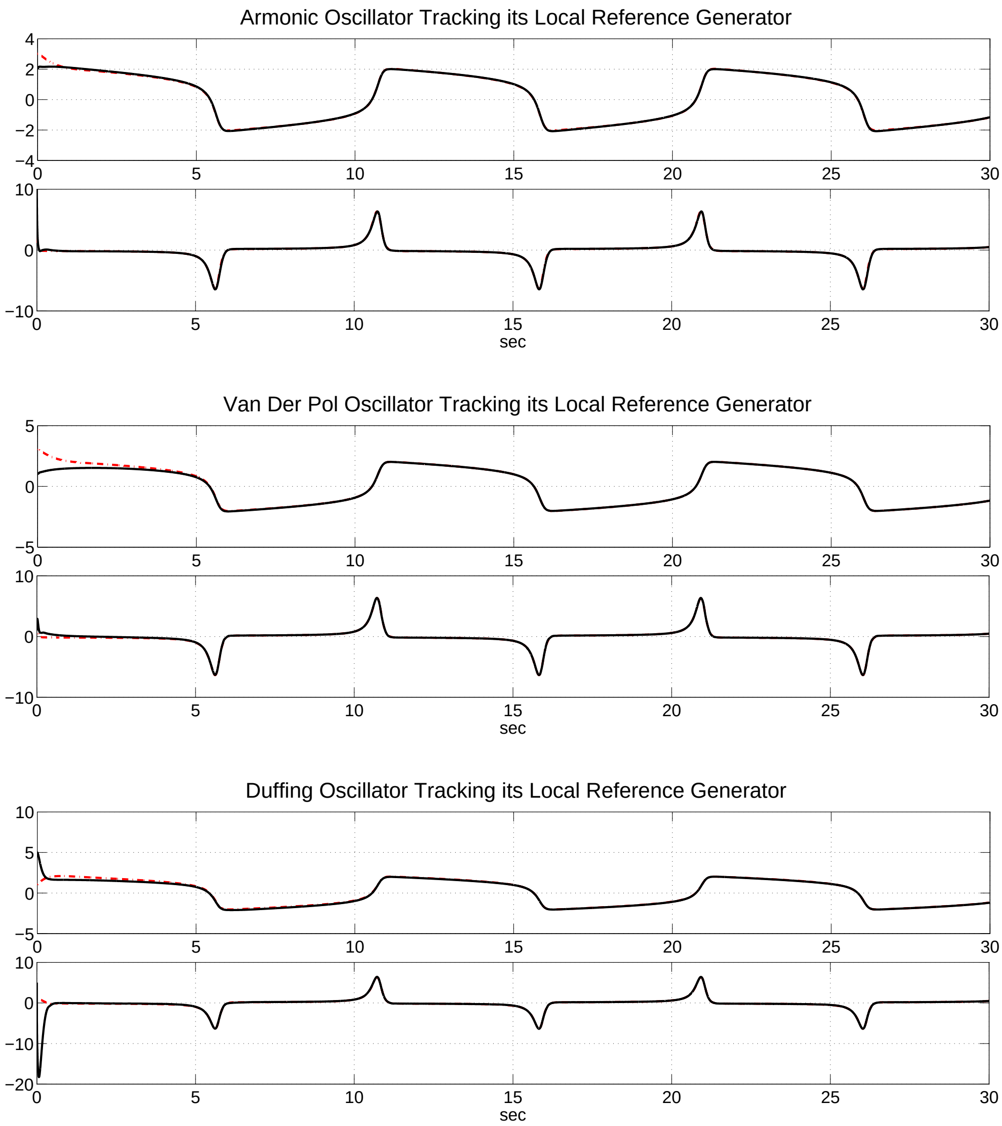

Figure 3.7: Tracking of the Local Reference generators for the three agents 3.37.

We cannot live only for ourselves. A thousand fibers connect us with our fellow men; and among those fibers, as sympathetic threads, our actions run as causes, and they come back to us as effects.

Herman Melville

4

Dynamical Edges

An interesting and challenging aspect of consensus and synchronization problems is to consider dynamical systems as links: the introduction of dynamics on the links complicates the problem consistently, since the agents are not allowed to communicate directly anymore. The information available at each agents is *filtered* by the links and this aspect has to be taken into account carefully.

The problem of dynamical links is motivated by applications in electrical, hydraulic and transportation networks framework, in which links cannot be considered ideal (see e.g. Torres et al. (2015), Dhople et al. (2014), for the case of electrical networks and Trip et al. (2014) for the case of hydraulic networks). Dynamical links can also appear as the result of a design process aiming to control the edge flow in distribution networks Bürger and De Persis (2015).

Driven by these motivations, a few other authors have investigated the synchronization problem over networks with dynamic links. In Hill and Chen (2006), a set of linear systems is interconnected via dynamic linear links and the node dynamics is controlled only through the outputs of edge dynamic systems, while the edge dynamics can not be directly controlled. In general, most of the results that can be found in literature rely on *passivity arguments*, assuring that the interconnection of systems and links preserves an overall passive (and with certain extent, incremental) property.

In this chapter we study the problem of synchronization of nonlinear oscillators with dynamic linear links: in contrast with the passivity arguments used in Hill and Chen (2006), Torres et al. (2015) and Dhople et al. (2014), the analysis here presented can be seen as an extension of the results of Section 3.1 to the case of dynamic links. The passivity assumption are thus substituted by ISS assumptions on the systems at hand and the synchronization analysis is performed by means of Lyapunov arguments.

As a first step, we consider the problem of dynamic links with algebraic connection between the their input and their output: this scenario can be seen as an extension of the results in Section 3.1 in presence of disturbances. In the second part of this chapter, we consider the case of a generic dynamical link dynamics: we modify consistently the structure of the control architecture, by introducing a set of local observers to retrieve the information necessary to achieve synchronization. The first part of this chapter is inspired by Casadei et al. (2014a), while the second part is totally novel.

In order to facilitate the understanding of the problems at hand and of the proposed solution, we will often refer to the case of electrical power networks. However, the results presented in this chapter are purely theoretical and thus can be adapted to all other possible scenarios.

4.1 Motivation to the problem

Kron reduction is a well known procedure in classical electrical theory which allows to simplify the complexity of a network. By separating the nodes into *boundary nodes* and *interior nodes*, it is possible to reduce the network dimension by eliminating the interior nodes and obtain a *reduced model* of the network.

In the electrical framework, boundary nodes represents power injection nodes while interior nodes are passive nodes of the network. The idea is to iteratively simplify the network eliminating the interior nodes, to obtain an equivalent network representation.

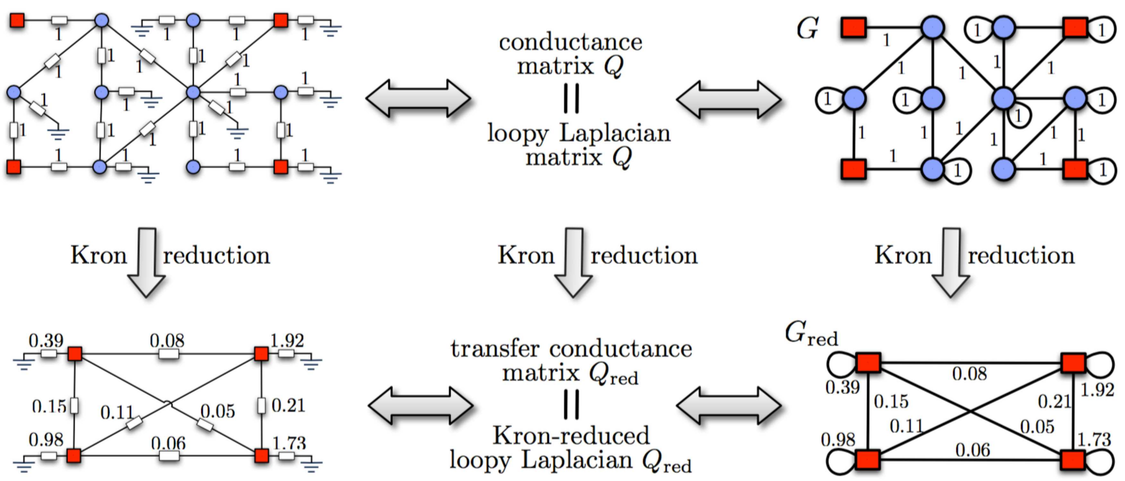

Figure 4.1: Example of Kron reduction and resulting graph representation of the network.

The example in Figure 4.1 (taken from Dörfler and Bullo (2011)) shows how Kron reduction simplifies the electrical network in the upper left corner towards the one below: on the left side of the figure instead, the graph representation is shown. The red *squares* are the boundary nodes while the blue *circles* are the interior nodes, which are eliminated to obtain an equivalent reduced model of the network.

The resulting network is a *loopy* graph, in which *self-loop* at each boundary node is allowed. The graph is thus described by a *loopy* Laplacian matrix L which can be though as being generated by two components

$$L = Q + \Xi$$

where Q is the conventional Laplacian matrix describing the interconnection between the boundary nodes

$$\begin{aligned} q_{kj} &= -a_{kj} \qquad \text{for } k \neq j \\ q_{kj} &= \sum_{i=1}^{N} a_{ki} \quad \text{for } k = j\,, \end{aligned}$$

while Ξ is a diagonal matrix

$$\Xi = \mathrm{diag}(\xi_1, \ldots, \xi_N)$$

with $\xi_i \in \mathbb{R}_{\geq 0}$, which describes the self-loop at each node.

Within the electrical network framework, Q describes the interconnection between power suppliers, while Ξ describes the equivalent load at each boundary node. On one hand, Q is the medium through which synchronization can be achieved, while Ξ can be seen as a the input matrix of a disturbance load to be rejected.

Motivated by this representation, we want to address the problem of synchronization of nonlinear agents connected via dynamical links. The impact of the loads will be investigated briefly in Section 4.2.4 as an extension of the result presented in Proposition 1 in Section 4.2

4.2 Synchronization of nonlinear oscillators with Dynamical Edges

In this section we study the problem of synchronization of nonlinear oscillators connected via dynamical links: first we present the problem and the assumptions under which the problem is solved. Then we give the main result. Simulations results are shown to illustrate the behavior of the network.

4.2.1 Preliminaries on the system and problem formulation

The N nonlinear agents are described by

$$\begin{array}{rcl} \dot{x}_j & = & Sx_j + B\phi(x_j) + u_j \\ y_j & = & Cx_j \end{array} \qquad (4.1)$$

with $x_j \in \mathbb{R}^d$, where u_j is the diffusive coupling control input to be defined, (S, B, C) is a triplet in prime form and $\phi(\cdot) : \mathbb{R}^d \to \mathbb{R}$ is globally Lipschitz, namely there exists a $\bar{\phi} > 0$ such that

$$\|\phi(x_1) - \phi(x_2)\| \leq \bar{\phi}\|x_1 - x_2\| \qquad \forall\, x_1,\, x_2 \in \mathbb{R}^d\,.$$

All the forthcoming results can be easily adapted to deal with the case in which the initial state of the agents ranges in a fixed (although arbitrary) compact set. In such a case the assumption on $\phi(\cdot)$ can be weakened by asking that it is just *locally* Lipschitz. For further details on the systems at hand, we refer the reader to Appendix A.

Furthermore, as in Section 3.1, it is supposed that system (4.1) is ISS with respect to the input u_j relative to a compact set.

Assumption 4.1. *There exists a compact set $X \subset \mathbb{R}^d$ such that the system*

$$\dot{x}_j = Sx_j + B\phi(x_j) + u_j$$

is input-to-state stable with respect to u_j relative to X, namely there exist a class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a class-$\mathcal{K}$ function $\gamma(\cdot)$ such that[1]

$$\|x_j(t, \bar{x}_j)\|_X \leq \max\{\beta(\|\bar{x}_j\|_X, t),\ \gamma(\sup_{\tau\in[0,t)} \|u_j(\tau)\|)\}\,.$$

[1] Here we denote by $\|x_j\|_X = \min_{w\in X} \|x - w\|$ the distance of x_j from X. Furthermore, $x_j(t, \bar{x}_j)$ denotes the solution of (4.1) at time t with initial condition $\bar{x}_j$ at time $t = 0$.

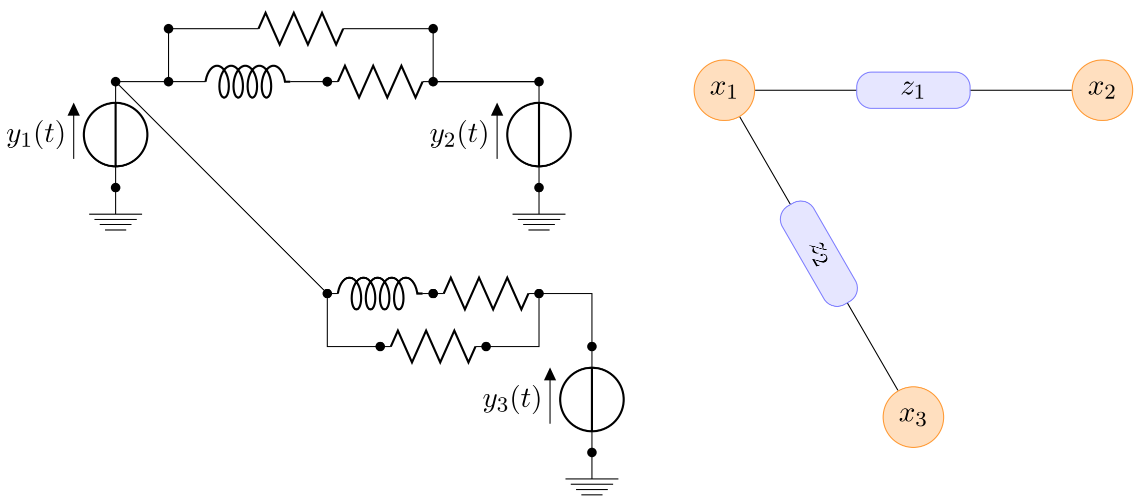

Figure 4.2: A simple example of connection of three agents with two dynamic links.

The M links are described by the linear dynamics

$$\begin{aligned} \dot{z}_k &= A_\ell z_k + B_\ell \sum_{j=1}^{N} \iota_{jk} y_j \\ p_k &= C_\ell z_k + D_\ell \sum_{j=1}^{N} \iota_{jk} y_j \end{aligned} \tag{4.2}$$

for $k = 1, \ldots, M$, with $z_k \in \mathbb{R}^l$, $p_k \in \mathbb{R}$ and ι_{kj} element of the incidence matrix $\mathcal{I}$.

We assume that the dynamics in (4.2) are Hurwitz with a well-defined stability margin. In particular we assume the following.

Assumption 4.2. *There exists a matrix $P_\ell = P_\ell^T > 0$ such that*

$$P_\ell A_\ell + A_\ell^T P_\ell \leq -a_\ell I$$

where a_ℓ is a positive stability margin.

The stability margin a_ℓ will play a crucial role in the following stability analysis.

Thinking about the electrical framework, the output of the N agents y_j, for $j = 1, \ldots, N$, represents a voltage applied to a link. The difference of voltages applied to a link, described by the input term $\sum_{j=1}^{N} \iota_{kj} y_j$, determines the current flowing through the link, namely the output of the link p_k. The outputs of the links are the available *quantity* to be processed to construct u_j, for $j = 1, \ldots, N$, in such a way that synchronization of the output of the N agents is achieved.

Figure 4.6 shows an example of possible interconnection of three agents with two dynamic links: the current z_1 flowing from agent 1 to agent 2, depends on the the difference of the outputs $y_1 - y_2$ through the incidence matrix $\mathcal{I}$. Similarly, the current z_2 flowing from agent 1 to agent 3, depends on the difference of the outputs $y_1 - y_3$ through the incidence matrix $\mathcal{I}$.

The result presented in the next section relies upon a connectivity assumption of the graph $\mathcal{G}$ precisely formulated in the following.

Assumption 4.3. *The graph $\mathcal{G}$ is connected, namely there exists a constant $\mu > 0$ such that*

$$\mu \leq \lambda_i(L) \quad \text{for all} \quad i = 2, \dots, N. \tag{4.3}$$

Furthermore, it is worth noticing that in this scenario the Laplacian matrix is intrinsically symmetric. Thus, it admits a purely diagonal Jordan form.

Our goal is to define the control input u_j, for $j = 1, \dots, N$, in such a way that the outputs of the N systems achieve synchronization. Namely, the positive orbit of the set of all admissible initial conditions of (4.1) is bounded and output consensus is reached, *i.e.* for each admissible initial condition $x_j(0) \in \mathbb{R}^d$, $j = 1, \dots, N$, $x_i^\ell(0) \in \mathbb{R}^l$ with $i = 1, \dots, M$, there is a function $y^\star : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{t\to\infty} |y_j(t) - y^\star(t)| = 0 \qquad \text{for all } j = 1, \dots, N.$$

4.2.2 Main Result

To achieve synchronization, we chose u_j of the form

$$u_j = -K \sum_{i=1}^{M} \iota_{ji} p_i \tag{4.4}$$

with K a design parameter to be defined and ι_{ji} is the ji-th element of the incidence matrix $\mathcal{I}$.

With an eye to the solution given in Chapter 3.1, the vector K is chosen as

$$K = D(g)K_0 \qquad K_0 = PC^T \tag{4.5}$$

where $D(g) \in \mathbb{R}^{d\times d}$ is a diagonal matrix $D(g) = \mathrm{diag}(g, g^2, \dots, g^d)$ with g a design parameter, and P is solution of the Riccati equation

$$SP + PS^T - 2\mu PC^TCP = -aI \tag{4.6}$$

with μ introduced in Assumption 4.3 and a a positive constant.

By collecting the nodes dynamics and outputs as $\mathbf{x} = \mathrm{col}(x_1, \dots, x_N)$, $\mathbf{y} = \mathrm{col}(y_1, \dots, y_N)$ and the link dynamics and outputs as $\mathbf{z} = \mathrm{col}(z_1, \dots, z_M)$, $\mathbf{p} = \mathrm{col}(p_1, \dots, p_M)$, we can rewrite the dynamics of the network in compact form as

$$\begin{aligned}
\dot{\mathbf{x}} &= (I_N \otimes S)\mathbf{x} + (I_N \otimes B)\Phi(\mathbf{x}) - (I_N \otimes K)\mathcal{I}\mathbf{p} \\
\mathbf{y} &= (I_N \otimes C)\mathbf{x} \\
\dot{\mathbf{z}} &= (I_M \otimes A_\ell)\mathbf{z} + (I_M \otimes B_\ell)\mathcal{I}^T\mathbf{y} \\
\mathbf{p} &= (I_N \otimes C_\ell)\mathbf{z} + \mathcal{I}^T\mathbf{y}
\end{aligned} \tag{4.7}$$

with $\mathbf{x} \in \mathbb{R}^{dN}$, $\mathbf{z} \in \mathbb{R}^{lM}$, where, without loss of generality, we considered $D_\ell = 1$.

It turns out that the resulting closed-loop networked system reaches synchronization for an appropriate tuning of the parameter g provided that the stability margin of the link dynamics is sufficiently large in relation to the Lipschitz constant of the nonlinearity $\phi(\cdot)$. This is formalised in the next proposition.

Proposition 4.1. *Let Assumptions 4.1-4.3 hold. Consider the networked systems (4.7) controlled by the diffusive coupling control law (4.4) with K chosen as in (4.5) with $g = \varepsilon a_\ell$ with $\varepsilon > 0$. Then, there exist an $\varepsilon^\star \leq 1$ and, for all positive $\varepsilon \leq \varepsilon^\star$, an $a_\ell^\star > 0$ (dependent on the Lipschitz constant $\bar{\phi}$ of $\phi(\cdot)$) such that, for all $a_\ell \geq a_\ell^*$ the compact set*

$$\begin{aligned}
\mathbf{X} &= \{(x_1, x_2, \ldots, x_N) \in X \times X \times \cdots \times X \\
&\qquad : \; x_1 = x_2 = \cdots = x_N\}
\end{aligned} \tag{4.8}$$

is globally asymptotically stable. $\triangleleft$

Proof. We consider (4.7) and focus our attention on the dynamics of the nodes. With some simple mathematical computation we obtain

$$\begin{aligned}
\dot{\mathbf{x}} &= (I_N \otimes S)\mathbf{x} + (I_N \otimes B)\Phi(\mathbf{x}) - (I_N \otimes K)\mathcal{I}\left((I_M \otimes C_\ell)\mathbf{z} + \mathcal{I}^T\mathbf{y}\right) \\
&= (I_N \otimes S)\mathbf{x} + (I_N \otimes B)\Phi(\mathbf{x}) - (\mathcal{I} \otimes KC_\ell)\mathbf{z} - (I_N \otimes K)\mathcal{I}\mathcal{I}^T\mathbf{y}.
\end{aligned}$$

By remembering that, by definition $\mathcal{I}\mathcal{I}^T = L$

$$\begin{aligned}
\dot{\mathbf{x}} &= (I_N \otimes S)\mathbf{x} + (I_N \otimes B)\Phi(\mathbf{x}) - (\mathcal{I} \otimes KC_\ell)\mathbf{z} - (I_N \otimes K)L(I_N \otimes C)\mathbf{x} \\
&= (I_N \otimes S)\mathbf{x} + (I_N \otimes B)\Phi(\mathbf{x}) - (\mathcal{I} \otimes KC_\ell)\mathbf{z} - (L \otimes KC)\mathbf{x} \\
&= \left[(I_N \otimes S) - (L \otimes KC)\right]\mathbf{x} + (I_N \otimes B)\Phi(\mathbf{x}) - (\mathcal{I} \otimes KC_\ell)\mathbf{z}.
\end{aligned}$$

We define the change coordinate

$$\mathbf{x} \mapsto \chi = (T^{-1} \otimes I_d)\mathbf{x}$$

with T defined as

$$T = \begin{bmatrix} 1 & \mathbf{0}_{1\times(N-1)} \\ \mathbf{1}_{(N-1)} & I_{N-1} \end{bmatrix}. \tag{4.9}$$

It turns out that

$$\tilde{L} = T^{-1}LT = \begin{bmatrix} 0 & L_{12} \\ 0 & L_{22} \end{bmatrix}.$$

Note that, by definition of T, $\chi = \mathrm{col}(x_1,\, x_2 - x_1,\, \ldots,\, x_N - x_1)$. Thus, we split the systems into

$$\begin{aligned} \dot{\chi}_1 &= S\chi_1 + (L_{12} \otimes KC)\boldsymbol{\chi}_2 + B\phi(z_1) + K\mathcal{I}_1\mathbf{z} \\ \dot{\boldsymbol{\chi}}_2 &= [(I_{N-1} \otimes S) + (L_{22} \otimes KC)]\,\boldsymbol{\chi}_2 + \Delta\Phi(\chi_1, \boldsymbol{\chi}_2) + (\mathcal{I}_2 \otimes KC_\ell)\mathbf{z} \end{aligned}$$

where $\boldsymbol{\chi}_2 = \mathrm{col}(\chi_2, \ldots, \chi_N)$ and $\mathcal{I}_1, \mathcal{I}_2$ are such that

$$T^{-1}\mathcal{I} = \begin{bmatrix} \mathcal{I}_1 \\ \mathcal{I}_2 \end{bmatrix}$$

according to the dimensions of χ_1 and $\boldsymbol{\chi}_2$, and $\Delta\Phi(\chi_1, \boldsymbol{\chi}_2)$ is

$$\Delta\Phi(\chi_1, \boldsymbol{\chi}_2) = (I_{N-1} \otimes B) \begin{pmatrix} \phi(\chi_1 + \chi_2) - \phi(\chi_1) \\ \phi(\chi_1 + \chi_3) - \phi(\chi_1) \\ \vdots \\ \phi(\chi_1 + \chi_N) - \phi(\chi_1) \end{pmatrix}.$$

Note that there exists a $\bar{\Phi} > 0$ such that

$$\|\Delta\Phi(\chi_1, \boldsymbol{\chi}_2)\| \le \bar{\Phi}\|\boldsymbol{\chi}_2\| \quad \forall\, \chi_1 \in \mathbb{R}^d,\, \boldsymbol{\chi}_2 \in \mathbb{R}^{(N-1)d}.$$

With simple reasoning, global asymptotic stability of $\boldsymbol{\chi}_2$ and the ISS property of χ_1 claimed in Assumption 4.1 would lead to synchronization of the N agents. In other words, global asymptotic stability of $\boldsymbol{\chi}_2$, leads to $y_k \to y^\star$ for $k = 1, \ldots, N$.

As far as the dynamics of the links are concerned, global asymptotic stability of

$\boldsymbol{\chi}_2$ would mean that, since $y_k \to y^\star$ for $k = 1, \ldots, N$,

$$\mathcal{I}^T \mathbf{y} = 0.$$

This and the stability property on the links dynamics claimed in Assumption 4.2 imply $\mathbf{z} = 0$.

The link dynamics are rewritten according to the change of coordinates (4.9) as

$$\begin{aligned} \dot{\mathbf{z}} &= (I_M \otimes A_\ell)\mathbf{z} + (I_M \otimes B_\ell)\mathcal{I}^T (I_N \otimes C)(T \otimes I_d)\boldsymbol{\chi} \\ &= (I_M \otimes A_\ell)\mathbf{z} + (\mathcal{I}^T T \otimes B_\ell C)\boldsymbol{\chi} \end{aligned}$$

and it turns out that the first column of $\mathcal{I}^T T$ is always zero, meaning that the dynamics of the links $\mathbf{z}$ do not depend on χ_1. We define the matrix $\mathcal{I}_\chi$ as the matrix obtained by removing the first column of $\mathcal{I}^T T$.

The analysis is thus reduced to the interconnected system

$$\begin{aligned} \dot{\boldsymbol{\chi}}_2 &= \left[(I_{N-1} \otimes S) + (L_{22} \otimes KC)\right] \boldsymbol{\chi}_2 \\ &\quad + \Delta\Phi(\chi_1, \boldsymbol{\chi}_2) + (\mathcal{I}_2 \otimes KC_\ell)\mathbf{z} \\ \dot{\mathbf{z}} &= (I_M \otimes A_\ell)\mathbf{z} + (\mathcal{I}_\chi \otimes B_\ell C)\boldsymbol{\chi}_2 \end{aligned}$$

We now change coordinates according to

$$\zeta = (I_{N-1} \otimes D_g^{-1})\boldsymbol{\chi}_2$$

and obtain

$$\begin{aligned} \dot{\zeta} &= g\left[(I_{N-1} \otimes S) + (L_{22} \otimes K_0 C)\right]\zeta + \tfrac{1}{g^d}\Delta\Phi(z_1, (I_{N-1} \otimes Dg)\zeta) \\ &\quad + (I_{N-1} \otimes D_g^{-1})(\mathcal{I}_2 \otimes KC_\ell)\mathbf{z} \\ \dot{\mathbf{z}} &= (I_M \otimes A_\ell)\mathbf{z} + (\mathcal{I}_\chi \otimes B_\ell C)(I_{N-1} \otimes D_g)\zeta \end{aligned} \tag{4.10}$$

which, after some manipulations, becomes

$$\begin{aligned}\dot{\zeta} &= g\left[(I_{N-1}\otimes S)+(L_{22}\otimes K_0C)\right]\zeta+\tfrac{1}{g^d}\Delta\Phi(z_1,(I_{N-1}\otimes Dg)\zeta)\\ &\qquad +(\mathcal{I}_2\otimes K_0C_\ell)\mathbf{z}\\ \dot{\mathbf{z}} &= (I_M\otimes A_\ell)\mathbf{z}+(\mathcal{I}_\chi\otimes gB_\ell C)\zeta.\end{aligned}$$

Note that there exists a $\bar{\Phi}''$ such that

$$\|\Delta\Phi(z_1,(I_{N-1}\otimes Dg)\zeta)\|\leq\bar{\Phi}''\|\zeta\|$$

for all $z_1\in\mathbb{R}^d$. This implies that, by properly tuning the parameter g it is possible to dominate the nonlinear term

$$\frac{1}{g^d}\Delta\Phi(z_1,(I_{N-1}\otimes Dg)\zeta)$$

with the linear and Hurwitz term

$$g\left[(I_{N-1}\otimes S)+(L_{22}\otimes K_0C)\right]\zeta.$$

Consider now a Lyapunov function

$$V=\zeta^T\mathbf{P}\zeta+\mathbf{z}^T\mathbf{P}_\ell\mathbf{z}$$

with $\mathbf{P}=(I_{N-1}\otimes P)$ and P solution of the Riccati (4.6) and $\mathbf{P}_\ell=(I_M\otimes P_\ell)$ and P_ℓ as in Assumption 4.2.

The derivative of V along solution of (4.10) *when* $\Delta\Phi(\cdot)\equiv 0$ becomes

$$\begin{aligned}\dot{V} &= g\zeta^T(I_{N-1}\otimes -aI_d)\zeta+2\zeta^T(\mathcal{I}_2\otimes PK_0C_\ell)\mathbf{z}\\ &\quad +\mathbf{z}^T\left(I_M\otimes(A_\ell^TP_\ell+P_\ell A_\ell)\right)\mathbf{z}+2\mathbf{z}^T(\mathcal{I}_\chi\otimes gB_\ell C)\zeta\end{aligned}$$

which, by remembering Assumption 4.2, can be bounded as

$$\begin{aligned}\dot{V} &\leq -ga\|\zeta\|^2-a_\ell\|\mathbf{z}\|^2+(c_1+gc_2)\|\zeta\|\|\mathbf{z}\|\\ &\leq -(ga-\frac{c_1}{2}-g\frac{\epsilon c_2}{2})\|\zeta\|^2-(a_\ell-\frac{c_1}{2}-g\frac{c_2}{2\epsilon})\|\mathbf{z}\|^2\end{aligned}$$

for some positive constants c_1, c_2 and an arbitrary ϵ. Taking $\epsilon=a/c_2$, $\varepsilon\leq a/c_2^2$ and

$g = \varepsilon a_\ell$, the previous bound can be used to conclude that the derivative of V along the solutions of (4.10) is

$$\begin{aligned} \dot{V} &\leq -(c_3 a_\ell - c_4)\|\zeta\|^2 - (c_5 a_\ell - \frac{c_1}{2})\|\mathbf{z}\|^2 \\ &\qquad +2\zeta^T(I_{N-1}\otimes P)\tfrac{1}{g^d}\Delta\Phi(z_1,(I_{N-1}\otimes Dg)\zeta) \\ &\leq -(c_3 a_\ell - c_4)\|\zeta\|^2 + 2\bar{\Phi}''\|I_{N-1}\otimes P\|\|\zeta\|^2 - (c_5 a_\ell - \frac{c_1}{2})\|\mathbf{z}\|^2 \\ &\leq -(c_3 a_\ell - c_6)\|\zeta\|^2 - (c_5 a_\ell - \frac{c_1}{2})\|\mathbf{z}\|^2 \end{aligned}$$

for some positive constants c_3, c_4, c_5 and c_6. Hence, the origin of system (4.10) is globally asymptotically stable if a_ℓ is sufficiently large. The claim of Proposition 4.1 immediately follows from Assumption 4.1 and the definition of the variables ζ, χ and x. □

4.2.3 Simulation Results

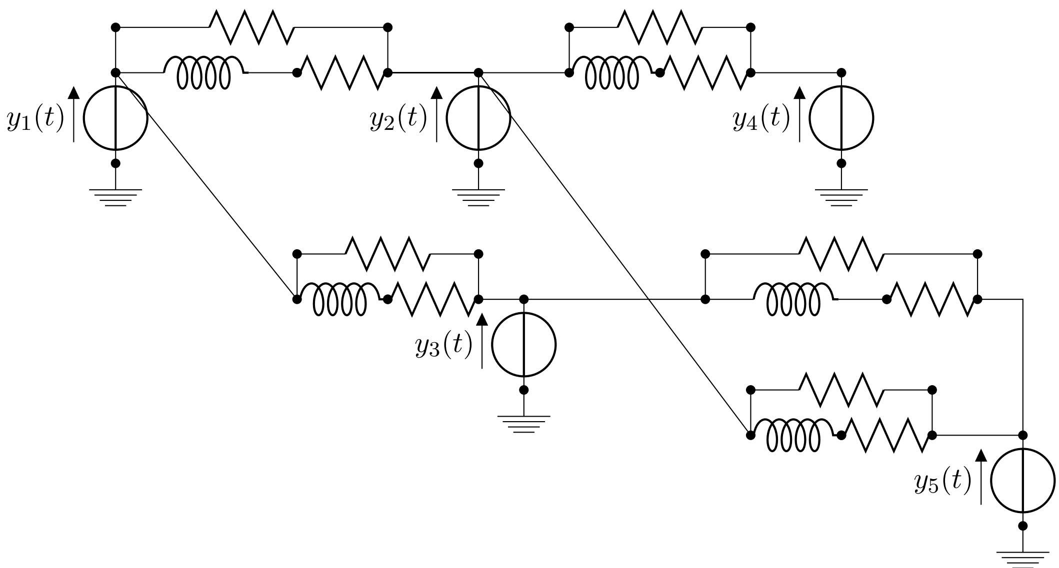

Figure 4.3: Circuit implemented in the simulation.

Simulation results are presented in this section, to show the behavior of the proposed solution: we considered 5 Van Der Pol oscillators

$$\begin{aligned} \dot{x}_{i_1} &= x_{i_2} \\ \dot{x}_{i_2} &= 2(1-x_{i_1}^2)x_{i_2} - x_{i_1} \end{aligned} \qquad y_i = x_{i_1} \tag{4.11}$$

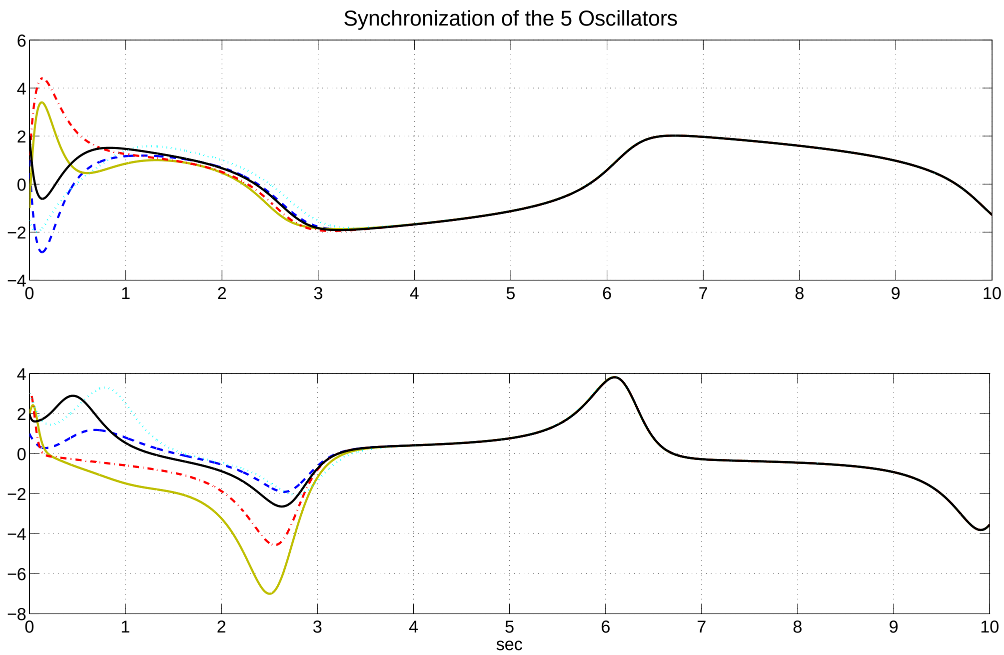

Figure 4.4: Synchronization of the 5 Van Der Pol oscillators.

connected according to Figure 4.3 via dynamic links, described by

$$\begin{aligned} \dot{z}_i &= -10z_i + \textstyle\sum_{j=1}^{N} \iota_{ji} y_j \\ p_i &= z_i \end{aligned} \tag{4.12}$$

The initial conditions of the agents are $w_1 = (1,1)^T, w_2 = (-1,2)^T, w_3 = (2,3), w_4 = (-1.5,3)$ and $w_5 = (2,2)$. The initial conditions of the links are $z_1 = 10, z_2 = 5, z_3 = 0, z_4 = -5$ and $z_5 = -10$. Furthermore, we choose K according to (4.5)-(4.6) with $g = 4$ and and $a = 1$.

The behavior of the oscillators is shown in Figure 4.4: after a transient, influenced by the different initial conditions on the links and their dynamics, the agents achieves synchronization.

4.2.4 The problem of synchronization in electrical networks with loads

In this section, we briefly describes how to recast the problem of synchronization of nonlinear oscillators connected with dynamic links in presence of loads at each agent. We want to show that, if the current drained from the load is measured, the problem can be tackled with the same approach described in the previous section.

Figure 4.5 shows an example of possible interconnection of three agents with two

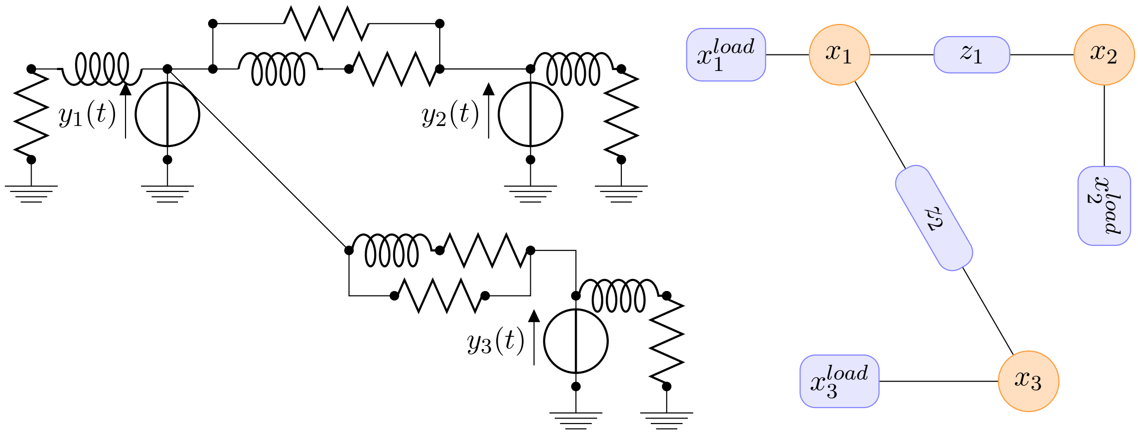

Figure 4.5: A simple example of connection of three agents with two dynamic links and a load at each agent.

dynamic links: the current flowing from agent 1 to agent 2, namely z_1, depends on the the difference of the outputs $y_1 - y_2$ (voltage drop) through the incidence matrix $\mathcal{I}$. At the same time, the current flowing from agent 1 to agent 3, namely z_2 depends on the difference of the outputs $y_1 - y_3$ (voltage drop) through the incidence matrix $\mathcal{I}$.

Similarly, the self-loop at each node can be seen as a dynamical link to which the output of each agent is applied: for the sake of simplicity, in order to maintain the notation as simple as possible, we model those dynamic links as in (4.2)

$$\begin{aligned} \dot{z}_k^\ell &= A_\ell z_k^\ell + B_\ell y_k \\ p_k^\ell &= C_\ell z_k^\ell \end{aligned} \tag{4.13}$$

for $k = 1, \dots, N$, with $z_k^\ell \in \mathbb{R}^l$, $p_k^\ell \in \mathbb{R}$ and ξ_k element of the *self-loop* matrix Ξ.

In Figure 4.5, the dynamics (4.13) are represented by the z_k^ℓ blocks, with $k = 1, 2, 3$, which are directly connected to each node of the network.

These leads to a set of $N + M$ identical links, which are respectively connected to and connecting the N nodes of the network.

Accordingly, we can rewrite (4.1) by coupling the agents and the loads as an augmented system

$$\begin{aligned} \dot{x}_k &= Sx_k + B\phi(x_k) - \xi_k F C_\ell z_k^\ell + u_k \\ \dot{z}_k^\ell &= A_\ell z_k^\ell + B_\ell C x_k \\ y_k &= Cx_k \end{aligned} \tag{4.14}$$

with $F \in \mathbb{R}^{d\times 1}$ an opportune matrix which establishes where the disturbance due to the loads is affecting the agents. The term $F\xi_k C_\ell z_k^\ell$ is indeed a disturbance to be compensated by the control.

In this framework, the outputs of links and loads are the available information to construct the control input u_k in (4.14) to simultaneously compensate the load request and to impose synchronization between the agents. In particular we chose u_k of the form

$$u_j = \xi_j F C_\ell z_k^\ell - K \sum_{i=1}^{M} \iota_{ji} p_i \tag{4.15}$$

with K a design parameter to be defined, ξ_j is the jj-th element of the self-loop matrix Ξ and ι_{ji} is the ji-th element of the incidence matrix $\mathcal{I}$.

By collecting the nodes dynamics and outputs as $\mathbf{x} = \mathrm{col}(x_1, \ldots, x_N)$, $\mathbf{y} = \mathrm{col}(y_1, \ldots, y_N)$, the control inputs as $\mathbf{u} = \mathrm{col}(u_1, \ldots, u_N)$ and the link dynamics and outputs as $\mathbf{z} = \mathrm{col}(z_1, \ldots, z_M)$, $\mathbf{p} = \mathrm{col}(p_1, \ldots, p_M)$, $\mathbf{z}^\ell = \mathrm{col}(z_1^\ell, \ldots, z_N^\ell)$, $\mathbf{p}^\ell = \mathrm{col}(p_1^\ell, \ldots, p_N^\ell)$, we can rewrite the dynamics of the network in compact form as

$$\begin{aligned}
\dot{\mathbf{x}} &= (I_N \otimes S)\mathbf{x} + (I_N \otimes B)\Phi(\mathbf{x}) - (\Xi \otimes F)\mathbf{p}^\ell + \mathbf{u} \\
\mathbf{y} &= (I_N \otimes C)\mathbf{x} \\
\dot{\mathbf{z}} &= (I_M \otimes A_\ell)\mathbf{z} + (I_M \otimes B_\ell)\mathcal{I}^T \mathbf{y} \\
\mathbf{p} &= (I_N \otimes C_\ell)\mathbf{z} + \mathcal{I}^T \mathbf{y} \\
\dot{\mathbf{z}}^\ell &= (I_N \otimes A_\ell)\mathbf{z}^\ell + (I_N \otimes B_\ell)\mathbf{y} \\
\mathbf{p}^\ell &= (I_N \otimes C_\ell)\mathbf{z}^\ell
\end{aligned} \tag{4.16}$$

where, without loss of generality, we considered $D_\ell = 1$.

Considering the definition of the control input given in (4.15), which in compact form reads as

$$\mathbf{u} = (\Xi \otimes F C_\ell)\mathbf{z}^\ell - (I_N \otimes K)\mathcal{I}\mathbf{p}$$

the overall system (4.16) becomes

$$\begin{aligned}
\dot{\mathbf{x}} &= (I_N \otimes S)\mathbf{x} + (I_N \otimes B)\Phi(\mathbf{x}) - (I_N \otimes K)\mathcal{I}\mathbf{p} \\
\mathbf{y} &= (I_N \otimes C)\mathbf{x} \\
\dot{\mathbf{z}} &= (I_M \otimes A_\ell)\mathbf{z} + (I_M \otimes B_\ell)\mathcal{I}^T\mathbf{y} \\
\mathbf{p} &= (I_M \otimes C_\ell)\mathbf{z} + \mathcal{I}^T\mathbf{y} \\
\dot{\mathbf{z}}^\ell &= (I_N \otimes A_\ell)\mathbf{z}^\ell + (I_N \otimes B_\ell)\mathbf{y} \\
\mathbf{p}^\ell &= (I_N \otimes C_\ell)\mathbf{z}^\ell
\end{aligned} \tag{4.17}$$

In the electrical framework, the control input (4.15) can be thought as composed by two terms: the first one is set to provide the required *power* by the load, the second is designed to achieve synchronization between the agents. It is worth mentioning, that in our framework we are not considering any limitation/saturation on the control input: this ideally means that we have sufficient *power* to guarantee both that the load request can be matched and the synchronization can be achieved.

Due to this, the stability analysis of (4.17) is reduced to the stability analysis of ($\mathbf{x}$,$\mathbf{z}$) dynamics, which thanks to the definition of the input (4.15) are not influenced by the $\mathbf{z}^\ell$ dynamics.

Proposition 4.2. *Let Assumptions 4.1-4.3 hold. Consider the networked systems (4.16) controlled by the diffusive coupling control law (4.15) with K chosen as in (4.5) with $g = \varepsilon a_\ell$ with $\varepsilon > 0$. Then, there exist an $\varepsilon^\star \leq 1$ and, for all positive $\varepsilon \leq \varepsilon^\star$, an $a_\ell^\star > 0$ (dependent on the Lipschitz constant $\bar{\phi}$ of $\phi(\cdot)$) such that, for all $a_\ell \geq a_\ell^\star$ the compact set*

$$\mathbf{X} \;=\; \{(x_1, x_2, \ldots, x_N) \in X \times X \times \cdots \times X \;:\; x_1 = x_2 = \cdots = x_N\} \tag{4.18}$$

is globally asymptotically stable.

The proof is the same of Proposition 4.1.

Remark 4.1. An interesting case of study could be to consider uncertain loads at the agents, namely the matrix Ξ is not known a priori. This scenario poses interesting questions on how to design (4.15) to compensate the unknown load *adaptively* and simultaneously to synchronize the systems. △

4.3 Dynamic links with no algebraic connection between input and output

In this section we will introduce a modification to the structure of the network and consequently of the regulator. In particular, we modify the link dynamics (4.2) according to

$$\begin{array}{rcl} \dot{z}_i & = & A_\ell z_i + B_\ell \sum_{j=1}^{N} \iota_{ji} y_j \\ p_i & = & C_\ell z_i \end{array} \tag{4.19}$$

for $i = 1, \ldots, M$, with $z_i \in \mathbb{R}^l$, $p_i \in \mathbb{R}$. In contrast with the previous definition, this dynamical link has no algebraic connection between the output of the link and the input applied to the link, namely the output of the agents.

Using the same notation of the previous sections, the node dynamics in compact form become

$$\begin{array}{rcl} \dot{\mathbf{x}} & = & (I_N \otimes S)\mathbf{x} + (I_N \otimes B)\Phi(\mathbf{x}) - (I_N \otimes K)\mathcal{I}(I_M \otimes C_\ell)\mathbf{z} \\ \mathbf{y} & = & (I_N \otimes C)\mathbf{x} \end{array} \tag{4.20}$$

with $\mathbf{x} \in \mathbb{R}^{dN}$, $\mathbf{z} \in \mathbb{R}^{lM}$. The absence of a *direct connection* between the input of the links and their outputs makes it hard to define the control parameters K in such a way that the system still achieves synchronization.

In several application, it would make sense to ask the agents to exchange information between each other directly, despite the filtering of the dynamic links. However, we want to propose a solution which does not require to add any further information exchange (physical or virtual).

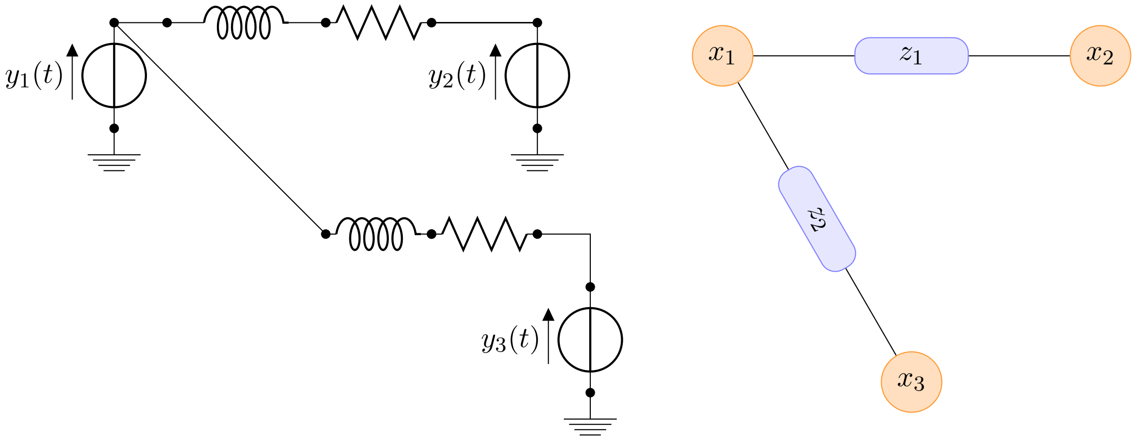

Figure 4.6: A simple example of connection of three agents with two dynamic links, with no direct connection between the output of the links and their inputs.

Hence, in order to solve the problem of synchronization, we introduce a dynamic regulator with the goal to observe the quantity $\sum_{j=1}^{N} \iota_{ji} y_j$ (the voltage drop applied to the link) from the output p_i (the output current of the link). A possible and intuitive design procedure for such an observer is a *dirty derivatives high gain observer* (see Teel and Praly (1995)). Assuming that the relative degree between the input $\sum_{j=1}^{N} \iota_{ji} y_j$ and the output p_i is r, by deriving r times p_i, it would be possible to reconstruct the input.

To understand the main idea behind this approach, consider the case of scalar links.

Assumption 4.4. *The link dynamic $z_i \in \mathbb{R}$. This implies that the relative degree r between the output of the links p_i and the input $\sum_{k=1}^{N} \iota_{ji}\ y_j$ is $r = 1$.*

Remark 4.2. It is worth noticing that the extension the case in which $z_i \in \mathbb{R}^l$ and $r > 1$ is just a mathematical complication. In order to maintain the concepts as clear as possible, we avoid *heavy notations* without loss of generality. In the scalar case, the link dynamics represents for instance an R-L filter

$$\begin{aligned} \dot{z}_i &= -\tfrac{R}{L} z_i + \tfrac{1}{L}(y_k - y_h) \\ p_i &= z_i \qquad\qquad\qquad i = 1, \ldots, M \end{aligned}$$

where $(y_k - y_h)$ is the voltage drop applied to the i-th link and p_i is the output current of the link. $\triangle$

From Section 3.1, we know that if the input to the agents can be constructed as

$$u_k = K \sum_{i=1}^{N} \ell_{ki} y_i \tag{4.21}$$

with ℓ_{ki} element of the Laplacian matrix L, synchronization can be achieved with an opportune tuning of the control gain K. The idea is to construct a dynamic control law which computes only the available information

$$\sum_{i=1}^{N} \iota_{ki} p_i$$

such that the information necessary to achieve synchronization is retrieved: in particular we want to compute the output of the links p_i in order to reconstruct the input applied to the links and consequently (4.21).

In order to illustrate the main idea of the design, consider two agents and a dynamic link connecting the two according to (see Figure 4.6)

$$\begin{aligned} \dot{z}_1 &= a_\ell z_1 + b_\ell C(x_1 - x_2) \\ p_1 &= z_1 \end{aligned}$$

By computing the derivative of the outputs p_1 it is possible to reconstruct, at each agent, the input applied by the others. Both agents x_1 and x_2 compute $\dot{p}_1$ to obtain respectively $y_2 = Cx_2$ and $y_1 = Cx_1$ and construct the control input as in (4.21).

Following the idea of an *high gain extended state observer*, we extend the dynamics of the link to its second derivative, obtaining

$$\ddot{z}_1 = a_\ell \dot{z}_1 + b_\ell C(\dot{x}_1 - \dot{x}_2)$$

By defining $\mathbf{z}_1 = [z_1, \dot{z}_1]$, we can write the extend link dynamic as

$$\begin{aligned} \dot{\mathbf{z}}_1 &= \begin{bmatrix} 0 & 1 \\ a_\ell^2 & 0 \end{bmatrix} \mathbf{z}_1 + \begin{bmatrix} 0 \\ a_\ell b_\ell \end{bmatrix} C(x_1 - x_2) + \begin{bmatrix} 0 \\ b_\ell \end{bmatrix} C(\dot{x}_1 - \dot{x}_2) \\ &= A_\ell \mathbf{z}_1 + B_\ell^{'} C(x_1 - x_2) + B_\ell^{''} C(\dot{x}_1 - \dot{x}_2) \qquad (4.22) \\ \varrho_1 &= \iota_{11} \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{z}_1 = \iota_{11}\, C_\ell \mathbf{z}_1 \\ \varrho_2 &= \iota_{21} \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{z}_1 = \iota_{21}\, C_\ell \mathbf{z}_1 \end{aligned}$$

We define the *extended state observer* at each agent of the kind

$$\begin{aligned} \dot{\zeta}_k &= S\zeta_k + K_o C_\ell(\zeta_k - \varrho_k) \\ \delta_k &= \tfrac{1}{b_\ell} \begin{bmatrix} -a_\ell & 1 \end{bmatrix} \zeta_k \end{aligned} \qquad (4.23)$$

for $k = 1, 2$, with δ_k output of the observer which recovers the *voltage drop* $C(x_1 - x_2)$ applied to the link z_1. Following this paradigm, the control input of the agents read as

$$u_k = -K\delta_k$$

for $k = 1, 2$, with K defined as in (4.5). If δ_k in (4.23) recovers (4.22), the control input just defined matchees (4.21) and thus guarantees synchronization.

4.3.1 Extension to the N-agents case

In the general case of N agents and M links one could think that the proposed control architecture is difficult to be analyzed. It could seem that each agent has to implement a set of observers, one for each dynamic link to which it is connected to. In this section we show that it is possible to consider an *equivalent link dynamics* which describes the dynamics of the sum of all incoming/outgoing currents at each node ϱ_k. We start by considering that the information available to the k-th agent is indeed

$$\varrho_k = \sum_{i=1}^{M} \iota_{ki}\, p_i$$

so the sum of all the currents *injected/drained* into/from the node. Following the same reasoning done before for a single link we extend the dynamics and obtain

$$\begin{aligned} \dot{\varrho}_k &= \textstyle\sum_{i=1}^{M} \iota_{ki} \dot{p}_i \\ &= \textstyle\sum_{i=1}^{M} \iota_{ki} \left(a_\ell z_i + b_\ell \sum_{j=1}^{N} \iota_{ji}\, y_j \right) \\ &= a_\ell \varrho_k + b_\ell \textstyle\sum_{i=1}^{M} \iota_{ki} \sum_{j=1}^{N} \iota_{ji}\, y_j \end{aligned} \tag{4.24}$$

$$\begin{aligned} \ddot{\varrho}_k &= a_\ell \dot{\varrho}_k + b_\ell \textstyle\sum_{i=1}^{M} \iota_{ki} \sum_{j=1}^{N} \iota_{ji}\, \dot{y}_j \\ &= a_\ell^2 \varrho_k + a_\ell b_\ell \textstyle\sum_{i=1}^{M} \iota_{ki} \sum_{j=1}^{N} \iota_{ji}\, y_j + b_\ell \sum_{i=1}^{M} \iota_{ki} \sum_{j=1}^{N} \iota_{ji}\, \dot{y}_j \end{aligned}$$

This represents an equivalent link, whose dynamic describes the dynamics of the *sum* all links connected to each node. Indeed, observing separately all the voltage drops applied at each link or observing the term

$$\sum_{i=1}^{M} \iota_{ki} \sum_{j=1}^{N} \iota_{ji}\, y_j$$

is equivalent. Furthermore, remembering that $L = \mathcal{I}\mathcal{I}^T$, we have

$$\sum_{i=1}^{M} \iota_{ki} \sum_{j=1}^{N} \iota_{ji}\, y_j = \sum_{j=1}^{N} \ell_{kj} y_j \tag{4.25}$$

which matches exactly the the term (4.21) we are aiming to *observe*.

With a small abuse of notation we define $\boldsymbol{\varrho}_k = \mathrm{col}(\varrho_k, \dot{\varrho}_k)$ and consequently obtain

$$\begin{aligned} \dot{\boldsymbol{\varrho}}_k &= \begin{bmatrix} 0 & 1 \\ a_\ell^2 & 0 \end{bmatrix} \boldsymbol{\varrho}_k + \begin{bmatrix} 0 \\ a_\ell b_\ell \end{bmatrix} \textstyle\sum_{j=1}^{N} \ell_{kj} y_j + \begin{bmatrix} 0 \\ b_\ell \end{bmatrix} \sum_{j=1}^{N} \ell_{kj} \dot{y}_j \\ &= A_\ell \boldsymbol{\varrho}_k + B_\ell' \textstyle\sum_{j=1}^{N} \ell_{kj} y_j + B_\ell'' \sum_{j=1}^{N} \ell_{kj} \dot{y}_j \\ \pi_k &= \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \boldsymbol{\varrho}_k \\ &= C_\ell \boldsymbol{\varrho}_k \end{aligned} \tag{4.26}$$

whit $\pi_k = \sum_{i=1}^{M} \iota_{ki}\, p_i$ output of the extended equivalent link.

Remark 4.3. It is worth noticing that (4.26) is a Brunovsky-*like* canonical form of the extended link, namely,

$$A_\ell = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ \star & \star & \dots & \star \end{bmatrix}, \quad B_\ell^{'} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \star \end{bmatrix}, \quad B_\ell^{''} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \star \end{bmatrix}, \quad C_\ell = \begin{bmatrix} \star & 0 & \dots & 0 \end{bmatrix} \tag{4.27}$$

The intrinsic structure of $A_\ell, B_\ell^{'}, B_\ell^{''}$ in (4.26) will play a fundamental role in the followings. △

Every agent in the network implements an observer for the $\boldsymbol{\varrho}_k$ dynamics of the kind

$$\begin{aligned} \dot{\zeta}_k &= A_\ell \zeta_k + K_o C_\ell (\zeta_k - \varrho_k) \\ \delta_k &= \Gamma \zeta_k \end{aligned} \tag{4.28}$$

for $k = 1, \dots, N$, where

$$K_o = D_o K_c \tag{4.29}$$

with $D_o = \mathrm{diag}(d_o, d_o^2)$ and $K_c = P_o C_\ell^T$ with P_o solution of

$$S P_o + P_o S^T - P_o C_\ell^T C_\ell P_o = -a_1 I \tag{4.30}$$

with $a_1 > 0$ and Γ opportunely defined in such a way that δ_k recovers (4.25) locally at each agent.

Then we define the control input, locally at each agents, as

$$u_k = K \mathrm{sat}(\delta_k) \tag{4.31}$$

where K is $K = D_g K_0$, with $D_g = \mathrm{diag}(g, \dots, g^d)$ and $K_0 = PC^T$ with P solution of the Riccati equation

$$SP + PS^T - \mu P C^T C P = -a_2 I$$

with $\mu \leq \lambda_2(L)$ and $a_2 > 0$ (see Section 3.1 for details on the design). Furthermore $\mathrm{sat}(\cdot) : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is defined in such a way that it preserves the direction of the vector to which it is applied. Taking inspiration from Teel and Praly (1994), we define $\mathrm{sat}(\cdot)$ as

$$\mathrm{sat}_{\mathcal{M}}(\cdot) = \min\left\{1, \frac{\mathcal{M}}{\|\cdot\|}\right\}(\cdot)$$

Letting $\mathbf{x} = \mathrm{col}(x_1, \dots, x_N)$, $\mathbf{z} = \mathrm{col}(z_1, \dots, z_M)$, $\boldsymbol{\varrho} = \mathrm{col}(\boldsymbol{\varrho}_1, \dots, \boldsymbol{\varrho}_N)$, $\boldsymbol{\zeta} = \mathrm{col}(\zeta_1, \dots, \zeta_N)$, $\boldsymbol{\delta} = \mathrm{col}(\delta_1, \dots, \delta_N)$ the overall system read as

$$
\begin{aligned}
\dot{\mathbf{x}} &= (I_N \otimes S)\mathbf{x} + (I_N \otimes B)\Phi(\mathbf{x}) - (I_N \otimes K)\hat{\boldsymbol{\delta}} \\
\mathbf{y} &= (I_N \otimes C)\mathbf{x} \\
\dot{\mathbf{z}} &= (I_M \otimes a_\ell)\mathbf{z} + (\mathcal{I}^T \otimes b_\ell C)\mathbf{x} \\
\dot{\boldsymbol{\varrho}} &= (I_N \otimes A_\ell)\boldsymbol{\varrho} + (L \otimes B_\ell^{'})\mathbf{y} + (L \otimes B_\ell^{''})\dot{\mathbf{y}} \\
\dot{\boldsymbol{\zeta}} &= (I_N \otimes A_\ell)\boldsymbol{\zeta} + (I_N \otimes K_o C_\ell)(\boldsymbol{\zeta} - \boldsymbol{\varrho}) \\
\boldsymbol{\delta} &= (I_N \otimes \Gamma)\boldsymbol{\zeta}
\end{aligned}
\tag{4.32}
$$

where $\hat{\boldsymbol{\delta}} = \mathrm{col}(\hat{\delta}_1, \dots, \hat{\delta}_N)$ with $\hat{\delta}_k = \mathrm{sat}(\delta_k)$,

Proposition 4.3. *Let Assumption 4.1-4.4 hold. Then there exists a $d_o^\star$ such that, for every $d_o > d_o^\star$ there exist $a_\ell^\star$ and $g^\star, \gamma^\star$ such that for every $a_\ell > a_\ell^\star$ and $g > \max\{g^\star, \gamma^\star\}$ and for every $\mathbf{x}_0 \in \mathbb{X} \subset \mathbb{R}^{dN}$ and $\mathbf{z}_0 \in \mathbb{Z} \subset \mathbb{R}^M$, the compact set*

$$
\mathbf{X} = \{(x_1, x_2, \dots, x_N) \in X \times X \times \dots \times X \;:\; x_1 = x_2 = \dots = x_N\} \tag{4.33}
$$

is globally asymptotically stable.

Proof. The proof is structured as follows:

- as a first step, we will show that under the assumptions of the framework, all the dynamics involved are bounded
- in the second part, we will prove that, after a certain time T_s, the observer error converges to an arbitrary small compact set
- in the third part, we will consider the interconnected systems after T_s and prove that synchronization is achieved

The first part is indeed instrumental to the second, in the sense that boundedness of the trajectories allows to conclude that the observer can be designed in such a way that the estimation error converges to an arbitrary small set. Once the estimation error is small, it will be possible to consider the stability of the interconnected system. The proof can be seen as a *separation principle* applied to the case of networks.

Boundedness of solutions

In order to prove boundedness of solutions, we will make use of ISS arguments. The reader is referred to Appendix B and Isidori (1999) for detailed explanations on the tools used in the following.

Consider system (4.32). Due to Assumption 4.1 on the agents, we know that there exists a *fixed* class $\mathcal{K}$ function $\gamma_x(\cdot) : \mathbb{R} \to \mathbb{R}$ and a class $\mathcal{KL}$ $\beta_x(\cdot, \cdot) \to \mathbb{R}$ such that

$$\|\mathbf{x}(t, \mathbf{x}_0)\|_X \leq \max\{\beta_x(\|\mathbf{x}_0\|, t), \gamma_x(\sup_{\tau\in[0,t)} \|\hat{\boldsymbol{\delta}}(\tau)\|)\} \tag{4.34}$$

Since the input to each agent is saturated, (4.34) becomes

$$\|\mathbf{x}(t, \mathbf{x}_0)\|_X \leq \max\{\beta_x(\|\mathbf{x}_0\|, t), \gamma_x(\mathcal{M})\} \tag{4.35}$$

and thus there exists a t_1 such that for $t > t_1$

$$\|\mathbf{x}(t_1, \mathbf{x}_0)\|_X \leq \gamma_x(\mathcal{M})$$

The fact that $\mathbf{x}$ is bounded and Assumption 4.2 imply that also $\mathbf{z}$ and thus $\boldsymbol{\varrho}$ are bounded. In particular, we can write that

$$\|\boldsymbol{\varrho}(t)\| \leq \max\{\beta_\varrho(\|\boldsymbol{\varrho}_0\|, t), \gamma_\varrho(\|\mathbf{x}\|)\} \tag{4.36}$$

and there exists a time t_2 such that, fro $t > t_2$

$$\|\boldsymbol{\varrho}(t)\| \leq \gamma_\varrho(\|\mathbf{x}\|)$$

In other words, there exists a time t_s such that for $t > t_s$

$$\begin{aligned} \|\mathbf{x}(t)\| &\leq \gamma_x(\mathcal{M}) \\ \|\boldsymbol{\varrho}(t)\| &\leq \gamma_\varrho \circ \gamma_x(\mathcal{M}) \end{aligned} \tag{4.37}$$

Last but not the least, boundedness of $\boldsymbol{\varrho}$ implies that also the observer state, namely $\boldsymbol{\zeta}$, is bounded.

These properties of boundedness are fundamental in order to deal with the transient behavior of the system towards synchronization. In the following part of the proof we will show that there exists a time T_s such that, for $t > T_s$ the estimation error can be arbitrarily bounded.

Observer convergence analysis

By introducing the agents dynamics and output (see (4.32)) into (4.26), $\boldsymbol{\varrho}$ dynamics reads as

$$\begin{aligned}\dot{\boldsymbol{\varrho}} &= (I_N \otimes A_\ell)\boldsymbol{\varrho} + (L \otimes B_\ell^{'})\mathbf{y} + (L \otimes B_\ell^{''})\dot{\mathbf{y}} \\ &= (I_N \otimes A_\ell)\boldsymbol{\varrho} + (L \otimes B_\ell^{'})(I_N \otimes C)\mathbf{x} \\ &\qquad +(L \otimes B_\ell^{''})(I_N \otimes C)\left[(I_N \otimes S)\mathbf{x} + (I_N \otimes B)\Phi(\mathbf{x}) - (I_N \otimes K)\hat{\boldsymbol{\delta}}\right] \\ &= (I_N \otimes A_\ell)\boldsymbol{\varrho} + \Big(L \otimes [B_\ell^{'}C + B_\ell^{''}CS]\Big)\,\mathbf{x} - (L \otimes B_\ell^{''}CK)\hat{\boldsymbol{\delta}} \\ &= (I_N \otimes A_\ell)\boldsymbol{\varrho} + \big(L \otimes \bar{B}_\ell\big)\,\mathbf{x} - (L \otimes B_\ell^{''}CK)\hat{\boldsymbol{\delta}}\end{aligned}$$

where we define $\bar{B}_\ell \in \mathbb{R}^{2\times d}$

$$\bar{B}_\ell = [B_\ell^{'}C + B_\ell^{''}CS]$$

By defining the observation error as $\boldsymbol{e} = \boldsymbol{\zeta} - \boldsymbol{\varrho}$, we get

$$\begin{aligned}\dot{\boldsymbol{e}} &= (I_N \otimes A_\ell)\boldsymbol{e} + (I_N \otimes K_o C_\ell)\boldsymbol{e} - \big(L \otimes \bar{B}_\ell\big)\,\mathbf{x} + (L \otimes B_\ell^{''}CK)\hat{\boldsymbol{\delta}} \\ &= \Big(I_N \otimes [S + K_o C_\ell]\Big)\boldsymbol{e} + (I_N \otimes \bar{A}_\ell)\boldsymbol{e} - (L \otimes \bar{B}_\ell)\mathbf{x} + (L \otimes B_\ell^{''}CK)\hat{\boldsymbol{\delta}}\end{aligned}$$

where we expressed $A_\ell = S + \bar{A}_\ell$, with

$$\bar{A}_\ell = \begin{bmatrix} 0 & 0 \\ a_\ell^2 & 0 \end{bmatrix}$$

We change coordinates according to

$$\boldsymbol{\varepsilon} = (I_N \otimes D_o^{-1})\boldsymbol{e}$$

and obtain

$$\begin{aligned}\dot{\boldsymbol{\varepsilon}} &= d_o\Big(I_N \otimes [S + K_c C_\ell]\Big)\boldsymbol{\varepsilon} + (I_N \otimes D_o^{-1}\bar{A}_\ell D_o)\boldsymbol{\varepsilon} \\ &\qquad -(I_N \otimes D_o^{-1})\left[(L \otimes \bar{B}_\ell)\mathbf{x} - (L \otimes B_\ell^{''}CK)\hat{\boldsymbol{\delta}}\right]\end{aligned}$$

Consider the candidate Lyapunov function W

$$W = \boldsymbol{\varepsilon}^T \mathbf{P}_o \boldsymbol{\varepsilon} = \sum_{i=1}^{N-1} \boldsymbol{\varepsilon}_i^T P_o \boldsymbol{\varepsilon}_i$$

with $\mathbf{P}_o = (I_N \otimes P_o)$, such that

$$\underline{\lambda}\|\boldsymbol{\varepsilon}\|^2 \leq W \leq \bar{\lambda}\|\boldsymbol{\varepsilon}\|^2$$

Its derivative with respect to time is

$$\dot{W} \leq -a_1 d_o \|\boldsymbol{\varepsilon}\|^2 + \|\bar{A}_\ell\|\|\boldsymbol{\varepsilon}\|^2 + 2\boldsymbol{\varepsilon}^T (I_N \otimes P_o D_o^{-1}) \left[-(L \otimes \bar{B}_\ell)\mathbf{x} + (L \otimes B_\ell'' CK)\hat{\boldsymbol{\delta}} \right]$$

and clearly for d_o sufficiently *large*, we get

$$\dot{W} \leq -a_1 d_o \|\bar{\varepsilon}_2\|^2 + \frac{1}{d_o} \left[p_1 \|\mathbf{x}\|^2 + p_2 \mathcal{M}^2 \right]$$

for some constant p_1 and p_2.

Due to the fact that (4.32) is bounded in all its dynamics, for $t > t_s$ it is possible to write that

$$\|\boldsymbol{\varepsilon}(t)\| \leq \sqrt{\frac{\bar{\lambda}}{\underline{\lambda}}} e^{-\frac{a_1 d_o}{2\bar{\lambda}} t} \|\boldsymbol{\varepsilon}(0)\| + \frac{1}{d_o} \max\{p_1 \gamma_x(\mathcal{M})^2, p_2 \mathcal{M}^2\}$$

which leads to the following Lemma.

Lemma 4.1. *For all $\epsilon \in \mathbb{R}_{>0}$ and for all $T_s > t_s$, there exists a $d_o^\star$ such that, for all $d_o > d_o^\star$*

$$\|(I_N \otimes D_o)\boldsymbol{\varepsilon}(t)\| \leq \epsilon$$

This Lemma allows now to consider the system (4.32) once the estimation error is small.

Study of the interconnected system

We have proved that for $t > T_s$,

$$\|(I_N \otimes D_o)\boldsymbol{\varepsilon}(t)\| \leq \epsilon$$

However, since the saturation depends also on the $\boldsymbol{\varrho}$ dynamics, this is not sufficient to conclude that the saturation is not active. We need to determine an opportune value of $\mathcal{M}$ such that

$$\frac{\mathcal{M}}{\|(I_N \otimes D_o)\boldsymbol{\varepsilon}(t) + \boldsymbol{\varrho}(t)\|} > 1 \qquad \text{for } \ t \geq T_S$$

For $t > T_s$, in virtue of (4.37), we can write that

$$\begin{aligned}\|(I_N \otimes D_o)\boldsymbol{\varepsilon}(t) + \boldsymbol{\varrho}(t)\| &\leq \|(I_N \otimes D_o)\boldsymbol{\varepsilon}(t)\| + \|\boldsymbol{\varrho}(t)\| \\ &\leq \epsilon + \|\boldsymbol{\varrho}(t)\| \\ &\leq \epsilon + \gamma_\varrho \circ \gamma_x(\mathcal{M})\end{aligned}$$

This leads to the following fact.

Fact 4.1. *By choosing the saturation bound $\mathcal{M}$ such that*

$$\mathcal{M} \geq \epsilon + \gamma_\varrho \circ \gamma_x(\mathcal{M}) \tag{4.38}$$

for $t \geq T_s$, the saturation is not active.

It is readily seen that (4.38) has a solution if and only if the gains $\gamma_\varrho(\cdot)$, $\gamma_x(\cdot)$ fulfills a *small-gain* like condition. Since the agents gain $\gamma_x(\cdot)$ is given *a priori*, it turns out that (4.38) has a solution if and only if $\gamma_\varrho(\cdot)$ is sufficiently small which implies that the stability margin a_ℓ has to be sufficiently big.

Thus we say, that there exists an $a_\ell^\star$ such that, for all $a_\ell \geq a_\ell^\star$, (4.38) has a solution $\mathcal{M}$.

Thus for $t > T_s$, due to the opportune choice of $\mathcal{M}$ the saturation is not active. Since Γ is defined in such a way that

$$-(I_N \otimes K\Gamma)\boldsymbol{\varrho} + (L \otimes KC)\mathbf{x} = 0$$

we can write the $\mathbf{x}$ dynamics according to

$$\begin{aligned}\dot{\mathbf{x}} &= (I_N \otimes S)\mathbf{x} + (I_N \otimes B)\Phi(\mathbf{x}) - (I_N \otimes K\Gamma D_o)\boldsymbol{\varepsilon} - (I_N \otimes K\Gamma)\boldsymbol{\varrho} \pm (L \otimes KC)\mathbf{x} \\ &= \big[(I_N \otimes S) - (L \otimes KC)\big]\mathbf{x} + (I_N \otimes B)\Phi(\mathbf{x}) - (I_N \otimes K\Gamma D_o)\boldsymbol{\varepsilon}\end{aligned} \tag{4.39}$$

Furthermore, by inserting (4.39) into the $\boldsymbol{\varrho}$ dynamics we get

$$\dot{\boldsymbol{\varrho}} = (I_N \otimes A_\ell)\boldsymbol{\varrho} + (L \otimes B_\ell^{'}C)\mathbf{x} + (L \otimes B_\ell^{''}C)\big[(I_N \otimes S) - (L \otimes KC)\big]\mathbf{x} \\ +(L \otimes B_\ell^{''}CK\Gamma D_o)\boldsymbol{\varepsilon}$$

The last equation leads also to rewrite the observation error $\boldsymbol{\varepsilon}$ according to

$$\begin{aligned}\dot{\boldsymbol{\varepsilon}} = d_o\,(I_N \otimes [S + K_c C_\ell])\,\boldsymbol{\varepsilon} + (I_N \otimes \bar{A}_\ell)\boldsymbol{\varepsilon} - (L \otimes D_o^{-1}B_\ell^{''}CK\Gamma D_o)\boldsymbol{\varepsilon} \\ -(L \otimes D_o^{-1}B_\ell^{'}C)\mathbf{x} - (L \otimes D_o^{-1}B_\ell^{''}C)\big[(I_N \otimes S) - (L \otimes KC)\big]\mathbf{x}\end{aligned} \tag{4.40}$$

We are now left with the study of the interconnection between (4.39)-(4.40) and the set of M links

$$\mathbf{z} = (I_M \otimes a_\ell)\mathbf{z} + (\mathcal{I}^T \otimes b_\ell C)\mathbf{x} \tag{4.41}$$

Thus we can *forget* about the fictional extended dynamics $\boldsymbol{\varrho}$ and focus on the synchronization of the agents and on the other *physical* components of the network. In order to do so, remembering that the Jordan form of the Laplacian is purely diagonal since the Laplacian is symmetric, we change coordinates according to

$$\begin{aligned}\bar{\mathbf{x}} &= (T_J^{-1} \otimes I_d)\mathbf{x} \\ \bar{\boldsymbol{\varepsilon}} &= (T_J^{-1} \otimes I_2)\boldsymbol{\varepsilon}\end{aligned}$$

where T_J is such that

$$L_J = T_J^{-1} L T_J = \begin{bmatrix} 0 & \mathbf{0}_{1\times N-1} \\ \mathbf{0}_{N-1\times 1} & L_{J-1} \end{bmatrix}$$

with $L_{J-1} = \mathrm{diag}(\lambda_2(L), \dots, \lambda_N(L))$.

Due to the structure of the change of coordinates, it is worth noticing that

$$\begin{aligned}\bar{\mathbf{x}} &= \begin{bmatrix} x_1 \\ \bar{\mathbf{x}}_2 \end{bmatrix} \\ \bar{\boldsymbol{\varepsilon}} &= \begin{bmatrix} \varepsilon_1 \\ \bar{\boldsymbol{\varepsilon}}_2 \end{bmatrix}\end{aligned}$$

For the $\bar{\mathbf{x}}$ subsystem we obtain

$$\begin{aligned}\dot{x}_1 &= Sx_1 + B\phi(x_1) + K\Gamma D_o \varepsilon_1 \\ \dot{\bar{\mathbf{x}}}_2 &= \big[(I_{N-1} \otimes S) - (L_{J-1} \otimes KC)\big]\bar{\mathbf{x}}_2 + (I_N \otimes B)\Phi(x_1, \bar{\mathbf{x}}_2) - (I_{N-1} \otimes K\Gamma D_o)\bar{\boldsymbol{\varepsilon}}_2\end{aligned}$$

where $\Delta\Phi(x_1, \bar{\mathbf{x}}_2)$ satisfies $\Delta\Phi(x_1, 0) = 0$ uniformly in x_1. Furthermore, note that due to the choice of K the matrix $\big[(I_{N-1}\otimes S)-(L_{J-1}\otimes KC)\big]$ is Hurwitz (see Lemma 3.1 in Section 3.1).

For the $\bar{\varepsilon}$ dynamics, letting $H_c = S + K_c C_\ell$ be Hurwitz thanks to the choice of K_c, we obtain

$$\begin{aligned}
\dot{\varepsilon}_1 &= d_o H_c \varepsilon_1 + A_\ell \varepsilon_1 \\
\dot{\bar{\boldsymbol{\varepsilon}}}_2 &= d_o(I_{N-1}\otimes H_c)\bar{\boldsymbol{\varepsilon}}_2 + (I_{N-1}\otimes \bar{A}_\ell)\bar{\boldsymbol{\varepsilon}}_2 - (L_{J-1}\otimes D_o^{-1}B_\ell^{''}CK\Gamma D_o)\bar{\boldsymbol{\varepsilon}}_2 \\
&\qquad -(L_{J-1}\otimes D_o^{-1}B_\ell^{'}C)\bar{\mathbf{x}}_2 - (L_{J-1}\otimes D_o^{-1}B_\ell^{''}C)\big[(I_N\otimes S)-(L\otimes KC)\big]\bar{\mathbf{x}}_2
\end{aligned}$$

The link dynamics instead reads as

$$\dot{\mathbf{z}} = (I_M\otimes a_\ell)\mathbf{z} + (\mathcal{I}^T T\otimes b_\ell C)\bar{\mathbf{x}}$$

and it turns out that the first column of $\mathcal{I}^T T$ is always zero, meaning that the dynamics of the links $\mathbf{z}$ do not depend on x_1. We define the matrix $\mathcal{I}_{\bar{x}}$ as the matrix obtained by removing the first column of $\mathcal{I}^T T$, obtaining

$$\dot{\mathbf{z}} = (I_M\otimes a_\ell)\mathbf{z} + (\mathcal{I}_{\bar{x}}\otimes b_\ell C)\bar{\mathbf{x}}_2$$

Proving that $\bar{\mathbf{x}}_2$ is GAS implies that synchronization between the agents is achieved. This in turn lead to conclude that the links dynamics $\mathbf{z}$ are asymptotically stable too. It is easy to see that for $d_o > d_o^\star$, ε_1 is exponentially stable. This also implies that, due to the ISS assumption on the agents, x_1 converges asymptotically to its invariant set X (see Assumption 4.1).

We restrict our attention on the interconnection of $\bar{\mathbf{x}}_2$, $\bar{\boldsymbol{\varepsilon}}_2$, and$\mathbf{z}$. We change coordinate according to

$$\boldsymbol{\chi}_2 = (I_{N-1}\otimes D_g^{-1})\bar{\mathbf{x}}_2$$

and obtain

$$\begin{aligned}
\dot{\boldsymbol{\chi}}_2 &= g\big[(I_{N-1}\otimes S)-(L_{J-1}\otimes K_0 C)\big]\boldsymbol{\chi}_2 + \tfrac{1}{g^d}(I_{N-1}\otimes B)\Delta\Phi^{'}\left(x_1,(I_{N-1}\otimes D_g)\boldsymbol{\chi}_2\right) \\
&\qquad +(I_{N-1}\otimes D_g^{-1}K\Gamma D_o)\bar{\boldsymbol{\varepsilon}}_2 \\
\dot{\bar{\boldsymbol{\varepsilon}}}_2 &= d_o(I_{N-1}\otimes H_c)\bar{\boldsymbol{\varepsilon}}_2 + (I_{N-1}\otimes \bar{A}_\ell)\bar{\boldsymbol{\varepsilon}}_2 - (L_{J-1}\otimes D_o^{-1}B_\ell^{''}CK\Gamma D_o)\bar{\boldsymbol{\varepsilon}}_2 \\
&\qquad -(L_{J-1}\otimes D_o^{-1}B_\ell^{'}CD_g)\boldsymbol{\chi}_2 - (L_{J-1}\otimes D_o^{-1}B_\ell^{''}CD_g)\big[(I_N\otimes S)-(L\otimes KC)\big]\boldsymbol{\chi}_2 \\
\dot{\mathbf{z}} &= (I_M\otimes a_\ell)\mathbf{z} + (\mathcal{I}_{\bar{x}}\otimes b_\ell CD_g)\boldsymbol{\chi}_2
\end{aligned}$$

Now we focus our attention separately on the three dynamics. We define the

Lyapunov function V_1 as

$$V_1 = \bar{\boldsymbol{\varepsilon}}_2^T \mathbf{P}_o \bar{\boldsymbol{\varepsilon}}_2 = \sum_{i=1}^{N-1} \bar{\varepsilon}_{2_i}^T P_o \bar{\varepsilon}_{2_i}$$

with $\mathbf{P}_o = (I_{N-1} \otimes P_o)$, satisfying

$$\underline{\lambda}_1 \|\bar{\varepsilon}_2\|^2 \leq V_1 \leq \bar{\lambda}_1 \|\bar{\varepsilon}_2\|^2$$

It turns out that

$$\begin{aligned}\dot{V}_1 \leq & -a_1 d_o \|\bar{\varepsilon}_2\|^2 + 2\bar{\boldsymbol{\varepsilon}}_2^T \mathbf{P}_o \Big((I_{N-1} \otimes \bar{A}_\ell) + (L_J \otimes D_o^{-1} B_\ell^{''} C K \Gamma D_o) \Big) \bar{\boldsymbol{\varepsilon}}_2 \\ & + \bar{\boldsymbol{\varepsilon}}_2^T \mathbf{P}_o \Big((L_{J-1} \otimes D_o^{-1} B_\ell^{'} C D_g) \boldsymbol{\chi}_2 \\ & \qquad + (L_{J-1} \otimes D_o^{-1} B_\ell^{''} C D_g) \big[(I_N \otimes S) + (L \otimes KC) \big] \boldsymbol{\chi}_2 \Big)\end{aligned}$$

which, using the fact that due to the structure of $B_\ell^{''}$

$$\|D_o^{-1} B_\ell^{''} C K \Gamma D_o\| \leq \|B_\ell^{''} C K \Gamma\|$$

for any $d_o > d_o^\star$ leads to

$$\begin{aligned}\dot{V}_1 \leq & -a_1 d_o \|\bar{\varepsilon}_2\|^2 + \bar{\boldsymbol{\varepsilon}}_2^T \mathbf{P}_o \Big((L_{J-1} \otimes D_o^{-1} B_\ell^{'} C D_g) \boldsymbol{\chi}_2 \\ & + (L_{J-1} \otimes D_o^{-1} B_\ell^{''} C D_g) \big[(I_N \otimes S) + (L \otimes KC) \big] \boldsymbol{\chi}_2 \Big)\end{aligned}$$

For the sake of clarity, remembering the structure of $B_\ell^{'}, B_\ell^{''}$ (see (4.27)), in compact form we write

$$\dot{V}_1 \leq -a_1 d_o \|\bar{\varepsilon}_2\|^2 + \frac{1}{d_o} g\, p_1 \|\bar{\varepsilon}_2\| \|\boldsymbol{\chi}_2\| \tag{4.42}$$

with $p_1 = 2\|P_o\| \|L_{J-1}\| (\|B_\ell^{'}\| + \|B_\ell^{''}\|)$.

As far as $\boldsymbol{\chi}_2$ is concerned, following Chapter 3.1, *when* $\bar{\varepsilon}_2 = 0$ we can rewrite

$$\begin{aligned}\dot{\boldsymbol{\chi}}_2 &= g(I_{N-1} \otimes S)\boldsymbol{\chi}_2 + \tfrac{1}{g^d}(I_{N-1} \otimes B)\Delta\Phi'\left(x_1, (I_{N-1} \otimes D_g)\boldsymbol{\chi}_2\right) + g(L_{J-1} \otimes K_0 C)\boldsymbol{\chi}_2 \\ &= g(I_{N-1} \otimes H_s)\boldsymbol{\chi}_2 + \tfrac{1}{g^d}(I_{N-1} \otimes B)\Delta\Phi'\left(x_1, (I_{N-1} \otimes D_g)\boldsymbol{\chi}_2\right)\end{aligned}$$

with $H_s = S + K_0 C$ Hurwitz. Standard high-gain arguments lead to conclude that also $\boldsymbol{\chi}_2$ is GAS, via an opportune choice of $g > g^\star$ (see Section 3.1.3). In particular there exists a Lyapunov function V_2

$$V_2 = \boldsymbol{\chi}_2^T \mathbf{P} \boldsymbol{\chi}_2 = \sum_{i=1}^{N-1} \chi_{2_i}^T P \chi_{2_i}$$

with $\mathbf{P} = (I_{N-1} \otimes P)$. It's derivative is

$$\dot{V}_2 \leq -a_2 g \|\boldsymbol{\chi}_2\|^2 + 2\boldsymbol{\chi}_2^T \mathbf{P} \left[(I_{N-1} \otimes D_g^{-1} K \Gamma D_o) \bar{\varepsilon}_2 \right]$$

which, in compact form can be written as

$$\dot{V}_2 \leq -a_2 g \|\boldsymbol{\chi}_2\|^2 + p_2 \|\boldsymbol{\chi}_2\| \|D_o \bar{\varepsilon}_2\| \tag{4.43}$$

with $p_2 = 2\|P\|\|K_0\|$.

Furthermore, we define

$$V_3 = \mathbf{z}^2$$

By deriving V_3, we obtain

$$\dot{V}_3 \leq -a_\ell \|\mathbf{z}\|^2 + 2\mathbf{z} \left[(\mathcal{I}_{\bar{x}}) \otimes b_\ell C D_g) \boldsymbol{\chi}_2 \right]$$

In compact form we write

$$\dot{V}_3 \leq -a_\ell \|\mathbf{z}\|^2 + g p_3 \|\mathbf{z}\| \|\boldsymbol{\chi}_2\|$$

with $p_3 = \|\mathcal{I}_{\bar{x}}\|$

Consider now

$$V_{tot} = \sqrt{V_1 + 1} + \sqrt{V_2 + 1} + \mu \sqrt{V_3 + 1}$$

Furthermore, for the sake of simplicity, given a constant c, with $\bar{c}$ we represent $\bar{c} = \dfrac{c}{\|Q\|}$ where Q is P_o, P respectively for the $\bar{\varepsilon}_2$ and $\boldsymbol{\chi}_2$ dynamics.

The derivative of V_{tot} is

$$\begin{aligned}
\dot{V}_{tot} \\
\leq \frac{-a_1 d_o \|\bar{\varepsilon}_2\|^2 + \frac{1}{d_o} g\, p_1 \|\bar{\varepsilon}_2\| \|\boldsymbol{\chi}_2\|}{\sqrt{\bar{\varepsilon}_2^T \mathbf{P}_o \bar{\varepsilon}_2 + 1}} + \frac{-a_2 g \|\boldsymbol{\chi}_2\|^2 + p_2 \|\boldsymbol{\chi}_2\| \|D_o \bar{\varepsilon}_2\|}{\sqrt{\boldsymbol{\chi}_2^T \mathbf{P} \boldsymbol{\chi}_2 + 1}} \\
+ \mu \frac{-a_\ell \|\mathbf{z}\|^2 + g p_3 \|\mathbf{z}\| \|\boldsymbol{\chi}_2\|}{\sqrt{\mathbf{z}^2}} \\
\leq -\bar{a}_1 d_o \|\bar{\varepsilon}_2\| + \frac{1}{d_o} g\, \bar{p}_1 \|\boldsymbol{\chi}_2\| - \bar{a}_2 g \|\boldsymbol{\chi}_2\| + \bar{p}_2 \|\boldsymbol{\chi}_2\| \|D_o \bar{\varepsilon}_2\| - \mu a_\ell \|\mathbf{z}\| + \mu g p_3 \|\boldsymbol{\chi}_2\|
\end{aligned}$$

Now, setting $\mu = \dfrac{\bar{\mu}}{g}$, and remembering that for $t > T_s$, $\|D_o\bar{\varepsilon}\| \leq \epsilon$, we obtain

$$\dot{V}_{tot} \leq -\bar{a}_1 d_o \|\bar{\boldsymbol{\varepsilon}}_2\| - \big(g(\bar{a}_2 - \bar{\mu} p_3) - \frac{1}{d_o} g\,\bar{p}_1 - \epsilon \bar{p}_2\big)\|\boldsymbol{\chi}_2\| - \mu \bar{a}_\ell \|\mathbf{z}\| \tag{4.44}$$

From (4.44), by defining

$$\begin{aligned} \bar{\mu} &< \frac{1}{2}\frac{\bar{a}_2}{p_3} \\ \gamma^\star &> \frac{2}{\bar{a}_2}\big(\frac{1}{d_o} g\,\bar{p}_1 + \epsilon \bar{p}_2\big) \end{aligned}$$

and setting $g \geq \max\{g^\star, \gamma^\star\}$, the result in Proposition 4.2 follows.

$\square$

4.3.2 Simulation Results

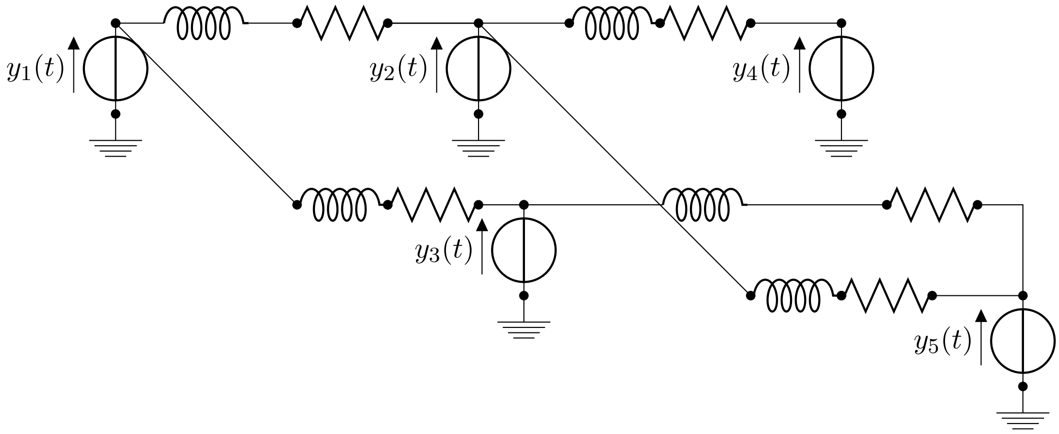

Figure 4.7: Circuit implemented in the simulation.

Simulation results are presented in this section, to show the behavior of the proposed control architecture: we considered 5 Van Der Pol oscillators

$$\begin{array}{rcl}\dot{x}_{i_1} & = & x_{i_2} \\ \dot{x}_{i_2} & = & 2(1-x_{i_1}^2)x_{i_2} - x_{i_1}\end{array} \qquad y_i = x_{i_1} \tag{4.45}$$

connected according to Figure 4.7 via dynamic links, described by

$$\begin{array}{rcl}\dot{z}_i & = & -10z_i + \sum_{j=1}^{N} \iota_{ji} y_j \\ p_i & = & z_i\end{array} \tag{4.46}$$

The initial conditions of the agents are $w_1 = (1,1)^T, w_2 = (-1,2)^T, w_3 = (2,3), w_4 = (-1.5,3)$ and $w_5 = (2,2)$. The initial conditions of the links are $z_1 = 4, z_2 = 1, z_3 = -1, z_4 = 2$ and $z_5 = -2$. The observers are designed according to (4.23), with $d_o = 100$ and K_c according to (4.29)-(4.30), with $a_1 = 1$. As far as the controller design is concerned, we choose K according to (4.5)-(4.6) with $g = 2$ and and $a_2 = 1$.

Figure 4.8 shows the behavior of the observers: due to the design and the *high gain* parameter d_o, the convergence is really fast but *peaking phenomena* is present. The peaking is the *practical* reason that leads to saturate the control input: it guarantees that the agents are not affected *excessively* by the transient of the observers.

Figure 4.9 shows the trajectories of the agents. After a transient, which is clearly influenced by the peaking of the observers, synchronization is achieved.

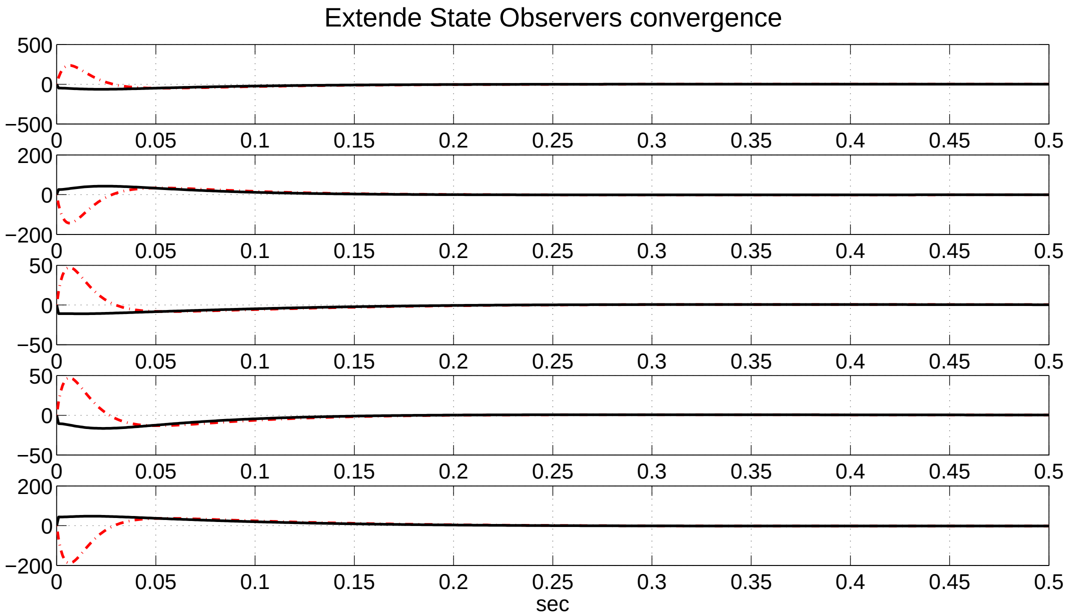

Figure 4.8: Observers behavior: convergence of the Extended State Observers to the desired value.

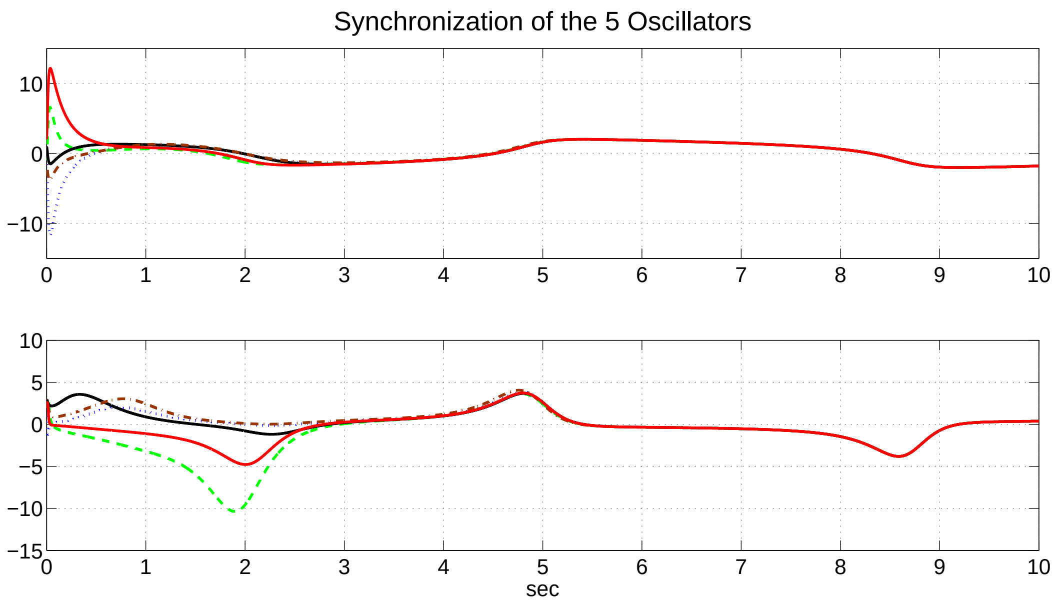

Figure 4.9: Synchronization of the 5 Van Der Pol oscillators.

Just as the wave cannot exist for itself, but is ever a part of the heaving surface of the ocean, so must I never live my life for itself, but always in the experience which is going on around me.

Albert Schweitzer

5

Disconnected and Switching Topologies

In this chapter we consider the scenario of networks on nonlinear oscillators when communication topologies are disconnected, persistently or just on some time intervals. The problem of disconnected topologies has been studied to understand if undesired or unstable behaviour might arise. Disconnected topologies typically lead to the so called clustering behaviour, in which agents achieve different synchronisation patterns, depending on connected sub-graphs composing the network. In Nabet et al. (2009), the clustering behaviour has been extensively studied and it is shown that this behaviour arises naturally in animal packs. A similar behaviour can be noticed also in opinion dynamics networks, where confidence is a time varying parameter based on which agents give trust only to a certain number of neighbors (see Morarescu and Girard (2011)).

Time varying topologies also attracted a lot of interests in the community and several results can be found. Among others, in Münz et al. (2011), the problem of switching network is addressed and consensus is achieved as long as the graph is uniformly quasi-strongly connected and fulfils a dwell-time condition. For the special case of integrators network, Moreau (2005) and Olfati-Saber and Murray (2004) give a consensus result based on the concept of *average connectivity* of the graph. In Li and Guo (2013) and Monshizadeh and van der Schaft (2014), the problem of consensus of linear systems with switching topology is considered: results are provided respectively by means of a hybrid control techniques and small gain/passivity arguments. In Jia and Tang (2012), fully nonlinear agents are considered and sufficient conditions to achieve consensus are cast in terms of linear matrix inequalities and frequency of switching, taking into account also the impact of communication delay.

A very general framework proposed in literature to deal with switching networks is the one based on the notion of *joint connectivity*. In contrast with the definition of connected graphs, joint connectivity does not require the graph to be instantaneously connected. Rather, the union on a certain time interval of all topologies among which the network is switching is required to be connected. Within this general framework, in Shi and Hong (2009), Yang et al. (2014a) different consensus results (such as target aggregation and state agreement) are proved under the assumption that the graph is jointly strongly connected and that each topology persist for a time period fulfilling a *dwell time* condition. The agents dynamics are linear in Yang et al. (2014a) while nonlinear dynamics for “informed agents", fulfilling an attractivity condition to the target set, are considered in Shi and Hong (2009).

Recently, the analysis developed in Yang et al. (2014a) has been extended in Yang et al. (2014b) to the case of nonlinear agents. In this work a general class of nonlinear systems assumed to be “non-expansive" is considered, and strong joint connectivity of the graph is shown to be sufficient condition for consensus. In the same paper it is shown that if the nonlinear dynamics just fulfill a globally Lipschitz condition then consensus is achieved under the assumption that graph is uniformly strongly joint connected and the Lipschitz constant of the nonlinear dynamics is sufficiently small.

With respect to all these topics, in the first part we show that the high gain design procedure, proposed in Chapter 3.1, succeed in achieving clustered consensus when disconnected topologies occur. In the second part of this chapter, we study a network of nonlinear oscillators when the topology is switching between a set of disconnected and connected topologies. By a Lyapunov analysis we prove that, if disconnected topologies last a limited amount of time and the switching law for connected topologies fulfill an average dwell time condition, synchronization is achieved.

The content of this chapter has been presented partially in Casadei et al. (2014b) and Casadei et al. (2015). However, most of the result presented are completely novel.

5.1 Disconnected topology and behavior of the network

In this chapter we study the behavior of a network of nonlinear oscillators with disconnected topology. We use the concept and notation of *independent connected topology* to group the agents into sub-graphs possessing some *stability property*. The reader is referred to Section 2.3 for details on the topic.

Consider a set of the $N_{\rm a}$ nonlinear agents is described by the following dynamics

$$\begin{array}{rcll} \dot{w}_k &=& s(w_k) + u_k & \quad w_k \in \mathbb{R}^d \\ y_k &=& c(w_k) & \end{array} \tag{5.1}$$

$k = 1, \ldots, N_{\rm a}$, in which u_k is the local control input, y_k is the local output whose value is transmitted to the neighbour agents, and

$$s(w_k) = Sw_k + B\phi(w_k)\,, \qquad c(w_k) = Cw_k \tag{5.2}$$

where (S, B, C) is a triplet of matrices in *prime* form (see Appendix A for further details on the systems at hand), coupled via

$$u_k = K \sum_{j=1}^{N} a_{kj}(y_j - y_k) \qquad \forall\, k = 1, \ldots, N \tag{5.3}$$

where a_{kj} are the elements of the adjacency matrix and K is a design parameter to be defined. Equivalently, using the Laplacian matrix, (5.3) reads as

$$u_k = K \sum_{j=1}^{N} \ell_{kj}\, y_j \qquad \forall\, k = 1, \ldots, N$$

with K chosen as illustrated in Section 3.1.

As usual, for the N homogeneous nonlinear systems (5.1) we assume the following.

Assumption 5.1. *The function $\phi(w_k)$ in (5.2) is globally Lipschitz and there exists a compact set $W \subset \mathbb{R}^d$ invariant for (5.1) such that the system*

$$\dot{w}_k = Sw_k + B\phi(w_k) + v_k$$

is input-to-state stable with respect to v_k relative to W, namely there exist a class-$\mathcal{KL}$ function

$\beta(\cdot,\cdot)$ *and a class-$\mathcal{K}$ function* $\gamma(\cdot)$ *such that*

$$\|w_k(t,\bar{w}_k)\|_W \le \max\{\beta(\|\bar{w}_k\|_W, t),\ \gamma(\sup_{\tau\in[0,t)} \|v_k(\tau)\|)\}\,.$$

We start by making this assumption about the generic disconnected topology describing the flow of information between the agents.

Assumption 5.2. *There exists a* $\mu > 0$ *such that, for all* $m = 1,\ldots,N_\mathrm{a}$ *such that* $\lambda_m(L) \neq 0$*, the following holds*

$$\mathrm{Re}\lambda_m(L) \ge \mu$$

Then the following result holds.

Proposition 5.1. *For all* $w_k(0) \in \mathbb{R}^d$*,* $k = 1,\ldots,N_\mathrm{a}$*, the trajectories of the agents are bounded. Furthermore, agents belonging to a independent connected component achieve consensus, namely there is a function* $y_i^\star : \mathbb{R} \to \mathbb{R}$*, for* $i = 1,\ldots,c$*, such that*

$$\lim_{t\to\infty} |y_k(t) - y_i^\star(t)| = 0\,,$$

for every k *such that the* k*-th agent belongs to the* i*-th independent connected component.*

Proof. The proof is divided in two parts. In the analysis of disconnected topologies, we will first focus on the independent connected components and show that inside each one of them consensus is achieved. Second, we will show that agents belonging to the residual component do not achieve consensus but have bounded trajectories.

Consensus inside independent connected components

By grouping all the agents according to $\mathbf{w} = \mathrm{col}(w_1,\ldots,w_{N_\mathrm{a}})$ and considering a fixed disconnected topology, we change coordinate according to $\mathbf{z} = (T^{-1}\otimes I_d)\mathbf{w}$, where T is defined in (2.1) relatively to a generic topology.

By relabeling the agents so that the agents within the same independent connected component are consecutive and the residual agents are confined at the end, $\mathbf{z}$ turns out to be

$$\mathbf{z} = \mathrm{col}\left(\mathrm{col}(z_{11},\mathbf{z}_{12}),\ldots,\mathrm{col}(z_{c1},\mathbf{z}_{c2}),\mathbf{z}_{res}\right) \tag{5.4}$$

with $z_{i1}\in\mathbb{R}^d$ and $\mathbf{z}_{i2}\in\mathbb{R}^{d(N_i-1)}$, for $i=1,\ldots,c$ and $\mathbf{z}_{res}\in\mathbb{R}^{dN_{res}}$.

The agents belonging to the c connected components are described by

$$\begin{aligned}\dot{z}_{j_1} &= Sz_{j_1} + B\phi(z_{j_1}) - (L_{12_j} \otimes KC)\mathbf{z}_{j_2} \\ \dot{\mathbf{z}}_{j_2} &= \left([I_{N_j-1} \otimes S] - [L_{22_j} \otimes KC]\right)\mathbf{z}_{j_2} + \Delta\Phi_j(z_{j_1}, \mathbf{z}_{j_2})\end{aligned}$$

for $j = 1, \ldots, c$, $z_{j_1} \in \mathbb{R}^d$ and $\mathbf{z}_{j_2} \in \mathbb{R}^{N_j-1}$, with $\Delta\Phi_j(z_{j_1}, \mathbf{z}_{j_2})$

$$\Delta\Phi_j(z_{j_1}, \mathbf{z}_{j_2}) = (I_{N_j-1} \otimes B)\begin{pmatrix}\phi(z_{j_1} + z_{j_2}) - \phi(z_{j_1}) \\ \vdots \\ \phi(z_{j_1} + z_{j_{N_i}}) - \phi(z_{j_1}))\end{pmatrix}$$

which is globally Lipschitz in $\mathbf{z}_{j_2}$ uniformly in z_{j_1} and $\Delta\Phi_j(z_{j_1}, 0) = 0$. As proposed in Isidori et al. (2014), we now rescale the variable $\mathbf{z}_{j_2}$ in the following way

$$\chi = (I_{N_j-1} \otimes D_g^{-1})\mathbf{z}_{j_2}$$

and obtain

$$\begin{aligned}\dot{z}_{j_1} &= Sz_{j_1} + B\phi(Z_1) - (L_{12_j} \otimes D_gK_0C)(I_{N_\mathrm{a}-1} \otimes D_g)\chi \\ \dot{\chi}_j &= gH_j\chi_j + \frac{1}{g^d}\Delta\Phi(z_{j_1}, (I_{N_\mathrm{a}-1} \otimes D_g)\chi_j)\end{aligned}$$

where $H_j = [(I_{N_j-1} \otimes S) - (L_{22_j} \otimes K_0C)]$. To show that the origin of the system with state χ_j is locally asymptotically stable, we consider the change of variable $\zeta_j = J_j\chi_j$ with J_j such that $\bar{H}_j = J_jH_jJ_j^{-1}$ is in Jordan form. We obtain a new system that is the cascade of system

$$\dot{\zeta}_j = g\bar{H}_j\zeta_j + \frac{1}{g^d}\Delta\Phi(z_{j_1}, (I_{N_\mathrm{a}-1} \otimes D_g)J_j^{-1}\zeta_j) \tag{5.5}$$

with system

$$\dot{z}_{j_1} = Sz_{j_1} + B\phi(z_{j_1}) - (L_{i,12} \otimes D_gK_0C)(I_{N_\mathrm{a}-1} \otimes D_g)J_i^{-1}\zeta_j\,. \tag{5.6}$$

To prove that consensus is achieved inside each of the c independent connected components (*i.e.* $j = 1, \ldots, c$), we now use Lypunov arguments. For the sake of simplicity in the notation, we drop the dependence on j. Consider the candidate Lyapunov function

$$V(\zeta) = \zeta^T(D(\ell) \otimes P^{-1})\zeta \tag{5.7}$$

where P is the solution of the Riccati equation

$$SP + PS^T - \mu PC^TCP = -aI$$

with $a > 0$ and $D(\ell) = \mathrm{diag}(1, \ell, \ell^2, \ldots, \ell^{N_a-2})$ with ℓ a positive design parameter. Note that there exist positive constants $\underline{\lambda} \leq \bar{\lambda}$, both dependent on ℓ, such that $\underline{\lambda}\zeta^T\zeta \leq V \leq \bar{\lambda}\zeta^T\zeta$.

The derivative of V along the solutions of (5.27) can be bounded as

$$\begin{aligned}
\dot{V} &= 2\zeta^T(D(\ell)\otimes P^{-1})[g\bar{H}\zeta + \frac{1}{g^d}\Delta\Phi(z_1, (I_{N_a-1}\otimes D_g)J_i^{-1}\zeta)] \\
&\leq 2\zeta^T(D(\ell)\otimes P^{-1})g\bar{H}\zeta + \frac{2}{g^d}\bar{\Phi}\|D(\ell)\otimes P^{-1}\|\|\|(I_{N_a-1}\otimes D_g)\|\|\|J_i^{-1}\|\zeta^T\zeta \\
&\leq 2\zeta^T(D(\ell)\otimes P^{-1})g\bar{H}\zeta + a_\phi\zeta^T\zeta
\end{aligned}$$

where a_ϕ is a positive constant not dependent on g (provided that the latter is taken $g \geq 1$).

From Theorem 2.1, we know that that if the graph is connected one eigenvalues of a Laplacian matrix is zero and the rest are all positive. We recall this crucial result (see Chapter 3.1.2).

Lemma 5.1. *Let Assumption 5.4 hold. Then, for each of the c independent connected component inside the graph, there exist a positive constants a'_c and $\ell^\star$ such that for all $\ell \geq \ell^\star$*

$$2\zeta^T(D(\ell)\otimes P^{-1})\bar{H}\,\zeta \leq -a'_c\zeta^T\zeta\,.$$

Using the previous lemma and taking $g^\star = (a_\phi + a_c\bar{\lambda})/a'_c$ with a_c an arbitrary positive constant ($g^\star \geq 1$ without loss of generality), it is immediately seen that for all $\ell \geq \ell^\star$ and $g \geq g^\star$ we have

$$\dot{V} \leq -(ga'_c - a_\phi)\zeta^T\zeta \leq -\frac{ga'_c - a_\phi}{\bar{\lambda}}V \leq -a_cV\,. \qquad (5.8)$$

By this we conclude that consensus is achieved within each cluster. It is worth noting that if the graph underlying the communication is connected, *i.e.* $c = 1$, the previous analysis shows that synchronization of the whole network is achieved.

Behavior of the residual agents

The residual subsystem, instead, reads as

$$\dot{\mathbf{z}}_{res} = \Big([I_{N_{res}} \otimes S] - [\tilde{L}_{res} \otimes KC]\Big)\, \mathbf{z}_{res} + \Delta\Phi_{res}(z_{1_1}, \dots, z_{c_1}, \mathbf{z}_{res}) + (\tilde{\Gamma} \otimes KC)\tilde{\mathbf{z}}_2$$

with $\tilde{\mathbf{z}}_2 = \mathrm{col}(\mathbf{z}_{1_2}, \dots, \mathbf{z}_{c_2})$ and where $\Delta\Phi_{res}(z_{1_1}, \dots, z_{c_1}, \mathbf{z}_{res})$ is

$$(I_{N_{res}} \otimes B) \begin{pmatrix} -\psi_{1_1}\phi(z_{1_1}) - \dots - \psi_{c_1}\phi(z_{c_1}) + \phi_{res_1} \\ \vdots \\ -\psi_{1_{N_{res}}}\phi(z_{1_1}) - \dots - \psi_{c_{N_{res}}}\phi(z_{c_1}) + \phi_{res_{N_{res}}} \end{pmatrix}$$

where $\phi_{res_1}, \dots, \phi_{res_c}$ are defined as

$$\begin{aligned} \phi_{res_1} &= \phi(\psi_{1_1} z_{1_1} + \dots + \psi_{c_1} z_{c_1} + z_{res_1}) \\ &\vdots \\ \phi_{res_c} &= \phi(\psi_{1_{N_{res}}} z_{1_1} + \dots + \psi_{c_{N_{res}}} z_{c_1} + z_{res_{N_{res}}}). \end{aligned}$$

Note that, in general, $\Delta\Phi_{res}(z_{1_1}, \dots, z_{c_1}, 0) \neq 0$.

We now change coordinate according to $\zeta_{res} = (I_{N_{res}} \otimes D_g^{-1})\mathbf{z}_{res}$ and obtain

$$\begin{aligned} \dot{\zeta}_{res} &= g\Big([I_{N_{res}} \otimes S] - [\tilde{L}_{res} \otimes K_0 C]\Big)\, \zeta_{res} + \tfrac{1}{g^d}\Delta\Phi_{res}(z_{1_1}, \dots, z_{c_1}, (I_{N_{res}} \otimes D_g)\zeta_{res}) \\ &\quad +(\tilde{\Gamma} \otimes K_0 C)\tilde{\mathbf{z}}_2 \end{aligned}$$

The fact that $\tilde{L}_{res}$ has positive eigenvalues leads to conclude that the matrix

$$[I_{N_{res}} \otimes S] - [\tilde{L}_{res} \otimes KC] = H_{res}$$

is Hurwitz. Furthermore, by adding and subtracting the term $\Delta\Phi_{res}(z_{1_1}, \dots, z_{c_1}, 0)$, we obtain

$$\begin{aligned} \dot{\zeta}_{res} &= gH_{res}\zeta_{res} + \tfrac{1}{g^d}\Delta\Phi_{res}(z_{1_1}, \dots, z_{c_1}, (I_{N_{res}} \otimes D_g)\zeta_{res}) \\ &\quad \pm\tfrac{1}{g^d}\Delta\Phi_{res}(z_{1_1}, \dots, z_{c_1}, 0) + (\tilde{\Gamma} \otimes K_0 C)\tilde{\mathbf{z}}_2 \\ &= gH_{res}\zeta_{res} + \tfrac{1}{g^d}\Delta\tilde{\Phi}_{res}(z_{1_1}, \dots, z_{c_1}, (I_{N_{res}} \otimes D_g)\zeta_{res}) \\ &\quad +\tfrac{1}{g^d}\Delta\Phi_{res}(z_{1_1}, \dots, z_{c_1}, 0) + (\tilde{\Gamma} \otimes K_0 C)\tilde{\mathbf{z}}_2 \end{aligned}$$

where

$$\Delta\tilde{\Phi}(\cdot) = \Delta\Phi_{res}(z_{1_1},\ldots,z_{c_1},(I_{N_{res}}\otimes D_g)\zeta_{res}) - \Delta\Phi_{res}(z_{1_1},\ldots,z_{c_1},0)$$

It is is easy to see that, $\zeta_{res} = 0$ implies $\Delta\tilde{\Phi}(z_{1_1},\ldots,z_{c_1},0) = 0$. From this and the fact that both $\Delta\Phi_{res}(z_{1_1},\ldots,z_{c_1},0)$ and $(\tilde{\Gamma}\otimes K_0 C)\tilde{\mathbf{z}}_2$ are bounded since

$$z_{1_1},\mathbf{z}_{1_2},\ldots,z_{c_1},\mathbf{z}_{c_2}$$

are all bounded (as shown in the first part of the proof), we can conclude that, for a sufficiently large g, ζ_{res} is bounded and thus $\mathbf{z}_{res}$ is bounded too, independently from the particular topology. However, nothing can be concluded about the particular asymptotic trajectory of the residual agents: in general it depends on the set of initial conditions and on the topology.

Following the definition of (2.1), the fact that $\mathbf{z}_{res}$ is bounded and that all the agents belonging to an independent connected components achieve consensus, leads to conclude that $\mathbf{w}$ is bounded too, despite the particular disconnected topology.

□

5.1.1 Simulation Results

In this section we show simulation results, considering 5 Lorentz oscillators. The Lorentz oscillators are described by

$$\begin{aligned}\dot{x}_{k_1} &= \sigma(x_{k_2}-x_{k_1})\\ \dot{x}_{k_2} &= x_{k_1}(\rho - x_{k_3}) - x_{k_2} \qquad y_k = x_{k_1}\,.\\ \dot{x}_{k_3} &= x_{k_1}x_{k_2} - \beta x_{k_3}\end{aligned} \tag{5.9}$$

for $k = 1,\ldots,5$. The values of parameters (σ,ρ,β) are $\sigma = 10$, $\rho = 28$ and $\beta = 8/3$. System (5.9) can be embedded into the fourth order system

$$\begin{aligned}\dot{w}_{k_1} &= w_{k_2}\\ \dot{w}_{k_2} &= w_{k_3}\\ \dot{w}_{k_3} &= w_{k_4}\\ \dot{w}_{k_4} &= \Phi(w_{k_1},w_{k_2},w_{k_3},w_{k_4})\end{aligned} \tag{5.10}$$

fitting into the structure of (5.1) and fulfilling the requested assumption.

In new coordinates, the agents' initial conditions are $w_1 = (1.5;1;0;0)$, $w_2 = (1;5;5;5)$, $w_3 = (2;10;10;10)$, $w_4 = (0.5;7;7;7)$ and $w_5 = (0;15;15;15)$. K is chosen as in Section 3.1 with the gain parameter $g = 50$ and $a = 1$.

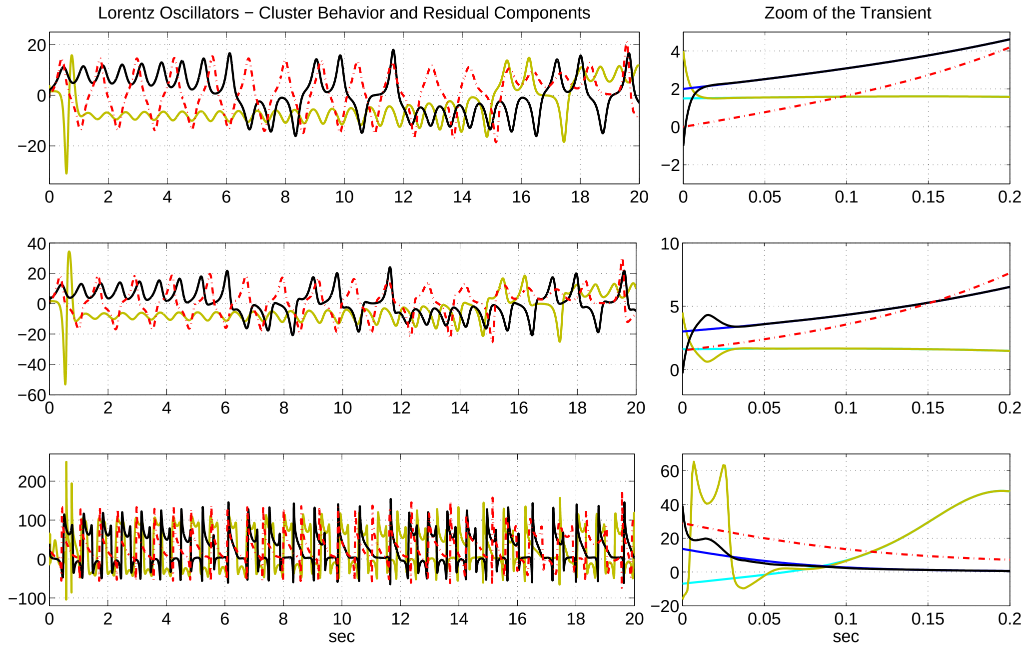

Figure 5.1: Lorentz Oscillator: behavior of the three components of the Lorentz oscillators with switching topology.

The fixed disconnected topology considered in this example is given by the Laplacian matrix

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix}$$

It is trivially seen that this topology is composed by two ICC: the first one is $ICC_1 = \{w_1, w_2\}$, while the second is $ICC_2 = \{w_3, w_4\}$. Agent w_5 is the residual component of the graph and receives information from all the other clusters.

Figure 5.1 shows the behavior of the system in the original coordinates (5.9). As expected, the agents belonging to the two ICC achieve clustered consensus (the black line and the green line in the plot), while the residual agent (the red dotted line) is simply bounded.

5.2 Cluster Consensus

In this section we analyze the problem of consensus of homogeneous nonlinear agents in case of disconnected topology. Specifically, we look for diffusive coupling strategies able to enforce the so-called cluster consensus within the networked agents. In order to avoid *undesired* behaviors of a certain group of agents (referred to as *residual agents*) not belonging to any connected components, we modify the conventional diffuse coupling by means of a time dependent strategy clustering the graph in the minimum number of connected components within which all the agents reach a consensus. Sufficient conditions for achieving cluster consensus are obtained by means of Lyapunov tools. Simulation results are also presented.

5.2.1 Problem formulation

Each of the N_a nonlinear agents is described by (5.1)-(5.2).

In order to achieve clustered-consensus between the agents the control input u_k of the k-th agent is chosen according to a diffusion coupling structure of the form

$$u_k = K\nu_k \quad \text{with} \quad \nu_k = \sum_{j=1}^{N} \tau_{kj} a_{kj} \left(Cw_j(t) - Cw_k(t) \right) \tag{5.11}$$

for $k = 1, \ldots, N_\mathrm{a}$, in which K is a vector (the same for all the agents) and $\tau_{kj} = \{0, 1\}$ are control parameters, all to be designed. The τ_{kj} act as "trust design parameters" chosen by the local k-th controller to use the information coming from the neighbor j-th agent (in such a case $\tau_{kj} = 1$) or to ignore it ($\tau_{kj} = 0$). As clarified next, the goal is to choose such parameters in order to force an empty residual set, by reassigning the agents that possibly belong to the residual set to one of the c independent connected components. The role of K, on the other hand, is to force consensus between the agents that belong to the same independent connect component. The final objective is to cluster all the N agents in c independent connected component achieving cluster consensus. It is worth noting that c is the minimum number of independent connected components that can be enforced with the actual graph topology. The k-th agent is thus coupled to the other agents via the control input (5.11), as

$$\dot{w}_k = s(w_k) + K\nu_k \,. \tag{5.12}$$

In practice, the parameters τ_{kj} affect the graph topology by possibly cutting the edge $(v_k, v_j) \in \mathcal{E}$ in the graph if τ_{kj} is chosen equal to zero. They thus modify the adjacency matrix A and, in turn, the laplacian matrix L. The diffusing coupling terms ν_k in (5.11) can be thus rewritten as

$$\nu_k(t) = -\sum_{j=1}^{N} \ell_{kj}(\tau_{kj}) C w_j(t) \qquad k = 1, \ldots, N\,. \tag{5.13}$$

where the values ℓ_{kj} of the Laplacian matrix depend on the particular choice of τ_{kj}.

We conclude this paragraph by noting that, by defining $\mathbf{w} = \mathrm{col}(w_1, \ldots, w_N)$, it is possible to rewrite the network of (5.12) as[1]

$$\dot{\mathbf{w}} = (I_N \otimes S)\mathbf{w} + (I_N \otimes B)\Phi(\mathbf{w}) + (I_N \otimes K)\mathcal{V}(\mathbf{w}) \tag{5.14}$$

where $\mathcal{V}(\mathbf{w}) = \mathrm{col}(\nu_1, \ldots, \nu_N)$, with ν_k introduced in (5.13), and

$$\Phi(\mathbf{w}) = \mathrm{col}(\phi(w_1), \ldots, \phi(w_N))$$

Many results can be found in literature proving that consensus can be achieved with a proper choice of the control parameters if the graph is connected (namely $c = 1$ and the residual graph is empty). In such a case, in fact, it has been shown that if the τ_{jk} are chosen all one, there exists a choice of K (based on high-gain arguments) such that trajectories of the agents are bounded and all the outputs y_k reach a consensus on a common signal $y^\star$.

Motivated by the result of previous section, the goal is now to design the degree-of-freedom τ_{kj} and K in such a way that the set of residual components is empty and the outputs of all the agents belonging to one of the c independent connected components reach a consensus on a common trajectory $y_i^\star$, $i = 1, \ldots, c$.

Problem. Design K and τ_{kj}, $k, j = 1, \ldots, N_\mathrm{a}$, in such a way that each of the N_a agents belongs to one of the c independent connected components in finite time (namely the set of residual agents is empty in finite time) and all the outputs of the agents belonging to a given independent connected components i reach a consensus on a common trajectory $y_i^\star$, with $i = 1, \ldots, c$.

The result we are aiming to solve is semiglobal, namely we assume the initial conditions of the k-th agents belong to a fixed (although arbitrary) compact set $W_k \in \mathbb{R}^{n_k}$, $k = 1, \ldots, N_\mathrm{a}$.

[1]Here and in what follows, $A \otimes B$ denotes the Kronecker product of the two matrices A and B.

5.2.2 A cluster consensus result

The design of the vector K follows the same paradigm presented in Isidori et al. (2014). In particular the vector K takes the form

$$K = D_g K_0 \,, \tag{5.15}$$

where $D_g = \mathrm{diag}(g, g^2, \ldots, g^d)$, with g a "gain" parameter and $K_0 = PC^T$ with P is solution of the Riccati equation

$$PS^T + SP - 2\mu PC^T CP = -aI \tag{5.16}$$

with $\mu \leq \min \mathrm{Re}(\lambda_i(L))$ (without considering $\lambda_i = 0$ for $i = 1, \ldots, c$) and $a > 0$.

The parameters τ_{jk}, on the other hand, are designed in such a way that the agent k that would belong to a residual component if τ_{kj} were chosen equal to 1 for all $j = 1, \ldots, N_{\mathrm{a}}$, is indeed associated to one of the c connected components. To achieve this goal we choose an update rule for τ_{kj} of the following: form

$$\left.\begin{array}{rcl} \dot{\tau}_{kj} & = & 0 \\ \dot{\beta}_k & = & -\gamma\beta_k \end{array}\right\} \quad \text{if } |K\nu_k| \leq \beta_k$$

$$\left.\begin{array}{rcl} \tau_{kj}^+ & = & \left\{ \begin{array}{ll} 1 & \text{for } \ j = \mathrm{argmin}_{j, j\neq k}(\|Cw_k - Cw_j\|) \\ 0 & \text{otherwise} \end{array} \right. \\ \beta_k^+ & = & \beta_0 \end{array}\right\} \quad \text{if } |K\nu_k| > \beta_k$$

where γ and β_0 are positive parameters yet to be fixed. It turns out that there exists a choice of the parameters γ, β_0 and g such that the previous controller asymptotically force all the agents to belong to one of the c independent connected components and reach a consensus therein.

Proposition 5.2. *For each agent k let τ_{kj}, $j = 1, \ldots, N_{\mathrm{a}}$, and K be fixed as detailed above (with K dependent on the parameter g). Furthermore, let the local initial conditions be taken as $\tau_{kj}(0) = 1$, for all $j = 1, \ldots, N_{\mathrm{a}}$, $\beta_k(0) = \beta_0$, $w_k(0) \in W_k \in \mathbb{R}^{n_k}$ with W_k a fixed compact set. Then, there exist positive $g^\star$ and $\gamma^\star$ and, for all $g \geq g^\star$, a $\beta^\star(g)$ such that for all $g \geq g^\star$, $\beta_0 \geq \beta^\star$ and positive $\gamma \leq \gamma^\star$ the following holds:*

- *there exists a time $T \geq 0$ such that the set of residual agents is empty for all $t \geq T$;*
- *the outputs y_k of the agents belonging to independent connected components reach a consensus. Namely, for each $i = 1, \ldots c$, there exists a $y_i^\star(t)$ generated by a solution of*

the system

$$\dot{w} = s(w), \qquad y = c(w)$$

such that if the agent k belongs to the i-th independent connected component the following holds

$$\lim_{t\to\infty} \|y_k(t) - y_i^\star(t)\| = 0\,.$$

Proof. Following the approach of Chapter 3.1, we change coordinates according to

$$\begin{bmatrix} w_1 \\ \zeta \end{bmatrix} = \begin{pmatrix} I_d & 0 \\ 0 & I_{N_\mathrm{a}-1} \otimes D_g^{-1} \end{pmatrix} (T_J^{-1} \otimes I_d)\mathbf{w}$$

where T_J is the change of coordinates which puts the Laplacian matrix in Jordan form and is defined as

$$T_J = \begin{bmatrix} 1 & 0_{1\times N_\mathrm{a}-1} \\ \mathbf{1}_{N_\mathrm{a}-1} & J \end{bmatrix}$$

with $J \in \mathbb{R}^{N_\mathrm{a}-1\times N_\mathrm{a}-1}$ a properly defined matrix whose inverse T_J^{-1} is

$$T_J^{-1} = \begin{bmatrix} 1 & 0_{1\times N_\mathrm{a}-1} \\ -J^{-1}\mathbf{1}_{N_\mathrm{a}-1} & J^{-1} \end{bmatrix}.$$

For $k = 2, \ldots, N_\mathrm{a}$, we can write w_k as

$$w_k = w_1 + (J_k \otimes D_g)\zeta$$

where J_k is the k-th row of J.

From Chapter 3.1.3, we know that, if the graph is connected, there would exist a $g^\star > 0$ such that for all $g \geq g^\star$ the Lyapunov function

$$V(\zeta) = \zeta^T(I_{N-1} \otimes P^{-1})\zeta$$

fulfills $\underline{\lambda}\|\zeta\|^2 \leq V(\zeta) \leq \overline{\lambda}\|\zeta\|^2$, $\dot{V}(\zeta) \leq -a_c V(\zeta)$, for some positive a_c, $\underline{\lambda}$ and $\overline{\lambda}$. Using the previous relations and the fact that $\zeta(0)$ belongs to a fixed compact set, standard Lyapunov arguments can be used to obtain

$$\|\zeta(t)\| \leq \bar{\beta}\, e^{-\bar{\gamma}t} \tag{5.17}$$

for some positive $\bar{\beta}$ and $\bar{\gamma}$. The previous relation, obtained under the assumption that the graph is connected, can be indeed used also to upper bound the state behavior of the agents belonging to a connected component characterizing a graph that is not connected. It is thus used in the following to bound the term $\|K\nu_k(t)\|$ of the agent k, and thus to update τ_{kj}. The control input $\|K\nu_k(t)\|$ can be bounded as

$$\begin{array}{rcl}\|K\nu_k(t)\| & \leq & \|K\sum_{j=1}^{N} a_{kj}C(w_j(t)-w_k(t))\| \\ & \leq & \|K\|\sum_{j=1}^{N}|a_{kj}|\|w_j(t)-w_k(t)\| \\ & \leq & \|K\|\sum_{j=1}^{N}|a_{kj}|\|(J_j-J_k)\otimes D_g\|\|\zeta(t)\| \\ & = & \Gamma_k\|\zeta(t)\|\end{array} \tag{5.18}$$

where Γ_k is defined as

$$\Gamma_k = \|K\|\sum_{j=1}^{N}(|a_{kj}|\|(J_j-J_k)\otimes D_g\|)\,.$$

The term $\|K\nu_k(t)\|$ of each agent belonging to an independent connected component is thus bounded by

$$\|K\nu_k(t)\| \leq \Gamma_k\bar{\beta}e^{-\bar{\gamma}t}\,. \tag{5.19}$$

Setting $\beta^\star = \Gamma_k\bar{\beta}$ and $\gamma^\star = \bar{\gamma}$, it follows that the local controller of agents that belongs to an independent connected component never switch the values of τ_{kj} and hence reach consensus with the other agents belonging to the same independent connected component, according to the analysis of Chapter 5.1. On the other hand, agents possibly belonging to the residual set will exhibit a term $\|K\nu_k(t)\|$ that necessarily overpass the value of $\beta_k(t)$ at a time T, since $\beta_k(t)$ is exponentially decreasing and ν_k not asymptotically vanishing (otherwise consensus would be achieved anyway). The proposed law for τ_{kj} guarantee that at time T the agent k cut the edges with all the neighbor agents except the j-th one that, at time $t = T$, has the closest output value. In this way the agent k has a single incoming edge from the agent j and thus, from $t \geq T$, belongs to the independent connected component of the agent j. The simultaneous reset of the value of β_k also guarantees, from the analysis above, that no further switches of τ_{kj} will occur. □

5.2.3 Simulation Results

As in Section 5.1.1, we consider 5 Lorentz oscillators with the same initial conditions. Figure 5.2 shows that, when the hybrid parameters τ_{kj} switches, agent w_5 belongs to one of the ICC and achieves synchronization within that subgroup of agents.

In order to show the difference of behavior with respect to the previous case in Section 5.1.1, we forced the hybrid parameters τ_{kj} to jump at 10 sec (otherwise they would have jumped after less then 0.5 sec). As the zoom in 5.2 clearly shows, after this *event* agent w_5 synchronizes with the *nearest* ICC.

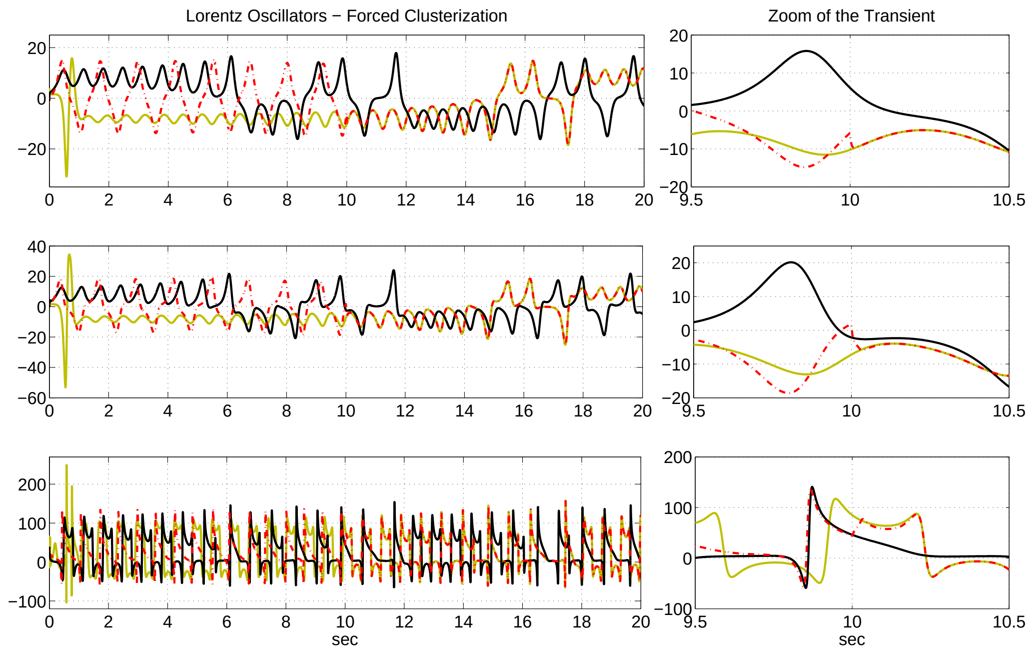

Figure 5.2: Lorentz Oscillator: behavior of the three components of the Lorentz oscillators with switching topology.

5.3 Switching networks

In previous sections, we showed that if the topology is disconnected agents do not achieve a *global* consensus: however, if the topology is changing in time, under certain conditions which involves the *average* connectedness of the graph it is possible to prove that synchronization is achieved, despite bounded time intervals in which the topology is not connected.

In this section, we show precisely under which conditions consensus is achieved if the topology switches between a set of connected and disconnected topologies. First we will define the switching topology conditions and then, give the main result of this section. We will make extensive use of hybrid systems tools and hybrid Lyapunov analysis: for further details on this aspects, the reader is referred to Appendix C and Goebel et al. (2008).

Each of the N_a agents is described by the nonlinear dynamics

$$\begin{array}{rcll} \dot{w}_k & = & s(w_k) + u_k & \qquad x_k \in \mathbb{R}^d \\ y_k & = & c(w_k) & \end{array} \tag{5.20}$$

in which, for each $k = 1, \ldots, N_\mathrm{a}$, $u_k \in \mathbb{R}^d$ is the control input, $y_k \in \mathbb{R}$ is the available measurement. Note that we deal with homogeneous nonlinear agents, namely $f(\cdot)$ and $h(\cdot)$ do not depend on k.

We rewrite (5.20) as

$$\dot{w}_k = Sw_k + B\phi(w_k) + u_k\,, \qquad y_k = Cw \tag{5.21}$$

with the triplet of matrices (S, B, C) that is in *prime* form. In the following we assume that the function $\phi(w_k)$ is globally Lipschitz, namely there exists a positive constant $\bar{\phi}$ such that $\|\phi(w)\| \leq \bar{\phi}\|w\|$ for all $w \in \mathbb{R}^d$. Such a globally Lipschitz condition is motivated by the fact of looking for “global" consensus results. The assumption in question could be weakened by just asking the previous function to be only locally Lipschitz if just semiglobal consensus results are of interest.

We look for a decentralized control structure in which the agents exchange only output information and the control law of each agent is taken as

$$u_k = K\nu_k\,, \qquad \nu_k = \sum_{j=1}^{N_\mathrm{a}} \ell_{kj} c(w_j) \tag{5.22}$$

with K to be designed in such a way that output consensus is reached among the agents.

Namely, for each initial condition $w_k(0) \in \mathbb{R}^d$, there is a function $y^\star : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{t\to\infty} |y_k(t) - y^\star(t)| = 0\,,$$

uniformly in the initial conditions, for all $k = 1, \dots, N_{\mathrm{a}}$. It is worth noting that, in the proposed framework, no leader is considered, and only the neighbor's information is available according to the underlying communication topology. Furthermore, local output of single agents rather than a *full state* information is assumed to be spread over the network.

We also assume that agents (5.21) have a robust compact attractor $W \subset \mathbb{R}^d$, where robustness is characterized in terms of Input-to-State Stability. This assumptions indeed guarantees that the network of (5.21) achieves synchronization on non-trivial trajectories.

Assumption 5.3. *There exists a compact set $W \subset \mathbb{R}^d$ invariant for (5.21) with $u = 0$ such that the system*

$$\dot{w} = Sw + B\phi(w) + u$$

is input-to-state stable with respect to u relative to W, namely there exist a class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a class-$\mathcal{K}$ function $\gamma(\cdot)$ such that[2]

$$\|w(t,\bar{w})\|_W \le \max\{\beta(\|\bar{w}\|_W, t),\ \gamma(\sup_{\tau\in[0,t)} \|u(\tau)\|)\}\,.$$

Finally, we fix a restriction on the communication topologies asking that the real part of the nontrivial eigenvalues of the Laplacian are uniformly bounded from below by a known constant μ.

Assumption 5.4. *There exists a $\mu > 0$ such that, for all $m = 1, \dots, N_{\mathrm{a}}$ such that $\lambda_m(L) \neq 0$, the following holds*

$$\mathrm{Re}\lambda_m(L) \ge \mu$$

5.3.1 Switching topology framework

We denote by $\mathcal{T} = \{\mathcal{T}_1, \dots, \mathcal{T}_{N_{\mathrm{t}}}\}$ the set of N_{t} possible communication topologies. This set of topologies $\mathcal{T}$ is also characterized by topologies that are not necessarily connected[3]. For this reason we split the set $\mathcal{T}$ in two disjoint sets $\mathcal{T}_{\mathrm{c}}$ and $\mathcal{T}_{\mathrm{nc}}$, which fulfill

[2]Here and in the following we denote by $\|w\|_W = \min_{x\in W} \|w - x\|$ the distance of w from W. Furthermore, $w(t,\bar{w})$ denotes the solution of (5.21) at time t with initial condition $\bar{w}$ at time $t = 0$.

[3]We recall that a communication topology is said to be *connected* if there is a node v from which any other node $v_k \in \mathcal{V} \setminus \{v\}$ can be reached, or equivalently if there is a path from v to all v_k. In the previous

$\mathcal{T} = \mathcal{T}_{\rm c} \cup \mathcal{T}_{\rm nc}$ and $\mathcal{T}_{\rm c} \cap \mathcal{T}_{\rm nc} = \emptyset$, collecting topologies that are, respectively, connected and disconnected.

For all $i = 1, \ldots, N_{\rm t}$, let $\Lambda_i = \{\lambda_1(L^i), \ldots, \lambda_{N_{\rm a}}(L^i)\}$ be the eigenvalues of L^i (the Laplacian of the i-th topology), ordered with increasing real part. As a consequence of Lemma 2.1, the following holds (see also Wieland (2010) and Ren and Beard (2005) for further details):

- if $\mathcal{T}_i \in \mathcal{T}_{\rm c}$ then $\lambda_1(L^i) = 0$ and ${\rm Re}\lambda_m(L^i) > 0$ for $m = 2, \ldots N_{\rm a}$;
- if $\mathcal{T}_i \in \mathcal{T}_{\rm nc}$ then there exists a $c_i \in [1, N_{\rm a}]$ such that $\lambda_m(L^i) = 0$ for $m = 1 \ldots, c_i$ and ${\rm Re}\lambda_m(L^i) > 0$ for $m = c_i + 1 \ldots N_{\rm a}$.

The different communication topologies alternates in time by forming an ordered sequence $\{\mathcal{T}_i\}_{i=1}^{\infty}$, with each $\mathcal{T}_i$ taken in the set $\mathcal{T}$. We denote by $\Delta T_i \geq 0$, $i = 1, \ldots, \infty$ the length of the time interval in which the i-th communication topology is active. Note that time intervals of zero length are allowed in the proposed framework. By this fact, without loss of generality, we can assume that the topologies alternates in time according to the rule that $\mathcal{T}_i \in \mathcal{T}_{\rm c}$ if i is odd and $\mathcal{T}_i \in \mathcal{T}_{\rm nc}$ is i is even. As a matter of fact, if two connected (disconnected) communication topologies occur in a row we can always "separate" them with a disconnected (connected) topology of zero length without practically changing the networked system dynamics. Note also that we do not assume that connected communication topologies persist for a guaranteed dwell time, namely connected topologies can last for arbitrarily small (indeed also of length zero) time interval. The kind of result we will prove (see next Proposition 5.3) is that consensus is reached if the intervals of time in which connected topologies govern the communication between the agents have a sufficiently long (in the average) duration and if the disconnected topologies duration is bounded.

We formulate now the assumption about the length of the time intervals in which disconnected topologies are active.

Assumption 5.5. *There exists a $T_0 > 0$ such that for all $i = 1, \ldots, N_{\rm t}$ such that $\mathcal{T}_i \in \mathcal{T}_{\rm nc}$ the following holds*

$$\Delta T_i \leq T_0\,.$$

The additional condition under which the main result will be proved asks that the time intervals in which the network is connected last, in the average, sufficiently long. More precisely, we asks that there exist positive $\tau \in \mathbb{R}_{\geq 0}$ and $n_0 \in \mathbb{N}$ such that, for all

definition a *path* from node v_j to node v_k in the i-th topology is a sequence of r distinct nodes $\{v_{\ell_1}, \ldots, v_{\ell_r}\}$ with $v_{\ell_1} = v_j$ and $v_{\ell_r} = v_k$ such that $(v_{i+1}, v_i) \in \mathcal{E}_i$

possible $n, i_0 \in \mathbb{N}$ with i_0 odd, we have

$$\sum_{i=i_0,\, i=i+2}^{i_0+2n} \Delta T_i \geq \tau\,(n - n_0) \tag{5.23}$$

The previous condition can be regarded as a average dwell-time condition (see Hespanha and Morse (1999)), with the time τ, in particular, that can be seen as an average length of the intervals in which the network is connected, and n_0 representing the number of "connected" intervals of zero duration that can occur in a row. The result formulated in the next proposition, in fact, claims that consensus is achieved if (5.23) is fulfilled for some n_0 and τ with the latter sufficiently large.

We conclude the section by remarking how the framework proposed in this section to model switching graphs is, from one hand, more restrictive than the one based on the property of *joint connectivity* used, for instance, in Shi and Hong (2009), Yang et al. (2014a). As a matter of fact, joint connectivity does not imply the existence of time intervals in which the graph is connected (as assumed in our framework) since it is the union of all possible network configurations that is required to have connectivity properties. On the other hand, all consensus results presented in literature that rely on a uniform joint connectivity condition ask that the different topologies persist a guaranteed dwell-time. In this respect the condition above, asking just a dwell-time in the average, is milder.

Furthermore, it is important to stress that in Shi and Hong (2009), Yang et al. (2014a) the graph is supposed to be uniformly *strongly* connected, a fact that implies that all agents are centroid of the graph, while in our case we simply ask the topolgies to be connected. In addition, to achieve state synchronization, in Yang et al. (2014a), the graph is not only asked to be uniformly strongly connected but also to be *fixed in the average*, a fact that imposes severe restrictions on the topology switching sequence.

5.3.2 Main Result

We now give the conditions under which the network of nonlinear agents achieve synchronization.

Proposition 5.3. *Consider the networked control system (5.20) controlled by (5.22) with K as in (5.15) under assumption listed in Chapter 5.3.1, with the length of the time interval of connected topology fulfilling the average dwell-time condition (5.23) for some $n_0 \geq 1$ and τ. Then, for all $w_k(0) \in \bar{\mathbb{W}}$ with $k = 1, \ldots, N_\mathrm{a}$ and $\bar{\mathbb{W}} \subset \mathbb{W}$, there exist a $\tau^\star$ and $g^\star$ such that* *for all* $\tau \geq \tau^\star$ *and* $g \geq g^\star$ *the compact invariant set*

$$\mathbf{W} \quad = \quad \{(w_1, w_2, \ldots, w_{N_\mathrm{a}}) \in W \times W \times \cdots \times W : \; w_1 = w_2 = \cdots = w_{N_\mathrm{a}}\} \tag{5.24}$$

is asymptotically stable for the closed-loop network system as long as $w_k \in \mathbb{W}$ *for* $k = 1, \ldots, N_\mathrm{a}$. ◁

Remark 5.1. The introduction of the set of initial condition $\bar{\mathbb{W}} \subset \mathbb{W}$ is due to the fact that we need to remain inside $\mathbb{W}$ to have the uniform observability condition and Lipschitz condition fulfilled. With this respect, the existence of a $g^\star$ and $\tau^\star$ should be implied. △

Proof. The proof of Proposition 5.3 is divided in three parts:

- in the first part of the proof we consider the behavior of the network for a a fixed connected topology. By recalling the results in Isidori et al. (2014) and in the proof of Proposition 1, we define a common Lyapunov function and we show that if the topology is connected agents converge towards synchronization
- in the second part of the proof, we consider disconnected topology, and analyze the behavior of the network by Lyapunov arguments
- in the third and final part of the proof, we consider the network under switching topologies and, by means of hybrid Lyapunov tools, we prove that, under an average dwell time condition, the agents achieve consensus despite *arbitrary* long time intervals in which the network is not connected

Connected topologies

Consider a generic fixed topology $\mathcal{T}_i \in \mathcal{T}$ and the change of coordinate

$$M = \begin{pmatrix} 1 & 0_{1\times(N_\mathrm{a}-1)} \\ 1_{N_\mathrm{a}-1} & I_{N_\mathrm{a}-1} \end{pmatrix} .$$

Defining $\mathbf{w} = \mathrm{col}(w_1, \ldots, w_{N_\mathrm{a}})$ and by bearing in mind the choice of K, the networked system can be compactly rewritten as

$$\dot{\mathbf{w}} = [(I_{N_a} \otimes S) - (L_i \otimes D_g K_0 C)]\, \mathbf{w} + (I_{N_a} \otimes B)\Phi(\mathbf{w}) \tag{5.25}$$

where $\Phi(\mathbf{w}) = \mathrm{col}(\phi(w_1), \ldots, \phi(w_{N_\mathrm{a}}))$.

Elementary computations show that

$$\tilde{L}^i = M^{-1}L^iM = \begin{pmatrix} 0 & L^i_{12} \\ 0_{(N_\mathrm{a}-1)\times 1} & L^i_{22} \end{pmatrix} \tag{5.26}$$

where $L^i_{12} = L^i_{[1,2:N_\mathrm{a}]}$ and $L^i_{22} = L^i_{[2:N_\mathrm{a},2:N_\mathrm{a}]} - 1_{N_\mathrm{a}-1}\, L^i_{[1,2:N_\mathrm{a}]}$. We consider now the change of variables

$$\mathbf{w} \quad \mapsto \quad \begin{pmatrix} \varrho_1 \\ \boldsymbol{\varrho} \end{pmatrix} = (M^{-1} \otimes I_d)\mathbf{w}\,,$$

with $\varrho_1 \in \mathbb{R}^d$ and $\boldsymbol{\varrho} \in \mathbb{R}^{(N_\mathrm{a}-1)d}$. Note that $\varrho_1 = w_1$ and

$$\boldsymbol{\varrho} = \begin{pmatrix} w_2 - w_1 & w_3 - w_1 & \dots & w_{N_\mathrm{a}} - w_1 \end{pmatrix}^T .$$

By using (5.26), an easy calculation shows that system (5.25) in the new coordinates reads as

$$\begin{aligned} \dot{\varrho}_1 &= S\varrho_1 + B\phi(\varrho_1) - (L^i_{12} \otimes D_gK_0C)\boldsymbol{\varrho} \\ \dot{\boldsymbol{\varrho}} &= \left[(I_{N_\mathrm{a}-1} \otimes S) - (L^i_{22} \otimes D_gK_0C)\right]\boldsymbol{\varrho} + \Delta\Phi(\varrho_1, \boldsymbol{\varrho}) \end{aligned}$$

where

$$\Delta\Phi(\varrho 1, \boldsymbol{\varrho}) = (I_{N-1} \otimes B) \begin{pmatrix} \phi(\varrho_2 + \varrho_1) - \phi(\varrho_1) \\ \vdots \\ \phi(\varrho_{N_\mathrm{a}} + \varrho_1) - \phi(\varrho_1) \end{pmatrix}$$

where $\boldsymbol{\varrho} = \mathrm{col}(\varrho_2, \dots, \varrho_{N_\mathrm{a}})$, with $\varrho_i \in \mathbb{R}^d$, $i = 2, \dots, N_\mathrm{a}$. Note that $\Delta\Phi(\varrho_1, \boldsymbol{\varrho})$ is globally Lipschitz in $\boldsymbol{\varrho}$ uniformly in ϱ_1 and $\Delta\Phi(\varrho_1, 0) = 0$ for all $\varrho_1 \in \mathbb{R}^d$. Namely, there exists a positive $\bar{\Phi}$ such that $\|\Delta\Phi(\varrho_1, \boldsymbol{\varrho})\| \leq \bar{\Phi}\|\boldsymbol{\varrho}\|$ for all $\varrho_1 \in \mathbb{R}^d$ and $\boldsymbol{\varrho} \in \mathbb{R}^{(N_\mathrm{a}-1)d}$.

We now rescale the variable $\boldsymbol{\varrho}$ in the following way

$$\chi = (I_{N_\mathrm{a}-1} \otimes D_g^{-1})\boldsymbol{\varrho}$$

and obtain

$$\begin{aligned} \dot{\varrho}_1 &= S\varrho_1 + B\phi(\varrho_1) - (L^i_{12} \otimes D_gK_0C)(I_{N_\mathrm{a}-1} \otimes D_g)\chi \\ \dot{\chi} &= gH_i\chi + \frac{1}{g^d}\Delta\Phi(z_1, (I_{N_\mathrm{a}-1} \otimes D_g)\chi) \end{aligned}$$

where $H_i = [(I_{N-c} \otimes S) - (L^i_{22} \otimes K_0C)]$. To show that the origin of the system with state χ is locally asymptotically stable, we consider the change of variable $\zeta = J_i\chi$ with J_i such that $\bar{H}_i = J_iH_iJ_i^{-1}$ is in Jordan form. We obtain a new system that is the cascade of system

$$\dot{\zeta} = g\bar{H}_i\zeta + \frac{1}{g^d}\Delta\Phi(z_1, (I_{N_{\rm a}-1} \otimes D_g)J_i^{-1}\zeta) \tag{5.27}$$

with system

$$\dot{z}_1 = Sz_1 + B\phi(z_1) - (L_{i,12} \otimes D_g K_0 C)(I_{N_{\rm a}-1} \otimes D_g)J_i^{-1}\zeta\,. \tag{5.28}$$

We consider the candidate Lyapunov function

$$V(\zeta) = \zeta^T(D(\ell) \otimes P^{-1})\zeta \tag{5.29}$$

where P is the solution of (5.16) and $D(\ell) = \mathrm{diag}(1, \ell, \ell^2, \dots, \ell^{N_{\rm a}-2})$ with ℓ a positive design parameter yet to be fixed. Note that there exist positive constants $\underline{\lambda} \leq \bar{\lambda}$, both dependent on ℓ, such that $\underline{\lambda}\zeta^T\zeta \leq V \leq \bar{\lambda}\zeta^T\zeta$.

Using the Lemma 5.1 and taking $g^\star = (a_\phi + a_{\rm c}\bar{\lambda})/a'_{\rm c}$ with $a_{\rm c}$ an arbitrary positive constant ($g^\star \geq 1$ without loss of generality), it is immediately seen that for all $\ell \geq \ell^\star$ and $g \geq g^\star$ we have

$$\dot{V} \leq -(ga'_{\rm c} - a_\phi)\zeta^T\zeta \leq -\frac{ga'_{\rm c} - a_\phi}{\bar{\lambda}}V \leq -a_{\rm c}V\,. \tag{5.30}$$

Hence, (5.30) shows that when connected topologies occur, the Lyapunov function (5.29) is strictly decreasing along solution of (5.27). The local asymptotic stability of ζ and the ISS properties of the z_1 guarantee that all the agents outputs y_k reach consensus on a common trajectory $y^\star$.

Disconnected topologies

We consider now time intervals in which $\mathcal{T}_i \in \mathcal{T}_{\rm nc}$ (i even) and study the behavior of the common Lyapunov function (5.29). For all i such that $\mathcal{T}_i \in \mathcal{T}_{\rm nc}$, it comes straightforward that the matrix $\bar{H}_i$ in (5.27) is not Hurwitz. Hence the result in Lemma 1 cannot be claimed.

The derivative of (5.29) during time intervals in which the topology is not connected is

$$
\begin{aligned}
\dot{V} &= 2\zeta^T(D(\ell)\otimes P^{-1})[gH_i\zeta + \frac{1}{g^d}\Delta\Phi(z_1,(I_{N_a-1}\otimes D_g)\zeta)] \\
&\leq g\,a'_{\rm nc}\,\zeta^T\zeta + \frac{2}{g^d}\bar{\Phi}\|D(\ell)\otimes P^{-1}\|\|(I_{N_{\rm a}-1}\otimes D_g)\|\zeta^T\zeta \\
&\leq (g\,a'_{\rm nc}+a_\phi)\zeta^T\zeta = a_{\rm nc}V
\end{aligned}
\tag{5.31}
$$

where $a'_{\rm nc} := 2\|(D(\ell)\otimes P^{-1})H_i\|$, a_ϕ is the positive constant introduced above, and $a_{\rm nc} = (ga'_{\rm c}+a_\phi)/\underline{\lambda}$.

Equation (5.31) means that, in general, $V(\zeta)$ is increasing when disconnected topologies occurs.

Hybrid analysis

We now proceed towards the proof of Proposition 5.3. We will consider the whole network of agents under switching topologies: we will consider the Lyapunov function (5.29) and analyze the behavior of the agents during connected flows, disconnected flows and jumps of topology.

First we estimate the jump in the value of $V(\zeta)$ when a change in the topology occurs, namely when $\mathcal{T}_{i+1}$ replaces $\mathcal{T}_i$. Denoting by ζ^+ and χ^+ the "next value" of the state variables ζ and χ when a jump in the topology occurs, we note that $\chi^+=\chi$ and

$$
\zeta^+ = J_i^+\,\chi^+ = J_{i+1}\,\chi = J_{i+1}\,J_i^{-1}\,\zeta \tag{5.32}
$$

Hence, by letting

$$
\bar{\upsilon} = \max_{i,j\in[1,\ldots N_{\rm t}]}\|J_j\,J_i^{-1}\|
$$

we can easily bound the jump of the Lyapunov function when the topology switches as

$$
\begin{aligned}
V^+ &= \zeta^{+T}(D(\ell)\otimes P^{-1})\zeta^+ \leq \bar{\lambda}\|\zeta^+\|^2 \\
&\leq \bar{\lambda}\bar{\upsilon}^2\|\zeta\|^2 \leq \frac{\bar{\lambda}}{\underline{\lambda}}\,\bar{\upsilon}^2\,V := a_{\rm j}V\,.
\end{aligned}
\tag{5.33}
$$

We will continue the analysis by considering the closed-loop networked system as an hybrid system flowing during the time intervals in which the communication topology is connected (i odd), and "instantaneously" jumping in the intervals in which the topology is disconnected. To this end, let i be odd and let t_i, t_{i+1} be, respectively, the times at which the topology switches from $\mathcal{T}_i\in\mathcal{T}_{\rm c}$ to $\mathcal{T}_{i+1}\in\mathcal{T}_{\rm nc}$, and from $\mathcal{T}_{i+1}$ to $\mathcal{T}_{i+2}\in\mathcal{T}_{\rm c}$. By bearing in mind (5.31) and (5.33), we have that the jump undergone by the Lyapunov function between two connected topologies can be estimated as

$$\begin{aligned} V(t_{i+1}^{+}) &\leq a_{\mathrm{j}} V(\zeta(t_{i+1}^{-})) \leq a_{\mathrm{j}} e^{a_{\mathrm{nc}} T_0} V(\zeta(t_i^{+})) \\ &\leq a_{\mathrm{j}} e^{a_{\mathrm{nc}} T_0} a_{\mathrm{j}} V(\zeta(t_i^{-})) = e^{\sigma_{\mathrm{j}}} V(\zeta(t_i^{-})) \end{aligned}$$

with $\sigma_{\mathrm{j}} := a_{\mathrm{nc}} T_0 + 2\ln(a_{\mathrm{j}})$.

We are thus left to study an hybrid system governed by (5.30) during flows and instantaneously jumping as $V^{+} \leq e^{\sigma_{\mathrm{j}}} V$, with the length of the flow intervals governed by an average dwell time of the form (5.23).

The fact that the time intervals satisfy an average dwell-time condition given by (5.23) allows one to say (see Cai et al. (2008)) that flow and jump times of the hybrid system can be thought of as governed by a clock variable ς flowing according to the differential inclusion $\dot{\varsigma} \in [0, 1/\tau]$ when $\varsigma \in [0, n_0]$ and jumping as $\varsigma^{+} = \varsigma - 1$ when $\varsigma \in [1, n_0]$. We thus endow the networked system with the clock variable and study the resulting hybrid system whose Lyapunov function flows and jumps according to the following rules

$$\left.\begin{aligned} \dot{\varsigma} &\in [0, 1/\tau] \\ \dot{V} &\leq -a_{\mathrm{c}} V \end{aligned}\right\} \quad (\varsigma, V) \in [0, n_0] \times \mathbb{R}$$
$$\left.\begin{aligned} \varsigma^{+} &= \varsigma - 1 \\ V^{+} &\leq e^{\sigma_{\mathrm{j}}} V \end{aligned}\right\} \quad (\varsigma, V) \in [1, n_0] \times \mathbb{R}$$

Specifically, following Cai et al. (2008), let

$$\mathcal{W}(\varsigma, \zeta) = e^{L\varsigma} V(\zeta)$$

with $L \in (\sigma_{\mathrm{j}}, \tau a_{\mathrm{c}}/2)$. During flows, by compactly writing (5.27) as $\dot{\zeta} = F(\zeta, z_1)$, we have that for all $v \in \operatorname{col}([0, 1/\tau], F(\zeta, z_1))$

$$\begin{aligned} \langle \nabla \mathcal{W}(\tau, \zeta), v \rangle &= L\, e^{L\varsigma}\, \dot{\varsigma}\, V(\zeta) + e^{L\varsigma} \langle \nabla V(\zeta), F(\zeta, z_1) \rangle \\ &\leq L e^{L\varsigma} \frac{1}{\tau} V(\zeta) - a_{\mathrm{c}} e^{L\varsigma} V(\zeta) \\ &\leq -(a_{\mathrm{c}} - \frac{L}{\tau}) \mathcal{W}(\varsigma, \zeta) \\ &\leq -\frac{a_{\mathrm{c}}}{2} \mathcal{W}(\varsigma, \zeta) \end{aligned}$$

for all $(\varsigma, \zeta, z_1) \in [0, n_0] \times \mathbb{R}^{(N_{\mathrm{a}}-1)d} \times \mathbb{R}^{d}$. On the other hand, during jumps, we have that

$$\begin{aligned}\mathcal{W}^{+} &= e^{L\varsigma^{+}}V(\zeta^{+}) \leq e^{L(\varsigma-1)}e^{\sigma_{\mathrm{j}}}V(\zeta) \\ &= e^{-L+\sigma_{\mathrm{j}}}e^{L\varsigma}V(\zeta) = e^{-L+\sigma_{\mathrm{j}}}\mathcal{W}(\varsigma,\zeta) \\ &= \varepsilon\mathcal{W}(\varsigma,\zeta)\end{aligned}$$

with $\varepsilon = e^{-L+\sigma_{\mathrm{j}}} \in (0,1)$. The Lyapunov function $\mathcal{W}(\cdot,\cdot)$ is thus decreasing both during flows and during jumps. This and the fact that $\mathcal{W}$ is positive definite with respect to the set $[0,n_0]\times\{0\}$ lead to the conclusion that the set $[0,n_0]\times\{0\}$ is globally asymptotically stable. This, by taking advantage from Assumption 1 and from the cascade structure of system (5.27)-(5.28), proves the result.

□

5.3.3 Simulation Results

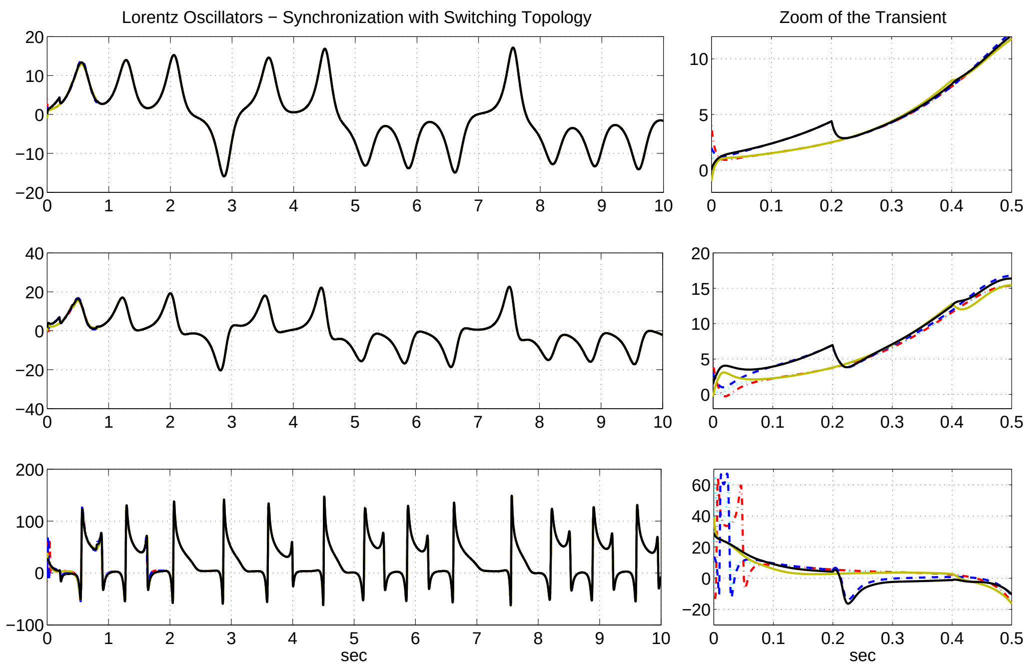

Figure 5.3: Lorentz Oscillator: behavior of the three components of the Lorentz oscillators with switching topology.

Again, we consider 5 Lorentz oscillators with the same initial condition of (5.9). In order to show simulation results in case of switching topology, we selected ten *random* topologies (odd disconnected, even connected), with $T_0 = 0.3$ *sec*, $N_0 = 3$ and $\delta = 2$ (see Figure 5.4 for a sample of 4 sec). The control parameter K is chosen as in Section 3.1 with the gain parameter $g = 50$ and $a = 1$.

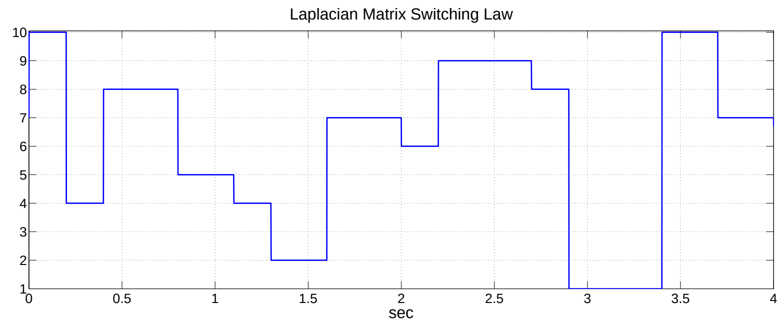

Figure 5.4: Laplacian Switching signal: a sample of 4 sec of the switching signal governing the change of topology.

Figure 5.3 shows the three components of the five Lorentz oscillators in the original coordinates (5.9) achieving synchronization. In the top right corner, the zoom shows the initial transient towards synchronization.

No experience has been too unimportant, and the smallest event unfolds like a fate, and fate itself is like a wonderful, wide fabric in which every thread is guided by an infinitely tender hand and laid alongside another thread and is held and supported by a hundred others.

Rainer Maria Rilke

6

Event triggered control of networks

One of the main challenges in network control is indeed to design a decentralized control architecture locally at each agent, by just computing the information available from neighbors. The recent advances in embedded controllers allows to easily design such an architecture in practical case: several commercial platforms are now able to compute locally the information from *neighbors* and calculate a suitable control input locally. Nevertheless, technological constraints requires to minimize the computational load and the exchange of information.

These aspects motivate the study of event triggered solutions for networked systems, with the aim to define communication *protocols* guaranteeing both the achievement of the desired task and the minimization of computation load and information exchange through the network.

The issue of event triggered control has been studied first in the case of single feedback loops (see Heemels et al. (2012) and all the references therein). With this respect, seminal results on the topic can be found in Anta and Tabuada (2010), Marzo and Tabuada (2011), Aström and Bernhardsson (1999) and Carnevale et al. (2007). As far as the multi-agent scenario is concerned, the challenges are particularly severe. In fact, these systems are intrinsically decentralized and can compute only their own and neighbors information.

This results in a distributed triggering rule, where each agents determines its sampling time sequence based only on the local information available. Several techniques and architecture have been proposed in literature, considering *event-triggered* and *self-triggered* solutions (see Liuzza et al. (2014), Demire and Lunze (2012), Dimarogonas et al. (2012), Gracia and Antsalkis (2013),Nowzari and Cortes (2012), De Persis et al. (2013)). However, most of the solutions that can be found literature target the case of linear systems (often single or double integrators dynamic). In the context of nonlinear systems, Liuzza et al. (2014) targets the problems of *linearizable* nonlinear systems: the main issue of the proposed architecture is that it requires the exchange a huge amount of information between the agents (full-state information, inputs).

Following the approach suggested in De Persis and Postoyan (2014)-Postoyan and De Persis (2016), we aim to study the problem for a general class of nonlinear systems which exchange only output information with their neighbors. In contrast with Postoyan and De Persis (2016), we make use of ISS arguments and observability conditions, rather than *passivity based* considerations. We propose an *event-triggered* solution which only requires the knowledge of the number of the agents in the network and guarantees that the agents achieve synchronization.

Summarizing, our architectures matches the following requirements:

- the agents exchange only just their outputs
- the agents achieves synchronization by means of a static *diffusive coupling*: they do not need to know the control input of the other agents or any other information, nor to estimate other agents behavior
- the agents do not need to know the initial condition of the other agents
- the solution is *event-triggered*, namely the agents determine their triggering sequence locally

The content of this chapter is completely novel. Since we make extensive use of hybrid systems tools and hybrid Lyapunov analysis, the reader is referred to Appendix C and Goebel et al. (2008) for further details on the topic.

6.1 Problem Formulation

Each of the N nonlinear agents is described by the following dynamics

$$\begin{aligned} \dot{w}_k &= s(w_k) + u_k \qquad w_k \in \mathbb{R}^d \\ y_k &= c(w_k) \end{aligned} \tag{6.1}$$

$k = 1, \ldots, N$, in which u_k is the local control input, y_k is the local output whose value is transmitted to the neighbour agents, and

$$s(w_k) = Sw_k + B\phi(w_k)\,, \qquad c(w_k) = Cw_k \tag{6.2}$$

where (S, B, C) is a triplet of matrices in *prime* form, that is S is a shift matrix (all 1's on the upper diagonal and all 0's elsewhere), $B^T = (0 \cdots 0\ \ 1)$ and $C = (1\ \ 0 \cdots 0)$.

Conventionally, in order to achieve consensus between the agents the control input u_k of the k-th agent is chosen according to a *diffusive coupling* structure of the form

$$u_k = -K\nu_k \quad \text{with} \quad \nu_k = \sum_{j=1}^{N} \ell_{kj}\, y_j \tag{6.3}$$

for $k = 1, \ldots, N$, with ℓ_{kj} elements of the Laplacian matri and K a vector (the same for all the agents) of control parameters.

However, in this framework the output of the agents is not continuously available. Instead each agent in the network samples the information from its neighbors only at certain time instants. We define[1] $\hat{\mathbf{y}}_k = \mathrm{col}(\hat{y}_1, \ldots, \hat{y}_N)_k$ as the outputs of the neighbors of agent k, sampled by agent k. Our goal is to define a control input $u_k = f(\hat{\mathbf{y}}_k)$ such that synchronization is achieved, namely there exists a $y^\star$ such that

$$\lim_{t \to \infty} y_k - y^\star = 0$$

for $k = 1, \ldots, N$.

6.2 Main Result

In order to deal with sampled information, we modify the diffusive coupling (6.3) and choose a *sampled diffusive coupling* of the form

$$u_k = -K\hat{\nu}_k \tag{6.4}$$

[1]Here and in the following, we denote by $\hat{x}$ the sampled value of $x(t)$ at a certain time instant t_k.

with

$$\begin{aligned}
\left.\begin{aligned}
\dot{\hat{\nu}}_k &= 0 \\
\dot{\beta}_k &= -\gamma\beta_k
\end{aligned}\right\} & \text{ if } \|K\hat{\nu}_k\| \leq \beta_k \\
\left.\begin{aligned}
\hat{\nu}_k^+ &= \nu_k \\
\beta_k^+ &= \beta_0
\end{aligned}\right\} & \text{ if } \|K\hat{\nu}_k\| > \beta_k
\end{aligned} \tag{6.5}$$

for $k = 1, \ldots, N$, with with γ and β_0 design parameters to be defined, where $\beta_k \in \mathbb{R}$ is a local *clock* for the k-th agent.

In (6.5), we correlate the sampling time instants to the *behavior* of the input: when the norm of the input $\|K\hat{\nu}_k\|$ is greater than β_k, the k-th agent samples the output of its neighbors. The clock is then reset to a constant value β_0 after each event and determine the following triggering instant when β_k is less or equal then a certain value.

For the design of K in (6.5), we follow the approach described in Section 3.1, in which K is chosen as

$$K = D_g K_0 \tag{6.6}$$

with $D_g = \text{diag}(g, \ldots, g^d)$ where g an *high-gain* parameter and $K_0 = PC^T$ with P solution of the Riccati equation

$$SP + PS^T - 2\mu PC^TCP + a_cI = 0 \tag{6.7}$$

With control input defined according to (6.4)-(6.5), each agent (6.1) reads as

$$\begin{aligned}
\left.\begin{aligned}
\dot{w}_k &= Sw_k + B\phi(w_k) + K\hat{\nu}_k \\
\dot{\hat{\nu}}_k &= 0 \\
\dot{\beta}_k &= -\gamma\beta_k
\end{aligned}\right\} & \text{ if } \|K\hat{\nu}_k\| \leq \beta_k \\
\left.\begin{aligned}
w_k^+ &= w_k \\
\hat{\nu}_k^+ &= \nu_k \\
\beta_k^+ &= \beta_0
\end{aligned}\right\} & \text{ if } \|K\hat{\nu}_k\| > \beta_k
\end{aligned} \tag{6.8}$$

By letting $\mathbf{w} = \text{col}(w_1, \ldots, w_N)$, $\boldsymbol{\beta} = \text{col}(\beta_1, \ldots, \beta_N)$ and $\hat{\boldsymbol{\nu}} = \text{col}(\hat{\nu}_1, \ldots, \hat{\nu}_N)$, we can rewrite the overall networked system as

$$\left.\begin{array}{rcl}\dot{\mathbf{w}} & = & (I_N \otimes S)\mathbf{w} + (I_N \otimes B)\Phi(\mathbf{w}) + K\hat{\boldsymbol{\nu}} \\ \dot{\hat{\boldsymbol{\nu}}} & = & 0 \\ \dot{\boldsymbol{\beta}} & = & -\gamma\boldsymbol{\beta}\end{array}\right\} \qquad \forall\, k \in [1,\ldots,N], \|K\hat{\nu}_k\| \le \beta_k$$
$$\left.\begin{array}{rcl}\mathbf{w}^+ & = & \mathbf{w} \\ \begin{pmatrix}\hat{\boldsymbol{\nu}}^+ \\ \boldsymbol{\beta}^+\end{pmatrix} & = & G(\mathbf{w},\hat{\boldsymbol{\nu}},\boldsymbol{\beta})\end{array}\right\} \qquad \exists\, k \in [1,\ldots,N], \|K\hat{\nu}_k\| > \beta_k \tag{6.9}$$

with

$$G(\mathbf{w},\hat{\boldsymbol{\nu}},\boldsymbol{\beta}) = \{G_k(\mathbf{w},\hat{\boldsymbol{\nu}},\boldsymbol{\beta}) : k = 1,\ldots,N \text{ and } \|K\nu_k\| > \beta_k\}$$

and

$$G_k(\mathbf{w},\hat{\boldsymbol{\nu}},\boldsymbol{\beta}) = (\hat{w}_1^k,\ldots,\hat{w}_i^k,\ldots,\hat{w}_N^k,\beta_1,\ldots,\beta_k = \beta_0,\ldots,\beta_N)$$

for all $K = 1,\ldots,N$ where $\hat{w}_j^k$ stands for all the sampled values of the neighbors of agent k. The defined jump map guarantees outer semi-continuity: in particular, when the triggering condition is met for two different agents, the jump map sequences the jumps so that the graph of solutions is closed.

Proposition 6.1. *Consider the networked control system* (6.9) *with K chosen according to* (6.6)-(6.7). *Then there exists a $g^\star$ such that, for any $g \ge g^\star$ there exist $\gamma^\star, \beta_0^\star$, such that for any $\gamma \ge \gamma^\star$, $\beta_0 \ge \beta_0^\star$ and for all $w_k(0) \in \mathbb{W} \subset \mathbb{R}^d$, the compact invariant set*

$$\begin{aligned}\mathbf{W} = \{(\mathbf{w},\hat{\boldsymbol{\nu}},\boldsymbol{\beta}) \ :\ \mathbf{w} \in W \times \cdots \times W, w_1 = \cdots = w_N,\\ \hat{\nu}_1 = \ldots = \hat{\nu}_N = 0,\\ \beta_1,\ldots,\beta_N \in [0,\beta_0]\}\end{aligned} \tag{6.10}$$

is asymptotically stable.

Proof. The proof of Proposition 1 is divided in two parts. First we show how to determine the design parameters for the triggering rule (6.5). Second, we show that under this triggering rule, the network of nonlinear oscillators reaches synchronization.

Triggering rule design

In order to find the design parameters for the triggering rule, we make a *continuous-time analysis* of the network. This analysis allows to define a *target behavior* of the nominal input. Then each agent compares its own input to this target behavior and determines when to trigger the neighbor's information.

To this end, consider the network of (6.1) controlled by (6.3). We obtain

$$\dot{\mathbf{w}} = (I_N \otimes S)\mathbf{w} + (I_N \otimes B)\Phi(\mathbf{w}) + (L \otimes KC)\mathbf{w}$$

We *rescale* $\mathbf{w}$ according to

$$\boldsymbol{\varsigma} = (I_N \otimes D_g^{-1})\mathbf{w}$$

which component-wise read as

$$\varsigma_k = D_g^{-1} w_k$$

The dynamics of $\boldsymbol{\varsigma}$ is

$$\dot{\boldsymbol{\varsigma}} \;\; = \;\; g\left[(I_N \otimes S) + (L \otimes K_0 C)\right]\boldsymbol{\varsigma} + \frac{1}{g^d}(I_N \otimes B)\Phi\left((I_N \otimes D_g)\boldsymbol{\varsigma}\right)$$

Then we define the *average dynamics*

$$\bar{\varsigma} = \frac{1}{N}\sum_{j=1}^{N} \varsigma_j$$

and accordingly the error with respect to this average dynamics as $\mathbf{e} = \mathrm{col}(e_1, \dots, e_N)$ with $e_k = \varsigma_k - \bar{\varsigma}$ for $k = 1, \dots, N$.

We choose the candidate Lyapunov function

$$V = \mathbf{e}^T (I_N \otimes P^{-1})\mathbf{e}$$

with P solution of the Riccati equation (6.7) such that

$$\underline{\lambda}\|\mathbf{e}\|^2 \leq V \leq \bar{\lambda}\|\mathbf{e}\|^2$$

Its derivative is

$$\dot{V} = \textstyle\sum_{k=1}^{N} 2e_k^T P^{-1}\dot{\varsigma}_k - \sum_{k=1}^{N} 2e_k^T P^{-1}\dot{\bar{\varsigma}} \tag{6.11}$$

By considering the fact that $\sum_{k=1}^{N} e_k^T = 0$, the term

$$\sum_{k=1}^{N} e_k^T P^{-1} \dot{\bar{\varsigma}} = 0$$

and thus, by subtracting

$$\sum_{k=1}^{N} e_k^T P^{-1} (gS\bar{\varsigma} + \frac{1}{g^d} B\phi(D_g\bar{\varsigma})) = 0$$

we obtain

$$\dot{V} = \sum_{k=1}^{N} 2e_k^T P^{-1} \left[gSe_k + \tfrac{1}{g^d} B\big(\phi(D_g\varsigma_k) - \phi(D_g\bar{\varsigma})\big) + gK_0 \sum_{j=1}^{N} \ell_{kj} C\varsigma_j \right] \quad (6.12)$$

Using the fact that, due to the zero-row sum property of the Laplacian $\sum_{j=1}^{N} \ell_{kj}\bar{\varsigma} = 0$, (6.12) can be written as

$$\begin{aligned} \dot{V} &= \sum_{k=1}^{N} 2e_k^T P^{-1} \left[gSe_k + \tfrac{1}{g^d} B\big(\phi(D_g\varsigma_k) - \phi(D_g\bar{\varsigma})\big) + gK_0 \sum_{j=1}^{N} \ell_{kj} Ce_j \right] \\ &= 2g\mathbf{e}^T (I_N \otimes P^{-1}) \Big((I_N \otimes S) - (L \otimes K_0 C) \Big) \mathbf{e} + \frac{2}{g^d} \mathbf{e} (I_N \otimes P^{-1})(I_N \otimes B) \Delta\Phi(\boldsymbol{\varsigma}, \bar{\varsigma}) \end{aligned} \quad (6.13)$$

where $\Delta\Phi(\boldsymbol{\varsigma}, \bar{\varsigma})$ is

$$\Delta\Phi(\boldsymbol{\varsigma}, \bar{\varsigma}) = \begin{pmatrix} \phi(D_g\varsigma_1) - \phi(D_g\bar{\varsigma}) \\ \vdots \\ \phi(D_g\varsigma_N) - \phi(D_g\bar{\varsigma}) \end{pmatrix}$$

Remembering that, by assumption, $\phi(\cdot)$ is globally Lipschitz

$$\|\phi(\varsigma) - \phi(\bar{\varsigma})\| \leq \bar{\phi} \|e_k\|$$

with $\bar{\phi}$ being the Lipschitz constant and thus $\Delta\Phi(\boldsymbol{\varsigma}, \bar{\varsigma})$ can be bounded as

$$\|\frac{1}{g^d} \Delta\Phi(\boldsymbol{\varsigma}, \bar{\varsigma})\| \leq \bar{\phi} \|\mathbf{e}\|$$

By considering this (6.13) becomes

$$\dot{V} = 2g\mathbf{e}^T (I_N \otimes P^{-1}) \Big((I_N \otimes S) - (L \otimes K_0 C) \Big) \mathbf{e} + 2\bar{\phi}\, \mathbf{e} (I_N \otimes P^{-1})(I_N \otimes B) \|\mathbf{e}\| \quad (6.14)$$

which, due to the choice of K_0 and following the arguments of the proof of Proposition 1 in Isidori et al. (2014), after some computations leads to

$$\dot{V} = -g\ a_c\|\mathbf{e}\|^2 + \bar{\phi}\|P^{-1}\|\|\mathbf{e}\|^2$$

which, clearly shows that there exist a $g^\star$ such that, for all $g \geq g^\star$

$$\dot{V} = -g\ a_c\|\mathbf{e}\|^2 \tag{6.15}$$

Remembering that $w_k(0) \in \mathbb{W}$, we can also conclude that $e_k(0) \in \mathbb{E}$, where $\mathbb{E}$ is closed subset of $\mathbb{R}^n$. This last relationship,(6.15) and standard Lyapunov arguments can be used to obtain

$$\begin{aligned} \|\mathbf{e}\| &\leq \sqrt{\tfrac{\bar{\lambda}}{\underline{\lambda}}}e^{-\frac{g\,a_c}{2}t}\|\mathbf{e}(0)\| \\ &\leq \bar{\beta}\ e^{-g\bar{\gamma}t} \end{aligned} \tag{6.16}$$

for some positive parameters $\bar{\beta}$ and $\bar{\gamma}$. Thus, using again the fact that $\sum_{j=1}^{N}\ell_{kj}\bar{\varsigma} = 0$, the control input $\|K\nu_k(t)\|$ can be bounded as

$$\begin{aligned} \|K\nu_k(t)\| &\leq \|K\|\|\textstyle\sum_{j=1}^{N}\ell_{kj}Cw_j(t)\| \\ &= \|K\|\|\textstyle\sum_{j=1}^{N}\ell_{kj}CD_g(\varsigma_j(t)-\bar{\varsigma})\| \\ &\leq g\|K\|\textstyle\sum_{j=1}^{N}|\ell_{kj}|\|e_j(t)\| \\ &= \Gamma_k\|\mathbf{e}(t)\| \end{aligned} \tag{6.17}$$

where Γ_k, which depends on g, is defined as

$$\Gamma_k = g\|K\|\sum_{j=1}^{N}|\ell_{kj}|\|D_g\|\,.$$

The term $\|K\nu_k(t)\|$ of each agent is thus bounded by

$$\|K\nu_k(t)\| \leq \Gamma_k\bar{\beta}e^{-g\bar{\gamma}t}\,. \tag{6.18}$$

It is readily seen that in the continuous case the input is strictly decreasing in time. We set $\beta_0^\star \geq \bar{\Gamma}\bar{\beta}$, with $\bar{\Gamma} = \max_{k=1,\ldots,N}\Gamma_k$ and $\gamma^\star \geq g\bar{\gamma}$ for the design of (6.5). Note that, by definition of the jump rule in (6.5), Zeno behaviours are avoided.

Remark 6.1. From the analysis just developed, it is clear that in order to define the clock β_k it is necessary to know global parameters of the network (namely the smallest eigenvalue $\lambda_2(L)$ of the Laplacian matrix and $\bar{\Gamma}$): it is worth mentioning that these parameters can be estimated by knowing the number of agents in the network N. △

Synchronization of nonlinear oscillators under triggering rule

In order to prove that under triggering rule (6.5), we rewrite (6.9) according to the defintion of $\boldsymbol{\varsigma}$ as

$$\left.\begin{aligned}
&\dot{\boldsymbol{\varsigma}} = g(I_N \otimes S)\boldsymbol{\varsigma} + \frac{1}{g^d}(I_N \otimes B)\Phi(I_N \otimes D_g)\boldsymbol{\varsigma} + gK\hat{\boldsymbol{\nu}} \\
&\dot{\hat{\boldsymbol{\nu}}} = 0 \\
&\dot{\boldsymbol{\beta}} = -\gamma\boldsymbol{\beta}
\end{aligned}\right\} \forall\, k \in [1, \ldots, N], \|K\hat{\nu}_k\| \leq \beta_k$$
$$\left.\begin{aligned}
\boldsymbol{\varsigma}^+ &= \mathbf{w} \\
\begin{pmatrix} \hat{\boldsymbol{\nu}}^+ \\ \boldsymbol{\beta}^+ \end{pmatrix} &= G(\mathbf{w}, \hat{\boldsymbol{\nu}}, \boldsymbol{\beta})
\end{aligned}\right\} \quad \exists\, k \in [1, \ldots, N], \|K\hat{\nu}_k\| > \beta_k \tag{6.19}$$

where we implicitly used the fact that, following (6.3), we can write

$$\begin{aligned}
\hat{\nu}_k &= \textstyle\sum_{j=1}^N \ell_{kj} C\hat{w}_j \\
&= \textstyle\sum_{j=1}^N \ell_{kj} CD_g\hat{\varsigma}_j \\
&= g\textstyle\sum_{j=1}^N \ell_{kj} C\hat{\varsigma}_j
\end{aligned}$$

By also considering the fact that it is always possible to write $\hat{\nu}_k = \nu_k + \Delta\nu_k$, (6.20) can be written as

$$\left.\begin{aligned}\dot{\boldsymbol{\varsigma}} &= g(I_N\otimes S)\boldsymbol{\varsigma}+\frac{1}{g^d}(I_N\otimes B)\Phi(I_N\otimes D_g)\boldsymbol{\varsigma}\\ &\qquad\qquad\qquad\qquad +gK(\boldsymbol{\nu}+\Delta\boldsymbol{\nu})\\ \dot{\hat{\boldsymbol{\nu}}} &= 0\\ \dot{\boldsymbol{\beta}} &= -\gamma\boldsymbol{\beta}\end{aligned}\right\}\forall\, k\in[1,\ldots,N], \|K\hat{\nu}_k\|\leq\beta_k$$
$$\left.\begin{aligned}\boldsymbol{\varsigma}^+ &= \mathbf{w}\\ \begin{pmatrix}\hat{\boldsymbol{\nu}}^+\\ \boldsymbol{\beta}^+\end{pmatrix} &= G(\mathbf{w},\hat{\boldsymbol{\nu}},\boldsymbol{\beta})\end{aligned}\right\}\qquad \exists\, k\in[1,\ldots,N], \|K\hat{\nu}_k\|>\beta_k \tag{6.20}$$

Then, following the same steps in the first part of the proof we define a *physical* Lyapunov function

$$V_{\mathrm{p}} = \mathbf{e}^T(I_N\otimes P^{-1})\mathbf{e}$$

with P solution of the Riccati equation (6.7). By performing the same calculation as before (which are now skipped, for sake of simplicity), during flows of (6.20) we obtain

$$\dot{V} = -g\ a_c\|\mathbf{e}\|^2+\textstyle\sum_{k=1}^{N} e_k^T P^{-1} gK\Delta\nu_k \tag{6.21}$$

The *perturbation term* due to the sampling

$$\sum_{k=1}^{N} e_k^T P^{-1} gK\Delta\nu_k$$

in (6.21) can potentially compromise the achievement of synchronization.

In order to compensate the unstable effect of this term, we define a *cyber* Lyapunov function

$$V_{\mathrm{c}} = \sum_{k=1}^{N} V_{\mathrm{c}}^k$$

where V_{c}^k is

$$V_{\mathrm{c}}^k = \frac{1}{2}(\beta_k-\beta_0)^2$$

which during flows of (6.20) leads to

$$\begin{aligned}\dot{V}_{\rm c}^{k} &= -\gamma^2\beta_k^2\\ V_{\rm c}^{k+} &= 0\end{aligned}$$

For the sake of convenience, we rewrite (6.20) as

$$\dot{q} = F(q) \ \text{ for } \ q \in C,\ q^+ = G(q) \ \text{ for } \ q \in D \tag{6.22}$$

where $q := (\varsigma, \hat{\boldsymbol{\nu}}, \boldsymbol{\beta})$, $C := \{q : \|K\hat{\nu}_k\| \leq \beta_k \ \forall\ k = 1, \ldots, N\}$ and $D := \{q : \|K\hat{\nu}_k\| > \beta_k$ for any $k = 1, \ldots, N\}$.

Then considering $V_{\rm tot} = \sqrt{V_{\rm p}+1} + \sqrt{V_{\rm c}+1}$, during flows we obtain

$$\begin{aligned}\langle V_{\rm tot}(q), F(q)\rangle &\leq \frac{-g\, a_c\|\mathbf{e}\|^2 + \sum_{k=1}^N e_k^T P^{-1} g K\Delta\nu_k}{\sqrt{\mathbf{e}^T(I_N\otimes P^{-1})\mathbf{e}}} + \frac{-\sum_{k=1}^N \gamma\beta_k^2}{\sqrt{\sum_{k=1}^N \frac{1}{2}(\beta_k-\beta_0)^2+1}}\\ &\leq -\frac{\underline{\lambda}}{\bar{\lambda}} g\, a_c\|\mathbf{e}\|^2 + \frac{\underline{\lambda}}{\bar{\lambda}} g \textstyle\sum_{k=1}^N K\nu_k - \gamma\sum_{k=1}^N \beta_k\end{aligned} \tag{6.23}$$

Now since $\|K\nu_k\| \leq \beta_k$ (see the first part of the proof) and $\|K\hat{\nu}_k\| \leq \beta_k$ due to the definition of (6.5),

$$\|K\Delta\nu_k\| \leq \|K\nu_k - K\hat{\nu}_k\|$$

leads to

$$\|K\Delta\nu_k\| \leq 2\beta_k$$

Thus, simple computations and the fact that $\gamma \geq g\frac{\underline{\lambda}}{\bar{\lambda}}$ leads to

$$\frac{\underline{\lambda}}{\bar{\lambda}} g\sum_{k=1}^N K\nu_k - \gamma\sum_{k=1}^N \beta_k \leq -\frac{\gamma}{2}\sum_{k=1}^N \beta_k \tag{6.24}$$

and by inserting (6.24) in (6.23) we obtain

$$\langle V_{\rm tot}(q), F(q)\rangle \leq -\frac{\underline{\lambda}}{\bar{\lambda}} g\, a_c\gamma\|\mathbf{e}\|^2 - \frac{\gamma^2}{2}\|\boldsymbol{\beta}\|^2 \tag{6.25}$$

Furthermore, $V_{\rm p}(G(q)) = V_{\rm p}(q)$, since $\mathbf{e}$ does not undergo any jump. Similarly, $V_{\rm c}(G(q)) \leq V_{\rm c}(q)$ since for $q \in D$ there is only one k such that $\beta_k^+ = \beta_0$ while all the other $\beta_j^+ = \beta_j$, for all $j = 1, \ldots, N$ and $j \neq i$. Hence we conclude that

$$V_{\text{tot}}(G(q)) \leq V_{\text{tot}}(q)$$

By applying Theorem 8.2 in Goebel et al. (2008), we also conclude that the solutions of (6.20) asymptotically converge to the largest invariant set contained in

$$\mathcal{A} = \{(\boldsymbol{\varsigma}, \hat{\boldsymbol{\nu}}, \boldsymbol{\beta}) \ : \ \varsigma_1 = \cdots = \varsigma_N, \|\hat{\nu}_1\|, \ldots, \|\hat{\nu}_N\| \leq \beta_0, \beta_1, \ldots, \beta_N \in [0, \beta_0]\}$$

Then, noting that $\varsigma_1 = \ldots = \varsigma_N$ implies $w_1 = \ldots = w_N$ and $\boldsymbol{\nu} = \hat{\boldsymbol{\nu}} = 0$, the claim of Proposition 6.1 follows.

□

6.3 Simulation Results

In order to prove that the result holds in practice, in this section we show simulation results. We considered 6 Van Der Pol oscillators described by

$$\begin{array}{rcl} \dot{w}_{i_1} & = & w_{i_2} \\ \dot{w}_{i_2} & = & 2(1 - w_{i_1}^2)w_{i_2} - w_{i_1} \end{array} \qquad y_i = w_{i_1} \tag{6.26}$$

for $i = 1, \ldots, 6$. The initial condition are taken as $w_1(0) = [-4, 1]^T, w_2(0) = [-2, 2]^T, w_3(0) = [-5, 1]^T, w_4(0) = [5, 1.5]^T, w_5(0) = [3.5, 2.5]^T, w_6(0) = [3, -2]^T$. The control parameter K of the *diffusive coupling* is chosen according to (6.6)-(6.7), with $a = 1$ and $g = 2$. The self-triggering rule is chosen with $\gamma = 50$ and $\beta_0 = 100$.

The Laplacian matrix describing the connection between the agents is

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Figure 6.1 shows the synchronization of the 6 agents: after approximately 3 sec the agents have *almost* reached synchronization, as shown in the zoom.

Figure 6.2 shows a *zoom* of 1 sec of the triggering rule: it can be seen that each agents samples independently from the others, every time the norm of its input $\|u_i\|$ *intersects* its bound β_i. Note that, due to *computational constraints*, Figure 6.3 shows that each

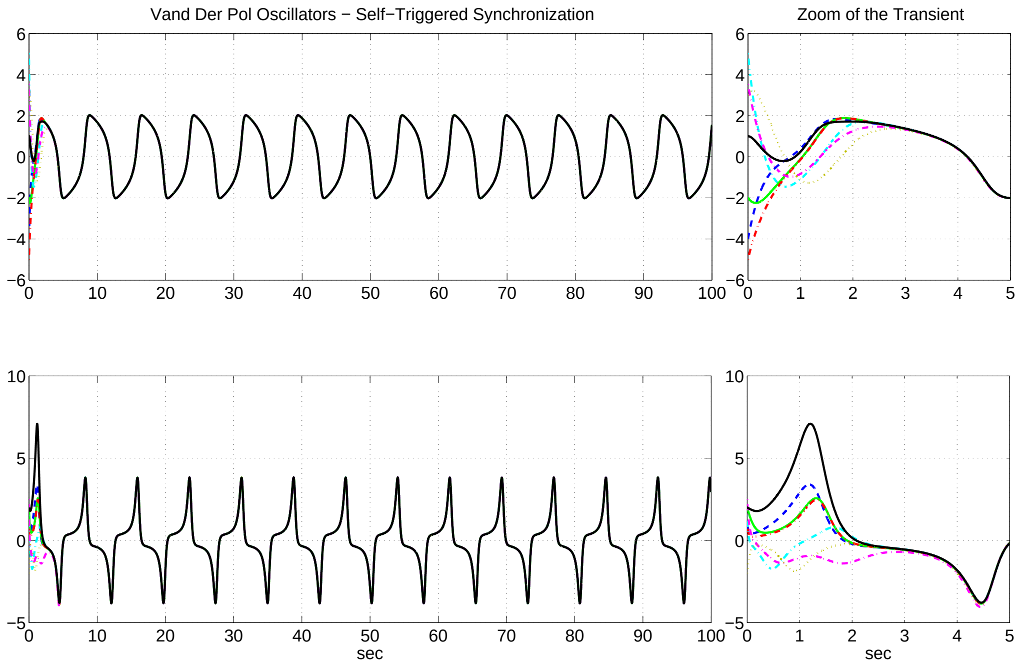

Figure 6.1: Synchronization of the 6 Van Der Pol oscillators.

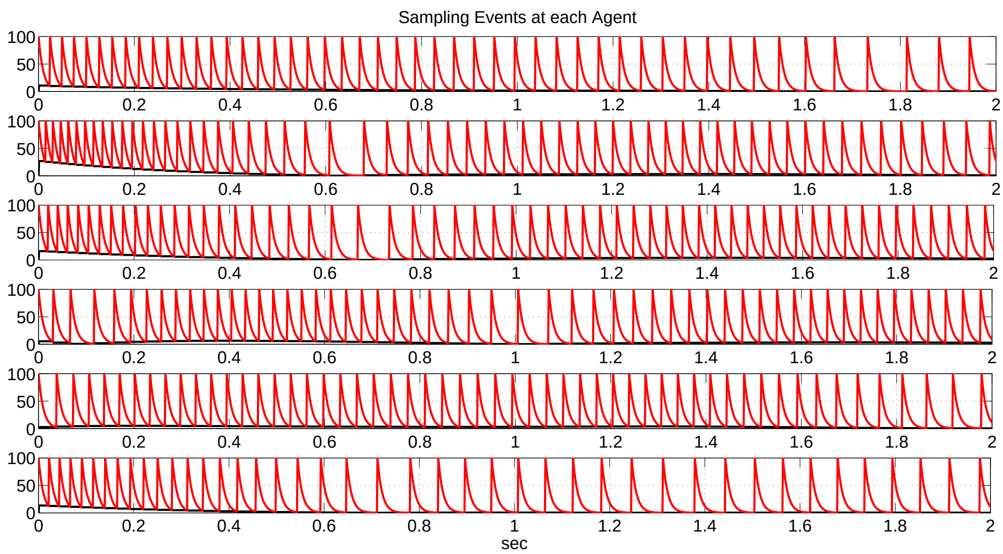

Figure 6.2: Sampling events for 1 sec for the 6 agents: as it is shown in the figure, all the agents have different sampling times which depends on the intersection between the norm of the control inputs $\|u_i\|$ (black line) and the bound β_i (red line).

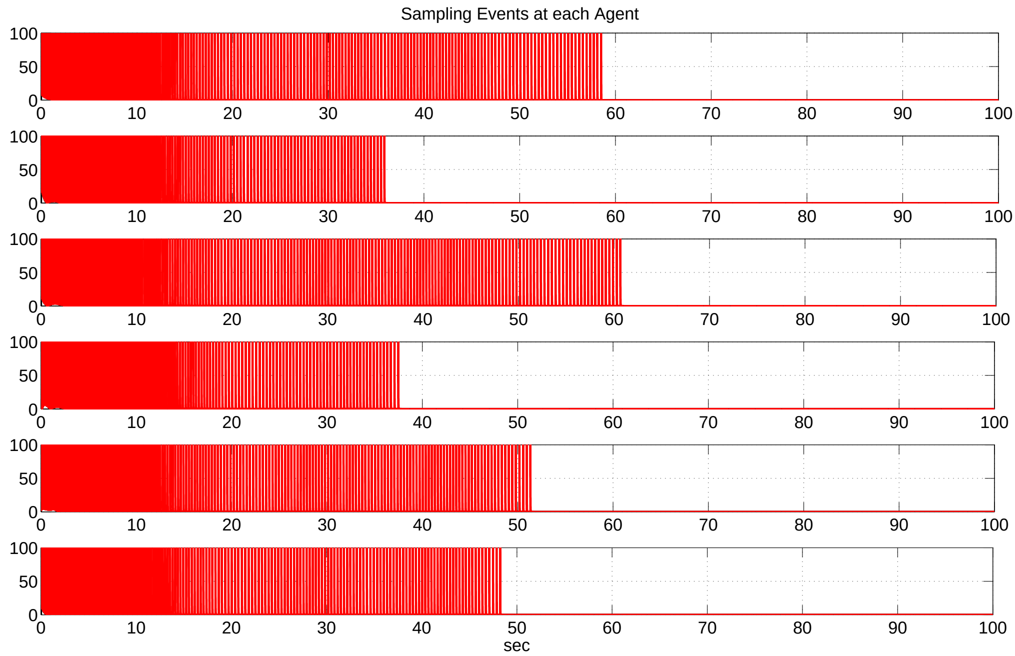

Figure 6.3: Sampling events for the 6 agents: as it is shown in the figure, all the agents have different sampling times and stop sampling neighbors outputs at different time instants.

agent stops sampling neighbors output at different time instants: these means that the error (with respect to the neighbors) has become so *small* that the triggering condition is not met anymore.

Reasoning draws a conclusion, but does not make the conclusion certain, unless the mind discovers it by the path of experience.

Roger Bacon

7

Conclusion

We have started this thesis by claiming that *"we're connected with everything"* and thus it is not possible to decompose complex problems into separated sub-problems, neglecting the influence each part of a system has on the others. This is the reason why in recent years, the scientific community has turned its attention towards networks and multi-agent systems, using consensus and synchronization techniques to model complex and interconnected behaviors.

The objective of this thesis was to consider synchronization and consensus problems for networks of nonlinear agents in different *operating* scenarios. In Chapter 3, we started by considering the classic problem of synchronization over a fixed connected topology for nonlinear homogeneous and heterogeneous systems. Then, in Chapter 4 we modified the classic framework by introducing dynamical systems as links: this scenario mimics the power networks framework, where the links cannot be considered as an ideal *line* through which agents exchange information. In Chapter 5, we also considered the problem of disconnected topologies and behavior of nonlinear systems when disconnected topology occurs: we designed a control architecture capable of enforcing cluster-synchronization and studied under which switching conditions agents reach synchronization over the whole network. Last, in Chapter 6, we considered the practical problem of sampled information exchange: we studied the impact of sampling over the network and designed a triggering rule, locally at each agent, such that each agent samples the information from the neighbors only when necessary while the overall network reaches synchronization.

The world of networks offers still a huge number of open problems which have not been taken into account in this thesis and *every day* new applications of synchronization and consensus theory rise on the research horizon. Furthermore, some the problems addressed in this thesis could be tackled with other techniques, yielding different (yet difficult to compare) results. For instance, as we pointed out in Chapter 5, the case of switching topology could be handled with different assumptions on the switching rule. However, we believe that the overall approach presented in this thesis gives a deep theoretical understanding of the behavior of networks and could be a reliable starting point for more *in-depth* analysis of particular problems.

One of the main challenge in dealing with networks is to consider a general class of systems from which the results obtained in the theoretical framework can be adapted to more practical scenarios without loss of generality. In fact, one of the goals of this thesis was to lay the *foundations* from which theoretical approaches to synchronization in the nonlinear framework could be extended to a huge set of *practical* problems. As engineers, we always have to keep in mind the theoretical results, even sophisticated and elegant, find their real meaning in the practical applications. With this respect, Chapter 4 and Chapter 6 are essentially motivated by realistic scenarios and the results present in those chapters can be considered as a novel approach to these problems. It is worth pointing out that the nonlinear dynamics considered in most of the literature are often *fairly* representative of realistic systems. Even though inside this thesis we considered a general class of nonlinear systems (characterized by an *observability* property and Lipschitz conditions), in most of the cases physical systems do not fulfill the requirements of such a class. Thus, a possible extension of the theoretical results presented in this thesis is to consider more *application-driven* representation of nonlinear systems and to re-design the control architecture according to the problem at hand.

Another key aspect in networks control is to design fully decentralized structure. As a matter of fact, most of the results that can be found in literature and the results presented in this thesis rely on knowledge of the *algebraic connectivity*, or more in general the smallest eigenvalue of the Laplacian matrix. It is well known that this parameter can be estimated by knowing the number of the agents in the network. However there are some scenarios in which the number of systems can be unknown or time varying: for instance, in the power-grids framework, the number of *users* and *suppliers* cannot be known a priori. Thus, one of the future challenges is to extend the well known results about synchronization and consensus towards adaptive solutions, not requiring the knowledge of any global parameter of the network.

A

Appendix: Prime Form and Observability for Nonlinear Systems

This Appendix is devoted to introduce the notion of observability and the observability canonical forms for nonlinear systems. For the sake of simplicity we considered the class of single-input single-output nonlinear systems.

By following Gauthier and Kupka (2001), the observability property is introduced with the notion of *canonical flag*. When this flag is regular and independent of u it is said to be *uniform*. In practice, it define a regularity property of the observation space (see (Besançon, 2007, Definition 6, Chapter 1))

A.1 Observability and canonical forms

Consider a nonlinear system of the form

$$\dot{x} = f(x,u)\,, \qquad y = h(x,u)\,, \tag{A.1}$$

where the state $x \in \mathbb{R}^n$, the input $u \in \mathbb{R}$ and the output $y \in \mathbb{R}$. The functions f, g, h are considered smooth enough and $f(0,0) = 0$, $h(0,0) = 0$. Now let define - recursively - a sequence of functions φ_i, $i = 1, \ldots, n$, as follows

$$
\begin{aligned}
\varphi_1(x,u) &:= h(x,u)\,, \\
\varphi_i(x,u) &:= \frac{\partial \varphi_{i-1}}{\partial x}(x,u) f(x,u)\ ,
\end{aligned}
\tag{A.2}
$$

and let define a sequence of i-vector-valued functions $\Phi_i(x,u)$ as follows

$$
\Phi_i(x,u) := \begin{pmatrix} \varphi_1(x,u) \\ \vdots \\ \varphi_i(x,u) \end{pmatrix}, \qquad \forall\, i = 1, \ldots, n\,. \tag{A.3}
$$

With these functions we have all the tools necessary to correctly define the so called "canonical flag" of a system (see (Gauthier and Kupka, 2001, Definition 2.1, Chapter 2).

Definition A.1. *The canonical flag of* (A.1) *is a set consisting of n distribution in $\mathbb{R}^n$, parametrized by u, defined by*

$$
D_i(u)\ :\ x\ \mapsto\ \ker\left[\frac{\partial \Phi_i}{\partial x}\right]_{(x,u)}, \qquad \forall\, i = 1, \ldots, n
$$

with the functions Φ_i defined by (A.3). *The canonical flag is said to be uniform if*

(i) all the $D_i(u)$, $i = 1, \ldots, n$, have constant dimension $n - i$ for all $u \in \mathbb{R}$ ("regularity" condition);

(ii) all the $D_i(u)$, $i = 1, \ldots, n$, are independent of u ("u-independence" condition), i.e. $\partial_u D(u) = 0$.

The notion of canonical flag is useful to characterize the local weak observability of a nonlinear system. We are not going to develop more along this line because, as already stressed, we are only interested in the existence of (global) normal forms for which we are able to design a tunable observer. A uniform canonical flag is in general not enough to guarantee the existence of global change of coordinates, and only local results can be achieved (see, for instance, Theorem 2.1 in (Gauthier and Kupka, 2001, Chapter 3)). Thus, to state stronger properties, we define the mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ as

$$
\Phi(x)\ :=\ \Phi_n(x,0)\ =\ \begin{pmatrix} h(x,0) \\ L_{f(x,0)} h(x,0) \\ \vdots \\ L_{f(x,0)}^{n-1} h(x,0) \end{pmatrix}. \tag{A.4}
$$

With the notion of uniform canonical flag and the mapping Φ above defined, we have all the ingredients in order to guarantee the existence of a global change of coordinates in which the system (A.1) as a triangular structure. The following result, is a direct consequence of the local result given in (Gauthier and Kupka, 2001, Theorem 2.1, Chapter 3).

Theorem A.1. *Consider the system* (A.1) *and suppose that its canonical flag is uniform (according to Definition A.1) and the mapping* $\Phi(\cdot)$ *defined in* (A.4) *is a global diffeomorphism. Then the system* (A.1) *is globally diffeomorphic, via* Φ*, to a system of the form*

$$\begin{aligned} \dot{z}_i &= f_i(z_1, \dots, z_i, z_{i+1}, u)\,, \qquad 1 \le i \le n-1\,, \\ \dot{z}_n &= f_n(z_1, \dots, z_n, u)\,, \\ y &= h(z_1, u)\,, \end{aligned} \tag{A.5}$$

with the functions f_i*,* $i = 1, \dots, n-1$ *and* h *that fulfil*

$$\frac{\partial h}{\partial z_1}(z_1, u) \neq 0\ , \qquad \frac{\partial f_i}{\partial z_{i+1}}(z_1, \dots, z_i, z_{i+1}, u) \neq 0\ , \quad i = 1, \dots, n-1\,,$$

for any $(z, u) \in \mathbb{R}^n \times \mathbb{R}$.

Proof. See (Marconi et al., 2004, Lemma 2). □

Consider now an input-affine single-input single-output nonlinear system of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)\,u \\ y &= h(x) \end{aligned} \tag{A.6}$$

where the state $x \in \mathbb{R}^n$, the input $u \in \mathbb{R}$ and the output $y \in \mathbb{R}$. Let the functions $\varphi_i(x, u)$ in (A.2), Φ_i in (A.3) and Φ in (A.4) be defined similarly for the system (A.6). For this class of system the result of Theorem A.1 can be further specialized, by obtaining a triangular structure with a well-defined linear part.

Theorem A.2. *Consider the system* (A.6) *and suppose that its canonical flag is uniform (according to Definition A.1) and the mapping* $\Phi(\cdot)$ *defined in* (A.4) *is a global diffeomorphism. Then the system* (A.6) *is globally diffeomorphic, via* Φ*, to a system of the form*

$$\begin{aligned} \dot{z} &= \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_{n-1} \\ \dot{z}_n \end{pmatrix} = \begin{pmatrix} z_2 \\ z_3 \\ \vdots \\ z_n \\ a(z) \end{pmatrix} + \begin{pmatrix} b_1(z_1) \\ b_2(z_1, z_2) \\ \vdots \\ b_{n-1}(z_1, \dots, z_{n-1}) \\ b_n(z_1, \dots, z_n) \end{pmatrix} u\,, \\ y &= z_1\,. \end{aligned} \tag{A.7}$$

Proof. See (Gauthier and Kupka, 2001, Theorem 4.1, Chapter 3). □

In the previous theorem we showed that for input-affine nonlinear systems we can find a change of coordinates in which the system has a linear part characterized by a sort of "chain of integrators". It is not hard to see that, from an observer-design perspective, the form (A.7) is easier to handle with respect to the form (A.5).

A trivial extension of Theorem (A.2) can be obtained for the class of systems analyzed in this Thesis. Consider a multi-input single-output nonlinear system of the form

$$\begin{aligned} \dot{x} &= f(x) + u \\ y &= h(x) \end{aligned} \tag{A.8}$$

where the state $x \in \mathbb{R}^n$, the input $u \in \mathbb{R}^n$ and the output $y \in \mathbb{R}$.

Theorem A.3. *Consider the system* (A.6) *and suppose that its canonical flag is uniform (according to Definition A.1) and the mapping* $\Phi(\cdot)$ *defined in* (A.4) *is a global diffeomorphism. Then the system* (A.8) *is globally diffeomorphic, via* Φ, *to a system of the form*

$$\begin{aligned} \dot{z} &= \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_{n-1} \\ \dot{z}_n \end{pmatrix} = \begin{pmatrix} z_2 \\ z_3 \\ \vdots \\ z_n \\ a(z) \end{pmatrix} + v\,, \\ y &= z_1\,. \end{aligned} \tag{A.9}$$

with

$$v = \frac{\partial \Phi(x)}{\partial x}^{-1} u$$

In the next section, we make use of this last result to introduce the structure of the networked systems considered in this thesis. We will use a slightly different notation, compatible with the one used in all the chapters of this thesis.

A.2 Nonlinear systems in prime form and control of networks

During the whole thesis we consider the problem of achieving synchronization between a set of nonlinear systems which could be represented in prime form. Having in mind the consideration of Appendix A.1, we now show that systems considered are indeed a general *family* of nonlinear uniformly observable systems.

Each of the N agents in the network is described by the nonlinear dynamics

$$\begin{array}{rcl} \dot{x}_k & = & f(x_k) + u_k \qquad x_k \in \mathbb{R}^d \\ \iota_k & = & h(x_k) \end{array} \tag{A.10}$$

in which, for each $k = 1, \ldots, N$, $u_k \in \mathbb{R}^d$ is the control input, $\iota_k \in \mathbb{R}$ is the available measurement. Note that (A.10) represents homogeneous nonlinear agents, namely $f(\cdot)$ and $h(\cdot)$ do not depend on k.

In general, we look for a decentralized control structure in which the agents exchange only output information and the control law of each agent is taken as

$$u_k = \mathcal{K}(x_k)\nu_k\,, \qquad \nu_k = \sum_{j=1}^{N} \ell_{kj} h(x_j) \tag{A.11}$$

with $\mathcal{K}(x_k)$ to be designed in such a way that output consensus is reached among the agents. Namely, for each initial condition $x_k(0) \in \mathbb{R}^d$, there is a function $\iota^\star : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{t\to\infty} |\iota_k(t) - \iota^\star(t)| = 0\,,$$

uniformly in the initial conditions, for all $k = 1, \ldots, N$. Indeed, only local output of single agents (rather than a *full state* information) is assumed to be spread over the network.

Since the goal is to achieve consensus by only processing the output ι_k of each agent, we ask that systems (A.10) are uniformly observable (see Isidori (1995)) as detailed next.

Assumption A.1. *The map $\Psi : \mathbb{R}^d \mapsto \mathbb{R}^d$ defined as*

$$\Psi(x_k) = \begin{pmatrix} h(x_k) \\ L_f h(x_k) \\ \vdots \\ L_f^{d-1} h(x_k) \end{pmatrix} \tag{A.12}$$

is a global diffeomorphism.

The requirement of the existence of a global diffeomorphism $\Psi(\cdot)$ is motivated by the fact that we look for consensus results that hold globally, namely without restrictions on the initial state of the agents. The previous assumption could be weakened by just asking that the map $\Psi(\cdot)$ is a local diffeomorphism on some given set at the price of obtaining just semiglobal consensus results, namely by restricting the initial state of the agents to some prescribed compact set. Details in this direction are omitted since they can be obtained by using tools that are customary in the literature of stabilization of nonlinear systems.

The existence of such a diffeomorphism allows us to define a change of coordinate $w_k = \Psi(x_k)$, which maps system (5.20) to

$$\dot{w}_k = Sw_k + B\phi(w_k) + v_k\,, \qquad y_k = Cw \tag{A.13}$$

with

$$v_k = \frac{\mathrm{d}\Psi(x_k)}{\mathrm{d}x_k} u_k$$

and with the triplet of matrices (S, B, C) that is in *prime* form, that is S is a shift matrix (all 1's on the upper diagonal and all 0's elsewhere), $B^T = (0 \cdots 0 \;\; 1)$ and $C = (1 \;\; 0 \cdots 0)$. In the following we assume that the function

$$\phi(w_k) = L_{\mathrm{f}}^d h(x_k)|_{x_k = \Psi^{-1}(w_k)}$$

is globally Lipschitz, namely there exists a positive constant $\bar{\phi}$ such that $\|\phi(w)\| \leq \bar{\phi}\|w\|$ for all $w \in \mathbb{R}^d$. Such a globally Lipschitz condition is motivated by the fact of looking for "global" consensus results. The assumption in question could be weakened by just asking the previous function to be only locally Lipschitz if just semiglobal consensus results are of interest.

By bearing in mind the definition of (A.13), we design $\mathcal{K}(x_k)$ in (A.11) as

$$\mathcal{K}(x_k) = \frac{\mathrm{d}\Psi(x_k)}{\mathrm{d}x_k}^{-1} K \tag{A.14}$$

and thus obtain

$$v_k = -K \sum_{j=1}^{N} \ell_{kj} C w_j \qquad k = 1, \ldots, N\,. \tag{A.15}$$

By collecting all the agents as $\mathbf{w} = \mathrm{col}(w_1, \ldots, w_N)$, this allows to represent the network of (A.10) as

$$\begin{aligned} \dot{\mathbf{w}} &= (I_N \otimes S)\mathbf{w} + (I_N \otimes B)\Phi(\mathbf{w}) + (L \otimes KC)\mathbf{w} \\ \mathbf{y} &= (I_N \otimes C)\mathbf{w} \end{aligned} \tag{A.16}$$

where $\Phi(\mathbf{w})$ is

$$\Phi(\mathbf{w}) = \begin{pmatrix} \phi(w_1) \\ \vdots \\ \phi(w_N) \end{pmatrix}$$

Remark A.1. It is worth noticing that the assumption that $v_k \in \mathbb{R}^d$ in (A.13) *acts* on all components of the state is in general non-restrictive. Consider for instance a system of the kind

$$\begin{aligned} \dot{w} &= Sw + B\Big(\phi(w) + v\Big) \\ y &= Cw \end{aligned} \tag{A.17}$$

with $w \in \mathbb{R}^d$ and $v \in \mathbb{R}^d$. The pair (S, B) is indeed controllable and (S, C) is observable. Thus it is always possible to define a dynamic regulator of the form of

$$\begin{aligned} \dot{\eta} &= \varphi(\eta, y) \\ v &= \gamma(\eta) \end{aligned} \tag{A.18}$$

that solve the problem of stabilization of (A.17). In the *network* scenario, a generic k-th system would read as

$$\begin{aligned} \dot{w}_k &= Sw_k + B\Big(\phi(w_k) + v_k\Big) \\ y_k &= Cw_k \end{aligned}$$

and the dynamic regulator

$$\begin{aligned} \dot{\eta}_k &= \varphi(\eta_k, \textstyle\sum_{i=1}^N \ell_{ki} Cw_i) \\ v_k &= \gamma(\eta_k) \end{aligned}$$

processes the information coming from the neighbors.

For the sake of simplicity, in this Thesis we consider systems of the form (A.13), but the results presented could be extended to the case of (A.17). △

B

Appendix: Input-to-State Stability

In this Appendix, we introduce basic concepts about *input-to-state stability*, which are extensively used during the thesis. For more details about the topic we refer the reader to Isidori (1999).

First, we recall some useful concepts about *comparison functions*.

Definition B.1. *A continuous function* $\alpha : [0, a) \to [0, \infty)$ *is said to belong to the class* $\mathcal{K}$ *if it is strictly increasing and* $\alpha(0) = 0$*. If* $a = \infty$ *and* $\lim_{r\to\infty} \alpha(r) = \infty$*, the function is said to belong to class* $\mathcal{K}_\infty$*.*

Definition B.2. *A continuous function* $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ *is said to belong to the class* $\mathcal{KL}$ *if, for each fixed* s*, the function*

$$\begin{array}{rrcl} \alpha : & [0, a) & \to & [0, \infty) \\ & r & \mapsto & \beta(r, s) \end{array}$$

belongs to class $\mathcal{K}$ *and, for each fixed* r*, the function*

$$\begin{array}{rrcl} \varphi : & [0, \infty) & \to & [0, \infty) \\ & s & \mapsto & \beta(r, s) \end{array}$$

is decreasing and $\lim_{s\to\infty} \varphi(s) = 0$*.*

Class $\mathcal{K}$ and $\mathcal{KL}$ function have the following properties:

- the composition of two class $\mathcal{K}$ (respectively class $\mathcal{K}_\infty$) functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ denoted as $\alpha_1(\alpha_2(\cdot))$ or $\alpha_1 \circ \alpha_2(\cdot)$, is a class $\mathcal{K}$ (respectively class $\mathcal{K}_\infty$) function

- if $\alpha(\cdot)$ is a class $\mathcal{K}$ function defined on $[0, a)$ and $b = \lim_{r \to a} \alpha(r)$, there exists a unique function $\alpha^{-1} : [0, b) \to [0, a)$ such that

$$\begin{aligned} \alpha^{-1}(\alpha(r)) &= r, \quad \text{for all } r \in [0, a) \\ \alpha(\alpha^{-1}(r)) &= r, \quad \text{for all } r \in [0, b) \end{aligned}$$

  Furthermore $\alpha^{-1}(\cdot)$ is a class $\mathcal{K}$ function. If $\alpha(\cdot)$ is a class $\mathcal{K}_\infty$ function, also $\alpha^{-1}(\cdot)$ belongs to class $\mathcal{K}_\infty$

- if $\beta(\cdot, \cdot)$ is a class $\mathcal{KL}$ function, there exists two class $\mathcal{K}_\infty$ functions $\gamma(\cdot)$ and $\theta(\cdot)$ such that

$$\beta(r, s) \leq \gamma(\exp^{-s} \theta(r))$$

As far as Lyapunov theory is concerned, *comparison functions* are useful tools to determine if a nonlinear system has stability properties with respect to a certain equilibrium. We can formulate these concepts as follows.

Theorem B.1. *Consider a nonlinear system*

$$\dot{x} = f(x) \tag{B.1}$$

with $x \in \mathbb{R}^n$ and $f(0) = 0$, with $f(\cdot)$ locally Lipschitz. Let $V : \mathcal{B}_d \to R$ be a C^1 function such that, for all $x : \|x\| < d$, there exists two class $\mathcal{K}$ functions $\bar{\alpha}(\cdot)$, $\underline{\alpha}(\cdot)$, defined on $[0, d)$ for which

$$\underline{\alpha}(x) \leq V(x) \leq \bar{\alpha}(x)$$

Then

- *if*

$$\frac{\partial V}{\partial x} f(x) \leq 0$$

  the equilibrium $x = 0$ is stable for (B.1)

- *if, there exists a class $\mathcal{K}$ function $\alpha(\cdot)$ such that*

$$\frac{\partial V}{\partial x} f(x) \leq \alpha(\|x\|)$$

  the equilibrium $x = 0$ is asymptotically stable for (B.1)

- *if* $d = \infty$ *and the functions* $\bar{\alpha}(\cdot)$, $\underline{\alpha}(\cdot)$ *and* $\alpha(\cdot)$ *belong to class* $\mathcal{K}_\infty$, *the equilibrium* $x = 0$ *is globally asymptotically stable for (B.1)*

Lemma B.1. *The equilibrium* $x = 0$ *of system (B.1) is globally asymptotically and locally exponentially stable if and only if there exists a smooth function* $V : \mathbb{R}^n \to \mathbb{R}$ *and class* $\mathcal{K}_\infty$ *functions* $\bar{\alpha}(\cdot)$, $\underline{\alpha}(\cdot)$ *and real numbers* $\delta, a, b \in \mathbb{R}_+$, *such that*

$$\underline{\alpha}(x) \leq V(x) \leq \bar{\alpha}(x)$$

and

$$\frac{\partial V}{\partial x} f(x) \leq -\alpha(\|x\|)$$

for all $x \in \mathbb{R}^n$ *and for* $s \in [0, \delta]$

$$\underline{\alpha}(s) = as^2, \qquad \alpha(s) = bs^2$$

With this concept in mind, we are able to introduce the basic concepts of ISS which turn out to be crucial in the analysis of interconnected systems. Consider the system

$$\dot{x} = f(x, u) \tag{B.2}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $f(0, 0) = 0$, with $f(x, u)$ locally Lipschitz. We ask the input u to be bounded, $u \in L^m_\infty$. Namely $u : [o\infty) \to \mathbb{R}^m$ satisfies

$$\|u(\cdot)\|_\infty = \sup_{t \geq 0} \|u(t)\|$$

Definition B.3. *System (B.2) is said to be input-to-state table if there exist a class* $\mathcal{KL}$ *function* $\beta(\cdot, \cdot)$ *and a class* $\mathcal{K}$ *function such that, for any input* $u \in L^m_\infty$ *and any initial condition* $x(0)$, *the response of the (B.2) satisfies*

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma(\|u(\cdot)\|_\infty) \tag{B.3}$$

for all $t \geq 0$.

Condition (B.3) can be also written in a more expressive form, namely

$$\|x(t)\| \leq \max\{\beta(\|x(0)\|, t), \gamma(\|u(\cdot)\|_\infty)\} \tag{B.4}$$

which explicitly shows that if (B.2) is ISS, its evolution can be bounded by the maximum between its *free evolution* (namely $\beta(\|x(0)\|, t)$) and a certain *gain* function of the input term (namely $\gamma(\|u(\cdot)\|_\infty)$). We now move towards the formulation of the previous definition in terms of Lyapunov analysis.

Definition B.4. *A C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ is called ISS-Lyapunov function for (B.2) if there exist class $\mathcal{K}_\infty$ functions $\bar{\alpha}(\cdot)$, $\underline{\alpha}(\cdot)$ and $\alpha(\cdot)$ and a class $\mathcal{K}$ function $\chi(\cdot)$ such that*

$$\underline{\alpha}(x) \leq V(x) \leq \bar{\alpha}(x) \tag{B.5}$$

and

$$\|x\| \geq \chi(\|u\|) \qquad \Rightarrow \qquad \frac{\partial V}{\partial x} f(x,u) \leq \alpha(\|x\|) \tag{B.6}$$

This definition of ISS-Lyapunov function is indeed really important in order to understand the concepts of ISS. First of all, notice that, $u(t) = 0$ implies global asymptotic stability of (B.2), or in other words

$$\|x(t)\| \leq \beta(\|x(0)\|, t)$$

Consider now the case of nonzero inputs, define

$$M = \|u(\cdot)\|_\infty$$

and set

$$c = \bar{\alpha}(\chi(M))$$

It is readily seen from (B.5) that the set

$$\Omega_c = \{x \in \mathbb{R}^n \ : \ V(x) \leq c\}$$

is such that $\mathcal{B}_{\chi(M)} \subset \Omega_c$. From (B.6) it can be concluded that, as long as $x(t)$ is on the boundary of Ω_c, $\|x(t)\| \geq \chi(\|u(t)\|)$ and thus

$$\frac{\partial V}{\partial x} f(x(t), u(t)) < 0$$

and thus it can be concluded that for any $x(0)$ in the interior of Ω_c the solution $x(t)$ of (B.2) is defined for all $t \geq 0$ and $x(t) \in \Omega_c$. By setting

$$\gamma(r) = \underline{\alpha}^{-1}(\bar{\alpha}(\chi(r)))$$

we can easily obtain that, for all $t \geq 0$,

$$\|x(t)\| \leq \gamma(\|u(\cdot)\|_\infty)$$

which confirms (B.3). If instead $x(0)$ does not belong to the interior of Ω_c, we observe that

$$V(x(t)) > c \qquad \Rightarrow \qquad \|x(t)\| \geq \chi(\|u(t)\|)$$

and thus

$$\frac{\partial V}{\partial x} f(x(t), u(t)) \leq -\alpha(\|x(t)\|) < 0$$

In other words, as long as $V(x(t)) > c$, the function $V(x(t))$ is decreasing, implying that $x(t)$ is bounded. Furthermore, there exists a time t_s such that

$$\begin{aligned} V(x(t)) &> c, \quad \text{for all } 0 \leq t < t_s \\ V(t_s) &= c \end{aligned}$$

Suppose that it is not the case, then $V(x(t)) > c$ for all $t \geq 0$ and thus

$$\frac{\partial V}{\partial x} f(x(t), u(t)) \leq -\alpha(\bar{\alpha}^{-1}(V(x(t))))$$

that leads to

$$\|x(t)\| \leq \beta(\|x(0)\|, t)$$

which however is in contradiction with $V(x(t)) > c$, since $x(t) \to 0$ would imply $V(x(t)) \to 0$.

Thus we can conclude that

$$\begin{aligned} \|x(t)\| &\leq \beta(\|x(0)\|, t) \;\; \text{for all } 0 \leq t < t_s \\ \|x(t)\| &\leq \gamma(\|u(\cdot)\|_\infty) \;\; \text{for all } t \geq t_s \end{aligned}$$

which is indeed (B.4).

These reasoning lead to the following Theorem.

Theorem B.2. *System (B.2) is input-to-state stable if and only is there exists an ISS-Lyapunov function.*

An simple way to check if a Lyapunov function is an ISS-Lyapunov function is given in the following Lemma.

Lemma B.2. *Consider system (B.2). A C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ is an ISS-Lyapunov for (B.2) if and only if there exist class $\mathcal{K}_\infty$ functions $\bar{\alpha}(\cdot)$, $\underline{\alpha}(\cdot)$ and $\alpha(\cdot)$ and a class $\mathcal{K}$ function $\sigma(\cdot)$ such that (B.5) holds and*

$$\frac{\partial V}{\partial x} f(x(t), u(t)) \leq -\alpha(\|x\|) + \sigma(\|u\|) \tag{B.7}$$

for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.

The relationship (B.7) is equivalent to (B.5)-(B.6). To see this suppose that it holds and define

$$\chi(r) = \alpha^{-1}(k\sigma(r))$$

Then, we can write

$$\frac{\partial V}{\partial x} f(x(t), u(t)) \leq -\frac{k-1}{k}\alpha(\|x\|)$$

which is indeed a relationship of the form of (B.6). Also, (B.6) implies that for $\|x\| \geq \chi(\|u\|)$, then (B.7) holds for any $\sigma(\cdot)$. By defining

$$\psi(r) = \max_{\|u\|=r, \|x\| \leq \chi(r)} \{\frac{\partial V}{\partial x} f(x, u) + \alpha(\chi(\|u\|))\}$$

it turns out that for $\|x\| \geq \chi(\|u\|)$,

$$\frac{\partial V}{\partial x} f(x, u) \leq -\alpha(\|x\|) + \psi(\|u\|)$$

By defining

$$\sigma(r) = \max\{0, \psi(r)\}$$

the result in Lemma B.2 come straightforward.

As far as the two functions $\alpha(\cdot)$ and $\sigma(\cdot)$ are concerned, it is seen that infinite pairs of functions could be defined. The Lemma we introduce allows to construct families of pairs fulfilling (B.7).

Lemma B.3. *Assume $\{\alpha(\cdot), \sigma(\cdot)\}$ is an ISS-pair for (B.2), fulfilling (B.7).*

i) Let $\tilde{\sigma}(\cdot)$ be a class $\mathcal{K}$ function such that $\sigma(r) = \mathcal{O}[\tilde{\sigma}(r)]$ as $r \to \infty$. Then there exists a class $\mathcal{K}_\infty$ function $\tilde{\alpha}(\cdot)$ such that $\{\tilde{\alpha}(\cdot), \tilde{\sigma}(\cdot)\}$ is an ISS-pair

ii) Let $\tilde{\alpha}(\cdot)$ be a class $\mathcal{K}_\infty$ function such that $\alpha(r) = \mathcal{O}[\tilde{\alpha}(r)]$ as $r \to 0^+$. Then there exists a class $\mathcal{K}$ function $\tilde{\sigma}(\cdot)$ such that $\{\tilde{\alpha}(\cdot), \tilde{\sigma}(\cdot)\}$ is an ISS-pair

Another important formulation of the ISS property (which will be useful in the following), is given in the following Theorem. It derives from the fact that

$$\beta(\|x(0)\|, t) \leq \beta(\|x(0)\|, 0)$$

and $\beta(\|x(0)\|, 0)$ is a class $\mathcal{K}$ function. Then, we could reformulate (B.4) as

$$\|x(t)\| \leq \max\{\gamma_0(\|x(0)\|), \gamma(\|u\|_\infty)\}$$

for some class $\mathcal{K}$ function $\gamma_0(\cdot)$ ($\gamma(\cdot)$ is the same defined before). Furthermore, since

$$\lim_{t\to\infty} \beta(\|x(0)\|, t) = 0$$

we can write that

$$\lim_{t\to\infty} \sup \|x(t)\| \leq \gamma(\|u(\cdot)\|_\infty)$$

or equivalently

$$\lim_{t\to\infty} \sup \|x(t)\| \leq \gamma(\lim_{t\to\infty} \sup(\|u(t)\|))$$

Theorem B.3. *System (B.1) is input-to-state stable if and only if there exists class $\mathcal{K}$ functions $\gamma_0(\cdot)$ and $\gamma(\cdot)$ such that, for any input $u \in L_\infty^m$ and $x(0) \in \mathbb{R}^n$, the evolution of the system satisfies*

$$\begin{aligned} \|x(\cdot)\|_\infty &\leq \max\{\gamma_0(\|x(0)\|), \gamma(\|u\|_\infty)\} \\ \lim_{t\to\infty} \sup \|x(t)\| &\leq \gamma(\lim_{t\to\infty} \sup \|u(t)\|) \end{aligned}$$

B.1 Input-to-State Stability for Cascade-Systems

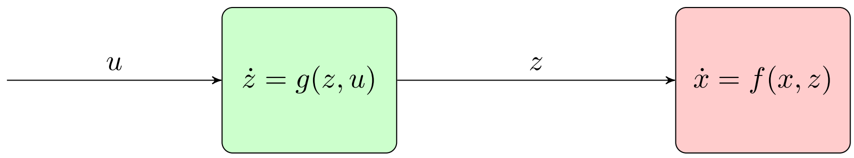

Figure B.1: Cascaded nonlinear systems

Consider now the cascade system

$$\begin{aligned} \dot{x} &= f(x, z) \\ \dot{z} &= g(z, u) \end{aligned} \tag{B.8}$$

with $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $u \in \mathbb{R}^p$, $f(0,0) = 0$, $g(0,0) = 0$ and $f(\cdot,\cdot)$, $g(\cdot,\cdot)$ locally Lipschitz.

Theorem B.4. *Suppose*

$$\dot{x} = f(x, z) \tag{B.9}$$

with state x and input z is input-to-state stable and that

$$\dot{z} = g(z, u) \tag{B.10}$$

with state z and input u is also input-to-state stable. Then (B.11) is input-to-state stable.

Proof. By assumptions, system (B.9) and (B.10) have ISS-pairs $\{\alpha(\cdot), \sigma(\cdot)\}$, $\{\eta(\cdot), \varrho(\cdot)\}$ respectively. Define

$$\tilde{\eta}(s) = \begin{cases} \eta(s) & \text{for small } s \\ \sigma(s) & \text{for large } s \end{cases}$$

Then, by Lemma B.3, there exists a $\tilde{\varrho}$ such that $\{\tilde{\eta}(\cdot), \tilde{\varrho}(\cdot)\}$ is an ISS-pair for (B.10). Define also $\tilde{\sigma}(\cdot)$ as

$$\tilde{\sigma}(s) = \frac{1}{2}\tilde{\eta}(s)$$

Again, by Lemma B.3, there exists a $\tilde{\alpha}(\cdot)$ such that $\{\tilde{\alpha}(\cdot), \frac{1}{2}\tilde{\eta}(\cdot)\}$ is an ISS-pair for (B.9). This implies the existence of two positive definite functions $V(x)$ and $W(z)$ such that

$$\begin{aligned} \tfrac{\partial V}{\partial x} f(x,z) &\leq -\tilde{\alpha}(\|x\|) + \tfrac{1}{2}\tilde{\eta}(\|z\|) \\ \tfrac{\partial W}{\partial z} g(z,u) &\leq -\tilde{\eta}(\|z\|) + \tilde{\varrho}(\|u\|) \end{aligned}$$

By considering $U(x,z) = V(x) + W(z)$, we obtain

$$\frac{\partial V}{\partial x} f(x,z) + \frac{\partial W}{\partial z} g(z,u) \leq -\tilde{\alpha}(\|x\|) - \frac{1}{2}\tilde{\eta}(\|z\|) + \tilde{\varrho}(\|u\|)$$

which shows that $U(x,z)$ is an ISS Lyapunov function for (B.11) in the sense of (B.2). □

B.2 Small gain theorem

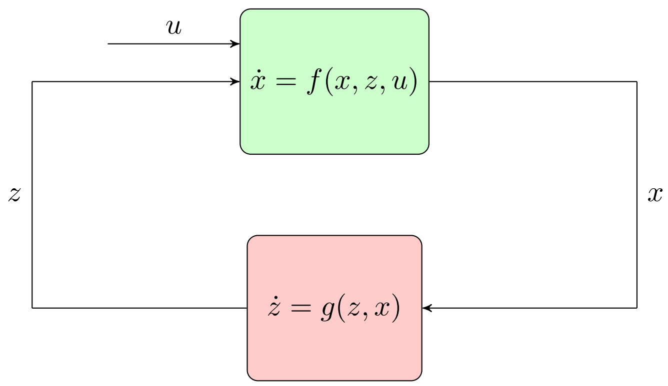

Figure B.2: Small Gain Theorem

The *Small Gain Theorem* concerns nonlinear systems connected in *feedback* form and gives an important insight on sufficient conditions to establish global asymptotic stability for *feedback* connected globally asymptotically stable systems. As shown in Figure

B.2, consider the following interconnection

$$\begin{aligned} \dot{x} &= f(x, z, u) \\ \dot{z} &= g(z, x) \end{aligned} \tag{B.11}$$

with $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $u \in \mathbb{R}^p$, $f(0,0,0) = 0$, $g(0,0) = 0$. Suppose that both systems are ISS, namely x is ISS with respect to z, u and z is ISS with respect to x. These assumption implies that there exist two class $\mathcal{K}$ functions $\gamma_{02}(\cdot)$ and $\gamma_2(\cdot)$ such that

$$\begin{aligned} \|z(t)\| &\leq \max\{\gamma_{02}(\|z(0)\|), \gamma_2(\|x(\cdot)\|_\infty)\} \\ \lim_{t\to\infty} \sup \|z(t)\| &\leq \gamma_2(\lim_{t\to\infty} \sup \|x(t)\|) \end{aligned} \tag{B.12}$$

Similarly, there exist class $\mathcal{K}$ functions $\gamma_{01}(\cdot)$, $\gamma_1(\cdot)$ and $\gamma_u(\cdot)$ such that

$$\begin{aligned} \|x(t)\| &\leq \max\{\gamma_{01}(\|x(0)\|), \gamma_1(\|z(\cdot)\|_\infty), \gamma_u(\|u(\cdot)\|_\infty)\} \\ \lim_{t\to\infty} \sup \|x(t)\| &\leq \max\{\gamma_1(\lim_{t\to\infty} \sup \|z(t)\|), \gamma_u(\lim_{t\to\infty} \sup \|u(t)\|)\} \end{aligned} \tag{B.13}$$

Theorem B.5. *If*

$$\gamma_1(\gamma_2(r)) < r \qquad \text{for all } r > 0 \tag{B.14}$$

system (B.11) with state $\zeta = (x, z)$ and input u is input to state stable. Furthermore, the class $\mathcal{K}$ functions

$$\begin{aligned} \gamma_0(r) &= \max\{2\gamma_{01}(r), 2\gamma_{02}(r), 2\gamma_1 \circ \gamma_{02}(r), 2\gamma_2 \circ \gamma_{01}(r)\} \\ \gamma(r) &= \max\{2\gamma_1 \circ \gamma_u(r), 2\gamma_u(r)\} \end{aligned}$$

are such that $\zeta(t)$ with any input $u(\cdot) \in L_\infty^m$ can be bounded according to

$$\begin{aligned} \|\zeta(\cdot)\|_\infty &= \max\{\gamma_0(\|\zeta(0)\|), \gamma(\|u(\cdot)\|_\infty)\} \\ \lim_{t\to\infty} \sup \|\zeta(t)\| &= \gamma(\lim_{t\to\infty} \sup \|u(t)\|) \end{aligned}$$

Proof. The proof of the Theorem can be divided in two parts. First boundednes of solution for x and z is shown, then the claim of the proposition is proved. In order to prove boundedness of solutions, consider $x(0) \in \mathbb{R}^n$, $x(0) \in \mathbb{R}^m$ and $u \in \mathcal{L}_\infty^p$. Suppose that they are not bounded: this implies that for every $R > 0$, there exists a time $T > 0$ such that trajectories are well defined and

$$\|x(T)\| > R \qquad \text{or} \qquad \|z(T)\| > R \tag{B.15}$$

Then, choose R as

$$\begin{aligned} R &> \max\{\gamma_{02}(\|z(0)\|), \gamma_2 \circ \gamma_{01}(\|x(0)\|), \gamma_2 \circ \gamma_u(\|u(\cdot)\|_\infty)\} \\ R &> \max\{\gamma_{01}(\|x(0)\|), \gamma_1 \circ \gamma_{02}(\|z(0)\|), \gamma_u(\|u(\cdot)\|_\infty)\} \end{aligned} \tag{B.16}$$

and consider T such that (B.15) is true. Consider

$$x(t)^T = \begin{cases} x(t) & \text{if} \quad t \in [0, T] \\ 0 & \text{if} \quad t > T \end{cases}$$

which is bounded for every t. The response $\bar{z}(t)$ of the lower system of (B.11) with initial condition $z(0)$ and input $x(\cdot)^T$ is

$$\|\bar{z}(t)\| \leq \max\{\gamma_{02}(\|z(0)\|), \gamma_2(\|x(\cdot)^T\|_\infty)\}$$

for all t. Since $\bar{z}(t) = z(t)$ for $t \in [0, T]$, we also can say that

$$\|z(\cdot)^T\|_\infty = \max_{t \in [0,T]} \|z(t)\| \leq \max\{\gamma_{02}(\|z(0)\|), \gamma_2(\|x(\cdot)^T\|_\infty)\} \tag{B.17}$$

With the same approach and notation, we can write that

$$\|\bar{x}(t)\| \leq \max\{\gamma_{01}(\|x(0)\|), \gamma_2(\|z(\cdot)^T\|_\infty), \gamma_u(\|u(\cdot)\|_\infty)\}$$

and, since $\bar{x}(t) = x(t)$ for $t \in [0, T]$

$$\|x(\cdot)^T\|_\infty = \max_{t \in [0,T]} \|x(t)\| \leq \max\{\gamma_{01}(\|x(0)\|), \gamma_1(\|z(\cdot)^T\|_\infty), \gamma_u(\|u(\cdot)\|_\infty)\} \tag{B.18}$$

By putting (B.18) into (B.17) and remembering that by hypothesis $\gamma_1 \circ \gamma_2(r) < r$ we have

$$\|z(\cdot)^T\|_\infty \leq \max\{\gamma_{02}(\|z(0)\|), \gamma_2 \circ \gamma_{01}(\|x(0)\|), \gamma_2 \circ \gamma_u(\|u(\cdot)\|_\infty)\}$$

Similarly one could write that

$$\|x(\cdot)^T\|_\infty \leq \max\{\gamma_{01}(\|x(0)\|), \gamma_1 \circ \gamma_{02}(\|z(0)\|), \gamma_u(\|u(\cdot)\|_\infty)\}$$

Thus, using (B.16), we have

$$\begin{aligned}\|z(T)\| &\leq \max\{\gamma_{02}(\|z(0)\|), \gamma_2 \circ \gamma_{01}(\|x(0)\|), \gamma_2 \circ \gamma_u(\|u(\cdot)\|_\infty)\} < R \\ \|x(T)\| &\leq \max\{\gamma_{01}(\|x(0)\|), \gamma_1 \circ \gamma_{02}(\|z(0)\|), \gamma_u(\|u(\cdot)\|_\infty)\} < R\end{aligned}$$

which clearly contradicts (B.15).

Having shown that the trajectories of (B.11) are bounded for all $t \geq 0$, by (B.12) and (B.13), we know that

$$\begin{aligned}\|z(t)\| &\leq \max\{\gamma_{02}(\|z(0)\|), \gamma_2(\|x(\cdot)\|_\infty)\} \\ \|x(t)\| &\leq \max\{\gamma_{01}(\|x(0)\|), \gamma_1(\|z(\cdot)\|_\infty), \gamma_u(\|u(\cdot)\|_\infty)\}\end{aligned}$$

which yields

$$\begin{aligned}\|z(t)\| &\leq \max\{\gamma_{02}(\|z(0)\|), \gamma_2 \circ \gamma_{01}(\|x(0)\|), \gamma_2 \circ \gamma_u(\|u(\cdot)\|_\infty)\} \\ \|x(t)\| &\leq \max\{\gamma_{01}(\|x(0)\|), \gamma_1 \circ \gamma_{02}(\|z(0)\|), \gamma_u(\|u(\cdot)\|_\infty)\}\end{aligned}$$

Similarly, one could write that

$$\begin{aligned}\lim_{t\to\infty} \sup \|z(t)\| &\leq \gamma_2 \circ \gamma_u(\lim_{t\to\infty} \sup \|u(t)\|) \\ \lim_{t\to\infty} \sup \|x(t)\| &\leq \gamma_u(\lim_{t\to\infty} \sup \|u(t)\|)\end{aligned}$$

From this, knowing that

$$\|\zeta(\cdot)\|_\infty \leq \max\{2\|z(\cdot)\|_\infty, 2\|x(\cdot)\|_\infty\}$$

it follows that

$$\lim_{t\to\infty} \sup \|\zeta(t)\| \leq \max\{2 \lim_{t\to\infty} \sup \|z(t)\|, \lim_{t\to\infty} \sup \|x(t)\|\}$$

and the result follows easily. □

C

Appendix: Hybrid Systems

In this Appendix we introduce basic facts about Hybrid systems and give basic notation to understand the analysis of Hybrid systems. These concepts are extensively used in Chapter 5 and Chapter 6. For a more exhaustive coverage of the topic the reader is referred to Goebel et al. (2008).

C.1 Hybrid systems modeling

A general Hybrid systems can be represented by

$$\begin{cases} x \in C & \dot{x} \in F(x) \\ x \in D & x^{+} \in G(x) \end{cases} \tag{C.1}$$

where $C \subset \mathbb{R}^n$ is the *flow set*, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the *flow map*, $D \subset \mathbb{R}^n$ is the *jump set* and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the *jump map*.

Solutions of hybrid systems are defined on the *so called* hybrid time domain, which is introduced in the next definition.

Definition C.1. *A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is compact hybrid domain if*

$$E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}, j) \tag{C.2}$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq \ldots \leq t_J$. It is a hybrid time domain if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, \ldots, J\})$ is a compact hybrid domain.

The previous definition simply states that E is a compact hybrid time domain if it is a union of a finite sequence of time intervals, while E is a hybrid time domain if the last interval is of the form $[t_j, T)$ with $T = \infty$. We now introduce the concept of a hybrid arc.

Definition C.2. *A function $\phi : E \to \mathbb{R}^n$ is a hybrid arc of E is a hybrid time domain and if for every $j \in \mathbb{N}$, the function $t \mapsto \phi(t, j)$ is locally absolutely continuous on the interval $I^j = \{t : (t, j) \in E\}$.*

In the previous definition, the requirement of absolute continuity is only referred to the intervals I^j which have nonempty interiors. In general these intervals could be empty or consist of only one point. Given the hybrid system (C, F, D, G), its solution is a series of hybrid arc satisfying certain conditions on the hybrid time domain. We formulate this concept in the next definition.

Definition C.3. *A hybrid arc ϕ is a solution for (C, F, D, G) if $\phi(0, 0) \in \bar{C} \cup D$ and*

- *for all $j \in \mathbb{N}$ such that $I^k = \{t : (t, j) \in \mathrm{dom}\phi\}$ has nonempty interior*

$$\begin{aligned} \phi(t, j) &\in C && \text{for} && t \in \mathrm{int} I^j \\ \dot{\phi}(t, j) &\in F(\phi(t, j)) && \text{for almost all} && t \in I^j \end{aligned} \tag{C.3}$$

- *for all $(t, j) \in \mathrm{dom}\phi$ such that $(t, j+1) \in \mathrm{dom}\phi$*

$$\begin{aligned} \phi(t, j) &\in D \\ \phi(t, j+1) &\in G(\phi(t, j)) \end{aligned} \tag{C.4}$$

Furthermore, a hybrid arc ϕ is said to be maximal if there does not exist another solution φ such that $\mathrm{dom}\phi$ is a proper subset of $\mathrm{dom}\varphi$ and $\phi(t, j) = \varphi(t, j)$ for all $(t, j) \in \mathrm{dom}\phi$.

The existence of nontrivial solutions to hybrid systems can be characterized as follows

Proposition C.1. *Consider the hybrid system $\mathcal{H} = (C, F, D, G)$. Let $\zeta \in \bar{C} \cup D$. If $\zeta \in D$ or there exists $\varepsilon > 0$ and absolutely continuous funtion $z : [0, \varepsilon] \to \mathbb{R}^n$ such that $z(0) = \zeta$*

and $\dot{z}(t) \in F(z(t))$ for almost all $t \in [0, \varepsilon]$ and $z(t) \in C$ for all $t \in (0, \varepsilon]$, then there exists a nontrivial solution ϕ for $\mathcal{H}$ with $\phi(0,0) = \zeta$. Furthermore, if z exists for all $\zeta \in \bar{C} \setminus D$, then there exist a nontrivial solution from every point of $\bar{C} \cup D$ and every solution ϕ satisfies one of these conditions:

- *ϕ is complete*
- $\mathrm{dom}\phi$ *is bounded and the interval I^J (with $J = \sup_j \mathrm{dom}\phi$) has nonempty interior and is open on the right. Also, there is no absolutely continuous function $z : [a, b] \to \mathbb{R}^n$ fulfilling $\dot{z}(t) \in F(z(t))$ for almost all $tin[a, b]$, $z(t) \in C$ for all $tin(a, b)$ and such that $I^J \subset [a, b)$ and $z(t) = \phi(t, J)$ for all $t \in I^J$*
- $\mathrm{dom}\phi$ *is bounded and $\phi(T, J) \notin \bar{C} \cup D$, where $(T, J) = \sup \mathrm{dom}\phi$*

Proof. See Goebel et al. (2008). □

The existence of unique solution is instead characterized in the following proposition.

Proposition C.2. *Consider the hybrid system $\mathcal{H} = (C, F, D, G)$. For every $\zeta \in \bar{C} \cup D$ there exists a unique maximal solution ϕ with $\phi(0,0) = \zeta$ provided that the following holds:*

- *for every $\zeta \in \bar{C} \setminus D$, $T > 0$, if two absolutely continuous functions $z_1, z_2 : [0, T] \to \mathbb{R}^n$ are such that $\dot{z}_i \in F(z_i(t))$ for almost all $t \in [0, T]$, $z_i(t) \in C$ for all $t \in (0, T]$ and $z_i(0) = \zeta$, then necessarily $z_1(t) = z_2(t)$ for all $t \in [0, T]$*
- *for every $\zeta \in \bar{C} \cap D$, there does not exist an $\varepsilon > 0$ and an absolutely continuous function $z : [0, \varepsilon] \to \mathbb{R}^n$ such that $z(0) = \zeta$, $\dot{z}(t) \in F(z(t))$ for almost all $t \in [0, \varepsilon]$ and $z(t) \in C$ for all $t \in (0, \varepsilon]$*
- *for every $\zeta \in \setminus D$, $G(\zeta)$ consists of one point*

The condition stated above respectively means that from no point there exist two flowing solutions, or a flowing solution and a jumping solution, or two jumping solutions.

C.2 Switching signals and hybrid systems

Switching systems are common in engineering and in particular networks application. A switching system can be represented as

$$\dot{z} = f_{\sigma}(z) \tag{C.5}$$

with σ taking values in $Q = \{1, 2, \ldots, q_{\max}\}$ and for every $\sigma \in Q$, $f_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Typically, σ is a piecewise constant function and *selects* the flow map governing $z(t)$.

In many cases, the *switching signal* σ is not a generic piecewise constant signal. The most common *switching signal* in networks applications are particular type of signal:

- σ is a *dwell-time signal* and the solution is a dwell-time solution with dwell time $\tau_D > 0$ it $t_{i+1} - t_i \geq \tau_D$ for all $i = 1, 2, \ldots$, that is jumps are separated by at least τ_D
- σ is a*persistent dwell-time signal* with persistent dwell time $\tau_D > 0$ and period of persistence $T > 0$ if there exists a subsequence $0 = t_{i_0}, t_{i_1}, \ldots$ of a sequence $\{t_i\}$ such that $t_{i_{k+1}} - t_{i_k} \geq \tau_D$ for $k = 1, 2, \ldots$ and $t_{i_{k+1}} - t_{i_k} \leq T$. That is, between two consecutive intervals of length at least τ_D passe at most T amount of time
- σ is a *weak dwell-time signal* with dwell time $\tau_D > 0$ if there exists a subsequence $0 = t_{i_0}, t_{i_1}, \ldots$ of a sequence $\{t_i\}$ such that $t_{i_{k+1}} - t_{i_k} \geq \tau_D$ for $k = 1, 2, \ldots$, that is there are infinitely many intervals of length τ_D with no switching
- σ is an *average dwell-time signal* with dwell-time $\tau_D > 0$ and offset $N_0 \in \mathbb{N}$ if, for all $0 \leq s \leq t$, the number of jumps $N(\cdot, \cdot)$ in the interval $[s, t]$ satisfies

$$N(t, s) \leq \frac{1}{\tau_D}(t - s) + N_0$$

C.3 Uniform asymptotic stability and Lyaounov theory for hybrid systems

For hybrid systems, when referring to stability properties we in general speak of *uniform global pre-asymptotic stability* of a closed set. Simply speaking we ask that the distance of every possible solution of the hybrid system with respect to the set can be bounded by a function depending on the *initial distance* to the set and the elapsed time. The term *pre* allows maximal solution not to be complete.

We formulate this concept in the following definition

Definition C.4. *Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and a closed set $\mathcal{X} \subset \mathbb{R}^n$. This set is said to be:*

- uniformly globally stable *if there exists a call-$\mathcal{K}_\infty$ function α such that any solution ϕ fulfills $|\phi(t,j)|_{\mathcal{X}} \leq \alpha(|\phi(0,0)|_{\mathcal{X}}$ for all $(t,j) \in \mathrm{dom}\phi$*
- uniformly globally pre-attractive *if for each $\varepsilon > 0$ and $r > 0$ there exists a $T > 0$ such that, for any solution $|\phi(0,0)|_{\mathcal{X}} \leq r$, $(t,j) \in \mathrm{dom}\phi$ and $t + j \geq T$ imply $|\phi(0,0)|_{\mathcal{X}} \leq \varepsilon$*

- uniformly globally pre-asymptotically stable *if it is both uniformly globally stable and pre-attractive*

As for nonlinear systems, Lypunov functions are a powerful tool to analyze stability properties for hybrid systems. Indeed, by means of Lyapunov functions it is possible to establish global pre-asymptotic stability of a set.

Definition C.5. *A function* $V : \mathrm{dom}V \to \mathbb{R}$ *is a Lyapunov candidate for the hybrid system* $\mathcal{H} = (C, F, G, D)$ *if:*

- $\bar{C} \cup D \cup G(D) \subset \mathrm{dom}V$

- V *is continuously differentiable on an open set containing* $\bar{C}$

Then, the following theorem gives the conditions under which the candidate Lyapunov functions guarantees uniform global pre-asymptotic stability.

Theorem C.1. *Let* $\mathcal{H} = (C, F, G, D)$ *be a hybrid system and* $\mathcal{X} \subset \mathbb{R}^n$ *a closed set. If* V *is a candidate Lyapunov function and there exist* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *and a continuous positive definite function* ρ *such that*

$$\begin{aligned} \alpha_1(|x|_{\mathcal{X}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{X}}) \qquad & \forall\, x \in C \cup D \cup G(D) \\ \langle \bigtriangledown V(x), f \rangle \leq -\rho(|x|_{\mathcal{X}}) \qquad & \forall\, x \in C, f \in F(x) \\ V(g) - V(x) \leq -\rho(|x|_{\mathcal{X}}) \qquad & \forall\, x \in D, g \in G(D) \end{aligned} \tag{C.6}$$

then $\mathcal{X}$ *is uniformly globally pre-asymptotically stable for* $\mathcal{H}$.

Previous theorem asks the Lypunov function to decrease both during flows and jumps. Of course conditions (C.6) can be weakened, for instance by in the case in which flows are nonincreasing during flows, strictly decreasing during jumps and jumps occur sufficiently often. These *weakened* Lyapunov condition are shown in the following theorems (the proof can be found in Goebel et al. (2008)).

Theorem C.2. (Persistent Jumping) *Let* $\mathcal{H} = (C, F, G, D)$ *be a hybrid system and* $\mathcal{X} \subset \mathbb{R}^n$ *a closed set. If* V *is a candidate Lyapunov function and there exist* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *and a continuous positive definite function* ρ *such that*

$$\begin{aligned} \alpha_1(|x|_{\mathcal{X}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{X}}) \qquad & \forall\, x \in C \cup D \cup G(D) \\ \langle \bigtriangledown V(x), f \rangle \leq 0 \qquad & \forall\, x \in C, f \in F(x) \\ V(g) - V(x) \leq -\rho(|x|_{\mathcal{X}}) \qquad & \forall\, x \in D, g \in G(D) \end{aligned} \tag{C.7}$$

If, for each $r > 0$ there exists a T, $\gamma_r \in \mathcal{K}_\infty$ and $N_r \geq 0$ such that for every solution ϕ is such that $|\phi(0,0)|_\mathcal{X} \in (0,r]$, $(t,j) \in \mathrm{dom}\phi$ and $t + j \geq T$ imply $j \geq \gamma_r(T) - N_r$, then $\mathcal{X}$ is uniformly globally pre-asymptotically stable.

Theorem C.3. (Persistent Flowing) *Let $\mathcal{H} = (C, F, G, D)$ be a hybrid system and $\mathcal{X} \subset \mathbb{R}^n$ a closed set. If V is a candidate Lyapunov function and there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a continuous positive definite function ρ such that*

$$\begin{aligned} \alpha_1(|x|_\mathcal{X}) \leq V(x) \leq \alpha_2(|x|_\mathcal{X}) \qquad & \forall\, x \in C \cup D \cup G(D) \\ \langle \nabla V(x), f \rangle \leq -\rho(|x|_\mathcal{X}) \qquad & \forall\, x \in C, f \in F(x) \\ V(g) - V(x) \leq 0 \qquad & \forall\, x \in D, g \in G(D) \end{aligned} \tag{C.8}$$

If, for each $r > 0$ there exists a T, $\gamma_r \in \mathcal{K}_\infty$ and $N_r \geq 0$ such that for every solution ϕ is such that $|\phi(0,0)|_\mathcal{X} \in (0,r]$, $(t,j) \in \mathrm{dom}\phi$ and $t + j \geq T$ imply $t \geq \gamma_r(T) - N_r$, then $\mathcal{X}$ is uniformly globally pre-asymptotically stable.

The following theorem, which is the clear base for Proposition 5.3 and Proposition 6.1, considers the case in which the system might increase during flows or jump: these increments are though balanced by decrements during jumps or flows respectively.

Theorem C.4. (Increase balanced by decrease) *Let $\mathcal{H} = (C, F, G, D)$ be a hybrid system and $\mathcal{X} \subset \mathbb{R}^n$ a closed set. If V is a candidate Lyapunov function and there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a continuous positive definite function ρ such that*

$$\begin{aligned} \alpha_1(|x|_\mathcal{X}) \leq V(x) \leq \alpha_2(|x|_\mathcal{X}) \qquad & \forall\, x \in C \cup D \cup G(D) \\ \langle \nabla V(x), f \rangle \leq \lambda_c V(x) \qquad & \forall\, x \in C, f \in F(x) \\ V(g) - V(x) \leq e^{\lambda_d} V(x) \qquad & \forall\, x \in D, g \in G(D) \end{aligned} \tag{C.9}$$

If, there exists $\gamma > 0$ and $M > 0$ such that, for each solution ϕ, $(t,j) \in \mathrm{dom}\phi$ implies

$$\lambda_c t + \lambda_d \leq M - \gamma(t + j)$$

then $\mathcal{X}$ is uniformly globally pre-asymptotically stable.

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