Alma Mater Studiorum – Università di Bologna

DOTTORATO DI RICERCA IN

AUTOMATICA E RICERCA OPERATIVA

Ciclo XXVII

Settore Concorsuale di afferenza: 09 / G1 – AUTOMATICA

Settore Scientifico disciplinare: ING-INF / 04 - AUTOMATICA

ROBUST NONLINEAR OUTPUT REGULATION BY IDENTIFICATION TOOLS

Presentata da: **Francesco Forte**

Coordinatore Dottorato **Relatore**

Prof. Daniele Vigo **Prof. Lorenzo Marconi**

Esame finale anno 2015

Declaration of Authorship

I, Francesco FORTE, declare that this thesis titled, 'Robust Nonlinear Output Regulation by Identification Tools' and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed: ____

Date: ____

"The most exciting phrase to hear in science, the one that heralds new discoveries, is not 'Eureka!' but 'That's funny...'"

Isaac Asimov

Abstract

The present thesis focuses on the problem of robust output regulation for minimum phase nonlinear systems by means of identification techniques. Given a controlled plant and an exosystem (an autonomous system that generates eventual references or disturbances), the control goal is to design a proper regulator able to process the only measure available, i.e the error/output variable, in order to make it asymptotically vanishing. In this context, such a regulator can be designed following the well known "internal model principle" that states how it is possible to achieve the regulation objective by embedding a replica of the exosystem model in the controller structure. The main problem shows up when the exosystem model is affected by parametric or structural uncertainties, in this case, it is not possible to reproduce the exact behavior of the exogenous system in the regulator and then, it is not possible to achieve the control goal. In this work, the idea is to find a solution to the problem trying to develop a general framework in which coexist both a standard regulator and an estimator able to guarantee (when possible) the best estimate of all uncertainties present in the exosystem in order to give "robustness" to the overall control loop. It is important to underline that the design procedure presented is valid when the steady state control law and its time derivatives up to a certain order are assumed to satisfy a regression formula with known regression vector. Speaking of structure, from one side, it is possible to design continuous internal model regulators by means of high-gain methods; this can be useful to cast everything in a semiglobal setting and to get really good performance also in presence of unsatisfying identification performances (the high gain keeps the regulation error bounded and small in any case). On the other side, the proposed control structure combines continuous time dynamics and "hybrid identifiers"; the fact of considering hybrid systems is essentially motivated by the goal of setting up a general framework where many design strategies can be used. The identifier can model classical continuous adaptive laws developed so far in literature and also a particular case, typical of the identification field, in which the designer deals with a sample data set and discrete prediction models. The final idea underlined in the thesis is the joined work done by the regulator and the identifier to achieve the control goal; in other words the main interest is the investigation of the interplay between control and identification in order to develop an overall control structure able to guarantee the best performances of the loop, estimating the steady state control law by minimizing the asymptotic regulation error.

Acknowledgements

While I was working on my bachelor degree, a colleague told me "I will not write the acknowledgments because I only have to thank myself for this work..." and he did it! I think this is not my case because all this work would not have been possible without the support and the assistance of some people.

First of all, the most important person is my advisor Prof. Lorenzo Marconi; he has believed in me since the beginning of this adventure, he has given me a lot of possibilities to express myself and he has allowed everything about this thesis. He has been my reference point since I knew him and I am absolutely sure he will be a great mentor for all students he will have in the future...really thanks Prof.

My sincere thanks also goes to Prof. Andrew Teel for offering me the opportunity to join his group at UCSB...I will never forget that wonderful experience!

A special mention goes to my labmates...thanks for all the great moments passed together, for all the fruitful discussions we have had, for all the fun...I cannot forget all the jokes done! Amazing...

Last but not the least, I would like to thank my family: my mother Teresa, my father Romolo and my sister Chiara for giving me continuous support and encouragement.

Contents

List of Figures

List of Tables

Notation

$\mathbb{R}$	set of real numbers
$\mathbb{R}_{\geq 0}$	set of nonnegative real numbers
$\mathbb{N}$	set of nonnegative integers
$\mathbb{R}^n$	the $n-$dimensional Euclidean space
$\in$	belong to
$\subset$	subset
$\supset$	superset
$\cup$	union
$\cap$	intersection
$\times$	Cartesian product
$\triangleleft$	end of assumption/theorem
$\square$	end of proof
$:=$	defined as
$\gg$	much greater than
$\ll$	much less than
$\lvert x\rvert$	norm of x, with $x \in \mathbb{R}^n$
$\lvert x\rvert_{\mathcal{A}}$	$\inf_{y\in\mathcal{A}} \lvert x-y\rvert$, distance of x from $\mathcal{A}$, with $x \in \mathbb{R}^n$ and $\mathcal{A}$ set of $\mathbb{R}^n$
$L_{f(x)}g(x)$	Lie derivative $\left(:= \frac{\partial g(x)}{\partial x} f(x)\right)$
I_n	$n \times n$ identity matrix
$\mathrm{col}(a_1, \ldots, a_n)$	column vector with elements $(a_1, \ldots, a_n)$
$\mathrm{diag}(a_1, \ldots, a_n)$	an $n \times n$ diagonal matrix with a_i as its i-th diagonal element
$A^\top$	transpose
A^{-1}	inverse of A
$A^\dagger$	pseudo inverse of A
$\det(A)$	determinant of A
$\lambda(A)$	eigenvalue of A
$\lvert\lvert A\rvert\rvert$	norm of A
$\otimes$	Kronecker product

Dedicated to my family

Chapter 1

Introduction

In this chapter the reader can find two main sections: the first is a general introduction about the work developed in the present thesis, in particular, the state of the art, some relevant references in the regulation field and the general idea of the problem faced and solved in the next chapters; the second part describes the structure of the thesis and the organization.

1.1 General Introduction

Although the nonlinear regulation theory has reached a maturity stage, there are some crucial aspects still open as far as the design of robust regulators are concerned. In particular, a systematic design of robust regulators having the so-called *internal model property* in presence of steady state laws affected by parametric or structural uncertainties is definitely an open research field. So far, many researchers dealt with this problem using adaptive techniques as in [1] and [2], while others faced the problem using techniques typically adopted in the context of adaptive observers design, achieving interesting results both in the linear and in the nonlinear case and in global and semi-global context, see [3], [4] and [5]. More recently, some authors have proposed regression-like methods by developing adaptive and learning algorithms for nonlinear internal models to deal with uncertainties in the steady state control law, see among others [6]. Relying on the same philosophy, in [7], the authors have shown how to design regression-based internal model regulators using static adaptation laws, instead of standard dynamical estimation schemes, to offset parametric uncertainties in the steady state control law. Something inherent to the field of hybrid systems has been developed in [8], in which the interconnection of a feed-forward model of the exosystem with a hybrid adaptive law is presented. In this thesis can be found a different perspective to the problem of adaptive regulation in which prediction error identification methods, which are routinely used in robust control contexts ([9], [10]), can be adopted to design robust nonlinear regulators. The point of departure is the design procedure presented in [11] in which the steady state control law and its time-derivatives up to a certain order are assumed to satisfy a regression formula (with known regression vector) by which internal model regulators can be designed by means of high-gain tools. The regression formula in this context is thought of as a prediction model relating the "next" time derivative of the steady state control law to the "previous" derivatives through a unknown regression vector. The proposed control structure combines continuous-time dynamics and "hybrid identifiers", the latter specifically designed to estimate the actual regression vector. The fact of considering hybrid systems as identifiers is essentially motivated by the goal of setting up a general framework where many design strategies can be cast. In fact, on one hand, the proposed approach aims to capture continuous-time adaptive regulator design procedures, proposed so far in literature, as particular cases. On the other hand, the aim is to open the doors to identification tools that typically rely on sampling the available data set and to update the prediction model in a discrete-time fashion. In this context the kind of result claimed is practical output regulation with an asymptotic error that depends on the prediction error.

The theory presented in this work is clearly linked to the literature of identification in connection to robust control design ([9], [10]), and specifically to the issue of interplay between identification and control (see [12]) according to which identification methods must be synergistically used with control design methods to optimize closed-loop performances. In the actual framework the transient performances of the closed-loop system are not taken in count. Rather the main interest is to optimize the steady state error by "synergistically" designing the internal model and the identifier in order to estimate the steady state control law by minimizing, in some sense, the asymptotic regulation error. The controller has essentially an high-gain structure with an high-gain observer estimating the "dirty derivatives" of the ideal steady state control law. The latter are then processed by an identifier adaptively tuning the internal model.

1.2 Thesis Organization

The thesis is organized as follows:

- In Chapter 2 the basics about nonlinear output regulation are given. Furthermore, the problem of robustness is analyzed and the main idea is presented;

- Chapter 3 shows all the mathematical details of the new framework that solves the problem of robust output regulation;
- In Chapter 4 there are two particular cases of study: the first is the identification of the uncertain parameters in case of linear parametric model, by means of the well-known Least Squares method, while the second case deals with a linear parametric model for what the identification procedure is nested in the regulator structure allowing a static adaptation;
- In Chapter 5 some simulations on simple examples are reported. The idea is to validate numerically the theory presented in the previous chapters of the thesis.
- In Chapter 6 there are the conclusions about the work done and eventual future developments concerning the remaining still open problems of the actual approach.

In the two Appendices there are auxiliary results useful to understand some parts of the work. In particular these results regard hybrid systems and hybrid input to state stable Lyapunov functions for that kind of systems and also the small-gain theorem governing the interconnections of hybrid systems in presence of average dwell-time.

Chapter 2

Nonlinear Output Regulation Theory

In this chapter, the basics of nonlinear output regulation are shown. In the first section the emphasis goes on the high-gain tools used to generalize the structure of the regulator, while the second part deals with the problem of the robustness, giving the idea at the basis of the overall thesis.

2.1 Nonlinear Output Regulation Background

In this section we briefly recall some basic concepts regarding the nonlinear output regulation with high gain methods ([13], [11]) that are instrumental for the main result of the overall work. It is possible to start by considering the following nonlinear system modeled by equations of the form

$$\dot{w} = s(w) \tag{2.1a}$$
$$\dot{z} = f(z, w, e_1) \tag{2.1b}$$
$$\dot{e}_i = e_{i+1} \qquad i = 1, \dots, r-1 \tag{2.1c}$$
$$\dot{e}_r = q(z, w, e) + b(z, w, e)u. \tag{2.1d}$$

In the previous system one can recognize two main subsystems: the first, described by (2.1a), is the so-called *exosystem* with state $w \in W \subset \mathbb{R}^s$ generating possible *references* signals to be tracked and/or possible *disturbances* that must be rejected. The set W is a compact set that is assumed to be invariant for the exosystem dynamics (2.1a). The second subsystem is the *controlled plant* given in (2.1b), (2.1c), (2.1d) in which

$(z, e_1, \dots, e_r) \in \mathbb{R}^n \times \mathbb{R}^r$ is the state, $u \in \mathbb{R}$ is the control input, and e is the regulation error. All functions in the overall system, i.e. $s(\cdot)$, $f(\cdot,\cdot,\cdot)$, $q(\cdot,\cdot,\cdot)$ and $b(\cdot,\cdot,\cdot)$ are smooth in their arguments, with the function $b(\cdot,\cdot,\cdot)$, the so-called "high-frequency gain" of the system, that is assumed to be bounded from below by a positive number $\underline{b}$, i.e

$$b(z, w, e) \geq \underline{b} > 0 \qquad \forall (z, w, e) \in \mathbb{R}^n \times W \times \mathbb{R}^r .$$

The main results presented in the thesis do not rely on perfect knowledge of the functions $s(\cdot)$, $f(\cdot,\cdot,\cdot)$, $q(\cdot,\cdot,\cdot)$ and $b(\cdot,\cdot,\cdot)$ but rather on certain structural properties that will be detailed next.

In this framework, the problem of *semiglobal asymptotic* output regulation can be formulated as follows: given the sets $W \subset \mathbb{R}^s$, $Z \subset \mathbb{R}^n$ and $E \subset \mathbb{R}^r$ of initial conditions for the system (2.1a)–(2.1d), design an error-feedback controller with state $\xi \in \mathbb{R}^d$, for some positive d, and initial condition in a compact set $\Xi \subset \mathbb{R}^d$ such that all trajectories of the closed-loop system starting from $W \times Z \times E \times \Xi$ are bounded and

$$\lim_{t \to \infty} e(t) = 0$$

uniformly in the initial conditions.

We shall approach the previous problem under assumptions that are customary in the literature of output regulation. In particular we assume the existence of a smooth function $\pi : \mathbb{R}^s \to \mathbb{R}^n$ that solves the so called "regulator equations"

$$L_{s(w)}\pi(w) = \frac{\partial \pi(w)}{\partial w} s(w) = f(\pi(w), w, 0), \tag{2.2}$$

for all $w \in W$. This implies the existence of a compact set

$$\mathcal{A} := \{(w, z) \in W \times \mathbb{R}^n \ : \ z = \pi(w)\}$$

that is invariant for the dynamics

$$\dot{w} = s(w), \qquad \dot{z} = f(z, w, 0) . \tag{2.3}$$

The previous system is the zero dynamics of system (2.1a)-(2.1d) relative to the input u and to the output e. As in most of the literature about output regulation, we make a *minimum-phase* assumption on system (2.3) that is formalized as follows.

Assumption. (Minimum Phase)
The set $\mathcal{A}$ is globally asymptotically and locally exponentially stable[1] for (2.3) with a domain of attraction of the form $W \times \mathbb{R}^n$. $\triangleleft$

In the design of the regulator a crucial role is played by the function $c : W \times \mathbb{R}^n \to \mathbb{R}$ defined as

$$c(w, z) = -q(w, z, 0)/b(w, z, 0)\,. \tag{2.4}$$

This function is the so-called "friend" associated to the zero dynamics of system (2.1a)–(2.1d) (see [14]). In the context of output regulation, the output signals generated by system (2.3) with output (2.4) with initial conditions ranging in $\mathcal{A}$ are the steady state control inputs that must be generated by the controller in order to keep the regulation error identically to zero. It is thus apparent that system (2.3) restricted to the set $\mathcal{A}$ with output (2.4) plays a crucial role in the design of the regulator.

All the considerations done so far, address the problem of output regulation in the case of any relative degree for the controlled system. In what follows, without loss of generality, the problem will be considered for the simpler case of $r = 1$, the reason this can be done, follows from classical results about output feedback stabilization briefly summarized here just for sake of completeness.

First of all, consider the following change of variable for the system (2.1a)-(2.1d)

$$\begin{aligned} e_i &\mapsto y_i := k_c^{-(i-1)} e_i, \qquad i = 1, \ldots, r-1, \\ e_r &\mapsto \theta_c := e_r + k_c^{r-1} a_0 e_1 + k_c^{r-2} a_1 e_2 + \cdots + k_c a_{r-2} e_{r-1}, \end{aligned}$$

where $k_c > 1$ is a design parameter and the all the other parameters a_i, $i = 0, \ldots, r-2$, are such that all roots of the polynomial

$$\lambda^{r-1} + a_{r-2}\lambda^{r-1} + \cdots + a_1\lambda + a_0 = 0$$

have negative real part. After this change of variable, system (2.1a)-(2.1d) becomes a system of the form

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= f(z, w, y_1) \\ \dot{y} &= k_c A_H y + B\theta_c \\ \dot{\theta}_c &= \tilde{q}(w, z, y, \theta_c, k_c) + \tilde{b}(w, z, y, \theta_c, k_c) u \end{aligned} \tag{2.5}$$

[1]The forthcoming result can be extended to cover the case in which the set $\mathcal{A}$ is only locally asymptotically stable with a domain of attraction of the form $W \times \mathcal{D}$ with $\mathcal{D}$ an open set of $\mathbb{R}^n$ such that $Z \subset \mathcal{D}$.

where $y = (y_1, \ldots, y_{r-1})$, A_H is a Hurwitz matrix and $\tilde{q}$, $\tilde{b}$ are smooth functions with

$$\tilde{b}(w, z, y, \theta_c, k_c) \geq \underline{b} > 0 \qquad \forall (z, w, y, \theta_c) \in \mathbb{R}^n \times W \times \mathbb{R}^{r-1} \times \mathbb{R}$$

and for all $k_c > 0$. Note that, by definition, $y_1 = e_1$. Let $\tilde{E} \in \mathbb{R}^{r-1}$ be a compact set such that $e \in E \to y \in \tilde{E}$ and note that, if $k_c > 1$, the set $\tilde{E}$ can be taken independent on k_c. System (2.5), regarded as a system with input u and output θ_c, has relative degree one and zero dynamics

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= f(z, w, y_1) \\ \dot{y} &= k_c A_H y. \end{aligned} \tag{2.6}$$

For such a system, under the minimum phase assumption, can be used classical results ([15]) to show the existence of a $k_c^\star > 1$ such that for all $k_c \geq k_c^\star$, the set $\mathcal{A} \times \{0\}$ is locally asymptotically stable for (2.6), with a domain of attraction of the form $W \times \tilde{\mathcal{D}}$, with $\tilde{\mathcal{D}} \supset Z \times \tilde{E}$. Suppose now that a controller, function of θ_c, solves the problem of output regulation for the system (2.5). This controller is driven by the regulated variable θ_c and not by the actual regulated output y_1. However, by construction, θ_c is a fixed linear combination of the components $y_, \ldots, y_r$ of the partial state e of the original system (2.1a)-(2.1d). In this case, e_i coincides with the $(i-1)-$th time derivative of the actual regulated output e_1. In order to secure asymptotic convergence to the desired target set, $e_1, \ldots, e_r$ can be replaced by appropriate estimates $\hat{e}_1, \ldots, \hat{e}_r$ provided by a high gain observer driven only by e_1. Using these estimates to replace the expression of θ_c in the controller, yields a final regulator able to solve the problem for the original plant (2.1a)-(2.1d).

On the basis of these arguments, in what follows, it is possible to restrict the discussion to the case of systems having *relative degree* $r = 1$, which, for notational convenience, are rewritten in the following *normal form*

$$\dot{w} = s(w) \tag{2.7}$$

$$\dot{z} = f(z, w, e) \tag{2.8}$$

$$\dot{e} = q(z, w, e) + b(z, w, e)u\,. \tag{2.9}$$

As a matter of fact it is a well-known fact ([16]) that the output regulation problem is solved by a *continuous-time* regulator if one is able to design smooth functions $M : \mathbb{R}^d \to \mathbb{R}^d$, $G : \mathbb{R}^d \to \mathbb{R}^{d\times 1}$, and $\gamma : \mathbb{R}^d \to \mathbb{R}$, such that, for some smooth function $\tau : \mathbb{R}^s \to \mathbb{R}^d$

and with $\mathcal{T}$ the compact set defined as

$$\mathcal{T} := \{(w,\xi) \in W \times \mathbb{R}^d \ : \ \xi = \tau(w)\},$$

the set $\mathcal{A} \times \mathcal{T}$ is locally asymptotically stable for the system

$$\dot{w} = s(w), \qquad \dot{z} = f(w,z,0), \qquad \dot{\xi} = M(\xi) + G(\xi)c(w,z) \tag{2.10}$$

with a domain of attraction $W \times \mathbb{R}^n \times \mathcal{C}$ with $\mathcal{C}$ an open set of $\mathbb{R}^d$ satisfying $\mathcal{C} \supset \Xi$, and, in addition,

$$\gamma(\tau(w)) = c(w,\pi(w)) \qquad \forall\, w \in W\,. \tag{2.11}$$

In this context, in fact, the continuous-time controller that solves the problem at hand is a system of the form

$$\begin{aligned} \dot{\xi} &= M(\xi) + G(\gamma(\xi) + v) \\ u &= \gamma(\xi) + v\,, \qquad v = -\kappa(e) \end{aligned} \tag{2.12}$$

where $\kappa(\cdot)$ is a properly defined class-$\mathcal{K}$ function. As a matter of fact, the closed loop system given by (2.7)–(2.9) and (2.12) is a system that has relative degree one relative to the input v and output e and has a zero dynamics precisely given by (2.10). Furthermore, due to (2.11), the set $\mathcal{A}\times\mathcal{T}\times\{0\}$ is an invariant set for the closed loop system with $v = 0$. Under these circumstances, standard high-gain arguments can be used to show that an "high-gain" function[2] $\kappa(\cdot)$ succeeds in making the set $\mathcal{A}\times\mathcal{T}\times\{0\}$ locally asymptotically stable with a domain of attraction containing the compact set of initial conditions.

As shown in [17], functions $M(\cdot)$, $G(\cdot)$ and $\gamma(\cdot)$ with the desired properties can be always constructed by following a design procedure that, however, is not, in general, constructive. A relevant context where a *constructive* design procedure can be given is the one originally presented in [11] in which, by letting $u^\star : W \to \mathbb{R}$ the restriction of (2.4) to the set $\mathcal{A}$ defined as

$$u^\star(w) = c(w,\pi(w))\,, \tag{2.13}$$

it is asked that the following *regression* formula

$$L^d_{s(w)} u^\star(w) = \varphi(u^\star(w), L_{s(w)}u^\star(w), \ldots, L^{d-1}_{s(w)}u^\star(w)), \qquad \forall\, w \in W \tag{2.14}$$

is fulfilled for some positive d and some *known* locally Lipschitz function $\varphi : \mathbb{R}^d \to \mathbb{R}$. In this case, in fact, the theory of high-gain observers ([18]) can be used to show that

[2]The $\kappa(e)$ can be indeed taken as a linear function ke with k a sufficiently large gain if the set $\mathcal{A}\times\mathcal{T}$ is also locally *exponentially* stable for (2.10).

the above properties are fulfilled with

$$G(\xi) = G := \mathrm{col}(\lambda_1 g,\ \lambda_2 g^2,\ \dots,\ \lambda_d g^d)\,, \tag{2.15}$$

where g is a design parameter and the λ_i's that are coefficients of an Hurwitz polynomial,

$$M(\xi) := \mathrm{col}(\xi_2,\ \dots,\ ,\xi_d,\ ,\varphi_s(\xi)) - G\xi_1\,, \tag{2.16}$$

where $\varphi_s(\cdot)$ is a uniformly bounded and locally Lipschitz function, and $\gamma(\xi) = \xi_1$. By choosing $M(\cdot)$, G, and $\gamma(\cdot)$ in this way, by letting

$$\tau(w) = \mathrm{col}(u^\star(w), \dots, L_{s(w)}^{d-1} u^\star(w))\,, \tag{2.17}$$

and by choosing $\varphi_s(\cdot)$ so that it agrees with $\varphi(\cdot)$ for all $\xi = \tau(w)$, $w \in W$, it turns out that there exists a $g^\star > 1$ (depended on the Lipschitz constant and on the bound of $\varphi_s(\cdot)$) such that the set $\mathcal{A} \times \mathcal{T}$ is locally asymptotically stable for (2.10) and (2.11) is fulfilled.

The previous high-gain design methodology can be also proved to be robust to possible residual bias in the relation (2.14). Specifically, in [7] it has been shown that if there exists a known locally Lipschitz function $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that, instead of (2.14),

$$L_s^d u^\star(w) = \varphi(u^\star(w), L_s u^\star(w), \dots, L_s^{d-1} u^\star(w)) + \nu(w), \qquad \forall\, w \in W \tag{2.18}$$

for some continuous function $\nu : W \to \mathbb{R}$, then there exists $g^\star > 1$, only dependent on the Lipschitz constant of $\varphi(\cdot)$, such that for all $g \geq g^\star$ the same regulator presented above guarantees that the closed-loop trajectories originating from the given compact sets are bounded and the regulation error fulfils

$$\limsup_{t\to\infty} \|e(t)\| \leq \frac{c}{g^{d+1}} \max_{w\in W} \|\nu(w)\| \tag{2.19}$$

where c is a positive constant. Practical, instead of asymptotic, regulation is thus achieved with a residual error that depends on the entity of the residual bias $\nu(w)$.

2.2 The Issue of Robustness and Main Idea

The previous high-gain framework and relation (2.14) are at the basis of the robust regulator design. The main idea developed in the thesis is to regard the function $\varphi(\cdot)$ in (2.14) as unknown and to estimate it on line by adopting *prediction error identification methods*, [9]. In particular, relation (2.14) is regarded as a *prediction model* of the d-th time derivative of the signal $u^\star(w(t))$ at time t using the *regression vector* $(u^\star(w(t)), L_s u^\star(w(t)), \ldots, L_s^{d-1} u^\star(w(t)))$, with the identification objective that is to estimate the function $\varphi(\cdot)$ "best fitting" the data associated to the actual $u^\star(w(t))$. The goal is to design a *practical* regulator in which the asymptotic bound on the closed-loop regulation error is a function of the asymptotic value of the prediction error between the actual value of the d-th time derivative of the signal $u^\star(w(t))$ and its estimated value obtained by processing the regression vector.

If the signal $u^\star(w(t))$ and its derivative up to the order d were known, the problem could be addressed by running identification algorithms to compute the best fitting function from the data set. Since $u^\star(w(t)), \ldots, L_s^d u^\star(w(t))$ are not measurable in our output regulation context, the idea that is pursued in the thesis is to estimate their value by employing the "dirty derivative" (using the terminology in [19]) features of the internal model of the form indicated in the previous section. Namely, the ability of the ξ-system in (2.10), with $M(\cdot)$ and G given in (2.16) and (2.15) to roughly estimate asymptotically the function $u^\star(t)$ and its time derivative up to the order $d-1$, with an estimation error that can be arbitrarily decreased by increasing g, *regardless* the specific form of $\varphi_s(\cdot)$ in (2.16) (provided that a bound on the Lipschitz constant is fixed). Since the identification problem potentially requires the knowledge also of $L_s^d u^\star(w)$, the regulator that is presented later has dimension $d+1$, namely one more with respect to the one presented above. The extra state variable ξ_{d+1}, that is redundant as far as the internal model property is concerned, has precisely the role of providing a "dirty estimate" of $L_s^d u^\star(w)$ that is used in the identification algorithm.

In the present approach, the dynamical system providing the estimation of the d-th derivative according to the regression vector is an *hybrid system* combining continuous and discrete-time dynamics, [20]. The fact of considering hybrid systems as identifiers is essentially motivated by the goal of setting up a general framework where many design strategies can be cast. In fact, on one hand, the proposed approach aims to capture continuous-time adaptive regulator design procedures, proposed so far in literature, as particular cases. On the other hand, the aim is to open the doors to identification tools that typically rely on sampling the available data set and to update the prediction model in a discrete-time fashion. The jump times and flow intervals of the hybrid identifier will be triggered by a clock variable that will be required to fulfill *average dwell time* and *reverse average dwell time* constraints (see [21]) in order to enforce appropriate asymptotic properties to the closed-loop system. From a formal view point the clock hybrid dynamics will be described by differential and algebraic inclusions able to model a number clock dynamics that are not necessarily uniform in time. From a practical view point the clock will be triggered by a "supervisor" selecting the appropriate flow and jump rule for the identifier according to real-time information. Fast or slow clock timing can be dynamically selected according to the available data (such as the actual value of regulation error) and the a priori knowledge of the exogenous dynamics. The requirement behind the hybrid identifiers will be given in a quite general setting as presented in the next chapters. A schematic of the control structure is shown in Figure 2.1. The final observation is that in the proposed framework the dimension d of the regulator can be regarded as an independent design parameter to be chosen also in relation to real-time and implementation constraints (which, very often, prevent one to choose large value of d). The choice of d, in general, entails a trade off between the minimization of the asymptotic error bound (typically asking for large values of d) and the computational burden that typically limits the maximum value of d.

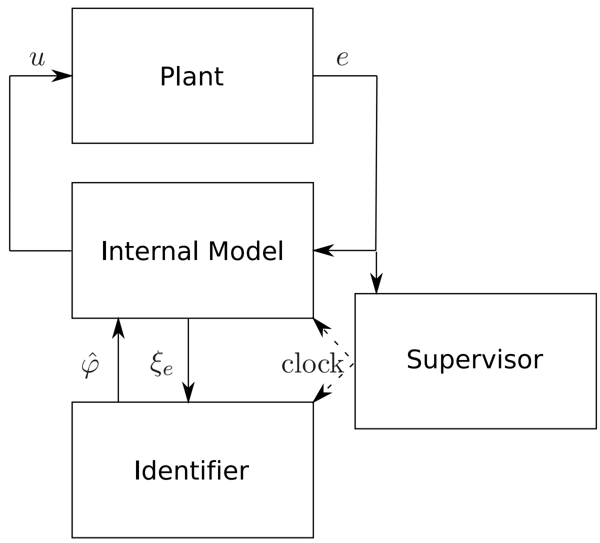

FIGURE 2.1: Schematic of the control structure.

Chapter 3

Robust Nonlinear Output Regulation

In the actual chapter the idea is to present all the mathematical details of the main idea shown in Section 2.2 for achieving robust regulation. The first section gives the tools for the construction of the overall regulator; the second section proposes the fundamental requirements for the identifier (used for the best estimation of all the uncertainties in the regulation scheme), while the third section analyzes the asymptotic properties of the overall control loop, i.e the interconnection between the controlled plant, the regulator and the identifier (Figure 2.1).

3.1 The Regulator Structure

In this paragraph it is shown the structure of the overall controller able to guarantee the regulation to zero of the considered plant. In details, first of all, consider the identifier as an hybrid dynamical system whose flow dynamics and jump map are described by

$$\left.\begin{array}{rcl} \dot{\eta}_{\mathrm{c}} & \in & F_c(\eta_c) \\ \dot{\eta}_{\mathrm{e}} & = & F_{\mathrm{e}}(\eta_{\mathrm{e}}, u_\eta) \end{array}\right\} (\eta_{\mathrm{c}}, \eta_{\mathrm{e}}, u_\eta) \in C_{\mathrm{c}} \times \mathbb{R}^m \times \mathbb{R}^{d+1}$$
$$\left.\begin{array}{rcl} \eta_{\mathrm{c}}^+ & \in & J_{\mathrm{c}}(\eta_c) \\ \eta_{\mathrm{e}}^+ & = & J_{\mathrm{e}}(\eta_{\mathrm{e}}, u_\eta) \end{array}\right\} (\eta_{\mathrm{c}}, \eta_{\mathrm{e}}, u_\eta) \in D_{\mathrm{c}} \times \mathbb{R}^m \times \mathbb{R}^{d+1} \tag{3.1}$$

with output

$$\hat{\varphi} = \Gamma_\eta(\eta_{\mathrm{e}}, u_{\eta 1}) \tag{3.2}$$

where $(\eta_c, \eta_e) \in \mathbb{R} \times \mathbb{R}^m$, $m > 0$, $F_c : C_c \rightrightarrows \mathbb{R}$ and $J_c : D_c \rightrightarrows \mathbb{R}$ are outer semicontinuous and locally bounded set-valued functions, C_c and D_c are closed intervals of $\mathbb{R}$, $u_\eta = \text{col}(u_{\eta 1}, u_{\eta 2})$, with $u_{\eta 1} \in \mathbb{R}^d$ and $u_{\eta 2} \in \mathbb{R}$, is a vector of inputs, and $F_e : \mathbb{R}^m \times \mathbb{R}^{d+1} \to \mathbb{R}^m$, $J_e : \mathbb{R}^m \times \mathbb{R}^{d+1} \to \mathbb{R}^m$ and $\Gamma_\eta : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ are smooth functions, with $J_e(\cdot)$ and $\Gamma_\eta(\cdot)$ that are *globally Lipschitz*. The scalar variable η_c plays the role of clock governing the length of the flow intervals and the jump times according to the definition of the flow and jump sets C_c and D_c. Both discrete-time and continuous-time dynamics can be captured by the previous description.

With $\tau(w)$ defined as in (2.17), the hybrid identifier (3.1) should be ideally driven by the inputs $u_{\eta 1} = \tau(w)$, representing the regression vector in the interpretation given in Section 2.2, and $u_{\eta 2} = L_s^d u^\star(w)$, representing the "next" derivative, yielding an estimate $\hat{\varphi}(t) = \Gamma_\eta(\eta(t), \tau(w(t)))$ able to best predict $L_s^d u^\star(w(t))$ on the basis of the values of the regression vector. Since $\tau(w)$ and $L_s^d u^\star(w)$ are not accessible, the hybrid identifier (3.1) is fed with the state $\xi_e = \text{col}(\xi, \xi_{d+1})$, $\xi \in \mathbb{R}^d$, $\xi_{d+1} \in \mathbb{R}$, of an "extended" internal model unit[1], namely

$$u_{\eta 1} = \xi\,, \qquad u_{\eta 2} = \xi_{d+1}\,, \tag{3.3}$$

governed by the hybrid system

$$\begin{aligned}
\dot{\xi}_e &= \begin{pmatrix} \dot{\xi} \\ \dot{\xi}_{d+1} \end{pmatrix} = \begin{pmatrix} S\xi + B\xi_{d+1} + Gv \\ \Gamma'_{\eta s}(\eta, \xi_e) + \lambda_{d+1} g^{d+1} v \end{pmatrix} \\
& \qquad (\xi_e, \eta, v) \in \mathbb{R}^{d+1} \times (C_c \times \mathbb{R}^m) \times \mathbb{R} \\
\xi_e^+ &= \begin{pmatrix} \xi^+ \\ \xi_{d+1}^+ \end{pmatrix} = \begin{pmatrix} \xi \\ \Gamma_\eta(J_e(\eta_e, \xi_e), \xi) \end{pmatrix} \\
& \qquad (\xi_e, \eta, v) \in \mathbb{R}^{d+1} \times (D_c \times \mathbb{R}^m) \times \mathbb{R} \\
u &= C\xi + v
\end{aligned} \tag{3.4}$$

where $(S, B, C) \in \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times 1} \times \mathbb{R}^{1\times d}$ is a triplet in prime form[2], G is defined in (2.15), g is a design parameter, the λ_i $i = 1, \ldots, d+1$, are coefficients of an Hurwitz polynomial, v is a residual input and $\Gamma'_{\eta s} : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ is a locally Lipschitz bounded function obtained by appropriately saturating the function

$$\Gamma'_\eta(\eta_e, \xi_e) = \frac{\partial \Gamma_\eta(\eta_e, \xi)}{\partial \eta} F_\eta(\eta_e, \xi_e) + \sum_{i=1}^{d} \frac{\partial \Gamma_\eta(\eta_e, \xi)}{\partial \xi_i} \xi_{i+1}\,. \tag{3.5}$$

[1]The adjective "extended" has to be interpreted with respect to the internal model considered in Section 2.1 of dimension d.

[2]That is S is a shift matrix (all 1's on the upper diagonal and all 0's elsewhere), $B^T = (0 \cdots 0\ \ 1)$ and $C = (1\ \ 0 \cdots 0)$.

Details on how the saturation level of $\Gamma'_{\eta s}(\cdot)$ has to be chosen are presented later. The regulator is thus (3.1), (3.3), (3.4) where v is the residual input that will be chosen as

$$v = -\kappa e$$

with κ a design parameter.

The flow time intervals and the times at which jumps occur are uniquely determined by the clock dynamics. The fact of modeling the latters as differential and algebraic inclusions allows one for considering a number of clock timing not necessarily "uniform" in time. Fast and slow clocks might be dynamically triggered according to real time information. The only constraint that will be imposed by the forthcoming analysis to the clock dynamics is to fulfill *average and reverse average dwell-time* conditions. In particular, to make sure that continuous-time dynamics present in the loop exhibit their asymptotic properties, the forthcoming stability analysis will rely upon a condition asking that flow intervals are "persistently" present and last "in the average" a guaranteed amount of time. From a formal viewpoint the notion of *average dwell-time* ([22]) is used to rigorously fix the required property. I would like to recall ([22]) that the clock subsystem satisfies an average dwell-time if there exist $N_0 > 1$ and $\delta > 0$ such that of all (t, j) and (s, i) belonging to the hybrid time domain of the clock with $t + j > s + i$ the following holds

$$j - i \leq \delta(t - s) + N_0 \,. \tag{3.6}$$

In the previous relation $1/\delta$ denotes the average dwell-time, while N_0 denotes the maximum number of consecutive jumps that might occur not separated by flow intervals. The average-dwell time condition expressed above might be eventually completed with a "reverse" condition asking that clocks are also "persistently" enforced. This condition might be crucial in order to design the hybrid identifier with the desired asymptotic properties detailed in the next Section 3.2 in certain discrete-time identification settings (as, for instance, in the case presented in Section 4.1). From a formal viewpoint the notion of *reverse average dwell-time* ([21]) is used to rigorously fix the required property. I would like to recall ([21]) that the clock subsystem satisfies a reverse average dwell-time if there exist $N_0 > 1$ and $\delta > 0$ such that of all (t, j) and (s, i) belonging to the hybrid time domain of the clock with $t + j > s + i$ the following holds

$$t - s \leq \delta(j - i) + N_0\delta \,. \tag{3.7}$$

3.2 Identifier Design Requirements

A crucial role in achieving small (possibly zero) asymptotic regulation error will be clearly played by the design of the hybrid identifier (3.1), namely by the design of the functions $F_{\mathrm{e}}(\cdot)$, $J_{\mathrm{e}}(\cdot)$ and $\Gamma_\eta(\cdot)$, and of the sets F_{c}, J_{c}, C_{c} and D_{c}. According to the identification literature ([9]), the design of the identifier entails the choice of a certain model structure for the function $\varphi(\cdot)$ and to choose an estimation method to select the "best" member in the family defined by the model structure. In the selection of the model structure, different approaches can be followed in relation to the amount of knowledge about the steady state input (and, specifically, about the fulfillment of a relation of the form (2.14)) one has a priori, and to the value of d governing the dimension of the regulator. *Gray box* models, in which the candidate model for $\varphi(\cdot)$ is properly parametrized by using, for instance, linear regression laws, as well as *black-box* models are possible alternatives ([9]). About the estimation method, minimization of estimation functional of some function of the prediction error, such as least squares methods, are routinely adopted. The methods are typically "trajectory based", namely optimization is performed with respect to a specific data set. In our context the specific data set with respect to which optimization is performed is given by the steady state input $u^\star(w(t))$ associated to the specific exosystem trajectory $w(t)$.

The requirements assumed for the design of this system are precisely presented below. The first requirement is existence of an exponentially stable "steady-state" for (3.1) driven by the "ideal" input $u_\eta = \mathrm{col}(\tau(w), L_s^d u^\star(w))$ (denoted by $\eta_{\mathrm{e}} = \sigma(\eta_{\mathrm{c}}, w)$ in the following). As (3.1) is not driven by the ideal input $(\tau(w), L_s^d u^\star(w))$ but, rather, by the available dirty derivatives state $\mathrm{col}(\xi, \xi_{d+1})$, a robustness property of such a steady state is required. It is given in terms of input-to-state stability with respect to a disturbance, referred to as d_e in the following, additive to the ideal input $(\tau(w), L_s^d u^\star(w))$. The previous properties are the ones playing a role in the asymptotic stability analysis. In addition, it is assumed that the output $\Gamma_\eta(\cdot)$ of (3.1) evaluated along the steady state trajectory of the identifier is the "best guess" of the "next" time derivatives $L_s^d u^\star(w)$, namely the function able to minimize the prediction error (which will be denoted by ε). In the following we refer to $J(\varepsilon)$ the functional that is behind the selection of the best guess. The expression of $J(\varepsilon)$ is deliberately left unspecified at this level of the analysis since it does not affect the stability analysis. A possible choice is then presented in Section 4.1 when a specific hybrid identifier is designed.

From sake of compactness, we rewrite system (3.1) as

$$
\begin{array}{rcll}
\dot{\eta} & \in & F_\eta(\eta, u_\eta) & \qquad (\eta, u_\eta) \in C_\eta \times \mathbb{R}^{d+1} \\
\eta^+ & \in & J_\eta(\eta, u_\eta) & \qquad (\eta, u_\eta) \in D_\eta \times \mathbb{R}^{d+1}
\end{array}
$$

where $\eta = \mathrm{col}(\eta_{\mathrm{c}}, \eta_{\mathrm{e}})$, and where the set-valued functions $F_\eta(\cdot)$, $J_\eta(\cdot)$, and the flow and jump sets C_η, D_η are suitably defined. Furthermore, we let $\tau_e : \mathbb{R}^s \to \mathbb{R}^{d+1}$ be the smooth function defined as

$$\tau_e(w) = \mathrm{col}(\tau(w),\, L_s^d u^\star(w))\,.$$

Identifier Design Requirement.

The hybrid system (3.1) with output (3.2) is said to satisfy an "Identifier Design Requirement" if the following properties hold:

(a) there exists a smooth function $\sigma : \mathbb{R} \times \mathbb{R}^s \to \mathbb{R}^m$ such that the hybrid system

$$\left.\begin{array}{rcl} \dot{w} & = & s(w) \\ \dot{\eta} & \in & F_\eta(\eta, \tau_e(w) + \mathrm{d}_e) \end{array}\right\} (w, \eta, \tau_e(w) + \mathrm{d}_e) \in W \times C_\eta \times \mathbb{R}^{d+1}$$
$$\left.\begin{array}{rcl} w^+ & = & w \\ \eta^+ & \in & J_\eta(\eta, \tau_e(w) + \mathrm{d}_e) \end{array}\right\} (w, \eta, \tau_e(w) + \mathrm{d}_e) \in W \times D_\eta \times \mathbb{R}^{d+1} \tag{3.8}$$

is pre-ISS (Input-to-State Stable) with respect to the input d_e relative to the set

$$\mathcal{B} = \{(w, \eta) \in W \times (C_\eta \cup D_\eta) \;:\; \eta_{\mathrm{e}} = \sigma(\eta_{\mathrm{c}}, w)\}$$

without restrictions on the initial state and non-zero restriction on the input, and with linear asymptotic gain. That is (see [23]), there exists a locally Lipschitz function $V_\eta : W \times \mathbb{R}^{m+1} \to \mathbb{R}_{\geq 0}$, such that the following holds:

- there exist locally linear $\mathcal{K}_\infty$ functions $\underline{\alpha}_\eta, \bar{\alpha}_\eta$ such that for all $(w, \eta) \in W \times \mathbb{R}^{m+1}$

$$\underline{\alpha}_\eta(\|(w, \eta)\|_{\mathcal{B}}) \leq V_\eta(w, \eta) \leq \bar{\alpha}_\eta(\|(w, \eta)\|_{\mathcal{B}})\,;$$

- there exist positive r, χ_η and c_η, such that for all $(w, \eta) \in W \times C_\eta$ and for all d_e fulfilling $\|\mathrm{d}_e\| \leq r$ we have

$$V_\eta(w, \eta) \geq \chi_\eta \|\mathrm{d}_e\| \quad \Rightarrow \quad V_\eta^o((w, \eta), \mathrm{v}) \leq -c_\eta(V_\eta(w, \eta)) \qquad \forall\, \mathrm{v} \in \begin{pmatrix} s(w) \\ F_\eta(\eta, \tau_e(w) + \mathrm{d}_e) \end{pmatrix};$$

- there exists a positive constant $\lambda_\eta < 1$ such that for all $(w, \eta) \in W \times D_\eta$ and for all d_e fulfilling $\|\mathrm{d}_e\| \leq r$ we have, with the same χ_η as in the previous item,

$$V_\eta(\mathrm{v}) \le \max\{\lambda_\eta V_\eta(w,\eta)\,,\ \chi_\eta\|\mathrm{d}_e\|\}$$
$$\forall\ \mathrm{v} \in \begin{pmatrix} s(w) \\ J_\eta(\eta, \tau_e(w) + \mathrm{d}_e) \end{pmatrix}.$$

(b) Let $\varepsilon : \mathbb{R}^s \times \mathbb{R} \to \mathbb{R}$ be the smooth prediction error function defined as

$$\varepsilon(\eta_c, w) = L_s^d u^\star(w) - \Gamma_\eta(\sigma(\eta_c, w), \tau(w))\,. \tag{3.9}$$

Then, for all $\eta_c(t,j) \in C_\mathrm{c} \cup D_\mathrm{c}$ solution of the clock subsystem in (3.1) and for all $w(t,j) \in W$ solution of the exosystem, the hybrid identifier is optimal with respect to some estimation functional $J(\varepsilon(\eta_c(t,j), w(t,j)))$.

With the function $\sigma(\cdot)$ introduced in the item (a) above, the tuning of the regulator (3.1), (3.3), (3.4) can be then completed by specifying $\Gamma'_{\eta\mathrm{s}}(\eta_\mathrm{e}, \xi_e)$ as any locally Lipschitz *bounded* function that agrees with $\Gamma'_\eta(\eta_\mathrm{e}, \xi_e)$ for all $(\eta, \xi_e) \in (C_\eta \cup D_\eta) \times \mathbb{R}^{d+1}$ such that $\|(w,\eta)\|_\mathcal{B} \le c$, $\|(w,\xi_e)\|_{\mathrm{gr}\tau_e} \le c$ for some positive c, where

$$\mathrm{gr}\,\tau_e = \{(w, \xi_e) \in W \times \mathbb{R}^{d+1}\ :\ \xi_e = \tau_e(w)\}\,.$$

3.3 Asymptotic Properties of the Closed-Loop System

In this section we are going to study the asymptotic properties of the closed-loop system. We will show how, for an appropriate tuning of the regulator, the resulting closed-loop hybrid system is pre-ISS relative to a compact set, whose projection on the error space is the origin, with respect to a "disturbance" input given by the prediction error $\varepsilon(\eta_c, w)$. The overall closed-loop system is a hybrid system flowing according to

$$\begin{aligned}
\dot w &= s(w) \\
\dot z &= f(z, w, e) \\
\dot\xi_e &= \begin{pmatrix} \dot\xi \\ \dot\xi_{d+1} \end{pmatrix} = \begin{pmatrix} S\xi + B\xi_{d+1} + Gv \\ \Gamma'_{\eta\mathrm{s}}(\eta_\mathrm{e}, \xi_e) + \lambda_{d+1} g^{d+1} v \end{pmatrix} \\
\dot\eta &\in F_\eta(\eta, \xi_e) \\
\dot e &= q(z, w, e) + b(z, w, e)(C\xi - \kappa e)\,.
\end{aligned}$$

when $(w, z, \xi_e, \eta, e) \in W \times \mathbb{R}^n \times \mathbb{R}^{d+1} \times C_\eta \times \mathbb{R}$, and jumping according to

$$\begin{aligned}
w^+ &= w & z^+ &= z \\
\xi^+ &= \xi & \xi_{d+1}^+ &= \Gamma_\eta(J_\mathrm{e}(\eta_\mathrm{e}, \xi_e), \xi) \\
\eta^+ &\in J_\eta(\eta, \xi_e) & e^+ &= e
\end{aligned}$$

when $(w, z, \xi_e, \eta, e) \in W \times \mathbb{R}^n \times \mathbb{R}^{d+1} \times D_\eta \times \mathbb{R}$. By letting $\mathbf{x} = \mathrm{col}(w, z, \xi_e, \eta_\mathrm{e}, e)$ such a system is rewritten in compact form as

$$\left.\begin{array}{rcl} \dot{\eta}_\mathrm{c} & \in & F_\mathrm{c}(\eta_\mathrm{c}) \\ \dot{\mathbf{x}} & = & F_\mathbf{x}(\mathbf{x}) \end{array}\right\} \quad (\eta_\mathrm{c}, \mathbf{x}) \in C_\mathrm{c} \times C_\mathbf{x}\,,$$

$$\left.\begin{array}{rcl} \eta_\mathrm{c}^+ & \in & J_\mathrm{c}(\eta_\mathrm{c}) \\ \mathbf{x}^+ & = & J_\mathbf{x}(\mathbf{x}) \end{array}\right\} \quad (\eta_\mathrm{c}, \mathbf{x}) \in D_\mathrm{c} \times D_\mathbf{x} \tag{3.10}$$

where the functions $F_\mathbf{x}(\cdot)$, $J_\mathbf{x}(\cdot)$ and the sets $C_\mathbf{x}$, $D_\mathbf{x}$ are appropriately defined. The main result is presented in the next theorem claiming that the regulation error is asymptotically bounded by a (linear) function of the prediction error provided that the clock subsystem satisfies an average dwell-time.

Theorem 3.1. *Consider the closed-loop system (3.10) with the zero dynamics of the regulated plant fulfilling the minimum-phase assumption and with system (3.1) fulfilling the identifier design requirement. Furthermore, for all (t, j) and (s, i) belonging to the hybrid time domain of (3.10) such that $t + j > s + i$, assume that the average dwell time condition (3.6) is fulfilled for some $\delta \geq 0$ and $N_0 \geq 1$. Then, for any compact set $\mathbf{X} \subset W \times \mathbb{R}^n \times \mathbb{R}^{d+1} \times \mathbb{R}^m \times \mathbb{R}$, there exist $\delta^\star > 0$, $g^\star > 0$ and $\kappa^\star(g) > 0$ such that for all $\delta \in (0, \delta^\star)$, $g \geq g^\star$, $\kappa \geq \kappa^\star(g)$, and for all (t, j) belonging to the hybrid time domain of (3.10) with flow and jump sets restricted to $C_\mathrm{c} \times (C_\mathbf{x} \cap \mathbf{X})$ and $D_\mathrm{c} \times (D_\mathbf{x} \cap \mathbf{X})$ the following holds*

$$\lim_{t+j\to\infty} \sup |e(t, j)| \leq \rho \lim_{t+j\to\infty} \sup |\varepsilon(\eta_\mathrm{c}(t, j), w(t, j))| \tag{3.11}$$

with ρ a positive constant. $\triangleleft$

It is worth noting that the asymptotic estimate (3.11) holds as long as the state of the closed-loop system remains in a fixed (arbitrarily large) compact set $\mathbf{X}$, with the latter that affects the value of δ, g and κ. Namely, the result is semiglobal in the state. Forward invariance of the set $\mathbf{X}$ by the closed-loop system is not claimed in the theorem. In case the trajectories exit from the restricted flow and jump set, the solution stop to exist according to the result above. However, arguments similar to the ones that are used below show that the same control structure can force the state of the closed-loop system, with initial value in any arbitrary compact set, to be bounded for sufficiently high value of δ, g and κ.

The proof of Theorem 3.1 is presented in the rest of the section.

The closed-loop system has unitary relative degree between the input v and the output e and zero dynamics described by the hybrid system flowing according to

$$\begin{array}{rcl} \dot{w} & = & s(w) \\ \dot{z} & = & f(z,w,0) \\ \dot{\xi} & = & S\xi + G(c(z,w) - C\xi) \\ \dot{\xi}_{d+1} & = & \Gamma'_{\eta\mathrm{s}}(\eta_\mathrm{e},\xi_e) + \lambda_{d+1} g^{d+1}(c(z,w) - C\xi) \\ \dot{\eta} & \in & F_\eta(\eta,\xi_e) \end{array}$$

when $(w,z,\xi_e,\eta) \in W \times \mathbb{R}^n \times \mathbb{R}^{d+1} \times C_\eta$, and jumping according to

$$\begin{array}{rcl} w^+ & = & w \\ z^+ & = & z \\ \xi^+ & = & \xi \\ \xi^+_{d+1} & = & \Gamma_\eta(J_\mathrm{e}(\eta_\mathrm{e},\xi_e),\xi) \\ \eta^+ & \in & J_\eta(\eta,\xi_e) \end{array}$$

when $(w,z,\xi_e,\eta) \in W \times \mathbb{R}^n \times \mathbb{R}^{d+1} \times D_\eta$. In the following we study the asymptotic properties of the zero dynamics. It is worth regarding such a system as the interconnection of three subsystems. The first is the hybrid system

$$\left.\begin{array}{rcl} \dot{\eta}_\mathrm{c} & \in & F_\mathrm{c}(\eta_\mathrm{c}) \\ \dot{w} & = & s(w) \\ \dot{\xi} & = & S\xi + G(c(z,w) - C\xi) \\ \dot{\xi}_{d+1} & = & \Gamma'_{\eta\mathrm{s}}(\eta_\mathrm{e},\xi_e) + \lambda_{d+1} g^{d+1}(c(z,w) - C\xi) \end{array}\right\} \quad (\eta,w,z,\xi_e) \in C_\eta \times W \times \mathbb{R}^n \times \mathbb{R}^{d+1}$$
$$\left.\begin{array}{rcl} \eta_\mathrm{c}^+ & \in & J_\mathrm{c}(\eta_\mathrm{c}) \\ w^+ & = & w \\ \xi^+ & = & \xi \\ \xi^+_{d+1} & = & \Gamma_\eta(J_\mathrm{e}(\eta_\mathrm{e},\xi_e),\xi) \end{array}\right\} \quad (\eta,w,z,\xi_e) \in D_\eta \times W \times \mathbb{R}^n \times \mathbb{R}^{d+1} \tag{3.12}$$

regarded as a system with state $(\eta_\mathrm{c},w,\xi_e)$ and exogenous inputs (w,z,η_e), interconnected to a second hybrid system that is the system (3.8) with state η and input d_e taken as $\mathrm{d}_e := \xi_e - \tau_\mathrm{e}(w)$. The previous interconnection is then in cascade with the continuous-time system given by (2.3), see Figure 3.1. We will show that such a interconnection is pre-ISS relative to a certain compact set with respect to an exogenous input represented by the prediction error (3.9). To this purpose we start by studying system (3.12) that, by letting $\varsigma = \mathrm{col}(w, \xi_e)$, is compactly rewritten as

$$\begin{array}{rlcl} \dot{\eta}_{\mathrm{c}} \in & F_{\mathrm{c}}(\eta_{\mathrm{c}})\,, & \dot{\varsigma} & = F_{\varsigma}(\varsigma, (w, \eta_{\mathrm{e}}, z)) \\ \eta_{\mathrm{c}}^{+} \in & J_{\mathrm{c}}(\eta_{\mathrm{c}})\,, & \varsigma^{+} & = J_{\varsigma}(\varsigma, (w, \eta_{\mathrm{e}}, z)) \end{array} \tag{3.13}$$

with flow and jump conditions respectively given by $(\eta, \varsigma, z) \in C_\eta \times C_\varsigma \times \mathbb{R}^n$ and $(\eta, \varsigma, z) \in D_\eta \times D_\varsigma \times \mathbb{R}^n$, where the functions $F_\varsigma(\cdot)$, $J_\varsigma(\cdot)$ and the sets C_ς and D_ς are properly defined. The next result shows that if g is taken large and if an average dwell-time constraint is fulfilled between consecutive jumps of the clock, then the system is pre-ISS relative to a compact set with respect to inputs given by the prediction error $\varepsilon(\eta_{\mathrm{c}}, w)$ and by the functions $\ell_1(w, z)$, $\ell_2(\eta, w)$ defined as

$$\ell_1(w, z) = c(w, z) - u^\star(w)\,, \qquad \ell_2(\eta, w) = \eta_{\mathrm{e}} - \sigma(\eta_{\mathrm{c}}, w)\,.$$

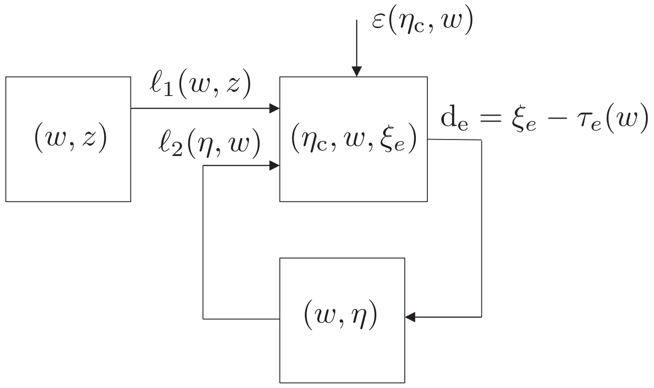

FIGURE 3.1: A graphical sketch of the zero dynamics hybrid interconnection.

Proposition 3.2. *With $\bar{\tau}_e : (C_{\mathrm{c}} \cup D_{\mathrm{c}}) \times W \to \mathbb{R}^{d+1}$ the locally Lipschitz function*

$$\bar{\tau}_e(\eta_{\mathrm{c}}, w) = col(\tau(w)\,,\ \Gamma_\eta(\sigma(\eta_{\mathrm{c}}, w), \tau(w)))\,,$$

let $\mathcal{C}$ be the compact set defined as

$$\mathcal{C} = \{(\eta_{\mathrm{c}}, w, \xi_e) \in (C_{\mathrm{c}} \cup D_{\mathrm{c}}) \times W \times \mathbb{R}^{d+1} \quad : \quad \xi_e = \bar{\tau}_e(\eta_{\mathrm{c}}, w)\}\,.$$

Furthermore, for all (t, j) and (s, i) belonging to the hybrid time domain of (3.10) such that $t+j > s+i$, assume that the average dwell time condition (3.6) is fulfilled for some $\delta \geq 0$ and $N_0 \geq 1$.

Then, there exist $\delta_1^\star > 0$ and $g_1^\star > 0$ such that for all positive $\delta \leq \delta_1^\star$ and $g \geq g_1^\star$, system (3.12) is pre-ISS relative to the set $\mathcal{C}$ with respect to the inputs $\varepsilon(\cdot)$, $\ell_1(\cdot)$ and $\ell_2(\cdot)$ with linear asymptotic gains. Furthermore, for all compact set $Z \subset \mathbb{R}^n$ and positive constants

T and ϵ, there exists a $g_2^\star > 0$ such that for all $g \geq g_2^\star$ the following holds

$$\|\xi_e(t,j) - \tau_e(w(t,j))\| \leq \epsilon$$

for all $t \geq T$ and (t,j) belonging to the hybrid time domain of (3.13) with flow and jump sets respectively given by $C_\eta \times C_\varsigma \times Z$ and $D_\eta \times D_\varsigma \times Z$. $\triangleleft$

The proof of the proposition is presented in Appendix A. For the following developments it is worth noting that the property of pre-ISS with respect to the set $\mathcal{C}$ claimed in the proposition is equivalent (see [21]) to the existence of a locally Lipschitz function $V_\varsigma : (C_\mathrm{c} \cup D_\mathrm{c}) \times (C_\varsigma \cup D_\varsigma) \to \mathbb{R}_{\geq 0}$, such that the following holds:

- there exist positive constants $\underline{\alpha}_\varsigma$, $\bar{\alpha}_\varsigma$ such that for all $(\eta_\mathrm{c}, \varsigma) \in (C_\mathrm{c} \cup D_\mathrm{c}) \times (C_\varsigma \cup D_\varsigma)$

$$\underline{\alpha}_\varsigma \,\|(\eta_\mathrm{c}, \varsigma)\|_\mathcal{C} \leq V_\varsigma(\eta_\mathrm{c}, \varsigma) \leq g^d \,\bar{\alpha}_\varsigma \,\|(\eta_\mathrm{c}, \varsigma)\|_\mathcal{C}\,; \tag{3.14}$$

- there exist positive χ_ς and c_ς, such that for all $(\eta, \varsigma, z) \in C_\eta \times C_\varsigma \times \mathbb{R}^n$ we have

$$\begin{aligned} V_\varsigma(\eta_\mathrm{c}, \varsigma) \geq \chi_\varsigma \max\{g^d|\ell_1(w,z)|\,,\ \frac{1}{g}\|\ell_2(\eta, w)\|\,, |\varepsilon(\eta_\mathrm{c}, w)|\} \\ \Rightarrow\ V_\varsigma^o((\eta_\mathrm{c}, \varsigma), \mathrm{v}) \leq -c_\varsigma V_\varsigma(\eta_\mathrm{c}, \varsigma)\,; \end{aligned}$$

  for all $\mathrm{v} \in \mathrm{col}(F_\mathrm{c}(\eta_\mathrm{c}),\ F_\varsigma(\varsigma, (w, \eta_\mathrm{e}, z)))$;

- there exists a positive λ_ς with $\lambda_\varsigma < 1$, such that for all $(\eta, \varsigma, z) \in D_\eta \times D_\varsigma \times \mathbb{R}^n$ we have, with the same χ_ς as in the previous item,

$$V_\varsigma(\mathrm{v}) \leq \max\{\lambda_\varsigma V_\varsigma(\eta_\mathrm{c}, \varsigma)\,, \chi_\varsigma|\ell_1(w,z)|\,,\ \chi_\varsigma\|\ell_2(\eta, w)\|\,,\ \chi_\varsigma|\varepsilon(\eta_\mathrm{c}, w)|\}$$

  for all $\mathrm{v} \in \mathrm{col}(J_\mathrm{c}(\eta_\mathrm{c}),\ J_\varsigma(\varsigma, (w, \eta_\mathrm{e}, z)))$.

We now consider the interconnection of system (3.12) and system (3.8) that, denoting by $\chi = \mathrm{col}(\varsigma, w, \eta_\mathrm{e})$ the combined state, is compactly rewritten as

$$\begin{array}{lll} \dot{\eta}_\mathrm{c} \in F_\mathrm{c}(\eta_\mathrm{c})\,, & \dot{\chi} = F_\chi(\chi, (w,z)) & (\eta_\mathrm{c}, \chi, w, z) \in C_\mathrm{c} \times C_\chi \times W \times \mathbb{R}^n \\ \eta_\mathrm{c}^+ \in J_\mathrm{c}(\eta_\mathrm{c})\,, & \chi^+ = J_\chi(\chi, (w,z)) & (\eta_\mathrm{c}, \chi, w, z) \in D_\mathrm{c} \times D_\chi \times W \times \mathbb{R}^n \end{array} \tag{3.15}$$

where the functions $F_\chi(\cdot)$, $J_\chi(\cdot)$ and the sets C_χ, D_χ ara properly defined. This system is studied by restricting the state χ to an arbitrary compact set denoted by K_χ, namely we restrict the flow and jump sets of (3.15) respectively to $C_\mathrm{c} \times (C_\chi \cap K_\chi) \times W \times \mathbb{R}^n$ and $D_\mathrm{c} \times (D_\chi \cap K_\chi) \times W \times \mathbb{R}^n$.

As far as system (3.8) is concerned, let $(\eta_{\rm cp}, w_{\rm p}) \in (C_{\rm c} \cup D_{\rm c}) \times W$ be such that

$$\|(\eta_{\rm c}, w, \xi_e)\|_{\mathcal{C}} = \|(\eta_{\rm c}, w, \xi_e) - (\eta_{\rm cp}, w_{\rm p}, \xi_{e,\rm p})\|$$

with $\xi_{e,\rm p} = \bar{\tau}_e(\eta_{\rm cp}, w_{\rm p})$ and note that the input $\rm d_e$ can be bounded as (using the fact that $\bar{\tau}_e(\cdot)$ is locally Lipschitz and that $C_{\rm c}$, $D_{\rm c}$ and W are compact)

$$\begin{array}{rcl}
\|{\rm d_e}\| & = & \|\xi_e - \tau_e(\eta_{\rm c}, w)\| = \|\xi_e - \bar{\tau}_e(\eta_{\rm c}, w) + \bar{\tau}_e(\eta_{\rm c}, w) - \tau_e(\eta_{\rm c}, w)\| \\
& \leq & \|\xi_e - \bar{\tau}_e(\eta_{\rm c}, w)\| + \|\bar{\tau}_e(\eta_{\rm c}, w) - \tau_e(\eta_{\rm c}, w)\| \\
& = & \|\xi_e - \bar{\tau}_e(\eta_{\rm c}, w)\| + |\varepsilon(\eta_{\rm c}, w)| \\
& = & \|\xi_e - \xi_{\rm ep} + \xi_{\rm ep} - \bar{\tau}_e(\eta_{\rm c}, w)\| + |\varepsilon(\eta_{\rm c}, w)| \leq \\
& & \|\xi_e - \xi_{\rm ep}\| + \|\xi_{\rm ep} - \bar{\tau}_e(\eta_{\rm c}, w)\| + |\varepsilon(\eta_{\rm c}, w)| \\
& \leq & \|(\eta_{\rm c}, \varsigma)\|_{\mathcal{C}} + \|\bar{\tau}_e(\eta_{\rm cp}, w_{\rm p}) - \bar{\tau}_e(\eta_{\rm c}, w)\| + |\varepsilon(\eta_{\rm c}, w)| \\
& \leq & \|(\eta_{\rm c}, \varsigma)\|_{\mathcal{C}} + \bar{\tau}\|(\eta_{\rm cp}, w_{\rm p}) - (\eta_{\rm c}, w)\| + |\varepsilon(\eta_{\rm c}, w)| \\
& \leq & (1+\bar{\tau})\|(\eta_{\rm c}, \varsigma)\|_{\mathcal{C}} + |\varepsilon(\eta_{\rm c}, w)| \leq \dfrac{(1+\bar{\tau})}{\underline{\alpha}_\varsigma} V_\varsigma(\eta_{\rm c}, \varsigma) + |\varepsilon(\eta_{\rm c}, w)| \,.
\end{array}$$

Using the previous bound and the conditions fulfilled by $V_\eta(w, \eta)$ according to the hybrid identifier requirements, it turns out that for all $(\eta_{\rm c}, \chi) \in C_{\rm c} \times (C_\chi \cap K_\chi)$ we have that if $\|{\rm d_e}\| \leq r$

$$V_\eta(w,\eta) \geq \chi'_\eta \max\{V_\varsigma(\eta_{\rm c}, \varsigma)\,, |\varepsilon(\eta_{\rm c}, w)|\} \;\Rightarrow V^o_\eta((w,\eta), {\rm v}) \leq -c_\eta V_\eta(w,\eta) \qquad (3.16)$$

for all ${\rm v} \in {\rm col}(s(w)\,,\; F_\eta(\eta, \tau_e(w) + {\rm d_e}))$, where χ'_η is a positive constant. Furthermore, for all $(\eta_{\rm c}, \chi) \in D_{\rm c} \times (D_\chi \cap K_\chi)$ we have that if $\|{\rm d_e}\| \leq r$ then

$$V_\eta({\rm v}) \leq \max\{\lambda_\eta V_\eta(w,\eta)\,,\; \chi'_\eta V_\varsigma(\eta_{\rm c}, \varsigma)\,,\; \chi'_\eta |\varepsilon(\eta_{\rm c}, w)|\} \qquad (3.17)$$

for all ${\rm v} \in {\rm col}(s(w)\,,\; J_\eta(\eta, \tau_e(w) + {\rm d_e}))$, where, without loss of generality, the constant χ'_η has been taken the same as the one considered during flows.

We consider now the ς-subsystem. By using the same arguments used above to bound $\rm d_e$, using this time the fact that $\sigma(\cdot)$ is locally Lipschitz and that $\underline{\alpha}_\eta(\cdot)$ is locally linear, it possible to claim the existence of constants $c_\ell > 0$ and a_ℓ such that

$$\|\ell_2(w,\eta)\| \leq c_\ell \|(w,\eta)\|_{\mathcal{B}} \leq c_\ell \underline{\alpha}_\eta^{-1}(V_\eta(w,\eta)) \leq a_\ell V_\eta(w,\eta)$$

for all $(\eta_{\rm c}, \chi) \in C_{\rm c} \times ((C_\chi \cup D_\chi) \cap K_\chi)$. Using this bound and Proposition 3.2, it follows that for all $(\eta_{\rm c}, \chi, (w,z)) \in C_{\rm c} \times (C_\chi \cap K_\chi) \times (W \times \mathbb{R}^n)$

$$\begin{array}{c}
V_\varsigma(\eta_{\rm c}, \varsigma) \geq \chi'_\varsigma \max\{g^d\; |\ell_1(w,z)|\,,\; \dfrac{1}{g} V_\eta(w,\eta)\,, |\varepsilon(\eta_{\rm c}, w)|\} \\
\Rightarrow V^o_\varsigma((\eta_{\rm c}, \varsigma), {\rm v}) \leq -c_\varsigma V_\varsigma(\eta_{\rm c}, \varsigma)
\end{array} \qquad (3.18)$$

for all $\mathrm{v} \in \mathrm{col}(F_{\mathrm{c}}(\eta_{\mathrm{c}}),\, F_\varsigma(\varsigma,(w,\eta,z)))$, where χ'_ς is a positive constant. Furthermore, for all $(\eta_{\mathrm{c}},\chi,(w,z)) \in D_{\mathrm{c}} \times (D_\chi \cap K_\chi) \times (W \times \mathbb{R}^n)$

$$V_\varsigma(\mathrm{v}) \quad \geq \quad \max\{\lambda_\varsigma V_\varsigma(\eta_{\mathrm{c}},\varsigma)\,,\; \chi'_\varsigma|\ell_1(w,z)|\,,\chi'_\varsigma V_\eta(w,\eta)\,,\chi'_\varsigma|\varepsilon(\eta_{\mathrm{c}},w)|\} \tag{3.19}$$

for all $\mathrm{v} \in \mathrm{col}(J_{\mathrm{c}}(\eta_{\mathrm{c}}), J_\varsigma(\varsigma,(w,\eta,z)))$, where χ'_ς is a positive constant taken, without loss of generality, equal to the one used during flows. Now let $g_3^\star$ be such that

$$g_3^\star > \chi'_\varsigma\,\chi'_\eta\,.$$

Using (3.16), (3.17), (3.18), (3.19), and the fact that the hybrid system under study satisfies an average dwell-time between two consecutive jumps, it turns out that for all $g \geq g_3^\star$ the interconnection of system (3.12) is pre-ISS relative to the set $\mathcal{B}\times\mathcal{C}$. As a matter of fact, by following [24], it turns out that a hybrid time domain of the clock subsystem that satisfies (3.6) necessarily coincides with the domain of some solution of the hybrid system flowing according to $\dot\eta_{\mathrm{c}} \in [0,\delta]$ if $\eta_{\mathrm{c}} \in [0,N_0]$, and jumping according $\eta_{\mathrm{c}}^+ = \eta_{\mathrm{c}} - 1$ if $\eta_{\mathrm{c}} \in [1,N_0]$. This implies that system (3.15), given by the interconnection of system (3.12) with system (3.8), fits in the framework of Theorem B.2 in Appendix B. In particular, there exist a[3] $\delta_2^\star > 0$ and a locally Lipschitz function $V_\chi : (C_{\mathrm{c}} \cup D_{\mathrm{c}}) \times ((C_\chi \cup D_\chi)\cap K_\chi) \to \mathbb{R}_{\geq 0}$ such that for all positive dwell-time $\delta \leq \delta_2^\star$ and for all $g \geq \max\{g_1^\star\,,\; g_3^\star\}$ the following holds

- the exist locally linear class-$\mathcal{K}_\infty$ functions $\underline{\alpha}_\chi(\cdot)$ and $\bar\alpha_\chi(\cdot)$ such that for all $(\eta_{\mathrm{c}},\chi,(w,z)) \in (C_{\mathrm{c}} \cup D_{\mathrm{c}}) \times (C_\chi \cup D_\chi) \cap K_\chi) \times (W \times \mathbb{R}^n)$

$$\underline{\alpha}_\chi(\|(\eta_{\mathrm{c}},\chi)\|_{\mathcal{B}\times\mathcal{C}}) \leq V_\chi(\eta_{\mathrm{c}},\chi) \leq \bar\alpha_\chi(\|(\eta_{\mathrm{c}},\chi)\|_{\mathcal{B}\times\mathcal{C}})\,;$$

- for all $(\eta_{\mathrm{c}},\chi,(w,z)) \in C_{\mathrm{c}} \times (C_\chi \cap K_\chi) \times (W \times \mathbb{R}^n)$ we have that if $\|\mathrm{d_e}\| \leq r$ then

$$V_\chi(\eta_{\mathrm{c}},\chi) \geq \chi_\chi \,\max\{g^d\;|\ell_1(w,z)|\,,\; |\varepsilon(\eta_{\mathrm{c}},w)|\} \;\Rightarrow V_\chi^o((\eta_{\mathrm{c}},\chi),\mathrm{v}) \leq -c_\chi V_\chi(\eta_{\mathrm{c}},\chi)$$

  for all $\mathrm{v} \in \mathrm{col}(F_{\mathrm{c}}(\eta_{\mathrm{c}}),\, F_\chi(\chi,(w,z)))$, for some positive constants χ_χ and c_χ;

- for all $(\eta_{\mathrm{c}},\chi,(w,z)) \in D_{\mathrm{c}} \times (D_\chi \cap K_\chi) \times (W \times \mathbb{R}^n)$ we have that if $\|\mathrm{d_e}\| \leq r$ then

$$V_\chi(\mathrm{v}) \leq \max\{\lambda_\chi V_\chi(\eta_{\mathrm{c}},\chi)\,,\chi_\chi|\ell_1(w,\eta)|\,,\; \chi_\chi|\varepsilon(\eta_{\mathrm{c}},w)|\}$$

[3]Since χ'_ς depends on the compact set K_χ, the value of $\delta_2^\star$ depends in general on the latter (see the poof of the result in the appendix). This is the first point motivating why the average dwell time $\delta^\star$ mentioned in the statement of the main theorem depends, in general, on the compact set $\mathbf{X}$. Note that such a dependence disappears if $\underline{\alpha}_\eta(\cdot)$ is linear.

for all $\mathrm{v} \in \mathrm{col}(J_{\mathrm{c}}(\eta_{\mathrm{c}}), J_\chi(\chi, (w, z)))$, for some positive $\lambda_\chi < 1$, with χ_χ the same positive constant specified in the previous item.

We consider now the cascade connection of χ and (w, z) subsystems, namely the zero dynamics of the closed-loop system. In the following we construct a locally Lipschitz ISS Lyapunov function for the cascade. In this study we restrict the state z to an arbitrary compact set $Z \subset \mathbb{R}^n$. Furthermore, with ϵ and T fixed so that $\epsilon \in (0, r)$ and T any possible positive constant, we let $g_2^\star$ the positive constant introduced in the second part of Proposition 3.2 and we *fix once for all* the constant $g \geq g^\star := \max\{g_1^\star, g_2^\star, g_3^\star\}$.

By letting $x = \mathrm{col}(\chi, w, z)$, the zero dynamics is compactly rewritten as

$$\begin{array}{rclrcl} \dot{\eta}_{\mathrm{c}} & \in & F_{\mathrm{c}}(\eta_{\mathrm{c}})\,, & \dot{x} & = & F_x(x)(\eta_{\mathrm{c}}, x) \in C_{\mathrm{c}} \times C_x \\ \eta_{\mathrm{c}}^+ & \in & J_{\mathrm{c}}(\eta_{\mathrm{c}})\,, & x^+ & = & J_x(x)(\eta_{\mathrm{c}}, x) \in D_{\mathrm{c}} \times D_x \end{array} \tag{3.20}$$

where the set valued functions $F_x(\cdot)$, $J_x(\cdot)$ are properly defined and the flow and jumps sets are respectively given by $C_x = (C_\chi \cap K_\chi) \times W \times Z$, $D_x = (D_\chi \cap K_\chi) \times W \times Z$. By the minimum-phase assumption and by converse Lyapunov results (see Theorem 4 in [16]), there exists a locally Lipschitz function $V_z : W \times \mathbb{R}^n \to \mathbb{R}$ such that

$$\underline{\alpha}_z(\|(w, z)\|_{\mathcal{A}}) \leq V_z(w, z) \leq \bar{\alpha}_z(\|(w, z)\|_{\mathcal{A}})$$

and

$$V^o((w, z), F_z(w, z)) \leq -c_z V_z(w, z)$$

for all $(w, z) \in W \times \mathbb{R}^n$, where $\underline{\alpha}_z(\cdot)$ and $\bar{\alpha}_z(\cdot)$ are locally linear class-$\mathcal{K}_\infty$ functions, c_z is a positive constant, and $F_z(w, z) = \mathrm{col}(s(w), f(w, z, 0))$. For $(w, z) \in W \times \mathbb{R}^n$, let $w_p \in \mathcal{A}$ be such that $\|(w, z)\|_{\mathcal{A}} = \|(w, z) - (w_p, \pi(w_p))\|$. By the fact that $c(\cdot, \cdot)$ and $\pi(\cdot)$ are locally Lipschitz functions and that W is a compact set, there exist a locally Lipschitz function $\rho_c : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a positive constant $\bar{\pi}$, such that the following holds

$$\begin{array}{rcl} |\ell_1(w, z)| & = & |c(w, z) - c(w, \pi(w))| \leq \rho_c(\|z - \pi(w)\|) \\ & = & \rho_c(\|z - \pi(w_p) + \pi(w_p) - \pi(w)\|) \\ & \leq & \rho_c(\|z - \pi(w_p)\| + \|\pi(w_p) - \pi(w)\|) \\ & \leq & \rho_c(\|(w, z)\|_{\mathcal{A}} + \bar{\pi}\|w - w_p\|) \\ & \leq & \rho_c(\|(w, z)\|_{\mathcal{A}} + \bar{\pi}\|(w, z)\|_{\mathcal{A}}) \\ & := & \rho_c((1 + \bar{\pi})\|(w, z)\|_{\mathcal{A}}) \\ & \leq & \rho_c((1 + \bar{\pi})\,\underline{\alpha}_z^{-1}(V_z(w, z))) \end{array}$$

for all $(w, z) \in W \times \mathbb{R}^n$. Using the previous estimate of V_χ, the fact that $\underline{\alpha}_z(\cdot)$ is linearly bounded and the compactness of Z, the bound on $|\ell_1(\cdot)|$ implies that for all

$(\eta_c, x) \in C_c \times C_x$ we have that if $\|d_e\| \leq r$ then

$$V_\chi(\eta_c, \chi) \geq \chi_\chi \max\{g^d \, \bar{c} \, V_z(w, z)\,,\; |\varepsilon(\eta_c, w)|\} \;\Rightarrow\; V_\chi^o((\eta_c, \chi), \mathrm{v}) \leq -c_\chi V_\chi(\eta_c, \chi)$$

for all $\mathrm{v} \in \mathrm{col}(F_c(\eta_c),\, F_\chi(\chi, (w, z)))$, for some positive constants $\bar{c}$. Similarly, for all $(\eta_c, x) \in D_c \times D_x$ we have that if $\|d_e\| \leq r$ then

$$V_\chi(\mathrm{v}) \leq \max\{\lambda_\chi V_\chi(\eta_c, \chi)\,, \chi_\chi \, \bar{c} \, V_z(w, z)\,,\; \chi_\chi |\varepsilon(\eta_c, w)|\}$$

for all $\mathrm{v} \in \mathrm{col}(J_c(\eta_c), J_\chi(\chi, (w, z)))$.

Now let $W_x : (C_c \cup D_c) \times (C_x \cup D_x) \to \mathbb{R}_{\geq 0}$ be the locally Lipschitz function defined as

$$W_x(\eta_c, x) = \max\{V_\chi(\eta_c, \chi)\,,\; g^d \, \rho \, V_z(w, z)\}$$

where ρ is a constant such that $\rho \geq \chi_\chi \bar{c}$. Simple arguments, show that there exist locally linear class-$\mathcal{K}_\infty$ functions $\underline{\alpha}'_x(\cdot)$ and $\bar{\alpha}'_x(\cdot)$ such that

$$\underline{\alpha}'_x(\|(\eta_c, x)\|_{\mathcal{A}\times\mathcal{B}\times\mathcal{C}}) \leq W_x(\eta_c, x) \leq \bar{\alpha}'_x(\|(\eta_c, x)\|_{\mathcal{A}\times\mathcal{B}\times\mathcal{C}})$$

for all $(\eta_c, x) \in (C_c \cup D_c) \times (C_x \cup D_x)$. We now study $W_x(\cdot)$ during flows. For all $(\eta_c, x) \in C_c \times C_x$ such that $V_\chi(\eta_c, \chi) > g^d \rho V_z(w, z)$ (namely $W_x(\eta_c, x) = V_\chi(\eta_c, \chi)$), we have that if $W_x(\eta_c, x) \geq \chi_\chi |\varepsilon(\eta_c, w)|$ then

$$V_\chi(\eta_c, \chi) \geq \max\{\chi_\chi |\varepsilon(\eta_c, w)|\,, g^d \rho V_z(w, z)\} \geq \max\{\chi_\chi |\varepsilon(\eta_c, w)|\,, g^d \, \chi_\chi \, \bar{c} \, V_z(w, z)\}$$

and hence, if $\|d_e\| \leq r$, $W_x^o((\eta_c, x), \mathrm{v}) \leq -c_\chi V_\chi(\eta_c, \chi) = -c_\chi W_x(\eta_c, x)$ for all $\mathrm{v} \in \mathrm{col}(F_c(\eta_c), F_x(x))$. On the other hand, for all $(\eta_c, x) \in C_c \times C_x$ such that $V_\chi(\eta_c, \chi) < g^d \rho V_z(w, z)$ (namely $W_x(\eta_c, x) = g^d \rho V_z(w, z)$) then

$$W_x^o((\eta_c, x), \mathrm{v}) \;\; = \;\; g^d \rho V^o((w, z), F_z(w, z)) \leq -g^d \rho c_z V_z(w, z) = -c_z W_x(\eta_c, x)$$

for all $\mathrm{v} \in \mathrm{col}(F_c(\eta_c), F_x(x))$. Finally, using the fact that, for all $(\eta_c, x) \in C_c \times C_x$ such that $V_\chi(\eta_c, \chi) = g^d \rho V_z(w, z)$, $W_x^o((\eta_c, x), \mathrm{v}) \leq \max\{V_\chi^o((\eta_c, \chi), \mathrm{v}_\chi),\, g^d \rho V_z^o((w, z), F_z(w, z))\}$ for all $\mathrm{v} = \mathrm{col}(\mathrm{v}_\chi, F_z(w, z)) \in \mathrm{col}(F_c(\eta_c), F_x(x))$ (see [25]), we conclude that for all $(\eta_c, x) \in C_c \times C_x$ if $\|d_e\| \leq r$ then

$$W_x(\eta_c, x) \geq \chi_\chi |\varepsilon(\eta_c, w)| \Rightarrow W_x^o((\eta_c, x), \mathrm{v}) \leq -c'_x W_x(\eta_c, x)$$

for all $\mathrm{v} \in \mathrm{col}(F_c(\eta_c), F_x(x))$, where $c'_x = \min\{c_\chi,\, c_z\}$. Consider now $W_x(\eta_c, x)$ during jumps. By bearing in mind the definition of W_x, the jump rule of $V_\chi(\eta_c, \chi)$, and the fact that $V_z(w, z)$ doesn't jump, we have that if $\|d_e\| \leq r$

$$\begin{aligned}
W_x(\mathrm{v}) &\leq \max\{\lambda_\chi V_\chi(\eta_\mathrm{c},\chi)\,,\ \chi_\chi \bar{c} V_z(w,z)\,,\ \chi_\chi |\varepsilon(\eta_\mathrm{c},w)|\, g^d \varrho V_z(w,z)\} \\
&\leq \max\{\lambda_\chi W_x(\eta_\mathrm{c},x)\,,\ \chi_\chi \bar{c} V_z(w,z)\,,\ \chi_\chi |\varepsilon(\eta_\mathrm{c},w)|\, W_x(\eta_\mathrm{c},x)\} \\
&\leq \max\{\lambda_\chi W_x(\eta_\mathrm{c},x)\,,\ \frac{\chi_\chi \bar{c}}{\rho g^d} W_x(\eta_\mathrm{c},x)\,,\ \chi_\chi |\varepsilon(\eta_\mathrm{c},w)|\, W_x(\eta_\mathrm{c},x)\} \\
&= \max\{W_x(\eta_\mathrm{c},x)\,,\ \chi_\chi |\varepsilon(\eta_\mathrm{c},w)|\}
\end{aligned}$$

for all $\mathrm{v} \in \mathrm{col}(J_\mathrm{c}(\eta_\mathrm{c}), J_x(x))$. When $\varepsilon(\eta_\mathrm{c}, w) = 0$ the function $W_x(\cdot)$ is decreasing during flows but not necessarily during jumps. As above we can use Proposition B.1 in Appendix B to construct an ISS-Lyapunov function. As a matter of fact the fulfillment of the average dwell-time condition (3.6) guarantees that the clock subsystem in system (3.20) can be thought of as flowing according to $\dot{\eta}_\mathrm{c} \in [0,\delta]$ if $\eta_\mathrm{c} \in [0, N_0]$, and jumping according $\eta_\mathrm{c}^+ = \eta_\mathrm{c} - 1$ if $\eta_\mathrm{c} \in [1, N_0]$. This implies that system (3.20) fits in the framework of Proposition B.1 in Appendix B that guarantees that for all $\delta > 0$ then the locally Lipschitz function $V_x = \exp(L\eta_\mathrm{c}) W_x(\eta_\mathrm{c}, x)$ with $L \in (0\,, c_x'/\delta)$ satisfies the following:

- the exist locally linear class-$\mathcal{K}_\infty$ functions $\underline{\alpha}_x(\cdot)$ and $\bar{\alpha}_x(\cdot)$ such that for all $(\eta_\mathrm{c}, x) \in (C_\mathrm{c} \cup D_\mathrm{c}) \times (C_x \cup D_x)$

  $$\underline{\alpha}_x(\|(\eta_\mathrm{c},x)\|_{\mathcal{A}\times\mathcal{B}\times\mathcal{C}}) \leq V_x(\eta_\mathrm{c},x) \leq \bar{\alpha}_x(\|(\eta_\mathrm{c},x)\|_{\mathcal{A}\times\mathcal{B}\times\mathcal{C}})\,;$$

- for all $(\eta_\mathrm{c}, x) \in C_\mathrm{c} \times C_x$ we have that if $\|\mathrm{d_e}\| \leq r$ then

  $$V_x(\eta_\mathrm{c},x) \geq \chi_x\ |\varepsilon(\eta_\mathrm{c},w)| \ \Rightarrow\ V_x^o((\eta_\mathrm{c},x),\mathrm{v}) \leq -c_x V_x(\eta_\mathrm{c},x)$$

  for all $\mathrm{v} \in \mathrm{col}(F_\mathrm{c}(\eta_\mathrm{c}), F_x(x))$, for some positive constants χ_x and c_x;

- for all $(\eta_\mathrm{c}, x) \in D_\mathrm{c} \times D_x$ we have that if $\|\mathrm{d_e}\| \leq r$ then

  $$V_x(\mathrm{v}) \leq \max\{\lambda_x V_x(\eta_\mathrm{c},x)\,,\ \chi_x|\varepsilon(\eta_\mathrm{c},w)|\}$$

  for all $\mathrm{v} \in \mathrm{col}(J_\mathrm{c}(\eta_\mathrm{c}), J_x(x))$, for some positive $\lambda_x < 1$, with χ_x the same positive constant specified in the previous item.

The final part of the proof addresses the interconnection of the zero dynamics with the error dynamics. We start by putting the flow dynamics of the closed-loop system in normal form ([14]) by considering the change of variables

$$\xi \ \ \rightarrow \bar{\xi} = \xi - G\int_0^e \frac{1}{b(z,w,\zeta)} d\zeta, \quad \xi_{d+1} \rightarrow \bar{\xi}_{d+1} = \xi_{d+1} - \lambda_{d+1} g^{d+1} \int_0^e \frac{1}{b(z,w,\zeta)} d\zeta\,.$$

Denoting by $\bar{x}$ the state variable that coincides with x except the ξ_e entry that is substituted with $(\bar{\xi}, \bar{\xi}_{d+1})$, simple computation shows that the closed-loop system in the new coordinates reads as

$$\left.\begin{array}{rcl} \dot{\eta}_{\rm c} & \in & F_{\rm c}(\eta_{\rm c}) \\ \dot{\bar{x}} & = & F_x(\bar{x}) + \Delta_F(\bar{x}, e)e \\ \dot{e} & = & q_0(\bar{x}) + q_1(\bar{x}, e)e + b(\bar{x}, e)v \\ v & = & -\kappa e \end{array}\right\} (\eta_{\rm c}, x, e) \in C_{\rm c} \times C_x \times \mathbb{R}$$
$$\left.\begin{array}{rcl} \eta_{\rm c}^+ & \in & J_{\rm c}(\eta_{\rm c}) \\ \bar{x}^+ & = & J_x(\bar{x}) + \Delta_J(\bar{x}, e)e \\ e^+ & = & e \end{array}\right\} (\eta_{\rm c}, x, e) \in D_{\rm c} \times D_x \times \mathbb{R} \tag{3.21}$$

where $\Delta_F(\cdot)$, $\Delta_J(\cdot)$ and $q_0(\cdot)$ are properly defined functions, $F_x(\cdot)$ and $J_x(\cdot)$ are the same of (3.20), $q_0(\bar{x}) = c(w, z) - C\bar{\xi}$ and where, with a mild abuse of notation, we let $b(x, e) = b(z, w, e)$. Note that $q_0(\bar{x}) = 0$ for all $\bar{x} \in \mathcal{A} \times \mathcal{B} \times \mathcal{C}$. We study the interconnection by restricting the error e to some compact set $E \subset \mathbb{R}$. We start showing that the $(\eta_{\rm c}, \bar{x})$ subsystem is ISS relative to the set $\mathcal{A} \times \mathcal{B} \times \mathcal{C}$ with respect to the inputs (ε, e). To this purpose we observe that, for all $(\eta_{\rm c}, \bar{x}, e) \in C_{\rm c} \times C_x \times E$ and for all ${\rm v} \in {\rm col}(F_{\rm c}(\eta_{\rm c}), F_x(\bar{x}) + \Delta_F(\bar{x}, e)e)$, if $V_x(\eta_{\rm c}, \bar{x}) \geq \chi_x |\varepsilon(\eta_{\rm c}, w)|$ and $\|{\rm d_e}\| \leq r$ then

$$\begin{aligned} & V_x^o((\eta_{\rm c}, \bar{x}), {\rm v}) = \\ & \lim_{(\eta_{\rm c}', \bar{x}') \to (\eta_{\rm c}, \bar{x}),} \sup_{h \to 0^+} \frac{V_x({\rm col}(\eta_{\rm c}', \bar{x}') + h{\rm v}) - V_x(\eta_{\rm c}', \bar{x}')}{h} \\ & = \lim_{(\eta_{\rm c}', \bar{x}') \to (\eta_{\rm c}, \bar{x}),} \sup_{h \to 0^+} \frac{V_x((\eta_{\rm c}', \bar{x}') + h{\rm v}_1 + h{\rm v}_2) - V_x(\eta_{\rm c}', \bar{x}')}{h} \\ & = \lim_{(\eta_{\rm c}', \bar{x}') \to (\eta_{\rm c}, \bar{x}),} \sup_{h \to 0^+} \frac{1}{h}(V_x((\eta_{\rm c}', \bar{x}') + h{\rm v}_1 + h{\rm v}_2) - \\ & \qquad V_x((\eta_{\rm c}', \bar{x}') + h{\rm v}_1) + V_x((\eta_{\rm c}', \bar{x}') + h{\rm v}_1) - V_x(\eta_{\rm c}', \bar{x}')) \\ & \leq \rho_V \|{\rm v}_2\| - c_x V_x(\eta_{\rm c}', \bar{x}) \end{aligned}$$

where ${\rm v}_1 \in {\rm col}(F_{\rm c}(\eta_{\rm c}), F_x(\bar{x}))$ and ${\rm v}_2 \in {\rm col}(0, \Delta_F(\bar{x}, e)e)$ are such that ${\rm v} = {\rm v}_1 + {\rm v}_2$, and ρ_V is the Lipschitz constant of $V_x(\cdot)$ on $C_{\rm c} \times C_x \times E$. Using the fact that $\|{\rm v}_2\| \leq \nu_\Delta e$ for all $(\bar{x}, e) \in C_x \times E$ with ν_Δ a positive constant, the previous expression immediately yields that for all $(\eta_{\rm c}, \bar{x}, e) \in C_{\rm c} \times C_x \times E$ if $\|{\rm d_e}\| \leq r$ then

$$V_x(\eta_{\rm c}, \bar{x}) \;\; \geq \;\; \max\{\chi_x |\varepsilon(\eta_{\rm c}, w)|\,,\; \frac{2\,\nu_\Delta\,\rho_V}{c_x}|e|\} \Rightarrow V_x^o((\eta_{\rm c}, \bar{x}), {\rm v}) \leq -\frac{c_x}{2} V_x(\eta_{\rm c}, \bar{x})$$

and for all $\mathrm{v} \in \mathrm{col}(F_{\mathrm{c}}(\eta_{\mathrm{c}}), F_x(\bar{x}) + \Delta_F(\bar{x}, e)e)$. We now study $V_x(\cdot)$ during jumps. For all $(\eta_{\mathrm{c}}, \bar{x}, e) \in D_{\mathrm{c}} \times D_x \times E$ and for all $\mathrm{v} \in \mathrm{col}(J_{\mathrm{c}}(\eta_{\mathrm{c}}), J_x(\bar{x}) + \Delta_J(\bar{x}, e)e)$, if $\|\mathrm{d_e}\| \leq r$

$$\begin{aligned} V_x(\mathrm{v}) &= V_x(\mathrm{v}_1 + \mathrm{v}_2) = V_x(\mathrm{v}_1) + V_x(\mathrm{v}_1 + \mathrm{v}_2) - V_x(\mathrm{v}_1) \\ &\leq \max\{\lambda_x V_x(\eta_{\mathrm{c}}, \bar{x}),\ \chi_x|\varepsilon(\eta_{\mathrm{c}}, w)|\} + \rho_V\|\mathrm{v}_2\| \\ &\leq \max\{2\lambda_x V_x(\eta_{\mathrm{c}}, \bar{x}),\ 2\chi_x|\varepsilon(\eta_{\mathrm{c}}, w)|, 2\rho_V\nu_\Delta|e|\} \end{aligned}$$

with v_1 and v_2 defined as above.

Consider now the e system endowed with the clock subsystem[4]. Let $V_e(\eta_{\mathrm{c}}, e) = |e|$ and note that, by simple computations, there exist positive constants $\kappa_1^\star$, χ_e, c_e such that for all $\kappa \geq \kappa_1^\star$ and for all $(\eta_{\mathrm{c}}, \bar{x}, e) \in C_{\mathrm{c}} \times C_x \times E$ we have

$$V_e(\eta_{\mathrm{c}}, e) \leq \frac{\chi_e}{\kappa}|q_0(\bar{x})| \Rightarrow V^o((\eta_{\mathrm{c}}, e), \mathrm{v}) \leq -c_e V_e(\eta_{\mathrm{c}}, e)$$

for all $\mathrm{v} \in \mathrm{col}(F_{\mathrm{c}}(\eta_{\mathrm{c}}), q_0(\cdot) + q_1(\cdot)e + b(\cdot)v)$ Furthermore, during jumps, $V_e^+(\eta_{\mathrm{c}}, e) = V_e(\eta_{\mathrm{c}}, e)$.

With the previous computations at hand, it is simple to cast the study of closed-loop system (3.21) in the framework of Theorem B.2 in Appendix B. Specifically, note that, by the fact that $q_0(\cdot)$ is locally Lipschitz and vanishing on the set $\mathcal{A} \times \mathcal{B} \times \mathcal{C}$, and the fact that $\underline{\alpha}_x(\cdot)$ is locally linear there exist positive constants $\bar{q}'$ and $\bar{q}$ such that for all $(\eta_{\mathrm{c}}, \bar{x}) \in (C_{\mathrm{c}} \cup D_{\mathrm{c}}) \times (C_x \cup D_x)$ we have

$$|q_0(\bar{x})| \leq \bar{q}'\|(\eta_{\mathrm{c}}, \bar{x})\|_{\mathcal{A}\times\mathcal{B}\times\mathcal{C}} \leq \bar{q}'\underline{\alpha}_x^{-1}(V_x(\eta_{\mathrm{c}}, \bar{x})) \leq \bar{q}V_x(\eta_{\mathrm{c}}, \bar{x})\,.$$

Furthermore, note that (see [24]) the fact that the hybrid time domain of the clock-subsystem fulfills (3.6) implies that the η_{c} dynamics can be thought of as flowing according to $\dot{\eta}_{\mathrm{c}} \in [0, \delta]$ if $\eta_{\mathrm{c}} \in [0, N_0]$, and jumping according $\eta_{\mathrm{c}}^+ = \eta_{\mathrm{c}} - 1$ if $\eta_{\mathrm{c}} \in [1, N_0]$. By letting $k_2^\star = (2\nu_\Delta\rho_V\chi_e\bar{q})/c_x$, it is then immediately seen that for all $k \geq \max\{\kappa_1^\star, \kappa_2^\star\}$ system (3.21) fits in the framework of Theorem B.2 in Appendix B. In particular, there exists[5] a $\delta_4^\star > 0$ such that for all $\delta \in (0, \delta_4^\star)$ there exists a locally Lipschitz function $V_{\mathbf{x}} : (C_{\mathrm{c}} \cup D_{\mathrm{c}}) \times (C_x \cup D_x) \times E \to \mathbb{R}_{\geq 0}$ such that the following holds:

- the exist locally linear class-$\mathcal{K}_\infty$ functions $\underline{\alpha}_{\mathbf{x}}(\cdot)$ and $\bar{\alpha}_{\mathbf{x}}(\cdot)$ such that for all $(\eta_{\mathrm{c}}, \bar{x}, e) \in (C_{\mathrm{c}} \cup D_{\mathrm{c}}) \times (C_x \cup D_x) \times E$

$$\underline{\alpha}_{\mathbf{x}}(\|(\eta_{\mathrm{c}}, \bar{x}, e)\|_{\mathcal{L}}) \leq V_{\mathbf{x}}(\eta_{\mathrm{c}}, \bar{x}, e) \leq \bar{\alpha}_{\mathbf{x}}(\|(\eta_{\mathrm{c}}, \bar{x}, e)\|_{\mathcal{L}})$$

[4]Formally the study of the interconnection (3.21) involves the study of the interconnection of the two subsystems with state $(\eta_{\mathrm{c}}, \bar{x})$ and (η_c, e), namely both the $\bar{x}$ and e subsystems are endowed with the clock dynamics.

[5]Since ρ_V and ν_Δ depend, in general, on E, by following the proof of the results in Appendix, it turns out that $\delta_4^\star$ depends, in the general, on E. This is the second point motivating why the average dwell time $\delta^\star$ introduced in the statement of the main theorem depends, in general, on the compact set $\mathbf{X}$.

with $\mathcal{L} = \mathcal{A} \times \mathcal{B} \times \mathcal{C} \times \{0\}$

- for all $(\eta_c, x, e) \in C_c \times C_x \times E$ we have that if $\|d_e\| \leq r$ then

$$V_{\mathbf{x}}(\eta_c, \bar{x}, e) \geq \chi_{\mathbf{x}} \, |\varepsilon(\eta_c, w)| \;\Rightarrow\; V^o_{\mathbf{x}}((\eta_c, \bar{x}, e), \mathrm{v}) \leq -c_{\mathbf{x}} V_{\mathbf{x}}(\eta_c, \bar{x}, e)$$

  for all $\mathrm{v} \in \mathrm{col}(F_c(\eta_c), F_x(x), q_0(\cdot) + q_1(\cdot)e + b(\cdot)v)$, for some positive constants $\chi_{\mathbf{x}}$ and $c_{\mathbf{x}}$;

- for all $(\eta_c, \bar{x}, e) \in D_c \times D_x \times E$ we have that if $\|d_e\| \leq r$ then

$$V_{\mathbf{x}}(\mathrm{v}) \leq \max\{\lambda_{\mathbf{x}} V_{\mathbf{x}}(\eta_c, \bar{x}, e)\,,\; \chi_{\mathbf{x}} |\varepsilon(\eta_c, w)|\}$$

  for all $\mathrm{v} \in \mathrm{col}(J_c(\eta_c), J_x(\bar{x}), e)$, for some positive $\lambda_{\mathbf{x}} < 1$, with $\chi_{\mathbf{x}}$ the same positive constant specified in the previous item.

Now note that, by the second part of Proposition 3.2 and by the tuning of the parameter $g^\star$, it turns out that for all (t, j) in the hybrid time domain of system (3.21) such that $t \geq T$ we have $\|d_e(t, j)\| \leq r$. Thus, in finite time, as long as trajectories of (3.21) stay in $(C_c \cup D_c) \times (C_x \cup D_x) \times E$, a Lyapunov function $V_{\mathbf{x}}$ with the properties above exists. From this the claim of Proposition 3.1 follows with the asymptotic bound (3.11) resulting from the arguments in [26, Proposition 2.7].

Chapter 4

Identification Tools for Robust Regulation

In this chapter we are going to study the particular case of linear parametric models for what concerns the regression law used for the adaptive part of the regulation framework. In details, the firs part shows how it is possible to design a discrete identifier based on the common Least Squares algorithm used in classical identification. The second part analyzes the case of nonlinear regression law affine in parameters using an alternative method, i.e. the estimation of the uncertainties is nested in the regulator structure.

4.1 The case of Linear Regression Law and Least Squares Method

In this section we develop the case in which the model structure relating $L_s^d u^\star(w(t))$ and the regression vector is assumed to be a linearly parametrized function of the form

$$L_s^d u^\star(w(t)) = \Psi^T(\tau(w))\theta \tag{4.1}$$

in which $\Psi : \mathbb{R}^d \to \mathbb{R}^p$, $p > 0$, is a locally Lipschitz known function, and $\theta \in \Theta \subset \mathbb{R}^p$ is a vector of uncertain parameters with Θ a known compact set. We are interested to design a hybrid identifier of the form (3.1) fulfilling the basic requirements specified in Section 3.2, in which the estimation method used to select the best $\theta \in \Theta$ is a discrete-time least squares criterion. Specifically, let us consider a hybrid clock subsystem such that for all initial conditions $\eta_{c0} = \eta_c(0,0) \in C_c \cup D_c$ the associated hybrid time domain $E_{\eta_{c0}} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ fulfills an average dwell-time condition of the form (3.6) (required by the analysis in Section 3.3) and a reverse average dwell-time condition of the form (3.7) for some $N_0 \geq 1$ and $\delta > 0$. The reverse condition is imposed in order to have persistent jumps required by the discrete-time nature of the estimator we are going to develop. With $N > 1$, let $\mathcal{I}_{\eta_{c0}} = \{(t_{j_1}, j_1), \ldots, (t_{j_N}, j_N)\}$ be an arbitrary set of N distinct hybrid times such that $(t_{j_i}, j_i) \in E_{\eta_{c0}}$, and jumps occur at (t_{j_i}, j_i), $i = 1, \ldots N$. Our goal is to develop an hybrid identifier of the form (3.1) such that the basic hybrid requirement in Section 3.2 are fulfilled with estimation functional given by

$$J(\varepsilon(\eta_c(t,j), w(t,j))) = \frac{1}{2N} \sum_{(t_j,j)\in\mathcal{I}_{\eta_{c0}}} \varepsilon(\eta_c(t_j,j), w(t_j,j))^2$$

by using (4.1) as prediction model structure. As usual in the context of least squares identification methods we make a persistence of excitation assumption formulated as follows.

Assumption. (Persistence of excitation)
There exists a $\bar{\upsilon} > 0$ such that for all $\eta_{c0} \in C_c \cup D_c$, for all sequence of N distinct jump hybrid times $\mathcal{I}_{\eta_{c0}}$, and for all exosystem trajectories $w(t,j) \in W$ with $(t,j) \in E_{\eta_{c0}}$, the following holds

$$\det \sum_{(t_j,j)\in\mathcal{I}_{\eta_{c0}}} \Psi(\tau(w(t_j,j)))\,\Psi(\tau(w(t_j,j)))^T \geq \bar{\upsilon}\,. \quad \triangleleft$$

Our identifier (3.1) has state (η_c, η_e), with $\eta_c \in \mathbb{R}$ the clock and $\eta_e = \mathrm{col}(\eta_1, \eta_2, \eta_3)$, $\eta_1 \in \mathbb{R}^N$, $\eta_2 = \mathrm{col}(\eta_{21}, \ldots, \eta_{2N}) \in \mathbb{R}^{pN}$, $\eta_{2i} \in \mathbb{R}^p$, $i = 1, \ldots, N$, $\eta_3 \in \mathbb{R}^p$ the state of the identifier. The system flows according to

$$\begin{array}{rcl} \dot{\eta}_c & \in & F_c(\eta_c) \\ \dot{\eta}_1 & = & 0 \\ \dot{\eta}_2 & = & 0 \\ \dot{\eta}_3 & = & 0 \end{array} \tag{4.2}$$

with flow conditions $(\eta_c, \eta_1, \eta_2, \eta_3) \in C_c \times \mathbb{R}^N \times \mathbb{R}^{pN} \times \mathbb{R}^p$, and jumps according to

$$\begin{array}{rcl} \eta_c^+ & \in & J_c(\eta_c) \\ \eta_1^+ & = & S\eta_1 + B\xi_{d+1} \\ \eta_2^+ & = & (S \otimes I_p)\eta_2 + (B \otimes I_p)\Psi(\xi) \\ \eta_3^+ & = & L(\eta_1^+, \eta_2^+) \end{array} \tag{4.3}$$

with jump conditions $(\eta_c, \eta_1, \eta_2, \eta_3) \in D_c \times \mathbb{R}^N \times \mathbb{R}^{pN} \times \mathbb{R}^p$, $D_c = [0, N_0\delta]$, and output

$$\Gamma_\eta(\eta, \xi) = \Psi(\xi)^T \eta_3$$

where

$$S = \begin{pmatrix} 0 & I_{N-1} \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$L(\eta_1, \eta_2) = R_{\rm sat}(\eta_2)^{-1} \sum_{i=1}^{N} \eta_{2i}\, \eta_{1i}$$

and $L(\cdot, \cdot)$ is any globally Lipschitz function fulfilling

$$\det(\sum_{i=1}^{N} \eta_{2i}\eta_{2i}^T) \geq 2\bar{v} \quad \Rightarrow \quad L(\eta_1, \eta_2) = \left(\sum_{i=1}^{N} \eta_{2i}\eta_{2i}^T\right)^{-1} \sum_{i=1}^{N} \eta_{2i}\, \eta_{1i}\,.$$

System (4.2)-(4.2) implements a classical discrete-time least squares algorithm for the estimation of the parameters θ. Specifically, the η_1 and η_2 dynamics describe two shift registers, the former storing the last N samples of the "next" derivatives ξ_{d+1}, and the latter storing the last N samples of the regressors $\Psi(\xi)$. The variable η_3, then, represents an estimate of the uncertain vector Θ obtained by properly processing the value of η_1 and η_2.

Since the hybrid clock time domain fulfills a reverse average dwell time condition, according to [21], the clock dynamics can be thought of as flowing according to $\dot{\eta}_{\rm c} = 1$ and jumping according to $\eta_{\rm c}^+ = \max\{0\,,\ \eta_{\rm c} - \delta\}$ with flow and jump sets coincident and equal to $C_{\rm c} = D_{\rm c} = [0, N_0\delta]$.

In the remaining part of the section we prove that the previous system fulfills the hybrid identifier requirements specified in Section 3.2. Partitioning the exogenous disturbance $\mathrm{d_e}$ as $\mathrm{d_e} = \mathrm{col}(\mathrm{d}, \mathrm{d}_{d+1})$, $\mathrm{d} \in \mathbb{R}^d$, $\mathrm{d}_{d+1} \in \mathbb{R}$, the η-subsystem of (3.8) reads as $\dot{w} = s(w)$, $\dot{\eta}_{\rm c} = 1$, $\dot{\eta}_1 = 0$, $\dot{\eta}_2 = 0$ and $\dot{\eta}_3 = 0$ during flows, and

$$\begin{array}{rcl} \eta_1^+ &=& S\eta_1 + B(\tau_{d+1}(w) + \mathrm{d}_{d+1}) \\ \eta_2^+ &=& (S \otimes I_p)\eta_2 + (B \otimes I_p)\psi(\tau(w) + \mathrm{d}) \\ \eta_3^+ &=& L(\eta_1^+, \eta_2^+) \end{array}$$

during jumps. We start analyzing the η_1 subsystem. For all $\eta_{\rm c} \in C_{\rm c} \cup D_{\rm c}$ let $\eta_{\rm c0} \in C_{\rm c} \cup D_{\rm c}$ be such that $\eta_{\rm c} = \eta_{\rm c}(t, N-1)$ for some $t \in \mathbb{R}_{\geq 0}$ such that $(t, N-1) \in E_{\eta_{\rm c0}}$ and let $\varphi_w(t, w)$ be the value of the trajectory of $\dot{w} = s(w)$ at time t with initial condition w at $t = 0$. Furthermore, with $(t_i, i) \in E_{\eta_{\rm c0}}$, $i = 0, \ldots, N-1$, the hybrid jump times, let

$$T_1(\eta_{\rm c}, w) = \mathrm{col}(T_{11}(\eta_{\rm c}, w)\,, \ldots, T_{1N}(\eta_{\rm c}, w))$$

with $T_{1i} : (C_\text{c} \cup D_\text{c}) \times W \to \mathbb{R}$, $i = 1, \ldots, N$ defined as (using the fact that $\dot{\eta}_\text{c} = 1$)

$$T_{1i}(\eta_\text{c}, w) = \tau_{d+1}(\varphi_w(t - t_{i-1}, w)) = \\ \tau_{d+1}(\varphi_w(-\eta_\text{c} + \eta_\text{c}(t_{N-1}, N-1) + (t_{N-i-1} - t_{N-1}), w))\,.$$

Note that, by definition of $\varphi_w(\cdot,\cdot)$, $\dot{T}_1(\eta_\text{c}, w) = 0$. Furthermore , $T_{1i}(\eta_\text{c}^+, w^+) = T_{1i+1}(\eta_\text{c}, w)$, $i = 1, \ldots, N-1$, and $T_{1N}(\eta_\text{c}^+, w^+) = \tau_{d+1}(w)$.

Consider $W_1(\eta_\text{c}, w, \eta_1) = \sum_{i=1}^N c_i|\eta_{1i} - T_{1i}(\eta_\text{c}, w)|$ where the constants c_i are such that $c_i = 8c_{i-1}$, $i = 2, \ldots, N$, and $c_1 > 0$. During flow, using the fact that $\dot{\eta}_1 = 0$ and that $\dot{T}_1(\eta_\text{c}, w) = 0$, we have $\dot{W}_1 = 0$. During jumps, bearing in mind the jumps rule for η_1,

$$\begin{aligned} W_1(\eta_\text{c}, w, \eta_1)^+ &= \sum_{i=1}^N c_i|\eta_{1i}^+ - T_{1i}(\eta_\text{c}^+, w^+)| \\ &= \sum_{i=2}^N c_{i-1}|\eta_{1i} - T_{1i-1}(\eta_\text{c}^+, w)| + c_N|\tau_{d+1}(w) + \mathrm{d}_{d+1} - T_{1N}(\eta_\text{c}^+, w)| \\ &= \sum_{i=2}^N \frac{c_{i-1}}{c_i} c_i|\eta_{1i} - T_{1i}(\eta_\text{c}, w)| + c_N|\tau_{d+1}(w) + \mathrm{d}_{d+1} - \tau_{d+1}(w)| \\ &= \frac{1}{8}W_1(\eta_\text{c}, w, \eta_1) - \frac{1}{8}c_1|\eta_{11} - T_{11}(\eta_\text{c}, w)| + c_N|\mathrm{d}_{d+1}| \\ &\le \frac{1}{8}W_1(\eta_\text{c}, w, \eta_1) + c_N|\mathrm{d}_{d+1}|\,. \end{aligned}$$

Furthermore note that $c_1\|\eta_1 - T_1(\eta_\text{c}, w)\| \le c_1\|\eta_1 - T_1(\eta_\text{c}, w)\|_1 \le W_1(\eta_\text{c}, w, \eta_1) \le c_N\|\eta_1 - T_1(\eta_\text{c}, w)\|_1 \le c_N\sqrt{N}\|\eta_1 - T_1(\eta_\text{c}, w)\|$. The $W_1(\cdot)$ decreases during jumps (if $\mathrm{d}_{d+1} = 0$) but not during flows. In order to obtain an ISS hybrid Lyapunov function we follow [23] and we take $V_1(\eta_\text{c}, w, \eta_1) = \exp(-L\eta_\text{c})W_1(\eta_\text{c}, w, \eta_1)$ with $L > 0$ such that $\exp(L\delta) < 2$. During flow, using the fact that $\dot{\eta}_\text{c} = 1$, we have $\dot{V}_1(\eta_\text{c}, w, \eta_1) = -LV_1(\eta_\text{c}, w, \eta_1)$. During jumps, using the fact that $\eta_\text{c}^+ \le \max\{0, \eta_\text{c} - \delta\}$ and that $\eta_\text{c} \le N_0\delta$, we have (using $\eta_\text{c} - \max\{0, \eta_\text{c} - \delta\} \le \delta$)

$$\begin{aligned} &V_1(\eta_\text{c}, w, \eta_1)^+ = \exp(-L\eta_\text{c}^+)W_1(\eta_\text{c}, w, \eta_1)^+ \\ &\le \frac{1}{8}\exp(-L\max\{0, \eta_\text{c} - \delta\})W_1(\eta_\text{c}, w, \eta_1) + \exp(-L\max\{0, \eta_\text{c} - \delta\})c_N|\mathrm{d}_{d+1}| \\ &\le \frac{1}{8}\exp(-L\max\{0, \eta_\text{c} - \delta\})\exp(L\eta_\text{c})V_1(\eta_\text{c}, w, \eta_1) + c_N|\mathrm{d}_{d+1}| \\ &\le \frac{1}{8}\exp(L\delta)V_1(\eta_\text{c}, w, \eta_1) + c_N|\mathrm{d}_{d+1}| \\ &= \lambda V_1(\eta_\text{c}, w, \eta_1) + c_N|\mathrm{d}_{d+1}| \end{aligned}$$

where $\lambda = \frac{1}{8}\exp(L\delta) < \frac{1}{4}$. Note that there exist $\underline{\alpha} > 0$ and $\bar{\alpha} > 0$ such that

$$\underline{\alpha}\|\eta_1 - T_1(\eta_\text{c}, w)\| \le V_1(\eta_\text{c}, w, \eta_1) \le \bar{\alpha}\|\eta_1 - T_1(\eta_\text{c}, w)\|.$$

We consider now the η_2 subsystem. With the definition above of hybrid time domain $E_{\eta_{c0}}$, of $(t, N-1) \in E_{\eta_{c0}}$, and of $(t_i, i) \in E_{\eta_{c0}}$, $i = 0, \dots, N-1$, in mind, we let

$$T_2(\eta_c, w) = \mathrm{col}(T_{21}(\eta_c, w), \dots, T_{2N}(\eta_c, w))$$

where $T_{2i} : (C_c \cup D_c) \times W \to \mathbb{R}^p$, $i = 1, \dots, N$ are defined as

$$\begin{aligned} T_{2i}(\eta_c, w) &= \Psi(\tau(\varphi_w(t - t_{i-1}, w))) \\ &= \Psi(\tau(\varphi_w(-\eta_c + \eta_c(t_{N-1}, N-1) + (t_{N-i-1} - t_{N-1}), w)) . \end{aligned}$$

As above, we observe that, during flows, $\dot{T}_2(\eta_c, w) = 0$, and, during jumps, $T_{2i}(\eta_c^+, w^+) = T_{2i+1}(\eta_c, w)$, $i = 1, \dots, N-1$, and $T_{2N}(\eta_c^+, w^+) = \Psi(\tau(w))$.

Moreover, having defined $\delta(w, \mathrm{d}) = \Psi(\tau(w) + \mathrm{d}) - \Psi(\tau(w))$, we note that $\Psi(\tau(w) + \mathrm{d}) = \Psi(\tau(w)) + \delta(w, \mathrm{d})$ and, for any $r > 0$, there exists a constant $\bar{\delta} > 0$ such that $\|\delta(w, \mathrm{d})\| \le \bar{\delta}\|\mathrm{d}\|$ for all $w \in W$ and $\mathrm{d} \in \mathbb{R}^d$ such that $\|\mathrm{d}\| \le r$. Consider now

$$W_2(\eta_c, w, \eta_2) = \sum_{i=1}^{N} c_i |\eta_{2i} - T_{2i}(\eta_c, w)|$$

where the constants c_i are defined as above. The same steps presented above for W_1 lead to conclude that $\dot{W}_2(\eta_c, w, \eta_2) = 0$ during flows and, by using the bound on $\delta(\cdot)$, that

$$W_2(\eta_c, w, \eta_2)^+ \le \frac{1}{8} W_2(\eta_c, w, \eta_2) + c_N \bar{\delta} \|\mathrm{d}\|$$

during jumps. As above, by letting

$$V_2(\eta_c, w, \eta_2) = \exp(-L\eta_c) W_2(\eta_c, w, \eta_2)$$

with the same L defined before, we obtain that

$$\dot{V}_2(\eta_c, w, \eta_2) = -L V_2(\eta_c, w, \eta_2)$$

during flows and

$$V_2(\eta_c, w, \eta_2)^+ \le \lambda V_2(\eta_c, w, \eta_2) + c_N \bar{\delta} \|\mathrm{d}\|$$

during jumps (with the same λ introduced before). Similarly to the analysis above, moreover, it turns out that

$$\underline{\alpha} \|\eta_2 - T_2(\eta_c, w)\| \le V_2(\eta_c, w, \eta_2) \le \bar{\alpha} \|\eta_2 - T_2(\eta_c, w)\|.$$

Finally we consider the η_3 subsystem. Let $T_3(\eta_c, w) = L(T_1(\eta_c, w), T_2(\eta_c, w))$, and let $W_3(\eta_c, w, \eta_3) = c\|\eta_3 - T_3(\eta_c, w)\|_1$ with c a positive constant yet to be fixed. During flows we have $\dot{W}_3 = 0$. During jumps, by letting ℓ the Lipschitz constant of $L(\cdot,\cdot)$ and bearing in mind the definition of $\bar{\delta}$ above,

$$\begin{aligned}
&W_3^+(\eta_c, w, \eta_3) = c\|\eta_3^+ - T_3(\eta_c, w)^+\|_1 \\
&= c\|L(\eta_1^+, \eta_2^+) - L(T_1(\eta_c, w)^+, T_2(\eta_c, w)^+)\|_1 \\
&\leq c\bar{\ell}\|(\eta_1^+, \eta_2^+) - (T_1(\eta_c, w)^+, T_2(\eta_c, w)^+)\|_1 \\
&\leq c\bar{\ell}(\sum_{i=2}^{N} |\eta_{1i} - T_{1i-1}(\eta_c^+, w)| + \sum_{i=2}^{N} \|\eta_{2i} - T_{2i-1}(\eta_c^+, w)\|_1 \\
&\qquad\qquad + |\tau_{d+1}(w) + \mathrm{d}_{d+1} - T_{1N}(\eta_c^+, w)| \\
&\qquad\qquad + \|\Psi(\tau(w) + \mathrm{d}) - T_{2N}(\eta_c^+, w)\|_1) \\
&\leq c\bar{\ell}(\sum_{i=2}^{N} |\eta_{1i} - T_{1i}(\eta_c, w)| + \sum_{i=2}^{N} \|\eta_{2i} - T_{2i}(\eta_c, w)\|_1 + \\
&|\tau_{d+1}(w) + \mathrm{d}_{d+1} - \tau_{d+1}(w)| + \|\Psi(\tau(w) + \mathrm{d}) - \Psi(\tau(w))\|_1) \\
&\leq c\bar{\ell}(\|\eta_1 - T_1(\eta_c, w)\|_1 + \|\eta_2 - T_2(\eta_c, w)\|_1 + \bar{\delta}\|d\|_1) \\
&\leq c\frac{\bar{\ell}\sqrt{N}}{\underline{\alpha}}(V_1(\eta_c, w, \eta_1) + V_2(\eta_c, w, \eta_2)) + c\bar{\ell}\sqrt{N}\bar{\delta}\|d_e\|
\end{aligned}$$

We now rescale W_3 in order to obtain a Lyapunov function for the η_3-subsystem that is decreasing during flow (and without any special property during jumps). In particular, by letting $V_3(\eta_c, w, \eta_3) = \exp(-L\eta_c)W_3(\eta_c, w, \eta_3)$, we have that $\dot{V}_3(\eta_c, w, \eta_3) = -LV_3(\eta_c, w, \eta_3)$ during flows and, during jumps,

$$\begin{aligned}
V_3(\eta_c, w, \eta_3)^+ &= c\frac{\bar{\ell}\sqrt{N}}{\underline{\alpha}}\exp(-L\max\{0, \eta_c - \delta\}) \\
&\qquad (V_1(\eta_c, w, \eta_1) + V_2(\eta_c, w, \eta_2) + \bar{\delta}\underline{\alpha}\|\mathrm{d}_e\|) \\
&\leq c\frac{\bar{\ell}\sqrt{N}}{\underline{\alpha}}(V_1(\eta_c, w, \eta_1) + V_2(\eta_c, w, \eta_2) + \delta\underline{\alpha}\|\mathrm{d}_e\|)
\end{aligned}$$

Finally, we construct a Lyapunov function for the whole η system as $V(\eta, w) = V_1(\eta_c, w, \eta_1) + V_2(\eta_c, w, \eta_2) + V_3(\eta_c, w, \eta_3)$. First note that there exist positive $\underline{\alpha}_\eta$ and $\bar{\alpha}_\eta$ such that $\underline{\alpha}_\eta\|(w, \eta)\|_{\mathcal{B}} \leq V(\eta, w) \leq \bar{\alpha}_\eta\|(w, \eta)\|_{\mathcal{B}}$ with $\mathcal{B}$ defined as in Section 3.2 with

$$\sigma(\eta_c, w) = \mathrm{col}(T_1(\eta_c, w), T_2(\eta_c, w), T_3(\eta_c, w))\,.$$

Furthermore, during flows, $\dot{V}(\eta, w) \leq -LV(\eta, w)$, while, during jumps,

$$\begin{aligned}
V(\eta, w)^+ &= (\lambda + c\frac{\bar{\ell}\sqrt{N}}{\underline{\alpha}})(V_1(\eta_c, w, \eta_1) + V_2(\eta_c, w, \eta_2)) \\
&\qquad + c_N|\mathrm{d}_{d+1}| + c_N\bar{\delta}\|\mathrm{d}\| + c\bar{\ell}\sqrt{N}\bar{\delta}\|\mathrm{d}_e\| \\
&\leq \lambda'_\eta V(\eta, w) + \bar{c}\|\mathrm{d}_e\| \\
&\leq \max\{2\lambda'_\eta V(\eta, w),\ 2\bar{c}\|\mathrm{d}_e\|\}
\end{aligned}$$

where $\bar{c}$ is a positive constant and $\lambda'_\eta = (\lambda + c\frac{\bar{\ell}\sqrt{N}}{\underline{\alpha}})$. The ISS property behind the identifier design requirement is thus fulfilled by taking c so that $c\frac{\bar{\ell}\sqrt{N}}{\underline{\alpha}} < 1/4$ with $\lambda_\eta = 2\lambda'_\eta < 1$.

The fact that proposed identifier is optimal with respect to the least squares functional specified above immediately comes from the definition of $\Gamma_\eta(\cdot)$ by using the persistence of excitation assumption and the definition of $\sigma(\cdot)$.

4.2 The case of Linear Regression Law and Implicit Adaptation

The starting point in this alternative design methodology, is the existence of a regression formula that governs the k-th time derivative of the desired steady state input $u^\star(w(t))$. The formula is specified in the next Assumption. For ease of notation, here and in the following we let $u^\star_{[a,b]} := (u^{\star(a)}, \dots, u^{\star(b)})^T$, with $0 \le a < b$, the vector of time derivatives of $u^\star$.

Assumption 1.
There exist $k > 0$, $p > 0$, locally Lipschitz functions $h : \mathbb{R}^k \to \mathbb{R}$ and $L : \mathbb{R}^k \to \mathbb{R}^p$ such that

$$u^{\star(k)}(w) = h(u^\star_{[0,k-1]}(w)) + L(u^\star_{[0,k-1]}(w))\, \theta^\star \quad \forall\, w \in W. \tag{4.4}$$

where $\theta^\star \in \mathbb{R}^p$ is a vector of uncertainties. ◁

In the second part of this section we show how the previous assumption is fulfilled in a number of relevant cases.

By differentiating $i \ge 0$ times relation (4.4) and collecting the resulting equations, we obtain the following set of equations

$$u^\star_{[k,k+i]}(w) = H_i(u^\star_{[0,k+i-1]}(w)) + A_i(u^\star_{[0,k+i-1]}(w))\, \theta^\star \tag{4.5}$$

where

$$\begin{aligned} A_i(u^\star_{[0,k+i-1]}) &= \text{col}\begin{bmatrix} L_0(u^\star_{[0,k-1]}) & \cdots & L_i(u^\star_{[0,k+i-1]}) \end{bmatrix} \\ H_i(u^\star_{[0,k+i-1]}) &= \text{col}\begin{bmatrix} h_0(u^\star_{[0,k-1]}) & \cdots & h_i(u^\star_{[0,k+i-1]}) \end{bmatrix} \end{aligned} \tag{4.6}$$

where $L_0(\cdot) = L(\cdot)$, $h_0(\cdot) = h(\cdot)$, $L_{j+1}(\cdot) = \dot{L}_j(\cdot)$, $h_{j+1}(\cdot) = \dot{h}_j(\cdot)$, $j = 0, \dots, i-1$, and where for compactness we have omitted the argument w of $u^\star$.

The proposed methodology relies upon the following crucial assumption.

Assumption 2.
There exists a $m \geq p$ and $\epsilon > 0$ such that

$$\det(A_m^T(u^\star_{[0,k+m-1]}(w))\, A_m(u^\star_{[0,k+m-1]}(w))) \geq \epsilon$$

for all $w \in W$. $\triangleleft$

The previous assumption implies that

$$\mathrm{rank}(A_m(u^\star_{[0,k+m-1]}(w))) = p \quad \forall\, w \in W$$

and, in turn, that the uncertain vector $\theta^\star$ can be obtained from (4.5) as a function of $u^\star$ and its first $(k+m)$-th time derivatives. In particular, taking the $(m+1)$-th time derivative of (4.4) and replacing $\theta^\star$ with the estimation ($\hat{\theta}$) obtained by left-inverting (4.5) for $i = m$, one obtains

$$\begin{aligned} u^{\star(m+k+1)} = {} & h_{m+1}(u^\star_{[0,k+m]}) + L_{m+1}(u^\star_{[0,k+m]})\cdot \\ & A_m^\dagger(u^\star_{[0,k+m-1]})[u^\star_{[k,k+m]} - h_m(u^\star_{[0,k+m-1]})] \end{aligned}$$

where $A_m^\dagger$ represents a pseudoinverse of A_m given by

$$A_m^\dagger(\cdot) = [A_m^T(\cdot)A_m(\cdot)]^{-1}A_m(\cdot)\,.$$

This relation, in turn, is equivalent to (2.14) for an appropriately defined $\varphi(\cdot)$ with $d = m + k + 1$.

In the remaining part of the section we show how the previous assumptions are fulfilled in a number of relevant cases in which $u^\star$ is generated by nonlinear oscillators. The three cases of Van der Pol, Duffing, and Lorentz *uncertain* oscillators are considered and are dealt with in the following subsections.

4.2.1 Van der Pol Oscillator

As exosystem, we consider the Van der Pol oscillator described by

$$\begin{aligned} \dot{w}_1 &= w_2 \\ \dot{w}_2 &= -\omega^2 w_1 + \epsilon(1 - w_1^2)w_2 \end{aligned} \tag{4.7}$$

in which ω and ϵ are uncertain parameters, and consider the case in which the desired steady state input $u^\star(w) = w_1$. the set W is the omega limit set where the steady state trajectories of the Van der Pol take place. It turns out that

$$\ddot{u}^\star(w) = -u^\star(w)\omega^2 + (1 - u^{\star 2}(w))\dot{u}^\star(w)\epsilon \tag{4.8}$$

and thus Assumption 1 is fulfilled with $\kappa = 2$, $h(\cdot) = 0$, $L(\cdot) = \big(-u^\star(w)\,,\ (1-u^{\star 2}(w))\dot{u}^\star(w)\big)$ and $\theta^\star = \big(\omega^2\,,\ \epsilon\big)^T$. We start now to take time derivatives of (4.8) to identify an $m \geq 2$ for which Assumption 2 is fulfilled. By differentiating once, we obtain

$$u^\star_{[2,3]}(w) = A_1(u_{[0,2]})\theta^\star \tag{4.9}$$

where

$$A_1(u^\star_{[0,2]}) = \begin{bmatrix} -u^\star & (1-u^{\star 2})\dot{u}^\star \\ -\dot{u}^\star & \ddot{u}^\star - 2u^\star\dot{u}^{\star 2} - u^{\star 2}\ddot{u}^\star \end{bmatrix}. \tag{4.10}$$

It turns out that there are points of W where A_1 is singular (see Fig. 4.1). By thus taking a further derivative we obtain

$$u^\star_{[2,4]}(w) = A_2(u^\star_{[0,3]}(w))\ \theta^\star \tag{4.11}$$

with

$$A_2(u^\star_{[0,3]}) = \begin{bmatrix} -u^\star & (1-u^{\star 2})\dot{u}^\star \\ -\dot{u}^\star & \ddot{u}^\star - 2u^\star\dot{u}^{\star 2} - u^{\star 2}\ddot{u}^\star \\ -\ddot{u}^\star & u^{\star(3)} - 2\dot{u}^{\star 3} - 6u^\star\dot{u}^\star\ddot{u}^\star - u^{\star 2}u^{\star(3)}\,. \end{bmatrix} \tag{4.12}$$

A numerical analysis of the minors of A_2 (see Fig. 4.2) reveals that the matrix has rank 2 for all $w \in W$ and thus Assumption 2 is fulfilled.

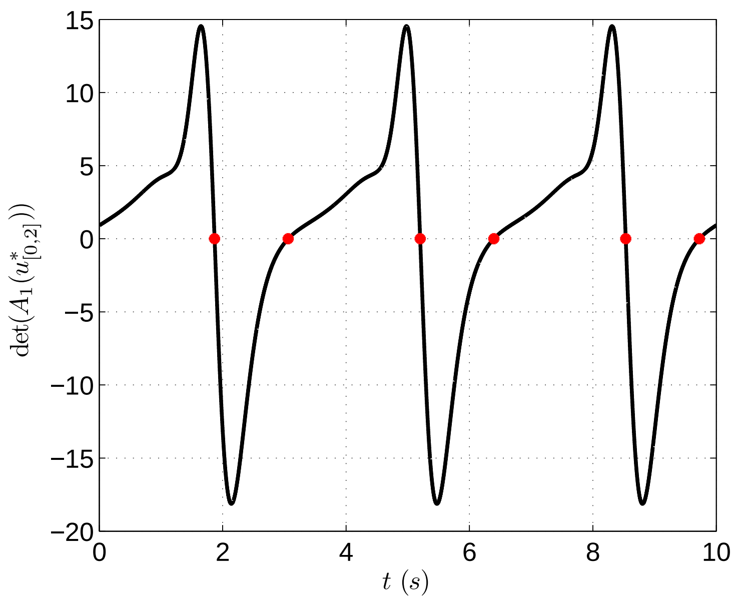

Figure 4.1: Determinant of $A_1(u^\star_{[0,2]})$ on the limit cycle ($\omega^2 = 1$ and $\epsilon = 1$).

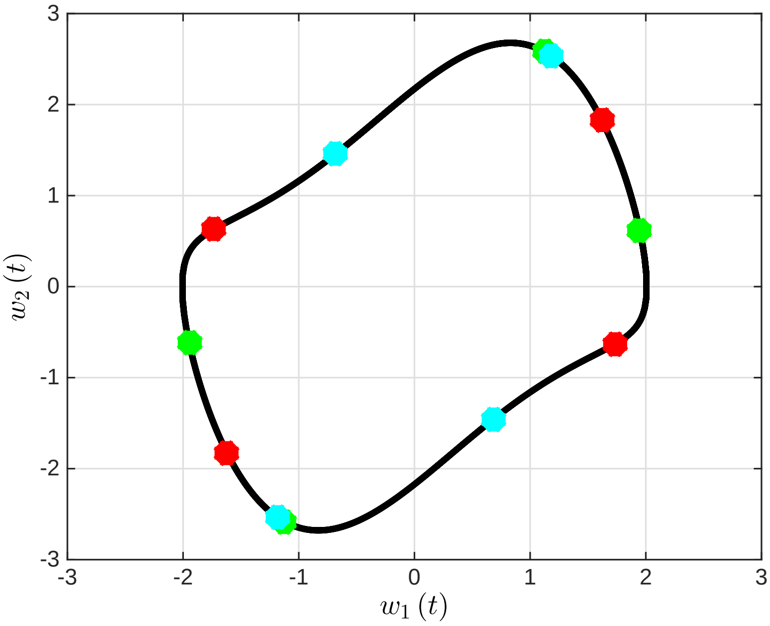

FIGURE 4.2: Limit cycle for the VdP oscillator with $\omega^2 = 1$, $\epsilon = 1$ and singularity points for each minors of matrix $A_2(u^\star_{[0,3]})$. The red points are the singularity points for the minor $A^1 := A^{12}$ having selected the first two rows of the starting matrix; the magenta points for the minor $A^2 := A^{13}$ (first and third rows) and the cyan points for the minor $A^3 := A^{23}$ (second and third rows).

4.2.2 Duffing Oscillator

We consider now the case in which $u^\star(w)$ is generated by the Duffing oscillator modeled by

$$\begin{aligned} \dot{w}_1 &= w_2 \\ \dot{w}_2 &= -w_1^3\alpha - w_1\beta \end{aligned} \tag{4.13}$$

where α and β are uncertain parameters and $u^\star(w) = w_1$. the set W is the limit cycle of the oscillator. It turns out that

$$\ddot{u}^\star(w) = -u^{\star 3}(w)\alpha - u^\star(w)\beta\,, \tag{4.14}$$

namely Assumption 1 is fulfilled with $k = 2$, $h(\cdot) = 0$, $L(\cdot) = \big(-u^{\star 3}(w),\ -u^\star(w)\big)$ and $\theta^\star = (\alpha\,,\ \beta)^T$. By differentiating once relation (4.14) we obtain

$$u^\star_{[2,3]} = A_1(u^\star_{[0,1]})\theta^\star$$

with

$$A_1(u^\star_{[0,1]}) = \begin{bmatrix} -u^{\star 3} & -u^\star \\ -3u^{\star 2}\dot{u}^\star & -\dot{u}^\star \end{bmatrix} \tag{4.15}$$

that is singular in some point of the limit cycle. Taking a further derivative we get

$$u^\star_{[2,4]} = A_2(u_{[0,2]})\theta^\star$$

with

$$A_2(u^\star_{[0,2]}) = \begin{bmatrix} -u^{\star 3} & -u^\star \\ -3u^{\star 2}\dot{u}^\star & -\dot{u}^\star \\ -3\ddot{u}^\star u^{\star 2} - 6u^\star \dot{u}^{\star 2} & -\ddot{u}^\star \end{bmatrix} \tag{4.16}$$

that is still rank-deficient. By thus taking a further derivative we get

$$u^\star_{[2,5]} = A_3(u^\star_{[0,3]})\theta^\star$$

with

$$A_3(u^\star_{[0,3]}) = \begin{bmatrix} -u^{\star 3} & -u^\star \\ -3u^{\star 2}\dot{u}^\star & -\dot{u}^\star \\ -3\ddot{u}^\star u^{\star 2} - 6u^\star \dot{u}^{\star 2} & -\ddot{u}^\star \\ -3u^{\star(3)}u^{\star 2} - 18u^\star \dot{u}^\star \ddot{u}^\star - 6\dot{u}^{\star 3} & -u^{\star(3)} \end{bmatrix} \tag{4.17}$$

which, finally, has rank 2 (see Figures 4.3-4.4).

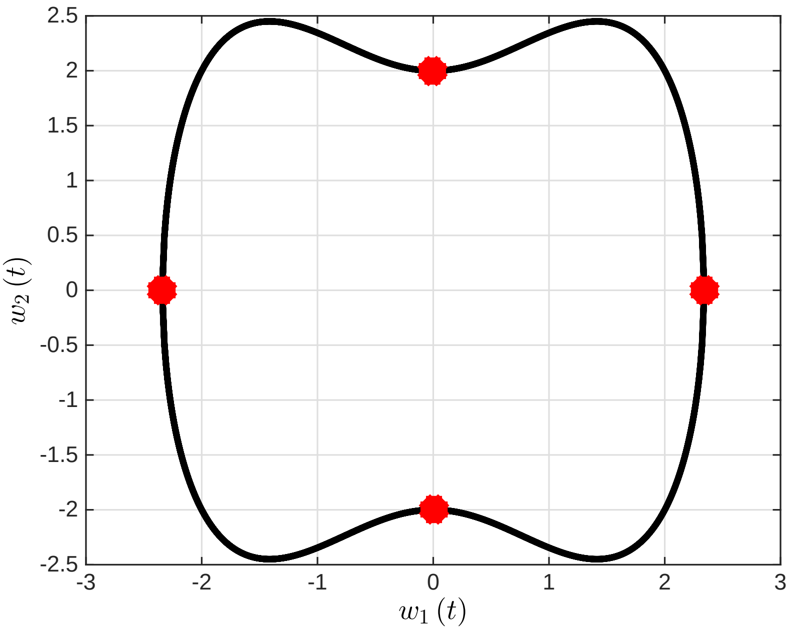

FIGURE 4.3: Limit cycle for the Duffing oscillator with $\alpha = 1$, $\beta = -2$. In the red points at least one minor out of six (of matrix $A_3(u^\star_{[0,3]})$) is not singular.

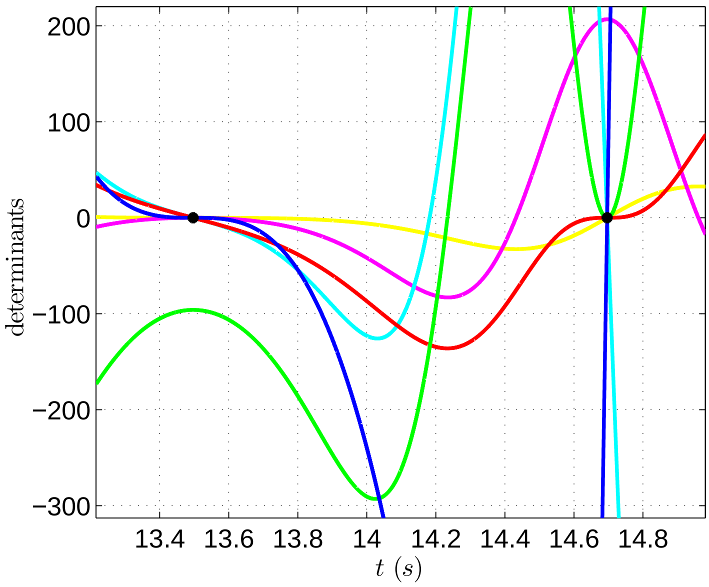

FIGURE 4.4: The plot shows two of four singularity points of Fig. 4.3 in which is visible that five out of six determinants pass always through zero but, in the same points, the remaining one is always different from zero.

4.2.3 Lorentz Oscillator

As a third example we consider the case in which $u^\star$ coincides with the w_1 component of the Lorentz oscillator described by

$$\begin{aligned} \dot{w}_1 &= \sigma(w_2 - w_1) \\ \dot{w}_2 &= w_1(\rho - w_3) - w_2 \\ \dot{w}_3 &= w_1 w_2 - \beta w_3 \end{aligned} \tag{4.18}$$

where (σ, ρ, β) are positive uncertain parameters. We let the set W coincide with the Lorentz attractor by assuming a persistence of excitation condition of the oscillator. Specifically we assume there exists a $\epsilon > 0$ such that

$$w_1^2 + \dot{w}_1^2 = \|u^\star_{[0,1]}\|^2 \geq \epsilon \qquad \forall w \in W\,.$$

We start differentiating $u^\star$ in order to obtain the regression formula (4.4) and to fulfill Assumption 1. We have $w_1 = u^\star(w)$ and $\dot{u}^\star(w) = \sigma(w_2 - u^\star(w))$ from which $w_2 = u^\star(w) + \dot{u}^\star(w)/\sigma$. By differentiating further $\dot{u}^\star$ we get

$$\begin{aligned} \ddot{u}^\star &= \sigma[u^\star(\rho - w_3) - w_2 - u^\star] \\ &= -\dot{u}^\star + c_1 u^\star + c_2 \dot{u}^\star + c_2 u^\star w_3 \end{aligned} \tag{4.19}$$

with $c_1 := \sigma(\rho - 1)$, $c_2 := -\sigma$. Furthermore,

$$\dot{w}_3 = u^{\star 2}(w) + \frac{u^\star(w)\dot{u}^\star(w)}{\sigma} - \beta w_3\,.$$

By differentiating once more (4.19) and using the previous expression of $\dot{w}_3$, we obtain

$$u^{\star(3)} = -u^{\star 2}\dot{u}^\star - \ddot{u}^\star + c_1\dot{u}^\star + c_2\ddot{u}^\star + c_2 u^{\star 3} + (c_2\dot{u}^\star - c_2\beta u^\star)w_3\,. \tag{4.20}$$

Relations (4.19) and (4.20) can be compactly rewritten as

$$u^\star_{[2,3]} = \rho(u^\star_{[0,2]}) + C(\rho,\sigma)\,\varphi(u^\star_{[0,2]}) + M(\sigma,\beta)\,u^\star_{[0,1]}\,w_3$$

where

$$\varphi = \begin{pmatrix} u^\star_{[0,2]} \\ u^{\star 3} \end{pmatrix},\ \rho = \begin{pmatrix} -\dot{u}^\star \\ -\ddot{u}^\star - u^{\star 2}\dot{u}^\star \end{pmatrix} \tag{4.21}$$

and

$$C := \begin{bmatrix} c_1 & c_2 & 0 & 0 \\ 0 & c_1 & c_2 & c_2 \end{bmatrix},\ M := \begin{bmatrix} c_2 & 0 \\ -c_2\beta & c_2 \end{bmatrix}. \tag{4.22}$$

By taking advantage from the persistence of excitation condition, the previous relation can be used to express w_3 as a function of $u^\star_{[0,3]}$, namely

$$w_3 = \frac{1}{\|u^\star_{[0,1]}\|^2} {u^\star_{[0,1]}}^T M^{-1}\left(\rho(u^\star_{[0,2]}) + C\,\varphi(u^\star_{[0,2]})\right)$$

or, equivalently,

$$\begin{aligned} w_3 \;=\; & \frac{{u^\star_{[0,1]}}^T \otimes \rho(u^\star_{[0,2]})^T}{\|u^\star_{[0,1]}\|^2}\mathrm{vect}(M(\sigma,\beta)^{-1}) + \\ & \frac{{u^\star_{[0,1]}}^T \otimes \varphi(u^\star_{[0,2]})^T}{\|u^\star_{[0,1]}\|^2}\mathrm{vect}(M(\sigma,\beta)^{-1}C(\rho,\sigma)) \end{aligned}$$

where $\otimes$ denotes the Kronecker product and $\mathrm{vect}(T)$ is the column vector obtained by taking row-wise the elements of the matrix T.

Furthermore, by taking another derivative of (4.20) we get

$$\begin{aligned} u^{\star(4)} \;=\; & -3u^\star\dot{u}^{\star 2} - u^{\star 2}\ddot{u}^\star - u^{\star(3)} + \\ & c_1\ddot{u}^\star + c_2(u^{\star(3)} + 4u^{\star 2}\dot{u}^\star) - c_2\beta u^{\star 3} + \beta u^{\star 2}\dot{u}^\star \\ & c_2(\ddot{u}^\star - 2\beta\dot{u}^\star + \beta^2 u^\star)w_3 \end{aligned} \tag{4.23}$$

by which, using the expression of w_3 above and compacting the terms, we obtain

$$u^{\star(4)} = h(u^\star_{[0,3]}) + L(u^\star_{[0,3]})\theta^\star \tag{4.24}$$

with $\theta^\star \in \mathbb{R}^{10}$ defined as

$$\theta := \left(\sigma,\ \beta\sigma\rho,\ \beta^2\sigma\rho,\ \beta^3\sigma\rho,\ \beta\sigma,\ \beta^2\sigma,\ \beta^3\sigma,\ \beta,\ \beta^2,\ \beta^3\right)^T$$

and where $h(\cdot)$ and $L(\cdot)$ are appropriately defined functions. This proves that Assumption 1 is fulfilled. To check if there exists a value of m such that Assumption 2 is fulfilled, we go further by simplifying a bit the analysis by assuming that the parameter β is known. This implies, by rearranging a bit the terms in (4.24), that the following relation

$$u^{\star(4)} = \tilde{h}(u^\star_{[0,3]}) + \tilde{L}(u^\star_{[0,3]})\tilde{\theta}^\star \tag{4.25}$$

holds, where $\tilde{h}$ and $\tilde{L}$ are known functions (dependent on β) and $\tilde{\theta}^\star \in \mathbb{R}^2$ is defined as $\tilde{\theta}^\star = (\sigma,\ \rho\sigma)^T$.

By differentiating once the equation (4.25) we get the following compact form

$$u^\star_{[4,5]} = \tilde{H}_1(u^\star_{[0,4]}) + \tilde{A}_1(u^\star_{[0,4]})\tilde{\theta}^\star$$

with

$$\tilde{H}_1(u^\star_{[0,4]}) := \begin{bmatrix} \tilde{h}(u^\star_{[0,3]}) \\ \tilde{h}_1(u^\star_{[0,4]}) \end{bmatrix} \text{ and } \tilde{A}_1(u^\star_{[0,4]}) := \begin{bmatrix} \tilde{L}(u^\star_{[0,3]}) \\ \tilde{L}_1(u^\star_{[0,4]}) \end{bmatrix}$$

To check whether the 2×2 matrix $\tilde{A}_1(u^\star_{[0,4]})$ fulfills Assumption 2, we ran simulations with different values of the parameters and of initial conditions and we found that the matrix is singular in certain points of the Lorentz attractor. A further time derivative is thus taken by obtaining

$$u^\star_{[4,6]} = \tilde{H}_2(u^\star_{[0,5]}) + \tilde{A}_2(u^\star_{[0,5]})\tilde{\theta}^\star$$

in which

$$\tilde{H}_2(u^\star_{[0,5]}) := \begin{bmatrix} \tilde{h}(u^\star_{[0,3]}) \\ \tilde{h}_1(u^\star_{[0,4]}) \\ \tilde{h}_2(u^\star_{[0,5]}) \end{bmatrix} \text{ and } \tilde{A}_2(u^\star_{[0,5]}) := \begin{bmatrix} \tilde{L}(u^\star_{[0,3]}) \\ \tilde{L}_1(u^\star_{[0,4]}) \\ \tilde{L}_2(u^\star_{[0,5]}) \end{bmatrix}$$

with $\tilde{A}_2(u^\star_{[0,5]})$ that is a 3×2 matrix. Numerical tests obtained with different values of the parameters and of the initial conditions showed that the three determinants of each minor of the matrix are never simultaneously zero, namely that the matrix has rank 2 on the Lorentz attractor for the numerical values used in the simulation. Assumption 2 is thus numerically verified and we obtain that relation (2.14) is fulfilled with a $\Psi(\cdot)$ of the form

$$u^{\star(7)} = \tilde{h}_3(u^\star_{[0,6]}) + \tilde{L}_3(u^\star_{[0,6]})\tilde{A}_2^\dagger(u^\star_{[0,5]})(u^\star_{[4,6]} - \tilde{H}_2(u^\star_{[0,5]})).$$

where $\tilde{A}_2^\dagger$ is the left inverse of $\tilde{A}_2$.

Chapter 5

Examples and Simulations

This chapter is completely dedicated to some examples in order to give a numerical validation to the presented theory. For the case of a hybrid identifier it is possible to distinguish the two sub-cases of asymptotic and practical regulation, while for the case of implicit identification is just given an example about the asymptotic scenario.

5.1 Output Regulation with Least Squares Method

In the actual section we show how it is possible to achieve both asymptotic and practical regulation by means of the theoretical concepts presented so far. In details, we are going to carry out some simulations by employing a simple but really effective example in which, first case, we know exactly the exosystem we are having to do with (there is no residual bias $\nu(w)$) and second case, under the minumum phase assumption, we suppose not a perfect knowledge of the map $\pi(w)$ solution of the regulator equation (2.2) (this leads to a defective regression law as in (2.18)).

We consider a controlled plant described by the following nonlinear system

$$\begin{aligned} \dot{x}_1(t) &= -x_1^3(t) - w_1^2(t) + x_2^3(t) & \qquad x_1(0) &= x_{10} \\ \dot{x}_2(t) &= \mu x_1(t) + u(t) - w_1(t) & \qquad x_2(0) &= x_{20} \end{aligned}$$

in which the parameter $\mu \in \{0, 1\}$ is just used to turn on/off the contribute of the term $x_1(t)$, that represents the presence of the map $\pi(w)$ in the steady state control law. The exosystem, i.e. the system that generates the exogenous signal we want to track or reject, is the following well known Van Der Pol nonlinear oscillator with dynamics given by

$$\begin{aligned} \dot{w}_1(t) &= w_2(t) & w_1(0) &= w_{10} \\ \dot{w}_2(t) &= -\theta_1^\star w_1(t) + \theta_2^\star(1 - w_1^2(t))w_2(t) & w_2(0) &= w_{20} \end{aligned}$$

where $\theta^\star := [\theta_1^\star,\, \theta_2^\star]^\top$ is a vector composed by two *unknown* parameters (for the Van Der Pol system they correspond to the natural frequency of oscillation and the nonlinear damping coefficient, respectively). By choosing the regulated variable as $e(t) := x_2(t)$ and $z(t) := x_1(t)$, we notice that the system is still in the normal form described by the equations (2.8)-(2.9), and in particular we can rewrite the plant as follows

$$\begin{aligned} \dot{z}(t) &= -z^3(t) - w_1^2(t) + e^3(t) & z(0) &= x_{10} \\ \dot{e}(t) &= \mu z(t) + u(t) - w_1(t) & e(0) &= x_{20}. \end{aligned}$$

5.1.1 Asymptotic Regulation

In this specific case ($\mu = 0$) it is quite simple to compute the steady state control law able to guarantee the zero regulation error, i. e. $u^\star(w(t)) = w_1(t)$. Furthermore, we note that the system is trivially minimum phase ($\pi(w) = 0$) and with $d = 2$, we have that $u^\star(w(t))$ satisfies *exactly* the regression formula stated in (2.14) (this fact implies a third order internal model controller as seen in Section 2.2). Our goal is to achieve asymptotic regulation by means of the internal model structure given in (3.4) and the identification unit given in (4.2)-(4.3). For sake of compactness, in Table 5.1 we list all the simulation parameters used to implement the overall regulation scheme.

$(\theta_1^\star,\, \theta_2^\star) = (1, 1)$	$(w_{10}, w_{20}) = (1, 1)$
$(x_{10}, x_{20}) = (1, 1)$	$(g, k, \eta_c) = (10, 100, 1)$
$(\xi_{10}, \xi_{20}, \xi_{30}) = (0, 0, 0)$	$(\eta_{c0}, \eta_{10}, \eta_{20}, \eta_{30}) = (0, 0, 0, 0)$

Table 5.1: Output regulation with Least Squares method: list of parameters used in the simulation.

In Figure 5.1 it is possible to see the error signals that approach zero asymptotically. In particular, the regulated error $e\,(t)$ (saturated only for visualization purposes) goes to zero very fast and becomes different from zero just during the parametric adaptation (from 0 to 15 s). In the remaining pictures it is possible to notice the satisfying behavior of the two estimation errors while the identification structure is performing the adaptation (for ease of notation the two estimation are represented by the variables $\hat{\theta}_1\,(t)$ and $\hat{\theta}_2\,(t)$).

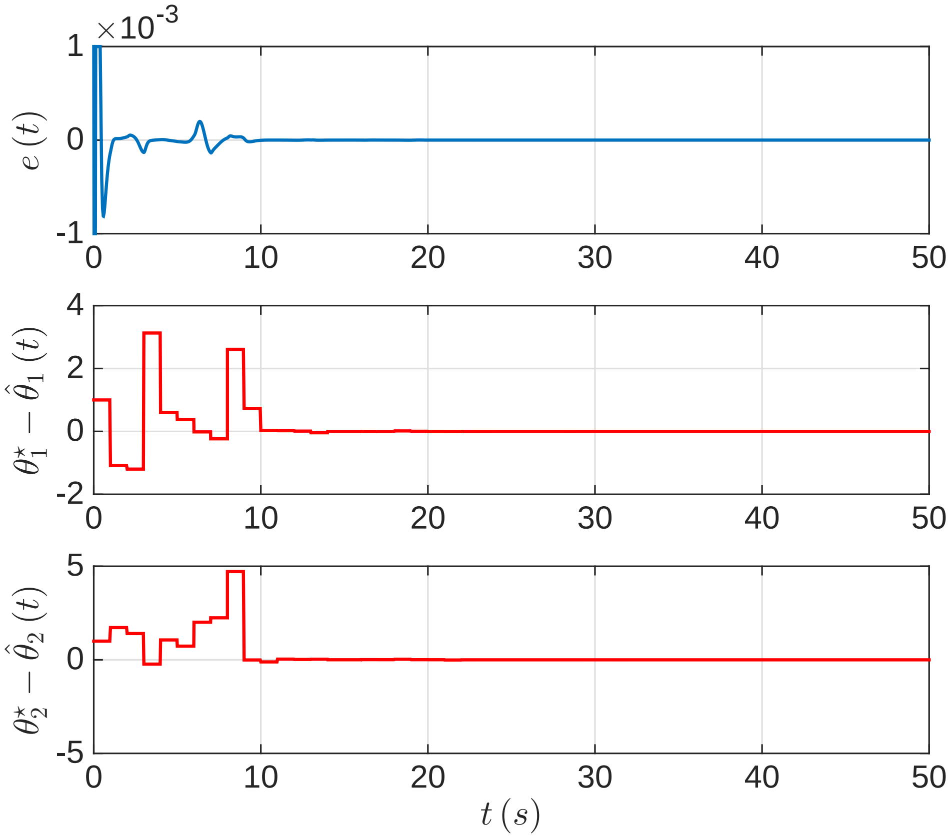

FIGURE 5.1: The pictures above show the behavior of the three main signals of the simulation scheme, i.e. the regulated error $e\,(t)$, the estimation error of the natural frequency of the Van Der Pol oscillator $\theta_1^\star - \hat{\theta}_1\,(t)$ and finally the nonlinear damping estimation error $\theta_2^\star - \hat{\theta}_2\,(t)$.

5.1.2 Practical regulation

Now, we want to show how it is possible to achieve at least practical regulation also when a perfect knowledge of the regression formula (2.14) is not possible. For this purpose, consider the same controlled plant as before but with a different value of the parameter μ, namely $\mu = 1$. In this case the steady state control law becomes

$$u^\star(w(t)) = w_1(t) - \pi(w(t))$$

that fulfills the regression formula listed in (2.18).

In Figure 5.2 we have reported the regulation error $e(t)$ and the estimation errors $\theta_1^\star - \hat{\theta}_1(t)$, $\theta_2^\star - \hat{\theta}_2(t)$ in presence of the uncertain map $\pi(w)$; obviously the values of the two unknown parameters do not approach zero asymptotically, because of the presence of an error on the model used for estimation, and also the regulated variable is different from zero but still small thanks to the effect of the high gain g in the internal model unit.

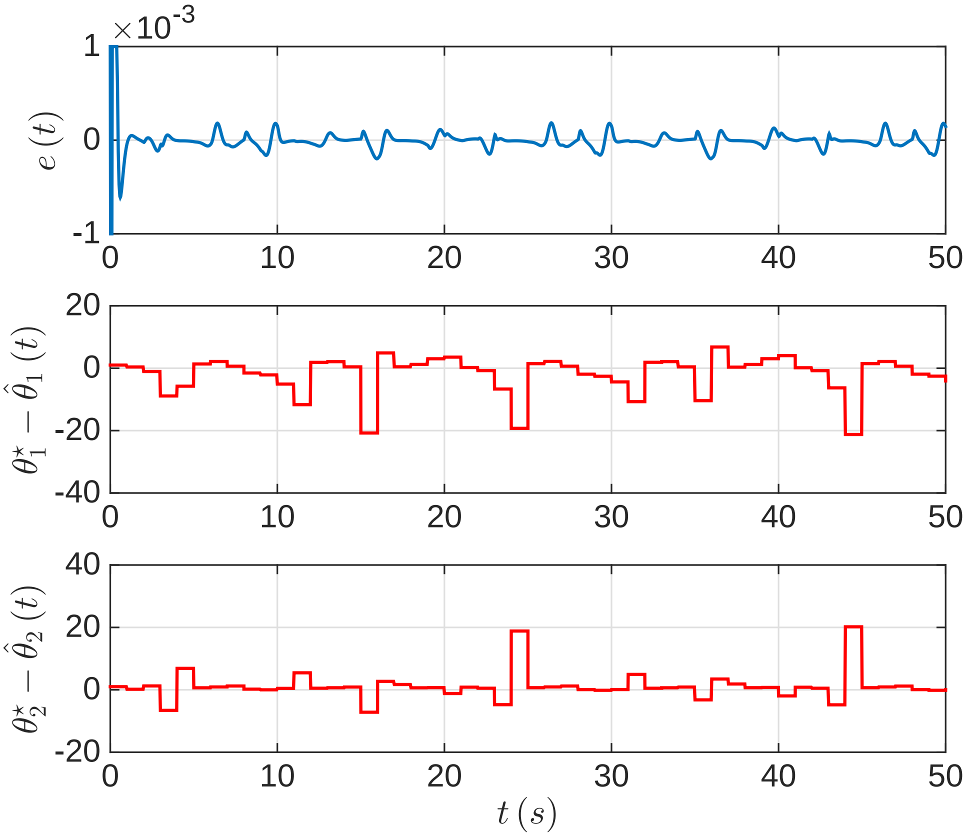

FIGURE 5.2: The picture shows the regulation error $e(t)$ and the estimation errors $\theta_1^\star - \hat{\theta}_1(t)$, $\theta_2^\star - \hat{\theta}_2(t)$ during practical regulation.

As final remark we want to focus the attention on the equation (2.19) showing some comparison between three different practical regulation scenarios where we increase the dimension d of the internal model regulator. With an eye to Figure 5.3, one can notice that increasing the parameter d the error assumes very small values, this means that also in presence of uncertain structure of the exosystem or in presence of uncertain map $\pi(w)$ it is possible to achieve good performance of regulation just adjusting the dimension of the controller. All parameters for the simulation are the same listed in Table 5.1, with the difference of the structure of the controller, that is, for $d = 2 \rightarrow \Psi = [-\xi_2; \xi_3 - \xi_3\xi_1^2 - 2\xi_1\xi_2^2]$, for $d = 3 \rightarrow \Psi = [-\xi_3; \xi_4 - 2\xi_2^3 - 6\xi_1\xi_2\xi_3 - \xi_1^2\xi_4]$ and finally for $d = 4 \rightarrow \Psi = [-\xi_4; \xi_5 - 12\xi_2^2\xi_3 - 8\xi_1\xi_2\xi_4 - \xi_1^2\xi_5 - 6\xi_1\xi_3^2]$.

5.2 Output Regulation with Implicit Adaptation Method

In the actual paragraph, the simulation shows the effective performance in case of robust regulation with the alternative method presented in Section 4.2, i.e. the method in

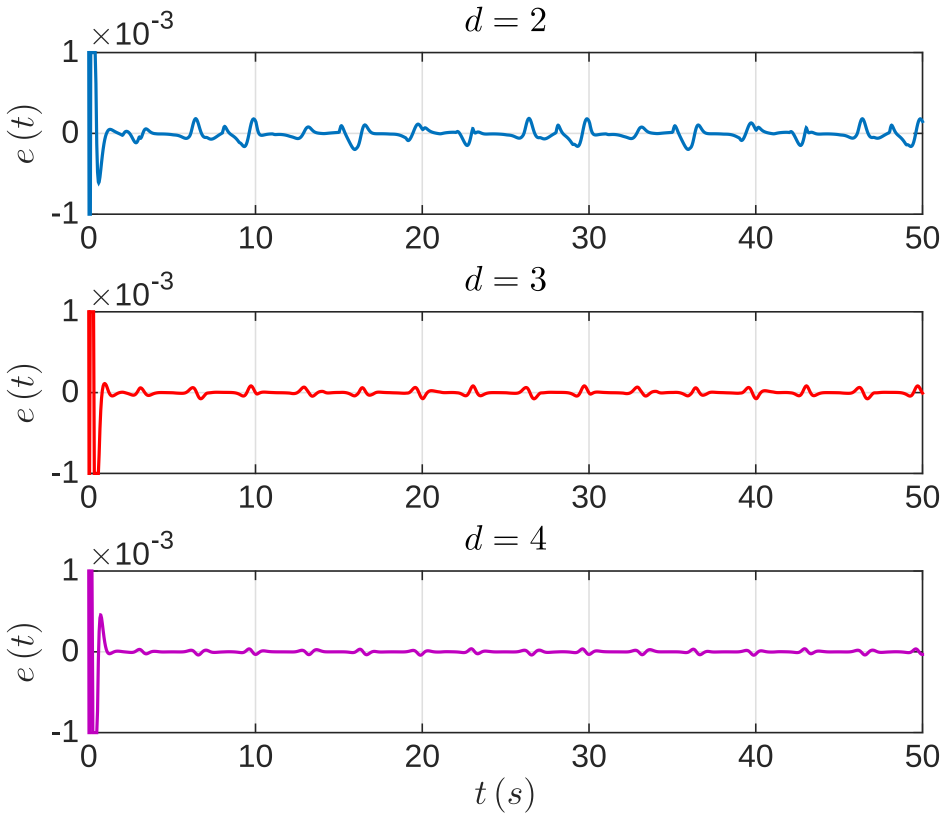

FIGURE 5.3: Note the behavior of the three error signals during practical regulation when the dimension of the internal model regulator is increased.

which the estimation of all uncertainties is nested in the structure of the internal model regulator.

In the example we consider as controlled plant a linear oscillator (for simplicity, it is possible to run simulation considering more complex nonlinear systems) described by the following equations

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) & x_1(0) &= x_{10} \\ \dot{x}_2(t) &= -x_1(t) + u(t) - w_1(t) & x_2(0) &= x_{20} \end{aligned}$$

forced by the control variable $u(t)$ and by a matched exogenous disturbance $w_1(t)$ generated by a Van der Pol exosystem modeled (as in the previous example) by the equations

$$\begin{aligned} \dot{w}_1(t) &= w_2(t) & w_1(0) &= w_{10} \\ \dot{w}_2(t) &= -\theta_1^\star w_1(t) + \theta_2^\star(1 - w_1^2(t))w_2(t) & w_2(0) &= w_{20} \end{aligned}$$

with the same two uncertain constant parameters collected in the vector $\theta^\star := [\theta_1^\star,\, \theta_2^\star]^\top$. The control goal is to regulate x_1 to zero by means of a state feedback control law.

Output feedback solutions can be easily obtained by the state feedback solution derived below by means of standard arguments that are here omitted. In this case we can define the error $e = x_1 + x_2$ and the variable $z = x_1$; with the the previous choice at hand we are able to write the system in the new coordinates

$$\begin{aligned} \dot{z}(t) &= -z(t) + e(t) & z(0) &= x_{10} \\ \dot{e}(t) &= e(t) + u(t) - w_1(t) - 2z(t) & e(0) &= x_{10} + x_{20} \end{aligned}$$

For this specific case, all the parameter used for running the simulation are listed in Table 5.2.

$(\theta_1^\star, \theta_2^\star) = (1, 1)$	$(w_{10}, w_{20}) = (2.5, 0)$
$(x_{10}, x_{20}) = (1, 0)$	$(g, k) = (10, 40)$
$(\xi_{10}, \xi_{20}, \xi_{30}, \xi_{40}, \xi_{50}) = (0, 0, 0, 0, 0)$	$(\lambda_1, \ldots, \lambda_5) = (4, 16, 25, 19, 7)$

TABLE 5.2: Asymptotic output regulation with Implicit Adaptation method: list of parameters used in the simulation.

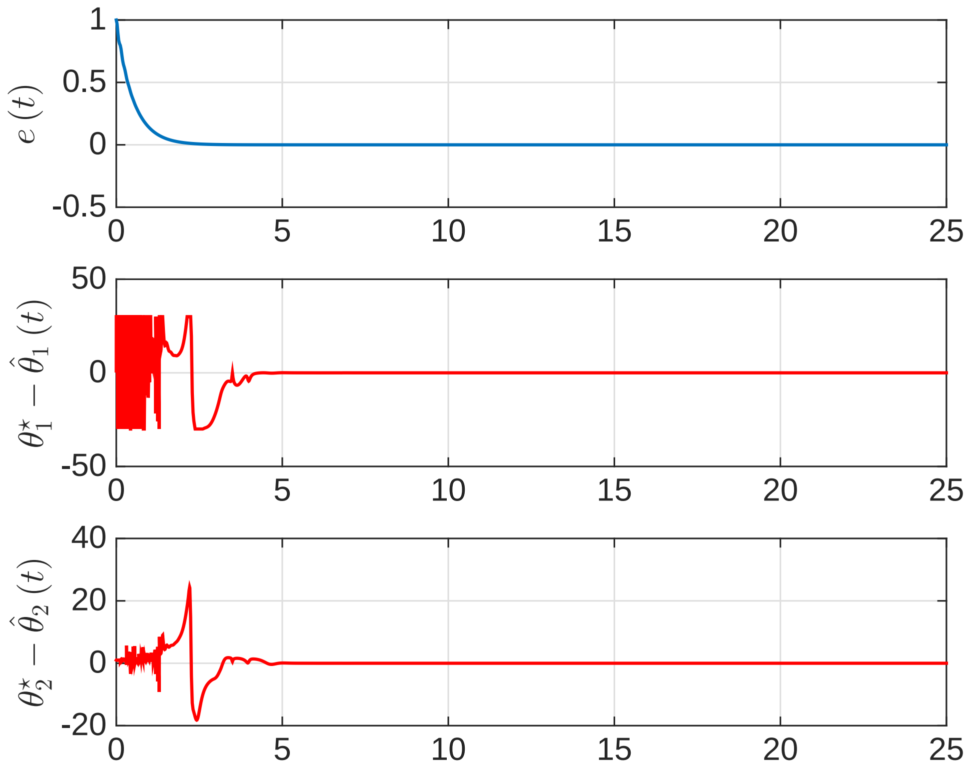

FIGURE 5.4: The picture shows the regulated error $e(t)$ and the two errors $\theta_1^\star - \hat{\theta}_1(t)$ and $\theta_2^\star - \hat{\theta}_2(t)$. The estimation is hidden in the regulator structure.

By following the theory in Section 4.2 the function $\varphi(\cdot)$ in (2.14) is of the form

$$\varphi(\xi) = \left[\xi_4, \quad \xi_5 - \xi_1^2\xi_5 - 12\xi_2^2\xi_3 - 6\xi_1\xi_3^2 - 8\xi_1\xi_2\xi_4\right]\hat{\theta}$$

with $\hat{\theta}$ the vector of estimated parameters given by

$$\hat{\theta} = A_2^{\dagger}(u^{\star}_{[0,3]})[\xi_3, \xi_4, \xi_5]^{\top}$$

where $A_2^{\dagger}$ is the left inverse of A_2. This function has been properly saturated outside $\tau(w)$ to avoid peaking phenomena. As shown in Figure 5.4, the harmonic oscillator starts from an initial condition with $x_1 = 1$ and asymptotically converges to zero, and also the "nested" estimation method exhibits really good performance as one can see from the picture of the two estimation errors $\theta_1^{\star} - \hat{\theta}_1(t)$ and $\theta_2^{\star} - \hat{\theta}_2(t)$.

Chapter 6

Conclusions

In this chapter, final considerations and observations about the overall work are reported. After the first section that describes briefly the work done so far, there is a second section with all the main conclusions and eventual future developments regarding the robust regulation topic.

6.1 Summary of the Thesis

The present thesis deals with the problem of robust output regulation for a particular class of nonlinear systems, namely, such a systems in normal form, single-input single-output, continuous and such that they satisfy the minimum phase assumption (that is a restrictive hypothesis but really important in order to use high-gain techniques). The work is based on the usage of such high-gain techniques introduced by Byrnes and Isidori in 2004, that solved the nonlinear output regulation problem; in details, they developed a constructive way to design a controller with the so called internal model property (a controller that satisfies the internal model principle). In that work, the two authors carried out an analysis of the asymptotic properties of the closed loop system with the above-mentioned regulator, in which is possible to note that just in case of perfect knowledge of the model of the exosystem, it is possible to obtain a regulator with the internal model property, basically, in case of parametric or structural uncertainties of the exosystem is possible solely to achieve practical regulation. In the new framework proposed in this thesis, the problem of robust regulation in case of uncertainties (so we deal with uncertain or completely unknown exosystems) has been studied, trying to construct and analyze, from an asymptotic point of view, a general setup that allow the designer to integrate an adaptive part able to help the existing controller; what we have done is to work on the synergistic union and cooperation of the two main parts of the overall control system: the proper regulation part and the identification part, modifying them, when possible, in order to generate an interplay between the two, able to guarantee the best possible final regulation result (minimizing the regulation error). In details, we have analyzed the interconnections between all the systems in the loop namely, the controlled plant, the exosystem, the regulator and the identifier, developing theorems to prove the good asymptotic properties of the closed loop system. Everything has been supported by several simulations carried out on simple benchmark examples and they have led to the validation of the presented theory also from a numerical point of view.

6.2 Conclusions and Future Works

As conclusions of the overall work done, it is possible to say that the developed framework is quite robust because is strongly based on high-gain techniques that intrinsically allow a sort of robustness, but obviously, the main drawbacks are exactly those related to that world: first of all, the poor robustness to the measurement noises (very significant in the reality, they could affect the overall control system) and the classical peaking phenomenon (the peak in the transient during the estimation of the time derivatives of a signal using an high-gain/dirty derivative observer); both these situations should be taken into account, in fact, they render the propose approach valid just from a theoretical point of view but weakly applicable from a practical one. Just from these considerations one could think about some new interesting ideas, among them, trying to develop and use techniques able to avoid the peaking problems and at the same time techniques able to decrease the sensitivity to noises related to the measures of signals that could be dangerous during an eventual implementation phase of the control algorithm. As underlined by the main idea of this work, the parameter "d" (the regulator dimension) can be choose, in theory, to bypass the problem linked to the asymptotic amplitude of the regulation error, in fact, we have proved that both the prediction error and the regulation error are inversely proportional to the gain of the controller, but also to its proper dimension; however, this is a drawback if one think to the fact that the dimension of the state of the regulator must be huge in order to ensure the smallest regulation error; this fact leads to an excessive computational weight because of the calculus of a big number of derivatives that implies, also, the worst case of peaking phenomenon. A possible solution could be to choose a small parameter "d" and to work instead, on the identifier side. In fact, thanks to the approximation theory, would be possible to estimate any regression law (linear or not) thanks to a surely linear model but with a "selected" number of parameters; the word "selected" means that the designer could choose the number of parameters for estimation such that the prediction error is as small as possible and, as direct consequence, the regulation error would be small; the conclusion is that we do not create other problems for the regulator side but we just increase the number of parameter for the identification side. The same kind of problems, obviously, exist for the approach with implicit adaptation, nested in the regulator, with the further applicative drawback of the *necessary* big dimension of the regulator; in fact that dimension is fixed one for all, first, by the number of derivatives we have to calculate in order to get the suitable regression law and second, by the number of derivatives we have to take into account to satisfy the hypothesis on the rank of the matrix we want to use for the estimation of all parameters. As happens in the case of the Lorentz oscillator, this approach can be used just with more than 20 states for the regulator (difficult to implement). For this kind of method, it is not so clear how to overcome the obstacle and this is the reason why it could be an interesting starting point for further research on the subject.

Appendix A

Proof of Proposition 3.2

Consider the change of variables

$$\xi_e \quad \mapsto \quad \tilde{\xi}_e := D_g(\xi_e - \bar{\tau}_e(\eta_c, w))$$

with $D_g = \mathrm{diag}(g^d\,, g^{d-1}\,, \ldots\,, 1)$. We first compute the flow dynamics of $\tilde{\xi}_e$ component-wise. The components $\tilde{\xi}_i$, $i = \ \ldots d-1$, are described by (adding and subtracting the term $u^\star(w)$ defined in (2.13))

$$\begin{aligned}
\dot{\tilde{\xi}}_i &= g^{d+1-i}(\xi_{i+1} + \lambda_i g^i(u^\star(w) - \xi_1) - \tau_{i+1}(w) + \lambda_i g^i(c(w,z) - u^\star(w))) \\
&= g(\tilde{\xi}_{i+1} - \lambda_i \tilde{\xi}_1) + \lambda_i g^{d+1} \ell_1(w,z)\,.
\end{aligned}$$

As far as $\tilde{\xi}_d$ is concerned, the following dynamics can be computed

$$\begin{aligned}
\dot{\tilde{\xi}}_d &= g(\xi_{d+1} + \lambda_d g^d(u^\star(w) - \xi_1) - \tau_{d+1}(w) + \lambda_d g^d \ell_1(w,z)) \\
&= g(\xi_{d+1} - \Gamma_\eta(\sigma(\eta_\mathrm{c}, w), \tau(w)) - \lambda_d \tilde{\xi}_1 \\
&\quad + \Gamma_\eta(\sigma(\eta_\mathrm{c}, w), \tau(w)) - \tau_{d+1}(w)) + \lambda_d g^{d+1} \ell_1(w,z) \\
&= g(\tilde{\xi}_{d+1} - \lambda_d \tilde{\xi}_1) + g(\Gamma_\eta(\sigma(\eta_\mathrm{c}, w), \tau(w)) - \tau_{d+1}(w)) + \lambda_d g^{d+1} \ell_1(w,z) \\
&= g(\tilde{\xi}_{d+1} - \lambda_d \tilde{\xi}_1) - g\varepsilon(\eta_\mathrm{c}, w) + \lambda_d g^{d+1} \ell_1(w,z)\,.
\end{aligned}$$

Finally, using the fact that $\dot{\sigma}(\eta_\mathrm{c}, w) = F_\mathrm{e}(\sigma(\eta_\mathrm{c}, w), \tau_e(w))$ for all $(\eta_\mathrm{c}, w) \in C_\mathrm{c} \times W$, $\dot{\tilde{\xi}}_{d+1}$ reads as

$$\dot{\tilde{\xi}}_{d+1} = \Gamma'_{\eta\mathrm{s}}(\eta_\mathrm{e}, \xi_e) - \lambda_{d+1} g \tilde{\xi}_1 - \Gamma'_\eta(\sigma(\eta_\mathrm{c}, w), \tau_e(w)) + \lambda_{d+1} g^{d+1} \ell_1(w,z)$$

with the term $\Gamma'_{\eta s}(\eta_e, \xi_e) - \Gamma'_{\eta}(\sigma(\eta_c, w), \tau_e(w))$ that can be elaborated as

$$\begin{aligned}
&\Gamma'_{\eta s}(\eta_e, \xi_e) - \Gamma'_{\eta}(\sigma(\eta_c, w), \tau_e(w)) = \\
&\quad \underbrace{\Gamma'_{\eta s}(\eta_e, \xi_e) - \Gamma'_{\eta s}(\eta_e, \bar{\tau}_e(\eta_c, w))}_{:= \varrho_1(\eta, \xi_e, w)} + \\
&\qquad \underbrace{\Gamma'_{\eta s}(\eta_e, \bar{\tau}_e(\eta_c, w)) - \Gamma'_{\eta s}(\eta_e, \tau_e(w))}_{:= \varrho_2(\eta, w)} + \\
&\qquad\quad \underbrace{\Gamma'_{\eta s}(\eta_e, \tau_e(w)) - \Gamma'_{\eta}(\sigma(\eta_c, w), \tau_e(w))}_{:= \varrho_3(\eta, w)} .
\end{aligned}$$

Note that, by the definition of $\Gamma'_{\eta s}(\cdot)$, there exist positive constants c_1, c_2 and c_3 such that

$$\begin{aligned}
|\varrho_1(\eta, \xi_e, w)| &\leq c_1 \|D_g^{-1}\tilde{\xi}_e\| \\
|\varrho_2(\eta, w)| &\leq c_2 |\varepsilon(\eta_c, w)| \\
|\varrho_3(\eta, w)| &\leq c_3 \|\ell_2(\eta, w)\|
\end{aligned}$$

for all $(\eta, w, \xi_e) \in (C_\eta \cup D_\eta) \times W \times \mathbb{R}^{d+1}$. Putting together the expressions of $\tilde{\xi}_i$, $i = 1, \ldots, d+1$, it turns out that the $\tilde{\xi}_e$ dynamics during flows can be written as

$$\dot{\tilde{\xi}}_e = gH\tilde{\xi}_e + \begin{pmatrix} 0 \\ \cdots \\ 0 \\ -g\varepsilon(\eta_c, w) \\ \varrho_1(\eta, \xi_e, w) + \varrho_2(\eta, w) + \varrho_3(\eta, w) \end{pmatrix} + g^{d+1} \begin{pmatrix} \lambda_1 \\ \cdots \\ \lambda_d \\ \lambda_{d+1} \end{pmatrix} \ell_1(w, z) \tag{A.1}$$

where H is an Hurwitz matrix. Consider the positive definite function $W_\varsigma(\eta_c, \varsigma) = \sqrt{\tilde{\xi}_e^T P \tilde{\xi}_e}$ with $P = P^T > 0$ such that $PH + H^T P = -I$, and note that $\underline{\lambda}_P \|\tilde{\xi}_e\| \leq W_\varsigma(\eta_c, \varsigma) \leq \bar{\lambda}_P \|\tilde{\xi}_e\|$ where $\underline{\lambda}_P$ and $\bar{\lambda}_P$ are, respectively, the square root of the lowest and the largest eigenvalues of P. Note that, by assuming without loss of generality that $g > 1$, $W_\varsigma(\eta_c, \varsigma) \geq \underline{\lambda}_P \|\tilde{\xi}_e\| \geq \underline{\lambda}_P \|\xi_e - \bar{\tau}_e(\eta_c, w)\| \geq \underline{\lambda}_P \|(\eta_c, \varsigma)\|_{\mathcal{C}}$. Moreover, let $(\eta_{cp}, w_p) \in (C_c \cup D_c) \times W$ be such that $\|(\eta_c, \varsigma)\|_{\mathcal{C}} = \|(\eta_c, w, \xi_e) - (\eta_{cp}, w_p, \xi_{ep})\|$ with $\xi_{ep} = \bar{\tau}_e(\eta_{cp}, w_p)$. Then, using the fact that $\bar{\tau}_e(\cdot)$ is locally Lipschitz and C_c, D_c, W are compact,

$$\begin{aligned} W_\varsigma(\eta_c, \varsigma) &\le \bar{\lambda}_P \|\tilde{\xi}_e\| \le g^d \bar{\lambda}_P \|\xi_e - \bar{\tau}_e(\eta_c, w)\| \\ &= g^d \bar{\lambda}_P \|\xi_e - \xi_{ep} + \bar{\tau}_e(\eta_{cp}, w_p) - \bar{\tau}_e(\eta_c, w)\| \\ &\le g^d \bar{\lambda}_P (\|\xi_e - \xi_{ep}\| + \|\bar{\tau}_e(\eta_{cp}, w_p) - \bar{\tau}_e(\eta_c, w)\|) \\ &\le g^d \bar{\lambda}_P (\|(\eta_c, \varsigma)\|_{\mathcal{C}} + \bar{\tau}\|(\eta_{cp}, w_p) - (\eta_c, w)\|) \\ &\le g^d \bar{\lambda}_P (1 + \bar{\tau}) \|(\eta_c, \varsigma)\|_{\mathcal{C}} , \end{aligned}$$

where $\bar{\tau}$ is positive constant. Namely, $\underline{\alpha}'_\varsigma \|(\eta_c, \varsigma)\|_{\mathcal{C}} \le W_\varsigma(\eta_c, \varsigma) \le g^d \bar{\alpha}'_\varsigma \|(\eta_c, \varsigma)\|_{\mathcal{C}}$ with $\underline{\alpha}'_\varsigma = \underline{\lambda}_P$ and $\bar{\alpha}'_\varsigma = \bar{\lambda}_P(1+\bar{\tau})$. We now consider W_ς during flows. By taking the derivative of W_ς along the solutions of the previous system, by using the previous bounds on $\varrho_1(\cdot)$, $\varrho_2(\cdot)$ and $\varrho_3(\cdot)$ and using $W_\varsigma(\eta_c, \varsigma) \le \bar{\lambda}_P \|\tilde{\xi}_e\|$, one obtains that there exists a $g_1^\star > 0$ (dependent on the constant c_1) such that for all $g \ge g_1^\star$ the following holds

$$\begin{aligned} & W_\varsigma(\eta_c, \varsigma) \ge \chi'_\varsigma \max\{\, |\varepsilon(\eta_c, w)|\,,\; g^d |\ell_1(w, z)|\,,\; \frac{1}{g}\|\ell_2(\eta, w)\|\,\} \\ & \Rightarrow\; \langle \nabla W_\varsigma(\eta_c, \varsigma), \bar{F}_\xi(\tilde{\xi}_e, \eta, w, z)\rangle \le -c'_\varsigma W_\varsigma(\eta_c, \varsigma) \end{aligned} \tag{A.2}$$

for some positive constant c'_ς and χ'_ς, for all $(\eta_c, \varsigma) \in C_c \times C_\varsigma$ and $(\eta, w, z) \in C_\eta \times W \times \mathbb{R}^n$, where $\bar{F}_\xi(\cdot)$ is the right-hand side of (A.1).

We now consider the $W_\varsigma(\eta_c, \varsigma)$ during jumps. By bearing in mind the jump rules for ξ, ξ_{d+1}, η and w in (3.12), and the fact that $\sigma(\eta_c, w)^+ = J_e(\sigma(\eta_c, w), \tau_e(w))$, it follows that $W_\varsigma(\mathrm{v}) = \sqrt{\zeta(\eta, w, \xi_e)^T P \zeta(\eta, w, \xi_e)}$ for all $\mathrm{v} \in \mathrm{col}(J_c(\eta_c), J_\varsigma(\varsigma, (w, \eta_e, z)))$ where

$$\zeta = \begin{pmatrix} \mathrm{diag}(g^d, \ldots, g)\, (\xi - \tau(w)) \\ \Gamma_\eta(J_e(\eta_e, \xi_e), \xi) - \Gamma_\eta(J_e(\sigma(\eta_c, w), \tau_e(w)), \tau(w)) \end{pmatrix}.$$

By using the fact that $\Gamma_\eta(\cdot)$ and $J_e(\cdot)$ are locally Lipschitz and bounded, the last element of $\zeta(\eta, w, \xi)$ can be bounded as

$$\begin{aligned} & |\Gamma_\eta(J_e(\eta_e, \xi_e), \xi) - \Gamma_\eta(J_e(\sigma(\eta_c, w), \tau_e(w)), \tau(w))| \le \\ & \nu_1 \|\ell_2(\eta, w)\| + \nu_2 \|\xi_e - \tau_e(w)\| + \nu_3 \|\xi - \tau(w)\| \le \\ & \nu_1 \|\ell_2(\eta, w)\| + (\nu_2 + \nu_3) \|\xi_e - \tau_e(w)\| \le \\ & \nu_1 \|\ell_2(\eta, w)\| + (\nu_2 + \nu_3) \|\xi_e - \bar{\tau}_e(w)\| \\ & \qquad\qquad + (\nu_2 + \nu_3) \|\bar{\tau}_e(w) - \tau_e(w)\| \le \\ & \nu_1 \|\ell_2(\eta, w)\| + (\nu_2 + \nu_3) \|\xi_e - \bar{\tau}_e(w)\| + (\nu_2 + \nu_3) |\varepsilon(\eta_c, w)| \end{aligned}$$

for some positive ν_i, $i=1,2,3$. Using $(a+b+c)^2 \leq 3a^2+3b^2+3c^2$ for all positive a, b, c, one easily obtains (by assuming, without loss of generality, that $g>1$)

$$\begin{aligned}
\|\zeta(\eta,w,\xi_e)\|^2 \leq & \sum_{i=1}^{d} g^{d-i+1}|\xi_i-\tau_i(w)|^2+\nu_1'\|\ell_2(\eta,w)\|^2 \\
& +\nu_2'\|\xi_e-\bar{\tau}_e(w)\|^2+\nu_2'|\varepsilon(\eta_c,w)|^2 \\
= & \sum_{i=1}^{d}(g^{d-i+1}+\nu_2')|\xi_i-\tau_i(w)|^2+\nu_2'|\xi_{d+1} \\
& -\Gamma_\eta(\sigma(\eta_c,w),\tau(w))|^2+\nu_1'\|\ell_2(\eta,w)\|^2+\nu_2'|\varepsilon(\eta_c,w)|^2 \\
\leq & (1+\nu_2')\sum_{i=1}^{d} g^{d-i+1}|\xi_i-\tau_i(w)|^2+(1+\nu_2')\|\xi_{d+1} \\
& -\Gamma_\eta(\sigma(\eta_c,w),\tau(w))\|^2+\nu_1'\|\ell_2(\eta,w)\|^2+\nu_2'|\varepsilon(\eta_c,w)|^2 \\
= & (1+\nu_2')\|\tilde{\xi}_e\|^2+\nu_1'\|\ell_2(\eta,w)\|^2+\nu_2'|\varepsilon(\eta_c,w)|^2\,,
\end{aligned}$$

namely, using the fact that $W_\varsigma(\eta_c,\varsigma) \geq \underline{\lambda}_P\|\tilde{\xi}_e\|$ and $W_\varsigma(\mathrm{v}) \leq \bar{\lambda}_P\|\zeta(\eta,w,\xi_e)\|$ for all $\mathrm{v}\in \mathrm{col}(J_c(\eta_c),J_\varsigma(\varsigma,(w,\eta_e,z)))$,

$$W_\varsigma(\mathrm{v}) \leq \chi_\varsigma' \max\{W_\varsigma(\eta_c,\varsigma)\,,\ \|\ell_2(\eta,w)\|\,,\ |\varepsilon(\eta_c,w)|\} \tag{A.3}$$

where χ_ς' is a constant taken, without loss of generality, equal to the one in (A.2). Relations (A.2), (A.3) do not prove yet the desired result as W_ς is not necessarily decreasing during jumps when $\ell_i(\cdot)=0$, $i=1,2$, and $\varepsilon(\cdot)=0$ (namely $\chi_\varsigma'>1$). The presence of an average dwell-time plays a role to complete the proof. As a matter of fact, following [24], it turns out that a hybrid time domain of the clock subsystem that satisfies (3.6) necessarily coincides with the domain of some solution to the hybrid system flowing according to $\dot{\eta}_c\in[0,\delta]$ if $\eta_c\in[0,N_0]$, and jumping according $\eta_c^+=\eta_c-1$ if $\eta_c\in[1,N_0]$. This implies that the existence of a ISS Lyapunov function $V_\varsigma(\eta_c,\varsigma)$ with the properties specified just after the statement of Proposition 3.2 directly follows from Proposition B.1 in Appendix B applied to the system flowing according to $\dot{\eta}_c\in[0,\delta]$, $\dot{\tilde{\xi}}_e=\bar{F}_e(\cdot)$ and jumping according to $\eta_c^+=\eta_c-1$, $\tilde{\xi}_e^+=\zeta(\cdot)$ by taking $V_\varsigma(\eta_c,\varsigma)=\exp(L\eta_c)W_\varsigma(\eta_c,\varsigma)$, with $L\in(\ln(\chi_\varsigma')\,,c_\varsigma'/\delta)$ and $\delta^\star=c_\varsigma'/\ln(\chi_\varsigma')$.

For the proof of the second part, note that there exists a $\bar{\ell}>0$ such that $|\ell_1(w,z)|\leq\bar{\ell}$ for all $(w,z)\in W\times Z$. The result then follows by standard continuous-time high-gain arguments by using now the change of coordinates $\tilde{\xi}_e=D_g(\xi_e-\tau_e(w))$ and using the fact that $\Gamma_{\eta s}'(\cdot)$ and $\ell_1(w,z)$ are bounded for all $(w,\eta,\xi_e,z)\in W\times(C_\eta\cup D_\eta)\times\mathbb{R}^{d+1}\times Z$.

Appendix B

Auxiliary Results

B.1 Hybrid ISS Lyapunov Functions in Presence of Average Dwell-Time

Let $\mathcal{H}$ be the hybrid system

$$\left.\begin{array}{rcl}\dot{\eta}_{\rm c} & \in & [0,\delta] \\ \dot{x} & \in & F(\eta_{\rm c},x,d)\end{array}\right\} \quad (\eta_{\rm c},x,d)\in[0,N_0]\times C_x\times C_d$$
$$\left.\begin{array}{rcl}\eta_{\rm c}^+ & = & \eta_{\rm c}-1 \\ x^+ & \in & J(\eta_{\rm c},x,d)\end{array}\right\} \quad (\eta_{\rm c},x,d)\in[1,N_0]\times D_x\times D_d$$

for some $\delta > 0$ and $N_0 \geq 1$. Assume that there exists a locally Lipschitz function $W : ([0,N_0]\times C_x)\cup([1,N_0]\times D_x)\to\mathbb{R}_{\geq 0}$ satisfying the following properties:

- there exist class-$\mathcal{K}_\infty$ functions $\underline{\alpha}'(\cdot)$, $\bar{\alpha}'(\cdot)$ such that for all x in the domain of W the following holds

$$\underline{\alpha}'(\|(\eta_{\rm c},x)\|_{\mathcal{S}})\leq W(\eta_{\rm c},x)\leq\bar{\alpha}'(\|(\eta_{\rm c},x)\|_{\mathcal{S}})$$

  where $\mathcal{S}$ is a compact set;

- there exist a class-$\mathcal{K}_\infty$ function $\chi_1'(\cdot)$ and a positive c_1 such that for all $(\eta_{\rm c},x,d)\in[0,N_0]\times C_x\times C_d$

$$W(\eta_{\rm c},x)\geq\chi_1'(|d|) \quad\Rightarrow\quad W^o((\eta_{\rm c},x),{\rm v})\leq -c_1 W(\eta_{\rm c},x)$$

  for all ${\rm v}\in{\rm col}([0,\delta],F(\eta_{\rm c},x,d))$;

- there exist a class-$\mathcal{K}_\infty$ function $\chi_2'(\cdot)$ and a positive constant c_2 and such that for all $(\eta_c, x, d) \in [1, N_0] \times D_x \times D_d$

$$W(\mathrm{v}) \leq \max\{\exp(c_2)\, W(\eta_c, x)\,,\ \chi_2'(|d|)\}$$

for all $\mathrm{v} \in \mathrm{col}(\{\eta_c - 1\}, J(\eta_c, x, d))$.

Proposition B.1. *Consider the hybrid system $\mathcal{H}$ and assume the existence of the locally Lipschitz function W with the properties specified before. If $\delta \leq c_1/c_2$ then $\mathcal{H}$ is pre-ISS relative to the set $\mathcal{S}$. In particular the locally Lipschitz function $V(\eta_c, x) = exp(L\eta_c)W(\eta_c, x)$ with $L \in (c_2\,,\ c_1/\delta)$ satisfies the following:*

- *for all (η_c, x) in the domain of V the following holds*

$$\underline{\alpha}(\|(\eta_c, x)\|_{\mathcal{S}}) \leq V(\eta_c, x) \leq \bar{\alpha}(\|(\eta_c, x)\|_{\mathcal{S}})$$

with class-$\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot) = \underline{\alpha}'(\cdot)$ and $\bar{\alpha}(\cdot) = exp(LN_0)\bar{\alpha}'(\cdot)$.

- *for all $(\eta_c, x, d) \in [0, N_0] \times C_x \times C_d$ the following holds*

$$V(\eta_c, x) \geq \chi_1(|d|) \quad \Rightarrow \quad V^o((\eta_c, x), \mathrm{v}) \leq -cV(\eta_c, x)$$

for all $\mathrm{v} \in col([0, \delta], F(\eta_c, x, d))$, with positive $c = c_1 - L\delta$ and class-$\mathcal{K}_\infty$ function $\chi_1(\cdot) = \chi_1'(\cdot)$;

- *for all $(\eta_c, x, d) \in [1, N_0] \times D_x \times D_d$ the following holds*

$$V(\mathrm{v}) \leq \max\{\lambda V(\eta_c, x)\,,\ \chi_2(|d|)\}$$

for all $\mathrm{v} \in col(\{\eta_c - 1\}, J(\eta_c, x, d))$, with positive $\lambda = exp(c_2 - L) < 1$ and class-$\mathcal{K}_\infty$ function $\chi_2(\cdot) = exp(L(N_0 - 1))\chi_2'(\cdot)$.

The proof of this proposition is in [24] (see Proposition IV.1).

B.2 Small-Gain Theorem for Hybrid Interconnections with Average Dwell-Time

Consider the hybrid interconnection

$$\left.\begin{array}{rcl}\dot{\eta}_c & \in & [0,\delta] \\ \dot{x}_i & \in & F_i(\eta_c, x_1, x_2, d)\end{array}\right\} (\eta_c, x_1, x_2, d) \in [0, N_0] \times C_1 \times C_2 \times C_d$$
$$\left.\begin{array}{rcl}\eta_c^+ & = & \eta_c - 1 \\ \xi_i^+ & \in & J_i(x_1, x_2, d)\end{array}\right\} (\eta_c, x_1, x_2, d) \in [1, N_0] \times D_1 \times D_2 \times D_d$$

where $i = 1, 2$, for some $\delta > 0$ and $N_0 \geq 1$, and assume that there exist two locally Lipschitz functions $V_i : [0, N_0] \times C_i \cup D_i \to \mathbb{R}_{\geq 0}$ such that the following holds

- there exists class-$\mathcal{K}_\infty$ functions $\underline{\alpha}_i(\cdot)$, $\bar{\alpha}_i(\cdot)$ such that for all $(\eta_c, x_i) \in [0, N_0] \times C_i \cup D_i$
$$\underline{\alpha}_i(\|(\eta_c, x_i)\|_{\mathcal{S}_i}) \leq V_i(\eta_c, x_i) \leq \bar{\alpha}_i(\|(\eta_c, x_i)\|_{\mathcal{S}_i})$$
where $\mathcal{S}_i$ are compact sets, $i = 1, 2$;
- there exist positive constants χ_{1i}, c_i and class-$\mathcal{K}_\infty$ functions $\sigma_{1i}(\cdot)$ such that for all $(\eta_c, x_1, x_2, d) \in [0, N_0] \times C_1 \times C_2 \times C_d$
$$V_i(\eta_c, x_i) \geq \max\{\chi_{1i} V_j(\eta_c, x_j)\,,\; \sigma_{1i}(|d|)\} \quad \Rightarrow \quad V_i^o((\eta_c, x_i), \mathrm{v}) \leq -c_i V_i(\eta_c, x_i)$$
$\forall\, \mathrm{v} \in \mathrm{col}([0,\delta], F_i(\eta_c, x_1, x_2, d))$, with $i, j = 1, 2$, $i \neq j$;
- there exist positive constants χ_{2i}, χ_{3i}, and class-$\mathcal{K}_\infty$ functions $\sigma_{2i}(\cdot)$ such that for all $(\eta_c, x_1, x_2, d) \in [1, N_0] \times D_1 \times D_2 \times D_d$
$$V_i(\mathrm{v}) \leq \max\{\chi_{2i} V_i(\eta_c, x_i)\,,\; \chi_{3i} V_j(\eta_c, x_j)\,,\; \sigma_{i2}(|d|)\}$$
for all $\mathrm{v} \in \mathrm{col}(\{\eta_c - 1\}, J_i(\eta_c, x_1, x_2, d))$, with $i, j = 1, 2$, $i \neq j$;
- the following holds: $\chi_{11}\chi_{12} < 1$.

In this framework the following result holds.

Theorem B.2. *There exists a $\delta^\star > 0$ such that if $\delta \leq \delta^\star$ the interconnection is pre-ISS relative to the set $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$ with respect to d. In particular, by letting $x = col(x_1, x_2)$, $C = C_1 \times C_2$, $D = D_1 \times D_2$, $F(\eta_c, x, d) = col(F_1(\eta_c, x_1, x_2, d), F_2(\eta_c, x_1, x_2, d))$, and $J(\eta_c, x, d) = col(J_1(\eta_c, x_1, x_2, d), J_2(\eta_c, x_1, x_2, d))$, there exists a locally Lipschitz function $V : [0, N_0] \times (C \cup D) \to \mathbb{R}_{\geq 0}$ such that the following holds*

- *for all* (η_c, x) *in the domain of* V *the following holds*

$$\underline{\alpha}(\|(\eta_c, x)\|_{\mathcal{S}}) \leq V(\eta_c, x) \leq \bar{\alpha}(\|(\eta_c, x)\|_{\mathcal{S}})$$

- *there exists a positive* c *and a class-*$\mathcal{K}_\infty$ *function* $\sigma(\cdot)$ *such that for all* $(\eta_c, x, d) \in [0, N_0] \times C \times C_d$

$$V(\eta_c, x) \geq \sigma(|d|) \quad \Rightarrow \quad V^o((\eta_c, x), \mathrm{v}) \leq -cV(\eta_c, x)$$

for all $\mathrm{v} \in$ *col*$([0, \delta], F(\eta_c, x, d))$*;*

- *there exists a positive* $\lambda < 1$ *such that for all* $(\eta_c, x, d) \in [1, N_0] \times D \times D_d$ *the following holds with the same* $\sigma(\cdot)$ *of the previous item*

$$V(\mathrm{v}) \leq \max\{\lambda V(\eta_c, x)\,,\ \sigma(|d|)\} \quad \forall\, \mathrm{v} \in \textit{col}(\{\eta_c - 1\}, J(\eta_c, x, d))\,.$$

Proof. Let $\rho > 0$ be such that $\chi_{11} < \rho < 1/\chi_{12}$. Let $W : [0, N_0] \times (C \cup D) \to \mathbb{R}_{\geq 0}$ be the locally Lipschitz function

$$W(\eta_c, x) = \max\{V_1(\eta_c, x)\,,\ \rho V_2(\eta_c, x)\}\,.$$

Simple arguments can be used to prove the existence of class-$\mathcal{K}_\infty$ functions $\underline{\alpha}'(\cdot)$ and $\bar{\alpha}'(\cdot)$ fulfilling $\underline{\alpha}'(|(\eta_c, x)|_{\mathcal{S}}) \leq W(\eta_c, x) \leq \bar{\alpha}'(|(\eta_c, x)|_{\mathcal{S}})$ for all $(\eta_c, x) \in [0, N_0] \times (C \cup D)$ (details are omitted). Following the small-gain Theorem III.1 of [23], it turns out that there exists a positive c' and a class-$\mathcal{K}_\infty$ function $\sigma_1'(\cdot)$ such that for all $(\eta_c, x, d) \in [0, N_0] \times C \times C_d$

$$W(\eta_c, x) \geq \sigma_1'(|d|) \ \Rightarrow \ W^o((\eta_c, x), \mathrm{v}) \leq -c'W(\eta_c, x)$$

for all $\mathrm{v} \in F(\eta_c, x, d)$. On the other hand, for all $(\eta_c, x, d) \in [1, N_0] \times D \times D_d$ and for all $\mathrm{v} \in J(\eta_c, x, d)$

$$\begin{aligned} W(\mathrm{v}) &\leq \max\{\chi_{21} V_1(\eta_c, x)\,,\ \chi_{31} V_2(\eta_c, x)\,,\ \sigma_{21}(|d|)\,, \\ &\qquad \rho\chi_{22} V_2(\eta_c, x)\,,\ \rho\chi_{32} V_1(\eta_c, x)\,,\ \sigma_{22}(|d|)\} \\ &\leq \max\{\chi' W(\eta_c, x)\,, \sigma_2'(|d|)\} \end{aligned}$$

where $\chi' = \max\{\chi_{21}\,,\ \chi_{22}\,,\ \rho\chi_{32}\,,\ \chi_{31}/\rho\}$, and $\sigma_2'(\cdot)$ is the class-$\mathcal{K}_\infty$ function defined as $\sigma_2'(s) = \max\{\sigma_{21}(s)\,,\ \sigma_{22}(s)\}$. The result then follows by Proposition B.1 taking $V(\eta_c, x) = \exp(L\eta_c)W(\eta_c, x)$, with $L \in (\ln(\chi')\,, c'/\delta)$ and $\delta^\star = c'/\ln(\chi')$. □

Bibliography

[1] A. Serrani, A. Isidori, and L. Marconi. Semiglobal nonlinear output regulation with adaptive internal model. *IEEE Trans. Automatic Control*, AC-46:1178–1194, 2001.

[2] J. Huang. Asymptotic tracking and disturbance rejection in uncertain nonlinear systems. *IEEE Trans. Automatic Control*, 40:1118–1122, 1995.

[3] A. Ding. Global stabilization and disturbance suppression of a class of nonlinear systems with uncertain internal model. *Automatica*, 39(3):471–479, 2003.

[4] F. Delli Priscoli, L. Marconi, and A. Isidori. A new approach to adaptive nonlinear regulation. *SIAM Journal on Control and Optimization*, 45(3):829–855, 2006.

[5] R. Marino and P. Tomei. Adaptive nonlinear regulation for uncertain minimum phase systems with unknown exosystem. *Proceedings of the 47th IEEE conference on decision and control*, Cancun, Mexico, 2008.

[6] R. Marino and P. Tomei. An adaptive learning regulator for uncertain minimum phase systems with undermodeled unknown exosystems. *Automatica*, 47:739–747, 2011.

[7] A. Isidori, L. Marconi, and L. Praly. Robust design of nonlinear internal models without adaptation. *Automatica*, 48:2409–2419, July 2012.

[8] A. Serrani. Rejection of harmonic disturbances at the controller input via hybrid adaptive external model. *Automatica*, 42:1977–1985, 2006.

[9] L. Ljung. System identification - theory for the user. PTR Prentice Hall, 2nd ed, Upper Saddle River, N.J. 1999.

[10] T. S. Soderstrom and P. Stoica. System identification. Prentice Hall International 1989.

[11] C.I. Byrnes and A. Isidori. Nonlinear internal models for output regulation. *IEEE Trans. Automatic Control*, 49(12), Dec. 2004.

[12] M. Gevers. Towards a joint design of identification and control? In Essays on Control: Perspectives in the Theory and its Applications, H. L. Trentelman and J. C. Willems Editors, Springer Science+Business Media, LLC, 1993.

[13] C.I. Byrnes and A. Isidori. Limit sets, zero dynamics, and internal models in the problem of nonlinear output regulation. *IEEE Trans. Automatic Control*, 48:1712–1723, 2003.

[14] A. Isidori. Nonlinear control systems 3rd ed. Springer-Verlag, New York, 1995.

[15] C. I. Byrnes and A. Isidori. Asymptotic stabilization of minimum phase nonlinear systems. *IEEE Transactions on Automatic Control*, 36(10):1122–1137, June 1991.

[16] L. Marconi, L. Praly, and A. Isidori. Output stabilization via nonlinear luenberger observers. *SIAM Journal on Control and Optimization*, 45(6):2277–2298, 2007.

[17] L. Marconi and L. Praly. Uniform practical output regulation. *IEEE Trans. Automatic Control*, 53(5):1184–1202, 2008.

[18] J.P. Gauthier and I. Kupka. Deterministic observation theory and applications. Cambridge University Press, Cambridge, UK, 2001.

[19] A.R. Teel and L. Praly. Tools for semiglobal stabilization by partial state and output feedback. *SIAM Journal on Control and Optimization*, 33:1443–1488, 1995.

[20] R. Goebel, R.G. Sanfelice, and A.R. Teel. Hybrid dynamical systems: modeling, stability and robustness. Princeton University Press 2012.

[21] J. P. Hespanha, D. Liberzon, and A. R. Teel. Lyapunov conditions for input-to-state stability of impulsive systems. *Automatica*, 44:2735–2744, 2008.

[22] J. P. Hespanha and A. S. Morse. Stability of switched systems with average dwell-time. *In proc. 38th IEEE Conf. on Decision and Control*, pages 2655–2660, 1999.

[23] D. Liberzon, D. Neŝić, and A.R. Teel. Lyapunov-based small-gain theorems for hybrid systems. *IEEE Trans. Automatic Control*, 59(6):1395–1410, June 2014.

[24] C. Cai, A.R. Teel, and R. Goebel. Smooth lyapunov functions for hybrid systems, part ii: (pre)-asymptotically stable compact sets. *IEEE Trans. Automatic Control*, 53:734–748, 2008.

[25] F. H. Clarke. Optimization and nonsmooth analysis. Interscience, New York 1983.

[26] C. Cai and A.R. Teel. Characterizations of input-to-state stability for hybrid systems. *Systems and Control Letters*, 58:47–53, 2009.