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OBSERVERS AND ROBUST OUTPUT REGULATION FOR NONLINEAR SYSTEMS

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"Do not worry about your difficulties in Mathematics. I can assure you mine are still greater."

Albert Einstein

Abstract

Observers and output regulation are two central topics in nonlinear control system theory. Although many researchers have devoted their attention to these issues for more than 30 years, there are still many open questions.

An observer is a dynamical system which estimates the state of a given system using the knowledge of the trajectory of its output. In the observer theory a key role is played by the high-gain observers. These observers may be used when the system dynamics can be expressed in specific coordinated under the so-called observability canonical form. These observers have also the property that their rate of convergence can be arbitrarily increased by acting on one parameter, called high-gain parameter. Despite the evident benefits of this class of observers, their use in real applications is questionable due to some drawbacks: difficulty in expressing the canonical observability form, numerical problems, the peaking phenomenon and sensitivity to measurement noise. The purpose of the first part of the thesis is to enrich the theory of high-gain observers with novel techniques to overcome or at least to mitigate the aforementioned problems. On one hand, we study the possibility of writing an observer for multi-input multi-output observable systems in the original coordinates when the observability form is not known. On the other hand, we propose a novel class of high-gain observers, denoted as "low-power", which allows to overcome numerical problems, to avoid the peaking phenomenon and to improve the sensitivity properties to high-frequency measurement noise.

The second part of the thesis addresses the output regulation problem, namely the scenario in which we want to regulate to zero a given output while keeping all the trajectories of the system bounded even if the system is affected by external signals assumed to belong to a known class of systems and representing a reference to be tracked and/or unknown disturbances. This problem has been solved for linear systems during the 70's by

Francis and Wonham who coined the celebrated “internal model principle”. Constructive solutions have also been proposed in the nonlinear framework but under restrictive assumptions that reduce the class of systems to which this methodology can be applied. In this thesis we focus on the output regulation problem in presence of periodic disturbances and we propose a novel approach which allows to consider a broader class of nonlinear systems. With the proposed design the stabilization problem and the regulation problem are substantially decoupled. Taking advantage of the property that a nonlinear system input-to-state stable, subject to periodic inputs, admits periodic solutions of the same period, we propose a linear internal model embedding linear oscillators for each harmonic frequency of the periodic phenomenon. Forwarding technique is used to stabilize the cascade system. The behavior analysis is based on the Fourier analysis. The resulting behavior is robust in the sense defined by Francis and Wonham, namely output regulation is achieved in presence of uncertainties or disturbances, as long as the trajectories of the resulting closed-loop system are bounded.

Sommario

Gli osservatori e la regolazione dell'uscita sono due temi centrali nella teoria dei sistemi nonlineari. Nonostante molti ricercatori abbiano dedicato la loro attenzione a queste problematiche da più di 30 anni, ci sono ancora molti problemi aperti.

Un osservatore è un sistema dinamico che stima lo stato di un sistema utilizzando la conoscenza della traiettoria dalla sua uscita. Nella teoria degli osservatori un ruolo chiave è giocato dagli osservatori ad alto guadagno. Questi osservatori possono essere utilizzati nel caso in cui la dinamica del sistema può essere espressa in coordinate specifiche, sotto la cosiddetta forma canonica di osservabilità. Questi osservatori hanno anche la proprietà che la loro velocità di convergenza può essere arbitrariamente aumentata agendo su un singolo parametro, chiamato parametro di alto guadagno. Nonostante gli evidenti benefici di questa classe di osservatori, il loro utilizzo in applicazioni pratiche è discutibile a causa di alcuni inconvenienti: difficoltà nell'espressione della forma canonica di osservabilità, problemi numerici, fenomeno del peaking e sensitività al rumore di misura. Lo scopo di questa prima parte della tesi è di arricchire la teoria degli osservatori ad alto guadagno con nuove tecniche che permettono di superare o per lo meno di mitigare i suddetti problemi. Da una parte, studiamo la possibilità di scrivere un osservatore per sistemi osservabili a ingressi multipli e a uscite multiple nelle coordinate originali quando la forma di osservabilità non è conosciuta. Dall'altra, proponiamo una nuova classe di osservatori ad alto guadagno, chiamati "low-power", che permette di superare i problemi numerici, evitare il fenomeno di peaking e migliorare le proprietà di sensitività al rumore di misura ad alta frequenza.

La seconda parte della tesi affronta il problema della regolazione dell'uscita, ovvero lo scenario in cui vogliamo regolare a zero una determinata uscita e allo stesso tempo mantenere le traiettorie del sistema limitate anche se il sistema è influenzato da segnali esogeni che si suppone appartengano a una classe conosciuta e che rappresentano un riferimento da seguire e/o dei disturbi sconosciuti. Questo problema è stato risolto per sistemi lineari durante gli anni 70 da Francis e Wonham i quali hanno formulato il celebre "principio del modello interno". Soluzioni costruttive sono state proposte nel contesto nonlineare ma sotto ipotesi restrittive che riducono la classe di sistemi a quali si può applicare questa metodologia. In questa tesi ci concentriamo sul problema di regolazione dell'uscita in presenza di disturbi periodici, e proponiamo un nuovo approccio che permette di considerare una classe più ampia di sistemi nonlineari. Con la sintesi proposta il problema di stabilizzazione e di regolazione sono sostanzialmente disaccoppiati. Traendo beneficio dalla proprietà che un sistema nonlineare stabile ingresso-uscita (input-to-state stable in inglese), alimentato da un ingresso periodico, ammette soluzioni periodiche dello stesso periodo, proponiamo un modello interno lineare che contiene oscillatori lineari per ogni frequenza armonica del fenomeno periodico. La tecnica di forwarding è utilizzata per stabilizzare la cascata. L'analisi del comportamento si base sull'analisi di Fourier. Il comportamento ottenuto è robusto nel senso definito da Francis e Wonham, ovvero la regolazione dell'uscita è mantenuta in presenza di incertezze o disturbi, fintanto che le traiettorie del sistema chiuso in retroazione sono limitate.

Résumé

Les observateurs et la régulation de sortie sont deux thèmes centraux de la théorie des système non linéaires. Bien que de nombreux chercheurs ont consacré leur attention à ces questions depuis plus de trente ans, il y a encore de nombreuses questions ouvertes.

Un observateur est un système dynamique qui estime l'état d'un système en utilisant la connaissance de la trajectoire de sa sortie. Dans la théorie des observateurs un rôle clé est joué par les observateurs à grand gain. Ces observateurs peuvent être utilisés lorsque la dynamique du système peut être exprimée dans des coordonnées spécifiques sous une forme canonique dite d'observabilité. Ils ont aussi la propriété que leur vitesse de convergence peut être arbitrairement augmentée en agissant sur un seul paramètre, appelé paramètre de grand gain. Malgré les avantages évidents de cette classe d'observateurs, leur utilisation dans des applications pratiques est douteuse en raison de certains inconvénients: difficultés dans l'expression de la forme canonique d'observabilité, des problèmes numériques, le phénomène de peaking et la sensibilité au bruit de mesure. Le but de la première partie de la thèse est d'enrichir la théorie des observateurs à grand gain avec de nouvelles techniques qui permettent de surmonter ou au moins d'atténuer les problèmes ci-dessus. D'une part, nous étudions la possibilité d'écrire un observateur pour les systèmes observables avec plusieurs entrées et plusieurs sorties dans les coordonnées d'origine lorsque la forme d'observabilité n'est pas connue. D'autre part, nous proposons une nouvelle classe d'observateurs à grand gain, appelé "low-power", qui permet de surmonter les problèmes numériques, d'éviter le phénomène de peaking et d'améliorer les propriétés de sensibilité aux bruit de mesure à haute fréquence.

La deuxième partie de la thèse aborde du problème de la régulation de sortie, c'est-à-dire le scénario dans lequel nous voulons imposer une sortie donnée de rester nulle et en même temps maintenir les trajectoires de système limitées même si le système est

influencé par des signaux externes supposées appartenir à une classe connue et représentant une références à suivre et / ou des perturbations inconnues. Ce problème a été résolu pour les systèmes linéaires au cours des années 70, par Francis et Wonham qui ont énoncé le célèbre “principe de modèle interne”. Des solutions constructives ont aussi été proposées dans le cadre non linéaire mais sous des hypothèses restrictives qui réduisent la classe des systèmes auxquels cette méthodologie peut être appliquée. Dans la thèse, nous nous concentrons sur le problème de la régulation de sortie en présence de perturbations périodiques, et nous proposons une nouvelle approche qui nous permet de considérer une classe plus large de systèmes non linéaires. Dans la synthèse proposée, le problème de stabilisation et de régulation sont sensiblement découplés. Tirant profit de la propriété qu'un système non linéaire stable entrée-état (input-to-state stable en anglais), alimenté par une entrée périodique, admet des solutions périodiques de même période, nous proposons un modèle interne linéaire qui contient des oscillateurs linéaires pour chacune des fréquence harmoniques du phénomène périodique. La technique d'ajout d'intégrateur (forwarding en anglais) est utilisée pour obtenir un bouclage stabilisant pour la cascade. L'analyse du comportement repose alors sur l'analyse de Fourier. Le comportement obtenu est robuste au sens défini par Francis et Wonham, c'est-à-dire que la régulation de sortie est maintenue en présence d'incertitudes ou de bruit, aussi longtemps que les trajectoires du système en boucle fermée sont limitées.

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Contents

Notation

$\mathbb{R}$	set of real numbers
$\mathbb{R}_{\geq 0}$	set of non negative real numbers
$\mathbb{R}_{>0}$	set of real numbers larger than zero
$\mathbb{N}$	set of non negative integers
$\mathbb{N}_{>0}$	set of integers larger than zero
$\mathbb{C}$	set of complex real numbers
ℓ^2	space of square-summable sequences
$\in$	belongs to
$\subset$	subset
$\supset$	superset
$:=$	defined as
$\mapsto$	maps to
$\square$	end of proof
$\lvert x \rvert$	Euclidean norm of x, with $x \in \mathbb{R}^n$
$\lvert x \rvert_{\mathcal{A}}$	$\inf_{y \in \mathcal{A}} \lvert x - y \rvert$, distance of x from $\mathcal{A}$, with $x \in \mathbb{R}^n$ and $\mathcal{A}$ subset of $\mathbb{R}^n$
$\lVert s(\cdot) \rVert_a$	$\limsup_{t \to \infty} \lvert s(t) \rvert$, asymptotic norm of $s \in \mathbb{R}^n$, with $t \mapsto s(t)$
$\lVert s(\cdot) \rVert_\infty$	$\sup_{t \in [0,\infty]} \lvert s(t) \rvert$, L^∞ norm of $s \in \mathbb{R}^n$, with $t \mapsto s(t)$
$\lVert s(\cdot) \rVert_2$	$\sqrt{\frac{1}{T} \int_0^T \lvert s(t) \rvert^2}$, L^2 norm of $s \in \mathbb{R}^n$, with $t \mapsto s(t)$ and $s(t+T) = s(t)$
$\lvert A \rvert$	$\sup_{x \in \mathbb{R}^n} \left\{ \frac{\lvert Ax \rvert}{\lvert x \rvert} \right\}$, induced matrix norm

$A^\top$	transpose
A^{-1}	inverse
$A > 0$	positive definite matrix
$A \geq 0$	positive semi-definite matrix
$\det(A)$	determinant
rank(A)	rank
$\lambda(A)$	eigenvalue of A
$\sigma(A)$	spectrum of A, the set of its eigenvalues
$0_{n\times m}$	matrix of dimension $n \times m$ whose entries are all zeros
I_n	an $n\times n$ identity matrix, also denoted with I when there is no need to emphasize the dimension
diag $(a_1, \ldots, a_n)$	an $n \times n$ diagonal matrix with a_i as its i-th diagonal element
col $(a_1, \ldots, a_n)$	column vector with elements $(a_1, \ldots, a_n)$
Hurwitz	matrix with all eigenvalues with strictly negative real part

prime form — a triplet (A, B, C) of dimension n defined as

$$\begin{aligned} A &= \begin{pmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ 0 & 0_{1\times(n-1)} \end{pmatrix} \qquad B = \begin{pmatrix} 0_{(n-1)\times 1} \\ 1 \end{pmatrix} \\ C &= \begin{pmatrix} 1 & 0_{1\times(n-1)} \end{pmatrix} \end{aligned}$$

$L_f h(x)$ — Lie derivative of $h(x)$ along the vector field $f(x)$

$$L_f h(x) \;:=\; \frac{\partial h}{\partial x}(x) f(x)$$

sat$_L(\cdot)$ — saturation function: any bounded continuous function satisfying

$$\text{sat}_L(x) \;=\; x \quad \forall |x| \leq L\,, \qquad |\text{sat}_L(x)| \;\leq\; L \quad \forall x \in \mathbb{R}^n$$

feedback form — a (nonlinear) system possessing a triangular structure:

- *non-strict feedback form*

$$\begin{aligned} \dot{x}_i &= f_i(x_1, \ldots, x_i, x_{i+1}, u) \qquad i = 1, \ldots, n-1 \\ \dot{x}_n &= f_n(x_1, \ldots, x_n, u) \\ y &= h(x_1, u) \end{aligned}$$

- *strict feedback form*

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \varphi_i(x_1, \ldots, x_i, u) \qquad i = 1, \ldots, n-1 \\ \dot{x}_n &= \varphi_n(x_1, \ldots, x_n, u) \\ y &= x_1 + \nu(t) \end{aligned}$$

with state $x \in \mathbb{R}^n$, input $u \in U$, output $y \in \mathbb{R}$

Introduction

Over the last decades, observers and internal model-based regulators for output regulation have been two of the most interesting and investigated topic in the nonlinear systems community. Although a considerable number of results can be found in the literature, the conventional theoretical approaches presents several drawbacks and are often very hard to be implemented. The purpose of this thesis is to provide novel techniques for the analysis and design of observers and internal model regulators, with a particular attention on a easy-to-implement structure.

In the first part we study the class of nonlinear observers denoted as *high-gain observers*. In Chapter 1 the reader can find an overview of the theory of high-gain observers for single-input single-output nonlinear systems where the main features and the main drawbacks are highlighted. In Chapter 2, we propose some novel tools of analysis and design which enrich the theory of high-gain observers. Among these, we pay particular attention to the design of high-gain observers in the original coordinates and we give a novel set of sufficient conditions for the existence of a high-gain observer in the multi-input multi-output case. Then, in Chapter 3, we introduce a novel class of high-gain observers, denoted as "low-power". This novel methodology, based on dynamic extension, helps in overcoming (or improving) some of the main drawbacks of the standard high-gain observers. The low-power high-gain observers may be used in place of standard high-gain observers without loss of generality. An example is given at the end of the chapter in order to show the performances of the low-power high-gain observer.

The second part of the thesis is devoted to the problem of robust output regulation for nonlinear systems. In Chapter 4 we recall the basic ingredients of output regulation for linear system and we introduce a design procedure based on forwarding technique. The proposed approach is instrumental to nonlinear case. Indeed, we show that with the same design-methodology it is possible to solve the practical output regulation problem for multi-input multi-output nonlinear systems which are affine in the input in presence of (small) periodic disturbances. The proposed control law is robust to model uncertainties and the norm of the output can be made arbitrarily small (in a L^2 sense) by enlarging the dimension of the dynamic regulator. In Chapter 5, the same approach is applied to minimum-phase single-input single-output nonlinear systems. We show that with a (non-implementable) regulator of infinite dimension it is possible to solve the structurally robust asymptotic output regulation problem. An example is given at the end of the chapter in order to show the performances of the proposed regulator.

Part I

Observer Design

"Accuracy of observation is the equivalent of accuracy of thinking."

Wallace Stevens

1

Highlights of High-Gain Observers

The problem of designing asymptotic state observers for nonlinear systems is a central topic in the control literature (see Besançon (2007) or Gauthier and Kupka (2004) for general surveys on the topic). A special role is played by the so-called high-gain observers in which the error trajectory has an exponential decay rate that can be imposed arbitrarily fast by acting on a design parameter, appearing in the observer structure, typically known as "high-gain parameter" (see for instance the survey Khalil and Praly (2014) and references therein). Such observers are routinely used in control contexts where fast observation is useful, such as contexts of nonlinear output feedback stabilization by means of the nonlinear separation principle in which fast observation is required in order to prevent finite escape time of the closed-loop system (see, for instance, Teel and Praly (1994) and Atassi and Khalil (1999)). After two seminal works appeared in 1992 (Esfandiari and Khalil (1992) and Gauthier et al. (1992)), the investigation of high-gain observer in nonlinear theory attracted the attention of many researchers and a huge number of papers have been published on the topic (we refer to Besançon (2007) and Khalil and Praly (2014) and references therein). High-gain observers are successfully applied in problems of estimation (see Besançon (2007) or Gauthier and Kupka (1994)), output feedback control (see, among the others, Tornambé (1992a), Teel and Praly (1994), Atassi and Khalil (1999) and Shim and Seo (2000)) and output regulation (see Byrnes and Isidori (2004) or Seshagiri and Khalil (2005)) although their use in practical applications is made hard by a certain number of drawbacks.

In this chapter we make an overview of the state of art of high-gain observers, highlighting its main features, drawbacks and its application to the output regulation framework. Section 1.1 is devoted to emphasize some important notions about the *observability canonical forms* and it is motivated by the fact that high-gain observers can be applied to systems having a particular triangular structure, denoted as *feedback form*. In Sections 1.2 and 1.4 we illustrate the design of high-gain observers for systems in strict feedback form and non strict feedback form. Section 1.3 is devoted to highlight the main drawbacks of this class of nonlinear observers, whereas in Section 1.5 we discuss about the possibility of a design in the original coordinates. Finally, in Section 1.6 we show how the high-gain observer theory can be applied to the framework of nonlinear output regulation. The output feedback stabilization problem by means of high-gain observers is not topic of this work and we refer, among the others, to Teel and Praly (1994), Atassi and Khalil (1999), (Isidori, 1995, Chapter 9.6), Khalil and Praly (2014) and references therein.

The contents of this chapter are a reformulation of the contribution of many authors and contain no novelty. We refer in particular to Gauthier and Kupka (2004), Isidori (1995) and Marconi et al. (2004) for Section 1.1, Khalil and Praly (2014) for Sections 1.2 and 1.3, Gauthier and Kupka (2004) for Section 1.4, Deza et al. (1992) for Section 1.5, Byrnes and Isidori (2004) for Section 1.6.

1.1 Observabilility canonical forms

This section is devoted to present an overview of sufficient conditions for the existence of *observability canonical forms* for nonlinear systems. These observability forms can be seen as a special cases of a *feedback form*. As a matter of fact, when the aforementioned sufficient conditions are verified, the nonlinear system is diffeomorphic to a system for which we know how to design an observer (see the forthcoming Section 1.2). For the sake of simplicity we consider the class of single-input single-output nonlinear systems. The results presented herein cannot be extended to the multi-output multi-input case in a trivial way. It is not the purpose of this section to give a complete picture of the notion of observability in the nonlinear framework, and only a short summary is presented (see books devoted to the subject as Gauthier and Kupka (2004) or Besançon (2007) for more details). The contents of this section are a reformulation of the results given in Gauthier and Kupka (2004), Isidori (1995) and Marconi et al. (2004).

1.1.1 Canonical flag and observability canonical forms

Consider a nonlinear system of the form

$$\dot{x} \;=\; f(x,u)\,, \qquad\qquad y \;=\; h(x,u)\,, \tag{1.1}$$

where the state $x \in \mathbb{R}^n$, the input $u \in \mathbb{R}$ and the output $y \in \mathbb{R}$. The functions f, g, h are considered smooth enough. Let us define - recursively - a sequence of functions φ_i, $i = 1, \ldots, n$, as follows

$$\begin{aligned} \varphi_1(x,u) &:= h(x,u)\,, \\ \varphi_i(x,u) &:= \frac{\partial \varphi_{i-1}}{\partial x}(x,u) f(x,u)\,, \end{aligned} \tag{1.2}$$

and a sequence of i-vector-valued functions $\Phi_i(x,u)$ as

$$\Phi_i(x,u) := \begin{pmatrix} \varphi_1(x,u) \\ \vdots \\ \varphi_i(x,u) \end{pmatrix}, \qquad \forall\, i = 1, \ldots, n\,. \tag{1.3}$$

Definition 1.1. (Gauthier and Kupka, 2004, Definition 2.1, Chapter 2) *The canonical flag of* (1.1) *is a family of n distribution in $\mathbb{R}^n$, parametrized by u, defined by*

$$D_i(u) \,:\, x \;\mapsto\; \ker\left[\frac{\partial \Phi_i}{\partial x}\right]_{(x,u)}, \qquad\qquad \forall\, i = 1, \ldots, n\,,$$

with the functions Φ_i *defined by* (1.3). *The canonical flag is said to be uniform on* $x \in \mathbb{R}^n$ *if*

(i) all the $D_i(u)$, $i = 1, \ldots, n$, *have constant dimension* $n - i$ *for all* $u \in \mathbb{R}$ *and* $x \in \mathbb{R}^n$ *("regularity" condition);*

(ii) all the $D_i(u)$, $i = 1, \ldots, n$, *are independent of* u *("u-independence" condition), i.e.* $\partial_u D(u) = 0$.

The canonical flag defines a geometric property of the observation space. If the canonical flag is uniform, then the observation space is independent of the control input and it is possible to find a (local) change of coordinates such that the system (1.1) can be written in the so-called *observability canonical form*.

Proposition 1.1 (Gauthier and Kupka (2004), Theorem 2.1, Chapter 3)**.** *The system* (1.1) *has a uniform canonical flag if and only if, for all* $x^\circ \in \mathbb{R}^n$ *there exists a coordinate neighbourhood* $V_x \subset \mathbb{R}^n$ *of* x°, *such that in these coordinates,* $(V_z, z_1, \ldots, z_n)$, *the system* (1.1) *can be written as follows*

$$\begin{array}{rcll} \dot{z}_i &=& f_i(z_1, \ldots, z_i, z_{i+1}, u)\,, & i = 1, \ldots, n-1\,, \\ \dot{z}_n &=& f_n(z_1, \ldots, z_n, u)\,, & \\ y &=& h(z_1, u)\,, & \end{array} \tag{1.4}$$

with the functions f_i, $i = 1, \ldots, n-1$ *and* h *that fulfil*

$$\frac{\partial h}{\partial z_1}(z_1, u) \neq 0\,, \qquad \frac{\partial f_i}{\partial z_{i+1}}(z_1, \ldots, z_i, z_{i+1}, u) \neq 0\,, \quad i = 1, \ldots, n-1\,,$$

for any $(z, u) \in V_z \times \mathbb{R}$.

Notice that the system (1.4) is in *non-strict feedback form*. If we want to achieve stronger results, namely a global change of coordinates, extra assumptions are needed. For this, we define the mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\Phi(x) \;:=\; \Phi_n(x, 0) \;=\; \begin{pmatrix} h(x,0) \\ L_{f(x,0)} h(x,0) \\ \vdots \\ L^{n-1}_{f(x,0)} h(x,0) \end{pmatrix} . \tag{1.5}$$

Theorem 1.1 (Marconi et al. (2004), Lemma 2)**.** *Consider the system* (1.1) *and suppose that its canonical flag is uniform (according to Definition 1.1) and the mapping* $\Phi(\cdot)$ *defined in* (1.5) *is a global diffeomorphism. Then the system* (1.1) *is globally diffeomorphic, via* Φ, *to a* *system of the form* (1.4) *with the functions* f_i, $i = 1, \ldots, n-1$ *and* h *that fulfil*

$$\frac{\partial h}{\partial z_1}(z_1, u) \neq 0 \,, \qquad \frac{\partial f_i}{\partial z_{i+1}}(z_1, \ldots, z_i, z_{i+1}, u) \neq 0 \,, \quad i = 1, \ldots, n-1 \,,$$

for any $(z, u) \in \mathbb{R}^n \times \mathbb{R}$.

Consider now an input-affine single-input single-output nonlinear system of the form

$$\begin{array}{rcl} \dot{x} &=& f(x) + g(x)\, u \\ y &=& h(x) \end{array} \tag{1.6}$$

where the state $x \in \mathbb{R}^n$, the input $u \in \mathbb{R}$ and the output $y \in \mathbb{R}$. Let the functions f, g, h be smooth enough and let the functions $\varphi_i(x, u)$ in (1.2), Φ_i in (1.3) and Φ in (1.5) be defined similarly for the system (1.6). For this class of systems the result of Theorem 1.1 can be further specialized, thus obtaining a triangular structure with a well-defined linear part (and therefore in *strict feedback form*).

Theorem 1.2 (Gauthier and Kupka (2004), Theorem 4.1, Chapter 3)**.** *Consider the system* (1.6) *and suppose that its canonical flag is uniform (according to Definition 1.1) and the mapping* $\Phi(\cdot)$ *defined in* (1.5) *is a global diffeomorphism. Then the system* (1.6) *is globally diffeomorphic, via* Φ, *to a system of the form*

$$\begin{array}{rcl} \dot{z} &=& \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_{n-1} \\ \dot{z}_n \end{pmatrix} = \begin{pmatrix} z_2 \\ z_3 \\ \vdots \\ z_n \\ a(z) \end{pmatrix} + \begin{pmatrix} b_1(z_1) \\ b_2(z_1, z_2) \\ \vdots \\ b_{n-1}(z_1, \ldots, z_{n-1}) \\ b_n(z_1, \ldots, z_n) \end{pmatrix} u \,, \\ y &=& z_1 \,. \end{array} \tag{1.7}$$

1.1.2 Phase-variable representation

Consider again the class of nonlinear systems (1.1) and let us define - recursively - a new sequence of functions[1] φ_i, $i = 1, \ldots, n$, as follows

$$\begin{array}{rcl} \varphi_1(x, u_0) &=& h(x, u_0) \,, \\ \varphi_i(x, u_0, \ldots, u_{i-1}) &=& \dfrac{\partial \varphi_{i-1}}{\partial x} f(x, u_0) + \displaystyle\sum_{k=0}^{i-2} \frac{\partial \varphi_{i-1}}{\partial u_k} u_{k+1} \,. \end{array} \tag{1.8}$$

Note that as opposed to the functions φ_i defined in Section 1.1.1, here we are also differentiating in u. It is immediate to realize that these mappings, if the input $u(t)$ of (1.1) is

[1] with abuse of notation with respect to (1.2)

a C^i function of t, are precisely the mappings which express, for each i and at any given time t, the dependence of the i-th derivative $y^{(i)}(t)$ of the output $y(t)$ on the state $x(t)$ and the input $u(t)$ and its derivative $\dot{u}(t), \ldots, u^{(i)}(t)$. Indeed

$$y^{(i)}(t) = \varphi_{i+1}\left(x(t), u(t), \ldots, u^{(i)}(t)\right) .$$

Note that the functions $\varphi_i(\cdot)$ defined above coincide with the functions $\varphi_i(\cdot)$ defined in (1.2) when u is a constant function. As before, let us also define a sequence of i-vector-valued functions $\Phi_i(x, u_0, \ldots, u_{i-1})$[2] as follows

$$\Phi_i(x, u_0, \ldots, u_{i-1}) = \begin{pmatrix} \varphi_1(x, u_0) \\ \vdots \\ \varphi_i(x, u_0, \ldots, u_{i-1}) \end{pmatrix} . \tag{1.9}$$

The functions $\Phi_i(\cdot)$ defined above coincide with the functions $\Phi_i(\cdot)$ defined in (1.3) when u is constant.

Theorem 1.3. *Consider the system* (1.1) *and suppose the mapping* $\overline{\Phi} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ *defined as*

$$\overline{\Phi}(x, u_0, \ldots, u_{n-1}) \;=\; \begin{pmatrix} u_0 \\ \vdots \\ u_{n-1} \\ \cdots\cdots\cdots\cdots \\ \Phi_n(x, u_0, \ldots, u_{n-1}) \end{pmatrix}$$

with Φ_n *defined in* (1.9)*, is a global diffeomorphism. Then the system* (1.1) *is globally diffeormophic, via* $\overline{\Phi}$*, to a system of the form*

$$\begin{aligned} \dot{z}_i &= z_{i+1}\,, \qquad i = 1, \ldots, n-1\,, \\ \dot{z}_n &= \phi_n(z_1, \ldots, z_n, u, \dot{u}, \ldots, u^{(n-1)}, u^{(n)})\,, \\ y &= z_1\,. \end{aligned} \tag{1.10}$$

Proof. The function $\overline{\Phi}$ is a global diffeomorphism. As a consequence, it admits a (globally defined) inverse mapping $\overline{\Psi} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$, defined as

$$\overline{\Psi}(z, v_n) \;=\; \begin{pmatrix} v_n \\ \cdots\cdots\cdots \\ \Psi_n(z, v_n) \end{pmatrix}$$

[2]with abuse of notation with respect to (1.3).

with the notation $v_i = (u_0, \ldots, u_{i-1})$, such that

$$(v_n, x) \;=\; \overline{\Psi}(\overline{\Phi}(x, v_n))\,, \qquad\qquad x \;=\; \Psi_n(\Phi_n(x, v_n), v_n)\,.$$

Now let the function φ_{n+1} be defined as

$$\varphi_{n+1}(x, v_{n+1}) = \frac{\partial \varphi_n}{\partial x} f(x, u_0) + \sum_{k=0}^{n-1} \frac{\partial \varphi_n}{\partial u_k} u_{k+1}\,,$$

with the function φ_n defined in (1.8). By defining the function ϕ_n as

$$\phi_n(z, v_{n+1}) \;=\; \varphi_{n+1}\left(\Psi_n(z, v_n), v_{n+1}\right)\;,$$

the result follows by applying the change of coordinates $z = \Psi_n(x, u, \dot{u}, \ldots, u^{(n-1)})$ to the system (1.1). □

The form (1.10) is usually known as *phase-variable representation* (see, for instance, (Gauthier and Kupka, 2004, Section II.2)). In the previous result we considered the case where the system is globally diffemorphic to a system of the form (1.10) with the same dimension n, but this result could be easily generalized by considering an "immersion assumption". It is worth noticing that systems in the phase-variable representation are in *strict-feedback form*.

1.2 The high-gain construction

In the previous section we have shown different observability forms which can be achieved under different observability assumptions. It is worth noticing that all the observability forms introduced above can be gathered in two class of nonlinear systems: systems in *strict feedback form* and systems in *non-strict feedback form*. The former recovers the forms (1.7) and (1.10), whereas the latter the form (1.4). In this section we deal with the design of high-gain observers for the class of multi-input single output nonlinear systems in *strict feedback form* described as

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \varphi_i(x_1, \dots, x_i, u) \,, \qquad i = 1, \dots, n-1 \,, \\ \dot{x}_n &= \varphi_n(x_1, \dots, x_n, u) \,, \\ y &= x_1 + \nu(t) \,, \end{aligned} \tag{1.11}$$

where the state $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ evolves in a given compact subset X of $\mathbb{R}^n$, the input u is any function assumed to be known evolving in a compact subset U of $\mathbb{R}^m$ and $y \in \mathbb{R}$ is the measured output. We suppose the functions φ_i, $i = 1, \dots, n$ are locally Lipschitz. The function $t \mapsto \nu(t)$ represents a measurement noise assumed to be unknown and bounded. For this class of systems we can design a *high-gain observer* as follows

$$\begin{aligned} \dot{\hat{x}}_i &= \hat{x}_{i+1} + \hat{\varphi}_i(\hat{x}_1, \dots, \hat{x}_i, u) + k_i \ell^i (y - \hat{x}_1) \,, \qquad i = 1, \dots, n-1 \,, \\ \dot{\hat{x}}_n &= \hat{\varphi}_n(\hat{x}_1, \dots, \hat{x}_n, u) + k_n \ell^n (y - \hat{x}_1) \,, \end{aligned} \tag{1.12}$$

with state $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n) \in \mathbb{R}^n$ and where $k_1, \dots, k_n$ are coefficients to be properly chosen. As concern the functions $\hat{\varphi}_i$, when the functions φ_i are perfectly known, we can choose

$$\hat{\varphi}_i(x_1, \dots, x_i, u) \;=\; \mathsf{sat}_{\vartheta_i}\big(\varphi_i(x_1, \dots, x_i, u)\big) \,.$$

with the positive real number ϑ_i defined as

$$\vartheta_i \;=\; \max_{x \in X,\, u \in U} |\varphi_i(x_1, \dots, x_i, u)| \,.$$

On the contrary, when no information is available, one may choose to pick $\hat{\varphi}_i(\cdot) = 0$. For the sake of generality, in the following we suppose that the functions $\hat{\varphi}_i$ satisfy

$$|\varphi_i(x_1, \dots, x_i, u) - \hat{\varphi}_i(\hat{x}_1, \dots, \hat{x}_i, u)| \;\le\; L_i \sum_{j=1}^{i} |x_j - \hat{x}_j| + R_i \tag{1.13}$$

for all $(x, \hat{x}, u) \in X \times \mathbb{R}^n \times U$, for some $L_i > 0$, representing the Lipschitz constant of φ_i, and $R_i > 0$, representing model uncertainties, for $i = 1, \dots, n$.

As shown in the forthcoming theorem, the high-gain observer (1.12) solves the problem of semi-global[3] estimation for the system (1.11).

Theorem 1.4. *Let consider the system* (1.11) *and suppose* $x(t) \in X$ *and* $u(t) \in U$ *for all* $t \geq 0$*. Consider the observer* (1.12) *satisfying the bond* (1.13) *and let* $k_1, \ldots, k_n$ *be chosen such that the roots of the polynomial*

$$\lambda^n + k_1\lambda^{n-1} + \ldots + k_{n-1}\lambda + k_n$$

have negative real part. Then, there exists a $\ell^\star \geq 1$ *and strictly positive constants* μ_i*,* $i = 1, \ldots, 4$ *such that, for any* $\ell > \ell^\star$ *and for any initial conditions* $(x(0), \hat{x}(0)) \in X \times \mathbb{R}^n$*, the following bounds hold*

$$|\hat{x}_i(t) - x_i(t)| \;\leq\; \mu_1\, \ell^{i-1} \exp(-\mu_2\, \ell\, t)\, |\hat{x}(0) - x(0)| \,+\, \mu_3 \sum_{j=1}^{n} \ell^{i-(j+1)} R_j \,+\, \mu_4 \ell^{i-1}\, \|\nu(\cdot)\|_\infty \tag{1.14}$$

for any $t \geq 0$ *and for* $i = 1, \ldots, n$.

Proof. The forthcoming proof is a rearrangement of the results in Khalil and Praly (2014). We are interested here in giving a detailed proof in order to highlight some interesting properties and introduce some notations which will be used throughout the text. Let us rewrite the system (1.11) and the observer (1.12) in the more compact form

$$\begin{aligned} \dot{x} &= Ax + \phi(x,u)\,, \\ y &= Cx + \nu(t)\,, \\ \dot{\hat{x}} &= A\hat{x} + \hat{\phi}(\hat{x},u) + D_\ell K(y - C\hat{x})\,, \end{aligned} \tag{1.15}$$

where the pair (A, C) is in *prime form* and

$$\phi(\cdot) \;=\; \begin{pmatrix} \varphi_1(\cdot) \\ \vdots \\ \varphi_n(\cdot) \end{pmatrix}, \quad \hat{\phi}(\cdot) \;=\; \begin{pmatrix} \hat{\varphi}_1(\cdot) \\ \vdots \\ \hat{\varphi}_n(\cdot) \end{pmatrix}, \quad D_\ell \;=\; \begin{pmatrix} \ell & & \\ & \ddots & \\ & & \ell^n \end{pmatrix}, \quad K \;=\; \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix}.$$

The core of the proof relies in the following change of coordinates

$$\hat{x}_i \;\mapsto\; \varepsilon_i \;:=\; \frac{\hat{x}_i - x_i}{\ell^{i-1}}\,, \tag{1.16}$$

[3] We refer with *semi-global* to the fact that the compact sets $X \in \mathbb{R}^n$ and $U \in \mathbb{R}^m$ can be taken arbitrarily large.

which can be written in the compact form

$$\hat{x} \;\mapsto\; \varepsilon \;:=\; \ell D_\ell^{-1}(\hat{x} - x)\,.$$

By using the following equalities

$$D_\ell^{-1} A D_\ell \;=\; \ell A\,, \qquad C D_\ell \;=\; \ell C\,. \tag{1.17}$$

the dynamics of (1.15) are transformed into

$$\dot{\varepsilon} \;=\; \ell(A - KC)\varepsilon + \ell D_\ell^{-1}\Delta(\varepsilon, x, u) + \ell K\nu(t)$$

where

$$\Delta(\varepsilon, x, u) \;=\; \left[\hat{\phi}(\ell^{-1} D_\ell\, \varepsilon + x, u) - \phi(x, u)\right].$$

The re-scaled error coordinates ε emphasize the effect of the high-gain parameter ℓ. While making faster the Hurwitz linear part of the system $(A - KC)$, it helps in rejecting the nonlinearities introduced by $\Delta(\varepsilon, x, u)$. Intuitively the high-gain parameter ℓ has to be large enough in order to overcome the Lipschitz constant introduced by Δ. Furthermore, note that in this framework it is fundamental the triangular structure of the functions φ_i. As a matter of fact, notice that by using (1.13) we get

$$\begin{aligned}
\left|\ell D_\ell^{-1}\, \Delta(\varepsilon, x, u)\right| \;&\leq\; \sum_{j=1}^{n} \frac{1}{\ell^{j-1}} \left|\hat{\varphi}_j(\varepsilon_1 + x_1, \ldots, \ell^{j-1}\varepsilon_j + x_j, u) - \varphi_j(x_1, \ldots, x_j, u)\right| \\
&\leq\; \sum_{j=1}^{n} \frac{1}{\ell^{j-1}} \left(L_j \sum_{k=1}^{j} |\ell^{k-1}\varepsilon_k| + R_j \right) \\
&\leq\; \sum_{j=1}^{n} L_j \sum_{k=1}^{j} \ell^{k-j}|\varepsilon_k| + \sum_{j=1}^{n} \ell^{1-j} R_j \\
&\leq\; \sum_{j=1}^{n} L_j |\varepsilon| + \sum_{j=1}^{n} \ell^{1-j} R_j
\end{aligned}$$

and therefore, by denoting $L = \sum_{j=1}^{n} L_j$, we get

$$\left|\ell D_\ell^{-1}\, \Delta(\varepsilon, x, u)\right| \leq L\; |\varepsilon| + \sum_{j=1}^{n} \ell^{1-j} R_j \qquad \forall (\varepsilon, x, u) \in \mathbb{R}^n \times X \times U\,. \tag{1.18}$$

Note that if the functions φ_i had depended on x_j, $j > i$, we would have introduced terms with positive powers of ℓ thus making impossible the stabilization of the ε dynamics.

The proof concludes by applying standard Lyapunov arguments that we recall here. First of all, note that as a consequence of the choice of k_i, $i = 1, \dots, n$, the matrix $(A - KC)$ is Hurwitz. Therefore let $P = P^\top > 0$ be defined as solution of the Lyapunov matrix equation

$$P(A - KC) + (A - KC)^\top P \;=\; -I\,,$$

and let us define the Lyapunov Function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ as

$$V \;=\; \sqrt{\varepsilon^\top P \varepsilon}\,.$$

By denoting with $\underline{\lambda}$ and $\bar{\lambda}$ the minimum and the maximum eigenvalue of P we have

$$\sqrt{\underline{\lambda}}\,|\varepsilon| \;\leq\; V \;\leq\; \sqrt{\bar{\lambda}}\,|\varepsilon|\;.$$

Note that V is only Lipschitz. As a consequence, when V is not zero, by evaluating the Dini derivative of V along the solutions of ε we get

$$\begin{aligned}
D^+V \;&=\; \frac{1}{\sqrt{\varepsilon^\top P \varepsilon}}\,\varepsilon^\top P\left[\ell(A-KC)\varepsilon + \ell D_\ell^{-1}\Delta_\ell(\varepsilon, x, u) + \ell K \nu(t)\right] \\
&\leq\; -\frac{1}{\sqrt{\varepsilon^\top P \varepsilon}}\,\frac{\ell}{2}\,|\varepsilon|^2 + \frac{1}{\sqrt{\varepsilon^\top P \varepsilon}}\,\varepsilon^\top P\left(L\,|\varepsilon| + \sum_{j=1}^{n} \ell^{1-j} R_j + \ell K \nu(t)\right) \\
&\leq\; -\left(\frac{\ell}{2} - L\,\frac{\bar{\lambda}}{\sqrt{\underline{\lambda}}}\right)\frac{|\varepsilon|^2}{\sqrt{\varepsilon^\top P \varepsilon}} + \frac{\bar{\lambda}}{\sqrt{\underline{\lambda}}}\left(\sum_{j=1}^{n} \ell^{1-j} R_j + \ell |K| \bar{\nu}\right)
\end{aligned}$$

where we denoted $\bar{\nu} = \|\nu(t)\|_\infty$. On the contrary, for $V = 0$ we get

$$D^+V \;\leq\; \frac{\bar{\lambda}}{\sqrt{\underline{\lambda}}}\left(\sum_{j=1}^{n} \ell^{1-j} R_j + \ell |K| \bar{\nu}\right)$$

hence the previous expression holds for both cases. Now let $\ell^\star = 2\ell\bar{\lambda}/\sqrt{\underline{\lambda}}$. As a consequence, there exists a $a_1 > 0$ such that, for any $\ell > \max\{\ell^\star, 1\}$,

$$\dot{V} \;\leq\; -\ell\,\frac{a_1}{\bar{\lambda}}\,V + \frac{\bar{\lambda}}{\sqrt{\underline{\lambda}}}\left(\sum_{j=1}^{n} \ell^{1-j} R_j + \ell |K| \bar{\nu}\right)$$

and therefore

$$V(t) \le \exp\left(-\ell \frac{a_1}{\bar{\lambda}} t\right) V(0) + \frac{1-\exp(-\ell\, a_1/\bar{\lambda} t)}{\ell\, a_1/\bar{\lambda}} \frac{\bar{\lambda}}{\sqrt{\underline{\lambda}}} \left(\sum_{j=1}^{n} \ell^{1-j} R_j + \ell |K| \bar{\nu}\right)$$

As a consequence

$$|\varepsilon(t)| \le \mu_1 \exp(-\ell\, \mu_2 t)\, |\varepsilon(0)| + \mu_3 \sum_{j=1}^{n} \ell^{-j} R_j + \mu_4\, \bar{\nu}$$

with $\mu_1 = 1/\sqrt{\underline{\lambda}}$, $\mu_2 = a_1/\bar{\lambda}$, $\mu_3 = \bar{\lambda}^2/(\sqrt{\underline{\lambda}}\, a_1)$ and $\mu_4 = |K|\mu_3$. Finally the claim of the proof follows immediately by noting that for $\ell \ge 1$ we have

$$\ell^{-(i-1)}\, |x_i - \hat{x}_i| \le |\varepsilon| \le |x - \hat{x}| \ . \qquad \square$$

Remarks:

- Global observation can be achieved by the high-gain observer (1.12) only by asking the functions φ_i to be globally Lipschitz and by selecting $\hat{\varphi}_i(\cdot) = \varphi_i(\cdot)$ for all $i = 1, \ldots, n$.

- The high-gain observer (1.12) is characterized by the nice feature of being extremely easy to tune. There are n design-parameters k_i, $i = 1, \ldots, n$, to select and one *high-gain parameter* ℓ to be chosen large enough in order to overcome the Lipschitz constants L_i of the nonlinear functions $\varphi_i(\cdot)$, $i = 1, \ldots, n$. Furthermore, the rate of convergence of the observer can be made arbitrarily fast by increasing the high-gain parameter ℓ.

- The presence of R_j, $j = 1, \ldots, n$, deteriorates the estimate of the variables $\hat{x}_i$, $i = j+1, \ldots, n$, since $\ell^{i-j+1} > 1$ for $i > j$. Evidently, asymptotic estimate can be achieved only when the functions φ_i are perfectly known, namely $R_j = 0$, $j = 1, \ldots, n$, and when the measurement noise is not present.

- In the case the functions $\varphi_i(\cdot) = 1, \ldots, n-1 = 0$, system (1.11) reduces to

$$\begin{aligned} \dot{x}_i &= x_{i+1} \qquad i = 1, \ldots, n-1, \\ \dot{x}_n &= \varphi_n(x, u) \end{aligned}$$

  For the latter, the observer (1.12) can be designed as

$$\begin{aligned} \dot{\hat{x}}_i &= \hat{x}_{i+1} + \ell^i k_i (y - \hat{x}_1)\ , \qquad i = 1, \ldots, n-1, \\ \dot{\hat{x}}_n &= \ell^n k_n (y - \hat{x}_1) \end{aligned}$$

and it is sometimes referred as “dirty-derivatives-observer” (see Teel and Praly (1994)) or “differentiator” (see Vasiljevic and Khalil (2006)). As shown in many works (see, for instance, Teel (1996) and Atassi and Khalil (2000)), this observer can be successfully used in output feedback stabilization problems when $\varphi_n(0,0) = 0$. In this case the gain ℓ has to be chosen large enough to “deal” with the bound ϑ_n and not to overcome the Lipschitz constant L_n of the nonlinear function φ_n

1.3 Drawbacks of high-gain observers

The high-gain observer design (1.12) and the bound (1.14) introduced in Theorem 1.4 highlight three main drawbacks of this approach:

1. implementation issues,
2. the "peaking phenomenon",
3. sensitivity to measurement noise.

1.3.1 Implementation issues

The high-gain observer (1.12) is characterized by having the gain of the output injection terms which is proportional to $\ell, \ell^2, \ldots, \ell^n$. Furthermore, as shown in the proof of Theorem 1.4, the minimum value of ℓ which guarantees asymptotic convergence of the observer, is proportional to the Lipschitz constant of the nonlinear functions φ_i. As a consequence, if the high-gain parameter ℓ or the dimension n of the observed system are large, we need to implement in the observer (1.12) a term ℓ^n which may be very harmful from a numerical point of view. So a natural question is if this term ℓ^n is *necessary* or only *sufficient*.

Consider for the time an observer for the system (1.11) of the form

$$\begin{aligned} \dot{\hat{x}}_i &= \hat{x}_{i+1} + \hat{\varphi}_i(\hat{x}_1, \ldots, \hat{x}_i, u) + c_1(y - \hat{x}_1), \qquad i = 1, \ldots, n-1, \\ \dot{\hat{x}}_n &= \hat{\varphi}_n(\hat{x}_1, \ldots, \hat{x}_n, u) + c_n(y - \hat{x}_1) \ , \end{aligned}$$

where $c_1, \ldots, c_n$ are coefficients to be properly chosen. By assuming in this framework that no measurement noise is present, namely $\nu(t) = 0$, and by making the change of coordinates $e_i := \hat{x}_i - x_i$, we get

$$\begin{aligned} \dot{e}_i &= -c_1 e_1 + \hat{e}_{i+1} + \hat{\varphi}_i(e_1 + x_1, \ldots, e_i + x_i, u) - \varphi_i(x_1, \ldots, x_i, u), \\ & \qquad\qquad i = 1, \ldots, n-1, \\ \dot{e}_n &= -c_n e_1 + \hat{\varphi}_n(e_1 + x_1, \ldots, e_n + x_n, u) - \varphi_n(x_1, \ldots, x_n, u). \end{aligned}$$

An easy answer to our problem is obtained by considering the case where

$$\begin{aligned} \varphi_i(x, u) &= 0 \qquad i = 1, \ldots, n-1 \ , \\ \varphi_n(x, u) &= L\, x_n \ . \end{aligned}$$

for some positive real number L. By choosing

$$\begin{aligned} \hat{\varphi}_i(\hat{x}, u) &= 0 \qquad i = 1, \ldots, n-1 \ , \\ \hat{\varphi}_n(\hat{x}, u) &= L\, \hat{x}_n \ . \end{aligned}$$

the closed-loop dynamics reduces to

$$\dot{e} \;=\; F\,e$$

where we denoted $e = (e_1, \ldots, e_n)$ and

$$F \;=\; \begin{pmatrix} -c_1 & 1 & 0 & \ldots & \ldots & 0 \\ -c_2 & 0 & 1 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ -c_{n-2} & 0 & \ldots & 0 & 1 & 0 \\ -c_{n-1} & 0 & \ldots & \ldots & 0 & 1 \\ -c_n & 0 & \ldots & \ldots & \ldots & L \end{pmatrix}.$$

Convergence of the observer is guaranteed only if the matrix F is Hurwitz. Its eigenvalues λ are roots of

$$\mathcal{P}(\lambda) \;=\; \det \begin{pmatrix} (\lambda + c_1) & -1 & 0 & \ldots & \ldots & 0 \\ c_2 & \lambda & -1 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ c_{n-2} & 0 & \ldots & \lambda & -1 & 0 \\ c_{n-1} & 0 & \ldots & \ldots & \lambda & -1 \\ c_n & 0 & \ldots & \ldots & \ldots & (\lambda - L) \end{pmatrix}$$

By using the multilinearity of the determinant, we have:

$$\begin{aligned} \mathcal{P}(\lambda) &= \det \begin{pmatrix} (\lambda + c_1) & -1 & 0 & \ldots & \ldots & 0 \\ c_2 & \lambda & -1 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ c_{n-2} & 0 & \ldots & \lambda & -1 & 0 \\ c_{n-1} & 0 & \ldots & \ldots & \lambda & -1 \\ c_n & 0 & \ldots & \ldots & \ldots & \lambda \end{pmatrix} + \det \left(\begin{array}{cccccc|c} (\lambda + c_1) & -1 & 0 & \ldots & \ldots & & 0 \\ c_2 & \lambda & -1 & \ddots & & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots \\ c_{n-2} & 0 & \ldots & \lambda & -1 & & 0 \\ c_{n-1} & 0 & \ldots & \ldots & \lambda & & 0 \\ \hline c_n & 0 & \ldots & \ldots & \ldots & & -L \end{array}\right) \\ &= \left[\lambda^n + c_1\lambda^{n-1} + \ldots + c_{n-1}\lambda + c_n\right] \;-\; L \det \begin{pmatrix} \lambda + c_1 & -1 & 0 & \ldots & 0 \\ c_2 & \lambda & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ c_{n-2} & 0 & \ddots & \ddots & -1 \\ c_{n-1} & 0 & \ldots & \ldots & \lambda \end{pmatrix} \end{aligned}$$

and therefore

$$\mathcal{P}(\lambda) = \lambda^n + (c_1 - L)\lambda^{n-1} + (c_2 - c_1 L)\lambda^{n-2} + \ldots + (c_{n-1} - c_{n-2}L)\lambda + (c_n - c_{n-1}L)$$

So a necessary condition for stability of the e-dynamics is:

$$c_1 \ > \ L\,, \qquad c_2 \ > \ c_1\,L\,, \qquad \ldots\,, \qquad c_{n-1} \ > \ c_{n-2}\,L\,, \qquad c_n \ > \ c_{n-1}\,L$$

which implies $c_n > L^n$. This prove that even in the linear case the last gain of the output injection term must be larger than the Lipschitz constant of the nonlinear term powered up to the state dimension of the system.

These simple arguments show that the choice

$$c_i \ = \ \ell^i\,k_i$$

is not restrictive and guarantees an easy tunability for the stability of the closed loop system, by delegating to the choice of k_i, $i = 1, \ldots, n$, the stability of the re-scaled error closed loop matrix $(A - KC)$ (see the notation introduced in the proof of Theorem 1.4) defined as

$$A - KC \ = \ \begin{pmatrix} -k_1 & 1 & 0 & \ldots & \ldots & 0 \\ -k_2 & 0 & 1 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ -k_{n-2} & 0 & \ldots & 0 & 1 & 0 \\ -k_{n-1} & 0 & \ldots & \ldots & 0 & 1 \\ -k_n & 0 & \ldots & \ldots & \ldots & 0 \end{pmatrix}$$

and to the choice of ℓ the convergence rate of the observer for values large enough (namely for values which overcomes the Lipschitz constant of the nonlinear term φ). It seems clear that if we want to avoid to implement terms $\ell, \ldots, \ell^n$ in the observer, we need some different strategy, as a nonlinear change of coordinates, the use of non-linear functions, or dynamic extension.

1.3.2 The peaking phenomenon

Convergence of the observer (1.12) has been proved in Theorem 1.4. In absence of measurement noise and of model uncertainties, namely $\nu(t) = 0$ and $R_i = 0$, $i = 1, \ldots, n$, the $\hat{x}_i$ dynamics can be bounded as

$$|x_i(t) - \hat{x}_i(t)| \ \leq \ \mu_1\,\ell^{i-1}\,\exp(-\mu_2\,\ell\,t)\,|x(0) - \hat{x}(0)|\ .$$

During the transient the decaying term $\exp(-\mu_2 \ell t)$ is closed to one. As a consequence the variable $\hat{x}_i$ shows a peak which is proportional to the error in the initial conditions and multiplied by a term ℓ^{i-1}, producing an estimate completely unreliable and which can be very large from a numerical point of view when ℓ or i are very large. The interaction of peaking with nonlinearities can induce finite escape time in output feedback scenarios. In particular in the lack of global growth conditions, high-gain observers can destabilize the closed-loop system as the observer gain is driven sufficiently high. Though finite escape time can be avoided by introducing saturations (see for instance Esfandiari and Khalil (1992), Teel and Praly (1994), Teel (1996), Tornambé (1992a)) either in the control law, either in the estimate $\hat{x}$, prevent or reduce peaking is still a challenge from a numerical point of view. A possible solution to remove peaking is to adopt the strategy which will be proposed in Section 2.2. The main idea consists in adding to the observer-dynamic a “modification term” which makes invariant some given compact set. However, since this design is ℓ-dependent and contains powers of ℓ which may be larger than n, this solution could not be the optimal choice from a numerical point of view when ℓ or n are very large.

The peaking phenomenon has been extensively studied in literature (see for instance Mita (1977), Polotskii (1979), Esfandiari and Khalil (1992), Teel and Praly (1994)) and different solutions have been proposed, based on re-scaling (Esfandiari and Khalil (1992)), projections (Maggiore and Passino (2003)), hybrid re-design (Prieur et al. (2012)) or time-varying gain approaches (El Yaagoubi et al. (2004)). Finally, a very recent publication (Teel (2016)) proposes an elegant solution based on a nested-saturation design.

1.3.3 Sensitivity to measurement noise

One of the main feature which questions the use of a high-gain observer in applications is its sensitivity to measurement noise. The bound (1.14) introduced in Theorem 1.12 shows that when $R_i = 0$, $i = 1, \ldots, n$, the asymptotic estimate is bounded by

$$\lim_{t\to\infty} |x_i(t) - \hat{x}_i(t)| \;\leq\; \mu_4 \ell^{i-1} \|\nu(\cdot)\|_\infty \tag{1.19}$$

The latter highlights the $\mathcal{H}_\infty$ bound between the estimate error and the measurement noise, showing that the estimates $\hat{x}_i$ are affected by the noise with a gain which is proportional to ℓ^{i-1}. As a consequence when $\|\nu(\cdot)\|_\infty$, n or ℓ are very large, the estimates may become completely unreliable, imposing some upper bound on the value of the high-gain parameter ℓ if estimation in presence of measurement noise is desired. This trade-off between the speed of the state estimation and the sensitivity to measurement noise is a well-known fact in the observer theory. In this respect, high-gain observers tuned to obtain fast estimation dynamics are necessarily very sensitive to high-frequency noise (see, for instance, Mahmoud and Khalil (2002)). As reported in Khalil and Praly (2014), “a sound strategy to achieve fast convergence while reducing the impact of measurement noise at steady state is to use a larger ℓ during the transient time and then decrease it at steady state”. Along this idea many techniques which vary the high-gain parameter with some scheme have been proposed (see, among others, Ahrens and Khalil (2009), Boizot et al. (2010), Marino and Santosuosso (2007), Prasov and Khalil (2013), Sanfelice and Praly (2011)).

In Section 2.4 we propose a new analysis tool which give a more precise bound in presence of high-frequency measurement noise.

1.4 High gain observers for systems in non-strict feedback form

In this section we focus on the design of observer for systems in the *non-strict feedback form*

$$\begin{aligned} \dot{x}_i &= f_i(\boldsymbol{x}_i, x_{i+1}, u)\,, \qquad 1 \le i \le n-1\,, \\ \dot{x}_n &= f_n(x_1, \dots, x_n, u)\,, \\ y &= h(x_1, u)\,, \end{aligned} \tag{1.20}$$

with state $x = \text{col}(x_1, \dots, x_n) \in \mathbb{R}^n$, input $u \in \mathbb{R}$ and output $y \in \mathbb{R}$, and where we used the notation $\boldsymbol{x}_i = \text{col}(x_1, \dots, x_i)$. Due to the nonlinear structure, the design (1.12) cannot be applied to this class of systems. As a consequence, in this section we show how to design a suitable observer under the forthcoming extra assumptions.

Assumption 1.1. *Each of the maps $f_i(\cdot)$, $i = 1, \dots, n$, is globally Lipschitz with respect to $\boldsymbol{x}_i$, uniformly with respect to u and x_{i+1}, namely there exists a $L > 0$ such that for all $\boldsymbol{x}_i \in \mathbb{R}^i$, $\boldsymbol{x}'_i \in \mathbb{R}^i$, $x_{i+1} \in \mathbb{R}$ and $u \in \mathbb{R}$, the following holds*

$$\begin{aligned} |f_i(\boldsymbol{x}_i,\, x_{i+1},\, u) - f_i(\boldsymbol{x}'_i,\, x_{i+1},\, u)| &\le L\, |\boldsymbol{x}_i - \boldsymbol{x}'_i|\,, \qquad 1 \le i \le n-1\,, \\ |f_n(\boldsymbol{x}_n,\, u) - f_n(\boldsymbol{x}'_n,\, u)| &\le L\, |\boldsymbol{x}_n - \boldsymbol{x}'_n|\,. \end{aligned}$$

Assumption 1.2. *There exist two positive constants $0 < \alpha \le \beta < \infty$, such that for all $(x, u) \in \mathbb{R}^n \times \mathbb{R}$ the following bounds hold*

$$\alpha \le \left|\frac{\partial h(x_1,\, u)}{\partial x_1}\right| \le \beta\,, \qquad \alpha \le \left|\frac{\partial f_i(\boldsymbol{x}_i,\, x_{i+1},\, u)}{\partial x_{i+1}}\right| \le \beta\,, \quad 1 \le i \le n-1\,.$$

Within this framework the main result proposed in Gauthier and Kupka (2004) is a systematic design of a high-gain observer that takes the form

$$\begin{aligned} \dot{\hat{x}}_i &= f_i(\hat{\boldsymbol{x}}_i,\, \hat{x}_{i+1},\, u) + \ell^i\, k_i\, (y - h(\hat{x}_1,\, u))\,, \qquad 1 \le i \le n-1\,, \\ \dot{\hat{x}}_n &= f_n(\hat{\boldsymbol{x}}_n,\, u) + \ell^n\, k_n\, (y - h(\hat{x}_1,\, u))\,, \end{aligned} \tag{1.21}$$

where $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n)^\top$ is the estimate state, with the notation $\hat{\boldsymbol{x}}_i = (\hat{x}_1, \dots, \hat{x}_i)^\top$. With respect to the high-gain observer (1.12) we can emphasixe a substantial difference. While the effect of the high-gain parameter ℓ is the same, *i.e.* speed up the dynamics of the observer, the coefficients k_i, $i = 1, \dots, n$ cannot be chosen in the same trivial way. In particular, as highlighted in Gauthier and Kupka (2004), a more sophisticated design is needed, summarized in the forthcoming technical lemma.

Lemma 1.1 (Gauthier and Kupka (2004), Lemma 2.1, Chapter 6). *Let $A(t) \in \mathbb{R}^{n\times n}$ and $C(t) \in \mathbb{R}^{1\times n}$ be time-varying matrices defined as*

$$\begin{aligned} A(t) &= \begin{pmatrix} 0 & \psi_2(t) & 0 & \cdots & 0 \\ 0 & 0 & \psi_3(t) & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \psi_n(t) \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \\ C(t) &= \begin{pmatrix} \psi_1(t) & 0 & 0 & \cdots & 0 \end{pmatrix}, \end{aligned}$$

in which $\psi_i(t)$ are continuous function satisfying

$$\alpha \;\le\; \psi_i(t) \;\le\; \beta \qquad \forall\, t \ge 0\,, \quad 1 \le i \le n\,, \tag{1.22}$$

for some constants $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$. Then there exist a $\lambda > 0$, a vector $K = \mathrm{col}(k_1, \ldots, k_n)$, and a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$(A(t) - KC(t))^\top P + P(A(t) - KC(t)) \;\le\; -\lambda\, I\,.$$

Under the previous assumptions the observer (1.21) can be tuned in order to obtain a global estimate of the state of (1.20).

Theorem 1.5 (Gauthier and Kupka (2004), Theorem 2.2, Chapter 6). *Consider the system* (1.20) *under the Assumptions 1.1 and 1.2. Let the coefficients $k_1, \ldots, k_n$ of the observer* (1.21) *be chosen according to Lemma 1.1 with α, β given by Assumption 1.2. There exist a $\ell^\star \ge 1$ and positive constants μ_1 and μ_2 such that for all $\ell > \ell^\star$ and for all $(x(0), \hat{x}(0)) \in \mathbb{R}^n \times \mathbb{R}^n$ the following bound holds*

$$|\hat{x}_i(t) - x_i(t)| \;\le\; \mu_1\, \ell^{i-1} \exp(-\mu_2 \ell\, t)\, |\hat{x}(0) - x(0)|\;.$$

Note that the globally Lipschitz condition in Assumption 1.1 is motivated by the fact that in Theorem 1.5 we look for a global observer. In case the state of the system ranges in a fixed known compact set, the condition of Assumption 1.1 can be weakened by asking the functions $f_i(\cdot)$ to be only locally Lipschitz with respect to $\boldsymbol{x}_i$. In this case the design of the observer (1.21) may follow the same design of observer (1.12), for instance by saturating the nonlinear functions $f_i(\cdot)$ outside the domain of interest.

1.5 Design in the original coordinates

This chapter concludes by illustrating some difficulties that may arise by following the prescriptions of Sections 1.1 and 1.2. As shown in Section 1.1 input-affine nonlinear systems of the form

$$\dot{x} \;=\; f(x) + g(x)u\,, \qquad y \;=\; h(x)\,, \tag{1.6 revisited}$$

can be put, by means of the (nonlinear) change of coordinates

$$\Phi(x) \;=\; \text{col}\Big(h(x) \quad L_f h(x) \quad \cdots \quad L_f^{n-1}h(x)\Big)$$

in the *strict feedback form* (1.7) that we rewrite here by using the compact notation

$$\dot{z} \;=\; Az + \phi(z,u)\,, \qquad y = Cz\,, \tag{1.7 revisited}$$

with

$$\phi(z,u) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a(z) \end{pmatrix} + \begin{pmatrix} b_1(z_1) \\ \vdots \\ b_i(z_1,\dots,z_i) \\ \vdots \\ b_n(z_1,\dots,z_n) \end{pmatrix} u\,.$$

Once the system is put in the coordinates (1.7), it is possible to design an asymptotic convergent observer by following the design proposed in Section 1.2. The observer (1.12) achieves asymptotic estimation of the system (1.7) and by applying the inverse mapping $\hat{x} = \Phi^{-1}(\hat{z})$ it is possible to recover an estimate of the system (1.6).

However, the main obstacle of this procedure is that we need to know the inverse mapping of $\Phi(x)$, which may be very difficult to compute in explicit way. By following the intuition of Deza et al. (1992), this problem can be overcame by implementing the high-gain observer in the original coordinates: instead of changing the coordinates for the original system and then designing the observer, it is possible to go in the opposite direction, namely to "bring-back" in the original coordinates the observer. The high-gain observer can be thus implemented as

$$\dot{\hat{x}} \;=\; f(\hat{x}) + g(\hat{x})u + \left(\frac{\partial\Phi}{\partial x}(\hat{x})\right)^{-1} D_\ell K(y - h(\hat{x}))$$

where we denote $D_\ell = \text{diag}(\ell,\dots,\ell^n)$[4] with $\ell \geq 1$ the high-gain parameter, and K is a

[4]We use the same notation introduced in the proof of Theorem 1.4.

$n \times 1$ matrix to be chosen such that there exists a positive definite symmetric matrx P satisfying

$$P(A - KC) + (A - KC)^{\top} \; = \; -I \, .$$

It is not hard to see that, by applying the change of coordinates $\hat{z} = \Phi(\hat{x})$, the proposed observer transforms as

$$\dot{\hat{z}} \; = \; A\hat{z} + \phi(\hat{z}, u) + D_{\ell} K(y - C\hat{z}) \; .$$

The latter coincides with the observer design proposed in (1.12). However, if the function $\phi(\hat{z}, u)$ does not satisfy the bound (1.13) Theorem 1.4 can not apply. As opposed to the design procedure illustrated in Section 1.2 for the functions $\hat{\varphi}_i$, the main issue here relies on the fact that we are not able (in a systematic way) to design a modified version of the functions f, g such that in the z-coordinates, the function $\phi(\hat{z}, u)$ (implicitly implemented in the observer in the original coordinates) is bounded in the strict feedback coordinates and well-defined outside of some compact set of interest. This problem has been solved, for instance, in Maggiore and Passino (2003) by using projection. An other possibility, which will be investigated in Section 2.2, is to modify the previous observer such that its state $\hat{x}$ remains invariant on some compact subset of $\mathbb{R}^n$.

1.6 Application to output regulation

The problem of output regulation for nonlinear systems, beginning with the work Isidori and Byrnes (1990), has been addressed by several authors and a corpus of results has been developed, the majority of which address the problem in question for single-input single-output minimum-phase systems (see for instance Byrnes et al. (1997), Isidori et al. (2003) Huang (2007), Pavlov et al. (2006) and references therein). For the class of systems in question, output regulation amounts to making a compact attractor, on which some regulated variables are zero, asymptotically stable. The distinguishing feature of the framework is that the attractor is not invariant for the original uncontrolled plant and has internal dynamics governed by the dynamics of an autonomous exogenous system (the so-called exosystem) whose state is not measurable. This, in turn, asks for the design of regulators that include an appropriate copy of the exosystem dynamics able to make the desired attractor invariant and leads to the celebrated internal model-based design strategy.

Starting with the contribution in Isidori and Byrnes (1990), many improvements have been proposed in the output regulation literature in the last twenty years or so with the aim of making the framework where internal model-based regulators could be systematically designed even more general (see, among others, Huang and Lin (1994a), Serrani et al. (2001), Marconi et al. (2007), Marconi and Praly (2008)). Related to the design methodology presented in this section, an interesting framework has been proposed in Byrnes and Isidori (2004) in which the so-called "friend", which is the ideal steady state input able to make the desired attractor invariant, and a certain number of its time derivatives are assumed to fulfil a regression law. The important observation made in Byrnes and Isidori (2004) is that, in this framework, tools typically adopted in the field of nonlinear high-gain observers can be successfully adopted in order to design internal model-based regulators. This observation opened an interesting research direction in which a high-gain observer can be used for the design of the internal model. The same framework has been also taken in Isidori et al. (2012) in order to design adaptive linear regulators, namely regulators with adaptive mechanisms able to cope with uncertainties in the exosystem. The fact that design methodologies typically used in the design of nonlinear observers could be successfully employed in the design of internal models has been further investigated and developed in Delli Priscoli et al. (2006) in which the theory of adaptive (not necessarily high-gain) nonlinear observers has been proposed for this purpose.

The presentation of this section has been strongly inspired by Astolfi et al. (2017) and the contents are well-known results that can be found in Byrnes and Isidori (2004), Marconi et al. (2007) and Isidori et al. (2012).

1.6.1 The framework of output regulation

In this section we recall briefly the framework considered by Byrnes and Isidori (2004) where the aforementioned high-gain methodology applies. We consider the class of systems in normal form with unitary relative degree[5] described by

$$\begin{array}{rcl} \dot{z} & = & f(w,z,e) \\ \dot{e} & = & q(w,z,e)+b(w,z,e)u \end{array} \tag{1.23}$$

in which $(z,e) \in \mathbb{R}^n \times \mathbb{R}$ is the state, $u \in \mathbb{R}$ is the control input and $w \in \mathbb{R}^\rho$ is a an exogenous variable that, in the context of output regulation, is thought of as generated by an autonomous system (typically referred to as exosystem) of the form

$$\dot{w} \;=\; s(w) \tag{1.24}$$

whose state ranges in a compact *invariant* set $W \subset \mathbb{R}^\rho$. The state component e represents the measured output and the regulation error to be steered to zero. It is assumed that $f(\cdot)$, $q(\cdot)$, $b(\cdot)$, $s(\cdot)$ are smooth enough functions and that the function $b(\cdot)$ is bounded from below, *i.e.* there exists a strictly positive real numbers $\underline{b}$ such that

$$b(w,z,e) \geq \underline{b} \qquad \forall\, (w,z,e) \in \mathbb{R}^\rho \times \mathbb{R}^n \times \mathbb{R}\,. \tag{1.25}$$

The initial initial condition of the system (1.23) is assumed to range in an arbitrary but known compact set $Z \times E \subset \mathbb{R}^n \times \mathbb{R}$. Within this framework the problem of output regulation amounts to designing a controller of the form

$$\begin{array}{rcl} \dot{\xi} & = & \psi(\xi,e) \\ u & = & \gamma(\xi,e) \end{array} \tag{1.26}$$

with initial conditions in a compact set Ξ, such that the trajectories of the closed-loop system originating from $W \times Z \times E \times \Xi$ are bounded and

$$\lim_{t\to\infty} e(t) = 0 \tag{1.27}$$

uniformly in the initial conditions. Very often asymptotic regulation is difficult to achieve in a general nonlinear context and it thus makes sense to relax (1.27) into a *practical* regulation objective, namely to ask that $\lim_{t\to\infty} \sup |e(t)| \leq \epsilon$ with ϵ a small positive number.

The previous problem is addressed under a number of assumptions that we briefly recall. The first regards the existence of the solution of the so-called *regulator equations*.

[5] Notice that the case with a relative degree higher than one can be reduced to the unitary relative degree case by means of a change of variables (see, among the others, Byrnes and Isidori (2004)).

In this framework, in particular, we assume the existence of a differentiable function $\pi : \mathbb{R}^\rho \to \mathbb{R}^n$ solution of

$$L_s\pi(w) \;=\; f(w,\pi(w),0) \qquad \forall\, w \in W\,.$$

This assumption guarantees that the set $\mathcal{A} \subset W \times \mathbb{R}^n$, defined as

$$\mathcal{A} \;=\; \{(w,z) \in W \times \mathbb{R}^n \;:\; z = \pi(w)\}\,,$$

is invariant for the *zero dynamics* of the system (1.23) with input u and output e that are described by

$$\begin{array}{rcl} \dot{w} &=& s(w) \\ \dot{z} &=& f(w,z,0)\,. \end{array} \tag{1.28}$$

The second assumption asks that system (1.23) is also minimum-phase. In our framework the minimum-phaseness assumption is formalised as follows.

Assumption 1.3. *The set $\mathcal{A}$ is asymptotically and locally exponentially stable for the system* (1.28) *with a domain of attraction of the form $W \times \mathcal{D}$ where $\mathcal{D}$ is an open set of $\mathbb{R}^n$ such that $Z \subset \mathcal{D}$.*

The local exponential stability requirement in the previous assumption is written just for sake of simplicity and it can be removed by properly adapting the design of the regulator presented in the following (see, for instance, Byrnes and Isidori (2004), Marconi et al. (2007),Isidori (2010)). In the design of the regulator solving the problem of output regulation, a crucial role is played by the so-called "friend", which is the function $c : W \to \mathbb{R}$ defined as

$$c(w) \;:=\; -\,\frac{q(w,\pi(w),0)}{b(w,\pi(w),0)}\;. \tag{1.29}$$

By bearing in mind (1.23), it turns out that such a function represents the ideal steady state input needed to keep the regulation error identically zero, namely the control input that must be applied to (1.23) to make the set $\mathcal{A} \times \{0\}$ invariant. In the following construction we do not assume a specific structure for $c(\cdot)$ as typically done, through the so-called immersion assumption, in most of the work on the subject (Byrnes and Isidori (2004), Isidori (2010) and references therein). Rather, the internal model-based regulator designed in the following relies on the knowledge of an integer $d > 0$ and of a function $\varphi : \mathbb{R}^d \to \mathbb{R}$ fulfilling

$$L_s^d c(w) \;=\; \varphi\Big(c(w),\, L_s\, c(w),\, \ldots,\, L_s^{d-1}\, c(w)\Big) + \nu(w) \qquad \forall\, w \in W \tag{1.30}$$

for some (unknown) function $\nu : W \to \mathbb{R}$. In case the previous relation is fulfilled with

$\nu \equiv 0$ asymptotic regulator will be achieved. Practical regulation, with an asymptotic error that is upper bounded by a function of $\sup_{w \in W} |\nu(w)|$, is otherwise obtained. The previous framework allows one to regard the parameter d as a degree-of-freedom by which the designer can tradeoff the dimension of the regulator (and thus its complexity) and the bound on the asymptotic error. As a matter of fact, larger values of d allow, in general, to identify a $\varphi(\cdot)$ that makes relation (1.30) fulfilled with a smaller bound of the residual term $|\nu(\cdot)|$, by thus obtaining a regulator able to guarantee smaller asymptotic errors.

In the remaining part of the section we illustrate the main framework under which a regulator can be designed (see Byrnes and Isidori (2004)), by highlighting how the theory of nonlinear observers, and in particular the one of high-gain observers, turns out to be useful in the regulator construction. The fact of dealing with regulated plant that are affine in the input suggests to consider regulator structures of the same kind, namely regulators of the form

$$\begin{array}{rcl} \dot{\xi} & = & \phi(\xi) + \Psi\, v \qquad \xi \in \mathbb{R}^m \\ u & = & \gamma(\xi) + v \\ v & = & -\kappa e \end{array} \tag{1.31}$$

where $\phi(\cdot)$ and $\gamma(\cdot)$ are smooth functions, Ψ is a column vector, and κ is a design parameter, all to be designed. The resulting closed-loop system, has a normal form that, having defined the change of variables

$$\xi \;\mapsto\; \chi \;:=\; \xi - \Psi \int_0^e \frac{1}{b(w,z,s)} ds\,,$$

reads as

$$\begin{array}{rcl} \dot{w} & = & s(w) \\ \dot{z} & = & f(w,z,e) \\ \dot{\chi} & = & \phi(\chi) - \Psi\left(\gamma(\chi) + \dfrac{q(w,z,e)}{b(w,z,e)}\right) + \Delta(w,z,\chi,e) \\ \dot{e} & = & q(w,z,e) + b(w,z,e)\gamma(\chi) + b(w,z,e)v + L(w,z,\chi,e) \end{array} \tag{1.32}$$

where $\Delta(\cdot)$ and $L(\cdot)$ are properly defined functions such that $\Delta(w,z,\chi,0) = 0$ and $L(w,z,\chi,0) = 0$ for all $(w,z,\chi) \in W \times \mathbb{R}^n \times \mathbb{R}^m$. This system, regarded as a system with input v and output e, has still unitary relative degree and, as an easy computation shows, zero dynamics described by

$$\begin{array}{rcl} \dot{w} & = & s(w) \\ \dot{z} & = & f(w,z,0) \\ \dot{\chi} & = & \phi(\chi) - \Psi\left(\gamma(\chi) + \dfrac{q(w,z,e)}{b(w,z,e)}\right) . \end{array} \tag{1.33}$$

Note that these dynamics have a cascade structure with system (1.28) driving the system with state χ. In the following we denote by $X \subset \mathbb{R}^m$ the compact set of initial conditions for the new variable χ. The problem of output regulation is then reformulated as a problem of output feedback stabilisation of system (1.32). In particular the problem at hand is solved if one is able to prove the existence of a compact set of $\mathbb{R}^\rho \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$, on which the regulation error e is identically zero, that is asymptotically stable for system (1.32) with a domain of attraction containing the set of initial conditions. To this purpose high-gain design paradigms for minimum-phase systems can be successfully adopted (Byrnes and Isidori (2004)). In particular, the following two requirements play a role in the design of the stabiliser:

a) there exists a set $\mathcal{B} \subset \mathbb{R}^\rho \times \mathbb{R}^n \times \mathbb{R}^m$ that is asymptotically and locally exponentially stable for system (1.33) with a domain of attraction of the form $W \times \mathcal{D}_e$ with $\mathcal{D}_e \subset \mathbb{R}^n \times \mathbb{R}^m$ an open set fulfilling $Z \times X \subset \mathcal{D}_e$.

b) the following holds

$$q(w,z,0) + b(w,z,0)\gamma(\chi) \;=\; 0 \quad \forall\, (w,z,\chi) \in \mathcal{B}\,.$$

Requirement (a), in turn, asks that system (1.32), regarded as a system with input v and output e, is minimum-phase. On the other hand, requirement (b) asks that the coupling term between the zero dynamics (1.33) and the error dynamics is vanishing on $\mathcal{B} \times \{0\}$, namely that the latter set is invariant for 1.32 with $v = 0$. That properties, in turn, make system (1.32) fitting into frameworks of stabilisation of minimum-phase nonlinear systems in which the choice $v = -\kappa e$, with κ sufficiently large, succeeds in asymptotically stabilising the set $\mathcal{B} \times \{0\}$. This is formalised in the next theorem whose proof can be found in Marconi et al. (2007).

Theorem 1.6. *Assume that the requirements* (a) *and* (b) *specified before are fulfilled for some compact set* $\mathcal{B}$*. Then, there exists a* $\kappa^\star > 0$ *such that for all* $\kappa \geq \kappa^\star$ *the set* $\mathcal{B} \times \{0\}$ *is asymptotically and locally exponentially stable for system* (1.23)-(1.24) *controlled by* (1.31) *with a domain of attraction of the form* $W \times \mathcal{D}_{cl}$ *with* $\mathcal{D}_{cl} \subset \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$ *an open set fulfilling* $Z \times X \times E \subset \mathcal{D}_{cl}$*.*

The high-gain paradigm is indeed robust in case requirement (a) above is only achieved practically rather than asymptotically. More specifically, requirement (a) above can be relaxed to the requirement (a') specified in the following at the price of achieving just practical instead of asymptotic regulation as claimed in the next Theorem 2.

a') There exists a set $\mathcal{B} \subset \mathbb{R}^\rho \times \mathbb{R}^n \times \mathbb{R}^m$ such that the trajectories of system (1.33) originating from $W \times Z \times X$ fulfil

$$|(w(t), z(t), \chi(t))|_\mathcal{B} \;\leq\; \max\left\{c_1 \exp(-c_2 t)|(w(0), z(0), \chi(0))|_\mathcal{B}\,,\; \epsilon\right\} \tag{1.34}$$

for some positive constants c_1, c_2 and ϵ.

Theorem 1.7. *Assume that the requirements* (a') *and* (b) *specified before are fulfilled for some compact set $\mathcal{B}$ and positive constants c_1, c_2 and ϵ. Then, there exist a $\kappa^\star > 0$ and a $c > 0$, such that for all $\kappa \geq \kappa^\star$ the trajectories of the resulting closed-loop* (1.23)-(1.24) *and* (1.31) *originating from the compact set of initial conditions $W \times Z \times X \times E$ are bounded and*

$$\lim_{t\to\infty} \sup |e(t)| \;\leq\; \frac{c}{\kappa}\,\epsilon\,.$$

The previous considerations shift the focus of the design on system (1.33) and, in particular, on the design of the triplet $(\phi(\cdot), \psi, \gamma(\cdot))$ fulfilling the requirements (a') and (b). In the next section the problem in question is solved by using the high-gain observer theory.

1.6.2 High-gain observers for the internal-model design

The problem of fulfilling requirement (a') and (b) introduced at the end of the previous section is now addressed by using design tools that are adopted in the literature of high-gain observers, by recalling the results presented in Byrnes and Isidori (2004). To this end, let the dimension of the regulator (1.31) be taken as $m = d$ and, by bearing in mind the definition in (1.29), let $\tau : W \to \mathbb{R}^d$ be defined as

$$\tau(w) \;:=\; \mathrm{col}\begin{pmatrix} c(w) & L_s c(w) & \ldots & L_s^{d-1} c(w) \end{pmatrix}\,,$$

and the triplet $(\phi(\cdot), \Psi, \gamma(\cdot))$ be taken as

$$\phi(\xi) \;:=\; \mathrm{col}\Big(\xi_2 \quad \cdots \quad \xi_{d-1} \quad \hat{\varphi}(\xi)\Big)\,, \qquad \Psi \;:=\; \mathrm{col}\Big(k_1 \quad \cdots \quad \ell^d k_d\Big)\,, \qquad \gamma(\xi) \;:=\; \xi_1 \tag{1.35}$$

where ℓ is a design parameter, the k_i's are coefficients of an Hurwitz polynomial, and $\hat{\varphi}(\cdot)$ is a bounded function that agrees with $\varphi(\cdot)$ on $\tau(W)$. Then, we have the following proposition whose proof can be obtained by slightly generalising the arguments in Byrnes and Isidori (2004) and can be found in Isidori et al. (2012).

Proposition 1.2. *Let* $c(\cdot)$ *in* (1.29) *be fulfilling* (1.30) *and let the triplet* $(\phi(\cdot), \Psi, \gamma(\cdot))$ *be taken as in* (1.35). *Then there exist a* $\ell^\star > 0$ *such that for all* $\ell \geq \ell^\star$ *requirements* (a') *and* (b) *of Section 1.6.1 are fulfilled with*

$$\mathcal{B} = \{(w, z, \chi) \in W \times \mathbb{R}^n \times \mathbb{R}^d\,, \quad z = \pi(w)\,,\ \chi = \tau(w)\}$$

and the ϵ *in* (1.34) *of the form*

$$\epsilon \;=\; \frac{r}{\ell^d} \sup_{w \in W} |\nu(w)|$$

with r *a positive number.*

By joining the result of Theorem 1.7 and the previous proposition it is then immediately concluded that there exists a $\kappa^\star$ (dependent on ℓ) such that for all $\kappa \geq \kappa^\star$ the regulator (1.31)) with $(\phi(\cdot), \Psi, \gamma(\cdot))$ taken as in (1.35) guarantees that the trajectories of the closed-loop systems originating from the given compact sets are bounded and

$$\lim_{t\to\infty} \sup |e(t)| \;\leq\; \frac{r'}{\kappa \ell^d} \sup_{w \in W} |\nu(w)| \tag{1.36}$$

for some positive constant r'. In particular, if the integer d and the function $\varphi(\cdot)$ can be taken so that relation (1.30) is fulfilled with $\nu(\cdot) = 0$, the proposed controller guarantees asymptotic regulation. Otherwise, just practical regulation is achieved with the bound on the asymptotic error that can be arbitrarily decreased by increasing κ or ℓ^d.

"When it is obvious that the goals cannot be reached, don't adjust the goals, adjust the action steps."

Confucius

2

Tools for High-Gain Observers

The topic of this chapter is the enrichment of the high-gain observer theory with new tools of design and analysis. The first section presents a novel observability form for single-input single-output non-input affine nonlinear systems. Necessary conditions under which this general class of nonlinear systems is diffeomorphic to a system having a *strict feedback form* are given. For the latter, the knowledge of the derivative of the input is needed. In Section 2.2 we propose a new design tool, based on a *gradient technique*, which may be applied to constrain the state of the observer in some given compact set. The proposed technique is coordinates independent, provided a convexity assumption holds. This tool can be successfully applied to the design of a high-gain observer in the original coordinates, or to prevent the peaking phenomenon (see Section 1.3.2). Carrying on the idea of the design in the original coordinates, in Section 2.3 we propose a novel set of sufficient conditions for the existence of an observer in the multi-input multi-output case. Some simple examples illustrate the main result. Finally, in Section 2.4, we propose a new analysis methodology which allows to characterize the steady-state behaviour of a high-gain observer in presence of high-frequency measurement noise. This chapter contains novel results published in Astolfi et al. (2013b) (Section 2.1), Astolfi and Praly (2013) (Section 2.2), Astolfi and Praly (2016-17) (Section 2.3) and Astolfi et al. (2016a) (Section 2.4).

2.1 Strict feedback form for non-input-affine systems

In Section 1.1 we showed that for input-affine nonlinear systems of the form

$$\begin{array}{rcl} \dot{x} & = & f(x) + g(x)u \\ y & = & h(x) \end{array} \qquad \text{(1.6 revisited)}$$

we can find a change of coordinates in which the system has a strict feedback form

$$\begin{array}{rcll} \dot{z}_i & = & z_{i+1} + b_i(z_1, \dots, z_i)u\,, & i = 1, \dots, n-1\,, \\ \dot{z}_n & = & a(z) + b_n(z_1, \dots, z_n)u\,, & \\ y & = & z_1\,, & \end{array} \qquad \text{(1.7 revisited)}$$

namely a linear part characterized by a sort of "chain of integrators" . From an observer-design perspective, the form (1.7) is easy to handle with respect to the form[1]

$$\begin{array}{rcll} \dot{z}_i & = & f_i(z_1, \dots, z_i, z_{i+1}, u)\,, & i = 1, \dots, i-1\,, \\ \dot{z}_n & = & f_n(z_1, \dots, z_n, u)\,, & \\ y & = & h(z_1, u)\,, & \end{array} \qquad \text{(1.4 revisited)}$$

Thus, one may wonder if a similar result can be achieved also for the general class of nonlinear systems (1.1)

$$\begin{array}{rcl} \dot{x} & = & f(x, u)\,, \\ y & = & h(x, u)\,, \end{array} \qquad \text{(1.1 revisited)}$$

which we know be, under some conditions, diffeomorphic to the form (1.4). As stated in the forthcoming theorem, under stronger assumptions, one may find a suitable change of coordinates by which the system (1.1) has the form (1.7) with the difference that the system is affine with respect to the input-derivative. The following result is an adaptation to the diffemorphic case of a more general result given in Astolfi et al. (2013b).

Theorem 2.1. *Consider the system* (1.1) *and suppose the mapping* $\Gamma : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}$ *defined as*

$$\Gamma(x, u) := \begin{pmatrix} u \\ \Phi_n(x, u) \end{pmatrix}$$

with $\Phi_n : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ *defined as*[2]

$$\Phi_n(x, u) := \mathrm{col}\Big(h(x, u) \quad L_{f(x,u)}h(x, u) \quad \cdots \quad L^{n-1}_{f(x,u)}h(x, u)\Big)$$

[1] see Sections 1.2 and 1.4

[2] See (1.3)

is a global diffeomorphism. Then, the system (1.1) *is globally diffemorphic, via* Φ_n, *to a system of the form*

$$\begin{aligned} \dot{z} &= \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_{n-1} \\ \dot{z}_n \end{pmatrix} = \begin{pmatrix} z_2 \\ z_3 \\ \vdots \\ z_n \\ a(u,z) \end{pmatrix} + \begin{pmatrix} b_1(u,z_1) \\ b_2(u,z_1,z_2) \\ \vdots \\ b_{n-1}(u,z_1,\dots,z_{n-1}) \\ b_n(u,z) \end{pmatrix} \dot{u}\,, \\ y &= z_1\,. \end{aligned} \tag{2.1}$$

Proof. This result is a special case of a more generic result given in Astolfi et al. (2013b). Therefore, for the kind of completeness, we give all the proof. Let us add an integrator to system (1.1), namely

$$\begin{aligned} \dot{x} &= f(x,u)\,, \\ \dot{u} &= v\,, \\ y &= h(x,u)\,, \end{aligned}$$

that is regarded as a system with input v, output y and state $\xi = \mathrm{col}(u,x)$. By letting $F(\xi) = \mathrm{col}(f(x,u), 0)$, $G = \mathrm{col}(0, 1)$, $H(\xi) = h(x,u)$, the previous system can be compactly rewritten as

$$\begin{aligned} \dot{\xi} &= F(\xi) + Gv \\ y &= H(\xi) \end{aligned}$$

In the new coordinates we get

$$\Phi_n(x,u) = \begin{pmatrix} H(\xi) \\ L_{F(\xi)}H(\xi) \\ \cdots \\ L^{n-1}_{F(\xi)}H(\xi) \end{pmatrix}.$$

The variable $z = \Phi_n(x,u)$ is governed by the dynamics

$$\begin{aligned} \dot{z}_1 &= z_2 + \tilde{g}_1(\xi)v \\ &\vdots \\ \dot{z}_{n-1} &= z_n + \tilde{g}_{n-1}(\xi)v \\ \dot{z}_n &= \tilde{f}(\xi) + \tilde{g}_n(\xi)v \end{aligned}$$

where $\tilde{f}(\xi) = L^n_{F(\xi)}H(\xi)$ and $\tilde{g}_i(\xi) = L_G L^{i-1}_{F(\xi)}H(\xi)$, $i = 1, \dots, n$. By assumption the function $\Gamma(x, u)$ is a global diffeomorphism so there exists a C^1 function $\Upsilon : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ defined as

$$\Upsilon(u, x) \;=\; \begin{pmatrix} u \\ \Psi_n(x, u) \end{pmatrix}$$

such that

$$\xi \;=\; \Upsilon(\Gamma(\xi)) \,, \qquad \mathfrak{z} \;=\; \Gamma(\Upsilon(\mathfrak{z})) \,,$$

for all $\xi \in \mathbb{R}^{n+1}$ and for all $\mathfrak{z} \in \mathbb{R}^{n+1}$, where we use the notation $\mathfrak{z} = \mathrm{col}(u, z)$. Now let $b_i : \mathbb{R}^{n+1} \to \mathbb{R}$, $i = 1, \dots, n$, be the continuous function defined as

$$b_i(\mathfrak{z}) \;=\; \tilde{g}_i(\Upsilon(\mathfrak{z})) \,.$$

For all $k = 1, \dots, n$, and each pair $\mathfrak{z}^a = (u, z^a)$ and $\mathfrak{z}^b = (u, z^b)$ in $\mathbb{R}^{n+1}$ satisfying $z_i^a = z_i^b$ for all $i = 1, \dots, k$, we have $b_k(\mathfrak{z}^a) = b_k(\mathfrak{z}^b)$. This fact follows by an elementary adaptation of the arguments in (Gauthier and Kupka, 2004, Theorem 4.1, chapter 3) we write here just for the case $k = 1$.

Let $u_* \in \mathbb{R}$, $x_*^a \in \mathbb{R}^n$ and $x_*^b \in \mathbb{R}^n$ be such that $\xi_*^a = (u_*, x_*^a)$ and $\xi_*^b = (u_*, x_*^b)$ satisfy $H(\xi_*^a) = H(\xi_*^b)$ or equivalently $z_{*0}^a = z_{*0}^b$, with $z_*^a = \Gamma(\xi_*^a)$ and $z_*^b = \Gamma(\xi_*^b)$. Assume we have

$$b_1(u_*, z_*^a) \;\neq\; b_1(u_*, z_*^b)$$

i.e.

$$L_G H(\xi_*^a) = \tilde{g}_1(\xi_*^a) \;\neq\; \tilde{g}_1(\xi_*^b) = L_G H(\xi_*^b) \,.$$

By continuity there exist neighbourhoods $\mathcal{N}^a$ and $\mathcal{N}^b \subset \mathbb{R} \times \mathbb{R}^n$ of ξ_0^a and ξ_0^b such that

$$L_G H(\xi^a) \neq L_G H(\xi^b) \qquad \forall\, (\xi^a, \xi^b) \in \mathcal{N}^a \times \mathcal{N}^b \,.$$

Consider now the system

$$\dot{x}^a \;=\; f(x^a, u) \,, \quad \dot{x}^b = f(x^b, u) \,, \quad \dot{u} = v_1$$

with output $\tilde{y} = h(x^a, u) - h(x^b, u)$ and input v_1 taken as the feedback

$$v_1 = \frac{L_{F(\xi^a)}H(\xi^a) - L_{F(\xi^b)}H(\xi^b)}{L_G H(\xi^a) - L_G H(\xi^b)} \,.$$

It is motivated by the fact that it gives $\dot{\tilde{y}} = 0$. And it is as many times differentiable as needed as long as (ξ^a, ξ^b) is in $\mathcal{N}^a \times \mathcal{N}^b$. Let $(\xi^a(t), \xi^b(t))$ be its solution with initial value (ξ_*^a, ξ_*^b). There exists a $T > 0$ such that for all $t \in [0, T)$ $(\xi^a(t), \xi^b(t)) \in \mathcal{N}^a \times \mathcal{N}^b$ and, as a consequence, the components $x^a(t)$ and $x^b(t)$ are in ${}^cX$ and $u(t)$ is in ${}^cU$ for all $t \in [0, T)$. Furthermore, since $t \mapsto \tilde{y}(t)$ is constant on $[0, T)$ and $\tilde{y}(0) = 0$, it is zero on the whole interval. So the same holds for its n first derivatives. By definition of the function Φ_n and by denoting $\boldsymbol{u}_i(t) = (u(t), u^{(1)}(t), \ldots, u^{(i-1)}(t))$, we get $\Phi_n(x^a(t), \boldsymbol{u}_n(t)) = \Phi_n(x^b(t), \boldsymbol{u}_n(t))$ and thus

$$\begin{aligned} x^a(t) &= \Psi_n(\Phi_n(x^a(t), \boldsymbol{u}_n(t)), \boldsymbol{u}_n(t)) \\ &= \Psi_n(\Phi_n(x^b(t), \boldsymbol{u}_n(t)), \boldsymbol{u}_n(t)) \\ &= x^b(t) \qquad \forall t \in [0, T) , \end{aligned}$$

This yields in particular $x_*^a = x_*^b$. So we have $\xi_*^a = \xi_*^b$ and thus $\tilde{g}_1(\xi_*^a) = \tilde{g}_1(\xi_*^b)$. This is a contradiction. In this way, we have shown that, for each pair $\mathfrak{z}^a = (u, z^a)$ and $\mathfrak{z}^b = (u, z^b)$ in $\mathbb{R} \times \mathbb{R}^n$ satisfying $z_0^a = z_0^b$, we have $b_1(\mathfrak{z}^a) = b_1(\mathfrak{z}^b)$. Similar arguments, can be used by induction for $k = 1, \ldots, n$, with an appropriate choice of the input derivative $u^{(k+1)}$. From the above, it follows that the functions $b_i(\mathfrak{z})$ presents a triangular structure in the z_i components of z. Finally, by defining

$$a(z, u) = \tilde{f}(\Upsilon(\mathfrak{z})) ,$$

with these functions we do have obtained the form (2.1). □

Notice that the form (2.1) is in the *strict-feedback form* (1.11) introduced in Section 1.2. As a consequence, if the inverse of the diffeomorphism $\Gamma(x, u)$ and the derivative of the input u are known[3], one can implement an observer for the system (1.1) by following the procedure of Section 1.2, thus resulting in a simpler design with respect to the one proposed in Section 1.4.

[3] For instance, when the control input u is obtained via a backstepping design, see Astolfi and Praly (2013).

2.2 Imposing state constrains in high-gain observers

In this section we present a design tool, based on a *gradient technique*, which can be applied to high-gain observers in order to constrain the state of the latter in a desired compact set. This tools results very effective for the observer design in the original coordinates introduced in Section 1.5, although it may be applied also to the standard design in the feedback coordinates (1.11) or (1.20) in order to remove the peaking phenomenon. For the sake of compactness, we are going to present the tool as an application for the original coordinates design, adding some remarks at the end of this section. The main contents of this section have been published in Astolfi and Praly (2013).

We consider here multi-input single-output input-affine nonlinear systems of the form

$$\begin{array}{rcl} \dot{x} & = & f(x) + g(x)u \\ y & = & h(x) \end{array} \tag{2.2}$$

where the state x belongs to an open subset $\mathcal{X}$ of $\mathbb{R}^n$, the input evolve in a compact subset U of $\mathbb{R}^m$ and $y \in \mathbb{R}$ is the measured output. The functions f, g, h are smooth enough and the system (2.2) satisfies the forthcoming assumptions[4]. Comments on the following assumptions are given at the end of this section.

Assumption 2.1. *There exists a diffeomorphism $\Phi : \mathcal{X} \to \mathbb{R}^n$ such that the system* (2.2) *is transformed, via $z = \Phi(x)$, into a system in strict-feedback form, namely*

$$\dot{z} \;=\; Az + \phi(z, u) \;, \qquad\qquad y \;=\; Cz \;, \tag{2.3}$$

where (A, C) is a pair in prime form *and $\phi : \mathcal{Z} \times \mathbb{R} \to \mathbb{R}^n$, with $\mathcal{Z} = \Phi(\mathcal{X})$, is a locally Lipschitz function possessing a triangular structure, namely $\phi = (\varphi_1, \ldots, \varphi_n)^\top$ with*

$$\varphi_i(z, u) \;=\; \varphi_i(z_1, \ldots, z_i, u) \;, \qquad\qquad 1 \leq i \leq n \,.$$

Assumption 2.2. *Given the set $\mathcal{X}$ and a diffeomorphism Φ, for any compact subset X of $\mathcal{X}$, there exists a C^1 function $h_2 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying:*

H1. *the set $\mathfrak{C}_0 = \{x \in \mathcal{X} : h_2(x) \leq 0\}$ contains X and has a non empty interior;*

H2. *the set $\{x \in \mathbb{R}^n \;:\; h_2(x) < 1\}$ is a subset of $\mathcal{X}$;*

H3. *the function $x \mapsto \frac{h_2(x)}{\left|\frac{\partial h_2}{\partial x}(x)\right|}$ is continuous on $\mathcal{X}$;*

[4] Necessary conditions under which Assumption 2.1 is satisfied in the single-input case ($m = 1$) are given in Theorem 1.2.

H4. *for any real number s in $[0,1]$ and any x_1 and x_2 in $\mathcal{X}$ satisfying $h_2(x_1) \leq s$ and $h_2(x_2) \leq s$ we have $h_2(x) \leq s$ for all x which satisfies for some λ in $[0,1]$:*

$$\Phi(x) \;=\; \lambda\Phi(x_1) + (1-\lambda)\Phi(x_2)\ .$$

This means nothing but the fact that, for any non negative real number s the image by Φ of the set $\{x \in \mathbb{R}^n : \ h_2(x) \leq s\}$ is convex.

H5. *the set $\mathfrak{C}_{\frac{1}{2}} = \{x \in \mathbb{R}^n \ : \ h_2(x) \leq \frac{1}{2}\}$ is compact.*

The point H1 of the previous assumption motivates us for introducing a dummy measured output

$$y_2 = h_2(x)$$

and to consider that its measured value is always 0, namely the state x evolves in X for all $t \geq 0$. With this, we can proposed the following observer in the original coordinates

$$\dot{\hat{x}} \;=\; f(\hat{x}) + g(\hat{x})u + \left(\frac{\partial \Phi}{\partial x}(\hat{x})\right)^{-1} D_\ell K(y - h(\hat{x})) + \mathcal{M}(\hat{x}, y, u) \tag{2.4}$$

with state $\hat{x} \in \mathbb{R}^n$ and where $D_\ell = \text{diag}(\ell, \ldots, \ell^n)$ with $\ell \geq 1$ the high-gain parameter, and K is a $n \times 1$ matrix chosen such that there exists a positive definite symmetric matrix P satisfying

$$P(A - KC) + (A - KC)^\top \;=\; -I\,. \tag{2.5}$$

The "modification term" $\mathcal{M}$ is defined as

$$\mathcal{M}(\hat{x}, y, u) \;=\; -\tau_\ell(\hat{x}, y, u)\ \ell^{-1} \left(\frac{\partial \Phi}{\partial x}(\hat{x})\right)^{-1} D_\ell P^{-1} D_\ell \left(\frac{\partial \Phi}{\partial x}(\hat{x})\right)^{-1T} \frac{\partial h_2}{\partial x}(\hat{x})^\top h_2(\hat{x}) \tag{2.6}$$

where $\tau_\ell : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ is a locally Lipschitz function to be chosen large enough. The function $\mathcal{M}$ is an extra output injection term that acts only when the state $\hat{x}$ goes outside the set X, and guarantees the set $\mathfrak{C}_{\frac{1}{2}}$ to be forward invariant for the observer (2.4), as stated by the forthcoming theorem.

Theorem 2.2. *Consider the system* (2.2) *under Assumptions 2.1 and 2.2 and suppose that $x(t) \in X$ and $u(t) \in U$ for all $t \geq 0$. Consider the observer* (2.4) - (2.6) *with initial conditions $\hat{x}(0) \in \mathfrak{C}_{\frac{1}{2}}$. There exist a function $\tau_\ell : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$, strictly positive real numbers $\ell^\star \geq 1$ and μ_1, μ_2, such that for any $\ell > \ell^\star$*

(i) *the set $\mathfrak{C}_{\frac{1}{2}}$ is forward invariant for the observer* (2.4) - (2.6),

(ii) $|\hat{x}(t) - x(t)| \;\leq\; \mu_1\, \ell^{n-1}\, \exp(-\mu_2 \ell\, t)\, |\hat{x}(0) - x(0)|\,.$

Proof. We start by proving the point (i). Note that

$$\frac{\partial h_2}{\partial x}(\hat{x})\,\dot{\hat{x}} \;=\; R(\hat{x},y,u) - \;\tau_\ell(\hat{x},y,u)\left|D_\ell P^{-\frac{1}{2}}\left(\frac{\partial \Phi}{\partial x}(\hat{x})\right)^{-1\top}\frac{\partial h_2}{\partial x}(\hat{x})^\top\right|^2 h_2(\hat{x})$$

where we have let

$$R(\hat{x},y,u) = \frac{\partial h_2}{\partial x}(\hat{x})\left[f(\hat{x}) +\, g(\hat{x})u + \left(\frac{\partial \Phi}{\partial x}(\hat{x})\right)^{-1} D_\ell K(y - h(\hat{x}))\right].$$

This motivates us for choosing τ_ℓ satisfying

$$\tau_\ell(\hat{x},y,u) \;\geq\; \frac{8R(y,u,\hat{x})\;h_2(\hat{x})^2}{\left|D_\ell P^{-\frac{1}{2}}\left(\frac{\partial \Phi}{\partial x}(\hat{x})\right)^{-1\top}\frac{\partial h_2}{\partial x}(\hat{x})^\top\right|^2}$$

which can be computed on-line. Thanks to H2, the function $x \mapsto \tau_\ell(x)$ defined this way is continuous on $\mathcal{X}$. It implies that the derivative of $h_2(\hat{x})$ along the solutions is non positive when $h_2(\hat{x})$ is strictly larger than $\frac{1}{2}$. This implies that, for each s in $[\frac{1}{2}, 1]$ the set $\{\hat{x} :\; h_2(\hat{x}) \leq s\}$ is forward invariant and so is the compact set $\mathfrak{C}_{\frac{1}{2}}$.

In order to prove (ii), let for the time ignore the modification term $\mathcal{M}(\hat{x})$ in the observer (2.4). Let $V : \mathcal{X}\times\mathcal{X} \to \mathbb{R}_{\geq}0$ be the Lyapunov function defined as

$$V \;=\; \ell\left(\Phi(\hat{x}) - \Phi(x)\right)^\top\, D_\ell^{-1}\,P\,D_\ell^{-1}\ell\,\left(\Phi(\hat{x}) - \Phi(x)\right).$$

Its derivative along the solution is given by

$$\begin{aligned}\dot{V} \;=\; 2\ell D_\ell^{-1}(\Phi(\hat{x}) - \Phi(x))^\top P D_\ell^{-1}\ell\Big[&A\Phi(\hat{x}) + D_\ell K(C\Phi(x) - C\Phi(\hat{x}))\\ &+ \phi(\Phi(\hat{x}),u) - A\Phi(x) - \phi(\Phi(x),u)\Big]\end{aligned}$$

Let L be a strictly positive real number satisfying

$$|\phi(\Phi(\hat{x}),u) - \phi(\Phi(x),u)| \;\leq\; L\,|\Phi(\hat{x}) - \Phi(x)|\,,\qquad \forall\,(\hat{x},x)\in,\mathfrak{C}_{\frac{1}{2}}\times\mathfrak{C}_{\frac{1}{2}}\quad\forall\,u\in U\,.$$

By denoting $e = \ell D_\ell^{-1}(\Phi(\hat{x}) - \Phi(x))$ and by using (1.17) and (2.5), we get

$$\dot{V} \;\leq\; -\ell|e|^2 + 2L\,|P|\,|e|^2\,.$$

With a straightforward application of the same arguments used in the proof of the Theorem 1.4, we can prove that there exists a $\ell^\star \geq 1$ and strictly positive real numbers m_1, m_2 such that

$$|e(t)| \;\leq\; m_1\, \ell^{n-1} \exp(-m_2\, \ell\, t)|e(0)|\,,$$

and therefore, by using the fact that for any $\ell \geq 1$

$$|e| \;\leq\; |\Phi(\hat{x}) - \Phi(x)| \;\leq\; \ell^{n-1}\, |e|$$

we get

$$|\Phi(\hat{x}(t)) - \Phi(x(t))| \;\leq\; m_1\, \ell^{n-1} \exp(-m_2\, \ell\, t)\, |\Phi(\hat{x}(0)) - \Phi(x(0))|\;,$$

for all $\ell > \ell^\star$. Now, recall that Φ is a diffeomorphism on $\mathcal{X}$. As a consequence for any compact set in $\mathcal{X}$, and in particular for $\mathfrak{C}_{\frac{1}{2}}$, there exist positive real numbers L_Φ and $L_{\Phi^{-1}}$ satisfying

$$\begin{aligned} |\hat{x} - x| \;=\; |\Phi^{-1}(\Phi(\hat{x})) - \Phi^{-1}(\Phi(x))| \;&\leq\; L_{\Phi^{-1}}|\Phi(\hat{x}) - \Phi(x)| \\ |\Phi(\hat{x}) - \Phi(x)| \;&\leq\; L_\Phi|\hat{x} - x| \end{aligned}$$

for any $(\hat{x}, x) \in \mathfrak{C}_{\frac{1}{2}} \times \mathfrak{C}_{\frac{1}{2}}$. The proof completes by defining $\mu_1 = m_1 L_{\Phi^{-1}} L_\Phi$ and $\mu_2 = m_2$.

Note that the additional term $\mathcal{M}(\hat{x})$ does not destroy the aforementioned asymptotic properties. This can be proved by noting that when it is acting, it augments the derivative of the Lyapunov function V with

$$-\tau_\ell(y, u, \hat{x})\, [\Phi(\hat{x}) - \Phi(x)]^\top \left(\frac{\partial \Phi}{\partial x}(\hat{x})\right)^{-1\top} \frac{\partial h_2}{\partial x}(\hat{x})^\top h_2(\hat{x})\;.$$

Recall that by definition a convex function f satisfies (see Boyd and Vandenberghe (2011))

$$f(x) \;\geq\; f(y) + f'(y)(x - y) \qquad \Longrightarrow \qquad f'(y)(y - x) \;\geq\; f(y) - f(x)\;.$$

Furthermore the properties H3 and H4 of the Assumption 2.2 imply that the function h_2 is monotonic and the image of the set $\{x \in \mathcal{X} : h_2(x) \leq s\}$ is convex. As a consequence, when $h_2(x)$ is zero, which is the case when the model state x remains in X, and when $h_2(\hat{x})$ is in $[0, 1]$, and in particular $\mathfrak{C}_{\frac{1}{2}}$, the convexity property aforementioned of h_2 gives

$$\frac{\partial h_2}{\partial x}(\hat{x}) \left(\frac{\partial \Phi}{\partial x}(\hat{x})\right)^{-1} [\Phi(\hat{x}) - \Phi(x)] \;\geq\; h_2(\hat{x}) - h_2(x) \;\geq\; h_2(\hat{x}) \;\geq\; 0$$

By recalling that $h_2(\hat{x}) \geq 0$ and $\tau_\ell(\cdot) > 0$ for any $\hat{x} \in \mathfrak{C}_{\frac{1}{2}}$ we get

$$0 \;\leq\; \tau_\ell(\hat{x}, y, u)\, [\Phi(\hat{x}) - \Phi(x)]^\top \left(\frac{\partial \Phi}{\partial x}(\hat{x})\right)^{-1\top} \frac{\partial h_2}{\partial x}(\hat{x})^\top h_2(\hat{x})$$

We conclude that, that the derivative of V remains negative and the bound claimed in the theorem holds as long as $x \in X$ and $\hat{x}$ in $\{x \in \mathcal{X} : h_2(x) \leq 1\}$ and so in $\mathfrak{C}_{\frac{1}{2}}$. □

Remarks

- There is a systematic way to define the function h_2 when, given the compact set X we know the existence of a positive definite symmetric matrix Q and a real number R satisfying

  $$\Phi(X) \;\subseteq\; \left\{z \in \mathbb{R}^n : z^\top Q z \;\leq\; R\right\} \;\subset\; \left\{z \in \mathbb{R}^n : z^\top Q z \;\leq\; R+1\right\} \;\subset\; \Phi(\mathcal{X})\,.$$

  In this case we the function h_2 can be chosen as

  $$h_2(x) \;=\; \max\left\{\Phi(x)^\top Q\, \Phi(x) - R\,,\, 0\right\}^2\,.$$

  It is easy to check that with this choice Properties H1 and H2 are satisfied:

  $$\begin{aligned} \{x \in \mathcal{X} : h_2(x) \leq 0\} &= \left\{x \in \mathcal{X} : \Phi(x)^\top Q\, \Phi(x) \leq R\right\} &\supseteq\; \Phi(X) \\ \{x \in \mathcal{X} : h_2(x) < 1\} &= \left\{x \in \mathcal{X} : \Phi(x)^\top Q\, \Phi(x) < R+1\right\} &\subset\; \Phi(\mathcal{X}) \end{aligned}$$

  The function claimed by H3 is given by

  $$\begin{aligned} \frac{h_2(x)}{\left|\frac{\partial h_2}{\partial x}(x)\right|} &= \frac{\max\left\{\Phi(x)^\top Q\, \Phi(x) - R,\, 0\right\}^2}{\left|4 \max\left\{\Phi(x)^\top Q\, \Phi(x) - R\,,\, 0\right\} \Phi(x)^\top Q\, \frac{\partial \Phi}{\partial x}(x)\right|} \\ &\leq \max\left\{\frac{\Phi(x)^\top Q\, \Phi(x) - R}{|\Phi(x)^\top Q|}\,,\, 0\right\} \left|\left(\frac{\partial \Phi}{\partial x}(x)\right)^{-1}\right| \end{aligned}$$

  which is continuous on $\mathcal{X}$ because $\Phi(x)$ is a diffeomorphism (and therefore its Jacobian is always non singular for all $x \in \mathcal{X}$). In order to prove property H4, recall that since Q is a positive definite symmetric matrix the following holds

$$z_1^\top Q z_2 \;=\; z_2^\top Q z_1 \;, \qquad 2 z_1^\top Q z_2 \;\leq\; z_1^\top Q z_1 + z_2^\top Q z_2 \;,$$

for any $z_1 \in \mathbb{R}^n$ and $z_2 \in \mathbb{R}^n$. Now let $s \in [0,1]$ be fixed. Let x_1 and x_2 in $\mathcal{X}$ satisfying $h_1(x_1) \leq s$, $h_2(x) \leq s$. By denoting $z_1 = \Phi(x_1)$ and $z_2 = \Phi(x_2)$ we have

$$z_1 Q z_1 \;\leq\; \sqrt{s} + R \;, \qquad z_2 Q z_2 \;\leq\; \sqrt{s} + R$$

Now let consider any $z = \Phi(x)$ of the form

$$z = \lambda z_1 + (1-\lambda) z_2$$

for any $\lambda \in [0,1]$. We have

$$\begin{aligned}
z^\top Q z &= (\lambda z_1 + (1-\lambda) z_2)^\top Q (\lambda z_1 + (1-\lambda) z_2) \\
&= \lambda^2 z_1^\top Q z_1 + (1-\lambda)^2 z_2^\top Q z_2 + 2\lambda(1-\lambda) z_1^\top Q z_2 \\
&\leq \lambda^2 z_1^\top Q z_1 + (1-\lambda)^2 z_2^\top Q z_2 + \lambda(1-\lambda)(z_1^\top Q z_1 + z_2^\top Q z_2) \\
&\leq \lambda z_1^\top Q z_1 + (1-\lambda) z_2^\top Q z_2 \\
&\leq \lambda(\sqrt{s} + R) + (1-\lambda)(\sqrt{s} + R) \\
&\leq (\sqrt{s} + R)
\end{aligned}$$

proving that also H4 holds.

- We may dislike the convexity property mentioned in H4 above. Unfortunately it is in some sense necessary. Indeed our objective with the modification $\mathcal{M}$ is to preserve the high-gain paradigm. This means in particular that we choose to keep an Euclidean distance in the image by Φ as a Lyapunov function for the error. Also we need an infinite gain margin, as defined in Definition 2.8 in Sanfelice and Praly (2012), since the correction term must dominate all the other ones in the expression of $\dot{\hat{x}}$ when h_2 becomes too large. Then, as proved in Lemma 2.7 of Sanfelice and Praly (2012), with such constraints, the convexity assumption is necessary. This implies that, if we want to remove the convexity assumption, we have to find another class of observers.

- As shown in Astolfi and Praly (2013), the proposed technique can be easily extended to the class of nonlinear systems

$$\dot{x} = f(x,u)\,, \qquad y = h(x,u)\,,$$

by exploiting the results of Theorem 2.1 presented in the Section 2.1. In this case the knowledge of the input-derivative is needed and the function h_2 claimed in Assumption 2.2 is in general u-dependent. In an output feedback scenario, we can always suppose the input-derivative $\dot{u}$ is known by making a backstepping design.

- Clearly the proposed tool can be applied to the high-gain observer (1.12) in order to prevent the peaking phenomenon. For instance, we can take the function h_2 as

$$h_2(x) = \max\left\{x^\top Q\, x - R\,,\, 0\right\}^2$$

for some $Q = Q^\top > 0$ and $R > 0$, and the modification term (2.6) as

$$\mathcal{M}(\hat{x}, y, u) = -\tau_\ell(y, u, \hat{x})\, \ell^{-1} D_\ell P^{-1} D_\ell \frac{\partial h_2}{\partial x}(\hat{x})^\top h_2(\hat{x})$$

with $\tau_\ell : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}_{>0}$ a locally Lipschitz function to be chosen (large enough) according to the analysis made in the proof of Theorem 2.2. As a consequence, if the set

$$\mathfrak{C}_0 = \{x \in \mathbb{R}^n \,:\, x^\top Q x \leq R\}$$

is invariant for the system (1.11), the set

$$\mathfrak{C}_{\frac{1}{2}} = \{x \in \mathbb{R}^n \,:\, x^\top Q x \leq R + \tfrac{1}{2}\}$$

becomes invariant for the observer (1.12), thus removing the annoying peaking phenomenon.

2.3 High-gain observer for multi-output nonlinear systems

The contents of this section have been published in Astolfi and Praly (2016-17). We focus in particular on the high-gain observer design for input affine multi-input multi-output nonlinear systems of the form

$$\begin{array}{rcl} \dot{x} &=& f(x) + g(x)u \\ y &=& h(x) \end{array} \tag{2.7}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$ and output $y \in \mathbb{R}^p$. The problem of observation with high-gain tools is strongly related to the existence of suitable triangular coordinates, and to the existence of a diffeomorphism Φ which puts the system (2.7) in the aforementioned form. As shown in Tornambé (1992b), a typical expression for Φ is

$$\Phi(x) = \begin{pmatrix} \Phi_1(x) \\ \vdots \\ \Phi_p(x) \end{pmatrix}, \qquad \Phi_i(x) = \begin{pmatrix} h_i(x) \\ L_f h_i(x) \\ \vdots \\ L_f^{q_i-1} h_i(x) \end{pmatrix},$$

where h_i is the i-th component of h, q_i are integers called "observability indexes" and $\sum_{i=1}^p q_i = n$. The dynamics of the system (2.7) expressed in the new coordinates reads

$$\begin{array}{rcl} \dot{z} &=& Az + B\psi(z) + \varphi(z)u \\ y &=& Cz \end{array} \tag{2.8}$$

where

$$\begin{array}{rcl} A &:=& \text{blckdiag}\big(A_1, \ldots, A_p\big) \in \mathbb{R}^{n\times n} \\ B &:=& \text{blckdiag}\big(B_1, \ldots, B_p\big) \in \mathbb{R}^{n\times p} \\ C &:=& \text{blckdiag}\big(C_1, \ldots, C_p\big) \in \mathbb{R}^{p\times n} \end{array}$$

where (A_i, B_i, C_i) is a triplet in *prime form* of dimension q_i and

$$\begin{array}{rcl} \psi(z) &=& \text{col}\Big(\psi_1(z) \quad \cdots \quad \psi_p(z)\Big), \\ \psi_i(z) &:=& L_f^{q_i} h_i(x)\Big|_{x=\Phi^{-1}(z)}, \qquad i = 1, \ldots, p, \\ \varphi(z) &=& \text{col}\Big(\Big(\varphi_1^1(z) \quad \cdots \quad \varphi_1^{q_1}(z)\Big) \quad \cdots \quad \Big(\varphi_p^1(z) \quad \cdots \quad \varphi_p^{q_p}(z)\Big)\Big), \\ \varphi_i^j(z) &:=& L_g L_f^{j-1} h_i\Big|_{x=\Phi^{-1}(z)}, \qquad i = 1, \ldots, p, \quad j = 1, \ldots, q_i. \end{array}$$

Sometimes the notions of observability indexes and relative degree indexes[5] coincide, namely when

$$L_g L_f^{j-1} h_i(x) = 0 \qquad i = 1, \dots, p, \quad j = 1, \dots, q_i - 1 \, .$$

In this case the system (2.8) reads as

$$\begin{array}{rcl} \dot{z} & = & Az + B(\psi(z) + \varphi_q(z)u) \\ y & = & Cz \end{array} \tag{2.9}$$

with $\varphi_q : \mathbb{R}^n \to \mathbb{R}^{p \times m}$ defined as

$$\varphi_q(z) \;=\; \text{col}\left(\varphi_1^{q_1}(z) \quad \cdots \quad \varphi_p^{q_p}(z)\right), \quad \varphi_i^{q_i} \;:=\; L_g L_f^{q_i - 1} h_i(x)\Big|_{x = \Phi^{-1}(z)}, \quad i = 1, \dots, p \, .$$

When $\psi(\cdot)$ and $\varphi(\cdot)$ are locally Lipschitz functions a high-gain observer for the class of nonlinear systems (2.8) can be designed as

$$\dot{\hat{z}} \;=\; A\hat{z} + B\hat{\psi}(\hat{z}) + \hat{\varphi}(\hat{z})u + D_\ell K(y - C\hat{z})$$

where $\hat{\psi}(\cdot)$ and $\hat{\varphi}(\cdot)$ are bounded functions that agrees with $\psi(\cdot)$ and $\varphi(\cdot)$ on some domain, and

$$\begin{array}{rclcrcl} K & = & \text{blckdiag}(K_1, \dots, K_p) & \qquad & K_i & = & \text{col}(k_1^i, \dots, k_{q_i}^i) \\ D_\ell & = & \text{blckdiag}(D_1(\ell), \dots, D_p(\ell)) & \qquad & D_i(\ell) & = & \text{blckdiag}(\ell^{\delta_i}, \dots, \ell^{q_i \delta_i}) \end{array}$$

with K_i designed so that $(A_i - K_i C_i)$ is Hurwitz for any $i = 1, \dots, p$, and where $\delta_i > 0$ are some indexes which depends on the structure of the nonlinearities. Similarly, a high-gain observer for the class of nonlinear systems (2.9) can be taken as

$$\dot{\hat{z}} \;=\; A\hat{z} + B(\hat{\psi}(\hat{z}) + \hat{\psi}_q(\hat{z})u) + D_\ell K(y - C\hat{z})$$

Furthermore, in an output stabilization frameworks (see, for instance, Seshagiri and Khalil (2005)) the latter can be even reduced to

$$\dot{\hat{z}} \;=\; A\hat{z} + D_\ell K(y - C\hat{z})$$

when $\varphi(0) = 0$ and $u = \alpha(z)$, with $\alpha : \mathbb{R}^n \to \mathbb{R}^m$ a stabilizing feedback law satisfying $\alpha(0) = 0$.

[5]See the Chapter 5 of Isidori (1995)

As extensively studied in literature, the functions $\psi(\cdot)$ and $\varphi(\cdot)$ may not have the triangular structure we like for the design of a high-gain observer. As already discussed in the proof of Theorem 1.4, in the single-input single-output case a necessary condition is

$$\frac{\partial \varphi_i}{\partial z_j} = 0 \qquad i = 1, \ldots, n-1, \qquad \forall\, j \geq i\ .$$

However this condition is in general not sufficient in the multi-input multi-output case because variables coming from other blocks may introduce "bad terms" multiplied by powers too large of the high-gain parameter. It is worth noticing that these conditions are coordinates-dependent: choosing a different diffeomorphism $\Phi(\cdot)$ may lead to "right" triangular structures. Conditions under which we do get the triangular dependence for the functions $\psi(\cdot)$ and $\varphi(\cdot)$ have been studied for instance in Bornard and Hammouri (1991) and Hammouri et al. (2010). Anyhow, it is worth noticing that in order to check the above conditions it is needed not only an expression for Φ, but also the knowledge of Φ^{-1}.

Again, by following the intuition of writing the observer in the original coordinates, we give in this section alternative conditions, for which the inverse of Φ is not needed, under which we do have an appropriate structure for an observer design.

Sufficient conditions

In the following we consider systems of the form (2.7) and we suppose the state x belongs to an open set $\mathcal{X} \subset \mathbb{R}^n$ and the input u evolves in a compact set $U \subset \mathbb{R}^m$.

Assumption 2.3. *There exist a C^1 function $\Phi : \mathcal{X} \to \mathbb{R}^n$, sequences of matrices $L_\ell \in \mathbb{R}^{n\times n}$, $M_\ell \in \mathbb{R}^{n\times n}$ and $N_\ell \in \mathbb{R}^{p\times p}$, a matrix $C \in \mathbb{R}^{p\times n}$, matrix functions $u \in \mathcal{U} \mapsto K(u) \in \mathbb{R}^{n\times p}$ and $u \in \mathcal{U} \mapsto A(u) \in \mathbb{R}^{n\times n}$, and a positive definite symmetric matrix $P \in \mathbb{R}^{n\times n}$ and real numbers $\gamma > 0$ and $d > 0$, such that*

O1) *the function Φ is a diffeomorphism on the set $\mathcal{X}$,*

O2) $C\,\Phi(x) \;=\; h(x)$,

O3) *the matrices $A(u), K(u), P, C$ satisfy, for any $u \in U$,*

$$P(A(u) - K(u)C) + (A(u) - K(u)C)^\top P \;\leq\; -2\gamma P\,,$$

$$A(u)\, L_\ell \;=\; L_\ell\, M_\ell\, A(u)\,, \qquad N_\ell\, C\, L_\ell \;=\; C\,,$$

O4) *the matrix M_ℓ is such that $M_\ell P^{-1}$ is symmetric and satisfies*

$$\lim_{\ell\to+\infty} \lambda_{\min}(M_\ell P^{-1}) \;=\; +\infty\,,$$

O5) $\lambda_{\max}\left(L_\ell M_\ell P^{-1} L_\ell^\top\right) \le \lambda_{\min}\left(M_\ell P^{-1}\right)^d$,

$1 \le \lambda_{\min}\left(L_\ell M_\ell P^{-1} L_\ell^\top\right) \lambda_{\min}\left(M_\ell P^{-1}\right)^d$.

Moreover, for any compact set $\mathfrak{C}$ and $\widehat{\mathfrak{C}}$ satisfying $\mathfrak{C} \subset \widehat{\mathfrak{C}} \subset \mathcal{X}$ there exists a sequence of positive real numbers c_ℓ such that

O6) $\lim_{\ell \to +\infty} c_\ell = 0$,

O7) *the function $B : \mathbb{R}^{n\times m} \to \mathbb{R}^n$ defined as*

$$B(\Phi(x), u) = L_f\Phi(x) + L_g\Phi(x)u - A(u)\,\Phi(x)\ ,$$

satisfies, for all $x_a \in \mathfrak{C}$, $x_b \in \widehat{\mathfrak{C}}$ and $u \in U$,

$$\left|P^{\frac{1}{2}} M_\ell^{-1} L_\ell^{-1}\left[B(\Phi(x_a), u) - B(\Phi(x_b), u)\right]\right| \le c_\ell \left|P^{\frac{1}{2}} L_\ell^{-1}\left[\Phi(x_a) - \Phi(x_b)\right]\right| .$$

Remarks:

- As shown in the forthcoming lemma, the existence of a high-gain observer for the system (2.7) is guaranteed if Assumption 2.3 holds. In particular the properties O1, O2, O3, O6 and O7 guarantee the existence of a converging observer in the original coordinates whereas properties O4 and O5 assure its tunability property.

- We remark that these conditions can be checked without need of finding formally the inverse mapping Φ^{-1}. In particular, given a system and a candidate diffeomorphism Φ satisfying O2 (linear dependence of the diffeomorphism on the output), it is possible to fix the degrees of freedom $K(u)$, M_ℓ, N_ℓ, L_ℓ, P which properly defines the high-gain observer (see (2.10)) according to O3 (which guarantees the convergence of the observer) and check the Lipschitz condition in O7.

- As introduced before, sometimes the nonlinear terms $B(\Phi(x, u), u)$ can be disregarded in the high gain observer design (usually also called dirty derivative observer). In this case, these nonlinear terms act through their bound and not their Lipschitzness. Unfortunately then a very specific structure is needed because otherwise the gain between these nonlinear terms and some estimation error is increasing with the observer gain. Here we intend to consider a broader class of systems and thus we do need to have these terms present in the observer.

Further clarifications about the formal conditions imposed by Assumption 2.3 will be given through some examples after stating the main result of this section.

Observer design

By following the same route of Section 2.2 the high-gain observer in the original coordinates is designed under Assumption 2.2 as

$$\dot{\hat{x}} = f(\hat{x}) + g(\hat{x})u + \left(\frac{\partial \Phi}{\partial x}(\hat{x})\right)^{-1} L_\ell M_\ell K(u) N_\ell\big(y - h(\hat{x})\big) + \mathcal{M}(\hat{x}, y, u) \tag{2.10}$$

with

$$\mathcal{M}(\hat{x}, y, u) \;=\; -\tau_\ell(\hat{x}, y, u) \; \left(\frac{\partial \Phi}{\partial x}(\hat{x})\right)^{-1} L_\ell M_\ell P_\ell^{-1} L_\ell^\top \left(\frac{\partial \Phi}{\partial x}(\hat{x})\right)^{-1\top} \frac{\partial h_2}{\partial x}(\hat{x})^\top h_2(\hat{x}) \tag{2.11}$$

with $\tau_\ell : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ a functions to be chosen large enough. The modification term $\mathcal{M}$ is an extra output injection term that acts only when the state $\hat{x}$ goes outside a given set $X \subset \mathcal{X}$, and guarantees the set $\mathfrak{C}_{\frac{1}{2}}$ to be forward invariant for the observer (2.10) as stated by the forthcoming theorem.

Theorem 2.3. *Consider system* (2.7) *with* $u(t) \in U$ *and* $x(t) \in X$ *for all* $t \geq 0$ *where* X *is a compact subset of* $\subset \mathcal{X}$*. Suppose Assumptions 2.2 and 2.3 hold and consider the observer* (2.10)-(2.11) *with initial conditions* $\hat{x}(0) \in \mathfrak{C}_{\frac{1}{2}}$*. There exist a function* $\tau_\ell : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$*, strictly positive real numbers* μ_1*,* μ_2 *and* $\ell^\star \geq 1$ *such that, for any* $\ell > \ell^\star$

(i) *the set* $\mathfrak{C}_{\frac{1}{2}}$ *is forward invariant for the observer* (2.10), (2.11),

(ii) *by defining* $a(\ell) \;=\; \lambda_{\min}(M_\ell P^{-1})$ *we have*

$$|\hat{x}(t) - x(t)| \;\leq\; \mu_1\, a(\ell)^d \, \exp(-\mu_2 a(\ell)\, t)\, |\hat{x}(0) - x(0)| \;.$$

Proof. First we observe that

$$\frac{\partial h_2}{\partial x}(\hat{x})\dot{\hat{x}} = \; R(\hat{x}, y, u) - \; \tau_\ell(\hat{x}, y, u) \left| (M_\ell P^{-1})^{\frac{1}{2}} L_\ell^\top \left(\frac{\partial \Phi}{\partial x}(\hat{x})\right)^{-1\top} \frac{\partial h_2}{\partial x}(\hat{x})^\top \right|^2 h_2(\hat{x})$$

where we have let

$$R(\hat{x}, y, u) = \frac{\partial h_2}{\partial x}(\hat{x}) \left[f(\hat{x}) + \; g(\hat{x})u + \left(\frac{\partial \Phi}{\partial x}(\hat{x})\right)^{-1} L_\ell M_\ell K(u) N_\ell [y - h(\hat{x})] \right].$$

This motivates us for choosing τ_ℓ satisfying

$$\tau_\ell(\hat{x},y,u) \geq \frac{8h_2(\hat{x})^2\, R(\hat{x},y,u)}{\left|(M_\ell P^{-1})^{\frac{1}{2}} L_\ell^\top \left(\frac{\partial \Phi}{\partial x}(\hat{x})\right)^{-1\top} \frac{\partial h_2}{\partial x}(\hat{x})^\top\right|^2}$$

which can be computed on-line. Thanks to H2, the function $\hat{x} \mapsto \tau_\ell(\hat{x},y,u)$ defined this way is continuous on $\mathcal{X}$. So we can use τ_ℓ as long as $\hat{x}$ is in $\mathcal{X}$. It implies that the derivative of $h_2(\hat{x})$ along the solutions is non positive when $h_2(\hat{x})$ is strictly larger than $\frac{1}{2}$. This implies that, for each s in $[\frac{1}{2},1]$ the set $\{(\hat{x}) \,:\, h_2(\hat{x}) \leq s\}$ is forward invariant and so is the compact set $\widehat{\mathfrak{C}}_{\frac{1}{2}}$ in particular.

Now consider the following change of coordinates

$$x \;\mapsto\; z \;:=\; \Phi(x) \;, \qquad\qquad \hat{x} \;\mapsto\; \hat{z} \;:=\; \Phi(\hat{x}) \;,$$

by which systems (2.7) and (2.10) are transformed in

$$\begin{array}{rcl} \dot{z} &=& A(u)z + B(z,u) \;, \\ \dot{\hat{z}} &=& A(u)\hat{z} + B(\hat{z},u) + L_\ell M_\ell K(u) N_\ell C(z-\hat{z}) \;, \end{array}$$

By using property O4, consider the candidate Lyapunov function

$$U \;=\; \frac{1}{2}(\hat{z}-z)^\top (L_\ell M_\ell P^{-1} L_\ell^\top)^{-1}(\hat{z}-z) \;. \tag{2.12}$$

Now suppose the modification term $\mathcal{M}$ is not present. We get

$$\begin{array}{rcl} \dot{U} &=& (\hat{z}-z)^\top L_\ell^{-\top} P M_\ell^{-1} L_\ell^{-1}\Big((A(u) - L_\ell M_\ell K(u) N_\ell C)(\hat{z}-z) \\ && \qquad\qquad\qquad\qquad\qquad + B(\hat{z},u) - B(z,u)\Big) \\ &=& (L_\ell^{-1}(\hat{z}-z))^\top P M_\ell^{-1} L_\ell^{-1}\Big((A(u)L_\ell - L_\ell M_\ell K(u) N_\ell C L_\ell) L_\ell^{-1}(\hat{z}-z) \\ && \qquad\qquad\qquad\qquad\qquad + B(\hat{z},u) - B(z,u)\Big) \end{array}$$

Furthermore, by applying properties O3 and O7 of Assumption 2.3,

$$\begin{array}{rcl} \dot{U} &=& (L_\ell^{-1}(\hat{z}-z))^\top \Big(P(A(u) - K(u)C) L_\ell^{-1}(\hat{z}-z) \\ && \qquad\qquad + P M_\ell^{-1} L_\ell^{-1}\big(B(\hat{z},u) - B(z,u)\big)\Big) \\ &\leq& -\gamma |P^{\frac{1}{2}} L_\ell^{-1}(\hat{z}-z)|^2 + c_\ell |P^{\frac{1}{2}} L_\ell^{-1}(\hat{z}-z)|^2 \end{array}$$

As a consequence, by using O6, there exists a $\ell^\star$ such that, for any $\ell \geq \ell^\star$ we have

$$\dot{U} \;\leq\; -\frac{\gamma}{2}(\hat{z}-z)^\top L_\ell^{-\top} P L_\ell^{-1}(\hat{z}-z)$$

Since we have

$$\begin{array}{rcl} P & \geq & \lambda_{\min}(P) \\ M_\ell P^{-1} & \geq & \lambda_{\min}(M_\ell P^{-1}) \end{array} \quad \Longrightarrow \quad P \;\geq\; \lambda_{\min}(P)\lambda_{\min}(M_\ell P^{-1}) P M_\ell^{-1}$$

we obtain

$$\begin{aligned} \dot{U} &\leq -\gamma\, \lambda_{\min}(P)\, \lambda_{\min}(M_\ell P^{-1})\, \frac{1}{2}\, (\hat{z}-z)^\top L_\ell^{-\top} P M_\ell^{-1} L_\ell^{-1}(\hat{z}-z) \\ &\leq -\gamma\, \lambda_{\min}(P)\, \lambda_{\min}(M_\ell P^{-1})\, U \end{aligned}$$

and therefore

$$U(t) \;\leq\; \exp(-\gamma\, \lambda_{\min}(P)\, a(\ell)\, t)\, U(0)\,.$$

Note that by definition of U we have

$$\frac{1}{\lambda_{\max}(L_\ell M_\ell P^{-1} L_\ell^\top)}\, |\hat{z}-z|^2 \;\leq\; U \;\leq\; \frac{1}{\lambda_{\min}(L_\ell M_\ell P^{-1} L_\ell^\top)}\, |\hat{z}-z|^2$$

Furthermore, by using O5 we have

$$\frac{\lambda_{\max}(L_\ell M_\ell P^{-1} L_\ell^\top)}{\lambda_{\min}(L_\ell M_\ell P^{-1} L_\ell^\top)} \;\leq\; a(\ell)^{2d}$$

As a consequence we get

$$|\hat{z}(t)-z(t)| \;\leq\; a(\ell)^d \exp(-\gamma\, \lambda_{\min}(P)\, a(\ell)\, t)\, |\hat{z}(0)-z(0)|\,.$$

Because Φ is a diffeomorphism defined on $\mathcal{X}$, for any compact subset $\mathfrak{C}$ and $\hat{\mathfrak{C}}$ of $\mathcal{X}$, there exist positive real numbers L_Φ and $L_{\Phi^{-1}}$ satisfying

$$\begin{aligned} |\hat{x}-x| \;=\; |\Phi^{-1}(\Phi(\hat{x})) - \Phi^{-1}(\Phi(x))| \;&\leq\; L_{\Phi^{-1}} |\Phi(\hat{x}) - \Phi(x)| \\ |\Phi(\hat{x}) - \Phi(x)| \;&\leq\; L_\Phi |\hat{x}-x| \end{aligned}$$

for any $(\hat{x}, x) \in \mathfrak{C}_{\frac{1}{2}} \times \mathfrak{C}_{\frac{1}{2}}$. The bound (ii) of the theorem would complete by defining $\mu_1 = L_{\Phi^{-1}} L_\Phi$ and $\mu_2 = \gamma\, \lambda_{\min}(P)$. On the other hand, the modification $\mathcal{M}$ augments

$\dot{U}$ in (2.12) with

$$-\tau_\ell(\hat{x})\,[\Phi(\hat{x}) - \Phi(x)]^\top \left(\frac{\partial \Phi}{\partial x}(\hat{x})\right)^{-1\top} \frac{\partial h_2}{\partial x}(\hat{x})^\top h_2(\hat{x}) \ .$$

But, when $h_2(x)$ is zero which is the case when the model state x remains in $\mathcal{X}$ and when $h_2(\hat{x}_\ell)$ is in $[0,1]$, the convexity property of h_2 in H4 gives

$$0 \ \leq \ [\Phi(\hat{x}) - \Phi(x)]^\top \left(\frac{\partial \Phi}{\partial x}(\hat{x}_\ell)\right)^{-1\top} \frac{\partial h_2}{\partial x}(\hat{x})^\top h_2(\hat{x}) \ .$$

proving that the bound (ii) still holds. □

Examples

In this section we show three example where the conditions of Assumption 2.3 are fulfilled.

Example 1 (single-input single-output input-affine systems):

The conditions of Assumption 2.3 are satisfied for systems of the form

$$\begin{array}{rcl} \dot{x} & = & f(x) + g(x)\,u \\ y & = & h(x) \end{array} \qquad \text{(1.6 revisited)}$$

satisfying Theorem 1.2 with

$$\Phi(x) = \begin{pmatrix} h(x) \\ \vdots \\ L_f^{n-1}h(x) \end{pmatrix}, \qquad A(u) = \begin{pmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ 0 & & \cdots & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^\top$$

$$B(\Phi(x), u) = \begin{pmatrix} 0 \\ \vdots \\ L_f^n h(x) \end{pmatrix} + \begin{pmatrix} L_g h(x) \\ \vdots \\ L_f^{n-1} L_g h(x) \end{pmatrix} u$$

$$L_\ell = \text{diag}\left(1, \ell, \dots, \ell^{n-1}\right) , \qquad M_\ell = \ell\, I \ , \qquad N_\ell = 1 \ , \qquad d = 2n \ , \qquad c_\ell = \frac{1}{\ell} \ ,$$

and $K(u) = K$ chosen such that $(A - KC)$ is Hurwitz. We recover the observer (2.4) with $D_\ell = L_\ell M_\ell N_\ell$.

Example 2 (a non triangular case):

In Assumption 2.3, $A(u)$ is allowed to be input-dependent to allow a broader class of nonlinear systems. Consider for instance the following input-affine nonlinear system

$$\begin{array}{rcl}\dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & u\end{array} \qquad y \;=\; -x_1 + x_2 + x_2^2 \tag{2.13}$$

where $x \in \mathbb{R}^2$ the state, $u \in \mathbb{R}$ the control input, $y \in \mathbb{R}$ the measured output. The conditions of Assumption 2.3 are satisfied with $\mathcal{X} = \mathbb{R}^2$, U any compact subset of $(-\infty, \frac{1}{2})$,

$$\Phi(x) \;=\; \begin{pmatrix} -x_1 + x_2 + x_2^2 \\ -x_2 \end{pmatrix}, \quad A(u) \;=\; \begin{pmatrix} 0 & 1-2u \\ 0 & 0 \end{pmatrix}, \quad B(\Phi(x), u) \;=\; \begin{pmatrix} u \\ -u \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}, \quad M_\ell = \begin{pmatrix} \ell & 0 \\ 0 & \ell \end{pmatrix}, \quad L_\ell = \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix}, \quad N_\ell = 1\,,$$

$d = 4$, $c_\ell = \dfrac{1}{\ell}$ with $k_1 > 0$, $k_2 > 0$. Note that by computing the inverse mapping of Φ

$$\Phi(x) \;=\; \begin{pmatrix} -x_1 + x_2 + x_2^2 \\ -x_2 \end{pmatrix}, \qquad \Phi^{-1}(z) \;=\; \begin{pmatrix} -z_1 - z_2 + z_2^2 \\ -z_2 \end{pmatrix},$$

system (2.13) is transformed via $z = \Phi(x)$

$$\begin{array}{rcl}\dot{z}_1 & = & z_2(1-2u) + u \\ \dot{z}_2 & = & u\end{array} \qquad y \;=\; z_1$$

which is not in the *feedback form* (1.11) and does not satisfies Assumption 1.2.

Example 3 (a multi-input multi-output case):

As an illustration of a multi-input multi-output nonlinear we consider a simplified model of the longitudinal dynamics of a fixed-wing vehicle flying at high speed, given (see Poulain and Praly (2010)) by

$$\begin{array}{rcl}\dot{v} & = & e - g\sin(\gamma) \\ \dot{\gamma} & = & \pounds\, v\sin(\theta - \gamma) - \dfrac{g\cos(\gamma)}{v} \\ \dot{\theta} & = & q\end{array} \tag{2.14}$$

where v is the modulus of the speed, γ is the path angle, θ is the pitch angle, q is the pitch rate, g is the standard gravitational acceleration and $\pounds$ is an aerodynamic lift coefficient.

This model makes sense for v strictly positive only. In practical application the problem is to regulate γ at 0, with v remaining close to a prescribed cruise speed v_0, using the pitch rate q and the thrust e as controls, and with γ and θ as only measurements. Here we want to show that system (2.14) satisfies the conditions of Assumption 2.3 and therefore it is possible to design an observer of the form (2.10). By defining

$$x = \begin{pmatrix} v \\ \gamma \\ \theta \end{pmatrix}, \qquad y = \begin{pmatrix} \theta \\ \gamma \end{pmatrix} \qquad u = \begin{pmatrix} q \\ e \end{pmatrix}$$

systems (2.14) reads in the more compact form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

with

$$\begin{aligned} f(x) &= \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} = \begin{pmatrix} -g\sin(x_2) \\ \pounds x_1 \sin(x_3 - x_2) - \dfrac{g\cos(x_2)}{x_1} \\ 0 \end{pmatrix}, \\ g(x) &= \begin{pmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad h(x) = \begin{pmatrix} h_1(x) \\ h_2(x) \end{pmatrix} = \begin{pmatrix} x_3 \\ x_2 \end{pmatrix}. \end{aligned}$$

The model (2.14) makes sense only when x is in the open set $\mathcal{X}_e$ defined as

$$\mathcal{X}_e = (0, +\infty) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \subset \mathbb{R}^3 .$$

We can check that the conditions of Assumptions 2.3 are satisfied by choosing

$$\Phi(x) = \begin{pmatrix} h_1(x) \\ h_2(x) \\ L_f h_2(x) \end{pmatrix} := \begin{pmatrix} x_3 \\ x_2 \\ f_2(x) \end{pmatrix}.$$

It is diffeomorphism on

$$\mathcal{X} = \left\{ (x_1, x_2, x_3) \in \mathcal{X}_e \ : \ x_1 < \sqrt{\frac{g}{\pounds} \frac{\cos(x_2)}{\sin(|x_3 - x_2|)}} \quad \text{if} \quad x_3 - x_2 < 0 \right\}$$

which is the set satisfying

$$\frac{\partial f_2}{\partial x_1}(x) \;>\; 0 \qquad \forall\, x \in \mathcal{X} \;.$$

Furthermore, A, B, C, L_ℓ, M_ℓ, N_ℓ and c_ℓ are defined as

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{array}{rcl} L_\ell & = & \text{diag}(1,1,\ell)\;, \\ M_\ell & = & \text{diag}(\ell,\ell,\ell)\;, \\ N_\ell & = & \text{diag}(1,1)\;, \end{array} \quad c_\ell = \frac{1}{\ell}\;,$$

$$B(\Phi(x),u) = \begin{pmatrix} u_1 \\ 0 \\ \frac{\partial f_2}{\partial x_1}(x)(-g\sin(x_2)+u_2) + \frac{\partial f_2}{\partial x_2}(x) f_2(x) + \frac{\partial f_2}{\partial x_3}(x) u_1 \end{pmatrix}.$$

Finally, for any strictly positive number γ, we can define P as symmetric positive definite matrix of the form

$$P = \begin{pmatrix} * & * & * \\ * & * & p_{23} \\ * & p_{23} & p_{33} \end{pmatrix}$$

where $2p_{23} \leq -\gamma p_{33}$. Note that there exists a real number ρ such that we have

$$PA + A^\top P - \rho\, C^\top C \;\leq\; -\gamma P$$

implying the existence of a real number $\underline{\gamma}_\rho$ such that, for any $\gamma_\rho \geq \underline{\gamma}_\rho$, with

$$K = \gamma_\rho P^{-1} C^\top$$

assumptions O3 is satisfied. With the functions above it is easy to check that also properties O1, O2, O4, O5, O6 and O7 are satisfied for any input $u \in U$, with U a compact subset of $\mathbb{R}^2$. Finally, in order to implement an observer of the form (2.10) for the system (2.14) we need to design the function h_2. For this, let us define the following open set [6] of $\Phi(\mathcal{X})$

$$\Xi \;=\; \Big\{ z \in \mathbb{R}^3 : z_1 \in \Big(-\frac{\pi}{2};\frac{\pi}{2}\Big),\; z_2 \in \Big(-\frac{\pi}{2};\frac{\pi}{2}\Big),\; z_3 < -2\sqrt{g\pounds|z_1 - z_2|}\;\text{ if }\;(z_1 - z_2) \leq 0 \Big\}.$$

It can be checked that any compact set of Ξ is in the image of $\Phi(\mathcal{X})$ by noting that when $(x_1, x_2, f_2(x)) \in \Xi$ then $\frac{\partial f_2}{\partial x_1} > 0$. As a consequence the function $h_2(x)$ can be defined as

$$h_2(x) \;=\; h_2^1(x) + h_2^2(x) + h_2^3(x) + h_2^4(x)$$

[6] We use $|z_1 - z_2|$ to upper bound $\cos z_2 \sin(z_1 - z_2)$.

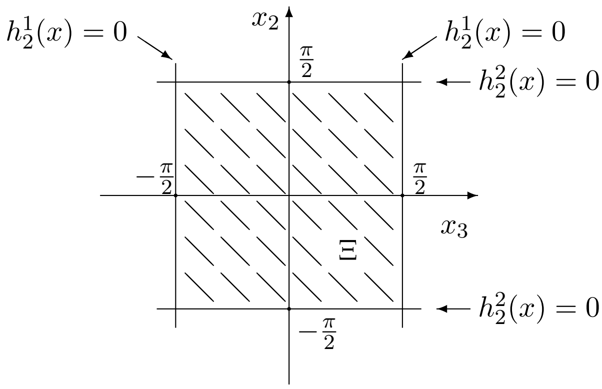

Figure 2.1: Design of the functions h_2^1, h_2^2.

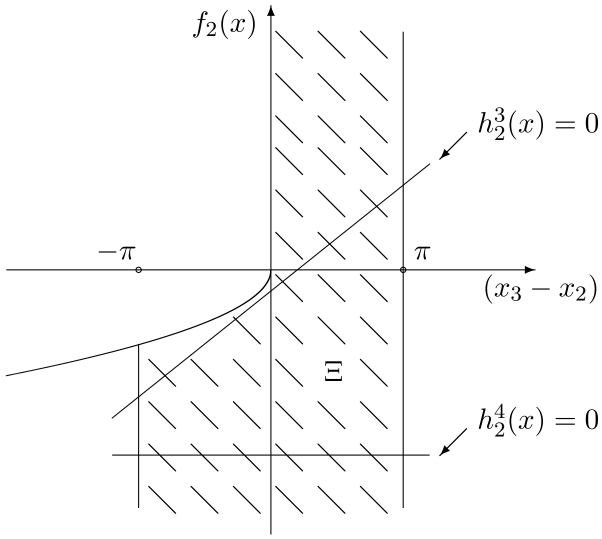

Figure 2.2: Design of the functions h_2^3, h_2^4.

$$h_2^1(x) = \max\left\{\frac{4x_3^2}{\pi^2} - \varepsilon_1; 0\right\}^2, \qquad h_2^2(x) = \max\left\{\frac{4x_2^2}{\pi^2} - \varepsilon_2; 0\right\}^2,$$

$$h_2^3(x) = \max\left\{\varepsilon_3\left(x_3 - x_2\right) - f_2(x) - \varepsilon_4; 0\right\}^2, \qquad h_2^4(x) = \max\left\{\frac{f_2(x)}{f_{2\max}} - \varepsilon_5; 0\right\}^2,$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5$ and $f_{2\max}$ are positive constants to be properly chosen. The functions $h_{2,1}$ and $h_{2,2}$ take care respectively of x_3 and x_2 to stay in the set Ξ as showed in Figure 2.1, whereas functions $h_{2,3}$ and $h_{2,4}$ take care of $f_2(x)$ as in Figure 2.2.

2.4 Asymptotic behaviour in presence of measurement noise

As already introduced in Section 1.3.3, one of the main feature which questions the use of a high-gain observer in applications is its sensitivity to measurement noise. Attempts of analysis have been done but only $\mathcal{H}_\infty$ bounds of the form (1.19) have been characterised (see, for instance, Ball and Khalil (2009), or Vasiljevic and Khalil (2008)). As already noticed in Vasiljevic and Khalil (2006), the $\mathcal{H}_\infty$ analysis is too conservative and fails to catch the "low-pass" filtering characteristics of the high-gain observer when ℓ is fixed to some (eventually large) value.

In this section we focus on the effect of the measurement noise on the steady state of the high-gain observer estimate. We propose a novel technique, which may be eventually applied to other frameworks, to analyse the error estimate steady-state behaviour of the high-gain observer in presence of high-frequency measurement noise, based on the approximation of a solution of a partial differential equation modelling the steady state of the estimate. For the sake of simplicity in the analysis, we consider system in the strict feedback form (1.11) satisfying

$$
\begin{array}{rcll}
\varphi_i(x_1,\ldots,x_i,u) & = & 0 & \qquad i=1,\ldots,n-1,\\
\varphi_n(x_1,\ldots,x_n,u) & = & \varphi(x), &
\end{array}
$$

with $\varphi(x)$ any known locally Lipschitz function. This class of systems can be compactly written with the notation

$$
\begin{array}{rcl}
\dot{x} & = & Ax + B\varphi(x)\\
y & = & Cx + \nu(t)
\end{array}
\tag{2.15}
$$

where (A, B, C) is a triplet in *prime form* of dimension n. For such nonlinear systems we consider the standard high-gain observer (1.12) that is is implemented in this particular framework as

$$
\dot{\hat{x}} \;=\; A\hat{x} + B\varphi_s(\hat{x}) + G(y - C\hat{x}) \tag{2.16}
$$

where $\hat{x} = \text{col}(\hat{x}_1, \ldots, \hat{x}_n) \in \mathbb{R}^n$ is the estimated state, (A, B, C) denotes as before a triplet in *prime form* of dimension n,

$$
G := \text{col}\begin{pmatrix}\ell k_1 & \cdots & \ell^n k_n\end{pmatrix},
$$

and $\varphi_s(\cdot)$ is a locally Lipschitz bounded function that agree with $\varphi(\cdot)$ on a bounded set $X_\delta \supset X$, namely there exists a $\bar{\varphi} > 0$ such that $|\varphi_s(x)| \leq \bar{\varphi}$ for all $x \in \mathbb{R}^n$ and $\varphi_s(x) = \varphi(x)$ for all $x \in X_\delta$. By considering the change of coordinates

$$
\hat{x} \;\mapsto\; e \;:=\; \hat{x} - x
$$

it turns out that system (2.16) transforms as

$$\dot{e} \;=\; Fe + B\Delta_{\varphi}(e,x) + G\nu(t) \tag{2.17}$$

where

$$F \;:=\; (A - GC) \;=\; \begin{pmatrix} -\ell k_1 & 1 & \cdots & 0 \\ -\ell^2 k_2 & & 1 & \\ \vdots & & \ddots & \\ -\ell^{n-1} k_{n-1} & & & 1 \\ -\ell^n k_n & 0 & \cdots & 0 \end{pmatrix},$$

and $\Delta_{\varphi}(e,x)$ is the locally Lipschitz function defined as

$$\Delta_{\varphi}(e,x) := \varphi_s(e+x) - \varphi(x)\,. \tag{2.18}$$

By applying the results of Theorem 1.4 it can be shown that for a *generic* bounded measurement noise, the observer guarantees bounded trajectories with a linear asymptotic gain. The asymptotic gain of the i-th state estimates depends on ℓ^{i-1} thus tending to be worst as long as “higher" components in (2.17) are considered.

The goal of this section is to better characterise the asymptotic gain in presence of *high-frequency* noise with ℓ that is fixed in order to have the above mentioned ISS property. Towards this end we model the measurement noise as

$$\varepsilon \dot{w} = Sw\,, \qquad\qquad \nu = Pw\,, \tag{2.19}$$

where S is a neutrally stable matrix, P is a row vector, and $\varepsilon \in (0,1)$ is a parameter that will be taken small in the forthcoming analysis. System (2.19) can be conveniently seen as generator of $\vartheta > 0$ harmonics at frequencies $\omega_i/\varepsilon > 0$, $i = 1,\ldots,\vartheta$, namely, the matrices S and P take the form

$$S = \text{blkdiag}(S_1,\ldots,S_{\vartheta})\,, \qquad S_i = \begin{pmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{pmatrix}$$

and $P = ((0\;1)\;(0\;1)\;\cdots\;(0\;1))$. In the following we assume that w ranges in a compact invariant set W.

As a preparatory step towards the nonlinear analysis, it is instructive to consider the linear case, namely the case in which $\varphi(x) = \Phi x$ with Φ a row vector. In this case the observer (2.16) can be taken[7] with $\varphi_s(\hat{x}) = \Phi\hat{x}$, thus resulting in an error system

[7]Because of linearity boundedness of the function $\varphi_s(x)$ is not needed.

(2.17)-(2.19) given by

$$\begin{aligned} \varepsilon\dot{w} &= Sw \\ \dot{e} &= (F+B\Phi)e + GPw \end{aligned}$$

with the matrix $F+B\Phi$ that is Hurwitz for ℓ sufficiently large. Using the fact that S is neutrally stable and that $F+B\Phi$ is Hurwitz it follows that the state of the previous system reaches a steady state fully described by the state of the noise generator. In particular, denoting by Π_ε the matrix solution of the Sylvester equation

$$\Pi_\varepsilon S \;=\; \varepsilon(F+B\Phi)\Pi_\varepsilon + \varepsilon GP$$

it turns out that

$$\lim_{t\to\infty}(e(t) - \Pi_\varepsilon w(t)) \;=\; 0\,. \tag{2.20}$$

The solution of the previous Sylvester equation can be characterised at high-frequency (namely for small value of ε) to have more insight about how the gain between the measurement noise and the j-th estimation error is affected by ℓ. In particular, using that the fact that S is not singular, it is easy to check that

$$\Pi_\varepsilon \;=\; \varepsilon GPS^{-1} + \varepsilon^2\bar{\Pi}_\varepsilon$$

with

$$\bar{\Pi}_\varepsilon \;:=\; \sum_{k=2}^{\infty}\varepsilon^{k-2}(F+B\Phi)^{k-1}GPS^{-k}\,,$$

is a solution of the Sylvester equation. In particular, the series defining $\bar{\Pi}_\varepsilon$ is convergent as long as ε is taken sufficiently small[8]. Namely, there exist $\varepsilon_1^\star(\ell) > 0$ and $\bar{\pi}(\ell) > 0$ such that $|\bar{\Pi}_\varepsilon| \le \bar{\pi}(\ell)$ for all positive $\varepsilon \le \varepsilon_1^\star(\ell)$. By bearing in mind how ℓ enters in G, and denoting by Π_i the i-th row of Π_ε, $i = 1,\ldots,n$, we have

$$|\Pi_i| \;\le\; \varepsilon\,\ell^i k_i|PS^{-1}| + \varepsilon^2|\bar{\Pi}_\varepsilon|$$

As a consequence, by choosing a positive $\varepsilon_2^\star(\ell) \le \varepsilon_1^\star(\ell)$ satisfying

$$\varepsilon_2^\star(\ell) \;\le\; \max\{k_i\}\frac{|PS^{-1}|}{|\bar{\Pi}_\varepsilon|}$$

we have that for all positive $\varepsilon \le \varepsilon_2^\star(\ell)$ the following holds

$$\lim_{\varepsilon\to 0}|\Pi_i| \;\le\; \mu\,\varepsilon\,\ell^i$$

[8]Observe that the term $(F+B\Phi)^{k-1}$ grows polynomially in ℓ.

where μ is a positive constant. From this, using (2.20) and the fact that W is compact, we can then conclude that for all positive $\varepsilon \leq \varepsilon_2^\star(\ell)$ the following holds

$$\lim_{t\to\infty} \sup |e_i(t)| \;\leq\; \mu\,\varepsilon\,\ell^i\; \|w(\cdot)\|_\infty \,.$$

The previous relation clearly shows the “low-pass" filtering properties of the high-gain observer, namely

$$\lim_{\varepsilon\to 0} \lim_{t\to\infty} \sup |e_i(t)| \;=\; 0\,,$$

and the fact that the asymptotic gain of the j-th error component at high-frequency depends on ℓ^i (whereas the $\mathcal{H}_\infty$ bound is proportional to ℓ^{i-1}, as shown by (1.19)).

Nonlinear Analysis

By compactly writing the system dynamics (2.15) as

$$\dot{x} = f(x)$$

the overall dynamics given by the observed system (2.15) , the observer error dynamics (2.17) and the noise generator (2.19) read as

$$\begin{array}{rcl} \varepsilon\dot{w} &=& Sw \\ \dot{x} &=& f(x) \\ \dot{e} &=& Fe + B\Delta_\varphi(e,x) + GPw\,. \end{array} \qquad (2.21)$$

Having tuned the parameters k_i, $i = 1,\dots,n$, and ℓ according to Theorem 1.4, the trajectories of this system are bounded. The system in question, thus, has a well-defined steady state that can be characterised with the tools proposed in Isidori and Byrnes (2008). More specifically, the triangular structure of the system (with the x and w subsystem driving the e subsystem) implies the existence (recall that W and X are compact) of a possibly set-valued function $\pi_\varepsilon : X \times W \rightrightarrows \mathbb{R}^n$ such that the set

$$\text{graph}(\pi_\varepsilon) = \big\{(w,x,e) \in W \times X \times \mathbb{R}^n \;\;:\;\; e \in \pi_\varepsilon(w,x)\big\}$$

is asymptotically stable for (2.21). Furthermore, the properties of the high-gain observer when the measurement noise is absent (*i.e.* when $w = 0$) show that

$$\pi_\varepsilon(0,x) = \{0\} \qquad \forall\, x \in X\,.$$

The following technical lemma provides an arbitrarily accurate approximation of a continuous selection of $\pi_\varepsilon(\cdot,\cdot)$.

Lemma 2.1. *Consider system* (2.17) *with* ℓ *fixed and let* r *be an arbitrary positive number. There exist continuous functions* $\bar{\psi}_{j,i} : X \times W \to \mathbb{R}$, $j = 1, \ldots, n$, $i = 1, \ldots, r$, *such that having defined*

$$\Psi_j(w,x) := \sum_{i=1}^{r} \ell^{j+i-1} \bar{\psi}_{j,i}(w,x)\, \varepsilon^i$$

$$\Psi_\varepsilon(w,x) := \mathrm{col}\begin{pmatrix} \Psi_1(w,x) & \cdots & \Psi_n(w,x) \end{pmatrix}.$$

and

$$E_\varepsilon(w,x) := \frac{\partial \Psi_\varepsilon(w,x)}{\partial w} Sw + \frac{\partial \Psi_\varepsilon(w,x)}{\partial x} f(x) - F\Psi_\varepsilon(w,x) - GPw - B\Delta_\varphi(\Psi_\varepsilon(w,x),x)\,,$$

the following holds

$$\lim_{\varepsilon\to 0^+} \frac{E_\varepsilon(w,x)}{\varepsilon^{r-1}} = 0\,, \qquad \forall\,(w,x) \in W \times X\,,$$

$$E_\varepsilon(0,x) = 0 \qquad \forall\,(\varepsilon,x) \in [0,1] \times X$$

Proof. First of all consider the case where $w = 0$. By recalling the definition of $\Delta_\varphi(\cdot,\cdot)$ in (2.18) it is easy to verify that $\Psi_\varepsilon(0,x) = 0$ makes $E_\varepsilon(0,x) = 0$. As a consequence in the following we will show that $\Psi_\varepsilon(w,x)$ can be chosen as a continuous function in w satisfying $\Psi_\varepsilon(0,x) = 0$. Now let

$$\psi_{j,i}(w,x) := \ell^{j+i-1} \bar{\psi}_{j,i}(w,x)$$

so that

$$\Psi_j(w,x) = \sum_{i=1}^{r} \psi_{j,i}(w,x)\, \varepsilon^i\,. \tag{2.22}$$

Since w and x range in bounded sets and the function $\psi_{j,i}(\cdot,\cdot)$ are continuous, we have that

$$\lim_{\varepsilon\to 0^+} \Psi_\varepsilon(w,x) = 0 \qquad \forall\,(w,x) \in W \times X\,.$$

Expanding $\Delta_\varphi(\Psi_\varepsilon, x)$ by Taylor around $\Psi_\varepsilon = 0$ we obtain

$$\Delta_\varphi(\Psi_\varepsilon, x) \;=\; \sum_{i=1}^{r} \varphi_i(x)[\Psi_\varepsilon]^i + \rho_r(\Psi_\varepsilon, x)$$

in which $\varphi_i(\cdot)$, $i = 1, \ldots, r$, are continuous functions to be made precise, $\rho_r(\cdot, \cdot)$ is a continuous remainder function, and the $[\Psi_\epsilon]^i$ are monomials of the form

$$[\Psi_\varepsilon]^i = \prod_{j=1}^{n} \Psi_j^{\kappa_j}\,, \qquad\qquad \sum_{j=1}^{n} \kappa_j \;=\; i\,.$$

By replacing Ψ_j with the expressions (2.22) and grouping the terms with the same power of ε, the Taylor expansion of $\Delta_\varphi(\cdot, \cdot)$ can be rewritten as

$$\Delta_\varphi(\Psi_\varepsilon, x) \;=\; \sum_{i=1}^{r} \varepsilon^i \phi_i(w, x) + \varepsilon^{r+1} R_\varepsilon(w, x) \tag{2.23}$$

where the functions $\phi_i(\cdot)$, $i = 1, \ldots, r$, and $R_\varepsilon(\cdot, \cdot)$ are appropriately defined continuous functions satisfying $\phi_(0, x) = 0$, $R_\varepsilon(0, x) = 0$ and $R_\varepsilon(\cdot, \cdot)$ is bounded on $W \times X$ for any $\varepsilon \in [0, 1]$. As far as the ϕ_i's are concerned, in particular, we observe that, because the Ψ_j are polynomials in ε and the $[\Psi_\varepsilon]^i$ are polynomials in the Ψ_j, only the coefficients of power smaller or equal to i in ε in the Ψ_j can be in ϕ_i. Namely, $\phi_i(\cdot, \cdot)$ depends only on $\psi_{j,k}$ with $k \leq i$, for all $i = 1, \ldots, r$ and $j = 1, \ldots, n$.

Let us now define $E_\varepsilon(\cdot, \cdot)$ with components $E_1(\cdot, \cdot)$, ..., $E_n(\cdot, \cdot)$ as

$$\begin{aligned}
E_1(w, x) \;&=\; \dot{\Psi}_1 + \ell k_1 \Psi_1 - \Psi_2 - \ell k_1 P w \\
&\;\vdots \\
E_n(w, x) \;&=\; \dot{\Psi}_n + \ell^n k_n \Psi_1 - \Delta_\varphi(\Psi_\epsilon, x) - \ell^n k_n P w
\end{aligned}$$

where, for sake of compactness, we omitted the argument (w, x) from the functions Ψ_j, $j = 1, \ldots, n$, and Ψ_ε. By embedding (2.22) and (2.23) in the previous expressions, the following is obtained

$$\begin{aligned}
E_j(w, x) \;&=\; \sum_{i=1}^{r} \left[L_f \psi_{j,i} + \frac{1}{\varepsilon} L_S \psi_{j,i} \right] \varepsilon^i + \ell^j k_j \sum_{i=1}^{r} \psi_{1,i}\, \varepsilon^i - \sum_{i=1}^{r} \psi_{j+1,i} \varepsilon^i - \ell^j k_j P w \\
&=\; \left[L_S \psi_{j,1} - \ell^j k_j P w \right] + \sum_{i=1}^{r} \varepsilon^i \left[L_f \psi_{j,i} + L_S \psi_{j,i+1} + \ell^j k_j \psi_{1,i} - \psi_{j+1,i} \right]
\end{aligned}$$

for $j = 1, \ldots, n-1$ and

$$\begin{aligned}
E_n(w, x) \;&=\; \sum_{i=1}^{r} \left[L_f \psi_{n,i} + \frac{1}{\varepsilon} L_S \psi_{n,i} \right] \varepsilon^i + \ell^n k_n \sum_{i=1}^{r} \psi_{1,i} \varepsilon^i \\
&\qquad\qquad - \sum_{i=1}^{r} \phi_i \varepsilon^i - \varepsilon^{r+1} R_\varepsilon(w, x) - \ell^n k_n P w
\end{aligned}$$

$$= \quad [L_S\psi_{n,1} - \ell^n k_n P w] - \varepsilon^{r+1} R_\varepsilon(w,x) + \sum_{i=1}^{r} \varepsilon^i \left[L_f\psi_{n,i} + L_S\psi_{n,i+1} + \ell^n k_n \psi_{1,i} - \phi_i\right]$$

in which $\psi_{j,r+1} := 0$, $j = 1, \ldots, n$, and

$$\dot{\psi}_{j,i} \;=\; L_f\psi_{j,i} + L_S\psi_{j,i} \;:=\; \frac{\partial \psi_{j,i}(w,x)}{\partial x} f(x) + \frac{\partial \psi_{j,i}(w,x)}{\partial w} S w\,.$$

The idea now is to iteratively select the functions $\psi_{j,i+1}(\cdot,\cdot)$ to annihilate, in the previous expressions, the terms in ε^i, $i = 0, \ldots, r-1$, $j = 1, \ldots, n$. We start by considering the terms of order 0 in ε which are all annihilated by taking

$$\psi_{j,1}(w,x) \;=\; \ell^j\, k_j\, P\, S^{-1}\, w \;:=\; \ell^j\, \bar{\psi}_{j,1}(w,x)\,, \qquad j = 1, \ldots, n\,,$$

solving $L_S\psi_{j,i} - \ell^j k_j P w = 0$. We observe that $\bar{\psi}_{j,1}(w,x)$, and thus $\psi_{j,1}(w,x)$, $j = 1, \ldots, n$, are polynomials in w of order 1 with constant coefficients and such that $\bar{\psi}_{j,1}(0,x) = 0$. Furthermore since $\phi_1(\cdot,\cdot)$ depends only on $\psi_{j,1}$, we assume that $\phi(\cdot,\cdot)$ is a polynomial in w of order 1 satisfying $\phi(0,x) = 0$.

We proceed now by induction by assuming that all the functions $\psi_{j,k}(\cdot,\cdot)$, $k = 1, \ldots, i$, $j = 1, \ldots, n$ have been fixed in the form

$$\psi_{j,k}(w,x) \;=\; \ell^{j+k-1}\, \bar{\psi}_{j,k}(w,x)$$

for some continuous $\bar{\psi}_{j,k}(w,x)$ that are polynomials in w of order k with coefficients dependent on x, so that to annihilate the terms in ε^{k-1} in E_j. Furthermore, since $\phi_i(\cdot,\cdot)$ only depends on $\psi_{j,k}$ with $k \leq i$, $j = 1 \ldots, n$, we assume that $\phi_i(\cdot,\cdot)$ is a polynomial in w of order i with coefficients dependent on x. In this case we see that the terms of order i in ε are annihilated if $\psi_{j,i+1}(\cdot,\cdot)$ can be chosen so that

$$\begin{aligned} -L_S\psi_{j,i+1} &= L_f\psi_{j,i} + \ell^j k_j \psi_{1,i} - \psi_{j+1,i} \qquad & j = 1, \ldots, n-1\,, \\ -L_S\psi_{n,i+1} &= L_f\psi_{n,i} + \ell^n k_n \psi_{1,i} - \phi_i\,. \end{aligned}$$

Using the induction assumptions on the functions $\psi_{j,i}(\cdot,\cdot)$, $j = 1, \ldots, n$, and $\phi_i(\cdot,\cdot)$, and the fact that S is invertible, it is easy to see that the previous PDEs admit solutions of the form

$$\psi_{j,i+1}(w,x) \;=\; \ell^{j+i}\, \bar{\psi}_{j,i+1}(w,x)$$

$j = 1, \ldots, n$, for some $\bar{\psi}_{j,i+1}(w,x)$ which, in turn, are polynomials in w of order $i+1$ with coefficients that are continuous functions of x and satisfying $\psi_{j,i+1}(0,x) = 0$.

The induction iteration can be then used to choose $\psi_{j,i}(\cdot,\cdot)$, $j = 1,\dots,n$, $i = 1,\dots,r$, of the form

$$\psi_{j,i}(w,x) = \ell^{j+i-1}\,\bar{\psi}_{j,i}(w,x)$$

where $\bar{\psi}_{j,i}(w,x)$ are polynomial functions in w of order i with coefficients that are continuous functions of x. By embedding those functions in the expressions of $E_j(\cdot,\cdot)$, $j = 1,\dots,n$, and bearing in mind the definition of $R_\varepsilon(\cdot,\cdot)$, it is readily seen that

$$\begin{aligned} E_j(w,x) &= \varepsilon^r[L_f\psi_{j,r} + \ell^j k_j\psi_{1,r} - \psi_{j+1,r}] \\ E_n(w,x) &= \varepsilon^r[L_f\psi_{n,r} + \ell^n k_n\psi_{1,r} - \phi_r] + \varepsilon^{r+1}w\bar{R}_\varepsilon(w,x) \end{aligned}$$

where $\psi_{j,r}$ where $\bar{R}_\varepsilon(\cdot,\cdot)$ is an appropriately defined continuous function, by which the claim of the lemma immediately follows. □

The previous lemma is instrumental to the proof of the next proposition which is the main result of the section. As in the linear case, the low-pass filter properties of the nonlinear high-gain observer are highlighted.

Proposition 2.1. *Consider system* (2.21) *with $x(t) \in X$ and $w(t) \in W$ for all $t \geq 0$ with X and W bounded sets. Let the function $\varphi_s(\cdot)$ embedded in $\Delta_\varphi(\cdot,\cdot)$ be chosen so that it is locally Lipschitz and it agrees with $\varphi(\cdot)$ on a set $X_\delta \supset X$. Let ℓ be fixed so that system (2.17) is ISS with respect to the input ν. Then, there exists a $\varepsilon^\star(\ell) > 0$ such that for all positive $\varepsilon \leq \varepsilon^\star(\ell)$ the following holds*

$$\limsup_{t\to\infty} |e_i(t)| \;\leq\; \mu\,\varepsilon\,\ell^i\,\|w(\cdot)\|_\infty \qquad i = 1,\dots,n$$

with μ a positive constant.

Proof. Let consider the change of variables

$$e \;\mapsto\; \tilde{e} \;:=\; e - \Psi_\varepsilon(w,x)$$

with $\Psi_\varepsilon(\cdot,\cdot)$ introduced in the previous lemma with an $r > 1$ and observe that, by bearing in mind the definition of $E_\varepsilon(\cdot,\cdot)$, we have

$$\dot{\Psi}_\varepsilon = F\Psi_\varepsilon + B\Delta_\varphi(\Psi_\varepsilon,x) + GPw + E_\varepsilon(w,x)\,.$$

In the new coordinates we get

$$\dot{\tilde{e}} = F\tilde{e} + B\bar{\Delta}(\tilde{e},\Psi_\varepsilon,x) + E_\varepsilon(w,x)\,, \tag{2.24}$$

with

$$\begin{aligned}\bar{\Delta}(\tilde{e},\Psi_\varepsilon,x) &= \Delta_\varphi(e,x)-\Delta_\varphi(\Psi_\varepsilon(w,x),x)\\ &= \Delta_\varphi(\tilde{e}+\Psi_\varepsilon(w,x),x)-\Delta_\varphi(\Psi_\varepsilon(w,x),x)\\ &= \varphi_s(\tilde{e}+\Psi_\varepsilon(w,x)+x)-\varphi(x)-(\varphi_s(\Psi_\varepsilon(w,x)+x)-\varphi(x))\\ &= \varphi_s(\tilde{e}+\Psi_\varepsilon(w,x)+x)-\varphi_s(\Psi_\varepsilon(w,x)+x)\end{aligned}$$

Observe that the function φ_s is bounded and agrees with φ on X_δ and therefore the Lipschitz constant of $\bar{\Delta}(\tilde{e},\Psi_\varepsilon,x)$ coincides with that of $\Delta(e,x)$ for any $(\tilde{e},w,x,\varepsilon)\in\mathbb{R}^n\times W\times X\times[0,1]$. As a consequence the same values of ℓ that make system (2.17) ISS with respect to the input $\nu(t)$ make also system (2.24) ISS with respect to the input $E_\varepsilon(\cdot,\cdot)$. In particular, there exists a positive constant μ_0 such that

$$\begin{aligned}\lim_{t\to\infty}\sup|\tilde{e}(t)| &= \lim_{t\to\infty}\sup|e(t)-\Psi_\varepsilon(w(t),x(t))|\\ &\le \mu_0\lim_{t\to\infty}\sup|E_\varepsilon(w(t),x(t))|\\ &\le \mu_0\left\|E_\varepsilon(w(\cdot),x(\cdot))\right\|_\infty\end{aligned}$$

Using the fact that, for any $r\ge 1$, $E_\varepsilon(w,x)$ is a term in ε^r, it follows that there exists a positive constant μ_1 such that

$$\lim_{t\to\infty}\sup|\tilde{e}(t)| \le \mu_1\varepsilon^r\left\|w(\cdot)\right\|_\infty .$$

Consider now the the expressions of the components $\Psi_i(\cdot,\cdot)$, $i=1,\ldots,n$, of $\Psi_\varepsilon(\cdot,\cdot)$ introduced in the previous lemma. It turns out that there exist a positive $\varepsilon_1^\star(\ell)\le\varepsilon_1^\star(\ell)$ and a positive constant μ_2 such that

$$|\Psi_i(w,x)| \le \mu_2\,\varepsilon\,\ell^i\,|w|$$

for all $i=1,\ldots,n$, for all positive $\varepsilon\le\varepsilon_1^\star(\ell)$ and for all $(w,x)\in W\times X$. From this,

$$\begin{aligned}\lim_{t\to\infty}\sup|e_j(t)| &= \lim_{t\to\infty}\sup|\tilde{e}_j(t)+\Psi_j(w(t),x(t))|\\ &\le \lim_{t\to\infty}\sup|\tilde{e}_j(t)|+\lim_{t\to\infty}\sup|\Psi_j(w(t),x)(t)|\\ &\le \lim_{t\to\infty}\sup|\tilde{e}_j(t)|+\left\|\Psi_j(w(\cdot),x(\cdot))\right\|_\infty\\ &\le \mu_1\varepsilon^r\left\|w(\cdot)\right\|_\infty+\mu_2\varepsilon\ell^j\left\|w(\cdot)\right\|_\infty\end{aligned}$$

by which the result follows by taking an appropriate $\varepsilon^\star(\ell)\le\varepsilon_1^\star(\ell)$. □

"There are three principal means of acquiring knowledge available to us: observation of nature, reflection, and experimentation. Observation collects facts; reflection combines them; experimentation verifies the result of that combination. Our observation of nature must be diligent, our reflection profound, and our experiments exact."

Denis Diderot

3

Low-Power High-Gain Observers

High-gain observers have been extensively used in nonlinear control for their *tunability* property, namely the fact that the rate of convergence of the observer can be tuned by acting with one single high-gain parameter ℓ. This important feature is motivated by the use of observers in output feedback control and it has been proved (see, among the others, the milestones Atassi and Khalil (1999) and Teel and Praly (1994)) that this tunability property plays a key role in establishing a nonlinear separation principle. However, as already highlighted in Chapter 1, this class of observer has several drawbacks when used in real applications. Mainly:

(i) the maximum gain to implement is increasing polynomially with the system dimension n, *i.e.* we need to implement a term ℓ^n, where ℓ is the high-gain parameter. When nonlinear systems are considered, the value of the gain has to be taken large enough to dominate the nonlinear terms and, therefore, if the Lipschitz constant is very large, or the system dimension is high, the term ℓ^n can be numerically harmful in computations (see Section 1.3.1);

(ii) during the transient the variables present a peaking phenomenon which grows polynomially in ℓ, *i.e.* the value of the variables have an order of magnitude $O(\ell^{n-1})$ (see Section 1.3.2);

(iii) high-gain observers are typically characterized by high sensitivity to measurement noise by thus making their use practically impossible in a realistic noisy environment (see Section 1.3.3).

Motivated by these considerations, in this chapter we propose a new class of nonlinear high-gain observers, denoted as "*low-power high-gain observer*s", that preserves the same high-gain features but which substantially overtakes the aforementioned drawbacks. The proposed structure solves the implementation problems (i), having a gain that grows up only to power two, regardless the dimension of the system. The new structure is characterized by a state-dimension which is larger than n. As a matter of fact, the relative degree between the measurement and the state estimates is increased with respect to standard high-gain observer, thus reflecting better sensitivity properties with respect to high-frequency gain and improving (iii). Finally, the presence of additional saturations helps in avoiding the peaking phenomenon (ii).

The contents of this chapter provides a novel "low-power methodology", based on dynamic extension, complementary to classical "high-gain one". The first two sections illustrate that the new low-power construction can be used without loss of generality in place of the standard high-gain construction, by showing its application to systems in the strict feedback form (1.11) (Section 3.1) and in the non strict feedback form (1.20) (Section 3.2). In Section 3.3 we show how to modify the construction presented in Section 3.1 in order to remove the annoying peaking phenomenon. In Section 3.4, we apply the novel methodology introduced in Section 2.4 to analyse the performances of the new low-power high-gain observer in presence of high-frequency measurement noise, highlighting the effects of the relative degree on the steady-state behaviour. In Section 3.5 the new low-power high-gain observer is successfully applied to the output regulation framework as a design tool for robust internal models. Simulations are proposed in Section 3.6 to show the performances of the new observer. We remark that Appendices A and B provides some technical lemmas which are essentials for the design of the new class of observers.

The results of Section 3.1 appeared on Astolfi and Marconi (2015) and have been successfully applied to the output feedback stabilization framework in Wang et al. (2015). Section 3.2 presents results that have been published in Wang et al. (2016-17) in collaboration with one of the co-authors (Lei Wang). The results of Section 3.3 were produced under the co-supervision of professor Andrew Teel and will appear on Astolfi et al. (2016b). Section 3.4 is the extension of the work proposed in Astolfi et al. (2016a) whereas the results proposed in Section 3.5 will appear in Astolfi et al. (2017) and have been written under the co-supervision of Alberto Isidori.

Basic ingredients of the low-power construction

In this brief introduction we present the main idea of the low-power construction. Consider a system of dimension $n = 2$ of the form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \varphi(x) \end{aligned}$$

where (x_1, x_2) evolves in a compact set on which $\varphi(\cdot)$ is bounded. Suppose to implement the following *dirty-derivative* observer

$$\begin{aligned} \dot{\hat{x}}_1 &= \eta_1 + \ell c_{11}(y - \hat{x}_1) \\ \dot{\hat{x}}_2 &= \ell^2 c_{12}(y - \hat{x}_1) \end{aligned}$$

where $\hat{x}_1$ is the estimate of x_1 and $\hat{x}_2$ is the estimate of x_2. Its transfer function is shown in Figure 3.1.

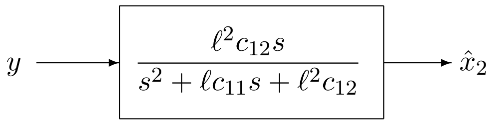

Figure 3.1: Dirty derivative observer of dimension 2 for a system in strict feedback form of dimension 2.

As shown in Section 1.2, this observer provides a "rough" estimate of x_1 and x_2, namely

$$\lim_{t\to\infty} |\hat{x}_1(t) - x_1(t)| + |\hat{x}_2(t) - x_2(t)| \;\leq\; \frac{\mu}{\ell} \max_{x\in X} |\varphi(x)| \;,$$

for some constant μ. Evidently, by augmenting ℓ we can increase the precision of the estimate. Now consider a system of dimension $n = 3$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= \varphi(x) \\ y &= x_1 \end{aligned}$$

and suppose to put in cascade of the previous observer a second *dirty-derivative* high-

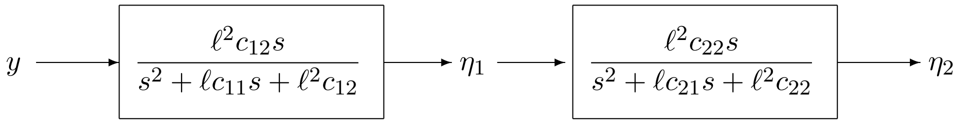

Figure 3.2: Cascade of two dirty derivative observers of dimension 2 for a system in strict feedback form of dimension 3.

gain observer driven by η_1 (see Figure 3.2), namely to design the following observer

$$
\begin{aligned}
\dot{\hat{x}}_1 &= \eta_1 + \ell c_{11}(y - \hat{x}_1) \\
\dot{\eta}_1 &= \ell^2 c_{12}(y - \hat{x}_1) \\
\dot{\hat{x}}_2 &= \eta_2 + \ell c_{21}(\eta_1 - \hat{x}_2) \\
\dot{\hat{x}}_3 &= \ell^2 c_{22}(\eta_1 - \hat{x}_2)
\end{aligned}
$$

where $\hat{x}_1$ represents an estimate of x_1, η_1 and $\hat{x}_2$ an estimate of x_2, and η_2 an estimate of x_3. The $(\hat{x}_2, \eta_2)$ dynamics are driven by η_1. It is not hard to see that $(\hat{x}_2, \eta_2)$ provide a "rough" estimate of η_1 and $\dot{\eta}_1$. But since η_1 is an estimate of x_2, we get that the second high-gain observer provides a "rough" estimate of x_2 and x_3, namely

$$
\lim_{\ell \to \infty} \lim_{t \to \infty} |\hat{x}_1(t) - x_1(t)| + |\eta_1(t) - x_2(t)| + |\hat{x}_2(t) - x_2(t)| + |\eta_2(t) - x_3(t)| \;=\; 0\,.
$$

Clearly the scheme above can be generalized by building a cascade of n *dirty-derivative* observers to get an estimate of a system of dimension n of the form

$$
\begin{aligned}
\dot{x}_i &= x_{i+1} \qquad i = 1, \ldots, n-1\,, \\
\dot{x}_n &= \varphi(x)
\end{aligned}
$$

by implementing an observer of dimension $2n - 2$ with a high-gain parameter which is raised up to power 2.

The scheme proposed above however can guarantee asymptotic estimation only for $\ell \to \infty$ because it misses the "consistency" terms which provide the correct steady-state values. As a consequence, let consider again a system of dimension three and suppose to modify the previous dirty derivative observer as shown in the Figure 3.3, namely we

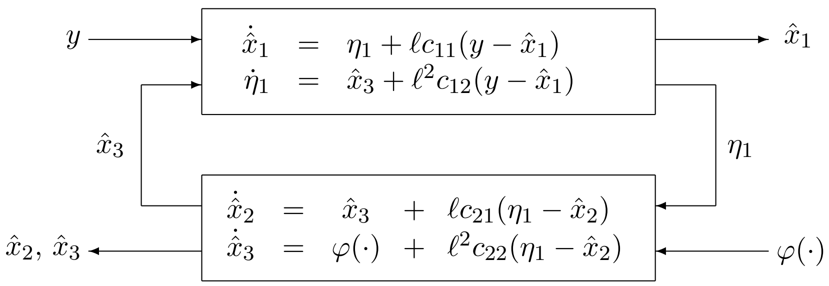

Figure 3.3: Low-power high-gain observer for a system in strict feedback form of dimension 3.

implement the forthcoming observer

$$\begin{aligned}
\dot{\hat{x}}_1 &= \eta_1 + \ell c_{11}(y - \hat{x}_1) \\
\dot{\eta}_1 &= \hat{x}_3 + \ell^2 c_{12}(y - \hat{x}_1) \\
\dot{\hat{x}}_2 &= \hat{x}_3 + \ell c_{21}(\eta_1 - \hat{x}_2) \\
\dot{\hat{x}}_3 &= \hat{\varphi}(\hat{x}_1, \hat{x}_2, \hat{x}_3) + \ell^2 c_{22}(\eta_1 - \hat{x}_2)
\end{aligned}$$

with $\hat{\varphi}(\cdot)$ a bounded function which agrees with φ on X. Now the derivative of η_1, which is the estimate of x_2, is fed by $\hat{x}_3$ which is the estimate of x_3. This scheme is "consistent" with the dynamics of the observed plant and it is not hard to prove that, for $\ell < \infty$ large enough, we get

$$\lim_{t\to\infty} |\xi_{11}(t) - x_1(t)| + |\xi_{12}(t) - x_2(t)| + |\xi_{21}(t) - x_2(t)| + |\xi_{22}(t) - x_3(t)| \;=\; 0\,,$$

namely the proposed observer provides an asymptotic estimate of the plant for finite values of ℓ. In the forthcoming sections we will exploit this structure in order to design an observer for a system in feedback form of dimension n which involves powers of ℓ up to order 2 regardless the dimension of the system n. Furthermore, we will show how the redundancy of the observer can be used in order to provide other benefits (higher relative degree between measurement noise and state estimates and removal of the peaking).

3.1 The low-power construction

Let consider nonlinear systems of the form (1.11), *i.e.*

$$\begin{array}{rcll} \dot{x}_i &=& x_{i+1} + \varphi_i(x_1,\ldots,x_i,u)\ , & i = 1,\ldots,n-1\ , \\ \dot{x}_n &=& \varphi_n(x_1,\ldots,x_n,u)\ , & \\ y &=& x_1 + \nu(t)\ , & \end{array} \tag{3.1}$$

where the state $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$ evolves in a given compact subset X of $\mathbb{R}^n$, the input u is any function assumed to be known evolving in a compact subset U of $\mathbb{R}^m$ and $y \in \mathbb{R}$ is the measured output. Furthermore, we suppose the functions φ_i are locally Lipschitz. The function $t \mapsto \nu(t)$ represents a bounded unknown measurement noise. The proposed *low-power high-gain observer* is

$$\begin{array}{rcl} \dot{\hat{x}}_1 &=& \eta_1 + \hat{\varphi}_1(\hat{x}_1,u) + c_{11}\ell(y-\hat{x}_1) \\ \dot{\eta}_1 &=& \eta_2 + \hat{\varphi}_2(\hat{x}_1,\hat{x}_2,u) + c_{12}\ell^2(y-\hat{x}_1) \\ &\vdots& \\ \dot{\hat{x}}_i &=& \eta_i + \hat{\varphi}_i(\hat{x}_1,\ldots,\hat{x}_i,u) + c_{i1}\ell(\eta_{i-1}-\hat{x}_i) \\ \dot{\eta}_i &=& \eta_{i+1} + \hat{\varphi}_{i+1}(\hat{x}_1,\ldots,\hat{x}_{i+1},u) + c_{i2}\ell^2(\eta_{i-1}-\hat{x}_i) \\ &\vdots& \\ \dot{\hat{x}}_{n-2} &=& \eta_i + \hat{\varphi}_{n-2}(\hat{x}_1,\ldots,\hat{x}_{n-2},u) + c_{(n-2)1}\ell(\eta_{n-3}-\hat{x}_{n-2}) \\ \dot{\eta}_{n-2} &=& \hat{x}_n + \hat{\varphi}_{n-1}(\hat{x}_1,\ldots,\hat{x}_{n-1},u) + c_{(n-2)2}\ell^2(\eta_{n-3}-\hat{x}_{n-2}) \\ \dot{\hat{x}}_{n-1} &=& \hat{x}_n + \hat{\varphi}_{n-1}(\hat{x}_1,\ldots,\hat{x}_{n-1},u) + c_{(n-1)1}\ell(\eta_{n-2}-\hat{x}_{n-1}) \\ \dot{\hat{x}}_n &=& \hat{\varphi}_n(\hat{x}_1,\ldots,\hat{x}_n,u) + c_{(n-1)2}\,\ell^2(\eta_{n-2}-\hat{x}_{n-1}) \end{array} \tag{3.2}$$

where $(\hat{x},\eta) \in \mathbb{R}^{2n-2}$ is the state of the observer, with $\hat{x} = \text{col}(\hat{x}_1,\ldots,\hat{x}_n) \in \mathbb{R}^n$ and $\eta = \text{col}(\eta_1,\ldots,\eta_{n-2}) \in \mathbb{R}^{n-2}$ and (c_{i1},c_{i2}), $i = 1,\ldots,n-1$ are coefficients to be properly chosen. As concern the functions $\hat{\varphi}_i$, by following the precepts of Section 1.2, when the functions φ_i are perfectly known, we can choose

$$\hat{\varphi}_i(x_1,\ldots,x_i,u) \;=\; \text{sat}_{\vartheta_i}\big(\varphi_i(x_1,\ldots,x_i,u)\big)\ .$$

with the positive real number ϑ_i defined as

$$\vartheta_i \;=\; \max_{x\in X,\ u\in U} |\varphi_i(x_1,\ldots,x_i,u)|\ .$$

On the contrary, when no information is available, one may choose to pick $\varphi_i(\cdot) = 0$. As a consequence, in the following we suppose that the functions $\hat{\varphi}_i$, $i = 1, \dots, n$ satisfy

$$|\varphi_i(x_1, \dots, x_i, u) - \hat{\varphi}_i(\hat{x}_1, \dots, \hat{x}_i, u)| \;\leq\; L_i|x - \hat{x}| + R_i \tag{3.3}$$

for all $(x, \hat{x}, u) \in X \times \mathbb{R}^n \times U$ and for some $L_i > 0$ and $R_i > 0$.

Theorem 3.1. *Consider system* (3.1) *and the observer* (3.2). *It is possible to choose the coefficients* $(c_{i1}\ c_{i2})$ *such that there exist a* $\ell^\star \geq 1$ *and strictly positive constants* μ_1, μ_2 *and* μ_3 *such that, for any* $\ell > \ell^\star$ *and for any initial conditions* $(x(0), \xi(0)) \in X \times \mathbb{R}^{2n-2}$, *the following bound holds*

$$\begin{aligned}
|\hat{x}_i(t) - x_i(t)| \;&\leq\; \mu_1\, \ell^{i-1} \exp(-\mu_2\, \ell\, t)\, \pi \,+\, \mu_3 \sum_{j=1}^{n} \ell^{i-(j+1)} R_j \,+\, \mu_4 \ell^{i-1} \left\|\nu(\cdot)\right\|_\infty \,, \\
&\qquad\qquad\qquad\qquad\qquad\qquad i = 1, \dots, n\,, \\
|\eta_i(t) - x_{i+1}(t)| \;&\leq\; \mu_1\, \ell^{i} \exp(-\mu_2\, \ell\, t)\, \pi \,+\, \mu_3 \sum_{j=1}^{n} \ell^{i-j} R_j \,+\, \mu_4 \ell^{i} \left\|\nu(\cdot)\right\|_\infty \,, \\
&\qquad\qquad\qquad\qquad\qquad\qquad i = 1, \dots, n-2\,,
\end{aligned} \tag{3.4}$$

where

$$\pi \;=\; \sum_{i=1}^{n} |\hat{x}_i(0) - x_i(0)| + \sum_{i=1}^{n-2} |\eta_i(0) - x_{i+1}(0)|\,. \tag{3.5}$$

as long as $x(t) \in X$ *and* $u(t) \in U$.

Proof. By using the notation

$$\begin{aligned}
\xi_i \;&=\; \mathrm{col}(\xi_{i1}, \xi_{i2}) &&:=\; \mathrm{col}(\hat{x}_i, \eta_i) \qquad i = 1, \dots, n-2, \\
\xi_{n-1} \;&=\; \mathrm{col}(\xi_{(n-1)1}, \xi_{(n-1)2}) &&:=\; \mathrm{col}(\hat{x}_{n-1}, \hat{x}_n)
\end{aligned}$$

the observer (3.2) can be written in the compact form

$$\begin{aligned}
\dot{\xi}_1 \;&=\; A\,\xi_1 + N\xi_2 + \phi_1 + D_2(\ell)\, K_1\, (y - C\xi_1)\,, \\
\dot{\xi}_i \;&=\; A\,\xi_i + N\xi_{i+1} + \phi_i + D_2(\ell)\, K_i\, (B^\top \xi_{i-1} - C\xi_i)\,, \qquad i = 2, \dots, n-2\,, \\
\dot{\xi}_{n-1} \;&=\; A\,\xi_{n-1} + \phi_{n-1} + D_2(\ell)\, K_i\, (B^\top \xi_{n-2} - C\xi_{n-1})\,,
\end{aligned} \tag{3.6}$$

where (A, B, C) is a triplet in *prime form* of dimension 2,

$$K_i := \mathrm{col}(c_{i1}, c_{i2})\,, \qquad N := BB^\top\,, \qquad D_2(\ell) := \mathrm{diag}\left(\ell, \ell^2\right)\,,$$

and

$$\phi_i \;:=\; \begin{pmatrix} \hat{\varphi}_i(\hat{x}_1,\ldots,\hat{x}_i,u) \\ \hat{\varphi}_{i+1}(\hat{x}_1,\ldots,\hat{x}_{i+1},u) \end{pmatrix}, \quad i = 1,\ldots,n-1\,.$$

Consider now the change of variables

$$\xi_i \;\mapsto\; \tilde{\xi}_i \;:=\; \xi_i - \text{col}\,(x_i,\; x_{i+1}) \qquad i = 1,\ldots,n-1\;,$$

by which the latter is transformed into

$$\begin{aligned} \dot{\tilde{\xi}}_1 &= H_1\,\tilde{\xi}_1 + N\,\tilde{\xi}_2 + \Delta_1 + D_2(\ell)K_1\nu(t)\,, \\ \dot{\tilde{\xi}}_i &= H_i\,\tilde{\xi}_i + N\tilde{\xi}_{i+1} + D_2(\ell)\,K_i\,B^\top\,\tilde{\xi}_{i-1} + \Delta_i\;, \qquad i = 2,\ldots,n-2, \\ \dot{\tilde{\xi}}_{n-1} &= H_{n-1}\,\tilde{\xi}_{n-1} + D_2(\ell)\,K_{n-1}\,B^\top\,\tilde{\xi}_{n-2} + \Delta_{n-1} \end{aligned}$$

where $H_i = A - D_2(\ell)\,K_i\,C$ and

$$\Delta_i \;:=\; \begin{pmatrix} \hat{\varphi}_i(\hat{x}_1,\ldots,\hat{x}_i,u) - \varphi_i(x_1,\ldots,x_i,u) \\ \hat{\varphi}_{i+1}(\hat{x}_1,\ldots,\hat{x}_{i+1},u) - \varphi_{i+1}(x_1,\ldots,x_{i+1},u) \end{pmatrix}, \qquad i = 1,\ldots,n-1\,.$$

By following the rescaling (1.16) and the property (1.17) used in the proof of Theorem 1.4, rescale now the variables $\tilde{\xi}_i$ as follows

$$\tilde{\xi}_i \;\mapsto\; \varepsilon_i \;:=\; \ell^{2-i}\,D_2(\ell)^{-1}\,\tilde{\xi}_i \qquad i = 1,\ldots,n-1\,,$$

which is, in compact notation,

$$\tilde{\xi} \;\mapsto\; \varepsilon \;:=\; \ell\mathcal{D}_\ell^{-1}\tilde{\xi}$$

with $\varepsilon = \text{col}(\varepsilon_1\ \ldots\ \varepsilon_{n-1})$ and

$$\mathcal{D}_\ell \;=\; \text{blckdiag}\left(D_2(\ell)\;, \ell D_2(\ell), \ldots\;, \ell^{n-2}D_2(\ell)\right)\,.$$

An easy calculation shows that

$$\dot{\varepsilon} = \ell M\varepsilon + \ell\mathcal{D}_\ell^{-1}\Delta(\varepsilon,x,u) + \ell\,\bar{K}\nu(t)$$

with M defined in the Appendix A, (A.1), $\bar{K} = \mathrm{col}(K_1, 0, \ldots, 0)$ and $\Delta(\varepsilon, x, u) := \mathrm{col}(\Delta_1, \ldots, \Delta_{n-1})$ that in the new coordinates read as

$$\Delta_i \;=\; \begin{pmatrix} \hat{\varphi}_i(C\varepsilon_1 + x_1, \ldots, \ell^{i-1}C\varepsilon_i + x_i, u) - \varphi_i(x_1, \ldots, x_i, u) \\ \hat{\varphi}_{i+1}(C\varepsilon_1, +x_1, \ldots, \ell^{i}C\varepsilon_{i+1} + x_{i+1}, u) - \varphi_{i+1}(x_1, \ldots, x_{i+1}u) \end{pmatrix},$$

for $i = 1, \ldots n-2$, and

$$\Delta_{n-1} \;=\; \begin{pmatrix} \hat{\varphi}_{n-1}(C\varepsilon_1 + x_1, \ldots, \ell^{n-2}C\varepsilon_{n-1} + x_{n-1}, u) - \varphi_{n-1}(x_1, \ldots, x_{n-1}, u) \\ \hat{\varphi}_{n}(C\varepsilon_1, +x_1, \ldots, \ell^{n-1}B^\top\varepsilon_{n-1} + x_{n}, u) - \varphi_{n}(x_1, \ldots, x_{n}u) \end{pmatrix}.$$

Note that as in the case of the standard high-gain observer (see Section 1.2), if the matrix M is Hurwitz, we get a stable linear system disturbed by the non-linearity $\Delta(\varepsilon, x, u)$. Furthermore, the gain ℓ can be arbitrarily increased in order to prove stability in the ε-coordinates. Evidently, from now on, we use the same arguments used in the proof Theorem 1.4. In particular, analogously to the bound (1.18) and by using (3.3), we get

$$\begin{aligned}
&\left|\ell\mathcal{D}_\ell^{-1}\Delta(\varepsilon, x, u)\right| \\
&\quad\leq \sum_{j=1}^{n-1}\frac{1}{\ell^{j-2}}D_2(\ell)^{-1}\Delta_i \\
&\quad\leq \sum_{j=2}^{n-1}\frac{2}{\ell^{j-1}}\left|\hat{\varphi}_j(C\varepsilon_1 + x_1, \ldots, \ell^{j-1}C\varepsilon_j + x_j, u) - \varphi_j(x_1, \ldots, x_j, u)\right| \\
&\qquad + \frac{1}{\ell^{n-1}}|\hat{\varphi}_n(C\varepsilon_1, +x_1, \ldots, \ell^{n-1}B^\top\varepsilon_{n-1} + x_n, u) - \varphi_n(x_1, \ldots, x_n u)| \\
&\qquad + |\hat{\varphi}_1(C\varepsilon_1 + x_1, u) - \varphi_1(x_1, u)| \\
&\quad\leq \sum_{j=2}^{n-1}\frac{2}{\ell^{j-1}}\left(L_j\sum_{k=1}^{j}|\ell^{k-1}C\varepsilon_k| + R_j\right) + L_1|C\varepsilon_1| + R_1 \\
&\qquad + \frac{L_n}{\ell^{n-1}}\left(\sum_{j=1}^{n-1}|\ell^{j-1}C\varepsilon_j| + |\ell^{n-1}B^\top\varepsilon_{n-1}|\right) + \frac{1}{\ell^{n-1}}R_n \\
&\quad\leq L_1|\varepsilon| + 2\sum_{j=2}^{n-1}L_j|\varepsilon| + L_n|\varepsilon| + R_1 + 2\sum_{j=2}^{n-1}\frac{1}{\ell^{j-1}}R_j + \frac{1}{\ell^{n-1}}R_n
\end{aligned}$$

and therefore, by denoting

$$L = L_1 + 2\sum_{j=2}^{n-1}L_j + L_n,$$

we have

$$\left|\ell D_\ell^{-1}\,\Delta(\varepsilon,x,u)\right| \le L\;|\varepsilon| + 2\sum_{j=1}^{n}\ell^{1-j}R_j \qquad \forall(\varepsilon,x,u)\in\mathbb{R}^{2n-1}\times X\times U\;.$$

Now let the coefficients (c_{i1},c_{i2}), $i=1,\ldots,n-1$, be chosen according to Lemma A.1 such that the matrix M is Hurwitz. As a consequence there exists a positive definite matrix P solution of the Lyapunov equation

$$PM + M^\top P \;=\; -I\,.$$

By following the proof of Theorem 1.4, let consider the Lyapunov Function $V:\mathbb{R}^{2n-2}\to\mathbb{R}_{\ge 0}$ defined as

$$V \;=\; \sqrt{\varepsilon^\top P\varepsilon}\,.$$

By denoting with $\underline{\lambda}$ and $\bar\lambda$ the minimum and the maximum eigenvalue of P we have

$$\sqrt{\underline{\lambda}}\,|\varepsilon| \;\le\; V \;\le\; \sqrt{\bar\lambda}\,|\varepsilon|\;\;.$$

Observe that V is only Lipschitz. As a consequence, when V is not zero, by evaluating the Dini derivative of V along the solutions of ε we get

$$\begin{aligned}
D^+V \;&=\; \frac{1}{\sqrt{\varepsilon^\top P\varepsilon}}\,\varepsilon^\top P\Big(\ell M\varepsilon + \ell\mathcal{D}_\ell^{-1}\Delta(\varepsilon,x,u) + \ell\bar K\nu(t)\Big)\\
&\le\; -\frac{1}{\sqrt{\varepsilon^\top P\varepsilon}}\,\frac{\ell}{2}\,|\varepsilon|^2 + \frac{1}{\sqrt{\varepsilon^\top P\varepsilon}}\,\varepsilon^\top P\left(L\,|\varepsilon| + 2\sum_{j=1}^{n}\ell^{1-j}R_j + \ell\bar K\nu(t)\right)\\
&\le\; -\left(\frac{\ell}{2} - L\,\frac{\bar\lambda}{\sqrt{\underline{\lambda}}}\right)\frac{|\varepsilon|^2}{\sqrt{\varepsilon^\top P\varepsilon}} + \frac{\bar\lambda}{\sqrt{\underline{\lambda}}}\left(2\sum_{j=1}^{n}\ell^{1-j}R_j + \ell|\bar K|\bar\nu\right)
\end{aligned}$$

where we denoted $\bar\nu = \|\nu(\cdot)\|_\infty$. On the contrary, for $V=0$ we get

$$D^+V \;\le\; \frac{\bar\lambda}{\sqrt{\underline{\lambda}}}\left(2\sum_{j=1}^{n}\ell^{1-j}R_j + \ell|\bar K|\bar\nu\right)$$

hence the previous expression holds for both cases. Now let $\ell^\star = 2\ell\bar\lambda/\sqrt{\underline{\lambda}}$. As a consequence, there exists a $a_1>0$ such that, for any $\ell>\max\{\ell^\star,1\}$,

$$\dot V \;\;\le\;\; -\ell\,\frac{a_1}{\bar\lambda}\,V + \frac{\bar\lambda}{\sqrt{\underline{\lambda}}}\left(2\sum_{j=1}^{n}\ell^{1-j}R_j + \ell|\bar K|\bar\nu\right)$$

and therefore

$$V(t) \;\leq\; \exp\left(-\ell\,\frac{a_1}{\bar\lambda}\,t\right) V(0) + \frac{1-\exp(-\ell\,a_1/\bar\lambda t)}{\ell\,a_1/\bar\lambda}\,\frac{\bar\lambda}{\sqrt{\underline{\lambda}}}\left(2\sum_{j=1}^{n}\ell^{1-j}R_j + \ell|\bar K|\bar\nu\right).$$

From this we get

$$|\varepsilon(t)| \;\leq\; \mu_1\,\exp(-\ell\,\mu_2 t)\;|\varepsilon(0)| + \mu_3\sum_{j=1}^{n}\ell^{-j}R_j + \mu_4\,\bar\nu$$

with $\mu_1 = 1/\sqrt{\underline{\lambda}}$, $\mu_2 = a_1/\bar\lambda$, $\mu_3 = 2\bar\lambda^2/(\sqrt{\underline{\lambda}}\,a_1)$ and $\mu_4 = |\bar K|\mu_3$. By noting that for $\ell \geq 1$ we have

$$|\varepsilon| \;\leq\; |\tilde\xi| \;\leq\; \sum_{i=1}^{n}|x_i-\hat x_i| + \sum_{i=1}^{n-2}|x_{i+1}-\eta_i|\,,$$

the proof concludes by using (3.5) and the following bounds

$$\begin{aligned}\ell^{-(i-1)}\,|x_i-\hat x_i| \;&\leq\; |\varepsilon| \qquad i=1,\ldots,n\,,\\ \ell^{-i}\,|x_{i+1}-\eta_i| \;&\leq\; |\varepsilon| \qquad i=1,\ldots,n-2\,.\end{aligned}$$

□

Remarks

- The bounds (3.4) shows that the new low-power structure guarantees asymptotic bounds which are comparable with the bounds (1.14) provided by the standard high-gain observer (1.12). The main difference relies in the fact that the high-gain observers (1.12) has a state dimension n with the high-gain parameter ℓ powered up to n, whereas the new low-power structure has dimension $2n-2$ with the gain ℓ powered up to 2. In view of the considerations introduced in Section 1.3.1, it is worth noticing that we are completely solving the implementation issues related to the gain ℓ^n when ℓ or n are very large.

- As already explicated at the end of Section 1.2, the same kind of considerations can be made about the bounds (3.4). In particular asymptotic estimation can be achieved only when $R_i = 0$, $i = 1,\ldots,n$ and when there is no measurement noise.

- Despite the low-power high-gain observer (3.2) presents no terms with powers of ℓ larger than 2, the estimates $\hat x_i$, $i = 1,\ldots,n$ are affected by the peaking phenomenon as the standard high-gain observer. As shown in Section 3.3, a more sophisticated design is needed to solve this problem, namely by introducing saturations functions. It is interesting to note that also the gain from the estimate $\hat x_i$

and the measurement noise is multiplied by a term ℓ^{i-1}. However, as studied in Section 3.4, the noise is affecting only the first two variables $\hat{x}_1$ and η_1: the relative degree between $\nu(t)$ and $\hat{x}_i$, $i > 1$ is always larger than one, thus resulting in better sensitivity to high-frequency noise.

3.2 The case of non-strict feedback form

While in Section 3.1 we have presented the novel low-power high-gain observer design for the class of systems in strict feedback form, in this section we deal with the case of systems in non-strict feedback form. In particular, by following the same design philosophy, we want to show that the high-gain observer (1.21) of Section 1.4 can be modified in order to design an observer with a high-gain term powered up to two regardless the dimension n of the system by substantially overtaking the problem of having the high-gain parameter powered to the order n. The results of Section 3.2 have been published in Wang et al. (2016-17) and the work has been done with the help of the co-author Lei Wang.

Consider a system in the non-strict feedback form (1.20) that we recall here

$$\begin{array}{rcl} \dot{x}_i &=& f_i(x_1,\ldots,x_i,x_{i+1},u)\,, \qquad 1\le i\le n-1\,,\\ \dot{x}_n &=& f_n(x_1,\ldots,x_n,u)\,,\\ y &=& h(x_1,u)\,, \end{array} \tag{1.20 revisited}$$

with state $x\in\mathbb{R}^n$, input $u\in\mathbb{R}$ and measured output $y\in\mathbb{R}$. Without loss of generality with respect to the proposed design (1.21), we suppose that Assumptions 1.1, 1.2 hold. By mimicking the structure of the low-power high-gain observer presented in Section 3.1, the structure of the observer has the following form

$$\begin{array}{rcl} \dot{\hat{x}}_1 &=& f_1(\hat{x}_1,\eta_1,u)+c_{11}\ell(y-\hat{x}_1)\\ \dot{\eta}_1 &=& f_2(\hat{x}_1,\hat{x}_2,\eta_2,u)+c_{12}\ell^2(y-\hat{x}_1)\\ &\vdots& \\ \dot{\hat{x}}_i &=& f_i(\hat{x}_1,\ldots,\hat{x}_i,\eta_i,u)+c_{i1}\ell(\eta_{i-1}-\hat{x}_i)\\ \dot{\eta}_i &=& f_{i+1}(\hat{x}_1,\ldots,\hat{x}_{i+1},\eta_{i+2},u)+c_{i2}\ell^2(\eta_{i-1}-\hat{x}_i)\\ &\vdots& \\ \dot{\hat{x}}_{n-2} &=& f_{n-2}(\hat{x}_1,\ldots,\hat{x}_{n-2},\eta_{n-2},u)+c_{(n-2)1}\ell(\eta_{n-3}-\hat{x}_{n-2})\\ \dot{\eta}_{n-2} &=& f_{n-1}(\hat{x}_1,\ldots,\hat{x}_{n-1},\hat{x}_n,u)+c_{(n-2)2}\ell^2(\eta_{n-3}-\hat{x}_{n-2})\\ \dot{\hat{x}}_{n-1} &=& f_{n-1}(\hat{x}_1,\ldots,\hat{x}_{n-1},\hat{x}_n,u)+c_{(n-1)1}\ell(\eta_{n-2}-\hat{x}_{n-1})\\ \dot{\hat{x}}_n &=& f_n(\hat{x}_1,\ldots,\hat{x}_n,u)+c_{(n-1)2}\ell^2(\eta_{n-2}-\hat{x}_{n-1}) \end{array} \tag{3.7}$$

where $(\hat{x},\eta)\in\mathbb{R}^{2n-2}$ is the state, $\hat{x}=\text{col}(\hat{x}_1,\ldots,\hat{x}_n)\in\mathbb{R}^n$ and $\eta=\text{col}(\eta_1,\ldots,\eta_{n-2})\in\mathbb{R}^{n-2}$, ℓ is the high-gain parameter and the coefficients (c_{i1},c_{i2}), $i=1,\ldots,n-1$ have to be properly chosen. In Section 1.4 we showed that the coefficients k_i, $i=1,\ldots,n$, of the observer (1.21) need to be chosen according to Lemma 1.1. Similarly, a special design procedure is needed for the observer (3.7). We refer to the Lemma B.1 given in the Appendix B for the technicalities.

Theorem 3.2. *Consider the system* (1.20) *satisfying Assumptions 1.1 and 1.2 and the observer* (3.7)*. There exists a choice of the coefficients* (c_{i1}, c_{i2}) *such that there exist a* $\ell^\star \geq 1$ *and strictly positive real numbers* μ_1 *and* μ_2 *such that, for all* $\ell > \ell^\star$*,the following bound holds*

$$|\hat{x}_i(t) - x_i(t)| \;\leq\; \mu_1\,\ell^{i-1}\,\exp(-\mu_2 \ell t)\,\pi\,,$$

with $\pi = \sum_{i=1}^{n} |\hat{x}_i(0) - x_i(0)| + \sum_{i=1}^{n-2} |\eta_i(0) - x_{i+1}(0)|$*, for all* $(x(0), \eta(0)) \in \mathbb{R}^n \times \mathbb{R}^{2n-2}$ *and for all* $t \geq 0$*.*

Proof. Throughout the proof we denote $B := (0,1)^\top$, $C := (1,0)$. Furthermore we will use the compact notation $\boldsymbol{x}_i = (x_1, \ldots, x_i)$ and $\hat{\boldsymbol{x}}_i = (\hat{x}_1, \ldots, \hat{x}_i)$. By using the notation

$$\begin{array}{rcllll} \xi_i &=& \mathrm{col}(\xi_{i1}, \xi_{i2}) & := & \mathrm{col}(\hat{x}_i, \eta_i) & i = 1, \ldots, n-2, \\ \xi_{n-1} &=& \mathrm{col}(\xi_{(n-1)1}, \xi_{(n-1)2}) & := & \mathrm{col}(\hat{x}_{n-1}, \hat{x}_n) & \end{array}$$

the observer (3.2) can be written in the compact form

$$\begin{array}{rcll} \dot{\xi}_1 &=& \begin{pmatrix} f_1(\hat{\boldsymbol{x}}_1, B^\top \xi_1, u) + \ell c_{11}(y - C\xi_1) \\ f_2(\hat{\boldsymbol{x}}_2, B^\top \xi_2, u) + \ell^2 c_{12}(y - C\xi_1) \end{pmatrix} & \\ \dot{\xi}_i &=& \begin{pmatrix} f_i(\hat{\boldsymbol{x}}_i, B^\top \xi_i, u) + \ell c_{i1}(B^\top \xi_{i-1} - C\xi_i) \\ f_{i+1}(\hat{\boldsymbol{x}}_{i+1}, B^\top \xi_{i+1}, u) + \ell^2 c_{i2}(B^\top \xi_{i-1} - C\xi_i) \end{pmatrix} & i = 2, \ldots, n-2\,, \\ \dot{\xi}_{n-1} &=& \begin{pmatrix} f_{n-1}(\boldsymbol{x}_{n-1}, B^\top \xi_{n-1}, u) + \ell c_{(n-1)1}(B^\top \xi_{n-2} - C\xi_{n-1}) \\ f_n(\boldsymbol{x}_{n-1}, B^\top \xi_{n-1}, u) + \ell^2 c_{(n-1)2}(B^\top \xi_{n-2} - C\xi_{n-1}) \end{pmatrix} & \end{array}$$

Consider now the change of variable

$$\xi_i \;\mapsto\; \tilde{\xi}_i \;:=\; \xi_i - \mathrm{col}\,(x_i,\; x_{i+1}) \qquad i = 1, \ldots, n-1\,, \tag{3.8}$$

We first start from $\tilde{\xi}_1$ dynamics. By using the mean value theorem, one can get

$$\begin{array}{rcl} \dot{\tilde{\xi}}_{11} &=& f_1(\hat{\boldsymbol{x}}_1,\, B^\top \xi_1,\, u) - f_1(\boldsymbol{x}_1, x_2, u) - \ell\, c_{11}\,(y - \hat{x}_1) \\ &=& f_1(\hat{\boldsymbol{x}}_1, B^\top \xi_1, u) - f_1(\boldsymbol{x}_1, B^\top \xi_1, u) + f_1(\boldsymbol{x}_1, B^\top \xi_1, u) \\ && \qquad -f_1(\boldsymbol{x}_1, x_2, u) - \ell\, c_{11}\,(h(C\xi_1,\, u) - h(x_1, u)) \\ &=& \dfrac{\partial f_1}{\partial x_2}(\boldsymbol{x}_1(t), \delta_1(t), u(t))\tilde{\xi}_{12} - \ell c_{11} \dfrac{\partial h}{\partial x_1}(\delta_0(t), u(t))\tilde{\xi}_{11} \\ && \qquad + f_1(\hat{\boldsymbol{x}}_1, B^\top \xi_1, u) - f_1(\boldsymbol{x}_1, B^\top \xi_1, u) \end{array}$$

$$
\begin{aligned}
\dot{\tilde{\xi}}_{12} &= f_2(\hat{\boldsymbol{x}}_2, B^\top \xi_2, u) - f_2(\boldsymbol{x}_2, x_3, u) - \ell^2\, c_{12}\,(y - \hat{x}_1) \\
&= f_2(\hat{\boldsymbol{x}}_2, B^\top \xi_2, u) - f_2(\boldsymbol{x}_2, B^\top \xi_2, u) + f_2(\boldsymbol{x}_2, B^\top \xi_2, u) \\
&\qquad\qquad - f_2(\boldsymbol{x}_2, x_3, u) - \ell^2\, c_{12}\,(h(C\xi_1,\, u) - h(x_1, u)) \\
&= \frac{\partial f_2}{\partial x_3}(\boldsymbol{x}_2(t), \delta_2(t), u(t))\tilde{\xi}_{22} - \ell^2 c_{12}\frac{\partial h}{\partial x_1}(\delta_0(t), u(t))\tilde{\xi}_{11} \\
&\qquad\qquad + f_2(\hat{\boldsymbol{x}}_2, B^\top \xi_2, u) - f_2(\boldsymbol{x}_2, B^\top \xi_2, u)
\end{aligned}
$$

for some $\delta_0(t)$ and $\delta_1(t)$. For the sake of compactness, we set

$$
a_1(t) \;=\; \frac{\partial f_1}{\partial x_2}(\boldsymbol{x}_1(t), \delta_1(t), u(t))\,, \qquad a_2(t) \;=\; \frac{\partial f_2}{\partial x_3}(\boldsymbol{x}_2(t), \delta_2(t), u(t))\,,
$$

and $b_1(t) = \dfrac{\partial h}{\partial x_1}(\delta_0(t), u(t))$, to obtain

$$
\begin{aligned}
\dot{\tilde{\xi}}_{11} &= a_1(t)\tilde{\xi}_{12} - \ell c_{11} b_1(t)\tilde{\xi}_{11} + \bar{f}_1(t)\,, \\
\dot{\tilde{\xi}}_{12} &= a_2(t)\tilde{\xi}_{22} - \ell^2 c_{12} b_1(t)\tilde{\xi}_{11} + \bar{f}_2(t)\,,
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{f}_1(t) &= f_1(\hat{\boldsymbol{x}}_1, B^\top \xi_1, u) - f_1(\boldsymbol{x}_1, B^\top \xi_1, u)\,, \\
\bar{f}_2(t) &= f_2(\hat{\boldsymbol{x}}_2, B^\top \xi_2, u) - f_2(\boldsymbol{x}_2, B^\top \xi_2, u)\,.
\end{aligned}
$$

Hence, we get the $\tilde{\xi}_1$ dynamics, described by

$$
\dot{\tilde{\xi}}_1 = H_1(t)\tilde{\xi}_1 + N_2(t)\tilde{\xi}_2 + \bar{F}_1(t)
$$

with

$$
H_1(t) = \begin{pmatrix} -\ell c_{11} b_1(t) & a_1(t) \\ -\ell^2 c_{12} b_1(t) & 0 \end{pmatrix}, \quad \bar{F}_1(t) = \begin{pmatrix} \bar{f}_1(t) \\ \bar{f}_2(t) \end{pmatrix}, \quad N_2(t) \;=\; \begin{pmatrix} 0 & 0 \\ 0 & a_2(t) \end{pmatrix}.
$$

Applying the same arguments to $\tilde{\xi}_i$-dynamics for $2 \le i \le n-2$, yields

$$
\begin{aligned}
\dot{\tilde{\xi}}_{i1} &= a_i(t)\tilde{\xi}_{i2} - \ell c_{i1}\tilde{\xi}_{i1} + \ell c_{i1}\tilde{\xi}_{i-1,2} + \bar{f}_i(t)\,, \\
\dot{\tilde{\xi}}_{i2} &= a_{i+1}(t)\tilde{\xi}_{i+1,2} - \ell^2 c_{i2}\tilde{\xi}_{i1} + \ell^2 c_{i2}\tilde{\xi}_{i-1,2} + \bar{f}_{i+1}(t)\,,
\end{aligned}
$$

where, for compactness, we have defined

$$
\begin{aligned}
a_{i+1}(t) &= \frac{\partial f_{i+1}}{\partial x_{i+2}}(\boldsymbol{x}_{i+1}(t), \delta_{i+1}(t), u(t))\,, \\
\bar{f}_i(t) &= f_i(\hat{\boldsymbol{x}}_i(t), B^\top \xi_i(t), u(t)) - f_i(\boldsymbol{x}_i(t), B^\top \xi_i(t), u(t))\,, \\
\bar{f}_{i+1}(t) &= f_{i+1}(\hat{\boldsymbol{x}}_{i+1}(t), B^\top \xi_{i+1}(t), u(t)) - f_{i+1}(\boldsymbol{x}_{i+1}(t), B^\top \xi_{i+1}(t), u(t))\,,
\end{aligned}
$$

for some $\delta_{i+1}(t)$. Thus, we get the $\tilde{\xi}_i$ dynamics

$$\dot{\tilde{\xi}}_i = H_i(t)\tilde{\xi}_i + N_{i+1}(t)\tilde{\xi}_{i+1} + D_2(\ell)Q_i\tilde{\xi}_{i-1} + \bar{F}_i(t)$$

with

$$H_i(t) = \begin{pmatrix} -\ell c_{i1} & a_i(t) \\ -\ell^2 c_{i2} & 0 \end{pmatrix}, \quad \bar{F}_i(t) = \begin{pmatrix} \bar{f}_i(t) \\ \bar{f}_{i+1}(t) \end{pmatrix},$$

$D_2(\ell) = \text{diag}(\ell, \ell^2)$ and Q_i defined as in (B.1). Applying the same trick to $\tilde{\xi}_{n-1}$ dynamics, yields

$$\begin{aligned} \dot{\tilde{\xi}}_{n-1,1} &= a_{n-1}(t)\tilde{\xi}_{n-1,2} - \ell c_{n-1,1}\tilde{\xi}_{n-1,1} + \ell c_{n-1,1}\tilde{\xi}_{n-2,2} + \bar{f}_{n-1}(t)\,, \\ \dot{\tilde{\xi}}_{n-1,2} &= -\ell^2 c_{n-1,2}\tilde{\xi}_{n-1,1} + \ell^2 c_{n-1,2}\tilde{\xi}_{n-2,2} + \bar{f}_n(t)\,, \end{aligned}$$

where we have defined

$$\begin{aligned} \bar{f}_{n-1}(t) &= f_{n-1}(\hat{\boldsymbol{x}}_{n-1}, B^\top\xi_{n-1}, u) - f_{n-1}(\boldsymbol{x}_{n-1}, B^\top\xi_{n-1}, u)\,, \\ \bar{f}_n(t) &= f_n(\hat{\boldsymbol{x}}_{n-1}, B^\top\xi_{n-1}, u) - f_n(\boldsymbol{x}_{n-1}, x_n, u)\,. \end{aligned}$$

Further, the $\tilde{\xi}_{n-1}$ dynamics can be rewritten into the compact form

$$\dot{\tilde{\xi}}_{n-1} = H_{n-1}(t)\tilde{\xi}_{n-1} + D_2(\ell)Q_{n-1}\tilde{\xi}_{n-2} + \bar{F}_{n-1}(t)\,,$$

in which

$$H_{n-1}(t) = \begin{pmatrix} -\ell c_{n-1,1} & a_{n-1}(t) \\ -\ell^2 c_{n-1,2} & 0 \end{pmatrix}, \qquad \bar{F}_{n-1}(t) = \begin{pmatrix} \bar{f}_{n-1}(t) \\ \bar{f}_n(t) \end{pmatrix}.$$

Note that for each element of $\bar{F}_i(t)$, being Lipschitz by Assumption 1.1, one can find that $\bar{f}_i(t)$ satisfy

$$|\bar{f}_i(t)| \leq L|\hat{\boldsymbol{x}}_i - \boldsymbol{x}_i|\,, \quad 1 \leq i \leq n\,.$$

Rescale now the variables $\tilde{\xi}_i$ as follows

$$\varepsilon_i = \ell^{2-i}D_2(\ell)^{-1}\tilde{\xi}_i\,, \quad i = 1, 2, \ldots, n-1\,.$$

By setting $\varepsilon = \text{col}(\varepsilon_1, \ldots, \varepsilon_{n-1})$, an easy calculation shows that

$$\dot{\varepsilon} = \ell M(t)\varepsilon + \bar{F}_\ell(t) \tag{3.9}$$

in which the time varying block tridiagonal matrix $M(t)$ is defined as in (B.1), where

$a_i(t)$ for $1 \leq i \leq n-1$ and $b_1(t)$ are bounded from below and from above for all $t \geq 0$ by Assumption 1.1, $b_i(t) = 1$ for $2 \leq i \leq n-1$ and the vector $\bar{F}_\ell(t)$ is defined by

$$\bar{F}_\ell(t) = \mathcal{D}_\ell \bar{F}(t)$$

with $\mathcal{D}_\ell$ and $\bar{F}(t)$ defined by

$$\begin{aligned} \mathcal{D}_\ell &= \text{blckdiag}\left(D_2(\ell), \ldots, \ell^{n-2} D_2(\ell)\right), \\ \bar{F}(t) &= \text{col}\left(\bar{F}_1, \bar{F}_2(t), \ldots, \bar{F}_{n-1}(t)\right). \end{aligned}$$

Inspection on the each element of $\bar{F}_\ell$ shows that, for $\ell > 1$

$$|\ell^{1-i} \bar{f}_i(t)| \;\leq\; L|\varepsilon|, \quad i = 1, 2, \ldots, n,$$

thus yielding that there exists a real number $\bar{L} > 0$, independent of ℓ, such that $|\bar{F}_\ell| \leq \bar{L}|\varepsilon|$. Now let the coefficients (c_{i1}, c_{i2}) be chosen according to Lemma B.1 for a given matrix P and positive constant λ. Therefore we can choose the Lyapunov candidate as $W(\varepsilon) = \varepsilon^\top P \varepsilon$, whose time derivative along the trajectories of system (3.9) is given by

$$\begin{aligned} \dot{W}(\varepsilon) &= \ell \varepsilon^\top (PM + M^\top P)\varepsilon + 2\varepsilon^\top P \bar{F}_\ell \\ &\leq -(\ell\lambda - 2\bar{L}|P|)|\varepsilon|^2 . \end{aligned}$$

Choosing $\ell^\star = \dfrac{2\bar{L}|P|}{\lambda}$, one can conclude that for any $\ell > \ell^\star$, there exists a positive constant α_1 such that

$$\dot{W} \;\leq\; -\alpha_1 \ell |\varepsilon|^2 .$$

Recalling the fact that there exist positive constants $\bar{\sigma}$ and $\underline{\sigma}$ such that

$$\underline{\sigma}\,|\varepsilon|^2 \;\leq\; W(\varepsilon) \;\leq\; \bar{\sigma}\,|\varepsilon|^2 ,$$

it can be further deduced that

$$|\varepsilon(t)| \;\leq\; \mu_1 \exp(-\mu_2 \ell t)\,|\varepsilon(0)|$$

for some positive constants μ_1 and μ_2, independent of ℓ. Finally the claim of the proof follows immediately by noting that for $\ell \geq 1$ we have

$$\ell^{-(i-1)}\,|\hat{x}_i - x_i| \;\leq\; |\varepsilon| \;\leq\; |\tilde{\xi}| \;\leq\; \sum_{i=1}^{n} |\hat{x}_i - x_i| + \sum_{i=1}^{n-2} |\eta_i - x_{i+1}| . \qquad \square$$

3.3 Peaking-free design

In Section 3.1 we illustrated the main features of the new low-power approach. One advantage with respect to the standard high-gain observer (1.12) comes from an implementation with the power of the gain ℓ which is raised only up to power 2 regardless the dimension of the system. However, it can be easily seen that the design (3.2) does not solve the problem of peaking during the transient. In this section we propose a modification of the observer (3.2) which solves completely the problem of peaking without loosing its main characteristics. Basically saturations functions are added at various levels: they prevent the peaking phenomenon during the transients and do not "act" during the asymptotic behaviour which coincides with the nominal one. Furthermore, one extra dynamics is added at the bottom of the chain in order to preserve the removal of the peaking at any step. As a matter of fact, the proposed observer is a modification of the one introduced in Astolfi et al. (2016b). Note that the same "saturation design" can be adopted also for the class of observer (3.7).

We consider here, for the sake of simplicity in the analysis, the class of nonlinear system (3.1) when

$$\begin{array}{rcll} \varphi_i(x_1,\dots,x_i,u) & = & 0 & i=1,\dots,n-1, \\ \varphi_n(x_1,\dots,x_n,u) & = & \varphi(x,u), & \end{array}$$

with $\varphi(x)$ any locally Lipschitz function, namely the system

$$\begin{array}{rcll} \dot{x}_i & = & x_{i+1}\ , & i=1,\dots,n-1\ , \\ \dot{x}_n & = & \varphi(x,u)\ , & \\ y & = & x_1\ , & \end{array} \tag{3.10}$$

where the state $x = (x_1,\dots,x_n)^\top$ evolves in a compact subset X of $\mathbb{R}^n$, the input u is assumed to be known and evolves in some compact set $U \subset \mathbb{R}^m$ and the output $y \in \mathbb{R}$. In the following we define the values $\vartheta_i > 0$ as follows

$$\begin{array}{rcll} \vartheta_i & := & \max\limits_{x\in X} |x_i| & i=1,\dots,n, \\ \vartheta_{n+1} & := & \max\limits_{x\in X, u\in U} |\varphi(x,u)| & \end{array}$$

From the results of Theorem 3.1 we know that the variables $\hat{x}_i$ during the transient shows a peak of order ℓ^{i-1}, whereas the variables η_i shows a peak of order ℓ^i. As a consequence the proposed "peaking-free low-power high-gain observer" reads as

$$\begin{array}{rcl}
\dot{\hat{x}}_1 & = & \eta_1 + \ell\, c_{11}\,(y - \hat{x}_1) \\
\dot{\eta}_1 & = & \mathrm{sat}_3(\eta_2) + \ell^2\, c_{12}(y - \hat{x}_1) \\
& \vdots & \\
\dot{\hat{x}}_i & = & \eta_i + \ell\, c_{i1}\,(\mathrm{sat}_i(\eta_{i-1}) - \hat{x}_i) \\
\dot{\eta}_i & = & \mathrm{sat}_{i+2}(\eta_{i+1}) + \ell^2\, c_{i2}\,(\mathrm{sat}_i(\eta_{i-1}) - \hat{x}_i) \\
& \vdots & \\
\dot{\hat{x}}_{n-1} & = & \eta_{n-1} + \ell\, c_{(n-1)1}\,(\mathrm{sat}_{n-1}(\eta_{n-2}) - \hat{x}_{n-1}) \\
\dot{\eta}_{n-1} & = & \mathrm{sat}_{n+1}(\hat{\varphi}(\hat{x}, u)) + \ell^2\, c_{(n-1)2}(\mathrm{sat}_{n-1}(\eta_{n-2}) - \hat{x}_{n-1}) \\
\dot{\hat{x}}_n & = & \mathrm{sat}_{n+1}(\hat{\varphi}(\hat{x}, u)) + \ell\, c_n(\mathrm{sat}_n(\eta_{n-1}) - \hat{x}_n)
\end{array} \tag{3.11}$$

where $(\hat{x}, \eta) \in \mathbb{R}^{2n-1}$ is the state of the observer, with $\hat{x} = \mathrm{col}(\hat{x}_1, \ldots, \hat{x}_n) \in \mathbb{R}^n$ and $\eta = \mathrm{col}(\eta_1, \ldots, \eta_{n-1}) \in \mathbb{R}^{n-1}$. The coefficients (c_{i1}, c_{i2}), $i = 1, \ldots, n-1$ and c_n have to be properly chosen as detailed later, $\hat{\varphi}$ is any bounded Lipschitz function satisfying

$$|\varphi(x, u) - \hat{\varphi}(\hat{x}, u)| \;\leq\; L|x - \hat{x}| + R \qquad \forall\, (x, \hat{x}) \in X \times \mathbb{R}^n$$

for some $L > 0$ and $R > 0$, and the functions sat_i are defined as

$$\mathrm{sat}_i(\cdot) \;:=\; \mathrm{sat}_{r_i}(\cdot)\,, \qquad i = 1, \ldots, n+1\,,$$

with the saturations level r_i chosen as

$$r_i \;:=\; \vartheta_i + \varrho\,,$$

with ϱ a small positive real number to be defined. It can be proved that the observer (3.11) has the same asymptotic properties of the observer (3.2) and moreover that the variables $\hat{x}_i(t)$ are bounded for all $t \geq 0$ by a number which is independent on the high-gain parameter ℓ, whereas the variables $\eta_i(t)$ shows a peak of order ℓ.

With respect to the choice required in Theorem 3.1 in which the parameters (c_{i1}, c_{i2}), $i = 1, \ldots, n-1$ were just asked to make the matrix M defined in (A.1) Hurwitz (see Lemma A.1), in this case the presence of saturation functions asks the parameters also fulfil additional "small-gain" conditions. We refer to Appendix A for the technicalities and in particular to Definition A.1.

Theorem 3.3. *Consider system* (3.10) *and the observer* (3.11). *There exists a choice of* $(c_{i1}\ c_{i2})$, $i = 1, \ldots, n-1$ *and* c_n *such that, for any compact sets* $X \subset \mathbb{R}^n$, $\widehat{X} \subset \mathbb{R}^n$ *and* $E \subset \mathbb{R}^{n-1}$, *and for any* $\varrho > 0$, *there exist* $\mu_1 > 0$, $\mu_2 > 0$, $\mu_3 > 0$, $\delta_i > 0$, $i = 1, \ldots, n$, γ_i, $i = 1, \ldots, n-1$ *and* $\ell^\star \geq 1$ *such that, for any* $\ell > \ell^\star$ *and for any* $(x(0), \hat{x}(0), \eta(0)) \in X \times \widehat{X} \times E$, *the following bounds hold*

$$\begin{aligned} |\hat{x}_i(t) - x_i(t)| &\leq \min\left\{\ell^{i-1}\mu_1 \exp(-\ell\,\mu_2 t)\,\pi + \frac{\mu_3}{\ell^{n-(i-1)}}R\ ,\ \delta_i\right\} \\ |\eta_i(t) - x_{i+1}(t)| &\leq \min\left\{\ell^{i}\mu_1 \exp(-\ell\,\mu_2 t)\,\pi + \frac{\mu_3}{\ell^{n-(i-1)}}R\ ,\ \gamma_i\,\ell\right\} \end{aligned} \tag{3.12}$$

with

$$\pi = \sum_{i=1}^{n} |\hat{x}_i(0) - x_i(0)| + \sum_{i=1}^{n-1} |\eta_i(0) - x_{i+1}(0)|\ ,$$

for all $t \geq 0$ *such that* $x(t) \in X$, $u(t) \in U$.

Proof. The proof is divided in two parts. First we prove the asymptotic convergence of the observer by proving that after some time the saturations are no more "working". In this proof we get a conservative bound. Then, we make a more detailed analysis to show that this bound can be refined and that, indeed, the estimates $\hat{x}_i$ are "peaking-free" whereas the variables η_i peak with an order of magnitude which grows with ℓ.

In the following (A, B, C) denotes a triplet of dimension 2 in *prime form*, and $D_2(\ell) = \mathrm{diag}(\ell, \ell^2)$. Also, we use the same notation introduced in Section 3.1 for the matrices B_i, K_i, E_i, $\bar{Q}_i$ and M_i. We define the following changes of variables

$$(\hat{x}, \eta) \mapsto \xi\ , \quad \begin{cases} \xi_{i1} := \hat{x}_i\ , & i = 1, \ldots, n-1\ , \\ \xi_{i2} := \eta_i\ , & i = 1, \ldots, n-1\ , \\ \xi_n := \hat{x}_n\ , & \end{cases} \tag{3.13}$$

$$\xi \mapsto \varepsilon\ , \quad \begin{cases} \varepsilon_{i1} := \ell^{-(i-1)}(\xi_{i1} - x_i)\ , & i = 1, \ldots, n-1\ , \\ \varepsilon_{i2} := \ell^{-i}(\xi_{i2} - x_{i+1})\ , & i = 1, \ldots, n-1\ , \\ \varepsilon_n := \ell^{-(n-1)}(\xi_n - x_n)\ , & \end{cases} \tag{3.14}$$

$$\xi \mapsto \zeta\ , \quad \begin{cases} \zeta_{i1} := (\xi_{i1} - x_i)\ , & i = 1, \ldots, n-1\ , \\ \zeta_{i2} := \ell^{-1}(\xi_{i2} - x_{i+1})\ , & i = 1, \ldots, n-1\ , \\ \zeta_n := \xi_n - x_n\ , & \end{cases} \tag{3.15}$$

with

$$\xi = (\xi_1, \ldots, \xi_{n-1}, \xi_n) \in \mathbb{R}^{2n-1}, \quad \xi_i = (\xi_{i1}, \xi_{i2}) \in \mathbb{R}^2, \quad i = 1, \ldots, n-1, \quad \xi_n \in \mathbb{R}$$
$$\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{n-1}, \varepsilon_n) \in \mathbb{R}^{2n-1}, \quad \varepsilon_i = (\varepsilon_{i1}, \varepsilon_{i2}) \in \mathbb{R}^2, \quad i = 1, \ldots, n-1, \quad \varepsilon_n \in \mathbb{R},$$
$$\zeta = (\zeta_1, \ldots, \zeta_{n-1}, \zeta_n) \in \mathbb{R}^{2n-1}, \quad \zeta_i = (\zeta_{i1}, \zeta_{i2}) \in \mathbb{R}^2, \quad i = 1, \ldots, n-1, \quad \zeta_n \in \mathbb{R},$$

Finally, we define the variables v_i, $i = 1, \ldots, n-1$ as

$$\begin{aligned} v_i &:= \mathsf{sat}_{i+2}(\eta_{i+1}) - x_{i+2} \qquad \forall\, i = 1, \ldots, n-2 \\ v_{n-1} &:= \mathsf{sat}_{n+1}(\hat{\varphi}(\hat{x}, u)) - \varphi(x, u)\,. \end{aligned}$$

Recall that by definition

$$|v_i(t)| \;\le\; \vartheta_{i+2} + R_{i+2} \qquad \forall\, i = 1, \ldots, n-1\,, \qquad \forall\, t \ge 0\,. \tag{3.16}$$

Exit from saturation and convergence

Case i = 1. Consider the change of coordinates (3.13) and its first variable $\xi_1 \in \mathbb{R}^2$. Its dynamics is given by

$$\dot{\xi}_1 \;=\; A\xi_1 + B\,\mathsf{sat}_3(B\xi_2) + D_2(\ell)K_1(y - C\xi_1)\,.$$

By using the change of coordinates (3.14) the latter is transformed into

$$\dot{\varepsilon}_1 = \ell E_1 \varepsilon_1 + \ell^{-1} B v_1$$

Since E_1 is Hurwitz, there exists a $P_1 = P_1^\top > 0$ solution of

$$P_1 E_1 + E_1^\top P_1 = -I\,.$$

As a consequence we can apply Lemma A.4 for any $T_1 > 0$ with $r = 1$ to get a $\underline{\ell}_1$ satisfying

$$|\ell\, \varepsilon_1(t)| \;\le\; \varrho \qquad \forall\, t \ge T_1\,,$$

for all $\ell \ge \underline{\ell}_1$. By noting that (for any $\ell \ge 1$)

$$\begin{aligned} \ell^{-1}|\xi_{12} - x_2| \;&\le\; |\varepsilon_{12}| \;\le\; |\varepsilon_1| \\ |\xi_{12}| \;&\le\; |\xi_{12} - x_2| + |x_2| \end{aligned}$$

we get

$$|\xi_{12}(t)| \;\le\; \ell|\varepsilon_1| + |x_2| \;\le\; \varrho + R_2 \;\le\; \vartheta_2$$

for any $\ell \geq \underline{\ell}_1$ and for all $t \geq T_1$. With this in mind, we have that $\text{sat}_2\eta_1(t) = \eta_1(t)$ for all $t \geq T_1$.

General Case i > 1. After time $T_{i-1} > 0$ we get $\text{sat}_{j+1}\eta_j(t) = \eta_j(t)$ for $j = 1, \ldots, i-1$ and, as a consequence, $Bv_j = N\xi_{j+1}$ for all $j = 1, \ldots, i-2$. With the change of coordinates (3.13) in mind, consider the cascade system made by the first i elements of ξ_i,

$$\begin{aligned}
\dot{\xi}_1 &= A\,\xi_1 + N\,\xi_2 + D_2(\ell)K_1(y - C\xi_1)\\
&\vdots\\
\dot{\xi}_{i-1} &= A\,\xi_{i-1} + B\,\text{sat}_{i+1}\eta_i + D_2(\ell)\,K_{i-1}(B^\top\xi_{i-2} - C\xi_{i-1})\\
\dot{\xi}_i &= A\,\xi_i + B\,\text{sat}_{i+2}\eta_{i+1} + D_2(\ell)\,K_{i-1}(B^\top\xi_{i-1} - C\xi_i)
\end{aligned}$$

By using the change of coordinates (3.14) the latter is transformed into

$$\begin{pmatrix}\dot{\boldsymbol{\varepsilon}}_{i-1}\\ \dot{\varepsilon}_i\end{pmatrix} = \ell\begin{pmatrix}M_{i-1} & 0\\ \bar{Q}_i & E_i\end{pmatrix}\begin{pmatrix}\boldsymbol{\varepsilon}_{i-1}\\ \varepsilon_i\end{pmatrix} + \frac{1}{\ell^i}\begin{pmatrix}B_{2(i-1)} & 0\\ 0 & B\end{pmatrix}\begin{pmatrix}\ell\,v_{i-1}(t)\\ v_i(t)\end{pmatrix}$$

with the notation $\boldsymbol{\varepsilon}_k = (\varepsilon_1, \ldots, \varepsilon_k)^\top$. Note that

$$\begin{aligned}
v_{i-1}(t) &= \text{sat}_{i+1}(B\eta_i) - x_{i+1}\\
&= \text{sat}_{i+1}(\ell^i\varepsilon_{i2} + x_{i+1}) - x_{i+1}
\end{aligned}$$

By the Lipschitz mean-value theorem, for each t sufficiently large, there exists a continuous function $\rho(t) \in [0, 1]$ such that

$$v_{i-1}(t) \;=\; \rho(t)\,\ell^i\varepsilon_{i2}\,.$$

As a consequence we get

$$\dot{\boldsymbol{\varepsilon}}_i \;=\; \ell\Lambda_i(t)\boldsymbol{\varepsilon}_i + \frac{1}{\ell^i}B_{2i}v_i(t)$$

where $\Lambda_i(t)$ is defined as

$$\Lambda_i(t) := \begin{pmatrix}M_{i-1} & \rho(t)\bar{N}_{i-1}\\ \bar{Q}_i & E_i\end{pmatrix}\;. \tag{3.17}$$

By applying Lemma A.3 we know the existence of a $P_i = P_i^\top > 0$ satisfying

$$P_i\Lambda_i(t) + \Lambda_i(t)^\top P_i \;\le\; -I$$

for all $t \ge T_{i-1}$. As a consequence we can apply Lemma A.4 with some $T_i > 0$ and $r = i$ to get a $\underline{\ell}_i \ge \underline{\ell}_{i-1}$ satisfying

$$|\ell^i \boldsymbol{\varepsilon}_i(t)| \;\le\; \varrho \qquad \forall\, t \ge T_1\,,$$

By recalling (for any $\ell \ge 1$)

$$\ell^{-i}|\xi_{i2} - x_{i+1}| \;\le\; |\varepsilon_{i2}| \;\le\; |\boldsymbol{\varepsilon}_i|$$

$$|\xi_{i2}| \;\le\; |\xi_{i2} - x_{i+1}| + |x_{i+1}| \;\le\; |\ell^i\varepsilon_{i2}| + |x_{i+1}| \;\le\; |\ell^i\boldsymbol{\varepsilon}_i| + |x_{i+1}|$$

we get

$$|\eta_i(t)| \;=\; |\xi_{i2}(t)| \;\le\; \ell^i|\boldsymbol{\varepsilon}_i| + |x_{i+1}| \;\le\; \varrho + R_{i+1} \;\le\; \vartheta_{i+1}$$

for any $\ell \ge \underline{\ell}_i$ and for all $t \ge T_i$. This prove that for $t \ge T_i$ also $\text{sat}_{i+1}\eta_i = \eta_i$ and therefore the dynamics of system $\boldsymbol{\varepsilon}_i$ read as

$$\dot{\boldsymbol{\varepsilon}}_i \;=\; \ell\, M_i\boldsymbol{\varepsilon}_i + \frac{1}{\ell^i}B_i v_i(t)$$

Overall convergence. After time $T_{n-1} > 0$ we get $\text{sat}_{j+1}\eta_j(t) = \eta_j(t)$ for $j = 1, \ldots, n-1$. As a consequence the observer (3.11) in the ε coordinates (3.14) reads as

$$\begin{pmatrix} \dot{\boldsymbol{\varepsilon}}_{n-1} \\ \dot{\varepsilon}_n \end{pmatrix} \;=\; \ell \begin{pmatrix} M_{n-1} & 0 \\ \bar{q}_n & -c_n \end{pmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}_{n-1} \\ \varepsilon_n \end{pmatrix} + \frac{1}{\ell^{n-1}} \begin{pmatrix} B_{2n-2} \\ 1 \end{pmatrix} v_{n-1}(t)$$

where $\bar{q}_n = (0, \ldots, 0, c_n)$. Now let the matrix $\overline{M}_n$ be defined as

$$\overline{M}_n \;:=\; \begin{pmatrix} M_{n-1} & 0 \\ \bar{q}_n & -c_n \end{pmatrix} \qquad \bar{B} = \begin{pmatrix} B_{2n-2} \\ 1 \end{pmatrix}.$$

It is Hurwitz for any $c_n > 0$. Therefore we can define the matrix $\overline{P}_n = \overline{P}_n^\top > 0$ solution of

$$\overline{P}_n\,\overline{M}_n + \overline{M}_n^\top\overline{P}_n \;=\; -I\,.$$

Note that

$$|v_{n-1}| \;=\; |\text{sat}_{n+1}\hat{\varphi}(\hat{x}, u) - \varphi(x, u)| \;\le\; L|\hat{x} - x| + R \;\le\; L\,\ell^{n-1}\,|\varepsilon| + R\,.$$

By applying standard arguments used in the proofs of Theorems 1.4 and 3.1 it is easy to get

$$|\varepsilon(t)| \quad \leq \quad \mu_1 \, \exp(-\ell \, \mu_2 t) \; |\varepsilon(0)| + \frac{\mu_3}{\ell^n} \, R$$

for some positive μ_1, μ_2 and μ_3. Finally, by using (for $\ell \geq 1$)

$$\begin{array}{rcccl} \ell^{-(i-1)} \, |x_i - \hat{x}_i| & \leq & |\varepsilon| & \leq & \sum_{i=1}^{n} |x_i - \hat{x}_i| + \sum_{i=1}^{n-1} |x_{i+1} - \eta_i| \\ \ell^{-i} \, |x_{i+1} - \eta_i| & \leq & |\varepsilon| & \leq & \sum_{i=1}^{n} |x_i - \hat{x}_i| + \sum_{i=1}^{n-1} |x_{i+1} - \eta_i| \end{array}$$

we get

$$|x_i(t) - \hat{x}_i(t)| \quad \leq \quad \ell^{i-1} \mu_1 \, \exp(-\ell \, \mu_2 t) \, \pi + \frac{\mu_3}{\ell^{n-(i-1)}} R \, , \tag{3.18}$$

$$|x_{i+1}(t) - \eta_i(t)| \quad \leq \quad \ell^{i} \mu_1 \, \exp(-\ell \, \mu_2 t) \, \pi + \frac{\mu_3}{\ell^{n-(i-1)}} R \tag{3.19}$$

with π defined as in the statement of the theorem. In the next part of the proof we show that the latter are too conservative and we show how to get (3.12).

Bound Estimate

Let consider again the ξ dynamics at time 0. Using the change of coordinates ζ_i defined in (3.15), we get

$$\begin{array}{rcll} \dot{\zeta}_1 & = & \ell E_1 \zeta_1 + \ell^{-1} B v_1(t) & \\ \dot{\zeta}_i & = & \ell E_i \zeta_i + \ell^{-1} B v_i(t) + \ell K_{i-1} \mathrm{sat}_i \eta_{i-1} & \qquad i = 2, \ldots, n-1 \\ \dot{\zeta}_n & = & -\ell c_n \zeta_n + v_{n-1}(t) + \ell c_n \, \mathrm{sat}_n \eta_{n-1} & \end{array}$$

Now let consider the Lyapunov functions V_i defined as

$$V_i \; := \; \zeta_i^\top \, S_i \, \zeta_i \; , \qquad i = 1, \ldots, n-1 \; , \qquad V_n \; := \; \frac{1}{2c_n} \, \zeta_n^2$$

with S_i given by

$$S_i E_i + E_i^\top S_i = -I \, .$$

We get

$$\dot{V}_1 \quad \leq \quad -\ell |\zeta_1|^2 + 2|\zeta_1||S_i|\ell^{-1} v_1$$

$$\begin{array}{rcl}\dot{V}_i & \leq & -\ell|\zeta_i|^2 + 2|\zeta_i||S_i|(\ell^{-1}v_i + \ell K_{i-1}\,\mathrm{sat}_i\eta_{i-1}) \qquad i=2,\ldots,n-1,\\ \dot{V}_n & \leq & -\ell|\zeta_n|^2 + c_n^{-1}\zeta_n\,v_{n-1} + \ell\;\zeta_n\,\mathrm{sat}_n\eta_{n-1}\end{array}$$

By using (3.16), standard Lyapunov arguments show that there exist positive constants a_{ij}, $j = 1,\ldots,4$, such that, if ℓ is taken sufficiently large, we get

$$\begin{array}{rcl}|\zeta_1(t)| & \leq & a_{11}\exp(-a_{12}\ell\,t)|\zeta_1(0)| + \dfrac{a_{13}}{\ell^2}(R_3+\vartheta_3)\\ |\zeta_i(t)| & \leq & a_{i1}\exp(-a_{i2}\ell\,t)|\zeta_i(0)| + \dfrac{a_{i3}}{\ell^2}(R_{i+2}+\vartheta_{i+2}) + a_{i4}|K_{i-1}|\vartheta_i\\ & & \qquad\qquad i=2,\ldots,n-1\,,\\ |\zeta_n(t)| & \leq & a_{n1}\exp(-a_{n2}\ell\,t)|\zeta_i(0)| + \dfrac{a_{n3}}{\ell}(R_{n+1}+\vartheta_{n+1}) + a_{i4}c_n\vartheta_n\end{array}$$

Furthermore, by recalling (for any $\ell \geq 1$)

$$\begin{array}{rcccl}|x_i-\hat{x}_i| & \leq & |\zeta_i| & \leq & |x_i-\hat{x}_i| + |x_{i+1}-\eta_i|\,,\\ \dfrac{1}{\ell}|x_{i+1}-\eta_i| & \leq & |\zeta_i| & \leq & |x_i-\hat{x}_i| + |x_{i+1}-\eta_i|\,,\end{array}$$

and by using the notation

$$\begin{array}{rcl}\pi_i & = & \displaystyle\max_{x\in X,\,\hat{x}\in\widehat{X},\,\eta\in E}|x_i-\hat{x}_i| + |x_{i+1}-\eta_i|\,, \qquad i=1,\ldots,n-1\\ \pi_n & = & \displaystyle\max_{x\in X,\,\hat{x}\in\widehat{X}}|x_n-\hat{x}_n|\,,\end{array}$$

we get the following set of bounds for $\hat{x}_i$

$$\begin{array}{rcl}|x_1(t)-\hat{x}_1(t)| & \leq & a_{11}\exp(-a_{12}\ell t)\,\pi_1 + \dfrac{a_{13}}{\ell^2}(R_3+\vartheta_3)\,,\\ |x_i(t)-\hat{x}_i(t)| & \leq & a_{i1}\exp(-a_{i2}\ell t)\,\pi_i + \dfrac{a_{i3}}{\ell^2}(R_{i+2}+\vartheta_{i+2}) + a_{i4}|K_{i-1}|\vartheta_i\,,\\ & & \qquad\qquad i=2,\ldots,n-1\\ |x_n(t)-\hat{x}_n(t)| & \leq & a_{n1}\exp(-a_{n2}\ell t)\,\pi_n + \dfrac{a_{n3}}{\ell}(R_{n+1}+\vartheta_{n+1}) + a_{n4}c_n\vartheta_n\,,\end{array}$$

and for η_i,

$$|\eta_i(t)| \leq a_{i1}\ell\exp(-a_{i2}\ell t)\,\pi_i + \frac{a_{i3}}{\ell}(R_{i+2}+\vartheta_{i+2}) + a_{i4}\ell|K_{i-1}|\vartheta_i\,,$$

for $i = 1,\ldots,n-1$. Now let $\delta_i > 0$, $i = 1,\ldots,n$ be defined as

$$\delta_1 \;=\; a_{11}\,\pi_1 + a_{13}(R_3+\vartheta_3)\,,$$

$$\begin{aligned} \delta_i &= a_{i1}\,\pi_i + a_{i3}(R_{i+2} + \vartheta_{i+2}) + a_{i4}|K_{i-1}|\vartheta_i\,, \qquad i = 2, \ldots, n-1 \\ \delta_n &= a_{n1}\,\pi_n + a_{n3}(R_{n+1} + \vartheta_{n+1}) + a_{n4}c_n\vartheta_n\,. \end{aligned}$$

As a consequence, for any $\ell \geq 1$ we get

$$|\hat{x}_i(t) - x_i(t)| \;\leq\; \delta_i \qquad \forall\, t \geq 0\,. \tag{3.20}$$

Furthermore, by letting $\gamma_i > 0$, $i = 1, \ldots, n-1$, be defined as

$$\gamma_i \;=\; a_{i1}\,\pi_i + a_{i3}(R_{i+2} + \vartheta_{i+2}) + a_{i4}|K_{i-1}|\vartheta_i\,,$$

we also get

$$|\eta_i(t) - x_{i+1}(t)| \;\leq\; \gamma_i\,\ell \qquad \forall\, t \geq 0 \tag{3.21}$$

for any $\ell \geq 1$. The proof concludes by combining (3.20)-(3.21) with (3.18)-(3.19). □

3.4 Asymptotic behaviour in presence of measurement noise

Aim of this section is to characterize the behaviour in presence of measurement noise of the low-power high-gain observer presented in this chapter. By following the same arguments used in Section 2.4, we consider in this section the class of autonomous systems described by

$$\begin{array}{rcll} \dot{x} & = & A_n x + B_n \varphi(x)\,, & x \in \mathbb{R}^n\,, \\ y & = & C_n x + \nu(t)\,, & y \in \mathbb{R} \end{array} \tag{3.22}$$

where (A_n, B_n, C_n) is a tripled in *prime form*, φ is a locally Lipschitz function, x is assumed to evolve in some given compact set $X \subset \mathbb{R}^n$ and the output is affected by a bounded noise $\nu(t)$. The low-power high-gain observer (3.2) for the system (3.22) has the form (by using the same compact notation of (3.6))

$$\begin{array}{rcll} \dot{\xi}_1 & = & A\,\xi_1 + N\,\xi_2 + D_2(\ell)\,K_i\,(y - C\xi_1) & \\ \dot{\xi}_i & = & A\,\xi_i + N\,\xi_{i+1} + D_2(\ell)\,K_i\,(B^\top \xi_{i-1} - C\xi_i) & i = 1, \ldots, n-2\,, \\ \dot{\xi}_{n-1} & = & A\,\xi_{n-1} + B\,\varphi_s(\hat{x}) + D_2(\ell)\,K_{(n-1)}\,(B^\top \xi_{n-2} - C\xi_{n-1}) & \end{array} \tag{3.23}$$

where (A, B, C) is a triplet in *prime form* of dimension 2, φ_s is a bounded locally Lipschitz function that agrees with φ on the set X, the matrix $D_\ell = \text{diag}(\ell, \ldots, \ell^n)$ with ℓ the high-gain parameter to be chose large enough (to "overcome" the Lipschitz constant of φ), $K_i = \text{col} c_{i1}, c_{i2}$ are coefficients which can be chosen according to Lemma A.1 and

$$\hat{x} \;:=\; \Gamma\xi, \qquad \Gamma \;:=\; \text{blkdiag}\,(\underbrace{C_2,\;\ldots,\;C_2}_{(n-2)\text{ times}}, I_2)\,.$$

By making the following change of coordinates

$$\xi_i \;\mapsto\; \tilde{\xi}_i \;:=\; \xi_i - \text{col}\,(x_i,\; x_{i+1}) \qquad i = 1, \ldots, n-1\,,$$

the observer (3.23) transforms as

$$\begin{array}{rcl} \dot{\tilde{\xi}}_1 & = & H_1\,\tilde{\xi}_1 + N\,\tilde{\xi}_2 + G_1\nu(t) \\ & \vdots & \\ \dot{\tilde{\xi}}_i & = & Q_i\,\tilde{\xi}_{i-1} + H_i\,\tilde{\xi}_i + N\tilde{\xi}_{i+1} \\ & \vdots & \\ \dot{\tilde{\xi}}_{n-1} & = & Q_{n-1}, \tilde{\xi}_{n-2} + H_{n-1}\,\tilde{\xi}_{n-1} + B\,\Delta_\varphi(\tilde{\xi}, x) \end{array}$$

which can be written in the more compact form

$$\dot{\tilde{\xi}} \;=\; F\tilde{\xi} + B\Delta_{\varphi}(\tilde{\xi}, x) + G\nu(t) \tag{3.24}$$

where

$$F = \begin{pmatrix} H_1 & N & & & \\ Q_2 & H_2 & N & & \\ & \ddots & \ddots & \ddots & \\ & & Q_{n-2} & H_{n-2} & N \\ & & & Q_{n-1} & H_{n-1} \end{pmatrix} \qquad G = \begin{pmatrix} G_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ B \end{pmatrix}.$$

$$\begin{array}{rcll} H_i & := & A - D_2(\ell)\, K_i\, C & \forall\, i = 1, \ldots, n-1\;, \\ Q_i & := & D_2(\ell)\, K_i\, B^T & \forall\, i = 2, \ldots, n-1\;, \\ G_1 & := & D_2(\ell) K_1 P\;, & \\ N & := & BB^\top & \end{array}$$

and

$$\Delta_{\varphi}(\tilde{\eta}, x) \;:=\; \varphi_s(\Gamma'\tilde{\eta} + x) - \varphi(x)\,. \tag{3.25}$$

If ℓ is taken sufficiently large then system (3.24) is input-to-state stable with respect to the input ν. In particular, standard Lyapunov arguments can be used to prove the existence of positive constants μ_1, μ_2 and μ_3 such that the following bounds hold

$$|e_j(t)| \;\leq\; \max\left\{\mu_1\, \ell^{j-1}\, \exp(-\mu_2 \ell\, t)\, |\tilde{\xi}(0)|\,,\; \mu_3\, \ell^{j-1}|\; \|\nu(\cdot)\|_{\infty}\right\} \tag{3.26}$$

where $e \in \mathbb{R}^n$, $e = \Gamma\tilde{\xi}$, for all $t \geq 0$, $j = 1, \ldots, n$. The goal of this note is to better characterise the asymptotic gain in presence of *high-frequency* noise as already done for the standard high-gain observer in Section 2.4. Towards this end we model the measurement noise as

$$\begin{array}{rcll} \varepsilon\dot{w} & = & Sw\,, & w \in \mathbb{R}^m \\ \nu & = & Pw\,, & \end{array} \tag{3.27}$$

where $S \in \mathbb{R}^{m\times m}$ is a neutrally stable matrix, P is a row vector, and $\varepsilon \in (0,1)$ is parameter that will be taken small in the forthcoming analysis. System (3.27) can be conveniently seen as generator of $m > 0$ harmonics at frequencies $\omega_i/\varepsilon > 0$, $i = 1, \ldots, m$, namely, the matrices S and P take the form

$$S = \text{blkdiag}(S_1, \ldots, S_m)\,, \qquad S_i = \begin{pmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{pmatrix}$$

and $P = ((0\ 1)\ (0\ 1)\ \cdots\ (0\ 1))$. In the following we assume that w ranges in a compact invariant set W.

As a preparatory step towards the nonlinear analysis, it is instructive to consider the linear case, namely the case in which $\varphi(x) = \Phi x$ with Φ a row vector. In this case the observer (3.23) can be taken[1] with $\varphi_s(\hat{x}) = \Phi\hat{x}$. By observing that

$$\Phi\Gamma\eta - \Phi x = \Phi\Gamma\tilde{\xi}$$

the resulting error system (3.23) -(3.27) is thus given by

$$\begin{array}{rcl} \varepsilon\dot{w} & = & Sw \\ \dot{\tilde{\xi}} & = & (F + B\Phi\Gamma)\tilde{\xi} + GPw \end{array}$$

with the matrix $F + B\Phi\Gamma$ that is Hurwitz for ℓ sufficiently large. Using the fact that S is neutrally stable and that $F + B\Phi\Gamma$ is Hurwitz it follows that the state of the previous system reaches a steady state fully described by the state of the noise generator. In particular, denoting by $\bar{\Pi}_\varepsilon$ the $(2n-2) \times m$ matrix solution of the Sylvester equation

$$\bar{\Pi}_\varepsilon S \;=\; \varepsilon(F + B\Phi\Gamma')\bar{\Pi}_\varepsilon + \varepsilon GP$$

it turns out that

$$\lim_{t\to\infty}(\tilde{\xi}(t) - \bar{\Pi}_\varepsilon w(t)) \;=\; 0\,. \qquad (3.28)$$

From this it follows that

$$\lim_{t\to\infty}(e(t) - \Pi_\varepsilon w(t)) \;=\; 0$$

with Π_ε the $n\times m$ matrix defined as $\Pi_\varepsilon := \Gamma\bar{\Pi}_\varepsilon$ The expression of Π_ε can be characterised at high-frequency (namely for small value of ε) to have more insight about how the gain between the measurement noise and the j-th estimation error is affected by ℓ. In particular, using that the fact that S is not singular, it is easy to check that

$$\Pi_\varepsilon \;=\; \sum_{k=1}^{\infty}\varepsilon^k\Pi^k$$

with

$$\Pi^k \;:=\; \Gamma(F + B\Phi\Gamma)^{k-1}GPS^{-k}\,, \qquad k = 1,\ldots,\infty\,.$$

In particular, the series defining Π_ε is convergent as long as ε is taken sufficiently small. Namely, for all $k^\star > 0$ there exist $\varepsilon^\star > 0$ and $\bar{\pi} > 0$ such that $\sum_{k=k^\star}^{\infty}\varepsilon^k\Pi^k \le \bar{\pi}$ for all positive $\varepsilon \le \varepsilon^\star$. By bearing in mind the expressions of F and G and how those

[1]Because of linearity boundedness of the function $\varphi_s(x)$ is not needed.

matrices depend on ℓ, easy computations reveal that the expressions of the Π^k, for $k = 1,\ldots,\lceil\frac{n+1}{2}\rceil$ have the following structure

$$\Pi^1 = \begin{pmatrix} \ell M_{1,1} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \Pi^2 = \begin{pmatrix} M_{2,1}(\ell) \\ \ell^3 M_{2,2} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \ell M_{2,n} \end{pmatrix}, \ \Pi^3 = \begin{pmatrix} M_{3,1}(\ell) \\ M_{3,2}(\ell) \\ \ell^5 M_{3,3} \\ 0 \\ \vdots \\ 0 \\ \ell M_{3,n-1} \\ M_{3,n}(\ell) \end{pmatrix}, \ \Pi^4 = \begin{pmatrix} M_{4,1}(\ell) \\ M_{4,2}(\ell) \\ M_{4,3}(\ell) \\ \ell^7 M_{4,4} \\ \vdots \\ \ell M_{4,n-2} \\ M_{4,n-1}(\ell) \\ M_{4,n}(\ell) \end{pmatrix}, \ldots$$

in which the $M_{j,i}$, are appropriately defined row vectors that are constant or dependent on ℓ as clear from the notation. That is, the j-th element, $j = 1,\ldots,n$, of Π^k, $k = 1,\ldots,\lceil\frac{n+1}{2}\rceil$, which is denoted by Π^k_j, is given by

$$\begin{aligned} \Pi^k_j &= 0 && j = k+1,\ldots n-k+1 \\ \Pi^k_k &= \ell^{2k-1} M_{k,k} && k \leq \lceil\tfrac{n+1}{2}\rceil \\ \Pi^k_j &= \ell M_{k,j} && k = 2,\ldots,\lceil\tfrac{n+1}{2}\rceil - 1,\ j = n-k+2 \end{aligned}$$

in which the specified $M_{k,j}$ are constant row vectors, and the elements of Π^k not specified in the previous formulae that are generic (ℓ dependent) row vectors. From this, using (3.28) and the fact that W is compact, we can then conclude that for all $j = 1,\ldots,n$, there exists a $\varepsilon^\star(\ell) > 0$ such that for all $\varepsilon \leq \varepsilon^\star(\ell)$ the following holds

$$\begin{aligned} \limsup_{t\to\infty} |e_j(t)| &\leq \mu\,\varepsilon^j\,\ell^{2j-1}\,\|w(\cdot)\|_\infty\,, && j = 1,\ldots,\lceil\tfrac{n+1}{2}\rceil \\ \limsup_{t\to\infty} |e_j(t)| &\leq \mu\,\varepsilon^{n-j+2}\,\ell\,\|w(\cdot)\|_\infty\,, && j = \lceil\tfrac{n+1}{2}\rceil + 1,\ldots,n \end{aligned}$$

for some positive constant μ. The previous relation clearly shows the “low-pass" filtering properties of the low-power high-gain observer, namely

$$\lim_{\varepsilon\to 0}\limsup_{t\to\infty} |e_j(t)| = 0\,.$$

In addition, the remarkable feature of the low-power observer is to have an asymptotic gain that is affected by ε powered at a value that increases as long as "higher" components of the errors are considered. This is consequence of the fact that the relative degree between the measurement noise and the j-th error component increases with j. This fact shows that the actual observer behaves better than standard high-gain observers as far as sensitivity to measurement noise is concerned. As a matter of fact, the results presented in Section 2.4 have shown that, in case of standard high-gain observers, the asymptotic gain between the measurement noise and the j-th error component depends on ε regardless the value of j. This is immediate consequence of the fact that the relative degree between the measurement noise and the j-th error component in standard high-gain observers is always one regardless the value of j.

Furthermore, the previous analysis reveals how the high-gain parameter ℓ affects the asymptotic gain of the j-th error component at high-frequency. In particular, it is worth noting that for the first $\lceil \frac{n+1}{2} \rceil$ error components the gain increases with ℓ according to a term of the form ℓ^{2j-1}, namely the effect of the high-gain parameter on the sensitivity to measurement noise becames worst as long as "higher" components of the error (namely higher values of j) are considered. On the other hand, the asymptotic gain for the "lower" component, namely for the components e_j for $j = \lceil \frac{n+1}{2} \rceil + 1, \ldots, n$, depends on ℓ no matter which j is considered. In this respect it is interesting to observe that, in case of standard high-gain observer, the analysis in Section 2.4 revealed that the asymptotic gain between the measurement noise and the j-th error component is affected by the high-gain parameter by a term of the form ℓ^j. The new low-power observer, thus, behaves better for the sensitivity of the error e_j for $j = \lceil \frac{n+1}{2} \rceil + 1, \ldots, n$, while it is worst for the first $\lceil \frac{n+1}{2} \rceil + 1$ components.

In the next section we present the theoretical tool that allows one to get the same kind of result also in the nonlinear setting.

Nonlinear Analysis

By compactly writing the system dynamics (3.22) as

$$\dot{x} = f(x)$$

the overall dynamics given by the observed system (3.22) , the observer error dynamics (3.23) and the noise generator (3.27) read as

$$\begin{array}{rcl} \varepsilon \dot{w} &=& Sw \\ \dot{x} &=& f(x) \\ \dot{\tilde{\xi}} &=& F\tilde{\xi} + B\Delta_\varphi(\tilde{\xi}, x) + GPw\,. \end{array} \tag{3.29}$$

Having tuned the parameters c_{i1}, c_{i2}, $i = 1, \ldots, n-1$, and ℓ as said before, the trajectories of this system are bounded. The system in question, thus, has a well-defined steady state that can be characterised with the tools proposed in Isidori and Byrnes (2008). More specifically, the triangular structure of the system (with the x and w subsystem driving the e subsystem) implies that the existence of a possibly set-valued function $\pi_\varepsilon : X \times W \rightrightarrows \mathbb{R}^{2n-2}$ such that the set

$$\text{graph}(\pi_\varepsilon) = \big\{(w, x, e) \in W \times X \times \mathbb{R}^n \ : \ \tilde{\xi} \in \pi_\varepsilon(w, x)\big\}$$

is asymptotically stable for (3.29). Furthermore, the properties of the high-gain observer when the measurement noise is absent (*i.e.* when $w = 0$) show that

$$\pi_\varepsilon(0, x) = \{0\} \qquad \forall\, x \in X\,.$$

The following technical lemma provides an arbitrarily accurate approximation of a continuous selection of $\pi_\varepsilon(\cdot,\cdot)$. The Lemma refers to a number of functions that enter in definition of the approximation. In order to keep compact the claim of the Lemma, we introduce those functions beforehand. In particular, let

$$\upsilon := \left\lceil \frac{n}{2} \right\rceil, \qquad m := \left\lceil \frac{n+1}{2} \right\rceil$$

and let r be an arbitrary number satisfying $r \geq m$. Observe that for any n we have $m \geq \upsilon$. The approximation, of order r, of the steady state is then a function $\Psi_\varepsilon : W \times X \to \mathbb{R}^{2n-2}$ defined as

$$\begin{aligned}\Psi_\varepsilon(w, x) \ \ := \ \ \text{col}\Big(\ & \Psi_1(w, x) \quad \Lambda_1(w, x) \quad \Psi_2(w, x) \quad \Lambda_2(w, x) \\ & \qquad\qquad \cdots \quad \Psi_{n-1}(w, x) \quad \Lambda_{n-1}(w, x) \ \Big)\end{aligned}$$

in which

$$\begin{aligned}\Psi_j(w, x) \ &:= \ \sum_{i=r_j}^{r} \psi_{j,i}(w, x)\,\varepsilon^i\,, \qquad j = 1, \ldots, n-1 \\ \Lambda_j(w, x) \ &:= \ \sum_{i=p_j}^{r} \lambda_{j,i}(w, x)\,\varepsilon^i\,, \qquad j = 1, \ldots, n-1\end{aligned} \tag{3.30}$$

where the r_j and p_j are defined as

$$\begin{aligned} r_j \ &= \ j\,, & j &= 1, \ldots, \upsilon\,, \\ r_j \ &= \ n - j + 2\,, & j &= \upsilon + 1, \ldots, n-1\,, \\ p_j \ &= \ j\,, & j &= 1, \ldots, \upsilon\,, \\ p_j \ &= \ n - j + 1\,, & j &= \upsilon + 1, \ldots, n-1\,, \end{aligned} \tag{3.31}$$

with

$$\begin{aligned} \psi_{j,i} \ &: \ X \times W \to \mathbb{R}, \qquad j \ = \ 1, \ldots, n-1, \ \ i \ = \ r_j, \ldots, r \\ \lambda_{j,i} \ &: \ X \times W \to \mathbb{R}, \qquad j \ = \ 1, \ldots, n-1, \ \ i \ = \ p_j, \ldots, r \end{aligned}$$

appropriately defined continuous functions. We have then the following result.

Lemma 3.1. *Consider system* (3.23) *and the notations introduced before. There exist continuous functions* $\psi_{j,i}(\cdot,\cdot)$ *and* $\lambda_{ji}(\cdot,\cdot)$ *such that, having defined*

$$E_\varepsilon(w,x) := \frac{\partial \Psi_\varepsilon(w,x)}{\partial w} Sw + \frac{\partial \Psi_\varepsilon(w,x)}{\partial x} f(x) - F\Psi_\varepsilon(w,x) - GPw - B\Delta_\varphi(\Psi_\varepsilon(w,x),x)$$

the following holds

$$\begin{aligned} \lim_{\varepsilon\to 0^+} \frac{E_\varepsilon(w,x)}{\varepsilon^{r-1}} = 0\,, \qquad & \forall\,(w,x) \in W \times X\,, \\ E_\varepsilon(0,x) = 0 \qquad & \forall\,(\varepsilon,x) \in [0,1] \times X \end{aligned}$$

Furthermore, there exist continuous functions $\bar\psi_{j,j}(\cdot,\cdot)$*, and* $\bar\lambda_{j,j}(\cdot,\cdot)$*, such that*

$$\begin{array}{rcll} \psi_{j,r_j}(w,x) & := & \ell^{2j-1}\,\bar\psi_{j,r_j}(w,x), & j = 1,\ldots,m, \\ \psi_{r_j}(w,x) & := & \ell\,\bar\psi_{j,r_j}(w,x), & j = m+1,\ldots,n-1, \\ \lambda_{n-1,2}(w,x) & := & \ell\,\bar\lambda_{n-1,2}(w,x), & \end{array}$$

Proof. First of all consider the case where $w = 0$. By recalling the definition of $\Delta_\varphi(\cdot,\cdot)$ in (3.25) it is easy to verify that $\Psi_\varepsilon(0,x) = 0$ makes $E_\varepsilon(0,x) = 0$. As a consequence in the following we will show that $\Psi_\varepsilon(w,x)$ can be chosen as a continuous function in w satisfying $\Psi_\varepsilon(0,x) = 0$.

Now, since w and x range in bounded sets and the function $\psi_{j,i}(\cdot,\cdot)$ and $\lambda_{j,i}(\cdot,\cdot)$ are continuous, we have that

$$\lim_{\varepsilon\to 0^+} \Psi_\varepsilon(w,x) = 0 \qquad \forall\,(w,x) \in W \times X\,.$$

Let us denote $\overline{\Psi}_\varepsilon = \Gamma'\Psi_\varepsilon$ and let denote with $\overline{\Psi}_j$ the j-th element of $\overline{\Psi}_\varepsilon$. Expanding $\Delta_\varphi(\overline{\Psi}_\varepsilon, x)$ by Taylor around $\overline{\Psi}_\varepsilon = 0$ we obtain

$$\Delta_\varphi(\overline{\Psi}_\varepsilon, x) \;=\; \sum_{i=1}^{r} \varphi_i(x)[\overline{\Psi}_\varepsilon]^i + \rho_r(\overline{\Psi}_\varepsilon, x)$$

in which $\varphi_i(\cdot)$, $i = 1,\ldots,r$, are properly defined continuous functions, $\rho_r(\cdot,\cdot)$ is a properly defined continuous remainder function, and the $[\overline{\Psi}_\varepsilon]^i$ are monomials of the form

$$[\overline{\Psi}_\varepsilon]^i = \prod_{j=1}^{n-1} \overline{\Psi}_j^{k_j}\,, \qquad \sum_{j=1}^{n-1} k_j \;=\; i\,.$$

By replacing the Ψ_j in the definition of $\overline{\Psi}_j$ with the expression (3.30) and grouping the terms with the same power of ε, the Taylor expansion of $\Delta_\varphi(\cdot,\cdot)$ can be rewritten as

$$\Delta_\varphi(\overline{\Psi}_\varepsilon, x) \;=\; \sum_{i=1}^{r} \varepsilon^i \phi_i(w,x) + \varepsilon^{r+1} R_\varepsilon(w,x) \tag{3.32}$$

where the functions $\phi_i(\cdot,\cdot)$, $i = 1,\ldots,r$, and $R_\varepsilon(\cdot,\cdot)$ are appropriately defined continuous functions satisfying $\phi_i(0,x) = 0$ and $R_\varepsilon(0,x) = 0$. As far as the ϕ_i's are concerned, in particular, we observe that, because the $\overline{\Psi}_j$ are polynomials in ε and the $[\overline{\Psi}_\varepsilon]^i$ are polynomials in the $\overline{\Psi}_j$, only the coefficients of power smaller or equal to i in ε in the $\overline{\Psi}_j$ can be in ϕ_i. Namely $\phi_i(\cdot,\cdot)$ depends only on $\psi_{j,k}$ and on $\lambda_{n-1,k}$ with $k \le i$ for all $i = 1,\ldots r$ and $j = 1,\ldots,n-1$.

Consider now the expression of $E_\varepsilon(\cdot,\cdot)$ and, by letting

$$E_\varepsilon(\cdot,\cdot) := \Big(E_1(\cdot,\cdot)\;\; \Xi_1(\cdot,\cdot)\;\; E_2(\cdot,\cdot)\;\; \Xi_2(\cdot,\cdot)\;\; \cdots\;\; E_{n-1}(\cdot,\cdot)\;\; \Xi_{n-1}(\cdot,\cdot)\Big)^\top$$

note that

$$\begin{aligned}
E_1 &= \dot{\Psi}_1 + \ell c_{11}\Psi_1 - \Lambda_1 - \ell c_{11} P w\\
\Xi_1 &= \dot{\Lambda}_1 + \ell^2 c_{12}\Psi_1 - \Lambda_2 - \ell^2 c_{12} P w\\
&\;\;\vdots\\
E_j &= \dot{\Psi}_j + \ell c_{j1}\Psi_j - \Lambda_j - \ell c_{j1}\Lambda_{j-1}\\
\Xi_j &= \dot{\Lambda}_j + \ell^2 c_{j2}\Psi_j - \Lambda_{j+1} - \ell^2 c_{j2}\Lambda_{j-1}\\
&\;\;\vdots\\
E_{n-1} &= \dot{\Psi}_{n-1} + \ell c_{(n-1)1}\Psi_{n-1} - \Lambda_{n-1} - \ell c_{(n-1)1}\Lambda_{n-2}\\
\Xi_{n-1} &= \dot{\Lambda}_{n-1} + \ell^2 c_{(n-1)2}\Psi_{n-1} - \Delta_\varphi(\overline{\Psi}_\varepsilon, x) - \ell^2 c_{(n-1)2}\Lambda_{n-2}
\end{aligned}$$

where, for the sake of compactness, we omitted the argument (w,x) from the functions Ψ_j, Λ_j, $j = 1,\ldots,n-1$ and $\overline{\Psi}_\varepsilon$. By embedding (3.30) and (3.32) in the previous expressions, the following is obtained

$$\begin{aligned}
E_1 &= \sum_{i=1}^{r}\left[L_f\psi_{1,i} + \frac{1}{\varepsilon}L_S\psi_{1,i}\right]\varepsilon^i + \ell c_{11}\sum_{i=1}^{r}\psi_{1,i}\varepsilon^i - \sum_{i=1}^{r}\lambda_{1,i}\varepsilon^i - \ell c_{11} P w\\
\Xi_1 &= \sum_{i=1}^{r}\left[L_f\lambda_{1,i} + \frac{1}{\varepsilon}L_S\lambda_{1,i}\right]\varepsilon^i + \ell^2 c_{12}\sum_{i=1}^{r}\psi_{1,i}\varepsilon^i - \sum_{i=2}^{r}\lambda_{2,i}\varepsilon^i - \ell^2 c_{12} P w
\end{aligned}$$

$$
\begin{aligned}
E_j &= \sum_{i=r_j}^{r}\left[L_f\psi_{j,i}+\frac{1}{\varepsilon}L_S\psi_{j,i}\right]\varepsilon^i+\ell c_{j1}\sum_{i=r_j}^{r}\psi_{j,i}\varepsilon^i-\sum_{i=p_j}^{r}\lambda_{j,i}\varepsilon^i-\ell c_{j1}\sum_{i=p_{j-1}}^{r}\lambda_{j-1,i}\varepsilon^i\\
\Xi_j &= \sum_{i=p_j}^{r}\left[L_f\lambda_{j,i}+\frac{1}{\varepsilon}L_S\lambda_{j,i}\right]\varepsilon^i+\ell^2 c_{j2}\sum_{i=r_j}^{r}\psi_{j,i}\varepsilon^i-\sum_{i=p_{j+1}}^{r}\lambda_{j+1,i}\varepsilon^i-\ell^2 c_{j2}\sum_{i=p_{j-1}}^{r}\lambda_{j-1,i}\varepsilon^i
\end{aligned}
$$

for $j = 2, \dots, n-2$ and

$$
\begin{aligned}
E_{n-1} &= \sum_{i=r_{n-1}}^{r}\left[L_f\psi_{n-1,i}+\frac{1}{\varepsilon}L_S\psi_{n-1,i}\right]\varepsilon^i+\ell c_{(n-1)1}\sum_{i=r_{n-1}}^{r}\psi_{n-1,i}\varepsilon^i\\
&\qquad -\sum_{i=2}^{r}\lambda_{n-1,i}\varepsilon^i-\ell c_{(n-1)1}\sum_{i=p_{n-2}}^{r}\lambda_{n-2,i}\varepsilon^i\\
\Xi_{n-1} &= \sum_{i=2}^{r}\left[L_f\lambda_{n-1,i}+\frac{1}{\varepsilon}L_S\lambda_{n-1,i}\right]\varepsilon^i+\ell^2 c_{(n-1)2}\sum_{i=r_{n-1}}^{r}\psi_{n-1,i}\varepsilon^i\\
&\qquad -\sum_{i=1}^{r}\varepsilon^i\phi_i-\varepsilon^{r+1}R_\varepsilon-\ell^2 c_{(n-1)2}\sum_{i=p_{n-2}}^{r}\lambda_{n-2,i}\varepsilon^i
\end{aligned}
$$

in which

$$
\begin{aligned}
L_f\psi_{j,i} &:= \frac{\partial\psi_{j,i}(w,x)}{\partial x}f(x), \qquad L_S\psi_{j,i} := \frac{\partial\psi_{j,i}(w,x)}{\partial w}S\,,\\
L_f\lambda_{j,i} &:= \frac{\partial\lambda_{j,i}(w,x)}{\partial x}f(x), \qquad L_S\lambda_{j,i} := \frac{\partial\lambda_{j,i}(w,x)}{\partial w}Sw.
\end{aligned}
$$

The idea now is to iteratively select the functions $\psi_{j,i+1}(\cdot,\cdot)$, $\lambda_{j,i+1}(\cdot,\cdot)$ to annihilate, in the previous expressions the terms in ε of order i, with $i = 0, \dots, r-1$, for $j = 1, \dots, n$. We start by considering the term of order 0 in ε in the expression of E_1 and Ξ_1 which are annihilated by taking

$$
\begin{aligned}
\psi_{1,1}(w,x) &= \ell c_{11}PS^{-1}w\,,\\
\lambda_{1,1}(w,x) &= \ell^2 c_{12}PS^{-1}w\,.
\end{aligned}
\tag{3.33}
$$

We observer that $\psi_{1,1}(w,x)$, $\lambda_{1,1}(w,x)$, are polynomials in w of order 1 with constant coefficients. Since $r_j = p_j = j$ for $j = 1, \dots, v$, having fixed the terms $\psi_{1,1}$ and $\lambda_{1,1}$ it is possible, iteratively, to select all the functions $\psi_{j,j}(w,x)$ and $\lambda_{j,j}(w,x)$ for $j = 2, \dots, v-1$ to annihilate the terms in ε of order $j-1$ in E_j and Ξ_j by solving the following PDEs

$$
\begin{aligned}
-L_S\psi_{j,j} &= \ell c_{j1}\lambda_{j-1,j-1}\\
-L_S\lambda_{j,j} &= \ell^2 c_{j2}\lambda_{j-1,j-1}
\end{aligned}
$$

Using the fact that S is invertible the previous PDEs admit a solution which is polynomial in w of order 1 with constant coefficients. With this we have fixed the terms

$\psi_{j,j}$ and $\lambda_{j,j}$ for $j = 1, \dots, \upsilon - 1$. Once the terms above have been fixed, it is possible to repeat the selection process for $i = 1, \dots, \upsilon - 1$ from the top in order to select the functions $\psi_{j,j+i}$ and $\lambda_{j,j+i}$ for $j = 1, \dots, \upsilon - i - 1$, to annihilate the terms in ε of order $j + i - 1$ in E_j and Ξ_j, by solving the following PDEs

$$\begin{array}{rcl} -L_S\psi_{1,1+i} &=& \ell c_{11}\psi_{1,i} - \lambda_{1,i} \\ -L_S\lambda_{1,1+i} &=& \ell^2 c_{12}\psi_{1,i} - \lambda_{2,i} \end{array}$$

and

$$\begin{array}{rcl} -L_S\psi_{j,j+i} &=& \ell c_{j1}(\psi_{j,j+i-1} - \lambda_{j-1,j+i-1}) - \lambda_{j,j+i-1} \\ -L_S\lambda_{j,j+i} &=& \ell^2 c_{j2}(\psi_{j,j+i-1} - \lambda_{j-1,j+i-1}) - \lambda_{j+1,j+i-1} \end{array}$$

for $j = 2, \dots, \upsilon - i - 1$. Again, using the fact that S is invertible the previous PDEs admit a solution which is polynomial in w of order 1 with constant coefficients. With this we have assigned all the terms $\psi_{j,k}$ and $\lambda_{j,k}$ for $j = 1, \dots, \upsilon - 1$ and $k = j, \dots, \upsilon$.

Now let us consider the terms E_j and Ξ_j for $j = \upsilon + 1, \dots, n - 1$ by starting from the bottom. Observer that $\phi_1(\cdot,\cdot)$ only depends on $\psi_{1,1}$. Hence we can assume that $\phi_1(w, x)$ is a polynomial in w of order 1 with coefficients dependent on x and vanishing when $w = 0$. As a consequence the terms in ε of order 1 in Ξ_{n-1} and the term in ε of order 2 in E_{n-1} are annihilated if $\lambda_{n-1,2}$ and $\psi_{n-1,3}$ can be chosen such that

$$\begin{array}{rcl} L_S\psi_{n-1,3} &=& \lambda_{n-1,2} \\ L_S\lambda_{n-1,2} &=& \phi_1 \end{array} \tag{3.34}$$

Using the fact that S is invertible and function ϕ_1 is a polynomial in w the previous PDEs admit a solution which is polynomial of order 1 in w with coefficients which depends on x. Recall that for $i = \upsilon+1, \dots, n-1$ we have $p_i > p_{i+1}$. As a consequence, once the term $\lambda_{n-1,2}$ has been fixed we can proceed iteratively in order to select (from the bottom) all the functions $\psi_{j,r_j}(w, x)$ and $\lambda_{j,p_j}(w, x)$ for $j = \upsilon + 1, \dots, n - 2$ by annihilating the terms in ε of order $r_j - 1$ in E_j and of order $p_j - 1$ in Ξ_j by solving the following PDEs

$$\begin{array}{rcl} L_S\psi_{j,r_j} &=& \lambda_{j,p_j} \\ L_S\lambda_{j,p_j} &=& \lambda_{j+1,p_{j+1}} \end{array}$$

Using the fact that S is invertible the previous PDEs admit a solution which is polynomial of order 1 in w with coefficients which depends on x. With this we have fixed all the terms ψ_{j,r_j} and λ_{j,p_j} for $j = \upsilon + 1, \dots, n - 1$. Now consider the term of order 3 in ε in Ξ_{n-1} and of order 4 in in E_{n-1}. We observe that the function ϕ_2 depends only on the terms $\psi_{j,i}$, $j = 1, \dots, n - 1$, and $\lambda_{n-1,i}$ with powers of ε smaller (or equal) to two, namely $i \leq 2$, and therefore on the functions $\psi_{1,1}$, $\psi_{1,2}$, $\psi_{2,2}$ and $\lambda_{n-1,2}$ which have already been fixed. More in general the term ϕ_i depends on the terms $\psi_{j,k}$, with $j \in [1, n-1]$ such that $r_j \leq i$ and $\lambda_{n-1,k}$, with $k \leq i$. As a consequence, we can proceed iteratively for $i = 1, \ldots, \upsilon - 2$ from the bottom, in order to select all the functions $\psi_{j,r_j+i}(w, x)$ and $\lambda_{j,p_j+i}(w, x)$ for $j = \upsilon + i, \ldots, n-1$ by annihilating the terms in ε of order $r_j + i - 1$ in E_j and of order $p_j + i - 1$ in Ξ_j by solving the following PDEs

$$\begin{array}{rcl} L_S \psi_{n-1,r_j+i} & = & \lambda_{n-1,p_j+i} - \ell c_{(n-1)1} \lambda_{n-2,p_j+i-1} \\ L_S \lambda_{n-1,p_j+i} & = & \phi_{i+1}(x) - \ell^2 c_{(n-1)2} \lambda_{n-2,p_j+i-1} \end{array}$$

for $j = n-1$ and

$$\begin{array}{rcl} L_S \psi_{j,r_j+i} & = & \lambda_{j,p_j+i} - \ell c_{j1} \lambda_{j-1,p_j+i-1} \\ L_S \lambda_{j,p_j+i} & = & \lambda_{j+1,p_{j+1}+i-1} - \ell^2 c_{j2} \lambda_{j-1,p_j+i-1} \end{array}$$

for $j = \upsilon + i, \ldots, n-2$, and where we considered

$$\lambda_{j,i} = 0$$

if $i < p_j$. Using the fact that S is invertible and function ϕ_i is a polynomial in w of order i the previous PDEs admit a solution which is polynomial in w with coefficients which depends on x. With this we have fixed all the terms $\psi_{j,i}$ and $\lambda_{j,k}$ for $j = \upsilon + 1, \ldots, n-1$, $i = r_j, \ldots, \upsilon$, $k = p_j, \ldots, \upsilon$.

As far as the terms Ξ_υ and E_υ are concerned, we have that its structure changes whether υ is even or odd. Anyhow, by letting

$$\lambda_{j,i} = 0$$

if $i < p_j$, consider the terms in ε of order $\upsilon - 1$. We can select the functions $\psi_{\upsilon,r_\upsilon}$ and $\lambda_{\upsilon,p_\upsilon}$ by solving the following PDEs

$$\begin{array}{rcl} -L_S \psi_{\upsilon,\upsilon} & = & -\ell c_{\upsilon 1} \lambda_{\upsilon-1,\upsilon-1} \\ -L_S \lambda_{\upsilon,\upsilon} & = & -\ell^2 c_{\upsilon 2} \lambda_{\upsilon-1,\upsilon-1} - \lambda_{\upsilon+1,p_{\upsilon+1}} \end{array}$$

Since the term $\lambda_{\upsilon+1,p_{\upsilon+1}}$ has been already fixed by the previous procedure, by using the fact that S is invertible and functions $\lambda_{\upsilon-1,\upsilon-1}$ and $\lambda_{\upsilon+1,p_{\upsilon+1}}$ are polynomial in w, the previous PDEs admit a solution which is polynomial of in w with coefficients which may depends on x. With this we have finally assigned all the terms $\psi_{j,i}$, $\lambda_{j,i}$

for $j = 1, \ldots, n-1$ and $i = 1, \ldots, v$. The forthcoming scheme can be helpful in understanding how the recursive procedure is held. Note that if n is odd we get

$$
\begin{array}{rcl}
\Psi_1 & = & \varepsilon\star + \varepsilon^2\star + \varepsilon^3\star + \ldots + \varepsilon^{v-1}\star + \varepsilon^v\star + \ldots \\
\Lambda_1 & = & \varepsilon\star + \varepsilon^2\star + \varepsilon^3\star + \ldots + \varepsilon^{v-1}\star + \varepsilon^v\star + \ldots \\
\Psi_2 & = & \varepsilon^2\star + \varepsilon^3\star + \ldots + \varepsilon^{v-1}\star + \varepsilon^v\star + \ldots \\
\Lambda_2 & = & \varepsilon^2\star + \varepsilon^3\star + \ldots + \varepsilon^{v-1}\star + \varepsilon^v\star + \ldots \\
& \vdots & \ddots \\
\Psi_{v-1} & = & \varepsilon^{v-1}\star + \varepsilon^v\star + \ldots \\
\Lambda_{v-1} & = & \varepsilon^{v-1}\star + \varepsilon^v\star + \ldots \\
\Psi_v & = & \varepsilon^v\star + \ldots \\
\Lambda_v & = & \varepsilon^v\star + \ldots \\
\Psi_{v+1} & = & \varepsilon^v\star \\
\Lambda_{v+1} & = & \varepsilon^{v-1}\star + \varepsilon^v\star + \ldots \\
& \vdots & \iddots \\
\Psi_{n-1} & = & \varepsilon^3\star + \ldots + \varepsilon^{v-1}\star + \varepsilon^v\star + \ldots \\
\Lambda_{n-1} & = & \varepsilon^2\star + \varepsilon^3\star + \ldots + \varepsilon^{v-1}\star + \varepsilon^v\star + \ldots
\end{array}
$$

where $\star$ denotes a term $\psi_{j,i}$ or $\lambda_{j,i}$, whereas in case n is even, we have

$$
\begin{array}{rcl}
\Psi_{v-1} & = & \varepsilon^{v-1}\star + \varepsilon^v\star + \varepsilon^{v+1}\star + \ldots \\
\Lambda_{v-1} & = & \varepsilon^{v-1}\star + \varepsilon^v\star + \varepsilon^{v+1}\star + \ldots \\
\Psi_v & = & \varepsilon^v\star + \varepsilon^{v+1}\star + \ldots \\
\Lambda_v & = & \varepsilon^v\star + \varepsilon^{v+1}\star + \ldots \\
\Psi_{v+1} & = & \varepsilon^{v+1}\star + \ldots \\
\Lambda_{v+1} & = & \varepsilon^v\star + \varepsilon^{v+1}\star + \ldots
\end{array}
$$

Once the terms $\psi_{j,i}$, $\lambda_{j,i}$ for $j = 1, \ldots, n-1$ and $i = 1, \ldots, v$, have been assigned, the iterative procedure may be lead similarly in order to select all the remaining terms $\psi_{j,i+1}$, $\lambda_{j,i+1}$ for $j = 1, \ldots, n-1$ and $i = v, \ldots, r$ by annihilating at each step the terms in ε of order i which depends on the terms $\psi_{j,i}$, $\lambda_{j,i}$ which have already been fixed in the previous iterative process.

Finally, by embedding those functions in the expressions of $E_j(\cdot,\cdot)$, $\Xi_j(\cdot,\cdot)$, $j = 1, \ldots, n-1$ and bearing in mind the definition of $R_\varepsilon(\cdot,\cdot)$, it is readily seen that

$$
\begin{array}{rcl}
E_1(w,x) & = & \varepsilon^r[L_f\psi_{1,r} + \ell c_{11}\psi_{1,r} - \lambda_{1,r}] \\
E_j(w,x) & = & \varepsilon^r[L_f\psi_{j,r} + \ell c_{j1}\psi_{j,r} - \lambda_{j,r} - \ell c_{j1}\lambda_{j-1,r}]
\end{array}
$$

for $j = 2, \dots, n-1$ and

$$\begin{array}{rcl}\Xi_1(w,x) & = & \varepsilon^r[L_f\lambda_{1,r} + \ell^2 c_{12}\psi_{1,r} - \lambda_{2,r}] \\ & \vdots & \\ \Xi_j(w,x) & = & \varepsilon^r[L_f\lambda_{j,r} + \ell^2 c_{j2}\psi_{j,r} - \lambda_{j+1,r} - \ell^2 c_{j2}\lambda_{j-1,r}] \\ & \vdots & \\ \Xi_{n-1}(w,x) & = & \varepsilon^r[L_f\lambda_{n-1,r} + \ell^2 c_{(n-1)2}\psi_{n-1,r} - \ell^2 c_{(n-1)2}\lambda_{n-2,r} - \phi_r] \\ & & \qquad + \varepsilon^{r+1} w \bar{R}_\varepsilon(w,x)\end{array}$$

where $\bar{R}_\varepsilon(\cdot,\cdot)$ is an appropriately defined continuous function. By using the previous expressions and the fact that $\psi_{j,i}(0,x) = 0$, $\lambda_{j,i}(0,x) = 0$ for any $j = 1, \dots, n-1$ and $i = 1, \dots, r$, the first part of the lemma immediately follows.

Now, by using (3.33), we have for $j = 1$

$$\psi_{1,1}(w,x) \ := \ \ell c_{11} P S^{-1} w \ = \ \ell \bar{\psi}_{1,1}(w,x) \ ,$$

For $j = 2, \dots, m-1$ we have $p_{j-1} < p_j$. As a consequence the term $\psi_{j,r_j}(w,x)$ is computed a solution of of the PDE

$$L_S \psi_{j,r_j} \ := \ \ell c_{j1} \lambda_{j-1,p_{j-1}}$$

namely it depends on the $\lambda_{j-1,p_{j-1}}$. Again, by using (3.33), we have that $\lambda_{1,1}$ is choosen as

$$\lambda_{1,1}(w,x) \ = \ \ell^2 c_{12} P S^{-1} w \ := \ \ell^2 \bar{\lambda}_{1,1}(w,x) \ .$$

For $j = 2, \dots, m-1$, since $p_{j-1} < p_{j+1}$, the λ_{j,p_j} are computed as solution of

$$L_S \lambda_{j,p_j} \ = \ \ell^2 c_{j2} \lambda_{j,p_{j-1}}$$

With this in mind, we can always select functions $\bar{\psi}_{j,j}(w,x)$ and $\bar{\lambda}_{j,j}(w,x)$ such that

$$\begin{array}{rcl}\psi_{j,r_j}(w,x) & := & \ell^{2j-1}\bar{\psi}_{j,r_j}(w,x) \\ \lambda_{j,p_j}(w,x) & := & \ell^{2j}\bar{\lambda}_{j,p_j}(w,x)\end{array} \qquad j = 1, \dots, m-1.$$

Now let start from the bottom by computing $\lambda_{n-1,2}$ and ψ_{j,r_j} for $j = m+1, \dots, n-1$. Observe that $\lambda_{n-1,2}$ is chosen as solution of (3.34). As a consequence, since ϕ_1 depends only on $\psi_{1,1}$ which is a function in ℓ, we can deduce that also $\lambda_{n-1,2}$ can be written as

$$\lambda_{n-1,2}(w,x) := \ell \bar{\lambda}_{n-1,2}(w,x) \ .$$

Also, for $j = m+1\dots, n-1$ we have that $p_j < p_{j-1}$. As a consequence the functions ψ_{j,r_j} are chosen as solution of

$$L_S\psi_{j,r_j} \;=\; \lambda_{j,p_j}$$

Therefore we need to compute before the terms λ_{j,p_j}. For $j = n-1$, $\lambda_{n-1,2}$ is a term in ℓ. For $j = m+1,\dots,n-2$ we can compute λ_{j,p_j} as solution of

$$L_S\lambda_{j,p_j} \;=\; \lambda_{j+1,p_{j+1}}$$

because $p_{j+1} < p_{j-1}$. As a consequence the λ_{j,p_j} are all functions of $\lambda_{n-1,2}$ which is a term in ℓ. As a consequence we have

$$\begin{array}{rcl} \psi_{j,r_j}(w,x) &=& \ell\bar{\psi}_{j,r_j}(w,x) \\ \lambda_{j,p_j}(w,x) &=& \ell\bar{\lambda}_{j,p_j}(w,x) \end{array} \qquad j = m+1,\dots,n-1$$

for some $\bar{\psi}_{j,r_j}(w,x)$ and $\bar{\lambda}_{j,p_j}(w,x)$. It remains to fix the terms ψ_{m,r_m}, λ_{m,p_m}.

If n is odd we have $r_{m-1}^2 < r_m^2$. As a consequence ψ_m can be computed as solution of

$$L_S\psi_{m,r_m} \;=\; \ell c_{m1}\lambda_{m-1,p_{m-1}}$$

But being $\lambda_{m-1,p_{m-1}}$ a term in $\ell^{2(m-1)}$ we get that

$$\psi_{m,r_m}(x,w) \;:=\; \ell^{2m-1}\bar{\psi}_{m,r_m}(x,w)$$

for some $\bar{\psi}_{m,r_m}(x,w)$.

If n is even we have $p_{m+1} < p_m = p_{m-1}$. As a consequence λ_{m,p_m} is chosen as solution of

$$L_S\lambda_{m,p_m} \;=\; \lambda_{m+1,p_{m+1}}$$

resulting in a term in ℓ, whereas ψ_{m,r_m} is chosen as solution of

$$L_S\psi_{m,r_m} \;=\; \ell c_{m1}\lambda_{m-1,p_{m-1}} + \lambda_{m,p_m}$$

But since λ_{m,p_m} is a term in ℓ whereas $\lambda_{m-1,p_{m-1}}$ is a term in $\ell^{2(m-1)}$ we get again

$$\psi_{m,r_m}(x,w) \;:=\; \ell^{2m-1}\bar{\psi}_{m,r_m}(x,w)$$

for some $\bar{\psi}_{m,r_m}(x,w)$. This concludes the proof. □

The previous lemma is instrumental to the proof of the next proposition which is the main result of this section.

Proposition 3.1. *Consider system* (3.29) *with* $x(t) \in X$ *and* $w(t) \in W$ *for all* $t \geq 0$ *with* X *and* W *bounded sets. Let the function* $\varphi_s(\cdot)$ *embedded in* $\Delta_\varphi(\cdot,\cdot)$ *be chosen so that it is locally Lipschitz and it agrees with* $\varphi(\cdot)$ *on a set* $X_\delta \supset X$. *Let* ℓ *be fixed so that system* (3.24) *is ISS with respect to the input* ν. *Then, there exists a* $\varepsilon^\star(\ell) > 0$ *such that for all positive* $\varepsilon \leq \varepsilon^\star(\ell)$ *the following holds*

$$\begin{aligned}\lim_{t\to\infty} \sup |e_j(t)| &\leq \mu\, \varepsilon^j\, \ell^{2j-1}\, \|w(\cdot)\|_\infty \qquad j = 1,\dots,m \\ \lim_{t\to\infty} \sup |e_j(t)| &\leq \mu\, \varepsilon^{n-j+2}\, \ell\, \|w(\cdot)\|_\infty \qquad j = m+1,\dots,n\end{aligned}$$

with μ *a positive constant.*

Proof. Let consider the change of variables

$$\tilde{\xi} \;\mapsto\; \eta \;:=\; \tilde{\xi} - \Psi_\varepsilon(w,x)\;,$$

with $\Psi_\varepsilon(\cdot,\cdot)$ introduced in the previous lemma with an $r > 1$ and note that, by bearing in mind the definition of $E_\varepsilon(\cdot,\cdot)$,

$$\dot{\Psi}_\varepsilon = F\Psi_\varepsilon + B\Delta_\varphi(\Psi_\varepsilon, x) + GPw + E_\varepsilon(w,x)\,.$$

Furthermore, note that

$$\begin{aligned}\Delta_\varphi(\tilde{\xi},x) - \Delta_\varphi(\Psi_\varepsilon(w,x),x) &= \Delta_\varphi(\eta + \Psi_\varepsilon(w,x),x) - \Delta_\varphi(\Psi_\varepsilon(w,x),x) \\ &= \varphi_s(\eta + \Psi_\varepsilon(w,x) + x) - \varphi(x) \\ &\qquad -(\varphi_s(\Psi_\varepsilon(w,x) + x) - \varphi(x)) \\ &= \varphi_s(\eta + \Psi_\varepsilon(w,x) + x) - \varphi_s(\Psi_\varepsilon(w,x) + x) \\ &= \Delta_\varphi(\eta, \Psi_\varepsilon + x)\,.\end{aligned}$$

Note that there exists a $\varepsilon_1^\star(\ell) \in (0,1]$ such that for all positive (the value of $\varepsilon^\star$ depends, besides other, on the choice of the set X_δ on which $\varphi_s(\cdot)$ coincides with $\varphi(\cdot)$) $\varepsilon \leq \varepsilon_1^\star(\ell)$

$$\Delta_\varphi(0, \Psi_\varepsilon(w,x) + x) = 0 \qquad \forall\,(w,x) \in W \times X\,.$$

By the previous facts the error dynamics in the new coordinates can be easily computed as

$$\dot{\eta} = F\eta + B\Delta_\varphi(\eta, \Psi_\varepsilon(w,x) + x) + E_\varepsilon(w,x)\,. \tag{3.35}$$

Since the Lipschitz constant of $\Delta_\varphi(\cdot,\cdot)$ is not affected by the value of the arguments, the same values of ℓ that make system (3.24) ISS with respect to the input $\nu(t)$ make also system (3.35) ISS with respect to the input $E_\varepsilon(\cdot,\cdot)$. In particular, there exists a positive constant μ_0 such that

$$\begin{aligned}\lim_{t\to\infty}\sup|\eta(t)| &= \lim_{t\to\infty}\sup|\tilde{\xi}(t)-\Psi_\varepsilon(w(t),x(t))| \\ &\leq \mu_0 \lim_{t\to\infty}\sup|E_\varepsilon(w(t),x(t))| \\ &\leq \mu_0 \in \|E_\varepsilon(w(\cdot),x(\cdot))\|_\infty\end{aligned}$$

Using the fact that, for any $r \geq m$, $E_\varepsilon(w,x)$ is a term in ε^r, it follows that there exists a positive constant μ_1 such that

$$\lim_{t\to\infty}\sup|\eta(t)| \leq \mu_1\varepsilon^r \|w(\cdot)\|_\infty \,.$$

Consider now the the expressions of the components $\Psi_j(\cdot,\cdot)$, $j=1,\dots,n-1$ and Λ_{n-1}, of $\Psi_\varepsilon(\cdot,\cdot)$ introduced in Lemma 3.1. It turns out that there exist a positive $\varepsilon_2^\star(\ell) \leq \varepsilon_1^\star(\ell)$ and a positive constant μ_2 such that

$$\begin{aligned}|\Psi_j(w,x)| &\leq \mu_2\,\varepsilon^j\,\ell^{2j-1}\,|w| \qquad j=1,\dots,m, \\ |\Psi_j(w,x)| &\leq \mu_2\,\varepsilon\,\ell\,|w| \qquad j=m+1,\dots,n-1,\end{aligned}$$

for all positive $\varepsilon \leq \varepsilon_2^\star(\ell)$ and for all $(w,x)\in W\times X$. From this, for all $j=1,\dots,m$, we have

$$\begin{aligned}\lim_{t\to\infty}\sup|e_j(t)| &= \lim_{t\to\infty}\sup|\eta_j(t)+\Psi_j(w(t),x(t))| \\ &\leq \lim_{t\to\infty}\sup|\eta_j(t)| + \|\Psi_j(w(\cdot),x(\cdot))\|_\infty \\ &\leq \mu_1\varepsilon^r\|w(\cdot)\|_\infty + \mu_2\varepsilon^j\ell^{2j-1}\|w(\cdot)\|_\infty\end{aligned}$$

for $j=m+1,\dots,n-1$

$$\begin{aligned}\lim_{t\to\infty}\sup|e_j(t)| &= \lim_{t\to\infty}\sup|\eta_j(t)+\Psi_j(w(t),x(t))| \\ &\leq \lim_{t\to\infty}\sup|\eta_j(t)| + \|\Psi_j(w(\cdot),x(\cdot))\|_\infty \\ &\leq \mu_1\varepsilon^r\|w(\cdot)\|_\infty + \mu_2\varepsilon^{r_j^1}\ell\,\|w(\cdot)\|_\infty \\ &\leq \mu_1\varepsilon^r\|w(\cdot)\|_\infty + \mu_2\varepsilon^{n-j+2}\ell\,\|w(\cdot)\|_\infty\end{aligned}$$

and finally for $j = n$ we have

$$\begin{aligned}\lim_{t\to\infty} \sup |e_n(t)| &= \lim_{t\to\infty} \sup |\eta_n(t) + \Lambda_{n-1}(w(t), x(t))| \\ &\leq \lim_{t\to\infty} \sup |\eta_n(t)| + \|\Lambda_{n-1}(w(\cdot), x(\cdot))\|_\infty \\ &\leq \mu_1 \varepsilon^r \|w(\cdot)\|_\infty + \mu_2 \varepsilon^2 \ell \|w(\cdot)\|_\infty\end{aligned}$$

by which the result follows by taking an appropriate $\varepsilon^\star(\ell) \leq \varepsilon_2^\star(\ell)$ and μ. □

Remark The same kind of noise analysis for the observer (3.2) in presence of measurement noise, still holds for the observer (3.11) if the saturation levels are chosen large enough. In particular, if asymptotically the saturations functions in (3.11) are no more acting (namely the effect of the noise is not too large) then we can apply off-the shelf all the arguments presented in this section where we showed the benefits related to the increase of the relative degree. Simulation results are given in the last section of this chapter.

3.5 Low-power tools in output regulation

The main criticisms that can be done to the high-gain design methodology for internal models introduced in Section 1.6.2 is the same mentioned above in the context of observation, namely the fact that the power of the high-gain parameter is raised up to the order of the internal model, with the latter than can be large to eventually have the friend and its time derivatives fulfilling the regression law said before. Motivated by this, in this section, we adapt the tools presented in Section 3.1 to develop a "low-power" methodology for the design of internal models. The content of this section is part of a book written in honour of Laurent Praly: Astolfi et al. (2017).

Our main goal is to show that the "low-power" tools introduced in Section 3.1 can be successfully adopted in order to design the triplet $(\phi(\cdot), \Psi, \gamma(\cdot))$ of the regulator (1.31) fulfilling the requirements (a') and (b) of Section 1.6.1. It is argued that it is known a positive $d > 0$ and a function $\varphi(\cdot)$ fulfilling (1.30) for some (unknown) function $\nu(\cdot)$. As a consequence, by following the observer design (3.6), let

$$m \;=\; 2d-2$$

and, let

$$\phi(\xi) \;:=\; \begin{pmatrix} \phi_1(\xi) \\ \phi_2(\xi) \\ \vdots \\ \phi_{d-1}(\xi) \end{pmatrix}, \qquad \Psi \;:=\; \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \vdots \\ \Psi_{d-1} \end{pmatrix}, \qquad \gamma(\xi) \;:=\; \xi_{11}\,, \tag{3.36}$$

where

$$\xi = \mathrm{col}(\xi_1, \ldots, \xi_{d-1}) \in \mathbb{R}^{2d-2}\,, \qquad \xi_i = \mathrm{col}(\xi_{i1}, \xi_{i2}) \in \mathbb{R}^2\,,$$

the functions $\phi_i : \mathbb{R}^{2d-2} \to \mathbb{R}^2$, $i = 1, \ldots, d-1$, are defined as

$$\phi_1(\xi) \;:=\; \begin{pmatrix} \xi_{12} \\ \xi_{22} \end{pmatrix}, \quad \phi_i(\xi) \;:=\; \begin{pmatrix} \xi_{i2} + \ell\, c_{i1}\,(\xi_{(i-1)2} - \xi_{i1}) \\ \xi_{(i+1)2} + \ell^2\, c_{i2}\,(\xi_{(i-1)2} - \xi_{i1}) \end{pmatrix}, \tag{3.37}$$

for $i = 2, \ldots, d-2$,

$$\phi_{d-1}(\xi) \;:=\; \begin{pmatrix} \xi_{(d-1)2} + \ell\, c_{(d-1)1}\,(\xi_{(d-2)2} - \xi_{(d-1)1}) \\ \hat{\varphi}(\Gamma\xi) + \ell^2\, c_{(d-1)2}\,(\xi_{(d-2)2} - \xi_{(d-1)1}) \end{pmatrix} \tag{3.38}$$

in which

$$\Gamma \;:=\; \text{blkdiag}\left(\begin{pmatrix} 1 & 0 \end{pmatrix} \; \begin{pmatrix} 1 & 0 \end{pmatrix} \; \cdots \; \begin{pmatrix} 1 & 0 \end{pmatrix} \; \begin{pmatrix} 1 & 1 \end{pmatrix} \right), \tag{3.39}$$

(c_{i1}, c_{i2}), $i = 1, \ldots, d-1$, are coefficients to be appropriately chosen, and the vectors Ψ_i, $i = 1, \ldots, d-1$ are defined as

$$\Psi_1 := \begin{pmatrix} \ell\, c_{11} \\ \ell^2\, c_{12} \end{pmatrix}, \quad \Psi_i := \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad i = 2, \ldots, d-1\,.$$

It will be shown that the previous choice of the triplet $(\phi(\cdot), \Psi, \gamma(\cdot))$ makes the requirements (a) and (b') fulfilled provided that the coefficients $(c_{i,1}, c_{i,2})$, $i = 1, \ldots, d-1$, are appropriately chosen and ℓ is taken sufficiently large. As a matter of fact, it is not hard to see that with this choice the regulator (1.31) in the χ-coordinates (1.32) reads as a low-power high-gain observer (3.2) driven by an the (disturbed) input $c(w)$, as detailed in the proof of the forthcoming proposition in which we refer to the function $\tau_e : W \to \mathbb{R}^{2d-2}$ defined as

$$\tau_e(w) := \begin{pmatrix} \tau_{e,1}(w) \\ \tau_{e,2}(w) \\ \vdots \\ \tau_{e,d-1}(w) \end{pmatrix}, \quad \tau_{e,i} = \begin{pmatrix} \tau_{e,i1} \\ \tau_{e,i2} \end{pmatrix} := \begin{pmatrix} L_s^{i-1} c(w) \\ L_s^{i} c(w) \end{pmatrix}, \quad i = 1, \ldots, d-1\,.$$

Proposition 3.2. *Let $c(\cdot)$ in* (1.29) *be fulfilling 1.30 and let the triplet $(\phi(\cdot), \Psi, \gamma(\cdot))$ be taken as in* (3.36)-(3.38). *There exist a choice of the coefficients $(c_{i,1}, c_{i,2})$, $i = 1, \ldots, d-1$, such that there exist a $\ell^\star > 0$ so that for all $\ell \geq \ell^\star$ requirements* (a') *and* (b) *of Section 1.6.1 are fulfilled with*

$$\mathcal{B} \;=\; \{(w, z, \chi) \in W \times \mathbb{R}^n \times \mathbb{R}^{2d-2}\,, \quad z = \pi(w)\,,\; \chi = \tau_e(w)\}$$

and the ϵ in (1.34) *of the form*

$$\epsilon \;=\; \frac{r}{\ell^d} \sup_{w \in W} |\nu(w)|$$

with r a positive number.

Proof. By the indicated choices of the triplet $(\phi(\cdot), \Psi, \gamma(\cdot))$ in (3.36)-(3.38), it turns out that the χ subsystem in (1.33) reads as

$$\begin{aligned}
\dot{\chi}_1 &= A\,\chi_1 + N\,\chi_2 + D_2(\ell)\,c_1\,\left(c(w) + \delta(w,z) - C\,\chi_1\right)\\
\dot{\chi}_i &= A\,\chi_i + N\,\chi_{i+1} + D_2(\ell)\,c_i\,(B^\top\,\chi_{i-1} - C\,\chi_i)\,, \quad i = 2,\dots,d-2\\
\dot{\chi}_{d-1} &= A\,\chi_{d-1} + B\,\hat{\varphi}(\Gamma\chi) + D_2(\ell)\,c_{d-1}\,(B^\top\,\chi_{d-2} - C\,\chi_{d-1})
\end{aligned}$$

where (A,B,C) is a triplet in *prime form* of dimension 2, $c_i = \mathrm{col}(c_{i1}, c_{i2})$, $D_2(\ell) = \mathrm{diag}(\ell,\, \ell^2)$, and $N = \mathrm{diag}(0,\, 1)$, and

$$\delta(w,z) \;:=\; \frac{q(w,z,0)}{b(w,z,0)} - c(w)\,.$$

By recalling that $c(w)$ satisfies (1.30), the previous system can be re-written in the form

$$\begin{aligned}
\dot{\chi}_1 &= A\,\chi_1 + N\,\chi_2 + D_2(\ell)\,c_1\,\left(y - C\,\chi_1\right)\\
\dot{\chi}_i &= A\,\chi_i + N\,\chi_{i+1} + D_2(\ell)\,c_i\,(B^\top\,\chi_{i-1} - C\,\chi_i)\,, \quad i = 2,\dots,d-2\\
\dot{\chi}_{d-1} &= A\,\chi_{d-1} + B\,\hat{\varphi}(\Gamma\chi) + D_2(\ell)\,c_{d-1}\,(B^\top\,\chi_{d-2} - C\,\chi_{d-1})
\end{aligned} \tag{3.40}$$

where the measured output y is generated by the system

$$\begin{aligned}
\dot{z} &= A_d z + B_d(\varphi(z) + v(w))\,,\\
y &= C_d z + \delta(w,z)\,,
\end{aligned} \qquad z_i(0) := L_s^{i-1}c(w(0))\,, \quad i = 1,\dots,d\,,$$

where $z \in \mathbb{R}^d$ and (A_d, B_d, C_d) is a triplet in *prime form* of dimension d. The system (3.40) coincides with the low-power high-gain observer (3.2) written in the coordinates (3.6). As a consequence the Theorem 3.1 can be applied off the shelf to show the existence of constants $\mu_1 > 0$, $\mu_2 > 0$, $r > 0$ and $\ell^\star > 0$ such that

$$\begin{aligned}
|\chi_{i1}(t) - \tau_{e,i1}(w(t))| &\le \mu_1\ell^{i-1}\exp(-\mu_2\,\ell\,t) + \frac{r}{\ell^{i-1}}|\nu(w)|_\infty + \ell^{i-1}|\delta(w,z)|_\infty\\
|\chi_{i2}(t) - \tau_{e,i2}(w(t))| &\le \mu_1\ell^{i}\exp(-\mu_2\,\ell\,t) + \frac{r}{\ell^{i}}|\nu(w)|_\infty + \ell^{i}|\delta(w,z)|_\infty
\end{aligned}$$

for any $\ell > \ell^\star$, and for all $i = 1,\dots,n-1$. The proof completes by using the fact that $|\delta(w,z)|_\infty = 0$ and by using

$$\lim_{t\to\infty}|\chi_{11}(t) - c(w(t))| \;\le\; \frac{r}{\ell^d}\sup_{w\in W}|\nu(w)| \qquad \square$$

In view of Theorem 1.7, similarly to the standard high-gain design presented in the Section 1.6.2, the regulator (1.31) with $(\phi(\cdot), \Psi, \gamma(\cdot))$ obtained from (3.36)-(3.38) guarantees asymptotic regulation if the function $\varphi(\cdot)$ fulfils (1.30) with $\nu(\cdot) = 0$. Otherwise just practical regulation is achieved with a bound on the asymptotic error that can be arbitrarily decreased by increasing κ or ℓ^d. With respect to the previous case, however, the remarkable feature of the proposed regulator is that the high-gain parameter ℓ is powered just up to the order 2 by making the design possible even in presence of large values of d. Note, in particular, that the asymptotic gain relating the term $\nu(\cdot)$ to the regulation error is still proportional to $1/\ell^d$ notwithstanding the regulator is implemented only with terms proportional to ℓ and ℓ^2. As a last remark, note that without loss of generality the peaking-free design of Section 3.3 can be adopted in this framework for the functions $(\phi(\cdot), \Psi, \gamma(\cdot))$.

3.6 Simulation results

In this last section we show an example of application of the peaking-free low-power high-gain observer introduced in Section 3.3. We consider as example an uncertain forced harmonic oscillator described as

$$\ddot{s} + \omega^2 s = u(t) , \qquad y = s + \nu(t)$$

where ω is the unknown frequency, $u(t)$ is an input belonging to an open bounded set $\mathcal{U} \subset \mathbb{R}$ and assumed to be known, and y is the measured output. We suppose that the output may be affected by high-frequency measurement noise $\nu(t)$. The purpose is to estimate the state $s, \dot{s}$ and the unknown frequency $\omega > 0$ from the output y. This system can be described in the state space by

$$\begin{array}{rcl} \dot{z}_1 &=& z_2 \\ \dot{z}_2 &=& -z_3 z_1 + u \\ \dot{z}_3 &=& 0 \\ y &=& z_1 + \nu(t) \end{array} \tag{3.41}$$

living in the open bounded subset of $\mathbb{R}^3$

$$\mathcal{Z} \;=\; \left\{(z_1, z_2, z_3) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \;:\; d_1 < z_1^2 + z_2^2 < d_2 \,,\; d_3 < z_3 < d_4\right\}$$

with $0 < d_1 < d_2$ and $0 < d_3 < d_4$, forced by the input $u \in \mathcal{U}$. In the state-space representation (3.41) z_1 coincides with s, z_2 with $\dot{s}$, and z_3 with ω^2. By following the results in Astolfi et al. (2013b), we consider the mappings

$$\varphi_0(z) \;:=\; z_1 \,, \quad \varphi_1(z) \;:=\; z_2 \,, \quad \varphi_2(z, v_0) \;:=\; -z_1 z_3 + v_0 \,, \quad \varphi_3(z, v_1) \;:=\; -z_2 z_3 + v_1 \,,$$

and

$$\Phi_4(z, v_0, v_1) \;:=\; \text{col}\,(\varphi_0, \varphi_1, \varphi_2, \varphi_3) \,.$$

It turns out that the function $\Psi_4 : \Phi_4(\mathcal{Z} \times \mathcal{V}) \to \mathcal{Z}$ defined as

$$\Psi_4(\cdot) \;:=\; \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \dfrac{(v_0 - \varphi_2)\varphi_0 + (v_1 - \varphi_3)\varphi_1}{\varphi_0^2 + \varphi_1^2} \end{pmatrix}$$

is such that

$$z \;=\; \Psi_4(\Phi_4(z, v), v)$$

for any $(z, v) \in \mathcal{Z} \times \mathcal{V}$, with $v = (v_0, v_1)$, and $\mathcal{V} = \mathcal{U} \times \mathbb{R}$. As a consequence we get that the system (3.41) can be immersed, via the mapping

$$T(z) \;:=\; \mathrm{col}\,(z_1\,,\; z_2\,,\; -z_1 z_3\,,\; -z_2 z_3)$$

into the following system in strict feedback form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 + u \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -\varpi(x)\, x_3 + \varpi(x) u \\ y &= x_1 + \nu(t) \end{aligned} \tag{3.42}$$

defined on $\mathbb{R}^4$ and where the function $\varpi : \mathbb{R}^4 \to \mathbb{R}$ is defined as

$$\varpi(x) \;:=\; \frac{x_1 x_3 + x_2 x_4}{\max\{d_1, x_1^2 + x_2^2\}}\,.$$

Numerical simulations can be used to verify that

$$\|x_1(\cdot)\|_\infty < 4\,, \qquad \|x_2(\cdot)\|_\infty < 5.5\,, \qquad \|x_3(\cdot)\|_\infty < 12\,, \qquad \|x_4(\cdot)\|_\infty < 16\,.$$

for any initial condition in the set $\Phi_4(\mathcal{Z}, \mathcal{U}, 0)$ with $d_1 = 0.5$, $d_2 = 2$, $d_3 = 0.5$, $d_4 = 3$ and $\mathcal{U} = (-3, 3)$. By following prescriptions of Section 3.3 we implement the low-power peaking free high-gain observer (3.11) for the system (3.42) as follows

$$\begin{aligned} \dot{\hat{x}}_1 &= \eta_1 + \ell\, c_{11}\,(y - \hat{x}_1) \\ \dot{\eta}_1 &= \mathrm{sat}_3(\eta_2) + u + \ell^2\, c_{12}(y - \hat{x}_1) \\ \dot{\hat{x}}_2 &= \eta_2 + u + \ell\, c_{21}\,(\mathrm{sat}_2(\eta_1) - \hat{x}_2) \\ \dot{\eta}_2 &= \mathrm{sat}_4(\eta_3) + \ell^2\, c_{22}\,(\mathrm{sat}_2(\eta_1) - \hat{x}_2) \\ \dot{\hat{x}}_3 &= \eta_3 + \ell\, c_{31}\,(\mathrm{sat}_3(\eta_2) - \hat{x}_3) \\ \dot{\eta}_3 &= \mathrm{sat}_5(\varpi(\hat{x})\,(u - \hat{x}_3)) + \ell^2\, c_{32}(\mathrm{sat}_3(\eta_2) - \hat{x}_3) \\ \dot{\hat{x}}_4 &= \mathrm{sat}_5(\varpi(\hat{x})\,(u - \hat{x}_3)) + \ell\, c_4(\mathrm{sat}_4(\eta_3) - \hat{x}_4) \\ \hat{\omega} &= \sqrt{|\varpi(\hat{x})|} \end{aligned} \tag{3.43}$$

The coefficients (c_{i1}, c_{i2}), $i = 1, \ldots, 4$ are chosen as in Table A.1 (see the case $n = 4$) and $c_4 = c_{31}$ and the saturations level are fixed to $r_1 = 5$, $r_2 = 6$, $r_3 = 14$, $r_4 = 18$, $r_5 = 55$.

We compared the observer (3.43) with a standard high-gain observer designed as

$$\begin{array}{rcl} \dot{\hat{x}}_1 & = & \hat{x}_2 + \ell\, c_1\,(y - \hat{x}_1) \\ \dot{\hat{x}}_2 & = & \hat{x}_3 + u + \ell^2\, c_2\,(y - \hat{x}_1) \\ \dot{\hat{x}}_3 & = & \hat{x}_4 + \ell^3\, c_3\,(y - \hat{x}_1) \\ \dot{\hat{x}}_4 & = & \mathrm{sat}_5(\varpi(\hat{x})(u - \hat{x}_3)) + \ell^4\, c_4(y - \hat{x}_1) \\ \hat{\omega} & = & \sqrt{|\varpi(\hat{x})|} \end{array} \qquad (3.44)$$

with $c_1 = 5.99$, $c_2 = 13.1778$, $c_3 = 12.6034$, $c_4 = 4.4156$ so that the roots of $\lambda^4 + c_1\lambda^3 + c_2\lambda^2 + c_3\lambda + c_4$ are in $(-1, -1.33, -1.66, -2)$.

In the simulation the initial conditions of the system (3.41) are $(s(0), \dot{s}(0)) = (1, 0)$ and $\omega = 1.58$ (namely $z_3 = 2.5$). The the time-varying signal $u(t)$ is generated as

$$\ddot{p} + p + 20p^3 = 0\ , \qquad (p(0), \dot{p}(0)) = (1, 0), \qquad u(t) = \dot{p}(t)\ ,$$

and the initial conditions of the two observers coincide with the origin. First we simulated the nominal behaviours of the two observer in absence of meausrement noise, namely $\nu(t) = 0$. The Table 3.1 shows the maximum peaking values of the state $(\hat{x}, \eta)$ of the low-power peaking free high-gain observer (3.43) and the time needed to convergence to an error sufficiently small, *i.e.* the time T_ϵ such that

$$\sqrt{|s(t) - \hat{x}_1(t)|^2 + |\dot{s}(t) - \hat{x}_2(t)|^2} \ < \ \epsilon \qquad \forall\, t \geq T_\epsilon\,,$$

for different values of ℓ. The Table 3.1 shows the maximum peaking values of the state $\hat{x}$ of the standard high-gain observer (3.44) and the time needed to convergence to an error sufficiently small for the same values of ℓ. Then we simulated the behaviour of the two observers when the measurement noise is chosen as $\nu(t) = 0.05\sin(200t)$ and the high-gain parameter is fixed to $\ell = 10$. The Figure 3.4 and the Table 3.3 compare the the low-power peaking free high-gain observer (3.43) and the standard high-gain observer (3.44) showing the transient and the asymptotic norm of the errors.

From the tables it can be seen that despite the eigenvalues of the low-power high-gain observer (3.43) and the standard high-gain observer (3.44) are in the same range (namely between -1 and -2) the rate of convergence of the observer (3.44) is faster when the gain ℓ is not to high. This phenomenon is caused by the largest dimension of the low-power construction. On the contrary, when the high-gain parameter is very large, the effect of the saturations helps in achieving faster convergence. With the novel observer we obtain numerical advantages (we do not have to implement a term ℓ^4 but only ℓ^2), the peaking phenomenon is completely overcome and it can be noticed a remarkable improvement of the sensitivity to high-frequency measurement noise.

Low-power peaking-free high-gain observer

	$\ell = 5$	$\ell = 10$	$\ell = 100$	$\ell = 1000$
$T_{0.01}$	4.154	1.437	0.062	0.009
$\|\hat{x}_1(\cdot)\|_\infty$	1.46	1.46	1.48	1.57
$\|\hat{x}_2(\cdot)\|_\infty$	5.05	5.55	5.79	6.26
$\|\hat{x}_3(\cdot)\|_\infty$	6.52	6.52	6.41	6.49
$\|\hat{x}_4(\cdot)\|_\infty$	9.95	15.1	13.9	14.0
$\|\eta_1(\cdot)\|_\infty$	6.62	12.9	128	1308
$\|\eta_2(\cdot)\|_\infty$	14.0	27.0	443	5080
$\|\eta_3(\cdot)\|_\infty$	9.15	16.1	158	1727

Table 3.1: Behaviour of the low-power peaking-free high-gain observer (3.43) when $\nu(t) = 0$ for different values of ℓ.

Standard high-gain observer

	$\ell = 5$	$\ell = 10$	$\ell = 100$	$\ell = 1000$
$T_{0.01}$	2.265	0.934	0.122	0.015
$\|\hat{x}_1(\cdot)\|_\infty$	1.46	1.46	1.51	1.56
$\|\hat{x}_2(\cdot)\|_\infty$	9.11	18.4	193	$1.89 \cdot 10^3$
$\|\hat{x}_3(\cdot)\|_\infty$	38.3	153	$1.63 \cdot 10^4$	$1.56 \cdot 10^6$
$\|\hat{x}_4(\cdot)\|_\infty$	65.1	504	$5.37 \cdot 10^5$	$5.09 \cdot 10^8$

Table 3.2: Behaviour of the standard high-gain observer (3.44) when $\nu(t) = 0$ for different values of ℓ.

Low-power peaking free high-gain observer	Standard high-gain observer
$\|\hat{x}_1(\cdot) - x_1(\cdot)\|_a = 0.016$	$\|\hat{x}_1(\cdot) - x_1(\cdot)\|_a = 0.016$
$\|\hat{x}_2(\cdot) - x_2(\cdot)\|_a = 0.024$	$\|\hat{x}_2(\cdot) - x_2(\cdot)\|_a = 0.360$
$\|\hat{\omega}(\cdot) - \omega\|_a = 0.034$	$\|\hat{\omega}(\cdot) - \omega\|_a = 2.406$

Table 3.3: Comparison between the behaviours of low-power peaking-free high-gain observer (3.43) and the standard high-gain observer (3.44) with $\ell = 10$ and measurement noise chosen as $\nu(t) = 0.05\sin(200t)$.

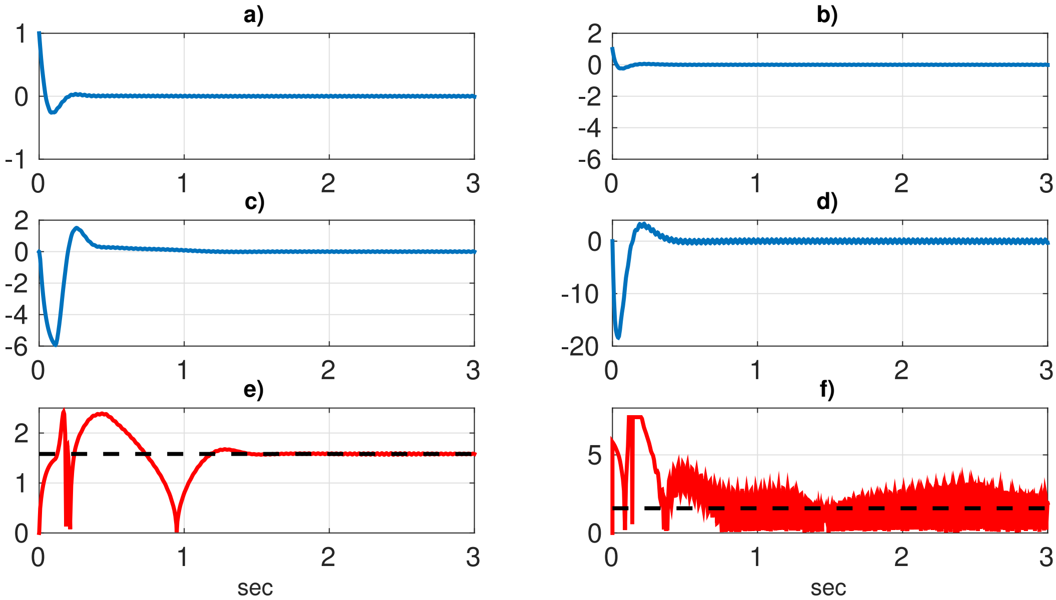

Figure 3.4: Comparison between the behaviours of low-power peaking-free high-gain observer (3.43) and the standard high-gain observer (3.44) with $\ell = 10$ and measurement noise chosen as $\nu(t) = 0.05\sin(200t)$. Plot **a)**: behaviour of $e_1(t) = \hat{x}_1(t) - s(t)$ for the low-power high-gain observer (3.43). Plot **b)**: behaviour of $e_1(t) = \hat{x}_1(t) - s(t)$ for the standard high-gain observer (3.44). Plot **c)**: behaviour of $e_2(t) = \hat{x}_2(t) - \dot{s}(t)$ for the low-power high-gain observer (3.43). Plot **d)**: behaviour of $e_2(t) = \hat{x}_2(t) - \dot{s}(t)$ for the standard high-gain observer (3.44). Plot **e)**: behaviour of $\hat{\omega}$ (red line) for the low-power high-gain observer (3.43) and value of ω (black line). Plot **f)**: behaviour of $\hat{\omega}$ (red line) for the standard high-gain observer (3.44) and value of ω (black line).

Conclusion

The first part of the dissertation has been devoted to the design of nonlinear observers which relies on high-gain techniques. The contribution of this work is twofold: on one hand the development of techniques which allow the design of a high-gain observer in the original coordinates; on the other hand the introduction of a new design methodology of fast estimation based on dynamic extension and that allows a "low-power" implementation.

Chapter 2 is mainly devoted to the design of high-gain observers in the original coordinates. This approach is particularly useful in those contexts in which the computation of the triangular coordinates (*feedback form*), in which standard high-gain observers may be applied, is not trivial. Though these coordinates are guaranteed to exist under an observability assumption, the computation of this observability form may be extremely complicated. The design of the observer in the original coordinates is made possible by modifying its dynamics so that its trajectories remain in some desired compact set. If the trajectories of the observed system remain in the same compact set, asymptotic fast estimation is achieved. The main restriction of the proposed approach is a convexity assumption which is necessary if we want to preserve the high-gain paradigm. The same technique is then applied to multi-input multi-output nonlinear systems. A novel set of sufficient conditions for the existence of an observer in the original coordinates is given. With respect to other conditions studied in literature, the proposed ones may be verified in the original coordinates and the computation of the inverse of a nonlinear mapping is not needed.

In Chapter 3 a novel *low-power* methodology for the design of high-gain observers is introduced. The main motivations of the new structure is that of overcoming the drawbacks which make questionable the use of high-gain observers in practical applications. The new class of low-power high-gain observers may be applied to all those contexts where standard high-gain observers apply. They can be used for the estimation of systems in *feedback form* and they are characterized by having a high-gain term which is powered up to two regardless the dimension of the system, introducing evident numerical benefits in the implementation. By using dynamic extension and saturations it is possible to design an exponential tunable convergent observer which guarantees the same performances of the standard high-gain observer, and moreover prevents the peaking phenomenon, beside the aforementioned numerical advantages. Furthermore, with respect to high-gain observers, the relative degree between the measure and the estimates is augmented thus reflecting in better sensitivity properties in presence of high-frequency measurement noise. The main drawback of the new class of observer is that of having a dimension which is larger than the system dimension. An application to the output regulation framework by means of high-gain tools is proposed.

Finally, two novel analysis tools are proposed. A new set of sufficient conditions under which a single-input single-output nonlinear systems can be put in strict-feedback coordinates is proposed (see Section 2.1). In the new coordinates the systems is characterized by being affine with respect to the derivative of the input. This transformations make easier the design of the observer and can be applied when the derivative of the control input is known (for instance, when a step of *backstepping* is made).

The second analysis tool introduced in this work is the analysis of the steady-state behaviour of the observer in presence of high-frequency measurement noise. This methodology allows to catch the low-pass filter properties of the observer that cannot be shown by a $\mathcal{H}_\infty$ analysis. The novel analysis is successfully applied to standard high-gain observers (Section 2.4) and to the new class of low-power high-gain observers (Section 3.4).

Part II

Robust Regulation

"Far better an approximate answer to the right question, which is often vague, than an exact answer to the wrong question, which can always be made precise."

John Tukey

4

Structurally Robust Output Regulation

The problem of regulating desired outputs to some references while rejecting other external signals is generically known as output regulation problem. Regulation in the linear (multi-input multi-output) framework has been completely solved by Francis and Wonham (1976) during the 70's. In this contribution the authors made also clear what is the *internal model principle*, *i.e.* the fact that output regulation property is insensitive to plant parameter variations "only if the controller utilizes feedback of the regulated variable, and incorporates in the feedback path a suitably reduplicated model of the dynamic structure of the exogenous signals which the regulator is required to process".

Regulation in the nonlinear case, however, is somehow still an open problem due to the difficulties of extending the linear paradigm to a more general framework. Equivalent formulation of the regulation problem and the internal model principle in the nonlinear case has been developed during the 80's and especially in the 90's by many authors (see, among the others, Byrnes et al. (1997)). A breakthrough in the direction of solving the problem of output regulation for uncertain nonlinear systems was the crucial observation made in Khalil (1992) (and independently in Huang and Lin (1993), Delli Priscoli (1993) and Delli Priscoli (1997)) that internal models must not only be able to generate inputs corresponding to the trajectories of the system, but also a number of *higher order nonlinear deformations*. For example, in the case of a cubic nonlinearity with unknown coefficient and sinusoidal reference output, the internal model must generate the sinusoid in question and its *third* harmonic. In particular Huang and Lin, appealing to concept of "regulation of order k" (namely, regulation up to a steady-state error which is infinitesimal of order k with respect to the amplitude of the disturbance input) introduced in an earlier paper Huang and Rugh (1992), provided in Huang and Lin (1994b) a methodology for the design of a controller which, regardless of small parameter perturbations, achieves regulation of order k. This methodology was proven Huang and Lin (1993) to yield exact regulation, regardless of small parameter variations, for some relevant classes of nonlinear systems, by designing an internal model which generates all the exogenous inputs as well all higher harmonics, up to order k. By denoting the "*friend*" as the right steady-state input which makes the regulated output constantly equal to zero, Delli Priscoli (1993) arrived at the similar conclusion that structurally stable regulation is possible if the family of all possible friends (which depends on a certain set of plant-parameters) can be seen as a subset of the set of all possible solutions of a fixed ordinary differential equation. More details can be found in Byrnes et al. (1997) where a survey on the problem is given.

Inspired by the design and analysis philosophy recently proposed in Poulain and Praly (2010), we propose a novel method to solve the problem of structurally robust output regulation in presence of periodic disturbances. We show that the proposed methodology is robust in the sense of Francis and Wonham (1976), namely (asymptotic/approximate) output regulation is achieved in presence of uncertainties or disturbances, as long as the resulting closed-loop system has bounded trajectories. The result is based on the property that a nonlinear ISS (Input-to-State Stable) system driven by a periodic input admits periodic solutions of the same period (see Agrachev et al. (2007)) and the analysis is strongly driven by Fourier analysis.

This chapter is organized as follows. In Section 4.1 we give some highlights on the linear robust output regulation problem. A constructive design based on forwarding techniques is given. The linear case is instrumental to the main results of the subsequent sections where the same forwarding philosophy is applied to nonlinear systems and in particular to the class of input-affine multi-input multi-output (possibly non-square) nonlinear systems. After introducing the main ideas (Section 4.2) we deal with the design of the internal model (Section 4.3) and the stabilizer (Section 4.4) and we show the main results on *structurally robust output regulation*. The results given in this chapter hold in case of small-disturbances.

4.1 Robust output regulation for linear systems via forwarding

The problem of output regulation in the linear framework has been completely solved in the 70's in the celebrated work "The internal model principle of control theory" of Francis and Wonham Francis and Wonham (1976). In the first part of this section, we make a review of some of the most relevant facts. The presentation is strongly inspired by (Byrnes et al., 1997, Chapter 1) and (Isidori et al., 2003, Sections 1.3, 1.4). Then, in the second part of this section we provide a constructive solution for the robust output regulation problem based on forwarding techniques with a certain number of technical lemmas that, as far as we know, are novel in the literature of linear output regulation.

The linear framework

Consider a linear system of the form

$$\begin{aligned} \dot{x} &= Ax + Bu + Pw \\ e &= Cx + Qw \end{aligned} \tag{4.1}$$

with the state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$ and regulated output $y \in \mathbb{R}^p$. In order to have a well-posed problem we suppose the number m of inputs is larger or equal than the number p of regulated output (see, for instance, Isidori et al. (2003) for further details), *i.e.* $m \geq p$. The plant (4.1) is affected by an exogenous signal $w \in \mathbb{R}^r$ which may represent disturbances to reject or references to track and it is generated by an autonomous exosystem of the form

$$\dot{w} = Sw\,. \tag{4.2}$$

We refer to the problem of *output regulation* as the problem of finding an output feedback law

$$\begin{aligned} \dot{\xi} &= F\xi + Gy \\ u &= H\xi \end{aligned} \tag{4.3}$$

such that

(a) the equilibrium $(x, \xi) = (0, 0)$ of the *unforced closed-loop system*

$$\begin{aligned} \dot{x} &= Ax + BH\xi \\ \dot{\xi} &= F\xi + GCx \end{aligned}$$

is asymptotically stable

(b) the trajectories of the *forced closed-loop system*

$$\begin{aligned}\dot{w} &= Sw \\ \dot{x} &= Ax + BH\xi + Pw \\ \dot{\xi} &= F\xi + GCx + GQw\end{aligned} \tag{4.4}$$

are bounded and such that

$$\lim_{t\to\infty} e(t) = 0$$

for every initial condition $(x(0), \xi(0), w(0))$.

The problem of *robust output regulation* for linear plants subject to parameter uncertainties can be setted in the following framework: the set of matrices $\{A, B, C, P, Q\}$ which characterize (4.1) can be viewed as an element of a *space of parameters*

$$\mathcal{P} = \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{n\times r} \times \mathbb{R}^{p\times r} .$$

As a consequence, uncertainties on the values of these parameters can be simply expressed by allowing the set of uncertain parameters $\{A, B, C, P, Q\}$ to range on a given neighbourhood $\mathcal{P}_0$ of a fixed element $\{A_0, B_0, C_0, P_0, Q_0\}$ of $\mathcal{P}$. In this set-up, we refer to the problem of *robust output regulation* as the problem of finding a feedback law of the form (4.3) such that

(i) the requirements (a) and (b) are satisfied for the plant characterized by the nominal set of parameters $\{A_0, B_0, C_0, P_0, Q_0\}$,

(ii) the requirements (a) and (b) are satisfied for each perturbed set of parameters $\{A, B, C, P, Q\}$.

The problem of robust output regulation can be solved under a certain number of assumptions, some of which are trivially necessary and some of which can be proven to be necessary if certain additional design goals are to to be obtained.

Assumption 4.1. *The pair (A, B) is stabilizable and the pair (C, A) is detectable.*

This is a very first well-known necessary and sufficient conditions for the existence of matrices F, G, H such that the closed-loop matrix

$$J = \begin{pmatrix} A & BH \\ GC & F \end{pmatrix}$$

has all eigenvalues with negative real part. Thus, this is a necessary condition for the fulfilment of requirement (a) of the output regulation problem and need not to be discussed further.

Assumption 4.2. *The exosystem* (4.2) *is neutrally stable.*

We refer to *neutral stability* as the fact that all eigenvalues of S have zero real part and multiplicity one in the minimal polynomial. As a consequence, in suitable coordinates, S can be always be expressed as a skew-symmetric matrix. If this assumption holds, all trajectories of the exosystem (4.2) are bounded in backward and forward time and none of them decays to zero as $t \to \infty$. Boundedness of the trajectories of w guarantees that, if requirement (a) of the design problem is fulfilled, then, for any $(x(0), \xi(0), w(0))$ the trajectories of the forced closed-loop system (4.4) are bounded, since $x(t), \xi(t)$ can be viewed as the response of an asymptotically stable linear system to a bounded input. The non-existence of trajectories of (4.2) which decay to zero as $t \to \infty$ on the other hand, singles-out non-interesting trajectories $w(t)$ for which the fulfilment of requirement (b) would be trivially implied by the fulfilment of requirement (a).

As a very first trivial result we have the forthcoming lemma which provides a basic necessary condition for the existence of solutions to the output regulation problem.

Lemma 4.1 (Byrnes et al. (1997), Corollary 1.5)**.** *Consider the plant* (4.1) *with exosystem* (4.2) *and suppose Assumption 4.2 holds. There exists a controller which solves the problem of output regulation only if there exists matrices* Π *and* Ψ *satisfying*

$$\begin{array}{rcl} \Pi S & = & A\Pi + B\Psi + P \\ 0 & = & C\Pi + Q \end{array} \tag{4.5}$$

We usually refer to (4.5) as the regulator equations. In this context $x = \Pi w$ represents the right-steady state on which the regulated output is zero, whereas $u = \Psi w$, usually referred as the "*friend*", represents the right steady-state of the input.

As a consequence of Lemma 4.1, the regulator equations (4.5) must have a solution for any perturbed set of parameters $\{A, B, C, P, Q\}$ in $\mathcal{P}_0$. In particular, the equations

$$\begin{array}{rcl} \Pi S & = & A_0\Pi + B_0\Psi + P \\ 0 & = & C_0\Pi + Q \end{array} \tag{4.6}$$

must have a solution for every P, Q such that $\{A_0, B_0, C_0, P, Q\}$ is in $\mathcal{P}_0$. But since the set of matrices $\{A_0, B_0, C_0, P_0, Q_0\}$ is an interior point of $\mathcal{P}_0$ and (4.6) are linear equations, it follows the that the equations in question must have a solution for all P, Q. This observation lends itself to the characterization of a basic necessary condition for the existence of a solution to the robust output regulation problem. To this end, it suffices to recall the following important result (see, for instance, Byrnes et al. (1997)) about the linear matrix equations of the form (4.6).

Lemma 4.2 (Byrnes et al. (1997), Proposition 1.6)**.** *The linear equations* (4.6) *have a solu-*

tion Π, Ψ *for all* P, Q *if and only if the matrix*

$$\begin{pmatrix} A_0 - \lambda I & B_0 \\ C_0 & 0 \end{pmatrix} \tag{4.7}$$

has independent rows for each λ *which is an eigenvalue of* S.

Motivated by this result, it is completely natural to add the following new assumption, usually known as *non-resonance condition*.

Assumption 4.3. *The matrix*

$$\begin{pmatrix} A - \lambda I & B \\ C & 0 \end{pmatrix}$$

has independent rows for each λ *which is an eigenvalue of* S.

We remark that the previous assumption is asking that none of the eigenvalues of S the exosystem (4.2) is a *transmission zero* of the unforced open-loop system (4.1), *i.e.*

$$\begin{aligned} \dot{x} &= Ax + Bu \\ e &= Cx \end{aligned}$$

It has been proved (see Francis and Wonham (1976), Byrnes et al. (1997)) that under the previous assumptions *the robust output regulation problem* does have a solution. In the forthcoming section we provide a constructive design based on forwarding techniques.

The internal model principle and forwarding design

The solution to the *robust output regulation problem* leads to the celebrated *internal model principle*, claiming that the controller (4.3) solving the problem necessarily embeds suitable copies of the exosystem. In rough words the recipe for designing the controller follows the following two steps: first add p copies of the exosystem processing the errors, one for each error e, which represent the internal model of the exosystem and then stabilize the resulting cascade system with $w = 0$. In more precise term, the dynamic regulator is consisting of two parts, as shown in Figure 4.1:

(1) an *internal model unit* which processes the regulated output e;

(2) a *stabilizer unit* which stabilizes the cascade system *plant - internal model unit*.

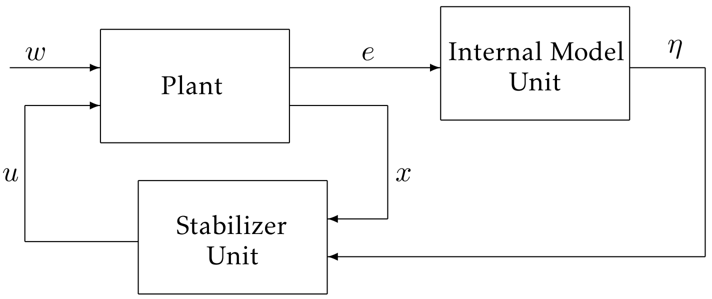

Figure 4.1: Robust Output Regulation Scheme

The internal model unit can be designed as

$$\dot{\eta} \;=\; \Phi\eta + \Gamma e$$

with $\eta \in \mathbb{R}^{r\times p}$, where (Φ, Γ) is a controllable pair and Φ is a "copy" of the matrix S. For instance we can select

$$\Phi = \begin{pmatrix} 0 & I_p & 0 & \cdots & 0 \\ 0 & 0 & I_p & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & & \cdots & I_p \\ -s_0 I_p & -s_1 I_p & -s_2 I_p & \cdots & -s_{r-1} I_p \end{pmatrix}_{(r\times p)\times(r\times p)} \qquad \Gamma \;=\; \begin{pmatrix} I_p \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{(r\times p)\times p}$$

where the real numbers $s_0, \ldots, s_{r-1}$ denote the coefficients of the minimal polynomial of the matrix S.

Once the *internal model unit* has been designed, we have to design a *stabilizer unit* for the *unforced extended open-loop system*

$$\begin{aligned} \dot{x} &= Ax + Bu \\ \dot{\eta} &= \Phi\eta + \Gamma C x \end{aligned} \tag{4.8}$$

Considering for the time begin the state x is measured, we are left with design a static state feedback law of the form

$$u \;=\; \mathcal{K}x + \mathcal{L}\eta$$

with $\mathcal{K}$ and $\mathcal{L}$ chosen such that the *unforced closed-loop system*

$$\begin{aligned} \dot{x} &= (A + B\mathcal{K})x + B\mathcal{L}\eta \\ \dot{\eta} &= \Phi\eta + \Gamma C x \end{aligned}$$

is asymptotically stable. Existence of such matrices $\mathcal{K}$, $\mathcal{L}$ is always guaranteed under the Assumptions 4.1, 4.3 when the numbers of inputs m is larger than the number of outputs p. To go from the state feedback above to the desired output feedback, we can design an observer of the form

$$\dot{\hat{x}} \;=\; A\hat{x} + Bu + L(y - C\hat{x})$$

and then implement the feedback law $u = \mathcal{K}\hat{x} + \mathcal{L}\eta$.

Though the design of the matrices $\mathcal{K}$ and $\mathcal{L}$ can be done by a pole - assignment, we are interested here into a methodology which may be extended to the nonlinear framework. For this, observer that system (4.8) is in feedforward form. As a consequence a possible strategy consists in adopting forwarding technique to design the *stabilizing unit*. For this, the following additional assumption is supposed.

Assumption 4.4. *The matrix A is Hurwitz.*

Remark *Note that the previous Assumption is not restrictive at all. Indeed, if A is not Hurwitz, under Assumption 4.1 it is possible to design a preliminary state feedback $u = Kx + v$ with K chosen so that the matrix $\bar{A} = (A + BK)$ is Hurwitz. Then, in place of* (4.8), *we consider the extended system*

$$\begin{aligned} \dot{x} &= \bar{A}x + Bv \\ \dot{\eta} &= \Phi x + \Gamma C x \end{aligned}$$

Recall that stabilizability and detectability are preserved by a state-feedback. Furthermore, a state-feedback does not change the zero-transmission of a system. We conclude that, under Assumptions 4.1 and 4.3, the pair $(\bar{A}, B)$ is stabilizable, the pair $(C, \bar{A})$ is detectable, and the non-resonance condition of the triplet $(\bar{A}, B, C,)$ holds. As a consequence, without loss of generality, in the following we will refer to the matrix A whether or not this preliminary state feedback has been designed.

With the previous assumption in mind, let the matrix M be defined as the solution of

$$MA \;=\; \Phi M + \Gamma C \tag{4.9}$$

Note that since the intersection of the spectrum of A (which is Hurwitz) and Φ (which is neutrally stable) is empty, the matrix M is uniquely defined. Now consider the Lyapunov function

$$V = x^\top R x + (\eta - Mx)^\top T(\eta - Mx)$$

where the matrices $R = R^\top > 0, T = T^\top > 0$ are solutions of

$$RA + A^\top R \;=\; -I\,, \qquad T\Phi + \Phi^\top T \;=\; 0\,.$$

Note that the matrix Φ can always be expressed in suitable coordinates as a skew-symmetric matrix. As a consequence, if Φ is a skew-matrix, T can be always chosen as the identity matrix. In the following, for the sake of simplicity, we will make this choice, recalling that $\Phi + \Phi^\top = 0$. Hence, the derivative of V is given by

$$\begin{aligned}
\dot V &= 2x^\top R(Ax + Bu) + 2(\eta - Mx)^\top(\Phi\eta + \Gamma Cx - M(A + Bu))\\
&= -|x|^2 + 2x^\top RBu + 2(\eta - Mx)^\top(\Phi\eta - \Phi Mx + \Phi Mx + \Gamma Cx - MAx - MBu)\\
&= -|x|^2 + 2x^\top RBu + 2(\eta - Mx)^\top[\Phi(\eta - Mx) - MBu]\\
&= -|x|^2 + 2(x^\top R - (\eta - Mx)^\top M)Bu
\end{aligned}$$

and by taking

$$u \;=\; -B^\top Rx + B^\top M^\top(\eta - Mx) \tag{4.10}$$

we get

$$\dot V \;=\; -|x|^2 - u^\top u\,.$$

By using La Salle's arguments, we can prove that the state of the system (4.8) in closed loop with (4.10) converges to the set

$$\{(x,\eta) \in \mathbb{R}^n \times \mathbb{R}^{r\times p} \;:\; x = 0,\; u = 0\} \;=\; \{0\} \times \{B^\top M^\top \eta = 0\}$$

The forthcoming lemma gives a sufficient condition for the asymptotically stability of the closed-loop system (4.8)-(4.10).

Lemma 4.3. *Under Assumption 4.4 the unforced closed-loop system*

$$\begin{aligned}
\dot x &= Ax + Bu\\
\dot\eta &= \Phi\eta + \Gamma Cx\\
u &= -B^\top Rx + B^\top M^\top(\eta - Mx)
\end{aligned}$$

is asymptotically stable if the pair $(B^\top M^\top, \Phi)$ is observable.

Proof. By using the previous arguments we can prove that x converges to zero and that η converges to the set

$$B^\top M^\top \eta \;=\; 0$$

where its dynamics reduces to

$$\dot{\eta} \;=\; \Phi\eta\,.$$

Since the pair $(B^\top M^\top, \Phi)$ is observable, $\eta(t)$ identically zero for all $t \geq 0$ is the only admissible solution. This prove that the solutions converge to the origin and therefore the closed-loop system is asymptotically stable. □

By using the previous lemma we can see that the solution to output regulation problems boils down in proving that the pair $(B^\top M^\top, \Phi)$ is observable. As shown in the forthcoming Lemma, this is always guaranteed under the non-resonance conditions.

Lemma 4.4. *Under Let M be solution of* (4.9). *If Assumptions 4.3 and 4.4 hold, the pair $(B^\top M^\top, \Phi)$ is observable.*

Proof. Let $-\lambda$ be an eigenvalue of Φ and let v be its associated eigenvector, *i.e.* $-\lambda v = \Phi v$. Since Φ is skew-symmetric also λ is an eigenvalue of Φ. Furthermore, $\Phi = -\Phi^\top$. As a consequence

$$(-\lambda v)^\top = (\Phi v)^\top = v^\top \Phi^\top = -v^\top \Phi \qquad \Longrightarrow \qquad \lambda v^\top = v^\top \Phi$$

Now let pre-multiply equation (4.9) by $v^\top$. We get

$$v^\top M A \;=\; v^\top \Phi M + v^\top \Gamma C \qquad \Longrightarrow \qquad v^\top M(\lambda I - A) + v^\top \Gamma C \;=\; 0\,.$$

Let assume that v is the in right-kernel of $M^\top B^\top$, *i.e.*

$$B^\top M^\top v = 0 \qquad \Longrightarrow \qquad v^\top M B = 0\,.$$

By collecting the previous relations we get

$$\begin{pmatrix} v^\top M & v^\top \Gamma \end{pmatrix} \begin{pmatrix} \lambda I - A & B \\ C & 0 \end{pmatrix} = 0$$

But this contradicts the Assumption 4.3. As a consequence there is non zero vector v satisfying

$$\begin{pmatrix} \lambda I - \Phi \\ B^\top M^\top \end{pmatrix} v = 0$$

and therefore the PBH observability test

$$\operatorname{rank} \begin{bmatrix} \lambda I - \Phi \\ B^\top M^\top \end{bmatrix} \;=\; n \qquad \forall\, \lambda \in \sigma(\Phi)$$

where $\sigma(\Phi)$ denotes the spectrum of Φ, is satisfied, concluding the proof. □

The observability of the pair $(B^\top M^\top, \Phi)$ is strictly connected to the existence of the regulator equations (4.5), as shown in the forthcoming lemma.

Lemma 4.5. *Under the Assumption 4.4, the pair* $(B^\top M^\top, \Phi)$ *is observable if and only if there exist matrices* Π, Ψ *solution of the regulator equations* (4.5) *for any matrices* P, Q.

Proof. The proof is divided in two part. First we prove that if the pair $(B^\top M^\top, \Phi)$ is observable then the regulator equations do exist. Then, we prove the converse. In the following we denote with J the matrix

$$J \;=\; \begin{pmatrix} (A - BB^\top R - BB^\top M^\top M) & BB^\top M^\top \\ \Gamma C & \Phi \end{pmatrix} . \tag{4.11}$$

By using Lemma 4.3, if the pair $(B^\top M^\top, \Phi)$ is observable then the matrix J defined is Hurwitz. As a consequence, since the spectrum of S and J are distinct, there always exist Π and Σ solution of the Sylvester equation

$$\begin{pmatrix} \Pi \\ \Sigma \end{pmatrix} S \;=\; J \begin{pmatrix} \Pi \\ \Sigma \end{pmatrix} + \begin{pmatrix} P \\ \Gamma Q \end{pmatrix}$$

The second equation in particular reads

$$\Sigma S \;=\; \Phi\Sigma + \Gamma(C\Pi + Q) \,.$$

By noting that S and Φ have the same eigenvalues and that the pair (Φ, Γ) is controllable we get $C\Pi + Q = 0$ (see Theorem 1.7 and pages 24-26 of Byrnes et al. (1997)). The first part of the proof concludes by setting

$$\Psi = -B^\top(R + M^\top M^\top)\Pi + B^\top M^\top \Sigma \,.$$

Now assume the regulator equations (4.5) exist. By pre-multiplying by M the first equation we get

$$\begin{aligned} M\Pi S &= MA\Pi + MB\Psi + MP \\ 0 &= C\Pi + Q \end{aligned}$$

By using (4.9) we get

$$\begin{aligned} M\Pi S &= (\Phi M + \Gamma C)\Pi + MB\Psi + MP \\ 0 &= C\Pi + Q \end{aligned}$$

and therefore, by multiplying the second equation by Γ we get

$$(M\Pi)S \;=\; \Phi(M\Pi) + (MP - \Gamma Q + MB\Psi)\ . \tag{4.12}$$

Now let $-\lambda$ be an eigenvector of Φ and suppose the pair $(B^\top M^\top, \Phi)$ is not observable, namely there exists v satisfying

$$\Phi v = -\lambda v\ , \qquad B^\top M^\top v = 0\ ,$$

and therefore $v^\top \Phi = \lambda v^\top$. Since Φ and S have the same spectrum, there exists a w satisfying

$$Sw = \lambda w\,.$$

As a consequence, by pre-multiplying (4.12) by $v^\top$ and by post-multiplying (4.12) by w we get

$$\begin{array}{rcl} v^\top(M\Pi)Sw &=& v^\top\Phi(M\Pi)w + v^\top(MP - \Gamma Q + MB\Psi)w \\ v^\top(M\Pi)\lambda w &=& \lambda v^\top(M\Pi)w + v^\top(MP - \Gamma Q + MB\Psi)w \end{array} \tag{4.13}$$

and therefore

$$v^\top(MP - \Gamma Q + MB\Psi)w \;=\; 0\ .$$

Since v, w, M, Γ does not depend on P and Q, this contradicts the fact that the regulator equations do have a solution for any pair of matrices P, Q. □

By collecting all the previous results we have finally the following theorem.

Theorem 4.1. *Consider the nominal plant*

$$\begin{array}{rcl} \dot{x} &=& A_0x + B_0u + P_0w \\ e &=& C_0x + Q_0w \end{array}$$

and suppose Assumption 4.2 holds, the pair (A_0, B_0) is stabilizable, the pair (C_0, A_0) is detectable and the non resonance condition (4.7) *holds. Then the regulator*

$$\begin{array}{rcl} \dot{\eta} &=& \Phi\eta + \Gamma e \\ u &=& -B_0^\top R_0 x + B_0^\top M_0^\top(\eta - Mx) \end{array}$$

with M_0, R_0 solution of

$$M_0A_0 \;=\; \Phi M_0 + \Gamma C_0\ , \qquad R_0A_0 + A_0^\top R_0 \;=\; -I\ ,$$

solves the robust output regulation problem.

Proof. By collecting the results of Lemma 4.3 and 4.4 we know that the matrix J_0 defined as

$$J_0 = \begin{pmatrix} (A_0 - B_0 B_0^\top R_0 - B_0 B_0^\top M_0^\top M_0) & B_0 B_0^\top M_0^\top \\ \Gamma C_0 & \Phi \end{pmatrix}$$

is Hurwitz. As a consequence there exists Π_0, Σ_0 solution of

$$\begin{pmatrix} \Pi_0 \\ \Sigma_0 \end{pmatrix} S = J_0 \begin{pmatrix} \Pi_0 \\ \Sigma_0 \end{pmatrix} + \begin{pmatrix} P_0 \\ \Gamma Q_0 \end{pmatrix}$$

The second equation in particular reads

$$\Sigma_0 S = \Phi \Sigma_0 + \Gamma (C_0 \Pi_0 + Q_0) \, .$$

By noting that S and Φ have the same eigenvalues and that the pair (Φ, Γ) is controllable we get $C_0 \Pi_0 + Q_0 = 0$ (see Theorem 1.7 and pages 24-26 of Byrnes et al. (1997)). Now, by applying the change of coordinates-

$$x \mapsto \tilde{x} \, , \qquad \tilde{x} := x - \Pi_0 w \, , \qquad \eta \mapsto \tilde{\eta} \, , \qquad \tilde{\eta} := \eta - \Sigma_0 w$$

the closed-loop system reads

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{\eta}} \end{pmatrix} = J_0 \begin{pmatrix} \tilde{x} \\ \tilde{\eta} \end{pmatrix}, \qquad e = C_0 \tilde{x} \, .$$

Since J_0 is Hurwitz the system is stable. As a consequence

$$\lim_{t \to \infty} \tilde{x}(t) = 0 \, , \quad \lim_{t \to \infty} \tilde{\eta}(t) = 0 \, , \qquad \Longrightarrow \qquad \lim_{t \to \infty} e(t) = \lim_{t \to \infty} C_0 \tilde{x}(t) = 0 \, ,$$

by which we conclude the output regulation problem is solved. Finally, in order to prove the robustness, we define $\mathcal{P}_0$ as the neighbourhood of $(A_0, B_0, C_0, P_0, Q_0)$ such that $(A, B, C, P, Q) \in \mathcal{P}_0$ satisfies Assumptions 4.1 4.3, 4.4, and the matrix J defined as

$$J = \begin{pmatrix} (A - B B_0^\top R_0 - B B_0^\top M_0^\top M_0) & B B_0^\top M_0^\top \\ \Gamma C_0 & \Phi \end{pmatrix} \, .$$

is Hurwitz. By applying the same arguments it is possible to prove that $\lim\limits_{t \to \infty} e(t) = 0$.
□

4.2 The issue of structural robustness in the nonlinear case

As shown in Section 4.1, the solution of the robust output regulation problem in the linear framework is made of two main ingredients (see also Figure 4.1):

(1) an *internal model unit* which processes the regulated output e;

(2) a *stabilizer unit* which stabilizes the cascade system *plant-internal model unit*.

In this regard, we want to point out a substantial difference in the *internal model unit* design between the approach used in the linear framework and the one commonly adopted in the nonlinear framework. In the linear case (see Section 4.1) the internal model is designed as a copy of the exosystem thus resulting not "model-based", *i.e.* its design does not depend on the matrices $\{A, B, C, P, Q\}$ which characterize the plant. On the contrary, in most of the solutions proposed for nonlinear systems (see, among others, Isidori et al. (2003), Byrnes and Isidori (2004), Marconi et al. (2007), Seshagiri and Khalil (2005), Memon and Khalil (2010) and more recently Astolfi et al. (2017)) the internal model unit plays the role of an observer for the "friend", *i.e.* the correct steady state of the input which makes the error identically zero. As a consequence, the design of the internal mode unit in the nonlinear case results is model-dependent. This fact reflects in three main considerations.

The first aspect regards the issue of robustness. As commonly explained (see (Isidori, 1995, Chapter 8.5)), by considering the uncertain parameters as part of the state of the exosystem, we can gain robustness with respect to constant disturbances, providing *parameter robustness*; nevertheless when model errors (namely in the modelled functions) are present we can not guarantee any more the asymptotic properties of the regulator, thus loosing in robustness. We refer to this second property as *structural robustness*, to stress the robustness property of the regulator with respect to errors model that can be non-constant.

The second consideration regards the "circular loop" which may happen between the internal model design and the stabilizer design, namely the issue of being able to design the two units independently and robustly, such that minimum perturbations of the plant model do not destroy the stability properties of the closed-loop system. In this respect it is not completely clear how to break this loop in which the friend to be observed may depend on the stabilizer design and the internal model unit design may depend on the stabilizer (see Isidori and Marconi (2012)).

Finally, we want to stress that most of the proposed techniques in the literature relies on a certain number of additional assumptions which are not necessary in the linear framework: *relative degree* and *normal form* which are not affected by the presence of the exogenous disturbance and *minimum-phaseness* (i.e. *zero-dynamics* which are asymptotically stable) are commonly assumed (see Byrnes and Isidori (2004), Marconi et al. (2007), Seshagiri and Khalil (2005), Memon and Khalil (2010), Li and Khalil (2013)) making restrictive the class of nonlinear systems to which the approach applies. All these assumptions make also difficult the extension to the multi-input multi-output case, as confirmed in the effort made in Astolfi et al. (2013a) and also previously noticed in Isidori and Marconi (2012).

These issues motivates an introduction of a new methodology to deal with the problem of output regulation. In the new approach we are driven by two main facts

(i) a nonlinear ISS (Input-to-State Stable) system, driven by a periodic input, admits periodic solutions of the same period (see Yoshizawa (1966) or more recently Agrachev et al. (2007)), though it may contain a number of higher frequencies (Khalil (1992), Huang and Lin (1993), Delli Priscoli (1993), Delli Priscoli (1997));

(ii) the cascade *plant - internal model unit* is in feedforward form and therefore forwarding tools can be applied in the design of the *stabilizer unit*;

and we have as a final goal

(a) a design which can be applied to multi-input multi-output (possibly non-square) nonlinear systems and in absence of normal forms;

(b) a design which is *structurally robust* (in a rigorous sense to be defined).

In the forthcoming two subsections we exploit (i) and (ii) showing how to merge these facts in order to obtain a novel design for the output regulation problem. In particular, in Section 4.3 we show how to design the internal model unit starting from (i), whereas Section 4.4 is devoted to the design of a stabilizer for the cascade plant - internal model unit.

4.3 Internal model design

The contents of this section are an extension to the multi-input multi-output case of the results presented in Astolfi et al. (2015).

Motivated by (i) we restrict ourselves to the class of disturbances/references which are T-periodic and we consider, for the sake of simplicity, multi-input multi-output input-affine nonlinear systems of the form

$$\begin{array}{rcl} \dot{x} & = & f(x,w) + g(x,w)u \\ e & = & h(x,w) \end{array} \tag{4.14}$$

with state $x \in \mathbb{R}^n$, control inputs $u \in \mathbb{R}^m$ and regulated outputs $e = (e_1, \dots, e_p) \in \mathbb{R}^p$, with[1] $m \geq p$ and exogenous signal $w \in \mathbb{R}^r$ which satisfies $w(t+T) = w(t)$. Also we suppose the functions f, g, h are smooth enough and $f(0,0) = 0$, $h(0,0) = 0$ and $g(0,0) \neq 0$. If $e(t+T) = e(t)$, then it can be expressed by a Fourier Series. We denote by $c_{j,k}$ the k-th Fourier coefficient of the j-th component of e, i.e.

$$c_{j,k} \;=\; \frac{1}{T}\int_0^T e_j(t) \exp\left(\mathrm{i}\, k\, \omega\, t\right) dt\,, \qquad \omega \;=\; \frac{2\pi}{T}\,. \tag{4.15}$$

We remark that if $\eta_{j,k}$ is a T-periodic solution of

$$\dot{\eta}_{j,k} \;=\; -\mathrm{i}\, k\, \omega \eta_{j,k} + e_j(t)$$

then

$$\eta_{j,k}(0) = \eta_{j,k}(T) = \eta_{j,k}(0) + \int_0^T \exp(-\mathrm{i}\, k\, \omega\, t)\, e_j(t)\, dt$$

meaning that the Fourier coefficients $c_{j,k}$ is zero. Motivated by this consideration we design the internal model unit as a set of p integrators plus a bunch of L oscillators whose frequency are ω and its multiples, namely

$$\begin{array}{rcl} \dot{\eta}_0 & = & e \\ \dot{\eta}_1 & = & \Phi_1 \eta_1 + \Gamma_1\, e \\ & \vdots & \\ \dot{\eta}_L & = & \Phi_L \eta_L + \Gamma_L\, e \end{array} \tag{4.16}$$

with state $\eta = (\eta_0, \eta_1, \dots, \eta_L) \in \mathbb{R}^{p(1+2L)}$, $\eta_0 \in \mathbb{R}^p$, $\eta_k \in \mathbb{R}^{2p}$ for $k = 1, \dots, L$, and where

[1] This conditions is necessary in the linear case. As a consequence we make the same assumption in this nonlinear framework without loss of generality.

the matrices Φ_k, Γ_k, for any $k \in \{1, \dots, L\}$, are defined as

$$\Phi_k \;=\; \text{blckdiag}\begin{pmatrix}\phi_k & \cdots & \phi_k\end{pmatrix}_{2p\times 2p} \qquad \Gamma_k \;=\; \text{blckdiag}\begin{pmatrix}\Gamma & \cdots & \Gamma\end{pmatrix}_{2p\times p}$$

with

$$\phi_k \;=\; k\,\phi\,, \qquad \phi \;=\; \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}, \qquad \Gamma \;=\; \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

By construction, the i-th component of η_0 is an integrator processing the i-th component of the outuput e, *i.e.*

$$\eta_0^i \;=\; e_i\;,$$

whereas the i-th component of η_k, $k \geq 1$, is an oscillator at frequency $k\omega$ processing the i-th component of the output e, namely

$$\eta_k^i \;=\; k\,\phi\,\eta_k^i + \Gamma e_i\;.$$

By using the a complex notation the internal model (4.16) can be also written as

$$\begin{array}{rcll} \dot\eta_0 & = & e\,, & \eta_0 \in \mathbb{C}^p\,, \\ \dot\eta_k & = & \mathrm{i}\,k\omega\,\eta_k + e\,, & \eta_k \in \mathbb{C}^p\,, \quad k \in \{1,\dots,L\}\,, \end{array} \qquad \eta = (\eta_0, \dots, \eta_L) \in \mathbb{C}^{L+1}\;.$$

Both representations are equivalent, and throughout the text we will use indifferently the real notation or the complex one in order to simplify the expressions.

Now suppose that we are able to design find a control law u such that the cascade (4.14)-(4.16) has bounded trajectories. The following result states that, as already observed above, if the η_k are T-periodic, then necessarily the Fourier coefficients of the output e associated to the frequencies embedded in the internal model are equal to zero. Furthermore, if the internal model is "rich" enough, *i.e.* its dimension is large enough, then the L^2 norm of the error will be small enough. In this regard, the number L of oscillators is considered as a design-parameter.

Proposition 4.1. *Let $(x(t), \eta(t))$ be a bounded trajectory of the cascade (4.14)-(4.16) corresponding to some bounded input $u(t)$ such that $\eta_k(t+T) = \eta_k(t)$ for all $t \geq 0$ and all $k = 0, 1, \dots, L$. Then necessarily*

$$c^0_{j,k} \;=\; 0\,, \qquad\qquad \forall\, k = 0, 1, \dots, L, \qquad \forall\, j = 1, \dots, p\,.$$

Moreover, if $t \mapsto u(t)$ and $t \mapsto w(t)$ are continuous, for any compact set $C_x \subset \mathbb{R}^n$, for any $d_1 > 0$, $d_2 > 0$, $\bar u > 0$ and $\epsilon > 0$ such that $x(t) \in C_x$ for all $t \geq 0$, $\|w(\cdot)\|_\infty \leq d_1$,

$\|\dot{w}(\cdot)\|_\infty \leq d_2$ and $\|u(\cdot)\|_\infty \leq \bar{u}$, there exists $L^\star \geq 1$ such that, if $L \geq L^\star$ then

$$\|e(t)\|_2 \;=\; \sqrt{\frac{1}{T}\int_0^T |e(t)|^2 dt} \;\leq\; \epsilon\,.$$

Proof. Consider the internal model unit (4.16). The component η_k^i of its solution satisfies

$$\eta_k^j(t+T) \;=\; \exp(-\mathrm{i}k\omega T)\,\eta_k^j(t) + \int_t^{T+t} \exp(-\mathrm{i}k\omega(T+t-s))\,e_j(s)\,\mathrm{d}s\,.$$

Since η_k^j is T-periodic, without loss of generality we pick $t=0$ and therefore

$$0 = \int_0^T \exp(\mathrm{i}k\omega s)\,e_j(s)\,\mathrm{d}s\,,$$

and by using definition (4.15) we get $c_{j,k} = 0$ for any $k \in \{0,1,\ldots,L\}$. The same argument can be used for any $i \in \{1,\ldots,p\}$ to complete the first part of the proof.

When f,g,h are C^1, $w(t)$ and $u(t)$ are C^1 functions of time, so is $e(t)$. So let H_1, F_0 and D_1 be the real numbers defined as

$$\begin{aligned} H_j &= \sup_{x\in C_x,\,|w|\leq d_1} \left\{\frac{\partial h_j}{\partial x}(x,w)\right\}\,, \\ F_0 &= \sup_{x\in C_x,\,|w|\leq d_1,\,|u|\leq\bar{u}} \{f(x,w)+g(x,w)u\}\,, \\ D_j &= \sup_{x\in C_x,\,|w|\leq d_1} d_2\left\{\frac{\partial h_j}{\partial w}(x,w)\right\}\,. \end{aligned} \tag{4.17}$$

It follows that $|\dot{e}_j(t)| \leq H_jF_0 + D_j$ for all $t \geq 0$. Moreover, along any solution, the function $t \mapsto (e_j(t),\dot{e}_j(t))$ is continuous and thus square integrable on $[0,T]$. It follows that $e_j(t)$ and $\dot{e}_j(t)$ can be expressed by a Fourier Series

$$\begin{aligned} e_j(t) &= \sum_{k=0}^{\infty} c_{j,k}\,\exp(\mathrm{i}\,k\omega t)\,, \\ \dot{e}_j(t) &= \sum_{k=0}^{\infty} \mathrm{i}\,k\omega c_{j,k}\,\exp(\mathrm{i}\,k\omega t) \;=\; \sum_{k=0}^{\infty} c'_{j,k}\,\exp(\mathrm{i}\,k\omega t)\,, \end{aligned}$$

where $c'_{j,k} = \mathrm{i}\, k\omega c_{j,k}$. By using Parseval's identity we get

$$\sqrt{\frac{1}{T}\int_0^T |\dot{e}_j(s)|^2\,\mathrm{d}s} \;=\; \sqrt{2\sum_{k=0}^{\infty}\left(c'_{j,k}\right)^2} \;\leq\; H_j F_0 + D_j\,.$$

From the previous result we know that $c_{j,k} = 0$ for all $k \in \{0, 1, \ldots L\}$. As a consequence (and by using the definition of $c'_{j,k}$) we get

$$\omega^2(L+1)^2 \sum_{k=L+1}^{\infty} (c_{j,k})^2 \;\leq\; \sum_{k=L+1}^{\infty} (k\omega\, c_{j,k})^2 \;\leq\; \frac{(H_j F_0 + D_j)^2}{2}\,.$$

Again, by using Parseval's identity, we get

$$\frac{1}{T}\int_0^T |e_j(s)|^2 \mathrm{d}s \;\leq\; \frac{(H_j F_0 + D_j)^2}{\omega^2(L+1)^2}\,.$$

Finally by using

$$\sqrt{\frac{1}{T}\int_0^T |e(s)|^2 \mathrm{d}s} \;\leq\; \sum_{j=1}^{p}\sqrt{\frac{1}{T}\int_0^T |e_j(s)|^2 \mathrm{d}s} \;\leq\; \sum_{j=1}^{p}\frac{(H_j F_0 + D_j)}{\omega(L+1)}$$

the proof completes by setting $L^* = \left\lceil \sum_{j=1}^{p} \frac{(H_j F_0 + D_j)T}{2\pi\,\epsilon} \right\rceil$. □

The previous proposition proves that when we add an oscillator at a certain frequency in the internal model unit, we are able to make zero the correspondent Fourier coefficient of the regulated output. Also, by enlarging the dimension of the internal model, *i.e.* by adding more and more oscillators at higher frequencies, we can reduce the L^2 norm of the output. As a consequence, we may need an internal model unit of infinite dimension (an infinite number of oscillator) to regulate e to zero. A number large enough of oscillator guarantees in any case an error on the regulated output small enough, *i.e. practical output regulation* is achieved.

We stress that the design of the internal model unit (4.16) does not depend on the functions f, g, h. As a consequence the results of Proposition 4.1 hold for any set of functions f, g, h and C^1 bounded control input $u(t)$ which guarantees bounded trajectories of the cascade (4.14)-(4.16) and $\eta(t) = \eta(t+T)$, *i.e. structurally robust (practical) output regulation* is achieved.

Finally we remark that the internal model (4.16) could be considerably generalized. As already noticed in Astolfi and Praly (2016-17) for the pure integrator case, the internal model unit can be taken as

$$\begin{array}{rcl}\dot{\eta}_0 & = & \gamma_0(\eta, x, e)\\ \dot{\eta}_1 & = & \Phi_1\eta_1 + \gamma_1(\eta, x, e)\\ & \vdots & \\ \dot{\eta}_L & = & \Phi_L\eta_L + \gamma_L(\eta, x, e)\end{array}$$

where $\gamma_0, \gamma_1, \ldots, \gamma_L$ are C^1 functions satisfying

$$\begin{array}{rcl}0 \;=\; \gamma_0(\eta, x, e) & \Longrightarrow & 0 \;=\; e\,,\\ 0 \;=\; \displaystyle\int_0^T \exp(\mathrm{i}k\omega s)\,\gamma_k(\eta(s), x(s), e(s))\,\mathrm{d}s & \Longrightarrow & 0 \;=\; \displaystyle\int_0^T \exp(\mathrm{i}k\omega s)\,e(s)\,\mathrm{d}s\,.\end{array}$$

In the next subsection we exploit the fact (ii) showing how to design with forwarding techniques the stabilizer unit for the extended system (4.14)-(4.16).

4.4 Forwarding design

In this section how to design a control law u for the cascade system (4.14)-(4.16) which ensure bounded trajectories for the closed-loop system. As already explained in the linear case (see Section 4.1), since this cascade is in *strict feedforward form*, a *forwarding* design may be applied for the stabilizer unit (see Figure 4.1). The forwarding techniques have been widely studied in the last 20 years and the contents of this section are well-known. See, for a detailed overview, chapter 6.2 of Sepulchre et al. (1997). The main results of this Section (Lemma 4.6 and Propositions 4.2 and 4.3) are an adaptation of the results in Mazenc (1996).

Let consider the unforced cascade system (4.14) - (4.16), namely the system

$$\begin{aligned} \dot{x} &= f(x,0) + g(x,0)u \\ \dot{\eta}_0 &= h(x,0) \\ \dot{\eta}_1 &= \Phi_1\eta_1 + \Gamma_1\, h(x,0) \\ &\vdots \\ \dot{\eta}_L &= \Phi_L\eta_L + \Gamma_L\, h(x,0) \end{aligned} \tag{4.18}$$

As already introduced, we are interested in the design of a function $\alpha(x, \eta_0, \eta_1, \ldots, \eta_L)$ such that the origin of the closed-loop systems is asymptotically stable and locally exponentially stable. As in the linear case (see Section 4.1) a certain number of assumptions is needed. Evidently, a necessary condition for the solvability of the problem is the controllability of the unforced system

$$\dot{x} \;=\; f(x,0) + g(x,0)u\ . \tag{4.19}$$

We suppose we know a function $\beta : \mathbb{R}^n \to \mathbb{R}^m$ such that the origin of

$$\dot{x} \;=\; f(x,0) + g(x,0)\beta(x)$$

is an asymptotically and locally exponentially stable equilibrium point with $\mathcal{S} \subset \mathbb{R}^n$ as a domain of attraction. By using the converse Lyapunov theorem from Kurzweil (1956), there exists a C^1 function $V : \mathcal{S} \to \mathbb{R}^+$ which is positive definite and proper on $\mathcal{S}$ such that the function W defined as

$$W(x) \;=\; -\frac{\partial V}{\partial x}(f(x,0) + g(x,0)\beta(x)) \tag{4.20}$$

is positive definite on $\mathcal{S}$. Note that if V is known from the design of β, it may not be proper on $\mathcal{S}$. As a consequence, to make it proper, we first define ϖ as

$$\varpi \;=\; \inf_{x \notin \mathcal{S}} V(x)$$

and then we replace $V(x)$ by $\dfrac{V(x)}{\varpi - V(x)}$, see Teel and Praly (1995). Unfortunately, in doing so, the domain of definition of this new function V may be a strict subset of $\mathcal{S}$. In the following we still call $\mathcal{S}$ this domain on which V is proper. Furthermore, by following the "linear recipe" of Section 4.1 (see Assumption 4.4), we suppose that the pre-state feedback $u = \beta(x)$ is already applied to system (4.19). We call with abuse of notation the function $f(x,0)$ as $f(x,0) + g(x,0)\beta(x)$ and the following assumption is made.

Assumption 4.5. *There exist a C^1 function $V : \mathcal{S} \to \mathbb{R}_{\geq 0}$ which is positive definite and proper on $\mathcal{S}$ and a positive definite function $W : \mathcal{S} \to \mathbb{R}^{\geq 0}$ such that*

$$\frac{\partial V}{\partial x}(x) f(x,0) \;\leq\; -W(x)$$

for all $x \in \mathcal{S}$.

Again, by following the linear results of Section 4.1, the following non-resonance condition, completely equivalent to Assumption 4.3, is supposed.

Assumption 4.6. *The non-resonance condition holds,* i.e. *the matrix*

$$\begin{pmatrix} A - \lambda I & B \\ C & 0 \end{pmatrix}$$

has independent rows for each $\lambda = \mathrm{i}k\omega$, $k \in \{0, 1, \ldots, L\}$, where

$$A = \frac{\partial f}{\partial x}(0,0)\;, \quad B = g(0,0)\;, \quad C = \frac{\partial h}{\partial x}(0,0)\;.$$

By mimicking the linear case, in the forthcoming lemma we define a function $\mathcal{M}_k(x)$ which is the nonlinear extension of the matrix M defined in (4.9). As shown below, this function $\mathcal{M}_k(x)$ plays a key role in the design of a stabilizer for the system (4.18). In this section we will use the real notation for the internal model unit (4.16).

Lemma 4.6. *Under Assumptions 4.5 and 4.6, for any $k \in \mathbb{N}$ the following expressions define properly C^2 functions $\mathcal{M}_0 : \mathbb{R}^n \to \mathbb{R}^p$, $\mathcal{M}_k : \mathbb{R}^n \to \mathbb{R}^{2p}$, $k > 0$,*

$$\begin{aligned} \mathcal{M}_0(x) &= \lim_{t\to\infty} \int_0^t h(x(s), 0)\, \mathrm{d}s \ , \\ \mathcal{M}_k(x) &= \lim_{t\to\infty} \int_0^t \exp(\Phi_k s)\, \Gamma_k\, h(x(s), 0)\, \mathrm{d}s \ . \end{aligned} \tag{4.21}$$

They are solution of

$$\begin{aligned} \frac{\partial \mathcal{M}_0}{\partial x}(x) f(x, 0) &= h(x, 0) \ , \\ \frac{\partial \mathcal{M}_k}{\partial x}(x) f(x, 0) &= \Phi_k \mathcal{M}_k(x) + \Gamma_k\, h(x, 0) \end{aligned} \tag{4.22}$$

Furthermore, the pair $(B^\top M_k^\top, \Phi)$ is observable, where we denote, for any $k \in \mathbb{N}$,

$$M_k = \frac{\partial \mathcal{M}_k}{\partial x}(0) \ .$$

Proof. Let simplify the notations in

$$f(x) := f(x, 0) \ , \qquad h(x) := h(x, 0) \ ,$$

and consider the following system

$$\begin{aligned} \dot{z}_k &= \Phi_k\, z_k + \Gamma_k h(x) \\ \dot{x} &= f(x) \end{aligned}$$

with Φ_k, Γ_k defined in (4.16) and where $x = 0$ is asymptotically stable and locally exponentially stable for $\dot{x} = f(x)$ with $x \in \mathcal{S}$. The solution of z_k is given by

$$Z_k((x, z_k), t) = \exp(\Phi_k\, t) z_k + \int_0^t \exp(\Phi_k(t - s))\, \Gamma_k\, h(X(x, s))\, \mathrm{d}s$$

Because of the exponential stability of the x-dynamics and the neutral stability of Φ, the origin has a stable manifold. It is the set of pairs (x, z_k) such that we have

$$\lim_{t\to\infty} Z_k((x, z_k), t) = 0 \ .$$

and therefore satisfying

$$z_k = -\lim_{t\to\infty}\int_0^t \exp(\Phi_k s)\,\Gamma_k\, h(X(x,s))\,\mathrm{d}s$$

We conclude that the function $\mathcal{M}_k(x)$ defined in (4.21) as

$$\mathcal{M}_k(x) = \lim_{t\to\infty}\int_0^t \exp(\Phi_k s)\,\Gamma_k\, h(X(x,s))\,\mathrm{d}s$$

is well defined. The same arguments hold for the case $k = 0$ completing the first part of the proof. Observability of the pair $(B^\top M_k^\top)$ follows by linearizing (4.22) around the origin

$$M_k A = \Phi_k M_k + \Gamma_k C$$

and by noting that the assumptions of Lemma 4.4 are satisfied. □

Under the previous assumptions and with the previous lemma in mind, we are able to design a state-feedback law for the unforced extended open-loop system (4.18). This result is completely analogous to the linear result of Lemma 4.3. Also, it can be noticed that the forthcoming proposed regulator (4.23) is a nonliner version of the linear regulator (4.10).

Proposition 4.2. *Consider the system* (4.18)*. Let $L \in \mathbb{N}$ be fixed and suppose Assumptions 4.5 and 4.6 holds. For any $\bar{u} > 0$ (maybe infinite), the origin of* (4.18) *in closed loop with $u = \alpha(x,\eta)$,*

$$\begin{aligned}\alpha(x,\eta) &= \min\{\bar{u}\,, |\alpha_0(x,\eta)|\,\}\frac{\alpha_0(x,\eta)}{|\alpha_0(x,\eta)|}\\ \alpha_0(x,\eta) &= -\left[\frac{\partial V}{\partial x}(x)g(x,0) - \sum_{k=0}^{L}(\eta_k - \mathcal{M}_k(x))^\top \frac{\partial \mathcal{M}_k}{\partial x}(x)g(x,0)\right]^\top\end{aligned} \tag{4.23}$$

is asymptotically stable and locally exponentially stable with $\mathcal{S} \times \mathbb{R}^{p(1+2L)}$ as domain of attraction.

Proof. With Assumption 4.5 let U be the Lyapunov Function defined as

$$U(x,z) = V(x) + \frac{1}{2}\sum_{k=0}^{L}(\eta_k - \mathcal{M}_k(x))^\top(\eta_k - \mathcal{M}_k(x))$$

where V satisfies (4.20). The function U is positive definite, proper on $\mathcal{S} \times \mathbb{R}^{p(1+2L)}$

and its derivative along the solutions satisfies

$$\dot U(x,\eta) \ \le$$
$$-W(x) \ + \ \frac{\partial V}{\partial x}(x) g(x,0) u +$$
$$\sum_{k=0}^{L} (\eta_k - \mathcal{M}_k(x))^\top \left(\Phi_k\, \eta_k + \Gamma_k\, h(x,0) - \frac{\partial \mathcal{M}_k}{\partial x}(x) f(x,0) - \frac{\partial \mathcal{M}_k}{\partial x}(x) g(x,0) u \right)$$

Recall that $\mathcal{M}_k$ satisfies (4.22). As a consequence we have

$$\Phi_k\, \eta_k + \Gamma_k\, h(x,0) - \frac{\partial \mathcal{M}_k}{\partial x}(x) f(x,0) \ = \ \Phi_k(\eta_k - \mathcal{M}_k(x)) \ .$$

Furthermore, by noting that

$$\Phi + \Phi^\top \ = \ 0$$

we get

$$\dot U(x,\eta) \ \le \ -W(x) + \left[\frac{\partial V}{\partial x}(x)\, g(x,0) - \sum_{k=0}^{L} (\eta_k - \mathcal{M}_k(x))^\top \frac{\partial \mathcal{M}_k}{\partial x}(x) g(x,0) \right] u$$

with the choice $u = \alpha(x,\eta)$, with α defined as in (4.23), we obtain

$$\dot U(x,\eta) \ \le \ -W(x) - \min\left\{ \bar u\, |\alpha_0(x,\eta)|, \ |\alpha_0(x,\eta)|^2 \right\} \ .$$

Applying La Salle theorem we conclude that that the largest invariant set contained in

$$\mathcal{A} \ := \ \left\{ (x,\eta) \in \mathbb{R}^n \times \mathbb{C}^{p(L+1)} \ : \ W(x) = 0\,, \ \alpha_0(x,\eta) = 0 \ \right\}$$

is asymptotically stable. By using the fact that $W(x) = 0$ if and only if $x = 0$, $\frac{\partial V}{\partial x}(0) = 0$ and moreover $\mathcal{M}_k(0) = 0$, we see the set $\mathcal{A}$ reduces to

$$\mathcal{A} \ = \ \left\{ (x,\eta) \in \mathbb{R}^n \times \mathbb{C}^{p(L+1)} \ : \ x = 0\,, \ B^\top M_k^\top\, \eta_k = 0 \ \right\} \ .$$

By using the fact that the pair $(B^\top, M^\top, \Phi)$ is observable (see Lemma 4.6) we conclude that $\mathcal{A}$ coincides with the origin. Therefore the origin is asymptotically stable with a domain of attraction $\mathcal{S} \times \mathbb{R}^{p(1+2L)}$. By noting that around the origin we have $u = \alpha_0(x,\eta)$, we conclude that the origin is also locally exponentially stable. □

The solution proposed in the previous proposition relies on the knowledge of the function $\mathcal{M}_k(x)$, solution of the partial differential equation (4.22). As shown in Lemma 4.6 the solution always exists, but formally it can be very hard to find. Therefore in the forthcoming proposition we propose a simpler design based on the linearization of (4.22).

Proposition 4.3. *Consider the system* (4.18). *Let* $L \in \mathbb{N}$ *be fixed and suppose Assumptions 4.5 and 4.6 holds. For any* $\bar{u} > 0$, *the origin of* (4.18) *in closed loop with* $u = \alpha(x, \eta)$,

$$\begin{aligned} \alpha(x,\eta) &= \min\{\bar{u}\ , |\alpha_1(x,\eta)|\ \}\frac{\alpha_1(x,\eta)}{|\alpha_1(x,\eta)|} \\ \alpha_1(x,\eta) &= -\left[\ell'(V(x))\frac{\partial V}{\partial x}(x)g(x,0) - \sum_{k=0}^{L}\frac{(\eta_k - M_k x)^\top}{\sqrt{1+\sum_{k=0}^{L}|\eta_k - M_k x|^2}}M_k g(x,0)\right]^\top \end{aligned} \tag{4.24}$$

with $\ell : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *a* C^1 *function with strictly positive derivative to be chosen large enough (see Mazenc and Praly (1996)), is asymptotically stable and locally exponentially stable with* $\mathcal{S} \times \mathbb{R}^{p(1+2L)}$ *as domain of attraction.*

Proof. First of all note that the matrix $M_k = \dfrac{\partial \mathcal{M}_k}{\partial x}(0)$ is solution of

$$M_k F \;=\; \Phi_k\, M_k + \Gamma_k\, H$$

With Assumption 4.5 in mind let U be the Lyapunov Function defined as

$$U(x,z) \;=\; \ell(V(x)) + \sqrt{1 + \sum_{k=0}^{L}|\eta_k - M_k\, x|^2 - 1}$$

where V satisfies (4.20) and ℓ is a function to be chosen large enough. The function U is positive definite, proper on $\mathcal{S} \times \mathbb{R}^{p(1+2L)}$ and its derivative along solutions satisfies

$$\begin{aligned} &\dot{U}(x,\eta) \;\leq \\ &\quad -\ell'(V(x))W(x) \;+\; \ell'(V(x))\frac{\partial V}{\partial x}(x)g(x,0)u \;+ \\ &\quad \frac{\sum_{k=0}^{L}(\eta_k - \mathcal{M}_k(x))^\top}{\sqrt{1+\sum_{k=0}^{L}|\eta_k - M_k(x)|^2}}\Big(\Phi_k\,\eta_k + \Gamma_k\, h(x,0) - M_k\big(f(x) + g(x,0)u\big)\Big) \end{aligned}$$

By adding e subtracting the term $\dfrac{\partial \mathcal{M}_k}{\partial x}(x)f(x)$, by defining

$$\Delta(x) \;=\; \left(M_k - \frac{\partial \mathcal{M}_k}{\partial x}(x)\right) f(x)\,,$$

and by recalling (4.22) we get

$$\begin{aligned}\dot U(x,\eta) &\leq -\ell'(V(x))W(x) + \ell'(V(x))\frac{\partial V}{\partial x}(x)g(x)u \\ &\qquad -\frac{\sum_{k=0}^{L}(\eta_k - \mathcal{M}_k(x))^\top}{\sqrt{1+\sum_{k=0}^{L}|\eta_k - M_k(x)|^2}}(\Delta(x) + M_k g(x,0)u)\end{aligned}$$

With the choice $u = \alpha(x,\eta)$ we obtain

$$\dot U(x,\eta) \leq -\ell'(V(x))W(x) + \Delta(x) - \min\left\{\bar u\,|\alpha_1(x,\eta)|,\ \alpha_1(x,\eta)^2\right\}.$$

By choosing the function ℓ such that

$$2|\Delta(x)| \leq \ell'(V(x))W(x) \qquad \forall\, x \in \mathcal{S}\,,$$

and by applying La Salle theorem we conclude that that the largest invariant set contained in the set

$$\left\{(x,\eta) \in \mathbb{R}^n \times \mathbb{C}^{p(L+1)}\ :\ W(x) = 0\,,\ \alpha_1(x,\eta) = 0\ \right\}$$

is asymptotically stable. The same arguments used in Proposition 4.2 can be used to conclude that the origin is asymptotically stable and locally exponentially stable with $\mathcal{S} \times \mathbb{R}^{p(1+2L)}$ as domain of attraction. □

The goal of this section was to show that forwarding techniques can be applied in order to design a stabilizing feedback law for the system (4.18). Nevertheless we recall that the literature on forwarding techniques is wide and many other different (and simpler) designs can be adopted. See, among others, Mazenc and Praly (1996), Teel (1996), Sepulchre et al. (1997). Finally, we remark that the system (4.18) falls exactly in the same framework considered by Kaliora and Astolfi (2001) and Kaliora and Astolfi (2004) and that their techniques can be directly applied under Input-to-State Stability assumptions.

4.5 Weak output regulation

In this section we combine the results of Sections 4.3 and 4.4 to show that the proposed regulator achieve *structurally robust weak practical output regulation*. We refer with *weak* the fact that the bound on the disturbance may depend on the dimension of the internal model and could shrink to zero when augmenting the number of oscillators to infinity. In Chapter 5 we will show a case where this phenomenon does not happen, namely the bound on the disturbance is independent on the number of oscillators, thus achieving *structurally robust practical output regulation*. To summarize the results, we consider the class of system (4.14) described as

$$\begin{array}{rcl} \dot{x} & = & f(x,w) + g(x,w)u \\ e & = & h(x,w) \end{array} \tag{4.14}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, and the dynamic regulator given by

$$\begin{array}{rcl} \dot{\eta}_0 & = & e \\ \dot{\eta}_1 & = & \Phi_1 \eta_1 + \Gamma_1 \, e \\ & \vdots & \\ \dot{\eta}_L & = & \Phi_L \eta_L + \Gamma_L \, e \\ u & = & \alpha(x,\eta) \end{array} \tag{4.25}$$

with $\eta = (\eta_0, \ldots, \eta_L) \in \mathbb{R}^{p(1+2L)}$ and where α can be designed for instance as shown in Section 4.4.

As a consequence of Proposition 4.2 and results on local exponential stability, we have that when w is small enough, the closed-loop system (4.14),(4.25) admits a unique periodic trajectory which is asymptotically stable.

Proposition 4.4 (Existence of periodic solutions)**.** *Consider the system* (4.14) *under Assumption 4.6, and suppose the origin of the closed-loop system* (4.14)-(4.25) *is an asymptotically and locally exponentially stable equilibrium point with domain of attraction* $\mathcal{A} = \mathcal{S} \times \mathbb{R}^{p(1+2L)}$ *when* $w = 0$*. Then, for any compact set* $C_{x\eta} \subset \mathcal{A}$*, containing the origin, there exists a real number* $d_1 > 0$ *such that, for any* T*-periodic function* $t \mapsto w(t)$ *satisfying* $\|w(\cdot)\|_\infty \leq d_1$*, there exists a unique* T*-periodic trajectory* $(x^\star(t), \eta^\star(t)) \in C_{x\eta}$ *which is asymptotically stable and locally exponentially stable with* $\mathcal{B}$ *as domain of attraction, with* $C_{x\eta} \subset \mathcal{B} \subset \mathcal{A}$*.*

The proof of this proposition is a direct application of well-known results on local exponential stability, local input-to-state stability and existence of periodic solution under small perturbations. See, for instance Yoshizawa (1966) or more recently Agrachev et al. (2007). Note that by augmenting L we add poles on the imaginary axe. Consequently the stability margin of the closed loop system may decrease with L and so the bound d_1 may decrease with augmenting L. In Chapter 5 we will show a case where this phenomenon does not happen. By combining the previous proposition with Proposition 4.1 we get the following theorem claiming that *weak practical output regulation* is achieved.

Theorem 4.2 (Weak practical output regulation problem)**.** *Consider the system* (4.14) *under Assumptions 4.5, 4.6, and suppose the origin of the closed-loop system* (4.14)-(4.25) *is an asymptotically and locally exponentially stable equilibrium point with domain of attraction* $\mathcal{A} = \mathcal{S} \times \mathbb{R}^{p(1+2L)}$ *when* $w = 0$*. Then the regulator* (4.25) *solves the problem of weak practical output regulation for the system* (4.14)*, namely given a compact set* $C_x \subset \mathcal{S}$ *and real numbers* $\bar{u} > 0$ *and* $d_2 > 0$*, for any* $\epsilon > 0$ *there exists* $L^\star > 0$ *and, for any* $L \geq L^\star$*, there exists a* $d_1 > 0$ *such that, for any initial condition* $(x(0), \eta(0)) \in C_x \times \{0\}$ *and for any* C^1 *function* $t \mapsto w(t)$ *T-periodic with* $\|w(\cdot)\|_\infty \leq d_1$ *and* $\|\dot{w}(\cdot)\|_\infty \leq d_2$*, the solutions of the closed-loop system* (4.14)-(4.25) *are bounded,* T*-periodic and such that*

(i) $c^0_{i,k} = 0$ *for all* $i = 1, \ldots, p$ *and* $k \leq L$*;*

(ii) $\|e(t)\|_2 \leq \epsilon$*.*

The proof of the theorem follows by direct application of Proposition 4.1 and 4.4. As already noticed in the proof of Proposition 4.4 the bound d_1 may shrink when augmenting L. As a consequence, given C_x, $\bar{u} > 0$, and $d_2 > 0$, one may implement an optimal recursive algorithm to find the smallest L and the largest d_1 which satisfies $\|e(t)\|_2 \leq \epsilon$.

With the previous theorems, we can prove that the regulator (4.25) is *structurally robust*, namely the weak practical output regulation problem is solved for all the systems

$$\begin{aligned} \dot{x} &= \xi(x, u, w) \\ e &= \zeta(x, u, w) \end{aligned} \tag{4.26}$$

"close enough" to (4.14). In order to state the result on output regulation, we need first to introduce the forthcoming technical Lemma, claiming persistence of the equilibrium for the closed loop system (4.26)-(4.25). This lemma relies on Lemma C.2 given in Appendix C.

Lemma 4.7 (Persistence of equilibrium). *For any compact sets $\underline{\mathcal{C}}$ and $\overline{\mathcal{C}}$, the latter being forward invariant for the closed-loop system* (4.14)-(4.25)*, which satisfy*

$$\{0\} \subsetneq \underline{\mathcal{C}} \subsetneq \overline{\mathcal{C}} \subsetneq \mathcal{A}\,,$$

and for any open neighborhood $\mathcal{N}_{\overline{\mathcal{C}}}$ of $\overline{\mathcal{C}}$, contained in $\mathcal{A}$, there exists a strictly positive real number $\bar{\delta}$ such that to any pair (ξ, ζ) of C^1 functions which satisfies

$$|\xi(x,u,0) - [f(x,0) + g(x,0)u]| + |\zeta(x,u,0) - h(x,0)| \;\le\; \bar{\delta} \qquad \forall (x,u) \in \underline{\mathcal{C}}_x \times U \quad (4.27)$$

and

$$\left|\begin{pmatrix} \frac{\partial \xi}{\partial x}(x,u,0) & \frac{\partial \xi}{\partial u}(x,u,0) \\ \frac{\partial \zeta}{\partial x}(x,u,0) & \frac{\partial \zeta}{\partial u}(x,u,0) \end{pmatrix} - \begin{pmatrix} \frac{\partial f}{\partial x}(0,x) + \frac{\partial g}{\partial x}(x,0)u & g(x,0) \\ \frac{\partial h}{\partial x}(x,0) & 0 \end{pmatrix}\right| \;\le\; \bar{\delta} \qquad (4.28)$$

for all $(x,u) \in \underline{\mathcal{C}}_x \times U$, where $U = \alpha(\overline{\mathcal{C}})$, we can associate a point $\mathcal{X}_e = (x_e, \eta_e)$ which is an exponentially stable equilibrium point of (4.26)-(4.25) *whose basin of attraction $\mathcal{B}$ contains $\overline{\mathcal{C}}$.*

Proof. The result follows by direct application of Lemma C.2. In particular the conditions of Lemma C.2 are satisfied if (4.27) and (4.28) implies (C.5) and (C.6). By using the notation $\mathcal{X} = (x, \eta_0, \dots, \eta_L)$,

$$\varphi_m(\mathcal{X}, w) = \begin{pmatrix} f(x,w) + g(x,w)u \\ \Gamma_0\, h(x,w) \\ \Phi_1\eta_1 + \Gamma_1\, h(x,w) \\ \vdots \\ \Phi_L\eta_L + \Gamma_L\, h(x,w) \end{pmatrix}, \quad \varphi_p(\mathcal{X}, w) = \begin{pmatrix} \xi(x,u,w) \\ \Gamma_0\, \zeta(x,u,w) \\ \Phi_1\eta_1 + \Gamma_1\, \zeta(x,u,w) \\ \vdots \\ \Phi_L\eta_L + \Gamma_L\, \zeta(x,u,w) \end{pmatrix},$$

and $\Gamma_0 = I$ with $u = \alpha(x, \eta)$, we see that

$$\begin{aligned} |\varphi_p(\mathcal{X},0) - \varphi_m(\mathcal{X},0)| \;&\le\; |\xi(x,\alpha(x,\eta),0) - [f(x,0) + g(x,0)\alpha(x,\eta)]| \\ &\qquad + \sum_{k=0}^{L} |\Gamma_k\zeta(x,\alpha(x,\eta),0) - \Gamma_k h(x,0)| \\ &\le\; (L+1)\,|\xi(x,\alpha(x,\eta),0) - [f(x,0) + g(x,0)\alpha(x,\eta)]| \\ &\qquad + (L+1)\,|\zeta(x,\alpha(x,\eta),0) - h(x,0)| \\ &\le\; (L+1)\bar{\delta} \end{aligned}$$

Moreover

$$\left|\frac{\partial \varphi_p}{\partial x}(x,0) - \frac{\partial \varphi_m}{\partial x}(x,0)\right| \leq$$

$$\left|\begin{pmatrix} \frac{\partial \xi}{\partial x} + \frac{\partial \xi}{\partial u}\frac{\partial \alpha}{\partial x} & \frac{\partial \xi}{\partial u}\frac{\partial \alpha}{\partial \eta} \\ \Gamma_0\left(\frac{\partial \zeta}{\partial x} + \frac{\partial \zeta}{\partial u}\frac{\partial \alpha}{\partial x}\right) & 0 \\ \Gamma_1\left(\frac{\partial \zeta}{\partial x} + \frac{\partial \zeta}{\partial u}\frac{\partial \alpha}{\partial x}\right) & \Gamma_1\frac{\partial \zeta}{\partial u}\frac{\partial \alpha}{\partial \eta} + \Phi_1 \\ \vdots & \vdots \\ \Gamma_L\left(\frac{\partial \zeta}{\partial x} + \frac{\partial \zeta}{\partial u}\frac{\partial \alpha}{\partial x}\right) & \Gamma_L\frac{\partial \zeta}{\partial u}\frac{\partial \alpha}{\partial \eta} + \Phi_L \end{pmatrix} - \begin{pmatrix} \frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\alpha + g\frac{\partial \alpha}{\partial x} & g\frac{\partial \alpha}{\partial \eta} \\ \Gamma_0\frac{\partial h}{\partial x} & 0 \\ \Gamma_1\frac{\partial h}{\partial x} & \Phi_1 \\ \vdots & \vdots \\ \Gamma_L\frac{\partial h}{\partial x} & \Phi_L \end{pmatrix}\right|_{w=0}$$

By recalling that α is C^1 and by using the compact notation

$$\begin{aligned} \Delta_{pm} &= \left|\begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial u} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial u} \end{pmatrix} - \begin{pmatrix} \frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}u & g \\ \frac{\partial h}{\partial x} & 0 \end{pmatrix}\right|_{w=0} \\ \Delta_\alpha &= \sup_{(x,\eta)\in\underline{\mathcal{C}}}\left(\left|\frac{\partial \alpha}{\partial x}\right| + \left|\frac{\partial \alpha}{\partial \eta}\right|\right) \end{aligned}$$

we get

$$\left|\frac{\partial \varphi_p}{\partial x}(x,0) - \frac{\partial \varphi_m}{\partial x}(x,0)\right| \leq (1+L)\Delta_\alpha\Delta_{pm} \leq (1+L)\Delta_\alpha\bar{\delta}$$

We conclude that the conditions (C.5) and (C.6) are verified for

$$\delta = (1+L)\,\bar{\delta}\,\max\{\Delta_\alpha, 1\}\,. \qquad \square$$

Finally, by combining the results of Theorem 4.2 and the Lemma 4.7 we can show that the structurally robust weak practical output regulation problem is solved, as stated in the forthcoming lemma. It can be seen as a generalization (but in the state-feedback case) of Proposition 3 of Astolfi and Praly (2016-17)

Lemma 4.8 (Structurally robust weak practical output regulation problem). *The dynamic regulator* (4.25) *solves the problem of structurally robust weak practical output regulation for the system* (4.14)*, namely it solves the weak practical output regulation problem for any pair* (ξ, ζ) *of* C^1 *functions satisfying the assumptions of Lemma 4.7 and such that the matrix*

$$\begin{pmatrix} A - \lambda I & B \\ C & D \end{pmatrix}$$

has independent rows for each $\lambda = \mathrm{i}k\omega$*,* $k \in \{0, 1, \ldots, L\}$*, where*

$$A = \left.\frac{\partial \xi}{\partial x}\right|_{\mathcal{X}_e}, \qquad B = \left.\frac{\partial \xi}{\partial u}\right|_{\mathcal{X}_e}, \qquad C = \left.\frac{\partial \zeta}{\partial x}\right|_{\mathcal{X}_e}, \qquad D = \left.\frac{\partial \zeta}{\partial u}\right|_{\mathcal{X}_e},$$

with the equilibrium $\mathcal{X}_e$ *given by Lemma 4.7.*

Proof. The proof follows by the exponential stability of the equilibrium $\mathcal{X}_e$ given by Lemma 4.7 and by checking that the conditions of Theorem 4.2 are satisfied. □

We want to stress that even if all the feedback-design has been highlighted for the state-feedback case, the results can be rephrased in the output feedback case without loss of generality. An example of this methodology is given in Astolfi and Praly (2016-17) where the internal model reduces to an integrator. Output feedback results in our practical output regulation problem can be achieved by using the dynamic regulator (4.25) and by replacing x by an estimate $\hat{x}$ which is provided a suitable tunable state-observer. The latter can be designed with any desired technique and for example with the high-gain observer presented in Section 2.3.

"You cannot always control what goes on outside. But you can always control what goes on inside."

Wayne Dyer

5

Structurally Robust Output Regulation for Minimum Phase Systems

The results of Section 4.5 applies to a very generic class of systems, that is input-affine multi-input multi-output nonlinear systems. As a consequence the results we are able to shown are *weak* in the sense that they apply only when the disturbances are very small. The main limitations of this approach rely on the theoretical tools used in the proof, mainly existence of periodic solutions under locally input-to-state stability condition. The latter is provided by the local exponential stability guaranteed by the forwarding design for the unforced system in cascade with the internal model. As a consequence the allowed magnitude of the disturbance depends on the dimension of the closed-loop system, which is $n + p(2L + 1)$ (where n is the dimension of the system, p is the dimension of the regulated output and $2L + 1$ the dimension of the internal model). As a matter of fact, by augmenting the dimension of the internal model we may reduce the L^2 norm of the regulated output, but we may also need to reduce the allowed disturbance magnitude.

In this section we want to apply the proposed approach to single-input single-output minimum-phase nonlinear systems which possesses a well-defined relative degree. As in Section 1.6.1 we consider systems with unitary relative degree, knowing that the extension to the case in which the relative degree is larger than one, can be easily handled by means of high-gain tools, as shown for instance in (Isidori, 1999, Section 12.1). For this class of system we are able to show results which are much more stronger than the ones proposed in Section 4.5. In particular *structurally robust asymptotic output regulation* is achieved and moreover the allowed disturbance magnitude is not affected by the dimension of the internal model.

5.1 Asymptotic regulation in the infinite dimensional case

By following a very standard framework of output regulation we consider in this section nonlinear systems with unitary relative degree, *i.e.* that can be written as

$$\begin{aligned} \dot{z} &= f(w,z,e) \\ \dot{e} &= q(w,z,e) + u \end{aligned} \tag{5.1}$$

with state $z \in \mathbb{R}^n$, control input $u \in \mathbb{R}$ and regulated output $e \in \mathbb{R}$. The initial conditions $(z(0), e(0))$ of the system range in a given compact set $Z \times E \subset \mathbb{R}^n \times \mathbb{R}$ and f and q are C^1 functions. We suppose the exogenous signal w evolves in a given compact set $W \subset \mathbb{R}^r$ and that $t \mapsto w(t)$ is a T-periodic C^1 function. Since (w, z, e) ranges in the compact set $W \times Z \times E$, let ρ be the Lipschitz constant of q, namely

$$|q(w,z',e'') - q(w,z',e'')| \;\leq\; \rho|z' - z''| + \rho|e' - e''| \tag{5.2}$$

for any $w \in W$ and any $(z', e') \in Z \times E$ and $(z'', e'') \in Z \times E$.

In this section we want to show that we can solve the robust output regulation problem for the class of systems described above by following the "recipe" of Chapter 4, but with a dynamic regulator of *infinite* dimension. As in Section 4.3, we extend the system (5.1) with a bunch of an infinite number of oscillators

$$\begin{aligned} \dot{\eta}_0 &= e \\ \dot{\eta}_k &= k\phi\,\eta_k + \Gamma e \qquad k \in \mathbb{N} \end{aligned} \tag{5.3}$$

with $\eta = (\eta_k)_{k\in\mathbb{N}}$, with $\eta_0 \in \mathbb{R}$ and $\eta_k \in \mathbb{R}^2$ for all $k \in \mathbb{N}_{>0}$ and

$$\phi = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}, \qquad \omega = \frac{2\pi}{T}, \qquad \Gamma = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We suppose the initial conditions of $\eta(0)$ are in ℓ^2, namely $\sqrt{\sum_{k=0}^{\infty} |\eta_k(0)|^2} < \infty$. Note that a trivial choice is $\eta_k(0) = 0$ for all $k \in \mathbb{N}$. The proposed regulator based on the forwarding techniques presented in Section 4.4, is

$$\begin{aligned} v &= \sum_{k=0}^{\infty} M_k^\top(\eta_k - M_k e) \\ u &= -g\,e + v \end{aligned} \tag{5.4}$$

for some $g > 0$ large enough and where the real number M_0 and the matrices $M_k \in \mathbb{R}^2$,

$k \in \mathbb{N} > 0$, are defined as solutions of

$$\begin{aligned} -M_0\, g &= 1 \\ -M_k\, g &= k\phi M_k + \Gamma \end{aligned} \tag{5.5}$$

Note that we are not using the $\mathcal{M}(\cdot)$ proposed in Lemma 4.6 which would be computed as solution of

$$\frac{\partial \mathcal{M}_k}{\partial e}(e)\big(-ge + q(0,0,e)\big) \;=\; k\phi\, \mathcal{M}_k(e) + \Gamma\, e$$

but we are making an approximation computing $\mathcal{M}_k$ as solution of

$$\frac{\partial \mathcal{M}_k}{\partial e}(e)\big(-ge\big) \;=\; k\phi\, \mathcal{M}_k(e) + \Gamma\, e$$

namely we decide to ignore the Lipschitz term $q(0,0,e)$, which is dominated by $g\, e$ if g is large enough.

The aim of this section is to prove that the dynamic regulator (5.3)-(5.4) solves the *structurally robust asymptotic output regulation problem* for the class of system (5.1), namely

$$\lim_{t\to\infty} e(t) = 0$$

uniformly in the initial conditions $(w(0), z(0), e(0)) \in W \times Z \times E$, $\eta(0) \in \ell^2$ and for any pair of functions f, q satisfying a certain number of assumptions that are customary in the literature of output regulation.

Assumption 5.1. *For any T-periodic C^1 function $t \mapsto w(t) \in W$ there exists a T-periodic C^1 function $\pi(t)$ satisfying*

$$\dot{\pi}(t) \;=\; f(w(t), \pi(t), 0)\,.$$

In the forthcoming assumption we use the forthcoming notation

$$\tilde{z} \;:=\; z - \pi(t)\,, \qquad \tilde{f}(\tilde{z}, e, t) \;:=\; f(w(t), \tilde{z} + \pi(t), e) - f(w(t), \pi(t), 0)\,.$$

Assumption 5.2. *For any T-periodic C^1 function $t \mapsto w(t) \in W$ there exists a positive definite function $U(\tilde{z})$ satisfying*

$$\frac{\partial U}{\partial \tilde{z}}(\tilde{z})\; \tilde{f}(\tilde{z}, e, t) \;\leq\; -\alpha|\tilde{z}|^2 + \gamma|e|^2 \tag{5.6}$$

for some real numbers $\alpha > 0$ and $\gamma > 0$.

Assumption 5.1 implies the existence of the regulator equations, whereas Assumption 5.2 implies the zero-dynamics are strongly minimum-phase (see, for instance, Byrnes et al. (2003), Byrnes and Isidori (2003), Byrnes and Isidori (2004)), namely the origin of

$\dot{\tilde{z}} = f(\tilde{z}, 0, t)$ is exponentially stable. Further comments about the (restrictive) condition (5.6) are given at the end of the section.

Note that the closed-loop systems (5.1)-(5.3)-(5.4) has the form (4.14)-(4.25), and it is not hard to see that, by considering an internal model of finite dimension, the origin is exponentially stable for g large enough when $w = 0$ (see Lemma 2.2 in Teel and Praly (1995)). Furthermore, notice that under the Assumption 5.2 the non-resonance conditions (see Assumption 4.6) are automatically fulfilled by definition of the zero-dynamics (see Isidori (1995)). As a consequence the conditions of Theorem 4.2 and Lemma 4.8 are satisfied, namely *structurally robust weak practical output regulation* is guaranteed when considering an internal model of finite dimension. However, due to the structure of system (5.1) a stronger result is achieved. In particular, the forthcoming theorem states that the regulator (5.3)-(5.4) solves the *structurally robust output regulation problem* for the system (5.1) with an internal model of infinite dimension.

Theorem 5.1. *The regulator* (5.3)-(5.4) *solves the Structurally Robust Output Regulation Problem for all the systems of the form* (5.1) *with f satisfying Assumption 5.1 and 5.2, and q satisfying* (5.2), *namely for any T-periodic C^1 function $t \mapsto w(t) \in W$ and for any set $Z \times E \subset \mathbb{R}^n \times \mathbb{R}$ there exist a $g^\star > 0$ such that, and for any $g \geq g^\star$ the trajectories of the closed-loop system* (5.1)-(5.3)-(5.4) *starting in $(z(0), e(0)) \in Z \times E$, $\eta(0) \in \ell^2$, are bounded (in ℓ^2 sense) and* $\lim_{t\to\infty} e(t) = 0$.

In order to prove the theorem we need to introduce some technical results. First of all note that the matrices M_0 and M_k, solutions of (5.5) can be computed as

$$M_k \;=\; -(k\phi + Ig)^{-1}\Gamma \;=\; -\begin{pmatrix} g & k\omega \\ -k\omega & g \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and therefore we obtain

$$M_0 \;=\; -\frac{1}{g}\,, \qquad\qquad M_k \;=\; \frac{1}{(k\omega)^2 + g^2} \begin{pmatrix} k\omega \\ -g \end{pmatrix}\,. \tag{5.7}$$

Lemma 5.1. *For any $g > 0$, there exists a real number $M > 0$ such that*

$$\sum_{k=0}^{\infty} M_k^\top M_k \;\leq\; M\,.$$

Proof. The result follows by computing

$$\begin{aligned}\sum_{k=0}^{\infty} M_j^\top M_j &= \frac{1}{g^2} + \sum_{k=1}^{\infty} \frac{1}{(k\omega)^2 + g^2} \\ &= \frac{1}{g^2} \sum_{k=0}^{\infty} \frac{1}{(\frac{k\omega}{g})^2 + 1} \\ &\le \frac{1}{g^2} \int_0^\infty \frac{1}{s^2+1} \mathrm{d}s \;=\; \frac{1}{g^2} \Big[\arctan(s)\Big]_0^\infty \;=\; \frac{1}{g^2}\frac{\pi}{2}\end{aligned}$$

and by setting $M = \dfrac{1}{g^2}\dfrac{\pi}{2}$. □

In order to solve the output regulation problem, for any $t \mapsto w(t)$ we must find a corresponding input, the "friend", such that the output e is constantly equal to zero, namely

$$0 \;=\; q(w(t), \pi(t), 0) + u \; .$$

A consequence of Assumption 5.1 is that the friend, denoted as $\psi : W \to \mathbb{R}$, is well-defined, and is computed as

$$\psi(t) \;:=\; -q(w(t), \pi(t), 0) \; . \tag{5.8}$$

Since the mapping $t \mapsto w(t)$ and $t \mapsto \pi(t)$ are C^1 and T-periodic, and q is C^1, so is $t \mapsto \psi(w(t))$. As a consequence it can be expressed by a Fourier series. In the forthcoming lemma we show that our dynamic regulator (5.3) is able to generate any possible T-periodic friend.

Lemma 5.2. *For any T-periodic C^1 function $t \mapsto w(t)$ and for any $g > 0$, there exist a number $\sigma_0^\circ \in \mathbb{R}$ and a set of vectors $\sigma_k^\circ \in \mathbb{R}^2$, $k \in \mathbb{N}_{>0}$ such that $\sigma^\circ = (\sigma_k^\circ)_{k\in\mathbb{N}} \in \ell^2$, namely there exists a number σ such that*

$$\sum_{k=0}^{\infty} |\sigma_k^\circ|^2 \;\le\; \sigma \, ,$$

and moreover σ° satisfies

$$\begin{aligned}\dot\sigma_0 &= 0 \, , & \sigma_0(0) &= \sigma_0^\circ \, , \\ \dot\sigma_k &= k\,\phi\,\sigma_k \, , & \sigma_k(0) &= \sigma_k^\circ \, , \\ \psi(w(t)) &= \sum_{k=0}^{\infty} M_k^\top \sigma_k(t) \; . \end{aligned} \tag{5.9}$$

with M_k defined by (5.7).

Proof. Since $t \mapsto w(t)$ is a C^1 T-periodic function and q is C^1, also $t \to \psi(w(t))$ is a T-periodic C^1 function. As a consequence it can be written in terms of Fourier series. In particular we have that

$$\psi(t) \;=\; \psi_0 + \sum_{k=1}^{\infty} \psi_k^c \,\cos(k\omega\, t) + \psi_k^s \,\sin(k\omega\, t)$$

We denote $\psi_k = (\psi_k^c, \psi_k^s)^\top$. The solution $\sigma_k(t)$ of (5.9) is given by

$$\sigma_k(t) \;=\; \exp(k\phi\, t)\sigma_k(0) \;=\; \begin{pmatrix} \sigma_k^c(0)\,\cos(k\omega\, t) + \sigma_k^s(0)\,\sin(k\omega\, t) \\ -\sigma_k^s(0)\,\cos(k\omega\, t) + \sigma_k^c(0)\,\sin(k\omega\, t) \end{pmatrix}$$

where we set $\sigma_k(0) = (\sigma_k^c(0), \sigma_k^s(0))^\top$. Also we have

$$M_k^\top \sigma_k(t) \;=\; \left(\frac{\sigma_k^c(0)k\omega - \sigma_k^s(0)g}{(k\omega)^2 + g^2}\right)\cos(k\omega\, t) + \left(\frac{\sigma_k^c(0)g + \sigma_k^s(0)k\omega}{(k\omega)^2 + g^2}\right)\sin(k\omega\, t)$$

By identifying all the terms of the two series $\psi(w(t))$ and $M_k^\top \sigma_k(t)$ we get

$$\frac{\sigma_k^c(0)k\omega - \sigma_k^s(0)g}{(k\omega)^2 + g^2} \;=\; \psi_k^c \,, \qquad \frac{\sigma_k^c(0)g + \sigma_k^s(0)k\omega}{(k\omega)^2 + g^2} \;=\; \psi_k^s \,,$$

which can be written in the compact notation

$$\frac{1}{(k\omega)^2 + g^2}\begin{pmatrix} k\omega & -g \\ g & k\omega \end{pmatrix}\begin{pmatrix} \sigma_k^c(0) \\ \sigma_k^s(0) \end{pmatrix} \;=\; \begin{pmatrix} \psi_k^c \\ \psi_k^s \end{pmatrix}.$$

As a consequence we see that the σ_k° claimed in the statement are given by

$$\sigma_k^\circ \;=\; \begin{pmatrix} k\omega & g \\ -g & k\omega \end{pmatrix}\psi_k$$

In order to prove that $\sigma^\circ \in \ell^2$ note that that

$$\sum_{k=0}^{\infty} |\sigma_k^\circ|^2 \;\leq\; \sum_{k=0}^{\infty} \left|\begin{pmatrix} k\omega & g \\ -g & k\omega \end{pmatrix}\right|^2 |\psi_k|^2 \;\leq\; 2\sum_{k=0}^{\infty} \left((k\omega)^2 + g^2\right)|\psi_k|^2$$

Since $t \to \psi(w(t))$ is C^1, $\frac{d\psi(w(t))}{dt}$ is square integrable and there exists are real number $d > 0$ such that

$$\sum_{k=0}^{\infty} k\,|\psi_k|^2 \;\leq\; d$$

Therefore there exists a σ such that

$$\sum_{k=0}^{\infty} |\sigma_k^{\circ}|^2 \;\leq\; 2\sum_{k=0}^{\infty} \left((k\omega)^2 + g^2\right) |\psi_k|^2 \;\leq\; \sigma \,.$$

□

Now we are in the position to give the proof of Theorem 5.1.

Proof. For the time being, let suppose that g has been fixed and let the $\sigma_k(t)$ be defined according to Lemma 5.2 with M_k computed as (5.7). Consider the change of coordinates

$$\eta_k \;\mapsto\; \tilde{\eta}_k \;:=\; \eta_k - \sigma_k(t)$$

by which the η_k-dynamics transforms as

$$\dot{\tilde{\eta}}_k \;=\; k\phi\, \tilde{\eta}_k + \Gamma e \,.$$

Furthermore, the control input v is modified into

$$v \;=\; \sum_{k=0}^{\infty} M_k^{\top}(\tilde{\eta}_k - M_k e + \sigma_k) \qquad\Longrightarrow\qquad v - \psi \;=\; \sum_{k=0}^{\infty} M_k^{\top}(\tilde{\eta}_k - M_k e) \,.$$

Consider now the Lyapunov function

$$V \;=\; e^2 + \sum_{k=0}^{\infty}(\tilde{\eta}_k - M_k e)^{\top}(\tilde{\eta}_k - M_k e) \,.$$

Since $(\eta_k(0))_{k\in\mathbb{N}} \in \ell^2$ and $(\sigma_k(0))_{k\in\mathbb{N}} \in \ell^2$ also $(\tilde{\eta}_k(0))_{k\in\mathbb{N}} \in \ell^2$. As a consequence $V(0)$ is bounded. Its derivative along the solution is (by dropping the arguments of the functions for compactness) given by

$$\begin{aligned}
\dot{V} \;&=\; -g\, e^2 + eq + ev + \sum_k (\tilde{\eta}_k - M_k e)^{\top}[k\phi\, \tilde{\eta}_k + \Gamma e - M_k(q - ge + v)] \\
&=\; -ge^2 + eq + ev \\
&\qquad + \sum_k (\tilde{\eta}_k - M_k e)^{\top}[k\phi(\tilde{\eta}_k - M_k e) + (\Gamma + \phi M_k + g\, M_k)e - M_k(q + v)] \\
&=\; -g\, e^2 + eq + ev - \sum_k (\tilde{\eta}_k - M_k e)^{\top} M_k\, (q + v) \\
&=\; -g\, e^2 + eq + ev - (v - \psi)(q + v) \\
&=\; -g\, e^2 + e(q + \psi) + e(v - \psi) - (v - \psi)^2 - (v - \psi)(q + \psi) \\
&\leq\; -(g - \tfrac{3}{2})e^2 - \tfrac{1}{2}(v - \psi)^2 + \tfrac{3}{2}(q + \psi)^2
\end{aligned}$$

where we used (5.5) and the fact that $x^\top \phi\, x = 0$ for any $x \in \mathbb{R}^2$. Also recall that

$$|q(w,z,e) + \psi(w)| \;=\; |q(w,z,e) - q(w,\pi(w),0)| \;\leq\; \rho|\tilde{z}| + \rho|e| \;.$$

As a consequence, we also get

$$\dot{V} \;\;\leq\;\; -(g - \tfrac{3}{2} - 2\rho)\, e^2 - \tfrac{1}{2}(v-\psi)^2 + 2\rho|\tilde{z}|^2 \;.$$

Now let consider the Lypaunov function $W = cU + V$ with $c > 0$ to be defined. By using (5.6), its derivative satisfies

$$\begin{aligned} \dot{W} \;&\leq\; -(g - \tfrac{3}{2} - 2\rho)\, e^2 - \tfrac{1}{2}(v-\psi)^2 + 2\rho|\tilde{z}|^2 - c\,\alpha|\tilde{z}|^2 + c\gamma|e|^2 \\ &\leq\; -(g - \tfrac{3}{2} - 2\rho - c\gamma)\, e^2 - \tfrac{1}{2}(v-\psi)^2 - (c\,\alpha - 2\rho)|\tilde{z}|^2 \end{aligned}$$

Finally, with c and $g^\star$ satisfying

$$c \;\geq\; 2\rho/\alpha\,, \qquad\qquad g^\star \;>\; \tfrac{3}{2} + 2\rho + c\gamma\,,$$

and by recalling that $W(0)$ is bounded, we conclude that for any $g \geq g^\star$, $W(t)$ is always decreasing for all $t \geq 0$, the solutions are bounded (in ℓ^2 sense) for all $t \geq 0$ and moreover converge to

$$\lim_{t\to\infty} e(t) = 0\;, \qquad \lim_{t\to\infty}(v(t) - \psi(w(t))) = 0\;, \qquad \lim_{t\to\infty}(z(t) - \pi(w(t))) = 0\;.$$

namely *asymptotic output regulation* is achieved uniformly in the initial conditions. Note that the value of $g^\star$ can be computed beforehand and does not depend on the matrices M_k, avoiding any "circular loop". The *structural robustness* property of regulator (5.4) is a direct consequence of the fact that the arguments used to prove asymptotic regulation does not depend on the particular choice of f and q but only on the values of γ, α and ρ. □

Remarks

- We stress that the proposed regulator (5.4) is *structurally robust* in the sense that it solves the *asymptotic output regulation problem* for a whole family of functions satisfying the previous assumptions. In this regard, with respect to the design procedure commonly proposed in literature (see Isidori et al. (2012), Marconi et al. (2007), Byrnes and Isidori (2004), Seshagiri and Khalil (2005), Memon and Khalil (2010)), the internal-model does not need to be re-designed if the functions f, q change structure.

- The class of nonlinear systems (5.1) could be generalized by considering the class of nonlinear systems with a relative degree μ larger than one (see, for instance, Isidori (1995) and the output regulation solution proposed in Byrnes and Isidori (2004)), namely

$$\begin{aligned}\dot{z} &= f(w, e, C\xi) \\ \dot{\xi} &= A\xi + Be \\ \dot{e} &= q(w, z, \xi, e) + b(w, z, \xi, e)u\end{aligned}$$

  where the state $\xi \in \mathbb{R}^{\mu-1}$, (A, B, C) is a triplet in prime form of dimension $\mu - 1$ and $b(\cdot)$ is any C^1 function lower and upper bounded for any $(w, z, \xi, e) \in \mathbb{R}^r \times \mathbb{R}^n \times \mathbb{R}^\mu \times \mathbb{R}$. Furthermore, also Assumption 5.2 may be relaxed by asking a milder ISS condition. In this case the regulator (5.4) needs to be modified for instance by choosing $u = -\kappa(e) + \text{sat}(v)$ with $\kappa : \mathbb{R} \to \mathbb{R}_{\geq 0}$ a continuous function to be chosen large enough and sat any saturation function with some saturation level to be defined (see for instance Kaliora and Astolfi (2004)). A detailed solution is currently under investigation.

5.2 Regulation in the finite dimensional case

In the previous section we showed how asymptotic regulation can be achieved by dynamic regulator with infinite (though countable) dimension, which is in general not implementable. As a consequence we consider the case in which the dynamic control law has a finite dimension. The aim of this section is to prove that the dynamic control law (5.4)-(5.3) with a finite number of "oscillators" solves the *structurally robust practical output regulation problem* for the system (5.1) that we recall here

$$\begin{aligned} \dot{z} &= f(w, z, e) \\ \dot{e} &= q(w, z, e) + u \end{aligned} \tag{5.1}$$

with $z \in \mathbb{R}^n$, regulated output $e \in \mathbb{R}$, control input $u \in \mathbb{R}$, initial conditions $(z(0), e(0)) \in Z \times E$, f, q C^1 functions and $t \mapsto w(t)$ any C^1 and T-periodic function. In other word, with g fixed, for any given $\epsilon > 0$, we want to show that we are able to find an integer $L \geq 0$ (by using the same notation introduced in Section 4.3) such that the dynamic regulator implemented as

$$\begin{aligned} \dot{\eta}_0 &= e \\ \dot{\eta}_k &= k\phi\, \eta_k + \Gamma e \qquad k = 1, \ldots, L \end{aligned} \tag{5.10}$$

with $\eta = (\eta_0, \ldots, \eta_L) \in \mathbb{R}^{1+2L}$, with $\eta_0 \in \mathbb{R}$ and $\eta_k \in \mathbb{R}^2$ and $\eta(0)$ ranging in some given compact set $\Xi \subset \mathbb{R}^{1+2L}$, and

$$\begin{aligned} v &= \sum_{k=0}^{L} M_k^\top (\eta_k - M_k e) \\ u &= -g\, e + v \end{aligned} \tag{5.11}$$

with the M_k computed as solution of (5.5), satisfies

$$\limsup_{t\to\infty} \sqrt{\frac{1}{T}\int_t^{t+T} |e(s)|^2\, ds} \;<\; \epsilon \tag{5.12}$$

for any T-periodic C^1 function $t \mapsto w(t) \in W$ and for any initial condition $(z(0), e(0)) \in Z \times E$. Recall that by applying Proposition 4.1 we know that if we are able to design u such that the solutions $\eta_k(t)$ are bounded then the first k Fourier coefficients of the output e are equal to zero, namely

$$\lim_{t\to\infty} \int_t^{t+T} \cos(k\omega\, t) e(t) dt \;=\; 0\,, \qquad \lim_{t\to\infty} \int_t^{t+T} \sin(k\omega\, t) e(t) dt \;=\; 0\,, \tag{5.13}$$

for $k = 0, 1, \ldots, L$. Furthermore, if L is large enough, then (5.12) is satisfied.

Motivated by the infinite-dimensional case, suppose that the closed loop system (5.1)-(5.10)-(5.11) has bounded trajectories and satisfies Proposition 4.1. By denoting with $(\pi^L(t), \varepsilon^L(t), \sigma^L(t))$ the asymptotic periodic solution of (z, e, η), we would get

$$\begin{aligned} \dot{\pi}^L(t) &= f(w(t), \pi^L(t), \varepsilon^L(t)) \\ \dot{\varepsilon}^L(t) &= q(w(t), \pi^L(t), \varepsilon^L(t)) - d\,\varepsilon^L(t) + \psi_\eta^L(t) \end{aligned} \tag{5.14}$$

where $d = g + \sum_{k=0}^{L} M_k^\top M_k$ and where

$$\begin{aligned} \dot{\sigma}_k^L &= k\phi\, \sigma_k^L(0)\ ,\ =\ \sigma_k^\circ\ , \\ \psi_\eta^L(w(t)) &= \sum_{k=0}^{L} M_k^\top \sigma_k^L(t)\ , \end{aligned} \tag{5.15}$$

for some σ_k° which can be computed as in Lemma 5.2. It is worth noticing that the previous solution may be in general not unique (for instance when $|w(\cdot)|_\infty$ is very large), namely there could be exists infinite solutions $(\pi^L(t),\ \varepsilon^L(t)\ \psi^L(t))$ satisfying

$$\begin{aligned} \dot{\pi}^L(t) &= f(w(t), \pi^L(t), \varepsilon^L(t)) \\ \dot{\varepsilon}^L(t) &= q(w(t), \pi^L(t), \varepsilon^L(t)) - d\,\varepsilon^L(t) + \psi^L(t) \end{aligned}$$

and equations (5.12) and (5.13). However, under Assumption 5.2, exponential stability of the solution $|z - \pi(t)|$ and of the equilibrium point $e = 0$ would guarantee uniqueness of the solution $(\pi^L(t),\ \varepsilon^L(t))$ for any chosen $\psi^L(t)$. As a consequence, in the following, we will assume that for any $L \in \mathbb{N}$ there exists only one periodic solution satisfying (5.14) - (5.15) and (5.12) - (5.13) and

$$(\pi^L(t), \varepsilon^L(t)) \in Z \times E \qquad \forall\, t \geq 0\ .$$

In the forthcoming theorem we will show that if the gain g is chosen large enough, a solution does exist for any L. In particular we will state that the choice of g is independent on L, which may be considered as an additional design parameter to fulfil equations (5.12) and (5.13).

Theorem 5.2. *The regulator* (5.10)-(5.11) *solves the structurally robust practical output regulation problem for all the systems of the form* (5.1) *with f satisfying Assumption 5.1 and 5.2, and q satisfying* (5.2), *namely for any T-periodic C^1 function $t \mapsto w(t) \in W$ and for any set $Z \times E \subset \mathbb{R}^n \times \mathbb{R}$ there exists a $g^\star > 0$, such that, for any fixed $g \geq g^\star$ we have:*

(i) *for any $L \in \mathbb{N}$ the trajectories $(z(t), e(t), \eta(t))$ of the closed-loop system* (5.1)-(5.10)-

(5.11) *starting in* $(z(0), e(0)) \in Z \times E\times$ *and* $\eta(0) \in \Xi \subset \mathbb{R}^{1+2L}$ *are bounded and satisfy* (5.13) *for all* $k = 0, 1, \ldots, L$;

(ii) *for any (small) real number* $\epsilon > 0$ *we can find a positive integer* $L^\star(\epsilon)$, $\lim_{\epsilon\to 0^+} L^\star(\epsilon) = \infty$, *such that, for any* $L \geq L^\star$, (5.12) *holds.*

Proof. Let for the time suppose that g has been fixed and let the M_k be computed as (5.7). We want to show that with this same g the closed loop system (5.1)-(5.10)-(5.11) has bounded trajectories for any $L \geq 0$. So, given any L, let $(\pi^L(t), \varepsilon^L(t), \sigma^L_\eta(t))$ be its corresponding periodic solution satisfying (5.14)- (5.15) and (5.13) with $d = g + \sum_{k=0}^{L} M_k^\top M_k$. Note that $d \leq g + M$ for any L with M given by Lemma 5.1. Recall that $\varepsilon^L(t)$ satisfies

$$\dot\varepsilon^L(t) \;=\; q(w(t), \pi^L(t), \varepsilon^L(t)) - g\,\varepsilon^L(t) + \psi^L(t) - \sum_{k=0}^{L} M_k^\top M_k\,\varepsilon^L(t)$$

whereas the functions σ^L_k satisfy

$$\dot\sigma^L(t) \;=\; k\phi\;\sigma^L_k(t) + \Gamma\,\varepsilon^L(t)$$

because

$$\int_0^T \exp(-k\phi\,t)\Gamma\,\varepsilon^L(t)\,\mathrm{d}t \;=\; 0$$

is always verified for all $k = 0, 1, \ldots, L$ by assumption. Now let consider the following change of coordinates

$$\begin{array}{rcrcl} e & \mapsto & \tilde e & := & e - \varepsilon^L(t)\ , \\ \eta_k & \mapsto & \tilde\eta_k & := & \eta_k - \sigma^L_k(t)\ , \\ z & \mapsto & \tilde z & := & z - \pi^L(t)\ , \end{array}$$

In the new coordinates we get

$$\begin{array}{rcl} \dot{\tilde e} & = & \tilde q(w, \tilde z, \tilde e) - g\tilde e + \tilde v \\ \dot{\tilde\eta}_k & = & k\phi\,\tilde\eta_k + \Gamma\tilde e \\ \tilde v & = & \displaystyle\sum_{k=0}^{L} M_k^\top(\tilde\eta_k - M_k\tilde e) \end{array}$$

where

$$\tilde q(w, \tilde z, \tilde e) \;=\; q(w(t), \tilde z + \pi^L(t), \tilde e + \varepsilon^L(t)) - q(w(t), \pi^L(t), \varepsilon^L(t))\ .$$

Consider now the Lyapunov function

$$V = \tilde{e}^2 + \sum_{k=0}^{L} (\tilde{\eta}_k - M_k \tilde{e})^\top (\tilde{\eta}_k - M_k \tilde{e})$$

Its derivative along the solution is (by dropping the arguments of the functions for compactness) given by

$$\begin{aligned}
\dot{V} &= -g\,\tilde{e}^2 + \tilde{e}\tilde{q} + \tilde{e}\tilde{v} + \sum_k (\tilde{\eta}_k - M_k\tilde{e})^\top [k\phi\tilde{\eta}_k + \Gamma\tilde{e} - M_k(\tilde{q} - g\tilde{e} + \tilde{v})] \\
&= -g\,\tilde{e}^2 + \tilde{e}\tilde{q} + \tilde{e}\tilde{v} \\
&\qquad + \sum_k (\tilde{\eta}_k - M_k\tilde{e})^\top [k\phi(\tilde{\eta}_k - M_k\tilde{e}) + (\Gamma + M_k + kM_k)\tilde{e} - M_k(\tilde{q} + \tilde{v})] \\
&= -g\,\tilde{e}^2 + \tilde{e}\tilde{q} + \tilde{e}\tilde{v} - \sum_k (\tilde{\eta}_k - M_k\tilde{e})^\top M_k\,(\tilde{q} + \tilde{v}) \\
&= -g\,\tilde{e}^2 + \tilde{e}\tilde{q} + \tilde{e}\tilde{v} - \tilde{v}(\tilde{q} + \tilde{v}) \\
&\leq -(g - \tfrac{3}{2})\tilde{e}^2 - \tfrac{1}{2}\tilde{v}^2 + \tfrac{3}{2}\tilde{q}^2 \\
&\leq -(g - \tfrac{3}{2} - 2\rho)\,\tilde{e}^2 - \tfrac{1}{2}\tilde{v}^2 + 2\rho|\tilde{z}|^2
\end{aligned}$$

where we used in the last step

$$|\tilde{q}(w, z, \tilde{e})| = |q(w, \tilde{z} + \pi^L(t), \tilde{e} + \varepsilon^L(t)) - q(w, \pi^L(t), \varepsilon^L(t))| \leq \rho|\tilde{z}| + \rho|\tilde{e}|$$

and the inequality $(a+b)^2 \leq 2a^2 + 2b^2$. As in Theorem 5.1 we have now to study the interconnection between the $(\tilde{e}, \tilde{\eta})$-dynamics and the $\tilde{z}$-zero dynamics. As a consequence, if the bound (5.6) in Assumption 5.2 holds with the $\tilde{f}$ in given by

$$\tilde{f}(\tilde{z}, \tilde{e}, t) := f(w(t), \tilde{z} + \pi^L(t), \tilde{e} + \varepsilon^L(t)) - f(w(t), \pi^L(t), \varepsilon^L(t))\ ,$$

namely

$$\frac{\partial U}{\partial \tilde{z}}(\tilde{z})\ \tilde{f}(\tilde{z}, \tilde{e}, t) \leq -\alpha|\tilde{z}|^2 + \gamma|\tilde{e}|^2\ ,$$

then we can show (see proof of Theorem 5.1) that there exists a $g^\star > 0$ such that, for any $g \geq g^\star$ the solutions are bounded and asymptotically converges to the set $\big\{(\tilde{z}, \tilde{v}, \tilde{\eta}) = 0\big\}$, implying

$$\lim_{t\to\infty} e(t) = \varepsilon^L(t)\ , \qquad \lim_{t\to\infty} \sum_{k=0}^{L} M_k^\top \left(\eta_k(t) - \sigma_k^L(t)\right) = 0\ , \qquad \lim_{t\to\infty} z(t) = \pi^L(t)$$

which proves the point (i) of the theorem by recalling that $\varepsilon^L(t)$ satisfies (5.13).

To prove the second part of the Theorem we can use off the shelf Proposition 4.1. With g fixed such that $g > g^\star$, let Ψ and F be the real numbers defined as

$$\begin{aligned} \Psi &= \sup_{w\in W,\, L\in\mathbb{N}} |\psi^L(w)|\,, \\ F &= \sup_{w\in W,\, z\in Z,\, e\in E,\, d\in(g,g+M)} |\, q(w,z,e) - de\,| \ . \end{aligned}$$

It follows that $|\dot{e}(t)| \leq F + \Psi$ for all $t \geq 0$. Suppose now there exists a differentiable T-periodic solution $e(t+T) = e(t)$. As a consequence $e(t)$ and $\dot{e}(t)$ can be expressed by a Fourier Series (we use here the complex notation for compactness)

$$\begin{aligned} e(t) &= \sum_{k=0}^{\infty} c_k^0 \exp(\mathrm{i}k\omega t)\ , \\ \dot{e}(t) &= \sum_{k=0}^{\infty} \mathrm{i}\omega c_k^0 \exp(\mathrm{i}k\omega t) \ = \ \sum_{k=0}^{\infty} c_k' \exp(\mathrm{i}k\omega t) \end{aligned}$$

where $c_k' \ = \ \mathrm{i}k\,\omega c_k^0$. By using Parseval's Identity we get

$$\sqrt{\frac{1}{T}\int_0^T |\dot{e}(s)|^2 ds} \ = \ \sqrt{2\sum_{k=0}^{\infty}(c_k')^2} \ \leq \ F + \Psi$$

Now assume that $c_k^0 \ = 0$ for any $k \ = \ 0, 1, \ldots, L$. As a consequence, by using the definition of c_k'

$$\omega^2(L+1)^2 \sum_{k=L+1}^{\infty} (c_k^0)^2 \ \leq \ \sum_{k=L+1}^{\infty} (k\omega c_k^0)^2 \ \leq \ \frac{(F+\Psi)^2}{2}$$

Again, by using Parseval's Identity we also have

$$\frac{1}{T}\int_0^T |e(s)|^2 ds \ \leq \ \frac{(F+\Psi)^2}{\omega^2(L+1)^2}$$

As a consequence let $L^\star$ be chosen as

$$L^\star \ = \ \left\lceil \frac{(F+\Psi)T}{2\pi\epsilon} \right\rceil$$

It is evident that $\lim_{\epsilon\to 0} L^\star(\epsilon) = \infty$. Finally, by using the previous bounds, we can easily prove that for any $L \geq L^\star$. the periodic solution $e(t+T) = e(t)$ satisfies $\displaystyle\frac{1}{T}\int_0^T |e(s)|^2 ds \ \leq \ \epsilon\ .$ □

5.3 Simulation Results

We consider the following minimum-phase nonlinear systems

$$\begin{aligned} \dot{z} &= f(w, z, e) \\ \dot{e} &= q(w, z, e) + u \end{aligned} \tag{5.16}$$

wtth $z \in \mathbb{R}$, $e \in \mathbb{R}$, $u \in \mathbb{R}$ and nominal functions

$$\begin{aligned} f(0, z, e) &= -z + z^2 e \ , \\ q(0, z, e) &= z + e \ . \end{aligned}$$

The control law is chosen as

$$\begin{aligned} u &= -ge + v \\ v &= c \sum_{k=0}^{L} M_k^\top (\eta_k - M_k e) \end{aligned} \tag{5.17}$$

where c is a positive number which may be used in order to improve the performances (namely the rate of convergence) of the forwarding control law. In the simulations we considered the case in which $g = 4$ and $c = 10$ and the initial conditions are chosen as $(z(0), e(0)) = (-\sqrt{2}, \sqrt{2})$. The exogenous signal w is generated as

$$w(t) = \sinh \left(\sum_{i=1}^{4} a_i \sin(\omega_i t + \phi_i) \right)$$

with

$$\begin{aligned} a_1 &= 0.25 \ , & \omega_1 &= \omega \ , & \phi_1 &= 0.1 \ , \\ a_2 &= 0.25 \ , & \omega_2 &= 2\omega \ , & \phi_2 &= 0.2 \ , \\ a_3 &= 0.25 \ , & \omega_3 &= 3\omega \ , & \phi_3 &= -0.4 \ , \\ a_4 &= 0.25 \ , & \omega_4 &= 5\omega \ , & \phi_4 &= 0.3 \ , \end{aligned}$$

and $\omega = 1$. The functions f and q are chosen as

$$\begin{aligned} f(w, z, e) &= -z + z^2 e + w - wz \ , \\ q(w, z, e) &= z + e - w + 0.3 w^2 \ . \end{aligned}$$

with the friend defined as

$$\begin{aligned} \dot{\pi}(t) &= f(w, \pi(t), 0) \\ \psi(w) &= -q(w(t), \pi(t), 0) \end{aligned}$$

As expected by Theorem 5.2, with $g > 0$ fixed, the trajectories of the closed-loop system remain bounded for any choice of L (numerical simulations have been done up to $L = 50$). Figure 5.1 **a)** shows the spectrum of the friend $\psi(t)$. It contains harmonics at frequencies $f = 0\text{Hz}, 1\text{Hz}, 2\text{Hz}, 3\text{Hz}, 4\text{Hz}, 5\text{Hz}$ and higher harmonics. Figure 5.1 **b)** shows the spectrum of the output e when the internal model is disconnected, namely $v = 0$. As expected it contains harmonics at all frequencies of the friend. Figure 5.1 **c)** shows the spectrum of the output e in presence of an internal model with $L = 2$. As expected by the result of Theorem 5.2the frequencies $f = 0\text{Hz}, 1\text{Hz}, 2\text{Hz}$ disappear from the spectrum. The Figure 5.2 shows **a)** the steady-state behaviour of the control input $u(t)$ when the internal model is not present, **b)** when $L = 2$, and **c)** when $L = 5$. It can be easily seen that by augmenting the number of harmonics the control input $u(t)$ approximates better and better the correct steady-state behaviour of the friend $\psi(t)$. Finally in Table 5.1 we show how the L^2 norm and the L^∞ norm of the error change in the aforementioned cases. The simulations confirm the results of Theorem 5.2.

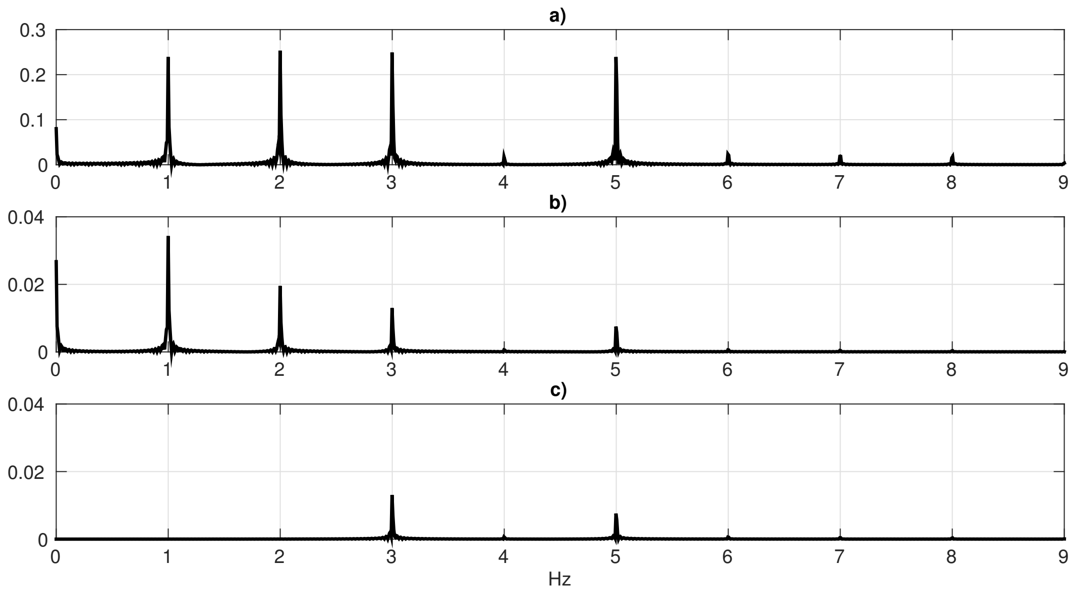

Figure 5.1: Plot **a)** shows the spectrum of the friend $\psi(t)$. Plot **b)** shows the spectrum of the output steady-state $e(t)$ when there is no internal model (namely $v = 0$). Plot **c)** shows the spectrum of the output steady-state $e(t)$ when the dimension of the internal model is $L = 2$.

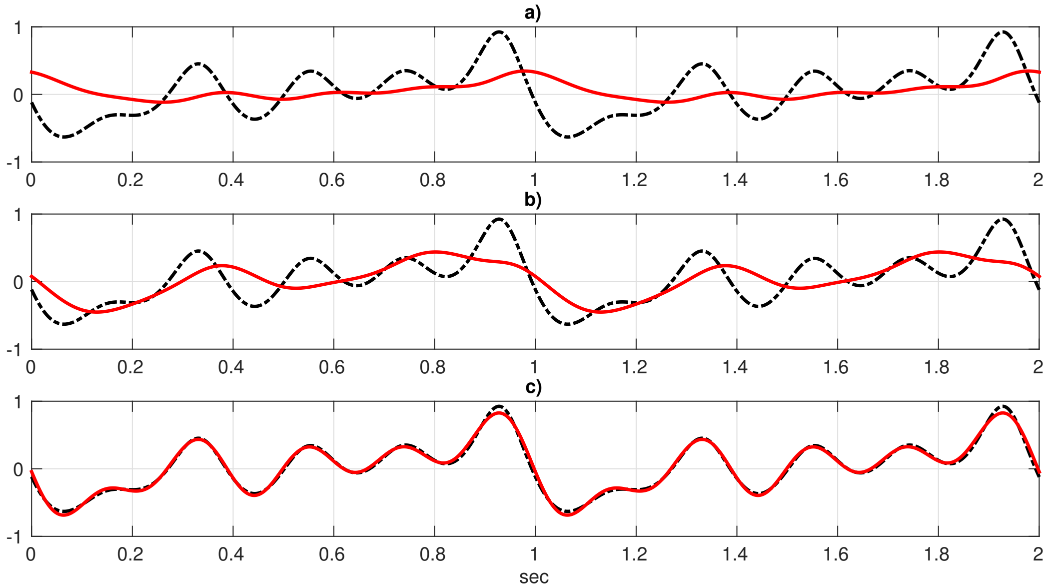

Figure 5.2: Behaviours of the friend and the control input. Black line: steady-state of the friend $\psi(t)$. Red line: steady-state of the control input $u(t)$ when **a)** there is no internal model (namely $v = 0$), **b)** the dimension of the internal model is $L = 2$, **c)** the dimension of the internal model is $L = 5$.

$v = 0$	$\|e\|_2 = 1.0266$	$\|e\|_\infty = 0.0864$
$L = 2$	$\|e\|_2 = 0.6030$	$\|e\|_\infty = 0.0224$
$L = 5$	$\|e\|_2 = 0.1536$	$\|e\|_\infty = 0.0021$

Table 5.1: L^2 norm and L^∞ norm of the error.

Conclusion

THE structurally robust output regulation problem in presence of periodic disturbance/references for nonlinear systems has been investigated. The purpose of the novel methodology introduced in Chapter 4 is to extend the linear design proposed by Francis and Wonham in the 70's to nonlinear systems. A crucial observation is that in the nonlinear case the internal model must not only be able to generate inputs corresponding to the trajectories of the system, but also a number of higher order nonlinear deformations. As a consequence the main idea of the proposed approach is to process the regulated output with a bench of linear oscillators whose frequencies correspond with the basic frequency of the disturbance and its multiples. Forwarding technique is proposed as design tool to stabilize the cascade system - internal model. This design is made for the unforced nominal system (namely when the disturbance is not acting and in absence of parameter uncertainties) and guarantees the origin to be locally exponentially stable. As a consequence, if the disturbance is small enough, the trajectories of the forced nominal closed loop system (namely when the disturbance is acting) are uniquely defined, bounded and periodic with the same period of the disturbance. It has been shown that the effect of each oscillator is to annihilate, in the output's spectrum, the Fourier coefficient corresponding to the oscillator frequency. By adding enough oscillators it is then possible to make arbitrarily small the L^2 norm of the regulated output and thus obtaining practical output regulation. Since by adding oscillators in the internal model we are increasing the number of poles in the imaginary axes, we may reduce the robustness margin with respect to the disturbance. As a consequence the maximum disturbance magnitude allowed may decrease while increasing the number of oscillators. We refer to this phenomenon by saying that we solve the *weak practical output regulation problem*.

The proposed approach require the knowledge of the unforced nominal model, but because of exponential stability, persistence of equilibrium under model uncertainties of the unforced closed loop system is preserved. As a consequence also existence of periodic solutions for the forced perturbed closed loop system is guaranteed and *structurally robust weak practical output regulation* is achieved. The proposed methodology can be applied to multi-input multi-output (with a number of regulated outputs smaller or equal than the number of control inputs) nonlinear systems, is robust to structural uncertainties, and does not require any additional (restrictive) assumption such as minimum-phase, relative degree of normal form.

In Chapter 5 this approach is specialized to a single-input single-output nonlinear system with unitary relative degree which is strongly minimum-phase namely its zero-dynamics are exponentially stable. It has been shown that with an internal model (not implementable) embedding an infinite number of oscillators, *structurally robust asymptotic output regulation* is achieved. Evidently, when the number of oscillators is finite, only practical regulation is achieved. A general framework where these assumptions are relaxed (for instance by considering the class of minimum-phase nonlinear systems with a relative degree possibly higher than one) is currently under study.

The research along this novel methodology is far from being concluded and only a first step has been done. The lack of the existence of a control Lyapunov function with forwarding technique does not help in understanding the aforementioned phenomenon of reducing of disturbance magnitude when augmenting the number of oscillators, although the minimum-phase case showed that under stronger conditions this phenomenon can be avoided. An interesting development is the case of periodic disturbances uncertain in the period. Adaptive techniques could be adopted in this framework in order to design an internal model which embeds a bench of oscillators with a basis frequency that may be adapted to the nominal one in order to reduce the norm of the error.

Appendices

A

Technical Lemmas for Block Tridiagonal Matrices

The following appendix is devoted to the study of stability of matrices possessing a particular block tridiagonal structure. Throughout this section we will use the following notation. The matrices $A \in \mathbb{R}^{2\times 2}$, $B \in \mathbb{R}^{2\times 1}$, $B_i \in \mathbb{R}^{i\times 1}$, and $C \in \mathbb{R}^{1\times 2}$ are defined as

$$A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C := \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad B_i := \begin{pmatrix} 0 & 0 & \dots 0 & 1 \end{pmatrix}^\top$$

Given the coefficients c_{i1} and c_{i2}, $i > 0$ we define the matrix $K_i = (c_{i1}, c_{i2})^\top$ and the following matrices $E_i \in \mathbb{R}^{2\times 2}$, $Q_i \in \mathbb{R}^{2\times 2}$ and $N \in \mathbb{R}^{2\times 2}$ as

$$E_i := A - K_i C = \begin{pmatrix} -c_{i1} & 1 \\ -c_{i2} & 0 \end{pmatrix}, \quad Q_i := K_i B^\top = \begin{pmatrix} 0 & c_{i1} \\ 0 & c_{i2} \end{pmatrix}, \quad N := BB^\top = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We define in a recursive manner the matrices M_i as

$$M_1 := E_1\,, \qquad M_i := \begin{pmatrix} M_{i-1} & \bar{N}_i \\ \bar{Q}_i & E_i \end{pmatrix}, \qquad i > 1,$$

with $\bar{N}_i := B_{2(i-1)}\, B^\top$, $\bar{Q}_i := K_i B_{2(i-1)}^\top$. The matrices M_i are characterized by a block-tridiagonal structure

$$M_i := \begin{pmatrix} E_1 & N & 0 & & \dots & \dots & 0 \\ Q_2 & E_2 & N & \ddots & & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & Q_j & E_j & N & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & Q_{i-1} & E_{i-1} & N \\ 0 & \dots & \dots & \dots & 0 & Q_i & E_i \end{pmatrix}. \tag{A.1}$$

We denote with M the matrix M_i of dimension $2n-2$.

A.1 Eigenvalues assignment

Lemma A.1. *Let $\mathcal{P}(\lambda) = \lambda^{2n-2} + m_1\lambda^{2n-3} + ... + m_{2n-3}\lambda + m_{2n-2}$ be an arbitrary Hurwitz polynomial. There exists a choice of positive (c_{i1}, c_{i2}), $i = 1, \dots, n-1$, such that the characteristic polynomial of M coincides with $\mathcal{P}(\lambda)$.*

Proof. The proof of this Lemma is done givin a constructive procedure, namely we show a procedure to assign the eigenvalues of M given an arbitrary Hurwitz polynomial for the characteristic polynomial of M.

Procedure to assign the eigenvalues of M Consider the matrices $M_i \in \mathbb{R}^{2i-2} \times \mathbb{R}^{2i-2}$ recursively defined as

$$M_1 = E_1\,, \qquad M_i = \begin{pmatrix} M_{i-1} & \bar{N}_i \\ \bar{Q}_i & E_i \end{pmatrix}, \qquad i = 2, \dots, n-1$$

where $\bar{N}_i = \mathrm{col}(0_{2(i-2)\times 2}, N)$, $\bar{Q}_i = (0_{2(i-2)\times 2}, Q_i)$ and E_i, $i = 1, \dots, n-1$, Q_i, $i = 2, \dots, n-1$, and N are defined as in the definition of M. Note that $M = M_{n-1}$ and, by letting $K_i = (c_{i1}\ c_{i2})^T$, note that Q_i and E_i depend on K_i, while M_i depends on $K_1, \dots, K_i$. We let $\mathcal{P}_{M_i}(\lambda) = \lambda^{2i} + m_1^i\lambda^{2i-1} + \dots + m_{2i-1}^i\lambda + m_{2i}^i$ and $\mathcal{P}_{M_{i-1}}(\lambda) = \lambda^{2i-2} + m_1^{i-1}\lambda^{2i-3} + \dots + m_{2i-3}^{i-1}\lambda + m_{2i-2}^{i-1}$ the characteristic polynomials of M_i and M_{i-1}, and we use the notation $m_{[1,j]}^i = \mathrm{col}(m_1^i, \dots, m_j^i) \in \mathbb{R}^j$, $m_{[1,k]}^{i-1} = \mathrm{col}(m_1^{i-1}, \dots, m_k^{i-1}) \in \mathbb{R}^k$ for some $j \leq 2i$ and $k \leq 2i-2$.
The characteristic polynomial of $\mathcal{P}_{M_i}(\lambda)$ can be deduced by decomposing $(\lambda I - M_i)$

as follows

$$\mathcal{P}_{M_i}(\lambda) \;=\; \det\left(\begin{array}{c|cc|cc} \lambda I - M_{i-2} & \begin{matrix} 0 & 0 \\ 0 & -1 \end{matrix} & & & \\ \hline \begin{matrix} 0 & -c_{(i-1)1} \\ 0 & -c_{(i-1)2} \end{matrix} & \lambda + c_{(i-1)1} & -1 & 0 & 0 \\ & c_{(i-1)2} & \lambda & 0 & -1 \\ \hline & 0 & -c_{i1} & \lambda + c_{i1} & -1 \\ & 0 & -c_{i2} & c_{i2} & \lambda \end{array}\right).$$

By expanding the determinant with respect to the last column we obtain

$$\begin{aligned} \mathcal{P}_{M_i}(\lambda) &= \lambda(\lambda + c_{i1})\mathcal{P}_{M_{i-1}}(\lambda) \\ &\quad + c_{i2}\left[\mathcal{P}_{M_{i-1}}(\lambda) - \lambda(\lambda + c_{(i-1)1})\mathcal{P}_{M_{i-2}}(\lambda)\right] . \end{aligned}$$

Hence, simple, although lengthy, computations show that the coefficients $m^i_{[1,2i]}$ of $\mathcal{P}_{M_i}(\lambda)$ and $m^{i-1}_{[1,2i-2]}$ of $\mathcal{P}_{M_{i-1}}(\lambda)$ are related as follow

$$\begin{aligned} m^i_{[1,2i-2]} &= (I_{2i-2} + c_{i1}F)m^{i-1}_{[1,2i-2]} + c_{i1}\,\mathrm{v}_1 \\ m^i_{2i-1} &= c_{i1}\, m^{i-1}_{2i-2} \\ m^i_{2i} &= c_{i2}\, m^{i-1}_{2i-2} \end{aligned} \tag{A.2}$$

where $\mathrm{v}_1 \in \mathbb{R}^{2i-2}$ is the zero vector with a 1 in the first position, and F is the zero matrix with the identity matrix I_{2i-3} in the lower left block, namely

$$F = \begin{pmatrix} 0_{1\times(2i-3)} & 0 \\ I_{2i-3} & 0_{(2i-3)\times 1} \end{pmatrix}_{(2i-2)\times(2i-2)} .$$

Note that $(I_{2i-2} + c_{i1}F)$ is invertible for all c_{i1}. Hence, from the first equation of (A.2), one obtains

$$m^{i-1}_{[1,2i-2]} = \Lambda(m^i_{[1,2i-2]},\, c_{i1})$$

where

$$\Lambda(m^i_{[1,2i-2]},\, c_{i1}) = (I_{2i-2} + c_{i1}F)^{-1}\,(m^i_{[1,2i-2]} - c_{i1}\,\mathrm{v}_1\,),$$

which, embedded in the second and in the third of (A.2), yields the relations

$$\sigma_1(m^i_{[1,2i-1]}, c_{i1}) = 0\,, \qquad c_{i2} = \sigma_2(m^i_{[1,2i]},\, c_{i1})$$

where

$$\sigma_1(m^i_{[1,2i-1]}, c_{i1}) = c_{i1}\, \mathrm{v}_2^T\, \Lambda(m^i_{[1,2i-2]},\, c_{i1}) - m^i_{2i-1}$$

$$\sigma_2(m^i_{[1,2i]},\, c_{i1}) = \frac{m^i_{2i}}{\mathrm{v}_2^T\, \Lambda(m^i_{[1,2i-2]},\, c_{i1})}$$

in which $\mathrm{v}_2 \in \mathbb{R}^{2i-2}$ is the zero vector with a 1 in the last position. We observe that $\sigma_1(\cdot,\cdot)$ is a polynomial in c_{i1} of odd order $2i-1$. As a consequence, for any $m^i_{[1,2i-1]}$ there always exists at least one real c_{i1} fulfilling $\sigma_1(m^i_{[1,2i-1]}, c_{i1}) = 0$.

The previous results can be used to set up a "basic assignment algorithm" that is then used iteratively to solve the eigenvalues assignment of the matrix M.

Basic assignment algorithm. Let $\bar{\mathcal{P}}_i(\lambda) = \lambda^{2i} + \bar{m}^i_1\lambda^{2i-1} + \ldots + \bar{m}^i_{2i}$ be an arbitrary polynomial. Then, there exist a real $\bar{K}_i = (\bar{c}_{i1}, \bar{c}_{i2})^T$ and a polynomial $\bar{\mathcal{P}}_{i-1}(\lambda) = \lambda^{2i-1} + \bar{m}^{i-1}_1\lambda^{2i-2} + \ldots + \bar{m}^{i-1}_{2i-2}$ such that

$$\begin{array}{rcl} K_i &=& \bar{K}_i \\ \mathcal{P}_{M_{i-1}} &=& \bar{\mathcal{P}}_{i-1}(\lambda) \end{array} \qquad \Rightarrow \qquad \mathcal{P}_{M_i}(\lambda) = \bar{\mathcal{P}}_i(\lambda)\,.$$

As a matter of fact, by letting $\bar{m}^i_{[1,2i-1]}$ the coefficients of $\bar{\mathcal{P}}_i(\lambda)$, it is possible to take $\bar{c}_{i1}$ as a real solution of $\sigma_1(\bar{m}^i_{[1,2i-1]}, c_{i1}) = 0$, $\bar{c}_{i2} = \sigma_2(\bar{m}^i_{[1,2i]},\, \bar{c}_{i1})$, and to take the coefficients $\bar{m}^{i-1}_{[1,2i-2]}$ of the polynomial $\bar{\mathcal{P}}_{i-1}(\lambda)$ as $\bar{m}^{i-1}_{[1,2i-2]} = \Lambda(\bar{m}^i_{[1,2i-2]},\, \bar{c}_{i1})$. □

With the previous algorithm in hand, the design of $K_1, \ldots, K_{n-1}$ to assign an arbitrary characteristic polynomial to M, can be then immediately done by the following steps:

1) With $\bar{\mathcal{P}}_{n-1}(\lambda)$ the desired characteristic polynomial of M, compute $(\bar{K}_{n-1}, \bar{\mathcal{P}}_{n-2}(\lambda))$ by running the basic assignment algorithm with $i = n-1$.

2) Compute iteratively $(\bar{K}_i, \bar{\mathcal{P}}_{i-1}(\lambda))$ by running the basic assignment algorithm for $i = n-2, \ldots, 2$.

3) Compute $\bar{K}_1 = (\bar{c}_{i1}, \bar{c}_{i2})^T$ so that $\lambda^2 + c_{i1}\lambda + c_{i2} = \bar{\mathcal{P}}_1(\lambda)$.

The procedure is summarized in the forthcoming Matlab code.

```
%% LOW POWER HIGH GAIN OBSERVER
%Procedure to find the coefficients of the observer,
%given a chosen set of eigenvalues.
%INITIALIZATION
n = ; %Dimension of the system. Set n>1.
dim = 2*n-2; %Dimension of the observer.
lambda_M = zeros(dim,1); %Eigenvalues of the observer.
c = zeros(dim,1); %Coefficients of the observer.
ell = 1; %High-gain parameter. Put 1 as default
%EXAMPLE TO SET THE EIGENVALUES (SKIP IF OTHER CHOICE IS MADE)
%In the following code we set all the eigenvalues
%between v_min and v_max with a constant step.
v_min = -1; % set a negative number
v_max = -1.9; % set a negative number smaller then v_min
step = (abs(v_max)-abs(v_min))/(dim-1);
lambda_M(1) = v_min;
for j = 2: dim
        lambda_M(j) = lambda_M(j-1)-step;
end
%PROCEDURE TO FIND COEFFICIENTS
p = poly(lambda_M);
m = zeros(dim,1);
for j = 1 : dim
    m(j) = p(j+1);
end
for j = 1 : dim
    m(j) = p(j+1);
end
if n == 2
    c(1) = m(1);
    c(2) = m(2);
elseif n > 2
    for iter = 1: n-2;
        k = dim-2*iter+2;
        pp = zeros(k,1);
        pp(1) = 1;
        for j = 2 : k
            pp(j) = (-1)^(j-1)*m(j-1);
        end
        r = roots(pp);
        c(k-1)=0;
        epsilon = 1e-4;
        for j=1:k-1
            if abs(imag(r(j))) < epsilon
                c(k-1) = r(j);
            end
```

```
        end
        c(k) = c(k-1)*m(k)/m(k-1);
        if iter < n-2
            t = c(k-1);
            T = eye(k-2);
            for j=2:k-2
                T_j = [zeros(j-1,k-2); eye(k-1-j) zeros(k-1-j,j-1)];
                T = T + (-1)^(j-1)*t^(j-1)*T_j;
            end
            new_m = T*(m(1:k-2,1)-[c(k-1); zeros(k-3,1)]);
            m = new_m;
        else
            c(k-3) = m(1)-c(k-1);
            c(k-2) = m(2) - c(k-1)*c(k-3);
        end
    end
end
%END PROCEDURE TO FIND COEFFICIENTS
%CONSTRUCTION OF THE MATRIX M
%Initialization
A2 = [0 1; 0 0];
C2 = [1 0];
H2 = [0 1];
D2 = diag([ell, ell^2]);
N2 = [0 0; 0 1];
K = zeros(2,n-1);
Q = zeros(2,dim);
E = zeros(2,dim);
M = zeros(dim,dim);
%Fill the matrices K,E,Q
for j = 1 : n-1
    K(:,j) = [c(2*j-1); c(2*j)];
    E(:,2*j-1:2*j) = A2 - D2*K(:,j)*C2;
    Q(:,2*j-1:2*j) = D2*K(:,j)*H2;
end
%Fill the matrix M
for j = 1 : n-1
    M(2*j-1:2*j,2*j-1:2*j) = E(:,2*j-1:2*j);
    if j< n-1
        M(2*j-1:2*j,2*(j+1)-1:2*(j+1)) = N2;
    end
    if j> 1
        M(2*j-1:2*j,2*(j-1)-1:2*(j-1)) = Q(:,2*j-1:2*j);
    end
end
```

A.2 Admissible coefficients

Let $G_i(\mathrm{i}\omega)$ and $H_i(\mathrm{i}\omega)$ be the transfer functions defined as

$$\begin{aligned} G_i(\mathrm{i}\omega) &:= B_i^\top(\mathrm{i}\omega I_{2i} - M_i)^{-1}B_i \ , \\ H_i(\mathrm{i}\omega) &:= B^\top(\mathrm{i}\omega I_2 - E_i)^{-1}K_i \ . \end{aligned} \tag{A.3}$$

and let γ_i and β_i their $\mathcal{H}_\infty$ gains, namely

$$\begin{aligned} \gamma_i &:= \max_{\omega\in\mathbb{R}} |G_i(\mathrm{i}\omega)| \, , \\ \beta_i &:= \max_{\omega\in\mathbb{R}} |H_i(\mathrm{i}\omega)| \, . \end{aligned} \tag{A.4}$$

Definition A.1. *The coefficients (c_{i1}, c_{i2}), $i = 1, \ldots, n-1$, are said to be admissible if*

- *M_i is Hurwitz for any $i = 1, \ldots, n-1$;*
- *$\gamma_i \cdot \beta_{i+1} < 1$, for $i = 1, \ldots, n-2$.*

The set of admissible coefficients is denoted by $\mathcal{G}_a$.

Lemma A.2. *The set $\mathcal{G}_a$ is non-empty.*

We show constructively a recursive procedure to assign the coefficients c_{i1}, c_{i2} which satisfies the Definition A.1.

i=1) Let c_{11} and c_{12} be any positive real numbers.

i>1) For $i > 1$, let $c_{i1} = c_{(i-1)1}$, and let $c_{i2} > 0$ be chosen such that $c_{i2} < \dfrac{c_{i1}}{\gamma_{i-1}}$, with γ_{i-1} defined as (A.4).

Proof. To verify that this choice satisfies the Definition A.1 let consider first the case $i = 1$. By choosing $c_{11} > 0$ and $c_{12} > 0$ the matrix E_1 is Hurwitz, and so is M_1. Now consider the case $i > 1$ and the following two systems

$$\begin{cases} \dot{x}_{i-1} = M_{i-1}x_{i-1} + B_{2(i-1)}u_{i-1} \\ y_{i-1} = B_{2(i-1)}^\top x_{i-1} \end{cases}$$
$$\begin{cases} \dot{z}_i = E_i z_i + K_i u_i \\ y_i = B^\top z_i \end{cases}$$

and let denote $x_i = (x_{i-1}, z_i)$. Let assume the matrix M_{i-1} is Hurwitz and note that the matrix E_i is Hurwitz for any choice of positive real numbers c_{i1}, c_{i2}. The previous systems can be represented by their transfer functions (A.3) with $\mathcal{H}_\infty$ gains

(A.4). Note that

$$H_i(\mathrm{i}\omega) \; = \; \frac{\mathrm{i}\omega c_{i2}}{(\mathrm{i}\omega)^2 + c_{i1}\mathrm{i}\omega + c_{i2}}\,, \qquad \beta_i \; = \; \frac{c_{i2}}{c_{i1}}\,.$$

By interconnecting the two subsystems with $u_{i-1} = y_i$ and $u_i = y_{-1}$, we obtain a closed loop system described by

$$\dot{x}_i \; = \; M_i x_i\,.$$

The choice of c_{i2} of the Step i satisfies the small-gain theorem (see, for instance, Isidori (1999))

$$\gamma_{i-1} \cdot \beta_i < 1$$

and therefore the interconnection (x_{i-1}, z_i) is asymptotically stable and M_i is Hurwitz. □

Note that, by using MATLAB, the values of γ_i can be easily calculated with the following command:

```
getPeakGain
```

Lemma A.3. *Let the coefficients c_{i1}, c_{i2}, $i = 1, \ldots, n$ be admissible according to Definition A.1 and let the matrix $\Lambda_i(t)$ be defined as*

$$\Lambda_i(t) := \begin{pmatrix} M_{i-1} & \rho(t)\bar{N}_{i-1} \\ \bar{Q}_i & E_i \end{pmatrix}$$

with $\rho(t) \in [0,1]$ any continuous function for all $t \geq 0$. Then, for any $i = 1, \ldots, n-1$, the origin of the system

$$\dot{x} \; = \; \Lambda_i(t)x \qquad \text{(A.5)}$$

is exponentially stablle.

Proof. The system (A.5) can be described as the interconnection of the system

$$\begin{aligned} \dot{x}_{i-1} &= M_{i-1}x_{i-1} + \rho(t)B_{2(i-1)}u_{i-1} \\ y_{i-1} &= B_{2(i-1)}^{\top} x_{i-1} \end{aligned}$$

with

$$\begin{aligned} \dot{z}_i &= E_i z_i + K_i u_i \\ y_i &= B^{\top} z_i \end{aligned}$$

when $u_{i-1} = y_i$ and $u_i = y_{-1}$. The $\mathcal{H}_\infty$ gain of the first system is given by $\rho(t)\gamma_{i-1}$

whereas the $\mathcal{H}_\infty$ gain of the second system is given by β_i. By assumptions we have

$$\rho(t)\gamma_{i-1}\,\beta_i \;<\; 1$$

and therefore the proof concludes by applying the small-gain theorem in Dragan (1993). □

Lemma A.4. *Let consider the system*

$$\dot{x} \;=\; kA(t)x + \frac{1}{k^r}Bu(t)\,. \tag{A.6}$$

with $x \in \mathbb{R}^n$ and for some $r \geq 0$ and any $k \geq 1$. Suppose that

(i) *the origin of the system $\dot{x}(t) \;=\; A(t)x(t)$ is globally exponentially stable;*

(ii) $\|u(\cdot)\|_\infty < \infty$.

Then for any compact set $X \subset \mathbb{R}^n$, real numbers $T > 0$ and $\epsilon > 0$ there exists $\underline{k} \geq 1$ such that, for any $k \geq \underline{k}$ and for any initial condition $x(0) \in X$

$$|k^r\, x(t)| \leq \epsilon \qquad \forall\, t \geq T\,.$$

Proof. Let denote with $\Phi(t,s)$ the transition matrix of $A(t)$. As consequence of assumption (i) there exist (see Theorem 6.7, Rugh (1996)) $\mu > 0$ and $\lambda > 0$ satisfying

$$|\Phi(t,t_0))| \;\leq\; \mu\exp(\lambda\,(t_0 - t))$$

for any $t \geq 0$, $t_0 \geq 0$. Let denote $\Phi_k(t,s)$ the transition matrix of $kA(t)$. We have

$$\Phi_k(t,s) \;=\; \Phi(t,s)^k\,, \qquad\qquad |\Phi_k(t,t_0))| \;\leq\; \mu\exp(\lambda\, k\,(t_0 - t))\,.$$

The solution of (A.6) is given by

$$x(t) \;=\; \Phi_k(t,t_0)x(t_0) + \frac{1}{k^r}\int_{t_0}^{t}\Phi_k(t,s)Bu(s)ds$$

and therefore the solution $x(t)$ starting from $t_0 = 0$ satisfies

$$\begin{aligned} |x(t)| \;&\leq\; \mu\exp(-\lambda\, k\, t)|x(0)| + \frac{1}{k^r}\int_0^t \Phi_k(t,s)Bu(s)ds \\ &\leq\; \mu\exp(-\lambda\, k\, t)|x(0)| + \frac{1}{k^r}\int_0^t \mu\exp(-\lambda k(t-s))\,ds\; B\;\|u(\cdot)\|_\infty \end{aligned}$$

$$\leq \quad \mu \exp(-\lambda\, k\, t)|x(0)| + \frac{1}{k^r}\frac{\mu}{\lambda\, k} B\ \|u(\cdot)\|_\infty$$

By noting that

$$\lim_{k\to\infty} k^r\, \mu \exp(-\lambda\, k\, T)|x(0)| = 0$$

for any $T > 0$, the proof concludes by choosing $\underline{k}$ satisfying

$$\underline{k}^r\, \mu \exp(-\lambda\, \underline{k}\, T) \max_{x\in X} |x(0)| < \frac{\epsilon}{2}\,, \qquad \frac{\mu}{\lambda\, \underline{k}}|B|\ \|u(\cdot)\|_\infty < \frac{\epsilon}{2}$$

□

A.3 Examples

We give here some examples in which we assign the coefficients according to Lemma A.1. The numerical results are computed by using the matlab code given at the end of Section A.1. It can be verified that these coefficients are also admissible according to Definition A.1.

n	dim(M)	$\sigma_i(\mathcal{P}_M(\lambda))$ $i = 1, \ldots, 2n-2$	(c_{i1}, c_{i2}) $i = 1, \ldots, n-1$
3	4	$-(0.4 + 0.2\,(i-1))$	$c_{i1} = 1.4$ $c_{i2} = \{0.88,\ 0.2182\}$
3	4	$-(1 + (i-1))$	$c_{i1} = 5$ $c_{i2} = \{10,\ 2.4\}$
4	6	$-(1 + 0.2\,(i-1))$	$c_{i1} = 3$ $c_{i2} = \{6.4,\ 2.131,\ 0.7095\}$
5	8	$-(1 + 0.2(i-1))$	$c_{i1} = 3.4$ $c_{i2} = \{10.72,\ 4.0137,\ 1.7810,\ 0.6667\}$
6	10	$-(1 + 0.1\,(i-1))$	$c_{i1} = 2.9$ $c_{i2} = \{10.1,\ 4.0387,\ 2.0187,\ 1.0090,\ 0.4035\}$

Table A.1: Examples of coefficients. We denote with $\sigma(\mathcal{P}_M(\lambda))$ the spectrum of the characteristic polynomial $\mathcal{P}(\lambda)$. We denote with $\sigma_i(\mathcal{P}_M(\lambda))$ the i-th eigenvalue of the characteristic polynomial $\mathcal{P}_M(\lambda)$.

B

Technical Lemmas for Time-Varying Block Tridiagonal Matrices

The following appendix is devoted to the study of stability of time-varying matrices possessing a particular block tridiagonal structure. We define the matrices $E_i(t) \in \mathbb{R}^{2\times 2}$, $Q_i \in \mathbb{R}^{2\times 2}$, and $N_i(t) \in \mathbb{R}^{2\times 2}$ as

$$E_i(t) = \begin{pmatrix} -c_{i1}b_i(t) & a_i(t) \\ -c_{i2}b_i(t) & 0 \end{pmatrix}, \qquad Q_i = \begin{pmatrix} 0 & c_{i1} \\ 0 & c_{i2} \end{pmatrix}, \qquad N_i(t) = \begin{pmatrix} 0 & 0 \\ 0 & a_i(t) \end{pmatrix},$$

for $1 \leq i \leq n-1$, where $a_i(t)$ and $b_i(t)$ are positive for all $1 \leq i \leq n-1$ and $t \geq 0$, and (c_{i1}, c_{i2}) are positive coefficients to be chosen. We define recursively the matrices $M_i(t) \in \mathbb{R}^{(2i)\times(2i)}$ as

$$\begin{aligned} M_1(t) &= E_{n-1}(t)\,, \\ M_{i+1}(t) &= \begin{pmatrix} E_{n-i-1}(t) & \bar{N}_{n-i}(t) \\ \bar{Q}_{n-i} & M_i(t) \end{pmatrix}, \quad i = 1, \ldots, n-2\,, \end{aligned}$$

where $\bar{N}_{n-i}(t) \in \mathbb{R}^{2\times 2i}$ and $\bar{Q}_{n-i} \in \mathbb{R}^{2i\times 2}$ are defined as

$$\begin{aligned} \bar{N}_{n-1}(t) &= N_{n-1}(t)\,, & \bar{N}_{n-i}(t) &= \begin{pmatrix} N_{n-i}(t) & 0 & \dots & 0 \end{pmatrix}, \\ \bar{Q}_{n-1} &= Q_{n-1}\,, & \bar{Q}_{n-i} &= \begin{pmatrix} Q_{n-i}^\top & 0 & \dots & 0 \end{pmatrix}^\top . \end{aligned}$$

Finally the the block-tridiagonal matrix $M(t) \in \mathbb{R}^{(2n-2)\times(2n-2)}$ is defined as

$$M(t) = \begin{pmatrix} E_1(t) & N_2(t) & 0 & \cdots & \cdots & 0 \\ Q_2 & E_2(t) & N_3(t) & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & Q_{n-2} & E_{n-2}(t) & N_{n-1}(t) \\ 0 & \cdots & \cdots & 0 & Q_{n-1} & E_{n-1}(t) \end{pmatrix}, \tag{B.1}$$

Lemma B.1. *Consider the matrix $M(t)$ in* (B.1) *with $a_i(t)$ and $b_i(t)$ continuous functions fulfilling*

$$\begin{aligned} \alpha &\le a_i(t) \le \beta\,, \\ \alpha &\le b_i(t) \le \beta\,, \end{aligned} \qquad \forall\, 1 \le i \le n-1\,,\ t \ge 0\,, \tag{B.2}$$

for some positive α and β. There exist coefficients (c_{i1}, c_{i2}), $i = 1, \dots, n-1$, a symmetric positive definite matrix P and a positive constant λ, such that, for all $t \ge 0$ the following holds

$$PM(t) + M(t)^\top P \le -\lambda I\,. \tag{B.3}$$

The proof of Lemma B.1 immediately comes by the forthcoming two lemmas. The idea of the proof is to iterate a small-gain theorem by starting from the block of $M_1(t)$ on the bottom. This result has been published in Wang et al. (2016-17) and we thank Lei Wang for his valuable help in the proofs.

Lemma B.2. *Consider the matrix $M_1(t)$. There exist coefficients $c_{n-1,1}$ and $c_{n-1,2}$ and a positive definite symmetric matrix P_1 such that*

$$P_1 M_1(t) + M_1^\top(t) P_1 \le -\lambda_1 I\,,$$

for some positive constant λ_1.

Proof. Consider the system

$$\dot{\xi}_1 = E_{n-1}(t)\xi_1 \, , \tag{B.4}$$

in which $\xi_1 = \mathrm{col}(\xi_{11}, \xi_{12}) \in \mathbb{R}^2$. Let $\Theta(r)$ be the matrix having the form

$$\Theta(r) = \begin{pmatrix} r & 0 \\ -r & 1 \end{pmatrix} \tag{B.5}$$

for all $r \in \mathbb{R}$, and then consider the following change of variables

$$\eta_1 = \Theta(\gamma_1)\xi_1 \quad i.e. \quad \eta_{11} = \gamma_1\xi_{11} \, , \quad \eta_{12} = \xi_{12} - \gamma_1\xi_{11} \, ,$$

with $\gamma_1 > 0$ to be chosen. As a consequence, the system (B.4) in the new coordinates can be rewritten as (from now on we omit the time-dependence in the variables for the purpose of compactness)

$$\begin{aligned} \dot{\eta}_{11} &= -\left[c_{n-1,1}b_{n-1,1} - \gamma_1 a_{n-1}\right]\eta_{11} + \gamma_1 a_{n-1}\eta_{12} \, , \\ \dot{\eta}_{12} &= -\left[(\gamma_1^{-1}c_{n-1,2} - c_{n-1,1})b_{n-1,1} + \gamma_1 a_{n-1}\right]\eta_{11} - \gamma_1 a_{n-1}\eta_{12} \, . \end{aligned}$$

By taking γ_1 satisfying $c_{n-1,2} = \gamma_1 c_{n-1,1}$, we get

$$\begin{aligned} \dot{\eta}_{11} &= -(c_{n-1,1}b_{n-1,1} - \gamma_1 a_{n-1})\eta_{11} + \gamma_1 a_{n-1}\eta_{12} \, , \\ \dot{\eta}_{12} &= -\gamma_1 a_{n-1}\eta_{11} - \gamma_1 a_{n-1}\eta_{12} \, . \end{aligned}$$

Now choose the Lyapunov function

$$V_1 \; = \; |\eta_1|^2 \; = \; \xi_1\Theta(\gamma_1)^\top\Theta(\gamma_1)\xi_1 \, ,$$

whose time derivative is given by

$$\dot{V}_1 = -2(c_{n-1,1}b_{n-1,1} - \gamma_1 a_{n-1})\eta_{11}^2 - 2\gamma_1 a_{n-1}\eta_{12}^2$$

By coming back in the ξ_1-coordinates and by using Young's inequality, the above equality can be rewritten as

$$\begin{aligned} \dot{V}_1 &\leq -2\gamma_1^2(c_{n-1,1}b_{n-1,1} - 2\gamma_1 a_{n-1})\xi_{11}^2 - \gamma_1 a_{n-1}\xi_{12}^2 \\ &\leq -2\gamma_1^2(c_{n-1,1}\alpha - 2\gamma_1\beta)\xi_{11}^2 - \gamma_1\alpha\xi_{12}^2 \end{aligned}$$

Given any positive γ_1, and choosing $c_{n-1,1} > 2\gamma_1 \frac{\beta}{\alpha}$, we can conclude that

$$\dot{V}_1 \leq -\lambda_1 |\xi_1|^2$$

with $\lambda_1 = \min\{2\gamma_1^2(c_{n-1,1}\alpha - 2\gamma_1\beta),\, \gamma_1\alpha\}$. In other words, given $P_1 = \Theta(\gamma_1)^\top \Theta(\gamma_1)$, the inequality $P_1 M_1(t) + M_1(t)^\top P_1 \leq -\lambda_1 I$ holds, which completes the proof of Lemma B.2. □

Lemma B.3. *Assume there exist a symmetric positive definite matrix P_i and a positive constant λ_i such that $P_i M_i(t) + M_i(t)^\top P_i \leq -\lambda_i I$. Then there exist coefficients $c_{n-i-1,1}$ and $c_{n-i-1,2}$ and a positive definite symmetric matrix P_{i+1} such that*

$$P_{i+1} M_{i+1}(t) + M_{i+1}^\top(t) P_{i+1} \leq -\lambda_{i+1} I\,, \quad 1 \leq i \leq n-2$$

for some positive constant λ_{i+1}.

Proof. Consider the interconnected system

$$\begin{aligned} \dot{\xi}_{i+1} &= E_{n-i-1}(t)\xi_{i+1} + \bar{N}_{n-i}(t)\chi_i\,, \\ \dot{\chi}_i &= M_i(t)\chi_i + \bar{Q}_{n-i}\xi_{i+1}\,, \end{aligned} \tag{B.6}$$

where $\xi_{i+1} = \mathrm{col}(\xi_{i+1,1}, \xi_{i+1,2}) \in \mathbb{R}^2$ and $\chi_i = \mathrm{col}(\xi_1, \ldots, \xi_i) \in \mathbb{R}^{2i}$. Let's make the following linear coordinate change for the state ξ_{i+1} in (B.6)

$$\eta_{i+1} := \mathrm{col}(\eta_{i+1,1}, \eta_{i+1,2}) = \Theta(\gamma_{i+1})\xi_{i+1}\,,$$

where $\Theta(\gamma_{i+1})$ has the form (B.5) and γ_{i+1} is a positive constant to be chosen. The system (B.6) in the new coordinates can be rewritten as (again, from now on we omit the time-dependence in the variables for the purpose of compactness)

$$\begin{aligned} \dot{\eta}_{i+1,1} &= -\left[c_{n-i-1,1}b_{n-i-1,1} - \gamma_{i+1}a_{n-i-1}\right]\eta_{i+1,1} + \gamma_{i+1}a_{n-i-1}\eta_{i+1,2}\,, \\ \dot{\eta}_{i+1,2} &= -[(\gamma_{i+1}^{-1}c_{n-i-1,2} - c_{n-i-1,1})b_{n-i-1,1} + \gamma_{i+1}a_{n-i-1}]\eta_{i+1,1} \\ &\qquad - \gamma_{i+1}a_{n-i-1}\eta_{i+1,2} + \bar{N}_{n-i}\chi_i\,, \\ \dot{\chi}_i &= M_i(t)\chi_i + \Gamma_i(\eta_{i+1,2} + \eta_{i+1,1})\,, \end{aligned}$$

where $\Gamma_i = \mathrm{col}(c_{n-i,1}, c_{n-i,2}, 0, \ldots, 0)$.

Let us take $c_{n-i-1,2} = \gamma_{i+1} c_{n-i-1,1}$, thus yielding

$$\begin{aligned}
\dot{\eta}_{i+1,1} &= -\left[c_{n-i-1,1} b_{n-i-1,1} - \gamma_{i+1} a_{n-i-1}\right] \eta_{i+1,1} + \gamma_{i+1} a_{n-i-1} \eta_{i+1,2}\,, \\
\dot{\eta}_{i+1,2} &= -\gamma_{i+1} a_{n-i-1} \eta_{i+1,1} - \gamma_{i+1} a_{n-i-1} \eta_{i+1,2} + \bar{N}_{n-i} \chi_i\,, \\
\dot{\chi}_i &= M_i(t) \chi_i + \Gamma_i (\eta_{i+1,2} + \eta_{i+1,1})\,.
\end{aligned}$$

First consider the positive definite function $V_i = \chi_i^\top P_i \chi_i$, whose time derivative is given by

$$\begin{aligned}
\dot{V}_i &= 2\chi_i^\top P_i [M_i(t)\chi_i + \Gamma_i(\eta_{i+1,2} + \eta_{i+1,1})] \\
&\leq -\lambda_i |\chi_i|^2 + 2\chi_i^\top P_i \Gamma_i \xi_{i+1,2} \\
&\leq -\tfrac{1}{2}\lambda_i |\chi_i|^2 + \delta_1 \xi_{i+1,2}^2
\end{aligned}$$

for some positive δ_1, independent of γ_{i+1} and $c_{n-i-1,1}$. Next consider the positive definite function

$$W_{i+1} \;=\; |\eta_{i+1}|^2 \;=\; \xi_{i+1} \Theta(\gamma_{i+1})^\top \Theta(\gamma_{i+1}) \xi_{i+1}\,,$$

whose time derivative is given by

$$\begin{aligned}
\dot{W}_{i+1} &= -2\left[c_{n-i-1,1} b_{n-i-1,1} - \gamma_{i+1} a_{n-i-1}\right] \eta_{i+1,1}^2 - 2\gamma_{i+1} a_{n-i-1} \eta_{i+1,2}^2 \\
&\qquad + 2\eta_{i+1,2} \bar{N}_{n-i} \chi_i \\
&\leq -2\left[c_{n-i-1,1} b_{n-i-1,1} - \gamma_{i+1} a_{n-i-1}\right] \eta_{i+1,1}^2 - \gamma_{i+1} a_{n-i-1} \eta_{i+1,2}^2 \\
&\qquad + \tfrac{\beta^2}{\alpha \gamma_{i+1}} |\chi_i|^2 \\
&\leq -2\gamma_{i+1}^2 \left[c_{n-i-1,1} b_{n-i-1,1} - 2\gamma_{i+1} a_{n-i-1}\right] \xi_{i+1,1}^2 \\
&\qquad - \tfrac{3}{4}\gamma_{i+1} a_{n-i-1} \xi_{i+1,2}^2 + \tfrac{\beta^2}{\alpha \gamma_{i+1}} |\chi_i|^2 \\
&\leq -2\gamma_{i+1}^2 \left[c_{n-i-1,1} \alpha - 2\gamma_{i+1} \beta\right] \xi_{i+1,1}^2 - \tfrac{3}{4}\gamma_{i+1} \alpha \xi_{i+1,2}^2 + \tfrac{\beta^2}{\alpha \gamma_{i+1}} |\chi_i|^2\,.
\end{aligned}$$

Then consider the Lyapunov function $V_i + W_{i+1}$. By choosing γ_{i+1} such that

$$\gamma_{i+1} = \max\left\{\frac{2\delta_1}{\alpha}, \frac{4\beta^2}{\lambda_i \alpha}\right\}\,,$$

and $c_{n-i-1,1}$ satisfying

$$c_{n-i-1,1} > 2\gamma_{i+1} \frac{\beta}{\alpha}\,.$$

we get

$$\dot{V}_i + \dot{W}_{i+1} \;\leq\; -\frac{\lambda_i}{4} |\chi_i|^2 - \frac{1}{4}\gamma_{i+1} \alpha \xi_{i+1,2}^2 - 2\gamma_{i+1}^2 \left[c_{n-i-1,1} \alpha - 2\gamma_{i+1} \beta\right] \xi_{i+1,1}^2\,.$$

Therefore, set $\chi_{i+1} = \mathrm{col}(\xi_{i+1}, \chi_i)$ and

$$P_{i+1} = \mathrm{blckdiag}(\Theta(\gamma_{i+1})^\top \Theta(\gamma_{i+1}), P_i) \, ,$$

and consider the positive definite function $V_{i+1,1} = \chi_{i+1}^\top P_{i+1} \chi_{i+1}$. Its time derivative satisfies

$$\dot{V}_{i+1} \leq -\lambda_{i+1} |\chi_{i+1}|^2$$

in which

$$\lambda_{i+1} = \min \left\{ \frac{\lambda_i}{4}, 2\gamma_{i+1}^2 (c_{n-i-1,1}\alpha - 2\gamma_{i+1}\beta), \frac{1}{4}\gamma_{i+1}\alpha \right\} \, .$$

That is,

$$P_{i+1} M_{i+1}(t) + M_{i+1}(t)^\top P_{i+1} \leq -\lambda_{i+1} I \, ,$$

which completes the proof of Lemma B.3. □

C

Total Stability Theorems

In this chapter we study how the stability properties of a given model described by

$$\dot{x} = \varphi_m(x) \tag{C.1}$$

are propagated to a process described by

$$\dot{x} \;=\; \varphi_p(x) \tag{C.2}$$

when they are close enough. First, in Lemma C.1, we show the the persistence of equilibria under small perturbations. Then, in Lemma C.2 we show that this equilibria is also unique under stronger conditions. The forthcoming results combine total stability and hyperbolicity and are a variation of (Poulain and Praly, 2010, Theorem 6). They are published in (Astolfi and Praly, 2016-17).

Lemma C.1. *Let a C^1 function $\varphi_m : \mathbb{R}^n \to \mathbb{R}^n$ be given such that the origin is an asymptotically stable equilibrium point of* (C.1)*, with $\mathcal{A}$ as domain of attraction. Let $\overline{\mathfrak{C}}$ be an arbitrary compact subset of $\mathcal{A}$ which admits the equilibrium as an interior point and is forward invariant for the system* (C.1)*. For any open neighbourhood $\mathcal{N}_{\partial\overline{\mathfrak{C}}}$ of the boundary set $\partial\overline{\mathfrak{C}}$, contained in $\mathcal{A}$, there exists a strictly positive real number δ such that, for any C^1 function φ_p satisfying*

$$|\varphi_m(x) - \varphi_p(x)| \;\leq\; \delta \qquad \forall\, x \in \mathcal{N}_{\partial\overline{\mathfrak{C}}}\,, \tag{C.3}$$

the system (C.2) *has an equilibria in the interior of* $\overline{\mathcal{C}}$.

Proof. To prove the existence of an equilibria we use (Hale, 1980, Theorem 8.2) which says that a forward invariant set which is homeomorphic to the closed unit ball of $\mathbb{R}^n$ contains an equilibrium. As a consequence of asymptotic stability we know the existence of a forward invariant set by using a converse Lyapunov theorem. It may not be homeomorphic to the closed unit ball. Therefore our first task is to show the existence of such set satisfying the required properties.

The equilibrium of (C.1) being asymptotically attractive and interior to $\overline{\mathcal{C}}$ which is forward invariant, $\overline{\mathcal{C}}$ is attractive. It is also stable due to the continuity of solutions with respect to initial conditions uniformly on compact time subsets of the domain of definition. So it is asymptotically stable with the same domain of attraction $\mathcal{A}$ as the equilibrium. It follows from (Wilson, 1969, Theorem 3.2) that there exist C^∞ functions $V : \mathcal{A} \to \mathbb{R}_{\geq 0}$ and $U : \mathcal{A} \to \mathbb{R}_{\geq 0}$ which are proper on $\mathcal{A}$ and a class $\mathcal{K}_\infty$ function α satisfying

$$
\begin{aligned}
\alpha(|x|) &\leq V(x)\,, & V(0) &= 0\,, \\
\alpha(d(x,\overline{\mathcal{C}})) &\leq U(x)\,, & U(x) &= 0 \qquad \forall x \in \overline{\mathcal{C}}\,, \\
\frac{\partial V}{\partial x}(x)\,\varphi_m(x) &\leq -V(x) & \forall x &\in \mathcal{A}\,, \\
\frac{\partial U}{\partial x}(x)\,\varphi_m(x) &\leq -U(x) & \forall x &\in \mathcal{A}\,.
\end{aligned}
$$

Since $\overline{\mathcal{C}}$ is compact and $\mathcal{N}_{\partial\overline{\mathcal{C}}}$ is a neighbourhood of its boundary, there exists a real number $\rho > 0$ such that the set $\{\, x \in \mathcal{A} : \ d(x,\overline{\mathcal{C}}) \in (0,\rho\,] \,\}$ is a subset of $\mathcal{N}_{\partial\overline{\mathcal{C}}}$. Then, with the notations

$$
\bar{u} \;=\; \sup_{x\in\mathcal{A}:\, d(x,\overline{\mathcal{C}})\,\leq\,\rho} V(x)\,, \qquad \gamma = \frac{\alpha(\rho)}{2\bar{u}}\,,
$$

and since α is of class $\mathcal{K}_\infty$, we obtain the implications

$$
\begin{aligned}
U(x) + \gamma V(x) \;=\; \alpha(\rho) \qquad &\Longrightarrow \qquad \alpha(d(x,\overline{\mathcal{C}})) \;\leq\; U(x) \;\leq\; \alpha(\rho)\,, \\
&\Longrightarrow \qquad d(x,\overline{\mathcal{C}}) \;\leq\; \rho\,, \\
&\Longrightarrow \qquad V(x) \leq \bar{u}\,.
\end{aligned}
$$

With our definition of γ, this yields also

$$\begin{aligned} \alpha(\rho) - \gamma V(x) \ = \ U(x) \quad & \Longrightarrow \quad 0 \ < \ \frac{\alpha(\rho)}{2} \ \leq \ U(x) \, , \\ & \Longrightarrow \quad 0 < d(x, \overline{\mathcal{C}}) \ \leq \ \rho \, , \\ & \Longrightarrow \quad x \in \mathcal{N}_{\partial \overline{\mathcal{C}}} \setminus \overline{\mathcal{C}} \, . \end{aligned}$$

On the other hand, with the compact notation

$$\mathcal{V}(x) = U(x) + \gamma V(x) \ ,$$

we have

$$\frac{\partial \mathcal{V}}{\partial x}(x) \, \varphi_m(x) \ < \ -\mathcal{V}(x) \qquad \forall x \in \mathcal{A} \, .$$

All this implies that $\mathcal{V}$ is a Lyapunov Function for (C.1) on $\mathcal{A}$ in the sense of (Wilson, 1967, Page 324) and that the sublevel set $\{x \in \mathcal{A} \, : \, \mathcal{V}(x) \leq \alpha(\rho)\}$ is contained in $\mathcal{N}_{\partial \overline{\mathcal{C}}} \cup \overline{\mathcal{C}}$. It follows from (Wilson, 1967, Corollary 2.3) (thanks to the contribution of Freedman Freedman (1982) and Perelman Morgan and Gang (2007) the restriction on the dimension is not needed) that the level set $\{x \in \mathcal{A} \, : \, \mathcal{V}(x) = \alpha(\rho)\}$ is homeomorphic to the unit sphere. But, with the fact that the origin is asymptotically stable and the arguments used in the proof of (Wilson, 1967, Theorem 1.2), this implies that the sublevel set $\{x \in \mathcal{A} \, : \, \mathcal{V}(x) \leq \alpha(\rho)\}$ is homeomorphic to the closed unit ball. Then, since the set

$$C = \{x \in \mathcal{N}_{\partial \overline{\mathcal{C}}} : d(x, \overline{\mathcal{C}}) \in [\, 0, \rho \,] \, \}$$

is a compact subset of $\mathcal{N}_{\partial \overline{\mathcal{C}}} \subset \mathcal{A}$, the real number

$$G = \sup_{x \in C} \left| \frac{\partial \mathcal{V}}{\partial x}(x) \right| \tag{C.4}$$

is well defined and strictly positive. We get, for all x in C,

$$\begin{aligned} \frac{\partial \mathcal{V}}{\partial x}(x) \varphi_p(x) \ & = \ \frac{\partial \mathcal{V}}{\partial x}(x) \varphi_m(x) + \frac{\partial \mathcal{V}}{\partial x}(x) [\varphi_p(x) - \varphi_m(x)] \, , \\ & \leq \ -\mathcal{V}(x) + G \sup_{x \in C} |\varphi_p(x) - \varphi_m(x)| \, . \end{aligned}$$

So, if φ_p satisfies (C.3), with δ given by

$$\delta \ = \ \frac{\inf_{x \in C} \mathcal{V}(x)}{2G} \, ,$$

we have, for all x in $\{x \in \mathcal{A} : \mathcal{V}(x) = \alpha(\rho)\}$

$$\frac{\partial \mathcal{V}}{\partial x}(x)\varphi_p(x) \;\leq\; -\frac{1}{2}\mathcal{V}(x)\ .$$

This implies the compact sublevel set $\{x : \mathcal{V}(x) \leq \alpha(\rho)\}$ is homeomorphic to the closed unit ball and forward invariant for the system (C.2). With (Hale, 1980, Theorem 8.2), we conclude that this sublevel set contains an equilibrium of this system. □

Lemma C.2. *Let a C^1 function $\varphi_m : \mathbb{R}^n \to \mathbb{R}^n$ be given such that the origin is an exponentially stable equilibrium point of* (C.1) *with $\mathcal{A}$ as domain of attraction. For any compact sets $\underline{\mathfrak{C}}$ and $\overline{\mathfrak{C}}$, the latter being forward invariant for the above system, which satisfy*

$$\{0\} \subsetneq \underline{\mathfrak{C}} \subsetneq \overline{\mathfrak{C}} \subsetneq \mathcal{A}\ ,$$

there exists a strictly positive real number δ such that, for any C^1 function $\varphi_p : \mathbb{R}^n \to \mathbb{R}^n$ which satisfies:

$$|\varphi_p(x) - \varphi_m(x)| \;\leq\; \delta\,, \qquad \forall\, x \in \overline{\mathfrak{C}}, \tag{C.5}$$

$$\left|\frac{\partial \varphi_p}{\partial x}(x) - \frac{\partial \varphi_m}{\partial x}(x)\right| \;\leq\; \delta\,, \qquad \forall\, x \in \underline{\mathfrak{C}}, \tag{C.6}$$

there exists an exponentially stable equilibrium point of (C.2) *the basin of attraction of which contains the compact set $\overline{\mathfrak{C}}$.*

Proof. From the arguments used in Lemma C.1, we know there exists a strictly positive real number δ such that if (C.5) holds with δ, then the system (C.2) has at least one equilibrium when δ is sufficiently small. It remains to show that this equilibrium is unique and asymptotically and locally exponentially stable.

Let Π be a positive definite symmetric matrix and a a strictly positive real number satisfying

$$\Pi\frac{\partial \varphi_m}{\partial x}(0) + \frac{\partial \varphi_m}{\partial x}(0)^\top\Pi \;\leq -a\Pi\ , \qquad \lambda_{\min}(\Pi) = 1\ ,$$

where $\lambda_{\max}$ and $\lambda_{\min}$ respectively stand for max and min eigenvalues. By continuity there exists a strictly positive real number p_0 such that we have, for all x satisfying $x^\top\Pi x \leq p_0$,

$$\Pi\frac{\partial \varphi_m}{\partial x}(x) + \frac{\partial \varphi_m}{\partial x}(x)^\top\Pi \leq -\frac{a}{2}\Pi$$

and

$$x^\top \Pi \varphi_m(x) \le -\frac{a}{4} x^\top \Pi x.$$

Let $\varphi_p : \mathbb{R}^n \to \mathbb{R}^n$ be any C^1 function satisfying

$$|\varphi_p(x) - \varphi_m(x)| \;\le\; \frac{a}{4}\sqrt{\frac{p_0}{12\lambda_{\max}(\Pi)}}\,, \qquad \forall\, x\,:\, x^\top \Pi x = \frac{p_0}{6}\,. \tag{C.7}$$

We obtain

$$\begin{aligned} x^\top \Pi \varphi_p(x) &= x^\top \Pi \varphi_m(x) + x^\top \Pi \left[\varphi_p(x) - \varphi_m(x)\right] \\ &\le x^\top \Pi \varphi_m(x) + \frac{a}{8} x^\top \Pi x + \frac{2}{a}\left[\varphi_p(x) - \varphi_m(x)\right]^\top \Pi \left[\varphi_p(x) - \varphi_m(x)\right] \end{aligned}$$

and therefore

$$x^\top \Pi \varphi_p(x) \;\le\; -\frac{a}{16} x^\top \Pi x\,, \qquad \forall x : x^\top \Pi x \;=\; \frac{p_0}{6}\,.$$

In this condition, it follows from (Hale, 1980, Theorem 8.2) that, for each function φ_p satisfying (C.7), there exits a point x_e satisfying

$$\varphi_p(x_e) \;=\; 0, \qquad (x_e)^\top \Pi x_e \;\le\; \frac{p_0}{6}\,. \tag{C.8}$$

Assume further that φ_p satisfies

$$\left|\frac{\partial \varphi_p}{\partial x}(x) - \frac{\partial \varphi_m}{\partial x}(x)\right| \;\le\; \frac{a}{8\lambda_{\max}(\Pi)}\,, \qquad \forall x\,:\, x^\top \Pi x \;\le\; p_0\,. \tag{C.9}$$

In this case, we have, for all x satisfying $x^\top \Pi x \le p_0$,

$$\begin{aligned} \Pi \frac{\partial \varphi_p}{\partial x}(x) + \frac{\partial \varphi_p}{\partial x}(x)^\top \Pi \;&=\; \left[\frac{\partial \varphi_p}{\partial x}(x) - \frac{\partial \varphi_m}{\partial x}(x)\right]^\top \Pi \\ &\qquad + \Pi \frac{\partial \varphi_m}{\partial x}(x) + \frac{\partial \varphi_m}{\partial x}(x)^\top \Pi + \Pi \left[\frac{\partial \varphi_p}{\partial x}(x) - \frac{\partial \varphi_m}{\partial x} x)\right] \\ &\le\; -\frac{a}{4}\Pi\,. \end{aligned}$$

Note also that we have

$$\begin{aligned} &[x_e + s(x - x_e)]^\top \Pi [x_e + s(x - x_e)] \le p_0\,, \\ &\qquad\qquad \forall (x, x_e, s) :\; s \in [0,1]\,,\; x_e^\top \Pi x_e \le \frac{p_0}{6}\,,\; x^\top \Pi x \le \frac{p_0}{3}\,. \end{aligned}$$

Then, with

$$\varphi_p(x) \;=\; \varphi_p(x) - \varphi_p(x_e) \;=\; \int_0^1 \frac{\partial \varphi_p}{\partial x}(x_e + s(x - x_e))\, \mathrm{d}s[x - x_e]$$

and (C.8), we get, for all x satisfying $x^\top \Pi x \leq \frac{p_0}{3}$,

$$\begin{aligned}[x - x_e]^\top \Pi \varphi_p(x) \;&=\; \int_0^1 \left([x - x_e]^\top \Pi \frac{\partial \varphi_p}{\partial x}(x_e + s(x - x_e))[x - x_e] \right) \mathrm{d}s \\ &\leq\; -\frac{a}{4}[x - x_e]^\top \Pi [x - x_e]\,.\end{aligned}$$

Let

$$\delta_1 = \min\left\{ \frac{a}{4}\sqrt{\frac{p_0}{12\lambda_{\max}(\Pi)}}\,,\, \frac{a}{8\lambda_{\max}(\Pi)} \right\}\,,$$

and reduce p_0 if necessary to have that x satisfying $(x_e)^\top \Pi x_e \leq p_0$ is in $\underline{\mathcal{C}}$. Then (C.5) and (C.6) with $\delta = \delta_1$ implies (C.7) and therefore (C.8). We have established that the system (C.2) has an exponentially stable equilibrium with basin of attraction containing the compact set $\{x \in \mathbb{R}^n : x^\top \Pi x \leq \frac{p_0}{3}\}$.

Now, with ρ and $\mathcal{V}(x) = U(x) + \gamma V(x)$ as defined in the proof of Lemma C.1, we let $\underline{u} > 0$ be a real number such that we have

$$x^\top \Pi x \;\leq\; \frac{p_0}{3} \qquad \forall x \in \mathcal{A} \;:\; \mathcal{V}(x) \;\leq\; \underline{u}\,. \tag{C.10}$$

Let also

$$C = \{x \in \mathcal{A} : \underline{u} \leq \mathcal{V}(x)\;,\; d(x, \overline{\mathcal{C}}) \in [\,0, \rho\,]\,\}\,.$$

It is a compact subset of $\mathcal{N}_{\overline{\mathcal{C}}} \subset \mathcal{A}$. By mimicking the same steps as in the proof of Lemma C.1, we can obtain that, if φ_p satisfies

$$|\varphi_p(x) - \varphi_m(x)| \;\leq\; \frac{\inf_{x \in C} \mathcal{V}(x)}{2G}\,, \qquad \forall x \in \overline{\mathcal{C}}\,, \tag{C.11}$$

we have

$$\frac{\partial \mathcal{V}}{\partial x}(x)\varphi_p(x) \;\leq\; -\frac{1}{2}\mathcal{V}(x) \qquad \forall x \in C\,.$$

This implies the compact set $\{x \in \mathcal{A} : \mathcal{V}(x) \leq \underline{u}\}$ is asymptotically stable for the system (C.2) with basin of attraction $\mathcal{B}$ containing the compact set $\{x \in \mathcal{A} : \mathcal{V}(x) \leq \alpha(\rho)\}$ which contains $\overline{\mathcal{C}}$. Since, with (C.10), we have

$$\{x \in \mathcal{A} : V(x) \leq \underline{u}\} \;\subset\; \left\{x \in \mathbb{R}^n : x^\top \Pi x \leq \frac{p_0}{3}\right\}.$$

with (C.7), (C.9), and (C.11) we have established our result with δ given as

$$\delta = \min \left\{ \frac{a}{4}\sqrt{\frac{p_0}{12\lambda_{\max}(\Pi)}}\, ,\ \frac{a}{8\lambda_{\max}(\Pi)}\, ,\ \frac{\inf\limits_{x\in C} V(x)}{2\sup\limits_{x\in C}\left|\dfrac{\partial V}{\partial x}(x)\right|} \right\} .$$

□

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