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STATISTICAL MECHANICS OF HARD-CORE PARTICLES WITH ATTRACTIVE INTERACTIONS

Presentata da: DIEGO ALBERICI

Coordinatore Dottorato:
Chiar.ma Prof.ssa
GIOVANNA
CITTI

Relatore:
Chiar.mo Prof.
PIERLUIGI
CONTUCCI

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Chapter 1

Introduction

1.1 Statistical Mechanics

Statistical Mechanics studies the collective behaviour of systems made up of a large number of elementary components. The interesting phenomena emerge from the interactions among the elementary components of the system, while if each elementary component acts independently of the other ones their collective behaviour is a trivial superposition. The Statistical Mechanics formalism reveals that very simple microscopic interactions, when sufficiently strong, can produce critical phenomena at the macroscopic level (phase transitions, namely discontinuities of some physical observable in the thermodynamic limit).

The classical examples come from Physics: the transition from water to ice is characterized by an abrupt change of the density of the system at 0 Celsius temperature; the transition from a paramagnetic to a ferromagnetic material is characterized by an abrupt change of the magnetisation at the Curie temperature. In both examples the elementary components of the system are the molecules constituting a certain material. But it can be interesting to apply the Statistical Mechanics models also to other fields. In Computer Science and Neuroscience, the (artificial) neural networks are systems of interconnected neurons that exchange messages [58]. In Biology the bird flocks are an example of collective behaviour [15]. A possible approach to Socio-Economic Sciences is to consider interconnected groups of people that exchange trends, information and opinions (see [22] for an example of application): the social networks actualize this concept. Bouchaud's approach to study Economics and economical crisis is of this type [20, 19] (in particular he affirms that a scientific revolution is needed: real data should be taken into account when they contradict classical economics assumptions). In a few words the Statistical Mechanics approach is actually multidisciplinary and can be interesting for a large number of applications.

1.1.1 Boltzmann-Gibbs measure

In the Statistical Mechanics of equilibrium, a probability is assigned to each possible configuration of the elementary components. Then the statistical behaviour of the physical observables is studied when the number of elementary components goes to infinity (thermodynamic limit).

Definition 1.1. The *elementary components* of the system are indexed by a finite set Λ. The possible configurations of each elementary component are collected in a set $\mathcal{S}$ (we assume $\mathcal{S}$ finite). Therefore the possible *microscopic configurations* of the system are represented by the vectors of $\Omega = \mathcal{S}^\Lambda$.

A *Hamiltonian* $\mathcal{H}\colon \Omega \to (-\infty, \infty]$ is introduced: it assigns an energy to each microscopic configuration, modelling all the interactions among the elementary components. At the equilibrium, the associated *Boltzmann-Gibbs probability measure* is considered on the space of microscopic configurations:

$$\mathrm{P}_{\mathrm{BG}}(\mathcal{C}) \;=\; \frac{1}{\mathcal{Z}}\, e^{-\beta\, \mathcal{H}(\mathcal{C})} \quad \forall\, \mathcal{C} \in \Omega \,, \tag{1.1}$$

where $\mathcal{Z}$ is the normalizing factor (called *partition function*) and $\beta \in [0, \infty)$ is a parameter (called *inverse temperature*). We will usually absorb β in the Hamiltonian. In this framework any observable $\mathcal{O}\colon \Omega \to \mathbb{R}$ is a random variable with respect to the Boltzmann-Gibbs probability measure.

Because of the definition (1.1), the more likely microscopic configurations are those with the lowest energy. On the contrary configurations with infinite energy have zero probability: they can be excluded from the space of configurations Ω at the beginning. We will usually do so in this thesis; but for this introductory discussion it is convenient to keep $\Omega = \mathcal{S}^\Lambda$.

A justification of the Boltzmann-Gibbs distribution is out of our purposes: for particle systems it is related to Boltzmann entropy, to the ergodic hypothesis and the equivalence of ensembles (see [42] for an analysis of foundations of Statistical Mechanics). However it is interesting to observe that the Boltzmann-Gibbs distribution maximizes the information entropy given the expected energy, namely P_{BG} realizes

$$\max\Big\{ -\sum_{\mathcal{C}} \mathrm{P}(\mathcal{C})\, \log \mathrm{P}(\mathcal{C}) \,\Big|\, \sum_{\mathcal{C}} \mathrm{P}(\mathcal{C}) = 1\,,\ \sum_{\mathcal{C}} \mathrm{P}(\mathcal{C})\, \mathcal{H}(\mathcal{C}) = U_0 \Big\}$$

for any fixed energy $U_0 \in [\min \mathcal{H},\, \frac{1}{|\Omega|} \sum_{\mathcal{C}} \mathcal{H}(\mathcal{C})]$ and a suitable $\beta \geq 0$. In other words, imagine that the expected energy of the system is measured: there are several probability distributions that are consistent with this expected energy, but the Boltzmann-Gibbs distribution describes the system as random as it can be satisfying only the energy constraint.

1.1.2 Phase transitions

The most interesting phenomena in Statistical Mechanics are phase transitions. At finite volume Λ any expected observable

$$\langle \mathcal{O} \rangle_{\mathrm{BG}} \;=\; \frac{\sum_{\mathcal{C}\in\mathcal{S}^\Lambda} \mathcal{O}(\mathcal{C})\, e^{-\beta\mathcal{H}(\mathcal{C})}}{\sum_{\mathcal{C}\in\mathcal{S}^\Lambda} e^{-\beta\mathcal{H}(\mathcal{C})}}$$

is obviously an analytic function of the inverse temperature β. But since the systems studied in Statistical Mechanics are made up of a large number of elementary components, one is interested in the *thermodynamic limit*, that is the limit as $|\Lambda| \to \infty$. After having chosen a proper way to let Λ go to infinity, the system is said to exhibit a *phase transition* if the limiting generating function (called *pressure*)

$$p(\beta) := \lim_{\Lambda\nearrow\infty} \frac{1}{|\Lambda|} \log \mathcal{Z}_\Lambda(\beta) \;=\; \lim_{\Lambda\nearrow\infty} \frac{1}{|\Lambda|} \log \sum_{\mathcal{C}\in\mathcal{S}^\Lambda} e^{-\beta\mathcal{H}_\Lambda(\mathcal{C})}$$

is not analytic at some critical inverse temperature $\beta = \beta_c$. Observe that the derivatives of $\log\mathcal{Z}(\beta)$ are the cumulants of minus the energy:

$$\frac{\partial}{\partial\beta} \log\mathcal{Z} \;=\; \langle -\mathcal{H}\rangle_{\mathrm{BG}} \;,$$

$$\frac{\partial^2}{\partial\beta^2} \log\mathcal{Z} \;=\; \langle \mathcal{H}^2\rangle_{\mathrm{BG}} - \langle\mathcal{H}\rangle_{\mathrm{BG}}^2 \;,\;\ldots$$

Therefore there is a phase transition if in the thermodynamic limit some cumulant of the energy is not continuous with respect to $\beta \in [0,\infty)$ (while the function p itself is always continuous being concave). In the case $\mathcal{H}$ depends analytically on some parameters (e.g. magnetic field, chemical potential, ...), also the analyticity of the generating function p with respect to those parameters can be investigated.

A beautiful link between Algebra, Complex Analysis and Statistical Mechanics is given by the fact that phase transitions are strictly related to complex zeros of the partition function (the main example is the Lee-Yang theorem [71, 88]). To get an idea of this fact, assume that the Hamiltonian takes non-negative integer values up to $N = O(|\Lambda|)$ and rewrite the partition function as a polynomial:

$$\mathcal{Z}_\Lambda(z) \;=\; \sum_{k=0}^{N} C_\Lambda(k)\, z^k \;,$$

by setting $z := e^{-\beta}$ and $C_\Lambda(k) = \mathrm{card}\{\mathcal{C}\in\Omega \,|\, \mathcal{H}(\mathcal{C}) = k\}$ (assume $\max_k \log C_\Lambda(k) = O(|\Lambda|)$). By the Fundamental Theorem of Algebra $\mathcal{Z}_\Lambda(z)$ has N complex zeros $z_{\Lambda,p}$ and rewrites as

$$\mathcal{Z}_\Lambda(z) \;=\; C_\Lambda(N) \prod_{p=1}^{N} (z - z_{\Lambda,p}) \;.$$

If a complex stripe centred along a positive real interval $(a,b)+\mathrm{i}(-\delta,\delta)$ is free of zeros of $\mathcal{Z}_\Lambda$ for any Λ, then $\frac{1}{|\Lambda|}\log\mathcal{Z}_\Lambda(z)$ is a uniformly bounded sequence of analytic functions on the compact subsets of the stripe, therefore by the Vitali-Porter theorem [91] its limit $p(z)$ is analytic for $z\in(a,b)$.

1.1.3 Interactions and graphs

When the space of configurations is a product space ($\Omega = \mathcal{S}^\Lambda$), and the Hamiltonian writes as a sum on the elementary components

$$\mathcal{H}(\mathcal{C}) = \sum_{i\in\Lambda}\mathcal{H}_i(\mathcal{C}_i) \quad \forall\, \mathcal{C} = (\mathcal{C}_i)_{i\in\Lambda}\in\Omega\ ,$$

then the system is called *non-interacting*, since its elementary components are independent according to the Boltzmann-Gibbs measure. An elementary but meaningful observation is that phase transitions do not occur in non-interacting systems:

$$p(\beta) = \lim_{\Lambda\nearrow\infty}\frac{1}{|\Lambda|}\sum_{i\in\Lambda} p_i(\beta)\ ,$$

where each $p_i(\beta) = \log\sum_{\mathcal{C}_1\in\mathcal{S}} e^{-\beta\mathcal{H}_i(\mathcal{C}_1)}$ is an analytic function, hence p is analytic provided that the Cesaro-limit can be interchanged with the series.

Often systems with pairwise interactions are considered, namely

$$\mathcal{H}(\mathcal{C}) = \sum_{i\in\Lambda}\mathcal{H}_i(\mathcal{C}_i)\ +\ \sum_{i,j\in\Lambda}\mathcal{H}_{i,j}(\mathcal{C}_i,\mathcal{C}_j) \quad \forall\, \mathcal{C} = (\mathcal{C}_i)_{i\in\Lambda}\in\Omega\ .$$

It is natural to represent these systems on *graphs*: Λ is the vertex set, while the pairs (i,j) such that $\mathcal{H}_{i,j}$ is not identically zero are the edges corresponding to the interactions. In Physics *2 or 3-dimensional regular lattices* are usually considered, since the particles interact according to their distance in the Euclidean space. For other collective phenomena instead *sparse random graphs* are more suitable: there are different methods to build random graphs [59] that have characteristic features observed in real-world networks [9, 77], as the presence of hubs and the small-word properties. Models on the *complete graph* (namely the graph where each pair of vertices is connected) are usually considered as a first approximation in Physics, while for the emerging applications they play an important role since the behaviour on the complete graph is often similar to the behaviour on sparse random graphs.

1.1.4 A fundamental example: Ising models

One of the most popular examples in Statistical Mechanics is the *Ising model*. Consider a system made up of spin variables that can take only two opposite values: the space of configurations is $\Omega = \{-1,+1\}^\Lambda$, where Λ indexes the spins. It can be shown that the most general Hamiltonian $\mathcal{H}\colon \Omega \to \mathbb{R}$ writes, in a unique way, as

$$\mathcal{H}(\sigma) \,=\, -\sum_{X\subseteq\Lambda} J_X \prod_{i\in X} \sigma_i \quad \forall\, \sigma\in\Omega\,,$$

with $J_X\in\mathbb{R}$. In the case of pairwise interactions, the Hamiltonian reduces to

$$\mathcal{H}(\sigma) \,=\, -\sum_{i\in\Lambda} h_i\,\sigma_i - \sum_{\substack{i,j\in\Lambda\\ i\neq j}} J_{i,j}\,\sigma_i\sigma_j \quad \forall\, \sigma\in\Omega\,,$$

with $h_i,\, J_{i,j}\in\mathbb{R}$. It is clear from the definition of the Hamiltonian that if $J_{ij}>0$ the configurations with $\sigma_i=\sigma_j$ are favoured, while if $J_{ij}<0$ the configurations with $\sigma_i=-\sigma_j$ are favoured.

Assume zero external field: $h_i=0$ for all $i\in\Lambda$. When $J_{ij}\geq 0$ for all pairs (i,j), the system is called a *ferromagnet*: all the spins tend to imitate with one another; when $J_{ij}\leq 0$ for all pairs (i,j) the system is called an *antiferromagnet*; when the J_{ij} follow a symmetric distribution around 0 the system is called a *spin glass*.

In ferromagnets a phase transition can occur [84,47] and it is characterized by the divergence of the second derivative of $p(\beta,h)$ with respect to the field h; precisely below the critical temperature two states coexist, one characterized by most positive spins and the other one by most negative spins. Many results are available for Ising ferromagnets: they satisfy useful correlation inequalities [48, 66, 44, 49], the complex zeros of the partition function in the variable $z = e^{-2\beta h}$ are located on the unit circle [71, 88]; the exact solutions (see [12] for a review) for uniform $J \geq 0$ are studied in details on the complete graph, on the 1 and 2-dimensional lattice [79, 65, 98], on a large class of random graphs [26, 29, 30].

Spin glasses are much more complicated. Usually the J_{ij} are taken as *i.i.d.* Gaussian random variables and this disorder is *quenched* with respect to the thermal fluctuations. The model on finite-dimensional lattices was introduced in [34]. The model on the complete graph (SK model [92]) has been studied for several decades by physicists and mathematicians (see [75,93,82] and references therein). On the complete graph a phase transition occurs: at low temperature several states coexist, they cannot be characterized in terms of simple symmetries and they are organized in a hierarchical (ultrametric) structure. The exact solution of the model was first proposed by Parisi, but it took several years to prove the validity of the Parisi's formula [94, 52, 2, 54] and the ultrametric picture [81, 51, 1, 89].

1.2 Monomer-dimer models

Monomer-dimer models were introduced in Physics to describe phenomena like the adsorption of a diatomic gas on a monoatomic layer [85] or the behaviour of a fluid mixture of molecules of two different sizes [40]. They are studied in Mathematical and Theoretical Physics [55, 64, 99], in Combinatorics and Computer Science [74, 63], in the applications to Social Sciences [11].

While the Ising model is equivalent to a lattice gas model where monoatomic particles deposit on the vertices of a graph, the monomer-dimer model can be informally described in terms of sticks (diatomic particles) that deposit on the edges of a graph. The occupied edges are the *dimers*, while the empty vertices are the *monomers*. The space of configurations is $\Omega = \{0,1\}^E$, where E indexes the edges; but since different sticks cannot overlap on the same vertex (*hard-core interaction*), some configurations are forbidden. Instead of defining a Hamiltonian that takes the value $+\infty$ at these forbidden configurations, we prefer to restrict the space of configurations, which will not be a product space anymore:

$$\Omega' = \left\{ \alpha \in \{0,1\}^E \;\middle|\; \alpha_e \alpha_{e'} = 0 \quad \forall\, e, e' \in E,\, |e \cap e'| = 1 \right\} .$$

It is crucial to keep in mind that the hard-core interaction is now encoded in the space of configurations, therefore even when the Hamiltonian is only a sum of individual contributions the system *is* interacting.

To introduce a Hamiltonian for monomer-dimer models, we analyse the physical phenomenon of a diatomic gas, let say oxygen, that is adsorbed on a surface of a monoatomic material X. When a molecule of oxygen O_2 is adsorbed, its two atoms deposit on two neighbouring atoms X of the surface. No overlapping of different molecules of oxygen is allowed: this is the hard-core interaction due to the repulsive part of the van der Waals potential. Now the surface can be represented as a planar graph G, where every vertex corresponds to an atom X. The adsorbed molecules of oxygen form a monomer-dimer configuration on G: the adsorbed molecules O_2 are the dimers, while the atoms X free of oxygen are the monomers. Encoding the hard-core interaction in the space of configurations, a simple Hamiltonian describes the system taking into account only the dimer potential of every edge:

$$\mathcal{H}(\alpha) = -\sum_{e \in E} h_e\, \alpha_e \quad \forall \alpha \in \Omega' . \tag{1.2}$$

As already said, the van der Waals potential is repulsive at very short distance, preventing different molecules of oxygen from overlapping. But at a longer distance the van der Waals potential becomes attractive, pushing the molecules of oxygen to remain close one another. At too long distance instead the van der Waals potential is negligible. See fig.1.1. Depending on the lattice spacing

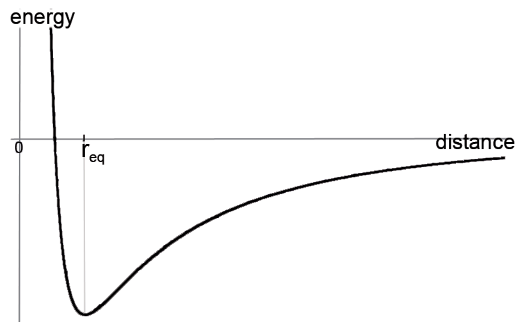

Figure 1.1: The van der Waals potential as a function of the distance r between two molecules.

r of the adsorbing material X, the attractive component of the van der Waals potential plays different roles. If r is larger than r_{eq} but not too much, then a Hamiltonian that take into account this attractive interaction is

$$\mathcal{H}(\alpha) = -\sum_{e \in E} h_e \, \alpha_e - \sum_{\substack{e,e' \in E \\ e \sim e'}} J_{ee'} \, \alpha_e \alpha_{e'} \quad \forall \alpha \in \Omega' \, , \tag{1.3}$$

with $J_{ee'} > 0$ ($e \sim e'$ means that the two edges are not incident but connected by a third edge). If instead r is much larger than r_{eq}, then this attractive interaction is too weak and can be neglected: the system is well described by the previous Hamiltonian (1.2). If r is smaller that r_{eq}, then there is even more repulsion among molecules and $J_{ee'}$ should be chosen as positive numbers. Finally if r oscillates around r_{eq}, some $J_{ee'}$ can be positive and some $J_{ee'}$ can be negative: one could observe an interesting phenomenon analogous to spin glasses.

The hard-core interaction alone ($J \equiv 0$) is not sufficient to cause a phase transition in monomer-dimer models. This remarkable fact was proved in great generality by Heilmann and Lieb [56, 55], by locating the complex zeros of the partition function. A probabilistic method instead was used by van den

Berg [14]. When an attractive component is added to the interaction, things change and phase transitions can emerge [57,6]. Also in the pure dimer problem (zero monomer activity: all the vertices occupied by a dimer) phase transitions are possible, as suggested by the Arctic Circle theorem [62].

On planar graphs the exact solution of the pure dimer problem was found by Kasteleyn, Fisher, Temperley [64, 39, 95], in terms of the Pfaffian of the adjacency matrix of a directed graph; the transfer matrix method was proposed by Lieb [73]. Recently this solution has been extended to include the presence of monomers at the boundary [45]. Heilmann and Lieb [55] provided a recursive formula for the monomer-dimer partition function, that permits to obtain some exact solutions, for example on the complete graph and on a large class of random graphs [4].

In this thesis we mainly deal with mean-field versions of the monomer-dimer model, with the exception of chapter 7. A non-zero monomer activity is always considered. For an introduction to pure dimer models (also known as perfect matching problems) on planar lattices we suggest [67].

1.2.1 Results obtained in this thesis

Mean-field monomer-dimer models, on sparse random graphs or on the complete graph, can be considered as an approximation of finite-dimensional physical models. On the other hand they have a particular interest for the emerging applications to Computer Science and Social Sciences [11], since the real-world networks are often modelled by particular families of random graphs [9, 77, 59].

Zdeborová and Mézard [99] gave a complete picture of the monomer-dimer model with pure hard-core interaction on sparse random graphs: using the theoretical physics approach of cavity method they compute the replica symmetric solution of the model and they verify its stability. In [4] we provided a complete rigorous proof of Zdeborová-Mézard's solution, starting from the previous work [18]. Our proof is based on the Heilmann-Lieb recursion [55] for the partition function, combined with some alternating correlation inequalities on trees.

As shown by Heilmann and Lieb [55, 56], the hard-core interaction is not sufficient to cause a phase transition in monomer-dimer models. In [8,6,7,5] we study monomer-dimer models on the complete graph and in particular in the first three of these works we add an attractive interaction to the hard-core one. We provide the solution of this model, showing that a phase transition occurs when the attractive interaction in sufficiently strong. The phase transition is studied in details: the monomer (or dimer) density is the order parameter and a coexistence curve separates a dimer phase from a monomer phase; at the critical point (where the coexistence curve stems) the critical exponents are the standard mean-field ones and the central limit theorem breakdowns, since the fluctuations are of order $N^{3/4}$. The study of these fluctuations is based on the works by Ellis and Newmann [35, 36] for mean-field spin models, with the fundamental difference that our space of configurations is not a product space due to the monomer-dimer hard-core interaction: to *decouple* this interaction we use a representation of the partition function in terms of Gaussian moments.

Finite-dimensional monomer-dimer models (and more general hard-rods models) are still interesting also for applications to Physics, in the theory of liquid crystals (see e.g. [32, 46, 83]). In [57] Heilmann and Lieb proposed some monomer-dimer models on $\mathbb{Z}^2$ with attractive interactions that favour the presence of clusters of neighbouring parallel dimers. They show by a reflection positivity argument that these systems exhibit a phase transition: at low temperatures a spontaneous order in the orientation of the dimers appears. On the other hand the authors conjecture that their models do not have a complete translational order: namely it is equally likely to observe a dimer attached to the left or to the right of a given vertex x, under local perturbations of the system sufficiently far from x. These two properties, the orientational order together with the absence of translational order, characterize liquid crystals. In [3] we did not solve the Heilmann-Lieb conjecture, which is open since 1979, but we proved the absence of translational order in a different framework, when the dimer potential favours one of the two orientations of dimers on the lattice $\mathbb{Z}^2$. Our proof is based on cluster expansion methods and starts from the Letawe's thesis [72].

Here there is a brief description of the chapters that follow. In the *chapter 2* we introduce the mathematical definitions and some general properties of monomer-dimer models, like the Gaussian representation for the partition function and the absence of phase transitions when only the hard-core interaction in considered. In the *chapters from 3 to 5* we study the monomer-dimer model on the complete graph: in chapter 3 the interaction is purely hard-core, while in chapters 4, 5 an imitative/attractive component of the interaction is added. In the *chapter 6* the pure hard-core monomer-dimer model is studied on a class of random graphs. Finally in the *chapter 7* we study a monomer-dimer model on the lattice $\mathbb{Z}^2$ with imitative interaction in the orientations, which origins from one of the Heilmann-Lieb liquid crystal models.

Chapter 2

Definitions and general results

Let $G = (V, E)$ be a finite undirected graph with vertex set V and edge set $E \subseteq \{ij \equiv \{i,j\} \,|\, i \in V,\, j \in V,\, i \neq j\}$.

Definition 2.1 (Monomer-dimer configurations)**.** A set of edges $D \subseteq E$ is called a *monomer-dimer configuration*, or a *matching*, if the edges in D are pairwise non-incident. The space of all possible monomer-dimer configurations on the graph G is denoted by $\mathscr{D}_G$.

Given a monomer-dimer configuration D, we say that every edge in D is occupied by a *dimer*, while every vertex that does not appear in D is occupied by a *monomer*. The set of monomers associated to D is denoted by $M_G(D)$ or simply $M(D)$.

Remark 2.2. We can associate an occupation variable $\alpha_{ij} \in \{0,1\}$ to each edge $ij \in E$: the edge ij is occupied by a dimer iff α_{ij} takes the value 1. It is clear that monomer-dimer configurations are in one-to-one correspondence with vectors $\alpha \in \{0,1\}^E$ satisfying the following constraint:

$$\forall\, i \in V \quad \sum_{j \underset{G}{\sim} i} \alpha_{ij} \,\leq\, 1 \,, \tag{2.1}$$

where $j \underset{G}{\sim} i$ means that $ij \in E$. Therefore, with a slight abuse of notation, we denote by $\mathscr{D}_G$ also the set of $\alpha \in \{0,1\}^E$ that satisfy eq. (2.1).

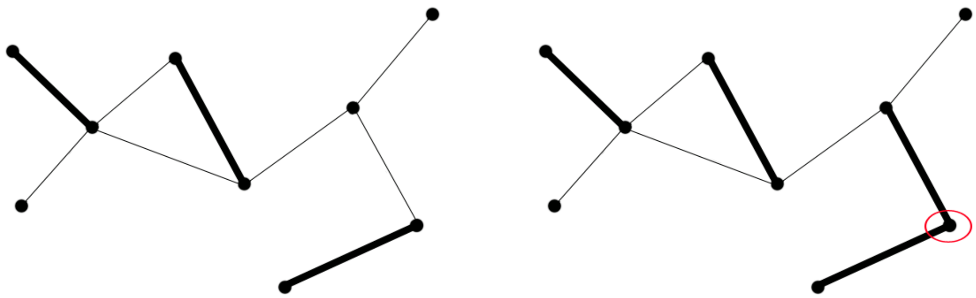

Figure 2.1: The bold edges in the left figure form a monomer-dimer configuration on the graph, while those in the right figure do not.

The condition (2.1) guarantees that at most one dimer can be incident to any given vertex i, namely two dimers cannot be incident. This fact is usually referred as *hard-core interaction* or *hard-core constraint* or *monogamy constraint*. We also introduce an auxiliary variable

$$\alpha_i := 1 - \sum_{j \sim i} \alpha_{ij} \in \{0, 1\} \tag{2.2}$$

for each vertex $i \in V$: the vertex i is occupied by a monomer iff α_i takes the value 1.

The definition of monomer-dimer configurations is already quite rich in itself. Indeed non-trivial combinatorial questions can be asked, as "How many monomer-dimer configurations, possibly with a given number of dimers, exist on a given graph G?". This combinatorial problem is known to be NP-hard in general, but there are polynomial algorithms and exact solutions for specific cases [64, 74, 63, 55, 4].

In Statistical Mechanics a further structure is introduced: we consider a Gibbs probability measure on the set of monomer-dimer configurations. There are several choices for the measure, depending on how we decide to model the interactions in the system.

2.1 Pure hard-core interaction

The first possibility is to take into account only the hard-core interaction among particles and assign a dimer activity $w_{ij} \geq 0$ to each edge $ij \in E$ and a monomer activity $x_i > 0$ to each vertex $i \in V$.

Definition 2.3 (Monomer-dimer models)**.** A *monomer-dimer model* on G is given by the following probability measure on $\mathscr{D}_G$:

$$\mu_G(D) := \frac{1}{Z_G} \prod_{ij \in D} w_{ij} \prod_{i \in M(D)} x_i \quad \forall\, D \in \mathscr{D}_G \;, \tag{2.3}$$

where the normalizing factor, called partition function, is

$$Z_G := \sum_{D \in \mathscr{D}_G} \prod_{ij \in D} w_{ij} \prod_{i \in M(D)} x_i \;. \tag{2.4}$$

The dependence of the measure on the activities w_{ij}, x_i is usually implicit in the notations. When we consider the complete graph with N vertices, the subscript $_G$ is usually substituted by $_N$.

Remark 2.4. It is worth to notice that the definition 2.3 is slightly redundant for two reasons. First one can consider without loss of generality monomer-dimer models on complete graphs only: a monomer-dimer model on the graph $G = (V, E)$ coincides with a monomer-dimer model on the complete graph with $N = |V|$ vertices, by taking $w_{ij} = 0$ for all pairs $ij \notin E$.

Secondly one can set without loss of generality all the monomer activities equal to 1: the monomer-dimer model with activities (w_{ij}, x_i) coincides with the monomer-dimer model with activities $(\frac{w_{ij}}{x_i x_j}, 1)$, since

$$\prod_{i \in M(D)} x_i = \left(\prod_{i \in V} x_i \right) \prod_{i \notin M(D)} \frac{1}{x_i} = \left(\prod_{i \in V} x_i \right) \prod_{ij \in D} \frac{1}{x_i\, x_j} \;.$$

Vice versa if the dimer activity is uniform on the graph then it can be set equal to 1: the monomer-dimer model with activities (w, x_i) coincides with the monomer-dimer model with activities $(1, \frac{x_i}{\sqrt{w}})$, since

$$w^{|D|} = w^{\frac{N - |M(D)|}{2}} = w^{N/2} \left(\frac{1}{\sqrt{w}} \right)^{|M(D)|} \;.$$

Remark 2.5. The following bounds for the pressure (logarithm of the partition function) will be used several times:

$$\sum_{i\in V}\log x_i \,\leq\, \log Z_G \,\leq\, \sum_{i\in V}\log x_i + \sum_{ij\in E}\log\big(1+\frac{w_{ij}}{x_i\,x_j}\big)\,. \tag{2.5}$$

The lower bound is obtained considering only the empty monomer-dimer configuration (a monomer on each vertex of the graph): $Z_G \geq \prod_{i\in V} x_i$. The upper bound is obtained using the fact that any monomer-dimer configuration is a (particular) set of edges:

$$\prod_{i\in V} x_i^{-1}\, Z_G \,=\, \sum_{D\in\mathscr{D}_G}\,\prod_{ij\in D}\frac{w_{ij}}{x_i\,x_j} \,\leq\, \sum_{D\subset E}\,\prod_{ij\in D}\frac{w_{ij}}{x_i\,x_j} \,=\, \prod_{ij\in E}\big(1+\frac{w_{ij}}{x_i\,x_j}\big)\,.$$

An interesting fact about monomer-dimer models is that they are strictly related to Gaussian random vectors.

Proposition 2.6 (Gaussian representation [8, 97])**.** *The partition function of any monomer-dimer model over N vertices can be written as*

$$Z_N \,=\, \mathbb{E}_{\boldsymbol{\xi}}\Big[\prod_{i=1}^{N}(\xi_i + x_i)\Big]\,, \tag{2.6}$$

where $\boldsymbol{\xi} = (\xi_1,\dots,\xi_N)$ is a Gaussian random vector with mean 0 and covariance matrix $W = (w_{ij})_{i,j=1,\dots,N}$ and $\mathbb{E}_{\boldsymbol{\xi}}[\,\cdot\,]$ denotes the expectation with respect to $\boldsymbol{\xi}$. Here the diagonal entries w_{ii} are arbitrary numbers, chosen in such a way that W is a positive semi-definite matrix.

Proof. The monomer-dimer configurations on the complete graph are all the partitions into pairs of any set $A \subseteq \{1,\dots,N\}$, hence

$$Z_N \,=\, \sum_{D\in\mathscr{D}_N}\,\prod_{ij\in D} w_{ij}\,\prod_{i\in M(D)} x_i \,=\, \sum_{A\subseteq\{1,\dots,N\}}\;\sum_{\substack{P \text{ partition}\\ \text{of } A \text{ into pairs}}}\;\prod_{ij\in P} w_{ij}\,\prod_{i\in A^c} x_i\,. \tag{2.7}$$

Now choose w_{ii} for $i = 1,\dots,N$ such that the matrix $W = (w_{ij})_{i,j=1,\dots,N}$ is positive semi-definite[1]. Then there exists an (eventually degenerate) Gaussian

[1] For example one can choose $w_{ii} \geq \sum_{j\neq i} w_{ij}$ for every $i = 1,\dots,N$.

vector $\boldsymbol{\xi} = (\xi_1, \dots, \xi_N)$ with mean 0 and covariance matrix W. And by the Isserlis-Wick rule (see theorem 3.10)

$$\mathbb{E}_{\boldsymbol{\xi}}\Big[\prod_{i\in A}\xi_i\Big] \;=\; \sum_{\substack{P \text{ partition} \\ \text{of } A \text{ into pairs}}} \;\prod_{ij\in P} w_{ij}\;. \tag{2.8}$$

Substituting (2.8) into (2.7) one obtains

$$Z_N \;=\; \mathbb{E}_{\boldsymbol{\xi}}\Big[\sum_{A\subseteq\{1,\dots,N\}}\;\prod_{i\in A}\xi_i\;\prod_{i\in A^c}x_i\Big] \;=\; \mathbb{E}_{\boldsymbol{\xi}}\Big[\prod_{i=1}^N(\xi_i+x_i)\Big]\;. \tag{2.9}$$

□

Heilmann and Lieb [55] provided a recursion for the partition functions of monomer-dimer models. As we will see this is a fundamental tool to obtain exact solutions and to prove general properties.

Proposition 2.7 (Heilmann-Lieb recursion [55])**.** *Fixing any vertex $i \in V$ it holds:*

$$Z_G \;=\; x_i\,Z_{G-i} \,+\, \sum_{j \underset{G}{\sim} i} w_{ij}\,Z_{G-i-j}\;. \tag{2.10}$$

Here $G - i$ denotes the graph obtained from G deleting the vertex i and all its incident edges.

The Heilmann-Lieb recursion can be obtained directly from the definition (2.4), exploiting the hard-core constraint: the first term on the r.h.s. of (2.10) corresponds to a monomer on i, while the following terms correspond to a dimer on ij for some j neighbour of i. Here we show a proof that uses Gaussian integration by parts.

Proof [8]. Set $N := |V|$. Introduce zero dimer weights $w_{hk} = 0$ for all the pairs $hk \notin E$, so that $Z_G \equiv Z_N$. Following the proposition 2.6, introduce an N-dimensional Gaussian vector $\boldsymbol{\xi}$ with mean 0 and covariance matrix W. Then write the identity (2.6) isolating the vertex i:

$$Z_G \;=\; \mathbb{E}_{\boldsymbol{\xi}}\Big[\prod_{k=1}^N(\xi_k+x_k)\Big] \;=\; x_i\,\mathbb{E}_{\boldsymbol{\xi}}\Big[\prod_{k\neq i}(\xi_k+x_k)\Big] + \mathbb{E}_{\boldsymbol{\xi}}\Big[\xi_i\,\prod_{k\neq i}(\xi_k+x_k)\Big]\;. \tag{2.11}$$

Now apply the Gaussian integration by parts (theorem 3.10) to the second term on the r.h.s. of (2.11):

$$\mathbb{E}_{\boldsymbol{\xi}}\Big[\xi_i \prod_{k\neq i}(\xi_k+x_k)\Big] \;=\; \sum_{j=1}^{N} \mathbb{E}_{\boldsymbol{\xi}}[\xi_i\xi_j]\;\mathbb{E}_{\boldsymbol{\xi}}\Big[\frac{\partial}{\partial\xi_j}\prod_{k\neq i}(\xi_k+x_k)\Big] \;=\; \sum_{j\neq i} w_{ij}\;\mathbb{E}_{\boldsymbol{\xi}}\Big[\prod_{k\neq i,j}(\xi_k+x_k)\Big]\,. \tag{2.12}$$

Notice that summing over $j\neq i$ in the r.h.s. of (2.12) is equivalent to sum over $j \underset{G}{\sim} i$, since by definition $w_{ij}=0$ if $ij\notin E$. Substitute (2.12) into (2.11):

$$Z_G \;=\; x_i\;\mathbb{E}_{\boldsymbol{\xi}}\Big[\prod_{k\neq i}(\xi_k+x_k)\Big] + \sum_{j\sim i} w_{ij}\;\mathbb{E}_{\boldsymbol{\xi}}\Big[\prod_{k\neq i,j}(\xi_k+x_k)\Big]\,. \tag{2.13}$$

To conclude the proof observe that $(\xi_k)_{k\neq i}$ is an $(N-1)$-dimensional Gaussian vector with mean 0 and covariance $(w_{hk})_{h,k\neq i}$. Hence by proposition 2.6

$$Z_{G-i} \;=\; \mathbb{E}_{\boldsymbol{\xi}}\Big[\prod_{k\neq i}(\xi_k+x_k)\Big]\,. \tag{2.14}$$

And similarly

$$Z_{G-i-j} \;=\; \mathbb{E}_{\boldsymbol{\xi}}\Big[\prod_{k\neq i,j}(\xi_k+x_k)\Big]\,. \tag{2.15}$$

□

The main general result about monomer-dimer models is the absence of phase transitions, proved by Heilmann and Lieb [55]. This result is obtained by localizing the complex zeros of the partition functions far from the positive real axes.

Theorem 2.8 (Zeros of the partition function [55])**.** *Consider uniform monomer activity x on the graph and arbitrary dimer activities w_{ij}. The partition function Z_G is a polynomial of degree N in x, where $N=|V|$. The complex zeros of Z_G are purely imaginary:*

$$\{x\in\mathbb{C}\,|\,Z_G(w_{ij},x)=0\}\;\subset\;\mathrm{i}\mathbb{R}\,. \tag{2.16}$$

Furthermore they interlace the zeros of Z_{G-i} for any given $i\in V$, that is:

$$a_1\leq a'_1\leq a_2\leq a'_2\leq\cdots\leq a'_{N-1}\leq a_N\,, \tag{2.17}$$

where $-\mathrm{i}a_1, \dots, -ia_N$ *are the zeros of* Z_G *and* $-\mathrm{i}a'_1, \dots, -\mathrm{i}a'_{N-1}$ *are the imaginary parts of the zeros of* Z_{G-i}*. The relation* (2.17) *holds with strict inequalities if* $w_{ij} > 0$ *for all* $i, j \in V$*.*

Proof. Set

$$Q_G(x) := \mathrm{i}^{-N} Z_G(\mathrm{i}x) \;=\; \sum_{D \in \mathscr{D}_G} (-1)^{|D|} \prod_{ij \in D} w_{ij}\; x^{|M(D)|}$$

which is a polynomial of degree N with real coefficients (and leading coefficient 1). It is sufficient to prove that the zeros of Q_G are real and that, given $i \in V$, they interlace the zeros of Q_{G-i}: that is (2.17) is satisfied where $a_1, \dots, a_N$ are the zeros of Q_G and $a'_1, \dots, a'_{N-1}$ are the zeros of Q_{G-i}.

Assume that $w_{ij} > 0$ for all $i, j \in V$; the general results will easily follow by a continuity argument.

The result can be proved by induction on the number of vertices N. For $N = 0$ and for $N = 1$ the result is trivially true. Let $N \geq 2$ and assume the result is true for every graph having at most $N-1$ vertices. The Heilmann-Lieb recursion gives for all $x \in \mathbb{C}$

$$Q_G(x) \;=\; x\, Q_{G-i}(x) \;-\; \sum_{j \in V \smallsetminus \{i\}} w_{ij}\; Q_{G-i-j}(x)\;. \tag{2.18}$$

From the induction assumption it follows that for every $j \in V \smallsetminus \{i\}$ the $N-2$ zeros of Q_{G-i-j} are real and strictly interlace the $N-1$ zeros of Q_{G-i}, which are also real. Since for every $j \in V \smallsetminus \{i\}$, $w_{ij} > 0$ and Q_{G-i-j} has same degree and same leading coefficient, it is easy to prove that also the $N-2$ zeros of $S_i := \sum_{j \in V \smallsetminus \{i\}} w_{ij} Q_{G-i-j}$ are real and strictly interlace the $N-1$ zeros of Q_{G-i}. We can easily deduce the sign of $S_i(x)$ when x is a zero of Q_{G-i}. As a consequence, using (2.18), we know the sign of $Q_G(x)$ when x is a zero of Q_{G-i}. It is easy to conclude that Q_G has N real zeros that are strictly interlaced by the zeros of Q_{G-i}. □

Corollary 2.9 (Absence of phase transitions)**.** *Consider dimer activities* $w\, w_{ij}^{(N)}$ *and monomer activities* $x\, x_i^{(N)}$ *and assume they are chosen in such a way that*

$p := \lim_{N\to\infty} \frac{1}{N} \log Z_N$ exists. Then the function p is analytic in the variables $(w,x) \in (0,\infty)^2$ and the derivatives $\frac{\partial^{h+k}}{\partial^h w\, \partial^k x}$ can be interchanged with the limit $N \to \infty$.

Proof. Let us consider $Z_N(w_{ij}, x)$. The general result will follow since $Z_N(ww_{ij}, xx_i)$ coincides up to a factor that does not affect the Gibbs measure with $Z_N(\frac{w_{ij}}{x_i x_j}, \frac{x}{\sqrt{w}})$.

Fix $x > 0$. Let $\varepsilon \in (0, x)$ and set $U := (\varepsilon, \infty) + \mathrm{i}\,\mathbb{R}$. By the theorem 2.8 the holomorphic function $x \mapsto Z_N(w_{ij}, x)$ does not vanish on the simply connected open set U, then $p_N := \frac{1}{N} \log Z_N(w_{ij}, x)$ is holomorphic on U. Furthermore the sequence p_N is uniformly bounded on the compact sets $K \subset U$, since

$$|Z_N(w_{ij}, x)| = \prod_{k=1}^{N} |x - \mathrm{i}a_k| \geq \varepsilon^N \ , \quad |Z_N(w_{ij}, x)| \leq Z_N\big(w_{ij}, \sup_K |x|\big) \ .$$

Then by the Vitali-Porter theorem [91], the sequence p_N converges uniformly on U to an holomorphic function p. And by the Weierstrass theorem the derivatives $\frac{\partial^k}{\partial^k x}$ can be interchanged with the limit $N \to \infty$. □

2.2 Hard-core and imitative interactions

Beyond the hard-core constraint it is possible to enrich monomer-dimer models with other kinds of interaction. Assign a monomer activity $x_i > 0$ to each vertex $i \in V$, a dimer activity $w_{ij} \geq 0$ and imitation coefficients $J'_{ij}, J''_{ij}, J'''_{ij} \in \mathbb{R}$ to each edge $ij \in E$.

Definition 2.10 (Monomer-dimer models)**.** A *monomer-dimer model* on G is given by the following Gibbs probability measure on $\mathscr{D}_G$:

$$\mu_G(D) := \frac{1}{Z_G} \prod_{ij\in D} w_{ij} \prod_{i\in M(D)} x_i \prod_{\substack{ij\in E:\\ i\notin M(D),\, j\notin M(D)}} e^{J'_{ij}} \prod_{\substack{ij\in E:\\ i\in M(D),\, j\in M(D)}} e^{J''_{ij}} \prod_{\substack{ij\in E:\\ i\notin M(D),\, j\in M(D)}} e^{J'''_{ij}} \tag{2.19}$$

for all $D \in \mathscr{D}_G$. The partition function Z_G is defined so that $\sum_{D\in\mathscr{D}_G} \mu_G(D) = 1$.

The dependence of the measure on the coefficients w_{ij}, x_i, J_{ij} is usually implicit in the notations. When we consider the complete graph with N vertices, the subscript $_G$ is usually substituted by $_N$.

When all the J_{ij}'s take the value zero this model is the pure hard-core model introduced in the previous section. Positive values of the J'_{ij}, J''_{ij} favour the configurations with clusters of dimers and clusters of monomers.

Sometimes it is convenient to rewrite to measure $\mu_G(D)$ in the Hamiltonian form $\frac{1}{Z_G}\, e^{-H_G(D)}$. This is possible by setting $x_i =: e^{h_i}$, $w_{ij} =: e^{h_{ij}}$ and

$$\begin{split} -\,H_G(\alpha) \,:=\, &\sum_{ij\in E} h_{ij}\alpha_{ij} + \sum_{i\in V} h_i\alpha_i \,+ \\ + \sum_{ij\in E}\Big(&J'_{ij}(1-\alpha_i)(1-\alpha_j) + J''_{ij}\alpha_i\alpha_j + J'''_{ij}(1-\alpha_i)\alpha_j + J'''_{ij}\alpha_i(1-\alpha_j)\Big) \end{split} \tag{2.20}$$

for all $\alpha \in \mathscr{D}_G$.

Remark 2.11. Let us analyse the redundancies in the definition 2.10. First a monomer-dimer model on the graph $G = (V, E)$ coincides with a monomer-dimer model on the complete graph with $N = |V|$ vertices, by taking $w_{ij} = 0$ and $J'_{ij} = J''_{ij} = J'''_{ij} = 0$ for all pairs $ij \notin E$.

Secondly the monomer and dimer activities can be reduced from (w_{ij}, x_i) to $(\frac{w_{ij}}{x_i\, x_j}, 1)$ or from (w, x_i) to $(1, \frac{x_i}{\sqrt{w}})$, as shown by the remark 2.4. Moreover also the imitation coefficients can be reduced from $(J'_{ij}, J''_{ij}, J'''_{ij})$ to $(J'_{ij} - J'''_{ij}, J''_{ij} - J'''_{ij}, 0)$, since

$$\alpha_i(1-\alpha_j) + (1-\alpha_i)\alpha_j \;=\; -\alpha_i\alpha_j - (1-\alpha_i)(1-\alpha_j) + 1\;.$$

If the imitation coefficients J', J'' are uniform on the graph G, then a further reduction is possible to remain with only one imitation coefficient, for example from (h_i, J', J'') to $\big(h_i + (J''-J')\deg_G i, \frac{J'+J''}{2}, \frac{J'+J''}{2}\big)$, since

$$\sum_{ij\in E}(\alpha_i + \alpha_j) \;=\; \sum_{i\in V} \alpha_i\, \deg_G i\;.$$

The Gaussian representation and the recursion relation found for the pure hard-core case can be extended to the imitative case, even if the resulting expressions are not as limpid as the previous ones.

Proposition 2.12. *The partition function of any monomer-dimer model over N vertices can be written as*

$$Z_N = \mathbb{E}_{\boldsymbol{\xi}}\Big[\sum_{A\subset\{1,\dots,N\}}\prod_{i\in A}\xi_i\prod_{i\in A^c}x_i\prod_{i\in A,j\in A}e^{J'_{ij}/2}\prod_{i\in A^c,j\in A^c}e^{J''_{ij}/2}\prod_{\substack{i\in A,j\in A^c\\ \text{or v.v.}}}e^{J'''_{ij}/2}\Big]\,, \quad (2.21)$$

where $\boldsymbol{\xi}=(\xi_1,\dots,\xi_N)$ is a Gaussian random vector with mean 0 and covariance matrix $W=(w_{ij})_{i,j=1,\dots,N}$ and $\mathbb{E}_{\boldsymbol{\xi}}[\,\cdot\,]$ denotes the expectation with respect to $\boldsymbol{\xi}$. The diagonal entries w_{ii} are arbitrary numbers, chosen in such a way that W is a positive semi-definite matrix. Moreover we set $J'_{ii}=J''_{ii}=J'''_{ii}=0$.

The proof is the same as proposition 2.6. It is interesting to observe that, when all the ξ_i's are positive, the sum inside the expectation on the r.h.s. of (2.21) is the partition function of an Ising model.

Proposition 2.13. *Fixing any vertex $i\in V$ it holds:*

$$Z_G = x_i\,\widetilde{Z}_{G-i} + \sum_{j\underset{G}{\sim}i} w_{ij}\,\widetilde{Z}_{G-i-j}\,, \quad (2.22)$$

where:

- *in the partition function $\widetilde{Z}_{G-i}$ the monomer activity x_k is replaced by $x_k\,e^{J''_{ik}}$ and the dimer activity $w_{kk'}$ is replaced by $w_{kk'}\,e^{J'''_{ik}+J'''_{ik'}}$ for all vertices k,k' (notice that only the neighbours of i really change their activities);*

- *in the partition function $\widetilde{Z}_{G-i-j}$ the monomer activity x_k is replaced by $x_k\,e^{J'''_{ik}+J'''_{jk}}$ and the dimer activity $w_{kk'}$ is replaced by $w_{kk'}\,e^{J'_{ik}+J'_{ik'}+J'_{jk}+J'_{jk'}}$ for all vertices k,k' (notice that only the neighbours of i or j really change their activities).*

The relation (2.22) can be obtained directly from the definition: the first term on the r.h.s. corresponds to a monomer on i, while the following terms correspond to a dimer on ij for some j neighbour of i.

The hard-core interaction is not sufficient to cause a phase transition, but adding also the imitative interaction the system can have phase transitions [23, 24,57,6]: in the chapters 4, 5 we will study this phase transition on the complete graph. To locate the complex zeros of the partition function in presence of imitation is an open problem.

Chapter 3

Hard-core interaction on the complete graph

This chapter is based on the joint work [8]. We consider a monomer-dimer model with **pure hard-core interaction** (see section 2.1), we fix a uniform dimer activity on the **complete graph**, while we choose **i.i.d. random monomer activities**.

Under quite general hypothesis on the distribution of the activities, we show that this model is exactly solvable and does not present a phase transition (in agreement with the general results by Heilmann and Lieb [55, 56]). Precisely we prove that in the thermodynamic limit the pressure density exists and is given by a one-dimensional variational principle, which admits a unique solution (theorem 3.2). The problem becomes accessible by the use of a Gaussian representation for the partition function, then a careful application of the Laplace method leads to the solution.

A particular case is obtained when the monomer activity is also uniform on the vertices. Thus we obtain the solution of the deterministic monomer-dimer model on the complete graph, which was previously studied by Heilmann and Lieb [55] using the Hermite polynomials.

Let $w > 0$. Let $x_i > 0$, $i \in \mathbb{N}$, be *independent identically distributed* random variables. In order to keep the logarithm of the partition function of order N, a normalization of the dimer activity as w/N is needed. Therefore in this chapter we will denote

$$Z_N \;=\; \sum_{D\in\mathscr{D}_N} \big(\frac{w}{N}\big)^{|D|} \prod_{i\in M(D)} x_i \;. \tag{3.1}$$

μ_N will denote the corresponding Gibbs measure and $\langle\,\cdot\,\rangle_N$ will be the expected value with respect to μ_N. Notice that now the partition function is a random variable and the Gibbs measure is a random measure.

Remark 3.1. Since the dimer weight is uniform, the Gaussian representation of (6.1) gives simply:

$$Z_N \;=\; \mathbb{E}_\xi\bigg[\prod_{i=1}^N(\xi + x_i)\bigg] \;, \tag{3.2}$$

where ξ is a one-dimensional Gaussian random variable with mean 0 and variance w/N.

Indeed by proposition 2.6, $Z_N = \mathbb{E}_{\boldsymbol{\xi}}\big[\prod_{i=1}^N(\xi_i+x_i)\big]$ where $\boldsymbol{\xi}=(\xi_1,\dots,\xi_N)$ is an N-dimensional Gaussian random vector with mean 0 and constant covariance matrix[1] $(w/N)_{i,j=1,\dots,N}$. It is easy to check that $\boldsymbol{\xi}$ has the same joint distribution of the constant random vector $(\xi,\dots,\xi)$. Therefore the identity (3.2) follows.

Z_N can be expressed as an expectation in the Gaussian variable ξ; but on the other hand Z_N is a random variable dependent on the monomer weights x_i's. To avoid confusion we rewrite (3.2) as an explicit integral in $\mathrm{d}\xi$:

$$Z_N \;=\; \frac{\sqrt{N}}{\sqrt{2\pi w}} \int_{\mathbb{R}} e^{-\frac{N}{2w}\,\xi^2} \prod_{i=1}^N(\xi + x_i)\;\mathrm{d}\xi \;. \tag{3.3}$$

Theorem 3.2. *Let $w > 0$. Let $x_i > 0$, $i \in \mathbb{N}$ be i.i.d. random variables. Denote by x a random variable distributed like x_i; suppose that $\mathbb{E}_x[x] < \infty$ and*

[1]Notice that setting also the diagonal entries to w/N, the resulting matrix is positive semi-definite: $\sum_{i=1}^N\sum_{j=1}^N(w/N)\,a_ia_j = (w/N)\,\big(\sum_{i=1}^N a_i\big)^2 \geq 0$ for every $a\in\mathbb{R}^N$.

$\mathbb{E}_x[(\log x)^2] < \infty$. *Then:*

$$\exists \lim_{N\to\infty} \frac{1}{N}\, \mathbb{E}_{\boldsymbol{x}}[\,\log Z_N] \;=\; \sup_{\xi\geq 0} \Phi(\xi) \;\in \mathbb{R} \tag{3.4}$$

where

$$\Phi(\xi) \;:=\; -\frac{\xi^2}{2w} + \mathbb{E}_x[\,\log(\xi + x)] \quad \forall\, \xi \geq 0\,. \tag{3.5}$$

Furthermore the function Φ *attains its maximum at a unique point* ξ^*. ξ^* *is the only solution in* $[0,\infty[$ *of the fixed point equation*

$$\xi^* \;=\; \mathbb{E}_x\left[\frac{w}{\xi^* + x}\right]\,. \tag{3.6}$$

Thus the following bounds hold:

$$\frac{-\mathbb{E}_x[x] + \sqrt{\mathbb{E}_x[x]^2 + 4w}}{2} \vee \sup_{t>0} \frac{-t + \sqrt{t^2 + 4w\,\mathbb{P}_x(x\leq t)}}{2} \;\leq\; \xi^* \;\leq\; \sqrt{w} \wedge \mathbb{E}_x\left[\frac{w}{x}\right]. \tag{3.7}$$

In consequence of the theorem 3.2 it is not hard to prove that the system does not present a phase transition in the parameter $w > 0$. It is also easy to compute the main macroscopic quantity of physical interest, that is the *dimer density*, in terms of the positive solution ξ^* of the fixed point equation (3.6). Therefore we state the following two corollaries before starting to prove the theorem.

Corollary 3.3. *In the hypothesis of the theorem 3.2, consider the limiting pressure density* $p(w) := \lim_{N\to\infty} \frac{1}{N}\, \mathbb{E}_{\boldsymbol{x}}\big[\log Z_N(w)\big]$. *Then* p *is a smooth function of* $w > 0$.

Proof. By the theorem 3.2 $p(w) = \Phi(w,\xi^*)$, where $\Phi(w,\xi) = -\xi^2/(2w) + \mathbb{E}_x[\,\log(\xi + x)]$ and $\xi^* = \xi^*(w)$ is *the only* positive solution of the equation $F(w,\xi) = 0$ with $F := \frac{\partial \Phi}{\partial \xi}$.

F is a smooth function on $]0,\infty[\,\times\,]0,\infty[$, because Φ is smooth as it will be proven in the lemma 3.6. In addition $\frac{\partial F}{\partial \xi}(w,\xi^*) \neq 0$ for all $w > 0$, by the lemma 3.6 equation (3.13).

As a consequence, by the implicit function theorem (see e.g. [86]), ξ^* is a smooth function of $w \in]0, \infty[$. Hence, by composition, also $p(w) = \Phi\big(w, \xi^*(w)\big)$ is a smooth function of $w \in]0, \infty[$. □

Corollary 3.4. *In the hypothesis of the theorem 3.2, the limiting dimer density*

$$d := \lim_{N\to\infty} \frac{1}{N}\, \mathbb{E}_{\boldsymbol{x}}\big[\langle\, |D|\, \rangle_N\big]$$

can be computed as

$$d \,=\, w\, \frac{\mathrm{d}\,p}{\mathrm{d}w} \,=\, \frac{(\xi^*)^2}{2w}\,. \tag{3.8}$$

Proof. Set $p_N := \frac{1}{N} \log Z_N$ and perform the change of parameter $w =: e^h$. Clearly $\frac{\mathrm{d}}{\mathrm{d}h} = w\, \frac{\mathrm{d}}{\mathrm{d}w}$ and it is easy to check that

$$\frac{\mathrm{d}\,\mathbb{E}_{\boldsymbol{x}}[p_N]}{\mathrm{d}h} \,=\, \mathbb{E}_{\boldsymbol{x}}\big[\langle\, |D|\, \rangle_N\big]\,.$$

By the theorem 3.2 and its corollary 3.3, $\mathbb{E}_{\boldsymbol{x}}[p_N]$ converges pointwise to a smooth function p as $N \to \infty$ for all values of $h \in \mathbb{R}$. A standard computation shows that $\mathbb{E}_{\boldsymbol{x}}[p_N]$ is a convex function of h. Therefore

$$\frac{\mathrm{d}\,\mathbb{E}_{\boldsymbol{x}}[p_N]}{\mathrm{d}h} \xrightarrow[N\to\infty]{} \frac{\mathrm{d}\,p}{\mathrm{d}h}\,.$$

Since $p(h) = \Phi\big(h, \xi^*(h)\big)$, where ξ^* is the critical point of Φ and is a smooth function of h, it is easy to compute

$$\frac{\mathrm{d}\,p}{\mathrm{d}h}(h) \,=\, \frac{\partial\Phi}{\partial h}(h,\xi^*) \,+\, \underbrace{\frac{\partial\Phi}{\partial\xi}(h,\xi^*)}_{=\,0}\, \frac{\mathrm{d}\xi^*}{\mathrm{d}h}(h) \,=\, \frac{(\xi^*)^2}{2\,e^h}\,.$$ □

Remark 3.5. In the deterministic case, namely when the distribution of the x_i's is a Dirac delta centred at a point x, the theorem 3.2 and its corollary 3.4 reproduce the results that have already been obtained in the Proposition 6 of [6] by a combinatorial computation. Indeed the fixed point equation (3.6) reduces to $\xi^* = \frac{w}{\xi^*+x}$, whose positive solution is

$$\xi^* = \frac{-x + \sqrt{x^2+4w}}{2}\,. \tag{3.9}$$

As a consequence, by (3.8), the limiting dimer and monomer density are respectively

$$d = \frac{x^2 - x\sqrt{x^2+4w} + 2w}{2w} \;, \quad m = 1 - 2\,d = \frac{-x^2 + x\sqrt{x^2+4w}}{2w} \;. \tag{3.10}$$

Moreover by and (3.8) the limiting pressure $p = \Phi(\xi^*)$ can be written as

$$p \;=\; -d - \frac{1}{2}\log\frac{2\,d}{w} \;=\; -\frac{1-m}{2} - \frac{1}{2}\log\frac{1-m}{w} \;. \tag{3.11}$$

3.1 Proof of the convergence

Now let us start to prove the theorem 3.2. The logic structure of the proof is divided in three main parts. First we study the basic properties of the function Φ. Then we use the uniform law of large numbers and other observations to show that for large N the integrated function in (3.3) can be well approximated by $e^{N\Phi}$. Finally we will be able to exploit the Laplace's method in order to compute a lower and an upper bound for $\frac{1}{N}\,\mathbb{E}_{\boldsymbol{x}}[\,\log Z_N]\,$.

Lemma 3.6. *Φ is continuous on $[0,\infty[\,$, it is smooth on $]0,\infty[$ and the derivatives can be taken inside the expectation. In particular for all $\xi > 0$ it holds*

$$\Phi'(\xi) \;=\; -\frac{\xi}{w} + \mathbb{E}_x\left[\frac{1}{\xi + x}\right] \;; \tag{3.12}$$

$$\Phi''(\xi) \;=\; -\frac{1}{w} - \mathbb{E}_x\left[\frac{1}{(\xi+x)^2}\right] < 0 \;. \tag{3.13}$$

As a consequence Φ has exactly one critical point ξ^* in $]0,\infty[\,$, that is the equation (3.6) has exactly one solution in $]0,\infty[\,$. ξ^* is the only global maximum point *of Φ on $[0,\infty[\,$.*

Proof. **I.** First of all $\Phi(\xi)$ is well-defined for all $\xi \geq 0$. Indeed for $\xi > 0$

$$\log(\xi + x) \begin{cases} \leq \xi + x - 1 \,\in L^1(\mathbb{P}_x) \\ \geq 1 - \frac{1}{\xi+x} \,\geq 1 - \frac{1}{\xi} \,\in L^1(\mathbb{P}_x) \end{cases} \;;$$

while for $\xi = 0$, $\mathbb{E}_x[|\log x|] \leq \mathbb{E}_x[(\log x)^2]^{1/2} < \infty$ by the Hölder inequality.

Φ is continuous at $\xi = 0$ by monotone convergence: $\log(\xi + x)$ decreases to $\log x$ as $\xi \searrow 0$ and $\mathbb{E}_x[\log(\xi + x)] < \infty$.

Let now $\xi > 0$ and let $\delta > 0$ such that $\xi - \delta > 0$. The first derivative of Φ at ξ can be computed inside the expectation, obtaining (3.12), since the difference quotient of $\xi \mapsto \log(\xi + x)$ satisfies the dominated convergence hypothesis. Indeed for all $\xi' \in]\xi - \delta, \xi + \delta[$

$$\left| \frac{\log(\xi' + x) - \log(\xi + x)}{\xi' - \xi} \right| \leq \sup_{\widetilde{\xi} \in [\xi, \xi']} \frac{1}{\widetilde{\xi} + x} \leq \sup_{\widetilde{\xi} \in [\xi, \xi']} \frac{1}{\widetilde{\xi}} \leq \frac{1}{\xi - \delta} \in L^1(\mathbb{P}_x) .$$

Now the second derivative of Φ at ξ can be computed inside the expectation, obtaining (3.13), since the difference quotient of $\xi \mapsto \frac{1}{\xi + x}$ satisfies the dominated convergence hypothesis. Indeed for all $\xi' \in]\xi - \delta, \xi + \delta[$

$$\left| \frac{\frac{1}{\xi' + x} - \frac{1}{\xi + x}}{\xi' - \xi} \right| \leq \sup_{\widetilde{\xi} \in [\xi, \xi']} \frac{1}{(\widetilde{\xi} + x)^2} \leq \sup_{\widetilde{\xi} \in [\xi, \xi']} \frac{1}{\left(\widetilde{\xi}\right)^2} \leq \frac{1}{(\xi - \delta)^2} \in L^1(\mathbb{P}_x) .$$

This reasoning can be iterated up to the derivative of any order, since $1/\left(\widetilde{\xi} + x\right)^k \leq 1/\left(\widetilde{\xi}\right)^k \leq 1/(\xi - \delta)^k \in L^1(\mathbb{P}_x)$ for all $\widetilde{\xi} \in]\xi - \delta, \xi + \delta[$ and all $k \geq 1$.

II. In virtue of (3.13) Φ is a strictly convex function on $]0, \infty[$. At the boundaries of this domain $\lim_{\xi \to 0+} \Phi'(\xi) = \mathbb{E}_x[x^{-1}] > 0$ and $\lim_{\xi \to \infty} \Phi'(\xi) = -\infty < 0$ by (3.12) and monotone converge. Therefore Φ has exactly one critical point ξ^* in $]0, \infty[$ and it is the only global maximum point of Φ. $\square$

Remark 3.7. Since ξ^* satisfies the fixed point equation (3.6), it is easy to obtain the bounds (3.7) for ξ^*. Since $\xi^* > 0$ and $x > 0$,

$$\xi^* = \mathbb{E}_x\left[\frac{w}{\xi^* + x}\right] \leq \frac{1}{\xi^*} \;\Rightarrow\; \xi^* \leq \sqrt{w}\,; \quad \xi^* = \mathbb{E}_x\left[\frac{w}{\xi^* + x}\right] \leq \mathbb{E}_x\left[\frac{w}{x}\right] .$$

Using the Jensen inequality,

$$\xi^* = \mathbb{E}_x\left[\frac{w}{\xi^* + x}\right] \geq \frac{w}{\xi^* + \mathbb{E}_x[x]} \;\Rightarrow\; (\xi^*)^2 + \xi^*\,\mathbb{E}_x[x] - w \geq 0 \;\Rightarrow\; \xi^* \geq \frac{-\mathbb{E}_x[x] + \sqrt{\mathbb{E}_x[x]^2 + 4w}}{2} .$$

Finally, since $\xi^* + x > 0$, it holds for all $t > 0$

$$\xi^* = \mathbb{E}_x\left[\frac{w}{\xi^* + x}\right] \geq \frac{w}{\xi^* + t}\,\mathbb{P}_x(x \leq t) \;\Rightarrow\; (\xi^*)^2 + \xi^*\,t - w\,\mathbb{P}_x(x \leq t) \geq 0 \;\Rightarrow$$
$$\Rightarrow\; \xi^* \geq \frac{-t + \sqrt{t^2 + 4w\,\mathbb{P}_x(x \leq t)}}{2}\,.$$

Lemma 3.8. *Define the random function*

$$\Phi_N(\xi) \;:=\; -\frac{\xi^2}{2w} + \frac{1}{N}\sum_{i=1}^{N} \log|\xi + x_i| \quad \forall\,\xi \in \mathbb{R}\,. \tag{3.14}$$

This function is defined also for negative values of ξ *and it takes the value* $-\infty$ *at the random points* $-x_1, \dots, -x_N$*. It is important to observe that*

$$\Phi_N(-\xi) \;<\; \Phi_N(\xi) \quad \forall\,\xi > 0\,. \tag{3.15}$$

i. *Let* $0 < M < \infty$*. Then for all* $\varepsilon > 0$

$$\mathbb{P}_{\boldsymbol{x}}\Big(\,\forall\,\xi\in[0,M]\;\;|\Phi_N(\xi) - \Phi(\xi)| < \varepsilon\,\Big) \xrightarrow[N\to\infty]{} 1\,. \tag{3.16}$$

ii. *Let* $0 < m < M < \infty$*. Then there exists* $\lambda_{m,M} > 0$ *such that*

$$\mathbb{P}_{\boldsymbol{x}}\Big(\,\forall\,\xi\in[m,M]\;\;\Phi_N(-\xi) < \Phi_N(\xi) - \lambda_{m,M}\,\Big) \xrightarrow[N\to\infty]{} 1\,. \tag{3.17}$$

iii. *Let* $C \in \mathbb{R}$*. Then there exists* $M_C > 0$ *such that*

$$\mathbb{P}_{\boldsymbol{x}}\Big(\,\forall\,\xi\in[M_C,\infty[\;\;\Phi_N(\xi) < C \;\text{ and }\; \Phi_N(\xi) < \varphi(\xi)\,\Big) \xrightarrow[N\to\infty]{} 1\,; \tag{3.18}$$

where φ *is the following deterministic function*

$$\varphi(\xi) \;:=\; -\frac{\xi^2}{2w} + \log\xi + \frac{1}{\xi}\,(\mathbb{E}_x[x] + 1) \quad \forall\,\xi > 0\,. \tag{3.19}$$

Notice that $\Phi_N(\xi) - \Phi(\xi) = \frac{1}{N}\sum_{i=1}^N \log(\xi + x_i) - \mathbb{E}_x[\,\log(\xi + x)]$ for all $\xi > 0$. Since the x_i, $i \in \mathbb{N}$ are i.i.d., the basic idea behind the lemma 3.8 is to approximate Φ_N with Φ by the law of large numbers. But this approximation is needed to hold at every ξ at the same time, hence a *uniform* law of large numbers is required.

To prove the theorem 3.2 it will be important to have found a good uniform approximation near the global maximum point ξ^* of Φ. Far from ξ^* instead such a uniform approximation cannot hold: for example Φ_N diverges to $-\infty$ at certain negative points, while, if the distribution of x is absolutely continuous and satisfies some integrability hypothesis, it is possible to show that $\Phi(\xi) = -\frac{\xi^2}{2w} + \mathbb{E}_x[\log|\xi + x|]$ is continuous on $\mathbb{R}$. But fortunately, far from ξ^*, it will be sufficient for our purposes to bound suitably Φ_N from above.

Proof. **i.** For every $x > 0$ the function $\xi \mapsto \log(\xi + x)$ is continuous on $[0, M]$ compact. Moreover there is domination:

$$\log(\xi + x) \begin{cases} \leq \log(M + x) \ \in L^1(\mathbb{P}_x) \\ \geq \log x \ \in L^1(\mathbb{P}_x) \end{cases} \qquad \forall \xi \in [0, M] \,.$$

Therefore (3.16) holds by the uniform weak law of large numbers (theorem 3.12).

ii. Clearly $\log(\xi + x) > \log|-\xi + x|$ for all $\xi, x > 0$. Furthermore an elementary computation shows that for all $\xi, x, \tau > 0$

$$\log(\xi + x) - \log|-\xi + x| \geq \tau \quad \Leftrightarrow \quad \frac{e^\tau - 1}{e^\tau + 1}\,\xi \leq x \leq \frac{e^\tau + 1}{e^\tau - 1}\,\xi \,.$$

Therefore for all $\xi \in [m, M]$ and all $\tau > 0$,

$$\begin{aligned} \Phi_N(\xi) - \Phi_N(-\xi) &= \frac{1}{N}\sum_{i=1}^N \big(\log(\xi + x_i) - \log|-\xi + x_i|\big) \geq \\ &\geq \frac{1}{N}\sum_{i=1}^N \tau\, \mathbb{1}\left(\frac{e^\tau - 1}{e^\tau + 1}\,\xi \leq x_i \leq \frac{e^\tau + 1}{e^\tau - 1}\,\xi\right) \geq \\ &\geq \tau\,\frac{1}{N}\sum_{i=1}^N \mathbb{1}\left(\frac{e^\tau - 1}{e^\tau + 1}\,M \leq x_i \leq \frac{e^\tau + 1}{e^\tau - 1}\,m\right) . \end{aligned} \tag{3.20}$$

Set $I^\tau_{m,M} := \big[\frac{e^\tau - 1}{e^\tau + 1}\,M\,,\,\frac{e^\tau + 1}{e^\tau - 1}\,m\big]$. Now by the weak law of large numbers, for all $\varepsilon > 0$

$$\mathbb{P}_{\boldsymbol{x}}\left(\frac{1}{N}\sum_{i=1}^N \mathbb{1}\big(x_i \in I^\tau_{m,M}\big) > \mathbb{P}_x\big(x \in I^\tau_{m,M}\big) - \varepsilon\right) \xrightarrow[N\to\infty]{} 1 \,. \tag{3.21}$$

Hence, using (3.20) and (3.21), for all $\tau, \varepsilon > 0$

$$\mathbb{P}_{\boldsymbol{x}}\bigg(\Phi_N(\xi) - \Phi_N(-\xi) \;>\; \tau\left(\mathbb{P}_x(x \in I^{\tau}_{m,M}) - \varepsilon\right)\bigg) \xrightarrow[N\to\infty]{} 1\;. \tag{3.22}$$

To conclude observe that $I^{\tau}_{m,M} \nearrow\,]0,\infty[$ (which is the support of the distribution of x) as $\tau \searrow 0$. Hence there exists $\tau_0 > 0$ such that $\mathbb{P}_x(x \in I^{\tau_0}_{m,M}) > 0$. Choose $0 < \varepsilon_0 < \mathbb{P}_x(x \in I^{\tau_0}_{m,M})$ and set

$$\lambda_{m,M} \;:=\; \tau_0\left(\mathbb{P}_x(x \in I^{\tau_0}_{m,M}) - \varepsilon_0\right) \;>0\;.$$

Then (3.17) follows from (3.22).

iii. For all $\xi > 0$ the following bound holds:

$$\begin{aligned}\Phi_N(\xi) \;&=\; -\frac{\xi^2}{2w} + \frac{1}{N}\sum_{i=1}^{N}\log(\xi + x_i) \;=\; -\frac{\xi^2}{2w} + \log\xi + \frac{1}{N}\sum_{i=1}^{N}\log\left(1 + \frac{x_i}{\xi}\right) \;\leq \\ &\leq\; -\frac{\xi^2}{2w} \;+\; \log\xi \;+\; \frac{1}{\xi}\,\frac{1}{N}\sum_{i=1}^{N}\;x_i\;.\end{aligned} \tag{3.23}$$

Now by the weak law of large numbers (no uniformity in ξ is needed here), for all $\varepsilon > 0$

$$\mathbb{P}_{\boldsymbol{x}}\bigg(\frac{1}{N}\sum_{i=1}^{N}\;x_i \;<\; \mathbb{E}_x[x] + \varepsilon\bigg) \xrightarrow[N\to\infty]{} 1\;. \tag{3.24}$$

Hence, using (3.23) and (3.24), for all $0 < \varepsilon < 1$

$$\mathbb{P}_{\boldsymbol{x}}\bigg(\forall\,\xi\!>\!0\;\;\; \Phi_N(\xi) < \varphi(\xi)\bigg) \xrightarrow[N\to\infty]{} 1\;. \tag{3.25}$$

Furthermore it holds $\varphi(\xi) \to -\infty$ as $\xi \to \infty$. Hence for all $C \in \mathbb{R}$ there exists $M_C > 0$ such that

$$\varphi(\xi) < C \quad \forall\,\xi > M_C\;. \tag{3.26}$$

In conclusion (3.18) follows from (3.25) and (3.26). □

Lemma 3.9. *There exists a constant $C_0 < \infty$ such that*

$$\mathbb{E}_{\boldsymbol{x}}\bigg[\bigg(\frac{\log Z_N}{N}\bigg)^2\bigg] \;\leq\; C_0 \quad \forall\,N \in \mathbb{N}\;. \tag{3.27}$$

Proof. Since $x \mapsto (\log x)^2$ is concave for $x \geq e$, the Jensen inequality can be used as follows:

$$\begin{aligned} \mathbb{E}_{\boldsymbol{x}}\big[(\log Z_N)^2\, \mathbb{1}(Z_N \geq e)\big] \;&=\; \mathbb{E}_{\boldsymbol{x}}\big[(\log Z_N)^2 \,\big|\, Z_N \geq e\big]\; \mathbb{P}_{\boldsymbol{x}}(Z_N \geq e) \;\leq \\ &\leq\; \big(\log \mathbb{E}_{\boldsymbol{x}}\big[Z_N \,\big|\, Z_N \geq e\big]\,\big)^2\; \mathbb{P}_{\boldsymbol{x}}(Z_N \geq e) \;= \\ &=\; \left(\log \frac{\mathbb{E}_{\boldsymbol{x}}\big[Z_N\, \mathbb{1}(Z_N \geq e)\big]}{\mathbb{P}_{\boldsymbol{x}}(Z_N \geq e)}\right)^2 \mathbb{P}_{\boldsymbol{x}}(Z_N \geq e) \;\leq \\ &\leq\; 2\,\big(\log \mathbb{E}_{\boldsymbol{x}}\big[Z_N\big]\big)^2 \;+\; 2 \max_{p\in[0,1]} (\log p)^2\, p\;. \end{aligned} \tag{3.28}$$

Since the x_i, $i \in \mathbb{N}$ are i.i.d. $\mathbb{E}_{\boldsymbol{x}}[Z_N]$ equals a deterministic partition function with uniform weights. Hence it is easy to bound it as follows:

$$\begin{aligned} \mathbb{E}_{\boldsymbol{x}}\big[Z_N\big] \;&=\; \sum_{D\in\mathscr{D}_N} \left(\frac{w}{N}\right)^{|D|} \mathbb{E}_x[x]^{|M(D)|} \;\leq\; \sum_{d=0}^{|E_N|} \binom{|E_N|}{d} \left(\frac{w}{N}\right)^{d} \mathbb{E}_x[x]^{N-2d} \;= \\ &=\; \mathbb{E}_x[x]^N \left(1 + \frac{w}{N}\,\mathbb{E}_x[x]^{-2}\right)^{|E_N|} \;\leq\; \mathbb{E}_x[x]^N\, \exp\left(\frac{N-1}{2}\,\frac{w}{\mathbb{E}_x[x]^2}\right) \end{aligned} \tag{3.29}$$

(here $|E_N| = \frac{N(N-1)}{2}$ denotes the number of edges in the complete graph over N vertices). Therefore, substituting (3.29) into (3.28),

$$\mathbb{E}_{\boldsymbol{x}}\big[(\log Z_N)^2\, \mathbb{1}(Z_N \geq e)\big] \;\leq\; 2\,N^2 \left(\log \mathbb{E}_x[x] + \frac{w}{2\,\mathbb{E}_x[x]^2}\right)^2 + \;2 \max_{p\in[0,1]} (\log p)^2\, p\;. \tag{3.30}$$

It remains to deal with the case $Z_N < e$. When $1 < Z_N < e$, it holds $0 < \log Z_N < 1$ hence trivially

$$\mathbb{E}_{\boldsymbol{x}}\big[(\log Z_N)^2\, \mathbb{1}(1 < Z_N < e)\big] \;\leq\; \mathbb{E}_{\boldsymbol{x}}\big[(\log e)^2\, \mathbb{1}(1 < Z_N < e)\big] \;\leq\; 1\;. \tag{3.31}$$

When instead $Z_N \leq 1$, it holds $\log Z_N \leq 0$ hence we need a lower bound for Z_N. For example, considering only the configuration with no dimers, $Z_N \geq \prod_{i=1}^N x_i$.

Therefore:

$$\mathbb{E}_{\boldsymbol{x}}\big[(\log Z_N)^2\, \mathbb{1}(Z_N \leq 1)\big] \,\leq\, \mathbb{E}_{\boldsymbol{x}}\Big[\Big(\log \prod_{i=1}^N x_i\Big)^2\, \mathbb{1}(Z_N \leq 1)\Big] \,\leq\, \mathbb{E}_{\boldsymbol{x}}\Big[\Big(\sum_{i=1}^N \log x_i\Big)^2\Big]$$
$$\leq\, N^2\, \mathbb{E}_x\big[\log x\big]^2 \,+\, N\, \mathbb{E}_x\big[(\log x)^2\big]\,. \tag{3.32}$$

In conclusion the lemma is proved splitting $\mathbb{E}_{\boldsymbol{x}}\big[(\log Z_N)^2\big]$ as $\mathbb{E}_{\boldsymbol{x}}\big[(\log Z_N)^2\, \mathbb{1}(Z_N \geq e)\big] + \mathbb{E}_{\boldsymbol{x}}\big[(\log Z_N)^2\, \mathbb{1}(1 < Z_N < e)\big] + \mathbb{E}_{\boldsymbol{x}}\big[(\log Z_N)^2\, \mathbb{1}(Z_N \leq 1)\big]$ and applying the bounds (3.30), (3.31), (3.32). □

Proof of the theorem 3.2. It remains to prove only the convergence (3.4). Fix $C < \Phi(\xi^*)$. Fix $0 < m < M_C =: M < \infty$ such that (3.18) holds and $m < \xi^* < M$: it is possible to make such a choice thanks to the bounds (3.7) for ξ^* proven in the remark 3.7. Fix $\lambda_{m,M} =: \lambda > 0$ such that (3.17) holds. Let $\varepsilon > 0$. Then consider the following random events depending on $x_1, \dots, x_N$

$$E^1_{N,\varepsilon} := \{\,\forall\, \xi \in [0, M]\ \ |\Phi_N(\xi) - \Phi(\xi)| < \varepsilon\,\}$$
$$E^2_N := \{\,\forall\, \xi \in [m, M]\ \ \Phi_N(-\xi) < \Phi_N(\xi) - \lambda\,\}$$
$$E^3_N := \{\,\forall\, \xi \in [M, \infty[\ \ \Phi_N(\xi) < C\,,\ \ \Phi_N(\xi) < \varphi(\xi)\,\}$$

and set $E_{N,\varepsilon} := E^1_{N,\varepsilon} \cap E^2_N \cap E^3_N$. It is convenient to split the expectation of $\log Z_N$ as follows:

$$\mathbb{E}_{\boldsymbol{x}}\Big[\frac{1}{N}\log Z_N\Big] \,=\, \mathbb{E}_{\boldsymbol{x}}\Big[\frac{1}{N}\log Z_N\ \mathbb{1}\big(E_{N,\varepsilon}\big)\Big] \,+\, \mathbb{E}_{\boldsymbol{x}}\Big[\frac{1}{N}\log Z_N\ \mathbb{1}\big((E_{N,\varepsilon})^c\big)\Big]\,. \tag{3.33}$$

In the following we are going to see that in the limit $N \to \infty$ the second term on the r.h.s. of (3.33) is negligible, while the first term can be computed using the Laplace's method.

By the lemma 3.8, using the Hölder inequality and the lemma 3.9,

$$\Big|\,\mathbb{E}_{\boldsymbol{x}}\Big[\frac{1}{N}\log Z_N\, \mathbb{1}\big((E_{N,\varepsilon})^c\big)\Big]\,\Big| \,\leq\, \mathbb{E}_{\boldsymbol{x}}\Big[\Big(\frac{1}{N}\log Z_N\Big)^2\Big]^{1/2}\ \mathbb{P}_{\boldsymbol{x}}\big((E_{N,\varepsilon})^c\big)^{1/2} \xrightarrow[N\to\infty]{} 0\,. \tag{3.34}$$

[*Upper bound*] Using the Gaussian representation (3.3), a simple upper bound for Z_N is

$$Z_N \,\leq\, \frac{\sqrt{N}}{\sqrt{2\pi w}} \int_{\mathbb{R}} e^{-\frac{N}{2w}\xi^2} \prod_{i=1}^{N} |\xi + x_i| \, \mathrm{d}\xi \,=\, \frac{\sqrt{N}}{\sqrt{2\pi w}} \int_{\mathbb{R}} e^{N\,\Phi_N(\xi)} \, \mathrm{d}\xi \,. \tag{3.35}$$

If the event $E_{N,\varepsilon}$ holds true, remembering also the inequality (3.15), then the following upper bound holds:

$$\begin{aligned} &\int_{\mathbb{R}} e^{N\,\Phi_N(\xi)} \, \mathrm{d}\xi \,\leq \\ &\leq\, 2\int_0^m e^{N\,\Phi_N(\xi)} \, \mathrm{d}\xi \,+\, \int_m^M e^{N\,\Phi_N(\xi)} \, \mathrm{d}\xi \,+\, \int_m^M e^{N\,(\Phi_N(\xi)-\lambda)} \, \mathrm{d}\xi \,+\, 2\int_M^\infty e^{N\,\Phi_N(\xi)} \, \mathrm{d}\xi \,\leq \\ &\leq\, 2\int_0^m e^{N\,(\Phi(\xi)+\varepsilon)} \, \mathrm{d}\xi \,+\, \int_m^M e^{N\,(\Phi(\xi)+\varepsilon)} \, \mathrm{d}\xi \,+\, \int_m^M e^{N\,(\Phi(\xi)+\varepsilon-\lambda)} \, \mathrm{d}\xi \,+\, 2\,e^{(N-1)\,C} \int_M^\infty e^{\varphi(\xi)} \, \mathrm{d}\xi \,= \\ &\underset{N\to\infty}{=}\, O\big(e^{N\,(\max_{[0,m]}\Phi+\varepsilon)}\big) \,+\, e^{N\,(\Phi(\xi^*)+\varepsilon)} \,\frac{\sqrt{2\pi}\,(1+o(1))}{\sqrt{-N\,\Phi''(\xi^*)}} \,+\, O\big(e^{N\,(\Phi(\xi^*)+\varepsilon-\lambda)}\big) \,+\, O\big(e^{N\,C}\big)\,; \end{aligned} \tag{3.36}$$

the last step is obtained by applying the Laplace's method (theorem 3.11) to the function Φ, which by lemma 3.6 satisfies all the necessary hypothesis. Now since $\max_{[0,m]}\Phi$, $\Phi(\xi^*)-\lambda$ and C are strictly smaller than $\Phi(\xi^*)$, it holds

$$\text{r.h.s. of (3.36)} \underset{N\to\infty}{\sim} e^{N\,(\Phi(\xi^*)+\varepsilon)} \,\frac{\sqrt{2\pi}}{\sqrt{-N\,\Phi''(\xi^*)}} \,. \tag{3.37}$$

As a consequence of (3.35), (3.36), (3.37),

$$\frac{1}{N}\log Z_N \;\mathbb{1}(E_{N,\varepsilon}) \,\leq\, \Phi(\xi^*)+\varepsilon+O\bigg(\frac{\log N}{N}\bigg)\,,$$

where the $O(\frac{\log N}{N})$ is deterministic. Therefore for all $\varepsilon > 0$

$$\limsup_{N\to\infty} \mathbb{E}_{\boldsymbol{x}}\bigg[\frac{1}{N}\log Z_N \;\mathbb{1}(E_{N,\varepsilon})\bigg] \,\leq\, \Phi(\xi^*)+\varepsilon\,. \tag{3.38}$$

[*Lower bound*] Observe that the product $\prod_{i=1}^{N}(\xi+x_i)$ is always positive for $\xi \geq 0$, while it is negative for some $\xi < 0$. Hence using the Gaussian representation

(3.3), a lower bound for Z_N is

$$
\begin{aligned}
Z_N \;\geq\; & \frac{\sqrt{N}}{\sqrt{2\pi w}}\left(\int_0^{\infty} e^{-\frac{N}{2w}\xi^2}\prod_{i=1}^{N}|\xi+x_i|\,\mathrm{d}\xi \;-\; \int_{-\infty}^{0} e^{-\frac{N}{2w}\xi^2}\prod_{i=1}^{N}|\xi+x_i|\,\mathrm{d}\xi\right) \;= \\
= \; & \frac{\sqrt{N}}{\sqrt{2\pi w}}\left(\int_0^{\infty} e^{N\,\Phi_N(\xi)}\,\mathrm{d}\xi \;-\; \int_{-\infty}^{0} e^{N\,\Phi_N(\xi)}\,\mathrm{d}\xi\right) .
\end{aligned}
\tag{3.39}
$$

If the event $E_{N,\varepsilon}$ holds true, remembering also the inequality (3.15), then the following lower bound holds:

$$
\begin{aligned}
& \int_0^{\infty} e^{N\,\Phi_N(\xi)}\,\mathrm{d}\xi \;-\; \int_{-\infty}^{0} e^{N\,\Phi_N(\xi)}\,\mathrm{d}\xi \;\geq \\
\geq\; & \int_m^M e^{N\,\Phi_N(\xi)}\,\mathrm{d}\xi \;-\; \int_m^M e^{N\,(\Phi_N(\xi)-\lambda)}\,\mathrm{d}\xi \;\geq \\
\geq\; & \int_m^M e^{N\,(\Phi(\xi)-\varepsilon)}\,\mathrm{d}\xi \;-\; \int_m^M e^{N\,(\Phi(\xi)+\varepsilon-\lambda)}\,\mathrm{d}\xi \;= \\
\underset{N\to\infty}{=}\; & e^{N\,(\Phi(\xi^*)-\varepsilon)}\,\frac{\sqrt{2\pi}\,(1+o(1))}{\sqrt{-N\,\Phi''(\xi^*)}} \;-\; e^{N\,(\Phi(\xi^*)+\varepsilon-\lambda)}\,\frac{\sqrt{2\pi}\,(1+o(1))}{\sqrt{-N\,\Phi''(\xi^*)}}\;;
\end{aligned}
\tag{3.40}
$$

the last step is obtained by applying the Laplace's method (theorem 3.11) to the function Φ, which by lemma 3.6 satisfies all the necessary hypothesis. Now since $\Phi(\xi^*)+\varepsilon-\lambda<\Phi(\xi^*)-\varepsilon$ for all $0<\varepsilon<\frac{1}{2}\lambda$, for such a choice of ε it holds

$$
\text{r.h.s. of (3.40)} \;\underset{N\to\infty}{\sim}\; e^{N\,(\Phi(\xi^*)-\varepsilon)}\,\frac{\sqrt{2\pi}}{\sqrt{-N\,\Phi''(\xi^*)}}\;. \tag{3.41}
$$

As a consequence of (3.39), (3.40), (3.41), for all $0<\varepsilon<\frac{1}{2}\lambda$

$$
\frac{1}{N}\log Z_N\;\mathbb{1}(E_{N,\varepsilon}) \;\geq\; \left(\Phi(\xi^*)-\varepsilon+O\!\left(\frac{\log N}{N}\right)\right)\mathbb{1}(E_{N,\varepsilon})\;,
$$

where the $O(\frac{\log N}{N})$ is deterministic. Therefore, using also the lemma 3.8, for all $0<\varepsilon<\frac{1}{2}\lambda$

$$
\liminf_{N\to\infty}\mathbb{E}_{\boldsymbol{x}}\!\left[\frac{1}{N}\log Z_N\;\mathbb{1}(E_{N,\varepsilon})\right] \;\geq\; \liminf_{N\to\infty}\left(\Phi(\xi^*)-\varepsilon+O\!\left(\frac{\log N}{N}\right)\right)\mathbb{P}_{\boldsymbol{x}}(E_{N,\varepsilon}) \;=\; \Phi(\xi^*)-\varepsilon\;. \tag{3.42}
$$

In conclusion the convergence $\mathbb{E}_{\boldsymbol{x}}[\frac{1}{N}\log Z_N]\to\Phi(\xi^*)$ as $N\to\infty$ is proven by considering (3.33) for $0<\varepsilon<\frac{1}{2}\lambda$, then letting $N\to\infty$ exploiting (3.34), (3.38), (3.42), and finally letting $\varepsilon\to0+$. □

3.2 Appendix

In this appendix we state the main technical results used in this chapter. We omit their proofs that can be found in the literature.

Theorem 3.10 (Gaussian integration by parts; Wick-Isserlis formula)**.** *Let* $(\xi_1,\ldots,\xi_n)$ *be a Gaussian random vector with mean* 0 *and positive semi-definite covariance matrix* $C=(c_{ij})_{i,j=1,\ldots,n}$ *. Let* $f\colon\mathbb{R}^{n-1}\to\mathbb{R}$ *be a differentiable function such that* $\mathbb{E}\big[\big|\xi_1\,f(\xi_2,\ldots,\xi_n)\big|\big]<\infty$ *and* $\mathbb{E}\big[\big|\frac{\partial f}{\partial\xi_j}(\xi_2,\ldots,\xi_n)\big|\big]<\infty$ *for all* $j=2,\ldots,n$*. Then:*

$$\mathbb{E}\big[\xi_1\,f(\xi_2,\ldots,\xi_n)\big]\;=\;\sum_{j=2}^{n}c_{1j}\,\mathbb{E}\bigg[\frac{\partial f}{\partial\xi_j}(\xi_2,\ldots,\xi_n)\bigg]\,. \tag{3.43}$$

As a consequence one can prove the following:

$$\mathbb{E}\bigg[\prod_{i=1}^{n}\xi_i\bigg]\;=\;\sum_{\substack{P\text{ partition of}\\\{1,\ldots,n\}\text{ into pairs}}}\;\prod_{\{i,j\}\in P}c_{ij}\,. \tag{3.44}$$

The Gaussian integration by parts (3.43) can be found in [93]. The Wick-Isserlis formula (3.44) follows by (3.43) using an induction argument; but it appeared for the first time in [60].

Theorem 3.11 (Laplace's method)**.** *Let* $\phi\colon[a,b]\to\mathbb{R}$ *be a function of class* C^2*. Suppose that there exists* $x_0\in\,]a,b[$ *such that*

i. $\phi(x_0)>\phi(x)$ *for all* $x\in[a,b]$ *(i.e.* x_0 *is the only global maximum point of* ϕ*);*

ii. $\phi''(x_0)<0\,.$

Then as $n\to\infty$

$$\int_a^b e^{n\,\phi(x)}\,\mathrm{d}x\;=\;e^{n\,\phi(x_0)}\,\frac{\sqrt{2\pi}}{\sqrt{-n\,\phi''(x_0)}}\,\big(1+o(1)\big)\,. \tag{3.45}$$

A formal proof of the Laplace's method can be found in [25].

Theorem 3.12 (uniform weak law of large numbers)**.** *Let $\mathcal{X}$, Θ be metric spaces. Let X_i, $i \in \mathbb{N}$ be i.i.d. random variables taking values in $\mathcal{X}$. Let $f : \mathcal{X} \times \Theta \to \mathbb{R}$ be a function such that $f(\cdot, \theta)$ is measurable for all $\theta \in \Theta$. Suppose that:*

i. Θ *is compact;*

ii. $\mathbb{P}\big(f(X_1, \cdot)$ *is continuous at* $\theta\big) = 1$ *for all* $\theta \in \Theta$*;*

iii. $\exists\, F : \mathcal{X} \to [0, \infty]$ *such that* $\mathbb{P}\big(|f(X_1, \theta)| \le F(X_1)\big) = 1$ *for all* $\theta \in \Theta$ *and* $\mathbb{E}[F(X_1)] < \infty$.

Then for all $\varepsilon > 0$

$$\mathbb{P}\left(\sup_{\theta \in \Theta} \left| \frac{1}{n} \sum_{i=1}^{n} f(X_i, \theta) - \mathbb{E}[f(X, \theta)] \right| \ge \varepsilon \right) \xrightarrow[n \to \infty]{} 0\, . \tag{3.46}$$

The uniform law of large number appeared in [61]. It is based on the (standard) law of large numbers and on a compactness argument.

Chapter 4

Hard-core and imitative interactions on the complete graph

This chapter is based on the joint works [6, 7]. We study a monomer-dimer model with **hard-core and imitative interactions** (see section 2.2) on the **complete graph**; we fix uniform dimer activity $w_{ij} \equiv 1/N$, uniform monomer activity $x_i \equiv e^h$ and uniform imitation coefficients $J'_{ij} \equiv J''_{ij} \equiv J/N \geq 0$, $J'''_{ij} \equiv 0$.

We show that this model is exactly solvable and, for J sufficiently large, presents a phase transition between a high monomer density phase and a high dimer density phase. The properties of the phase transition are studied in details. Precisely we prove that in the thermodynamic limit the pressure density p exists and is given by a one-dimensional variational principle in the monomer density m (theorem 4.1), which admits two solutions when (h, J) belongs to a curve Γ or one solution otherwise (propositions 4.5, 4.9). The order parameter $m(h, J)$ presents a jump discontinuity along Γ, while it is continuous but not differentiable at the critical point (h_c, J_c), its critical exponents are the mean-field ones: $\boldsymbol{\beta} = 1/2$ along the direction of Γ and $1/\boldsymbol{\delta} = 1/3$ along any other direction (theorem 4.14).

To *decouple* the imitative interaction we adopt the strategy used by Guerra for the Curie-Weiss model [53], while the exact solution of the pure hard-core model is obtained by a combinatorial argument.

4.1 Solution of the model

Let $h \in \mathbb{R}$ and $J > 0$. Since the number of edges is of order N^2, in order to keep the logarithm of the partition function of order N, a normalisation of the dimer activity as $1/N$ and of the imitation coefficient as J/N are needed. Therefore in this chapter we will consider the Hamiltonian

$$H_N(\alpha) := -h \sum_{i=1}^{N} \alpha_i - \frac{J}{N} \sum_{1 \le i < j \le N} \big(\alpha_i\, \alpha_j + (1-\alpha_i)\,(1-\alpha_j)\big) \tag{4.1}$$

for every monomer-dimer configuration on the complete graph $\alpha \in \mathscr{D}_N$, and the partition function

$$Z_N := \sum_{\alpha \in \mathscr{D}_N} N^{-|D|} \exp(-H_N(\alpha)) \,, \tag{4.2}$$

where $|D| = \sum_{1 \le i < j \le N} \alpha_{ij} = \big(N - \sum_{i=1}^{N} \alpha_i\big)/2\,.$ The corresponding Gibbs measure is

$$\mu_N(\alpha) := \frac{N^{-|D|} \exp(-H_N(\alpha))}{Z_N} \quad \forall\, \alpha \in \mathscr{D}_N \tag{4.3}$$

and the expectation with respect to the measure μ_N is denoted by $\langle\,\cdot\,\rangle_N$. In particular, setting $m_N(\alpha) := \frac{1}{N} \sum_{i=1}^{N} \alpha_i$, the monomer density is

$$\langle m_N \rangle_N = \sum_{\alpha \in \mathscr{D}_N} \frac{\sum_{i=1}^{N} \alpha_i}{N}\, \frac{\exp(-H_N(\alpha))}{Z_N} = \frac{\partial}{\partial h} \frac{\log Z_N}{N} \,. \tag{4.4}$$

The main result of this section is the following theorem, where in the limit $N \to \infty$ the model is solved in terms of a one-dimensional variational principle.

Theorem 4.1. *Let $h \in \mathbb{R}$, $J \ge 0$. Then*

$$\exists\, p := \lim_{N \to \infty} \frac{\log Z_N}{N} = \sup_m \widetilde{p}(m) \in \mathbb{R} \,, \tag{4.5}$$

where the sup *can be taken indifferently over* $m \in [0,1]$ *or* $m \in \mathbb{R}$, *and*

$$\widetilde{p}\,(m) \;:=\; -J\,m^2 \,+\, \frac{1}{2}J \,+\, p^{(0)}\big((2m-1)\,J + h\big) \quad \forall\, m \in \mathbb{R}\;, \tag{4.6}$$

$$p^{(0)}(t) \;:=\; -\,\frac{1-g(t)}{2} \,-\, \frac{1}{2}\log(1-g(t)) \;=\; -\frac{1-g(t)}{2} - \log g(t) + t\;, \tag{4.7}$$

$$g(t) \;:=\; \frac{1}{2}\,(\sqrt{e^{4t} + 4\,e^{2t}} - e^{2t}) \;=\; \big(p^{(0)}\big)'(t) \quad \forall\, t \in \mathbb{R}\;. \tag{4.8}$$

Furthermore the function $m \mapsto \widetilde{p}\,(m)$ *attains its maximum in (at least) one point* $m^* \in]0,1[$, *which is a solution of the the consistency equation*

$$m \;=\; g\big((2m-1)\,J + h\big)\;. \tag{4.9}$$

At each value of the parameters (h, J) *such that* $h \mapsto m^*(h, J)$ *is differentiable, the monomer density admits thermodynamic limit and precisely:*

$$\exists \lim_{N\to\infty} \langle m_N \rangle_N \;=\; m^* \;\in]0,1[\;. \tag{4.10}$$

This result relies on two main facts:

1) for $J = 0$ the thermodynamic limit of the pressure per particle can be computed explicitly and turns out to be $p^{(0)}(h)$;

2) for $J > 0$ the Hamiltonian (4.1) can be expressed as a quadratic form in the hamiltonian with $J = 0$.

Therefore before proving the theorem we state and prove these two results.

Denote by $Z_N^{(0)}$ and $\langle m_N \rangle_N^{(0)}$ respectively the partition function and the monomer density at $J = 0$, namely when the system presents only the hard-core interaction.

Proposition 4.2. *Let* $h \in \mathbb{R}$. *Then*

$$\exists \lim_{N\to\infty} \frac{\log Z_N^{(0)}}{N} \;=\; p^{(0)}(h)\;, \tag{4.11}$$

and

$$\exists \lim_{N\to\infty} \langle m_N \rangle_N^{(0)} \;=\; g(h)\;. \tag{4.12}$$

The analytic functions $p^{(0)}$ *and* g *are defined respectively by (4.7) and (4.8).*

Proof. This proposition has already been proven in the remark 3.5, by using the Gaussian representation of the partition function and the Laplace method. Another way to prove it is to write an explicit combinatorial expression for the pure hard-core partition function on the complete graph (see [6]). □

On the complete graph the Hamiltonian (4.1) admits a useful rewriting, which shows that it depends on the monomer-dimer configuration α only via the fraction of monomers $m_N(\alpha)$.

Lemma 4.3. *For all* $\alpha \in \mathscr{D}_N$,

$$H_N(\alpha) \,=\, -N\left(J\, m_N(\alpha)^2 \,+\, (h-J)\, m_N(\alpha) \,+\, c_N\right) \tag{4.13}$$

with $c_N := \frac{N-1}{2\,N}\, J$.

Proof. Using the identities analysed in the remark 2.11, the Hamiltonian (4.1) rewrites as

$$\begin{aligned} H_N \,=\, & -\frac{N(N-1)}{2}\,\frac{J}{N} \,+ \\ & -\left(h-(N-1)\,\frac{J}{N}\right)\sum_{i=1}^N \alpha_i \,-\, 2\,\frac{J}{N}\sum_{1\le i<j\le N} \alpha_i\alpha_j \;. \end{aligned}$$

Then on the complete graph it holds

$$2\sum_{1\le i<j\le N} \alpha_i\alpha_j \,=\, \left(\sum_{i=1}^N \alpha_i\right)^2 - \sum_{i=1}^N \alpha_i \;. \tag{4.14}$$

Substituting in the previous expression one obtains

$$H_N = -\frac{N-1}{2}\,J \,-\, (h-J)\sum_{i=1}^N \alpha_i \,-\, \frac{J}{N}\left(\sum_{i=1}^N \alpha_i\right)^2$$

and since $\sum_{i=1}^N \alpha_i = N\, m_N(\alpha)$ the identity (4.13) is proved. □

Now using proposition 4.2 and lemma 4.3 we are able to prove theorem 4.1. Our technique is the same used by Guerra [53] to solve the Curie-Weiss model (namely the ferromagnetic Ising model on the complete graph).

Proof of Theorem 4.1. The proof is done providing a lower and an upper bound for the pressure per particle.

[**LowerBound**] Fix $m \in \mathbb{R}$. As $\big(m_N(\alpha) - m\big)^2 \geq 0$, clearly $m_N(\alpha)^2 \geq 2\,m\,m_N(\alpha) - m^2$. Hence by lemma 4.3, using the hypothesis $J \geq 0$,

$$\begin{aligned} -H_N(\alpha) \;&=\; N\,\big(J\,m_N(\alpha)^2 \,+\, (h-J)\,m_N(\alpha) \,+\, c_N\big) \;\geq \\ &\geq\; N\,\big((2J\,m + h - J)\,m_N(\alpha) \,-\, J\,m^2 + c_N\big) \end{aligned}$$

thus

$$\begin{aligned} Z_N \;&=\; \sum_\alpha N^{-|D|}\exp(-H_N(\alpha)) \;\geq\; \sum_\alpha N^{-|D|}\exp N\big((2J\,m+h-J)\,m_N(\alpha) - J\,m^2 + c_N\big) \;= \\ &=\; e^{N\,\gamma_N(m)}\;Z_N^{(0)}\big(t(m)\big) \end{aligned}$$

where $\gamma_N(m) := -J\,m^2 + \frac{N-1}{2N}\,J$ and $t(m) := 2J\,m + h - J\,.$

[**UpperBound**] m_N takes values in the set $\mathcal{A}_N := \{0, \frac{1}{N}, \dots, \frac{N-1}{N}, 1\}$. Clearly, writing δ for the Kronecker delta, $\sum_{m\in\mathcal{A}_N}\delta_{m,m_N(\alpha)} = 1$ and $F(m_N(\alpha)^2)\,\delta_{m,m_N(\alpha)} = F(2\,m\,m_N(\alpha) - m^2)\,\delta_{m,m_N(\alpha)}$ for any function F. Hence by lemma 4.3,

$$\begin{aligned} \delta_{m,m_N(\alpha)}\,\exp(-H_N(\alpha)) \;&=\; \delta_{m,m_N(\alpha)}\,\exp N(J\,m_N(\alpha)^2 + (h-J)\,m_N(\alpha) + c_N) \;= \\ &=\; \delta_{m,m_N(\alpha)}\,\exp N\big((2J\,m + h - J)\,m_N(\alpha) \,-\, J\,m^2 + c_N\big) \end{aligned}$$

thus

$$\begin{aligned} Z_N \;&=\; \sum_\alpha N^{-|D|}\sum_{m\in\mathcal{A}_N}\delta_{m,m_N(\alpha)}\;\exp(-H_N(\alpha)) \;= \\ &=\; \sum_\alpha N^{-|D|}\sum_{m\in\mathcal{A}_N}\delta_{m,m_N(\alpha)}\;\exp N\big((2J\,m+h-J)\,m_N(\alpha) \,-\, J\,m^2 + c_N\big) \;\leq \\ &\leq\; \sum_{m\in\mathcal{A}_N}\sum_\alpha N^{-|D|}\;\exp N\big((2J\,m+h-J)\,m_N(\alpha) \,-\, J\,m^2 + c_N\big) \;= \\ &=\; \sum_{m\in\mathcal{A}_N} e^{N\,\gamma_N(m)}\;Z_N^{(0)}\big(t(m)\big) \;\leq\; (N+1)\sup_{m\in[0,1]}\big\{e^{N\,\gamma_N(m)}\;Z_N^{(0)}\big(t(m)\big)\big\}\,. \end{aligned}$$

Therefore putting together lower and upper bound we have found:

$$\sup_{m\in[0,1]}\big\{e^{N\,\gamma_N(m)}\;Z_N^{(0)}\big(t(m)\big)\big\} \;\leq\; Z_N \;\leq\; (N+1)\sup_{m\in[0,1]}\Big\{e^{N\,\gamma_N(m)}\;Z_N^{(0)}\big(t(m)\big)\Big\}\,.$$

Then, taking the logarithm and dividing by N,

$$0 \le \frac{\log Z_N}{N} - \sup_{m\in[0,1]} \left\{\gamma_N(m) + \frac{\log Z_N^{(0)}\big(t(m)\big)}{N}\right\} \le \frac{\log(N+1)}{N} \xrightarrow[N\to\infty]{} 0 \, .$$

Now the pressure per particle $h \mapsto \frac{\log Z_N^{(0)}(h)}{N}$ is a convex function, hence as $N \to \infty$ the convergence $\frac{\log Z_N^{(0)}(h)}{N} \to p^{(0)}(h)$ of proposition 4.2 is uniform in h on compact sets. Moreover $\gamma_N(m) \to \gamma(m) := -J\,m^2 + \frac{1}{2}\,J$ uniformly in m as $N \to \infty$. Therefore

$$\gamma_N(m) + \frac{\log Z_N^{(0)}\big(t(m)\big)}{N} \xrightarrow[N\to\infty]{} \gamma(m) + p^{(0)}\big(t(m)\big)$$

and the convergence is uniform in m on compact sets. As a consequence also

$$\sup_{m\in[0,1]} \left\{\gamma_N(m) + \frac{\log Z_N^{(0)}\big(t(m)\big)}{N}\right\} \xrightarrow[N\to\infty]{} \sup_{m\in[0,1]} \left\{\gamma(m) + p^{(0)}\big(t(m)\big)\right\} .$$

This concludes the proof of (4.5).

It remains to prove (4.9) and (4.10). First of all observe that

$$\frac{\partial\widetilde{p}}{\partial m}(m) = -2J\,m + 2J\,g\big((2m-1)\,J + h\big)\ ,$$

since $(p^{(0)})' = g$ (see proposition 4.2). It holds $\frac{\partial\widetilde{p}}{\partial m}(m) > 0$ for all $m \le 0$ and $\frac{\partial\widetilde{p}}{\partial m}(m) > 0$ for all $m \ge 1$, therefore the function $m \mapsto \widetilde{p}\,(m)$ attains its global maximum inside the interval $]0,1[$ and any global maximum point m^* is a critical point of $\widetilde{p}$, i.e. satisfies equation (4.9).

Now $\frac{1}{N}\log Z_N(h,J)$ is a convex function of h and, as shown before, it converges to $p(h,J) = \widetilde{p}\,(m^*(h,J),h,J)$ as $N \to \infty$. Therefore, assuming that $m^*(h,J)$ is differentiable in h, the monomer density $m_N = \frac{\partial}{\partial h}\frac{1}{N}\log Z_N$ converges to $\frac{\partial}{\partial h}\,p$. Thus to prove (4.10) it suffices to compute this derivative:

$$\frac{\partial p}{\partial h} = \frac{\mathrm{d}}{\mathrm{d}h}\,\widetilde{p}\,\big(m^*(h,J),h,J\big) = \underbrace{\frac{\partial\widetilde{p}}{\partial m}(m^*)}_{=\,0}\,\frac{\partial m^*}{\partial h} + \underbrace{\frac{\partial\widetilde{p}}{\partial h}}_{=\,(p^{(0)})'} = g\big((2m^*{-}1)\,J{+}h\big) = m^* \, .$$

□

4.2 Study of the phase transition

In this section we study the properties of the solution provided by theorem 4.1. We divide the analysis in three subsections. In subsection 4.2.1 we study all the *stationary points* of the function $m \mapsto \widetilde{p}(m,h,J)$. One of them will be the *global maximum point* m^* we are interested in, since it represents the monomer density. We provide their complete classification, regularity properties and asymptotic behaviour as functions of the parameters h and J. As a consequence in subsection 4.2.2 we are able to identify the region where there exists a unique global maximum point m^*. The function m^* is single-valued and smooth on the plane (h,J) with the exception of a implicitly defined curve Γ union its endpoint (h_c,J_c); along Γ the order parameter m^* presents a jump discontinuity: this fact has a crucial physical role since it represents the coexistence of two different thermodynamic phases and in physical jargon we say that a *phase transition* occur. The point (h_c,J_c) is the *critical point* of the system, where m^* is continuous but not differentiable. In subsection 4.2.3 we compute the *critical exponents* that characterizes the behaviour of m^* near (h_c,J_c).

4.2.1 Analysis of the stationary points

Let us identify the stationary points of the function $\widetilde{p}(m,h,J)$ defined by (4.6). Remembering that $(p^{(0)})' = g$, one computes

$$\frac{\partial \widetilde{p}}{\partial m}(m,h,J) \;=\; -2J\,m \;+\; 2J\;g\big((2m-1)J+h\big) \tag{4.15}$$

$$\frac{\partial^2 \widetilde{p}}{\partial m^2}(m,h,J) \;=\; -2J \;+\; (2J)^2\;g'\big((2m-1)J+h\big) \tag{4.16}$$

Since $0<g<1$, it follows that for every $J>0$, $h\in\mathbb{R}$

$$\frac{\partial \widetilde{p}}{\partial m}(m,h,J) > 0 \quad \forall\, m \in \,]-\infty,0]\;, \qquad \frac{\partial \widetilde{p}}{\partial m}(m,h,J) < 0 \quad \forall\, m \in [1,\infty[\;. \tag{4.17}$$

Therefore $\widetilde{p}\,(\cdot\,, h, J)$ attains its maximum in (at least) one point $m = m^*(h, J) \in\,]0, 1[$, which satisfies

$$\frac{\partial \widetilde{p}}{\partial m}\,(m, h, J) = 0 \qquad \text{i.e.} \quad m = g\big((2m-1)J + h\big)\ , \tag{4.18}$$

$$\frac{\partial^2 \widetilde{p}}{\partial m^2}\,(m, h, J) \leq 0 \qquad \text{i.e.} \quad g'\big((2m-1)J + h\big) \leq \frac{1}{2J}\ . \tag{4.19}$$

The stationary points are characterized by equation (4.18), which can not be explicitly solved. Anyway their properties and a rough approximation of their values can be determined by studying inequality (4.19), which admits explicit solution.

The next proposition displays the intervals of concavity/convexity of the function $m \mapsto \widetilde{p}\,(m, h, J)$. Set

$$J_c := \frac{1}{4\,(3 - 2\sqrt{2})}\ \approx\ 1.4571\ . \tag{4.20}$$

Proposition 4.4. *For $0 < J < J_c$ and $h \in \mathbb{R}$*

$$\frac{\partial^2 \widetilde{p}}{\partial m^2}\,(m, h, J)\ <\ 0 \quad \forall\, m \in \mathbb{R}\ .$$

For $J \geq J_c$ and $h \in \mathbb{R}$

$$\frac{\partial^2 \widetilde{p}}{\partial m^2}\,(m, h, J) \begin{cases} < 0 & \textit{iff } m < \phi_1(h, J) \textit{ or } m > \phi_2(h, J) \\ > 0 & \textit{iff } \phi_1(h, J) < m < \phi_2(h, J) \end{cases} ,$$

where for $i = 1, 2$

$$\phi_i(h, J)\ :=\ \frac{1}{2} - \frac{h}{2J} + \frac{1}{4J}\ \log a_i(J)\ , \tag{4.21}$$

$$a_i(J)\ :=\ \frac{-\big(\frac{1}{(2J)^2} + \frac{8}{2J} - 4\big)\ + (-1)^i\,\big(2 - \frac{1}{2J}\big)\sqrt{\frac{1}{(2J)^2} - \frac{12}{2J} + 4}}{\frac{4}{2J}}\ . \tag{4.22}$$

Observe that $\phi_1(h, J) \leq \phi_2(h, J)$ for all $h \in \mathbb{R}$, $J \geq J_c$ and equality holds iff $J = J_c$ (since $a_1(J_c) = a_2(J_c)$).

Proof. It follows from the expression (4.16) through a direct computation done in lemma 4.17 of the Appendix, taking $t = (2m-1)J + h$ and $c = \frac{1}{2J}$. □

Using the previous proposition we can determine *how many*, of *what kind* and *where* the stationary points of $\widetilde{p}\,(\cdot\,, h, J)$ are.

Proposition 4.5 (Stationary points: classification)**.** *The equation (4.18) in m has the following properties:*

1. *If $0 < J \leq J_c$ and $h \in \mathbb{R}$, there exists only one solution $m(h, J)$. It is the maximum point of $\widetilde{p}\,(\cdot\,, h, J)$.*

2. *If $J > J_c$ and $\psi_2(J) < h < \psi_1(J)$, then there exist three solutions $m_1(h, J), m_0(h, J), m_2(h, J)$. Moreover $m_1(h, J) < \phi_1(h, J)$ and $m_2(h, J) > \phi_2(h, J)$ are two local maximum points, while $\phi_1(h, J) < m_0(h, J) < \phi_2(h, J)$ is a local minimum point of $\widetilde{p}\,(\cdot\,, h, J)$.*

3. *If $J > J_c$ and $h > \psi_1(J)$, there exists only one solution $m_2(h, J)$. Moreover $m_2(h, J) > \phi_2(h, J)$ and it is the maximum point of $\widetilde{p}\,(\cdot\,, h, J)$.*

4. *If $J > J_c$ and $h = \psi_1(J)$, there exist two solution $m_1(h, J)$, $m_2(h, J)$. Moreover $m_1(h, J) = \phi_1(h, J)$ is a point of inflection, while $m_2(h, J) > \phi_2(h, J)$ is the maximum point of $\widetilde{p}\,(\cdot\,, h, J)$.*

5. *If $J > J_c$ and $h < \psi_2(J)$, there exists only one solution $m_1(h, J)$. Moreover $m_1(h, J) < \phi_1(h, J)$ and it is the maximum point of $\widetilde{p}\,(\cdot\,, h, J)$.*

6. *If $J > J_c$ and $h = \psi_2(J)$, there exist two solutions $m_1(h, J)$, $m_2(h, J)$. Moreover $m_2(h, J) = \phi_2(h, J)$ is a point of inflection, while $m_1(h, J) < \phi_1(h, J)$ is the maximum point of $\widetilde{p}\,(\cdot\,, h, J)$.*

Here ϕ_1, ϕ_2 are defined by (4.21), while for $i = 1, 2$ and $J \geq J_c$

$$\psi_i(J) \;:=\; J + \frac{1}{2}\log a_i(J) - 2J\, g\big(\frac{1}{2}\log a_i(J)\big)\,, \tag{4.23}$$

where a_i and g are defined respectively by (4.22) and (4.8). Observe that $\psi_2(J) \leq \psi_1(J)$ for all $J \geq J_c$ and equality holds iff $J = J_c$.

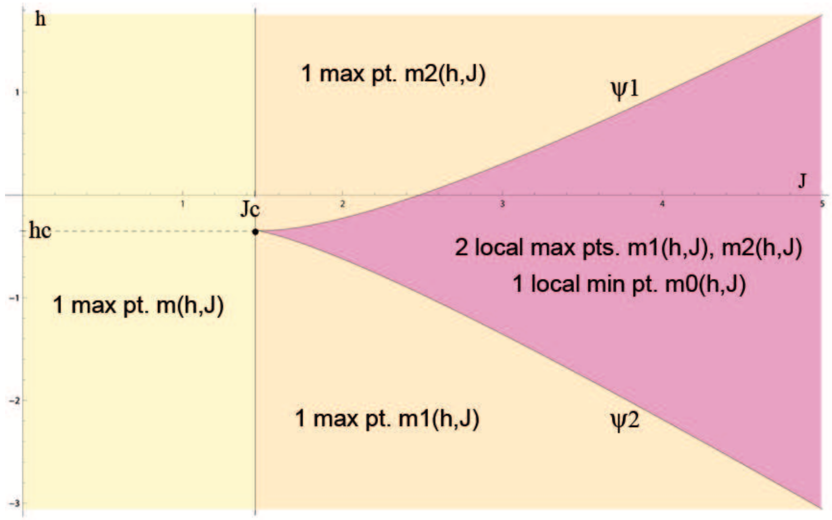

Figure 4.1: Number and nature of the stationary points of the function $m \mapsto \widetilde{p}(m,h,J)$ in the regions of the plane (h,J).

Proof. Fix $h \in \mathbb{R}$, $J > 0$ and to shorten the notation set $G(m) := \frac{\partial \widetilde{p}}{\partial m}(m,h,J)$, observing it is a continuous (smooth) function.

• Suppose $J \leq J_c$. By proposition 4.4, $G'(m) \leq 0$ for all $m \in \mathbb{R}$ and equality holds iff ($J = J_c$ and $m = \phi_1(h,J_c) = \phi_2(h,J_c)$). Hence G is strictly decreasing on $\mathbb{R}$. On the other hand by (4.17), $G(m) < 0$ for all $m \leq 0$ and $G(m) > 0$ for all $m \geq 1$. Therefore there exists a unique point m ($m \in]0,1[$) such that $G(m) = 0$.

• Suppose $J > J_c$. By proposition 4.4, G is strictly decreasing for $m \leq \phi_1(h,J)$, strictly increasing for $\phi_1(h,J) \leq m \leq \phi_2(h,J)$ and again strictly decreasing for $m \geq \phi_2(h,J)$. On the other hand by (4.17), $G(m_+) > 0$ for some point $m_+ < \phi_1(h,J)$ and $G(m_-) > 0$ for some point $m_- > \phi_2(h,J)$. Therefore:

$$(\,\exists\,(\text{a unique})\; m_1 \in]-\infty, \phi_1(h,J)] \text{ s.t. } G(m_1) = 0\,) \;\Leftrightarrow\; G(\phi_1(h,J)) \leq 0\,;$$

$$(\,\exists\,(\text{a unique})\; m_2 \in [\phi_2(h,J), \infty[\text{ s.t. } G(m_2) = 0\,) \;\Leftrightarrow\; G(\phi_2(h,J)) \geq 0\,;$$

$$(\,\exists\,(\text{a unique})\; m_0 \in [\phi_1(h,J), \phi_2(h,J)] \text{ s.t. } G(m_0) = 0\,) \;\Leftrightarrow\; G(\phi_1(h,J)) \leq 0\,,\; G(\phi_2(h,J)) \geq 0\,.$$

And now, using identity (4.15) and definitions (4.21), (4.23)

$$G(\phi_1(h,J)) \underset{(=)}{<} 0 \;\Leftrightarrow\; g\big((2\phi_1(h,J)-1)J+h\big) \underset{(=)}{<} \phi_1(h,J) \;\Leftrightarrow\; h \underset{(=)}{<} \psi_1(J)$$

and similarly $G(\phi_2(h,J)) \underset{(=)}{>} 0 \;\Leftrightarrow\; h \underset{(=)}{>} \psi_2(J)\,.$

The first • allows to conclude in case *1.*, while the second • allows to conclude in all the other cases. Notice that the nature of the stationary points of $\widetilde{p}\,(\cdot\,,h,J)$ is determined by the sign of the second derivative $\frac{\partial^2 \widetilde{p}}{\partial m^2}$ studied in proposition 4.4. □

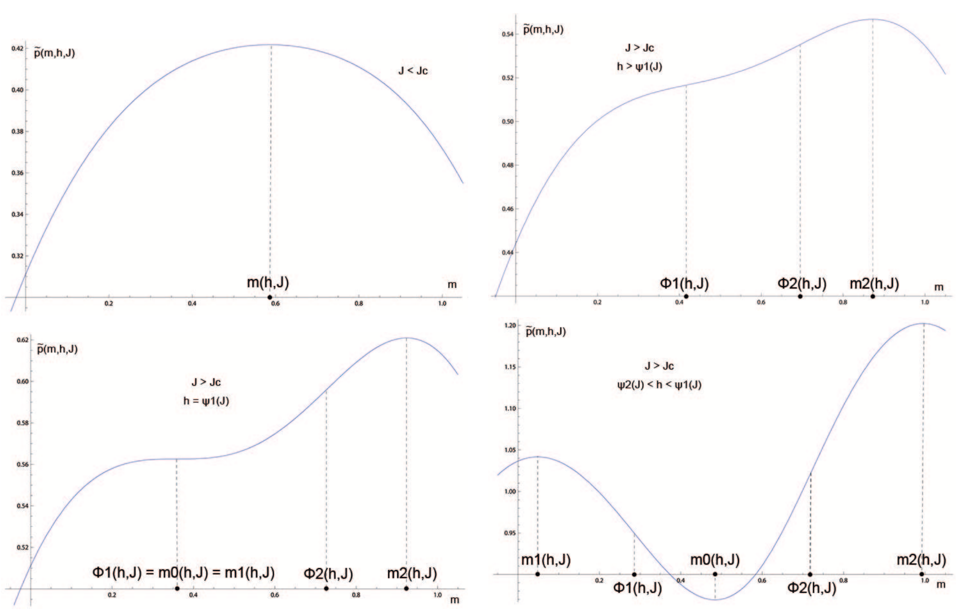

Figure 4.2: Plots of the function $m \mapsto \widetilde{p}\,(m,h,J)$ for different values of the parameters h, J. In particular cases *1.*, *3.*, *4.*, *2.* of proposition 4.5 are represented.

A special role is played by the point (h_c, J_c), where we set

$$h_c := \psi_1(J_c) = \psi_2(J_c) \;=\; \frac{1}{2}\,\log(2\sqrt{2}-2) - \frac{1}{4} \;\approx\; -0.3441\;, \qquad (4.24)$$

indeed in the next sub-sections it will turn out to be the *critical point* of the system. It is also useful to define

$$m_c := \phi_1(h_c, J_c) = \phi_2(h_c, J_c) \;=\; 2 - \sqrt{2} \;\approx\; 0.5857\;, \tag{4.25}$$

$$t_c := (2m_c - 1)J_c + h_c \;=\; \frac{1}{2}\,\log(2\sqrt{2} - 2) \;\approx\; -0.0941\;. \tag{4.26}$$

The computations are done observing that $a_1(J_c) \;=\; a_2(J_c) \;=\; 2\sqrt{2} - 2$ and $g(\frac{1}{2}\log(2\sqrt{2} - 2)) = 2 - \sqrt{2}$.

Remark 4.6. We notice that m_c is the (unique) solution of equation (4.18) for $h = h_c$ and $J = J_c$, that is $m(h_c, J_c) = m_c$. Indeed a direct computation using (4.8) shows

$$g\big((2m_c - 1)J_c + h_c\big) \;=\; g\big(t_c\big) \;= m_c.$$

Observe that as a consequence m_c is a solution of equation (4.18) for all (h, J) such that $h - h_c = (1 - 2m_c)(J - J_c)$.

In the next proposition we analyse the regularity of the solutions of equation (4.18).

Proposition 4.7 (Stationary points: regularity properties)**.** *Consider the stationary points of* $\widetilde{p}\,(\cdot\,, h, J)$ *defined in proposition 4.5:* $m(h, J)$, $m_1(h, J)$, $m_0(h, J)$, $m_2(h, J)$ *for suitable values of* h, J*. The functions*

$$\mu_1(h, J) := \begin{cases} m(h, J) & \textit{if}\;\; 0 < J \le J_c\,,\;\; h \in \mathbb{R} \\ m_1(h, J) & \textit{if}\;\; J > J_c\,,\;\; h \le \psi_1(J) \end{cases}, \tag{4.27}$$

$$\mu_2(h, J) := \begin{cases} m(h, J) & \textit{if}\;\; 0 < J \le J_c\,,\;\; h \in \mathbb{R} \\ m_2(h, J) & \textit{if}\;\; J > J_c\,,\;\; h \ge \psi_2(J) \end{cases}, \tag{4.28}$$

$$\mu_0(h, J) := \begin{cases} m(h, J) & \textit{if}\;\; 0 < J \le J_c\,,\;\; h \in \mathbb{R} \\ m_0(h, J) & \textit{if}\;\; J > J_c\,,\;\; \psi_2(J) \le h \le \psi_1(J) \end{cases} \tag{4.29}$$

have the following properties:

i) are continuous on the respective domains;

ii) are C^∞ in the interior of the respective domains;

iii) for $i = 0, 1, 2$ and (h, J) in the interior of the domain of μ_i

$$\frac{\partial}{\partial h}\widetilde{p}\,(\mu_i(h,J),h,J) \;=\; \mu_i\;, \quad \frac{\partial}{\partial J}\widetilde{p}\,(\mu_i(h,J),h,J) \;=-\,\mu_i\,(1-\mu_i)\;; \tag{4.30}$$

$$\frac{\partial \mu_i}{\partial h} \;=\; \frac{2\,\mu_i\,(1-\mu_i)}{2-\mu_i-4J\,\mu_i\,(1-\mu_i)}\;, \qquad \frac{\partial \mu_i}{\partial J} \;=\; (2\mu_i-1)\,\frac{\partial \mu_i}{\partial h}\;. \tag{4.31}$$

Proof. i) First prove the continuity of μ_1. Observe that by propositions 4.5, 4.4:

- for (h, J) in $D_1 := \{(h,J)\,|\,(0 < J \leq J_c,\ h \in \mathbb{R})$ or $(J > J_c,\ h \leq \psi_2(J))\}$, $\mu_1(h, J)$ is the *only* maximum point of $\widetilde{p}\,(\cdot\,,h,J)$ on the interval $[0,1]$;

- for (h, J) in $D_2 := \{(h,J)\,|\,J \geq J_c,\ h \leq \psi_1(J)\}$, $\mu_1(h,J)$ is the *only* maximum point of $\widetilde{p}\,(\cdot\,,h,J)$ on the interval $[0,\phi_1(h,J)]$.

Hence by the Berge's maximum theorem [78], continuity of the functions $\widetilde{p}$ and ϕ_1 implies continuity of the function μ_1 on the sets D_1 and D_2. As D_1 and D_2 are both closed subsets of $\mathbb{R}\times\mathbb{R}_+$, by the pasting lemma μ_1 is continuous on their union

$$D_1 \cup D_2 \;=\; \{(h,J)\;|\;(0 < J \leq J_c,\ h \in \mathbb{R}) \text{ or } (J > J_c,\ h \leq \psi_1(J))\}\;.$$

A similar argument proves the continuity of μ_2 and μ_0.

ii) Now prove the smoothness of μ_1, μ_2, μ_0 in the *interior* of their domains. Set $G(m,h,J) := \frac{\partial \widetilde{p}}{\partial m}(m,h,J)$. As just seen $m = \mu_1(h,J)$, $\mu_2(h,J)$, $\mu_0(h,J)$ are *continuous* solutions of

$$G(m,h,J) = 0\;,$$

for values of h, J in the respective domains. Observe that $G \in C^\infty(\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+)$

and by propositions 4.4, 4.5 it can happen

$$\begin{cases} \dfrac{\partial G}{\partial m}(m,h,J)=0 \\ G(m,h,J)=0 \end{cases} \Leftrightarrow \begin{cases} J \geq J_c\,,\ (m=\phi_1(h,J) \text{ or } m=\phi_2(h,J)) \\ G(m,h,J)=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} J \geq J_c,\, m=\phi_1(h,J) \\ h=\psi_1(J) \end{cases} \text{or} \quad \begin{cases} J \geq J_c,\, m=\phi_2(h,J) \\ h=\psi_2(J) \end{cases} .$$

$m=\mu_1(h,J)$ can fall only within the first case, while $m=\mu_2(h,J)$ can fall only within the second case. Therefore by the implicit function theorem [86] μ_1, μ_2, μ_0 are C^∞ on the interior of the respective domains.

iii) Let $i=0,1,2$ and (h,J) in the interior of the domain of μ_i. Using (4.6), $(p^{(0)})'=g$ and the fact that $\mu_i(h,J)$ satisfies equation (4.18), compute

$$\begin{aligned} \frac{\partial}{\partial h}\widetilde{p}\,(\mu_i,h,J) \;&= -2J\,\frac{\partial \mu_i}{\partial h} + (p^{(0)})'\big((2\mu_i-1)J+h\big)\,(2J\,\frac{\partial \mu_i}{\partial h}+1) \\ &= -2J\,\frac{\partial \mu_i}{\partial h} + \mu_i\,(2J\,\frac{\partial \mu_i}{\partial h}+1) \;=\; \mu_i\,; \end{aligned}$$

and similarly $\frac{\partial}{\partial J}\widetilde{p}\,(\mu_i,h,J)=\mu_i^2-\mu_i\,.$

Using the fact that $\mu_i(h,J)$ satisfies equation (4.18) compute

$$\begin{aligned} \frac{\partial \mu_i}{\partial h} \;&=\; \frac{\partial}{\partial h}\,g\big((2\mu_i-1)J+h\big) \;=\; g'\big((2\mu_i-1)J+h)\big)\,(1+2J\,\frac{\partial \mu_i}{\partial h}) \\ \Rightarrow\ \frac{\partial \mu_i}{\partial h} \;&=\; \frac{g'\big((2\mu_i-1)J+h\big)}{1-2J\,g'\big((2\mu_i-1)J+h\big)}\;; \end{aligned}$$

and similarly $\frac{\partial \mu_i}{\partial J}=\frac{(2\mu_i-1)\,g'\big((2\mu_i-1)J+h\big)}{1-2J\,g'\big((2\mu_i-1)J+h\big)}$. Then observe that $g'=2\,g\,(1-g)/(2-g)$ (identity (4.49) in the Appendix), hence since $\mu_i(h,J)$ satisfies equation (4.18)

$$g'\big((2\mu_i-1)J+h\big) \;=\; \frac{2\,\mu_i\,(1-\mu_i)}{2-\mu_i}\;;$$

substituting this in the previous identities concludes the proof. □

To end this subsection we study the asymptotic behaviour of the stationary points of $\widetilde{p}\,(\cdot\,,h,J)$ for large J.

Proposition 4.8 (Stationary points: asymptotic behaviour)**.** *Consider the stationary points $m_1(h,J)$, $m_0(h,J)$, $m_2(h,J)$ defined in proposition 4.5 for suitable values of h, J.*

i) For all fixed $h \in \mathbb{R}$

$$m_1(h,J) \xrightarrow[J\to\infty]{} 0\,, \quad m_2(h,J) \xrightarrow[J\to\infty]{} 1\,, \quad m_0(h,J) \xrightarrow[J\to\infty]{} \frac{1}{2}\,.$$

ii) Moreover for all fixed $h \in \mathbb{R}$

$$J\, m_1(h,J) \xrightarrow[J\to\infty]{} 0\,, \quad J\,(1-m_2(h,J)) \xrightarrow[J\to\infty]{} 0\,.$$

iii) And taking the sup *and* inf *over $h \in [\psi_2(J), \psi_1(J)]$*

$$\sup_h m_1(h,J) \xrightarrow[J\to\infty]{} 0\,, \quad \inf_h m_2(h,J) \xrightarrow[J\to\infty]{} 1\,.$$

Proof. i) First observe from the definition (4.23) that $\psi_2(J) \to -\infty$, $\psi_1(J) \to \infty$ as $J \to \infty$. Hence for any fixed $h \in \mathbb{R}$ there exists $\bar{J} > 0$ such that $\psi_2(J) < h < \psi_1(J)$ for all $J > \bar{J}$. This means that the limits in the statement make sense.

Now remind that by proposition 4.5, for $J > \bar{J}$

$$m_1(h,J) < \phi_1(h,J) < m_0(h,J) < \phi_2(h,J) < m_2(h,J)\,.$$

Observe from the definition (4.21) that $\phi_1(h,J) \to \frac{1}{2}$, $\phi_2(h,J) \to \frac{1}{2}$ as $J \to \infty$. It follows immediately that also $m_0(h,J) \to \frac{1}{2}$ as $J \to \infty$.

Moreover definition (4.21) entails that $J\big(\frac{1}{2} - \phi_1(h,J)\big) \to \infty$, $J\big(\phi_2(h,J) - \frac{1}{2}\big) \to \infty$ as $J \to \infty$. Exploit the fact that $m_1(h,J)$ is a solution of equation (4.18):

$$\begin{aligned} m_1(h,J) &= g\big((2m_1(h,J)-1)\,J + h\big) \,\le\, g\big((2\phi_1(h,J)-1)\,J + h\big) = \\ &= g\big(-2J\,(\frac{1}{2} - \phi_1(h,J)) + h\big) \xrightarrow[J\to\infty]{} 0\,, \end{aligned}$$

where also the facts that the function g is increasing and $g(t) \to 0$ as $t \to -\infty$ are used. Since m_1 takes values in $]0,1[$, conclude that $m_1(h,J) \longrightarrow 0$ as

$J \to \infty$. Similarly it can be shown that $m_2(h, J) \longrightarrow 1$ as $J \to \infty$.

ii) Start observing that, by a standard computation from the definition (4.8), $t\, g(-t) \longrightarrow 0$ and $t\,\big(1 - g(t)\big) \longrightarrow 0$ as $t \to +\infty$. Then exploit the fact that, for fixed h and J sufficiently large, $m_1 = m_1(h, J)$ is a solution of equation (4.18):

$$J\, m_1 \;=\; J\, g\big((2m_1 - 1)J + h\big) \;=$$
$$= \frac{\big((1-2m_1)J - h\big)\; g\big(-\,(1-2m_1)J + h\big)}{1 - 2m_1} + \frac{h\; g\big(-\,(1-2m_1)J + h\big)}{1 - 2m_1} \xrightarrow[J\to\infty]{} \frac{0}{1} + \frac{h\,0}{1} = 0\,,$$

using also that $m_1 \to 0$ as $J \to \infty$ by *i)*. Similarly it can be shown that $J\,(1 - m_2) \longrightarrow 0$ as $J \to \infty$.

iii) Start observing that, by a standard computation from the definition (4.23), $-J + \psi_1(J) \longrightarrow -\infty$ and $J + \psi_2(J) \longrightarrow \infty$ as $J \to \infty$. Then exploit the fact that, for $J > J_c$ and $h \in [\psi_2(J), \psi_1(J)]$, $m_1 = m_1(h, J)$ is a solution of equation (4.18):

$$\sup_{h\in[\psi_2,\psi_1]} m_1 \;=\; \sup_{h\in[\psi_2,\psi_1]} g\big((2m_1 - 1)J + h\big) \;\leq\; g\big((2m_1 - 1)J + \psi_1(J)\big) \;=$$
$$=\; g\big(2J\, m_1 - J + \psi_1(J)\big) \xrightarrow[J\to\infty]{} 0\ ,$$

using also the facts that g is an increasing function, $g(t) \to 0$ as $t \to -\infty$, and $J\, m_1 \to 0$ as $J \to \infty$ by *ii)*. Similarly it can be shown that $\inf_{h\in[\psi_2,\psi_1]} m_2 \longrightarrow 1$ as $J \to \infty$. □

4.2.2 Coexistence curve

In the previous subsection we have studied all the solutions of equation (4.18), that is all the stationary points of $m \mapsto \widetilde{p}\,(m, h, J)$. One of them is the point where the global maximum is attained and, because of theorem 4.1, we are interested in this one.

Consider the points m, m_1, m_0, m_2 defined in proposition 4.5 and look for the global maximum point of $m \mapsto \widetilde{p}\,(m, h, J)$:

- for $0 < J < J_c$ and $h \in \mathbb{R}$, $m(h, J)$ is the only local maximum point, hence it is the global maximum point;

- for $J > J_c$ and $h \leq \psi_2(J)$, $m_1(h,J)$ is the only local maximum point, hence it is the global maximum point;

- for $J > J_c$ and $h \geq \psi_1(J)$, $m_2(h,J)$ is the only local maximum point, hence it is the global maximum point;

- for $J > J_c$ and $\psi_2(J) < h < \psi_1(J)$, there are two local maximum points $m_1(h,J) < m_2(h,J)$, hence at least one of them is the global maximum point.

To answer which one is the global maximum point in the last case, we have to investigate the sign of the following function

$$\Delta(h,J) := \widetilde{p}\big(m_2(h,J),h,J\big) - \widetilde{p}\big(m_1(h,J),h,J\big) \tag{4.32}$$

for $J > J_c$ and $\psi_2(J) \leq h \leq \psi_1(J)$.

Proposition 4.9 (The wall: existence and uniqueness)**.** *For all $J > J_c$ there exists a unique $h = \gamma(J) \in]\psi_2(J), \psi_1(J)[$ such that $\Delta(h,J) = 0$. Moreover*

$$\Delta(h,J) \begin{cases} < 0 & \text{if } J > J_c,\ \psi_2(J) \leq h < \gamma(J) \\ > 0 & \text{if } J > J_c,\ \gamma(J) < h \leq \psi_1(J) \end{cases}.$$

Proof. It is an application of the intermediate value theorem. Fix $J > J_c$. It suffices to observe that

i. $\Delta\big(\psi_2(J),J\big) < 0$, because for $h = \psi_2(J)$ the only maximum point of the function $\widetilde{p}\,(\cdot\,,h,J)$ is $m_1(h,J)$;

ii. $\Delta\big(\psi_1(J),J\big) > 0$, because for $h = \psi_1(J)$ the only maximum point of the function $\widetilde{p}\,(\cdot\,,h,J)$ is $m_2(h,J)$;

iii. $h \mapsto \Delta(h,J)$ is a continuous function, by continuity of $\widetilde{p}$, m_1, m_2 (see proposition 4.7);

iv. $h \mapsto \Delta(h, J)$ is strictly increasing; indeed it is C^∞ on $]\psi_2(J), \psi_1(J)[$ by smoothness of $\widetilde{p}$, m_1, m_2 (see proposition 4.7) and, by formula (4.30),

$$\frac{\partial \Delta}{\partial h}(h, J) = \frac{\partial}{\partial h}\widetilde{p}\big(m_2(h,J), h, J\big) - \frac{\partial}{\partial h}\widetilde{p}\big(m_1(h,J), h, J\big) =$$
$$= m_2(h,J) - m_1(h,J) > \phi_2(h,J) - \phi_1(h,J) > 0$$

for all $h \in]\psi_2(J), \psi_1(J)[$. □

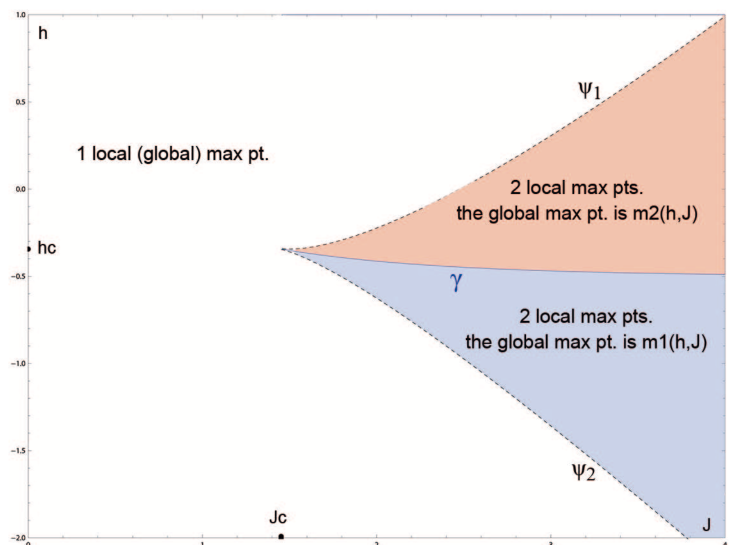

Figure 4.3: γ separates the values of h, J for which $m_1(h,J)$ is the global maximum point from those for which $m_2(h,J)$ is the global maximum point of $m \mapsto \widetilde{p}(m,h,J)$. As $m_1(h,J) < m_2(h,J)$, this entails a discontinuity of the global maximum point $m^*(h,J)$ along the "*wall*" Γ.

Remark 4.10. By the previous results the global maximum point of $m \mapsto \widetilde{p}(m,h,J)$ is

$$m^*(h,J) := \begin{cases} m(h,J) & \text{if } 0 < J \leq J_c\,,\ h \in \mathbb{R} \\ m_1(h,J) & \text{if } J > J_c\,,\ h < \gamma(J) \\ m_2(h,J) & \text{if } J > J_c\,,\ h > \gamma(J) \end{cases} \tag{4.33}$$

where the function γ is defined by proposition 4.9. Set also

$$\Gamma := \{(h,J) \mid J > J_c,\ h = \gamma(J)\}\,, \qquad \overline{\Gamma} := \Gamma \cup \{(h_c, J_c)\}\,. \tag{4.34}$$

Notice that proposition 4.9 guarantees that there is only a curve Γ in the plane (h, J) where the global maximum point of $m \mapsto \widetilde{p}\,(m, h, J)$ is not unique.

Remark 4.11. The techniques developed in this work do not allow us to conclude the existence of the monomer density on the wall. Nevertheless it is easy to show that, using Theorem 4.1, its limsup and liminf are included between m_1 and m_2. In the standard mean-field ferromagnetic model (Curie-Weiss) the existence of the magnetization on the wall ($h = 0$) is achieved by symmetry, a property that we do not have in the present case.

By proposition 4.7 it follows that the function m^* is continuous on its domain $(\mathbb{R} \times \mathbb{R}_+) \smallsetminus \Gamma$ and it is C^∞ on $(\mathbb{R} \times \mathbb{R}_+) \smallsetminus \overline{\Gamma}$. The behaviour of m^* at the critical point (h_c, J_c) will be investigated in the next subsection.

Now we investigate the main properties of the curve $\overline{\Gamma}$, which we call "*the wall*". Extend the function γ defined by proposition 4.9 by

$$\overline{\gamma}(J) := \begin{cases} \gamma(J) & \text{if } \; J > J_c \\ h_c & \text{if } \; J = J_c \end{cases} . \tag{4.35}$$

Proposition 4.12 (The wall: regularity properties)**.** *The function $\overline{\gamma}$ is C^∞ on $]J_c, \infty[$ and (at least) C^1 on $[J_c, \infty[$. In particular*

$$\gamma'(J) \;=\; 1 - m_1\big(\gamma(J), J\big) - m_2\big(\gamma(J), J\big) \quad \forall\, J > J_c \;,$$

and

$$\overline{\gamma}\,'(J_c) \;=\; 1 - 2\, m_c \;=\; -(3 - 2\sqrt{2}) \;.$$

Proof. I. First prove that the function $\gamma \in C^\infty(\,]J_c, \infty[\,)$.
By proposition 4.9 for all $J > J_c$, $h = \gamma(J)$ is the *unique* solution of equation

$$\Delta(h, J) = 0$$

where Δ is defined by (4.32). Moreover $\psi_2(J) < \gamma(J) < \psi_1(J)$. Observe that Δ is C^∞ on $\{(h, J) \,|\, J > J_c,\; \psi_2(J) < h < \psi_1(J)\}$ by smoothness of $\widetilde{p}$ and m_1, m_2

on this region (see proposition 4.7). And furthermore, as shown in the proof of proposition 4.9,

$$\frac{\partial\Delta}{\partial h}(h,J) \neq 0 \quad \forall\,(h,J) \text{ s.t. } h = \gamma(J)\,.$$

Therefore by the implicit function theorem [86] $\gamma \in C^\infty(\,]J_c,\infty[\,)$. Now

$$\Delta(\gamma(J),J)) \equiv 0 \;\Rightarrow\; 0 = \frac{\mathrm{d}}{\mathrm{d}J}\,\Delta(\gamma(J),J) = \frac{\partial\Delta}{\partial h}(\gamma(J),J)\,\gamma'(J) + \frac{\partial\Delta}{\partial J}(\gamma(J),J)$$
$$\Rightarrow\; \gamma'(J) \;=-\,\frac{\partial\Delta}{\partial J}\Big/\frac{\partial\Delta}{\partial h}\,(\gamma(J),J)\,;$$

by formulae (4.30) $\frac{\partial\Delta}{\partial h} = m_2 - m_1$ and $\frac{\partial\Delta}{\partial J} = (m_2^2 - m_2) - (m_1^2 - m_1)$; therefore

$$\gamma'(J) \;=\; 1 - (m_2 + m_1)\,(\gamma(J),J)\;.$$

II. Now prove that the extended function $\overline{\gamma} \in C^1([J_c,\infty[)$.

First observe that $\overline{\gamma}$ is continuous also in J_c, indeed:

$$\psi_2(J) < \gamma(J) < \psi_1(J) \;\; \forall\, J > J_c \quad\Rightarrow\quad \lim_{J\to J_c+}\gamma(J) = h_c$$

by definition of h_c (4.24) and continuity of ψ_1, ψ_2. Then observe that

$$\gamma'(J) \;=\; 1 - (m_2 + m_1)\,(\gamma(J),J) \;\xrightarrow[J\to J_c+]{}\; 1 - 2m_c$$

because $m(h_c,J_c) = m_c$ (remark 4.6) and the functions μ_1, μ_2 defined in proposition 4.7 are continuous. By an immediate application of the mean value theorem, this proves that there exists $\overline{\gamma}\,'(J_c) = 1 - 2m_c$. □

Proposition 4.13 (The wall: asymptote)**.** *The function $\overline{\gamma}$ has an asymptote, precisely*

$$\gamma(J) \;\xrightarrow[J\to\infty]{}\; -\frac{1}{2}\;.$$

Proof. I. Consider the function Δ defined by (4.32). The first step is to prove that $\Delta(h,J) \longrightarrow 0$ as $J \to \infty$, $h = -\frac{1}{2}$. Use definitions (4.6), (4.7) and the fact that for fixed h and J sufficiently large $m_1 = m_1(h,J)$, $m_2 = m_2(h,J)$ satisfy equation (4.18), in two different ways:

$$\widetilde{p}\,(m_1,h,J) \;=\; -J\,m_1^2 + \frac{J}{2} - \frac{1-m_1}{2} - \log g\big((2m_1-1)J + h\big) + (2m_1-1)J + h\;,$$
$$\widetilde{p}\,(m_2,h,J) \;=\; -J\,m_2^2 + \frac{J}{2} - \frac{1-m_2}{2} - \log m_2 + (2m_2-1)J + h\;.$$

Hence, reminding that $m_1 \to 0$ and $m_2 \to 1$ as $J \to \infty$ by proposition 4.8 part *i)*,

$$\Delta(h,J) \;=\; \widetilde{p}\,(m_2,h,J) - \widetilde{p}\,(m_1,h,J) \;= \\ \;=\; J\,(-m_2^2 + 2m_2 + m_1^2 - 2m_1) + \log g\big((2m_1-1)J + h\big) + \frac{1}{2} + o(1)\;,$$

Set $\delta := -m_2^2 + 2m_2 + m_1^2 - 2m_1$ and $t := (2m_1-1)J + h$ and prove that in general

$$J\,\delta + \log g(t) \;\xrightarrow[J\to\infty]{}\; h\;; \tag{4.36}$$

in particular it will follow that for $h = -\frac{1}{2}$

$$\Delta\big(-\frac{1}{2},\,J\big) \;\xrightarrow[J\to\infty]{}\; 0\;. \tag{4.37}$$

Now proving (4.36) is equivalent to prove $\exp(J\delta)\,g(t) \longrightarrow \exp(h)$ as $J \to \infty$; and using definition (4.8)

$$e^{J\delta}\,g(t) \;=\; e^{J\delta}\,\frac{\sqrt{e^{4t}+4e^{2t}} - e^{2t}}{2} \;=\; \frac{\sqrt{e^{2(J\delta+2t)} + 4e^{2(J\delta+t)}} - e^{J\delta+2t}}{2} \;\xrightarrow[J\to\infty]{}\; e^h\;,$$

because, since $J\,m_1 \to 0$ and $J\,(1-m_2) \to 0$ as $J \to \infty$ by proposition 4.8 part *ii)*,

$$J\delta + 2t \;=\; J\,\big(-\,(1-m_2)^2 + m_1^2 - 2m_1 - 1\,\big) + 2h \;\xrightarrow[J\to\infty]{}\; -\infty\;,$$

$$J\delta + t \;=\; J\,\big(-\,(1-m_2)^2 + m_1^2\,\big) + h \;\xrightarrow[J\to\infty]{}\; h\;.$$

II. Remember that by definition of γ in proposition 4.9

$$\Delta\big(\gamma(J),J\big) \;=\; 0 \quad \forall\, J > J_c\;; \tag{4.38}$$

hence using (4.37) will not be hard to prove that $\gamma(J) \longrightarrow -\frac{1}{2}$ as $J \to \infty$. Let $\epsilon > 0$. By (4.37) there exists $\bar{J}_\epsilon > J_c$ such that

$$\big|\Delta\big(-\frac{1}{2},\,J\big)\big| < \epsilon \quad \forall\, J > \bar{J}_\epsilon\;. \tag{4.39}$$

Now by the mean value theorem for all $J > J_c$ and $h \in [\psi_2(J), \psi_1(J)]$,

$$\big|\Delta(h,J) - \Delta\big(-\frac{1}{2},\,J\big)\big| \;\geq\; \inf_{[\psi_2(J),\psi_1(J)]} \big|\frac{\partial\Delta}{\partial h}\,(\cdot\,,J)\big|\;\big|h + \frac{1}{2}\big|\;.$$

Furthermore by identity (4.30) and proposition 4.8 part *iii)*

$$\inf_{[\psi_2(J),\psi_1(J)]} \Big|\frac{\partial \Delta}{\partial h}(\cdot\,,J)\Big| \;=\; \inf_{[\psi_2(J),\psi_1(J)]}(m_2-m_1)\,(\cdot\,,J)\;\geq$$

$$\geq \inf_{[\psi_2(J),\psi_1(J)]} m_2(\cdot,J) \;-\; \sup_{[\psi_2(J),\psi_1(J)]} m_1\,(\cdot,J) \;\xrightarrow[J\to\infty]{}\; 1\,.$$

Therefore there exist $\bar{J}$ such that

$$\big|\Delta(h,J)-\Delta\big(-\frac{1}{2},\,J\big)\big| \;\geq\; \frac{1}{2}\,\big|h+\frac{1}{2}\big| \qquad \forall\, J>\bar{J},\; h\in[\psi_2(J),\psi_1(J)]\,. \tag{4.40}$$

Choosing $h=\gamma(J)$ in (4.40), by (4.38), (4.39) one obtains that for all $J>\max\{\bar{J},\,\bar{J}_\epsilon\}$

$$\big|\gamma(J)+\frac{1}{2}\big| \;\leq\; 2\,\big|\Delta(\gamma(J),J)-\Delta\big(-\frac{1}{2},J\big)\big| < 2\epsilon\,. \qquad \square$$

4.2.3 Critical exponents

As observed in remark 4.10 the global maximum point $m^*(h,J)$ is a continuous function on $(\mathbb{R}\times\mathbb{R}^+)\smallsetminus\Gamma$, but it is smooth only outside the critical point (h_c,J_c). In this section we study the behaviour of the solutions of equation (4.18) near the critical point, with particular interest in the function m^*.

As usual the notation $f=\mathcal{O}(g)$ as $x\to x_0$ means that there exists a neighbourhood U of x_0 and a constant $C\in\mathbb{R}$ such that $|f(x)|\leq C\,|g(x)|$ for all $x\in U$. The notation $f\sim g$ as $x\to x_0$ means that $f(x)/g(x)\longrightarrow 1$ as $x\to x_0$. Finally $f=o(g)$ as $x\to x_0$ means that $f(x)/g(x)\longrightarrow 0$ as $x\to x_0$.

We call *critical exponent* of a function f at a point x_0 the following limit

$$\lim_{x\to x_0}\frac{\log|f(x)-f(x_0)|}{\log|x-x_0|}\,.$$

The main result of this section is the following:

Theorem 4.14. *Consider the global maximum point $m^*(h,J)$ of the function $m\mapsto\widetilde{p}\,(m,h,J)$ defined by (4.6).*

i) m^* is continuous on $(\mathbb{R}\times\mathbb{R}_+)\smallsetminus\Gamma$ and smooth on $(\mathbb{R}\times\mathbb{R}_+)\smallsetminus\overline{\Gamma}$, where $\overline{\Gamma}=\Gamma\cup\{(h_c,J_c)\}$ and the "wall" curve Γ is the graph of the function γ defined by proposition 4.9.

ii) The critical exponents of m^* *at the critical point* (h_c, J_c) *are:*

$$\boldsymbol{\beta} = \lim_{J \to J_c+} \frac{\log |m^*(\delta(J), J) - m_c|}{\log(J - J_c)} = \frac{1}{2}$$

along any curve $h = \delta(J)$ *with* $\delta \in C^2([J_c, \infty[)$, $\delta(J_c) = h_c$, $\delta'(J_c) = 1 - 2m_c$ *(i.e. if the curve is tangent to the "wall" in the critical point);*

$$\frac{1}{\boldsymbol{\delta}} = \lim_{J \to J_c} \frac{\log |m^*(\delta(J), J) - m_c|}{\log |J - J_c|} = \frac{1}{3}$$

$$\frac{1}{\boldsymbol{\delta}} = \lim_{h \to h_c} \frac{\log |m^*(h, \delta(h)) - m_c|}{\log |h - h_c|} = \frac{1}{3}$$

along any curve $h = \delta(J)$ *with* $\delta \in C^2(\mathbb{R}_+)$, $\delta(J_c) = h_c$, $\delta'(J_c) \neq 1 - 2m_c$ *or along a curve* $J = \delta(h)$ *with* $\delta \in C^2(\mathbb{R})$, $\delta(h_c) = J_c$, $\delta'(h_c) = 0$ *(i.e. if the curve is not tangent to the "wall" in the critical point).*

iii) Denote by $m^*(h^\pm, J) := \lim_{h' \to h\pm} m^*(h', J)$. *The critical exponent of* $m^*(h^+, J)$ *and* $m^*(h^-, J)$ *at the critical point* (h_c, J_c) *along the "wall"* $h = \gamma(J)$ *is still*

$$\boldsymbol{\beta} = \lim_{J \to J_c+} \frac{\log |m^*(\gamma(J)^+, J) - m_c|}{\log(J - J_c)} = \frac{1}{2}$$

$$\boldsymbol{\beta} = \lim_{J \to J_c+} \frac{\log |m^*(\gamma(J)^-, J) - m_c|}{\log(J - J_c)} = \frac{1}{2}$$

Proof. As observed in remark 4.10, the global maximum point m^* is expressed piecewise using the two local maximum points μ_1, μ_2 and inherits their continuity properties outside Γ and their regularity properties outside $\overline{\Gamma}$. Thus part *i)* of the theorem is already proved by proposition 4.7.
The proof of the other parts of the theorem, regarding the behaviour of m^* at the critical point (h_c, J_c), is long and rather technical, then we sketch only the major points. For the benefit to the reader, the remaining parts of the proof are given in Appendix B.

In the following proposition we find the fundamental equation characterizing the behaviour of the solutions of equation (4.18) near the critical point (h_c, J_c).

Proposition 4.15. *Here for $h \in \mathbb{R}$, $J > 0$ let $m = m(h, J)$ be any solution of the consistency equation (4.18):*

$$m = g\big((2m-1)J + h\big) \ .$$

Then m is continuous at (h_c, J_c) and furthermore, setting $t := (2m-1)J + h$, it satisfies

$$(t-t_c)^3 - \kappa_1\,(J-J_c)\,(t-t_c) - \kappa_2\,\varrho(h,J) + \mathcal{O}\big((t-t_c)^4\big) = 0 \qquad (4.41)$$

as $(h, J) \to (h_c, J_c)$, where we set $\kappa_1 := 3\,\frac{J_c}{J}\,(2-m_c)$, $\kappa_2 := 3\,\frac{J_c^2}{J}\,(2-m_c)$ and

$$\varrho(h,J) := h - h_c + (2m_c - 1)(J - J_c) \ . \qquad (4.42)$$

Proof. I. First show that m is continuous at (h_c, J_c). Exploit equation (4.18) for $m(h, J)$ and use continuity and monotonicity of g: as $(h, J) \to (h_c, J_c)$

$$\limsup m(h,J) = \limsup g\big((2\,m(h,J)-1)\,J + h\big) = g\big((2\limsup m(h,J)-1)\,J_c + h_c\big) \ ,$$

$$\liminf m(h,J) = \liminf g\big((2\,m(h,J)-1)\,J + h\big) = g\big((2\liminf m(h,J)-1)\,J_c + h_c\big) \ .$$

Thus $\limsup m(h, J)$ and $\liminf m(h, J)$ are both solution of equation $\mu = g\big((2\mu + 1)J_c + h_c\big)$. But this solution is unique by proposition 4.5, and it is m_c by remark 4.6. Therefore

$$\limsup_{(h,J)\to(h_c,J_c)} m(h,J) \;=\; \liminf_{(h,J)\to(h_c,J_c)} m(h,J) \;=\; m_c \ .$$

II. Make a Taylor expansion of the smooth function g at the point t_c (see (4.8), (4.26)). By identities (4.49), (4.50), (4.51) and since $g(t_c) = m_c$ it is easy to find

$$g(t) \;=\; m_c + \frac{1}{2J_c}\,(t-t_c) - \frac{1}{6J_c^2(2-m_c)}\,(t-t_c)^3 + \mathcal{O}\big((t-t_c)^4\big) \qquad (4.43)$$

as $t \to t_c$. Now choose $t := (2m-1)J + h$. Then $g(t) = m$ and

$$t - t_c \;=\; \varrho(h,J) + 2J\,(m - m_c) \ , \qquad (4.44)$$

where $\varrho(h, J) := h - h_c + (2m_c - 1)(J - J_c)$. Now (4.41) follows from (5.73). ☐

Given the previous expansion, the proof of part *ii)* of the theorem 4.14 is rather technical and it is contained in the proposition 4.21 of the Appendix (and in the other results of the Appendix B).

The part *ii)* of the theorem describes the critical behaviour of the local maximum points along curves of class C^2. Notice that "the wall" $\overline{\gamma}$ belongs to $C^1([J_c,\infty[) \cap C^\infty(]J_c,\infty[)$ by proposition 4.12, but we did not manage to prove that it is C^2 up to J_c. Anyway we are interested in the behaviour along this coexistence curve, which separates two different phases of the system. This is provided by part *iii)* of the theorem 4.14. To prove it let start with the following proposition, which is bases on corollary 4.19 and lemma 4.18 in the Appendix.

Proposition 4.16. *Consider the "wall" curve $h = \overline{\gamma}(J)$ defined by (4.35) and proposition 4.9. There exist $r > 0$, $C_1 < \infty$, $C_2 > 0$ such that for all $J \in\,]J_c, J_c + r[$.*

$$C_2 \leq \frac{\mu_2(\overline{\gamma}(J), J) - m_c}{\sqrt{J - J_c}} \leq C_1\,, \quad C_2 \leq \frac{m_c - \mu_1(\overline{\gamma}(J), J)}{\sqrt{J - J_c}} \leq C_1$$

Proof. Observe that by definition, on the curve $h = \overline{\gamma}(J)$, $J \geq J_c$, both the local maximum points $\mu_1(h, J)$, $\mu_2(h, J)$ exist.

As $\overline{\gamma} \in C^1([J_c,\infty[)$ (see proposition 4.12), the existence of the lower bound $C_2 > 0$ is guaranteed by corollary 4.19 part *2)*.

Only the existence of an upper bound $C_1 < \infty$ has to be proven. Fix $J > J_c$ and shorten the notation by $m_i = m_i(\overline{\gamma}(J), J) = \mu_i(\overline{\gamma}(J), J)$ and $t_i := (2m_i - 1)\,J + \gamma(J)$ for $i = 1, 2$. By proposition 4.15, t_1, t_2 satisfy equation (4.41). The Taylor expansion with Lagrange remainder of $\overline{\gamma}$ is (see proposition 4.12)

$$\gamma(J) = h_c + (1 - 2m_c)\,(J - J_c) + \gamma''(\bar{J})\,(J - J_c)^2\,, \quad \text{with } \bar{J} \in\,]J_c, J[\,;$$

notice $\gamma''(\bar{J})\,(J - J_c)^2$ is not necessarily a $\mathcal{O}\big((J - J_c)^2\big)$, because we do not know the behaviour of γ'' as $J \to J_c$, but for sure it is a $o(J - J_c)$ as $J \to J_c$.
Thus (see identities (4.42), (4.44)):

$$\varrho(h, J) = \gamma''(\bar{J})\,(J - J_c)^2 \quad \text{and} \quad t_i - t_c = 2J\,(m_i - m_c) + \gamma''(\bar{J})\,(J - J_c)^2$$

and equation (4.41) becomes:

$$(t_i - t_c)^3 - \kappa_1 \,(J - J_c)\,(t_i - t_c) - \kappa_2\, \gamma''(\bar{J})\,(J - J_c)^2 + \mathcal{O}\big((t_i - t_c)^4\big) = 0\;,$$

which entails

$$(m_i - m_c)^3 - \frac{\kappa_1}{(2J)^2}\,(J - J_c)\,(m_i - m_c) - \frac{\kappa_2}{(2J)^3}\,\gamma''(\bar{J})\,(J - J_c)^2\,(1 + o(1)) + \\ + \mathcal{O}\big((m_i - m_c)^4\big) = 0\,. \quad (4.45)$$

Distinguish two cases.

1) If $\gamma''(\bar{J})\,(J - J_c)^2 = \mathcal{O}\big((m_i - m_c)^4\big)$ (along a sequence), then (4.45) rewrites

$$(m_i - m_c)^3 - \frac{\kappa_1}{(2J)^2}\,(J - J_c)\,(m_i - m_c) + \mathcal{O}\big((m_i - m_c)^4\big) = 0\,, \qquad (4.46)$$

which, dividing by $m_i - m_c$ and solving, gives

$$m_i - m_c = \pm\,\frac{\sqrt{\kappa_1}}{2J}\,(J - J_c)^{\frac{1}{2}} + \mathcal{O}\big((m_i - m_c)^{\frac{3}{2}}\big)\;;$$

hence $m_i - m_c \sim \sqrt{\kappa_1}/(2J)\,(J - J_c)^{1/2}$, proving the result (along the sequence).

2) Now suppose $(m_i - m_c)^4 = o\big(\gamma''(\bar{J})\,(J - J_c)^2\big)$ (along a sequence), then (4.45) rewrites

$$(m_i - m_c)^3 \underbrace{- \frac{\kappa_1}{(2J)^2}\,(J - J_c)}_{=:\,p}\,(m_i - m_c) \underbrace{- \frac{\kappa_2}{(2J)^3}\,\gamma''(\bar{J})\,(J - J_c)^2\,(1 + o(1))}_{=:\,q} = 0\,. \qquad (4.47)$$

Claim $\Delta := (\frac{q}{2})^2 + (\frac{p}{3})^3 \leq 0$. Suppose by contradiction $\Delta > 0$. Then the cubic equation (4.47) has only one real solution: for $i = 1, 2$

$$m_i - m_c = u_+ + u_- \quad \text{with} \;\; u_\pm = \sqrt[3]{-\frac{q}{2} \pm \sqrt[2]{\Big(\frac{q}{2}\Big)^2 + \Big(\frac{p}{3}\Big)^3}}\;.$$

Observe that q and p are written only in terms of J, so that $u_+ + u_-$ at the main order do not depend implicitly on m_i. Therefore $m_1 - m_c$ and $m_2 - m_c$ must have the same sign for every $J > J_c$ small enough. But this contradicts proposition 4.5 and lemma 4.18, which ensures that in a right neighbourhood of J_c

$$m_2 - m_c > \phi_2 - m_c > 0 \quad \text{while} \;\; m_1 - m_c < \phi_2 - m_c < 0 \, .$$

This proves $\Delta \leq 0$. And now adapting to equation (4.47) the step *ii.* of the proof of corollary 4.19, $\Delta \leq 0$ entails (along the sequence)

$$m - m_c \;=\; \mathcal{O}\big((J - J_c)^{\frac{1}{2}}\big) \, .$$

This completes the proof of the proposition. □

To conclude the proof of the part *iii)* of theorem 4.14, it suffices to have the previous proposition and observe that

$$m^*(\gamma(J)^+, J) = m_2(\gamma(J), J) \, , \quad m^*(\gamma(J)^-, J) = m_1(\gamma(J), J)$$

for all $J > J_c$, by proposition 4.9 and continuity of m_1, m_2. □

4.3 Appendix: properties of the function g

We study the main properties of the function g defined by (4.8), which are often used in the chapter 4. Remind

$$g(t) \;=\; \frac{1}{2}\,(\sqrt{e^{4t} + 4\,e^{2t}} - e^{2t}) \quad \forall\, t \in \mathbb{R} \, .$$

Standard computations show that g is analytic on $\mathbb{R}$, $0 < g < 1$, $\lim_{h\to-\infty} g(t) = 0$, $\lim_{h\to\infty} g(t) = 1$, g is strictly increasing, g is strictly convex on $]-\infty, \frac{\log(2\sqrt{2}-2)}{2}]$ and strictly concave on $[\frac{\log(2\sqrt{2}-2)}{2}, \infty[$, $g(\frac{\log(2\sqrt{2}-2)}{2}) = 2 - \sqrt{2}$.

Solving in h the equation $g(t) = k$ for any fixed $k \in]0,1[$, one finds the inverse function:

$$g^{-1}(k) \;=\; \frac{1}{2} \log \frac{k^2}{1-k} \quad \forall k \in]0,1[\, . \tag{4.48}$$

It is useful to write the derivatives of g in terms of lower order derivatives of g itself. For the first derivative, think g as $(g^{-1})^{-1}$ and exploit (4.48):

$$g'(h) \;=\; \frac{1}{(g^{-1})'(k)}\Big|_{k=g(t)} \;=\; \frac{2\,k\,(1-k)}{2-k}\Big|_{k=g(t)} \;=\; \frac{2\,g(t)\,(1-g(t))}{2-g(t)} \tag{4.49}$$

Then for the second derivative, differentiate the rhs of (4.49) and substitute (4.49) itself in the expression:

$$g'' = \frac{2\,g'}{2-g}\big(1 - 2\,g + \frac{g\,(1-g)}{2-g}\big) \;=\; \frac{2\,g'\,(1-2\,g) + (g')^2}{2-g}\;. \tag{4.50}$$

The same for the third derivative: differentiate the rhs of (4.50) and substitute (4.50) itself in the expression:

$$\begin{aligned} g''' &= \frac{1}{2-g}\,\big(2\,g''(1-2\,g+g') - 4\,(g')^2 + g'\,\frac{2\,g'\,(1-2\,g)+(g')^2}{2-g}\,\big) \;= \\ &= \frac{g''\,(2-4\,g+3\,g') - 4\,(g')^2}{2-g}\;. \end{aligned} \tag{4.51}$$

Lemma 4.17. *For* $c > 6 - 4\sqrt{2}\,$,

$$g'(t) < c \quad \forall\, t \in \mathbb{R}\;.$$

For $0 < c \leq 6 - 4\sqrt{2}\,$,

$$g'(t) \begin{cases} < c & \textit{iff}\;\; t < \frac{1}{2}\,\log\alpha_-(c) \;\; \textit{or} \;\; t > \frac{1}{2}\,\log\alpha_+(c) \\ > c & \textit{iff}\;\; \frac{1}{2}\,\log\alpha_-(c) < t < \frac{1}{2}\,\log\alpha_+(c) \end{cases}\;,$$

where

$$\alpha_\pm(c) := \frac{-(c^2+8c-4)\;\pm\;(2-c)\sqrt{c^2-12c+4}}{4\,c}\;.$$

Proof. Investigate for example the inequality $g'(t) < c$. By (4.49) clearly $0 < g' < 2$, hence the inequality is trivially true for $c \geq 2$ and false for $c \leq 0$.

Using identity (4.49) one finds

$$g' < c \quad\Leftrightarrow\quad 2\,g^2 - (2+c)\,g + 2c > 0\;;$$

this is a second degree inequality in g with $\Delta = c^2 - 12c + 4$.

If $6 - 4\sqrt{2} < c < 6 + 4\sqrt{2}$, it is verified for any value of g.

If instead $c \leq 6 - 4\sqrt{2}$ or $c \geq 6 + 4\sqrt{2}$, it is verified if and only if

$$g(t) < \frac{2+c-\sqrt{c^2-12c+4}}{4} =: s_-(c) \quad \text{or} \quad g(t) > \frac{2+c+\sqrt{c^2-12c+4}}{4} =: s_+(c)\;.$$

For $0 < c < 2$, $s_\pm(c) \in]0,1[$ hence one can apply g^{-1}, which is strictly increasing:

$$t < g^{-1}(s_-(c)) \quad \text{or} \quad t > g^{-1}(s_+(c)) \,.$$

This concludes the proof because identity (4.48) and standard computations show that

$$g^{-1}(s_\pm(c)) \;=\; \frac{1}{2}\,\log \alpha_\pm(c) \,. \qquad \square$$

4.4 Appendix: critical exponents

Let us prove the results used in subsection 4.2.3 to compute the critical exponents.

Lemma 4.18. *Consider the inflection points ϕ_1, ϕ_2 of $\widetilde{p}$ defined by (4.21). Their behaviour at the critical point (h_c, J_c) along any curve $\delta \in C^1([J_c,\infty[)$, with $\delta(J_c) = h_c$, is*

$$\frac{\phi_1(\delta(J),J) - m_c}{\sqrt{J-Jc}} \xrightarrow[J\to J_c+]{} -C \,, \qquad \frac{\phi_2(\delta(J),J) - m_c}{\sqrt{J-Jc}} \xrightarrow[J\to J_c+]{} C \,.$$

where $C = \sqrt[4]{2}/(2J_c) > 0$.

Proof. For $i = 1,2$ and $J \geq J_c$ definition (4.21), observing that $(2m_c - 1)J = -h_c + (2m_c - 1)\,(J - J_c) + t_c$, gives

$$2J\left(\phi_i(\delta(J),J) - m_c\right) \;=\; \frac{1}{2}\log a_i(J) - t_c - (\delta(J) - h_c) - (2m_c - 1)(J - J_c) \,.$$

Now the definition (4.22) may be rewritten as

$$a_i(J) \;=\; \underbrace{(2J - 2 - \frac{1}{8J})}_{=:\,b(J)} \mp \underbrace{4\,(\frac{1}{2} - \frac{1}{8J})\sqrt{J - \tfrac{3-2\sqrt{2}}{4}}}_{=:\,c(J)}\;\sqrt{J - Jc}\,.$$

Thus, exploiting $\log(x + y) = \log x + \log(1 + y/x) = \log x + y/x + \mathcal{O}((y/x)^2)$ as $y/x \to 0$, $\frac{1}{2}\log b(J_c) = t_c$ and $\log b(J)$ differentiable at $J = J_c$,

$$\begin{aligned}\frac{1}{2}\log a_i(J) - t_c \;&=\; \frac{1}{2}\,\frac{\log b(J) - \log b(J_c)}{(J - J_c)}\,(J - J_c) \mp \frac{1}{2}\,\frac{c(J)}{b(J)}\,\sqrt{J - Jc} + \mathcal{O}(J - J_c)\\ &=\; \mp\frac{1}{2}\,\frac{c(J)}{b(J)}\,\sqrt{J - J_c} + \mathcal{O}(J - J_c) \,.\end{aligned}$$

To conclude put things together and use also δ differentiable at J_c:

$$2J\,\frac{\phi_i(\delta(J),J)-m_c}{\sqrt{J-Jc}} = \frac{\frac{1}{2}\log a_i(J)-t_c}{\sqrt{J-J_c}} - \frac{\delta(J)-h_c}{\sqrt{J-J_c}} - (2m_c-1)\sqrt{J-J_c}$$
$$= \mp\frac{1}{2}\,\frac{c(J)}{b(J)} + \mathcal{O}(\sqrt{J-J_c}\,) \xrightarrow[J\to J_c+]{} \pm\sqrt[4]{2}\,. \qquad \square$$

Next corollary gives a first bound for the critical exponents.

Corollary 4.19. *Here for $h\in\mathbb{R}$, $J>J_c$ let $m=m(h,J)$ be any solution of the consistency equation (4.18).*

1) There exist $r_1>0$, $C_1<\infty$ such that for all $(h,J)\in B\big((h_c,J_c),r_1\big)$ with $J>J_c$

$$|m-m_c| \,\le\, C_1\big(\,|h-h_c|^{\frac{1}{3}} + |J-J_c|^{\frac{1}{3}}\big)\,.$$

2) Assume that m pointwise coincides with one of the local maximum points m_1, m_2 (see proposition 4.5). There exist $r_2>0$, $C_2>0$ such that for all $(h,J)\in B\big((h_c,J_c),r_2\big)$ with $J>J_c$ and $h=\delta(J)$ for some $\delta\in C^1([J_c,\infty[)$, $\delta(J_c)=h_c$

$$|m-m_c| \,\ge\, C_2\,|J-J_c|^{\frac{1}{2}}\,.$$

Proof. 1) Set $t:=(2m-1)J+h$. By proposition 4.15, t satisfies equation (4.41), which can be treated as a third degree algebraic equation in $t-t_c$:

$$(t-t_c)^3\underbrace{-\kappa_1\,(J-J_c)}_{=:p}(t-t_c)\underbrace{-\kappa_2\,\varrho(h,J)+\mathcal{O}\big((t-t_c)^4\big)}_{=:q}=0\,.$$

Analyse the real solutions of this equation. Set $\Delta:=(\frac{q}{2})^2+(\frac{p}{3})^3$ and observe that $(\frac{q}{2})^2>0$ while $(\frac{p}{3})^3<0$ as we are assuming $J>J_c$.

i. If $\Delta>0$, the only real solution of (4.41) is

$$t-t_c=u_+ + u_- \quad\text{with}\;\; u_\pm=\sqrt[3]{-\frac{q}{2}\pm\sqrt[2]{\Delta}}\,.$$

On the other hand

$$\Delta>0\;\Rightarrow\;\Big(\frac{p}{3}\Big)^3=\mathcal{O}\Big(\Big(\frac{q}{2}\Big)^2\Big)\;\Rightarrow\;\Delta=\mathcal{O}\Big(\Big(\frac{q}{2}\Big)^2\Big)\,.$$

Therefore, reminding also definition (4.42),

$$t - t_c \,=\, \mathcal{O}\big((\tfrac{q}{2})^{\frac{1}{3}} \big) \,=\, \mathcal{O}\big((h-h_c)^{\frac{1}{3}} \big) + \mathcal{O}\big((J-J_c)^{\frac{1}{3}} \big) + \mathcal{O}\big((t-t_c)^{\frac{4}{3}} \big) \,,$$

hence $t - t_c = \mathcal{O}\big((h-h_c)^{\frac{1}{3}} \big) + \mathcal{O}\big((J-J_c)^{\frac{1}{3}} \big)$ because $(t-t_c)^{\frac{4}{3}-1} \to 0$ as $(h,J) \to (h_c, J_c)$.

ii. If $\Delta = 0$ or $\Delta < 0$ there are respectively two or three distinct real solutions of (4.41) and, from their explicit form, it is immediate to see that they all satisfy

$$t - t_c \,=\, \mathcal{O}\big(\sqrt[2]{-\tfrac{p}{3}}\, \big) \,=\, \mathcal{O}\big((J-J_c)^{\frac{1}{2}} \big) \,.$$

Conclude that for any possible value of Δ,

$$t - t_c = \mathcal{O}\big((h-h_c)^{\frac{1}{3}} \big) + \mathcal{O}\big((J-J_c)^{\frac{1}{3}} \big) \,.$$

Now, as observed in (4.44), $t - t_c = h - h_c + (2m_c - 1)(J - J_c) + 2J\,(m - m_c)$. Therefore also $m - m_c = \mathcal{O}\big((h-h_c)^{\frac{1}{3}} \big) + \mathcal{O}\big((J-J_c)^{\frac{1}{3}} \big)$, and this concludes the proof of the first statement.

2) Now consider the two maximum points m_1, m_2. By proposition 4.5

$$m_1 < \phi_1 < \phi_2 < m_2$$

where ϕ_1, ϕ_2 are the inflection points defined by (4.21). Hence applying lemma 4.18 one finds:

$$\frac{m_2 - m_c}{\sqrt{J - J_c}} > \frac{\phi_2 - m_c}{\sqrt{J - J_c}} \longrightarrow C\,, \quad \frac{m_c - m_1}{\sqrt{J - J_c}} > \frac{m_c - \phi_1}{\sqrt{J - J_c}} \longrightarrow C\,,$$

as $J \to J_c+$ and $h = \delta(J)$ with $\delta(J_c) = h_c$ and δ differentiable in J_c. And this proves the second statement. □

The next lemma tells us in which region of the plane (h, J) described by figure 4.1 a curve passing through the point (h_c, J_c) lies.

Lemma 4.20. *Let* $\delta \in C^2([J_c, \infty[)$ *such that* $\delta(J_c) = h_c$, $\delta'(J_c) =: \alpha$. *There exists* $r > 0$ *such that for all* $J \in]J_c, J_c + r[$

- *if* $\alpha = 1 - 2m_c$, $\psi_2(J) < \delta(J) < \psi_1(J)$ *;*

- *if* $\alpha < 1 - 2m_c$, $\delta(J) < \psi_2(J)$ *;*

- *if* $\alpha > 1 - 2m_c$, $\delta(J) > \psi_1(J)$ *.*

Proof. I. Observe that $a_i(J)$ is continuous for $J \geq J_c$ and smooth for $J > J_c$. Moreover $g'(\frac{1}{2}\log a_i(J)) = \frac{1}{2J}$ by definition (4.22) and lemma 4.17, and $g(\frac{1}{2}\log a_i(J_c)) = g(t_c) = m_c$ by definition (4.26) and remark 4.6. Then differentiating definition (4.23) at $J > J_c$,

$$\psi_i'(J) \;=\; 1 - 2\,g(\frac{1}{2}\log a_i(J)) + \frac{1}{2}\,\frac{a_i'(J)}{a_i(J)}\,\underbrace{\big(1 - 2J\,g'(\frac{1}{2}\log a_i(J))\big)}_{=0} \;\xrightarrow[J\to J_c]{}\; 1 - 2m_c\,.$$

Hence an immediate application of the mean value theorem shows that for $i = 1, 2$ there exits $\psi_i'(J_c) = 1 - 2m_c$.

II. Differentiating definition (4.22) at $J > J_c$ shows that $a_1'(J) \to -\infty$, $a_2'(J) \to +\infty$ as $J \to J_c+$, while $a_i(J) \to 2\sqrt{2} - 2$ as $J \to J_c$. Hence

$$\psi_i''(J) \;=\; -\,g'(\frac{1}{2}\log a_i(J))\,\frac{a_i'(J)}{a_i(J)} \;=\; -\frac{1}{2J}\,\frac{a_i'(J)}{a_i(J)} \;\xrightarrow[J\to J_c+]{}\; \begin{cases} +\infty & \text{for } i = 1 \\ -\infty & \text{for } i = 2 \end{cases}\,.$$

The result is provided comparing the first order Taylor expansions at J_c with Lagrange remainder of ψ_1, ψ_2 and δ. □

The following proposition essentially contain the proof of part *ii)* of theorem 4.14.

Proposition 4.21. *Let* $(h, J) \to (h_c, J_c)$ *along a curve* $h = \delta(J)$ *with* $\delta \in C^2(\mathbb{R}_+)$, $\delta(J_c) = h_c$, $\delta'(J_c) =: \alpha$ *or along a curve* $J = \delta(h)$ *with* $\delta \in C^2(\mathbb{R})$,

$\delta(h_c) = J_c$, $\delta'(h_c) = 0$, then

$$\mu_1(h,J) - m_c \sim \begin{cases} -C\,(J-J_c)^{\frac{1}{2}} & \textit{if } h=\delta(J),\ \alpha = 1-2m_c \textit{ and } J > J_c \\ C_\alpha\,(J-J_c)^{\frac{1}{3}} & \textit{if } h=\delta(J),\ \alpha < 1-2m_c \\ C_\infty\,(h-h_c)^{\frac{1}{3}} & \textit{if } J=\delta(h) \end{cases}$$

$$\mu_2(h,J) - m_c \sim \begin{cases} C\,(J-J_c)^{\frac{1}{2}} & \textit{if } h=\delta(J),\ \alpha = 1-2m_c \textit{ and } J > J_c \\ C_\alpha\,(J-J_c)^{\frac{1}{3}} & \textit{if } h=\delta(J),\ \alpha > 1-2m_c \\ C_\infty\,(h-h_c)^{\frac{1}{3}} & \textit{if } J=\delta(h) \end{cases}$$

where $C = \frac{1}{2J_c}\sqrt{3(2-m_c)}$, $C_\alpha = \frac{1}{2J_c}\sqrt[3]{\frac{3}{2}J_c(2-m_c)(2m_c-1+\alpha)}$, $C_\infty = \frac{1}{2J_c}\sqrt[3]{3J_c(2-m_c)}$. To complete the cases, along the line $h = h_c+(1-2m_c)(J-J_c)$, when $J \le J_c$

$$\mu_1(h,J) = \mu_2(h,J) = m_c.$$

Proof. Fix (h,J) on the curve given by the graph of δ and in the rest of the proof denote by m a solution of the consistency equation (4.18), i.e. $m = g\big((2m-1)J+h\big)$. Furthermore when necessary m is assumed to be a local maximum point of $\widetilde{p}$. Set $t := (2m-1)J+h$. By proposition 4.15, $t - t_c \to 0$ as $(h,J) \to (h_c,J_c)$ and it satisfies (4.41). Solve this equation in the different cases.

i) Suppose $h = \delta(J)$ with $\alpha = 1-2m_c$. Hence $h - h_c = (1-2m_c)(J-J_c) + \mathcal{O}\big((J-J_c)^2\big)$. Observe that by (4.42), (4.44)

$$\varrho(h,J) = \mathcal{O}\big((J-J_c)^2\big) \quad \text{and} \quad t-t_c = 2J\,(m-m_c) + \mathcal{O}\big((J-J_c)^2\big)\,.$$

Hence equation (4.41) becomes

$$(t-t_c)^3 - \kappa_1\,(J-J_c)\,(t-t_c) + \mathcal{O}\big((J-J_c)^2\big) + \mathcal{O}\big((t-t_c)^4\big) = 0\,.$$

Observe that if $J > J_c$ by corollary 4.19 part *2)*,

$$(J-J_c)^{\frac{1}{2}} \;=\; \mathcal{O}(t-t_c)\,;$$

therefore when $J > J_c$ the previous equation rewrites

$$(t-t_c)^3 - \kappa_1\,(J-J_c)\,(t-t_c) + \mathcal{O}\big((t-t_c)^4\big) = 0\ .$$

This one simplifies in

$$t = t_c \quad \text{or} \quad (t-t_c)^2 - \kappa_1\,(J-J_c) + \mathcal{O}\big((t-t_c)^3\big) = 0\ ,$$

giving $t = t_c$ or, as we are assuming $J > J_c$,

$$t - t_c \;=\; \pm\sqrt{\kappa_1}\,(J-J_c)^{\frac{1}{2}} + \mathcal{O}\big((t-t_c)^{\frac{3}{2}}\big)\ .$$

This entails

$$m - m_c \;=\; \pm\frac{\sqrt{\kappa_1}}{2J}\,(J-J_c)^{\frac{1}{2}} + \mathcal{O}\big((J-J_c)^2\big) + \mathcal{O}\big((m-m_c)^{\frac{3}{2}}\big)$$

and dividing both sides by $m - m_c$, since $(m-m_c)^{\frac{1}{2}} \to 0$, one finds

$$m - m_c \;\sim\; \pm\frac{\sqrt{\kappa_1}}{2J}\,(J-J_c)^{\frac{1}{2}}\ . \tag{4.52}$$

ii) Suppose $J = \delta(h)$ with $\delta'(h_c) = 0$. Hence $J - J_c = \mathcal{O}\big((h-h_c)^2\big)$. (4.42) and (4.44) give

$$\varrho(h,J) = h - h_c + \mathcal{O}\big((h-h_c)^2\big) \quad \text{and} \quad t - t_c = 2J\,(m-m_c) + h - h_c + \mathcal{O}\big((h-h_c)^2\big)\ .$$

Hence equation (4.41) becomes

$$(t-t_c)^3 - \kappa_2\,(h-h_c) + \mathcal{O}\big((h-h_c)^2\big) + \mathcal{O}\big((t-t_c)^4\big) = 0\ .$$

giving

$$t - t_c \;=\; \sqrt[3]{\kappa_2}\,(h-h_c)^{\frac{1}{3}} + \mathcal{O}\big((h-h_c)^{\frac{2}{3}}\big) + \mathcal{O}\big((t-t_c)^{\frac{4}{3}}\big)\ .$$

This entails

$$m - m_c \;=\; \frac{\sqrt[3]{\kappa_2}}{2J}\,(h-h_c)^{\frac{1}{3}} + \mathcal{O}\big((h-h_c)^{\frac{2}{3}}\big) + \mathcal{O}\big((m-m_c)^{\frac{4}{3}}\big)$$

and dividing both sides by $m - m_c$, since $(m-m_c)^{\frac{1}{3}} \to 0$, one finds

$$m - m_c \;\sim\; \frac{\sqrt[3]{\kappa_2}}{2J}\,(h-h_c)^{\frac{1}{3}}\ . \tag{4.53}$$

iii) Suppose $h = \delta(J)$ with $\alpha \neq 1-2m_c$. Hence $h-h_c = \alpha\,(J-J_c)+\mathcal{O}\big((J-J_c)^2\big)$. Observe that by (4.42), (4.44)

$$\begin{aligned} \varrho(h,J) &= (\alpha+2m_c-1)(J-J_c)+\mathcal{O}\big((J-J_c)^2\big)\,, \\ t-t_c &= 2J\,(m-m_c)+(\alpha+2m_c-1)(J-J_c)+\mathcal{O}\big((J-J_c)^2\big)\,. \end{aligned}$$

Hence equation (4.41) becomes

$$(t-t_c)^3 \underbrace{-\kappa_1\,(J-J_c)}_{=:p}(t-t_c)\underbrace{-\kappa_2\,(\alpha+2m_c-1)\,(J-J_c)+\mathcal{O}\big((J-J_c)^2\big)+\mathcal{O}\big((t-t_c)^4\big)}_{=:q}=0\,.$$

This third order equation has $\Delta := (\frac{q}{2})^2+(\frac{p}{3})^3>0$ for $|J-J_c|$ small enough, indeed if $J<J_c$ then $p>0$, while if $J>J_c$ then by corollary 4.19 part *1)* $(t-t_c)^4=\mathcal{O}\big((J-J_c)^{\frac{4}{3}}\big)=o(J-J_c)$ hence

$$\begin{aligned} & q=-\kappa_2\,(\alpha+2m_c-1)\,(J-J_c)+o\big(J-J_c\big) \quad\Rightarrow \\ & \Big(\frac{q}{2}\Big)^2+\Big(\frac{p}{3}\Big)^3 \;=\; \frac{\kappa_2^2}{4}\,(\underbrace{\alpha+2m_c-1}_{\neq 0})^2\,(J-J_c)^2\,(1+o(1))-\frac{\kappa_1^3}{27}\,(J-J_c)^3\;>\;0\,. \end{aligned}$$

Then, using Cardano's formula for cubic equations: $t-t_c=u_+ + u_-$ with

$$u_\pm \;=\; \sqrt[3]{-\frac{q}{2}\pm\sqrt[2]{\Big(\frac{q}{2}\Big)^2+\Big(\frac{p}{3}\Big)^3}} \;=\; \sqrt[3]{-\frac{q}{2}\pm\Big|\frac{q}{2}\Big|}\;+\mathcal{O}\big(\Big|\frac{p}{3}\Big|^{\frac{1}{2}}\big)\;;$$

hence

$$\begin{aligned} t-t_c \;&=\; \sqrt[3]{-q}\;+\mathcal{O}\big(\Big|\frac{p}{3}\Big|^{\frac{1}{2}}\big)\;= \\ &=\;\sqrt[3]{\kappa_2\,(\alpha+2m_c-1)}\,(J-J_c)^{\frac{1}{3}}+\mathcal{O}\big((J-J_c)^{\frac{2}{3}}\big)+\mathcal{O}\big((t-t_c)^{\frac{4}{3}}\big)+\mathcal{O}\big((J-J_c)^{\frac{1}{2}}\big)\,. \end{aligned}$$

This entails

$$m-m_c \;=\; \frac{\sqrt[3]{\kappa_2\,(\alpha+2m_c-1)}}{2J}\,(J-J_c)^{\frac{1}{3}}+\mathcal{O}\big((J-J_c)^{\frac{1}{2}}\big)+\mathcal{O}\big((m-m_c)^{\frac{4}{3}}\big)$$

and dividing both sides by $m-m_c$, since $(m-m_c)^{\frac{1}{2}}\to 0$, one finds

$$m-m_c\;\sim\;\frac{\sqrt[3]{\kappa_2\,(\alpha+2m_c-1)}}{2J}\,(J-J_c)^{\frac{1}{3}}\,. \tag{4.54}$$

Now by propositions 4.5, 4.7 and lemma 4.20, μ_1 and μ_2 are solutions of the consistency equation (4.18) defined near (h_c, J_c) along the curves $h = \delta(J)$ respectively with $\alpha \leq 1 - 2m_c$ and $\alpha \geq 1 - 2m_c$. Moreover for $\alpha = 1 - 2m_c$ and $J > J_c$ sufficiently small, by lemma 4.18,

$$\mu_2 - m_c > \phi_2 - m_c > 0 \quad \text{while} \quad \mu_1 - m_c < \phi_1 - m_c < 0 \; .$$

These facts together with (4.52), (4.53), (4.54) allow to conclude the proof. $\square$

Chapter 5

Law of large numbers, central limit theorem and violations

This chapter is based on the joint work [5]. We continue the study of the monomer-dimer model with **hard-core and imitative interactions** on the **complete graph** that has been presented in the chapter 4. While in the chapter 4 the deterministic limit of the average number of monomers is studied, here we study the distributional limits of the number of monomers (with respect to the Gibbs measure). Precisely we prove that a *law of large numbers* holds outside the coexistence curve Γ, where instead the limiting distribution is a convex combination of two Dirac deltas representing the two phases (theorems 5.1, 5.2). Moreover we prove that a *central limit theorem* holds outside $\Gamma \cup (h_c, J_c)$: at the critical point a normalisation by $N^{-3/4}$ is required and the limiting distribution is $Ce^{-cx^4}\mathrm{d}x$ (theorems 5.1, 5.3).

We follow the Gaussian convolution method introduced by Ellis and Newman for the mean-field Ising model (Curie-Weiss) in [35, 36, 37] in order to deal with the imitative potential. Here an additional difficulty stems from the fact that even in the absence of imitation the system keeps an interacting nature: to *decouple* the hard-core interaction we use the Gaussian representation of the partition function (see 2.6). We mention that recently the fluctuations of the

Ising model on random graphs have been studied in [31].

The fundamental quantity is the number of monomers in a given monomer-dimer configuration, hence we set

$$S_N(\alpha) := \sum_{i=1}^{N} \alpha_i \quad \forall \alpha \in \mathscr{D}_N \,. \tag{5.1}$$

The fraction of monomers is $m_N(\alpha) := \frac{1}{N}\, S_N(\alpha)\,.$

Let $h \in \mathbb{R}$ and $J \geq 0$. In this chapter, like in the previous one, we consider the Hamiltonian

$$H_N(\alpha) := \, -N \left((h - J)\, m_N(\alpha) + J\, m_N(\alpha)^2 \right) \tag{5.2}$$

for every monomer-dimer configuration on the complete graph $\alpha \in \mathscr{D}_N$, and the partition function

$$Z_N := \sum_{\alpha \in \mathscr{D}_N} N^{-|D|} \exp(-H_N(\alpha)) \;, \tag{5.3}$$

where $|D| = \sum_{1 \leq i < j \leq N} \alpha_{ij} = \big(N - \sum_{i=1}^{N} \alpha_i\big)/2\,.$ The corresponding Gibbs measure is

$$\mu_N(\alpha) := \frac{N^{-|D|} \exp(-H_N(\alpha))}{Z_N} \quad \forall\, \alpha \in \mathscr{D}_N \tag{5.4}$$

and the expectation with respect to the measure μ_N is denoted by $\mathbb{E}[\,\cdot\,]_{\mu_N}$. The pressure density is denoted by $p_N := \frac{1}{N}\, \log Z_N\,.$

The aim of the present chapter is to describe the limiting distribution of the random variable S_N with respect to the measure μ_N, in a suitable scaling when $N \to \infty\,.$ From now on δ_x is the Dirac measure centered at x, $\mathcal{N}\,(m, \sigma^2)$ denotes the Gaussian distribution with mean m and variance σ^2 and $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution with respect to the Gibbs measure μ_N as $N \to \infty\,.$

First of all consider the case $J = 0$, where the only interaction is the hard-core one. It is convenient to introduce the following notations:

$$Z_N^{(0)} := Z_N\Big|_{J=0} \,, \qquad p_N^{(0)} := p_N\Big|_{J=0} \,, \qquad \mu_N^{(0)} := \mu_N\Big|_{J=0} \,. \tag{5.5}$$

Recall from the proposition 4.2 that for all $h \in \mathbb{R}$

$$\lim_{N\to\infty} p_N^{(0)} = p^{(0)}(h) \tag{5.6}$$

and

$$\lim_{N\to\infty} \mathbb{E}_{\mu_N^{(0)}}[m_N] = \lim_{N\to\infty} \frac{\partial p_N}{\partial h}\frac{\partial p^{(0)}}{\partial h} = g(h) \; , \tag{5.7}$$

where the limiting functions are analytic and defined by

$$p^{(0)}(h) := -\frac{1-g(h)}{2} - \frac{1}{2}\log(1-g(h)) = -\frac{1-g(h)}{2} - \log g(h) + h \tag{5.8}$$

and

$$g(h) := e^h\,\frac{\sqrt{e^{2h}+4}-e^h}{2} \; . \tag{5.9}$$

At $J = 0$ the law of large numbers and the central limit theorem hold true. Precisely

Theorem 5.1. *At $J = 0$ the followings hold:*

$$m_N \xrightarrow{\mathcal{D}} \delta_{g(h)} \tag{5.10}$$

and

$$\frac{S_N - N\,g(h)}{\sqrt{N}} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0,\, \frac{\partial g}{\partial h}(h)\right) \; . \tag{5.11}$$

Notice that, even if $J = 0$, (5.11) is not a consequence of the standard central limit theorem, indeed S_N is not a sum of i.i.d. random variables because of the presence of the hard-core interaction. The theorem 5.1 follows from the recent results of Lebowitz-Pittel-Ruelle-Speer [70]. A different proof is given in the next section.

Now consider the case $J > 0$. Recall from the theorem 4.1 that

$$\lim_{N\to\infty} p_N = \sup_m \widetilde{p}(m) \tag{5.12}$$

where

$$\widetilde{p}(m) := -J\,m^2 + p^{(0)}\big((2m-1)J + h\big) \quad \forall m \in \mathbb{R} \; . \tag{5.13}$$

The points where the function $\widetilde{p}$ reaches its maximum satisfy the following consistency equation:

$$m = g((2m-1)J + h)\,. \tag{5.14}$$

The analysis of (5.14) allows to identify the region where there exists a unique global maximum point $m^*(h,J)$ of $\widetilde{p}$. The function m^* is single-valued and continuous on the plane (h,J) with the exception of an open curve Γ defined by an implicit equation $h = \gamma(J)$. Moreover m^* is smooth outside $\Gamma \cup (h_c, J_c)$. Instead on Γ there are two global maximum points $m_1(J) = m_1(\gamma(J), J)$ and $m_2(J) = m_2(\gamma(J), J):$ choosing $m_1 < m_2$, they correspond to the *dimer* and the *monomer phase* respectively. The curve Γ plays a crucial physical role since it represents the coexistence of two different thermodynamic phases and its endpoint (h_c, J_c) is the critical point of the system. Outside Γ, by differentiating the expression (5.12) with respect to the external field h, one obtains that the maximum point m^* is the limit of the average monomer density m_N with respect to the Gibbs measure:

$$\lim_{N\to\infty} \mathbb{E}_{\mu_N}[m_N] \;=\; \lim_{N\to\infty} \frac{\partial p_N}{\partial h} \;=\; \frac{\mathrm{d}}{\mathrm{d}h}\,\widetilde{p}(m^*) \;=\; m^*(h,J)\,. \tag{5.15}$$

We observe that in the standard mean-field ferromagnetic model (Curie-Weiss model) the existence of the limiting magnetization on the coexistence curve (zero external field) is achieved by a symmetry argument (spin flip), a property that we do not have in the present case.

Consider the asymptotic behaviour of the distribution of the number of monomers S_N with respect to the Gibbs measure μ_N. The *law of large numbers* holds outside the coexistence curve Γ, while on Γ its breakdown results in a convex combination of two Dirac deltas. Precisely

Theorem 5.2. *i) In the uniqueness region* $(h,J) \in \big(\mathbb{R}\times\mathbb{R}_+\big) \setminus \Gamma$, *it holds*

$$m_N \overset{\mathcal{D}}{\to} \delta_{m^*}\,. \tag{5.16}$$

ii) On the coexistence curve $(h, J) \in \Gamma$, *it holds*

$$m_N \overset{\mathcal{D}}{\to} \varrho_1\, \delta_{m_1} + \varrho_2\, \delta_{m_2}\ , \tag{5.17}$$

where $\varrho_l = \frac{b_l}{b_1+b_2}$, $b_l = (-\lambda_l(2-m_l))^{-1/2}$ *and* $\lambda_l = \frac{\partial^2 \widetilde{p}}{\partial m^2}(m_l)$, *for* $l = 1, 2$.

Remark 5.1. We notice that, on the contrary of what happens for the Curie-Weiss model, the statistical weights ϱ_1 and ϱ_2 on the coexistence curve are in general different, furthermore they are not simply given in terms of the second derivative of the variational pressure $\widetilde{p}$.

The first fact can be seen numerically, and analytically one can compute

$$\lim_{J\to\infty} \frac{\varrho_1(J)}{\varrho_2(J)} = \frac{1}{\sqrt{2}}\ . \tag{5.18}$$

Indeed, by exploiting the formula $(p^{(0)})'' = g' = 2g(1-g)/(2-g)$ (see Appendix 4.3), the ratio ϱ_1/ϱ_2 rewrites as

$$\frac{\varrho_1}{\varrho_2} = \sqrt{\frac{(2-m_2) - 4J\, m_2\,(1-m_2)}{(2-m_1) - 4J\, m_1\,(1-m_1)}}\ . \tag{5.19}$$

The second fact can be interpreted as follows: the relative weights ϱ_l have two contributions reflecting the presence of two different kinds of interaction. The first contribution λ_l is given by the second derivative of the variational pressure (5.13), while the second contribution $2 - m_l$ comes from the second derivative of the pressure of the pure hard-core model.

The central limit theorem holds outside the coexistence curve Γ and the critical point (h_c, J_c). At the critical point its breakdown results in a different scaling $N^{3/4}$ and in a different limiting distribution $Ce^{-cx^4}\mathrm{d}x$. Precisely

Theorem 5.3. *i) Outside the coexistence curve and the critical point* $(h, J) \in \big(\mathbb{R}\times\mathbb{R}^+\big) \setminus \big(\Gamma \cup (h_c, J_c)\big)$, *it holds*

$$\frac{S_N - Nm^*}{N^{1/2}} \overset{\mathcal{D}}{\to} \mathcal{N}\Big(0, \sigma^2\Big) \tag{5.20}$$

where $\sigma^2 = -\lambda^{-1} - (2J)^{-1} > 0$ *and* $\lambda = \frac{\partial^2 \widetilde{p}}{\partial m^2}(m^*) < 0$.

ii) At the critical point (h_c, J_c)*, it holds*

$$\frac{S_N - Nm_c}{N^{3/4}} \xrightarrow{\mathcal{D}} C\,\exp\left(\frac{\lambda_c}{24}\,x^4\right)\mathrm{d}x \tag{5.21}$$

where $\lambda_c = \frac{\partial^4 \tilde{p}}{\partial m^4}(m_c) < 0$*,* $m_c \equiv m^*(h_c, J_c)$ *and* $C^{-1} = \int_{\mathbb{R}} \exp(\frac{\lambda_c}{24}x^4)\mathrm{d}x$ *.*

5.1 Hard-core interaction on the complete graph

A basic ingredient of all the proofs is the knowledge of the properties of the moment generating function of S_N with respect to the Gibbs measure at $J = 0$. However, compared with spin models, monomer-dimer models have an additional feature: the problem at $J = 0$ is itself non trivial in the sense that the Gibbs measure is not a product measure. We start by deriving the properties of the partition function of the model at $J = 0$ that will be used during all the proofs.

For given $u, t \in \mathbb{R}$ and $\eta \geq 0$, let us consider

$$Z_N^{(0)}\big(u + \frac{t}{N^\eta}\big) \;=\; \sum_{D \in \mathscr{D}_N} N^{-|D|} \exp\Big(\big(u + \frac{t}{N^\eta}\big)\, S_N(D)\Big) \tag{5.22}$$

In order to obtain an asymptotic expansion of (5.22), which allows us to obtain its scaling properties, we will use a connection between the monomer-dimer problem and Gaussian moments. The Gaussian representation of the partition function on the complete graph (see eq. (3.3)) in this case gives:

Proposition 5.2. *The following representation of the partition function holds*

$$Z_N^{(0)}\big(u + \frac{t}{N^\eta}\big) \;=\; \sqrt{\frac{N}{2\pi}} \int_{\mathbb{R}} \Big(\Psi_N(x)\Big)^N \mathrm{d}x\;, \tag{5.23}$$

where

$$\Psi_N(x) \;:=\; \big(x + e^{u + \frac{t}{N^\eta}}\big)\, e^{-\frac{x^2}{2}}\;. \tag{5.24}$$

The above integral representation allows to use an extension of the Laplace method (see theorem 5.4 in the Appendix), to obtain a useful asymptotic expansion of $Z_N^{(0)}(u + \frac{t}{N^\eta})$. Precisely

Proposition 5.3. *For a given $u, t \in \mathbb{R}$ and $\eta \geq 0$*

$$Z_N^{(0)}(u+\frac{t}{N^\eta}) \equiv \exp\Big(N\, p_N^{(0)}(u+\frac{t}{N^\eta})\Big) \underset{N\to\infty}{\sim} \exp\Big(N\, p^{(0)}(u+\frac{t}{N^\eta})\Big)\sqrt{\frac{1}{2-g(u)}} \tag{5.25}$$

where $p^{(0)}$ and g are defined respectively in (5.8) *and* (5.9)*.*

Proof. Use proposition 5.2 and check that the function Ψ_N defined in (5.24) satisfies the hypothesis of Theorem 5.4, with $\hat{x}_N = e^{-(u+t/N^\eta)}g(u+t/N^\eta)$. By means of the stationarity condition $\hat{x}_N^2 + e^{u+t/N^\eta}\hat{x}_N - 1 = 0$, one finds $\log\Psi_N(\hat{x}_N) = p^{(0)}(u+t/N^\eta)$ and $\frac{\partial^2}{\partial x^2}\log\Psi_N(\hat{x}_N) = -2 + g(u+t/N^\eta)$. □

We will show that the previous proposition gives immediately Theorem 5.1. On other hand, in the case $J > 0$ we need additional information about the convergence of $p_N^{(0)}$ to $p^{(0)}$.

Proposition 5.4. *For each $k \in \{0,1,2,\ldots\}$, $\frac{\partial^k}{\partial h^k}p_N^{(0)}(h)$ converges uniformly to $\frac{\partial^k}{\partial h^k}p^{(0)}(h)$ on compact subsets of $\mathbb{R}$.*

Proof. The location of the complex zeros $h \in \mathbb{C}$ of the partition function $Z_N^{(0)}(h)$ was described in the work of Heilmann and Lieb in [55]: the theorem 2.8 shows that these zeros satisfy $\Re(e^h) = 0$, that is $\Im(h) \in \frac{\pi}{2} + \pi\mathbb{Z}$. Set $U := \mathbb{R} + i\left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \subset \mathbb{C}$. The analytic function $Z_N^{(0)}(h)$ does not vanish on the simply connected open set U, hence $p_N^{(0)}(h) \equiv \frac{1}{N}\log Z_N^{(0)}(h)$ is a well-defined analytic function on U. Moreover the sequence $\left(p_N^{(0)}(h)\right)_{N\in\mathbb{N}}$ is bounded uniformly in N and in $h \in K$, for every K compact subset of U; indeed

$$\left|p_N^{(0)}(h)\right| \leq \frac{1}{N}\left|\log\left|Z_N^{(0)}(h)\right|\right| + \frac{2\pi}{N},$$

from the definition of $Z_N^{(0)}$ it follows immediately

$$\frac{1}{N}\log\left|Z_N^{(0)}(h)\right| \leq \frac{1}{N}\log Z_N^{(0)}\Big(\sup_{h\in K}\Re(h)\Big),$$

and on the other hand, since $Z_N^{(0)}$ is a polynomial in the variable e^h, using the Fundamental Theorem of Algebra and thank to the choice of U, it follows

$$\frac{1}{N}\log\left|Z_N^{(0)}(h)\right| \geq \inf_{h\in K} e^{\Re(h)}\,\frac{\sqrt{2}}{2}.$$

Thus, the claim is a consequence of the Vitali-Porter and Weiestrass Theorems [91]. □

Proof of the Theorem 5.1. For each $u \in \mathbb{R}$ and $\eta \geq 0$ we define

$$S_{N,\eta,u} := \frac{S_N - u}{N^\eta}. \tag{5.26}$$

In order to prove the two statements of the Theorem 5.1, namely the law of large numbers (5.10) and the central limit theorem (5.11), it is enough to compute the limit of the moment generating function of $S_{N,\eta,u}$ for $\eta = 1, u = 0$ and for $\eta = \frac{1}{2}, u = g(h)$ respectively.

Consider the moment generating function of $S_{N,\eta,u}$ with respect to the Gibbs measure $\mu_N^{(0)}$ with external field h, namely for all $t \in \mathbb{R}$

$$\phi_{S_{N,\eta,u}}(t) := \sum_{D \in \mathscr{D}_N} \mu_N^{(0)}(D)\, e^{t\, S_{N,\eta,u}(D)} . \tag{5.27}$$

By (5.22) one can rewrite (5.27) as

$$\phi_{S_{N,\eta,u}}(t) = e^{-tu/N^\eta} \, \frac{Z_N^{(0)}(h + \frac{t}{N^\eta})}{Z_N^{(0)}(h)} . \tag{5.28}$$

Using proposition 5.3 for the numerator and the denominator of (5.28) one gets

$$\frac{Z_N^{(0)}(h + \frac{t}{N^\eta})}{Z_N^{(0)}(h)} \underset{N\to\infty}{\sim} \exp N\Big(p^{(0)}\big(h + \frac{t}{N^\eta}\big) - p^{(0)}(h)\Big) \tag{5.29}$$

Setting $\eta = 1$ and $u = 0$ and using the Taylor expansion $p^{(0)}(h + \frac{t}{N}) - p^{(0)}(h) = \frac{t}{N}\frac{\partial}{\partial h}p^{(0)}(h) + O(N^{-2})$ and $\frac{\partial}{\partial h}p^{(0)} = g$, we obtain

$$\lim_{N\to\infty} \phi_{S_{N,1,0}}(t) = e^{t\, g(h)} \quad \forall\, t \in \mathbb{R} \tag{5.30}$$

which implies (5.10).

In the case of the central limit theorem, setting $\eta = \frac{1}{2}$ and $u = g(h)$, the leading order is provided by the Taylor expansion of $p^{(0)}(h + \frac{t}{\sqrt{N}})$ up to the second order

$$p^{(0)}\big(h + \frac{t}{\sqrt{N}}\big) = p^{(0)}(h) + \frac{t}{\sqrt{N}}\frac{\partial}{\partial h}p^{(0)}(h) + \frac{t^2}{2N}\frac{\partial^2}{\partial h^2}p^{(0)}(h) + O(N^{-\frac{3}{2}}) ,$$

and then we obtain

$$\lim_{N\to\infty} \phi_{S_{N,\frac{1}{2},g(h)}}(t) = e^{\frac{t^2}{2}\frac{\partial}{\partial h}g(h)} \quad \forall\, t \in \mathbb{R} \tag{5.31}$$

which implies (5.11) and completes the proof. □

5.2 Hard-core and imitative interactions on the complete graph

The strategy in the case $J > 0$ follows the general method of Ellis and Newman [35], namely, in order to overcome the obstacle of the quadratic term in the interaction, we consider the convolution of the Gibbs measure μ_N with a suitable Gaussian random variable. Let us start by two simple lemmas.

Lemma 5.5. *For all integer N, let W_N and Y_N be two independent random variables. Assume that $W_N \xrightarrow{\mathcal{D}} W$, where*

$$\mathbb{E}\, e^{itW} \neq 0 \quad \forall t \in \mathbb{R}\,.$$

Then $Y_N \xrightarrow{\mathcal{D}} Y$ if and only if $W_N + Y_N \xrightarrow{\mathcal{D}} W + Y$.

Lemma 5.6. *Let $W \sim \mathcal{N}(0, (2J)^{-1})$ be a random variable independent of S_N for all $N \in \mathbb{N}$. Then given $\eta \geq 0$ and $u \in \mathbb{R}$, the distribution of*

$$\frac{W}{N^{1/2-\eta}} + \frac{S_N - Nu}{N^{1-\eta}} \tag{5.32}$$

is

$$C_N \exp\Big(N\, F_N\Big(\frac{x}{N^\eta} + u\Big)\Big)\, dx\,, \tag{5.33}$$

where $C_N^{-1} = \int_{\mathbb{R}} \exp\big(N\, F_N(\frac{x}{N^\eta} + u)\big) dx$ and

$$F_N(x) := -Jx^2 + p_N^{(0)}(2Jx + h - J)\,. \tag{5.34}$$

Proof. Given $\theta \in \mathbb{R}$,

$$\mathbb{P}\Big\{\frac{W}{N^{1/2-\eta}} + \frac{S_N - Nm}{N^{1-\eta}} \leq \theta\Big\} = \mathbb{P}\Big\{\sqrt{N}W + S_N \in E\Big\} \tag{5.35}$$

where $E = (-\infty, \theta N^{1-\eta} + Nm]$.

The law of $\sqrt{N}W + S_N$ is given by the convolution of the Gaussian $\mathcal{N}(0, N(2J)^{-1})$ with the distribution of S_N w.r.t. the Gibbs measure μ_N :

$$\begin{aligned}
&\mathbb{P}\Big\{\sqrt{N}W + S_N \in E\Big\} = \\
&\left(\frac{J}{\pi N}\right)^{\frac{1}{2}} \int_E dt\; \mathbb{E}_{\mu_N} \exp\left(-\frac{J}{N}\Big(t - S_N\Big)^2\right) = \\
&\frac{1}{Z_N}\left(\frac{J}{\pi N}\right)^{\frac{1}{2}} \int_E dt\; \exp\left(-\frac{J}{N}t^2\right) Z_N^{(0)}\left(\frac{2Jt}{N} + h - J\right),
\end{aligned} \tag{5.36}$$

where the last equality follows from (5.22). Making the change of variable $x = (t - Nu)/N^{1-\eta}$ in (5.36), we obtain:

$$P\Big\{\sqrt{N}W + S_N \in E\Big\} = C_N \int_{-\infty}^{\theta} dx\; \exp\left(-JN\Big(\frac{x}{N^\eta} + u\Big)^2\right) Z_N^{(0)}\left(2J\Big(\frac{x}{N^\eta} + u\Big) + h - J\right) \tag{5.37}$$

and the integrated function can be rewritten as (5.33). □

The core of the problem is the convergence of the sequence of measures determined by (5.33) for suitable values of η and u. Thus, we are interested in the limit of quantities of the form

$$\int_{\mathbb{R}} \exp\left(N\, F_N\Big(\frac{x}{N^\eta} + u\Big)\right) \psi(x)\, dx \tag{5.38}$$

where ψ is an arbitrary bounded continuous function. Clearly, the results depend crucially on the scaling properties of F_N near its maximum point(s). By (5.34), (5.13) and (5.6) we know that

$$\lim_{N\to\infty} F_N(x) = \widetilde{p}(x) \quad \forall\, x \in \mathbb{R}\,. \tag{5.39}$$

However, the study of the asymptotic behaviour of the integral (5.38) requires stronger convergence results provided by propositions 5.3 and 5.4.

Given a sequence of functions $f_N : \mathbb{R} \to \mathbb{R}$, for any $x, y \in \mathbb{R}$ we define

$$\Delta f_N(x; y) := f_N(x + y) - f_N(y). \tag{5.40}$$

Let $\mu \equiv \mu(h,J)$ be a maximum point of $\widetilde{p}$ and denote by $2k$ the order of the first non zero derivative at μ. Hence, making a Taylor expansion, one finds as $N \to \infty$

$$N\, \Delta\widetilde{p}(x\, N^{-\frac{1}{2k}};\mu) \;=\; \frac{\lambda}{(2k)!}\, x^{2k} + O\Big(N^{-\frac{1}{2k}}\Big) \tag{5.41}$$

where $\lambda = \frac{\partial^{2k}}{\partial m^{2k}}\widetilde{p}(\mu) < 0$.

The next proposition relates the asymptotic behaviors of $N\,\Delta F_N$ and $N\,\Delta\widetilde{p}$.

Proposition 5.7. *For any $x, y \in \mathbb{R}$ and $\eta \geq 0$,*

$$\lim_{N\to\infty} \exp\Big(N\,\Big(F_N(x\,N^{-\eta}+y) - \widetilde{p}(x\,N^{-\eta}+y)\Big)\Big) \;=\; c(y) \tag{5.42}$$

where $c(y) := \big(2 - g(2Jy+h-J)\big)^{-1/2}$. Hence,

$$N\Big(\Delta F_N(x\,N^{-\eta};y) - \Delta\widetilde{p}(x\,N^{-\eta};y)\Big) \underset{N\to\infty}{\longrightarrow} 0\,. \tag{5.43}$$

Proof. Keeping in mind the definitions (5.34), (5.13) and using Proposition 5.3 we get (5.42). Then (5.43) is a straightforward consequence. □

The next two propositions allow us to control the integral (5.38) in the limit $N \to \infty$.

Proposition 5.8. *Set $M := \max\{\tilde{p}(x) | x \in \mathbb{R}\}$, let $\mathcal{C}$ be a closed subset of $\mathbb{R}$ which contains no global maximum points of $\tilde{p}$. Then there exists $\varepsilon > 0$ such that*

$$e^{-NM}\int_{\mathcal{C}} e^{NF_N(x)}dx \;=\; O(e^{-N\varepsilon}) \quad \text{as } N\to\infty. \tag{5.44}$$

Proof. Observe that the sequence of functions $(p_N^{(0)})_{N\in\mathbb{N}}$ is uniformly Lipschitz with constant 1, namely for all $h, h' \in \mathbb{R}$ and $N \in \mathbb{N}$

$$|p_N^{(0)}(h) - p_N^{(0)}(h')| \;\leq\; |h-h'|\,, \tag{5.45}$$

since $\frac{\partial}{\partial h}p_N^{(0)} = \mathbb{E}_{\mu_N^{(0)}}(m_N) \in [0,1]$. From (5.45) and definition (5.34) we get

$$\lim_{|x|\to\infty} \sup_N F_N(x) = -\infty \tag{5.46}$$

and

$$\sup_N \int_{\mathbb{R}} e^{F_N(x)} dx < \infty \, . \tag{5.47}$$

Fixed $\varepsilon_1 > 0$, by (5.46) we can pick a number $A \in \mathbb{R}$ sufficiently large such that

$$\sup_{x \in \mathcal{O}_A} F_N(x) - M \leq -\varepsilon_1 \quad \forall\, N \in \mathbb{N} \tag{5.48}$$

where $\mathcal{O}_A \equiv \{x \in \mathbb{R} : |x| > A\}$. Furthermore $\mathcal{C} \setminus \mathcal{O}_A$ is compact (or possibly empty) and then, by proposition 5.4, there exist $\varepsilon_2 > 0$ and $\bar{N}$ such that

$$\sup_{x \in \mathcal{C} \setminus \mathcal{O}_A} F_N(x) - M \leq -\varepsilon_2 \quad \forall\, N > \bar{N} \, . \tag{5.49}$$

Thus setting $\varepsilon := \min(\varepsilon_1, \varepsilon_2)$ we get

$$\sup_{x \in \mathcal{C}} F_N(x) - M \leq -\varepsilon \quad \forall\, N > \bar{N} \tag{5.50}$$

Hence, for $N > \bar{N}$,

$$\begin{aligned} e^{-NM} \int_{\mathcal{C}} e^{N\,F_N(x)} dx \;&\leq\; e^{-NM}\, e^{(N-1)(M-\epsilon)} \int_{\mathcal{C}} e^{F_N(x)} dx \\ &\leq\; e^{-N\varepsilon}\, e^{-(M-\varepsilon)} \int_{\mathbb{R}} e^{F_N(x)} dx \, . \end{aligned} \tag{5.51}$$

The last is uniformly bounded in N by (5.47) and this completes the proof. □

In the rest of this section $\partial^k f(x)$ denotes the k^{th}-derivative of a function f at the point x.

Proposition 5.9. *Let μ be a maximum point of $\widetilde{p}$, let $2k$ be the order of the first non-zero derivative of $\widetilde{p}$ at μ. Given $\delta, \varepsilon > 0$, there exists $\overline{N}_\varepsilon$ such that for all $N > \overline{N}_\varepsilon$*

$$N\, \Delta F_N\big(x\, N^{-\frac{1}{2k}}; \mu\big) \;\leq\; \sum_{j=1}^{2k-1} \varepsilon\, x^j + L_{\delta,\varepsilon}\, x^{2k} \qquad \forall\, x,\ |x| < \delta N^{\frac{1}{2k}} \tag{5.52}$$

where

$$L_{\delta,\varepsilon} \;:=\; \frac{\partial^{2k}\widetilde{p}(\mu) + \varepsilon}{(2k)!} \;+\; \delta\, \frac{\sup_{[\mu-\delta,\mu+\delta]} |\partial^{2k+1}\widetilde{p}| + \varepsilon}{(2k+1)!} \, . \tag{5.53}$$

In particular, since $\partial^{2k}\widetilde{p}(\mu) < 0$, one can choose $\delta, \varepsilon > 0$ such that $L_{\delta,\varepsilon} < 0$, and then the sequence of functions

$$\exp\left(N\,\Delta F_N(x\,N^{-\frac{1}{2k}};\mu)\right)\,\mathbb{1}\left(|x| < \delta N^{\frac{1}{2k}}\right) \tag{5.54}$$

turns out to be dominated by an integrable function of x .

Proof. The Taylor expansion of F_N at the point μ gives

$$N\,\Delta F_N(x\,N^{-\frac{1}{2k}};\mu) \;=\; \sum_{j=1}^{2k-1} \frac{\partial^j F_N(\mu)}{j!}\,N^{1-j/2k}\,x^j + \frac{\partial^{2k} F_N(\mu)}{(2k)!}\,x^{2k} + \frac{\partial^{2k+1} F_N(\xi)}{(2k+1)!}\,N^{-\frac{1}{2k}}\,x^{2k+1} \tag{5.55}$$

where $\xi \in (\mu, \mu + x\,N^{-\frac{1}{2k}})$. We claim that for any $j \in \{1, \dots, 2k-1\}$

$$\partial^j F_N(\mu)\,N^{1-j/2k} \underset{N\to\infty}{\longrightarrow} 0\,. \tag{5.56}$$

Indeed, by (5.43)

$$N\left(\Delta F_N(x\,N^{-\frac{1}{2k}};\mu) - \Delta\widetilde{p}(x\,N^{-\frac{1}{2k}};\mu)\right) \underset{N\to\infty}{\longrightarrow} 0\,, \tag{5.57}$$

that is, by substituting (5.55) and (5.41) into (5.57),

$$\sum_{j=1}^{2k-1} \frac{\partial^j F_N(\mu)}{j!}\,N^{1-j/2k}\,x^j \;+\; \frac{\partial^{2k} F_N(\mu) - \partial^{2k}\widetilde{p}(\mu)}{(2k)!}\,x^{2k} \;+\; O\left(N^{-\frac{1}{2k}}\right) \underset{N\to\infty}{\longrightarrow} 0\,, \tag{5.58}$$

hence using proposition 5.4, we get

$$\sum_{j=1}^{2k-1} \frac{\partial^j F_N(\mu)}{j!}\,N^{1-j/2k}\,x^j \underset{N\to\infty}{\longrightarrow} 0 \tag{5.59}$$

which implies (5.56) since x is arbitrary. Thus (5.56) gives the control of the terms of order up to $2k-1$ in (5.55). The last two terms in (5.55) can be grouped together observing that $|x|^{2k+1} < x^{2k}\delta N^{\frac{1}{2k}}$; then the estimate (5.52) is obtained using the uniform convergence of $\partial^{2k} F_N$, $\partial^{2k+1} F_N$ on the compact set $[\mu-\delta, \mu+\delta]$, which is guaranteed by proposition 5.4. □

Proof of the Theorem 5.2. Denote by $\mathcal{M} = \{\mu_l\}_{l=1,\dots,P}$ the set global maximum points of $\widetilde{p}$ and let k_l and λ_l be as in (5.41). Set $M := \max_m \widetilde{p}(m) = \widetilde{p}(\mu_l)$ for each $l = 1,\dots,P$. From the analysis of $\widetilde{p}$ and using the properties of the function g, it turns out that k_l do not depend on l and precisely

$$(\mathcal{M}, k) = \begin{cases} \big(\{m^*(h,J)\}, 1\big) & \text{if } (h,J) \in (\mathbb{R}\times\mathbb{R}^+) \setminus \big(\gamma \cup (h_c, J_c)\big) \\ \big(\{m_c\}, 2\big) & \text{if } (h,J) = (h_c, J_c) \\ \big(\{m_1(J), m_2(J)\}, 1\big) & \text{if } (h,J) \in \gamma\,. \end{cases} \tag{5.60}$$

The argument described below applies in all the cases proving respectively (5.16) and (5.17). Keeping in mind (5.60), we proceed with the computation of the limiting distribution of the monomer density $m_N = S_N/N$. By lemmas 5.5 and 5.6 with $\eta = 0$ and $u = 0$, it suffices to prove that for any bounded continuous function ψ

$$\frac{\displaystyle\int_{\mathbb{R}} e^{N F_N(x)} \psi(x) dx}{\displaystyle\int_{\mathbb{R}} e^{N F_N(x)} dx} \to \frac{\sum_{l=1}^{P} \psi(\mu_l) b_l}{\sum_{l=1}^{P} b_l}\,. \tag{5.61}$$

For each $l = 1,\dots,P$ let $\delta_l > 0$ such that the sequence of functions (5.54), with μ_l in place of μ, is dominated by an integrable function. We choose $\bar{\delta} = \min\{\delta_l \mid l = 1,\dots,P\}$, decreasing it (if necessary) to assure that $0 < \bar{\delta} < \min\{|\mu_l - \mu_s| : 1 \le l \ne s \le P\}$. Denote by $\mathcal{C}$ the closed set

$$\mathcal{C} := \mathbb{R} \setminus \bigcup_{l=1}^{P} (\mu_l - \bar{\delta}, \mu_l + \bar{\delta})\,;$$

by proposition 5.8 there exists $\varepsilon > 0$ such that as $N \to \infty$

$$e^{-NM} \int_{\mathcal{C}} e^{N F_N(x)} \psi(x) dx \;=\; O(e^{-N\varepsilon})\,. \tag{5.62}$$

For each $l = 1, \ldots, P$ we have

$$N^{\frac{1}{2k}}\, e^{-NM} \int_{\mu_l-\bar{\delta}}^{\mu_l+\bar{\delta}} e^{N\,F_N(x)}\psi(x)\,dx \;= \\ = e^{N(F_N(\mu_l)-M)} \int_{|w|<\bar{\delta}N^{\frac{1}{2k}}} \exp\Big(N\,\Delta F_N\big(wN^{-\frac{1}{2k}};\mu_l\big)\Big)\,\psi\big(wN^{-\frac{1}{2k}}+\mu_l\big)\,dw \tag{5.63}$$

where the equality follows from the change of variable $x = \mu_l + wN^{-\frac{1}{2k}}$ and ΔF_N is defined in (5.40).

Since $M \equiv \widetilde{p}(\mu_l)$, from (5.42) we know that

$$\lim_{N\to\infty} e^{N\,(F_N(\mu_l)-M)} = \frac{1}{\sqrt{2-g(2J\mu_l+h-J)}} = \frac{1}{\sqrt{2-\mu_l}} \tag{5.64}$$

where the last equality follows from the fact that μ_l must satisfy the consistency equation (5.14).

By Proposition 5.9 we can apply the dominated convergence theorem to the integral on the r.h.s. of (5.63), then by (5.43) and (5.41) we obtain

$$\lim_{N\to\infty} N^{\frac{1}{2k}}\, e^{-NM} \int_{\mu_l-\bar{\delta}}^{\mu_l+\bar{\delta}} e^{N\,F_N(x)}\,\psi(x)\,dx \;= \\ = \frac{1}{\sqrt{2-\mu_l}} \int_{\mathbb{R}} \exp\Big(\frac{\lambda_l}{(2k)!}w^{2k}\Big)\,\psi(\mu_l)\,dw\;. \tag{5.65}$$

Making the change of variable $x = w(-\lambda_l)^{\frac{1}{2k}}$ in the r.h.s. of (5.65) and using (5.62) we obtain

$$\lim_{N\to\infty} N^{\frac{1}{2k}}\, e^{-NM} \int_{\mathbb{R}} e^{N\,F_N(x)}\psi(x)dx \;=\; \sum_{l=1}^{P} \frac{1}{\sqrt{2-\mu_l}}\,(-\lambda_l)^{-\frac{1}{2k}}\,\psi(\mu_l) \int_{\mathbb{R}} \exp\Big(-\frac{x^{2k}}{(2k)!}\Big)\,dx\;. \tag{5.66}$$

The analogous limit for the denominator of (5.61) follows from (5.66) by choosing $\psi = 1$. This concludes the proof of the Theorem 5.2. □

Proof of the Theorem 5.3. Keeping in mind (5.60), let us start by proving the following

$$\frac{\int_{\mathbb{R}} \exp\Big(N\,F_N\big(xN^{-\frac{1}{2k}}+m^*\big)\Big)\,\psi(x)\,dx}{\int_{\mathbb{R}} \exp\Big(N\,F_N\big(xN^{-\frac{1}{2k}}+m^*\big)\Big)\,dx} \;\to\; \frac{\int_{\mathbb{R}} \exp\Big(\frac{\lambda}{(2k)!}\,x^{2k}\Big)\,\psi(x)\,dx}{\int_{\mathbb{R}} \exp\Big(\frac{\lambda}{(2k)!}\,x^{2k}\Big)\,dx} \tag{5.67}$$

for any bounded continuous function ψ. We pick $\delta > 0$ such that the sequence of functions (5.54) is dominated by a integrable function. By proposition 5.8 there exists $\varepsilon > 0$ such that as $N \to \infty$

$$e^{-NM} \int_{|x| \geq \delta N^{\frac{1}{2k}}} \exp\Big(N\, F_N\big(xN^{-\frac{1}{2k}} + m^*\big)\Big)\, \psi(x)\, dx \;=\; O\big(N^{\frac{1}{2k}}\, e^{-N\varepsilon}\big) \tag{5.68}$$

where $M = \max_m \widetilde{p}(m)$. On the other hand as $|x| < \delta N^{1/2k}$

$$\begin{aligned} &e^{-NM} \int_{|x| < \delta N^{\frac{1}{2k}}} \exp\Big(N\, F_N\big(xN^{-\frac{1}{2k}} + m^*\big)\Big)\, \psi(x)\, dx \;= \\ &= e^{(F_N(m^*) - M)} \int_{|x| < \delta N^{\frac{1}{2k}}} \exp\Big(N\, \Delta F_N\big(xN^{-\frac{1}{2k}}; m^*\big)\Big)\, \psi(x)\, dx\;. \end{aligned} \tag{5.69}$$

Thus, by proposition 5.9 we can apply the dominated convergence theorem, and then by (5.64), (5.43) and (5.41) we obtain

$$\begin{aligned} &\lim_{N\to\infty} e^{-NM} \int_{|x| < \delta N^{\frac{1}{2k}}} \exp\Big(N\, F_N\big(xN^{-\frac{1}{2k}} + m^*\big)\Big)\, \psi(x)\, dx \;= \\ &\qquad = \frac{1}{\sqrt{2 - m^*}} \int_{\mathbb{R}} \exp\Big(\frac{\lambda}{(2k)!}\, x^{2k}\Big)\, \psi(x)\, dx \end{aligned} \tag{5.70}$$

which, combined with (5.68), implies (5.67).

For $k = 2$, by lemmas 5.5 and 5.6 with $\eta = 1/4$ and $u = m^*$, the convergence (5.67) is enough to obtain (5.21).

For $k = 1$, by lemmas 5.5 and 5.6 with $\eta = 1/2$ and $u = m^*$, since $W \sim \mathcal{N}(0, (2J)^{-1})$, the equation (5.67) implies that the random variable S_N converges to a Gaussian whose variance is $\sigma^2 = (-\lambda)^{-1} - (2J)^{-1}$, provided that

$$(-\lambda)^{-1} - (2J)^{-1} = \frac{\lambda + 2J}{-2\lambda J} > 0 \tag{5.71}$$

where $\lambda = \frac{\partial^2}{\partial m}\widetilde{p}(m^*)$. Considering the function g defined in (5.9), we have that $\frac{\partial^2}{\partial m}\widetilde{p}(m^*) + 2J = (2J)^2\, g'(2Jm^* + h - J)$. Since $g' > 0$ and $\lambda < 0$ the inequality (5.71) holds true. □

5.3 Appendix

The usual Laplace method [25] deals with integrals of the form

$$\int_{\mathbb{R}} \big(\psi(x)\big)^n\, dx$$

as n goes to infinity. In this appendix we prove a slight extension of the previous method where ψ can depend on n.

Theorem 5.4. *For all $n \in \mathbb{N}$ let $\psi_n : \mathbb{R} \to \overline{\mathbb{R}}$. Suppose there exists a compact interval $K \subset \mathbb{R}$ such that $\psi_n > 0$ on K, so that*

$$\psi_n(x) \;=\; e^{f_n(x)} \quad \forall\, x \in K \,.$$

Suppose that $f_n \in C^2(K)$ and

a) $f_n \underset{n\to\infty}{\to} f$ *uniformly on K ;*

b) $f_n'' \underset{n\to\infty}{\to} f''$ *uniformly on K .*

Moreover suppose that:

1) $\max_K f_n$ *is attained in a point $\hat{x}_n \in int(K)$;*

2) $\limsup_{n\to\infty} \big(\sup_{\mathbb{R}\setminus K} \log|\psi_n| - \max_K f_n\big) \;<\; 0$ *;*

3) $\max_K f$ *is attained in a unique point $\hat{x} \in K$;*

4) $f''(\hat{x}) < 0$ *;*

5) $\limsup_{n\to\infty} \int_{\mathbb{R}} |\psi_n(x)|\,\mathrm{d}x \;<\; \infty$ *.*

Then,

$$\int_{\mathbb{R}} \big(\psi_n(x)\big)^n\,\mathrm{d}x \underset{n\to\infty}{\sim} e^{n f_n(\hat{x}_n)} \sqrt{\frac{2\pi}{-n\, f''(\hat{x})}} \;. \tag{5.72}$$

In the proof we use the following elementary fact:

Lemma 5.10. *Let $(f_n)_n$ be a sequence of continuous functions uniformly convergent to f on a compact set K. Let $(I_n)_n$ and I be subsets of K such that $\max_{x\in I_n,\, y\in I} \mathrm{dist}(x,y) \to 0$ as $n \to \infty$. Then*

- $\max_{I_n} f_n \underset{n\to\infty}{\to} \max_I f$ *;*
- $\arg\max_{I_n} f_n \underset{n\to\infty}{\to} \arg\max_I f$ *, provided that f has a unique global maximum point on I.*

Proof of the Theorem 5.4. Since $\hat{x}_n$ is an internal maximum point for f_n *(hypothesis 1)*, $f_n'(\hat{x}_n) = 0$ and for all $x \in K$

$$f_n(x) \;=\; f_n(\hat{x}_n) + \frac{1}{2} f_n''(\xi_{x,n})\,(x-\hat{x}_n)^2 \quad \text{with } \xi_{x,n} \in (\hat{x}_n, x) \subset K \;. \tag{5.73}$$

Fix $\varepsilon > 0$. Since $f_n'' \underset{n\to\infty}{\longrightarrow} f''$ uniformly on K, there exists N_ε such that

$$|f_n''(\xi) - f''(\xi)| \;<\; \varepsilon \quad \forall\, \xi \in K \quad \forall\, n > N_\varepsilon \;. \tag{5.74}$$

Since f'' is continuous in $\hat{x}$, there exists $\delta_\varepsilon > 0$ such that $B(\hat{x}, \delta_\varepsilon) \subset K$ and

$$|f''(\xi) - f''(\hat{x})| \;<\; \varepsilon \quad \forall\, \xi :\, |\xi - \hat{x}| < \delta_\varepsilon \;. \tag{5.75}$$

By the lemma 5.10 $\hat{x}_n \underset{n\to\infty}{\longrightarrow} \hat{x}$, because $\hat{x}$ is the unique maximum point of f on K *(hypothesis 3)*. Thus there exists $\bar{N}_{\delta_\varepsilon}$ such that

$$|\hat{x}_n - \hat{x}| \;<\; \frac{\delta_\varepsilon}{2} \quad \forall\, N > \bar{N}_{\delta_\varepsilon} \;. \tag{5.76}$$

Therefore for $n > N_\varepsilon \vee \bar{N}_{\delta_\varepsilon}$ and $x \in B(\hat{x}, \delta_\varepsilon)$ it holds:

$$|\xi_{x,n} - \hat{x}| \;\le\; |\xi_{x,n} - x| + |x - \hat{x}| \;\le\; |\hat{x}_n - x| + |x - \hat{x}| \;\overset{(5.76)}{<}\; \frac{\delta_\varepsilon}{2} + \frac{\delta_\varepsilon}{2} = \delta_\varepsilon \;\Rightarrow$$

$$|f_n''(\xi_{x,n}) - f''(\hat{x})| \;\le\; |f_n''(\xi_{x,n}) - f''(\xi_{x,n})| + |f''(\xi_{x,n}) - f''(\hat{x})| \;\overset{(5.74),(5.75)}{<}\; \varepsilon + \varepsilon = 2\varepsilon \;.$$

By substituting into (5.73) we obtain that for $n > N_\varepsilon \vee \bar{N}_{\delta_\varepsilon}$ and $x \in B(\hat{x}, \delta_\varepsilon)$

$$f_n(x) \begin{cases} \le\; f_n(\hat{x}_n) + \frac{1}{2}\big(f''(\hat{x}) + 2\varepsilon\big)\,(x - \hat{x}_n)^2 \\[2ex] \ge\; f_n(\hat{x}_n) + \frac{1}{2}\big(f''(\hat{x}) - 2\varepsilon\big)\,(x - \hat{x}_n)^2 \end{cases} . \tag{5.77}$$

Now split the integral into two parts:

$$\int_{\mathbb{R}} \big(\psi_n(x)\big)^n \,\mathrm{d}x \;=\; \int_{B(\hat{x}_n,\delta_\varepsilon)} e^{n f_n(\hat{x}_n)}\,\mathrm{d}x \;+\; \int_{\mathbb{R}\setminus B(\hat{x}_n,\delta_\varepsilon)} \big(\psi_n(x)\big)^n \,\mathrm{d}x \;. \tag{5.78}$$

• To control the second integral on the r.h.s. of (5.78) we claim that there exists $\eta_{\delta_\varepsilon} > 0$ and $N^*_{\delta_\varepsilon}$ such that

$$\log|\psi_n(x)| < f_n(\hat{x}_n) - \eta_{\delta_\varepsilon} \quad \forall\, x \in \mathbb{R} \setminus B(\hat{x}_n, \delta_\varepsilon) \quad \forall\, N > N^*_{\delta_\varepsilon} \;; \tag{5.79}$$

namely $\limsup_{n\to\infty} \sup_{x\in\mathbb{R}\setminus B(\hat{x}_n,\delta_\varepsilon)} \log|\psi_n(x)| - f_n(\hat{x}_n) < 0$. Indeed:

$$\limsup_{n\to\infty} \sup_{x\in\mathbb{R}\setminus B(\hat{x}_n,\delta_\varepsilon)} \log|\psi_n(x)| - f_n(\hat{x}_n) =$$
$$\left(\limsup_{n\to\infty} \sup_{x\in K\setminus B(\hat{x}_n,\delta_\varepsilon)} f_n(x) - f_n(\hat{x}_n)\right) \vee \left(\limsup_{n\to\infty} \sup_{x\in\mathbb{R}\setminus K} \log|\psi_n(x)| - f_n(\hat{x}_n)\right) =$$
$$\left(\sup_{x\in K\setminus B(\hat{x},\delta_\varepsilon)} f(x) - f(\hat{x})\right) \vee \left(\limsup_{n\to\infty} \sup_{x\in\mathbb{R}\setminus K} \log|\psi_n(x)| - f_n(\hat{x}_n)\right)$$

where the last identity holds true by the lemma 5.10. Moreover $\sup_{x\in K\setminus B(\hat{x},\delta_\varepsilon)} f(x) - f(\hat{x}) < 0$ since $\hat{x}$ is the unique maximum point of the continuous function f on the compact set K *(hypothesis 3)*; while $\limsup_{n\to\infty} \sup_{x\in\mathbb{R}\setminus K} \log|\psi_n(x)| - f_n(\hat{x}_n) < 0$ by the *hypothesis 2*. This proves the claim.

Now using (5.79) and the *hypothesis 5*, there exist C and N such that for all $n > N \vee N^*_{\delta_\varepsilon}$

$$\begin{aligned} \int_{\mathbb{R}\setminus B(\hat{x}_n,\delta_\varepsilon)} \left|\psi_n(x)\right|^n \mathrm{d}x \;&\leq\; e^{(n-1)(f_n(\hat{x}_n)-\eta_{\delta_\varepsilon})} \int_{\mathbb{R}} |\psi_n(x)|\, \mathrm{d}x \\ &\leq\; C\, e^{n(f_n(\hat{x}_n)-\eta_{\delta_\varepsilon})}\,. \end{aligned} \tag{5.80}$$

• To study the first integral on the r.h.s. of (5.78), choose $\varepsilon \in (0, \varepsilon_0]$, where $f''(\hat{x}) + 2\varepsilon_0 < 0$ *(hypothesis 4)*. By (5.77), since we can compute Gaussian integrals, we find an upper bound:

$$\begin{aligned} \int_{B(\hat{x}_n,\delta_\varepsilon)} e^{nf_n(x)}\, \mathrm{d}x \;&\leq\; e^{nf_n(\hat{x}_n)} \int_{\mathbb{R}} e^{\frac{n}{2}\,(f''(\hat{x})+2\varepsilon)\,(x-\hat{x}_n)^2}\, \mathrm{d}x \\ &=\; e^{nf_n(\hat{x}_n)}\, \frac{1}{\sqrt{-\frac{n}{2}\,(f''(\hat{x})+2\varepsilon)}} \int_{\mathbb{R}} e^{-x^2}\, \mathrm{d}x \\ &=\; e^{nf_n(\hat{x}_n)} \sqrt{\frac{2\pi}{-n\,(f''(\hat{x})+2\varepsilon)}} \end{aligned} \tag{5.81}$$

and a lower bound:

$$
\begin{aligned}
\int_{B(\hat{x}_n,\delta_\varepsilon)} e^{n f_n(x)}\,\mathrm{d}x \;&\geq\; e^{n f_n(\hat{x}_n)} \int_{B(\hat{x}_n,\delta_\varepsilon)} e^{\frac{n}{2}\,(f''(\hat{x})+2\varepsilon)\,(x-\hat{x}_n)^2}\,\mathrm{d}x \\
&=\; e^{n f_n(\hat{x}_n)} \,\frac{1}{\sqrt{-\frac{n}{2}\,(f''(\hat{x})+2\varepsilon)}}\,\int_{B\left(0,\,\delta_\varepsilon\sqrt{-\frac{n}{2}(f''(\hat{x})-2\varepsilon)}\right)} e^{-x^2}\,\mathrm{d}x \\
&=\; e^{n f_n(\hat{x}_n)} \sqrt{\frac{2\pi}{-n\,(f''(\hat{x})-2\varepsilon)}}\,(1+\omega_{n,\varepsilon,\delta_\varepsilon})
\end{aligned}
\tag{5.82}
$$

where $\omega_{n,\varepsilon,\delta_\varepsilon} \to 0$ as $n\to\infty$ and ε is fixed.

In conclusion, by (5.78), (5.80), (5.81), (5.82) we obtain that for $\varepsilon \in (0,\varepsilon_0]$ and $n > N_\varepsilon \vee \bar{N}_{\delta_\varepsilon} \vee N \vee N^*_{\delta_\varepsilon}$ it holds:

$$
\begin{aligned}
\frac{\int_{\mathbb{R}} \big(\psi_n(x)\big)^n\,\mathrm{d}x}{e^{n f_n(\hat{x}_n)}\sqrt{\frac{2\pi}{-n\,f''(\hat{x})}}} \;&\leq\; \sqrt{\frac{f''(\hat{x})}{f''(\hat{x})+2\varepsilon}} \,+\, C\,\sqrt{-\frac{n\,f''(\hat{x})}{2\pi}}\,e^{-n\,\eta_{\delta_\varepsilon}} \\
&\underset{n\to\infty}{\longrightarrow}\; \sqrt{\frac{f''(\hat{x})}{f''(\hat{x})+2\varepsilon}} \underset{\varepsilon\to 0}{\longrightarrow}\; 1\;;
\end{aligned}
$$

and:

$$
\begin{aligned}
\frac{\int_{\mathbb{R}} \big(\psi_n(x)\big)^n\,\mathrm{d}x}{e^{n f_n(\hat{x}_n)}\sqrt{\frac{2\pi}{-n\,f''(\hat{x})}}} \;&\geq\; \sqrt{\frac{f''(\hat{x})}{f''(\hat{x})-2\varepsilon}}\,(1+\omega_{n,\varepsilon,\delta_\varepsilon}) \,-\, C\,\sqrt{-\frac{n\,f''(\hat{x})}{2\pi}}\,e^{-n\,\eta_{\delta_\varepsilon}} \\
&\underset{n\to\infty}{\longrightarrow}\; \sqrt{\frac{f''(\hat{x})}{f''(\hat{x})-2\varepsilon}} \underset{\varepsilon\to 0}{\longrightarrow}\; 1\;;
\end{aligned}
$$

hence (5.72) is proved. □

Chapter 6

Hard-core interaction on locally tree-like random graphs

This chapter is based on the work [4]. We consider monomer-dimer models with **pure hard-core interaction** (see section 2.1) living on some particular **random graphs**, that we will define properly in the following. The class of diluted graphs that we cover is the same for which the ferromagnetic Ising model was rigorously solved by Dembo and Montanari [26, 28], using the local weak convergence strategy developed in [10]. The fundamental feature of these random graphs is that locally they have a tree-like structure. Examples of "famous" graphs in this class are the Erdős-Rényi graphs and the configuration models, which provide a random graph with any prescribed degree sequence [59].

We fix a uniform monomer activity x and uniform dimer activity $w = 1$. We show that these monomer-dimer models are exactly solvable and do not present a phase transition (in agreement with the general results by Heilmann and Lieb [55,56]). Precisely we prove that in the thermodynamic limit the monomer density exists and is expressed as the expectation of a random variable X, whose distribution is determined as the unique solution of a fixed point equation (theorem 6.1). Moreover we deduce the existence of the pressure density in the thermodynamic limit and we obtain an expression in terms of X (theorem 6.2).

Therefore we provide a rigorous proof of the conjectures made by Zdeborová and Mézard [99], and partially studied in [18, 90]. Previously the problem of matchings on sparse random graphs had been already considered in [63, 13].

In order to exploit the locally tree-like structure of the considered graphs, we use some alternating correlations inequalities for monomer-dimer models (lemma 6.3): they are a great tool to pass from global quantities to local quantities. In this way we reduce ourselves to study the root monomer probability on a random tree. This problem is approached by means of the Heilmann-Lieb recursion on trees.

Let $x > 0$. Let $G = (V, E)$ be a finite graph. In this chapter we will denote

$$Z_G(x) \;=\; \sum_{D \in \mathscr{D}_G} x^{N-2|D|} \;, \tag{6.1}$$

$\mu_{G,x}$ will denote the corresponding Gibbs measure and $\langle\,\cdot\,\rangle_{G,x}$ will be the expected value with respect to $\mu_{G,x}$. As usual the pressure density is

$$p_G(x) \;:=\; \frac{1}{|V|}\,\log Z_G(x) \;, \tag{6.2}$$

and the monomer density is

$$m_G(x) \;:=\; \Big\langle \frac{|V| - 2|D|}{|V|} \Big\rangle_{G,x} \;=\; x\,\frac{\partial p_G}{\partial x}(x) \;, \tag{6.3}$$

Notice that when G is a random graph the partition function, the pressure density and the monomer density are random variables and the Gibbs measure is a random measure.

We state here the main results of this chapter, even if the definitions about the class of graphs that we treat will be clarified later.

Theorem 6.1 (Monomer density limit, see also [90])**.** *Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of finite random graphs, which:*

i. is locally convergent to the unimodular Galton-Watson tree $\mathcal{T}(P, \varrho)$;

ii. has asymptotic degree distribution P with finite second moment (equivalently $\overline{\varrho} < \infty$).

Consider the monomer-dimer model on the graphs G_n, $n \in \mathbb{N}$. Then almost surely for all $x > 0$ the monomer density

$$m_{G_n}(x) \xrightarrow[n\to\infty]{} \mathbb{E}[Y(x)] \,. \tag{6.4}$$

The function $x \mapsto \mathbb{E}[Y(x)]$ is analytic on $\mathbb{R}_+$. The law of the random variable $Y(x)$ is defined as:

$$Y(x) \stackrel{\mathcal{D}}{=} \frac{x^2}{x^2 + \sum_{i=1}^{\Delta} X_i} \,, \tag{6.5}$$

where Δ has distribution P and is independent of $(X_i)_{i\in\mathbb{N}}$, $(X_i)_{i\in\mathbb{N}}$ are i.i.d. copies of X, the distribution of X is the only solution supported in $[0,1]$ of the following fixed point distributional equation:

$$X \stackrel{\mathcal{D}}{=} \frac{x^2}{x^2 + \sum_{i=1}^{K} X_i} \,, \tag{6.6}$$

where K has distribution ϱ and is independent of $(X_i)_{i\in\mathbb{N}}$.

Theorem 6.2 (Pressure density limit)**.** *Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of random graphs, which:*

i. is locally convergent to the unimodular Galton-Watson tree $\mathcal{T}(P, \varrho)$;

ii. has asymptotic degree distribution P with finite second moment;

iii. is uniformly sparse.

Then almost surely for every $x > 0$

$$p_{G_n}(x) \xrightarrow[n\to\infty]{} \mathbb{E}\Big[\log\big(x + \sum_{i=1}^{\Delta} \frac{X_i(x)}{x}\big)\Big] - \frac{\overline{P}}{2}\, \mathbb{E}\Big[\log\big(1 + \frac{X_1(x)}{x}\, \frac{X_2(x)}{x}\big)\Big] \tag{6.7}$$

where Δ has distribution P and is independent of $(X_i)_{i\in\mathbb{N}}$, $(X_i)_{i\in\mathbb{N}}$ are i.i.d. copies of X, the distribution of X is the only solution supported in $[0,1]$ of the fixed point distributional equation

$$X \stackrel{\mathcal{D}}{=} \frac{x^2}{x^2 + \sum_{i=1}^{K} X_i} \,,$$

where K has distribution ϱ and is independent of $(X_i)_{i\in\mathbb{N}}$.

6.1 Preliminary results

It is useful to introduce the following notation for the probability of a monomer on a given vertex $o \in V$:

$$\mathcal{M}_x(G,o) := \langle \mathbb{1}_{o\in M(D)} \rangle_{G,x} \,. \tag{6.8}$$

Now the monomer density can be rewritten as

$$m_G(x) \;=\; \frac{1}{|V|}\sum_{o\in V}\mathcal{M}_x(G,o)\,. \tag{6.9}$$

Following [18] we introduce a recursion relation for the probability $\mathcal{M}_x(\cdot)$ that will be extensively used in the sequel; this is a rewriting of the recursion relation for the partition function $Z_\cdot(x)$ that appears in [55].

Lemma 6.1. *The family of functions $\mathcal{M}_x(G,o)$ fulfils the relation*

$$\mathcal{M}_x(G,o) = \frac{x^2}{x^2 + \sum_{v\sim o}\mathcal{M}_x(G-o,v)}\,. \tag{6.10}$$

Proof. Following the Heilmann-Lieb recursion (see proposition 2.7), it holds:

$$Z_G(x)\,\mathcal{M}_x(G,o) \;=\; \sum_{D\in\mathscr{D}_G\,:\,o\in M(D)} x^{|M(D)|} \;=\; x\,Z_{G-o}(x)\,,$$

and:

$$Z_G(x) \;=\; x\,Z_{G-o}(x) + \sum_{v\sim o} Z_{G-o-v}(x)\,.$$

Therefore one obtains:

$$\mathcal{M}_x(G,o) \;=\; \frac{x\,Z_{G-o}(x)}{x\,Z_{G-o}(x)+\sum_{v\sim o}Z_{G-o-v}(x)} \;=\; \Big(1+\sum_{v\sim o}\frac{Z_{G-o-v}(x)}{x\,Z_{G-o}(x)}\Big)^{-1} \;=$$

$$=\; \Big(1+x^{-2}\sum_{v\sim o}\mathcal{M}_x(G-o,v)\Big)^{-1} \;=\; \frac{x^2}{x^2+\sum_{v\sim o}\mathcal{M}_x(G-o,v)}\,. \qquad \square$$

Iterating the recursion relation (6.10), one obtains the *squared* recursion relation

$$\mathcal{M}_x(G,o) = \Big(1+\sum\nolimits_{v\sim o}\frac{1}{x^2+\sum_{u\sim v,\,u\neq o}\mathcal{M}_x(G-o-v,u)}\Big)^{-1}\,. \tag{6.11}$$

In the next lemma we allow the monomer activity to take complex values, precisely those of the open half-plane

$$\mathbb{H}_+ = \{z \in \mathbb{C} \,|\, \Re(z) > 0\}\,.$$

This has no physical or probabilistic meaning, but it is a technique to obtain powerful results at real positive monomer activities by exploiting complex analysis. This lemma already appeared in [18] and in particular point *ii* can be seen also as a consequence of theorem 4.2 in [55].

Lemma 6.2. *i. If $z \in \mathbb{H}_+$, then $z^{-1}\,\mathcal{M}_z(G,o) \in \mathbb{H}_+$*

ii. The function $z \mapsto \mathcal{M}_z(G,o)$ is analytic on $\mathbb{H}_+$

iii. If $z \in \mathbb{H}_+$, then $|\mathcal{M}_z(G,o)| \leq |z|/\Re(z)$

Proof. Note that $\mathbb{H}_+$ is closed with respect to the operations $w \mapsto w^{-1}$ and $(w_1, w_2) \mapsto w_1 + w_2$.

[*i*, *ii*] Proceed by induction on the number $N = |V|$ of vertices of the graph G. For $N = 1$ the graph G coincides with its vertex o, hence $\mathcal{M}_z(G,o) = 1$. Therefore for $z \in \mathbb{H}_+$, $z^{-1}\,\mathcal{M}_z(G,o) = z^{-1} \in \mathbb{H}_+$ and $\mathcal{M}_z(G,o) \equiv 1$ is obviously an analytic function of z.

Suppose now the statements *i* and *ii* hold for any graph of $N-1$ vertices and prove them for the graph G of N vertices. By lemma 6.1:

$$\mathcal{M}_z(G,o) = \frac{z^2}{z^2 + \sum_{v\sim o} \mathcal{M}_z(G-o,v)} = \frac{z}{z + \sum_{v\sim o} z^{-1}\,\mathcal{M}_z(G-o,v)}$$

By inductive hypothesis, for $z \in \mathbb{H}_+$ and for every $v \sim o$, $z^{-1}\,\mathcal{M}_z(G-o,v) \in \mathbb{H}_+$ and $\mathcal{M}_z(G-o,v)$ is an analytic function of z. Therefore

$$z + \textstyle\sum_{v\sim o} z^{-1}\,\mathcal{M}_z(G-o,v) \in \mathbb{H}_+ \quad \text{(in particular it is} \neq 0)$$

so that $z^{-1}\mathcal{M}_z(G,o) \in \mathbb{H}_+$ and $\mathcal{M}_z(G,o)$ is an analytic function of z (as it is the quotient of non-zero analytic functions).

[*iii*] Use lemma 6.1, then apply the elementary inequality $|z+w| \geq \Re(z+w)$ and conclude using point i:

$$|\mathcal{M}_z(G,o)| = \left|\frac{z}{z+\sum_{v\sim o} z^{-1}\,\mathcal{M}_z(G-o,v)}\right| \leq \frac{|z|}{\Re(z)+\sum_{v\sim o}\underbrace{\Re\big(z^{-1}\,\mathcal{M}_z(G-o,v)\big)}_{>0}}$$
$$\leq \frac{|z|}{\Re(z)}\,.$$

□

In the graph G, given $o \in V$ and $r \in \mathbb{N}$, we denote by $[G,o]_r$ the *ball of center* o *and radius* r, that is the (connected) subgraph of G induced by the vertices at graph-distance $\leq r$ from the vertex o. A *tree* is a connected graph with no cycles. If the graph G is locally a tree near the vertex o, the next lemma allows to bound the operator $\mathcal{M}_x(\cdot,o)$ from above/below by cutting away the "non-tree" part of G at even/odd distance from o.

Lemma 6.3 (Correlation inequalities on a locally tree-like graph)**.**

If $[G,o]_{2r}$ *is a tree, then* $\mathcal{M}_x(G,o) \leq \mathcal{M}_x([G,o]_{2r})$.

If $[G,o]_{2r+1}$ *is a tree, then* $\mathcal{M}_x(G,o) \geq \mathcal{M}_x([G,o]_{2r+1})$.

Proof. Proceed by induction on the distance $r \in \mathbb{N}$ from the origin o.
For $r=0$, the graph $[G,o]_0$ is the isolated vertex o hence

$$\mathcal{M}_x(G,o) \leq 1 = \mathcal{M}_x([G,o]_0)\,.$$

Assume now the result holds for $2r$ and prove it for $2r+1$ (with $r \geq 0$).
Suppose $[G,o]_{2r+1}$ is a tree. Note that $[G,o]_{2r+1} - o = \bigsqcup_{v\sim o}[G-o,v]_{2r}$, where $[G-o,v]_{2r}$ is a tree for every $v \sim o$. As in general $\mathcal{M}_x(H,v)$ depends only on the connected component of the graph H which contains the vertex v, it follows:

$$\mathcal{M}_x([G,o]_{2r+1} - o, v) = \mathcal{M}_x([G-o,v]_{2r})\,.$$

And by the induction hypothesis

$$\mathcal{M}_x(G-o,v) \leq \mathcal{M}_x([G-o,v]_{2r})\,.$$

Then using lemma 6.1 two times one obtains:

$$\begin{aligned}\mathcal{M}_x(G,o) &= \frac{x^2}{x^2 + \sum_{i\sim o}\mathcal{M}_x(G-o,i)} \geq \frac{x^2}{x^2 + \sum_{i\sim o}\mathcal{M}_x([G-o,i]_{2r})} \\ &= \frac{x^2}{x^2 + \sum_{i\sim o}\mathcal{M}_x([G,o]_{2r+1}-o,i)} = \mathcal{M}_x([G,o]_{2r+1})\,.\end{aligned}$$

Induction from $2r-1$ to $2r$ (with $r \geq 1$) is done analogously. □

6.2 Solution on trees

The next proposition describes the behaviour of our model on any finite tree. It is an easy consequence of lemma 6.3.

Proposition 6.4. *Let T be a locally finite tree rooted at o. Consider the monomer-dimer model on the finite sub-trees induced by the vertices in the first r generations $T(r) \equiv [T,o]_r$, $r \in \mathbb{N}$. Then:*

i. $r \mapsto \mathcal{M}_x(T(2r),o)$ *is monotonically decreasing*

ii. $r \mapsto \mathcal{M}_x(T(2r+1),o)$ *is monotonically increasing*

iii. $\mathcal{M}_x(T(2r),o) \geq \mathcal{M}_x(T(2s+1),o) \quad \forall\, r,s \in \mathbb{N}$

Proof. Let $r,s \in \mathbb{N}$.

[i] Consider the graph $T(2r+2)$. Cutting at distance $2r$ from o, one obtains $[T(2r+2),o]_{2r} = T(2r)$ which is a tree. Hence by lemma 6.3

$$\mathcal{M}_x(T(2r+2),o) \leq \mathcal{M}_x(T(2r),o)\,.$$

[ii] Consider the graph $T(2r+3)$. Cutting at distance $2r+1$ from o, one obtains $[T(2r+3),o]_{2r+1} = T(2r+1)$ which is a tree. Hence by lemma 6.3

$$\mathcal{M}_x(T(2r+3),o) \geq \mathcal{M}_x(T(2r+1),o)\,.$$

[iii] Consider the graph $T(2r+1)$. Cutting at distance $2r$ from o, one obtains $[T(2r+1),o]_{2r} = T(2r)$ which is a tree. Hence by lemma 6.3

$$\mathcal{M}_x(T(2r+1),o) \leq \mathcal{M}_x(T(2r),o)\,.$$

Now if $r \leq s$, combining point *i.* and this third inequality, one finds

$$\mathcal{M}_x(T(2r),o) \geq \mathcal{M}_x(T(2s),o) \geq \mathcal{M}_x(T(2s+1),o)\,;$$

while if $s \leq r$, combining point *ii.* and the third inequality, one finds

$$\mathcal{M}_x(T(2s+1),o) \leq \mathcal{M}_x(T(2r+1),o) \leq \mathcal{M}_x(T(2r),o)\,. \qquad \square$$

As a consequence of proposition 6.4 we obtain that on any locally finite rooted tree there exist $\lim_{r\to\infty}\mathcal{M}_x(T(2r),o)$, $\lim_{r\to\infty}\mathcal{M}_x(T(2r+1),o)$ and moreover

$$0 \leq \lim_{r\to\infty}\mathcal{M}_x(T(2r+1),o) = \sup_{r\in\mathbb{N}}\mathcal{M}_x(T(2r+1),o) \leq$$
$$\leq \inf_{r\in\mathbb{N}}\mathcal{M}_x(T(2r),o) = \lim_{r\to\infty}\mathcal{M}_x(T(2r),o) \leq 1\,.$$

A natural question is if these two limits coincide or not. In the next proposition we prove that they are analytic functions of the monomer activity x, therefore it will suffice to show that they coincide on a set of x's admitting a limit point to conclude that they coincide for all $x > 0$. First we state the following lemma of general usefulness.

Lemma 6.5. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of complex analytic functions on $U \subseteq \mathbb{C}$ open. Suppose that*

- *for every compact $K \subset U$ there exists a constant $C_K < \infty$ such that*

$$\sup_{z\in K}|f_n(z)| \leq C_K \quad \forall\, n\in\mathbb{N}\,;$$

- *there exist $U_0 \subseteq U$ admitting a limit point and a function f on U_0 such that*

$$f_n(z) \xrightarrow[n\to\infty]{} f(z) \quad \forall\, z\in U_0\,.$$

Then f can be extended on U in such a way that

$$f_n(z) \xrightarrow[n\to\infty]{} f(z) \quad \forall\, z\in U\,;$$

moreover the convergence is uniform on compact sets and f is analytic on U.

Proof. By hypothesis $(f_n)_{n\in\mathbb{N}}$ is a family of complex analytic functions on U, which is uniformly bounded on every compact subset $K \subset U$. Therefore by Montel's theorem (e.g. see theorems 2.1 p. 308 and 1.1 p. 156 in [69]), each sub-sequence $(f_{n_m})_{m\in\mathbb{N}}$ admits a further sub-sequence $(f_{n_{m_p}})_{p\in\mathbb{N}}$ that uniformly converges on every compact subset $K \subset U$ to an analytic function $f^{(\sigma)}$, where $\sigma = (n_{m_p})_{p\in\mathbb{N}}$. On the other hand by the second hypothesis one already knows that

$$\forall\, z \in U_0 \;\; \exists \lim_{n\to\infty} f_n(z)\,.$$

Thus by uniqueness of the limit, all the $f^{(\sigma)}$'s coincide on U_0 . Hence, as U_0 admits a limit point in U, by uniqueness of analytic continuation all the $f^{(\sigma)}$'s coincide on the whole U. Denoting f their common value, this entails that

$$\forall\, z\in U \;\; \exists \lim_{n\to\infty} f_n(z) = f(z)\,.$$

□

Proposition 6.6. *Let T be a locally finite tree rooted at o. Consider the monomer-dimer model on the sub-trees $T(r)$, $r \in \mathbb{N}$. Then the functions*

$$x \mapsto \lim_{r\to\infty} \mathcal{M}_x(T(2r), o) \quad , \quad x \mapsto \lim_{r\to\infty} \mathcal{M}_x(T(2r+1), o)$$

are analytic on $\mathbb{R}_+$.

Proof. Set $f_r(z) := \mathcal{M}_z(T(2r), o)$ and $g_r(z) := \mathcal{M}_z(T(2r + 1), o)$. By lemma 6.2 $(f_r)_{r\in\mathbb{N}}$ is a family of complex analytic functions on $\mathbb{H}_+$, and it is uniformly bounded on every compact subset $K \subset \mathbb{H}_+$:

$$\sup_{z\in K} |f_r(z)| \,\le\, \sup_{z\in K} \frac{|z|}{\Re(z)} \,<\infty \quad \forall\, r \in \mathbb{N}\,.$$

On the other hand by proposition 6.4 one already knows that

$$\forall\, x>0 \;\; \exists \lim_{r\to\infty} f_r(x)\,.$$

The result for $(f_r)_{r\in\mathbb{N}}$ then follows by lemma 6.5. The same reasoning holds for the sequence $(g_r)_{r\in\mathbb{N}}$.

□

Now we define an important class of random trees. We will prove that for these trees the previous limits on even and odd depth almost surely coincide at every monomer activity.

Definition 6.7 (Galton-Watson random tree)**.** Let $P = (P_k)_{k\in\mathbb{N}}$, $\varrho = (\varrho_k)_{k\in\mathbb{N}}$ be two probability distributions over $\mathbb{N}$. A *Galton-Watson tree* $\mathcal{T}(P, \varrho)$ is a random tree rooted at o and defined constructively as follows.
Let Δ be a random variable with distribution P, let $(K_{r,i})_{r\geq 1,\, i\geq 1}$ be i.i.d. random variables with distribution ϱ and independent of Δ.

1) Connect the root o to Δ offspring, which form the 1^{st} generation

2) Connect each node (r, i) in the r^{th} generation to $K_{r,i}$ offspring; all together these nodes form the $(r + 1)^{th}$ generation

Repeat recursively the second step for all $r \geq 1$ and obtain $\mathcal{T}(P, \varrho)$. We denote $\mathcal{T}(P, \varrho, r)$ the finite sub-tree of $\mathcal{T}(P, \varrho)$ induced by the first r generations.

A special case of Galton-Watson tree is when $\varrho = P$, which we simply denote $\mathcal{T}(\varrho) := \mathcal{T}(\varrho, \varrho)$ and $\mathcal{T}(\varrho, r) := \mathcal{T}(\varrho, \varrho, r)$. If instead the offspring distributions satisfy $\overline{P} := \sum_{k=0}^{\infty} k\, P_k < \infty$ and

$$\varrho_k = \frac{(k+1)\, P_{k+1}}{\overline{P}} \quad \forall k \in \mathbb{N}\,,$$

we call $\mathcal{T}(P, \varrho)$ a *unimodular* Galton-Watson tree.

In the following when we consider a Galton-Watson tree we suppose it is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and we denote $\mathbb{E}[\,\cdot\,]$ the expectation with respect to the measure $\mathbb{P}$. Remember that when the monomer-dimer model is studied on a random graph G, then the measure $\mu_{G,x}$ is a random measure and therefore the probability $\mathcal{M}_x(G, o)$ is a random variable.

Theorem 6.3. *Let $\mathcal{T}(\varrho)$ be a Galton-Watson tree such that $\overline{\varrho} := \sum_{k\in\mathbb{N}} k\, \varrho_k < \infty$. Consider the monomer-dimer model on the finite sub-trees $\mathcal{T}(\varrho, r)$, $r \in \mathbb{N}$.*

Then almost surely for every $x > 0$

$$\exists \lim_{r\to\infty} \mathcal{M}_x(\mathcal{T}(\varrho, r), o) =: X(x)\,.$$

The random function $x \mapsto X(x)$ is almost surely analytic on $\mathbb{R}_+$.

The distribution of the random variable $X(x)$ is the only solution supported in $[0,1]$ of the following fixed point distributional equation:

$$X \stackrel{\mathcal{D}}{=} \frac{x^2}{x^2 + \sum_{i=1}^{K} X_i}\,, \tag{6.12}$$

where $(X_i)_{i\in\mathbb{N}}$ are i.i.d. copies of X, K has distribution ϱ, $(X_i)_{i\in\mathbb{N}}$ and K are independent.

Proof. To ease the notation we drop the symbol ϱ as $\mathcal{T} := \mathcal{T}(\varrho)$ and $\mathcal{T}(r) := \mathcal{T}(\varrho, r)$. By proposition 6.4 there exist the two limits

$$X^+(x) := \lim_{r\to\infty} \mathcal{M}_x(\mathcal{T}(2r), o) \quad , \quad X^-(x) := \lim_{r\to\infty} \mathcal{M}_x(\mathcal{T}(2r+1), o)\,,$$

moreover $0 \le X^- \le X^+ \le 1$ and by proposition 6.6 the functions $x \mapsto X^+(x)$ and $x \mapsto X^-(x)$ are analytic on $\mathbb{R}_+$. The theorem is obtained by the following lemmas.

Lemma 6.8. *Given $x > 0$, $X^+(x)$ and $X^-(x)$ are both solutions of the following fixed point distributional equation:*

$$X \stackrel{\mathcal{D}}{=} \left(1 + \textstyle\sum_{i=1}^{K} (x^2 + \sum_{j=1}^{H_i} X_{i,j})^{-1}\right)^{-1}\,, \tag{6.13}$$

where $(X_{i,j})_{i,j\in\mathbb{N}}$ are i.i.d. copies of X, $(H_i)_{i\in\mathbb{N}}$ are i.i.d. with distribution ϱ, K has distribution ϱ, $(X_{i,j})_{i,j\in\mathbb{N}}$, $(H_i)_{i\in\mathbb{N}}$ and K are mutually independent.

We will write $u \leftarrow v$ to denote "u son of v in the rooted tree $(\mathcal{T}, o)$". We will indicate $\mathcal{T}_u(r)$ the sub-tree of $\mathcal{T}$ induced by the vertex u and its descendants until the r^{th} generation (starting counting from u). Using lemma 6.1 and precisely equation (6.11) one finds, with the notations just introduced,

$$\begin{aligned}
\mathcal{M}_x(\mathcal{T}(2r+2), o) &= \left(1 + \textstyle\sum_{v\leftarrow o} \left(x^2 + \sum_{u\leftarrow v} \mathcal{M}_x(\mathcal{T}(2r+2) - o - v, u)\right)^{-1}\right)^{-1} \\
&= \left(1 + \textstyle\sum_{v\leftarrow o} \left(x^2 + \sum_{u\leftarrow v} \mathcal{M}_x(\mathcal{T}_u(2r), u)\right)^{-1}\right)^{-1} \\
&\stackrel{\mathcal{D}}{=} \left(1 + \textstyle\sum_{i=1}^{K} \left(x^2 + \sum_{j=1}^{H_i} \mathcal{M}_x(\mathcal{T}_{i,j}(2r), o)\right)^{-1}\right)^{-1}\,,
\end{aligned}$$

where $(\mathcal{T}_{i,j}(2r))_{i,j\in\mathbb{N}}$ are i.i.d. copies of $\mathcal{T}(2r)$, independent of $(H_i)_{i\in\mathbb{N}}$ and K. Now since $\mathcal{M}_x(\mathcal{T}(2r),o) \xrightarrow[r\to\infty]{a.s.} X^+(x)$, it holds also

$$\mathcal{M}_x(\mathcal{T}(2r),o) \xrightarrow[r\to\infty]{\mathcal{D}} X^+(x)\,,$$

and moreover, thanks to the mutual independence of $\big(\mathcal{M}_x(\mathcal{T}_{i,j}(2r),o)\big)_{i,j\in\mathbb{N}}$, $(H_i)_{i\in\mathbb{N}}$, K , by standard probability arguments[1]

$$\Big(\big(\mathcal{M}_x(\mathcal{T}_{i,j}(2r),o)\big)_{i,j\in\mathbb{N}}\,,\,(H_i)_{i\in\mathbb{N}}\,,\,K\,\Big) \xrightarrow[r\to\infty]{\mathcal{D}} \Big(\,(X^+_{i,j})_{i,j\in\mathbb{N}}\,,\,(H_i)_{i\in\mathbb{N}}\,,\,K\,\Big)\,,$$

where $(X^+_{i,j})_{i,j\in\mathbb{N}}$ are i.i.d. copies of $X^+(x)$, independent of $(H_i)_{i\in\mathbb{N}}$ and K. Then for any bounded continuous function $\phi:[0,1]\to\mathbb{R}$

$$\begin{aligned}\mathbb{E}[\phi(X^+(x))] &= \lim_{r\to\infty}\mathbb{E}[\phi\big(\mathcal{M}_x(\mathcal{T}(2r+2),o)\big)]\\ &= \lim_{r\to\infty}\mathbb{E}\Bigg[\phi\Bigg(\Big(1+\textstyle\sum_{i=1}^{K}\big(x^2+\sum_{j=1}^{H_i}\mathcal{M}_x(\mathcal{T}_{i,j}(2r),o)\big)^{-1}\Big)^{-1}\Bigg)\Bigg]\\ &= \mathbb{E}\Bigg[\phi\Bigg(\Big(1+\textstyle\sum_{i=1}^{K}(x^2+\sum_{j=1}^{H_i}X^+_{i,j})^{-1}\Big)^{-1}\Bigg)\Bigg]\,.\end{aligned}$$

Namely $X^+(x)$ is a solution of distributional equation (6.13).
In an analogous way it can be proven that also $X^-(x)$ is a solution of distributional equation (6.13).

Lemma 6.9. *Almost surely for all $x>0$ $X^+(x)=X^-(x)$.*

By proposition 6.4 $X^+(x)\geq X^-(x)$. By lemma 6.8 $X^+(x)$ and $X^-(x)$ are both solutions of equation (6.13). Therefore, taking $\big((X^+_{i,j})_{i,j\in\mathbb{N}}\,,\,(X^-_{i,j})_{i,j\in\mathbb{N}}\big)$

[1] equivalence between convergence in distribution and convergence of the characteristic functions (e.g. see theorems 26.3 p. 349 and 29.4 p. 383 in [16]) can be used.

independent of $\big((H_i)_{i\in\mathbb{N}}\,, K\big)$, one obtains:

$$\begin{aligned}
&\mathbb{E}[|X^+(x) - X^-(x)|] \;=\; |\mathbb{E}[X^+(x)] - \mathbb{E}[X^-(x)]| \;= \\
&= \big|\,\mathbb{E}\big[\big(1+\textstyle\sum_{i=1}^K (x^2+\sum_{j=1}^{H_i} X^+_{i,j})^{-1}\,\big)^{-1}\big] + \\
&\qquad\qquad\qquad - \mathbb{E}\big[\big(1+\textstyle\sum_{i=1}^K (x^2+\sum_{j=1}^{H_i} X^-_{i,j})^{-1}\,\big)^{-1}\big]\big| \\
&= \big|\,\mathbb{E}\big[\frac{\sum_{i=1}^K (x^2+\sum_{j=1}^{H_i} X^-_{i,j})^{-1} - \sum_{i=1}^K (x^2+\sum_{j=1}^{H_i} X^+_{i,j})^{-1}}{\big(1+\sum_{i=1}^K (x^2+\sum_{j=1}^{H_i} X^+_{i,j})^{-1}\big)\big(1+\sum_{i=1}^K (x^2+\sum_{j=1}^{H_i} X^-_{i,j})^{-1}\big)}\big]\big| \\
&= \big|\,\mathbb{E}\big[\big(\textstyle\sum_{i=1}^K \dfrac{\sum_{j=1}^{H_i} (X^+_{i,j} - X^-_{i,j})}{(x^2+\sum_{j=1}^{H_i} X^-_{i,j})\,(x^2+\sum_{j=1}^{H_i} X^+_{i,j})}\big)\,\cdot \\
&\qquad \cdot\,\big(1+\textstyle\sum_{i=1}^K (x^2+\sum_{j=1}^{H_i} X^+_{i,j})^{-1}\big)^{-1}\,\big(1+\sum_{i=1}^K (x^2+\sum_{j=1}^{H_i} X^-_{i,j})^{-1}\big)^{-1}\big]\big| \\
&\le \frac{1}{x^4}\,\mathbb{E}\big[\textstyle\sum_{i=1}^K\sum_{j=1}^{H_i} |X^+_{i,j} - X^-_{i,j}|\,\big] \;=\; \dfrac{\overline{\varrho}^{\,2}}{x^4}\,\mathbb{E}[|X^+(x) - X^-(x)|\,]\,,
\end{aligned}$$

where the last equality is true by independence.
If $x > \sqrt{\overline{\varrho}}$, the contraction coefficient is $\overline{\varrho}^2/x^4 < 1$. Therefore for all $x > \sqrt{\overline{\varrho}}$

$$\mathbb{E}[|X^+(x) - X^-(x)|] = 0\,, \quad \text{i.e. } \; X^+(x) = X^-(x) \; a.s.$$

As $\mathbb{Q}$ is countable it follows that

$$\big(X^+(x) = X^-(x)\;\forall\, x \in]\sqrt{\overline{\varrho}}, \infty[\,\cap\mathbb{Q}\,\big)\; a.s.$$

Now remind that by proposition 6.6 $X^+(x)$, $X^-(x)$ are analytic functions of $x > 0$. Hence, as $\mathbb{Q}$ is dense in $\mathbb{R}$, this entails that

$$\big(X^+(x) = X^-(x)\;\forall\, x > 0\,\big)\; a.s.$$

by uniqueness of the analytic continuation.
As a consequence $\big(\exists \lim_{r\to\infty}\mathcal{M}_x(\mathcal{T}(r), o) = X^+(x) = X^-(x)\;\forall\, x>0\big)$ $a.s.$ We call $X(x)$ this random analytic function of x.

Lemma 6.10. *Given $x > 0$, the random variable $X(x)$, satisfying the distributional equation (6.13), satisfies also the distributional equation (6.12).*

Using lemma 6.1 and precisely equation (6.10), one finds

$$\begin{aligned}\mathcal{M}_x(\mathcal{T}(r+1),o) &= \frac{x^2}{x^2+\sum_{v\leftarrow o}\mathcal{M}_x(\mathcal{T}(r+1)-o,v)} = \frac{x^2}{x^2+\sum_{v\leftarrow o}\mathcal{M}_x(\mathcal{T}_v(r),v)} \\ &\stackrel{\mathcal{D}}{=} \frac{x^2}{x^2+\sum_{i=1}^{K}\mathcal{M}_x(\mathcal{T}_i(r),o)}\,,\end{aligned}$$

where $(\mathcal{T}_i(r))_{i\in\mathbb{N}}$ are i.i.d. copies of $\mathcal{T}(r)$, independent of K.

Now since $\mathcal{M}_x(\mathcal{T}(r),o) \xrightarrow[r\to\infty]{a.s.} X(x)$ (by definition, which is possible thanks to lemma 6.9), it holds also

$$\mathcal{M}_x(\mathcal{T}(r),o) \xrightarrow[r\to\infty]{\mathcal{D}} X(x)\,,$$

and moreover, thanks to the independence of $\big(\mathcal{M}_x(\mathcal{T}_i(r),o)\big)_{i\in\mathbb{N}}\,,\,K$,

$$\Big(\big(\mathcal{M}_x(\mathcal{T}_i(r),o)\big)_{i\in\mathbb{N}}\,,\,K\Big) \xrightarrow[r\to\infty]{\mathcal{D}} \big((X_i)_{i\in\mathbb{N}}\,,\,K\big)\,,$$

where $(X_i)_{i\in\mathbb{N}}$ are i.i.d. copies of $X(x)$, independent of K.

Then for any bounded continuous function $\phi:[0,1]\to\mathbb{R}$

$$\begin{aligned}\mathbb{E}[\phi(X(x))] &= \lim_{r\to\infty}\mathbb{E}[\phi\big(\mathcal{M}_x(\mathcal{T}(r+1),o)\big)] = \lim_{r\to\infty}\mathbb{E}\big[\phi\big(\frac{x^2}{x^2+\sum_{i=1}^{K}\mathcal{M}_x(\mathcal{T}_i(r),o)}\big)\big] \\ &= \mathbb{E}\big[\phi\big(\frac{x^2}{x^2+\sum_{i=1}^{K}X_i}\big)\big]\,.\end{aligned}$$

Namely $X(x)$ is a solution of distributional equation (6.12).

Lemma 6.11. *For a given $x>0$, the distributional equation (6.12) has a unique solution supported in* $[0,1]$.

Let Y be a random variable taking values in $[0,1]$ and such that

$$Y \stackrel{\mathcal{D}}{=} \frac{x^2}{x^2+\sum_{i=1}^{K}Y_i}\,,$$

where $(Y_i)_{i\in\mathbb{N}}$ are i.i.d. copies of Y, independent of K. Observe that:

$$\begin{array}{ccc} \dfrac{x^2}{x^2+\sum_{i=1}^{K}Y_i} & \leq & 1 \\ \rotatebox{90}{$\stackrel{\mathcal{D}}{=}$} & & \| \\ Y & & \mathcal{M}_x(\mathcal{T}(0),o) \end{array}$$

Therefore there exist $(Y_i')_{i\in\mathbb{N}}$ i.i.d. copies of Y and $(\mathcal{M}_x(\mathcal{T}(0),o)_i)_{i\in\mathbb{N}}$ i.i.d. copies of $\mathcal{M}_x(\mathcal{T}(0),o)$ such that

$$Y_i' \leq \mathcal{M}_x(\mathcal{T}(0),o)_i \quad \forall\, i \in \mathbb{N}\,.$$

Let $K' \overset{\mathcal{D}}{\sim} \varrho$ independent of $(Y_i')_{i\in\mathbb{N}}$, $(\mathcal{M}_x(\mathcal{T}(0),o)_i)_{i\in\mathbb{N}}$. Applying the function $\frac{x^2}{x^2+\sum_{i=1}^{K'}(\,\cdot\,)}$, which is monotonically decreasing in each argument, to each term of the previous inequality one finds

$$\begin{array}{ccc} \dfrac{x^2}{x^2+\sum_{i=1}^{K'}\mathcal{M}_x(\mathcal{T}(0),o)_i} & \leq & \dfrac{x^2}{x^2+\sum_{i=1}^{K'}Y_i'} \\ \overset{\mathcal{D}}{\|} & & \overset{\mathcal{D}}{\|} \\ \mathcal{M}_x(\mathcal{T}(1),o) & & Y \end{array}$$

Therefore there exist $(\mathcal{M}_x(\mathcal{T}(1),o)_i)_{i\in\mathbb{N}}$ i.i.d. copies of $\mathcal{M}_x(\mathcal{T}(1),o)$ and $(Y_i'')_{i\in\mathbb{N}}$ i.i.d. copies of Y such that

$$\mathcal{M}_x(\mathcal{T}(1),o)_i \leq Y_i'' \quad \forall\, i \in \mathbb{N}\,.$$

Let $K'' \overset{\mathcal{D}}{\sim} \varrho$ independent of $(\mathcal{M}_x(\mathcal{T}(1),o)_i)_{i\in\mathbb{N}}$, $(Y_i'')_{i\in\mathbb{N}}$. Applying the function $\frac{x^2}{x^2+\sum_{i=1}^{K''}(\,\cdot\,)}$, which is monotonically decreasing in each argument, to each term of the previous inequality one finds

$$\begin{array}{ccc} \dfrac{x^2}{x^2+\sum_{i=1}^{K''}Y_i''} & \leq & \dfrac{x^2}{x^2+\sum_{i=1}^{K''}\mathcal{M}_x(\mathcal{T}(1),o)_i'} \\ \overset{\mathcal{D}}{\|} & & \overset{\mathcal{D}}{\|} \\ Y & & \mathcal{M}_x(\mathcal{T}(2),o) \end{array}$$

Proceeding with this reasoning one obtains that for any $r \in \mathbb{N}$ there exist $\mathcal{M}_x(\mathcal{T}(r),o)^{\sim} \overset{\mathcal{D}}{=} \mathcal{M}_x(\mathcal{T}(r),o)$, $Y^{(r)} \overset{\mathcal{D}}{=} Y$ such that

$$\begin{array}{ccccccc} \mathcal{M}_x(\mathcal{T}(2r+1),o)^{\sim} & \leq & Y^{(2r+1)} & \text{and} & Y^{(2r)} & \leq & \mathcal{M}_x(\mathcal{T}(2r),o)^{\sim} \\ \big\downarrow{\scriptstyle\mathcal{D}} & & & \text{as } r\to\infty & & & \big\downarrow{\scriptstyle\mathcal{D}} \\ X^-(x) & & & & & & X^+(x) \end{array}$$

Since by lemma 6.9 $X^+(x) = X^-(x) = X(x)$ *a.s.*, it follows[2] that $Y \overset{\mathcal{D}}{=} X(x)$.

□

[2]A squeeze theorem for convergence in distribution holds: if $X_n \leq Y_n$, $Y_n' \leq X_n'$,

Corollary 6.12. *Let $\mathcal{T}(P,\varrho)$ be a Galton-Watson tree such that $\overline{\varrho} := \sum_{k\in\mathbb{N}} k\,\varrho_k < \infty$. Consider the monomer-dimer model on the sub-trees $\mathcal{T}(P,\varrho,r)$, $r\in\mathbb{N}$. Then almost surely for every $x>0$*

$$\exists \lim_{r\to\infty} \mathcal{M}_x(\mathcal{T}(P,\varrho,r),o) =: Y(x)\,.$$

The random function $x\mapsto Y(x)$ is a.s. analytic on $\mathbb{R}_+$.

The distribution of the random variable $Y(x)$ is

$$Y(x) \overset{\mathcal{D}}{=} \frac{x^2}{x^2+\sum_{i=1}^{\Delta} X_i}\,,$$

where Δ has distribution P and is independent of $(X_i)_{i\in\mathbb{N}}$, $(X_i)_{i\in\mathbb{N}}$ are i.i.d. copies of X, the distribution of X is the only solution supported in $[0,1]$ of the following fixed point distributional equation:

$$X \overset{\mathcal{D}}{=} \frac{x^2}{x^2+\sum_{i=1}^{K} X_i}\,,$$

where K has distribution ϱ and is independent of $(X_i)_{i\in\mathbb{N}}$.

Proof. We drop the symbols P, ϱ as $\mathcal{T}^* := \mathcal{T}(P,\varrho)$ and $\mathcal{T}^*(r) := \mathcal{T}(P,\varrho,r)$. Observe that $\mathcal{T}^*-o = \bigsqcup_{v\leftarrow o}\mathcal{T}^*_v$ and the random trees $(\mathcal{T}^*_v)_{v\leftarrow o}$ are i.i.d. Galton-Watson trees of the type $\mathcal{T}(\varrho)$. Using lemma 6.1

$$\mathcal{M}_x(\mathcal{T}^*(r+1),o) = \frac{x^2}{x^2+\sum_{v\leftarrow o}\mathcal{M}_x(\mathcal{T}^*(r+1)-o,v)} = \frac{x^2}{x^2+\sum_{v\leftarrow o}\mathcal{M}_x(\mathcal{T}^*_v(r),v)}$$

By theorem 6.3 for any v son of o, $\lim_{r\to\infty}\mathcal{M}_x(\mathcal{T}^*_v(r),o)$ almost surely exists, it is analytic, and its distribution satisfies equation (6.12). Therefore

$Y_n \overset{\mathcal{D}}{=} Y'_n \overset{\mathcal{D}}{=} Y$ for all $n\in\mathbb{N}$ and $X_n \xrightarrow[n\to\infty]{\mathcal{D}} X,\ X'_n \xrightarrow[n\to\infty]{\mathcal{D}} X$ then $Y \overset{\mathcal{D}}{=} X$.
To prove it work with the CDFs: $F_{X'_n}(x) \le F_{Y'_n}(x) = F_{Y_n}(x) \le F_{X_n}(x)$ $\forall x\in\mathbb{R}$, $F_{X'_n}(x) \xrightarrow[n\to\infty]{} F_X(x)$ and $F_{X_n}(x) \xrightarrow[n\to\infty]{} F_X(x)$ for every x continuity point of F_X. Since $F_{Y'_n} = F_{Y_n} = F_Y$, by the classical squeeze theorem it follows that $F_Y(x) = F_X(x)$ for every x continuity point of F_X. Now since F_X and F_Y are right-continuous and the continuity points of F_X are dense in $\mathbb{R}$, one concludes that $F_Y = F_X$.

$\lim_{r\to\infty} \mathcal{M}_x(\mathcal{T}^*(r), o)$ almost surely exists and is analytic, in fact

$$\lim_{r\to\infty} \mathcal{M}_x(\mathcal{T}^*(r), o) = \frac{x^2}{x^2 + \sum_{v\leftarrow o} \lim_{r\to\infty} \mathcal{M}_x(\mathcal{T}^*_v(r), v)}$$
$$\stackrel{\mathcal{D}}{=} \frac{x^2}{x^2 + \sum_{i=1}^{\Delta} X_i} \,,$$

where $(X_i)_{i\in\mathbb{N}}$ are i.i.d. copies of the solution supported in $[0,1]$ of equation (6.12) , Δ has distribution P and is independent of $(X_i)_{i\in\mathbb{N}}$. □

Corollary 6.13. *In the hypothesis of corollary 6.12, almost surely for every* $z \in \mathbb{H}_+$

$$\exists \lim_{r\to\infty} \mathcal{M}_z(\mathcal{T}(P, \varrho, r), o) =: Y(z)\,.$$

The random function $z \mapsto Y(z)$ *is almost surely analytic on* $\mathbb{H}_+$ *and the convergence is uniform on compact subsets of* $\mathbb{H}_+$.

Proof. Set $f_r(z) := \mathcal{M}_z(\mathcal{T}(P, \varrho, r), o)$. By lemma 6.2 $(f_r)_{r\in\mathbb{N}}$ is a sequence of complex analytic functions on $\mathbb{H}_+$, uniformly bounded on compact subsets. On the other hand by corollary 6.12 $(f_r)_{r\in\mathbb{N}}$ a.s. converges pointwise on $\mathbb{R}_+$. Then the result follows from lemma 6.5. □

6.3 From trees to graphs

Let $G_n = (V_n, E_n)$, $n \in \mathbb{N}$ be a sequence of finite random graphs, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We introduce now the main class of graphs studied in this paper. The idea is to fix a radius r and draw a vertex v uniformly at random from the graph G_n: for n large enough we want a large fraction of the balls $[G_n, v]_r$ to be (truncated) Galton-Watson trees.

Definition 6.14 (Locally tree-like random graphs)**.** The random graphs sequence $(G_n)_{n\in\mathbb{N}}$ *locally converges* to the unimodular Galton-Watson tree $\mathcal{T}(P, \varrho)$ if for any $r \in \mathbb{N}$ and for any (T, o) finite rooted tree with at most r generations

$$\frac{1}{|V_n|} \sum_{v\in V_n} \mathbb{1}\big([G_n, v]_r \cong (T, o)\big) \xrightarrow[n\to\infty]{a.s.} \mathbb{P}\big(\mathcal{T}(P, \varrho, r) \cong (T, o)\big)\,. \tag{6.14}$$

Here $\cong$ denotes the isomorphism relation between rooted graphs.

Remark 6.15. The following statements are equivalent:

i. $(G_n)_{n\in\mathbb{N}}$ locally converges to $\mathcal{T}(P,\varrho)$

ii. *a.s.* for all $r \in \mathbb{N}$ and (T,o) finite rooted tree with at most r generations

$$\frac{1}{|V_n|}\sum_{v\in V_n} \mathbb{1}\big([G_n,v]_r \cong (T,o)\big) \xrightarrow[n\to\infty]{} \mathbb{P}\big(\mathcal{T}(P,\varrho,r) \cong (T,o)\big)$$

iii. *a.s.* for all $r \in \mathbb{N}$ and F bounded function, invariant under rooted graph isomorphisms,

$$\frac{1}{|V_n|}\sum_{v\in V_n} F\big([G_n,v]_r\big)\ \mathbb{1}([G_n,v]_r \text{ is a tree}) \xrightarrow[n\to\infty]{} \mathbb{E}\big[F\big(\mathcal{T}(P,\varrho,r)\big)\big]$$

iv. *a.s.* for all $r \in \mathbb{N}$ and (B,o) finite rooted graph with radius $\leq r$

$$\frac{1}{|V_n|}\sum_{v\in V_n} \mathbb{1}\big([G_n,v]_r \cong (B,o)\big) \xrightarrow[n\to\infty]{} \mathbb{P}\big(\mathcal{T}(P,\varrho,r) \cong (B,o)\big)$$

v. *a.s.* for all $r \in \mathbb{N}$ and F bounded function, invariant under rooted graph isomorphisms,

$$\frac{1}{|V_n|}\sum_{v\in V_n} F\big([G_n,v]_r\big) \xrightarrow[n\to\infty]{} \mathbb{E}\big[F\big(\mathcal{T}(P,\varrho,r)\big)\big]$$

Observe that local convergence of random graphs $(G_n)_{n\in\mathbb{N}}$ to the random tree $\mathcal{T}(P,\varrho)$ is, in measure theory language, *a.s.*−weak convergence of random measures $(\nu_{r,n})_{n\in\mathbb{N}}$ to the measure ν_r for all $r \in \mathbb{N}$, where:

$$\nu_{r,n}(B,o) := \frac{1}{|V_n|}\sum_{v\in V_n} \mathbb{1}\big([G_n,v]_r \cong (B,o)\big) \quad \forall\,(B,o) \in \mathscr{G}(r)\,,$$

$$\nu_r(B,o) := \mathbb{P}\big(\mathcal{T}(P,\varrho,r) \cong (B,o)\big) \quad \forall\,(B,o) \in \mathscr{G}(r)\,.$$

and $\mathscr{G}(r)$ is the countable set of finite rooted graphs with radius $\leq r$, considered up to isomorphism. From this point of view, this remark gives different characterisations of the weak convergence of measures, valid in general for measures defined on a discrete countable set (in particular the equivalences $ii \Leftrightarrow iii$

and $iv \Leftrightarrow v$ can be seen as consequences of the Portmanteau theorem, e.g. see theorem 2.1 p.16 in [17]).

Remark 6.16. In a graph G the degree of a vertex v, denoted $\deg_G(v)$, is the number of neighbours of v. If $(G_n)_{n\in\mathbb{N}}$ locally converges to $\mathcal{T}(P,\varrho)$, then P is the *empirical degree distribution* of G_n in the limit $n\to\infty$. Indeed the degree is a local function ($\deg_G(v) = \deg_{[G,v]_1}(v)$) and clearly an indicator function is bounded, hence by remark 6.15 almost surely for every $k\in\mathbb{N}$

$$\frac{1}{|V_n|}\sum_{v\in V_n}\mathbb{1}(\deg_{G_n}(v)=k) \xrightarrow[n\to\infty]{} \mathbb{P}(\deg_{\mathcal{T}(P,\varrho)}(o)=k) \;=\; P_k\,.$$

Definition 6.17. The random graphs sequence $(G_n)_{n\in\mathbb{N}}$ is *uniformly sparse* if

$$\lim_{l\to\infty}\limsup_{n\to\infty}\frac{1}{|V_n|}\sum_{v\in V_n}\deg_{G_n}(v)\,\mathbb{1}(\deg_{G_n}(v)\geq l) \;=\; 0 \quad a.s.$$

Remark 6.18. If $(G_n)_{n\in\mathbb{N}}$ is uniformly sparse and locally convergent to $\mathcal{T}(P,\varrho)$,

$$\frac{|E_n|}{|V_n|} \xrightarrow[n\to\infty]{} \frac{1}{2}\,\overline{P} \quad a.s.$$

To prove it write $2\,\frac{|E_n|}{|V_n|} = \frac{1}{|V_n|}\sum_{v\in V_n}\deg_{G_n}(v)$, then fix $l\in\mathbb{N}$ and split the right-hand sum in two parts, concerning the degrees respectively smaller and grater than l. To the first part we may apply the local convergence hypothesis (remark 6.15):

$$\frac{1}{|V_n|}\sum_{v\in V_n}\deg_{G_n}(v)\,\mathbb{1}(\deg_{G_n}(v)\leq l) \xrightarrow[n\to\infty]{a.s.} \mathbb{E}[\,\deg_{\mathcal{T}(P,\varrho)}(o)\,\mathbb{1}(\deg_{\mathcal{T}(P,\varrho)}(o)\leq l)]$$
$$\xrightarrow[l\to\infty]{} \mathbb{E}[\deg_{\mathcal{T}(P,\varrho)}(o)] \;=\; \overline{P}\,.$$

To the second part we apply the uniform sparsity hypothesis:

$$\lim_{l\to\infty}\limsup_{n\to\infty}\frac{1}{|V_n|}\sum_{v\in V_n}\deg_{G_n}(v)\,\mathbb{1}(\deg_{G_n}(v)\geq l{+}1) \;=\; 0 \quad a.s.$$

Example 6.19. An *Erdős-Rényi random graph* G_n is a graph with n vertices, where each pair of vertices is linked by an edge independently with probability c/n. The sequence $(G_n)_{n\in\mathbb{N}}$ is uniformly sparse and locally converges to the unimodular Galton-Watson tree $\mathcal{T}(P, \varrho)$ with $P = \varrho = \text{Poisson}(c)$. For proof and further examples see [27, 26].

We are now able to prove the main result of this chapter.

Proof of the Theorem 6.1. Set $\mathcal{T}^* := \mathcal{T}(P, \varrho)$ and $\mathcal{T}^*(r) := \mathcal{T}(P, \varrho, r)$.
Let $r \in \mathbb{N}$ and $v \in V_n$. If $[G_n, v]_{2r+1}$ is a tree, then lemma 6.3 permits to localize the problem:

$$\mathcal{M}_x(G_n, v)\ \mathbb{1}([G_n, v]_{2r+1} \text{ is a tree}) \begin{cases} \leq \mathcal{M}_x([G_n, v]_{2r}, v)\ \mathbb{1}([G_n, v]_{2r+1} \text{ is a tree}) \\ \geq \mathcal{M}_x([G_n, v]_{2r+1}, v)\ \mathbb{1}([G_n, v]_{2r+1} \text{ is a tree}) \end{cases}$$

Now work with the right-hand bounds and take the averages over a uniformly chosen vertex v. First let $n \to \infty$ using the hypothesis of local convergence (see remark 6.15) and then let $r \to \infty$ using the results on Galton-Watson trees (corollary 6.12) and dominated convergence: almost surely for all $x > 0$

$$\frac{1}{|V_n|} \sum_{v \in V_n} \mathcal{M}_x([G_n, v]_{2r}, v)\ \mathbb{1}([G_n, v]_{2r+1} \text{ is a tree}) \ \xrightarrow[n \to \infty]{} \ \mathbb{E}\big[\mathcal{M}_x(\mathcal{T}^*(2r), o)\big] \ \underset{r \to \infty}{\searrow} \ \mathbb{E}[Y(x)]$$

and similarly

$$\frac{1}{|V_n|} \sum_{v \in V_n} \mathcal{M}_x([G_n, v]_{2r+1}, v)\ \mathbb{1}([G_n, v]_{2r+1} \text{ is a tree}) \ \xrightarrow[n \to \infty]{} \ \mathbb{E}\big[\mathcal{M}_x(\mathcal{T}^*(2r+1), o)\big] \ \underset{r \to \infty}{\nearrow} \ \mathbb{E}[Y(x)]\,.$$

On the other hand observe that $a.s.$ for all $x > 0$

$$\Big|\frac{1}{|V_n|} \sum_{v \in V_n} \mathcal{M}_x(G_n, v) - \frac{1}{|V_n|} \sum_{v \in V_n} \mathcal{M}_x(G_n, v)\ \mathbb{1}([G_n, v]_{2r+1} \text{ is a tree})\Big| \leq \frac{1}{|V_n|} \sum_{v \in V_n} \big(1 - \mathbb{1}([G_n, v]_{2r+1} \text{ is a tree})\big) \ \xrightarrow[n \to \infty]{} \ 1 - \mathbb{P}(\mathcal{T}^*(2r+1) \text{ is a tree}) = 0\,.$$

Therefore, reasoning with lim inf and lim sup, one finds that almost surely for all $x > 0$ there exists

$$\lim_{n \to \infty} \frac{1}{|V_n|} \sum_{v \in V_n} \mathcal{M}_x(G_n, v) \ = \ \mathbb{E}[Y(x)]\,.$$

Remembering the identity (6.9), the proof is concluded, except for the analyticity of $x \mapsto \mathbb{E}[Y(x)]$ which will follow from the next corollary. $\square$

Corollary 6.20. *In the hypothesis of theorem 6.1, almost surely for all* $z \in \mathbb{H}_+$

$$m_{G_n}(z) \xrightarrow[n\to\infty]{} \mathbb{E}[Y(z)]\ , \tag{6.15}$$

where the random variable $Y(z)$ *is defined in corollary 6.13. The function* $z \mapsto \mathbb{E}[Y(z)]$ *is analytic on* $\mathbb{H}_+$ *and the convergence is uniform on compact subsets of* $\mathbb{H}_+$ *. As a consequence almost surely for all* $k \geq 1$ *and* $z \in \mathbb{H}_+$

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}\, p_{G_n}(z) \xrightarrow[n\to\infty]{} \frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}\, \frac{\mathbb{E}[Y(z)]}{z}\ . \tag{6.16}$$

Proof. By lemma 6.2 $(m_{G_n})_{n\in\mathbb{N}}$ is a sequence of complex analytic functions on $\mathbb{H}_+$, which is uniformly bounded on compact subsets $K \subset \mathbb{H}_+$:

$$\sup_{z\in K} |m_{G_n}(z)| \leq \frac{1}{|V_n|} \sum_{v\in V_n} \sup_{z\in K} |\mathcal{M}_z(G_n, o)| \leq \sup_{z\in K} \frac{|z|}{\Re(z)} < \infty \quad \forall n \in \mathbb{N}\,.$$

On the other hand by theorem 6.1 $(m_{G_n}(x))_{n\in\mathbb{N}}$ a.s. converges pointwise on $\mathbb{R}_+$ to $\mathbb{E}[Y(x)]$. Then lemma 6.5 applies: $\mathbb{E}[Y(z)]$ is analytic in $z \in \mathbb{H}_+$ and a.s.

$$m_{G_n}(z) \xrightarrow[n\to\infty]{} \mathbb{E}[Y(z)] \quad \text{uniformly in } z \in K \text{ for every compact } K \subset \mathbb{H}_+.$$

This entails also the convergence of derivatives (e.g. see theorem 1.2 p. 157 in [69]). $\square$

The existence and analyticity of the monomer density in the thermodynamic limit entails the same properties for the pressure per particle. Only the additional assumption of uniform sparsity is required.

Corollary 6.21. *Let* $(G_n)_{n\in\mathbb{N}}$ *be a sequence of random graphs, which:*

i. is locally convergent to the unimodular Galton-Watson tree $\mathcal{T}(P, \varrho)$*;*

ii. has asymptotic degree distribution P *with finite second moment;*

iii. is uniformly sparse.

Then almost surely for every $x > 0$

$$p_{G_n}(x) \xrightarrow[n\to\infty]{} p(a) + \int_a^x \frac{\mathbb{E}[Y(t)]}{t}\,\mathrm{d}t \tag{6.17}$$

where $a > 0$ *is arbitrary,* $p(a) = \lim_{n\to\infty} p_{G_n}(a)$ *a.s., and* $Y(t)$ *is the random variable defined in theorem 6.1.*

The function $x \mapsto p(a) + \int_a^x \frac{\mathbb{E}[Y(t)]}{t}\,\mathrm{d}t$ *is analytic on* $\mathbb{R}_+$ *.*

Proof. From theorem 6.1, using the fundamental theorem of calculus and dominated convergence, it follows immediately that $a.s.$ for every $x > 0$, $a > 0$

$$p_{G_n}(x) - p_{G_n}(a) = \int_a^x \frac{\partial p_{G_n}}{\partial t}(t)\,\mathrm{d}t \xrightarrow[n\to\infty]{} \int_a^x \frac{\mathbb{E}[Y(t)]}{t}\,\mathrm{d}t \tag{6.18}$$

By theorem 6.1 the function $x \mapsto \mathbb{E}[Y(x)]$ is analytic on $\mathbb{R}_+$, therefore the integral function $x \mapsto \int_a^x \frac{\mathbb{E}[Y(t)]}{t}\,\mathrm{d}t$ is analytic on $\mathbb{R}_+$ too.

To conclude it remains to prove that almost surely for all $x > 0$

$$\exists \lim_{n\to\infty} p_{G_n}(x)\ .$$

Use the bounds for the pressure of remark 2.5 to estimate

$$p_{G_n}(x) - p_{G_n}(a) \begin{cases} \leq p_{G_n}(x) - \log a \\ \geq p_{G_n}(x) - \log a - \frac{|E_n|}{|V_n|}\log(1+\frac{1}{a^2}) \end{cases} \tag{6.19}$$

Put together (6.18), (6.19), remind $|E_n|/|V_n| \xrightarrow[n\to\infty]{a.s.} \overline{P}/2$ and obtain that $a.s.$ for all $x > 0$

$$\liminf_{n\to\infty} p_{G_n}(x) \geq \log a + \int_a^x \frac{\mathbb{E}[Y(t)]}{t}\,\mathrm{d}t\ ,$$

$$\limsup_{n\to\infty} p_{G_n}(x) \leq \log a + \frac{\overline{P}}{2}\log(1+\frac{1}{a^2}) + \int_a^x \frac{\mathbb{E}[Y(t)]}{t}\,\mathrm{d}t\ .$$

Therefore $a.s.$ for all $x > 0$

$$0 \leq \limsup_{n\to\infty} p_{G_n}(x) - \liminf_{n\to\infty} p_{G_n}(x) \leq \frac{\overline{P}}{2}\log(1+\frac{1}{a^2}) \xrightarrow[a\to\infty]{} 0\,,$$

which entails existence of $\lim_{n\to\infty} p_{G_n}(x)$ and completes the proof. □

Corollary 6.22. *In the hypothesis of corollary 6.21, if* $\overline{P} > 0$*, almost surely the pressure density* $\lim_{n\to\infty} p_{G_n}$ *is an analytic function of the monomer density* $\lim_{n\to\infty} m_{G_n}$.

Proof. Set $p_n := p_{G_n}$, $p := \lim_{n\to\infty} p_n$ and $m_n := m_{G_n}$, $m := \lim_{n\to\infty} m_n$.
By theorem 6.1 and corollary 6.21 on an event of probability 1 the monomer density m and the pressure p are analytic functions of the monomer activity $x > 0$. Now a direct computation shows that

$$x \,\frac{\partial m_n}{\partial x}(x) \;=\; \frac{\langle |M(D)|^2\rangle_{G_n,x} - \langle |M(D)|\rangle^2_{G_n,x}}{|V_n|} \;\geq\; 0 \;.$$

But a more precise lower bound is provided by theorems 7.3 and 7.6 in [55]:

$$x \,\frac{\partial m_n}{\partial x}(x) \;\geq\; \frac{|V_n|}{|E_n|}\, x^2 \left(1 - m_n(x)\right)^2 \quad \text{and} \quad 1 - m_n(x) \;\geq\; \frac{2}{x^2+2}\,\frac{|E_n|}{|V_n|} \;,$$

hence

$$x \,\frac{\partial m_n}{\partial x}(x) \;\geq\; \frac{4\,x^2}{(x^2+2)^2}\,\frac{|E_n|}{|V_n|} \;\xrightarrow[n\to\infty]{}\; \frac{2\,x^2}{(x^2+2)^2}\,\overline{P} \;.$$

By corollary 6.20 it follows:

$$x \,\frac{\partial m}{\partial x}(x) \;\geq\; \frac{2\,x^2}{(x^2+2)^2}\,\overline{P} \;>\; 0 \;.$$

Thus m is an analytic function of x with non-zero derivative, so that it is invertible and its inverse is analytic (e.g. see theorem 6.1 p. 76 of [69]). In other words x can be seen as an analytic function of m. Since the composition of analytic functions is analytic, it is proved that p is an analytic function of m. □ □

We are ready to proof the second main theorem of this chapter.

Proof of the Theorem 6.2. By theorem 6.1 and corollary 6.21 one already knows that almost surely there exist $\lim_{n\to\infty} x\, \frac{\partial p_{G_n}}{\partial x}(x) =: m(x)$ and $\lim_{n\to\infty} p_{G_n}(x) =: p(x)$ and that

$$p(x) = p(a) + \int_a^x \frac{m(t)}{t}\,\mathrm{d}t \;, \quad \text{i.e.} \;\; x\,\frac{\partial p}{\partial x}(x) = m(x) \;. \tag{6.20}$$

Applying remark 2.5 to G_n and passing to the limit exploiting remark 6.18, one obtains the following bounds

$$\log x \;\le\; p(x) \;\le\; \log x + \frac{\overline{P}}{2}\,\log(1+\frac{1}{x^2})\;, \quad \text{thus} \lim_{x\to+\infty} p(x) - \log x = 0\;. \tag{6.21}$$

Now set

$$\widetilde{p}(x) \;:=\; \mathbb{E}\big[\log\big(x+\sum_{i=1}^{\Delta}\frac{X_i}{x}\big)\big] \;-\; \frac{\overline{P}}{2}\,\mathbb{E}\big[\log\big(1+\frac{X_1}{x}\frac{X_2}{x}\big)\big]\;.$$

In order to prove that $p(x)=\widetilde{p}(x)$ it will suffice to show that $\widetilde{p}$ shares the two previous properties. Hence split the proof in two lemmas.

Lemma 6.23. *For every $x>0$*

$$x\,\frac{\partial\widetilde{p}}{\partial x}(x) \;=\; m(x)\;.$$

The random complex function $z\mapsto X(z)=\lim_{r\to\infty}\mathcal{M}_z(\mathcal{T}(\varrho,r),o)$ is a.s. analytic on $\mathbb{H}_+$ by corollary 6.13 and it is bounded by a deterministic function by lemma 6.1: $|X(z)|\le\frac{|z|}{\Re(z)}$. As a consequence also its derivative at $z_0\in\mathbb{H}_+$ is bounded by a deterministic constant, precisely fixing $r>0$ such that $\overline{B}(z_0,r)\subset\mathbb{H}_+$ the integral representation (e.g. see theorem 7.3 p. 128 in [69]) gives

$$\big|\frac{\mathrm{d}X}{\mathrm{d}z}(z_0)\big| \;=\; \big|\frac{1}{2\pi i}\int_{S(z_0,r)}\frac{X(z)}{(z-z_0)^2}\,\mathrm{d}z\big| \;\le\; \frac{1}{r}\,\max_{S(z_0,r)}\frac{|z|}{\Re(z)} \;=:c(z_0)\;.$$

It follows that the random functions under expectation in the expression of $\widetilde{p}$ are differentiable with integrable derivatives:

$$\big|x\,\frac{\partial}{\partial x}\log\big(x+\sum_{i=1}^{\Delta}\frac{X_i}{x}\big)\big| = \big|\frac{x+\sum_{i=1}^{\Delta}(\frac{\partial X_i}{\partial x}-\frac{X_i}{x})}{x+\sum_{i=1}^{\Delta}\frac{X_i}{x}}\big| \le \frac{x+\Delta\,(c(x)+\frac{1}{x})}{x} \;\in L^1(\mathbb{P}),$$

$$\big|x\,\frac{\partial}{\partial x}\log\big(1+\frac{X_1}{x}\frac{X_2}{x}\big)\big| = \big|\frac{\frac{\partial X_1}{\partial x}\frac{X_2}{x}+\frac{X_1}{x}\frac{\partial X_2}{\partial x}-2\,\frac{X_1}{x}\frac{X_2}{x}}{1+\frac{X_1}{x}\frac{X_2}{x}}\big| \le 2c(x)\,\frac{1}{x}+2\frac{1}{x^2}\,.$$

Thus one may apply Lebesgue's dominated convergence theorem and take the derivative under expectation, finding:

$$x\,\frac{\partial\widetilde{p}}{\partial x}(x) \;=\; \mathbb{E}\Big[\frac{x+\sum_{i=1}^{\Delta}(\frac{\partial X_i}{\partial x}-\frac{X_i}{x})}{x+\sum_{i=1}^{\Delta}\frac{X_i}{x}}\Big] \;-\; \frac{\overline{P}}{2}\,\mathbb{E}\Big[\frac{\frac{\partial X_1}{\partial x}\frac{X_2}{x}+\frac{X_1}{x}\frac{\partial X_2}{\partial x}-2\,\frac{X_1}{x}\frac{X_2}{x}}{1+\frac{X_1}{x}\frac{X_2}{x}}\Big]\,.$$

Now reordering terms and setting

$$
\begin{aligned}
I_0 &:= \mathbb{E}\Big[\frac{x}{x+\sum_{i=1}^{\Delta}\frac{X_i}{x}}\Big] \\
I_1 &:= -\,\mathbb{E}\Big[\frac{\sum_{i=1}^{\Delta}\frac{X_i}{x}}{x+\sum_{i=1}^{\Delta}\frac{X_i}{x}}\Big] + \overline{P}\,\mathbb{E}\Big[\frac{\frac{X_1}{x}\frac{X_2}{x}}{1+\frac{X_1}{x}\frac{X_2}{x}}\Big] \\
I_2 &:= \mathbb{E}\Big[\frac{\sum_{i=1}^{\Delta}\frac{\partial X_i}{\partial x}}{x+\sum_{i=1}^{\Delta}\frac{X_i}{x}}\Big] - \overline{P}\,\mathbb{E}\Big[\frac{\frac{X_1}{x}\frac{\partial X_2}{\partial x}}{1+\frac{X_1}{x}\frac{X_2}{x}}\Big]
\end{aligned}
$$

one may write $x\,\frac{\partial \widetilde{p}}{\partial x} = I_0 + I_1 + I_2$. Observe that $I_0 = m(x)$ by theorem 6.1. Then showing that $I_1 = I_2 = 0$ will prove the lemma.

Start proving that $I_1 = 0$. First condition on the values of Δ, use the fact that $(X_i)_{i\in\mathbb{N}}$ are i.i.d. and independent of Δ and K, and exploit the hypothesis of unimodularity (i.e. $d\,P_d = \overline{P}\varrho_{d-1}\ \forall\, d \geq 1$):

$$
\begin{aligned}
\mathbb{E}\Big[\frac{\sum_{i=1}^{\Delta}\frac{X_i}{x}}{x+\sum_{i=1}^{\Delta}\frac{X_i}{x}}\Big] &= \sum_{d=0}^{\infty}\sum_{i=1}^{d}\mathbb{E}\Big[\frac{\frac{X_i}{x}}{x+\sum_{i=1}^{d}\frac{X_i}{x}}\Big]\,P_d = \sum_{d=0}^{\infty} d\,\mathbb{E}\Big[\frac{\frac{X_d}{x}}{x+\sum_{i=1}^{d}\frac{X_i}{x}}\Big]\,P_d \\
&= \sum_{d=1}^{\infty}\overline{P}\,\mathbb{E}\Big[\frac{\frac{X_d}{x}}{x+\sum_{i=1}^{d}\frac{X_i}{x}}\Big]\,\varrho_{d-1} = \overline{P}\,\mathbb{E}\Big[\frac{\frac{X_{K+1}}{x}}{x+\sum_{i=1}^{K+1}\frac{X_i}{x}}\Big]\,,
\end{aligned}
$$

then exploit the fact that $X/x \overset{\mathcal{D}}{=} (x+\sum_{i=1}^{K} X_i/x)^{-1}$:

$$
\overline{P}\,\mathbb{E}\Big[\frac{\frac{X_{K+1}}{x}}{x+\sum_{i=1}^{K+1}\frac{X_i}{x}}\Big] = \overline{P}\,\mathbb{E}\Big[\frac{\frac{X_2}{x}}{(\frac{X_1}{x})^{-1}+\frac{X_2}{x}}\Big] = \overline{P}\,\mathbb{E}\Big[\frac{\frac{X_1}{x}\frac{X_2}{x}}{1+\frac{X_1}{x}\frac{X_2}{x}}\Big]\,.
$$

This proves $I_1 = 0$. An analogous reasoning proves that $I_2 = 0$; one should only observe that the family of couples $(X_i\,,\,\frac{\partial X_i}{\partial x})_{i\in\mathbb{N}}$ can be chosen i.i.d. and independent of Δ and K (it suffices to work on i.i.d. trees $(\mathcal{T}(\varrho)_i)_{i\in\mathbb{N}}$).

Lemma 6.24.

$$
\lim_{x\to+\infty}\widetilde{p}(x) - \log x = 0\,.
$$

A direct computation and the dominated convergence theorem give

$$
\widetilde{p}(x) - \log x = \mathbb{E}\Big[\log\Big(1+\sum_{i=1}^{\Delta}\frac{X_i}{x^2}\Big)\Big] - \frac{\overline{P}}{2}\,\mathbb{E}\Big[\log\Big(1+\frac{X_1\,X_2}{x^2}\Big)\Big] \xrightarrow[x\to\infty]{} 0\,,
$$

indeed the function $x \mapsto X(x)/x$ is bounded in $[0,1]$ for all $x \geq 1$.

Now lemmas 6.23, 6.24 together with formulae (6.20), (6.21) allow immediately to conclude the proof of the theorem:

$$p(x) - p(a) \;=\; \int_a^x \frac{m(t)}{t}\,\mathrm{d}t \;=\; \widetilde{p}(x) - \widetilde{p}(a) \quad\Rightarrow$$

$$p(x) \underbrace{-p(a) + \log a}_{\to 0 \text{ as } a\to\infty} \;=\; \widetilde{p}(x) \underbrace{-\widetilde{p}(a) + \log a}_{\to 0 \text{ as } a\to\infty} \quad\Rightarrow\quad p(x) = \widetilde{p}(x)\ . \qquad \square$$

6.4 Numerical estimates

To conclude we consider the particular case when the graphs sequence $(G_n)_{n\in\mathbb{N}}$ locally converges to $\mathcal{T}(P,\varrho)$ with $P = \varrho = \text{Poisson}(2)$ (e.g. this is the case of G_n Erdős-Rényi with $c = 2$), and we show an approximate plot of the monomer density $m(x) := \lim_{n\to\infty} m_{G_n}(x)$.

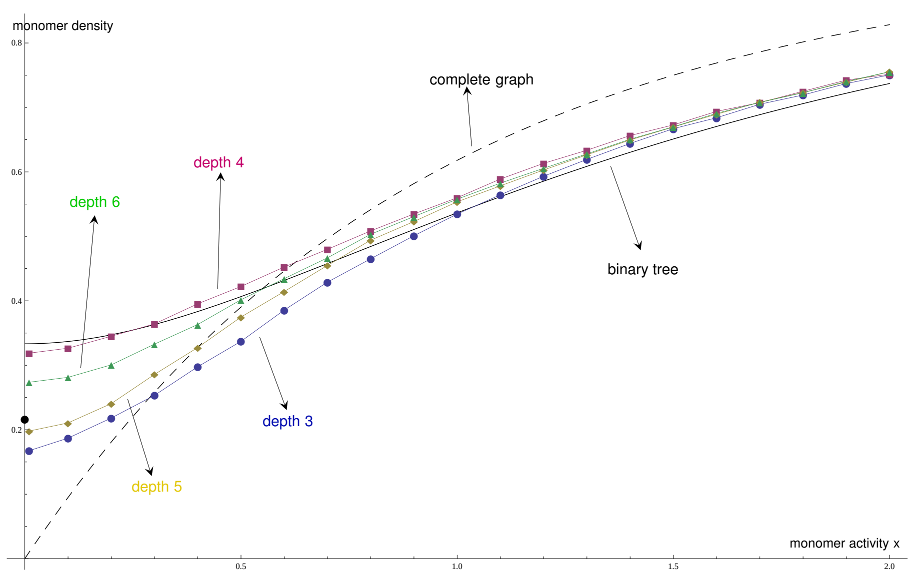

Figure 6.1: The figure displays upper (even depths) and lower (odd depths) bounds for the monomer density m versus the monomer activity x, in the Erdős-Rényi case with $c = 2$. The binary tree (continuous line) and the complete graph (dashed line) cases (treated in [55]) are also shown.

We describe briefly how to obtain it. The distributional recursion $X \overset{d}{=} x^2/(x^2 + \sum_{i=1}^{K} X_i)$ with $K \sim P = \text{Poisson}(2)$ is iterated a finite number r of times with initial values $X_i \equiv 1$. The obtained random variable $X(r)$ represents the monomer density on a truncated Galton-Watson tree $\mathcal{T}(P, P, r)$ (lemma 6.1). If X is the fixed point of the equation, we know that $X(2r) \searrow X$, $X(2r+1) \nearrow X$ as $r \to \infty$ (proposition 6.4, theorem 6.3) and that $\mathbb{E}[X]$ is the asymptotic monomer density on $(G_n)_{n\in\mathbb{N}}$ (theorem 6.1).

For values of $x = 0.01$, 0.1, 0.2, $\ldots$, 2, the random variables $X(r)$, $r = 3, 4, 5, 6$ are simulated numerically 10000 times and an empirical mean is done in order to approximate $\mathbb{E}[X(r)]$. The results are plotted as circles, squares, diamonds, triangles connected by straight lines.

The dot at 0.216074 on the vertical axes corresponds to the exact value of the monomer density when the monomer activity $x \to 0$, supplied by the Karp-Sipser formula [63] or by its extension due to Bordenave, Lelarge, Salez [18]. Therefore the graph of the monomer density $x \mapsto \mathbb{E}[X] = \lim_{n\to\infty} m_{G_n}(x)$ starts from (0, 0.216074) and lays between the diamonds and triangles curves.

Chapter 7

A liquid crystal model on the 2D-lattice

This chapter is based on the work [3]. We study a particular monomer-dimer model with **hard-core and imitative interactions** on the **2-dimensional lattice**[1] $\mathbb{Z}^2$. This model favours one orientation of the dimers (e.g. the horizontal one), both via a chemical potential and via a short-range imitation: we choose different potentials μ_h, μ_v for horizontal and vertical dimers and we consider an imitation potential $J > 0$ for pairs of neighbouring collinear dimers. We prove that when the parameters satisfy

$$\mu_\mathrm{h} > -J \quad \text{and} \quad \mu_\mathrm{v} < -\frac{5}{2}\,J\,, \tag{7.1}$$

the system has the properties of a **liquid crystal**: namely, at low temperatures, it displays a long-range order in the orientation of its molecules, while there is no complete ordering in their positions. In other words: clearly the choice of the dimer potentials results in more horizontal than vertical dimers, on the other hand a local perturbation of the system does not influence the position (left or right) of the horizontal dimer attached to a distant vertex.

[1] By the lattice $\mathbb{Z}^2$, we mean the graph with vertex set $\mathbb{Z}^2$ and edge set $E(Z^2)$ composed by the pairs of vertices having euclidean distance 1

Onsager [80] was the first to propose hard-rods models in order to explain the existence of liquid crystals. In 1979 Heilmann and Lieb [57] proposed two monomer-dimer models (named I and II) on the lattice $\mathbb{Z}^2$, where short-range attractive interactions among parallel dimers are considered beyond the hard-core interaction. They claimed that these systems are liquid crystals. In particular they proved the presence of a phase transition, by means of a reflection positivity argument [41]: at low temperature there is orientational order. Moreover they conjectured the absence of complete translational ordering for their models. A proof of this conjecture for the model I was announced in [57] by Heilmann and Kjær, but never appeared. Letawe, in her thesis [72], claimed to prove the conjecture by cluster expansion methods, but some proofs are missing and the result was not published. Numerical simulations related to the Heilmann-Lieb conjecture are performed in [83]. We also mention that, in absence of attractive interaction, systems of sufficiently long hard-rods were proved to display a phase transition and behave like liquid crystals by Disertori and Giuliani [32], using a two scales cluster expansion and the Pirogov-Sinai theory. In presence of attractive interaction, but without monomers, a quantum dimer model was recently proved to have a crystalline phase by Giuliani and Lieb [46]. Our result is in agreement with the Heilmann-Lieb conjecture. Indeed the model studied in this chapter is obtained from the model I of Heilmann and Lieb [57], but while they suppose

$$\mu_{\mathrm{h}} = \mu_{\mathrm{v}} =: \mu \quad \text{and} \quad \mu > -J \,, \tag{7.2}$$

we assume very different horizontal and vertical potentials as in (7.1). This choice of the parameters allows us to work with cluster expansion methods, by defining our polymers starting from regions of vertical dimers, instead of contours.

A *monomer-dimer configuration* on the lattice $\mathbb{Z}^2$ is represented by an occupation vector $\alpha \in \{0,1\}^{E(\mathbb{Z}^2)}$ satisfying the hard-core constraint

$$\sum_{y:\,(x,y)\in E(\mathbb{Z}^2)} \alpha_{(x,y)} \;\leq\; 1 \quad \forall x \in \mathbb{Z}^2\ . \tag{7.3}$$

Dimers on $\mathbb{Z}^2$ may have two different orientations: vertical (*v-dimers*) or horizontal (*h-dimers*), according to the orientation of the occupied edge. Let Λ be a finite sub-lattice of $\mathbb{Z}^2$. Consider a *horizontal boundary condition*[2], namely we assume that every site of $\mathbb{Z}^2 \setminus \Lambda$ has a h-dimers (with either free or fixed positions). Denote by $\mathscr{D}_\Lambda^{\mathrm{h}}$ the set of monomer-dimer configurations on Λ (we allow also dimers toward the exterior[3]) which are compatible with the selected horizontal boundary condition. The Hamiltonian, or energy, of a monomer-dimer configuration is defined as

$$\begin{aligned} H_\Lambda := \frac{\mu_{\mathrm{h}}+J}{2}\,\#\left\{\begin{matrix}\text{sites of }\Lambda\text{ with}\\ \text{monomer}\end{matrix}\right\} + \frac{\mu_{\mathrm{h}}-\mu_{\mathrm{v}}}{2}\,\#\left\{\begin{matrix}\text{sites of }\Lambda\text{ with}\\ \text{v-dimer}\end{matrix}\right\} + \\ + \frac{J}{2}\left(\#\left\{\begin{matrix}\text{sites of }\bar{\Lambda}\text{ with h-dimer}\\ \text{but h-neighbor also to a v-}\\ \text{dimer or a monomer}\end{matrix}\right\} + \#\left\{\begin{matrix}\text{sites of }\bar{\Lambda}\text{ with v-dimer}\\ \text{but v-neighbor also to a h-}\\ \text{dimer or a monomer}\end{matrix}\right\}\right). \end{aligned} \tag{7.4}$$

We assume that the parameters appearing in the Hamiltonian satisfy

$$\mu_{\mathrm{h}} > -J\,, \quad \mu_{\mathrm{h}} \geq \mu_{\mathrm{v}}\,, \quad J > 0\ . \tag{7.5}$$

In this way, if the horizontal boundary condition with free positions is chosen[4], then the *ground states* in $\mathscr{D}_\Lambda^{\mathrm{h}}$ (i.e. the configurations minimizing the energy under the given condition) are exactly the configurations where every site has a h-dimer. The partition function of the system is

$$Z_\Lambda^{\mathrm{h}} := \sum_{\alpha\in\mathscr{D}_\Lambda^{\mathrm{h}}} e^{-\beta H_\Lambda(\alpha)} \tag{7.6}$$

[2]The *external boundary* of Λ is $\partial^{\mathrm{ext}}\Lambda := \{x \in \mathbb{Z}^2 \setminus \Lambda \,|\, x \text{ neighbor of } y \in \Lambda\}$. The *internal boundary* of Λ is instead $\partial\Lambda \equiv \partial^{\mathrm{int}}\Lambda := \{x \in \Lambda \,|\, x \text{ neighbor of } y \in \mathbb{Z}^2 \setminus \Lambda\}$. We set $\bar{\Lambda} := \Lambda \cup \partial^{\mathrm{ext}}\Lambda$.

[3]Namely we allow dimers having one endpoint in Λ and one in $\mathbb{Z}^2 \setminus \Lambda$.

[4]Also fixed positions work, provided that the positions of the two h-dimers at the endpoints of each horizontal line of Λ allow a pure dimer configuration on that line.

where the parameter $\beta > 0$ is the inverse temperature.

Remark 7.1. We want to show that the Hamiltonian (7.4) essentially corresponds to the model I introduced by Heilmann and Lieb in [57], except for the important fact that we allow the horizontal and vertical dimer potentials μ_{h}, μ_{v} to be different, while they take $\mu_{\mathrm{h}} = \mu_{\mathrm{v}} = \mu$. We can introduce another Hamiltonian (that maybe is written in a more natural way; see fig.7.1):

$$\widetilde{H}_\Lambda := -\ \mu_{\mathrm{h}}\,\#\{\text{h-dimers in }\Lambda\ \} - \mu_{\mathrm{v}}\,\#\{\text{v-dimers in }\Lambda\ \} + \\ -\ J\,\#\left\{\begin{matrix}\text{pairs of neighboring} \\ \text{collinear dimers in }\Lambda\end{matrix}\right\} \tag{7.7}$$

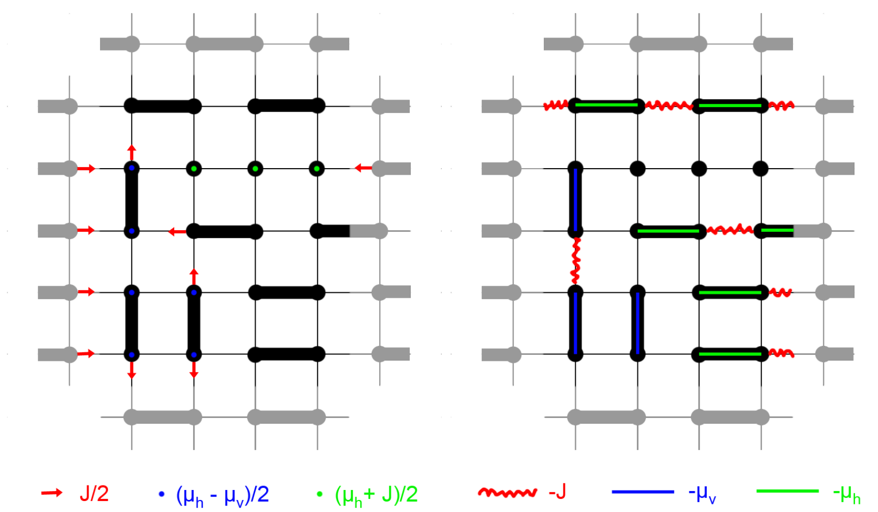

Figure 7.1: The same monomer-dimer configuration on the lattice Λ and the corresponding energies in accordance to the Hamiltonian (7.4) (on the left) and to the Hamiltonian (7.7) (on the right). A horizontal boundary condition is drawn in grey.

The monomer-dimer model I in [57] is given by the Hamiltonian (7.7) with $\mu_{\mathrm{h}} = \mu_{\mathrm{v}} = \mu$, when Λ is a rectangular lattice of even sides lengths with periodic boundary conditions (torus). It is easy to show that when Λ is a torus the two Hamiltonians (7.4), (7.7) describe the same model; indeed they only differ by an additive constant which does not affect the Gibbs measure:

$$\widetilde{H}_\Lambda + \frac{\mu_{\mathrm{h}} + J}{2}\,|\Lambda| \ = \ H_\Lambda \tag{7.8}$$

since

$$|\Lambda| - 2\,\#\{\text{h-dimers in } \Lambda\ \} = |\Lambda| - \#\begin{Bmatrix}\text{sites in } \Lambda \text{ with} \\ \text{h-dimer}\end{Bmatrix} =$$
$$= \#\begin{Bmatrix}\text{sites in } \Lambda \text{ with} \\ \text{monomer}\end{Bmatrix} + \#\begin{Bmatrix}\text{sites in } \Lambda \text{ with} \\ \text{v-dimer}\end{Bmatrix}\ ;$$

$$2\,\#\{\text{v-dimers in } \Lambda\ \} = \#\begin{Bmatrix}\text{sites in } \Lambda \text{ with} \\ \text{v-dimer}\end{Bmatrix}\ ;$$

$$|\Lambda| - 2\,\#\begin{Bmatrix}\text{pairs of neighboring} \\ \text{collinear dimers in } \Lambda\end{Bmatrix} = |\Lambda| - \#\begin{Bmatrix}\text{sites in } \Lambda \text{ with h-dimer (v-dimer)} \\ \text{and h-neighbor (v-neighbor) to an-} \\ \text{other h-dimer (v-dimer)}\end{Bmatrix} =$$
$$= \#\begin{Bmatrix}\text{sites in } \Lambda \text{ with} \\ \text{monomer}\end{Bmatrix} + \#\begin{Bmatrix}\text{sites in } \Lambda \text{ with h-dimer (v-dimer)} \\ \text{and h-neighbor (v-neighbor) also to} \\ \text{something different}\end{Bmatrix}\ .$$

On the other hand when Λ has horizontal boundary conditions the two Hamiltonians (7.4), (7.7) are not exactly equivalent. Indeed it holds[5]

$$\widetilde{H}_\Lambda + \frac{\mu_h + J}{2}\,|\Lambda| + \frac{J}{2}\,\#\begin{Bmatrix}\text{sites in } \partial_v^{\text{int}}\Lambda \text{ with-} \\ \text{out h-dimer}\end{Bmatrix} = H_\Lambda \tag{7.9}$$

when the following conventions are adopted in the definition (7.7): if only half a dimer is in Λ while the other half is in $\mathbb{Z}^2 \setminus \Lambda$, it counts $\frac{1}{2}$; if only one dimer of a pair of neighboring collinear dimers is in Λ, while the other one is in $\mathbb{Z}^2 \setminus \Lambda$, this pair counts $\frac{1}{2}$.

The monomer-dimer model that we have introduced, in a certain region of the parameters corresponding to large horizontal potential, small vertical potential and low temperature, behaves like a *liquid crystal*. This means that the model exhibits an order in the orientation of the molecules (dimers), while there is no complete order in their positions.

The following results will give a precise mathematical meaning to these statements. First we introduce some observables attached to the sites, asking questions as "Is there a horizontal dimer at site x?", "If so, is it positioned to

[5] ∂_v, ∂_h denote respectively the *vertical*, *horizontal component* of the boundary; e.g. $\partial_v\Lambda := \{x \in \Lambda \,|\, x \text{ h-neighbor of } y \in \mathbb{Z}^2 \setminus \Lambda\}$ and $\partial_h\Lambda := \{x \in \Lambda \,|\, x \text{ v-neighbor of } y \in \mathbb{Z}^2 \setminus \Lambda\}$.

the left or to the right of x?". To measure the absence or presence of some kind of order, at a microscopic level we study the expectations and the covariances of these quantities according to the Gibbs measure, while at a macroscopic level we introduce a suitable *order parameter* and study its expectation and possibly its variance[6].

Define the following local observables[7]

$$f_{\mathrm{h},x} := \mathbb{1}\big(x \text{ has a h-dimer}\big)\ , \quad f_{\mathrm{v},x} := \mathbb{1}\big(x \text{ has a v-dimer}\big)\ ; \tag{7.10}$$

$$f_{\mathrm{l},x} := \mathbb{1}\big(x \text{ has a left-dimer}\big)\ , \quad f_{\mathrm{r},x} := \mathbb{1}\big(x \text{ has a right-dimer}\big)\ . \tag{7.11}$$

Clearly $f_{\mathrm{h},x} = f_{\mathrm{l},x} + f_{\mathrm{r},x}$ and $f_{\mathrm{h},x} + f_{\mathrm{v},x} \leq 1$. In the following we denote the Gibbs expectation of any observable f by

$$\langle f \rangle^{\mathrm{h}}_{\Lambda} := \frac{1}{Z^{\mathrm{h}}_{\Lambda}} \sum_{\alpha \in \mathscr{D}^{\mathrm{h}}_{\Lambda}} f(\alpha)\, e^{-\beta H_{\Lambda}(\alpha)}\ .$$

We denote by N the minimal distance between any two vertical components of the boundary of Λ and our only assumption on the shape of Λ is that $N \to \infty$ as $\Lambda \nearrow \mathbb{Z}^2$. To fix ideas one could think that Λ is a rectangle (in this case N would be simply its horizontal side length), but actually we will need to consider also non-simply connected regions.

There exists $\beta_0 > 0$ depending on μ_{h}, μ_{v}, J only and $N_0(\beta)$ depending on β, μ_{h}, J only such that the following results hold true.

Theorem 7.1 (Microscopic expectations). *Assume that* $J > 0$, $\mu_{\mathrm{h}} + J > 0$ *and* $2\mu_{\mathrm{v}} + 5J < 0$. *Let* $\beta > \beta_0$. *Let* $\Lambda \subset \mathbb{Z}^2$ *finite having* $N > N_0(\beta)$. *Let* $x \in \Lambda$ *such that* $\mathrm{dist}_{\mathrm{h}}(x, \partial\Lambda) > N_0(\beta)$. *Then*

$$\langle f_{\mathrm{l},x} \rangle^{\mathrm{h}}_{\Lambda} \geq \frac{1}{2} - e^{-\beta \frac{\mu_{\mathrm{h}}+J}{2}} \quad , \quad \langle f_{\mathrm{r},x} \rangle^{\mathrm{h}}_{\Lambda} \geq \frac{1}{2} - e^{-\beta \frac{\mu_{\mathrm{h}}+J}{2}}\ . \tag{7.12}$$

[6]When the expectation of the order parameter is zero but the variance is not, a small perturbation can lead to a spontaneous order of the system.

[7]We say that the site x has a *left-dimer* if there is a dimer on the bond $\big(x, x-(1,0)\big)$, a *right-dimer* if there is a dimer on the bond $\big(x, x+(1,0)\big)$.

As a consequence:

$$\langle f_{\mathrm{h},x} \rangle_\Lambda^{\mathrm{h}} \geq 1 - 2\, e^{-\beta \frac{\mu_{\mathrm{h}}+J}{2}} \; ; \tag{7.13}$$

$$\left| \langle f_{\mathrm{r},x} \rangle_\Lambda^{\mathrm{h}} - \langle f_{\mathrm{l},x} \rangle_\Lambda^{\mathrm{h}} \right| \leq 2\, e^{-\beta \frac{\mu_{\mathrm{h}}+J}{2}} \; . \tag{7.14}$$

Theorem 7.2 (Microscopic covariances)**.** *Assume that* $J > 0$*,* $\mu_{\mathrm{h}} + J > 0$ *and* $2\mu_{\mathrm{v}} + 5J < 0$*. Let* $\beta > \beta_0$*. Let* $\Lambda \subset \mathbb{Z}^2$ *finite such that* $N > N_0(\beta)$*. Let* $x, y \in \Lambda$ *such that* $\mathrm{dist}_{\mathrm{h}}(x, \partial\Lambda) > N_0(\beta)$*,* $\mathrm{dist}_{\mathrm{h}}(y, \partial\Lambda) > N_0(\beta)$ *and* $\mathrm{dist}_{\mathrm{h}}(x, y) > N_0(\beta)$*. Then:*

$$\left| \langle f_{\mathrm{l},x}\, f_{\mathrm{l},y} \rangle_\Lambda^{\mathrm{h}} - \langle f_{\mathrm{l},x} \rangle_\Lambda^{\mathrm{h}} \langle f_{\mathrm{l},y} \rangle_\Lambda^{\mathrm{h}} \right| \leq \frac{9m}{16}\, e^{-\frac{m}{4}(\mathrm{dist}_{\mathbb{Z}^2}(x,y)-1)} \; , \tag{7.15}$$

$$\left| \langle f_{\mathrm{r},x}\, f_{\mathrm{r},y} \rangle_\Lambda^{\mathrm{h}} - \langle f_{\mathrm{r},x} \rangle_\Lambda^{\mathrm{h}} \langle f_{\mathrm{r},y} \rangle_\Lambda^{\mathrm{h}} \right| \leq \frac{9m}{16}\, e^{-\frac{m}{4}(\mathrm{dist}_{\mathbb{Z}^2}(x,y)-1)} \; , \tag{7.16}$$

$$\left| \langle f_{\mathrm{l},x}\, f_{\mathrm{r},y} \rangle_\Lambda^{\mathrm{h}} - \langle f_{\mathrm{l},x} \rangle_\Lambda^{\mathrm{h}} \langle f_{\mathrm{r},y} \rangle_\Lambda^{\mathrm{h}} \right| \leq \frac{9m}{16}\, e^{-\frac{m}{4}(\mathrm{dist}_{\mathbb{Z}^2}(x,y)-1)} \; . \tag{7.17}$$

The definition of m *is clarified in the Appendix (lemma 7.15); anyway it can be sufficient to know that* $m = e^{-\beta \frac{\mu_{\mathrm{h}}+3J}{2}} \, (1 + o(1))$ *as* $\beta \to \infty$*.*

The density of lattice sites occupied by h-dimers/v-dimers is respectively:

$$\nu_{\mathrm{h}} := \frac{1}{|\Lambda|} \sum_{x \in \Lambda} f_{\mathrm{h},x} \quad , \quad \nu_{\mathrm{v}} := \frac{1}{|\Lambda|} \sum_{x \in \Lambda} f_{\mathrm{v},x} \; . \tag{7.18}$$

A parameter measuring the orientational order of the dimers is

$$\Delta_{\mathrm{orient.}} := \nu_{\mathrm{h}} - \nu_{\mathrm{v}} \; . \tag{7.19}$$

Corollary 7.2 (Orientational Order Parameter)**.** *Assume that* $J > 0$*,* $\mu_{\mathrm{h}} + J > 0$ *and* $2\mu_{\mathrm{v}} + 5J < 0$*. Let* $\beta > \beta_0$*. Let* $\Lambda \subset \mathbb{Z}^2$ *finite, having* $N > 2\, N_0(\beta)$ *. Then*

$$\langle \Delta_{\mathrm{orient.}} \rangle_\Lambda^{\mathrm{h}} \geq \left(1 - 2\, \frac{N_0(\beta)}{N} \right) \left(1 - 4\, e^{-\beta \frac{\mu_{\mathrm{h}}+J}{2}} \right) . \tag{7.20}$$

Hence

$$\lim_{\beta \nearrow \infty} \liminf_{\Lambda \nearrow \mathbb{Z}^2} \langle \Delta_{\mathrm{orient.}} \rangle_\Lambda^{\mathrm{h}} = 1 \; . \tag{7.21}$$

The corollary 7.2 shows that fixing β sufficiently large and then choosing Λ sufficiently big (more precisely the distance N between vertical components of $\partial\Lambda$ must be large enough), the average density of sites occupied by h-dimers is arbitrarily close to $1:$ in other terms the system is oriented along the horizontal direction. The majority of sites is occupied by h-dimers. But there can still be some freedom, indeed we may distinguish the h-dimers in two classes according to their positions: a *h-dimer* is called *even* (resp. *odd*) if its left endpoint has even (resp. odd) horizontal coordinate. The density of lattice sites occupied by even/odd h-dimers is respectively:

$$\begin{aligned}
\nu_{\text{even}} &:= \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \mathbb{1}\big(x \text{ has an even h-dimer}\big) \;=\; \frac{2}{|\Lambda|} \sum_{\substack{x\in\Lambda \\ x_{\text{h}} \text{ even}}} f_{\text{r},x} \;, \\
\nu_{\text{odd}} &:= \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \mathbb{1}\big(x \text{ has an odd h-dimer}\big) \;=\; \frac{2}{|\Lambda|} \sum_{\substack{x\in\Lambda \\ x_{\text{h}} \text{ even}}} f_{\text{l},x} \;.
\end{aligned} \tag{7.22}$$

A parameter measuring the translational order of the h-dimers is

$$\Delta_{\text{transl.}} := \nu_{\text{even}} - \nu_{\text{odd}} \;. \tag{7.23}$$

Corollary 7.3 (Translational Order Parameter. Part I)**.** *Assume that $J > 0$, $\mu_{\text{h}} + J > 0$ and $2\mu_{\text{v}} + 5J < 0$. Let $\beta > \beta_0$. Let $\Lambda \subset \mathbb{Z}^2$ finite such that $N > 2\, N_0(\beta)$. Then*

$$\big|\langle \Delta_{\text{transl.}} \rangle^{\text{h}}_{\Lambda}\big| \;\le\; \Big(1 - 2\frac{N_0(\beta)}{N}\Big)\, 2\, e^{-\beta \frac{\mu_{\text{h}}+J}{2}} + 2\, \frac{N_0(\beta)}{N} \tag{7.24}$$

Hence

$$\lim_{\beta\nearrow\infty} \limsup_{\Lambda\nearrow\mathbb{Z}^2} \;\big|\langle \Delta_{\text{transl.}} \rangle^{\text{h}}_{\Lambda}\big| \;=\; 0 \;. \tag{7.25}$$

Corollary 7.4 (Translational Order Parameter. Part II)**.** *Assume that $J > 0$, $\mu_{\text{h}} + J > 0$ and $2\mu_{\text{v}} + 5J < 0$. Let $\beta > \beta_0$. Let $\Lambda \subset \mathbb{Z}^2$ finite such that $N > 2\, N_0(\beta)$. Then*

$$\big\langle (\Delta_{\text{transl.}})^2 \big\rangle^{\text{h}}_{\Lambda} \;-\; \big(\langle \Delta_{\text{transl.}} \rangle^{\text{h}}_{\Lambda}\big)^2 \;\le\; \frac{1}{|\Lambda|}\, \frac{9m}{(1 - e^{-\frac{m}{4}})^2} \;+\; \frac{N_0(\beta)}{N} \Big(6 - 8\frac{N_0(\beta)}{N}\Big) \;. \tag{7.26}$$

Hence for fixed $\beta > \beta_0$

$$\lim_{\Lambda \nearrow \mathbb{Z}^2} \left\langle (\Delta_{\text{transl.}})^2 \right\rangle_\Lambda^{\text{h}} - \left(\langle \Delta_{\text{transl.}} \rangle_\Lambda^{\text{h}} \right)^2 = 0 \,. \tag{7.27}$$

The corollaries 7.3, 7.4 show that fixing β sufficiently large and then choosing Λ sufficiently big (in particular the distance between different components of $\partial_{\text{v}}\Lambda$ must be big enough), the mean value and the variance of the difference between the density of even h-dimers and the density of odd h-dimers are arbitrarily close to zero. In other terms, at large but finite β, there is not a spontaneous translational order for the h-dimers.

Remark 7.5. The bounds (7.24) hold for any kind of horizontal boundary conditions, but in some particular cases it is possible to obtain a better result by a symmetry argument. Assume that Λ is a rectangle with $N+1$ sites in each horizontal side. If $N+1$ is odd, by choosing *horizontal dimers with free positions at the boundary* one obtains

$$\langle \Delta_{\text{transl.}} \rangle_\Lambda^{\text{h}} = \langle \nu_{\text{even}} \rangle_\Lambda^{\text{h}} - \langle \nu_{\text{odd}} \rangle_\Lambda^{\text{h}} = 0 \tag{7.28}$$

for all parameters β, J, μ_{h}, μ_{v}. To prove it consider the reflection on Λ with respect to the vertical axis at distance $\frac{N}{2}$ from $\partial_{\text{v}}\Lambda$: this transformation induces a bijection $T\colon \mathscr{D}_\Lambda^{\text{h}} \to \mathscr{D}_\Lambda^{\text{h}}$. It is easy to check that $H_\Lambda(T(\alpha)) = H_\Lambda(\alpha)$, $\nu_{\text{even}}(T(\alpha)) = \nu_{\text{odd}}(\alpha)$, $\nu_{\text{odd}}(T(\alpha)) = \nu_{\text{even}}(\alpha)$ for all $\alpha \in \mathscr{D}_\Lambda^{\text{h}}$.

On the other hand if $N+1$ is even, by choosing *periodic boundary conditions* one still obtains

$$\langle \Delta_{\text{transl.}} \rangle_\Lambda^{\text{per.}} = 0 \tag{7.29}$$

for all parameters β, J, μ_{h}, μ_{v}. To prove it one can consider the reflection on Λ with respect to two vertical axis at distance $\frac{N+1}{2}$ from each other: it induces a bijection from $\mathscr{D}_\Lambda^{\text{per.}}$ to itself having all the previous properties.

7.1 Polymer representation

In this section we show how to rewrite the partition function Z_Λ^{h} as a polymer partition function of type (7.94). This representation will be suitable for applying the cluster expansion machinery (see Appendix 7.5) in a regime of large horizontal potential, small vertical potential and low temperature.

We start by isolating the "few" vertical dimers. Associate to each monomer-dimer configuration $\alpha \in \mathscr{D}_\Lambda^{\mathrm{h}}$ the set

$$V = V(\alpha) := \{x \in \Lambda \mid x \text{ has a v-dimer according to } \alpha\} \, .$$

Partition V into its connected components (as a sub-graph of the lattice[8] $\mathbb{Z}^2$):

$$V = \bigcup_{i=1}^{n} S_i \quad , \quad S_i \in \mathscr{S}_\Lambda \; \forall\, i \quad , \quad \mathrm{dist}_{\mathbb{Z}^2}(S_i, S_j) > 1 \; \forall\, i \neq j$$

where the family $\mathscr{S}_\Lambda$ is defined by

$$\begin{aligned} S \in \mathscr{S}_\Lambda \;\stackrel{\text{def}}{\Leftrightarrow}\; & S \subseteq \Lambda \, , \; S \neq \emptyset \, , \; S \text{ connected (as a sub-graph of } \mathbb{Z}^2) \, , \\ & \text{every maximal vertical segment of } S \text{ has an even number} \\ & \text{of sites} \, , \\ & S \text{ does not contains those sites of } \partial_{\mathrm{v}}^{\mathrm{int}}\Lambda \text{ that necessarily} \\ & \text{have a h-dimer because of the boundary conditions.} \end{aligned} \tag{7.30}$$

The knowledge of the set V (or equivalently of $S_1, \dots, S_n$) does not determine completely the configuration α of the system, since on $\Lambda \setminus V$ there can be both h-dimers and monomers. Anyway a fundamental feature of the model is that the system on $\Lambda \setminus V$ can be partitioned into independent 1-dimensional systems. Introduce the family $\mathscr{L}_\Lambda(V)$ defined by

$$L \in \mathscr{L}_\Lambda(V) \;\stackrel{\text{def}}{\Leftrightarrow}\; L \text{ is a maximal horizontal line of } \Lambda \setminus V \, . \tag{7.31}$$

[8]On any graph the distance between two objects is defined as the length of the shortest path connecting them. In particular $\mathrm{dist}_{\mathbb{Z}^2}(S, S') := \inf_{x \in S,\, y \in S'} \mathrm{dist}_{\mathbb{Z}^2}(x, y)$ for all $S, S' \subset \mathbb{Z}^2$ and $\mathrm{dist}_{\mathbb{Z}^2}(x, y) := |x_{\mathrm{h}} - y_{\mathrm{h}}| + |x_{\mathrm{v}} - y_{\mathrm{v}}|$ for all $x = (x_{\mathrm{h}}, x_{\mathrm{v}})$, $y = (y_{\mathrm{h}}, y_{\mathrm{v}}) \in \mathbb{Z}^2$.

The Hamiltonian (7.4) rewrites as

$$H_\Lambda = \sum_{i=1}^{n} \left(\frac{\mu_{\mathrm{h}} - \mu_{\mathrm{v}}}{2} \, |S_i| + \frac{J}{2} \, |\partial_{\mathrm{h}} S_i| + \frac{J}{2} \, |\partial_{\mathrm{v}} S_i \cap \partial\Lambda| \right) + \\ + \sum_{L \in \mathscr{L}_\Lambda(\cup_i S_i)} \left(\frac{\mu_{\mathrm{h}} + J}{2} \, \# \begin{Bmatrix} \text{sites of } L \text{ with} \\ \text{monomer} \end{Bmatrix} + \frac{J}{2} \, \# \begin{Bmatrix} \text{sites of } L \text{ with h-dimer} \\ \text{but h-neighbor also to a} \\ \text{monomer or to } \cup_i S_i \end{Bmatrix} \right) .$$

Hence the partition function (7.6) rewrites as (see fig.7.2)

$$Z^{\mathrm{h}}_\Lambda = \sum_{n \geq 0} \frac{1}{n!} \sum_{\substack{S_1, \ldots, S_n \in \mathscr{S}_\Lambda \\ \mathrm{dist}(S_i, S_j) > 1 \, \forall i \neq j}} \prod_{i=1}^{n} e^{-\beta \left(\frac{\mu_{\mathrm{h}} - \mu_{\mathrm{v}}}{2} |S_i| + \frac{J}{2} |\partial_{\mathrm{h}} S_i| + \frac{J}{2} |\partial_{\mathrm{v}} S_i \cap \partial\Lambda| \right)} \prod_{L \in \mathscr{L}_\Lambda(\cup_i S_i)} Z_L \tag{7.32}$$

where Z_L is the monomer-dimer partition function of the line L, considered as a sub-lattice of the 1-dimensional lattice $\mathbb{Z}$, with suitable boundary conditions:

$$Z_L := \sum_{\alpha_L \in \mathscr{D}_L} e^{-\beta H_L(\alpha_L)} \; e^{I_{\mathrm{l}, x_{\mathrm{l}}}(\alpha_{x_{\mathrm{l}}})} \; e^{I_{\mathrm{r}, x_{\mathrm{r}}}(\alpha_{x_{\mathrm{r}}})} . \tag{7.33}$$

An explanation of the notations introduced in (7.33) is required. $\mathscr{D}_L$ denotes the set of monomer-dimer configurations on L (dimers can only be horizontal, external dimers at the endpoints of L are allowed);

$$H_L := \frac{\mu_{\mathrm{h}} + J}{2} \, \# \begin{Bmatrix} \text{sites of } L \text{ with} \\ \text{monomer} \end{Bmatrix} + \frac{J}{2} \, \# \begin{Bmatrix} \text{sites of } L \text{ with dimer} \\ \text{but h-neighbor also to a} \\ \text{monomer} \end{Bmatrix} ;$$

x_{l}, x_{r} denote respectively the left, right endpoint of the line L (which eventually may coincide): observe[9] that because of (7.31)

$$\bigcup_{L \in \mathscr{L}_\Lambda(\cup_i S_i)} x_{\mathrm{l}}(L) = \left(\left(\cup_i \, \partial^{\mathrm{ext}}_{\mathrm{r}} S_i \right) \cap \Lambda \right) \sqcup \left(\partial_{\mathrm{l}}\Lambda \setminus \cup_i \, \partial_{\mathrm{l}} S_i \right) , \tag{7.34}$$

$$\bigcup_{L \in \mathscr{L}_\Lambda(\cup_i S_i)} x_{\mathrm{r}}(L) = \left(\left(\cup_i \, \partial^{\mathrm{ext}}_{\mathrm{l}} S_i \right) \cap \Lambda \right) \sqcup \left(\partial_{\mathrm{r}}\Lambda \setminus \cup_i \, \partial_{\mathrm{r}} S_i \right) ; \tag{7.35}$$

[9] ∂_{l}, ∂_{r} denote respectively the *left*, *right component* of the vertical boundary; e.g. $\partial_{\mathrm{l}}\Lambda := \{x \in \Lambda \,|\, x - (1,0) \in \mathbb{Z}^2 \setminus \Lambda\}$ and $\partial_{\mathrm{r}}\Lambda := \{x \in \Lambda \,|\, x + (1,0) \in \mathbb{Z}^2 \setminus \Lambda\}$.

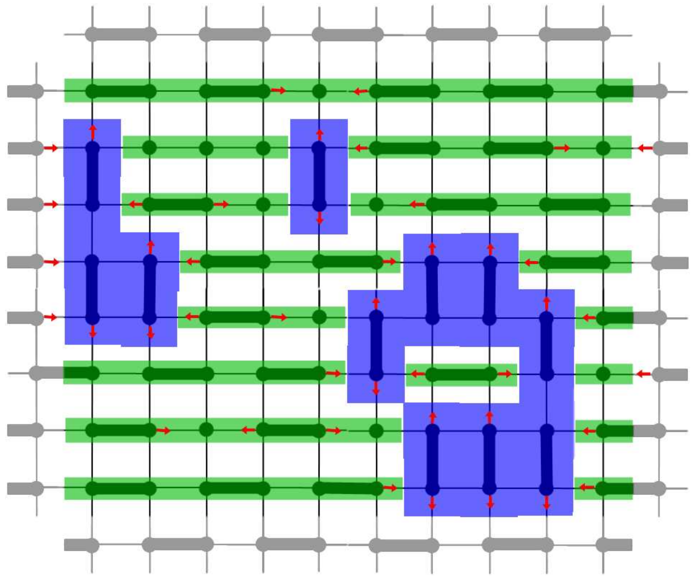

Figure 7.2: A monomer-dimer configuration on Λ and the corresponding regions S_1, S_2, S_3 and lines $L_1, \dots, L_{15} \in \mathscr{L}_\Lambda(\cup_i S_i)$. Given the positions of the regions, the configurations on the lines are mutually independent: the arrows represent the energy contributions of type $J/2$. A horizontal boundary condition is drawn.

finally[10]

$$\begin{aligned}
&\text{if } x_l \in \cup_i \partial_r^{\text{ext}} S_i \;\Rightarrow\; I_{l,x_l} := \begin{pmatrix} -\infty & -\beta\frac{J}{2} & 0 \end{pmatrix} \\
&\text{if } x_l \in \partial_l\Lambda\,,\ \text{on } x_l-(1,0) \text{ it is fixed a l-dimer} \;\Rightarrow\; I_{l,x_l} := \begin{pmatrix} -\infty & 0 & -\beta\frac{J}{2} \end{pmatrix} \\
&\text{if } x_l \in \partial_l\Lambda\,,\ \text{on } x_l-(1,0) \text{ it is fixed a r-dimer} \;\Rightarrow\; I_{l,x_l} := \begin{pmatrix} 0 & -\infty & -\infty \end{pmatrix} \\
&\text{if } x_l \in \partial_l\Lambda\,,\ \text{on } x_l-(1,0) \text{ there is a free h-dimer} \;\Rightarrow\; I_{l,x_l} := \begin{pmatrix} 0 & 0 & -\beta\frac{J}{2} \end{pmatrix}
\end{aligned} \tag{7.36}$$

[10]The possible states of a site $x \in L$ are three: "l"=*left-dimer* namely a dimer on the bond $\big(x, x-(1,0)\big)$, "r"=*right-dimer* namely a dimer on the bond $\big(x, x+(1,0)\big)$, "m"=*monomer*. Here we think I_{l,x_l}, I_{r,x_r} as vectors: $I_{l,x_l} = \begin{pmatrix} I_{l,x_l}(l) & I_{l,x_l}(r) & I_{l,x_l}(m) \end{pmatrix}$ and $I_{r,x_r} = \begin{pmatrix} I_{r,x_r}(l) & I_{r,x_r}(r) & I_{r,x_r}(m) \end{pmatrix}$.

and, similarly,

$$
\begin{aligned}
&\text{if } x_{\rm r} \in \cup_i \partial_{\rm l}^{\rm ext} S_i \;\Rightarrow\; I_{{\rm r},x_{\rm r}} := \begin{pmatrix} -\beta\frac{J}{2} & -\infty & 0 \end{pmatrix} \\
&\text{if } x_{\rm r} \in \partial_{\rm r}\Lambda\,,\ \text{on } x_{\rm r}+(1,0) \text{ it is fixed a r-dimer} \;\Rightarrow\; I_{{\rm r},x_{\rm r}} := \begin{pmatrix} 0 & -\infty & -\beta\frac{J}{2} \end{pmatrix} \\
&\text{if } x_{\rm r} \in \partial_{\rm r}\Lambda\,,\ \text{on } x_{\rm r}+(1,0) \text{ it is fixed a l-dimer} \;\Rightarrow\; I_{{\rm r},x_{\rm r}} := \begin{pmatrix} -\infty & 0 & -\infty \end{pmatrix} \\
&\text{if } x_{\rm r} \in \partial_{\rm r}\Lambda\,,\ \text{on } x_{\rm r}+(1,0) \text{ there is a free h-dimer} \;\Rightarrow\; I_{{\rm r},x_{\rm r}} := \begin{pmatrix} 0 & 0 & -\beta\frac{J}{2} \end{pmatrix}.
\end{aligned}
\tag{7.37}
$$

The 1-dimensional systems described by Z_L, $L \in \mathscr{L}_\Lambda(\cup_i S_i)$, are studied in the Appendix 7.4.

In the form (7.32) of $Z_\Lambda^{\rm h}$, the weight of the regions $(S_1, \ldots, S_n)$ is not a product of the weights of each region S_i, because of the lines L connecting different regions. Therefore the regions $S_i \in \mathscr{S}_\Lambda$ are not a good choice for a polymer representation of the model. In order to decouple some regions from some other ones, it is possible to do a simple trick. It is convenient to deal in different ways with the endpoints lying on $\partial^{\rm ext} S_i$ and those on $\partial\Lambda$; hence given a line $L \in \mathscr{L}_\Lambda(\cup_i S_i)$ we set

$$
\begin{aligned}
\varepsilon_{{\rm l},x_{\rm l}} &:= \mathbb{1}\left(x_{\rm l} \in (\cup_i \partial_{\rm r}^{\rm ext} S_i) \cap \Lambda\right)\,, \quad \eta_{{\rm l},x_{\rm l}} := 1 - \varepsilon_{{\rm l},x_{\rm l}} \overset{(7.34)}{=} \mathbb{1}\left(x_{\rm l} \in (\partial_{\rm l}\Lambda) \setminus \cup_i \partial_{\rm l} S_i\right)\;; \\
\varepsilon_{{\rm r},x_{\rm r}} &:= \mathbb{1}\left(x_{\rm r} \in (\cup_i \partial_{\rm l}^{\rm ext} S_i) \cap \Lambda\right)\,, \quad \eta_{{\rm r},x_{\rm r}} := 1 - \varepsilon_{{\rm r},x_{\rm r}} \overset{(7.35)}{=} \mathbb{1}\left(x_{\rm r} \in (\partial_{\rm r}\Lambda) \setminus \cup_i \partial_{\rm r} S_i\right)\,.
\end{aligned}
$$

Using the notations of the Appendix 7.4, given a line $L \in \mathscr{L}_\Lambda(\cup_i S_i)$ we introduce the two vectors representing the boundary conditions outside its endpoints $x_{\rm l}$, $x_{\rm r}$:

$$
B_{{\rm l},x_{\rm l}} := \begin{pmatrix} e^{I_{{\rm l},x_{\rm l}}(l)} & e^{I_{{\rm l},x_{\rm l}}(r)} & e^{-\beta\frac{\mu_{\rm h}+J}{4} + I_{{\rm l},x_{\rm l}}(m)} \end{pmatrix}\,, \; B_{{\rm r},x_{\rm r}} := \begin{pmatrix} e^{I_{{\rm r},x_{\rm r}}(l)} \\ e^{I_{{\rm r},x_{\rm r}}(r)} \\ e^{-\beta\frac{\mu_{\rm h}+J}{4} + I_{{\rm r},x_{\rm r}}(m)} \end{pmatrix}\;;
$$

then to shorten the notation we set

$$
b_{{\rm l},x_{\rm l}} := \frac{1}{\sqrt{\lambda_1}} B_{{\rm l},x_{\rm l}} E_{\rm r}^{(1)}\,, \quad b_{{\rm r},x_{\rm r}} := \frac{1}{\sqrt{\lambda_1}} E_{\rm l}^{(1)} B_{{\rm r},x_{\rm r}}\,.
$$

Now define

$$R_L := \frac{Z_L}{\lambda_1^{|L|}\, b_{\mathrm{l},x_\mathrm{l}}^{\eta_{\mathrm{l},x_\mathrm{l}}}\, b_{\mathrm{r},x_\mathrm{r}}^{\eta_{\mathrm{r},x_\mathrm{r}}}} - b_{\mathrm{l},x_\mathrm{l}}^{\varepsilon_{\mathrm{l},x_\mathrm{l}}}\, b_{\mathrm{r},x_\mathrm{r}}^{\varepsilon_{\mathrm{r},x_\mathrm{r}}} \tag{7.38}$$

and, using $\mathscr{L}$ as an abbreviation for $\mathscr{L}_\Lambda(\cup_i S_i)$, rewrite the quantity $\prod_{L\in\mathscr{L}} Z_L$ by means of elementary algebraic tricks:

$$\begin{aligned}
\prod_{L\in\mathscr{L}} \frac{Z_L}{\lambda_1^{|L|}} &= \prod_{L\in\mathscr{L}} \left(\left(R_L + b_{\mathrm{l},x_\mathrm{l}}^{\varepsilon_{\mathrm{l},x_\mathrm{l}}}\, b_{\mathrm{r},x_\mathrm{r}}^{\varepsilon_{\mathrm{r},x_\mathrm{r}}}\right) b_{\mathrm{l},x_\mathrm{l}}^{\eta_{\mathrm{l},x_\mathrm{l}}}\, b_{\mathrm{r},x_\mathrm{r}}^{\eta_{\mathrm{r},x_\mathrm{r}}}\right) \\
&= \left(\prod_{L\in\mathscr{L}} b_{\mathrm{l},x_\mathrm{l}}^{\eta_{\mathrm{l},x_\mathrm{l}}}\, b_{\mathrm{r},x_\mathrm{r}}^{\eta_{\mathrm{r},x_\mathrm{r}}}\right) \sum_{\mathscr{K}\subseteq\mathscr{L}} \left(\prod_{L\in\mathscr{K}} R_L\right) \left(\prod_{L\in\mathscr{L}\setminus\mathscr{K}} b_{\mathrm{l},x_\mathrm{l}}^{\varepsilon_{\mathrm{l},x_\mathrm{l}}}\, b_{\mathrm{r},x_\mathrm{r}}^{\varepsilon_{\mathrm{r},x_\mathrm{r}}}\right) .
\end{aligned}$$

By identities (7.34), (7.35) it holds

$$\begin{aligned}
\prod_{L\in\mathscr{L}} b_{\mathrm{l},x_\mathrm{l}}^{\eta_{\mathrm{l},x_\mathrm{l}}}\, b_{\mathrm{r},x_\mathrm{r}}^{\eta_{\mathrm{r},x_\mathrm{r}}} &= \left(\prod_{x\in\partial_\mathrm{l}\Lambda\setminus\cup_i\partial_\mathrm{l}S_i} b_{\mathrm{l},x}\right) \left(\prod_{x\in\partial_\mathrm{r}\Lambda\setminus\cup_i\partial_\mathrm{r}S_i} b_{\mathrm{r},x}\right) \\
\prod_{L\in\mathscr{L}\setminus\mathscr{K}} b_{\mathrm{l},x_\mathrm{l}}^{\varepsilon_{\mathrm{l},x_\mathrm{l}}}\, b_{\mathrm{r},x_\mathrm{r}}^{\varepsilon_{\mathrm{r},x_\mathrm{r}}} &= \left(\prod_{\substack{x\in(\cup_i\partial_\mathrm{r}^{\mathrm{ext}}S_i)\cap\Lambda \\ x\notin \operatorname{supp}\mathscr{K}}} b_{\mathrm{l},x}\right) \left(\prod_{\substack{x\in(\cup_i\partial_\mathrm{l}^{\mathrm{ext}}S_i)\cap\Lambda \\ x\notin \operatorname{supp}\mathscr{K}}} b_{\mathrm{r},x}\right) ;
\end{aligned}$$

By substituting into the previous formula and thinking $\mathscr{K} = \{L_1, \ldots, L_p\}$, we find out[11]

$$\begin{aligned}
\prod_{L\in\mathscr{L}} \frac{Z_L}{\lambda_1^{|L|}} &= \left(\prod_{x\in\partial_\mathrm{v}\Lambda\setminus\cup_i\partial_\mathrm{v}S_i} b_{\mathrm{l/r},\,x}\right) \cdot \\
&\cdot \sum_{p\geq 0} \frac{1}{p!} \sum_{\substack{L_1,\ldots,L_p\in\mathscr{L} \\ L_h\neq L_k\ \forall h\neq k}} \left(\prod_{k=1}^{p} R_{L_k}\right) \left(\prod_{\substack{x\in(\cup_i\partial_\mathrm{v}^{\mathrm{ext}}S_i)\cap\Lambda \\ x\notin\cup_k L_k}} b_{\mathrm{r/l},\,x}\right) .
\end{aligned} \tag{7.39}$$

Now substitute (7.39) into (7.32), using also the fact that $|\Lambda| = \sum_{i=1}^{n} |S_i| +$

[11] In the first product on the r.h.s. of (7.39) the shorten notation $b_{\mathrm{l/r},x}$ means: take $b_{\mathrm{l},x}$ if $x \in \partial_\mathrm{l}\Lambda$, take $b_{\mathrm{r},x}$ if $x \in \partial_\mathrm{r}\Lambda$; notice that $\partial_\mathrm{l}\Lambda$ and $\partial_\mathrm{r}\Lambda$ are disjoint for $N > 1$. In the last product instead the shorten notation $b_{\mathrm{r/l},x}$ means: take $b_{\mathrm{r},x}$ if $x \in \partial_\mathrm{l}^{\mathrm{ext}}S_i$ only, take $b_{\mathrm{l},x}$ if $x \in \partial_\mathrm{r}^{\mathrm{ext}}S_i$ only, and take the product $b_{\mathrm{r},x}\, b_{\mathrm{l},x}$ in the case that x belongs to both $\partial_\mathrm{l}^{\mathrm{ext}}S_i$ and $\partial_\mathrm{r}^{\mathrm{ext}}S_j$.

$\sum_{L\in\mathscr{L}_\Lambda(\cup_i S_i)} |L|$, and obtain:

$$
\begin{aligned}
Z^{\mathrm{h}}_\Lambda &= \lambda_1^{|\Lambda|}\left(\prod_{x\in\partial_{\mathrm{v}}\Lambda} b_{\mathrm{l/r},\,x}\right)\cdot \\
&\quad\cdot\sum_{n\geq 0}\frac{1}{n!}\sum_{\substack{S_1,\dots,S_n\in\mathscr{S}_\Lambda\\ \mathrm{dist}(S_i,S_j)>1\,\forall i\neq j}}\prod_{i=1}^{n}\left(\frac{e^{-\beta\left(\frac{\mu_{\mathrm{h}}-\mu_{\mathrm{v}}}{2}|S_i|+\frac{J}{2}|\partial_{\mathrm{h}}S_i|\right)}}{\lambda_1^{|S_i|}}\prod_{x\in\partial_{\mathrm{v}}\Lambda\cap\partial_{\mathrm{v}}S_i}\frac{e^{-\beta\frac{J}{2}}}{b_{\mathrm{l/r},\,x}}\right)\cdot \\
&\quad\cdot\sum_{p\geq 0}\frac{1}{p!}\sum_{\substack{L_1,\dots,L_p\in\mathscr{L}_\Lambda(\cup_i S_i)\\ L_k\neq L_h\,\forall k\neq h}}\left(\prod_{k=1}^{p}R_{L_k}\right)\left(\prod_{\substack{x\in(\cup_i\partial_{\mathrm{v}}^{\mathrm{ext}}S_i)\cap\Lambda\\ x\notin\cup_k L_k}} b_{\mathrm{r/l},\,x}\right).
\end{aligned}
\tag{7.40}
$$

The next step is to partition $\bigcup_{i=1}^n S_i\cup\bigcup_{k=1}^p L_k$ into connected components as a sub-graph of $\widetilde{\mathbb{Z}}^2$, where $\widetilde{\mathbb{Z}}^2$ is the lattice obtained from $\mathbb{Z}^2$ by removing all the vertical bonds incident to the lines L_k:

$$
\bigcup_{i=1}^{n}S_i\cup\bigcup_{k=1}^{p}L_k=\bigcup_{t=1}^{q}\mathrm{supp}\,P_t\quad ,
$$

$$
P_t\in\mathscr{P}_\Lambda\ \forall t\quad,\quad \mathrm{dist}_{\widetilde{\mathbb{Z}}^2}(\mathrm{supp}\,P_t,\mathrm{supp}\,P_s)>1\ \forall t\neq s
$$

where the family $\mathscr{P}_\Lambda$ (yes, it is finally our family of polymers! see fig.7.3) is defined by:

$$
\mathscr{P}_\Lambda := \left\{P\equiv\left((S_i)_{i\in I},(L_k)_{k\in K}\right)\,\Big|\,(S_i)_i\in\mathscr{P}\mathscr{S}_\Lambda\,,\ (L_k)_k\in\mathscr{P}\mathscr{L}_\Lambda(\cup_i S_i)\right\}\ , \tag{7.41}
$$

$$
(S_i)_{i\in I}\in\mathscr{P}\mathscr{S}_\Lambda \overset{\mathrm{def}}{\Leftrightarrow}\begin{cases}0\leq|I|<\infty\\ S_i\in\mathscr{S}_\Lambda\ \forall i\\ \mathrm{dist}_{\mathbb{Z}^2}(S_i,S_j)>1\ \forall i\neq j\ ,\end{cases} \tag{7.42}
$$

$$
(L_k)_{k\in K}\in\mathscr{P}\mathscr{L}_\Lambda(\cup_{i\in I}S_i) \overset{\mathrm{def}}{\Leftrightarrow}\begin{cases}0\leq|K|<\infty,\ |I|+|K|\geq 1\\ L_k\in\mathscr{L}_\Lambda(\cup_i S_i)\ \forall k\\ L_k\neq L_h\,\forall k\neq h\\ (\cup_i S_i)\cup(\cup_k L_k)\ \text{connected in }\widetilde{\mathbb{Z}}^2\ .\end{cases} \tag{7.43}
$$

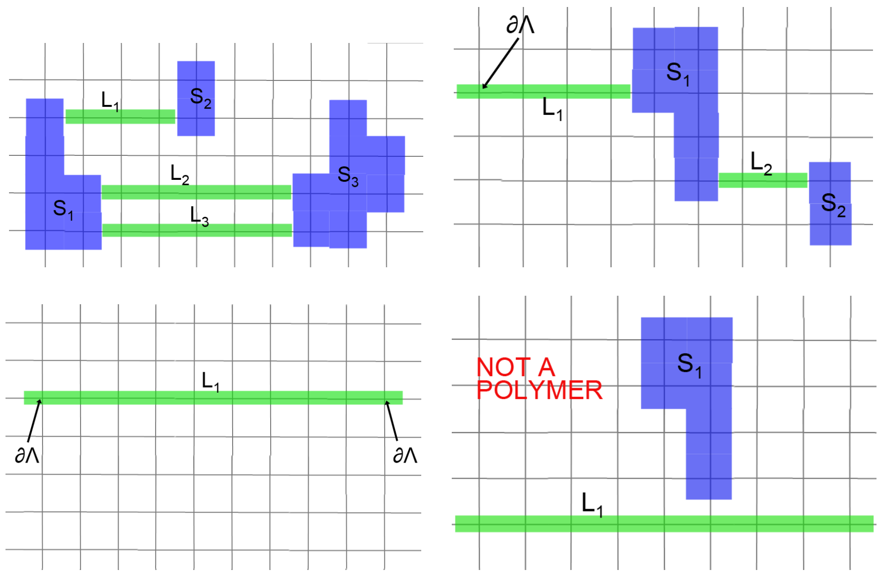

Figure 7.3: The first three pictures represent three different examples of polymers $P \in \mathscr{P}_\Lambda$. The set represented in the last picture is not a unique polymer since it is not connected in $\widetilde{\mathbb{Z}}^2$ (even if it is connected in $\mathbb{Z}^2$).

The identity (7.40) now rewrites as

$$Z_\Lambda^{\mathrm{h}} = C_\Lambda \sum_{q\geq 0} \frac{1}{q!} \sum_{P_1,\ldots,P_q \in \mathscr{P}_\Lambda} \prod_{t=1}^{q} \varrho_\Lambda(P_t) \prod_{t<s} \delta(P_t, P_s) \tag{7.44}$$

by setting, for all $P, P' \in \mathscr{P}_\Lambda$ with $P = \big((S_i)_{i\in I}, (L_k)_{k\in K}\big)$,

$$C_\Lambda := \lambda_1^{|\Lambda|} \prod_{x\in\partial_{\mathrm{v}}\Lambda} b_{\mathrm{l/r},x} \ , \tag{7.45}$$

$$\begin{aligned} \varrho_\Lambda(P) := &\left(\frac{1}{|I|!} \prod_{i\in I} \left(\frac{e^{-\beta\left(\frac{\mu_{\mathrm{h}}-\mu_{\mathrm{v}}}{2}|S_i| + \frac{J}{2}|\partial_{\mathrm{h}} S_i|\right)}}{\lambda_1^{|S_i|}} \prod_{x\in\partial_{\mathrm{v}}\Lambda\cap\partial_{\mathrm{v}} S_i} \frac{e^{-\beta\frac{J}{2}}}{b_{\mathrm{l/r},x}} \right)\right) \cdot \\ &\cdot \left(\frac{1}{|K|!} \prod_{k\in K} R_{L_k}\right) \left(\prod_{\substack{x\in(\bigcup_{i\in I}\partial_{\mathrm{v}}^{\mathrm{ext}} S_i)\cap\Lambda \\ x\notin\bigcup_{k\in K} L_k}} b_{\mathrm{r/l},x} \right) , \end{aligned} \tag{7.46}$$

$$\delta(P, P') := \begin{cases} 1\,, & \text{if } \operatorname{dist}_{\widetilde{\mathbb{Z}}^2}(P, P') > 1 \\ 0\,, & \text{otherwise} \end{cases} . \tag{7.47}$$

The identity (7.44) finally shows that the partition function Z_Λ^{h}, up to a factor C_Λ, admits a polymer representation of the form (7.94).

It is convenient to bound the polymer activity ϱ_Λ by a simpler quantity. Using the proposition 7.18 plus the lemmas 7.16, 7.17 and the fact that $|\partial_{\mathrm{h}} S_i| \geq 2$, one finds:

$$\varrho_\Lambda(P) \,\leq\, \widetilde{\varrho}(P) \,:=\, \left(\frac{1}{|I|!}\prod_{i\in I} e^{-\beta\left(\frac{\mu_{\mathrm{h}}-\mu_{\mathrm{v}}}{2}|S_i|+J\right)}\right)\left(\frac{1}{|K|!}\prod_{k\in K} e^{-m|L_k|}\,\gamma_{L_k}\right) \tag{7.48}$$

with the γ_L's defined by the equation (7.93).

7.2 Convergence of the cluster expansion

In the previous section we rewrote our partition function Z_Λ^{h} as a polymer partition function up to a factor C_Λ (see formula (7.44)). In this section we will find a region of the parameters space $\mu_{\mathrm{h}}, \mu_{\mathrm{v}}, J$ where the condition (7.95) is verified by our model at low temperature, so that the general theorem 7.4 about the convergence of the cluster expansion will apply to our case.

Theorem 7.3. *Assume that $J > 0$, $\mu_{\mathrm{h}} + J > 0$ and $2\mu_{\mathrm{v}} + 5J < 0$. By choosing*

$$a(P) := \frac{m}{2}\,|\operatorname{supp} P| \quad \forall\, P \in \mathscr{P}_\Lambda \tag{7.49}$$

the conditions

$$\sum_{\substack{P\in\mathscr{P}_\Lambda \\ \operatorname{supp} P\ni x}} \widetilde{\varrho}(P)\, e^{a(P)} \,\leq\, \frac{m}{8} \quad \forall\, x \in \Lambda\,, \tag{7.50}$$

$$\sum_{\substack{P\in\mathscr{P}_\Lambda \\ \delta(P,P^*)=0}} \widetilde{\varrho}(P)\, e^{a(P)} \,\leq\, a(P^*) \quad \forall\, P^* \in \mathscr{P}_\Lambda \tag{7.51}$$

hold true, provided that $\beta > \beta_0$ and $N > N_0(\beta)$ (N is the minimum distance between two vertical components of $\partial\Lambda$). Here $\beta_0 > 0$ depends on $\mu_{\mathrm{h}}, \mu_{\mathrm{v}}, J$ only, while $N_0(\beta)$ depends on $\beta, \mu_{\mathrm{h}}, J$ only; they do not depend on Λ, P^*, x .

Corollary 7.6. *Assume that $J > 0$, $\mu_{\mathrm{h}}+J > 0$ and $2\mu_{\mathrm{v}}+5J < 0$. Suppose also that $\beta > \beta_0$ and $N > N_0(\beta)$. Denote by $\mathscr{CP}_\Lambda$ the set of clusters*[12] *composed by polymers of $\mathscr{P}_\Lambda$. Then the partition function* (7.6) *rewrites as*

$$Z_\Lambda^{\mathrm{h}} = C_\Lambda \exp\Big(\sum_{(P_t)_t \in \mathscr{CP}_\Lambda}^{*} U_\Lambda\big((P_t)_t\big)\Big) \tag{7.52}$$

where we denote $\sum^*_{(P_t)_t\in\mathscr{CP}_\Lambda} := \sum_{q\ge 0} \frac{1}{q!} \sum_{(P_t)_{t=1}^q \in \mathscr{CP}_\Lambda}$ *and*

$$U_\Lambda(P_1,\dots,P_q) := u(P_1,\dots,P_q) \prod_{t=1}^{q} \varrho_\Lambda(P_t) \,. \tag{7.53}$$

Remind that C_Λ is defined by (7.45)*, ϱ_Λ is defined by* (7.46) *and u is defined by* (7.97), (7.47)*. Furthermore for all $\mathscr{E} \subseteq \mathscr{P}_\Lambda$ it holds*

$$\sum_{\substack{(P_t)_t\in\mathscr{CP}_\Lambda \\ \exists t:\, P_t\in\mathscr{E}}}^{*} \big|U_\Lambda\big((P_t)_t\big)\big| \le \sum_{\substack{P\in\mathscr{P}_\Lambda \\ P\in\mathscr{E}}} |\varrho_\Lambda(P)|\, e^{a(P)} \tag{7.54}$$

where a is defined by (7.49).

Proof. The corollary follows from the general theory of cluster expansion (theorem 7.4), since Z_Λ^{h} admits a polymer representation (7.44) and satisfies the Kotecky-Preiss condition ((7.51), $|\varrho_\Lambda| \le \widetilde{\varrho}\,$). □ □

For ease of reading, in the following of this section we will denote

$$\sum_{(S_i)_i}^{*} := \sum_n \frac{1}{n!} \sum_{(S_i)_{i=1}^n \in \mathscr{PS}_\Lambda} \qquad \text{and} \qquad \sum_{(L_k)_k}^{*} := \sum_p \frac{1}{p!} \sum_{(L_k)_{k=1}^p \in \mathscr{PL}_\Lambda(\cup_i S_i)}$$

where $\mathscr{PS}_\Lambda$, $\mathscr{PL}_\Lambda(\cup_i S_i)$ are the projections of the polymer set P_Λ defined in (7.42), (7.43). The next lemmas provide the entropy estimates that will be needed in the proof of theorem 7.3.

Lemma 7.7. *If $\cup_i S_i \neq \emptyset$, namely $n \ge 1$, then*

$$\sum_{(L_k)_k}^{*} 1 \le 4^{\sum_i |S_i|} \,. \tag{7.55}$$

[12] As explained in the Appendix 7.5, using the definition (7.47) for δ, a family of polymers $(P_1,\dots,P_q)$ is a *cluster* iff $\cup_{t=1}^q \operatorname{supp} P_t$ is connected in $\widetilde{\mathbb{Z}}^2$.

Proof. Fix $p \geq 0$ and denote by $\mathscr{PL}_\Lambda^{(p)}(\cup_i S_i)$ the set of $(L_k)_{k=1}^p \in \mathscr{PL}_\Lambda(\cup_i S_i)$. Given $(L_k)_{k=1}^p \in \mathscr{PL}_\Lambda^{(p)}(\cup_i S_i)$, each line L_k has at least one endpoint on $\cup_i \partial_{\mathrm{v}}^{\mathrm{ext}} S_i$, since $(\cup_i S_i) \cup (\cup_k L_k)$ have to be connected in $\widetilde{\mathbb{Z}}^2$. Therefore the number of ways to choose each L_k is at most $\sum_i |\partial_{\mathrm{v}}^{\mathrm{ext}} S_i| \leq 2 \sum_i |S_i|$. Since the L_k, $k = 1, \dots, p$, must be all distinct, it follows that

$$\left| \mathscr{PL}_\Lambda^{(p)}(\cup_i S_i) \right| \leq \left(2 \sum\nolimits_i |S_i|\right) \left(2 \sum\nolimits_i |S_i| - 1\right) \cdots \left(2 \sum\nolimits_i |S_i| - p + 1\right) .$$

Therefore

$$\sum_{(L_k)_k}^{*} 1 = \sum_p \frac{1}{p!} \left| \mathscr{PL}_\Lambda^{(p)}(\cup_i S_i) \right| \leq \sum_p \binom{2 \sum_i |S_i|}{p} = 2^{2 \sum_i |S_i|} .$$

□ □

Lemma 7.8. *Let $x \in \mathbb{Z}^2$. For all $s \geq 2$*

$$\#\left\{ S \subset \mathbb{Z}^2 \text{ connected} \,\middle|\, |S| = s ,\ S \ni x \right\} \leq \frac{16}{3} 4^{4s} . \tag{7.56}$$

Proof. Given a connected graph G and one of its vertices x, there exists a walk in G that starts from x and crosses each edge exactly twice[13]. Therefore

$$\begin{aligned} &\#\left\{ S \subset \mathbb{Z}^2 \text{ connected} \,\middle|\, |S| = s ,\ S \ni x \right\} \leq \\ &\leq \sum_{e=s-1}^{2s} \#\left\{ S \text{ connected sub-graph of } \mathbb{Z}^2 \,\middle|\, |\text{edges of } S| = e ,\ S \ni x \right\} \\ &\leq \sum_{e=s-1}^{2s} \#\left\{ \text{walks in } \mathbb{Z}^2 \text{ that start from } x \text{ and have lenght } 2e \right\} \\ &\leq \sum_{e=s-1}^{2s} 4^{2e} \leq \frac{4^{4s+2}}{3} . \end{aligned}$$

□ □

Lemma 7.9. *Let $A \subset \mathbb{Z}^2$ finite. For all $s \geq 2$, $1 \leq d < \infty$*

$$\#\left\{ S \subset \mathbb{Z}^2 \text{ connected} \,\middle|\, |S| = s ,\ \mathrm{dist}_{\mathrm{h}}(S, A) = d \right\} \leq \frac{32}{3} |A|\, 4^{4s} . \tag{7.57}$$

[13]This can be easily proven by induction on the number of edges.

Here $\mathrm{dist_h}(S,A) := \inf_{x\in S,\, y\in A} \mathrm{dist_h}(x,y)$ *and the horizontal distance between* $x=(x_\mathrm{h},x_\mathrm{v})$, $y=(y_\mathrm{h},y_\mathrm{v})\in\mathbb{Z}^2$ *is defined as*

$$\mathrm{dist_h}(x,y) := \begin{cases} |x_\mathrm{h}-y_\mathrm{h}| & \text{if } x_\mathrm{v}=y_\mathrm{v} \\ +\infty & \text{if } x_\mathrm{v}\neq y_\mathrm{v} \end{cases} . \tag{7.58}$$

Proof. Observe that $\mathrm{dist_h}(S,A)=d$ if and only if there exists a horizontal line L, $|L|=d{+}1$, having one endpoint on $\partial_\mathrm{v}A$ and the other one on $\partial_\mathrm{v}S$. Therefore:

$$\begin{aligned}
&\#\big\{S\subset\mathbb{Z}^2 \text{ connected} \;\big|\; |S|=s\,,\ \mathrm{dist_h}(S,A)=d\big\} \le \\
&\le \sum_{\substack{L \text{ horiz. line, } |L|=d+1,\\ \partial_\mathrm{v}A\ni \text{ one endpt. of } L}} \#\big\{S\subset\mathbb{Z}^2 \text{ connected} \;\big|\; |S|=s\,,\ \partial_\mathrm{v}S\ni \text{other endpt. of } L\big\} \\
&\le 2|\partial_\mathrm{v}A|\ \#\big\{S\subset\mathbb{Z}^2 \text{ connected} \;\big|\; |S|=s\,,\ S\ni 0\big\} \le 2|A|\,\frac{16}{3}\,4^{4s}\,.
\end{aligned}$$

For the last inequality we have used the lemma 7.8. □ □

Lemma 7.10. *Let* $n\ge 1$. *Let* $\mathcal{T}$ *be a tree over the vertices* $\{1,\dots,n\}$. *Let* $s_i\ge 2$ *for all* $i=1,\dots,n$ *and* $d_{ij}\ge 2$ *for all* $(i,j)\in\mathcal{T}$.
Then given $A\subset\mathbb{Z}^2$ *and* $1\le d<\infty$

$$\begin{aligned}
&\#\big\{(S_i)_{i=1}^n\in\mathscr{PS}_\Lambda \;\big|\; \mathrm{dist_h}(S_1,A)=d\,,\ |S_i|=s_i\,\forall i\,, \\
&\qquad\qquad \mathrm{dist_h}(S_i,S_j)=d_{ij}\,\forall(i,j)\in\mathcal{T}\big\} \le \\
&\le |A|\prod_{i=1}^n\left(\frac{32}{3}\,4^{4s_i}\,s_i^{\deg_\mathcal{T}(i)}\right)\ ;
\end{aligned} \tag{7.59}$$

while given $x\in\mathbb{Z}^2$

$$\begin{aligned}
&\#\big\{(S_i)_{i=1}^n\in\mathscr{PS}_\Lambda \;\big|\; S_1\ni x\,,\ |S_i|=s_i\,\forall i\,, \\
&\qquad\qquad \mathrm{dist_h}(S_i,S_j)=d_{ij}\,\forall(i,j)\in\mathcal{T}\big\} \le \\
&\le \prod_{i=1}^n\left(\frac{32}{3}\,4^{4s_i}\,s_i^{\deg_\mathcal{T}(i)}\right)\ .
\end{aligned} \tag{7.60}$$

Here $\deg_\mathcal{T}(i)$ *denotes the degree of the vertex* i *in the tree* $\mathcal{T}$.

Proof. Let start by proving the inequality (7.59) by induction on n. If $n=1$, then the tree $\mathcal{T}$ is trivial and (7.59) is already provided by the lemma 7.9. Now let $n \geq 2$, assume that (7.59) holds for at most $n-1$ vertices and prove it for n. It is convenient to think that the tree $\mathcal{T}$ is rooted at the vertex 1 and denote by $j \leftarrow i$ the relation "vertex j is son of vertex i in $\mathcal{T}$" and by $\mathcal{T}(i)$ the sub-tree of $\mathcal{T}$ induced by the vertex i together with its descendants. Then, denoting by $N_{\mathcal{T},1}\big(A, d; (s_i)_{i\in\mathcal{T}}, (d_{ij})_{(i,j)\in\mathcal{T}}\big)$ the cardinality on the l.h.s. of (7.59), it holds

$$N_{\mathcal{T},1}\big(A, d; (s_i)_{i\in\mathcal{T}}, (d_{ij})_{(i,j)\in\mathcal{T}}\big) \;=\; = \sum_{\substack{S_1\in\mathscr{S}_\Lambda,\ |S_1|=s_1 \\ \mathrm{dist_h}(S_1,A)=d}} \prod_{v\leftarrow 1} N_{\mathcal{T}(v),v}\big(S_1, d_{1v}; (s_i)_{i\in\mathcal{T}(v)}, (d_{ij})_{(i,j)\in\mathcal{T}(v)}\big)\ .$$

Since $\mathcal{T}(v)$ has at most $n-1$ vertices, the induction hypothesis gives

$$N_{\mathcal{T}(v),v}\big(S_1, d_{1v}; (s_i)_{i\in\mathcal{T}(v)}, (d_{ij})_{(i,j)\in\mathcal{T}(v)}\big) \;\leq\; s_1 \prod_{i\in\mathcal{T}(v)} \left(\frac{32}{3}\, 4^{4s_i}\, s_i^{\deg_{\mathcal{T}(v)}(i)}\right)\ .$$

Then by substituting in the previous identity, bounding $\deg_{\mathcal{T}(v)}(i)$ by $\deg_{\mathcal{T}}(i)$ and using the lemma 7.9, one obtains:

$$N_{\mathcal{T},1}\big(A, d; (s_i)_{i\in\mathcal{T}}, (d_{ij})_{(i,j)\in\mathcal{T}}\big) \;\leq\; |A| \prod_{i\in\mathcal{T}} \left(\frac{32}{3}\, 4^{4s_i}\, s_i^{\deg_{\mathcal{T}}(i)}\right)\ .$$

This concludes the proof of (7.59).

In order to prove the inequality (7.60), denote by $N'_{\mathcal{T},1}\big(x; (s_i)_{i\in\mathcal{T}}, (d_{ij})_{(i,j)\in\mathcal{T}}\big)$ the cardinality on the l.h.s. of (7.60) and observe that

$$N'_{\mathcal{T},1}\big(x; (s_i)_{i\in\mathcal{T}}, (d_{ij})_{(i,j)\in\mathcal{T}}\big) \;=\; \sum_{\substack{S_1\in\mathscr{S}_\Lambda,\ |S_1|=s_1 \\ S_1\ni x}} \prod_{v\leftarrow 1} N_{\mathcal{T}(v),v}\big(S_1, d_{1v}; (s_i)_{i\in\mathcal{T}(v)}, (d_{ij})_{(i,j)\in\mathcal{T}(v)}\big)\ .$$

By (7.59) we already know that

$$N_{\mathcal{T}(v),v}\big(S_1, d_{1v}; (s_i)_{i\in\mathcal{T}(v)}, (d_{ij})_{(i,j)\in\mathcal{T}(v)}\big) \;\leq\; s_1 \prod_{i\in\mathcal{T}(v)} \left(\frac{32}{3}\, 4^{4s_i}\, s_i^{\deg_{\mathcal{T}(v)}(i)}\right)\ .$$

Then by substituting in the previous identity, bounding $\deg_{\mathcal{T}(v)}(i)$ by $\deg_{\mathcal{T}}(i)$ and using the lemma 7.8, one obtains:

$$N'_{\mathcal{T},1}\big(x; (s_i)_{i\in\mathcal{T}}, (d_{ij})_{(i,j)\in\mathcal{T}}\big) \;\leq\; \prod_{i\in\mathcal{T}} \left(\frac{32}{3}\, 4^{4s_i}\, s_i^{\deg_{\mathcal{T}}(i)}\right)\ ,$$

which proves (7.60). □ □

of the theorem 7.3. According to the definition (7.47), the condition $\delta(P,P^*) = 0$ implies that $\operatorname{supp} P \cap [\operatorname{supp} P^*]_1 \neq \emptyset$, where $[A]_1 := \{x \in \mathbb{Z}^2 \,|\, \operatorname{dist}_{\mathbb{Z}^2}(x,A) \leq 1\}$. Therefore

$$\sum_{\substack{P\in\mathscr{P}_\Lambda \\ \delta(P,P^*)=0}} \widetilde{\varrho}(P)\, e^{a(P)} \;\leq\; \sum_{x\in[\operatorname{supp} P^*]_1} \sum_{\substack{P\in\mathscr{P}_\Lambda \\ \operatorname{supp} P\ni x}} \widetilde{\varrho}(P)\, e^{a(P)}$$
$$\leq\; 4\,|\operatorname{supp} P^*|\; \max_{x\in\Lambda} \sum_{\substack{P\in\mathscr{P}_\Lambda \\ \operatorname{supp} P\ni x}} \widetilde{\varrho}(P)\, e^{a(P)}\;.$$

Thus, by choosing $a(P) := \frac{m}{2}|\operatorname{supp} P|$ for all $P \in \mathscr{P}_\Lambda$, the inequality (7.51) will be a consequence of (7.50).

We have to prove the inequality (7.50). It is worth to write down explicitly the quantity we will work with (see the definitions (7.48) and (7.49)):

$$\widetilde{\varrho}(P)\, e^{a(P)} \;=\; \left(\frac{1}{n!}\prod_{i=1}^{n} e^{-\left(\beta\frac{\mu_{\rm h}-\mu_{\rm v}}{2}-\frac{m}{2}\right)|S_i|-\beta J}\right)\left(\frac{1}{p!}\prod_{k=1}^{p} e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k}\right)$$

for all $P\in\mathscr{P}_\Lambda$, $P = \left((S_i)_{i=1}^n, (L_k)_{k=1}^p\right)$. Notice that if $\operatorname{supp} P \ni x$, the site x may belong either to a region S_i or to a line L_k; hence we can split the sum on the l.h.s. of (7.50) into two parts:

$$\sum_{\substack{P\in\mathscr{P}_\Lambda \\ \operatorname{supp} P\ni x}} \widetilde{\varrho}(P)\, e^{a(P)} \;=\; \Sigma_1 \,+\, \Sigma_2 \quad \text{with:} \tag{7.61}$$

$$\Sigma_1 \;:=\; \sum_{\substack{(S_i)_i \\ \cup_i S_i\ni x}}^{*} \left(\prod_i e^{-\left(\beta\frac{\mu_{\rm h}-\mu_{\rm v}}{2}-\frac{m}{2}\right)|S_i|-\beta J}\right) \sum_{(L_k)_k}^{*} \prod_k e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k} \tag{7.62}$$

$$\Sigma_2 \;:=\; \sum_{(S_i)_i}^{*} \left(\prod_i e^{-\left(\beta\frac{\mu_{\rm h}-\mu_{\rm v}}{2}-\frac{m}{2}\right)|S_i|-\beta J}\right) \sum_{\substack{(L_k)_k \\ \cup_k L_k\ni x}}^{*} \prod_k e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k}\;. \tag{7.63}$$

During all the proof $o(1)$ will denote any function $\omega = \omega(\beta,\mu_{\rm h},J)$ such that $\omega\to 0$ as $\beta\to\infty$ and ω depends only on β, $\mu_{\rm h}$, J (in particular it does not depend on the choices of $\Lambda\subset\mathbb{Z}^2$, $x\in\mathbb{Z}^2$, $P\in\mathscr{P}_\Lambda$).

***I**. Study of the term Σ_1.*

We fix a family of regions $(S_i)_{i=1}^n$ that contains the point x; we also assume that $\mathscr{PL}_\Lambda(\cup_i S_i)$ is non-empty, otherwise the contribution to Σ_1 is zero. By the lemma 7.7 it holds

$$\sum_{(L_k)_k}^{*} \prod_k e^{-\frac{m}{2}|L_k|}\, \gamma_{L_k} \;\leq\; 4^{\sum_i |S_i|} \max_{(L_k)_k} \prod_k e^{-\frac{m}{2}|L_k|}\, \gamma_{L_k} \tag{7.64}$$

where the maximum is taken over all $(L_k)_k \in \mathscr{PL}_\Lambda(\cup_i S_i)$. The factor γ_{L_k} can take two values (see formula (7.93)), both smaller than 1 for β sufficiently large (uniformly with respect to L_k), since each line L_k must have at least one endpoint on $\cup_i \partial_{\mathrm{v}}^{\mathrm{ext}} S_i$ to ensure that $(\cup_i S_i) \cup (\cup_k L_k)$ is connected in $\widetilde{\mathbb{Z}}^2$.

Obviously $n \geq 1$ in order for $\cup_{i=1}^n S_i$ to contain the point x. It is convenient to consider separately the case $n = 1$ and the case $n \geq 2$:

$$\Sigma_1 \;=\; \Sigma_1' + \Sigma_1'' \,.$$

The case $n = 1$ is easy to deal with, simply by bounding the r.h.s. of (7.64) by $4^{|S|}$ and using the lemma 7.8. Precisely:

$$\begin{aligned} \Sigma_1' \;:=\; & \sum_{\substack{S \in \mathscr{S}_\Lambda \\ S \ni x}} e^{-\left(\beta \frac{\mu_\mathrm{h} - \mu_\mathrm{v}}{2} - \frac{m}{2}\right)|S| - \beta J} \sum_{(L_k)_k}^{*} \prod_k e^{-\frac{m}{2}|L_k|} \gamma_{L_k} \\ \leq\; & \sum_{\substack{S \in \mathscr{S}_\Lambda \\ S \ni x}} e^{-\left(\beta \frac{\mu_\mathrm{h} - \mu_\mathrm{v}}{2} - \frac{m}{2}\right)|S| - \beta J}\; 4^{|S|} \\ \leq\; & \sum_{\substack{s \geq 2 \\ \text{even}}} \frac{16}{3}\, 4^{4s}\, e^{-\left(\beta \frac{\mu_\mathrm{h} - \mu_\mathrm{v}}{2} - \frac{m}{2}\right)s - \beta J}\; 4^{s} \\ =\; & \frac{16}{3}\, 4^{10}\, e^{-\beta\,(\mu_\mathrm{h} - \mu_\mathrm{v} + J)}\, (1 + o(1)) \,. \end{aligned} \tag{7.65}$$

Now assume $n \geq 2$. Fix a family of lines $(L_k)_{k=1}^p \in \mathscr{PL}_\Lambda(\cup_i S_i)$. We can consider the graph $G \equiv G\big((S_i)_i, (L_k)_k\big)$ with vertices $i \in \{1, \dots, n\}$ and edges $k \in \{1, \dots, p\}$: the edge k joins the two vertices i, j iff the line L_k has one endpoint on $\partial_{\mathrm{v}}^{\mathrm{ext}} S_i$ and the other one on $\partial_{\mathrm{v}}^{\mathrm{ext}} S_j$. In the graph G there can be multiple edges, loops and pseudo-edges with a single endpoint. The graph G is connected (it follows from definition 7.43), hence G admits at least one spanning sub-tree $\mathcal{T}$. And clearly, since each factor $e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k}$ is smaller than 1,

$$\prod_{k=1}^{p} e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k} \;\leq\; \prod_{k\in\mathcal{T}} e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k} \;\leq\; \prod_{(i,j)\in\mathcal{T}} e^{-\frac{m}{2}(\mathrm{dist_h}(S_i,S_j)-1)}\,\gamma_{S_i,S_j}$$

where $\gamma_{S,S'} := \left(\frac{1}{2}e^{-\beta J} + e^{-\beta\frac{\mu_\mathrm{h}+J}{2}(\mathrm{dist_h}(S,S')-1)}\right)(1+o(1))$. Therefore:

$$\max_{(L_k)_k} \prod_k e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k} \;\leq\; \max_{\substack{\mathcal{T}\text{ tree over}\\ \{1,\dots,n\}}} \prod_{(i,j)\in\mathcal{T}} e^{-\frac{m}{2}(\mathrm{dist_h}(S_i,S_j)-1)}\,\gamma_{S_i,S_j} \tag{7.66}$$

Now using (7.64) and (7.66) we can bound Σ_1'':

$$\begin{aligned}\Sigma_1'' &:= \sum_{n\geq 2}\frac{1}{n!}\sum_{\substack{(S_i)_{i=1}^n\\ \cup_i S_i\ni x}}\left(\prod_{i=1}^n e^{-\left(\beta\frac{\mu_\mathrm{h}-\mu_\mathrm{v}}{2}-\frac{m}{2}\right)|S_i|-\beta J}\right)\sum_{(L_k)_k}^{*}\prod_k e^{-\frac{m}{2}|L_k|}\,\gamma_{L_k}\\
&\leq \sum_{n\geq 2}\sum_{\substack{\mathcal{T}\text{ tree over}\\ \{1,\dots,n\}}}\frac{1}{n!}\sum_{\substack{(S_i)_{i=1}^n\\ \cup_i S_i\ni x}}\left(\prod_{i=1}^n e^{-\left(\beta\frac{\mu_\mathrm{h}-\mu_\mathrm{v}}{2}-\frac{m}{2}-\log 4\right)|S_i|-\beta J}\right)\cdot\\
&\qquad\qquad\cdot\prod_{(i,j)\in\mathcal{T}} e^{-\frac{m}{2}(\mathrm{dist_h}(S_i,S_j)-1)}\,\gamma_{S_i,S_j}\end{aligned} \tag{7.67}$$

where in the sums we keep implicit that $(S_i)_{i=1}^n \in \mathscr{PS}_\Lambda$.

Substitute into (7.67) the entropy bound[14] (7.60). Since $\cup_i S_i \ni x$, but not necessarily $S_1 \ni x$, an extra factor n appears. Moreover observe that $|S_i|$ is even and ≥ 2 (see the definition (7.30)) and $\mathrm{dist_h}(S_i,S_j)\geq 2$. Then:

$$\begin{aligned}\Sigma_1'' \leq &\sum_{n\geq 2}\sum_{\substack{\mathcal{T}\text{ tree over}\\ \{1,\dots,n\}}}\frac{n}{n!}\sum_{\substack{(s_i)_{i=1,\dots,n}\\ s_i\text{ even }\geq 2}}\sum_{\substack{(d_{ij})_{ij\in\mathcal{T}}\\ d_{ij}\geq 2}}\left(\prod_{i=1}^n \frac{32}{3}\,4^{4s_i}\,s_i^{\deg_\mathcal{T}(i)}\right)\cdot\\
&\cdot\left(\prod_{i=1}^n e^{-\left(\beta\frac{\mu_\mathrm{h}-\mu_\mathrm{v}}{2}-\frac{m}{2}-\log 4\right)s_i-\beta J}\right)\prod_{(i,j)\in\mathcal{T}} e^{-\frac{m}{2}(d_{ij}-1)}\,\gamma_{d_{ij}}\end{aligned} \tag{7.68}$$

where $\gamma_d := \left(\frac{1}{2}e^{-\beta J} + e^{-\beta\frac{\mu_\mathrm{h}+J}{2}(d-1)}\right)(1+o(1))$.

Given $n\geq 2$ and $\delta_1,\dots,\delta_n\geq 1$, the number of trees $\mathcal{T}$ over the vertices $\{1,\dots,n\}$ with given degrees $\deg_\mathcal{T}(i)=\delta_i\ \forall i=1,\dots,n$ is exactly[15]

$$\frac{(n-2)!}{(\delta_1-1)!\cdots(\delta_n-1)!}$$

[14] The families of regions $(S_i)_{i=1}^n$ such that $\mathrm{dist_h}(S_i,S_j)=\infty$ for at least one edge $(i,j)\in\mathcal{T}$ give zero contribution to the sum, therefore we do not need to worry about them.

[15] This is an improvement of the well-known Cayley's formula.

if $\sum_{i=1}^{n}(\delta_i - 1) = n-2$ and zero otherwise. Furthermore the number of edges of $\mathcal{T}$ is $n-1$. Therefore the bound (7.68) leads to

$$\begin{aligned}\Sigma_1'' \le \sum_{n\ge 2} \left(\frac{32}{3}\, e^{-\beta J} \sum_{\substack{s\ge 2\\ \text{even}}} e^{-\left(\beta\frac{\mu_{\rm h}-\mu_{\rm v}}{2} - \frac{m}{2} - 5\log 4\right)s} \sum_{\delta\ge 1} \frac{s^\delta}{(\delta-1)!} \right)^n \cdot \\ \cdot \left(\sum_{d\ge 2} e^{-\frac{m}{2}(d-1)}\, \gamma_d \right)^{n-1} .\end{aligned} \tag{7.69}$$

The sum over s gives, as $\beta \to \infty$,

$$\begin{aligned}&\sum_{\substack{s\ge 2\\ \text{even}}} e^{-\left(\beta\frac{\mu_{\rm h}-\mu_{\rm v}}{2} - \frac{m}{2} - 5\log 4\right)s} \sum_{\delta\ge 1} \frac{s^\delta}{(\delta-1)!} = \\ &= \sum_{\substack{s\ge 2\\ \text{even}}} s\, e^{-\left(\beta\frac{\mu_{\rm h}-\mu_{\rm v}}{2} - \frac{m}{2} - 5\log 4 - 1\right)s} = 2\, e^2\, 4^{10}\, e^{-\beta(\mu_{\rm h}-\mu_{\rm v})} \left(1+o(1)\right) .\end{aligned} \tag{7.70}$$

The sum over d gives, as $\beta \to \infty$,

$$\begin{aligned}&\sum_{d\ge 2} e^{-\frac{m}{2}(d-1)}\, \gamma_d = \\ &= \left(\sum_{d\ge 2} e^{-\frac{m}{2}(d-1)}\, \frac{e^{-\beta J}}{2} + \sum_{d\ge 2} e^{-\frac{m}{2}(d-1)}\, e^{-\beta\frac{\mu_{\rm h}+J}{2}(d-1)} \right) (1+o(1)) \\ &= \left(\frac{1}{1-e^{-\frac{m}{2}}}\, \frac{e^{-\beta J}}{2} + o(1) \right) (1+o(1)) = e^{\beta\frac{\mu_{\rm h}+J}{2}}\, (1+o(1))\end{aligned} \tag{7.71}$$

where we used the fact that $1-e^{-\frac{m}{2}} = \frac{1}{2}\, e^{-\beta\frac{\mu_{\rm h}+3J}{2}}\, (1+o(1))$ (see lemma 7.15). Substituting (7.70), (7.71) into (7.69), one obtains

$$\Sigma_1'' \le \sum_{n\ge 2} \left(\frac{2^{26}e^2}{3}\, e^{-\beta(\mu_{\rm h}-\mu_{\rm v})+\beta\frac{\mu_{\rm h}+J}{2}}\, (1+o(1)) \right)^n e^{-\beta\frac{\mu_{\rm h}+J}{2}}\, (1+o(1)) . \tag{7.72}$$

Assume $\boldsymbol{\mu_{\rm h} - \mu_{\rm v} > \frac{\mu_{\rm h}+J}{2}}$. Then for β sufficiently large (7.72) becomes:

$$\begin{aligned}\Sigma_1'' &\le \left(\frac{2^{26}e^2}{3}\, e^{-\beta(\mu_{\rm h}-\mu_{\rm v})+\beta\frac{\mu_{\rm h}+J}{2}} \right)^2 e^{-\beta\frac{\mu_{\rm h}+J}{2}}\, (1+o(1)) \\ &= \frac{2^{52}e^4}{9}\, e^{-\beta\, 2(\mu_{\rm h}-\mu_{\rm v})+\beta\, \frac{\mu_{\rm h}+J}{2}}\, (1+o(1)) .\end{aligned} \tag{7.73}$$

II. *Study of the term Σ_2.*

The ideas are not far from those seen for Σ_1. We fix a family of regions $(S_i)_{i=1}^n$ and we assume that there exists $(L_k)_k \in \mathscr{PL}_\Lambda(\cup_i S_i)$ such that $\cup_k L_k \ni x$, otherwise the contribution to Σ_2 is zero. Clearly the line $L^x \in \mathscr{L}_\Lambda(\cup_i S_i)$ that contains x is unique. It is convenient to consider separately four cases:

$$\Sigma_2 = \Sigma_2' + \Sigma_2'' + \Sigma_2''' + \Sigma_2'''' .$$

In Σ_2' we assume $n = 0$, namely $\cup_i S_i = \emptyset$; then L^x have to be a maximal horizontal line of Λ. In Σ_2'' we assume $n = 1$, namely there is a unique region S and L^x may have one endpoint on $\partial_{\mathrm{v}}^{\mathrm{ext}} S$ and one on $\partial_{\mathrm{v}}\Lambda$ or both on $\partial_{\mathrm{v}}^{\mathrm{ext}} S$. In Σ_2''' we assume $n \geq 2$ and L^x has one endpoint on $\cup_i \partial_{\mathrm{v}}^{\mathrm{ext}} S_i$ and one on $\partial_{\mathrm{v}}\Lambda$ or both on the same $\partial_{\mathrm{v}}^{\mathrm{ext}} S_i$. In Σ_2'''' we assume $n \geq 2$ and L^x has one endpoint on $\partial_{\mathrm{v}}^{\mathrm{ext}} S_i$ and one on $\partial_{\mathrm{v}}^{\mathrm{ext}} S_j$ with $i \neq j$.

By methods similar to those already seen for Σ_1, one can prove that

$$\Sigma_2' \leq e^{-\frac{m}{2}N}\,(1 + o(1))\,; \tag{7.74}$$

$$\Sigma_2'' \leq \frac{2^{25}\sqrt{2}}{3}\, e^{-\beta\,(\mu_{\mathrm{h}} - \mu_{\mathrm{v}}) + \beta\,\frac{\mu_{\mathrm{h}}+2J}{2}}\,(1 + o(1))\,; \tag{7.75}$$

$$\Sigma_2''' \leq \frac{2^{52}e^4\sqrt{2}}{9}\, e^{-\beta\,2(\mu_{\mathrm{h}} - \mu_{\mathrm{v}}) + \beta\frac{2\mu_{\mathrm{h}}+3J}{2}}\,(1 + o(1))\,; \tag{7.76}$$

$$\Sigma_2'''' \leq \frac{2^{52}e^4}{9}\, e^{-\beta\,2(\mu_{\mathrm{h}} - \mu_{\mathrm{v}}) + \beta(\mu_{\mathrm{h}}+2J)}\,(1 + o(1))\,. \tag{7.77}$$

We refer to the Arxiv version of the paper for the details.

In conclusion, by using the estimates (7.65), (7.73), (7.74), (7.75), (7.76), (7.77), and the fact that $m = e^{-\beta\frac{\mu_{\mathrm{h}}+3J}{2}}\,(1 + o(1))$ (see lemma 7.15), if we assume

$\boldsymbol{\mu}_{\rm h} - \boldsymbol{\mu}_{\rm v} > \frac{\boldsymbol{\mu}_{\rm h}+\boldsymbol{J}}{\boldsymbol{2}}$, we find that:

$$
\begin{aligned}
&\frac{1}{m} \sum_{\substack{P\in\mathscr{P}_\Lambda \\ \text{supp}\, P \ni x}} \tilde{\varrho}(P)\, e^{a(P)} = \\
&= e^{\beta\frac{\mu_{\rm h}+3J}{2}} \left(\Sigma_1' + \Sigma_1'' + \Sigma_2' + \Sigma_2'' + \Sigma_2''' + \Sigma_2''''\right)(1+o(1)) \\
&\le \left(\frac{2^{24}}{3}\, e^{-\beta(\mu_{\rm h}-\mu_{\rm v})+\beta\frac{\mu_{\rm h}+J}{2}} + \frac{2^{52}\, e^4}{9}\, e^{-\beta\, 2(\mu_{\rm h}-\mu_{\rm v})+\beta\frac{\mu_{\rm h}+2J}{2}} + \frac{1}{m}\, e^{-\frac{m}{2}N} \right. \\
&\quad + \frac{2^{25.5}}{3}\, e^{-\beta(\mu_{\rm h}-\mu_{\rm v})+\beta\frac{2\mu_{\rm h}+5J}{2}} + \frac{2^{52.5}\, e^4}{9}\, e^{-\beta\, 2(\mu_{\rm h}-\mu_{\rm v})+\beta\frac{3\mu_{\rm h}+6J}{2}} \\
&\quad \left. + \frac{2^{52}\, e^4}{9}\, e^{-\beta\, 2(\mu_{\rm h}-\mu_{\rm v})+\beta\frac{3\mu_{\rm h}+7J}{2}} \right)(1+o(1)) \\
&= \left(\frac{1}{m}\, e^{-\frac{m}{2}N} + \frac{2^{25.5}}{3}\, e^{\beta(\mu_{\rm v}+\frac{5J}{2})} \right)(1+o(1))
\end{aligned}
\tag{7.78}
$$

where N is the minimum distance between two different vertical components of $\partial\Lambda$ and $o(1) \to 0$ as $\beta \to \infty$ (uniformly with respect to N).

Now we assume that $\boldsymbol{\mu}_{\rm v} + \frac{\boldsymbol{5J}}{\boldsymbol{2}} < \boldsymbol{0}$. Thus there exists $\beta_0 > 0$ such that for all $\beta > \beta_0$ the function $1 + o(1)$ on the r.h.s. of (7.78) is < 2 and the term $\frac{2^{25.5}}{3}\, e^{\beta(\mu_{\rm v}+\frac{5J}{2})} \le 1/32$. There exists[16] also $N_0(\beta)$ such that for all $N > N_0(\beta)$ the term $\frac{1}{m}\, e^{-\frac{m}{2}N} \le 1/32$. Therefore if $\mu_{\rm v} + \frac{5J}{2} < 0$ (which entails also the previous condition $\mu_{\rm h} - \mu_{\rm v} > \frac{\mu_{\rm h}+J}{2}$), then the inequality (7.78) implies that

$$
\sum_{\substack{P\in\mathscr{P}_\Lambda \\ \text{supp}\, P \ni x}} \tilde{\varrho}(P)\, e^{a(P)} \le \frac{m}{8}
$$

for $\beta > \beta_0$ and $N > N_0(\beta)$. This concludes the proof. □ □

7.3 Proofs of the liquid crystal properties

In this section we will finally prove that the model behaves like a liquid crystal, as stated at the beginning of this chapter, by means of the cluster expansion results obtained in the previous sections.

[16] $N_0 = \frac{2}{m} \log \frac{32}{m}$.

Proof of the theorem 7.1. We will prove the inequality (7.12) for $f_{\mathrm{l},x}$. That one for $f_{\mathrm{r},x}$ can be proved analogously; then (7.13) and (7.14) follow since $f_x = f_{\mathrm{l},x} + f_{\mathrm{r},x}$.

Observe that

$$\langle f_{\mathrm{l},x} \rangle_\Lambda^{\mathrm{h}} = \frac{Z^{\mathrm{h}}_{\Lambda\setminus x}}{Z^{\mathrm{h}}_\Lambda} ,$$

where $Z^{\mathrm{h}}_{\Lambda\setminus x}$ is the partition function over the lattice $\Lambda \setminus x$ with horizontal boundary conditions including a left-dimer at the site x. Since $N > N_0(\beta)$ and $\mathrm{dist_h}(x, \partial\Lambda) > N_0(\beta)$, both partition functions satisfy the hypothesis of the corollary 7.6. Hence by the cluster expansion (7.52) the partition functions rewrite as

$$Z^{\mathrm{h}}_\Lambda = C_\Lambda \exp\Bigg(\sum^*_{(P_t)_t \in \mathscr{C}\mathscr{P}_\Lambda} U_\Lambda\big((P_t)_t\big) \Bigg) ,$$

$$Z^{\mathrm{h}}_{\Lambda\setminus x} = C_{\Lambda\setminus x} \exp\Bigg(\sum^*_{(P_t)_t \in \mathscr{C}\mathscr{P}_{\Lambda\setminus x}} U_{\Lambda\setminus x}\big((P_t)_t\big) \Bigg) .$$

By applying the definition (7.45),

$$\frac{C_{\Lambda\setminus x}}{C_\Lambda} = \frac{b_{\mathrm{r},x-(1,0)}\ b_{\mathrm{l},x+(1,0)}}{\lambda_1} .$$

Now consider a polymer $P \in P_\Lambda \cup \mathscr{P}_{\Lambda\setminus x}$. Keeping in mind the definitions of polymer (7.41) and polymer activity (7.46), observe that[17]

$$\text{if } \mathrm{dist_h}(\mathrm{supp}\, P, x) > 1 \quad \Rightarrow \quad P \in \mathscr{P}_\Lambda \cap \mathscr{P}_{\Lambda\setminus x} \ , \ \varrho_\Lambda(P) = \varrho_{\Lambda\setminus x}(P) .$$

Therefore:

$$\sum^*_{(P_t)_t \in \mathscr{C}\mathscr{P}_{\Lambda\setminus x}} U_{\Lambda\setminus x}\big((P_t)_t\big) - \sum^*_{(P_t)_t \in \mathscr{C}\mathscr{P}_\Lambda} U_\Lambda\big((P_t)_t\big) \ \geq$$

$$\geq - \sum^*_{\substack{(P_t)_t \in \mathscr{C}\mathscr{P}_{\Lambda\setminus x} \\ \exists t:\, \mathrm{dist_h}(\mathrm{supp}\, P_t, x) \leq 1}} \big| U_{\Lambda\setminus x}\big((P_t)_t\big) \big| - \sum^*_{\substack{(P_t)_t \in \mathscr{C}\mathscr{P}_\Lambda \\ \exists t:\, \mathrm{dist_h}(\mathrm{supp}\, P_t, x) \leq 1}} \big| U_\Lambda\big((P_t)_t\big) \big| .$$

[17] The condition $\mathrm{dist_h}(\mathrm{supp}\, P, x) > 1$ guarantees that $\mathrm{supp}\, P \subseteq \Lambda \setminus x$ and that the polymer P does not include any line L_k having one endpoint on $x \pm (1, 0)$, nor any region S_i containing these points.

And by the inequalities (7.54) and (7.50) applied to both Z^{h}_{Λ}, $Z^{\mathrm{h}}_{\Lambda\setminus x}$,

$$\sum^{*}_{\substack{(P_t)_t\in\mathscr{CP}_\Lambda\\ \exists t:\,\mathrm{dist_h}(\mathrm{supp}\,P_t,x)\le 1}} \left|U_\Lambda\big((P_t)_t\big)\right| \;\le\; \sum_{\substack{P\in\mathscr{P}_\Lambda\\ \mathrm{dist_h}(\mathrm{supp}\,P,x)\le 1}} \widetilde{\varrho}(P)\,e^{a(P)} \;\le\; 3\,\frac{m}{8}\;;$$

$$\sum^{*}_{\substack{(P_t)_t\in\mathscr{CP}_{\Lambda\setminus x}\\ \exists t:\,\mathrm{dist_h}(\mathrm{supp}\,P_t,x)\le 1}} \left|U_{\Lambda\setminus x}\big((P_t)_t\big)\right| \;\le\; \sum_{\substack{P\in\mathscr{P}_{\Lambda\setminus x}\\ \mathrm{dist_h}(\mathrm{supp}\,P,x)\le 1}} \widetilde{\varrho}(P)\,e^{a(P)} \;\le\; 2\,\frac{m}{8}\;.$$

In conclusion one obtains:

$$\begin{aligned}\langle f_{\mathrm{l},x}\rangle^{\mathrm{h}}_{\Lambda} \;=\; \frac{Z^{\mathrm{h}}_{\Lambda\setminus x}}{Z^{\mathrm{h}}_{\Lambda}} \;&\ge\; \frac{b_{\mathrm{r},\,x-(1,0)}\,b_{\mathrm{l},\,x+(1,0)}}{\lambda_1}\;\exp\Big(-5\,\frac{m}{8}\Big)\\ &=\;\frac{1}{2}\left(1-e^{-\beta\frac{\mu_{\mathrm{h}}+J}{2}}\big(1+o(1)\big)\right)\,,\end{aligned}$$

where the last identity follows from the fact that $\lambda_1\, b_{\mathrm{r},x-(1,0)}\, b_{\mathrm{l},x+(1,0)} \;=\; E^{(1)}_{\mathrm{l}} B_{\mathrm{r},x-(1,0)}\, B_{\mathrm{l},x+(1,0)} E^{(1)}_{\mathrm{r}} \;=$ $\frac{1}{\sqrt{2}}(1-\frac{a}{2}\,(1+o(1)))\,\frac{1}{\sqrt{2}}(1-\frac{a}{2}\,(1+o(1)))$ (by lemma 7.17, since there is a left-dimer fixed at x according to $Z^{\mathrm{h}}_{\Lambda\setminus x}$), $\lambda_1 = 1+\frac{ab}{2}\,(1+o(1))$ (proposition 7.13), and $e^{-5m/8} = 1-\frac{5}{8}ab\,(1+o(1))$ (lemma 7.15). Finally, since $o(1)\to 0$ as $\beta\to\infty$ and $o(1)$ does not depend on the choice of x and Λ, one may obtain the desired inequality eventually increasing β_0 . □

Proof of the corollary 7.2. Set $\varphi_{\Lambda,N_0} := \#\{x\in\Lambda\,|\,\mathrm{dist_h}(x,\partial\Lambda) > N_0\}\,/\,|\Lambda|$. By the theorem 7.1, bound (7.13), using also $f_{\mathrm{v},x}\le 1-f_{\mathrm{h},x}$, one obtains:

$$\langle\Delta_{\mathrm{orient.}}\rangle^{\mathrm{h}}_{\Lambda} \;=\; \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\left(\langle f_{\mathrm{h},x}\rangle^{\mathrm{h}}_{\Lambda}-\langle f_{\mathrm{v},x}\rangle^{\mathrm{h}}_{\Lambda}\right) \;\ge\; \varphi_{\Lambda,N_0(\beta)}\left(1-4\,e^{-\beta\frac{\mu_{\mathrm{h}}+J}{2}}\right)\,.$$

On the other hand:

$$\varphi_{\Lambda,N_0} \;\ge\; \min_{\substack{L\text{ maximal}\\ \text{horiz. line of }\Lambda}} \varphi_{L,N_0} \;=\; \min_{\substack{L\text{ maximal}\\ \text{horiz. line of }\Lambda}} \frac{|L|-2N_0(\beta)}{|L|} \;=\; 1-2\,\frac{N_0(\beta)}{N}\,.$$ □

Proof of the corollary 7.3. Set $\varphi_{\Lambda,N_0} := \#\{x\in\Lambda\,|\,\mathrm{dist_h}(x,\partial\Lambda) > N_0\}\,/\,|\Lambda|$. By the theorem 7.1, bound (7.14),

$$\left|\langle\Delta_{\mathrm{transl.}}\rangle^{\mathrm{h}}_{\Lambda}\right| \;\le\; \frac{2}{|\Lambda|}\sum_{\substack{x\in\Lambda,\\ x_{\mathrm{h}}\text{ even}}}\left|\langle f_{\mathrm{r},x}\rangle^{\mathrm{h}}_{\Lambda}-\langle f_{\mathrm{l},x}\rangle^{\mathrm{h}}_{\Lambda}\right| \;\le\; \varphi_{\Lambda,N_0(\beta)}\,2e^{-\beta\frac{\mu_{\mathrm{h}}+J}{2}}+1-\varphi_{\Lambda,N_0(\beta)}\,.$$

On the other hand we have already observed in the proof of the corollary 7.2 that $\varphi_{\Lambda,N_0}\ge 1-2N_0/N$. □

Proof of the theorem 7.2. We will prove the inequality (7.15). (7.16) and (7.17) can be proved analogously. First of all observe that, since $0 \leq f_{\mathrm{l},x}, f_{\mathrm{l},y} \leq 1$,

$$\left|\langle f_{\mathrm{l},x}\, f_{\mathrm{l},y}\rangle^{\mathrm{h}}_{\Lambda} - \langle f_{\mathrm{l},x}\rangle^{\mathrm{h}}_{\Lambda}\, \langle f_{\mathrm{l},y}\rangle^{\mathrm{h}}_{\Lambda}\right| \;\leq\; \log\left(\frac{\langle f_{\mathrm{l},x}\, f_{\mathrm{l},y}\rangle^{\mathrm{h}}_{\Lambda}}{\langle f_{\mathrm{l},x}\rangle^{\mathrm{h}}_{\Lambda}\, \langle f_{\mathrm{l},y}\rangle^{\mathrm{h}}_{\Lambda}} \;\vee\; \frac{\langle f_{\mathrm{l},x}\rangle^{\mathrm{h}}_{\Lambda}\, \langle f_{\mathrm{l},y}\rangle^{\mathrm{h}}_{\Lambda}}{\langle f_{\mathrm{l},x}\, f_{\mathrm{l},y}\rangle^{\mathrm{h}}_{\Lambda}}\right) \;. \tag{7.79}$$

Now observe that:

$$\langle f_{\mathrm{l},x}\, f_{\mathrm{l},y}\rangle^{\mathrm{h}}_{\Lambda} \;=\; \frac{Z^{\mathrm{h}}_{\Lambda\setminus x,y}}{Z^{\mathrm{h}}_{\Lambda}}\;, \quad \langle f_{\mathrm{l},x}\rangle^{\mathrm{h}}_{\Lambda} \;=\; \frac{Z^{\mathrm{h}}_{\Lambda\setminus x}}{Z^{\mathrm{h}}_{\Lambda}}\;, \quad \langle f_{\mathrm{l},y}\rangle^{\mathrm{h}}_{\Lambda} \;=\; \frac{Z^{\mathrm{h}}_{\Lambda\setminus y}}{Z^{\mathrm{h}}_{\Lambda}}\;,$$

where $Z^{\mathrm{h}}_{\Lambda\setminus x}$, $Z^{\mathrm{h}}_{\Lambda\setminus y}$, $Z^{\mathrm{h}}_{\Lambda\setminus x,y}$ are the partition function respectively over the lattices $\Lambda\setminus x$, $\Lambda\setminus y$, $\Lambda\setminus x,y$, with horizontal boundary conditions including a left-dimer respectively at the site x , at the site y , at both sites x,y . Therefore

$$\frac{\langle f_{\mathrm{l},x}\, f_{\mathrm{l},y}\rangle^{\mathrm{h}}_{\Lambda}}{\langle f_{\mathrm{l},x}\rangle^{\mathrm{h}}_{\Lambda}\, \langle f_{\mathrm{l},y}\rangle^{\mathrm{h}}_{\Lambda}} \;=\; \frac{Z^{\mathrm{h}}_{\Lambda}\; Z^{\mathrm{h}}_{\Lambda\setminus x,y}}{Z^{\mathrm{h}}_{\Lambda\setminus x}\; Z^{\mathrm{h}}_{\Lambda\setminus y}}\;. \tag{7.80}$$

Since $N > N_0(\beta)$, $\mathrm{dist}_{\mathrm{h}}(x,\partial\Lambda) > N_0(\beta)$, $\mathrm{dist}_{\mathrm{h}}(y,\partial\Lambda) > N_0(\beta)$, $\mathrm{dist}_{\mathrm{h}}(x,y) > N_0(\beta)$, all four partition functions satisfy the hypothesis of the corollary 7.6. Hence by the cluster expansion (7.52) the partition functions rewrites as

$$\begin{aligned}
Z^{\mathrm{h}}_{\Lambda} \;&=\; C_{\Lambda}\, \exp\Bigg(\sum^{*}_{(P_t)_t\in\mathscr{C}\mathscr{P}_{\Lambda}} U_{\Lambda}\big((P_t)_t\big)\Bigg)\;,\\
Z^{\mathrm{h}}_{\Lambda\setminus x} \;&=\; C_{\Lambda\setminus x}\, \exp\Bigg(\sum^{*}_{(P_t)_t\in\mathscr{C}\mathscr{P}_{\Lambda\setminus x}} U_{\Lambda\setminus x}\big((P_t)_t\big)\Bigg)\;,\\
Z^{\mathrm{h}}_{\Lambda\setminus y} \;&=\; C_{\Lambda\setminus y}\, \exp\Bigg(\sum^{*}_{(P_t)_t\in\mathscr{C}\mathscr{P}_{\Lambda\setminus y}} U_{\Lambda\setminus y}\big((P_t)_t\big)\Bigg)\;,\\
Z^{\mathrm{h}}_{\Lambda\setminus x,y} \;&=\; C_{\Lambda\setminus x,y}\, \exp\Bigg(\sum^{*}_{(P_t)_t\in\mathscr{C}\mathscr{P}_{\Lambda\setminus x,y}} U_{\Lambda\setminus x,y}\big((P_t)_t\big)\Bigg)\;.
\end{aligned} \tag{7.81}$$

By applying the definition (7.45), it holds

$$\frac{C_{\Lambda}\; C_{\Lambda\setminus x,y}}{C_{\Lambda\setminus x}\; C_{\Lambda\setminus y}} \;=\; 1\;. \tag{7.82}$$

Now consider a polymer $P \in \mathscr{P}_{\Lambda} \cup \mathscr{P}_{\Lambda\setminus x} \cup \mathscr{P}_{\Lambda\setminus y} \cup \mathscr{P}_{\Lambda\setminus x,y}$. Keeping in mind the definitions of polymer (7.41) and polymer activity (7.46), observe that:

if $\mathrm{dist}_{\mathrm{h}}(\mathrm{supp}\, P, x) > 1$, $\mathrm{dist}_{\mathrm{h}}(\mathrm{supp}\, P, y) > 1 \quad\Rightarrow$

$P \in \mathscr{P}_{\Lambda} \cap \mathscr{P}_{\Lambda\setminus x} \cap \mathscr{P}_{\Lambda\setminus y} \cap \mathscr{P}_{\Lambda\setminus x,y}$, $\varrho_{\Lambda}(P) = \varrho_{\Lambda\setminus x}(P) = \varrho_{\Lambda\setminus y}(P) = \varrho_{\Lambda\setminus x,y}(P)$;

and that[18]:

$$\text{if } \operatorname{dist}_{\mathrm{h}}(\operatorname{supp} P, x) \le 1 \ , \ \operatorname{dist}_{\mathrm{h}}(\operatorname{supp} P, y) > 1 \quad \Rightarrow$$
$$P \in \big(\mathscr{P}_\Lambda \cap \mathscr{P}_{\Lambda\setminus y}\big) \setminus \big(\mathscr{P}_{\Lambda\setminus x} \cup \mathscr{P}_{\Lambda\setminus x,y}\big) \ , \ \varrho_\Lambda(P) = \varrho_{\Lambda\setminus y}(P) \ \text{ or}$$
$$P \in \big(\mathscr{P}_{\Lambda\setminus x} \cap \mathscr{P}_{\Lambda\setminus x,y}\big) \setminus \big(\mathscr{P}_{\Lambda} \cup \mathscr{P}_{\Lambda\setminus y}\big) \ , \ \varrho_{\Lambda\setminus x}(P) = \varrho_{\Lambda\setminus x,y}(P) \ \text{ or}$$
$$P \in \mathscr{P}_\Lambda \cap \mathscr{P}_{\Lambda\setminus x} \cap \mathscr{P}_{\Lambda\setminus y} \cap \mathscr{P}_{\Lambda\setminus x,y} \ , \ \varrho_\Lambda(P) = \varrho_{\Lambda\setminus y}(P) \ , \ \varrho_{\Lambda\setminus x}(P) = \varrho_{\Lambda\setminus x,y}(P) \ ;$$

and the case $\operatorname{dist}_{\mathrm{h}}(\operatorname{supp} P, x) > 1$, $\operatorname{dist}_{\mathrm{h}}(\operatorname{supp} P, y) \le 1$ is clearly symmetric to the previous one. Therefore:

$$\begin{aligned}
&\sideset{}{^*}\sum_{(P_t)_t \in \mathscr{CP}_\Lambda} U_\Lambda\big((P_t)_t\big) \ - \sideset{}{^*}\sum_{(P_t)_t \in \mathscr{CP}_{\Lambda\setminus x}} U_{\Lambda\setminus x}\big((P_t)_t\big) \ + \\
&- \sideset{}{^*}\sum_{(P_t)_t \in \mathscr{CP}_{\Lambda\setminus y}} U_{\Lambda\setminus y}\big((P_t)_t\big) \ + \sideset{}{^*}\sum_{(P_t)_t \in \mathscr{CP}_{\Lambda\setminus x,y}} U_{\Lambda\setminus x,y}\big((P_t)_t\big) \ \le \\
&\le \sideset{}{^*}\sum_{\substack{(P_t)_t \in \mathscr{CP}_\Lambda \\ \exists t:\, \operatorname{dist}_{\mathrm{h}}(\operatorname{supp} P_t, x) \le 1 \\ \exists t':\, \operatorname{dist}_{\mathrm{h}}(\operatorname{supp} P_{t'}, y) \le 1}} \big|U_\Lambda\big((P_t)_t\big)\big| \ + \sideset{}{^*}\sum_{\substack{(P_t)_t \in \mathscr{CP}_{\Lambda\setminus x} \\ \exists t:\, \operatorname{dist}_{\mathrm{h}}(\operatorname{supp} P_t, x) \le 1 \\ \exists t':\, \operatorname{dist}_{\mathrm{h}}(\operatorname{supp} P_{t'}, y) \le 1}} \big|U_{\Lambda\setminus x}\big((P_t)_t\big)\big| \ + \\
&+ \sideset{}{^*}\sum_{\substack{(P_t)_t \in \mathscr{CP}_{\Lambda\setminus y} \\ \exists t:\, \operatorname{dist}_{\mathrm{h}}(\operatorname{supp} P_t, x) \le 1 \\ \exists t':\, \operatorname{dist}_{\mathrm{h}}(\operatorname{supp} P_{t'}, y) \le 1}} \big|U_{\Lambda\setminus y}\big((P_t)_t\big)\big| \ + \sideset{}{^*}\sum_{\substack{(P_t)_t \in \mathscr{CP}_{\Lambda\setminus x,y} \\ \exists t:\, \operatorname{dist}_{\mathrm{h}}(\operatorname{supp} P_t, x) \le 1 \\ \exists t':\, \operatorname{dist}_{\mathrm{h}}(\operatorname{supp} P_{t'}, y) \le 1}} \big|U_{\Lambda\setminus x,y}\big((P_t)_t\big)\big| \ .
\end{aligned} \tag{7.83}$$

It is crucial to observe that given a cluster $(P_t)_t \in \mathscr{CP}_\Lambda$, since $\cup_t \operatorname{supp} P_t$ have to be connected in $\mathbb{Z}^2$,

$$\operatorname{dist}_{\mathbb{Z}^2}(x,y) \ \le \ \operatorname{dist}_{\mathbb{Z}^2}(\cup_t \operatorname{supp} P_t, x) + \sum_t |\operatorname{supp} P_t| - 1 + \operatorname{dist}_{\mathbb{Z}^2}(\cup_t \operatorname{supp} P_t, y) \ .$$

Hence, assuming that $\operatorname{dist}_{\mathbb{Z}^2}(\cup_t \operatorname{supp} P_t, x) \le 1$, $\operatorname{dist}_{\mathbb{Z}^2}(\cup_t \operatorname{supp} P_t, y) \le 1$, it

[18]The first possibility, namely P polymer only of the lattices that contain x, happens when $\operatorname{supp} P \ni x$ or P includes a region S_i containing $x - (1,0)$. The second possibility, namely P polymer only of the lattices that do not contain x, happens when P includes a line L_k with one endpoint on $x \pm (1,0)$. The last possibility happens when P includes a region S_i containing $x + (1,0)$ (and does not verify the other conditions).

follows

$$
\begin{aligned}
&\prod_t \widetilde{\varrho}(P_t) \; = \\
&= \prod_t \frac{1}{n_t!\, p_t!} \exp\left(-\beta \frac{\mu_{\mathrm{h}} - \mu_{\mathrm{v}}}{2} \sum_{i=1}^{n_t} |S_i| - m \sum_{k=1}^{p_t} |L_k| - \beta J \, n_t \right) \\
&= \exp\left(-\frac{m}{4} \sum_t |\operatorname{supp} P_t| \right) \cdot \\
&\quad \cdot \prod_t \frac{1}{n_t!\, p_t!} \exp\left(-\left(\beta \frac{\mu_{\mathrm{h}} - \mu_{\mathrm{v}}}{2} - \frac{m}{4} \right) \sum_{i=1}^{n_t} |S_i| - \frac{3\,m}{4} \sum_{k=1}^{p_t} |L_k| - \beta J \, n_t \right) \\
&\leq \exp\left(-\frac{m}{4} \left(\operatorname{dist}_{\mathbb{Z}^2}(x,y) - 1 \right) \right) \prod_t \widetilde{\varrho}_*(P_t)
\end{aligned}
$$

where $P_t = \big((S_i)_{i=1}^{n_t}, (L_k)_{k=1}^{p_t} \big)$ for all t and $\widetilde{\varrho}_*(P_t)$ is defined as the factor appearing in the product over t at the penultimate step. By defining $a_*(P) := \frac{m}{4} |\operatorname{supp} P|$, we have that $\widetilde{\varrho}_*(P)\, e^{a_*(P)}$ is essentially equivalent to $\widetilde{\varrho}(P)\, e^{a(P)}$: we can follow exactly the proof of the theorem 7.3 up to the inequality (7.78) and prove that the Kotecky-Preiss conditions (7.50), (7.51) hold also with $\widetilde{\varrho}_*$, a_* and $m/16$ in place of $\widetilde{\varrho}$, a and $m/8$ (eventually increasing β_0). Therefore, defining $\widetilde{U}_*\big((P_t)_t\big) := u\big((P_t)_t\big) \prod_t \widetilde{\varrho}_*(P_t)$, by the general theory of cluster expansion the inequality (7.54) holds also with $\widetilde{U}_*$, $\widetilde{\varrho}_*$ and a_* in place of U_Λ, ϱ_Λ and a. As a consequence:

$$\sum^{*}_{\substack{(P_t)_t \in \mathscr{CP}_\Lambda \\ \exists t:\, \mathrm{dist_h}(\mathrm{supp}\, P_t, x) \le 1 \\ \exists t':\, \mathrm{dist_h}(\mathrm{supp}\, P_{t'}, y) \le 1}} \left| U_\Lambda\big((P_t)_t\big)\right| \;\le\; \sum^{*}_{\substack{(P_t)_t \in \mathscr{CP}_\Lambda \\ \exists t:\, \mathrm{dist_h}(\mathrm{supp}\, P_t, x) \le 1 \\ \exists t':\, \mathrm{dist_h}(\mathrm{supp}\, P_{t'}, y) \le 1}} \left| u\big((P_t)_t\big)\right| \prod_t \widetilde{\varrho}(P_t) \;\le$$

$$\le\; e^{-\frac{m}{4}(\mathrm{dist}_{\mathbb{Z}^2}(x,y)-1)} \sum^{*}_{\substack{(P_t)_t \in \mathscr{CP}_\Lambda \\ \exists t:\, \mathrm{dist_h}(\mathrm{supp}\, P_t, x) \le 1 \\ \exists t':\, \mathrm{dist_h}(\mathrm{supp}\, P_{t'}, y) \le 1}} \left| u\big((P_t)_t\big)\right| \prod_t \widetilde{\varrho}_*(P_t)$$

$$=\; e^{-\frac{m}{4}(\mathrm{dist}_{\mathbb{Z}^2}(x,y)-1)} \sum^{*}_{\substack{(P_t)_t \in \mathscr{CP}_\Lambda \\ \exists t:\, \mathrm{dist_h}(\mathrm{supp}\, P_t, x) \le 1 \\ \exists t':\, \mathrm{dist_h}(\mathrm{supp}\, P_{t'}, y) \le 1}} \left| \widetilde{U}_*\big((P_t)_t\big)\right| \tag{7.84}$$

$$\overset{(7.54)}{\le}\; e^{-\frac{m}{4}(\mathrm{dist}_{\mathbb{Z}^2}(x,y)-1)} \sum_{\substack{P \in \mathscr{P}_\Lambda \\ \mathrm{dist_h}(\mathrm{supp}\, P, x) \le 1}} \widetilde{\varrho}_*(P)\, e^{a_*(P)}$$

$$\overset{(7.50)}{\le}\; e^{-\frac{m}{4}(\mathrm{dist}_{\mathbb{Z}^2}(x,y)-1)}\; 3\,\frac{m}{16}\,.$$

The same reasoning can be repeated also for the clusters in $\mathscr{CP}_{\Lambda\setminus x}$, $\mathscr{CP}_{\Lambda\setminus y}$ and $\mathscr{CP}_{\Lambda\setminus x,y}$. Thus, by (7.80), (7.81), (7.82), 7.83, (7.84), one finally obtains:

$$\frac{\langle f_{\mathrm{l},x}\, f_{\mathrm{l},y}\rangle^{\mathrm{h}}_\Lambda}{\langle f_{\mathrm{l},x}\rangle^{\mathrm{h}}_\Lambda\, \langle f_{\mathrm{l},y}\rangle^{\mathrm{h}}_\Lambda} \;=\; \frac{Z^{\mathrm{h}}_\Lambda\; Z^{\mathrm{h}}_{\Lambda\setminus x,y}}{Z^{\mathrm{h}}_{\Lambda\setminus x}\; Z^{\mathrm{h}}_{\Lambda\setminus y}} \;\le\; \exp\Big(e^{-\frac{m}{4}(\mathrm{dist}_{\mathbb{Z}^2}(x,y)-1)}\; (3+2+2+2)\,\frac{m}{16}\Big)\;.$$

The same bound can be shown to hold also for the inverse ratio $\frac{\langle f_{\mathrm{l},x}\rangle^{\mathrm{h}}_\Lambda\, \langle f_{\mathrm{l},y}\rangle^{\mathrm{h}}_\Lambda}{\langle f_{\mathrm{l},x}\, f_{\mathrm{l},y}\rangle^{\mathrm{h}}_\Lambda}$, hence by (7.79) we conclude that:

$$\left|\langle f_{\mathrm{l},x}\, f_{\mathrm{l},y}\rangle^{\mathrm{h}}_\Lambda \,-\, \langle f_{\mathrm{l},x}\rangle^{\mathrm{h}}_\Lambda\, \langle f_{\mathrm{l},y}\rangle^{\mathrm{h}}_\Lambda\right| \;\le\; e^{-\frac{m}{4}(\mathrm{dist}_{\mathbb{Z}^2}(x,y)-1)}\; \frac{9m}{16}\;. \qquad \square$$

Proof of the corollary 7.4. Since $\Delta_{\mathrm{transl.}} = \frac{2}{|\Lambda|}\sum_{\substack{x\in\Lambda, \\ x_{\mathrm{h}}\ \mathrm{even}}} (f_{\mathrm{r},x} - f_{\mathrm{l},x})$, the variance of Δ rewrites as:

$$\big\langle (\Delta_{\mathrm{transl.}})^2 \big\rangle^{\mathrm{h}}_\Lambda \,-\, \big(\langle \Delta_{\mathrm{transl.}}\rangle^{\mathrm{h}}_\Lambda\big)^2 \;=\; \frac{4}{|\Lambda|^2} \sum_{\substack{x,y\in\Lambda \\ x_{\mathrm{h}}, y_{\mathrm{h}}\ \mathrm{even}}} C_{x,y}$$

with

$$C_{x,y} \,:=\, \big(\langle f_{\mathrm{r},x}\, f_{\mathrm{r},y}\rangle^{\mathrm{h}}_\Lambda - \langle f_{\mathrm{r},x}\rangle^{\mathrm{h}}_\Lambda\, \langle f_{\mathrm{r},y}\rangle^{\mathrm{h}}_\Lambda\big) + \big(\langle f_{\mathrm{r},x}\rangle^{\mathrm{h}}_\Lambda\, \langle f_{\mathrm{l},y}\rangle^{\mathrm{h}}_\Lambda - \langle f_{\mathrm{r},x}\, f_{\mathrm{l},y}\rangle^{\mathrm{h}}_\Lambda\big) +$$
$$+\, \big(\langle f_{\mathrm{l},x}\rangle^{\mathrm{h}}_\Lambda\, \langle f_{\mathrm{r},y}\rangle^{\mathrm{h}}_\Lambda - \langle f_{\mathrm{l},x}\, f_{\mathrm{r},y}\rangle^{\mathrm{h}}_\Lambda\big) + \big(\langle f_{\mathrm{l},x}\, f_{\mathrm{l},y}\rangle^{\mathrm{h}}_\Lambda - \langle f_{\mathrm{l},x}\rangle^{\mathrm{h}}_\Lambda\, \langle f_{\mathrm{l},y}\rangle^{\mathrm{h}}_\Lambda\big)\;.$$

By the theorem 7.2, for $x, y \in \Lambda$ such that $\mathrm{dist_h}(x, \partial\Lambda) > N_0(\beta)$, $\mathrm{dist_h}(y, \partial\Lambda) > N_0(\beta)$ and $\mathrm{dist_h}(x, y) > N_0(\beta)$, it holds

$$C_{x,y} \,\le\, 4\,\frac{9m}{16}\; e^{-\frac{m}{4}\,(\mathrm{dist}_{\mathbb{Z}^2}(x,y)-1)} \;.$$

Hence:

$$\big\langle (\Delta_{\mathrm{transl.}})^2 \big\rangle^{\mathrm{h}}_{\Lambda} - \big(\langle \Delta_{\mathrm{transl.}} \rangle^{\mathrm{h}}_{\Lambda}\big)^2 \,\le\, 4\,\frac{9m}{16|\Lambda|^2} \sum_{\substack{x,y\in\Lambda\\ x\neq y}} e^{-\frac{m}{4}\,(\mathrm{dist}_{\mathbb{Z}^2}(x,y)-1)} + 1 - \varphi_{\Lambda,\Lambda,N_0(\beta)} \;,$$

where we set

$$\varphi_{\Lambda,\Lambda',N_0} := \frac{\#\{(x,y)\in\Lambda\times\Lambda' \,|\, \mathrm{dist_h}(x,\partial\Lambda) \vee \mathrm{dist_h}(y,\partial\Lambda') \vee \mathrm{dist_h}(x,y) > N_0\}}{|\Lambda|\,|\Lambda'|} \;.$$

Now observe that

$$\begin{aligned} \varphi_{\Lambda,\Lambda,N_0} \,&\ge \min_{\substack{L,L'\text{ maximal}\\ \text{horiz. lines of }\Lambda}} \varphi_{L,L',N_0} \,\ge \min_{\substack{L,L'\text{ maximal}\\ \text{horiz. lines of }\Lambda}} \frac{(|L|-2N_0)\,(|L'|-4N_0)}{|L|\,|L'|} \\ &\ge \left(1-2\frac{N_0}{N}\right)\left(1-4\frac{N_0}{N}\right) \;, \end{aligned}$$

hence $1-\varphi_{\Lambda,\Lambda,N_0} \le N_0/N\,(6-8N_0/N)$. And on the other hand:

$$\begin{aligned} \sum_{\substack{x,y\in\Lambda\\ x\neq y}} e^{-\frac{m}{4}\,(\mathrm{dist}_{\mathbb{Z}^2}(x,y)-1)} \,&\le\, |\Lambda| \sum_{\substack{x\in\mathbb{Z}^2\\ x\neq 0}} e^{-\frac{m}{4}\,(\mathrm{dist}_{\mathbb{Z}^2}(x,0)-1)} \,= \\ &= |\Lambda| \sum_{d\ge1} 4d\, e^{-\frac{m}{4}(d-1)} \,=\, |\Lambda|\,\frac{4}{(1-e^{-\frac{m}{4}})^2} \;. \end{aligned}$$

□

7.4 Appendix: 1D systems

Consider a finite line L, that is a finite connected sub-lattice of $\mathbb{Z}$. Consider a monomer-dimer model on L given by the following partition function:

$$Z_L \,=\, \sum_{\alpha\in\mathscr{D}_L} e^{-\beta H_L(\alpha)}\; e^{I_{\mathrm{l}}(\alpha_{x_{\mathrm{l}}})}\; e^{I_{\mathrm{r}}(\alpha_{x_{\mathrm{r}}})} \;.$$

$\mathscr{D}_L$ denotes the set of monomer-dimer configurations on L (allowing also external dimers at the endpoints of L); the Hamiltonian is defined as

$$H_L \,=\, \frac{\mu_{\mathrm{h}}+J}{2}\,\#\left\{\begin{matrix}\text{sites of } L \text{ with}\\ \text{monomer}\end{matrix}\right\} \,+\, \frac{J}{2}\,\#\left\{\begin{matrix}\text{sites of } L \text{ with dimer but neigh-}\\ \text{bor to monomer in } L\end{matrix}\right\} \;.$$

$x_{\rm l}, x_{\rm r}$ denote the left and the right endpoint of L respectively; $I_{\rm l}, I_{\rm r}$ represent the interaction among the configuration on L and the boundary condition outside its endpoints.

This one-dimensional system can be described by a *transfer matrix* T over the three possible states of a site, $l \equiv$"left-dimer", $r \equiv$"right-dimer", $m \equiv$"monomer":

$$T \;\equiv\; \begin{pmatrix} T(l,l) & T(l,r) & T(l,m) \\ T(r,l) & T(r,r) & T(r,m) \\ T(m,l) & T(m,r) & T(m,m) \end{pmatrix} \;:=\; \begin{pmatrix} 0 & 1 & \sqrt{ab} \\ 1 & 0 & 0 \\ 0 & \sqrt{ab} & a \end{pmatrix} \;, \tag{7.85}$$

where to shorten the notation we set $\sqrt{a} := e^{-\beta\frac{\mu_{\rm h}+J}{4}}$ the transfer contribution of a monomer[19], $\sqrt{b} := e^{-\beta\frac{J}{2}}$ the transfer contribution of a site with a dimer but neighbor to a monomer. Two vectors are also needed to encode the boundary conditions:

$$\begin{aligned} B_{\rm l} \;&\equiv\; \begin{pmatrix} B_{\rm l}(l) & B_{\rm l}(r) & B_{\rm l}(m) \end{pmatrix} \;:=\; \begin{pmatrix} e^{I_{\rm l}(l)} & e^{I_{\rm l}(r)} & \sqrt{a}\,e^{I_{\rm l}(m)} \end{pmatrix} \;, \\ B_{\rm r} \;&\equiv\; \begin{pmatrix} B_{\rm r}(l) \\ B_{\rm r}(r) \\ B_{\rm r}(m) \end{pmatrix} \;:=\; \begin{pmatrix} e^{I_{\rm r}(l)} \\ e^{I_{\rm r}(r)} \\ \sqrt{a}\,e^{I_{\rm r}(m)} \end{pmatrix} \;. \end{aligned} \tag{7.86}$$

Proposition 7.11. *The partition function of the system rewrites as a bilinear form:*

$$Z_L \;=\; B_{\rm l}\, T^{|L|-1}\, B_{\rm r} \;. \tag{7.87}$$

Proof. According to the previous definitions it is clear that for every configuration $\alpha \in \{l,r,m\}^{|L|}$

$$\begin{aligned} &\mathbb{1}(\alpha \in \mathscr{D}_L)\; e^{-\beta H_L(\alpha)} \;= \\ &\qquad = \;\sqrt{a}^{\,\mathbb{1}(\alpha_1 = m)}\; T(\alpha_1,\alpha_2)\; T(\alpha_2,\alpha_3)\; \dots\; T(\alpha_{|L|-1},\alpha_{|L|})\; \sqrt{a}^{\,\mathbb{1}(\alpha_{|L|}=m)} \;. \end{aligned}$$

[19] The transfer energy of a monomer is half the energy of a monomer because it appears during two "transfers".

Therefore

$$\begin{aligned} Z_L &= \sum_{\alpha\in\{l,r,m\}^{|L|}} B_{\mathrm{l}}(\alpha_1)\, T(\alpha_1,\alpha_2)\, T(\alpha_2,\alpha_3)\, \ldots\, T(\alpha_{|L|-1},\alpha_{|L|})\, B_{\mathrm{r}}(\alpha_{|L|}) \\ &= B_{\mathrm{l}}\, T^{|L|-1}\, B_{\mathrm{r}}\ . \end{aligned}$$

□ □

Assume for the moment that the transfer matrix T is diagonalizable. Denote by $\lambda_1, \lambda_2, \lambda_3$ its eigenvalues and by $E_{\mathrm{r}}^{(1)}, E_{\mathrm{r}}^{(2)}, E_{\mathrm{r}}^{(3)}$, $E_{\mathrm{l}}^{(1)}, E_{\mathrm{l}}^{(2)}, E_{\mathrm{l}}^{(3)}$ the corresponding right (column) eigenvectors and left (row) eigenvectors, normalized so that $E_{\mathrm{l}}^{(i)} E_{\mathrm{r}}^{(i)} = 1$ for $i = 1, 2, 3$.

Corollary 7.12.

$$Z_L = \sum_{i=1,2,3} \lambda_i^{|L|-1}\, B_{\mathrm{l}}\, E_{\mathrm{r}}^{(i)}\, E_{\mathrm{l}}^{(i)}\, B_{\mathrm{r}}\ . \tag{7.88}$$

Proof. Since we are assuming that T is diagonalizable, it holds $T = P\, D\, P^{-1}$ where D is the diagonal matrix of eigenvalues, P is the matrix with the right eigenvectors on the columns, P^{-1} has the left eigenvectors on the rows. Then $T^{|L|-1} = P\, D^{|L|-1}\, P^{-1}$ and

$$B_{\mathrm{l}}\, T^{|L|-1}\, B_{\mathrm{r}} = (B_{\mathrm{l}}\, P)\, D^{|L|-1}\, (P^{-1}\, B_{\mathrm{r}}) = \sum_{i=1}^{3} (B_{\mathrm{l}}\, E_{\mathrm{r}}^{(i)})\, \lambda_i^{|L|-1}\, (E_{\mathrm{l}}^{(i)} B_{\mathrm{r}})\ .$$

□ □

Now our purpose is to diagonalise the transfer matrix T when β is large.

Proposition 7.13. *For all $\beta > 0$ the transfer matrix T is diagonalizable over $\mathbb{R}$. Its eigenvalues are*

$$\begin{aligned} \lambda_1 &= 1 + \frac{ab}{2}\,(1 + o(1)) \\ \lambda_2 &= -1 + \frac{ab}{2}\,(1 + o(1)) \\ \lambda_3 &= a - ab - a^3 b\,(1 + o(1)) \end{aligned} \tag{7.89}$$

as $\beta \to \infty$.

Proof. The eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are the (complex) roots of the characteristic polynomial of T, that is

$$p(\lambda) := \det(\lambda I - T) = -ab + (\lambda - a)(\lambda^2 - 1) \,.$$

For all $\beta > 0$ it turns out that p has 3 distinct real roots[20], hence T is diagonalizable over the reals.
As $\beta \to \infty$, $p(\lambda) \to \lambda(\lambda^2 - 1)$ hence $\lambda_1 \to 1\,,\ \lambda_2 \to -1\,,\ \lambda_3 \to 0\,.$ Thus it is convenient to write $\lambda_1 = 1 + \varepsilon_1\,,\ \lambda_2 = -1 + \varepsilon_2\,,\ \lambda_3 = a + \varepsilon_3$ with $\varepsilon_i \to 0$ as $\beta \to \infty$ for $i = 1, 2, 3$. Now expand the polynomial p in powers of ε_i and truncate it at the first order:

$$\begin{aligned}
0 \;&=\; p(\lambda_1) \;=\; -ab + (1 - a + \varepsilon_1)\,(2\varepsilon_1 + \varepsilon_1^2) \;=\; -ab + 2\varepsilon_1\,(1 + o(1)) \\
&\Rightarrow\; \varepsilon_1 = \frac{ab}{2}\,(1 + o(1))\;; \\
0 \;&=\; p(\lambda_2) \;=\; -ab + (-1 - a + \varepsilon)\,(-2\varepsilon_2 + \varepsilon_2^2) \;=\; -ab + 2\varepsilon_2\,(1 + o(1)) \\
&\Rightarrow\; \varepsilon_2 = \frac{ab}{2}\,(1 + o(1))\;; \\
0 \;&=\; p(\lambda_3) \;=\; -ab + \varepsilon_3\,\big((a + \varepsilon_3)^2 - 1\big) \;=\; -ab - \varepsilon_3\,(1 + o(1)) \\
&\Rightarrow\; \varepsilon_3 = -ab\,(1 + o(1))\;.
\end{aligned}$$

In order to find the following order of λ_3, now one can write $\lambda_3 = a - ab\,(1 + \varepsilon_3')$ with $\varepsilon_3' \to 0$ as $\beta \to \infty$ and repeat the procedure:

$$\begin{aligned}
0 \;&=\; \frac{p(\lambda_3)}{-ab} \;=\; 1 + (1 + \varepsilon_3')\,\big(a^2\,(1 + o(1)) - 1\big) \;=\; a^2\,(1 + o(1)) - \varepsilon_3'\,(1 + o(1)) \\
&\Rightarrow\; \varepsilon_3' = a^2\,(1 + o(1))\;.
\end{aligned}$$

□ □

[20]The discriminant of the cubic is $\Delta = 18a(1-b) + 4a^2(1-b) + a^2 + 4 - 27a^2(1-b)$, which is strictly positive for all $0 \leq a, b \leq 1$, $(a, b) \neq (1, 0)$.

Proposition 7.14. *The right eigenvectors of the transfer matrix T are*

$$
\begin{aligned}
E_{\mathrm{r}}^{(1)} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 - \frac{a}{2}\,(1+o(1)) \\ 1 - \frac{a}{2}\,(1+o(1)) \\ \sqrt{ab}\,(1+o(1)) \end{pmatrix} \\
E_{\mathrm{r}}^{(2)} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + \frac{a}{2}\,(1+o(1)) \\ -1 - \frac{a}{2}\,(1+o(1)) \\ \sqrt{ab}\,(1+o(1)) \end{pmatrix} \\
E_{\mathrm{r}}^{(3)} &= \begin{pmatrix} -a\sqrt{ab}\,(1+o(1)) \\ -\sqrt{ab}\,(1+o(1)) \\ 1 + a\,(1+o(1)) \end{pmatrix}
\end{aligned}
\tag{7.90}
$$

and moreover

$$
\begin{aligned}
E_{\mathrm{r}}^{(2)}(1) + E_{\mathrm{r}}^{(2)}(2) + \sqrt{ab}\,E_{\mathrm{r}}^{(2)}(3) &= \frac{ab}{2\sqrt{2}}\,(1+o(1)) \\
E_{\mathrm{r}}^{(3)}(2) + \sqrt{ab}\,E_{\mathrm{r}}^{(3)}(3) &= -a^2\sqrt{ab}\,(1+o(1))
\end{aligned}
$$

as $\beta \to \infty$. The left eigenvectors are obtained by a simple transformation: $E_{\mathrm{l}}^{(i)} = \sigma\big(E_{\mathrm{r}}^{(i)}\big)$ for $i = 1, 2, 3$, where

$$
\sigma \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} := \begin{pmatrix} v_2 & v_1 & v_3 \end{pmatrix} .
$$

Proof. The right eigenvectors E_{r} associated to the eigenvalue λ are the non-zero solutions of the linear system

$$
(\lambda I - T)\,E_{\mathrm{r}} = 0 \;\Leftrightarrow\; E_{\mathrm{r}} = \begin{pmatrix} \lambda(\lambda - a) \\ \lambda - a \\ \sqrt{ab} \end{pmatrix} t \;,\; t \in \mathbb{R} \,.
$$

And the left eigenvectors E_{l} associated to the same eigenvalue λ are the non-zero solutions of the linear system

$$
E_{\mathrm{l}}\,(\lambda I - T) = 0 \;\Leftrightarrow\; E_{\mathrm{l}} = \begin{pmatrix} \lambda - a & \lambda(\lambda - a) & \sqrt{ab} \end{pmatrix} t \;,\; t \in \mathbb{R} \,.
$$

The desired normalization $E_{\mathrm{l}}\, E_{\mathrm{r}} = 1$ can be obtained by choosing

$$t = \sqrt{2\lambda(\lambda - a) + ab}$$

in both cases. Now to conclude the proof it is sufficient to exploit the estimates of the eigenvalues given by the proposition 7.13. □ □

The formula (7.88) together with the estimates of propositions 7.13, 7.14 give us a complete control on the one-dimensional system on L at low temperature, for every choice of the boundary conditions.

We concentrate on providing an estimation of the quantity R_L defined by (7.38), since it is needed in the section 7.1. We have to distinguish three cases, according to where the endpoints of L lie.

Lemma 7.15. *The ratios of the eigenvalues of the transfer matrix T are*

$$\frac{\lambda_2}{\lambda_1} = -1 + ab\,(1 + o(1)) \quad , \quad \frac{\lambda_3}{\lambda_2} = -a + ab\,(1 + o(1))$$

as $\beta \to \infty$. In particular setting $m := -\log\left|\lambda_2/\lambda_1\right|$ it holds

$$e^{-m} = 1 - e^{-\beta\frac{\mu_{\mathrm{h}}+3J}{2}}\,(1 + o(1)) \quad \text{as } \beta \to \infty\,. \tag{7.91}$$

Proof. It follows immediately from the proposition 7.13. □ □

Lemma 7.16. *If $x_{\mathrm{l}} \in \partial^{\mathrm{ext}}_{\mathrm{r}} S_j$, then as $\beta \to \infty$*

$$\begin{aligned} B_{\mathrm{l}}\, E_{\mathrm{r}}^{(1)} &= \frac{\sqrt{b}}{\sqrt{2}}\,(1 + o(1)) \\ B_{\mathrm{l}}\, E_{\mathrm{r}}^{(2)} &= -\frac{\sqrt{b}}{\sqrt{2}}\,(1 + o(1)) \\ B_{\mathrm{l}}\, E_{\mathrm{r}}^{(3)} &= \sqrt{a}\,(1 + o(1))\,. \end{aligned}$$

If $x_{\mathrm{r}} \in \partial^{\mathrm{ext}}_{\mathrm{l}} S_j$, then the same estimates hold for $E_{\mathrm{l}}^{(1)} B_{\mathrm{r}}$, $E_{\mathrm{l}}^{(2)} B_{\mathrm{r}}$, $E_{\mathrm{l}}^{(3)} B_{\mathrm{r}}$ respectively.

Proof. If $x_{\mathrm{l}} \in \partial^{\mathrm{ext}}_{\mathrm{r}} S_j$ then by (7.36) and (7.86) the vector describing the boundary condition on the left side of the line L is $B_{\mathrm{l}} = \begin{pmatrix} 0 & \sqrt{b} & \sqrt{a} \end{pmatrix}$. Then the estimates for $B_{\mathrm{l}}\, E_{\mathrm{r}}^{(i)}$, $i = 1, 2, 3$, are computed using the proposition 7.14. □ □

Lemma 7.17. *If $x_{\mathrm{l}} \in \partial_{\mathrm{l}}\Lambda$, then as $\beta \to \infty$*

$$B_{\mathrm{l}}\, E_{\mathrm{r}}^{(1)} = \begin{cases} \frac{1}{\sqrt{2}}\left(1 - \frac{a}{2}\,(1+o(1))\right) & \textit{if the h-dimer on } x_{\mathrm{l}}-(1,0) \textit{ has fixed position} \\ \sqrt{2}\left(1 - \frac{a}{2}\,(1+o(1))\right) & \textit{if the h-dimer on } x_{\mathrm{l}}-(1,0) \textit{ has free position} \end{cases}$$

$$B_{\mathrm{l}}\, E_{\mathrm{r}}^{(2)} = \begin{cases} -\frac{1}{\sqrt{2}}\left(1 + \frac{a}{2}\,(1+o(1))\right) & \textit{if the h-dimer on } x_{\mathrm{l}}-(1,0) \textit{ is fixed to the left} \\ \frac{1}{\sqrt{2}}\left(1 + \frac{a}{2}\,(1+o(1))\right) & \textit{if the h-dimer on } x_{\mathrm{l}}-(1,0) \textit{ is fixed to the right} \\ \frac{ab}{2\sqrt{2}}\,(1+o(1)) & \textit{if the h-dimer on } x_{\mathrm{l}}-(1,0) \textit{ has free position} \end{cases}$$

$$B_{\mathrm{l}}\, E_{\mathrm{r}}^{(3)} = \begin{cases} -a^2\sqrt{ab}\,(1+o(1)) & \textit{if the h-dimer on } x_{\mathrm{l}}-(1,0) \textit{ is fixed to the left} \\ -a\sqrt{ab}\,(1+o(1)) & \textit{if the h-dimer on } x_{\mathrm{l}}-(1,0) \textit{ is fixed to the right or free} \end{cases}$$

If $x_{\mathrm{r}} \in \partial_{\mathrm{r}}\Lambda$, then the same estimates hold respectively for $E_{\mathrm{l}}^{(1)} B_{\mathrm{r}}$, $E_{\mathrm{l}}^{(2)} B_{\mathrm{r}}$, $E_{\mathrm{l}}^{(3)} B_{\mathrm{r}}$ after substituting: $x_{\mathrm{l}}-(1,0)$ by $x_{\mathrm{r}}+(1,0)$, "left" by "right" and "right" by "left".

Proof. If $x_{\mathrm{l}} \in \partial_{\mathrm{l}}\Lambda$ then by (7.36) and (7.86) the vector describing the boundary condition on the left side of the line L is: $B_{\mathrm{l}} = \begin{pmatrix} 0 & 1 & \sqrt{ab} \end{pmatrix}$ if a left-dimer is fixed on $x_{\mathrm{l}}-(1,0)$; $B_{\mathrm{l}} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$ if a right-dimer is fixed on $x_{\mathrm{l}}-(1,0)$; $B_{\mathrm{l}} = \begin{pmatrix} 1 & 1 & \sqrt{ab} \end{pmatrix}$ if on $x_{\mathrm{l}}-(1,0)$ there is a h-dimer with free position. Then the estimates for $B_{\mathrm{l}}\, E_{\mathrm{r}}^{(i)}$, $i = 1,2,3$, are computed using the proposition 7.14.

□ □

Proposition 7.18. *Denote by $o(1)$ any function $\omega(\beta, \mu_{\mathrm{h}}, J)$ that goes to zero as $\beta \to \infty$ and does not depend on the choice of the line L nor on Λ. Then for every line $L \in \mathscr{L}_\Lambda(\cup_j S_j)$, $S_j \in \mathscr{S}_\Lambda$ pairwise disconnected, $\Lambda \subset \mathbb{Z}^2$ finite, it holds*

$$|R_L| \,\leq\, e^{-m|L|}\,\gamma_L \tag{7.92}$$

where the quantity γ_L can be chosen as follows:

$$\gamma_L := \begin{cases} \left(\frac{e^{-\beta J}}{2} + e^{-\beta \frac{\mu_{\rm h}+J}{2}|L|}\right)(1+o(1)) & \text{if } x_{\rm l} \in \cup_i \partial_{\rm r}^{\rm ext} S_i\,,\ x_{\rm r} \in \cup_i \partial_{\rm l}^{\rm ext} S_i \\ \frac{e^{-\beta \frac{J}{2}}}{\sqrt{2}}(1+o(1)) & \text{if } x_{\rm l} \in \cup_i \partial_{\rm r}^{\rm ext} S_i\,,\ x_{\rm r} \in \cup_i \partial_{\rm r}\Lambda \\ & \text{or vice versa } x_{\rm l} \in \cup_i \partial_{\rm l}\Lambda\,,\ x_{\rm r} \in \cup_i \partial_{\rm l}^{\rm ext} S_i \\ 1+o(1) & \text{if } x_{\rm l} \in \partial_{\rm l}\Lambda\,,\ x_{\rm r} \in \partial_{\rm r}\Lambda \end{cases} \tag{7.93}$$

Proof. • Suppose $x_{\rm l} \in \partial_{\rm r}^{\rm ext} S_i$ and $x_{\rm r} \in \partial_{\rm l}^{\rm ext} S_j$. The definition (7.38) and the corollary 7.12 give

$$\begin{aligned} \lambda_1 R_L &= \frac{Z_L}{\lambda_1^{|L|-1}} - B_{\rm l} E_{\rm r}^{(1)} E_{\rm l}^{(1)} B_{\rm r} \\ &= \left(\frac{\lambda_2}{\lambda_1}\right)^{|L|-1} B_{\rm l} E_{\rm r}^{(2)} E_{\rm l}^{(2)} B_{\rm r} + \left(\frac{\lambda_3}{\lambda_1}\right)^{|L|-1} B_{\rm l} E_{\rm r}^{(3)} E_{\rm l}^{(3)} B_{\rm r}\,. \end{aligned}$$

By the lemma 7.15 $|\lambda_3/\lambda_1| \le a\,|\lambda_2/\lambda_1|$ when β is sufficiently large. Therefore, using also the estimates of lemma 7.16, one finds

$$|R_L| \le \left|\frac{\lambda_2}{\lambda_1}\right|^{|L|-1} \left(\frac{b}{2} + a^{|L|}\right)(1+o(1))\,.$$

• Suppose now $x_{\rm l} \in \partial_{\rm r}^{\rm ext} S_j$ and $x_{\rm r} \in \partial_{\rm r}\Lambda$. The definition (7.38) and the corollary 7.12 give

$$\begin{aligned} \lambda_1^{1/2} R_L &= \frac{Z_L}{\lambda_1^{|L|-1} E_{\rm l}^{(1)} B_{\rm r}} - B_{\rm l} E_{\rm r}^{(1)} \\ &= \left(\frac{\lambda_2}{\lambda_1}\right)^{|L|-1} \frac{B_{\rm l} E_{\rm r}^{(2)} E_{\rm l}^{(2)} B_{\rm r}}{E_{\rm l}^{(1)} B_{\rm r}} + \left(\frac{\lambda_3}{\lambda_1}\right)^{|L|-1} \frac{B_{\rm l} E_{\rm r}^{(3)} E_{\rm l}^{(3)} B_{\rm r}}{E_{\rm l}^{(1)} B_{\rm r}}\,. \end{aligned}$$

By the lemma 7.15 $|\lambda_3/\lambda_1| \le a\,|\lambda_2/\lambda_1|$ when β is sufficiently large. Therefore, using also the estimates of lemmas 7.16, 7.17, one finds

$$|R_L| \le \left|\frac{\lambda_2}{\lambda_1}\right|^{|L|-1} \frac{\sqrt{b}}{\sqrt{2}}(1+o(1))\,.$$

• Suppose now $x_{\rm l} \in \partial_{\rm l}\Lambda$ and $x_{\rm r} \in \partial_{\rm r}\Lambda$. The definition (7.38) and the corollary

7.12 give

$$
\begin{aligned}
R_L &= \frac{Z_L}{\lambda_1^{|L|-1} B_{\mathrm{l}} E_{\mathrm{r}}^{(1)} E_{\mathrm{l}}^{(1)} B_{\mathrm{r}}} - 1 \\
&= \left(\frac{\lambda_2}{\lambda_1}\right)^{|L|-1} \frac{B_{\mathrm{l}} E_{\mathrm{r}}^{(2)} E_{\mathrm{l}}^{(2)} B_{\mathrm{r}}}{B_{\mathrm{l}} E_{\mathrm{r}}^{(1)} E_{\mathrm{l}}^{(1)} B_{\mathrm{r}}} + \left(\frac{\lambda_3}{\lambda_1}\right)^{|L|-1} \frac{B_{\mathrm{l}} E_{\mathrm{r}}^{(3)} E_{\mathrm{l}}^{(3)} B_{\mathrm{r}}}{B_{\mathrm{l}} E_{\mathrm{r}}^{(1)} E_{\mathrm{l}}^{(1)} B_{\mathrm{r}}} .
\end{aligned}
$$

By the lemma 7.15 $|\lambda_3/\lambda_1| \leq a\,|\lambda_2/\lambda_1|$ when β is sufficiently large. Therefore, using also the estimates of lemma 7.17, one finds

$$
|R_L| \leq \left|\frac{\lambda_2}{\lambda_1}\right|^{|L|-1} (1 + o(1)) .
$$

□ □

7.5 Appendix: cluster expansion

In this Appendix we state the main results about the general theory of cluster expansion used in this paper. The cluster expansion method permits to rewrite the logarithm of the partition function of a polymer system as a power series of the polymer activities. This expansion entails analyticity results and simplifies considerably the study of the correlation functions, that can be expressed in terms of ratios of partition functions. Clearly the cluster expansion cannot hold in general on the whole space of parameters: it converges only when the polymer activities are small enough to compete with the entropy. A rigorous study of the conditions of convergence dates back to [43, 50, 87], by means of Kirkwood-Salsburg type of equations. In this paper we use a criterion proposed by Kotecky and Preiss [68] in 1986. Afterwards this criterion was compared to the previous ones, was improved and simplified in [21, 33, 38, 76, 96] (for a clear and modern treatment we suggest for example the last work).

Let $\mathscr{P}$ be a finite set, called the *set of polymers*. Let $\varrho : \mathscr{P} \to \mathbb{C}$, called the *polymer activity*, and $\delta : \mathscr{P} \times \mathscr{P} \to \{0, 1\}$, called the *polymer hard-core* *interaction*, such that $\delta(P,P) = 0$ and $\delta(P,P') = \delta(P',P)$ for all $P, P' \in \mathscr{P}$. Consider the *polymer partition function*:

$$\begin{aligned} \mathcal{Z} &:= \sum_{\mathscr{P}' \subseteq \mathscr{P}} \prod_{P \in \mathscr{P}'} \varrho(P) \prod_{\substack{P,P' \in \mathscr{P}' \\ P \neq P'}} \delta(P,P') \\ &= \sum_{q \geq 0} \frac{1}{q!} \sum_{P_1,\dots,P_q \in \mathscr{P}} \prod_{t=1}^{q} \varrho(P_t) \prod_{t<s} \delta(P_t,P_s) \,. \end{aligned} \tag{7.94}$$

A family of polymers $(P_1,\dots,P_q)$ is called *compatible* if $\delta(P_t,P_s) = 1$ for all $t \neq s$; otherwise it is called *incompatible*. Observe that in the partition function $\mathcal{Z}$ only the compatible families of polymers give non-zero contributions.

A family of polymers $(P_1,\dots,P_q)$ is called a *cluster* if the graph with vertex set $\{1,\dots,q\}$ and edge set $\{(t,s) \,|\, \delta(P_t,P_s) = 0\}$ is connected.

Theorem 7.4. *Suppose that there exists $a\colon \mathscr{P} \to [0,\infty[$, called size function, such that the Kotecky-Preiss condition is satisfied, namely:*

$$\sum_{\substack{P \in \mathscr{P} \\ \delta(P,P^*)=0}} |\varrho(P)|\, e^{a(P)} \leq a(P^*) \quad \forall\, P^* \in \mathscr{P} \,. \tag{7.95}$$

Then:

$$\log \mathcal{Z} = \sum_{q \geq 0} \frac{1}{q!} \sum_{P_1,\dots,P_q \in \mathscr{P}} \left(\prod_{t=1}^{q} \varrho(P_t) \right) u(P_1,\dots,P_q) \tag{7.96}$$

where the series on the r.h.s. is absolutely convergent and

$$u(P_1,\dots,P_q) := \sum_{\substack{G=(V,E) \text{ connected graph} \\ V=\{1,\dots,q\} \\ E \subseteq \{(t,s) \,|\, \delta(P_t,P_s)=0\}}} (-1)^{|E|} \,. \tag{7.97}$$

Moreover, for all $\mathscr{E} \subseteq \mathscr{P}$

$$\sum_{q \geq 0} \frac{1}{q!} \sum_{\substack{P_1,\dots,P_q \in \mathscr{P} \\ \exists\, t\colon P_t \in \mathscr{E}}} \left| \prod_{t=1}^{q} \varrho(P_t) \right| |u(P_1,\dots,P_q)| \leq \sum_{\substack{P \in \mathscr{P} \\ P \in \mathscr{E}}} |\varrho(P)|\, e^{a(P)} \,. \tag{7.98}$$

It is worth to observe that if $(P_1,\dots,P_q)$ is not a cluster then $u(P_1,\dots,P_q) = 0$. Therefore only the clusters of polymers (that are infinitely many) give non-zero contributions to the expansion (7.96) of $\log \mathcal{Z}$.

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