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# Quantum Integrability in Non-Linear Sigma Models related to Gauge/String Correspondences

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To my loving parents, for having made this possible.
To Michela, for her unconditional love and support.
To my friends (Very and not), for being brothers.

*One has to grow hard but without ever losing tenderness.*

Ernesto “Che” Guevara

# Abstract

The Thermodynamic Bethe Ansatz analysis is carried out for the *extended*-$\mathbb{CP}^{N-1}$ class of integrable 2-dimensional Non-Linear Sigma Models. The principal aim of this program is to obtain further non-perturbative consistency check to the $S$-matrix proposed to describe the scattering processes between the fundamental excitations of the theory by analyzing the structure of the Renormalization Group flow. As a noteworthy byproduct we eventually obtain a novel class of TBA models, the $(\mathbb{CP}^{N-1})_p \times U(1)$ models, which fits in the known classification but with several important differences. The TBA framework allows the evaluation of some exact quantities related to the conformal UV limit of the model: effective central charge, conformal dimension of the perturbing operator and field content of the underlying CFT. The knowledge of this physical quantities has led to the possibility of conjecturing a perturbed CFT realization of the integrable models in terms of coset Kac-Moody CFT. The set of numerical tools and programs developed ad hoc to solve the problem at hand is also discussed in some detail with references to the code.

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# Preface

The work presented in this thesis is based on the last half of my graduate studies as a PhD student at Universitá di Bologna which resulted in a pubblication [146]. The reason behind this choice are basically those dictated by self-consistency together with the will to give a survey, in a somehow pedagogical way, on the physical framework and the mathematical tools of the modern approach to integrability in Quantum Field Theory.

# Chapter 1

# Introduction

Since the birth of *Quantum Field Theories* the role of symmetry principles has been of crucial importance in our understanding of fundamental physics. The origin of this perspective can be dated back to the well-known fundamental discoveries of Maxwell and Einstein and have reached their actual formulation thanks to the works of Yang and Mills [1], Higgs [2], t'Hooft [3] and many others. What stems from their work is one of the highest point of the human effort in understanding the universe: *the Standard Model* (SM) of particle physics. The SM is a local QFT based on the gauge symmetry group $SU(3) \times SU(2) \times U(1)$ and represents the result of a unification between the theory of strong interactions, the so-called *Quantum Chromodynamics* (QCD), with the Weinberg-Salam-Glashow gauge theory of electroweak interaction. It constitutes, remarkably, the best example of theory consistent with the experimental data as it has been verified with incredible precision in the last decades. However, despite the impressive success of the perturbative approach to the asymptotically-free QCD [4, 5] to explain, for example, deep inelastic scattering experiments and hadronic jet production in high-energy reactions, a satisfatory theoretical explanation of the most manifest experimental fact, namely the *confinement* of quarks and gluons inside hadrons, is still lacking. The reason is, clearly, the strongly non-perturbative nature of the confinement phenomenon whose formal proof, in a general gauge theory, requires the employment of non-perturbative theoretical tools which are very difficult to develop. It was only recently observed by Seiberg and Witten [6, 7] that supersymmetry and duality may be the path for the proof of confinement. Another approach is represented by *String theories*. These theories were discovered precisely in the attempt of describing the strong interactions, once the dual models

[8] were proposed to incorporate the so-called Regge trajectories observed in experiments. The advent of the description of strong interaction in terms of YM theory reduced the importance of string theory as a model of strong interactions, and for a while the interest in string theory was mainly related to the appealing possibility of giving an unified description of all the known interactions, including gravity. In more recent years the idea of duality between string and gauge theories opened the possibility of employing all the technology developed in the string context to obtain a description of gauge theories in the regime in which they are not directly accessible by perturbation theory. Building on the ideas developed by many authors [9, 10], in 1997 Maldacena conjectured [11] that the $\mathcal{N} = 4$ SYM theory in 4 dimensions is exactly dual to type IIB superstring in the $AdS_5 \times S^5$ curved background. In this picture, which much owns to *holography* ideas [12], the usual Minkowski space is recovered as the boundary of the $AdS_5$ space, while the 5-sphere is associated with the internal symmetry of the Yang-Mills gauge fields.

To be more concrete about the Ads/CFT correspondence we can consider the gauge theory as characterized by two parameters: the coupling constant $g_{YM}$ and and the number of colors $N$. In the large-$N$ limit, or planar limit, they can be combined into a single parameter, the 't Hooft coupling constant $\lambda = g_{YM}^2 N$. The bare gauge parameters can then be related to the strings parameters by [11]

$$4\pi g_s = g_{YM}^2 = \frac{\lambda}{N}\,, \quad \frac{R^4}{\alpha'^2} = \lambda\,, \tag{1.1}$$

with $g_s$ being the string coupling constant, $\alpha'$ the string tension and $R$ the radius of both $AdS_5$ and $S^5$. On the other side, the quantum $\mathcal{N} = 4$ SYM theory is a conformal theory, i.e. invariant under dilatations, so that the Poincaré symmetry is enhanced to the full superconformal group in 4 dimensions. From the AdS/CFT perspective a particularly important role is played by the dilatation operator, associated to the Hamiltonian of the gauge theory. This is so because the eigenvalues of the dilatation operator are the (quantum) dimensions of the local operators and, under the correspondence, they are associated to the states of the string, while their conformal dimensions are associated to the string energy. The dilatation operator corresponds to a non-compact generator of the global symmetry group, which can get quantum corrections and depends on the coupling constant, the corresponding conformal dimension is then known as *anomalous dimension*. As can be easily guessed from equation 1.1, the Maldacena duality is a strong/weak coupling duality, associating the perturbative regime in the $\mathcal{N} = 4$ SYM theory to

the strongly coupled regime of the superstring model. This is the reason why the duality is not easy to prove, since the regimes accessible by perturbative methods in the two theories do not match, and, at the same time, its more promising feature since, in principle, it would allow the inverse process: using the perturbative regime of a theory to obtain information on the strong coupling regime of the other. Other examples of the AdS/CFT correspondence were discovered recently, relating the $\mathcal{N} = 6$ *superconformal Chern-Simons theory* in 3 dimensions and string theory in $AdS_4 \times \mathbb{CP}^3$ [13] and this particular model motivated the main investigations of this thesis.

The main results obtained in the context of the AdS/CFT correspondence relies on two of the greatest revolution of contemporary theoretical physics: *conformal field theories* (CFT) and *integrability*.

In the last decades the introduction of *conformal invariance* has led to a remarkable understanding of the theory of critical phenomena both in statistical systems and quantum field theories [14, 15, 16, 17]. The tools developed within this framework has led to the possibilty of solving exactly, i.e. non-perturbatively, a large class of physical systems. This symmetry becomes especially powerful in 2 dimensions where the underlying generating algebra turns out to be infinite dimensional, the celebrated *Virasoro algebra*. A conformally invariant theory in 2 dimensions result then so constrained that the form of the correlators results fixed to a large extent by the sole symmetry requirements.

The other important development we were mentioning is constituted by the study of *integrability*. At the most intuitive level a physical system is integrable whenever it is endowed with a number of conserved quantities equal to the number of degrees of freedom. In quantum field theories this then amounts to the existence of infinite conserved charges associated to infinite conservation laws. The path which led to the modern understanding of integrability in 2-dimensional QFT originates from the notable result of Coleman and Mandula [18] which is a *no-go* theorem for the structure of the symmetry associated to a generic QFT. More precisely the theorem states that in 2 dimensions it is possible to combine space-time and internal transformation in a non-trivial way, as opposite to the 4-dimensional case in which the only possibility is represented by the trivial one, that is direct product. The requirement of integrability, together with the results of the Coleman-Mandula theorem, then constrain severely the form of the $S$-matrix for integrable QFT. In a series of remarkable works [19, 20, 21, 22] the authors, basing their considerations

both on specific models and general arguments, were able to formulate the crucial properties of a 2-dimensional integrable QFT which are:

- **Absence of particle production** in any scattering process.
- **Strict momentum conservation**, i.e. each momentum is conserved in the process.
- **Factorizability** of the $n$-particle scattering amplitude into 2-body processes.

Later, by refining the arguments of [21, 22], Parke showed in [23] that the existence of only two non-trivial conserved quantities in sufficient to ensure the properties listed above and therefore the integrability of the model.

A fundamental link between CFT and integrable models was established by A.B. Zamolodchikov in [24, 25, 26]. This result has been extensively employed in the study of 2-dimensional integrable QFTs over the last decades and can be summarised as follows: an integrable QFT may be viewed as the perturbation of a CFT by means of a particular operator of the CFT itself. The perturbation is such that the infinite conservation laws of the CFT are not spoiled but just deformed, giving rise to a new infinite family of conserved quantities, namely the conserved quantities of the integrable model. In this case the perturbation is said to be *integrable*.

Among the tool of integrability the *Thermodynamic Bethe Ansatz* (TBA for short) was originally introduced by C.N. Yang and C.P. Yang in [27], in the context of statistical mechanics, and formulated in the present form by A.B. Zamolodchikov [87]. TBA provides a non-perturbative framework with which, thanks to the sole knowledge of the $S$-matrix, it is possible to analyze integrable systems in their relation with the perturbed CFT realization, in the sense precised above. Within the TBA analysis it is possible to obtain a description of the thermodynamic of the system in terms of a set of non-linear integral equations whose solution allows to evaluate physically relevant quantities. Moreover, this analytic description is well suited for non-perturbative *Renormalization Group flow* analysis as it embodies the dependence on the energy scale in a very convenient analytic way. In relation with the perturbed CFT realization then the TBA approach allows for the calculation of the main informations related to the underlying CFT, which is

recovered as the *Ultra-Violet* limit of the RG flow. In this sense, the TBA enables, for instance, to evaluate the central charge of the CFT, the dimension of the perturbing operator and also gives the possibility of exploring the field content of the CFT. The purpose of the work presented in this thesis will be the application of these methods to the study of a concrete family of 2-dimensional massive integrable QFT's together with the further investigation of the mentioned approaches themselves.

In view of its importance in modern theoretical physics and for the crucial role played for the aim of this thesis we include a brief overview about conformal field theories to set up the notation and illustrate their elegance.

## Conformal symmetry

For the fundamental importance of conformal field theories in the understanding of integrable models we decided to give a brief overview of the most important results in order to fix the notation. A more general and exhaustive discussion can be found for example in [15, 17, 16].

The *classical conformal group* in arbitrary dimensions $d$ is the subset of coordinate transformations which leave the metric $g_{\mu\nu}$ invariant up to scale transformation of the form

$$g_{\mu\nu}(x) \to e^{\Lambda(x)} g_{\mu\nu}(x)\,, \tag{1.2}$$

this property has the consequence of changing the scale of the space but leaving untouched the angles defined through scalar products, henceforth the name conformal. By considering an infinitesimal transformation of the type $x'_\mu = x_\mu + \epsilon_\mu$ we can translate (1.2) into a condition for the infinitesimal displacement $\epsilon_\mu$, namely

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} \partial_\rho \epsilon^\rho \eta_{\mu\nu}\,, \tag{1.3}$$

where we are supposing to work in a flat space, $g_{\mu\nu} = \eta_{\mu\nu}$.

The full power of conformal symmetry emerges in considering the case $d = 2$ where this beautiful picture becomes sublime by virtue of a very particular symmetry enhancement. In this case conditions (1.3) become the Cauchy-Riemann theorem for holomorphic complex valued functions. Provided we introduce the complex coordinates $z = x^0 + ix^1$ and $\bar{z} = x^0 - ix^1$, the conformal group becomes the group

of analytic coordinate transformation in the complex plane of the form $z \to f(z)$ and $\bar{z} \to \bar{f}(\bar{z})$, for independent $f$ and $\bar{f}$. This way of looking immediately provides the infinite family of *classical infinitesimal conformal generators*

$$l_n = -z^{n+1}\partial\,, \quad \text{and} \quad \bar{l}_n = -\bar{z}^{n+1}\bar{\partial}\,, \text{ with } n \in \mathbb{Z}\,, \tag{1.4}$$

associated to the conformal algebra in 2 dimension which, therefore, turns out to be infinite dimensional. The classical conformal generators are known to satisfy the *Witt algebra*

$$[l_n, l_m] = (n-m)l_{n+m}\,, \quad [\bar{l}_n, \bar{l}_m] = (n-m)\bar{l}_{n+m}\,, \quad [l_n, \bar{l}_m] = 0\,. \tag{1.5}$$

Having (1.4) in mind it is easy to see that in the $z \to 0, \infty$ only the generators $l_0, l_{\pm 1}$ remain well defined and moreover, in view of the Witt algebra (1.5), they form a subalgebra which is isomorphic to $SL(2,\mathbb{C})/\mathbb{Z}_2$. These subalgebra is special also because with its elements we can construct the conformal generators of: translations $z \to z + a$, ($l_{-1}$ and $\bar{l}_{-1}$), dilatation $z \to \lambda z$, ($l_0 + \bar{l}_0$) and rotations $z \to e^{i\theta}z$, ($il_0 - i\bar{l}_0$).

In the past 40 years, conformal field theory has became one of the most active and fruitful topic of theoretical physics. The reason of its success relies on the fact that conformal invariance turns out to be an extremely powerful tool to analyse problems which are very difficult to treat for general QFT's as it is capable of giving exact solutions. In the context of physical systems with local interactions, conformal invariance can be understood as an immediate consequence of scale invariance. This observation was originally made by A.M. Polyakov [29]. However, the seminal paper which really boosts the modern approach to conformal invariance dates back to 1984 and is due to A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov [14]. In [14] the authors showed how to construct completely solvable CFT's, the minimal models, which thereafter have been extensively studied in the literature [30]. In particular they were able to formulate, and in some cases to explicitly solve, the differential equations satisfed by correlation functions. The latter being the manifestation at the level of the correlators of the Ward identities of the theory.

The quantization procedure known as *radial quantization*, see [16] for details, allows to formulate the quantum conformal theory in the complex $z$-plane. The central objects is represented by the energy-momentum tensor $T_{\mu\nu}$ which is always

symmetric and traceless for conformally invariant theories. The components of $T_{\mu\nu}$ in complex coordinates are given by

$$\begin{aligned} T(z) \equiv T_{zz} &= \frac{1}{4}\left(T_{00} - 2iT_{10} - T_{11}\right) , \\ \bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}} &= \frac{1}{4}\left(T_{00} + 2iT_{10} - T_{11}\right) , \\ \frac{1}{4}\Theta(z,\bar{z}) \equiv T_{z\bar{z}} = T_{\bar{z}z} &= \frac{1}{4}\left(T_{00} + T_{11}\right) . \end{aligned}$$

The conservation of $T$ amounts, in this language, to the holomorphicity conditions $\bar{\partial}T_{zz} = 0$ and $\partial T_{\bar{z}\bar{z}} = 0$, which justify the definitions $T(z)$ and $\bar{T}(\bar{z})$ for the diagonal components. It seems therefore natural to introduce an infinite set of generators $L_n$ and $\bar{L}_n$ which arises as operator-valued coefficients in the Laurent expansion of the corresponding energy-momentum component. For the holomorphic part we have

$$T(z) = \sum_{n=-\infty}^{+\infty} z^{-n-2} L_n \quad \Longleftrightarrow \quad L_n = \oint_z \frac{dw}{2\pi i}(w-z)^{n+1} T(w)\,, \tag{1.6}$$

an analogous relation holding for the anti-holomorphic component $\bar{T}(\bar{z})$ as well. According to the radial quantization procedure the algebra of these modes can be obtained by calculating some commutators of loop integrals in the complex plane and the main tool to perform this calculation is represented by the *operator product expansions* (OPE). OPE are a way of extracting the singular behavior of products of operators in the small distance regime or, seen in another way, an effective way of defining the *regular* part of the product of operators, which usually bears the name of *normal ordered product*. The OPE for the energy-momentum tensor reads

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + :T(z)T(w): , \tag{1.7}$$

where : ... : stands for the normal-ordered part of the, otherwise, ill-defined product of operators. The coefficient $c$ of the most singular term in the OPE of the energy-momentum tensor is known as *central charge* of the CFT. This parameter depends strongly on the model considered and it is one of the most characteristic quantities that describes a CFT.

The problem of finding the algebra of the modes $L_n$ amounts to the application of the complex analysis and contour integration techniques which can be used to

derive the celebrated *Virasoro algebra*

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0}\,. \tag{1.8}$$

The latter algebra can be viewed as an extension of the classical Witt algebra (1.5), which is still recovered for the subalgebra formed by $L_0, L_{\pm 1}$, because, in general, terms like the one proportional to $c$ are referred to as *central extension* of an algebra, being proportional to the identity operator.

The field content of the theory can be fully specified by the introduction of a particular class of fields: the *primary fields* of a CFT are those which, under holomorphic mappings $z = z(w)$ and $\bar{z} = \bar{z}(\bar{w})$, transform as

$$\phi(z,\bar{z}) = \left(\frac{\partial w}{\partial z}\right)^{\Delta} \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\bar{\Delta}} \phi(w,\bar{w})\,, \tag{1.9}$$

with $\Delta$ ($\bar{\Delta}$) being the *(anti)-holomorphic conformal dimension* of the operator $\phi(z,\bar{z})$. All the other operators which do not satisfy (1.9) are said *secondary or descendants fields*. The strong requirements for the transformation properties of a primary field severely constrain the form of the correlators between these type of fields. In particular the two-point function of a primary field is completely fixed by conformal invariance and (1.9), it reads

$$\langle \phi(z,\bar{z})\phi(w,\bar{w})\rangle = \frac{\mathcal{C}}{(z-w)^{2\Delta}(z-w)^{2\bar{\Delta}}} \tag{1.10}$$

It is common at this point to define the *spin* $s = \Delta - \bar{\Delta}$ and *scale or engineering dimension* $d = \Delta + \bar{\Delta}$ for the primary field $\phi$. Also the form of the 3 and 4-point function is rigidly constrained by the conformal simmetry and we defer to [14, 16, 31] for more details on this point.

Another relevant feature of primary fields consists in the special form taken by their OPE with the energy-momentum tensor components. It is given by

$$\begin{aligned} T(z)\phi(w,\bar{w}) &= \frac{\Delta\phi(w,\bar{w})}{(z-w)^2} + \frac{\partial\phi(w,\bar{w})}{(z-w)} + :T(z)\phi(w,\bar{w}):\,, \\ \bar{T}(\bar{z})\phi(w,\bar{w}) &= \frac{\bar{\Delta}\phi(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\partial\phi(w,\bar{w})}{(\bar{z}-\bar{w})} + :\bar{T}(\bar{z})\phi(w,\bar{w}):\,. \end{aligned} \tag{1.11}$$

We stress the fact that the concrete utility of this relation relies on the fact that it can be applied to identify the primary fields of theory, once the OPE is evaluated

by other techniques. Combining the OPE (1.11) with the modes definition (1.6) we can easily derive the action of the operators $L_n$ on a holomorphic primary field

$$[L_n, \phi(z)] = \Delta(n+1)z^n\phi(z) + z^{n+1}\partial\phi(z) \tag{1.12}$$

a relation which will reveal crucial in determining the space of state of the CFT. We can start by considering the vacuum state $|0\rangle$ of the theory, by imposing the regularity of $T(z)|0\rangle$ as $z \to 0$ we obtain, as a consequence of equations (1.6),

$$L_n|0\rangle = 0 \quad \text{for} \quad n \geq -1\,. \tag{1.13}$$

The introduction of the vacuum state allows the complete construction of the space of state in terms of the *highest weight states*, i.e. the eigenstates of the Virasoro generators $L_n, \bar{L}_n$, which are known to form a highest weight representation of the Virasoro algebra (1.8). To explicitly construct this representation let us start by considering a holomorphic primary field $\phi/z)$ of conformal dimensions $(\Delta, 0)$, which we use to construct the state

$$|\Delta\rangle = \phi(0)|0\rangle\,. \tag{1.14}$$

The state $|\Delta\rangle$ turns out to be a highest weight state since, by means of (1.12), it satisfies

$$L_0|\Delta\rangle = \Delta|\Delta\rangle \quad \text{and} \quad L_n|\Delta\rangle = 0\,, \forall n < 0\,, \tag{1.15}$$

these relations together allow to show that, in *unitary* CFT, the conformal dimensions and the central charge must be positive numbers. This has led to the remarkable matching between highest weight states and primary operators, a property which is ultimately responsible for the effectiveness of the CFT formulation we are outlining.

The rest of the Hilbert space can then be constructed starting from the heighest weight states $\Delta$ and is constituted by *descendant states* of the form

$$L_{-n_1}L_{-n_2}\cdots L_{-n_r}|\Delta\rangle\,, \quad \text{with} \quad n_i > 0 \,\text{and} \quad i = 1, 2, \cdots, r\,,$$

and turn out to be eigenstates of $L_0$ corresponding to the eigenvalue $n = \Delta + \sum_i n_i$ as can be easily computed by using (1.12) and the Virasoro algebra (1.8). The

construction we have just explained produces, for each primary field, an infinite tower of states known in mathematics as *Verma module*. The states composing the Verma module are not guaranteed to be linearly independent or even physical states; as a matter of fact there exist combinations of descendant states, at each level of the every Verma module, which vanish and are therefore dubbed *null states*. These states are spurious and unphysical states which must be removed from the Verma module in order to obtain an irreducible representation of the Virasoro algebra built over the heighest weight state $|\Delta\rangle$.

Once this has been achieved succefully the state-operator matching is complete. Now the descendant states can be put in one-to-one correspondence with the descendant fields completing the isomorphism between the two classes. A prominent example of descendant field is represented by the energy-momentum tensor. By considering the OPE (1.7) as a particular case of (1.11) we immediately conclude that the energy-momentum tensor cannot be a primary. This should not be a surprise since the transformation properties of $T$ are not those of a primary field, but neither those of a tensor, as the name may, uncorrectly, suggest. More precisely it can be shown that $T$ is actually a descendant of the identity, namely $(L_{-2}I)(z) = T(z)$. The actual transformation law of the energy-momentum tensor under the usual conformal mapping $z, \bar{z} \to w, \bar{w}$ is

$$T(w) = \left(\frac{z}{w}\right) T(z) + \frac{c}{12}\mathbf{S}(z,w)\,, \tag{1.16}$$

where we recognize the central charge $c$ of the theory and we introduced the *Schwartzian derivative*

$$\mathbf{S}(z,w) = \frac{\partial z}{\partial w}\frac{\partial^3 z}{\partial w^3} - \frac{3}{2}\left(\frac{\partial^2 z}{\partial w"}\right)^2\,. \tag{1.17}$$

The explicit transformation law for the energy momentum can be applied to obtain the form of the Hamiltonian in a desired geometry in terms of the modes of the plane geometry. In view of the TBA analysis we are going to perform it is thus useful to obtain the explicit form of the Hamiltonian on a cylinder of radius $R$. The conformal mapping $z = e^{2\pi w/R} = \exp\frac{2\pi}{R}(\sigma^0 + i\sigma^1)$, in terms of the cylinder coordinates $\sigma^0 \in \mathbb{R}$ and $\sigma^1 \in [0, R]$, once plugged into (1.16) results in the transformation law for the energy-momentum tensor holomorphic component

$T(z)$. The latter is written as

$$T^{(cyl)}(w) = \left(\frac{2\pi}{R}\right)^2 \left(z^2 T(z) - \frac{c}{24}\right) , \tag{1.18}$$

and an analogous relation holds for the anti-holomorphic component $\bar{T}(\bar{z})$. From the last relation the corresponding transformation law for the modes $L_n$ can be easily derived. The identification of the cylinder Hamiltonian can then be completed by noting that the plane dilatations $z' = \lambda z$, corresponding to the generator $L_0 + \bar{L}_0$ in the plane, are mapped into time shifts on the cylinder. This can be easily seen by noting that a dilatation in the $z$-plane becomes, under the exponential mapping, a shift in the time direction $w' = w + r/2\pi \log \lambda$ on the cylinder. Since the Hamiltonian is, by definition, the operator which generates time translations we can infer that

$$H^{(cyl)} = \frac{2\pi}{R}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right) . \tag{1.19}$$

The consequences of this relation will reveal essential in the non-perturbative RG flow analysis of integrable models of Chapter 4.

A remarkable generalization of the Virasoro algebra is constituted by the celebrated Kac-Moody algebras [32, 33, 34, 35] which can be used to formulate a very large class of CFT. These type of theories are known to correspond to a class of models known as *Wess-Zumino-Novikov-Witten* (WZNW) models [37, 38, 39], see Chapter 3 for more details. In brief WZNW models are 2-dimensional non-linear sigma models on the (compact) group manifold $G$ which contains a topological term, the Wess-Zumino term [37]. Kac-Moody algebras can be constructed by adding to the field content of a CFT two holomorphic currents of spin 1, $J(z)$ and $\bar{J}(\bar{z})$, which take values in a finite dimensional Lie algebra $G$. Choosing the generators $T^a$, with $a = 1, 2, \cdots, \dim(G)$, to be hermitean and given their general commutation relations

$$\left[T^a, T^b\right] = f^{ab}_c T^c , \tag{1.20}$$

the component $J^a(z)$ satisfy the OPE

$$J^a(z)J^b(w) = \frac{k d^{ab}}{(z-w)^2} + f^{ab}_c \frac{J^c(w)}{(z-w)} + : J^a(z)J^b(w) : , \tag{1.21}$$

where $d^{ab} = \mathrm{Tr}(T^a T^b)$ is the Cartan-Killing metric of the algebra $G$ used to raise and lower the group indices $a, b, c, \cdots$. By the usual contour integration technology we can introduce the Laurent modes $J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1}$ and recast (1.21) as

the *Kac-Moody algebra* commutation relations

$$\left[J_n^a, J_m^b\right] = f_c^{ab} J_{n+m}^c + knd^{ab}\delta_{n+m,0}\,, \tag{1.22}$$

denoted by $\hat{G}_k$ and where the integer number $k$ is called the *level*. Note that the modes $J_n^0$ form a subalgebra which satisfies the original Lie algebra commutation relations (1.20).

The principal application of the Kac-Moody current algebra to CFT is through the so-called *Sugawara construction* [40] which enables to write explicitly the energy-momentum tensor in terms of the current of the model, explicitly

$$T(z) = \frac{1}{2(k+\mathrm{cox}(G))} d_{ab}(J^a J^b)(z)\,, \tag{1.23}$$

where $\mathrm{cox}(G)$ is the dual Coxeter number of the algebra $G$ and the notation $(AB)(z)$ is another more comfortable notation to indicate the normal ordering $:AB:$. It is moreover possible to obtain the OPE (1.7) in this case, by employing the current OPE (1.21), and read from it the central charge of the CFT which reads

$$c(\hat{G}_k) = \frac{k\,\dim(G)}{k+\mathrm{cox}(G)}\,. \tag{1.24}$$

The space of states/operators can instead be constructed by identifying the heighest weight $\Lambda$ with the state created by the action of a Virasoro primary field $\phi_\Lambda$, of conformal dimension

$$\Delta(\phi_\Lambda) = \frac{c_\Lambda}{2(k+\mathrm{cox}(G))}\,, \tag{1.25}$$

with $c_\Lambda = (\Lambda, \Lambda + 2\rho)$ being the eigenvalue of the second order Casimir in the $\Lambda$-representation of $G$.

In view of the underlying WZNW model the level $k$ is required to be integer, by the presence of the Wess-Zumino topological term, and the relative coefficient between the Lagrangian and the topological term must be fine-tuned in order to provide a conformal theory [39]. Then in the paper [35], the authors applied the techniques of [14] to obtain the differential equations for all the correlators of the theory solving, in principle, the WZNW models.

To conclude our overview upon CFT we want to outline the relation that can be established by these theories and integrable models. This conception originated from a series of work by Al. B. Zamolodchikov [24, 25, 26] in which he explored the

possibility of perturbing a CFT by one of its primary operators with the precise aim of constructing a theory with infinite conserved charges associated. The action describing such a theory can then be written

$$S = S_{CFT} + \lambda \int d^x \Phi(x) \,, \tag{1.26}$$

where $S_{CFT}$ is the unperturbed CFT action and we assumed the perturbing operator $\Phi(x)$ to be a primary of conformal dimension $(\Delta, \Delta)$, i.e. of spin 0 and dimension $d = 2\Delta$. Moreover, in order to define a consistent theory, the perturbation must be relevant $\Delta < 1$ and the coupling $\lambda$ must be of dimension $2 - 2\Delta$ for dimensional considerations. Following Zamolodchikov's arguments we can consider a field $\mathcal{T}_s \in \hat{\mathcal{D}}_s$.

The space $\hat{\mathcal{D}}_s$ can be described as follows. Define $\mathcal{D}$ as the space of all the descendant fields of the identity, namely the space of composite fields made up from the holomorphic component of the energy momentum tensor and its derivatives. The space $\mathcal{D}$ admits an orthogonal decomposition

$$\mathcal{D} = \bigoplus_{s=-\infty}^{+\infty} \mathcal{D}_s \,,$$

in terms of the subspaces $\mathcal{D}_s$ spanned by holomorphic fields of conformal dimensions $(s, 0)$, i.e. spin $s$. The space $\hat{\mathcal{D}}_s$ is then obtained by removing from $\mathcal{D}_s$ all the fields which are total derivatives, namely of the form $L_{-1}\mathcal{D}_s$; this space can therefore be defined by $\hat{\mathcal{D}}_s = \mathcal{D}_s / L_{-1}\mathcal{D}_s$.

The field $\mathcal{T}_s \in \hat{\mathcal{D}}_s$ considered by Zamolodchikov satisfies, by definition, $\bar{\partial}\mathcal{T}_s = 0$. In considering the perturbed version of the CFT we will observe that the previous conservation law becomes modified as

$$\bar{\partial}\mathcal{T}_s = \sum_n \lambda^n \mathcal{R}^{(n)}_{s-1} \,, \tag{1.27}$$

where $\mathcal{R}^{(n)}_{s-1}$ are primaries of the unperturbed CFT of spin $s - 1$ and therefore conformal dimensions $(s - n(1 - \Delta), 1 - n(1 - \Delta))$. We leave on purpose the summation over $n$ in (1.27) unspecified since, when restricting our attention to unitary CFT's, by imposing the non-negativeness of the conformal dimensions we obtain an upper bound for $n$, namely $(1 - \Delta)^{-1}$. For simplicity we can consider a version of (1.27) in which at r.h.s. there's only the term with $n = 1$.

Next logical step is then the identification of the field $\mathcal{R}_{s-1}$ in terms of the field of the CFT, this requires the *conformal perturbation theory* (CPT) around the unperturbed CFT. The correlation functions involving the field $\mathcal{T}_s$ will have the perturbative form

$$\langle \mathcal{T}_s \cdots \rangle = \langle \mathcal{T}_s \cdots \rangle_{CFT} + \lambda \int dw d\bar{w} \, \langle \Phi(w, \bar{w}) \mathcal{T}_s \cdots \rangle_{CFT} + \cdots , \tag{1.28}$$

with $\langle \cdots \rangle_{CFT}$ being the correlation functions computed in the unperturbed CFT. Using the known OPE and a bit of complex analysis it is possible to recast the anomalous conservation law in the fundamental relation

$$\mathcal{R}_{s-1}(z, \bar{z}) = \oint_z \frac{dw}{2\pi i} \Phi(w, \bar{z}) \mathcal{T}_s(z) \, .$$

This relation expresses the anomalous term in terms of the field content of the unperturbed CFT, more precisely in terms of the perturbing field and some conserved quantity. Now if we can turn the anomalous term into a total holomorphic derivative, namely if $\mathcal{R}_{s-1} = \partial \Theta_{s-2}$, we can establish the following conservation law for the perturbed CFT

$$\bar{\partial} \mathcal{T}_s = \partial \Theta_{s-2} \, . \tag{1.29}$$

Finally, by recalling the *counting argument* of [25], we can interpret the anomalous conservation law as a map $\varphi_s : \mathcal{D}_s \to \mathcal{D}_{s-1}$. Consequently the existence of spin $s$ conserved quantities in the perturbed CFT will be ensured as long as $\dim \mathrm{Ker} \varphi_s \neq 0$, which is a necessary and sufficient condition for the existence of the conserved charges. Moreover, taking into account the obvious relation

$$\dim \hat{\mathcal{D}}_s = \dim \mathrm{Ker} \varphi_s + \dim \mathrm{Im} \varphi_s \, , \tag{1.30}$$

the previous condition reads

$$\dim \hat{\mathcal{D}}_s > \dim \hat{\mathcal{D}}_{s-1} \, . \tag{1.31}$$

The counting argument provides therefore a sufficient condition which allows for indications towards the quantum integrability of the perturbed CFT (1.26). The main advantage of this procedure is that it requires the knowledge of the only dimensionalities of the subspaces $\hat{\mathcal{D}}_s$ and $\hat{\mathcal{D}}_{s-1}$ without the necessity of explicitly computing the conserved charges.

Having summarized the main motivations and the fundamental concepts behind this thesis we proceed in outlining the material covered in each chapter.

**In Chapter 2** we start reviewing the notion of integrability. After a brief discussion on the historical and classical aspects, we proceed by describing the quantum aspects of integrable models. We do this by the particular perspective of $1+1$-dimensional QFT. After a brief account for the historical aspects related to the progress made in understanding the role of symmetry in QFT we introduce the necessary tool to develop the analytic $S$-matrix theory and translate the notion of integrability in this language.

**In Chapter 3** we introduce the general concept of Non-Linear Sigma Models in QFT by giving a geometrical brief overview on the topic. Then we present two classes of NLSM which will be of main importance in the following analysis: symmetric space NLSM and Wess-Zumino-Novikov-Witten NLSM. The rest of the chapter is then devoted to the formulation of the model of interest in terms of its Lagrangian formulation. After performing the necessary perturbative renormalization of the model, the explicit procedure for the proof of classical integrability is outlined and some remarks and observation are formulated for what concerns the quantum integrability of the model.

**In Chapter 4** we give a quite detailed introduction to the Thermodynamic Bethe Ansatz approach applied to $1+1$-dimensional integrable models. The core of the procedure, namely the study of mirror thermodynamic, is performed for simplicity but with great details for the most simple theory, of which the well known Lee-Yang model [95, 96] can be considered a prototype. We then follow the unifying spirit of [101, 88, 89] and introduce a general class of TBA models which can be put in correspondence with the simply laced Lie algebras of the $A, D, E$ and $T$ type. A specific model of the latter type is then used to introduce and explicitly perform the non-perturbative computations relative to the study of the Renormalization Group flow through the finite-size scaling function $c(r)$. This non-perturbative analysis proves itself fundamental in the operation of identifying the specific TBA model in terms of its perturbed CFT realization [87] by knowledge of the UV data. Then we generalize further the class of TBA models [107] by introducing some sort of tensor product of diagrams. In this context we also introduce the elegant and powerful formalism of $Y$-systems and how it can be used to obtain non-perturbative informations on the model. We conclude the chapter by outlining how the previous framework can be used to describe integrable non-linear sigma

model in $1+1$-dimensions and formulate the so-called Fendley's conjecture about symmetric space sigma model realizations in terms of WZNW CFT [37, 39, 38].

**In Chapter 5** we introduce the main result of this thesis, namely the TBA formulation of the *extended*-$\mathbb{CP}^{N-1}$ sigma model. The first part contains the non-perturbative analysis of the model from a more detailed point of view and result in the conjecture of an $S$-matrix for fundamental excitations with the relative Bethe Ansatz set of equations. Then the TBA analysis is carried over by performing the usual step in some details concluding with the formulation of a new class of TBA equations and corresponding $Y$-system. The chapter is then concluded by the RG flow analysis of the TBA equation from which will stem the possibility of conjecturing a perturbed coset CFT realization (close in spirit to those of [103]) of the sigma model at hand.

**In Chapter 6** we develop the numerical technology required to solve the TBA equations presented in the rest of the thesis. Motivated by the very good accuracy that a simple implementation has made possible we decided to include a section in which the numerical setup for these type of equations is formulated and explained in some details. The central part of the chapter contains the result obtained by applying the numerical algorithm just introduced to the novel family of TBA which is the central interest of this work. At the end we describe in some details the numerical strategies that has been employed to conjecture some exact result related to Rogers dilogarithm sum rule associated to the $Y$-system presented in this thesis.

**In Chapter 7** we review the key aspects of the derivation of the novel class of TBA and discuss the conjectures that have been formulated throughout the chapters. Then open questions will be outlined and possible interesting future research scenario are drawn.

# Chapter 2

# Integrable quantum field theories in two dimensions

In this chapter we introduce the notion of *Integrability* both at classical and quantum level and develop the modern mathematical technology to deal with it. As mentioned in the Introduction we will adopt the perspective of $1+1$-dimensional relativistic quantum field theory. Following this path will lead us to the formulation of the integrability property in terms of the properties of the most fundamental object in this framework: the $S$-matrix theory.

The content of the chapter is divided as follows: in Section 2.1 we introduce the basic idea of the notion of integrability for classical systems. Then, in Section 2.2, we provide an introduction to the idea of quantum integrability in connection with quantum field theories with the specific point of view of $1+1$-dimensional field theories. The chapters concludes with Section 2.3 in which the analytic $S$-matrix is briefly exposed and the property of integrability is translated in this formalism.

## 2.1 Overview

From the historical point of view the discovery of integrability is intertwined with the theory of physics since is first steps. In Hamiltonian Mechanics the state of a system is described by a set of generalized coordinates $q_i$ and their canonically conjugated momenta $p_i$, where $i = 1, \cdot, n$ labels the degrees of freedom of the system. This even-dimensional space is usually called *phase space*. There exists

a preferred dynamical function, i.e. a function defined on phase space, $H(q_i, p_i)$, called the *Hamiltonian*, which encodes the time evolution of the system through the first-order system of ordinary differential equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{2.1}$$

For any dynamical function $F(q, p)$ this implies that

$$\dot{F} = \{H, F\} \tag{2.2}$$

where we made use of the *Poisson brackets*, namely

$$\{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i} \tag{2.3}$$

The coordinates themselves obey the so-called *canonical Poisson brackets*, namely

$$\{q_i, q_j\} = 0 \qquad \{p_i, p_j\} = 0 \qquad \{p_i, q_j\} = \delta_{ij} \tag{2.4}$$

The Hamiltonian is somehow special because it is a conserved quantity under time evolution, as a trivial consequence of (2.2), therefore the motion takes place on the subvariety of phase space $H(q, p) = E$, namely the *Isoenergetic variety*.

The systems for which equations (2.1) can be solved exactly are very few. Nonetheless, there is a general framework where the explicit solutions can be obtained by solving a finite number of algebraic equations and computing finite number of integrals, i.e. the solution is obtained by quadratures.

A dynamical system on a phase space of dimension $2n$ is *Liouville integrable* if one knows $n$ independent dynamical functions $F_i$ on the phase space which are:

- conserved quantities under time evolution

$$\dot{F}_i = \{H, F_i\} = 0 \tag{2.5}$$

  In other words these functions are *Integrals of Motion*.

- in *Involution* among themselves with respect to the Poisson brackets, that is

$$\{F_i, F_j\} = 0 \tag{2.6}$$

If this family of conserved dynamical quantities exists then the system can be solved by quadratures. This is so because we can use a special system of coordinates related to the original one through a canonic transformation which preserves the Hamiltonian structure of the equations of motion: the *Action-Angle variables*. In these particular variables the momenta are actually the $n$ conserved quantities and the coordinates becomes angles parametrizing a $n$-dimensional torus which is the variety onto which the motion takes place.

### 2.1.1 Classical Integrability in Field Theories

The next logical step is to define the notion of integrability for field theories. The situation now is more involved since field theories can be thought of as dynamical system with an infinite, though countable, number of degrees of freedom. In this sense finding all the integrals of motion is a very different task and we will see how the Lax formalism is efficient in doing this.

The history of classical integrable systems is strictly related to the discovery, in the late 1960s, of the existence of *solitons*. The latter are strongly stable, localized solutions of non-linear partial differential equations. They could be understood by viewing these equations as infinite dimensional integrable Hamiltonian systems. Their study leads to a very fruitful approach for "integrating" such systems, the inverse scattering transform and more general inverse spectral methods (often reducible to Riemann-Hilbert problems), which generalize local linear methods like Fourier analysis. The basic idea of this method is to introduce a linear operator that is determined by the position in phase space and which evolves in time in such a way that its "spectrum" (in a suitably generalized sense) is invariant under the evolution. This provides, in certain cases, enough invariants, or "integrals of motion" to make the system completely integrable.

For historical and pedagogical reasons we present a prototypical example: the *Korteweg-de Vries equation*, KdV for short. KdV is a non-linear differential equation used to mathematically model the waves on shallow water surfaces. There exixt several equivalent forms of KdV and we decided to present it as follows

$$\partial_t u = 6u\,\partial_x u - \partial_x^3 u \tag{2.7}$$

where $u = u(t, x)$ is the height of the 1-dimensional wave profile propagating along the $x$-direction.

The modern approach to integrable systems relies on the notion of *Lax Pairs*. To introduce the Lax formalism we can consider the general non-linear evolution equation

$$\partial_t u = N(u)\,, \tag{2.8}$$

with initial condition $u(x, 0) = f(x)$ and where the operator $N$ is some non-linear operator which may depend on $x$ and on its derivatives but is independent on $t$.

Next suppose that the evolution equation (2.8) can be expressed in the form

$$\partial_t L = ML - LM\,, \tag{2.9}$$

where $L$ and $M$ are some *linear* operators in the variable $x$ which may depend on $u$ and its derivatives and act on some Hilbert space $\mathcal{H}$. We moreover assume the operator $L$ to be self-adjoint with respect to the inner product $\mathcal{H}$ is endowed with. Such a set of operators forms a *Lax pair*. The Lax pair for KdV is given by

$$\begin{aligned} L &= -\partial_x^2 + u \\ M &= -4\partial_x^3 + 6u\partial_x + 3\partial_x\,(u) \end{aligned} \tag{2.10}$$

and the KdV equation (2.7) is equivalent to the Lax equation (2.9).

Now we introduce the *spectral equation* associated to the non-linear problem (2.8) by means of

$$L\psi = \lambda\psi\,,$$

where $\psi \in \mathcal{H}$. The spectral parameter $\lambda$ can be proved to be independent of $t$ and the linear evolution equation for the associated wave function $\psi$ turns out to be given by

$$\partial_t \psi = M\psi\,,$$

which recall the familiar Schoroedinger evolution equation. On the basis of simple functional analysis it is then possible to show that the *monodromy matrix* $\mathcal{M}(\lambda)$

which is defined by means of the monodromy of two independent solutions of the linear evolution equation. It can be defined by

$$(\psi_1(x+2\pi), \psi_2(x+2\pi)) = (\psi_1(x), \psi_2(x))\mathcal{M}(\lambda)\,.$$

The fundamental role played by the monodromy matrix is that is represents a generating function for the infinite conserved quantities associated with the KdV equation. The first few of them are

$$\begin{aligned} I_1 &= \int \frac{dx}{2\pi} u(x)\,, \\ I_3 &= \int \frac{dx}{2\pi} u(x)^2\,, \\ I_3 &= \int \frac{dx}{2\pi} \left( u(x)^3 - \frac{(u')^2}{2} \right)\,. \end{aligned}$$

Having given a very brief overview of the flavor of integrable in classical physics let us now turn the attention to the quantum world.

## 2.2 Quantum Integrability

We now turn our attention to the quantum mechanical formulation of the notion of integrability. Rather than following the standard approach to this topic we will introduce the idea of integrable from a very particular but far reaching perspective: $1+1$ dimensional QFT. As discussed in the Introduction the variety of physical systems which can be described in terms of these low dimensional theory is huge and, moreover, this is the construction which fit better with the aim of this work which, ultimately, amounts in the non-perturbative analysis of a 2-dimensional sigma model.

The history of integrability in QFT moves its first steps in 1967 with the *Coleman-Mandula theorem* [18], a very powerful *no-go* theorem pertaining the allowed symmetry of the $S$-matrix of a QFT. The authors were able to obtain, under several assumptions, the maximum symmetry group associated to Poincaré invariant $1+3$-dimensional massive QFT. More precisely they found that for any massive, relativistic and local $1+3$-dimensional QFT the maximal symmetry group, $\mathcal{G}$, which contains the Poincaré group $\mathcal{P}$ and another group of internal symmetries $\mathcal{I}$

as subgroups is given by the direct product of the two

$$\mathcal{G} = \mathcal{P} \otimes \mathcal{I};$$

we stress that the $\mathcal{G}$ is assumed to be a Lie group, whose generators obey an algebra based on commutators, a hypothesis which is crucial in the derivation of the resul. On physical grounds the Coleman-Mandula theorem states that the only possible way of mixing the space-time and internal transformations is the trivial one. This strong result can be generalized, with very different results [41], by the introduction of anticommutators in the symmetry algebra of the $S$-matrix, which turns the latter into a supersymmetry algebra. More precisely, the theorem ensures, under certain assumptions, that the existence of any conserved charge which transforms under the Lorentz group like a tensor of spin higher than one in a $1+d$-dimensional QFT (with $d > 1$) is sufficient to conclude that its $S$-matrix is trivial, i.e. there's no interaction between particles. Such a rigid constraint for the form of the $S$-matrix in $1+3$ dimensions may be indication that lowering the number dimension can possibly result in a modification of the constraint such that the form of the $S$-matrix is fixed by the symmetry alone. The $1+1$-dimensional version of the theorem states, in fact, precisely the opposite with respect to the $1+3$-dimensional case: in bidimensional QFT it is possible to combine in a non-trivial way the space-time and internal symmetries. This very powerful result suggests that the existence of higher spin conserved quantities turns out to be deeply tied with the symmetry structure of QFT. In the subsequent series of papers [19, 20, 44, 21, 22] the authors pointed by out that two relevant properties were emerging: the absence of particle production in multiparticle scattering processes and the factorisability property of the associated scattering amplitudes. The problem was then solved by Parke in a remarkable paper [23] in which he was able to definitevely closed the question by showing how the existence of only two higher spin conserved charges, the spins of the latter being different, is enough to ensure the two crucial properties mentioned above. In this sense Parke theorem become an extremely powerful tool in determining whether or not a theory is integrable, it suffices in fact to find two higher spin conserved charges to prove the integrability of the QFT. Having introduced the general idea behind $1+1$-dimensional integrable field theory we can enters the details of what outlined so far.

### 2.2.1 Massive integrable quantum field theories

In what follows we will consider a $1+1$-dimensional QFT whose spectrum is composed by a certain number of asymptotic massive particles whose masses are denoteb by $m_a$. Taking advantage of the 2-dimensional relativistic kinematics we can conveniently parametrize the energy-momentum vector $p_a^\mu$ of the $a$-th particle in terms of the so-called *rapidity*

$$p_a^0 = E_a = m_a \cosh\theta_a\,, \qquad p_a^1 = p_a = m_a \sinh\theta_a\,, \tag{2.11}$$

where the rapidity is explicitly given by $\tanh\theta_a = p_a/E_a$. A very useful feature of the rapidity formalism is represented by the fact that this quantity behaves additively under Lorentz boosts and therefore the difference $\theta_{ab} = \theta_a - \theta_b$ of different particles' rapidity constitutes a Lorentz invariant quantity. It is therefore reasonable to assume that the scattering amplitude, being a Lorentz invariant objects, must depend only on the rapidity difference of the incoming particles. This fact nicely accomodates, as we will see below, for the absence of particle production in 2-dimensional integrable QFT, since the momenta of the incoming particles are separately conserved. To construct the $S$-matrix we need to introduce, as preliminary fundamental assumption, the existence of a set of vertex operator of the creation/destruction type, denoted by $V_a(\theta_a)$ representing an asymptotic particle of quantum numbers $a$ and rapidity $\theta_a$. We can then write the 2-body scattering amplitude for the process $a+b \to c+d$ as

$$S_{ab}^{cd}(\theta_{12}) = {}_{out}\langle V_c(\theta_1)V_d(\theta_2)|V_a(\theta_1)V_b(\theta_2)\rangle_{in}\,. \tag{2.12}$$

The operators $V_a(\theta_a)$ are a generalization of the usual bosonic and fermionic ladder operators, in the sense that they do provide a generalization of the corresponding algebra of operators, and can be used to construct the space of physical states. The algebra satisfied by the vertex operators bears the name *Faddeev-Zamolodchikov algebra* [120] and is constituted by a set of highly non-trivial relations among them, involving the $S$-matrix element $S_{ab}^{cd}$. The Faddeev-Zamolodchikov algebra is given

by

$$V_a(\theta_1)V_b(\theta_2) = \sum_{c,d} S_{ab}^{cd}(\theta_{12})V_d(\theta_2)V_c(\theta_1)\,, \tag{2.13}$$

$$V_a^\dagger(\theta_1)V_b^\dagger(\theta_2) = \sum_{c,d} S_{ab}^{cd}(\theta_{12})V_d^\dagger(\theta_2)V_c^\dagger(\theta_1)\,, \tag{2.14}$$

$$V_a(\theta_1)V_b^\dagger(\theta_2) = \sum_{c,d} S_{ab}^{cd}(-\theta_{12})V_d(\theta_2)V_c(\theta_1) + 2\pi\delta_{ab}\delta(\theta_{12})\,. \tag{2.15}$$

The algebra was originally formulated by the Zamolodchikov brothers [119, 120] and owns his name to the observation made by Faddeev in [42] of introducing the last term in (2.15).

In the same spirit of the construction of the Fock for ordinary QFT, we can build up the space of states by means of the vertex operators. We start by defining the vacuum state $|0\rangle$ as the state which is annhilated by all the vertex operators

$$V_a(\theta)|0\rangle = \langle 0|V_a^\dagger = 0\,,$$

then a generic $n$-particle state can be obtained by the repeated action of the creation operators $V_a^\dagger$ on the vacuum, namely

$$|V_{a_1}(\theta_1)V_{a_2}(\theta_2)\cdots V_{a_n}(\theta_n)\rangle = V_{a_1}^\dagger(\theta_1)V_{a_2}^\dagger(\theta_2)\cdots V_{a_n}^\dagger(\theta_n)|0\rangle\,. \tag{2.16}$$

The linear combinations of states of the form (2.16) do generate the Hilbert space of the theory. Clearly, in view of (2.13-2.15), not all the states obtained in this way are linearly independent and we should provide a prescription to select a basis of independent physical states. We characterize a $n$-particle *in*-state by requiring the rapidities to be ordered decreasingly from left to right, thus giving

$$|V_{a_1}(\theta_1)V_{a_2}(\theta_2)\cdots V_{a_n}(\theta_n)\rangle_{in} \text{ with } \quad \theta_1 > \theta_2 > \cdots > \theta_n\,. \tag{2.17}$$

While an $n$-particle *out*-state can be obtained by inverting the previous prescription, we then get

$$|V_{a_1}(\theta_1)V_{a_2}(\theta_2)\cdots V_{a_n}(\theta_n)\rangle_{out} \text{ with } \quad \theta_1 < \theta_2 < \cdots < \theta_n\,. \tag{2.18}$$

These prescriptions sound physically reasonable, as can be seen in Figure 2.1, because the *in*/*out* scattering states are defined as those for which the $t \to \mp\infty$

limit is non interacting and this naturally translates into the different orderings of (2.17) and (2.18).

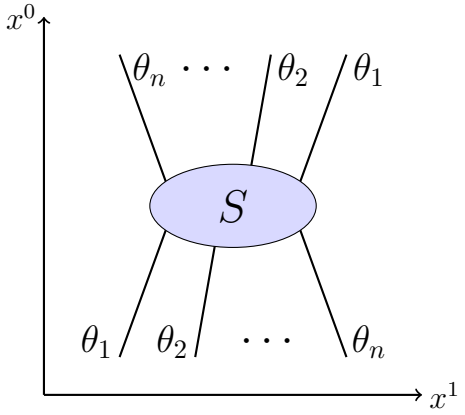



FIGURE 2.1: A representation of the generic $n$-particle scattering process with the rapidity ordering prescription.

As mentioned above, in any QFT, there may exist *higher spin conserved charges*; the latter are conserved quantities which transform under the Lorentz group as spin $s$ objects

$$\mathcal{Q}_s \to \Lambda^s \mathcal{Q}_s \,,$$

$\Lambda$ being a generic element of the group. Moreover we shall assume that these charges are of the *local* type, i.e. they can be expressed as the integral of an associated density

$$\mathcal{Q}_s = \int dx \mathcal{T}_{s+1} \,, \tag{2.19}$$

where $\mathcal{T}_{s+1}$ is one of the conserved currents introduced above, see equation (1.27). Whenever (2.19) holds true it is possible to show that the higher spin conserved charges $\mathcal{Q}_s$ are in *involution* among themselves

$$[\mathcal{Q}_s, \mathcal{Q}_{s'}] = 0 \,, \quad \forall s, s' \,. \tag{2.20}$$

Since the mass is a spin 0 conserved quantity we can deduce from (2.20) that its eigenstates constitute a basis for the eigenstates of the higher spin charges, in other words, the latter eigenstates are given by linear combination of the states belonging to the same mass multiplet. The basis of states (2.16) reveals particularly useful to diagonalize the charges $\mathcal{Q}_s$, in fact Lorentz invariance suffices to fix the form

for the 1-particle states

$$\mathcal{Q}_s|V_a(\theta)\rangle = \eta_a^s \left(m_a e^{\theta}\right)^s |V_a(\theta)\rangle \,, \tag{2.21}$$

while the locality condition (2.19) allows for the $n$-particle version

$$\mathcal{Q}_s|V_{a_1}(\theta_1)V_{a_2}(\theta_2)\cdots V_{a_n}(\theta_n)\rangle = \sum_{i=1}^{n} \eta_{a_i}^s \left(m_{a_i} e^{\theta_i}\right)^s |V_{a_1}(\theta_1)V_{a_2}(\theta_2)\cdots V_{a_n}(\theta_n)\rangle \,. \tag{2.22}$$

Under the general assumptions made throughout this section we are now in the position of giving evidence of one the key aspects of 2-dimensional integrable theory: the absence of particle production. As mentioned in the Introduction this constraint reveals essential in determining the form of the exact $S$-matrix. This can be seen as follows. Consider a general scattering process with $n$ *in*-particles and $m$ *out*-states, the corresponding scattering element is given by

$$S_{a_1 a_2 \cdots a_m}^{b_1 b_2 \cdots b_n} = {}_{out}\langle V_{b_1}(\theta_{b_1})V_{b_2}(\theta_{b_2})\cdots V_{b_n}(\theta_{b_n})|V_{a_1}(\theta_{a_1})V_{a_2}(\theta_{a_2})\cdots V_{a_m}(\theta_{a_m})\rangle_{in} \,.$$

The conservation of the charges under this process implies, taking (2.22) into account, that

$$\sum_{i=1}^{m} \eta_{a_i}^s \left(m_{a_i} e^{\theta_{a_i}}\right)^s = \sum_{i=1}^{n} \eta_{b_i}^s \left(m_{b_i} e^{\theta_{b_i}}\right)^s \tag{2.23}$$

must hold. If we study the solution of (2.23) under the condition of integrability, i.e. by assuming an infinite number of charges, it turns out that the resulting infinite system of non linear equation admits only the trivial solution. The latter is characterized by $n = m$ and $\theta_{a_i} = \theta_{b_i}$ therefore we can directly observe the absence of particle production and the individual conservation of the momenta. The argument just exposed can be found in [43, 44, 21].

Now we turn our attention to the crucial properties of integrable theories, namely the factorisability of the $S$-matrix. This result goes under the name of *Parke theorem* and can be found in [23], see also [43] for a recent review, and we will outline its statement without entering the technical details of the proof which could constitute a chapter on their own. The theorem states that any 2-dimensional theory which possesses two higher spin conserved charges present the absence of

particle production and the strict momentum conservation and, moreover, its $n$-particle processes $S$-matrix factorises in the product of 2-body scattering matrices. This can be expressed as

$$S_{a_1 a_2 \cdots a_m}^{b_1 b_2 \cdots b_n}(\theta_{b_1}, \theta_{b_2}, \cdots, \theta_{b_n}; \theta_{a_1}, \theta_{a_2}, \cdots, \theta_{a_m}) = \\ = \delta_{nm} \prod_{i=1}^{n} \delta(\theta_{a_i} - \theta_{b_i}) \prod_{i<j,k<l,\,1}^{n} S_{a_i a_j}^{b_l b_k}(\theta_{ij}) \,, \tag{2.24}$$

where the absence of particle production and the strict momentum conservation are explicitly imposed.

The proof of the absence of particle production is not so easy to obtain under the assumptions of Parke theorem. More precisely the existence of only two higher spin conserved charges imposes modifications of the arguments given in the previous section, see equation (2.23) and the discussion there. Without entering the details, for which we refer to [23, 43], we simply assume this constraint to be true and focus on the other key aspect: factorisability.

In order to prove factorisability we have to introduce the other fundamental assumption at the basis of Parke theorem, namely the possibility of describing the 1-particle asymptotic states by means of localized wave packets $\psi_a(x)$. It is known that a wave function description is not possible for relativistic quantum theories and the very reason behind this is ultimately the property of the latter of being multiparticle theories. In this sense integrable theories are special as, in view of the absence of particle production in scattering processes, they do not behave as the typical relativistic QFT but rensemble more the classical behavior. Moreover, we can resort to wave packet description for particles which are assumed to be sufficiently separated in space to be considered as free particles, at least for theories endowed with short range interactions. A remarkable consequence of the wave packet formulation is that, in this basis, the action of the transformation generated by the conserved charges amounts to a rapidity-dependent shift of the center of the packet. It is customary to analyze the typical $3 \to 3$ processes, as those of Figure 2.2. One of the byproducts of Parke's proof is the fact that, provided two higher spin charges exist, the two amplitudes must be equal, unlike for a general QFT for which they do not have to be necessariy related. This can be understood within the wave packet formulation in which, we recall, the isometries associated to the conserved charges are realized as shifts of the center of the packet. Within this formulation it can be proved [23] that the amplitudes of Figure (2.2) can

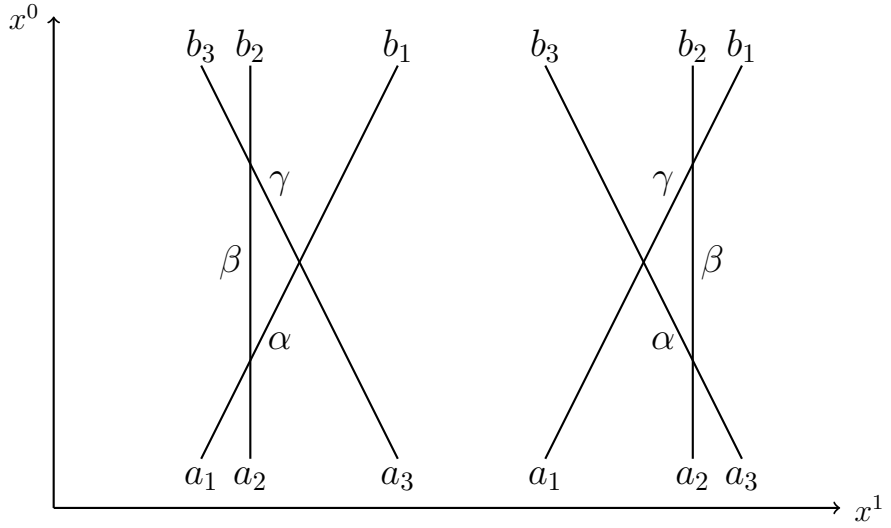



FIGURE 2.2: Graphical representation of the Yang-Baxter equation (2.25).

be mapped into each other by acting with the transformation generated by the charges which, being isometries, leave the amplitude unchanged. This statement results, in the $S$-matrix language, in the famous *Yang-Baxter equation* [45] which reads

$$S^{\alpha\beta}_{a_1a_2}(\theta_{12})S^{b_1\gamma}_{\alpha a_3}(\theta_{13})S^{b_2b_3}_{\beta\gamma}(\theta_{23}) = S^{\gamma b_3}_{a_1\alpha}(\theta_{13})S^{\beta\alpha}_{a_2a_3}(\theta_{23})S^{b_1b_2}_{\gamma\alpha}(\theta_{12})\,, \tag{2.25}$$

and was originally introduced in the study of statistical mechanics systems, more precisely the 1-dimensional anisotropic Heisemberg chain [45].

A particularly simple class of solution which are anyway non-trivial is represented by the so called *diagonal* scattering theories, namely theories for which the massive multiplet is constituted by one single particle. In this case not only the rapidities but also the individual quantum numbers are conserved, the form of the $S$-matrix is then given by

$$S^{cd}_{ab}(\theta) = \delta^c_a\delta^d_b S_{ab}(\theta)\,, \tag{2.26}$$

which trivially solves (2.25). Diagonal scattering theories represent a simplification of more realistic color-changing scattering but constitute, nonetheless, a very useful framework into which investigate the analytic properties of the associated $S$-matrix.

The $S$-matrix theory has been applied successfully to a large number of integrable 2-dimensional model in many fields of physics, from QFT to statistical mechanics [46, 47, 48, 49, 50]. It is therefore worthwhile to overview the principal properties

of the 2-body scattering amplitude both from the physical and analytic point of view.

## 2.3 The analytic $S$-matrix theory

The form of the $S$-matrix in integrable models is severely constrained by the Yang-Baxter equation (2.25) as a reflection, at the level of 2-body amplitudes, of the factorisation property induced by integrability. It is however possible to impose further physical constraints which often result in a complete determination of the analytic form of the scattering amplitudes [50, 119, 120]. The physical constraints we impose on the $S$-matrix are:

- **Lorentz invariance :** We have seen that the $S$-matrix is required to be a function of Lorentz invariant quantities, being itself a Lorentz invariant quantity. It is known in 4 dimensions that the fundamental kinematical invariants are represented by the so-called *Mandelstam variables*

$$s = (p_a + p_b)^2 \,, \quad t = (p_a - p_c)^2 \,, \quad u = (p_a - p_d)^2 \,,$$

for a general $a+b \to c+d$ scattering process. In 1+1-dimensions these quantity are dependent among each other and we can select one representative, usually $s$, onto which the $S$-matrix will depend. It reads explicitly

$$s = m_a^2 + m_b^2 + 2m_a m_b \cosh\theta_{ab} \,, \tag{2.27}$$

where $\theta_{ab} = \theta_a - \theta_b$, as usual. We can now interpret the amplitude as a function of $s$, or $\theta$, and perform its analytic continuation in the corresponding complex plane. It will turn out, see below, that the amplitude $S_{ab}(s)$ will display square root branch cuts starting at $s_\pm = (m_a \pm m_b)^2$, which correspond to the points $\theta = 0, i\pi$ in the complex rapidity plane, and therefore is not a meromorphic function. On the other side, meaning when considering the amplitude $S_{ab}(\theta)$ as a function of the rapidity, the analytic properties are better and we can restrict our analysis to the physical values of the rapidity located inside the so-called *physical strip* $0 < \mathrm{Im}\theta < \pi$.

- **Analiticity :** The analiticity condition we impose can be stated as the fact that the scattering amplitude $S_{ab}(s)$ and its complex conjugated $[S_{ba}(s)]^*$ must be the boundary values of the same analytic function on the two sides of the branch cut in the complex $s$-plane. Namely we define the physical amplitudes as

$$S_{ab}^{(ph.)}(s) \equiv \lim_{\epsilon \to 0} S_{ab}(s + i\epsilon)\,,$$
$$\left[S_{ab}^{(ph.)}(s)\right]^* \equiv \lim_{\epsilon \to 0} S_{ab}(s - i\epsilon)\,,$$

and taking into account that $S_{ab}(s \pm i\epsilon) \to S_{ab}(\pm\theta)$ as $\epsilon \to 0$ we obtain the *hermitean analiticity condition*

$$S_{ab}(\theta) = [S_{ab}(-\theta^*)]^*\,, \tag{2.28}$$

therefore the 2-body $S$-matrix in $1+1$-dimensional integrable theories is a complex analytic function of the rapidity $\theta$ on the physical strip $0 < \mathrm{Im}\theta < \pi$

- **Unitarity :** The unitarity of the $S$-matrix is expression of the overall normalization to one of the probability for a generic *in*-state to be found in anyone of the *out*-states. This translates on in the $S$-matrix language in the condition $SS^\dagger = S^\dagger S = 1$ and moreover, by taking analiticity (2.28) into account, can be written as

$$S_{ab}(\theta)S_{ba}(-\theta) = 1\,, \tag{2.29}$$

and by analytic continuation this result holds true for any value of $\theta$ in the whole complex plane.

- **Crossing symmetry :** This symmetry expresses the fact that the same process described in the $s$- and $t$-channel is found to describe two processes in which one of the incoming particle and the outgoing particle of opposite momentum are interchanged. This leads to the constraint

$$S_{ab}(i\pi - \theta) = S_{b\bar{a}}(\theta)\,,$$

where $\bar{a}$ denotes the anti-particle of quantum numbers $a$. Note that changing the sign of the momentum $p_a$ results in the exchange of $s$ and $t$ or the usual QFT interpretation.

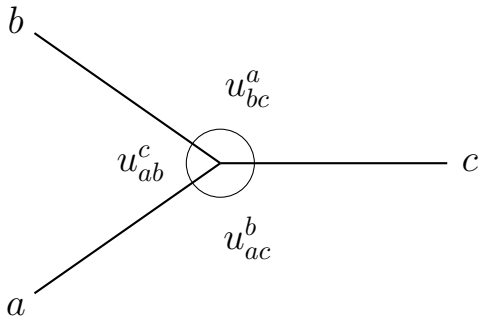



FIGURE 2.3: Geometrical representation of the fusing angles.

The listed properties and the requirements of integrability strongly constrains the form of the $S$-matrix but an intrinsic degree of ambiguity remains anyway when trying to construct scattering amplitudes. These multiplicative ambiguities are named *CDD factors* and are represented by scalar factors which automatically satisfies the physical constraints [119, 120, 88]. The general solution to the constraints we require for the physical $S$-matrix has, in the diagonal case [51], the structure

$$f(\theta) = \prod_{\alpha \in \mathcal{A}} f_\alpha(\theta)\,, \quad \text{where} \quad f_\alpha(\theta) = \frac{\sinh(\theta + i\alpha)/2}{\sinh(\theta - i\alpha)/2}\,.$$

where $\mathcal{A}$ is a conjugation-invariant subset of the complex plane. Clearly to select in a unique way the proper $\mathcal{A}$ some physical information must be supplied and this will ultimately result in the complete determination of the $S$-matrix form. Another extremely important data carried by the analytic $S$-matrix is represented by the *bound states* content of the theory. It can be shown that the presence of a simple pole of the amplitude in the physical strip signals the formation of a bound state; the latter turns out to be stable if the pole lies on the imaginary axis. To be more precise let us consider the amplitude $S_{ab}(\theta)$ and suppose it has a simple pole of the form $iu^c_{ab}$. This signals the presence of a stable bound state, created in the process $a + b \to c$, whose mass is given by

$$m_c^2 = m_a^2 + m_b^2 + 2m_a m_b \cos u^c_{ab}\,, \tag{2.30}$$

relation that comes from the very definition of $s$ (2.27) evaluated in the rest frame. The process is depicted in Figure 2.3. The angle $u^c_{ab}$ is known as *fusing angle* and invoking crossing symmetry we can interpret any of the particle $a, b$ and $c$ as a bound state formed by the remaining two. To each of this interpretation we can associate the corresponding version of equation (2.30) with the proper permutation

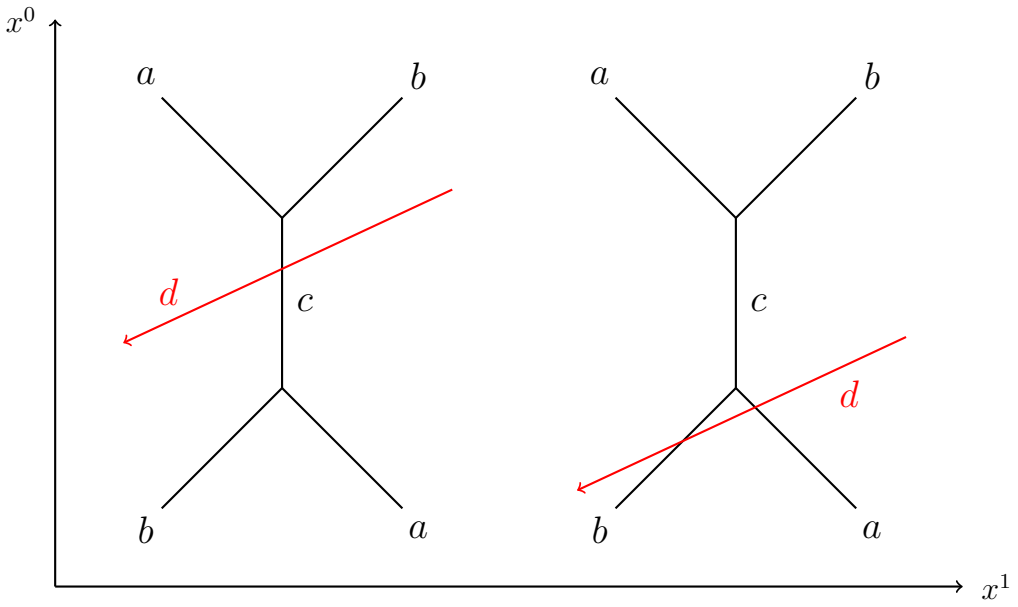



FIGURE 2.4: An illustration of the scattering process identified through the bootstrap equation (2.31)

of indices and the relative fusing angle. The three fusing angles are then seen to satisfy

$$u_{bc}^{a} + u_{ac}^{b} + u_{ab}^{c} = 2\pi \,,$$

a relation that again is quite constraining for the masses of the spectrum.

Clearly the appearance of new excitations in the spectrum calls for the possibility of obtaining the corresponding scattering amplitudes within the $S$-matrix theory we are outlining. This turns into the so-called *bootstrap equations* [47, 50, 75] which establishes the equivalence of the two diagram of Figure 2.4. We would like to stress that the wave packet framework is again the one proper to derive the mentioned relation. From the analytic point of view the bootstrap equation is given by

$$S_{cd}(\theta) = S_{ad}(\theta + i\bar{u}_{a\bar{c}}^{b}) S_{bd}(\theta - i\bar{u}_{b\bar{c}}^{a}) \,, \tag{2.31}$$

with $\bar{u} = \pi - u$.

The bootstrap equation represents a very powerful analytic tool to generate the amplitudes for all the particle of the spectrum. Its application, the *bootstrap program*, consists of various step: one starts by determining all the the bound

states of the theory by classifing the pole of the amplitudes between fundamental excitations. Then using (2.31) one evaluates all the scattering amplitudes between the bound states and the fundamental particles and among themselves. Finally, with the full spectrum and $S$-matrix at hand, one should check if the bootstrap procedure has "closed" by verifying that the poles of the $S$-matrix are those and only those that produce the known spectrum. If new particle appear at this stage one should use again (2.31) to obtain the new amplitudes and check if the bootstrap has closed and so on.

Having introduced the main tools and ideas of integrability for 2-dimensional QFT we can now turn our attention to the discussion of a very general class of QFT: Non-Linear Sigma Models.

# Chapter 3

# Non-Linear Sigma Models

The *Non-Linear Sigma Models* (NLSM), or simply sigma models, are a particular class of quantum field theories whose fields take values in a suitable Riemannian manifold $M$. The field values can therefore be considered as a set of cooordinate in the internal manifold $M$ whose curvature, described by a, generally, field-dependent metric, is a function of the fields. This feature allows the formulation of interacting quantum theories in which the structure of the interactions is governed by the geometry of the manifold in which the fields take values. Such an elegant way of formulating QFT relies its importance on the fundamental role played by symmetry principles in Physics, which, ultimately, is the main reason for many successful applications of NLSM in field theory, string theory and statistical mechanics. The geometrical nature of the interaction in NLSM translates into the geometrical interpretation of the counterterms, the existence of non-trivial field configurations (solitons), asymptotic freedom and the dynamical generations of vector bosons, i.e. all the familiar features of 4-dimensional quantum gauge theories [53]. In view of the particular applications on which this thesis focuses on, we wanto to stress that NLSM also provides a fundamental theoretical laboratory for the study of 2-dimensional exaclty solvable systems from all the fields of theoretical physics: the Ising model and the Heisenberg ferromagnet, in statistical mechanics; modern superstring theories, namely supersymmetric string theories on a curved gravitational background, are formulated in terms of supersymmetric NLSM. Even the Standard Model formulation can be translated in the NLSM language, in particular for what concerns the interpretation of the *spontaneous symmetry breaking* phenomenon.

The chapter is divided as follows: in Section 3.1 we give a quick overview on sigma models and their explicit formulation. We also give some examples which are relevant for the discussion of this thesis. Instead, Section 3.2 is devoted to the formulation of the family of models stemmed from [74]. The model is analyzed in terms of usual perturbative analysis and found to be renormalizable and asymptotically free. Then we address the question of integrability. Starting with the classical model we briefly describe how to construct the objects that ensures the former property. At last we summarize the so-called counting argument of [81] and apply it to the case at hand. This will ultimately result in the definition of an integrable subclass of models.

## 3.1 General sigma models

The most general (bosonic) NLSM can be written in terms of a set of $D$ scalar fields $\phi^a(x^\mu)$, therefore $a = 1, 2, \cdots, D$, in a $d$-dimensional flat spacetime, $\mu = 0, 1, \cdots, d$. The corresponding action reads

$$S = \frac{1}{2\lambda^2} \int d^d x \, g_{ab}(\phi) \partial^\mu \phi^a \partial_\mu \phi^b \,, \tag{3.1}$$

where $\partial^\mu = \eta^{\mu\nu}\partial_\nu$ and $\eta_{\mu\nu} = \mathrm{diag}(+, -, \cdots, -)$ are the usual covariant quantities in a $d$-dimensional Minkowski flat spacetime and $\lambda$ is the overall coupling constant, a counting device for loop expansions in this case. Usually, the metric $g_{ab}$ is also assumed to be *positive-definite*, to ensure the appearance of states with only positive norm, i.e. the absence of *ghost*.

From the QFT point of view, the highly non-linear NLSM action (3.1) can be rewritten thanks to the expansion of the metric, in powers of $\phi$ around the $\phi = 0$ configuration, which reads

$$g_{ab}(\phi) = g_{ab}(0) + \partial_c g_{ab}(0)\phi^c + \frac{1}{2}\partial_c \partial_d g_{ab}(0)\phi^c \phi^d \,, \tag{3.2}$$

and can be interpreted as the existence of an infinite number of coupling constants related to the higher derivatives of the metric $g_{ab}$ in $\phi = 0$.

Returning to the geometric point of view we immediately see that the action (3.1) is invariant under *coordinate diffeomorphism* of $M$, whose infinitesimal form is

give by the field reparametrization

$$\phi' = \phi + \xi(\phi)\,, \tag{3.3}$$

provided the metric transform as a rank-2 tensor with respect to the manifold indices. This invariance is not of the usual Noether-type because the associated transformation acts not only on the fields but also on the coupling constants, changing in this way the whole theory at once. Therefore, two different NLSM are physically equivalent if they are related through a field redefinition as (3.3), which implies $g' = g$ at the level of the metrics. Taking this observation into account the NLSM field theory is thus defined on the equivalence class of metrics, related via (3.3). In string theory, on the contrary, this formal invariance is kept, in some particular cases [54], manifest in order to properly deal with the geometrical interpretation of (3.1).

The isometries of the manifold $M$ can be sufficient to reduce the number of couplings in the field-theoretical interpretation of (3.1), see (3.2), to a finite number. The most known example of this type of circumstance is represented by the $O(n)$ NLSM [118], whose euclidean action can be written as

$$S_{O(n)} = \frac{1}{2\lambda^2}\int d^dx\, \partial^\mu \mathbf{n}\cdot\partial_\mu \mathbf{n}\,, \tag{3.4}$$

for the real scalar fields $\mathbf{n}(x^\mu)$ which take values on the $n-1$ dimensional sphere $\mathbf{n}\cdot\mathbf{n} = 1$. The latter form is not manifestly the same as (3.1) because in writing (3.4) we used some spurious degrees of freedom, namely the constrained vector $\mathbf{n}(x^\mu)$. According to [118], we can obtain the general form (3.1) associated to the $O(n)$ NLSM by solving the constraints in terms of some, suitably choosen, set of independent field variables and then substituting the result back in (3.4) to obtain explicitly the non-linear form of the associated model. The specific $O(3)$ case in $d = 2$, for instance, is known to govern the continuum limit of the isotropic ferromagnet in statistical mechanics. Moreover it displays classical [55] and quantum integrability [119, 120] and the explicit $S$-matrix between fundamental excitations has been constructed [120], we report it here for future reference

$$S_{ij}^{kl}(\beta) = \frac{\pi + i\beta}{(2\pi + i\beta)(\pi - i\beta)}\left(\frac{2\pi i\beta}{\pi + i\beta}\delta_{ij}\delta_{kl} - 2\pi\delta_{il}\delta_{jk} - i\beta\delta_{ik}\delta_{jl}\right)\,. \tag{3.5}$$

We will return later on this model, and also its generalization to arbitrary even $n$, in relation to the TBA approach to integrable models, see Chapter 4.

Another important example, which will constitute the starting point for the analysis presented in this thesis, is represented by the $\mathbb{CP}^{N-1}$ NLSM [56]. Apart from being interesting on its own, this type of model introduces the *gauging* procedure within the formalism of NLSM. The relative action can be written, in the spirit of $O(n)$ models, in terms of $N$ complex scalar fields $\mathbf{z} = (z_1, z_2, \cdots, z_N)$ subjet to the constraint $\bar{\mathbf{z}} \cdot \mathbf{z} = 1$ and reads explicitly

$$S_{\mathbb{CP}^{N-1}} = \frac{1}{2\lambda^2} \int d^d x \, \partial^\mu \left(D^\mu \mathbf{z}\right)^\dagger \cdot D_\mu \mathbf{z} \,, \tag{3.6}$$

in terms of the covariant derivative $D_\mu \mathbf{z} = (\partial_\mu + i A_\mu)\, \mathbf{z}$. In this case we speak of *gauged* model because (3.6) is manifestly invariant under the abelian gauge transformation

$$\mathbf{z}' = e^{i\Lambda(x)} \mathbf{z} \,, \quad A'_\mu = A_\mu - \partial_\mu \Lambda \,,$$

with gauge parameter $\Lambda(x)$. The absence of a kinetic term for the gauge field $A_\mu$ makes possible to eliminate it from the action (3.6). Thanks to the associated algebraic equation of motion, namely $A_\mu = i\bar{\mathbf{z}} \cdot \partial_\mu \mathbf{z}$, it is possible to replace the dummy gauge field in terms of the NLSM field $\mathbf{z}$, an operation which, ultimately, amounts to translate (3.6) into the general form (3.1).

The 2-dimensional NLSM will play a crucial role in this thesis and we will analyze some example to enlighten the fundamental features. From general point of view the immediate consequence of the choice $d = 2$ it that all the main ingredients of the action (3.1), namely the field $\phi$, the metric $g_{ab}$ and the infinite couplings, become dimensionless. In the quantum formulation this will result in the renormalizability of the model by means of the introduction of counterterms which have the same dimension as the Lagrangian. Since we know that (3.1) represents the most general (parity conserving) form of action, last observation implies that all UV counterterms can therefore be absorbed into the metric as its quantum deformations. This procedure goes under the name of (generalized) on-shell renormalizability [57, 58] and is explained in full details in [59]. Instead for a formal proof of NLSM renormalizability see [57] and also [60] for a discussion on the general structure of NLSM renormalization.

### 3.1.1 WZNW models

Here we introduce a particular class of NLSM which goes under the name of *Wess-Zumino-Novikov-Witten* models [37, 38, 39]. These models are characterized by the fact of being 2-dimensional quantum *conformal* NLSM whose underlying CFT is represented by the coset model discussed in the Introduction in relation with Kac-Moody algebras [32, 33]. The associated euclidean action can be written as

$$S_{WZNW} = \frac{k}{8\pi}\int_{S^2} d^2x\, \mathrm{Tr}\left(\partial_\mu g \partial^\mu g^{-1}\right) + \Gamma[g]\,, \tag{3.7}$$

where the matrix-valued field $g$ takes values in a (semisimple and compact) Lie group $G$. The term $\Gamma(g)$ in (3.7) is the *Wess-Zumino topological term* which is given by

$$\Gamma(g) = \frac{k}{12\pi}\int_B d^3y\, \varepsilon^{\mu\nu\rho}\mathrm{Tr}\left(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\rho g\right)\,. \tag{3.8}$$

The Wess-Zumino term (3.8) is given as a 3-dimensional integral over a ball $B$ whose boundary $\partial B$ has to be identified with the compactified 2-dimensional euclidean spacetime $S^2$. A more careful analysis reveals that the integrand in (3.8) is locally a total derivative and therefore the WZ term is a topological term. The coefficient appearing in (3.7) are normalized in the proper way [] to produce the associated $G_k$ Kac-Moody algebra for the conformal currents $J$, see equation (1.21) for the notational convention, which are explicitly realized in terms of the field $g$ as

$$\begin{aligned} J(z) &= J^a(z)T^a = kg^{-1}\partial g\,,\\ \bar{J}(\bar{z}) &= \bar{J}^a(\bar{z})T^a = -k\left(\bar{\partial}g\right)g^{-1}\,. \end{aligned}$$

We conclude this brief overview on NLSM by mentioning the possibility of introducing *gauged* NLSM [61]. The idea is to obtain an explicit construction for the NLSM over *symmetric spaces* of the type $G/H$, $G$ being a generic Lie group with anomaly-free subgroup $H$. In writing (3.7) for a generic Lie group $G$ we can assume that $\mathcal{G}$, the algebra associated to $G$, admits an $\mathrm{ad}(\mathcal{H})$-invariant orthogonal decomposition

$$\mathcal{G} = \mathcal{H}\oplus\mathcal{C}\,,\quad [\mathcal{H},\mathcal{H}]\subset\mathcal{H}\,,\qquad [\mathcal{H},\mathcal{C}]\subset\mathcal{C}\,,$$

where $\mathcal{C} = \mathcal{G}/\mathcal{H}$. A $\mathcal{G}$-valued field $J_{\mathcal{G}}$ can then be decomposed as $J_{\mathcal{G}} = J_{\mathcal{H}} + J_{\mathcal{C}}$. The explicit construction of the gauged model can then be performed by cosetting away the subgroup $H$. Concretely this amounts to minimally couple the $H$-valued gauge fields and make the action invariant under the gauge transformations

$$\begin{aligned} g' &= hgh^{-1}\,, \\ A'_\mu &= \partial_\mu h h^{-1} + hA_\mu h^{-1}\,, \end{aligned}$$

for gauge parameter $h \in H$. These type of models turns out to describe the $G/H$ coset CFT of [34].

Having given a quick glimpse of the main features of NLSM and having introduced some simple example we can now turn to the model which is of interest for this thesis: the *extended*-$\mathbb{CP}^{N-1}$ model.

## 3.2 The Basso-Rej sigma models

In this section we will introduce and describe the properties of a particular class of NLSM related to the $\mathbb{CP}^{N-1}$ complex manifold. Ordinary $\mathbb{CP}^{N-1}$ NLSM have been studied for long time [62, 63, 65] since they reproduce the main features of expected for 4-dimensional gauge theories: charge screening, asymptotic freedom and confinement, to list some of them. The $\mathbb{CP}^{N-1}$ models are known to be pertubatively renormalizable [66, 67] and asymptotically free. Moreover they have been analyzed in the large $N$ limit, in which the model is solvable [63, 65] and, similarly to the $O(2N)$ NLSM [68], they dynamically develop a mass scale. What is different in this case is that the fundamental excitations are confined by the long-range Coulomb potential induced non-perturbatively by the gauge field they are endowed with. The associated spectrum has a gap and the excitations fall into representation of the corresponding global symmetry group, namely $SU(N)$, cosetted by its center, $Z_N$ in this case.

From the non-pertubative point of view much less is known about the $\mathbb{CP}^{N-1}$ models beyond the large-$N$ limit. At the classical level, the model is integrable, as it possesses an infinite number of conserved charges [62], but these are spoiled

at the quantum level by the appearance of anomalies [70] and thus the integrability is broken by the quantization[1]. Nonetheless quantum integrability can be restored [71, 72] by the introduction of minimally coupled massless Dirac fermions. In [72] the authors were able to give arguments in favor of the restored quantum integrability and even to propose a candidate $S$-matrix, namely the minimal reflectionless $U(N)$-invariant $S$-matrix of [75]. Further, they give more robustness to the result by showing how the leading order large-$N$ expansion of the perturbative $S$-matrix agrees with the leading order of the integrable $S$-matrices proposed. Further support to the quantum integrability of the model were found in [76, 77]. An interesting minimal modification of the Koberle and Kurak models was recently proposed in [74] and will constitute the basis of our analysis.

### 3.2.1 The Basso-Rej model

The so-called *extended*-$\mathbb{CP}^{N-1}$ NLSM introduced in [74] are defined by the Lagrangian

$$L = \kappa \left(\partial_\mu - iA_\mu\right) \bar{z} \left(\partial^\mu + iA^\mu\right) z + i\bar{\psi}\gamma^\mu \left(\partial_\mu - iqA_\mu\right) \psi - \frac{\lambda}{2} \left(\bar{\psi}\gamma^\mu\psi\right)^2 , \qquad (3.9)$$

written in terms of:

- $z = (z_1, \ldots, z_N)$ : $SU(N)$ multiplet of complex bosons subject to the constraint $\bar{z}z = 1$.
- $\psi$ : massless Dirac fermion.
- $A_\mu$ : $U(1)$ Gauge Field.
- $1/\kappa$ : dimensionless coupling.
- $q$ : dimensionless fermion charge.
- $\lambda$ : dimensionless Thirring coupling controlling the fermion quartic self-interaction.

The *extended*-$\mathbb{CP}^{N-1}$ model enjoys the following symmetry:

[1] This is not true for the case $N = 2$ in which the model reduces to the $O(3)$ NLSM and hence is quantum-integrable [69, 119, 120, 19].

- $SU(N)$ **Rigid Symmetry** acting on the bosons $z$ and is related to the global isometry group of the symmetric space $\mathbb{CP}^{N-1}$.

- $U(1)$ **Gauge Symmetry** implemented by the presence of the (non-dynamic) gauge field $A_\mu$ and given explicitly by

$$\begin{aligned} A_\mu &\to A_\mu - \partial_\mu \omega \\ z &\to e^{i\omega} z \\ \bar{z} &\to e^{-i\omega} \bar{z} \,. \end{aligned} \tag{3.10}$$

  The lack of a kinetic term for the gauge field makes possible to integrate it out and plug back the result to reveal the highly non-linear nature of the model under consideration in term of the dynamic fields $z$ and $\psi$. The gauge field can be expressed as

$$A_\mu = i\bar{z}\partial_\mu z - \frac{q}{2\kappa}\bar{\psi}\gamma_\mu\psi \,. \tag{3.11}$$

  As unfolded by the large-$N$ analysis [65, 63] the gauge symmetry is non-perturbatively spontaneously broken. The effect is similar to what observed in the Schwinger model (i.e. euclidean 2-dimensional $QED$) [78]: the massless fermion is eaten up by the gauge field which acquires a mass and becomes dynamical. This lead to the screening of Coulomb interaction at long distances and the fundamental excitations, called *spinons*, are formed.

- $U(1)$ **Rigid Symmetry (Vector)** associated to the conservation of the fermionic number.

- $U(1)$ **Rigid Symmetry (Axial)** associated to the chiral rotation of the fermion. This is spoiled at the quantum level and breaks down to the discrete $\mathbb{Z}_{2k}$ which spontaneously breaks to $\mathbb{Z}_2$.

### 3.2.2 Perturbative renormalization

The model (3.9) comes endowed with three dimensionless parameters. The fermionic $U(1)$ charge is the only one which does not renormalize. To see this one can compute the Gauss law, namely

$$i2\kappa\bar{z}D_\mu z = q\bar{\psi}\gamma_\mu\psi \,, \tag{3.12}$$

where we immediately identify at l.h.s the $U(1)$ bosonic current $J_\mu = i2\kappa\bar{z}D_\mu z$. Relation (3.12) provides an identity between the bosonic and fermionic currents which, being physical observables, flow to a finite value in the UV limit. The charge $q$ plays the role of a constant of proportionality between the two currents and therefore should not renormalize.

The Thirring coupling $\lambda$ instead is a running coupling constant. To see this one should carefully compute the corresponding *1-loop* $\beta$-function, as done in [74] for instance. There we learn that to obtain the RG renormalization equation we have to consider the correlator of two fermionic currents $j_\mu$. The result of the analysis, in the $\overline{MS}$ renormalization scheme, boils down to the RG equations

$$\begin{aligned} \mu\frac{\partial\kappa}{\partial\mu} &= \frac{N}{2\pi} + O\left(\frac{1}{\kappa}\right), \\ \mu\frac{\partial\lambda}{\partial\mu} &= \frac{q^2}{4\pi\kappa^2} + O\left(\frac{1}{\kappa^3}\right), \\ \mu\frac{\partial q}{\partial\mu} &= 0, \end{aligned} \tag{3.13}$$

whose solutions reads

$$\kappa(\mu)|_{\text{1-loop}} = \frac{N}{2\pi}\log(\mu/\Lambda), \qquad \lambda(\mu)|_{\text{1-loop}} = \lambda_\infty - \frac{q^2}{2N\kappa}, \tag{3.14}$$

in terms of the renormalization scale $\Lambda$ and the UV Thirring coupling value $\lambda_\infty$. A quick look at the form of running bosonic coupling $\kappa(\mu)$ reveals that the model enjoys asymptotic freedom, as $\kappa \to 0$ in the UV limit.

Having established that (5.8) is a properly renormalizable model we can turn the attention to its integrability properties.

### 3.2.3 Classical integrability

As seen in Chapter 2 the classical integrability of a field theory relies on the existence of a non-abelian flat and conserved current $j^{ab}_\mu$, namely the current should satisfy

$$\partial_\mu j^{ab}_\nu - \partial_\nu j^{ab}_\mu + [j_\mu, j_\nu]^{ab} = 0, \qquad \partial_\mu j^{ab}_\mu = 0. \tag{3.15}$$

Such a current is then employed to construct the Lax connection

$$L_\mu(x) = \frac{1}{1-x^2}\text{Tr } J_\mu + \frac{x^2}{1-x^2}\epsilon_{\mu\nu}\text{Tr } J^\nu \,, \tag{3.16}$$

depending on a spectral parameter $x$ and which is flat by construction. With-the path-ordered exponential of the Lax connection we can eventually obtain the Monodromy matrix,

$$M(x) = P\exp\int d\sigma\, L_\sigma(x)\,. \tag{3.17}$$

The existence of the latter object guarantees the integrability of the model as can be expanded in $x$ to generate all the infinitely many non-abelian conserved charges.

In [74] the existence of such a flat and conserved current is discussed in details. The proof is slightly technical and we simply give a brief account of the idea. The most general conserved current can be constructed by linear combination of the conserved currents of the theory and a topological term [79, 80]. The former are given explicitly by the Noether procedure applied to the Lagrangian (3.9); $\mathcal{J}$, the current associated to the global $U(N)$ isometry, is then given by

$$\mathcal{J}^{ab}_\mu = \bar{z}^a D_\mu z^b - D_\mu \bar{z}^a z^b \,, \tag{3.18}$$

out of which we can extract the $U(1)$ component

$$\mathcal{J}'_\mu = \text{Tr}\mathcal{J}_\mu = 2\bar{z}D_\mu z\,. \tag{3.19}$$

Thus the most general conserved current can be written as the combination

$$j^{ab}_\mu = a\mathcal{J}^{ab}_\mu + b\mathcal{J}'_\mu\delta^{ab} + \epsilon_{\mu\nu}\partial^\nu f^{ab}\,, \tag{3.20}$$

with $f$ being an arbitrary matrix of functions. The general current (3.20), once plugged into (3.15), gives the anomalous flatness equation

$$\partial_\mu j^{ab}_\nu - \partial_\nu j^{ab}_\mu + [j_\mu, j_\nu]^{ab} = \epsilon_{\mu\nu}\sum_{i=1}^{4}\alpha_i O^{ab}_i - \epsilon_{\mu\nu}(1-a)\partial^\nu\partial_\nu f^{ab}\,, \tag{3.21}$$

where the explicit form of the coefficients $\alpha_i$ and of the operators $O^{ab}_\mu$, in terms of constants $a, b$ and the bosonic field $z$, can be found in [74]. By imposing the vanishing of the r.h.s. of (3.21) we obtain that the current (3.20) is flat for $c =$

$a = 1$ and for any value of $b$. The insensitiveness to the parameter $b$ is related to the fact that it controls the $U(1)$ part of the monodromy matrix which is not of interest here. To conclude the sketch of the proof we can then choose $b = 0$ and, by expanding the monodromy matrix (3.17) at large spectral parameter $x$, it is possible to generate the infinitely many non-abelian conserved charges of the model, matching those of [70].

#### 3.2.3.1 Towards quantum integrability

Having established the classical integrability of the model (3.9) we now turn our attention to the quantum version. Classical integrability may be, and it actually is in most cases, broken by quantization as the infinite conservation laws associated to the conserved charges may develop quantum anomalies. A elegant way of providing evidences for quantum integrability relies on the *counting argument* proposed in [81]. The underlying idea is as simple as powerful and we will give a sketch of it.

The starting point is the observation that the classical theory is scale invariant which implies, in light-cone coordinates, that the energy-momentum tensor conservation law takes the form $\partial_+ T_{--} = 0$. Thus it is easy to imagine how to construct an infinite series of conservation laws associated to the energy-momentum tensor, for instance it is trivial to verify that $\partial_+ (T_{--})^n = 0$, for any integer $n$. The broken scale invariance of the quantum model is then believed not to spoil these conservation laws but to deform them, at least for what concerns integrable models. As long as the deformation can be casted in the form

$$\partial_+ \left[ (T_{--})^n + F_1 \right] = \partial_- F_2 \,, \tag{3.22}$$

for some operators $F_1$ and $F_2$, the existence of a conserved charge is again ensured. In [81] it has been proposed that a correct way to determine quantum integrability will pass through the classification of all the possible anomalous term that can appear at the level of conservation laws. The next step is to classify all the admissible operator of the form $\partial_\pm(\cdots)$ modulo kinematic constraints and equations of motion. The spirit of the arguments goes then as follows: if the operators appearing in the two classifications match then this means that all the possible anomalies may be expressed as the divergence of some operator. Then at the quantum level, this will imply that any anomalous conservation law can be cast in the form (3.22), restoring the infinite number of conserved charges.

In [74] all the anomalies and divergences for two of the higher order conservation laws have been classified and the authors found that, in both cases, there's an unmatched anomaly for arbitrary values of the parameters. Considering the variety of models [81] that can be argued to be integrable according to this argument, the unmatched anomaly seems to signal that the integrability of (3.9) is lost at the quantum level. Nonetheless we will see in Chapter 5 how there exist an integrable subclass of (3.9), which can be obtained by fine-tuning the parameter of the model. This fine-tuning will require the use of non-perturbative techniques and will leave little doubt about the actual integrability of the fine-tuned subclass of models. These non-perturbative techniques, going under the name of *Thermodynamic Bethe Ansatz*, will be introduced in the next chapter.

# Chapter 4

# Thermodynamic Bethe Ansatz

In this chapter we describe the main aspects of the *Thermodynamic Bethe Ansatz* procedure as developed during the 90's by various groups and pioneered by A. B. Zamolodchikov [87, 88, 89, 90, 91, 92, 93].

Before entering the details of this approch we want to give a breif overview of the ideas underlying. We will focus on $1+1$-dimensional integrable QFT, in other words those for which the S-matrix factorises in two-particle scattering amplitudes and the absence of particle production is guaranteed, as discussed in Chapter 2.

We may look at integrable theories as particular perturbations of an underlying conformal theory. In this sense the infinite number of conserved quantities associated with the CFT can be combined in such a way to produce an infinite set of charges even for the perturbed theory. These theories, for which the perturbation is said *integrable*, can be analysed non-perturbatively by the TBA technique. The analysis performed within the TBA framework is precisely in this sense, for example one of the quantity which is accessible is related to the high energy RG flow of the theory in the parameter space. More precisely it is possible to compute the central charge of the conformal theory reached in the, so-called, UV limit.

The chapter is organized as follows. In Section 4.1 the TBA standard derivation is presented and applied to the most simple type of models, for instance the Lee-Yang model [95, 96]. In Section 4.2 the concept of Y-System, a system of functional equations which represents an alternative way of describing the system, is introduced; moreover the non-perturbative RG flow analysis is illustrated with

examples. In Section 4.3 more general TBA related to some types of Non-Linear Sigma Model are introduced and analysed.

## 4.1 The TBA Framework

The TBA analysis is a multi-step procedure which requires both physical intuition and sophisticated mathematical tools in order to be performed. In what follows we will develop the subject by starting with simple models, in the initial step for which the physical sense should not be spoiled by technical aspects, then, letting the models become more general, we will lead the discussion to the more advanced non-perturbative technique that can be applied in this framework.

### 4.1.1 The Bethe Wave Function

For the time being we will consider a $1+1$-dimensional integrable quantum field theory which possesses a factorizable S-matrix and moreover is defined on a compactified (length $L$) spatial dimension.

Since the glorious times of Perturbative QFT, it is well known that the *wave function formalism* cannot be applied to relativistic[1] theories. This fact is related to the effects of pair creation and virtual particles production, processes which force the theory to have a number of particle which is not fixed. Another way of looking at this is that the very definition of a *relativistic quantum particle* makes sense only in the asymptotic regions where the inter-particles distances are large, compared to the range of the interactions involved. In this conditions we can consider an asymptotic configuration composed of $N$ particles and associate to each of them a position $\{x_1, x_2, \cdots, x_N\}$. Such a situation can be described by means of a wave function $\Psi(x_1, x_2, \cdots, x_N)$, known as *Bethe Wave Function* and originally proposed in [97].

Now we can pick a specific particle $a$ and move it along the compactified space direction back to its original position. In doing so we clearly lose the wave function description whenever the $a$-particle is closed to one of the others, say of type $b$, since the interaction are then no longer negligible and the free particle pictures cannot

[1] Here the term *relativistic* is used to denote that we are considering quantum theories which are Lorentz invariant and are therefore intrinsically multiparticle theories.

be used. Nonetheless we know that for diagonal integrable scattering theories, such the one we are considering, the interaction is entirely encoded in the 2-particle S-matrix $S_{ab}(\theta_{ab})$. The amplitudes can be thought,at this stage, as numerical factors picked up by the wave function once the two particle are again well separated, and can be considered as free. Moreover the space is compactified and the wave function should be periodic in the variable $x_a$

$$\Psi(x_1, \cdots, x_a, \cdots, x_N) = \Psi(x_1, \cdots, x_a + L, \cdots, x_N)\,.$$

If we take the $a$-particle all around the circumference L and use the scattering data we end up with a consistency equation of the type

$$\Psi(x_1, \cdots, x_a + L, \cdots, x_N) = e^{iP_aL} \prod_{a \neq b} S_{ab}(\theta_{ab}) \Psi(x_1, \cdots, x_a, \cdots, x_N)\,,$$

which, in view of the periodicity of the wave function defined on a compactified spatial dimension, becomes a *quantization condition* for the particles rapidities

$$e^{iP_aL} \prod_{a \neq b} S_{ab}(\theta_{ab}) = 1\,. \tag{4.1}$$

This set of equations is known as *Bethe Ansatz* (BA) equations. The study of BA equations is indeed really tied with the study of integrable models as we will have appreciated if we had followed the customary approach to integrability. This very elegant and fruitful field of research has stemmed from the seminal work of A. Bethe on the 1 dimensional spin 1/2 chains [83]. Then the BA formulation culminated in the work of the St. Petersburg group [84] on the *Quantum Inverse Scattering Method*. Not having the possibility of covering this interesting topic we defer to the very good reviews and lecture notes on the algebraic BA and its application in relation to statistical systems [85]. From the mathematical point of view the BA procedure provides the analytic tool to perform the diagonalization of tensor products of matrices like the one appearing in (4.1). We will comment later on the various form of these equation in association to the various models we are going to meet.

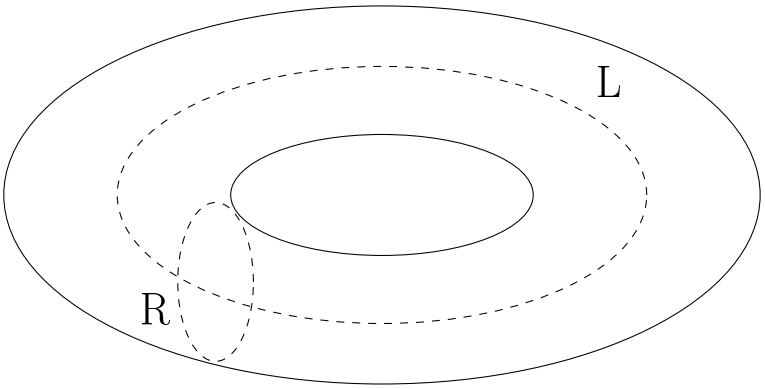



FIGURE 4.1: The TBA torus.

### 4.1.2 Mirror Thermodynamics

The next idea to deform the geometry of the space-time, i.e. the cylinder, not by bending it but by deforming the boundaries conditions: we compactify the time direction. From the geometrical point of view we can imagine of generating the cylinder on which the theory is defined as the limit of a torus with increasing external radius. Letting $L$ and $R$ be the generating circumferences, as in Figure 4.1 we can know make use of relativistic invariance and decide which direction to pick up as time. First we can pick the $L$-direction as time and send it to infinity keeping $R$ finite. Denoting with $H_R$ the Hamiltonian of the quantum system defined on the periodic compactified space direction $R$ and by $\mathcal{H}_R$ its Hilbert space, we have that the partition function associated to the field theory $\mathcal{Z}(R, L)$ is, in the $L \to \infty$ limit, dominated by

$$\mathcal{Z}(R, L) \approx \lim_{L\to\infty} \mathrm{Tr}_{\mathcal{H}_R} e^{-H_R L} \approx e^{-E_0(R)L}\,, \tag{4.2}$$

$E_0(R)$ being the *Casimir Energy* of the ground state of the theory which represents the dominating term in the partition function is such a limit.

Alternatively, we can consider the $R$-direction as the time and send $L$ to infinity anyway. In this case we end in a theory defined on a compactified time direction. We can then interpret the compactification length $R$ as the inverse of the temperature at which we are placing the theory. This is the *mirror thermodynamic* picture and will be provided of an Hamiltonian $H_L$ acting on an Hilbert space $\mathcal{H}_L$. In this setup the partition function is dominated by

$$\mathcal{Z}(R, L) \approx \lim_{L\to\infty} \mathrm{Tr}_{\mathcal{H}_L} e^{-H_L R} \approx e^{-RLf(R)} \tag{4.3}$$

where $f(R)$ is the free-energy per unit length of the system at finite temperature $T = 1/R$. By comparing equation (4.2) with (4.3) we then have

$$E_0(R) = Rf(R) \tag{4.4}$$

Last equation enables to access information about the underlying CFT by performing the UV limit $R \to 0$. Since at the UV point the theory should reduce to a CFT we can employ a result [98] about CFT to state that

$$\lim_{R\to 0} Rf(R) = -\frac{\pi}{6R} c_{\text{eff}}\,, \qquad\qquad c_{\text{eff}} = c - 24\Delta\,, \tag{4.5}$$

where $c_{\text{eff}}$ is the so-called the *effective central charge* of the underlying conformal field theory, i.e. the difference between $c$, the central charge of the underlying conformal field theory, and $\Delta$, the engineering dimension of the lowest operator[2]. Therefore, the introduction of periodic boundary conditions in a unitary CFT, see equation (1.19), has the effect of generating a non-zero value for the ground state energy in the UV-limit and in this sense the effective central charge can be interpreted as Casimir energy.

### 4.1.3 The TBA approach

The most simple example of how the *TBA procedure* works is represented by a theory in which there is just one kind of fundamental scalar excitations. Scattering amplitudes are thus ordinary functions $S(\theta)$ and the momentum quantization condition for $N$ particle lying on a cylinder of (spatial) circumference $L$ is given by

$$e^{ip_i L} \prod_{j\neq i} S(\theta_i - \theta_j) = 1 \qquad i = 1, 2, \ldots, N\,. \tag{4.6}$$

Here $p_i = m \sinh\theta_i$ is the relativistic momentum of the $i$-th particle. Parametrizing $S(\theta) = \exp\left(i\Delta(\theta)\right)$ and taking the logarithm we obtain

$$mL \sinh\theta_i + \sum_{j\neq i} \Delta(\theta_i - \theta_j) = 2\pi n_i\,, \tag{4.7}$$

[2]This quantity coincides with the Virasoro central charge for unitary models, the ones for which the dimension $\Delta$ is zero.

which is a more explicit expression since we introduced some sort of *occupation numbers* $n_i$ which describe the allowed states. Each possible state is thus characterized by the set of rapidities that solve (4.7) for each choice of the occupation numbers. The energy of the system is given by

$$H = \sum_{i=1}^{N} m \cosh \theta_i \,, \tag{4.8}$$

since the theory is relativistic.

The formidable analytical difficulty of the problem calls for some physical idea. We can analyse the system in the *thermodynamic limit*

$$L \to \infty \qquad N \to \infty \qquad N/L = \text{fixed} \,. \tag{4.9}$$

In this limit the growing number of particles forces the spectrum of rapidities to condense and the spacing between adjacent rapidities scales as $\theta_i - \theta_{i-1} \sim 1/mL$. It makes sense to introduce a continuous function by considering a certain number of particle $k_i$ whose rapidities lie in the interval $[\theta_{i-1}, \theta_i]$ and taking the thermodynamic limit; this translates in the definitions

$$\bar{\rho}(\theta) = \lim_{Th} \frac{k_i - k_{i-1}}{\theta_i - \theta_{i-1}} \,, \tag{4.10}$$

for the *density of occupied states* and

$$\rho(\theta) = \lim_{Th} \frac{n_i - n_{i-1}}{\theta_i - \theta_{i-1}} \,. \tag{4.11}$$

for the *density of allowed states*.

These density are useful to estimate the discrete sums over rapidities as integrals over the occupied states density

$$\lim_{Th} \sum_{i=1}^{N} f(\theta_i) = \int d\theta \, \rho(\theta) f(\theta) \tag{4.12}$$

Relation (4.12) can be used, for instance, to convert the energy of the system (4.8) into a functional of the occupation density

$$H \to H[\bar{\rho}] = \int d\theta \, \bar{\rho}(\theta) \cosh \theta \,. \tag{4.13}$$

The thermodynamic limit of (4.7) can be obtained by first recasting the sum as an integral

$$mL\sinh\theta_i + \int d\theta'\,\bar{\rho}(\theta')\Delta(\theta_i-\theta') = 2\pi n_i\,,$$

then subtracting the equation for $i-1$ to the previous, dividing by $\theta_i-\theta_{i-1}$

$$mL\frac{\sinh\theta_i-\sinh\theta_{i-1}}{\theta_i-\theta_{i-1}} + \int d\theta'\,\bar{\rho}(\theta')\frac{\Delta(\theta_i-\theta')-\Delta(\theta_{i-1}-\theta')}{\theta_i-\theta_{i-1}} = 2\pi\frac{n_i-n_{i-1}}{\theta_i-\theta_{i-1}}\,.$$

Eventually we can take the thermodynamic limit to obtain

$$mL\cosh\theta + \Big(\varphi * \bar{\rho}\Big) = 2\pi\rho(\theta) \tag{4.14}$$

where

$$\Big(\varphi * \bar{\rho}\Big) = \int d\theta'\,\bar{\rho}(\theta')\,\varphi(\theta-\theta')\,, \qquad \varphi(\theta) = \partial_\theta\Delta(\theta)\,. \tag{4.15}$$

Now it is all about thermodynamics. In fact the equilibrium distribution is the one that extremize the free energy functional $E-TS$. We can safely choose the functional (4.13) as a good candidate for the energy $E$. But two key ingredients are still not specified: the temperature and the entropy. By considering the theory as defined on a compactified imaginary time direction of circumference $R$ we are led to interpret $1/R$ as the temperature. For what concerns the entropy we consider the number of different distributions in the interval $\Delta\theta_i$. In the "fermionic" case it is given by

$$\frac{k_i!}{n_i!\,(k_i-n_i)!}$$

while in the "bosonic" we have

$$\frac{(k_i+n_i+1)!}{(n_i)!\,(k_i-1)!}\,.$$

In the limiting behavior $L\to\infty$ these numbers become functionals $N[\rho,\rho_1]$ related to the entropy by $S[\rho,\rho_1]=\log N[\rho,\rho_1]$. We explicitly have

$$S_F[\rho,\bar{\rho}] = \int d\theta\;\left[\rho\log\rho-\bar{\rho}\log\bar{\rho}-(\rho-\bar{\rho})\log(\rho-\bar{\rho})\right]\,,$$

$$S_B[\rho,\bar{\rho}] = \int d\theta\;\left[(\rho+\bar{\rho})\log(\rho+\bar{\rho})-\rho\log\rho-\bar{\rho}\log\bar{\rho}\right]\,.$$

**Fermionic case:** The free energy must be minimized under the constraint of the Bethe equations thermodynamic limit. To this purpose we introduce a Lagrange

multiplier $\lambda(\theta)$ to minimize

$$\begin{aligned}\Phi[\rho,\bar\rho,\lambda] = RF[\rho,\bar\rho] + \Lambda[\rho,\bar\rho,\lambda] = RH[\bar\rho] - S_F[\rho,\bar\rho] + \Lambda[\rho,\bar\rho,\lambda] = \\ = \int d\theta\,[mR\,\bar\rho\cosh\theta - \rho\log\rho + \bar\rho\log\bar\rho + (\rho-\bar\rho)\log(\rho-\bar\rho) + \\ + \lambda\left(2\pi\rho - mL\cosh\theta - \left(\varphi * \bar\rho\right)\right)\Big]\end{aligned}$$

The extremum configuration is then obtained by solving

$$\frac{\delta\Phi}{\delta\rho} = -\log\frac{\rho}{\rho-\bar\rho} + 2\pi\lambda = 0\,, \tag{4.16}$$

$$\frac{\delta\Phi}{\delta\bar\rho} = mR\cosh\theta + \log\frac{\bar\rho}{\rho-\bar\rho} - \left(\varphi * \lambda\right) = 0\,, \tag{4.17}$$

$$\frac{\delta\Phi}{\delta\lambda} = 2\pi\rho - mL\cosh\theta - \left(\varphi * \bar\rho\right) = 0\,. \tag{4.18}$$

Without employing equation (4.18), we can solve for $\lambda$ equation (4.16) and plug it into (4.17) to obtain

$$mR\cosh\theta + \log\frac{\bar\rho}{\rho-\bar\rho} - \left(\varphi * \log\left(1+\frac{\bar\rho}{\rho-\bar\rho}\right)\right) = 0\,,$$

which, with the aid of a more suitable variable the so-called *pseudoenergy*

$$e^{-\varepsilon} = \frac{\bar\rho}{\rho-\bar\rho}\,, \tag{4.19}$$

becomes the usual *Thermodynamic Bethe Ansatz Equation*

$$\varepsilon(\theta) = mR\cosh\theta - \frac{1}{2\pi}\left(\varphi * \log\left(1+e^{-\varepsilon}\right)\right). \tag{4.20}$$

We want to emphasize that, up to now, no explicit use of the constraint has been made. It comes at hand once we want to write the maximum value of the free

energy (4.4) (i.e. the Casimir Energy of the relativistic theory)

$$
\begin{aligned}
E(R) &= \frac{RF[\rho,\bar\rho]}{L} = \\
&= \frac{1}{L}\int d\theta\,[mR\,\bar\rho\cosh\theta - \rho\log\rho + \bar\rho\log\bar\rho + (\rho-\bar\rho)\log(\rho-\bar\rho)] = \\
&= \frac{1}{L}\int d\theta\left[mR\,\bar\rho\cosh\theta - \rho\log\frac{\rho}{\rho-\bar\rho} + \bar\rho\log\frac{\bar\rho}{\rho-\bar\rho}\right] = \\
&= \frac{1}{L}\int d\theta\left[\log\frac{\rho}{\rho-\bar\rho}\left(-\rho + \frac{1}{2\pi}\Big(\varphi * \bar\rho\Big)\right)\right] = \\
&= -\frac{m}{2\pi}\int d\theta\cosh\theta\,\log\frac{\rho}{\rho-\bar\rho}\,,
\end{aligned}
$$

which in terms of the pseudoenergy becomes

$$
E(R) = -\frac{m}{2\pi}\int d\theta\cosh\theta\log\left(1+e^{-\varepsilon(\theta)}\right)\,. \tag{4.21}
$$

**Bosonic case:** We decided to go through some details also in bosonic case in order to show a perhaps more quicker but less formal method. The quantity we want to maximize is represented by

$$
\begin{aligned}
f[\rho,\bar\rho] &= RF[\rho,\bar\rho] = RH[\bar\rho] - S_B[\rho,\bar\rho] = \\
&= \int d\theta\,[mR\,\bar\rho\cosh\theta - (\rho+\bar\rho)\log(\rho+\bar\rho) + \rho\log\rho + \bar\rho\log\bar\rho]\,,
\end{aligned}
$$

and we can evaluate the first order variation $\delta f = f[\rho+\delta\rho,\bar\rho+\delta\bar\rho] - f[\rho,\bar\rho]$, which is given by

$$
\delta f = \int d\theta\left[\left(mR\,\cosh\theta + \log\frac{\bar\rho}{\rho+\bar\rho}\right)\delta\bar\rho + \log\frac{\rho}{\rho+\bar\rho}\delta\rho\right]\,.
$$

Thus by setting $\delta f = 0$ we will obtain the equation corresponding to the bosonic case. To do this we have to take also into account the first order variation of the constraint, which is given by $2\pi\delta\rho = \Big(\varphi * \bar\rho\Big)$ and exchanging the order of convolution we find the TBA equation in the bosonic case

$$
mR\,\cosh\theta + \log\frac{\bar\rho}{\rho+\bar\rho} + \frac{1}{2\pi}\Big(\varphi * \log\frac{\rho}{\rho+\bar\rho}\Big) = 0\,,
$$

which, by defining a bosonic pseudoenergy

$$
e^{-\varepsilon} = \frac{\bar\rho}{\rho+\bar\rho}\,, \tag{4.22}
$$

becomes

$$\varepsilon(\theta) = mR\cosh\theta + \frac{1}{2\pi}\Big(\varphi * \log\left(1 - e^{-\varepsilon}\right)\Big) = 0\,. \tag{4.23}$$

The corresponding Casimir energy is found to be in this case

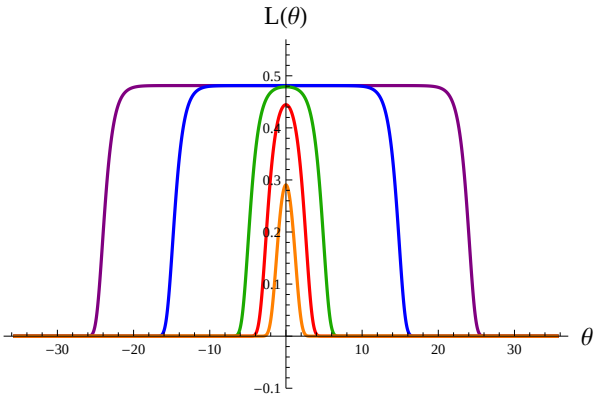
$$E(R) = \frac{m}{2\pi}\int d\theta\,\cosh\theta\,\log\left(1 - e^{-\varepsilon(\theta)}\right)\,. \tag{4.24}$$

The general structure of the TBA equations can be seen immediately from equations (4.20) and (4.23). Indeed the TBA procedure of implementing the mirror thermodynamic for an integrable relativistic theory on a finite-size geometry ultimately result in a non-linear integral equation for the unknown pseudoenergy $\varepsilon(\theta)$. As we will see later on this is the key mathematical aspects of TBA description the topic of integral equations is very well studied in mathematics since XVIII century. Having developed a formalism capable of condensing the information of a whole scattering theory, although a not so common scattering theories as integrable ones turn out to be, into a single integral equation is something remakable which allows the exact solution of integrable QFT. With the term exact we mean that the result obtainable, as we will see, are genuinely non-perturbative. Moreover, the TBA equations naturally incorporates the RG scale at which the theory is considered and allow for non-trivial RG flow analysis in term of non-perturbative computations. Establishing the way in which the TBA allows to follow the RG flow is the aim of next section.

### 4.1.4 UV limit and kink limit

We immediately realize that the dimensionful parameters of the problem, namely the mass scale $m$ and the finite-size length $R$, appear only in the dimensionless combination $r = mR$. For dimensional considerations we can then safely assume that the quantities relevant in the asymptotic regions will be functions of $r$ alone. The parameter $r$ will allow us to explore the RG flow regimes. In the deep UV region, for instance, the mass scale goes to zero and thus we can alternatively send $r$ to zero: the TBA equations behave in the same way. Unfortunately this limit is not safe to take straightforwardly because the vanishing of $r$ will imply in turn the vanishing of the driving terms resulting in a ill-defined set of equations.

Keeping $r$ finite but small and analyzing the behavior of solutions reveals that they develop a central region in which the solution is almost constant. It is customary to denote this region with the name *plateau region* and the constant values by *plateau values*. Moreover the width of the plateau grows as $r$ becomes smaller and smaller, see Figure 4.2, and the plateau region can be estimated to be roughly $[-\log(2/r), \log(2/r)]$.




| r | $c_{\text{eff}}$ (TBA) | Error | Color |
|---|---|---|---|
| $1.\times10^{-10}$ | 0.8 | 0. | |
| $1.\times10^{-6}$ | 0.7999999999994495 | $5.5\times10^{-13}$ | |
| 0.02 | 0.7998078836893896 | $1.9\times10^{-4}$ | |
| 0.2 | 0.7850829987007197 | $1.5\times10^{-2}$ | |
| 0.9 | 0.6153008497735487 | 0.18 | |

FIGURE 4.2: The profile of function $L(\theta)$ for several values of $r$. In the table, a numerical comparison of the relative effective central charge and the theoretical value $c_{th} = 4/5$ and the legend.

It is possible to get analytical access to UV limit by performing a simple manipulation. Namely we can hide the presence of $r$ by shifting the functions in such a way to send the edge of the falloff to the origin and to keep only the part of the driving term which will contributes in the, say, $\theta \gg 0$ regime. Doing this results in the so-called *kink limit*, or *massless limit*, of the TBA equations, namely

$$\epsilon_{\text{kink}}(\theta) = e^{\theta} - \Big(\varphi * L_{\text{kink}}\Big)(\theta) \tag{4.25}$$

where the *kink* quantities are defined by shifting the usual ones

$$\begin{aligned} \epsilon_{\text{kink}}(\theta) &= \epsilon(\theta + \log(2/r))\,, \\ L_{\text{kink}}(\theta) &= L(\theta + \log(2/r))\,. \end{aligned} \tag{4.26}$$

These shifts implies as well a modification for the Casimir energy of the model but we want to present a more convenient function, namely the *finite size scaling function*

$$c(r) = -\frac{6}{\pi} R E_0(R) = \frac{3r}{\pi^2} \int L(\theta) \cosh\theta \, d\theta\,, \tag{4.27}$$

which depends only on the dimensionless parameter $r$ and is governed in the $r \to 0$ limit by the solution of (4.25). The limit can now be taken safely and we are left with

$$c_{\text{eff}} = \lim_{r\to 0} c(r) = \frac{3}{\pi^2} \int L_{\text{kink}}(\theta)\, e^{\theta}\, d\theta\,,$$

where, as in (4.5), $c_{\text{eff}} = c - 24\Delta$.

Before entering the details of the exact and non-perturbative calculation of the RG flow UV limit we stop the TBA procedure for a while to generalize the concept developed so far and to introduce a way of converting the typical integral equations which arise in this context into a system of functional equations.

## 4.2 Universal TBA's

Since for many integrable systems the analytical form of the 2-body S-matrix is known, whether it is proved or conjectured, large efforts had been made by the comunity to carry out the TBA program for a great number of $1+1$-dimensional integrable systems [87, 88, 99, 100]. In this section we want to address to a particular class of systems, namely the integrable perturbations of some CFT related to the $A, D$ and $E$ affine Lie algebras. We will not enter in a detailed derivation of the scattering amplitudes and for a complete analysis we defer to the seminal papers on the topic [88, 89, 90, 91]. We stress that in this situation the scattering matrix is diagonal, a feature which will allow for a very elegant final form of the equations.

For the time being we can simply denote by $S_{ab}(\theta)$ the scattering amplitude for the scattering a particle of type $a$ with one of type $b$ at rapidity $\theta = \theta_1 - \theta_2$. With this writing we refer to the $S$-matrix which have been computed exactly for the aforementioned models that can be found in [88, 101].

### 4.2.1 ADET TBA's

The TBA analysis, including the standard RG flow analysis, for these type of systems is carried out in [89] and the result can be compactly written as a system of non-linear integral equations some unknown *pseudoenergies* $\epsilon_a(\theta)$, where $a = 1, \cdots, n$ and $n$ is the rank of the associated algebra and $\theta$ is the rapidity of the particle. Owing to the symmetry properties of $S_{ab}$ each system is naturally mapped into its Dynkin diagram and we can associate to each node the corresponding pseudoenergy. From now on we will thus perform the analysis with a generic group, denoted by $G$, in the families $A, D$ and $E$. To each node of diagram we also associate a *driving term* $\nu_a(\theta) = m_a R \cosh\theta$ in which the masses $m_a$ are organized in the so-called Perron-Frobenius eigenvector[3] of $G$. With these premises the *TBA physical form* reads

$$\epsilon_a(\theta) = \nu_a(\theta) - \frac{1}{2\pi} \sum_b \Big(\phi_{ab} * L_b\Big)(\theta)\,. \tag{4.28}$$

The corresponding function $c(r)$ is given by

$$c(r) = \frac{3}{\pi^2} \sum_a \int \nu_a(\theta) L_a(\theta)\, d\theta \tag{4.29}$$

where, as standard,

$$L_b(\theta) = \log\left(1 + e^{-\epsilon_b(\theta)}\right)\,, \qquad \phi_{ab}(\theta) = -i\frac{d}{d\theta}\log S_{ab}(\theta)\,.$$

Thanks to the very constrained form of the S-matrices involved it has been shown by several authors, see [90, 101] for instance, how these type of equations in which the form of the kernels involved varies from node to node being related to the phase of the corresponging physical $S$-matrix, can be brought into a *universal form*. The

[3] For a generic Lie algebra the Perron-Frobenius eigenvectors is an eigenvector of the incidence matrix with all positive entries. It can be proven to be unique and to correspond to the highest eigenvalue.

latter form involves just a single type of kernel and encodes its model-dependency only in the coupling among the nodes.

The key relation in this direction is represented by the following matrix identity

$$\Big(\delta_{ab} - \frac{1}{2\pi}\hat{\phi}_{ab}(\omega)\Big)^{-1} = \delta_{ab} - \frac{1}{2\cosh(\pi\omega/g)} G_{ab}\,, \tag{4.30}$$

in terms of the Fourier Transform of the scattering kernel

$$\hat{\phi}_{ab}(\omega) = \int d\theta\, e^{i\omega\theta}\phi_{ab}(\theta)\,,$$

and having incidentally made use of some group theoretical quantities, namely the *Dual Coxeter Number* $g$ and the *Coxeter Adjacency Matrix* $G = 2 - C$. The latter is an integer-valued matrix which encodes the Dynkin diagram structure of the *simple* Lie algebras, essentially it has a 1 at position $(i, j)$ if the node $i$ and $j$ are linked and a 0 otherwise.

Without entering the details of the proof of relation (4.30) we can get some confidence by evaluating it at $\omega = 0$ to obtain the well known identity [88, 89]

$$N = C(2 - C)^{-1}\,,$$

between the normalization matrix of the kernels and the Cartan matrix of the corresponding algebra $G$. Moreover,and most importantly, by applying (4.30) to the physical TBA equations (4.28) we obtain the celebrated *TBA Universal Form*

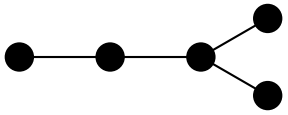
$$\epsilon_a(\theta) = \nu_a(\theta) - \frac{1}{2\pi}\sum_b G_{ab}\Big(\phi_h * (\nu_b - \Lambda_b)\Big)(\theta)\,, \tag{4.31}$$

with $\Lambda_a = \log(1 + e^{\epsilon_a})$. As anticipated all the convolutions are taken with the same universal kernel

$$\phi_g(\theta) = \frac{g}{2\cosh(g\,\theta/2)}\,, \tag{4.32}$$

which depends on $g$, the Coxeter number of the algebra $G$, and the coupling informations are carried by the adjacency matrix $G_{ab}$. As a matter of notation the massive nodes are drawn as black circles and the presence of a convolution in equation for node $i$ with the node $j$ is represented as link between the nodes. Since this structure is encoded in the adjacency matrix there's no suprise if the

TBA diagram matches the one of the corresponding Lie algebra. To be explicit TBAs like (4.31) for the choice $G = D_5$, for instance, is encoded in the following diagram

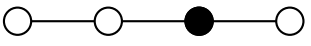
$D_5$ 

$$G_{D_5} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Another choice for the distribution of masses is possible. This was proposed for the firts time in [91, 92, 93] to describe the RG flows of minimal models perturbed by their least relevant operator $\phi_{13}$. The energy terms can be choosen as $\nu_a(\theta) = \delta_{ak} mR \cosh\theta$, for fixed $k$. In this case, denoted conventionally by $(G)_k$, the TBA universal form is given by

$$\epsilon_a(\theta) = \nu_a(\theta) - \frac{1}{2\pi} \sum_b G_{ab} \Big( \phi_2 * L_b \Big)(\theta) \,. \tag{4.33}$$

The massless nodes which do not carry energy nor momentum are usually called *magnons* in this context. Their presence is a relic of the auxiliary rapidities that have been introduced to diagonalize the color part of the trasfer matrix in the Bethe Ansatz technique. The only difference from the diagrammatic point of view is that magnonic nodes are drawn as white circles in order to distinguish them. As an illustration we display the situation for the $(A_4)_3$ case

$(A_4)_3$ 

$$G_{A_4} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

### 4.2.2 Y-Systems

One of the most remarkable facts of equations like (4.33,4.31) is that we can cast the integral form of the equation into a set of *functional equations*, the so-called

Y-system. This recasting requires a careful analysis since it involves integrals performed on the complex plane and thus the analytic continuation of the pseudoenergies. In order to shift properly the contour of integration one has to know the exact location of pole and zeros of the functions all over the complex plane and this might be a difficult task. To avoid complications we can perform the calculation for the ground state which is known to posses an analyticity strip $|\mathrm{Im}\theta| < \pi$ free of zero and poles. Once the complex shifts of the kernels are properly evaluated one is left with the following set of functional equations

$$Y\left(\theta + i\frac{\pi}{g}\right) Y\left(\theta - i\frac{\pi}{g}\right) = \prod_b \left(1 + Y_b(\theta)\right)^{G_{ab}} \tag{4.34}$$

for the Y-functions $Y_a(\theta) = e^{\epsilon_a(\theta)}$.

Despite its derivation, rigorously proved for the ground state, the Y-system's (4.34) validity is extended to all the states of the theory. This is reasonable since the choice of a state of the theory amounts, in the TBA language, to the analyticity and asymptotic considitions imposed on the Y-functions. We can clarify the role of the asymptotic conditions by making a comparison with respect to the theory of partial differential equations. In both context the equations give a general solution which becomes unique only when asymptotics (or boundary or initial value) conditions are supplied.

### 4.2.3 RG Flows and Rogers Dilogarithm

We can now carry over the TBA analysis with aim of reaching the UV point of the RG flow. The analytic access to UV quantities is strictly related to the appearance of the *Rogers Dilogarithm function*, see Appendix A for more details. This function is ubiquitous in TBA analysis and encodes the UV quantities thanks to the so called *plateau values* as we will see later on. The latter are a set of *stationary* solution to the Y-system, or alternatively to the TBA integral equations, which are used as arguments for the Rogers Dilogarithm to produce, for instance, the UV central charge thanks to some very specific *sum rule*.

These type of sum rules have been studied intensely both from the mathematical [155, 102, 145] and physical [52, 101, 107] point of view. The most simple cases can be treated analitically since for them the plateau value are exactly known but

for the general cases one has to resort to numerical analysis, which are usually extremely accurate, in order to conjecture the proper value for the sum rule.

To see how these type of calculations are performed we can consider the TBA for the $(A_{n+1})_1$ model of Section 4.2.1, see Figure 4.3 for the corresponding diagram. The explicit form of the TBA can be read from equation (4.33) by choosing the

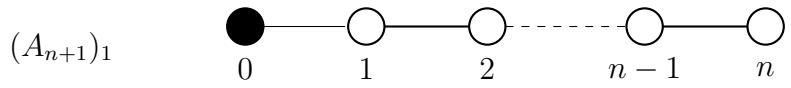



FIGURE 4.3: Dynkin diagram for the family of models $(A_{n+1})_1$ of the *ADE* type.

appropriate parameters. What is relevant is instead the kink, or massless, limit of the TBA equations. As discussed in Section 4.1.3, the kink limit is a way to safely perform the $r \to 0$ limit which would be otherwise ill-defined. Thus, shifting the rapidities by the quantity $\log(2/r)$ we can get rid of the explicit dependence from $r$ in the equations. Selecting one type of "chiral" kinks amounts in the choice of the sign of the exponential appearing as driving term. With these choices the kink limit[4] of the TBA equation is given by

$$\epsilon_0(\theta) + \Big(\varphi * L_1\Big)(\theta) = e^{\theta}\,, \tag{4.35}$$

$$\epsilon_k(\theta) + \Big(\varphi * L_{k-1}\Big)(\theta) + \Big(\varphi * L_{k+1}\Big)(\theta) = 0\,, \tag{4.36}$$

$$\epsilon_n(\theta) + \Big(\varphi * L_{n-1}\Big)(\theta) = 0\,, \tag{4.37}$$

where $k = 1, 2, \ldots, n-1$ and $\varphi(\theta) = \phi_{n+2}(\theta)$ is the universal kernel since $g = \mathrm{cox}(A_{n+1}) = n+2$, see (4.32). Remarkably the calculation is pretty much insensitive to the specific form of the kernel involved, at least for what concern the appearance of the Rogers Dilogarithm. The importance of the kernel will emerge in determining the plateau values which will be ultimately responsible for the sum rule value.

[4] We dropped the suffix in kink quantities to enlighten the notation.

For computational purposes, it will be useful to take a derivative of equations (4.35-4.37), we get[5]

$$\epsilon_0'(\theta) + \Big(\varphi * L_1'\Big)(\theta) = e^\theta \,, \tag{4.38}$$

$$\epsilon_k'(\theta) + \Big(\varphi * L_{k-1}'\Big)(\theta) + \Big(\varphi * L_{k+1}'\Big)(\theta) = 0 \,, \tag{4.39}$$

$$\epsilon_n'(\theta) + \Big(\varphi * L_{n-1}'\Big)(\theta) = 0 \,, \tag{4.40}$$

where we used

$$L_m' = \frac{d}{d\theta} L_m(\theta) = -\frac{1}{1+e^{\epsilon_m}} \frac{d\epsilon_m}{d\theta} \,.$$

According to (4.29) and (4.5) the effective central charge can now be written as

$$c_{\text{eff}} = \frac{6}{\pi^2} \int d\theta \, e^\theta L_0(\theta) \,,$$

which is exactly evaluable by means of some identity concerning Rogers Dilogarithms. In order to extract explicitly the integral representation for these special function we need some preliminary manipulations.

**Observation:** The peculiar asymptotic behavior displayed by $\epsilon_0$ is given by

$$\epsilon_0(\theta) = \begin{cases} \text{const.} + \dots & , \quad \theta \to -\infty \\ e^\theta + \dots & , \quad \theta \to +\infty \end{cases} \,, \tag{4.41}$$

while, on the other side, magnonic nodes will be characterized by a constant behavior in both limits. The asymptotics (4.41) allows to integrate by parts "half" of the integrand without picking up a finite contribution

$$I = \int d\theta \, e^\theta L_0(\theta) = \frac{1}{2} \int d\theta \, \left( L_0 e^\theta - L_0' e^\theta \right) \,, \tag{4.42}$$

since the boundary term $[L_0 e^\theta]_{-\infty}^{+\infty}$ vanishes.

[5]The convolution product enjoys the property $\frac{d}{dx}\Big(f * g\Big)(x) = \Big(f' * g\Big)(x) = \Big(f * g'\Big)(x)$

By substituting the right hand sides of equations (4.35) and (4.38) in (4.42) we come to

$$
\begin{aligned}
I &= \frac{1}{2}\int d\theta\, \left(L_0 e^{\theta} - L_0' e^{\theta}\right) = \\
&= \underbrace{\frac{1}{2}\int d\theta\, \left(L_0\epsilon_0 - L_0'\epsilon_0'\right)}_{I_0} + \frac{1}{2}\int d\theta\, \Big[L_0\Big(\varphi * L_1'\Big) - L_0'\Big(\varphi * L_1\Big)\Big]\,.
\end{aligned}
$$

Exchanging the convolution order[6] and using $n-1$ times equations (4.36) and (4.39) with $k = 1, 2, \ldots, n-1$ we get

$$
\begin{aligned}
I &= I_0 + I_1 + \frac{1}{2}\int d\theta\, \Big[L_1\Big(\varphi * L_2'\Big) - L_1'\Big(\varphi * L_2\Big)\Big] = \\
&= I_0 + I_1 + I_2 + \frac{1}{2}\int d\theta\, \Big[L_2\Big(\varphi * L_3'\Big) - L_2'\Big(\varphi * L_3\Big)\Big] = \cdots = \\
&= \sum_{k=0}^{n-1} I_k + \frac{1}{2}\int d\theta\, \Big[L_{n-1}\Big(\varphi * L_n'\Big) - L_{n-1}'\Big(\varphi * L_n\Big)\Big] = \sum_{k=0}^{n} I_k\,,
\end{aligned}
$$

where in the last step we made use of equations (4.37) and (4.40). Explicitly we obtain

$$
I = \sum_{k=0}^{n} I_k = \sum_{k=0}^{n} \frac{1}{2}\int_{-\infty}^{+\infty} d\theta \left[\log\left(1 + e^{-\epsilon_k(\theta)}\right) + \frac{\epsilon_k(\theta)}{1 + e^{\epsilon_k(\theta)}}\right]\frac{d\epsilon_k}{d\theta}\,.
$$

Now we can trade the rapidity variable for a more suitable integration variable, namely $t = 1/(1 + e^{\epsilon_k(\theta)})$, and, denoting by $T_{\pm}^{(k)}$ its asymptotic values, we obtain the fundamental result [91]

$$
I_k = \frac{1}{2}\int_{T_-^{(k)}}^{T_+^{(k)}} dt \left[\frac{\log(1-t)}{t} + \frac{\log t}{1-t}\right]\,,
$$

What obtained so far can be rewritten in terms of the *Rogers Dilogarithm function*, see Appendix A, which we recall for convenience

$$
\mathcal{L}(x) = -\frac{1}{2}\int_0^x \left[\frac{\log(1-t)}{t} + \frac{\log t}{1-t}\right] dt \quad , \quad 0 < x < 1\,.
$$

The final result can then be written, in terms of

$$
I_k = \mathcal{L}\left(T_-^{(k)}\right) - \mathcal{L}\left(T_+^{(k)}\right)\,,
$$

[6] Namely $\int dx\, f(x)\Big(g * h\Big)(x) = \int dx\, h(x)\Big(g * f\Big)(x)$, which is true assuming $g(x) = g(-x)$.

as

$$I = I_- - I_+ = \sum_{k=0}^{n} \mathcal{L}\left(T_-^{(k)}\right) - \sum_{k=0}^{n} \mathcal{L}\left(T_+^{(k)}\right) .$$

We immediately realize that a neat pattern has emerged. The quantity $I$, related to the effective central charge by $c = 6/\pi^2 I$, is splitted in two contributions and each of them is a *Rogers Dilogarithm sum rule*. The latter is inteded in the sense of a sum of Rogers dilogarithms, evaluated in particular values which solve a certain, in most cases highly non-linear, algebraic equation. The change of variable we used defines these quantities for us

$$T_\pm^{(k)} = \frac{1}{1 + e^{\epsilon_k(\pm\infty)}}$$

The defining equations for the last quantities are then given by the asymptotic behavior of the TBA equations (4.35-4.37) which are different in the two asymptotic regions. A central role is played by the magnonic nodes for which we saw earlier that the asymptotics were not those of (4.41) but rather are characterized by pseudoenergies whose profiles interpolate between two asymptotic finite plateau values. In this way we see that the terms $T_+^{(k)}$ vanish for a massive node and since $\mathcal{L}(0) = 0$ the massive node do not contribute in the $\theta \to +\infty$ regime. This suggests to interpret the $\theta \to +\infty$ contribution as coming from the Infra-Red (IR) RG limit since the massive particles decouple from the rest of the theory as their masses goes to infinity. In the other limit, namely $\theta \to -\infty$, the situation is reversed since the driving terms are all equal to zero. We can interpret this behavior as that of the UV regime in which the all the masses go to zero and the theory becomes conformal.

The kink, or massless, limit of the TBA equations is a very useful manipulation to explicitly extract the Rogers dilogarithm functional form but, for what concerns the asymptotic values, it may seem a little artificial. A remarkable properties of these type of system is that if one solves the full integral equations is left with a set of functions which display a behavior similar to that of the kink solutions. More precisely the very definition of the kink limit relies on the idea of hiding the $r$-dependence of the equations in such a way to obtain a sensible $r \to 0$ limit. This procedure has been introduced in order to cure the ill-definiteness of the equation for $r$ strictly null. Nonetheless from the numerical point of view, see Chapter 6, as

$r$ gets smaller and smaller the behavior of the solution of the full TBA rensembles very closely that of the kink solutions. Roughly speaking the full solution is obtained as "superposition" of the left and right kink solutions properly shifted back to their original position. The result for the case at hand, we choose $n = 6$, is displayed in Figure 4.4 where we can appreciate the plateau regions mentioned above. Moreover we can also notice how the full solutions interpolate between the constant plateau value in the two regimes. The expected symmetry pattern of the plateau values is respected as well, in fact we can see how the even $Z_2$ symmetry (6 nodes) of the UV region, central plateau in Figure 4.4, is reduced to a odd $Z_2$ symmetry (5 nodes) in the IR regime, asymptotics regions in Figure 4.4. For instance consider the first magnonic node $k = 1$, green line in Figure 4.4; in the central plateau region its value matches that of the UV symmetric partner, namely the $k = 5$ node (purple line in the picture), while in the asymptotic plateau region the profile of the solution interpolates to the value of the IR symmetric partner $k = 6$ (orange line in the picture).

In the case at hand, but as customary in these type of calculations, the asymptotic equations have to be analyzed separately.

**IR contribution:** In this regime there are only $n$ nodes since the massive one, the 0-th, has decoupled. Defining the quantity $x_a = e^{-\epsilon_a(-\infty)}$ for $a = 1, \cdots, n$ (i.e. ranging only over magnons) we have

$$x_a^2 = \prod_{b=1}^{n} (1 + x_b)^{I_{ab}} \,, \tag{4.43}$$

where $I_{ab} = \delta_{a,b+1} + \delta_{a,b-1}$ is the incidence matrix of $A_n$. The solution to this type of equations are known in literature [88, 93] and reads

$$x_a = -1 + \frac{\sin^2 \frac{\pi(a+1)}{n+3}}{\sin^2 \frac{\pi}{n+3}} \,. \tag{4.44}$$

**UV contribution:** The massive node now reaches a finite limit which in given in terms of the quantities $y_\alpha = e^{-\epsilon_\alpha(+\infty)}$ for $\alpha = 0, \cdots, n$ by

$$y_\alpha^2 = \prod_{\beta=1}^{n-1} (1 + y_\beta)^{I_{\alpha\beta}} \,, \tag{4.45}$$

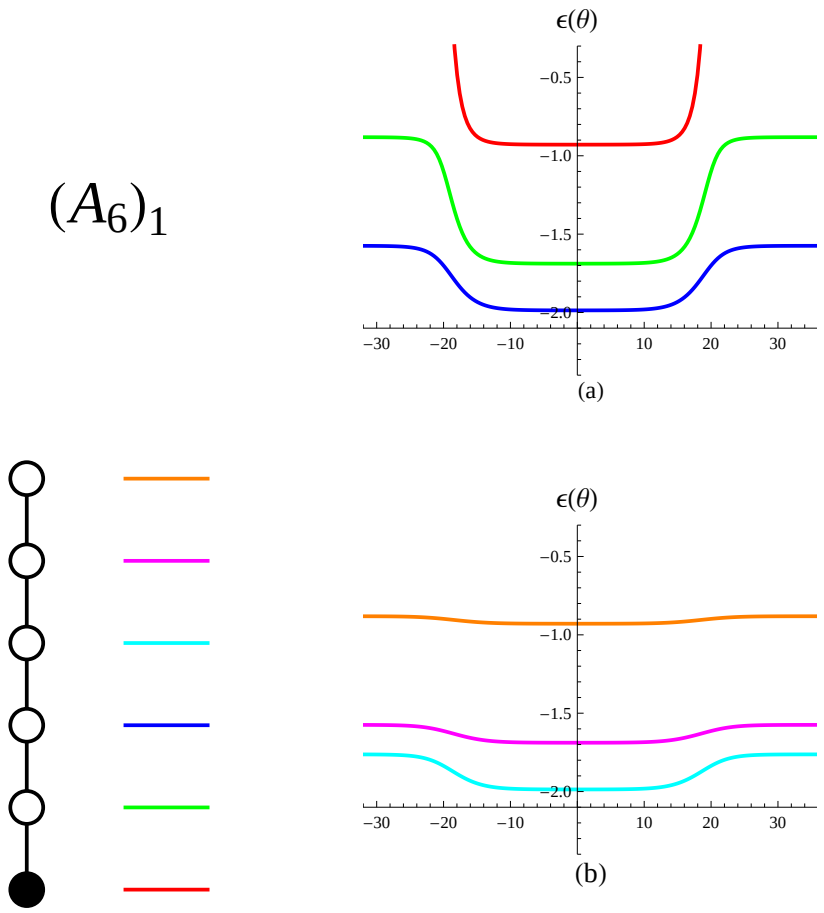



FIGURE 4.4: The result of the numerical integration of the $(A_6)_1$ TBA for $r = 10^{-8}$.

where and $I_{\alpha\beta} = \delta_{\alpha,\beta+1} + \delta_{\alpha,\beta-1}$ is the incidence matrix of $A_{n+1}$. Their explicit values can be obtained by adapting the parameters of (4.44).

Thanks to relation (A.9) and (A.4) of Appendix A both contribution can be evaluted exactly and become

$$c_+ = \frac{6}{\pi^2}\sum_{k=0}^{n}\mathcal{L}\left(T_+^{(k)}\right) = \frac{6}{\pi^2}\sum_{a=1}^{n}\mathcal{L}\left(1-\frac{1}{1+x_a}\right) = \frac{n(n+1)}{n+3}\,,$$

and

$$c_- = \frac{6}{\pi^2}\sum_{k=0}^{n}\mathcal{L}\left(T_-^{(k)}\right) = \frac{6}{\pi^2}\sum_{\alpha=0}^{n}\mathcal{L}\left(1-\frac{1}{1+y_\alpha}\right) = \frac{(n+1)(n+2)}{n+4}\,.$$

It is more interesting to analyze the result in term of the effective central charge which now can be written for the $(A_n)_1$ model as

$$c_{\text{eff}} = c\,(A_n) - c\,(A_{n-1}) = \frac{n(n+5)}{(n+2)(n+3)} \tag{4.46}$$

having introduced as notation the *central charge function* $c(G)$ denoting objects like (4.2.3) and (4.2.3), i.e. Rogers dilogarithm sums evaluated in values defined by equations like (4.43) with the incidence matrix of the algebra $G$. As we will see in the next section this is a very general behavior emerging even in more complicated and physically relevant models. Even the analysis of such a result is deferred to the next section where it could be collocated in a more general picture.

## 4.3 More General TBA's

In this section we will generalize the framework developed so far and put it to work in some non-trivial cases in which we will exploit the far reaching consequences and elegance of the TBA approach.

### 4.3.1 Product of graphs

The next logical step will be to generalize the construction of Section 4.2. In doing this we can start by considering a very broad class of conformal theories, namely the *Coset CFT* $G_k/H_{Ik}$ [34]. To set up the notation we consider a compact Lie algebra $G$ and one of its proper subalgebras $H$, embedded in $G$ with index $I$. The parameter $k$ stands for the level of the corresponding Kac-Moody algebra [32, 33].

The integrable models we want to consider are realized as perturbation of a coset CFT by some relevant operator $\Phi(x)$ of conformal dimension $\Delta_\Phi < 1$. The action for such models can thus be written as

$$S = \left[\frac{G_k}{H_{Ik}}\right] + \lambda \int d^2x \Phi(x) \tag{4.47}$$

where $[\cdots]$ denotes the action for the $G_k/H_{Ik}$ coset CFT and $\lambda$ is a bare coupling of dimension $2 - 2\Delta_\Phi$.

The knowledge of many exact S-matrices has led to intensive application of the TBA framework to many models [91, 93, 115]. All of them tend to the very same structure which we depict in general. There is an elegant way to obtain a unified form for all these type of TBA's [103, 104, 105, 106, 107, 108]. This idea of generalization stems from the study [105] of the $\Phi^{\mathrm{id,id}}_{\mathrm{adj}}$ perturbations of the coset models

$$\frac{G_k \times G_l}{G_{k+l}}\,, \tag{4.48}$$

which represent the most straightforward generalization of the minimal models $\phi_{1,3}$ perturbations.

We introduce two diagram: one encoding the particle species ($G$) and the other related to the magnonic structure of colors ($H$). The result is a diagram which can be considered a sort of tensor product of the two diagrams and is denoted by $G\Diamond H$. In doing this we conventionally define the diagram of $G$ to lie in the vertical direction and then to obtain the full diagram we attach to each node of $G$ a diagram of $H$ in the horizontal direction. At last we reproduce the link pattern of the base at each row for the corresponding new nodes. The situation has generalized and now we need two indices to label the nodes of the bidimensional diagram we just generated, we therefore introduce the usual pseudoenergies $\epsilon_{a,i}(\theta)$ where the index $a = 1, \cdots, \mathrm{rank}(G)$ runs (vertically) along the nodes of the copies of the diagram of $G$-type, whereas the index $i = 1, \cdots, \mathrm{rank}(H)$ runs (horizontally) along the diagrams of $H$-type. We need also to specify the driving terms $\nu_{a,i}(\theta)$ and we will see later under which constraints we can make the choices. This allows to write the general class of $G/H$ TBA's in the form

$$\epsilon_{a,i}(\theta) = \nu_{a,i}(\theta) - \Big[\varphi_g * \Big(\sum_b G_{ab}(\Lambda_{b,i} - \nu_{b,i}) - \sum_j H_{ij}\, L_{a,j}\Big)\Big](\theta)\,, \tag{4.49}$$

here $\varphi_g$ is the very same kernel of (4.32) in terms of the coxeter $g$ of $G$ and the matrices $G_{ab}$ and $H_{i,j}$ are the incidence matrices of the diagram of $G$ and $H$ respectively. The associated scaling function results to be given by

$$c(r) = \frac{3}{\pi^2} \sum_{a,i} \int d\theta\, \nu_{a,i}(\theta)\, L_{a,i}(\theta)\,. \tag{4.50}$$

As explained in detail in [107] the choices we can make for the mass terms $\nu_{a,i}$ are

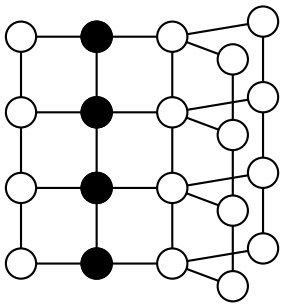


FIGURE 4.5: The two dimensional diagram encoding the TBA for the model $(A_4 \Diamond D_5)_2$

not completely free but must be picked in a particular way. Namely one is free to put masses, organized as the corresponding Perron-Frobenius eigenvector's entries, along the vertical direction provided we pick all the masses along the diagram of $G$ or none. Instead in the horizontal direction we have no such a constraint. This is the reason why the diamond product is not commutative and also why is the $G$ diagram which carries the informations related to massive particles. As an illustration we depict in Figure 4.5 the diagram for the case $(A_4 \Diamond D_5)_2$.

### 4.3.2 General Y-systems and sum rules

We can now present a very general and unified form of Y-systems. Starting from the integral equations (4.49) and performing the same manipulations of Section 4.2.2 we end up with the following system of functional equations

$$Y_{a,i}\left(\theta + i\frac{\pi}{g}\right) Y_{a,i}\left(\theta - i\frac{\pi}{g}\right) = \prod_b \left(1 + Y_{b,i}(\theta)\right)^{G_{ab}} \prod_b \left(1 + Y_{a,j}(\theta)^{-1}\right)^{-H_{ij}} , \quad (4.51)$$

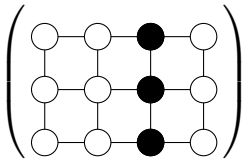
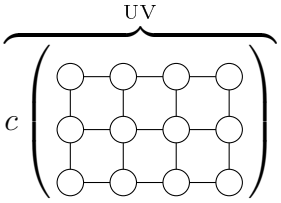
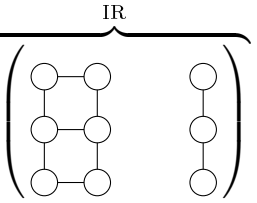
in terms of the functions $Y_{a,i}(\theta) = e^{\epsilon_{a,i}(\theta)}$. In this form the connection with the product of graphs is even more manifest and moreover we can also reverse-engineer the form of the graph by looking at the functional equations alone in a very straightforward way.

In view of following the RG flow we avoid the complications of an exact calculation, which will anyway be model dependent, and simply state the very general structure of the resulting central charge. As grasped in the explicit calculation of Section 4.2.3 the effective central charge is obtained, at this level, as the difference between the UV and IR contributions $c_{\text{eff}} = c_- - c_+$. Both the contribution are evaluated

according to a general Rogers dilogarithm sum rule

$$s\,(G\lozenge H)=\sum_{a=1}^{r(G))}\sum_{i=1}^{r(H)}\mathcal{L}\left(\frac{1}{1+y_{a,i}}\right)=\frac{\pi^2}{6}\frac{r(G)\,r(H)}{\mathrm{cox}(G)+\mathrm{cox}(H)}\mathrm{cox}(H)\,, \tag{4.52}$$

having explicitly denoted by $r(G)$ and $\mathrm{cox}(G)$ respectively the rank and coxeter number for a general Lie algebra $G$. Clearly the set of quantities $\{y_{a,i}\}$ are the solution of the plateau equations for the corresponding diagram. The latter is obtained for the UV part by treating all the nodes on the same footing while, for the IR part, we cancel the massive node and consider the diagram(s) remaining. To see this machinery at work we can apply the rule just stated to the following system

$c$  $= c$  $- c$  , (4.53)

where we dropped the subscripts for simplicity.

The universal sum rules (4.52) have been proved rigorously only in the case $A_n\lozenge A_m$ [102] but several arguments, most of all the very high accuracy of the numerical computation for a very broad range of the parameters, confirm its robustness. See Section 6.4 for more details about numerical simulations in this direction.

Another important information carried by the Y-system formulation is the dimension of the perturbation operator which generates the integrable model by perturbing the CFT. As a matter of fact the solutions of Y-systems of type (4.51) possess a natural periodicity in the complex plane. More precisely it can be shifted by rational multiples of $\pi$ in the purely imaginary direction of the complexified rapidity plane. It turns out that a general relation occurs for the $ADE\lozenge ADE$ under consideration, namely

$$Y_{a,i}(\theta+i\pi P)=Y_{\bar{a},\bar{i}}(\theta)\qquad\text{with}\qquad P=\frac{\mathrm{cox}(G)+\mathrm{cox}(H)}{\mathrm{cox}(G)} \tag{4.54}$$

and the barred indices $\bar{a}$ and $\bar{i}$ denotes particle-antiparticle coniugation.

The period 4.54 can then be used to obtain the dimension of the perturbing operator by means of [90, 91, 92]

$$\Delta_\Phi = 1 - \frac{1}{P} \tag{4.55}$$

This further data that can be extracted by the $Y$-system allow, in most cases, to identify the integrable model in terms of the underlying conformal theory and the corresponding perturbing operators. This operation is, obviously, very different according to the specific model we are considering and in the next section we give an overview of the possible identifications for the theories described by (4.49).

### 4.3.3 Models identification

In the following we want to give an overview on the consequences that one can draw from the non-perturbative considerations allowed by the TBA analysis. In particular we will identify, in some notable cases, the perturbed CFT realization of the TBA models discussed so far according to their diagrammatic interpretation.

#### 4.3.3.1 $A_1 \Diamond ADET$

These cases have been completely classified in [101] and belongs to the class of TBA's of Section 4.2. The corresponding CFTs are:

- $A_1 \Diamond A_n$ : The analysis presented in [91, 92, 93] classifies the special unitary group-related models. It is convenient to set $n = k + l - 1$ and consider $A_{k+l-1}$. The integrable model is then realized as the $(k, l)$-th $SU(2)$ coset CFT
$$\frac{SU(2)_k \times SU(2)_l}{SU(2)_{k+l}},$$
with the subgroup at the denominator diagonally embedded, perturbed by the operator of dimension $\Delta = 1 - 2/(k + l + 1)$. The choice of the forcing terms produces two different RG flows according to the choice of forcing terms. For the so-called *massive* choice, namely $\nu_{1,i} = \delta_{k,i} r \cosh\theta$, the theory flows to the massive theory described by the non-diagonal S-matrices of [110]. Another possibility arises by virtue of the $Z_2$ symmetry of the diagram, we can pick the energy terms as explained in [101] (the so-called left- and

right-movers) and obtain a flow which is massles and in the IR limit ends instead in the $(k-l,l)$-th $SU(2)$ CFT. Establishing in this way a deeply non-perturbative massless flows between the two theories.

- $A_1 \Diamond D_n$ : The choice $\nu_{1,i} = \delta_{k,i} r \cosh\theta$ for $k = 1, 2, \cdots, n-2$, i.e. by putting the mass on the tail of the $D_n$ diagram, produces a massive flow to the CFT of central charge

$$c = \frac{3k}{k+2}\,,$$

which agrees with the $k$-th critical line of [111]. The latter can be obtained as the UV limit of a FSSG model [148] whose perturbing operator is therefore given by $\Phi\bar{\Phi}$ with

$$\Phi = \psi_1 e^{\frac{i\beta}{\sqrt{4\pi}}\phi}\,; \tag{4.56}$$

$\psi_1$ being the $Z_k$ generating parafermion, whose dimension reads $(k-1)/k$, and $\phi$ a free massless boson, whose vertex operator appearing in (4.56) has dimension $\beta^2/8\pi$. Thus in order to identify the two descriptions we have to match the dimension of the perturbation $\Phi$ with the perturbing dimension as coming from the $Y$-system analysis which, according to (4.54), is given by $\Delta = 1 - 1/n$. This results in the identification

$$\frac{\beta^2}{8\pi} = \frac{1}{k} - \frac{1}{n}\,,$$

which allows to reconstruct the $S$-matrix for the perturbed massive theory as

$$S = S_k \otimes S_{SG}\left(\frac{1}{k} - \frac{1}{n}\right)\,.$$

where $S_k$ is the $S$-matrix of the $k$-th minimal model of [112] perturbed by the corresponding operator $\phi_{13}$ and $S_{SG}(\beta/8\pi)$ is the Sine-Gordon $S$-matrix [120] at coupling $\beta$.

If the mass is located just before the fork, i.e. $k = n-2$, we obtain the $\mathcal{N} = 2$ supersymmetric point of the corresponding line of models introduced and detailed explained in [113].

The last configuration is when the mass is located on the fork, $k = n$ or $k = n-1$ equivalently, and has been studied in [114]. The UV central charge of this model is given by

$$c_{UV} = \frac{3(n-1)}{n+2}$$

and matches with that of the $Z_n$ parafermionic models, whose realization in terms of coset CFT is $SU(2)_n/U(1)$. As usual the perturbation is identified through the $Y$-system data to be the operator $\psi_1\bar{\psi}_1 + \psi_1^\dagger\bar{\psi}_1^\dagger$.

Instead by putting the left and righ-movers on the nodes of the fork we allow for a massless TBA which flows in the opposite direction and in the IR limit reaches the central charge

$$c_{IR} = 1 - \frac{6}{(n+1)(n+2)}\,,$$

which matches the central charge of celebrated $\mathcal{M}_{n+1,n+2}$ conformal minimal model. In other words we have established a non-perturbative motivation for the RG flow between the $Z_n$ parafermions and the $n+1$-th minimal model, as originally proposed in [91].

- $A_1 \Diamond E_n$ : These types of models are the most exotic of the group and the TBA analysis produces the UV central charges displayed in Figure 4.6. Most of the cases can be identified with direct tensor product of non interacting minimal models. A specific cases which should be mentioned is the $(A_1 \Diamond E_6)_2$ model. Its UV central charge reads $c_{UV} = 25/14$ which can be obtained as the tensor product of two $\mathcal{M}_{8,9}$ minimal models; also the dimension of the integrable perturbation $\Delta = 6/7$ reveals that we can realize the perturbing operator as the product of operators with dimensions $3/28$ and $3/4$ respectively. Since the model enjoys a $Z_2$ symmetry we can also make the massless choice for the driving terms. With this choice the UV behavior is the same while the IR behavior becomes non-trivial and the model flows to a $c = 81/70$ theory which can be identified with the $m = 5$ model in the $\mathcal{N} = 1$ superconformal series [115]. In [101] particular attention is paid to the sequence

$$\begin{aligned} c(E_6) &= \frac{8}{7} \qquad \Delta(E_6) = \frac{6}{7}\,, \\ c(E_7) &= \frac{13}{10} \qquad \Delta(E_7) = \frac{9}{10}\,, \\ c(E_8) &= \frac{3}{2} \qquad \Delta(E_8) = \frac{15}{16}\,. \end{aligned}$$

when the massive term $r\cosh\theta$ is placed in the first node of the diagram, the $(A_1 \Diamond E_n)_1$ models in the language of the previous section. There, after formulating a particular extension of the last series, these models are discussed and related to the generalized parafermionic algebra with $Z_k$ grading.

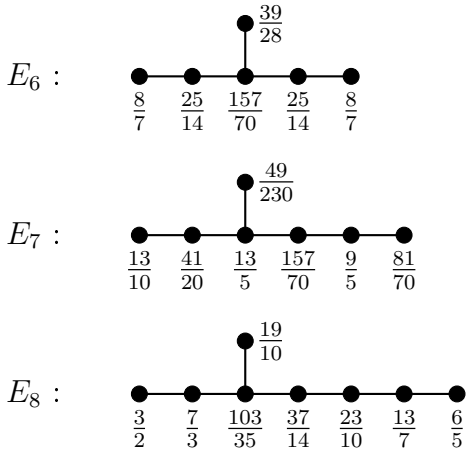



FIGURE 4.6: The Dynkin diagrams for the $E_{6,7,8}$ algebras are displayed with the central charge evaluated when the mass is placed on the corresponding node leaving the others magnonic.

These algebras [116], denoted by $SZ_k$, can be obtained by the fusion rules $\psi_i \times \psi_j = \psi_{i+j \bmod k}$ among the parafermions $\psi_i$ and contains a distinguished generating parafermion $\psi_1$ of dimension $1 + 1/k$. More precisely we have, for the first two cases, the following correspondences: the $SZ_2$ algebra is the $\mathcal{N} = 1$ superconformal algebra, generated by a field of spin 3/2 and the $SZ_3$ algebra is the spin 4/3 algebra introduced in [117].

- $A_1 \Diamond T_n$ : Only the massive choice is possible in this case. Putting the mass term $r \cosh\theta$ on the $l$-th node, the corresponding central charge is given by

$$c = \frac{3l}{l+2}\left(1 - \frac{2(l+2)}{p(p+2l)}\right),$$

in terms of the parameter $p = 2n + 1$. This is a case in which the computed charge is of the effective type $c = c_{UV} - 24\Delta_0$, $\Delta_0$ being the lowest conformal dimension in the underlying CFT, negative if the model is non-unitary. This observation allows to identify the UV limiting model as the non-unitary

$$\frac{SU(2)_k \times SU(2)_l}{SU(2)_{k+l}}$$

series, where $l$ is an integer and $k = p/2-2$ is an half-integer. The perturbing operator with dimension

$$\Delta = \frac{p-2}{p+2},$$

turns out to be the usual $\phi^{id,id,adj}$. For $l = 1$ we obtain the non-unitary minimal models series $\mathcal{M}_{p,p+2}$ perturbed by their $\phi_{13}$ operator while for $l = 2$ we obtain the $\mathcal{M}_{p,p+4}$ supersymmetric series.

#### 4.3.3.2 $ADE \Diamond A_n$

This case is discussed in details in [115]. We denote by $G$ a generic algebra of the $A, D$ or $E$ type. From the notational point of view it is useful to set again $n = k + l - 1$. The resulting TBA describes the integrable model corresponding to the coset $G$-WZW model

$$\frac{G_k \times G_l}{G_{k+l}}\,, \tag{4.57}$$

perturbed by the operator $\phi^{id,id}_{adj}$. The UV central charge computed from the TBA can be written as

$$c_{UV} = c(G_k) + c(G_l) - c(G_{k+l})\,, \quad \text{with} \quad c(G_m) = \frac{m\,\dim(G)}{m + \text{cox}(G)}\,, \tag{4.58}$$

in terms of the central charge of the $G$-WZW model at integer level in perfect agreement with what expected from the GKO construction for the coset (4.57). The IR RG flow instead can be obtained by applying the arguments of [92, 93] and produces the charge

$$c_{UV} = c(G_{k-l}) + c(G_l) - c(G_k)\,,$$

signal of the flow between the models

$$\frac{G_k \times G_l}{G_{k+l}} \quad \to \quad \frac{G_{k-l} \times G_l}{G_{k+l}}\,.$$

#### 4.3.3.3 $ADE \Diamond D_n$

This case has been analyzed in [107] in the particular configurations in which a mass is located on one of the nodes of the fork, namely the $n$-th or $n-1$-th node, or in which the left and righ-movers are placed on the fork. The motivation behind this choice is to generalize the analysis of $A_1 \Diamond D_n$ to higher level algebras.

These results shows how the TBA approach is effective when trying to identify the UV and IR limits and establishing RG flows between models: it can give

evidence and hints of highly non-trivial results on the structure of the RG flow in two dimensions.

### 4.3.4 Non-linear Sigma Models

Now we turn our attention to the TBA description of Non-Linear Sigma Models (NLSM). As introduced in Chapter 3 an integrable class of NSLM is constituted by the $O(n)$ NLSM and this property still holds at the quantum level. The S-matrix theory has been applied to these type of models in a series of remarkable papers by the Zamolodchikov's brothers [119, 120] which derived the exact form of the 2-body scattering amplitudes between the fundamental excitations of the model. Once the S-matrix is given the TBA procedure goes through straightforwardly, modulo technical complications, along the lines of Section 4.1.3.

#### 4.3.4.1 The $O(3)$ case

This is one of the simplest NLSM and its TBA description has been subjected to various investigations during the years [121, 122, 131, 129].

By trying to apply the TBA procedure we immediately face the biggest computational difference. In this case the things are a little more involved because of the presence of S-matrices which are rational functions of the rapidities. In this case the resulting Bethe Ansatz Equations (BAE) will not be periodic in the imaginary rapidity direction allowing for an unbounded number of magnonic configuration: *the magnonic strings*. These type of objects arise in considering the termodynamical limit of certain type of Bethe Ansatz equations. We will now briefly illustrate the various points of the TBA derivation for the $O(3)$ NLSM

- **Bethe Ansatz Equations:** Exploting the isomorphism $SU(2) \sim O(3)$, strictly true between algebras, and reformulating the model in a fermionic way, Wiegmann [124, 125] has been able to diagonalize the transfer matrix

for us, the result reads

$$
\begin{aligned}
&e^{imL\sinh\theta_i}\prod_{j=1,j\neq i}^{N_r}\frac{\theta_i-\theta_j-i\pi}{\theta_i-\theta_j+i\pi}\prod_{k=1}^{M}\frac{\theta_i-\lambda_k-i\pi}{\theta_i-\lambda_k+i\pi}=1\,,\ i=1,2,\cdots,N_r\,,\\
&\prod_{j=1,j\neq i}^{M}\frac{\lambda_i-\lambda_k-i\pi}{\lambda_i-\lambda_k+i\pi}\prod_{k=1}^{N_r}\frac{\lambda_i-\theta_k-i\pi}{\lambda_i-\theta_k+i\pi}=1\,,\ i=1,2,\cdots,M\,,
\end{aligned}
\tag{4.59}
$$

where the $\theta$'s are the $N_r$ particles' root whereas the $\lambda$'s are the $M$ magnonic auxiliary rapidities introduced, as customary, in the application of the Bethe Ansatz technique to non-diagonal scattering.

- **String Hypothesis:** In the thermodynamic limit the magnonic rapidities organize themselves in particular configurations called *strings* [126, 127] whose form reads in this case

$$
\lambda_{j,\alpha}^{(m)}=\lambda_j^{(m)}+\frac{i\pi}{2}\left(m+1-2\alpha\right)\,,\ \alpha=1,2,\ldots,m \tag{4.60}
$$

where $\lambda_j^{(m)}$ are the real center of the string. The most remarkable feature of this sort of *magnonic bound states* is that one can multiply the BAE corresponding to all the roots of the same string and obtain after a number of non-trivial simplification a set of equations only for the center of the string. This is the usual fusion procedure of S-matrices applied directly at the level of the Bethe equations. In considering the various string as single objects we have to make the following replacements in the BAE (4.59)

$$
\prod_{j=1}^{M}\ \rightarrow\ \prod_{m\in\mathcal{M}}\prod_{j=1}^{M_m}\prod_{\alpha=1}^{m}
$$

where

  - $\mathcal{M}$ is the set of possible root types
  - $M_m$ is the number of $m$-type roots
  - $\alpha$ runs over the string's elements in the complex rapidity plane

The result of this operation is a new set of BAE for the center of magnonic strings and particles' roots alone.

- **Thermodynamics:** This point is two-folded. Firstly, following the conventions and notations of [121], we perform the thermodynamic limit of the

BAE for the roots and strings' center, namely to let

$$L, N_r, M_m \to \infty\,, \quad \text{with} \qquad N_r/L = const.\,, \quad M_m/L = const. \tag{4.61}$$

together with the definitions of the continuous distributions of roots and strings rapidities

$$\bar{\sigma}_0(\beta) \equiv \lim \frac{n_i - n_{i-1}}{L(\beta_i - \beta_{i-1})}\,, \quad \bar{\sigma}_m(\beta) \equiv \lim \frac{s_i^{(m)} - s_{i-1}^{(m)}}{L(\beta_i - \beta_{i-1})}$$

where the quantum numbers $n_i$ and $s_i^{(m)}$ appeared in taking the logarithm of BAE. Note that since we are considering the thermodynamics of the system we have to consider every possible type of magnonic string. In other words the set $\mathcal{M}$ becomes the set of natural numbers $\mathbb{N}$ representing the fact that string of any length can be excited. This result in the integral form of the BAE which read

$$\begin{cases} \bar{\sigma}_0(\theta) - \Big(\Psi_1 * \sigma_0\Big)(\theta) - \displaystyle\sum_{m=1}^{\infty} \Big(\Psi_m * \sigma_m\Big)(\theta) = \frac{m \cosh\theta}{2\pi} \\ \bar{\sigma}_m(\lambda) + \displaystyle\sum_{k=1}^{\infty} \Big(\Phi_{mk} * \sigma_k\Big)(\lambda) - \Big(\Psi_m * \sigma_0\Big)(\lambda) = 0\,,\ m = 1, 2, \dots \end{cases},$$

in terms of the kernels

$$\Psi_m(x) = \frac{d}{dx}\Big[\varphi_{m+1}(x) - \varphi_{m-1}(x)\Big]\,, \text{with} \quad \varphi_k(x) = 2\arctan\left(\frac{2}{\pi}\frac{x}{k}\right)\,,$$

and

$$\Phi_{mk}(x) = \frac{d}{dx}\Big[\varphi_{m+k}\,(x) - \varphi_{m-k}\,(x) + 2\sum_{\alpha=1}^{m-1} \varphi_{m+k-2\alpha}\,(x)\Big]\,.$$

The real novelty of such a system of equation is the infinite number of magnonic nodes, trace of the different nature of the underlying S-matrix.

Then we can go through the minimization of the free energy $\mathcal{E} - \mathcal{S}/R$ in the mirror theory (at temperature $T = 1/R$) by picking

$$\begin{aligned} \mathcal{E}[\sigma_\mu] &= \int d\beta\, \sigma_0(\beta) \cosh\beta\,, \\ \mathcal{S}[\sigma_\mu] &= \sum_{\mu=0}^{\infty} \int d\beta \Big[\bar{\sigma}_\mu \log\bar{\sigma}_\mu - \sigma_\mu \log\sigma_\mu - (\bar{\sigma}_\mu - \sigma_\mu)\log(\bar{\sigma}_\mu - \sigma_\mu)\Big]\,, \end{aligned}$$

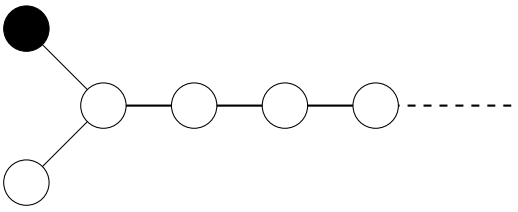


FIGURE 4.7: Dynkin-like diagram for the $O(3)$ NLSM TBA formulation.

we end up in the following *physical TBA* for the $O(3)$ NLSM

$$\begin{cases} \epsilon_0(\beta) = mR\cosh\beta - \Big(\Psi_1' * L_0\Big)(\beta) - \displaystyle\sum_{m=1}^{\infty}\Big(\Psi_m' * L_m\Big)(\beta) \\ L_m(\beta) = \Big(\Psi_m' * L_0\Big)(\beta) + \displaystyle\sum_{k=1}^{\infty}\Big(A_{mk} * L_k\Big)(\beta) \end{cases} \tag{4.62}$$

in terms of the non-linear functions $L_\mu = \log(1+e^{-\epsilon_\mu})$, of the pseudoenergies

$$\frac{\sigma_\mu}{\bar\sigma_\mu - \sigma_\mu} = \exp(-\epsilon_\mu)\,, \quad \mu = 0, 1, 2, \cdots,$$

and where the only novel convolution kernel is given by

$$A_{mk}(x) = \frac{d}{dx}\Phi_{mk}(x) + 2\pi\delta_{mk}\delta(x)$$

- **Y-system and RG flow:** Thanks to some identities among the kernels, which rensemble very closely the inversion relation (4.30) of the *ADE* case, which we report for completeness

$$\begin{aligned} A^{-1}_{mk}(x) &= 2\pi\delta_{mk}\delta(x) - \phi(x)\left(\delta_{m,k+1} + \delta_{m,k-1}\right)\,, \\ \Psi'_m(x) &= \Big(\phi * A_{m2}\Big)(x)\,, \qquad \phi(x) = \frac{1}{\cosh(x)}\,, \end{aligned} \tag{4.63}$$

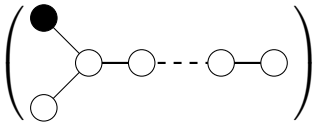
the TBA (4.62) can be brought to the *universal form* involving only the universal kernel $\phi(x)$ and the incidence matrix of what turns out to be a "$D_\infty$" type infinite diagram. The mass is located on the fork of the diagram which extends indefinitely in the other direction. We can then try to interpret the NLSM, i.e. a TBA with an infinite number of magnonic nodes, as the limit of a certain family of truncated model which we control from the side of CFT. In this case it seems natural to construct the NLSM TBA as the $N \to \infty$ of a $D_{N+1}$ series, as depicted in Figure 4.7. The corresponding

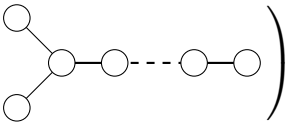
Y-system thus reads

$$Y_\mu\left(\theta+i\frac{\pi}{2}\right)Y_\mu\left(\theta-i\frac{\pi}{2}\right)=\prod_\kappa\left(1+Y_\kappa(\theta)\right)^{I_{\mu\kappa}}\ , \tag{4.64}$$

where $I_{\mu\kappa}$ is the incidence matrix of a $D_{N+1}$ Lie algebra.

From the point of view of central charges matching one found perfect agreement with the expected result, in fact if we compute the effective central charge with the methods of Section 4.2.3 we obtain

$c_{\text{eff}}\left(O(3)\right) = c\Big($  $\Big) =$

$= c\Big($  $\Big) - c\Big($  $\Big) =$

$$= c\left(D_{N+1}\right) - c\left(A_N\right) = N - \frac{N(N+1)}{N+3} \overset{N\to\infty}{\longrightarrow} 2\,.$$

This calculation confirms the expected result. In fact the RG flow of the sphere sigma model can be seen as a renormalization of the radius and reaching the UV limit amounts to send the radius to infinity, restoring a flat geometry. The limiting theory will thus be the theory of two free bosons on flat space, each of them will therefore give a contribution $c = 1$ to the effective central charge.

#### 4.3.4.2 $O(2r)$ TBA and Y-system

The $O(3)$ case reviewed in the previuos section is not the only $n$-dimensional sphere NLSM which can be treated by the TBA approach. In a series of important papers [103, 128, 129, 130] various authors showed how to formulate the TBA equations which governs the behavior of the $O(2r)$, $r \geq 2$, sigma models. The odd-dimensional cases are, up to now, not yet known due to the technical problems related to the non-simply laced algebras involved which make the string hypothesis a formidable task to perform.

According to [103, 128, 130] we can write the TBA system for the $O(2r)$ $(r \geq 2)$ NLSM as the limit of a certain sequence of coupled non-linear integral equations,

which read

$$\epsilon_0(\theta) = MR\cosh\theta - \sum_{a=1}^{r}\int \frac{d\theta'}{2\pi}\,\psi_{a,1}(\theta-\theta')\,\log\left(1+e^{-\epsilon_{a,1}(\theta')}\right)\,, \tag{4.65}$$

$$\begin{aligned}\epsilon_{a,m}(\theta) = &-\delta_{m1}[\delta_{a1}+\delta_{a2}\delta_{r2}]\,A_0(\theta)+\\ &-\,A_{a,m-1}(\theta)-A_{a,m+1}(\theta)+\sum_{b=1}^{p} I_{a,b}\,B_{b,m}(\theta)\,,\end{aligned} \tag{4.66}$$

where $g=2(r-1)$ and $I_{ab}$ are respectively the Coxeter number and the incidence matrix associated to the $D_r$ Lie algebra. In writing (4.65-4.66) we incidentally introduced the quantities

$$\begin{aligned}A_0(\theta) &\equiv \frac{g}{4\pi}\int_{-\infty}^{+\infty}\frac{d\theta'}{\cosh\frac{g(\theta-\theta')}{2}}\,\log\left(1+e^{-\epsilon_0(\theta')}\right)\,,\\ A_{a,m}(\theta) &\equiv \frac{g}{4\pi}\int_{-\infty}^{+\infty}\frac{d\theta'}{\cosh\frac{g(\theta-\theta')}{2}}\,\log\left(1+e^{-\epsilon_{a,m}(\theta')}\right)\,,\\ B_{a,m}(\theta) &\equiv \frac{g}{4\pi}\int_{-\infty}^{+\infty}\frac{d\theta'}{\cosh\frac{g(\theta-\theta')}{2}}\,\log\left(1+e^{\epsilon_{a,m}(\theta')}\right)\,,\end{aligned}$$

and the kernels $\psi_{a,1}$ which can be explicitly given in terms of their Fourier Transforms

$$\psi_{a,1}(\theta) = \int d\omega\, e^{i\omega\theta}\,\mathcal{N}_{a,1}(\omega)\,,$$

with

$$\begin{aligned}\mathcal{N}_{a,1}(\omega) &= \frac{\cosh\left(\omega\frac{(r-1-a)\pi}{g}\right)}{\cosh\left(\omega\pi/2\right)} \qquad\qquad a=1,2,\ldots,r-2\\ \mathcal{N}_{r-1,1}(\omega) &= \mathcal{N}_{r,1}(\omega) = \frac{1}{2}\frac{1}{\cosh\left(\omega\pi/2\right)}\,.\end{aligned}$$

A general Fourier pair relation, namely

$$\mathcal{R}(\alpha,\beta;\theta) \equiv \frac{1}{\alpha}\frac{\cos\left(\frac{\pi\beta}{2\alpha}\right)\cosh\left(\frac{\pi\theta}{2\alpha}\right)}{\cosh\left(\frac{\pi\theta}{\alpha}\right)+\cos\left(\frac{\pi\beta}{\alpha}\right)} = \frac{1}{2\pi}\int d\omega\, e^{i\omega\theta}\,\frac{\cosh(\beta\omega)}{\cosh(\alpha\omega)}\,,$$

allows to recast (4.65) as

$$\begin{aligned}\epsilon_0(\theta) = MR\cosh\theta - \sum_{a=1}^{r-2}\int d\theta'\,\mathcal{R}\left(\frac{\pi}{2},\frac{(r-1-a)\pi}{g}\,;\,\theta-\theta'\right)\,L_{a,1}(\theta')+\\ -\frac{1}{2}\int d\theta'\,\mathcal{R}\left(\frac{\pi}{2},0\,;\,\theta-\theta'\right)\,\left[L_{r-1,1}(\theta')+L_{r,1}(\theta')\right]\,.\end{aligned} \tag{4.67}$$

Now we can obtain the associated $Y$-system by a slightly different approach than that of Section 4.2.2) which turns out to reveal itself more useful for practical purpose. In fact in real life calculations, which customarily involve more general models than those of Section (4.3.1), obtaining general inversion relations, see equations (4.30) and (4.63), for the convolution kernels is a very difficult task. Nonetheless $Y$-systems can still be derived by analyzing the behavior of the pseudoenergies under shift of the rapidity in the imaginary direction. The integral form of the TBA equations provides the necessary analytic tools to control correctly these shifting operation. In fact, by carefully analyzing the behavior of the convolution integrals in the complex plane, it turns out that the role played by the eventual poles of the kernels is fundamental. Indeed it can happen that, in shifting the integration contour in the imaginary direction, a pole of the kernel located in the physical strip is encountered. As a consequence the contour must be deformed properly giving rise to the so-called *contact terms* which can be evaluated by making use of the residue theorem. As an application of this type of circumstances we present the result for the case at hand concerning the convolution kernel $\mathcal{R}(\alpha,\beta;\theta)$. The result can be written as

$$\begin{aligned}\mathcal{R}\left(\frac{\pi}{2},\frac{(r-1-a)\pi}{g};\,\theta+\frac{i\pi}{2}\right)+\mathcal{R}\left(\frac{\pi}{2},\frac{(r-1-a)\pi}{g};\,\theta-\frac{i\pi}{2}\right)=\\ =\delta\left(\theta+\frac{i(r-1-a)\pi}{g}\right)+\delta\left(\theta-\frac{i(r-1-a)\pi}{g}\right)\,;\end{aligned}$$

this result, along the lines of the previous discussion, allows to recast equation (4.67) as

$$\begin{aligned}&\epsilon_0\left(\theta+\frac{i\pi}{2}\right)+\epsilon_0\left(\theta-\frac{i\pi}{2}\right)=\\ &=-\sum_{a=1}^{r-2}\left[L_{a,1}\left(\theta-\frac{i(p-1-a)\pi}{g}\right)+L_{a,1}\left(\theta+\frac{i(r-1-a)\pi}{g}\right)\right]+\\ &\quad-L_{r-1,1}(\theta)-L_{r,1}(\theta)\,,\end{aligned}$$

and eventually, by defining

$$Y_{a,m}(\theta) \equiv e^{-\epsilon_{a,m}(\theta)}\,,$$
$$Y_0(\theta) \equiv e^{-\epsilon_0(\theta)}\,,$$

to obtain the following $Y$-system of functional equations

$$\begin{aligned} Y_0\left(\theta+\frac{i\pi}{2}\right)Y_0\left(\theta-\frac{i\pi}{2}\right) &= \prod_{a=1}^{r-2}\left[\left(1+Y_{a,1}(\theta-\frac{i(r-1-a)\pi}{g})\right)\times\right. \\ &\left.\times\left(1+Y_{a,1}(\theta+\frac{i(r-1-a)\pi}{g})\right)\right]\left(1+Y_{r-1,1}(\theta)\right)\left(1+Y_{r,1}(\theta)\right)\,, \\ Y_{a,m}\left(\theta+\frac{i\pi}{g}\right)Y_{a,m}\left(\theta-\frac{i\pi}{g}\right) &= [1+\delta_{1m}(\delta_{a1}+\delta_{r2}\delta_{a2})Y_0(\theta)]\times \\ &\times\frac{(1+Y_{a,m+1}(\theta))(1+Y_{a,m-1}(\theta))}{\prod_{b=1}^{r}\left(1+\frac{1}{Y_{b,m}(\theta)}\right)^{I_{ab}}}\,. \end{aligned} \tag{4.68}$$

Even in this case the functional $Y$-system (4.68) can be encoded in a diagram similar to that of Figure 4.8 but with an infinite number of magnonic rows, as typical form sigma models TBA description.

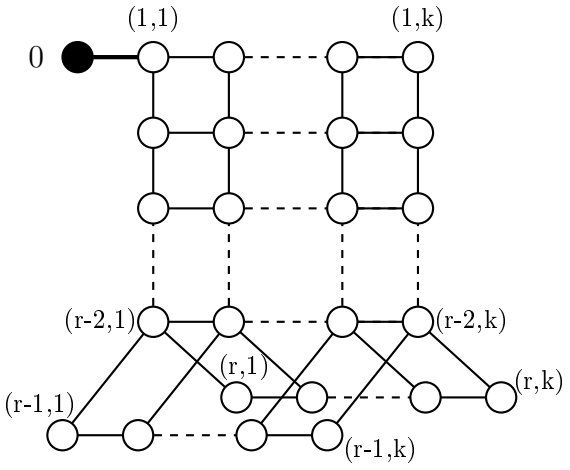



FIGURE 4.8: The $O(2r)$ diagram. Again the labels of each node are associated to the functions $Y$ in (4.68)

The TBA formulation allows to reconstruct the effective central charge of the model. In doing this we have to truncate the model to a certain level $k$, which represents the width of the diagram of Figure 4.8, in order to deal with a finite number of equations. We will denote this truncated version of the model by $O(2r)_k$.

Thanks to the methods illustrated in Section 6.4 it is possible to solve the plateau equations corresponding to the stationary solutions of 4.68 and to evaluate the dilogarithms sum rule [128, 131]. Unfortunately an analytic close form is still lacking for the plateau solution but extensive and highly accurate numerical analysis have led to conjecture the following expression

$$c_{\text{eff}}\Big(O(2r)_k\Big) = \frac{k(k+r-2)(2r-1)}{(k+2r-3)(k+2r-2)} \overset{k\to\infty}{\longrightarrow} 2r-1\,. \tag{4.69}$$

The limit is exactly the number of degrees of freedom of a flatten $2r-1$-dimensional sphere and the analysis of the RG flow for this TBA confirms the calculation based on counting arguments of the renormalized theory.

#### 4.3.4.3 Fendley's Conjecture

Other interesting NLSM which have been analysed within the TBA approach are those of [103, 104]. They consists of various examples of *symmetric spaces* NSLM which we will briefly define to fix the notation. For example, these type of NLSM arise in theories of interacting fermions invariant under some group $G$. If some fermion bilinear gets an expectation value manifestly invariant under some subgroup $H$, then the excitations at low energy can be described by a field taking values in $G/H$. In other words the fermions get endowed with a mass scale $M$, by integrating out the fermions we are then left with only bosonic excitations in $G/H$ with masses below $M$. The action can be written, in terms of symmetric matrix field $\Phi$ taking value in $G$, as

$$S = \frac{1}{g}\int d^2x \partial_\mu \Phi^\dagger \partial^\mu \Phi\,,$$

under the constraint $\Phi^\dagger\Phi = 1$. In theories with interacting fermions, this often results from introducing a bosonic field to replace four-fermion interaction terms with Yukawa terms (interactions between a boson and two fermions). Integrating out the fermions then gives such a potential for the bosons and hence the sigma model.

The first example is represented by the $SU(N)/SO(N)$ model. As seen before the peculiarity of the TBA description of NLSM is that they usually enjoy an infinite number of magnons and are realized as the limit of a particular sequence of truncated models. The analysis of [104] reveals that this NLSM is encoded in

the TBA $A_N \Diamond D_\infty$ model with a single row of masses on the fork. Truncating the model at finite level $k$ allows for the identification with the $SU(N)_k/SO(N)_{2k}$ model whose central charge correctly reproduces the NSLM degrees of freedom counting in the $k \to \infty$ limit. In other words these models present the same type of connection between the central charge calculation and the limit of the corresponding affine Lie algebras displayed by the $O(2n)$ NLSM of Section 4.3. There's a relevant amount of other known examples some of which, namely those of [104], are displayed in Table 4.1. All these consideration suggests that a unifying

| G/H NSLM | TBA | Truncated model |
|---|---|---|
| $SU(N)$ Gross-Neveu | $(A_N \Diamond A_k)_1$ | $\frac{SU(N)_k \times SU(N)_1}{SU(N)_{k+1}}$ |
| $O(2P)$ Gross-Neveu | $(D_P \Diamond A_k)_1$ | $\frac{O(2P)_k \times O(2P)_1}{O(2P)_{k+1}}$ |
| $\frac{O(2P)}{O(P) \times O(P)}$ | $(D_P \Diamond D_k)_1$ | $\frac{O(2P)_k}{O(P)_k \times O(P)_k}$ |

TABLE 4.1: Known examples of Fendley's conjecture.

scenario could be drawn. This is precisely the object of the so-called *Fendley's conjecture*:

*A symmetric space $G/H$ integrable NLSM is equivalent to the $k \to \infty$ limit of the corresponding WZW model $G_k/H_{Ik}$ perturbed by some operator.*

The first observation which confirms this general statement comes from central charges considerations. The naive degrees of freedom counting shows that the *UV* limiting theory should be that of $\dim(G) - \dim(H)$ free massless boson and since each boson carries a unit of central charge the same number also provides the central charge of the model. From the point of view of the WZNW it is known that they have a central charge of

$$c(G_k) = \frac{k \dim(G)}{k + \mathrm{cox}(G)} .$$

Thus the GKO construction [34] of coset CFT provides the general UV central charge

$$c\left(\frac{G_k}{H_{Ik}}\right) = \frac{k \dim(G)}{k + \mathrm{cox}(G)} - \frac{Ik \dim(H)}{Ik + \mathrm{cox}(H)} ,$$

which correctly tends to the degrees of freedom counting as $k \to \infty$.

The operatorial part of the identification can then be performed by recalling that the spectrum of fields in the coset theory can be extracted by the knowledge of

those of $G_k$ and $H_{Ik}$ respectively. The fields in the $G_k/H_{Ik}$ coset are constructed by decomposing the fields $\phi_G$ in $G_k$ into representations of $H_{Ik}$. Because the energy-momentum tensor obeys the orthogonal decomposition $T_G = T_H + T_{G/H}$, that of $\phi_G$ should be of the form

$$\phi_G = \bigoplus_a \phi^a_{G/H} \otimes \phi^a_H \,,$$

where the operator-valued coefficients $\phi^a_{G/H}$ are the fields of the coset model $G_k/H_{Ik}$. This field identification is needed in order to obtain the perturbing field explicitly by decomposing the WZW currents $J^A$, those of the $G_k$ theory, into fields of the coset theory. A consequence of $G/H$ being a symmetric space is that the generators of $G$ not in $H$ form a real irreducible representation of $H$ [118]. Therefore when a field $J^A(z)$ is decomposed into representations of $H$ there is only one term on the r.h.s which we denote by $\mathcal{J}^A$. The fields $\mathcal{J}^A$ form a real irreducible representation of $H$, of dimension $c_\sigma = \dim(G) - \dim(H)$ and are used to explicitly construct the perturbing operators

$$\Phi_{\text{pert.}} = \sum_{A=1}^{C_\sigma} \mathcal{J}^A(z)\mathcal{J}^A(\bar{z}) \,.$$

The field $\mathcal{J}^A$ has dimension 1 in the NSLM limit, so the perturbation $\Phi_{\text{pert.}}$ is of dimension 2 and so is naively marginal; the fact it is not exactly marginal is known as the phenomenon of dimensional transmutation in relation to asymptotic freedom. Therefore the coset and its perturbation have the general properties of a sigma model. Further support for this conjecture is discussed in [103, 128].

In the next chapter we will apply the TBA approach to a novel class of NLSM which will not fit in Fendley's classification and seems to enlarge it to even more interesting types of theories.

# Chapter 5

# The $(\mathbb{CP}^{N-1})_p$ Family

In this chapter we introduce a novel class of TBAs related to the *extended*-$\mathbb{CP}^{N-1}$ non-linear sigma model. The model has been introduced in Chapter 3 in relation with the $AdS/CFT$ correspondence. More precisely, a distinguished member of the integrable family, is believed to govern the low-energy limit of the $AdS_4 \times \mathbb{CP}^3$ type IIA superstring sigma model. After a more precise discussion about the quantum integrability of the models the TBA analysis is performed. All the steps from the $S$-matrix to the RG flows, i.e. the material covedered in the previous chapter, will be carried over and explained in some detail. The aim will be to conjecture an exact result for some physically interesting quantities associated to the UV limit of the RG flow of the model and, moreover, its identification in terms of perturbed coset CFT. The content of the chapter is based on the pubblication [146].

The structure of the chapter is the following: in Section 5.1 we explore further the integrability of the model and present the conjectured $S$-matrix along with the corresponding computation of the free-energy. In Section 5.2 we enter the details of the TBA procedure for the model at hand and we construct a truncated two parameter infinite family of novel TBA model. Section 5.3 is devoted to the RG flow analysis of the TBA equations and the successive interpretation in terms of perturbed conformal field theories.

## 5.1 Integrability consideration

To give more substantial argument to the quantum integrability of the model we can consider its elementary 2-body scattering. For these type of processes the 2-dimensional kinematic is constrained in such a way that the incoming momenta are separately conserved and the scattering amplitudes are functions of a single Mandelstam variable. Letting $\theta$ be the incoming particles' rapidity difference, the scattering we are considering are given by the $S$-matrix elements

$$\begin{aligned}\langle p_{3k}, p_{4l}|S|p_{1i}, p_{2j}\rangle &= S_{ij}^{kl}(\theta)\delta(p_1 - p_3)\delta(p_2 - p_4) + S_{ij}^{lk}(\theta)\delta(p_1 - p_4)\delta(p_2 - p_3)\,,\\ \langle p_{3k}, \bar{p}_{4l}|S|p_{1i}, \bar{p}_{2j}\rangle &= F_{ij}^{kl}(\theta)\delta(p_1 - p_3)\delta(\bar{p}_2 - \bar{p}_4) + B_{ij}^{lk}(\theta)\delta(p_1 - \bar{p}_4)\delta(\bar{p}_2 - p_3)\,,\end{aligned}$$

for the spinon-spinon and spinon-antispinon scattering processes. The $S$-matrix elements under the symmetry constraints imposed by the $U(N)$ invariance have been found long ago in [75]. There we learn that there are only six scalar amplitudes in terms of which the matrices $S, F$ and $B$ read

$$\begin{aligned}S_{ij}^{kl}(\theta) &= u_1(\theta)\delta_i^k\delta_j^l + u_2(\theta)\delta_i^l\delta_j^k\,,\\ F_{ij}^{kl}(\theta) &= t_1(\theta)\delta_i^k\delta_j^l + t_2(\theta)\delta_{ij}\delta^{kl}\,,\\ B_{ij}^{lk}(\theta) &= r_1(\theta)\delta_i^k\delta_j^l + r_2(\theta)\delta_{ij}\delta^{kl}\,.\end{aligned}$$

Moreover thanks to crossing symmetry these functions are not independent and we have

$$u_\alpha(i\pi - \theta) = t_\alpha(\theta)\,, \qquad\qquad r_1(i\pi - \theta) = r_2(\theta)\,. \tag{5.1}$$

The direct of computation of the large-$N$ expansion of these amplitudes is a direct application of the Feynman rules, which are the same as the $\mathbb{CP}^{N-1}$ model [63] except for the modified gauge field propagator [74]. In view of crossing symmetry (5.1) it suffices to evaluate the processes relative to the amplitudes $u_1, u_2$ and $r_1$; their expansion turns out to be given by

$$\begin{aligned}u_1(\theta) &= 1 - \frac{i\pi}{N}\left(\frac{2m^2}{sqrt{-}st} + \frac{s - 2m^2}{psqrt{-}st}\right) + O(N^{-2})\,,\\ u_2(\theta) = r_1(\theta) &= -\frac{i\pi}{N}\left(\frac{1}{\theta} + \frac{s}{s\theta + 2(p-1)sqrt{-}st}\right) + O(N^{-2})\,,\end{aligned} \tag{5.2}$$

in terms of the crucial parameter $p$ given by

$$p = \frac{q^2}{N(1+\lambda_\infty/\pi)}\,. \tag{5.3}$$

A closer look to (5.2) reveals that interesting limits in $p$ can be considered. As $p \to \infty$ the amplitudes reduce to those of the $O(2N)$ model written in $SU(N)$ variables. While if $p \to 0$ the spinon-spinon amplitude diverges, signaling the presence of the charge confinement effect in the $\mathbb{CP}^{N-1}$ model.

For what concerns integrability it is possible to show [75] that the factorizability of the scattering processes amounts, in this particularly symmetric case, to the constraint

$$u_2(\theta) = -\frac{i\nu}{\theta} u_1(\theta)\,, \tag{5.4}$$

for some constant $\nu$. Only two choices of $p$ are possible in order to satisfy condition (5.4) : $p = \infty$, the $O(2N)$ model limit, and $p = 1$. At this point, following the analysis of [72], it is possible to show that the amplitudes matches, in the large-$N$ leading order, those of the minimal reflectionless $SU(N)$-invariant $S$-matrix of [75].

### 5.1.1 Free energy matching

The large-$N$ analysis gives arguments in favor of the integrability of the model and thus it is worthwhile to try a non-perturbative approach to confirm the integrability condition $p = 1$. This can be done by evaluating the free energy of the theory at finite chemical potential, an operation that, remarkably, can be performed following two independent approaches allowing, in this way, for the comparison of the two calculation. On one side we can evaluate the free energy by perturbation theory, on the other hand by the $S$-matrix approach we can obtain the same quantity in an independent way. See [132, 133, 134] for the analogue computation for the $O(N)$ model. This technique has applied successfully to a number of theories [135, 136, 137] and consists of introducing the *ground state energy density* $\varepsilon$ of a gas of spinons with a finite density $\rho$. The free energy $f(h)$, $h$ being the chemical potential, can then be obtained by Legendre transforming the energy density,

namely

$$f(h) = \min_{\rho} \{\varepsilon(\rho) - \rho h\} \quad \rightarrow \quad \begin{cases} f(h) = \varepsilon(\rho) - \rho h \\ \\ h = \dfrac{d\varepsilon}{d\rho} \end{cases} .$$

The perturbative computation in this case has been performed in [74] and without entering the details of the computation we recall the $1/N$ expansion, which is given by

$$\begin{aligned} f(h) = -\frac{ph^2 N}{2\pi} \Bigg[ 1 - \frac{p(N-1)}{N \log(h/\Lambda)} + \\ - \frac{p(N-1)(N+p-2)\log\log(h/m) + pN(N-1)d_N}{N^2 \log^2(h/\Lambda)} + \cdots \Bigg] . \end{aligned} \tag{5.5}$$

where

$$d_N = -\log p + \frac{1-2p}{2} + \frac{3p-2}{2N} .$$

The result (5.5) of the perturbative computation must be matched with the computation coming from the $S$-matrix approach. To this end we recall that the relevant $S$-matrix amplitudes can be found in Appendix B. We remark that $S(\theta) = u_1(\theta) + u_2(\theta)$, i.e. the symmetric channel spinon-spinon scattering element, is the only relevant for the purpose of evaluating the free energy of the spinon gas. To this end notice that $S(\theta)$ enjoys, under the replacement $1/N \to 1 - 1/N$, the same functional form as in the $SU(N)$ Gross-Neveu model, treated in [135]. Borrowing their result and performing the same substitution into the free energy we obtain, in the $h \gg m$ limit,

$$\begin{aligned} f(h) = -\frac{h^2 N}{2\pi} \Bigg[ 1 - \frac{(N-1)}{N \log(h/\Lambda)} + \\ - \frac{(N-1^2)\log\log(h/m) + N(N-1)D_N}{N^2 \log^2(h/\Lambda)} + \cdots \Bigg] , \end{aligned} \tag{5.6}$$

with

$$D_N = \log\Gamma\left(1 + \frac{1}{N}\right) - \frac{\log 2}{N} - \frac{1}{2} + \frac{3}{2N} .$$

Remarkably the two logarithmic pattern shared by the forms (5.5) and (5.6) are compatible and, moreover, they match exactly if $p = 1$ for any $N$. This non-perturbative calculation confirms that the integrability is a property which goes beyond the $1/N$ order even though the parameters must be fine tuned in order to achieve it.

Further matching of the remaining terms allows for the identification of the relation between the spinon mass $m$ and the renormalization scale $\Lambda$, namely

$$\frac{m}{\Lambda} = \frac{(2/e)^{1/N}}{\Gamma\left(1+\frac{1}{N}\right)}\,, \tag{5.7}$$

in the $\overline{MS}$ subtraction scheme. From this further identification we can compare the mass gap with the known case $N = 2$ [135] and again discover that it is achieved if the integrability condition $p = 1$ holds.

### 5.1.2 The integrable model

According to the previous analysis, which parallels that of [74], setting $p = 1$ and fine-tuning the RG invariant $\lambda_\infty = 0$ result in an integrable *extended*-$\mathbb{CP}^{N-1}$ non-linear sigma model whose Lagrangian reads

$$L = \kappa\left(\partial_\mu - iA_\mu\right)\bar{z}\left(\partial^\mu + iA^\mu\right)z + i\bar{\psi}\gamma^\mu\left(\partial_\mu - iqA_\mu\right)\psi + \frac{1}{4\kappa}\left(\bar{\psi}\gamma^\mu\psi\right)^2\,. \tag{5.8}$$

The strong indications towards integrability of this $U(1)\times SU(N)$ symmetric model seems to be even more appealing if one focus on the $N = 4$ case, i.e. the low energy limit of the conjectured $AdS_4 \times \mathbb{CP}^3$ superstring sigma model.

Having collected sufficient convincing arguments in favor of the integrability of (5.8), including the explicit form for the $S$-matrix of the scattering of fundamental excitations, we now want to apply the TBA procedure of Chapter 4 to this model in order to establish further non-perturbative proof of its integrability.

## 5.2 A new family of TBA's

In this section we review the main step of the TBA procedure applied to the model 5.8. This will eventually lead to the formulation of a new family of TBA equations

related to the *extended*-$\mathbb{CP}^{N-1}$ gauged NLSM.

### 5.2.1 ABA and string hypothesis

The starting point of the TBA analysis are the Asymptotic Bethe Ansatz (ABA) equations in the NS sector of the $SU(4)\times U(1)$ symmetric model proposed in [74]. The explicit form is reported here for reference

$$
\begin{aligned}
e^{-imL\sinh\theta_k} &= \prod_{j\neq k}^{M} S(\theta_k-\theta_j)\prod_{j=1}^{\bar{M}} t_1(\theta_k-\bar{\theta}_j)\prod_{j=1}^{M_1}\left(\frac{\theta_k-\lambda_j+\frac{i\pi}{4}}{\theta_k-\lambda_j-\frac{i\pi}{4}}\right),\\
1 &= \prod_{j\neq k}^{M_1}\left(\frac{\lambda_k-\lambda_j+\frac{i\pi}{2}}{\lambda_k-\lambda_j-\frac{i\pi}{2}}\right)\prod_{j=1}^{M_2}\left(\frac{\lambda_k-\mu_j-\frac{i\pi}{4}}{\lambda_k-\mu_j+\frac{i\pi}{4}}\right)\prod_{j=1}^{M}\left(\frac{\lambda_k-\theta_j-\frac{i\pi}{4}}{\lambda_k-\theta_j+\frac{i\pi}{4}}\right),\\
1 &= \prod_{j\neq k}^{M_2}\left(\frac{\mu_k-\mu_j+\frac{i\pi}{2}}{\mu_k-\mu_j-\frac{i\pi}{2}}\right)\prod_{j=1}^{M_1}\left(\frac{\mu_k-\lambda_j-\frac{i\pi}{4}}{\mu_k-\lambda_j+\frac{i\pi}{4}}\right)\prod_{j=1}^{M_3}\left(\frac{\mu_k-\nu_j-\frac{i\pi}{4}}{\mu_k-\nu_j+\frac{i\pi}{4}}\right),\\
1 &= \prod_{j\neq k}^{M_3}\left(\frac{\nu_k-\nu_j+\frac{i\pi}{2}}{\nu_k-\nu_j-\frac{i\pi}{2}}\right)\prod_{j=1}^{M_2}\left(\frac{\nu_k-\mu_j-\frac{i\pi}{4}}{\nu_k-\mu_j+\frac{i\pi}{4}}\right)\prod_{j=1}^{\bar{M}}\left(\frac{\nu_k-\bar{\theta}_j-\frac{i\pi}{4}}{\nu_k-\bar{\theta}_j+\frac{i\pi}{4}}\right),\\
e^{-imL\sinh\bar{\theta}_k} &= \prod_{j\neq k}^{\bar{M}} S(\bar{\theta}_k-\bar{\theta}_j)\prod_{j=1}^{M} t_1(\bar{\theta}_k-\theta_j)\prod_{j=1}^{M_3}\left(\frac{\bar{\theta}_k-\nu_j+\frac{i\pi}{4}}{\bar{\theta}_k-\nu_j-\frac{i\pi}{4}}\right),
\end{aligned}
\tag{5.9}
$$

where the S-matrix elements are those of [75] in the reflectionless case, see Section 5.1. The spinon-spinon scattering amplitude $S$ is given by (B.1) while the spinon-antispinon element $t_1$ can be found in (B.3). In (5.9) $M$, $\bar{M}$ and $M_l$ with $l=1,2,3$ indicate the number of spinons, antispinons and flavour-$l$ magnons, respectively.

We stress the fact that with respect to [74], we have chosen the twist factor $q=1$, and redefined the magnonic rapidities as

$$
\lambda_k=\frac{\pi}{2}u_{1,k},\quad \mu_k=\frac{\pi}{2}u_{2,k},\quad \nu_k=\frac{\pi}{2}u_{3,k}\ .
$$

As for the $O(2N)$ case, and in general for sigma models, the analytic form of the $S$-matrix (i.e. its lack of periodicity) allows the existence of magnonic strings bound states [126], introduced previously in Section 4.3.4.1. In performing the thermodynamic limit, the dominant contribution to the free energy comes from

the excitation of magnonic strings of the form

$$\begin{aligned}
\lambda_{ka}^{(l)} &= \lambda_k^{(l)} + \frac{i\pi}{4}(l+1-2a), \quad (a=1,\ldots,l),\\
\mu_{kb}^{(m)} &= \mu_k^{(m)} + \frac{i\pi}{4}(m+1-2b), \quad (b=1,\ldots,m),\\
\nu_{kc}^{(n)} &= \nu_k^{(n)} + \frac{i\pi}{4}(n+1-2c), \quad (c=1,\ldots,n).
\end{aligned} \tag{5.10}$$

The strings are then treated as single objects which scatters among themselves and with the spinons and antispinons. The effective computational power of such a formulation is related to the simplifications which occurs if one tries to obtain the ABA for the strings. In doing this we multiply the ABA corresponding to different roots belonging to the same string, see Section 4.3.4.1, and obtain

$$\begin{aligned}
e^{-imL\sinh\theta_k} &= \prod_{j\neq k}^{M} S(\theta_k-\theta_j)\prod_{j=1}^{\bar{M}} t_1(\theta_k-\bar{\theta}_j)\prod_{l=1}^{\infty}\prod_{j=1}^{M^{(l)}}\left[S_{1,l}\left(\theta_k-\lambda_j^{(l)}\right)\right]^{-1},\\
1 &= \prod_{j=1}^{M} S_{l,1}\left(\lambda_k^{(l)}-\theta_j\right)\prod_{m=1}^{\infty}\prod_{j=1}^{M^{(m)}} S_{l,m}\left(\lambda_k^{(l)}-\mu_j^{(m)}\right)\\
&\quad\times\prod_{l'=1}^{\infty}\prod_{j=1}^{M^{(l')}}\left[S_{l,l'+1}\left(\lambda_k^{(l)}-\lambda_j^{(l')}\right)\right]^{-1}\left[S_{l,l'-1}\left(\lambda_k^{(l)}-\lambda_j^{(l')}\right)\right]^{-1},\\
1 &= \prod_{m'=1}^{\infty}\prod_{j=1}^{M^{(m')}}\left[S_{m,m'+1}\left(\mu_k^{(m)}-\mu_j^{(m')}\right)\right]^{-1}\left[S_{m,m'-1}\left(\mu_k^{(m)}-\mu_j^{(m')}\right)\right]^{-1}\\
&\quad\times\prod_{n=1}^{\infty}\prod_{j=1}^{M^{(n)}} S_{m,n}\left(\mu_k^{(m)}-\nu_j^{(n)}\right)\prod_{l=1}^{\infty}\prod_{j=1}^{M^{(l)}} S_{m,l}\left(\mu_k^{(m)}-\lambda_j^{(l)}\right),\\
1 &= \prod_{j=1}^{\bar{M}} S_{n,1}\left(\nu_k^{(n)}-\bar{\theta}_j\right)\prod_{m=1}^{\infty}\prod_{j=1}^{M^{(m)}} S_{n,m}\left(\nu_k^{(n)}-\mu_j^{(m)}\right)\\
&\quad\times\prod_{n'=1}^{\infty}\prod_{j=1}^{M^{(n')}}\left[S_{n,n'+1}\left(\nu_k^{(n)}-\nu_j^{(n')}\right)\right]^{-1}\left[S_{n,n'-1}\left(\nu_k^{(n)}-\nu_j^{(n')}\right)\right]^{-1},\\
e^{-imL\sinh\bar{\theta}_k} &= \prod_{j\neq k}^{\bar{M}} S(\bar{\theta}_k-\bar{\theta}_j)\prod_{j=1}^{M} t_1(\bar{\theta}_k-\theta_j)\prod_{n=1}^{\infty}\prod_{j=1}^{M^{(n)}}\left[S_{1,n}\left(\bar{\theta}_k-\nu_j^{(l)}\right)\right]^{-1},
\end{aligned} \tag{5.11}$$

where $M^{(q)}$ is the number of length-$q$ strings, and we have introduced the magnonic bound states scattering amplitudes

$$S_{l,m}(\theta) = \prod_{a=\frac{|l-m|+1}{2}}^{\frac{l+m-1}{2}} \left(\frac{\theta - i\frac{\pi a}{2}}{\theta + i\frac{\pi a}{2}}\right) = \prod_{a=1}^{l} \left(\frac{\theta - \frac{i\pi}{4}(l+m+1-2a)}{\theta + \frac{i\pi}{4}(l+m+1-2a)}\right). \tag{5.12}$$

The thermodynamic limit can then be performed upon introducing the densities of accessible states for spinons $\sigma$, antispinons $\bar{\sigma}$, for magnonic strings $\sigma_n^{(1)}$, $\sigma_n^{(2)}$, $\sigma_n^{(3)}$, likewise the occupied state densities $\rho$, $\bar{\rho}$, $\rho_n^{(1)}$, $\rho_n^{(2)}$, $\rho_n^{(3)}$. This can be done by a suitable modification of the definitions (4.11) and (4.10). Taking the logarithm of 5.11 and passing to the continuum we end up with the following system of *linear* integral equations for the densities

$$\begin{aligned}
\sigma(\theta) &= m\cosh\theta + \mathcal{K} * \rho(\theta) + G * \bar{\rho}(\theta) - \sum_{l=1}^{\infty} K_{1,l} * \rho_l^{(1)}(\theta), \\
\sigma_n^{(1)}(\theta) &= K_{n,1} * \rho(\theta) + \sum_{l=1}^{\infty} \left(K_{n,l} * \rho_l^{(2)}(\theta) - (K_{n,l+1} + K_{n,l-1}) * \rho_l^{(1)}(\theta)\right), \\
\sigma_n^{(2)}(\theta) &= \sum_{l=1}^{\infty} \left(K_{n,l} * \rho_l^{(3)}(\theta) + K_{n,l} * \rho_l^{(1)}(\theta) - (K_{n,l+1} + K_{n,l-1}) * \rho_l^{(2)}(\theta)\right), \\
\sigma_n^{(3)}(\theta) &= K_{n,1} * \bar{\rho}(\theta) + \sum_{l=1}^{\infty} \left(K_{n,l} * \rho_l^{(2)}(\theta) - (K_{n,l+1} + K_{n,l-1}) * \rho_l^{(3)}(\theta)\right), \\
\bar{\sigma}(\theta) &= m\cosh\theta + \mathcal{K} * \bar{\rho}(\theta) + G * \rho(\theta) - \sum_{l=1}^{\infty} K_{1,l} * \rho_l^{(3)}(\theta),
\end{aligned} \tag{5.13}$$

where $n = 1, 2, \ldots$. For what concerns the convolution kernels $\mathcal{K}(\theta)$, $G(\theta)$ and $K_{l,m}(\theta)$, they are listed and described in Appendix B. These integral equation constitutes the thermodynamic limit of the ABA (5.11) and can be interpreted as integral constraints imposed by integrability on the continuous densities. Having completed the preliminary operations of determining the thermodynamic behavior of the model we can now turn to the study of its mirror thermodynamics.

### 5.2.2 TBA equations and Y-System

The usual mirror thermodynamic procedure of minimizing the free energy $E - TS$ at the temperature $T = 1/R$ produces, as usual, the *physical TBA* equations for

the model. Without entering the details of the calculation, which can be found in [146], we display the resulting system of infinite non-linear coupled integral equations They read

$$\begin{aligned}
\epsilon_0(\theta) &= i\alpha + mR\cosh\theta - \mathcal{K} * L_0(\theta) - G * \bar{L}_0(\theta) - \sum_{l=1}^{\infty} K_{1,l} * L_{(1,l)}(\theta), \\
\epsilon_{(1,n)}(\theta) &= K_{n,1} * L_0(\theta) - \sum_{l=1}^{\infty} \left( K_{n,l} * L_{(2,l)}(\theta) - (K_{n,l+1} + K_{n,l-1}) * L_{(1,l)}(\theta) \right), \\
\epsilon_{(2,n)}(\theta) &= \sum_{l=1}^{\infty} \left( (K_{n,l+1} + K_{n,l-1}) * L_{(2,l)}(\theta) - K_{n,l} * L_{(1,l)}(\theta) - K_{n,l} * L_{(3,l)}(\theta) \right), \\
\epsilon_{(3,n)}(\theta) &= K_{n,1} * \bar{L}_0(\theta) - \sum_{l=1}^{\infty} \left( K_{n,l} * L_{(2,m)}(\theta) - (K_{n,l+1} + K_{n,l-1}) * L_{(3,l)}(\theta) \right), \\
\bar{\epsilon}_0(\theta) &= -i\alpha + R\cosh\theta - \mathcal{K} * \bar{L}_0(\theta) - G * L_0(\theta) - \sum_{l=1}^{\infty} K_{1,l} * L_{(3,l)}(\theta) \ .
\end{aligned} \tag{5.14}$$

in terms of

$$\frac{\rho(\theta)}{\sigma(\theta) - \rho(\theta)} = e^{-\epsilon_0(\theta)} \ , \quad \frac{\bar{\rho}(\theta)}{\bar{\sigma}(\theta) - \bar{\rho}(\theta)} = e^{-\bar{\epsilon}_0(\theta)} \ , \quad \frac{\rho_m^{(i)}(\theta)}{\sigma_m^{(i)}(\theta) - \rho_m^{(i)}(\theta)} = e^{-\epsilon_{(i,m)}(\theta)},$$

and

$$\begin{aligned}
L_0(\theta) = \ln\left(1 + e^{-\epsilon_0(\theta)}\right) \ , \quad \bar{L}_0(\theta) = \ln\left(1 + e^{-\bar{\epsilon}_0(\theta)}\right), \\
L_{(i,m)}(\theta) = \ln\left(1 + e^{-\epsilon_{(i,m)}(\theta)}\right),
\end{aligned}$$

with $i, m = 1, 2, \ldots$.

We should emphasize that in (5.14) we have included the chemical potential [88, 89] $\lambda = e^{i\alpha} = 1$ for the groundstate, while $\lambda = e^{i\alpha} = -1$ corresponds to the first excited state [138, 104] associated to the lifting, due to tunneling [139], of a two-fold vacuum degeneracy of the model. These two $\alpha$-vacua will be the main object of our investigation for what concerns the RG flow structure.

The expression for the $\alpha$-vacuum Casimir energy can then be obtained by employing equations (5.14) to obtain a form similar to (4.21), suitably adapted for the case at hand; The result turns out to be given by

$$E_\lambda(m, R) = -\frac{m}{2\pi} \int d\theta \, \cosh\theta \left( L_0(\theta) + \bar{L}_0(\theta) \right) \ . \tag{5.15}$$

The behavior of the energy in the far infrared $r = Rm \gg 1$ region can be extracted to be

$$E_{\pm 1}(m,R) \simeq \mp \frac{2m}{\pi} C_{(4,\infty)} K_1(mR), \tag{5.16}$$

where $K_1(x)$ is the modified Bessel function. The coefficient $C_{(4,\infty)}$ will be directly obtained from the TBA equations in Section 5.3.1 and it should match the number of $SU(4)$ flavours: $C_{(4,\infty)} = 4$ [74].

By making use of the kernel identities of Appendix B (see Section B.2) we can derive from the integral equations (5.14) a set of functional equation: the $Y$-system. They read

$$\begin{aligned}
Y_0(\theta + i\frac{\pi}{2})\, Y_0(\theta - i\frac{\pi}{2}) &= e^{-i4\alpha} \frac{\bar{Y}_0(\theta)}{Y_0(\theta)} \left(1 + Y_{(1,1)}(\theta + i\frac{\pi}{4})\right) \left(1 + Y_{(1,1)}(\theta - i\frac{\pi}{4})\right) \times \\
&\qquad \times \left(1 + Y_{(2,1)}(\theta)\right), \\
\bar{Y}_0(\theta + i\frac{\pi}{2})\, \bar{Y}_0(\theta - i\frac{\pi}{2}) &= e^{i4\alpha} \frac{Y_0(\theta)}{\bar{Y}_0(\theta)} \left(1 + Y_{(3,1)}(\theta + i\frac{\pi}{4})\right) \left(1 + Y_{(3,1)}(\theta - i\frac{\pi}{4})\right) \times \\
&\qquad \times \left(1 + Y_{(2,1)}(\theta)\right),
\end{aligned} \tag{5.17}$$

for the massive $Y$-functions and

$$\begin{aligned}
Y_{(1,l)}(\theta + i\frac{\pi}{4})\, Y_{(1,l)}(\theta - i\frac{\pi}{4}) &= \left(1 + \delta_{l1} Y_0(\theta)\right) \frac{\left(1 + Y_{(1,l-1)}\right)(\theta) \left(1 + Y_{(1,l+1)}(\theta)\right)}{\left(1 + \frac{1}{Y_{(2,l)}}(\theta)\right)} \\
Y_{(2,l)}(\theta + i\frac{\pi}{4})\, Y_{(2,l)}(\theta - i\frac{\pi}{4}) &= \frac{\left(1 + Y_{(2,l-1)}(\theta)\right) \left(1 + Y_{(2,l+1)}(\theta)\right)}{\left(1 + \frac{1}{Y_{(1,l)}(\theta)}\right) \left(1 + \frac{1}{Y_{(3,l)}(\theta)}\right)} \\
Y_{(3,l)}(\theta + i\frac{\pi}{4})\, Y_{(3,l)}(\theta - i\frac{\pi}{4}) &= \left(1 + \delta_{l1} \bar{Y}_0\right) \frac{\left(1 + Y_{(3,l-1)}(\theta)\right) \left(1 + Y_{(3,l+1)}(\theta)\right)}{\left(1 + \frac{1}{Y_{(2,l)}(\theta)}\right)},
\end{aligned} \tag{5.18}$$

for the magnonic ones. The $Y$-functions are related to the corresponding pseudoenergy by means of

$$Y_0(\theta) = e^{-\epsilon_0(\theta)}, \quad \bar{Y}_0(\theta) = e^{-\bar{\epsilon}_0(\theta)}, \quad Y_{(i,l)}(\theta) = e^{\epsilon_{(i,l)}(\theta)}\ . \tag{5.19}$$

In trying to construct a diagrammatic representation of such a system in terms of the Dynkin-like picture outlined in Chapter 4 we immediately identify for the

magnonic nodes a structure of type $A_3 \Diamond A_\infty$, as typical for NLSM [91, 121, 122]. Due to the presence of the product of massive $Y$-functions at r.h.s. it is harder to give a graphical prescription for the massive equations. Things become easier if one consider the following key relation

$$\frac{Y_0(\theta + i\frac{\pi}{4})\, Y_0(\theta - i\frac{\pi}{4})}{\bar{Y}_0(\theta + i\frac{\pi}{4})\, \bar{Y}_0(\theta - i\frac{\pi}{4})} = e^{-i4\alpha} \frac{1 + Y_{(1,1)}(\theta)}{1 + Y_{(3,1)}(\theta)}. \tag{5.20}$$

which allow us the *crossing* of the massive functional equations. We obtain the following more standard looking pair of equations

$$\begin{aligned} Y_0^+ \bar{Y}_0^- &= \left(1 + Y_{(1,1)}\left(\theta + i\frac{\pi}{4}\right)\right)\left(1 + Y_{(2,1)}(\theta)\right)\left(1 + Y_{(3,1)}\left(\theta - i\frac{\pi}{4}\right)\right), \\ \bar{Y}_0^+ Y_0^- &= \left(1 + Y_{(3,1)}\left(\theta + i\frac{\pi}{4}\right)\right)\left(1 + Y_{(2,1)}(\theta)\right)\left(1 + Y_{(1,1)}\left(\theta - i\frac{\pi}{4}\right)\right), \end{aligned} \tag{5.21}$$

where we introduce the short-hand notation $Y^{\pm}(\theta) = Y(\theta \pm i\pi/2)$. We can encode these relation in the diagram of Figure 5.1 by suitably adapting the graphical rules stated in Chapter 4. By looking at (5.21) we immediately notice that the shifted

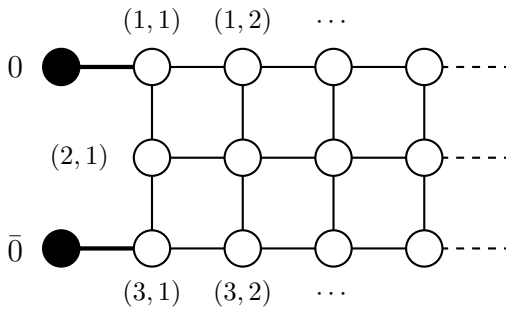



FIGURE 5.1: Dynkin-like diagram of the *extended*-($\mathbb{CP}^3$) sigma model TBA description.

$Y$-functions become crossed for the massive nodes. The possibility of this kind of crossed form has been already noticed in [140, 141] in the non-relativistic context of the full theory of which the model under consideration is the (relativistic) low energy reduction. In all known cases apparently only massive nodes enjoy crossed form equations. This novel type of "crossed" Y-system, without shifts on the r.h.s., was first obtained in [140] and [142], in the context of the TBA for anomalous dimensions in the planar $\mathcal{N} = 6$ superconformal Chern-Simons, i.e. $AdS_4/CFT_3$. Pictorially, the related $Y$-system diagram [140, 142] may be obtained from that for planar $AdS_5/CFT_4$ by means of some sort of "folding" process of the two wings with doubling of the fixed row of massive nodes; the same relation seems to hold between their low energy decoupled models, namely the present $\mathbb{CP}^3 \times U(1)$ [82]

and the $O(6)$ nonlinear sigma models [143], respectively. At last but not least, an intriguing example of "crossed" $Y$-system describes the strong coupling behaviour of the gluon scattering amplitudes in $SYM_4$ [141]. The key observation that can

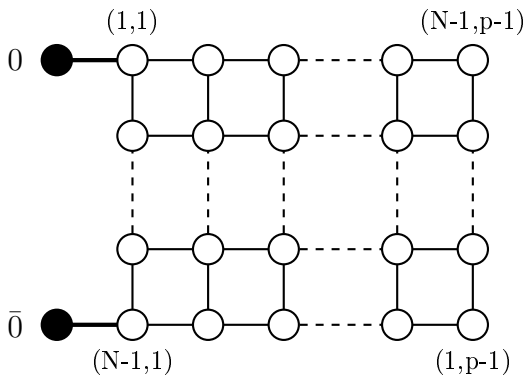



FIGURE 5.2: The diagrammatic representation of the generic $(\mathbb{CP}^{N-1})_p$ model $Y$-system formulation.

be drawn from (5.21) is that, thanks to crossing, we do not have any unusual term in the r.h.s., but just a chain of movements on the diagram and in $\theta$ in such a way that they respect the *conservation of the* algebraic[1] *sum of the displacements*. In other words, for any $Y$ the *algebraic sum* of all the displacements stays the same; moreover its absolute value coincides also with the absolute value of the total amount of displacement in the l.h.s., *i.e.*, in suitable units, exactly 2. Because of the algebraic sum, the path starting from $\mathbf{0}$ can reach $\bar{\mathbf{0}}$ and viceversa, contrarily to what happens in the (usual) non-crossed form.

Getting inspired by the empirical conservation rule just stated we have been able to postulate a relevant generalization of $Y$-system (5.21-5.17). Since the emerging pattern begins to rensemble the scenarios outlined in Section 4.3.1, up to possible generalization, it seems natural to consider a more general family of systems, stemming from the introduction of two positive integers $N$ and $p$, so that we conjecture for the massive nodes the equations

$$\begin{aligned} Y_0\left(\theta+i\frac{\pi}{2}\right)\bar{Y}_0\left(\theta-i\frac{\pi}{2}\right) &= \prod_{l=1}^{N-1}\left(1+Y_{(l,1)}\left(\theta+i\frac{\pi}{2}-i\frac{\pi l}{N}\right)\right), \\ Y_0\left(\theta-i\frac{\pi}{2}\right)\bar{Y}_0\left(\theta+i\frac{\pi}{2}\right) &= \prod_{l=1}^{N-1}\left(1+Y_{(l,1)}\left(\theta-i\frac{\pi}{2}+i\frac{\pi l}{N}\right)\right), \end{aligned} \tag{5.22}$$

[1] Not the absolute values sum, as in the above uncrossed form.

while, for the magnonic nodes, the straightforward generalization of (5.18) is given by

$$Y_{(i,j)}\left(\theta+i\frac{\pi}{N}\right)Y_{(i,j)}\left(\theta-i\frac{\pi}{N}\right)=\left(1+\delta_{i,1}\delta_{j,1}Y_0(\theta)+\delta_{i,N-1}\delta_{j,1}\bar{Y}_0(\theta)\right)\times \\ \times\prod_{l=1}^{p-1}\left(1+Y_{(i,l)}(\theta)\right)^{A^{(p-1)}_{l,j}}\prod_{l'=1}^{N-1}\left(1+\frac{1}{Y_{(l',j)}(\theta)}\right)^{-A^{(N-1)}_{l',i}}\quad ; \tag{5.23}$$

where $A^n$ is the incidence matrix of a $A_n$-type Lie algebra; explicitly we have

$$A^n_{ij}=\delta_{i,j+1}+\delta_{i,j-1}\qquad\text{for}\qquad i,j=1,2,\cdots,n\,.$$

The $Y$-system (5.22-5.23) can now be pictorially represented as in Figure 5.2 where the only difference is represented by the solid link connecting the massive node to the magnonic block. The meaning of such a graphical notation is to remind that since the shifts in the arguments of the $Y$-functions are different one should proceed along the first magnonic row.

Obviously, the system studied so far is recovered by fixing $(N,p)$ to $(4,\infty)$. With this simple generalisation, we are able to describe a previously unknown infinite family of Y-systems naturally associated to a generic $SU(N)$ algebra with quantum reduced coset level $p$. As we shall see in the following section, the obtained truncated family of Y-systems exhibit all the important features common to more standard types of Y-systems. In particular, they can be interpreted as periodic sets of discrete recursion relations [90] and their solutions lead to sum-rules [102] and functional identities for the Rogers dilogarithm [145].

We should stress that the results presented so far have been rigorously derived only for $(N,p)=(4,\infty)$, from now on we shall leave the two positive integers $N$ and $p$ unconstrained to explore the consistency of our congecture in terms of RG flow analysis. By somehow reversing the arguments of Section 4.2.2, it is convenient, for later purpose, to transform the Y-system into the Zamolodchikov's universal TBA form [90]. Thanks to the Fourier Transforms in (B.8), we obtain

the $(\mathbb{CP}^{N-1})_p \times U(1)$ TBA

$$\begin{aligned}
\epsilon_0(\theta) + \bar{\epsilon}_0(\theta) &= 2mR\cosh\theta - \sum_{l=1}^{N-1} \chi_{(1-\frac{2l}{N})} * \Lambda_{(l,1)}(\theta), \\
\epsilon_0(\theta) - \bar{\epsilon}_0(\theta) &= i2\alpha - \sum_{l=1}^{N-1} \psi_{(1-\frac{2l}{N})} * \Lambda_{(l,1)}(\theta), \\
\epsilon_{(i,j)}(\theta) &= \delta_{i,1}\delta_{j,1}\phi_{\frac{N}{2}} * L_0(\theta) + \delta_{i,N-1}\delta_{j,1}\phi_{\frac{N}{2}} * \bar{L}_0(\theta) + \\
&\quad + \sum_{l=1}^{p-1} A^{(p-1)}_{l,j}\phi_{\frac{N}{2}} * \Lambda_{(i,l)}(\theta) - \sum_{l=1}^{N-1} A^{(N-1)}_{l,i}\phi_{\frac{N}{2}} * L_{(l,j)}(\theta),
\end{aligned} \tag{5.24}$$

with $\alpha \in \{0, \pi\}$, $\Lambda_A(\theta) = \ln(1 + e^{\epsilon_A(\theta)})$. The corresponding $\alpha$-vacuum energy is given by equation (5.15), in which the dependence on $\lambda$ enters through the implicit dependence on $\alpha$ of the solutions to equation (5.24). We can generalize as well the IR limiting behavior of the free energy to

$$E_{\pm 1}(m, R) \simeq \mp \frac{2m}{\pi} C_{(N,p)} K_1(mR),$$

in the $Rm \gg 1$ infrared region. The coefficient $C_{(N,p)}$, which contains information on the $SU(N)$-related vacuum structure of the model at $(N, p)$ generic [91, 144], its central charge and underlying CFT, will be determined in the following section.

## 5.3 Perturbed coset realization

In this section we want to study the RG flow to the UV point of the two $\alpha$-vacua described by TBA (5.24) for $\alpha = 0, \pi$. The result obtained in this section rely mostly on numerical simulations due to the analytical difficulty of the equations involved. Our aim will be the identification of the model in terms of a suitable coset CFT and its perturbation primary field.

As customary, the $(\mathbb{CP}^{N-1})_p \times U(1)$ model can be thought of as a CFT perturbed by a marginally relevant operator, whose vacuum energy is given by the expression (5.15) endowed with the (vacuum) TBA solution. In particular, the CFT is characterised by the value of its conformal anomaly, $c$, which peculiarly enters the vacuum energy (5.15) in the UV limit $mR \ll 1$ [98]:

$$E_{UV} = -\frac{\pi}{6R}(c - 12d) \tag{5.25}$$

where $d = \Delta + \bar{\Delta}$ is the (conformal) dimension of the ground state of the theory and disappear for unitary theories, as here: $x = \Delta = \bar{\Delta} = 0$.

### 5.3.1 Central charge

As explained in Section 4.2.3 the quest for evaluating the effective central charge amounts to solve the plateau equations, i.e. find the stationary solution of the $Y$-system (5.22-5.23), associated to the UV and IR regimes of the TBA equations. The effective central charge splits into two parts

$$c_\lambda(N,p) = c_\lambda^{(UV)} - c_\lambda^{(IR)}\,, \tag{5.26}$$

where the two contributions are explicitly given by

$$c_\lambda^{(UV)} = \frac{6}{\pi^2}\left[\mathcal{L}\left(\frac{y_0}{1+y_0}\right) + \mathcal{L}\left(\frac{\bar{y}_0}{1+\bar{y}_0}\right) + \sum_{i=1}^{N-1}\sum_{l=1}^{p-1}\mathcal{L}\left(\frac{y_{i,l}}{1+y_{i,l}}\right)\right] \tag{5.27}$$

and

$$c_\lambda^{(IR)} = \frac{6}{\pi^2}\sum_{i=1}^{N-1}\sum_{l=1}^{p-1}\mathcal{L}\left(\frac{z_{i,l}}{1+z_{i,l}}\right) \tag{5.28}$$

in terms of the Rogers Dilogarithm function $\mathcal{L}(x)$, see Appendix A for more details. The constants $y$s and $z$s are given by the stationary solutions of the Y-system (5.22 - 5.23) and thus we expect different solutions according to the value of the chemical potential $\lambda = e^{i\alpha}$. We start by analysing the 0-vacuum, i.e. the ground state of the theory.

### 5.3.2 The ground state

The 0-vacuum represents the true ground state of the theory and since we expect a coset CFT as underlying theory we can assume that the theory will be *unitary*. This will imply in turn that the value of the effective central charge will coincide with the central charge of the limiting UV conformal theory.

The constant values $y$s we need to plug into sum rule (5.27) to produce the charge are the stationary solutions of

$$
\begin{aligned}
y_0\,\bar{y}_0 &= \prod_{l=1}^{N-1}\left(1+y_{(l,1)}\right)\,,\\
y_{(i,j)}^2 &= (1+\delta_{i,1}\delta_{j,1}y_0+\delta_{i,N-1}\delta_{j,1}\bar{y}_0)\prod_{l=1}^{p-1}\left(1+y_{(i,l)}\right)^{A^{(p-1)}_{l,j}}\prod_{l'=1}^{N-1}\left(1+\frac{1}{y_{(l',j)}}\right)^{-A^{(N-1)}_{l',i}}\,;
\end{aligned}
\tag{5.29}
$$

they have been calculated only numerically (see Section 6.4.2 for a detailed account) except for some low dimensional cases [146]. Letting

$$
\varphi=\frac{\pi}{2(N+p+1)} \tag{5.30}
$$

the results for lower ranks are the following

- $N=2$:

$$
y_{(1,i)}=(p-i)(p-i+2),\;\; y_0=\bar{y}_0=p,
$$

  with $i=1,2,\ldots,p-1$.

- $N=3$:

$$
y_{(1,i)}=y_{(2,i)}=\frac{\sin((p-i)\varphi)\sin((p-i+3)\varphi)}{\sin(\varphi)\sin(2\varphi)},
$$

  with $i=0,1,\ldots,p-1$ and $y_0=\bar{y}_0=y_{(1,0)}=y_{(2,0)}$.

- $N=4$:

$$
\begin{aligned}
y_{(1,p-1)}=y_{(3,p-1)}&=\frac{2\sin(2\varphi)+\sin(6\varphi)+\sin(10\varphi)}{2\sin(6\varphi)},\\
y_{(2,p-1)}&=\frac{2\sin(2\varphi)+\sin(6\varphi)+3\sin(10\varphi)}{2\sin(2\varphi)+3\sin(6\varphi)+\sin(10\varphi)}.
\end{aligned}
$$

Despite the analytic form of the solutions for generic $N$ and $p$ is yet unknown we were able to conjecture a result for the sum rule (5.27) at $\lambda=1$. Nonetheless we resort to a very high precision numerical computation, see Section 6.4 for details. Starting from $p=2$ and $N=2$ we were able to obtain the constants $y$s with a precision of about $10^{-15}$, for $p<20$ and $N<5$. Due to intrinsic loss of precision

with increasing size of the system the accuracy progressively decreased down to $10^{-12}$ for values around $p = 61$ and $N = 4$. The numerical results lead to the following precise conjecture for the sum rule (5.27)

$$c_1^{(UV)}(N,p) = \frac{p(1+pN-p)}{p+N-1}. \tag{5.31}$$

For what concern the IR regime instead the situation is well known in literature and the resulting model falls in the classification of Section 4.3.1. More precisely by the usual procedure of removing the massive nodes and the corresponding links we end up with a TBA model of type $A_{N-1}\Diamond A_{p-1}$, according to the classification of [107]. The constant $z$s are instead the solutions of

$$z_{(i,j)}^2 = \prod_{l=1}^{p-1} \left(1+z_{(i,l)}\right)^{A_{l,j}^{(p-1)}} \prod_{l'=1}^{N-1} \left(1+\frac{1}{z_{(l',j)}}\right)^{-A_{l',i}^{(N-1)}}, \tag{5.32}$$

they moreover are analytically known to be [102]:

$$z_{(i,j)} = \frac{\sin((j+N)\phi)\sin(j\phi)}{\sin((i+p)\phi)\sin(i\phi)}, \tag{5.33}$$

with $\phi = \pi/(p+N)$, and the corresponding Rogers dilogarithm sum-rule is

$$c_1^{(IR)}(N,p) = \frac{6}{\pi^2}\sum_{i=1}^{N-1}\sum_{l=1}^{p-1} \mathcal{L}\left(\frac{z_{(i,l)}}{1+z_{(i,l)}}\right) = \frac{p(N-1)(p-1)}{p+N}. \tag{5.34}$$

Equations (5.32), (5.33) and (5.34) constitutes the only known example of fully analytic proved in all the steps example of dilogarithm sum rule, as far as concerns the knowledge of who writes. This total analytic control has made possible to develop, check and tune the algorithms we relied on for the numerical computations. These numerical routines, described in Chapter 6, were then employed to analyze the novel case (5.29-5.27).

Finally, subtracting (5.34) from (5.31) we obtain the central charge of the underlying CFT, which reads

$$c_1(N,p) = \frac{p(1-p-N+N^2+2Np)}{(N+p)(N+p-1)}. \tag{5.35}$$

It turns out that the central charge (5.35) deduced from equations (5.22,5.23), can be written as

$$c_{(N,p)} = \frac{p\,\dim[SU(N)]}{p+N} - \frac{p\,\dim[SU(N-1)])}{p+N-1}\,,$$

which incidentally is that of the coset model

$$\frac{SU(N)_p}{SU(N-1)_p \times U(1)} \times U(1) \equiv \frac{SU(p)_{N-1} \times SU(p)_1}{SU(p)_N} \times U(1)\,. \tag{5.36}$$

This identification is the result of a series of trial and check attempts which have been progressively tested with the know data obtained from numerical simulations, as explained below. For the moment we simply mention the fact that we can see that the known Fendley's conjecture holds even in this case which, by virtue of its very bosonic and fermionic formulation, is not included in the classification of [104]. In this sense we believe that the particular functional form of the $Y$-system associated may be in some way related to the necessary introduction of fermions to provide the integrability of the model. The subject is now under current investigation.

Another non-trivial test that central charge (5.35) should pass is represented by the sigma model limit $p \to \infty$. In this limit we recover the $\mathbb{CP}^{N-1} \times U(1)$ sigma model to which remains associated the central charge

$$c_{(N,\infty)} = \dim[SU(N)] - \dim[SU(N-1)] = 2N-1\,, \tag{5.37}$$

a result that coincides with the value predicted in [74], through a naive degree of freedom counting argument. The argument is essentially based on the analysis of the renormalization flow of the manifold which is known to flow to a flat space in the UV regime and we will outline it.

The $\mathbb{CP}^{N-1}$ manifold is usually defined by introducing, on the element of the complex unit sphere $B = \left\{z \in \mathbb{C}^N | z^\dagger z = 1\right\}$, the equivalence relation $z' = e^{i\beta} z$. This operation result in complex manifold with boundary of real dimension

$$\dim\left(\mathbb{CP}^{N-1}\right) = 2(N-1)\,, \tag{5.38}$$

as can be obtained by the direct counting of the constraints imposed by the definition. Alternatively we could look at the coset realization of $\mathbb{CP}^{N-1}$ which reads

$$\mathbb{CP}^{N-1} = \frac{SU(N)}{SU(N-1) \times U(1)} \tag{5.39}$$

and by counting the dimensions of the unitary algebra involved obtain the same result of (5.38). At this point it is easy to obtain the UV bosonic degrees of freedom of the manifold $\mathbb{CP}^{N-1} \times U(1)$ to be exactly given by (5.39). In terms of the UV CFT this can be understood because the bosonic degrees of freedom carries a unit of central charge, and there are $2n-2$ of them according to (5.38), as well as the Dirac fermion, which is constitutes by two chiral components of central charge $1/2$.

The observation of equation (5.36) is crucial but not sufficient in trying to identify the family of models we are considering but gives a good candidate which we can test. Identifications based only on central charge are by no means unique, for example the two $U(1)$ factors in (5.36) yield compensating contributions to $c_1(N,p)$ leading to an equivalently good match with the central charge of the $\frac{SU(N)_p}{SU(N-1)_p}$ coset.

To further support the identification (5.36), following [90], we have determined the conformal dimension $\Delta_{(N,p)}$ of the perturbing operator using the intrinsic periodicity properties of the $Y$-system at finite $N$ and $p$. Assuming arbitrary initial conditions and using the $Y$-system as a recursion relation, as explained in detail in Section 6.4, we discovered that the following periodicity property holds

$$Y_A\Big(\theta + i\pi P(N,p)\Big) = Y_A(\theta) \quad \text{with} \quad P(N,p) = \frac{2(p+N-1)}{N}\,, \tag{5.40}$$

for any $Y$-function of the system, labelled generically by the multi-index $A$. Thus, according to [90] (cf. also [91, 101]), we can conclude that

$$\Delta_{\text{per}}(N,p) = 1 - \frac{1}{P_{(N,p)}} = 1 - \frac{N}{2(p+N-1)}, \tag{5.41}$$

is the conformal dimension of the operator which perturbs the conformal field theory at finite $p$ and generic $N$. The immediate consequence of this result is that the possibility $S(N)_p/SU(N-1)_p$ is discarded since in the corresponding table of the conformal dimensions of primary fields there's no sign of $\Delta_{\text{per}}(N,p)$.

To proceed further, we have made the simplest assumption that the two CFTs, originally disconnected and respectively related to the coset part and the $U(1)$, are tied together by the perturbing operator $\phi_{(N,p)}$ in the simplest possible way, namely

$$\phi_{(N,p)} = \phi_{[(\mathbb{CP}^{N-1})_p]} \times \phi_{[U(1)]}, \quad \Delta_{(N,p)} = \Delta_{[(\mathbb{CP}^{N-1})_p]} + \Delta_{[U(1)]}.$$

For the identification of $\Delta_{[(\mathbb{CP}^{N-1})_p]}$ and $\Delta_{[U(1)]}$, the presence of two independent integer parameters was very important as both $\Delta_{[(\mathbb{CP}^{N-1})_p]}$ and $\Delta_{[U(1)]}$ depend nontrivially on $N$ and $p$. This has given us the possibility of matching our results in particular limit in which it reproduces known models:

- At $p = 1$, the TBA equations (5.24) reduce to those for a pair of free fermion. This fact leads to

$$\Delta_{[(\mathbb{CP}^{N-1})_1]} = 0 \ , \quad \Delta_{[U(1)]} = \Delta_{(N,1)} = 1/2. \tag{5.42}$$

  which is a standard consequence of conformal symmetry. Also the central charge displays the expected behavior since $c_1(N, 1) = 1$ and it is well know that each free fermion carries a central charge of $1/2$.

- At $N = 2$, the TBA equations coincide with the $D_{p+1}$ models with two massive nodes placed on the fork and a tail of magnons. These ground state TBA equations were identified in [147] (see, also [101]), up to possible orbifold ambiguities, with a particular series of points of the Fractional Supersymmetric Sine-Gordon (FSSG) model [148]. The latter identification leads to the further constraint

$$\Delta_{[(\mathbb{CP}^1)_p]} = \frac{(p-1)}{p} \ , \quad \Delta_{[U(1)]} = \frac{1}{p(p+1)}. \tag{5.43}$$

  The effective central charge in this limit, namely $c_1(2, p) = 3p/(p + 2)$, reproduces correctely that of the FSSG [147].

Relations (5.42) and (5.43) together, allow to select the conformal dimension uniquely:

$$\Delta_{[(\mathbb{CP}^{N-1})_p]} = \frac{(p-1)(N+2p)}{2p(N+p-1)} \ , \quad \Delta_{[U(1)]} = \frac{N}{2p(N+p-1)} \ . \tag{5.44}$$

It is interesting to notice that for $p = 2$ the dimension $\Delta_{[(\mathbb{CP}^{N-1})_p]}$ corresponds to the field $\phi_{21}$ of the $c < 1$ minimal models $\mathcal{M}_{N+1,N+2}$, while for generic $N$ and $p$ it coincides precisely with the conformal dimension of the field $(p, \bar{p}, 1)+(\bar{p}, p, 1)$ in the $W^{(p)}$ minimal model $\frac{SU(p)_{N-1}\times SU(p)_1}{SU(p)_N}$, mentioned by Fendley [103] while discussing integrability issues related to the purely-bosonic $\mathbb{CP}^{N-1}$ sigma model.

Finally, following [91, 144], studying equations (5.24) in the infrared $mR \gg 1$ regime

$$\epsilon_0(\theta) - i\alpha \simeq \bar{\epsilon}_0(\theta) + i\alpha \simeq mR\cosh\theta - \frac{1}{2}\sum_{l=1}^{N-1}\ln(1 + z_{(l,1)}), \tag{5.45}$$

we find

$$E_{\pm 1}(m, R) \simeq \mp\frac{2m}{\pi}C_{(N,p)}K_1(mR),$$

with

$$C_{(N,p)} = \sqrt{\prod_{l=1}^{N-1}\left(1 + z_{(l,1)}\right)} = \frac{\sin(N\phi)}{\sin(\phi)}\ , \tag{5.46}$$

where we defined $\phi = \pi/(N + p)$ as before. In the sigma model limit $p \to \infty$: $\phi \to 0$ and (5.46) gives $C_{(N,\infty)} = N$, as expected.

### 5.3.3 The first excited state

The $\pi$-vacuum is the first excited state of the theory, more precisely it can be considered as the quantum lifting due to tunneling of a second ground state of the classical theory. The first excited state is described by the TBA (5.24) at chemical potential $\lambda = -1$ or, equivalently, at $\alpha = \pi$. A glance at the equations reveals that an important modification has occurred: the massive driving terms developed an imaginary part becoming complex-valued objects which calls for the analytic continuation of the TBA in the complex rapidity plane. Once more the problem is of formidable technical difficulty from the analytic point of view and we had to resort to numerical simulations, as for the ground state. In particular the usual procedure to evaluate the effective central charge for this state, namely to find the stationary solutions of the associated $Y$-system and plug them in the corresponding dilogarithm sum rule, has revealed unsuccessful. This is mainly related to the presence of complex-valued objects which spoils the efficiency of the algorithm. For the time being it seems that a direct computation of the effective central charge by means of a sum rule is out of scope.

We decided then to change perspectives and return to the very definition (4.5) of the central charge and rewrite it as

$$c_{-1}(N,p) = \lim_{r\to 0} c_{-1}(r) = \lim_{R\to 0} -\frac{6}{\pi R} E_{-1}(R).$$

where the central anomaly $c_{-1}(N,p)$ is the quantity we are interested in. Last expression has to be intended in a computational sense, i.e. as a procedure to reach the desired quantity numerically. Without entering the details, which are deferred to Section 6.3, we just want to highlight some milestone of the analysis. The idea we used is to start integrating numerically the TBA integral equations (5.24) (at $\lambda = -1$) at large values of the dimensionless parameter $r = mR \gg 1$. In this region we know the asymptotic behavior of the massive solutions, see (5.45), which are not spoiled by the introduction of a constant, although complex, shift in the driving terms. Having the analytic of the solution we can test the region of $r$ in which the numerical algorithm are reliable and stable, as we compare the numerical results with (5.45). Then we progressively decrease the value of $r$ by using the result of the previous computation as initial guess for the next one. In doing this we follow the correct solution from the IR region down to the UV regime. For what concerns the precision, in this case we obtained a neat loss of accuracy: for the smaller systems, namely $N, p < 3$, we were able to obtain precisions of the order of $10^{-4}$ which progressively increases to $10^{-2}$ for larger systems, $N \cdot p \sim 15$. The reason of this has to be traced to the process of solution itself which starts in the IR asympotic region computing solutions which are intrinsically affected by numerical errors. Thus in progressively reducing the value of $r$ these errors accumulate spoiling the correct result. Nonetheless thanks to the implementation techniques and numerical strategies developed in Section 6.3 we manage to conjecture a result for the effective central charge even in this case.

Again we stress that this result is obtained as continuation in the parameter $r$ for the TBA integral system (5.24) (at $\lambda = -1$) and in this sense it lacks a cross check with respect to the computation of plateau solutions and dilogarithm sum rule, as was the case for the ground state.

The numerical work has led us to conjecture the following value for the effective central charge of the $\pi$-vacuum

$$c_{-1}(N,p) = \lim_{r\to 0} c_{-1}(r) = -\frac{p\,(n+1)(2n+p-1)}{(n+p-1)(n+p)}\,. \tag{5.47}$$

The lowered accuracy of the numerical simulations called for some theoretical checks in order to verify the robustness of (5.47). As with the ground state, two particular limits can be compared with known results: $p = 1$ at generic $N$ and $N = 2$ at generic $p$. These limits can be identified with a pair of free fermions, the former, and with the FSSG, the latter.

To compare result (5.47) with the known models we need to know their behavior in presence of chemical potential $\lambda = -1$. In this regime a free fermion is known to develop an effective central charge of value $-1$. This can be seen by analitic continuation in the argument of the Rogers dilogarithm function and can be represented as

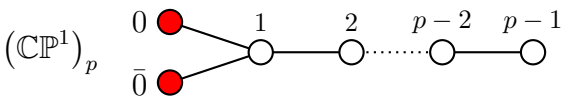
$$c\Big(\bullet\Big) = -1\,,$$

where the red coloration of the node reminds that the chemical potential is set to $-1$.

The $\pi$-vacuum of the $\left(\mathbb{CP}^{N-1}\right)_p$ family reduces at $p = 1$ and for every $N$ to a couple of free fermions of the type described above. It is thus non trivial that (5.47) reproduces exactly this result, as a matter of fact

$$c_{-1}(N,1) = c\Big(\begin{smallmatrix}\bullet\\ \bullet\end{smallmatrix}\Big) = 2\times c\Big(\bullet\Big) = -2\,.$$

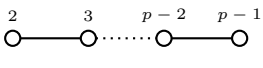
For the FSSG the match with (5.47) is still there but we need a few more elements beacause of the appearance of the chemical potential. To this aim let us consider a particular FSSG encoded in a $D_{p+1}$ diagram with the masses on the fork and at chemical potential $\lambda = -1$. Using the colored notation it can be represented as

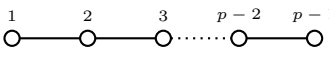


For which the effective central charge reads [147]

$$c_{-1}(2,p) = -\frac{3p\,(3+p)}{(1+p)(2+p)}\,. \tag{5.48}$$

With the introduction of the chemical potential the FSSG modify its ground state behavior. The way of computing the effective central charge is the same but the asymptotics for the nodes result slightly modified. In the UV limit the node labelled by 1 decouples from the system while in the IR regime, as usual, the massive nodes decouple. This lead to the following effective central charge

$$c_{-1}(D_{p+1}) = c\Big($$


$$\Big) - c\Big($$


$$\Big) =$$

$$= 2 \times c\Big($$

$$\Big) + c(A_{p-2}) - c(A_{p-1}) =$$

$$= -2 + \frac{(p-2)(p-1)}{p+1} - \frac{(p-1)p}{p+2} = c_{-1}(2,p)\,.$$

This is another non trivial analytical check for the conjectured effective central charge.

The dimension of the operator which creates the state described by (5.24) at negative chemical potential, i.e. the $\pi$-vacuum, reads

$$\Delta_{\pi-\text{vac}}(N,p) = \frac{n\,p}{8(n+p-1)}$$

obtained by making use of (4.5) and (5.35). A detailed analysis of the conformal wieght of this state is currently in progress along the lines of Section 5.3.2 and is reserved for a future pubblication.

#### 5.3.3.1 Conclusions

In this chapter we have proposed the Thermodynamic Bethe Ansatz equations and the Y-systems for a two-parameter infinite family of perturbed conformal field theories related to the $\mathbb{CP}^{N-1}$ sigma models coupled to a massless fermion.

Although the main motivation of the work was the recently discovered link [82] between the specific $N = 4$ case and a truncation of the $AdS_4 \times \mathbb{CP}^3$ type IIA string sigma model in the Alday-Maldacena decoupling limit [143], most of the results presented here are of a much wider mathematical and physical interest.

In particular, we have introduced a novel family of periodic Y-systems classified in terms of a pair of integers $(N,p)$. These functional relations differ from the standard Lie-algebra related ones, discussed for example in [90, 101, 155], in a non trivial way.

Some of the mathematical results presented here correspond to numerical-supported conjectures and, although we have little doubt on their exact validity, it would be still important to prove them rigorously. The main mathematical conjectures are: the Y-system periodicity (5.40), the stationary dilogarithm identities (5.31) and the following non stationary sum-rules

$$\sum_{n=1}^{2(N+p-1)} \left( \mathcal{L}\left(\frac{\bar{Y}_0(n)}{1+\bar{Y}_0(n)}\right) + \mathcal{L}\left(\frac{Y_0(n)}{1+Y_0(n)}\right) + \sum_{i=1}^{N-1}\sum_{j=1}^{p-1} \mathcal{L}\left(\frac{Y_{(i,j)}(n)}{1+Y_{(i,j)}(n)}\right)\right) = \\ = 2p(1+pN-p)\frac{\pi^2}{6}, \tag{5.49}$$

where $Y_A(n) = Y_A\left(\theta + i\frac{\pi}{N}n\right)$ are the solutions of the Y-system, obtained recursively from (5.18, 5.22) with arbitrary initial conditions [145].

Concerning the specific $\mathbb{CP}^3 \times U(1)$ sigma model, we have computed the value of the ultraviolet effective conformal central charge, confirming the results predicted in [74] through a naive degree of freedom counting argument. Our conclusion was instead reached using highly non trivial dilogarithm identities and by considering the sigma model as the $p \to \infty$ representative in the family of perturbed coset conformal field theories $\frac{SU(4)_p}{SU(3)_p \times U(1)} \times U(1)$.

Apart from the mathematical aspects mentioned above, there are many other issues that we would like to address in the near future: the kink vacuum structure, the exact S-matrix and the mass-coupling relation for the quantum truncated models, the numerical study of the TBA equations for the other excited states [149] and the derivation of simpler non-linear integral equations for both the groundstate and the excited states [150] are only a small sample of important open problems that deserve further attention.

After having introduced the necessary technical background we turn now to the description of the structure and the implementation of the numerical set of codes

which have made possible to conjecture all the new results presented in this chapter. Since the tone of the discussion changes drastically to a more technical language we decided to deal with it in the conclusive chapter.

# Chapter 6

# Numerical Work

This chapter is devoted to the numerical analysis of the TBA equations derived so far. Its aim is to present the detailed technical aspects required to numerically extract physical informations from TBA integral equations like (4.67) or $Y$-system functional equations as (4.51). As immediate check of the reliability of the algorithms we inform that all the numerical plots in this thesis have been produced with the data obtained by the simulations. We say this to emphasize the easiness with which the code can be implemented in such a way that minimum modifications are required for passing from a model to another. Moreover, all the RG flow analysis results of Section 5.3 have been conjectured thanks to the algorithms and the numerical investigations explained and developed below.

The algorithms and numerical strategies we present here are related specifically to the $(\mathbb{CP}^{N-1})_p \times U(1)$ TBA and $Y$-system but, as stressed above, their scope and generality goes beyond. The codes developed have been applied, with minor model-dependent modification such as the different adjacency structure and the different functional form of kernels, to all the models presented in this work. The incredible accuracy and speed of computation of the simulations give us the confidence to conjecture analytic result which are too well constrained, as we will see below, to be accidental. This can be achieved by solving the numerical implementation specific problems which are treated in the more model-independent way.

The chapter is divided as follows: in Section 6.1 we introduce the equations to be solved numerically and describe the implementation of the algorithm in all its aspects. Section 6.2 is devoted to the analysis of the results obtained by solving the TBA equation for the ground state of the $(\mathbb{CP}^{N-1})_p \times U(1)$ TBA. While in Section

6.3 the same analysis, after the required modification, is performed on the first excited state. At last, in Section 6.4, we introduce the numerical tools and ideas to deal with dilogarithm sum rules and central charge numerical identification.

## 6.1 Numerical Setup

According to [146], see also Chapter 5, the mirror thermodynamics in finite volume of the $\mathbb{CP}^{N-1} \times U(1)$ sigma model can be encoded in the TBA framework. The relevant system of integral equations is (5.24) which we report here for convenience

$$\begin{aligned}
\epsilon_0(\theta) + \bar{\epsilon}_0(\theta) &= 2mR\cosh\theta - \sum_{l=1}^{N-1} \chi_{(1-\frac{2l}{N})} * \Lambda_{(l,1)}(\theta), \\
\epsilon_0(\theta) - \bar{\epsilon}_0(\theta) &= i2\alpha - \sum_{l=1}^{N-1} \psi_{(1-\frac{2l}{N})} * \Lambda_{(l,1)}(\theta), \\
\epsilon_{(i,j)}(\theta) &= \delta_{i,1}\delta_{j,1}\phi_{\frac{N}{2}} * L_0(\theta) + \delta_{i,N-1}\delta_{j,1}\phi_{\frac{N}{2}} * \bar{L}_0(\theta) + \\
&\quad + \sum_{l=1}^{p-1} A_{l,j}^{(p-1)} \phi_{\frac{N}{2}} * \Lambda_{(i,l)}(\theta) - \sum_{l=1}^{N-1} A_{l,i}^{(N-1)} \phi_{\frac{N}{2}} * L_{(l,j)}(\theta)\,.
\end{aligned} \tag{6.1}$$

Finding analytical solutions of such a system of coupled non-linear integral equations is a formidable task and thus we decided to resort to numerical analysis.

TBA equations are known to be well suited for numerical simulations [87, 88, 89]. This is true mainly because of the Leray-Schauder-Tychonoff fixed point theorem which ensures not only the existence but also other important properties (uniqueness, reality, analyticity) of the solutions. This is an extremely powerful tool also for the numerical analysis since it ensures that a *recursive* procedure will converge, sooner or later, to the correct set of solutions.

Another aspect that makes the numerical analysis very effective is related to the physical behavior of the solutions. As a matter of fact as the dimensionless parameter $r$ goes to zero, the model is dragged to the conformal UV point (a regime in which the conformal field theory framework can be employed both on the quantitative and qualitative level) and the solutions develop a central plateau region which possesses an almost compact support [87]. This feature will be of crucial importance in the construction of the numerical algorithm.

To illustrate the recursive procedure we are adopting we can schematically rewrite the TBA equations as

$$\epsilon_a = \nu_a + \sum_b \Phi_{ab} * F[\epsilon_b] \tag{6.2}$$

where $a$ is a proper multi-index, $\nu$ is the *forcing term* of the corresponding node, $\Phi_{ab}$ is a matrix of *generalized universal kernels* and $F[\epsilon_b]$ is a certain non-linear function of the pseudoenergies. Now we can introduce a sequence of functions, one for each node, $\epsilon_a^{(n)}$ and use the TBA equations to find the iteration at step $n+1$; as initial guess we decided to pick the forcing term. Thus we can write the recursive procedure in a inductive-like way as

$$\begin{cases} \epsilon_a^{(0)} = \nu_a \\ \\ \epsilon_a^{(n)} = \nu_a + \sum_b \Phi_{ab} * F[\epsilon_b^{(n-1)}] \end{cases} \tag{6.3}$$

Clearly the true solution can be rigorously reached only in the $n \to \infty$ limit and we will develop a criterion to stop the iteration once the desired level of precision is reached.

### 6.1.1 The algorithm

Having defined the general iterative scheme of solution we can analize the details of the algorithm. Since this is the first incursion of the author in the world of numerical computation we decided to keep the implementation as simple as possible in order to focus on other aspects, such as stability, portability and time performance.

The process of numerical solution is intrinsically affected by some loss of information and in the case at hand the first drastic modification we have to make is to reduce the range of the rapidity $\theta$, naturally defined on the real axis (more precisely the whole complex plane), to some finite interval. The peculiar behavior of the massive solutions discussed before gives a crucial hint in this sense: the central plateau region is confined in the symmetric interval $[-\log(2/r), \log(2/r)]$ and outside this region the behavior is qualitatively an exponential growth. Taking

this into account we decided to define our numerical TBA on the interval

$$\Omega = [-A, A] \quad \text{with} \quad A = 2\log(2/r)\,,$$

and to discard everything that lies outside. These considerations account for the simulations performed in the $r \ll 1$ region where, as emphasized before, the plateau approximation holds true. In the $r$-asymptotic region, corresponding to the IR regime of the theory, this approximation fails to be reliable and the interval $\Omega$ must be choosen differently. Taking into account that the behavior of the solution at large $r$, see equation (5.45), receives exponentially small correction we can then set the proper value of $A$. By imposing the exponential correction to be of the order of the precision of the algorithm we establish a bound for the value $A$. It turns out that the more broad, in terms of the $r$ range, precise and stable computation required the maximum value of $A = 40$. This in turn implies the necessity of increasing the sampling number $N_s$, an operation which ultimately results in a consistent enlargement of the time required by the simulation.

The finite-support approximation can then be written, for a generic convolution term, as

$$\Big(f * g\Big)(\theta) = \int d\theta'\, f(\theta - \theta')g(\theta') \approx \int_{-A}^{A} d\theta'\, f(\theta - \theta')g(\theta') \tag{6.4}$$

Such a replacement can be motivated, at least for the massive nodes, by considering the asymptotic behavior $\epsilon_{\text{massive}}(\theta) \overset{\theta\gg 1}{\approx} e^{\theta}$ which enforces the so-called *double exponential* damping of the non-linear terms, namely

$$L_{\text{massive}} = \log\left(1 + e^{-\epsilon_{\text{massive}}(\theta)}\right) \overset{\theta\gg 1}{\approx} e^{-e^{\theta}} \tag{6.5}$$

Taking this into account one can safely consider the massive $L$ terms as (almost) compact supported, with support contained in the domain $\Omega$. For the magnonic nodes the situation is different because their asymptotics are finite or, more precisely, they tend to the IR plateau values (5.45) and receive exponentially small corrections. Nonetheless approximation (6.4) is still reliable, see next section for more details about this point.

The second substantial modification in switching to numerics is the *discretization* process. In fact continuous quantities as such we are dealing with cannot be treated numerically and one is forced to introduce some form of sampling. We discretized

the $\theta$ interval introducing a finite set of sampled values according to

$$\theta_i = -A + i\Delta\theta \quad \text{with} \quad i = 0, 1, 2, \ldots, N_s - 1 \tag{6.6}$$

clearly $\Delta$, the rapidity step, and $N_s$, the number of steps, are not independent quantities. They obey a simple consistence equation, $(N_s - 1)\Delta\theta = 2A$, which allows to choose the free parameter to set. For small values of $r$, around $10^{-10}$, we took $N_s = 1000$ to which corresponds $\Delta\theta \approx 0.09$. See below for a discussion on this point.

The computing core of the algorithm concerns the evaluation of the convolution products appearing in the LHS of (6.3). We decided to choose the simplest implementation, namely to discretize the integral using the rectangular approximation. Despite its simplicity the approximation is very accurate thanks to the good convergence properties of the equations mentioned above.

### 6.1.2 Technical aspects

In this section we want to highlight the technical points we faced and overcame in order to obtain a working code.

#### 6.1.2.1 Doubling the kernels

In discretizing the convolution products on a compact support we immediately realized that the kernels must be defined on an interval of double length. This is so because of the difference appearing in the argument of the integrand. To grasp this subtlety it suffices to consider a generic convolution and evaluate it at $\theta = -A$

$$\int_{-A}^{A} d\theta' \, f(-A - \theta') g(\theta') \tag{6.7}$$

Now letting $\theta'$ span the interval $[-A, A]$ forces the argument of $f$ to span the interval $[-2A, 0]$. The solution to this problem is simple: define the kernels of the convolutions on the interval $[-2A, 2A]$ and then use the index map

$$k(i, j) = N_s - 1 + |i - j| \tag{6.8}$$

to obtain the discretized convolution as

$$\int_{-A}^{A} d\theta'\, f(\theta-\theta')g(\theta') \quad \Rightarrow \quad \sum_{j=0}^{N_s-1} f_{k(i,j)}\, g_j\, \Delta\theta \tag{6.9}$$

#### 6.1.2.2 Analytical vs. Numerical Equivalence

A very profound lesson to learn when dealing with numerical simulation is that the analytical equivalence of two expression does not ensure their numerical equivalence. In our case this comes in play when considering the non-linear function, consider the $\Lambda$'s term appearing in TBA, it is trivial to check the equivalence

$$\Lambda_a(\theta) = \log(1+e^{\epsilon_a(\theta)}) = \log\left[e^{\epsilon_a(\theta)}\left(1+e^{-\epsilon_a(\theta)}\right)\right] = \epsilon_a(\theta) + L_a(\theta) \tag{6.10}$$

so one is naively tempted to treat the first and last expressions on equal footing. Numerically this is strongly wrong because of the exponential nature of the forcing term. Despite the $\Lambda$'s are of the same order of magnitude of the corresponding $\epsilon$'s, beacause the presence of the exp and log functions somehow balance themselves, the process of numerical calculation goes through the exponentiation of an exponential. This quantity easily escapes the range of the real number on any ordinary machine for values of $\theta$ of the order of 10.

#### 6.1.2.3 Enhancing the magnonic convolutions

Due to intrinsic systematic error in the numerical computation we found that near the boundary of the domain the magnonic pseudoenergies fail to reach the proper plateau value. We take advantage of the known analytical form of the plateau solutions [102, 146] to smoothen the convolutions involving magnons. By means of the equality

$$\phi * L_a = \phi * \left(L_a - L_a^{(\infty)}\right) + cL_a^{(\infty)}\,, \tag{6.11}$$

where $c$ is the normalization constant of the kernel and $L_a^{(\infty)}$ stands for the proper non-linear function evaluated in the plateau values, we were able not only to cure the boundary numerical problem but also to increase the overall precision of the computation of a factor $10^{-3}-10^{-1}$ at the cost of a about the 10% of the computation time.

## 6.2 The ground state TBA

This section is devoted to present the results of the numerical computation related to the 0-vacuum (i.e. the actual ground state) of the $(\mathbb{CP}^{N-1})_p \times U(1)$ theory. The simulation were performed with a C++ code in which we implemented the numerical technology developed in Section 6.1 specifically to solve the TBA system of integral equation (6.1) with $\alpha = 0$.

As typical in these circumstances the ground state is often the most stable and efficient state for what concerns numerical simulations. The reality and positivity of the solutions, joined with the very neat double exponential damping wich helps in countering the artificial finiteness of the domain, is the main reason behind the good numerical performances of the code.

We begin our numerical investigations by performing single-$r$ computation with the aim of testing the results both at the qualitative and quantitative level.

In Figure 6.1 the pseudoenergies are displayed for the choice $N = 4$ and $p = 5$ and their behavior is very nice. The simulation is performed at $r = 10^{-8}$ in deep UV regime and in fact we can recognize the characteristic plateau patterns which are interpolated by very sharp region near $\log(2/r)$.

We can appreciate that the different asymptotic behaviors for massive and magnonic nodes is indeed reproduced correctly. The analysis of the symmetry pattern in the displacements of the central values reveals that is the one expected by the $UV$ plateau equation (5.29) establishing a qualitative match between the analytical and numerical analysis. Moreover the set of central values fits those obtained by solving numerically (5.29) within an average numerical error which ranges from $10^{-6}$, for the large $N$ and $p$ ($Np \sim 30$) computations performed at $r = 10^{-8}$, to $10^{-11}$ with the most refined simulations, for $Np \sim 15$) computations at $r = 10^{-12}$. These results confirms the well known good behavior of the TBA ground state integral equations under numerical investigations [87, 89]. The last strong quantitative check we can perform is represented by the comparison of the asymptotics behavior with the theoretically expected ones, see equation (5.33). Even in this case the agreement is impressive, between $10^{-8}$ to $10^{-12}$ for the aforementioned simulations, and it constitutes an even stronger non-trivial consistency check.

After having verified the consistency of our results we can then proceed in performing multiple $r$ computation. This will result in the profiles of the associated

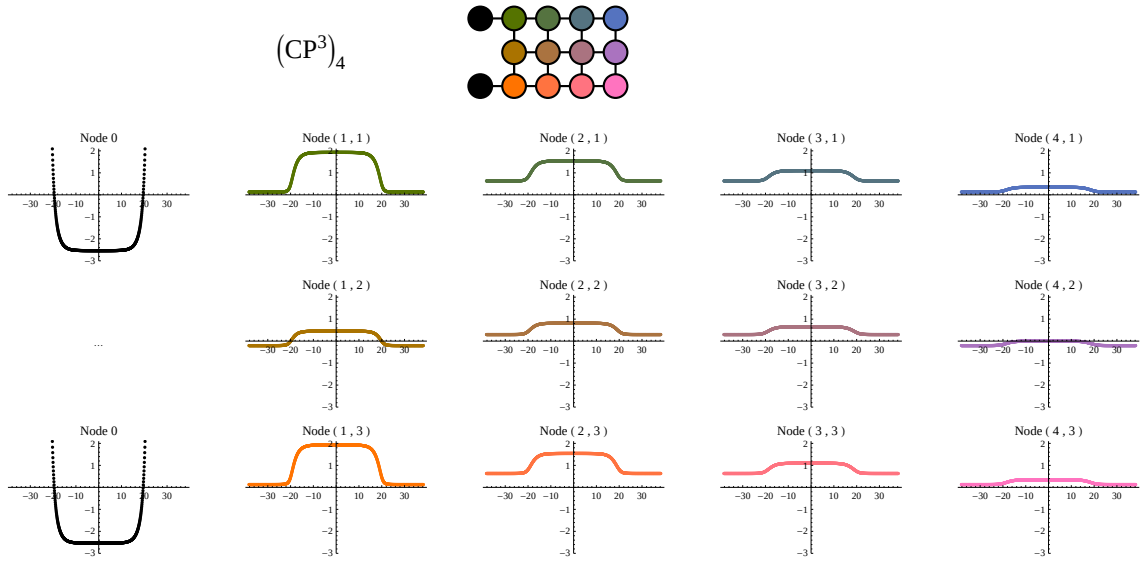



FIGURE 6.1: Pseudoenergies profiles for the choiche $N = 4$ and $p = 5$ at $r = 10^{-8}$.

finite-size scaling function $c(r)$ of (4.29) computed in the rectangular approximation. We mention this fact to emphasize, once more, the extremely good behavior of the TBA-like equation under numerical analysis and because we initially tried more sophisticated way of performing the integral and they result in a negligible increase in the precision. From the technical point of view, the simulations have been performed independently for each value of $r$ in view of the excellent result of single-valued run. We have further been able of controlling the growth of the computation time by taking advantage of multi-core machines onto which distribute the various instances of the program. This shrewdness has turned into a reduction of the computation time to the 12% of the expected one.

In Figure 6.2 and 6.3 are displayed the profiles of the $c$-functions for the member of the family reported. The $r$-range spans from $10^{-8}$ to 5 with a sampling step $\Delta r = 0.05$. The $r \sim 0$ value has then be compared with the conjectured effective central charge with an agreement ranging between $10^{-6}$ to $10^{-11}$. The high accuracy of the cross checks we have been able to perform on the results produced by the simulations for the ground state motivated us to try to modify it to solve the TBA for the first excited state, the $\pi$-vacuum.

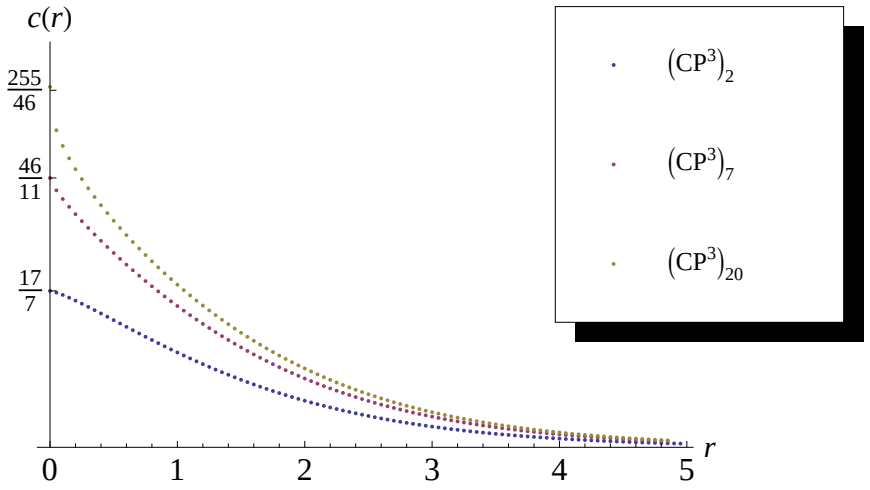



FIGURE 6.2: Several plots of the $c$-function for various values of the parameter $p$.

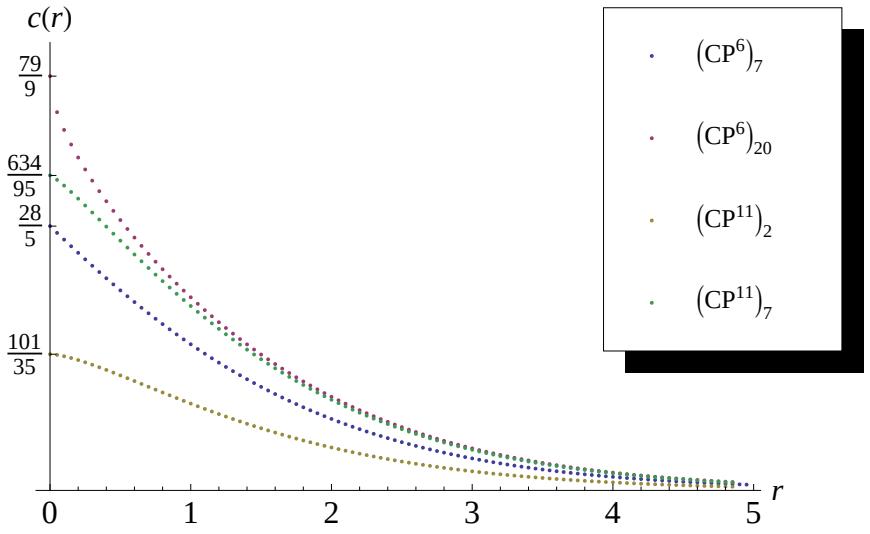



FIGURE 6.3: Several plots of the $c$-function for various values of the parameters $N$ and $p$.

## 6.3 First excited state TBA

In this section we present the results obtained from the numerical analysis of the second vacuum of the effective sigma model (5.8) in the TBA framework.

### 6.3.1 The model

The TBA description of the model is encoded again in the system (6.1) at $\alpha = \pi$. The only difference with respect to the ground state is represented by the

appearance of an imaginary chemical potential.

In this case the numerical analysis has not been as straightforward as for the ground state. The new main feature is represented by the imaginary part developed by the massive forcing terms. We decided to deal with the complex-valued nature of this system of equations not by brute force, i.e by implementing a complex-valued system of integral equation, but rather by re-defining the massive pseudoenergies as

$$\tilde{\epsilon}_0(\theta) = \epsilon_0(\theta) - i\pi\,, \qquad\qquad \tilde{\epsilon}_{\bar{0}}(\theta) = \epsilon_{\bar{0}}(\theta) + i\pi\,.$$

This implies in turn a modification of the non-linear terms related to massive nodes appearing in the convolutions of (6.1), namely

$$\tilde{L}_0(\theta) = \log\left(1 - e^{\tilde{\epsilon}_0(\theta)}\right)\,, \qquad\qquad \tilde{L}_{\bar{0}}(\theta) = \log\left(1 - e^{\tilde{\epsilon}_{\bar{0}}(\theta)}\right)\,.$$

These definitions allow to recast the TBA for the $\pi$-vacuum in the very same functional form of the ground state TBA (apart from the modification of massive functions with the tilded ones) and we report it here for convenience

$$\begin{aligned}
\tilde{\epsilon}_0(\theta) + \tilde{\epsilon}_{\bar{0}}(\theta) &= 2mR\cosh\theta - \sum_{l=1}^{N-1} \chi_{(1-\frac{2l}{N})} * \Lambda_{(l,1)}(\theta),\\
\tilde{\epsilon}_0(\theta) - \tilde{\epsilon}_{\bar{0}}(\theta) &= \sum_{l=1}^{N-1} \psi_{(1-\frac{2l}{N})} * \Lambda_{(l,1)}(\theta),\\
\epsilon_{(i,j)}(\theta) &= \delta_{i,1}\delta_{j,1}\phi_{\frac{N}{2}} * \tilde{L}_0(\theta) + \delta_{i,N-1}\delta_{j,1}\phi_{\frac{N}{2}} * \tilde{L}_{\bar{0}}(\theta) +\\
&\quad + \sum_{l=1}^{p-1} A^{(p-1)}_{l,j}\phi_{\frac{N}{2}} * \Lambda_{(i,l)}(\theta) - \sum_{l=1}^{N-1} A^{(N-1)}_{l,i}\phi_{\frac{N}{2}} * L_{(l,j)}(\theta),
\end{aligned} \tag{6.12}$$

Also the free energy of the system gets modified by such a redefinition

$$E^{(\pi)}(R) = -\frac{\pi c^{(\pi)}(r)}{6R} = -\frac{m}{2\pi}\int d\theta\cosh(\theta)\left[\tilde{L}_0(\theta) + \tilde{L}_{\bar{0}}(\theta)\right]\,.$$

The main advantage of the last forms is the fact that now we have a purely *real* system of equations. The price we paid in trading the manifestly complex form to the real one is that now the arguments of the logarithms in the convolutions are no longer positive definite and will possibly produce singularities and complex-valued pseudoenergies. We will see that this fact will have important numerical

consequences. The situation is well known in literature, see [151, 152] for instance, and physically corresponds to the introduction of a chemical potential $\lambda = -1$. In particular the analysis of two systems in this regime will provide some non trivial analytical checks for the conjectured quantities. These systems are the free fermion and the *Fractional Supersymmetric Sine Gordon* (FSSG) [147, 148] at negative chemical potential.

### 6.3.2 Numerical Analysis

We now turn to the numerical analysis of the TBA equations (6.12). We decided to test the performances of the computational setup described in section 6.1 on the $\pi$-vacuum as well. This choice was motivated by the very high precision of the simulations for the ground state. In particular the cut-off trick for magnonic convolutions is justified by the fact that in the large $r$ regime, corresponding to large $\theta$ values, the asymptotics are exactly the same: a constant plateau value, see (5.45), which receives exponentially small corrections of order $O\left(e^{-r}\right)$.

However the strategy in case is a little different. In trying to explore the behaviour of the solution for various $r$ we face the fact that the algorithm becomes instable as $r$ approaches zero. This is due to the presence of complex-valued logarithms which cannot be trated by a purely real implementation. Nonetheless, to avoid these numerical inconsistecies, we can start from large value of $r$ and decrease it progressively of a step $\Delta r$. In doing this we keep track of the solution at the previous value and use it as initial guess for the computation at the present value. The aim of the procedure is to follow the correct solution from the asymptotic regions, where can be checked analytically and the code is stable, down to $r = 0$.

This idea has been implemented with two additional features:

- *an adaptive step* $\Delta r$ :

  With the term *adaptive* we intend a code functionality which works as a self-consistency check and can recognize if the computed solution is reliable or not. If the consistency check fails the code make a step back and tries a new run after having dynamically modified the internal parameters.

- *an adaptive weight* $w$ :

The solution at step $n$ is evaluated by weighting the contribution coming from step $n-1$ with the convolutions terms, namely

$$\epsilon^{(n)}(\theta) = w\,\epsilon^{(n-1)}(\theta) + (1-w)\Big(\text{conv. terms}\Big) \tag{6.13}$$

This approach has revealed essential to reach small values of $r$ and we were able to numerically obtain the $c^{(\pi)}$-function profile for the first members of the family. The result are displayed in Figure 6.4. The picture reveals how drastically the minimum reachable value of $r$ is increased as $p$ grows. We noticed that the

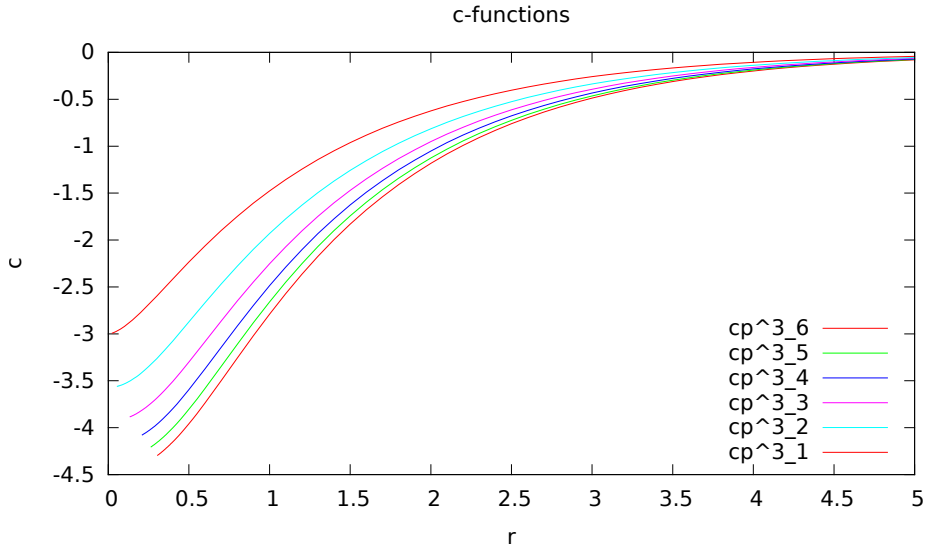



FIGURE 6.4: The *finite-size scaling function* $c(r)$ profiles for some members of the $(\mathbb{CP}^3)_p \times U(1)$ for various values of $p$.

numerical difficulties encountered in solving this type of problem is very sensitive to the parameter $p$ and more permissive with respect to changes in $N$. Therefore performing simulations to explore the other direction of the graph, letting $p = 1$ to minimize the lowest value of $r$ reachable, we obtained the functions displayed in Figure 6.5

In this case the precision was drastically decreased to range from $10^{-2}$ to $10^{-4}$. The possibility of conjecturing the analytic expression (5.47) for the value of the effective central charge has to rely on the method illustrated in the next section.

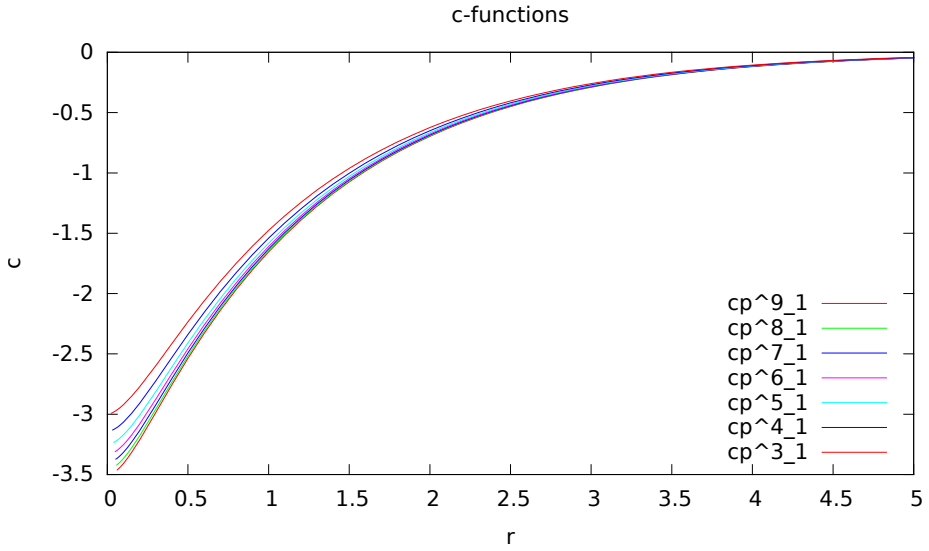



FIGURE 6.5: The *finite-size scaling function* $c(r)$ profiles for some members of the $(\mathbb{CP}^3)_p \times U(1)$ for various values of $p$.

## 6.4 Y-systems numerical analysis

This section is devoted to the numerical analysis of general $Y$-systems, as those encountered in Chapters 4 and 5, with the precise aim of obtaining a candidate for the associated effective central charge. From the more strict mathematical point of view this amounts to the numerical evaluation of a specific dilogarithm sum rule corresponding to the plateau equations of the associated $Y$-system. Since most of the $Y$-system presented in this work fall in families labelled by one, or more, integer numbers in such a way that the object of our research is almost always a *sequence* of rational numbers. This is by no means a surprise since we are considering integrable models realized as perturbations of some CFT and there's a large subset of CFT, known as *Rational Conformal Field Theories* which enjoys rational-valued central charges. This observation greatly constrains the form of the solutions and thus we have at our disposal a very selective criterion to test our results. At the same time the large amount of mathematically proved and, where not yet achieved, solidly conjectured results known in literature is impressive [52, 128, 130, 144, 145, 153, 155] and allows for cross-checks and comparisons.

The key observation is that, in general, $Y$-systems display a periodicity in the imaginary direction of the rapidity plane [90, 101] which, from the physical point of view, is related to the conformal dimension of some operator of the underlying

CFT, namely

$$Y_A(\theta + i\pi P) = Y_{A'}(\theta) \tag{6.14}$$

for general multi-index $A$ and $A'$ which may, or may not, be different. The mapping $A \to A'$ is model-dependent and every choice leads to different patterns according to the symmetry structure of the associated algebra. For the $ADET$ and $ADET\Diamond ADET$ cases of Chapter 4, $A'$ is the anti-particle index. Moreover, in the diagramatic representation of such $Y$-systems, there's a nice way of reading the particle-antiparticle inversion, they simply correspond to index which are symmetric with respect to symmetry of the diagram. Other type of more general $Y$-systems, such as the ones introduced in [146, 128] and related to NSLM, instead do not display this type of behavior although still enjoy the periodicity in the imaginary direction.

For the rest of the section we will focus our analysis on the $Y$-system (5.22,5.23) describing the ground state of the $(\mathbb{CP}^{N-1})_p \times U(1)$ model and only for what concerns the UV contribution. We report the explicit form of the $Y$-system here for convenience

$$\begin{aligned}
Y_0\left(\theta + i\frac{\pi}{2}\right)\bar{Y}_0\left(\theta - i\frac{\pi}{2}\right) &= \prod_{l=1}^{N-1}\left(1 + Y_{(l,1)}\left(\theta + i\frac{\pi}{2} - i\frac{\pi l}{N}\right)\right), \\
Y_0\left(\theta - i\frac{\pi}{2}\right)\bar{Y}_0\left(\theta + i\frac{\pi}{2}\right) &= \prod_{l=1}^{N-1}\left(1 + Y_{(l,1)}\left(\theta - i\frac{\pi}{2} + i\frac{\pi l}{N}\right)\right), \\
Y_{(i,j)}\left(\theta + i\frac{\pi}{N}\right)Y_{(i,j)}\left(\theta - i\frac{\pi}{N}\right) &= \left(1 + \delta_{i,1}\delta_{j,1}Y_0(\theta) + \delta_{i,N-1}\delta_{j,1}\bar{Y}_0(\theta)\right) \times \\
&\quad \times \prod_{l=1}^{p-1}\left(1 + Y_{(i,l)}(\theta)\right)^{A^{(p-1)}_{l,j}} \prod_{l'=1}^{N-1}\left(1 + \frac{1}{Y_{(l',j)}(\theta)}\right)^{-A^{(N-1)}_{l',i}};
\end{aligned} \tag{6.15}$$

this form is general enough (it is bidimensional, presents shifts at both members and displays both massive and magnonic nodes) to give a taste of the general procedure. The choice of the ground state was not an accident in fact the approach we are outlining is known [145, 153] to give the best performances for these type of states, i.e. those for which the numerics converge in the fastest and most stable way to the correct solutions. As anticipated in Section 5.3.2 after having eliminated the $\theta$-dependencies we are left with the system of non-linear algebraic equations (5.29). Taking the logarithm we obtain a form more suitable for numerical iterations which

explicitly reads

$$\begin{aligned}
\varepsilon_0 &= \bar{\varepsilon}_0 - \sum_{l=1}^{N-1} \log\left(1+e^{\varepsilon_{l,1}}\right), \\
\varepsilon_0 &= \bar{\varepsilon}_0, \\
\varepsilon_{(i,j)} &= \log\left(1+\delta_{i,1}\delta_{j,1}e^{-\varepsilon_0}+\delta_{i,N-1}\delta_{j,1}e^{-\bar{\varepsilon}_0}\right) + \\
&\quad + \sum_{l=1}^{p-1} A^{(p-1)}_{l,j} \log\left(1+e^{\varepsilon_{(i,l)}}\right) - \sum_{l'=1}^{N-1} A^{(N-1)}_{i,l'} \log\left(1+e^{\varepsilon_{(l',k)}}\right);
\end{aligned} \tag{6.16}$$

where the $\varepsilon$'s are the stationary values of the pseudoenergies. By using (6.16) recursively starting from arbitrary initial guess, mimicking somehow the numerical integration setup of Section 6.1, the convergence is impressively fast and stable and the runs can be performed on an ordinary laptop. For what concerns the more technical aspects we implement the iteration by weighting with $w = 0.5$ the old and new solutions (see equation (6.13)). The escape condition for the iteration loops is implemented by evaluating the absolute value of the average difference between the $\varepsilon$'s calculated at step $n$ and at step $n-1$; if this value is found to be less then $10^{-15}$ the solution is considered sufficiently close to the exact one and the iterative procedure stops.

Having succeeded in determining numerically the plateau values the next step is to evaluate the Rogers dilogarithm sum rule

$$c = \frac{6}{\pi^2}\left[\mathcal{L}\left(\frac{1}{1+e^{\varepsilon_0}}\right) + \mathcal{L}\left(\frac{1}{1+e^{\bar{\varepsilon}_0}}\right) + \sum_{i=1}^{N-1}\sum_{l=1}^{p-1}\mathcal{L}\left(\frac{1}{1+e^{-\varepsilon_{(i,l)}}}\right)\right] \tag{6.17}$$

Many numerical softwares, as Mathematica that we decide to employ, offer a numerical implementation of the usual Dilogarithm $Li_2(z)$ which is needed to compute the function $\mathcal{L}$, see relation (A.3) for the precise link between the two special functions. By plugging the solutions of (6.16) into the sum rule (6.17) we obtain a table of (approximated) rational numbers representing the central charges of the various members of the family. Our goal is therefore to obtain a closed expression, in term of rational functions of $N$ and $p$, for the central charges.

The numerical outcome for the central charges at $N = 4$ for the $p$-truncated models are compared with equation (5.35) in Table 6.1: the match is very good and leaves little doubt on the correctness of conjecture (5.31).

| Level $p$ | Numerics | Exact | Error |
|---|---|---|---|
| 2 | 1.8000000000000014 | $9/5$ | $1.3 \times 10^{-16}$ |
| 3 | 2.428571428571437 | $17/7$ | $8.4 \times 10^{-15}$ |
| 4 | 2.928571428571431 | $41/14$ | $2.6 \times 10^{-15}$ |
| 5 | 3.333333333333345 | $10/3$ | $6.7 \times 10^{-15}$ |
| 6 | 3.666666666666656 | $11/3$ | $1.1 \times 10^{-14}$ |
| 7 | 3.945454545454537 | $217/55$ | $8.4 \times 10^{-15}$ |
| 8 | 4.181818181818161 | $46/11$ | $2.0 \times 10^{-14}$ |
| 9 | 4.384615384615358 | $57/13$ | $2.7 \times 10^{-14}$ |
| 10 | 4.56043956043953 | $415/91$ | $3.0 \times 10^{-14}$ |
| 11 | 4.7142857142856 | $33/7$ | $1.1 \times 10^{-13}$ |
| 41 | 6.212121212124 | $205/33$ | $2.8 \times 10^{-12}$ |
| 51 | 6.35353535324 | $629/99$ | $2.9 \times 10^{-10}$ |
| 61 | 6.4519230761 | $671/104$ | $8.2 \times 10^{-10}$ |

TABLE 6.1: $N = 4$: comparison between numerics and equation (5.35).

### 6.4.1 Periodicity

Because of its link with the conformal dimension of the perturbing operator [90, 101, 107] it seems reasonable to look for a rational period $P$ of the form $R/Q$ with $R$ and $Q$ integers not necessarily prime between themselves. In practice the denominator $Q$ is usually suggested by the explicit form of the $Y$-system under consideration. In our case (6.15) the $Y$-system present to natural shifts: $\pi/2$, for massive nodes, and $\pi/N$, for magnonic ones; it is therefore natural to choose the denominator as $Q_{(N,p)} = N$ since it represents the smallest units of shift. This assumption is important at the numerical level because we can introduce a discretized version of $Y$-system by means of

$$y_A(n) = Y_A\left(i\frac{\pi}{2N}n\right) , \tag{6.18}$$

in terms of which the discretized and properly offsetted $Y$-system becomes

$$\begin{aligned} y_0(n) &= \frac{1}{\bar{y}_0(n-N)} \prod_{a=1}^{N-1} \Big(1 + y_{a,1}(n-a)\Big) \\ y_{a,l}(n) &= \frac{1 + \delta_{a1}\delta_{l1} y_0(n-1) + \delta_{aN}\delta_{l1}\bar{y}_0(n-1)}{y_{a,l}(n-2)} \times \\ &\qquad \times \frac{\Big(1 + y_{a,l-1}(n-1)\Big)\Big(1 + y_{a,l+1}(n-1)\Big)}{\left(1 + \dfrac{1}{y_{a+1,l}(n-1)}\right)\left(1 + \dfrac{1}{y_{a-1,l}(n-1)}\right)} \\ \bar{y}_0(n) &= \frac{1}{y_0(n-N)} \prod_{a=1}^{N-1} \Big(1 + y_{N-a,1}(n-a)\Big) \end{aligned} \tag{6.19}$$

By using (6.19) as a recursive relation starting from the necessary number of arbitrary initial conditions we have been able to check, up to $N$ and $p$ around 100, that the follwing *discrete* periodicity holds

$$y_A(n + R_{(N,p)}) = y_A(n) \qquad \text{with} \qquad R_{(N,p)} = 2(N + p - 1)\,. \tag{6.20}$$

This implies in turn a periodicity for the $Y$-function of the form (6.14) with $P_{(N,p)} = 2(N + p - 1)/N$. This numerical analysis has made possible the identification of the perturbing operator of Section 5.3.2.

For what concerns the precision of this numerical simulation we implement the escape condition in a fashion similar to what we have done for the numerical integrator; namely we check that the absolute value of the average difference between all the nodes at step $n$ and at step $n + R_{(N,p)}$ was less than $10^{-15}$. In order to avoid accidental escaping we perform the escape check on multiple, namely 3, independent values of $n$.

From a more general point of view this type of approach has revelead extremely successful and has been tested on all the $Y$-systems presented in this thesis with positive outcome and a very high accuracy.

### 6.4.2 Central Charge Numerics

After having discovered the periodicity of the $Y$-functions we are in the position of understanding more in detail the structure of the central charge coming from

the dilogarithms sum rule. To this aim consider a non-stationary version of the sum rule (6.17) in the form

$$\sum_{n=1}^{2(N+p-1)} \left( \mathcal{L}\left(\frac{y_0(n)}{1+y_0(n)}\right) + \mathcal{L}\left(\frac{\bar{y}_0(n)}{1+\bar{y}_0(n)}\right) + \sum_{i=1}^{N-1}\sum_{j=1}^{p-1} \mathcal{L}\left(\frac{y_{(i,j)}(n)}{1+y_{(i,j)}(n)}\right)\right) . \tag{6.21}$$

This can be interpreted as a discretized version of an integral over a period of the function in bracket and since it is a periodic quantity we know that the average does not depend on the starting point and thus the result is independent of $n$. Moreover we know from *Cluster Algebras* [154] and general $Y$-system [153, 155] considerations that the number represented by (6.21) must be an integer, say $I_{(N,p)}$, since it is related to the topological structure of the underlying algebra.

In view of these considerations the stationary sum rules, as (6.17) for instance, acquire a new meaning: they are average over a period of non-stationary sum rules. In other words the rational result of such sums can be represented as the ratio

$$c = \frac{I_{(N,p)}}{P_{(N,p)}} \tag{6.22}$$

where the denominator is exactly the already known discretized period.

This is the last ingredient needed to reconstruct the central charge. In fact now we can multiply the table of rational numbers obtained numerically in Section 6.4.2 by the corresponding period to obtain a two-parameter sequence of integer numbers which are extremely easier to indentify.

In order to fix uniquely the factor $I_{(N,p)}$ we resorted to numerical interpolation. Since we are dealing with integers we can safely assume the quantity $I_{(N,p)}$ to be a polynomial in $N$ and $p$. More precisely by keeping one of the parameter, say $N$, fixed, i.e. interpolating $I_{(N,p)}$ row by row, we can use interpolating built-in function to obtain several sequences of integers corresponding to different values of $N$. The full result is then obtained by selecting the coefficient of the same powers of $p$ and interpolating once more in the same way.

The methods presented in this chapter are very general and can be applied successfully to all the TBAs presented in this thesis. They have proved sufficiently accurate to allow for the conjecture of several non-trivial results concerning the RG

flow of integrable theories and to test them by analytic and numerical cross checks. This chapter concludes the research material covered in this thesis and we address to the draw some conclusions and outline future directions of investigation.

# Chapter 7

# Conclusions

The main object of the material presented in this thesis has been the development of a non-perturbative consistency check for the quantum integrability of the *extended*-$\mathbb{CP}^{N-1}$ family of $1+1$-dimensional Non-Linear Sigma Models, recently proposed in [74]. The model is an interesting generalization of a well studied class of sigma models, namely the (gauged) $\mathbb{CP}^{N-1}$, which is coupled to a self-interacting massless Dirac fermion. The physical interest in this class of models has been recently renewed in the AdS/CFT correspondence context where, the distinguished member of the family with $N = 4$, is believed to govern the low energy behavior, more precisely the Alday-Maldacena decoupling limit [143], of the $AdS_4 \times \mathbb{CP}^3$ type IIA superstring sigma model [82]. We recall that this model is conjectured to be quantum integrable thus the possibility of a detailed knowledge of its low-energy behavior, with particular focus on the informations that the TBA allows to extract, may shed some light on the full superstring theory. Moreover the $AdS_4 \times \mathbb{CP}^3$ model is conjectured to be dual to the $\mathcal{N} = 6$ superconformal Chern-Simons theory in $1+2$ dimensions, for which a set of ABA has been proposed in [28] to describe the 2-loop integrability of the model. The work presented here is, in this sense, a further evidence in the direction of this, less studied, AdS/CFT correspondence example.

Beyond this point, the quantum integrability problem for $\mathbb{CP}^{N-1}$-related NLSM is a long standing one [62, 63]. Since the proof of its classical integrability, through the construction of the explicit Lax operator [73, 74], and the corresponding lack of the quantum counterpart, spoiled by anomalies at quantum level [70], many attempts have been made to deform the model in order to restore integrability

[71, 72]. In this sense the class of model proposed by Basso and Rej constitutes a minimal estension of the $\mathbb{CP}^{N-1}$ NLSM which, thanks to the addition of a self-interacting massless Dirac fermion, can be fine-tuned, from the renormalization point of view, to define a subclass of integrable models.

These considerations motivated us in performing further non-perturbative analytic tests on the integrability of these models. One of the main results of [74] is that the authors have been able to conjecture an explicit form for the 2-body $S$-matrix between the fundamental excitations of the model. The scattering amplitudes turn out to be given by the reflectionless $SU(N)$-invariant $S$-matrix, those of type II according to the classification of [75]. The evidence of quantum integrability for these type of models, as described in [74], are supported, among the other arguments, by imposing the (non-trivial) matching between the free energy of a gas of spinons, as are named the fundamental excitations of the model, computed with two independent approaches: perturbation theory [74] and the $S$-matrix theory [135]. As a result they have been able to fine-tune the parameter of the general class of models in order to extract a subclass of integrable ones and, moreover, to obtain an exact expression for the mass gap of the spectrum. With the idea of giving more support to this observation we applied the Thermodynamic Bethe Ansatz analysis in order to test this conjecture by further non-perturbative considerations. Starting from the knowledge of the ABA equations in the NS sector we carried over all the various analytical steps which constitutes the TBA analysis. As explained in detail in Chapter 4 (see in particular the Section (4.3.4.1) related to the application of TBA to NLSM), the TBA procedure we applied has been constituted by the steps which we list in the following, along with the main general results and consequences that can be drawn, and outlining the open questions and possible research directions which have emerged in each step. We stress that the actual analytic derivation has been explicitly performed in all its parts only in the $N = 4$ case, the relevant one in the AdS/CFT context [74, 28].

- **String hypothesis:** We have been able to obtain the explicit form for the string hypothesis, along the lines of [126, 127]. These magnonic bound states have then been analyzed by the bootstrap procedure, at the level of the ABA, which ultimately resulted in the evaluation of the exact $S$-matrix for these states, see equation (5.12). The effectiveness and strength of the string hypothesis are thus confirmed by this non-trivial application.

- **Thermodynamic limit:** The ABA have then been considered in the thermodynamic limit and casted as linear system of integral equations (5.13) which is written, for some unknown densities of states, in terms of convolutions with some physical kernels, those obtained in Section B.1. The physical kernels have been analyzed in the complex plane to produce the shift relation of Section B.2 which have revealed essential to proceed in the analysis.

- **Mirror thermodynamics:** The theory has then been formulated on the TBA torus [87], which generates the finite-size geometry of the cylinder by a suitable limiting procedure, and the minimization of the free energy functional has been performed. The minimization ultimately resulted in the TBA system of non-linear integral equations (5.14) which enjoys a pair of massive nodes, corresponding to the spinon and anti-spinon massive excitations, and a three-layer infinity tail of magnonic nodes, corresponding to the three types of magnonic strings introduced.

- **$Y$-systems:** The kernels shift identities of Section B.2 has then allowed to recast the integral TBA into the system of functional relations known as $Y$-system [90]. The resulting $Y$-system fits the well-established classification of [107] even though it presents some new interesting features. The first difference concerns the presence of shifts on both sides of the functional equation, a features already outlined in [107] and observed for instance in the $O(2r)$ case [128]. While the second, and more important, new feature is related to the structure of the equations; indeed we obtained two different way of writing the associated $Y$-system: the uncrossed form (5.17) and the crossed one (5.21). This choice regards the massive nodes and it has firstly been observed in [140, 142], in relation to the $Y$-system associated to the $AdS_4 \times \mathbb{CP}^3$ superstring sigma model. Since the *extended*-$\mathbb{CP}^{N-1}$ model is believed to govern the low energy behavior of this particular string model the compatibility of the two $Y$-systems structures, apart from providing a qualitative evidence of this limit, opens the interesting possibility of investigate deeper the relation between the two models within the TBA approach. We also mention that, thanks to the crossed formulation of the $Y$-system, it has been possible to encode the TBA description into a Dynkin-like diagram, that of Figure 5.1. This behavior constitutes an interesting general feature

of $Y$-system that can be explored further in relation to the possibility of interpreting the crossed form as a mean of analytic continuation to a different Riemann sheet of the same function.

- **Truncated model:** As typical for sigma models the magnonic color structure is described in terms of an infinite collection of pseudoenergies, non-trivially coupled to each other [91, 104]. The solution of an infinite set of functional equations constitutes a formidable analytic task thus some sort of truncation, as those outlined in Section 4.3.4.1 for the $O(2r)$ case [128], to deal with a finite number of equations is required. Moreover, a careful analysis of the structure of the shifts in the massive equations of the $Y$-system has suggested the possibility of formulating a consistent empirical conservation rule for the shifts in the functional equations. These two observation has led to the a remarkable generalization of the equations rigorously derived so far. Namely, by introducing the rank $N$ and the level $p$ we have been able to generalize the description to an entire two-parameter infinite family of models: the $(\mathbb{CP}^{N-1})_p \times U(1)$ models. The corresponding two-parameter $Y$-system was obtained in (5.22-5.23) and enjoys the features discussed above for the previous model, which corresponds to the choice $N = 4$ and $p = \infty$.

- **Universal form:** Thanks to the Fourier analysis, along the lines of [90, 123], performed on the functional equation, which provides, as a byproduct, the introduction of the generalized universal kernels of (B.8), we eventually ended up with an non-linear system of integral equations which represents the TBA description for the model, see equation (5.24), in the, suitably extended, Zamolodchikov's universal form. This has allowed both analytic and numerical investigation on the whole novel family of integrable models. The physical content of the formulation is related to the ground state of the theory and to its first excited state, a second degenerate vacuum which is lifted by quantization. The dependence of the integral equations on the state is entirely encoded in the discrete parameter $\alpha \in \{0, \pi\}$, related to the chemical potential by $\lambda = e^{i\alpha}$ and appearing explicitly in (5.24).

- **RG flow:** The equation obtained allowed for a non-perturbative analysis of the Renormalization Group flow structure for the system. More specifically we have been able to conjecture, based on the high accuracy results obtained by employing a suite of numerical algorithms developed ad hoc, an exact form for the UV effective central charge, equation (5.35), of the model and also an

expression for the dimension of the perturbing operator, see equation (5.41). From a more strictly technical point of view the object of the conjecture is represented by the non-stationary Rogers dilogarithm sum rule (5.49), which can be suitably adapted to the case at hand to give the exact form of the UV central charge for the model, equation (5.31). Aside from being interesting from the physical point of view in relation to the possibility of model identifications, our conjectured result gives a further example of accessible Rogers dilogarithm sum rule [102, 145]. The mentioned non-stationary sum rule should be explored further from the analytic point of view in relation to the topic of cluster algebras [154]. The analysis has been performed for both the $\alpha$-vacua and the results passed several non-trivial checks by matching the known result [147] for particular limit of the parameters, in which the theory is known to reduce to previously known models: the free fermion and the Fractional Supersymmetric Sine-Gordon model [148].

For what concerns the interpretation of the exact results related to the $(\mathbb{CP}^{N-1})_p \times U(1)$ family of TBA, we followed the path delineated in [101, 107] and constructed an equivalent formulation in terms of perturbed coset CFT. Explicitly we associated to the general quantum truncated integrable model $(\mathbb{CP}^{N-1})_p \times U(1)$ the Kac-Moody coset CFT

$$\frac{SU(N)_p}{SU(N-1)_p \times U(1)} \times U(1)\,, \tag{7.1}$$

where the coset part account for the bosonic degrees of freedom while the $U(1)$ part is related to the Dirac fermion. In other words, the CFT we are considering is obtained as the direct product of the theories appearing in (7.1) which are a Kac-Moody related coset CFT [34] and a compactified free boson, known to be equivalent to a free Dirac fermion in 2 dimensions. We moreover checked that the perturbation which generates the integrable model is constituted by an operator which takes values in both the CFTs, see equation (5.44). The field content of the theories has then been used to split the perturbing dimension, as coming from the TBA analysis, in a unique way, defining in this way the exact form of the perturbing operator. The results coming from the TBA analysis of the $\pi$-vacuum confirm this scenario and the conjecture (7.1) but a unambiguous match with the operator content of the underlying CFT is still under current numerical and analytic investigations. We reserve the analysis and discussion of these results for future pubblications.

As mentioned before a great deal of informations has come from the numerical analysis of the equations discussed so far. Motivated by the well-known excellent behavior of TBA-related equations under numerical analysis we decided to develop a whole suite of softwares and scripts, based on several different languages (C++, Mathematica), to deal with the TBA equations. The accuracy of the results obtained was impressive, as extensively discussed in Chapter (6), and the non-trivial cross-checks, both at the analytic and numerical level, leave little doubt about the correctness of the conjectured formulas. This has been performed with a bottom-up approach, i.e. starting from simpler systems, such as those of the *ADET* type, see Section 4.2, whose analysis has allowed to tune and debug the codes. Then we proceed to implement more refined versions which have given the possibility of perform the analysis discussed above for the $(\mathbb{CP}^{N-1})_p \times U(1)$ models. The legacy of such an approach is constituted by a series of numerical tool for the solution of TBA-related problems with which we will start, as step 0, the creation of an open-source online database for the state-of-art of numerics in this field. The project is at an embrional stage but we are confident that, in view of the broad fields of applicability of the TBA formulation, its scope will expand in the future. For the moment the material is hosted at the author's web page.

At last we mention some of possible research directions opened by this work. At the most basic level the result obtained for the $\pi$-vacuum requires a deeper analysis both at the numerical and analytical level. From the analytic side a deeper understanding of the underlying CFT will shed some light on the operator content and provide a theoretical framework into which test conjecure (7.1). More support to the model identification could also be given by the study of other states of the theory, in order to establish a precise correspondence between the TBA description and the CFT formulation, thanks to the analytic continuation in the complex plane of the TBA equations, see for instance [162].

Another interesting open perspective is represented by the so-called *magnon resummation problem*. In recent years some techniques have been developed with the precise aim of reducing the tail of magnonic TBA nodes to a single node of different type, which can be indeed interpreted as a new node stemming from resummation of magnonic ones. The equation satisfied by these new type of node is still a non-linear integral equation which is defined on the complex plane and rensembles those of [163, 164], obtained independently by other means. The aim of this recasting is obviously related to the possibility of giving a unified description

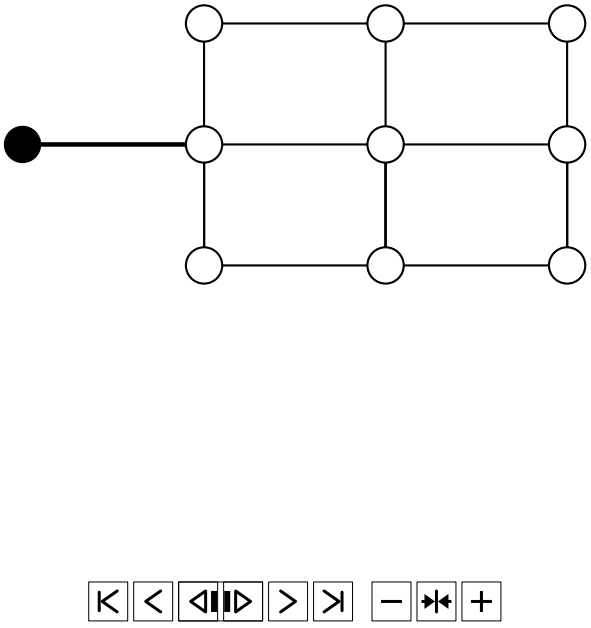


FIGURE 7.1: A graphical representation of the folding operation between the $O(6)$ and the $(\mathbb{CP}^3)_p \times U(1)$ TBAs.

of the whole theory in term of single non-linear integral equation. This aspect is currently under research.

At last we want to mention a remarkable formal link that can be drawn between integrable models. The operation that could possibly related two different TBA model is the so-called folding [101]. This operation can be defined rigorously for some type of simple Lie algebra related TBA, for instance the $T_n$ TBAs of [101], and correspond to introduce an equivalence class, based on the action of the Weyl group element, in the weight lattice of the algebra. This ultimately results in a morphism between different TBA models of which a very neat graphical interpretation is possible. Namely one can consider the (possible) symmetry of the diagram and fold the diagram along this axis of symmetry. In doing this nodes and links symmetric with respect to the axis should be identified, while those lying on the axis should be doubled. This graphical picture allows the introduction of this type of relation between the model we are dealing with and the corresponding $O(6)$ TBA. The folding operation applied to the $O(6)$ TBA produces exactly the diagram found in this work as can be seen in Figure 7.1. Apart from being on its own an interesting and promising tool for generating new TBA model starting from known ones, the study of this approach could shed some light on the relations between the two models. In view of their superstring full counterpart, namely the

well studied $AdS_5 \times S^5$ and $AdS_4 \times \mathbb{CP}^3$ models, this interesting techniques could also shed some light on the possible relations between the two most relevant models in the context of the AdS/CFT correspondence. The situation we have described is summarized in Figure 7.2 This interesting intertwining deserves, in our opinion,

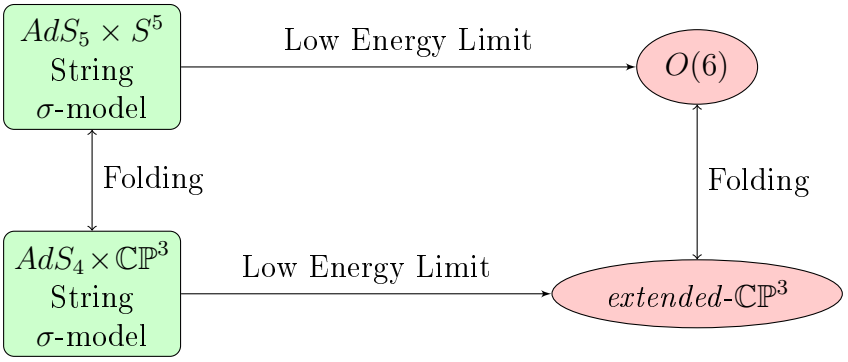



FIGURE 7.2: A pictorial representation of the possible relations between $AdS/CFT$-related models and their low energy behaviors.

more attention and is left for future investigations.

# Appendix A

# The Rogers Dilogarithm Function

In this appendix we review the main analytical properties of the *Rogers dilogarithm function*. The presence of this function in mathematical physics is very broad: volumes of polytopes in curved geometries, number theory, algebraic $K$-theory, representation theory of infinite dimensional algebras, low dimensional tolopogical problems, affine (or not) conformal field theories and TBA analysis.

Clearly we are mainly interested in presenting the properties and relations related and useful to the non-perturbative RG flow analysis that can be performed within the TBA framework. This is usually connected with the topic of dilogarithms *accessible sum rule*, namely the (finite) sum of dilogarithms evaluated in particular values which are given as the solutions of a highly non trivial algebraic system of equations.

## A.1 Definition and principal properties

The Rogers dilogarithm function $\mathcal{L}(x)$ can be defined for $0 \leq x \leq 1$ by the integral representation

$$\mathcal{L}\,(x) = -\frac{1}{2}\int_0^x \left(\frac{\log(1-t)}{t} + \frac{\log t}{1-t}\right) dt\,. \tag{A.1}$$

The symmetric form of the integrand suggests a relation with the usual dilogarithm (Euler's dilogarithm, to be precise) $Li_2(x)$, which can be defined, for $0 \leq x \leq 1$,

as

$$Li_2\left(x\right) = -\frac{1}{2}\int_0^x \frac{\log(1-t)}{t}dt = \sum_{n=1}^{\infty}\frac{x^n}{n^2}\,. \tag{A.2}$$

The relation between the two trascendental functions is then given by

$$\mathcal{L}(x) = Li_2\left(x\right) + \frac{1}{2}\log x\,\log(1-x)\,. \tag{A.3}$$

as can be checked by direct integration.

The function $\mathcal{L}(x)$ can be seen [102] to satisfy the *reflection* functional relation

$$\mathcal{L}(x) + \mathcal{L}\left(1-x\right) = \frac{\pi^2}{6}\,, \qquad 0 \le x \le 1\,, \tag{A.4}$$

and the so-called *five-terms relation*

$$\mathcal{L}(x) + \mathcal{L}\left(y\right) = \mathcal{L}\left(xy\right) + \mathcal{L}\left(\frac{x(1-y)}{1-xy}\right) + \mathcal{L}\left(\frac{y(1-x)}{1-xy}\right)\,,\ x>0\,, y<1\,, \tag{A.5}$$

wich, evaluated at $y = x$, becomes the well known *Abel duplication formula*

$$\mathcal{L}(x^2) = 2\mathcal{L}\left(x\right) - 2\mathcal{L}\left(\frac{x}{1+x}\right)\,. \tag{A.6}$$

The function $\mathcal{L}(x)$ can be continued analytically to the complex plane and presents two cuts on the real axis: from $-\infty$ to 0 and from 1 to $+\infty$. The analytic continuation can then be performed by means of the following relations

$$\begin{aligned}
\mathcal{L}(x) &= \frac{\pi^2}{3} - \mathcal{L}\left(\frac{1}{x}\right)\,,\ \text{for } x>1\,,\\
\mathcal{L}\left(x\right) &= \mathcal{L}\left(\frac{1}{1-x}\right) - \frac{\pi^2}{6}\,,\ \text{for } x<0\,,
\end{aligned}$$

together with the special values

$$\mathcal{L}\left(0\right) = 0\,, \quad \mathcal{L}\left(1\right) = \frac{\pi^2}{6}\,, \quad \mathcal{L}\left(+\infty\right) = \frac{\pi^2}{3}\,, \quad \mathcal{L}\left(+\infty\right) = -\frac{\pi^2}{6}\,.$$

In [156], using analytical techniques, Rogers was able to prove the following theorem. Define the polynomial

$$p(x,t) = 1 - t - \prod_{i=1}^{n}(1-\alpha_i x)$$

of degree $n$ in $x$, for non vanishing complex numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$. Then, denoting by $x_j$ the roots of $p(x,t) = 0$, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\mathcal{L}\left(\frac{\alpha_i}{\alpha_j}\right) - \mathcal{L}\left(\alpha_i x_j\right)\right] = \mathcal{L}(1-t)\,. \tag{A.7}$$

This relation can be remarkably generalized as can be seen in [102, 158]. This very general example constitutes the prototype of the sum rule encountered in Chapter 4 and 5. Namely it provides the value of a particular sum of dilogarithm evaluated at values which satisfy a certain algebraic equation. The main goal of such a theorem relies on the possibilities of evaluating the sum rule without the actual knowledge of the solutions to the equation involved.

Relations such (A.7) constitues also an example of *accessible identities*, namely relations that can be proved by means of the functional relation (A.5).

## A.2 Accessible identities

By making use of relations (A.4) and (A.6) we can easily obtain the following special values

$$\mathcal{L}\left(1\right) = \frac{\pi^2}{6}\,, \quad \mathcal{L}\left(-1\right) = -\frac{\pi^2}{12}\,, \quad \mathcal{L}\left(\frac{1}{2}\right) = \frac{\pi^2}{12}\,, \qquad \text{(L.Euler 1768);}$$

$$\mathcal{L}\left(\frac{1}{2}\left(\sqrt{5}-1\right)\right) = \frac{\pi^2}{10}\,, \quad \mathcal{L}\left(\frac{1}{2}\left(3-\sqrt{5}\right)\right) = \frac{\pi^2}{15}\,, \qquad \text{(J.Landen 1780).}$$

One of most complete list of the accessible identities involving Rogers dilogarithm function can be found in [102].

The most important result concerning the relations between the Rogers dilogarithm and the TBA description of RG flows comes from the analysis of the following dilogarithm sum rule

$$s(j,n,r) = \frac{6}{\pi^2}\sum_{k=1}^{n-1}\sum_{m=1}^{r}\mathcal{L}\left(\frac{\sin(k\varphi)\sin(n-k)\varphi}{\sin(m+k)\varphi\cdot\sin(m+n-k)\varphi}\right), \tag{A.8}$$

where $\varphi = (j+1)\pi/(n+r)$ and $0 \leq 2j \leq n+r-2$. In [160, 161] the function $s(j,n,r)$ has been evaluated explicitly and found to obey the level-rank duality property $s(n,r)+s(r,n) = (n-1)r$. It is given by

$$s(j,n,r) = c_r^n - 24\Delta_j^{r,n},$$

where

$$c_r^n = \frac{r(n^2-1)}{n+r} \qquad \Delta_j^{r,n} = \frac{n(n^2-1)}{24}\frac{j(j+2)}{n+r},$$

are respectively the central charge and primary fields conformal dimensions of a $SU(n)_r$ WZW conformal field theory. It is remarkable that the values appearing in A.8 can be reduced to those which solve the $A_{n-1}\Diamond A_r$ UV plateau equations in the TBA formulation of Chapter 4.

As last example we mention a sum rule which has been employed in the text, see Section 4.2.3). Firstly proved in [52, 159] it reads explicitly

$$\sum_{a=1}^{n}\mathcal{L}\left(\frac{\sin^2\frac{\pi}{n+3}}{\sin^2\frac{\pi(a+1)}{n+3}}\right) = \frac{\pi^2}{6}\frac{2n}{n+3} \tag{A.9}$$

and constitutes another example of physically relevant dilogarithm sum rule.

# Appendix B

# Scattering kernels and Bootstrap relations

In this appendix we are in going to write down explicitly the scattering matrices we used throughout Chapter 5 and the associated scattering kernels appearing in the integral equations. Then we deduce the kernels identities necessary to switch from the integral to the functional formulation of the TBA for the $(\mathbb{CP}^{N-1})_p \times U(1)$ family.

## B.1 Kernels

We list here the relevant $S$-matrices elements and associated kernels. For a complete derivation of these amplitudes in terms of pure $S$-matrix theory and symmetries we refer to [75].

### B.1.1 Spinon-Spinon Scattering

The spinon-spinon $S$-matrix [74] reads

$$S(\theta) = -\frac{\Gamma\left(1 + i\frac{\theta}{2\pi}\right)\Gamma\left(\frac{1}{4} - i\frac{\theta}{2\pi}\right)}{\Gamma\left(1 - i\frac{\theta}{2\pi}\right)\Gamma\left(\frac{1}{4} + i\frac{\theta}{2\pi}\right)} \tag{B.1}$$

Related to that, we define the kernel $\mathcal{K}$ as

$$\mathcal{K}(\theta) \equiv \frac{1}{2\pi i}\frac{d}{d\theta}\log S(\theta)$$

which may be representated in several worthwhile ways [1]:

$$\begin{aligned}\mathcal{K}(\theta) &= \frac{1}{4\pi^2}\left\{\psi\left(1+i\frac{\theta}{2\pi}\right)+\psi\left(1-i\frac{\theta}{2\pi}\right)-\psi\left(\frac{1}{4}+i\frac{\theta}{2\pi}\right)-\psi\left(\frac{1}{4}-i\frac{\theta}{2\pi}\right)\right\}\\ &= \sum_{n=0}^{\infty}\left(\frac{1}{\pi}\frac{2\pi(n+1/4)}{\theta^2+(2\pi(n+1/4))^2}-\frac{1}{\pi}\frac{2\pi(n+1)}{\theta^2+(2\pi(n+1))^2}\right)=\\ &= \int\frac{d\omega}{2\pi}e^{i\omega\theta}\,\frac{q-q^4}{1-q^4} \qquad \text{con} \qquad q=\exp\left(-\frac{\pi}{2}|\omega|\right)\end{aligned}$$

It is straightforward to get

$$\int_{-\infty}^{+\infty} d\theta\,\mathcal{K}(\theta) = \lim_{\omega\to 0}\hat{\mathcal{K}}(\omega) = \frac{3}{4}$$

### B.1.2 Spinon-Antispinon Scattering

The $S$-matrix associated to the spinon-antispinon scattering is

$$t_1(\theta) = \frac{\Gamma\left(\frac{1}{2}-i\frac{\theta}{2\pi}\right)\Gamma\left(\frac{3}{4}+i\frac{\theta}{2\pi}\right)}{\Gamma\left(\frac{1}{2}+i\frac{\theta}{2\pi}\right)\Gamma\left(\frac{3}{4}-i\frac{\theta}{2\pi}\right)} \tag{B.3}$$

Consequently the kernel $G(\theta)$ is

$$G(\theta) \equiv \frac{1}{2\pi i}\frac{d}{d\theta}\log t_1(\theta) \qquad ,$$

[1] It could be of any use to remind that

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma_E - \sum_{n=0}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n+1}\right) \tag{B.2}$$

where $\gamma_E$ stands for the Euler constant

explicitly

$$G(\theta) = \frac{1}{4\pi^2}\left\{\psi\left(\frac{3}{4}+i\frac{\theta}{2\pi}\right)+\psi\left(\frac{3}{4}-i\frac{\theta}{2\pi}\right)-\psi\left(\frac{1}{2}+i\frac{\theta}{2\pi}\right)-\psi\left(\frac{1}{2}-i\frac{\theta}{2\pi}\right)\right\}$$
$$= \sum_{n=0}^{\infty}\left(\frac{1}{\pi}\frac{2\pi(n+1/2)}{\theta^2+(2\pi(n+1/2))^2}-\frac{1}{\pi}\frac{2\pi(n+3/4)}{\theta^2+(2\pi(n+3/4))^2}\right) =$$
$$= \int\frac{d\omega}{2\pi}e^{i\omega\theta}\,\frac{q^2-q^3}{1-q^4} \qquad \text{con} \qquad q=\exp\left(-\frac{\pi}{2}|\omega|\right) \qquad .$$

Then

$$\int_{-\infty}^{+\infty} d\theta\, G(\theta) = \lim_{\omega\to 0}\hat{G}(\omega) = \frac{1}{4}$$

### B.1.3 Magnon Bound State Scattering

Magnonic solution strings scatter according to the matrices

$$S_{lm}(\theta) \equiv \prod_{\alpha=\frac{|l-m|+1}{2}}^{\frac{l+m-1}{2}} \frac{\theta - i\frac{\pi\alpha}{2}}{\theta + i\frac{\pi\alpha}{2}} \tag{B.4}$$

from which

$$K_{lm}(\theta) \equiv \frac{1}{2\pi i}\frac{d}{d\theta}\log S_{lm}(\theta) = \sum_{\alpha=\frac{|l-m|+1}{2}}^{\frac{l+m-1}{2}} \frac{1}{\pi}\frac{\alpha\pi/2}{\theta^2+(\alpha\pi/2)^2} \qquad ,$$

Fourier Transformed as

$$\hat{K}_{lm}(\omega) = \sum_{\alpha=\frac{|l-m|+1}{2}}^{\frac{l+m-1}{2}} e^{-\alpha|\omega|\pi/2} = \frac{e^{-\frac{|\omega|\pi}{4}|l-m|}-e^{-\frac{|\omega|\pi}{4}(l+m)}}{2\sinh(\pi|\omega|/4)} \qquad .$$

We obtain the matrix

$$N_{lm} = \int d\theta K_{lm}(\theta) = \hat{K}_{lm}(0) = \min\{l,m\} = \frac{l+m-|l-m|}{2}$$

whose inverse reads

$$\hat{K}^{-1}_{nl}(\omega) = 2\cosh\left(\frac{|\omega|\pi}{4}\right)\delta_{nl} - (\delta_{n,l-1}+\delta_{n,l+1}) \qquad ,$$

satisfying

$$\sum_l \hat{K}^{-1}_{nl}(\omega)\hat{K}_{lm}(\omega) = \delta_{nm}$$

## B.2 Helpful Relations in Bootstrapping Matrices and Kernels

Here we are reviewing the identities between scattering matrices (*cfr* [90, 101]) required in order to write down the $Y$-system and universal form TBA

$$\begin{aligned}
&S_{lm}\left(\theta+\frac{i\pi}{4}\right)\, S_{lm}\left(\theta-\frac{i\pi}{4}\right) = S_{l-1,m}\left(\theta\right)\, S_{l+1,m}\left(\theta\right)\, e^{2\pi i\Theta(\theta)\,\delta_{lm}}\\
&t_1\left(\theta+\frac{i\pi}{4}\right)\, t_1\left(\theta-\frac{i\pi}{4}\right) = -S\left(\theta+\frac{i\pi}{4}\right)\, S\left(\theta-\frac{i\pi}{4}\right)\, [S_{11}(\theta)]^{-1}\\
&S\left(\theta+\frac{i\pi}{2}\right)\, S\left(\theta-\frac{i\pi}{2}\right) = -\frac{t_1(\theta)}{S(\theta)} S_{12}(\theta)\, e^{2\pi i\Theta(\theta)}\\
&t_1\left(\theta+\frac{i\pi}{2}\right)\, t_1\left(\theta-\frac{i\pi}{2}\right) = -\frac{S(\theta)}{t_1(\theta)}\\
&S_{lm}\left(\theta+\frac{i\pi}{2}\right)\, S_{lm}\left(\theta-\frac{i\pi}{2}\right) = S_{l-2,m}\left(\theta\right)\, S_{l+2,m}\left(\theta\right)\, e^{2\pi i\Theta(\theta)\, I_{lm}}
\end{aligned} \tag{B.5}$$

($\Theta(x)$ stands for the Heaviside step function, while $I_{lm} = \delta_{l-1,m} + \delta_{l+1,m}$ ). These relations are reflected into the following ones, involving the kernels:

$$\begin{aligned}
&K_{lm}\left(\theta+\frac{i\pi}{4}\right) + K_{lm}\left(\theta-\frac{i\pi}{4}\right) = K_{l-1,m}\left(\theta\right) + K_{l+1,m}\left(\theta\right) + \delta(\theta)\,\delta_{lm}\\
&G\left(\theta+\frac{i\pi}{4}\right) + G\left(\theta-\frac{i\pi}{4}\right) = \mathcal{K}\left(\theta+\frac{i\pi}{4}\right) + \mathcal{K}\left(\theta-\frac{i\pi}{4}\right) - K_{11}(\theta)\\
&\mathcal{K}\left(\theta+\frac{i\pi}{2}\right) + \mathcal{K}\left(\theta-\frac{i\pi}{2}\right) = -\mathcal{K}(\theta) + G(\theta) + K_{12}(\theta) + \delta(\theta)\\
&G\left(\theta+\frac{i\pi}{2}\right) + G\left(\theta-\frac{i\pi}{2}\right) = \mathcal{K}(\theta) - G(\theta)\\
&K_{lm}\left(\theta+\frac{i\pi}{2}\right) + K_{lm}\left(\theta-\frac{i\pi}{2}\right) = K_{l-2,m}\left(\theta\right) + K_{l+2,m}\left(\theta\right) + \delta(\theta)\, I_{lm} +\\
&\qquad + \delta_{l1}\,\delta_{m1}\left[\delta(\theta+\frac{i\pi}{4}) + \delta(\theta-\frac{i\pi}{4})\right]
\end{aligned} \tag{B.6}$$

(the last relation makes sense [2] provided we define $K_{l,0} = 0$ , $K_{l,-1} = -K_{l,1}$). Moreover, we find:

$$\begin{aligned}
\mathcal{K}(\theta + \frac{i\pi}{2}) + G(\theta - \frac{i\pi}{2}) - K_{11}(\theta + \frac{i\pi}{4}) &= 0 \\
\mathcal{K}(\theta - \frac{i\pi}{2}) + G(\theta + \frac{i\pi}{2}) - K_{11}(\theta - \frac{i\pi}{4}) &= 0 \\
\mathcal{K}(\theta + \frac{i\pi}{2}) + G(\theta - \frac{i\pi}{2}) + K_{11}(\theta - \frac{i\pi}{4}) &= K_{12}(\theta) + \delta(\theta) \\
\mathcal{K}(\theta - \frac{i\pi}{2}) + G(\theta + \frac{i\pi}{2}) + K_{11}(\theta + \frac{i\pi}{4}) &= K_{12}(\theta) + \delta(\theta)
\end{aligned} \tag{B.7}$$

## B.3 Useful Fourier Transforms

We conclude this Appendix displaying some useful Fourier pair used in dealing with the $(\mathbb{CP}^{N-1})_p$ TBA (5.24)

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\cosh(\frac{\pi}{2}a\omega)}{\cosh(\frac{\pi\omega}{2})} e^{i\omega\theta} &= \frac{2}{\pi} \frac{\cos(a\pi/2)\cosh\theta}{\cos(a\pi) + \cosh(2\theta)} = \chi_a(\theta), \\
\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\sinh(\frac{\pi}{2}a\omega)}{\sinh(\frac{\pi\omega}{2})} e^{i\omega\theta} &= \frac{1}{\pi} \frac{\sin(a\pi)}{\cos(a\pi) + \cosh(2\theta)} = \psi_a(\theta), \\
\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{2\cosh(\frac{\pi\omega}{2a})} e^{i\omega\theta} &= \frac{a}{2\pi\,\cosh(a\theta)} = \phi_a(\theta).
\end{aligned} \tag{B.8}$$

From which we can easily read the normalization constants appearing in the plateau equations (5.29)

$$\begin{aligned}
\int_{-\infty}^{\infty} d\theta\, \chi_a(\theta) &= 1\,, \\
\int_{-\infty}^{\infty} d\theta\, \psi_a(\theta) &= \frac{1}{a}\,, \\
\int_{-\infty}^{\infty} d\theta\, \phi_a(\theta) &= \frac{1}{2}\,.
\end{aligned}$$

[2] Actually, the contact terms $\delta(\theta \pm i\frac{\pi}{4})$ are but a pretty formal scripture: relations (B.6) always appear in integrals and it is to be taken into account a residue calculation, whose net result is equivalent to the effect of some kind of complex-argument defined delta function.

# Bibliography


[1] C. -N. Yang and R. L. Mills, "*Conservation of Isotopic Spin and Isotopic Gauge Invariance*", Phys. Rev. **96** (1954) 191.

[2] P. W. Higgs, "*Broken Symmetries and the Masses of Gauge Bosons*", Phys. Rev. Lett. **13** (1964) 508.

[3] G. 't Hooft, "*The Confinement phenomenon in quantum field theory*", In *'t Hooft, G. (ed.): Under the spell of the gauge principle* 514-546, and in *Cargese 1981, Proceedings, Fundamental interactions* 639-671.
G. 't Hooft, "*Topology of the Gauge Condition and New Confinement Phases in Nonabelian Gauge Theories*", Nucl. Phys. B **190**, 455 (1981).

[4] D. J. Gross and F. Wilczek, "*Ultraviolet Behavior of Nonabelian Gauge Theories*", Phys. Rev. Lett. **30** (1973) 1343.

[5] H. D. Politzer, Phys. Rev. Lett. **30** (1973) 1346.

[6] N. Seiberg and E. Witten, "*Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory*", Nucl. Phys. B **426** (1994) 19 [Erratum-ibid. B **430** (1994) 485] [hep-th/9407087].

[7] N. Seiberg and E. Witten, "*Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD*", Nucl. Phys. B **431** (1994) 484 [hep-th/9408099].

[8] G. Veneziano, "*Construction of a crossing - symmetric, Regge behaved amplitude for linearly rising trajectories*", Nuovo Cim. A **57** (1968) 190.

[9] G. 't Hooft, "*A Planar Diagram Theory for Strong Interactions*", Nucl. Phys. B **72** (1974) 461.

[10] K. G. Wilson, "*Confinement of Quarks*", Phys. Rev. D **10** (1974) 2445.

[11] J. M. Maldacena, "*The Large N limit of superconformal field theories and supergravity*", Adv. Theor. Math. Phys. **2** (1998) 231 [hep-th/9711200].

[12] G. 't Hooft, gr-qc/9310026.

[13] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, "$\mathcal{N} = 6$ *superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*", JHEP **0810** (2008) 091 [arXiv:0806.1218 [hep-th]].

[14] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, "*Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory*", Nucl. Phys. B **241** (1984) 333.

[15] P. H. Ginsparg, "*Applied Conformal Field Theory*", hep-th/9108028.

[16] P.Di. Francesco, P. Mathieu and D. Senechal, "*Conformal field theory*", (Springer, New York, 1997).

[17] J.L. Cardy, "*Conformal invariance and statistical mechanics, in Fields, Strings and Critical Phenomena*", ed. E. Brezin and J. Zinn-Justin, *Les Houches 1988, Session XLIX* (North-Holland, Amsterdam, 1990) p. 169.

[18] S. R. Coleman and J. Mandula, "*All Possible Symmetries of the S Matrix*", Phys. Rev. **159** (1967) 1251.

[19] A. M. Polyakov, "*Hidden Symmetry of the Two-Dimensional Chiral Fields*", Phys. Lett. B **72** (1977) 224.

[20] A. B. Zamolodchikov, "*Exact Two Particle s Matrix of Quantum Sine-Gordon Solitons*", Pisma Zh. Eksp. Teor. Fiz. **25** (1977) 499 [Commun. Math. Phys. **55** (1977) 183].

[21] R. Shankar and E. Witten, "*The S Matrix of the Supersymmetric Nonlinear Sigma Model*", Phys. Rev. D **17** (1978) 2134.

[22] D. Iagolnitzer, "*Factorization of the Multiparticle s Matrix in Two-Dimensional Space-Time Models*", Phys. Rev. D **18** (1978) 1275.

[23] S. J. Parke, "*Absence of Particle Production and Factorization of the S Matrix in (1+1)-dimensional Models*", Nucl. Phys. B **174** (1980) 166.

[24] A. B. Zamolodchikov, "*Integrable field theory from conformal field theory*", Adv. Stud. Pure Math. **19** (1989) 641.

[25] A. B. Zamolodchikov, "*Integrals of Motion in Scaling Three State Potts Model Field Theory*", Int. J. Mod. Phys. A **3** (1988) 743.

[26] A. B. Zamolodchikov, "*Higher Order Integrals of Motion in Two-Dimensional Models of the Field Theory with a Broken Conformal Symmetry*", JETP Lett. **46** (1987) 160 [Pisma Zh. Eksp. Teor. Fiz. **46** (1987) 129].

[27] C. -N. Yang and C. P. Yang, "*One-dimensional chain of anisotropic spin spin interactions. 1. Proof of Bethe's hypothesis for ground state in a finite system*", Phys. Rev. **150** (1966) 321.
C. N. Yang and C. P. Yang, "*One-dimensional chain of anisotropic spin spin interactions. 2. Properties of the ground state energy per lattice site for an infinite system*", Phys. Rev. **150** (1966) 327.

[28] J. A. Minahan, W. Schulgin and K. Zarembo, "*Two loop integrability for Chern-Simons theories with N=6 supersymmetry*", JHEP **0903** (2009) 057 [arXiv:0901.1142 [hep-th]].

[29] A.M. Polyakov, JETP Lett. **12** 381 (1970).

[30] D. Friedan, Z. -a. Qiu and S. H. Shenker, "*Conformal Invariance, Unitarity and Two-Dimensional Critical Exponents*", Phys. Rev. Lett. **52** (1984) 1575.

[31] V. S. Dotsenko and V. A. Fateev, "*Conformal Algebra and Multipoint Correlation Functions in Two-Dimensional Statistical Models*", Nucl. Phys. B **240** (1984) 312.

[32] V. G. Kac, "*Simple graded algebras of finite growth*", Funct. Anal. Appl. **1** (1967) 328.

[33] R. V. Moody, "*Lie Algebras associated with generalized Cartan matrices*", Bull. Am. Math. Soc. **73** (1967) 217.

[34] P. Goddard, A. Kent and D. I. Olive, "*Virasoro Algebras and Coset Space Models*", Phys. Lett. B **152** (1985) 88.

[35] V. G. Knizhnik and A. B. Zamolodchikov, "*Current Algebra and Wess-Zumino Model in Two-Dimensions*", Nucl. Phys. B **247** (1984) 83.

[36] D. Gepner and E. Witten, "*String Theory on Group Manifolds*", Nucl. Phys. B **278** (1986) 493.

[37] J. Wess and B. Zumino, "*Consequences of anomalous Ward identities*", Phys. Lett. B **37** (1971) 95.

[38] S. P. Novikov, "*The Hamiltonian formalism and a many valued analog of Morse theory*", Usp. Mat. Nauk **37N5** (1982) 3.

[39] E. Witten, "*Global Aspects of Current Algebra*", Nucl. Phys. B **223** (1983) 422.
E. Witten, "*Nonabelian Bosonization in Two-Dimensions*", Commun. Math. Phys. **92** (1984) 455.

[40] H. Sugawara, "*A Field theory of currents*", Phys. Rev. **170** (1968) 1659.

[41] R. Haag, J. T. Lopuszanski and M. Sohnius, "*All Possible Generators of Supersymmetries of the s Matrix*", Nucl. Phys. B **88** (1975) 257.

[42] L. D. Faddeev, "*Quantum completely integral models of field theory*" Sov. Sci. Rev. C **1** (1980) 107.

[43] P. Dorey, "*Exact S matrices*", hep-th/9810026.

[44] R. Shankar and E. Witten, "*The S Matrix of the Kinks of the* $(\bar{\psi}\psi)^2$ *Model*", Nucl. Phys. B **141** (1978) 349 [Erratum-ibid. B **148** (1979) 538].

[45] C. -N. Yang, "*Some exact results for the many body problems in one dimension with repulsive delta function interaction*", Phys. Rev. Lett. **19** (1967) 1312.
R. J. Baxter, "*One-dimensional anisotropic Heisenberg chain*", Annals Phys. **70** (1972) 323.

[46] B. Schroer, T. T. Truong and P. Weisz, "*Towards an Explicit Construction of the Sine-Gordon Field Theory*", Phys. Lett. B **63** (1976) 422.

[47] M. Karowski, H. J. Thun, T. T. Truong and P. H. Weisz, "*On the Uniqueness of a Purely Elastic s Matrix in (1+1)-Dimensions*", Phys. Lett. B **67** (1977) 321.

[48] R.J. Eden, P.V. Landshoff, D.I. Olive and J.C. Polkinghorne, "*The analytic S-Matrix*, (CUP, Cambridge, 1966).

[49] S. Weinberg, "*The Quantum Theory of Fields*", (CUP, Cambridge, 1995).

[50] A. B. Zamolodchikov, "*Exact s Matrix of Quantum Sine-Gordon Solitons*", JETP Lett. **25** (1977) 468.

[51] P. Mitra, "*Elasticity, Factorization and s Matrices in (1+1)-Dimensions*", Phys. Lett. B **72** (1977) 62.

[52] A. N. Kirillov and N. Y. Reshetikhin, "*Exact solution of the integrable XXZ Heisenberg model with arbitrary spin. I. The ground state and the excitation spectrum*", J. Phys. A **20** (1987) 1565.

[53] A. M. Polyakov, "*Gauge Fields and Strings*", Contemp. Concepts Phys. **3** (1987) 1.

[54] A. A. Tseytlin, "*Sigma Model Approach To String Theory*", Int. J. Mod. Phys. A **4** (1989) 1257.

[55] M. Luscher and K. Pohlmeyer, "*Scattering of Massless Lumps and Nonlocal Charges in the Two-Dimensional Classical Nonlinear Sigma Model*", Nucl. Phys. B **137** (1978) 46.

[56] V. L. Golo and A. M. Perelomov, "*Solution of the Duality Equations for the Two-Dimensional SU(N) Invariant Chiral Model*", Phys. Lett. B **79** (1978) 112.

[57] D. Friedan, "*Nonlinear Models in Two Epsilon Dimensions*", Phys. Rev. Lett. **45** (1980) 1057.

[58] D. H. Friedan, "*Nonlinear Models in Two + Epsilon Dimensions*", Annals Phys. **163** (1985) 318.

[59] S.V. Ketov "*Quantum Non-Linear Sigma-Models: From Quantum Field Theory to Supersymmetry, Conformal Field Theory, Black Holes and Strings*", Texts and Monographs in Physics, Springer-Verlag, (2001).

[60] H. Osborn, "*Renormalization and Composite Operators in Nonlinear $\sigma$ Models*", Nucl. Phys. B **294** (1987) 595.
H. Osborn, "*Renormalization Group and Two Point Functions in Bosonic $\sigma$ Models*", Phys. Lett. B **214** (1988) 555.
H. Osborn, "*String Theory Effective Actions From Bosonic $\sigma$ Models*", Nucl. Phys. B **308** (1988) 629.

[61] K. Bardakci, E. Rabinovici and B. Saering, "*String Models with $c < 1$ Components*", Nucl. Phys. B **299** (1988) 151.
Q-H. Park, "*Lagrangian Formulation of Coset Conformal Field Theory*",

Phys. Lett. B **223** (1989) 175.
D. Karabali, Q-H. Park, H. J. Schnitzer and Z. Yang, "*A GKO Construction Based on a Path Integral Formulation of Gauged Wess-Zumino-Witten Actions*", Phys. Lett. B **216** (1989) 307.

[62] H. Eichenherr, "*SU(N) Invariant Nonlinear Sigma Models*", Nucl. Phys. B **146** (1978) 215 [Erratum-ibid. B **155** (1979) 544].

[63] A. D'Adda, M. Luscher and P. Di Vecchia, "*A 1/n Expandable Series of Nonlinear Sigma Models with Instantons*", Nucl. Phys. B **146** (1978) 63.

[64] A. D'Adda, P. Di Vecchia and M. Luscher, "*Confinement and Chiral Symmetry Breaking in CP**n-1 Models with Quarks*", Nucl. Phys. B **152** (1979) 125.

[65] E. Witten, "*Instantons, the Quark Model, and the 1/n Expansion*", Nucl. Phys. B **149** (1979) 285.

[66] G. Valent, "*Perturbative Renormalization Of The Cp(n) Model*", Nucl. Phys. B **238** (1984) 142.

[67] S. Hikami, "*Renormalization Group Functions of CP**1N Nonlinear Sigma Model and N Component Scalar QED Model*", Prog. Theor. Phys. **62** (1979) 226.

[68] A. M. Polyakov, "*Interaction of Goldstone Particles in Two-Dimensions. Applications to Ferromagnets and Massive Yang-Mills Fields*", Phys. Lett. B **59** (1975) 79.

[69] K. Pohlmeyer, "*Integrable Hamiltonian Systems and Interactions Through Quadratic Constraints*", Commun. Math. Phys. **46** (1976) 207.

[70] E. Abdalla, M. C. B. Abdalla and M. Gomes, "*Anomaly in the Nonlocal Quantum Charge of the $CP^{(n-1)}$ Model*", Phys. Rev. D **23** (1981) 1800.

[71] R. Koberle and V. Kurak, "*Liberation Of U(n) Solitons In The Cp**(n-1) Model By Massless Quarks*" IFUSP-P-202.

[72] R. Koberle and V. Kurak, "*Deconfinement in the CP(1N) Model by Massless Fermions*", Phys. Rev. Lett. **58** (1987) 627.

[73] I. Kawaguchi and K. Yoshida, "*Hybrid classical integrable structure of squashed sigma models: A Short summary*", J. Phys. Conf. Ser. **343** (2012) 012055 [arXiv:1110.6748 [hep-th]].

[74] B. Basso and A. Rej, "*On the integrability of two-dimensional models with U(1)xSU(N) symmetry*", Nucl. Phys. B **866** (2013) 337 [arXiv:1207.0413 [hep-th]].

[75] B. Berg, M. Karowski, P. Weisz and V. Kurak, "*Factorized U(n) Symmetric s Matrices in Two-Dimensions*", Nucl. Phys. B **134** (1978) 125.

[76] E. Abdalla, M. C. B. Abdalla and M. Gomes, "*Anomaly Cancellations in the Supersymmetric $CP^{(N-1)}$ Model*", Phys. Rev. D **25** (1982) 452.

[77] E. Abdalla and M. Forger, "*Integrable Nonlinear $\sigma$ Models With Fermions*", Commun. Math. Phys. **104** (1986) 123.

[78] J. S. Schwinger, "*The Theory of quantized fields. I.*", Phys. Rev. **82** (1951) 914.
J. S. Schwinger, "*The Theory of quantized fields. II.*", Phys. Rev. **91** (1953) 713.
J. S. Schwinger, "*Gauge Invariance and Mass. II.*", Phys. Rev. **128** (1962) 2425.

[79] J. Balog, P. Forgacs and L. Palla, "*A Two-dimensional integrable axionic sigma model and T duality*", Phys. Lett. B **484** (2000) 367 [hep-th/0004180].

[80] I. Kawaguchi and K. Yoshida, "*Hidden Yangian symmetry in sigma model on squashed sphere*", JHEP **1011** (2010) 032 [arXiv:1008.0776 [hep-th]].

[81] Y. Y. Goldschmidt and E. Witten, "*Conservation Laws in Some Two-dimensional Models*", Phys. Lett. B **91** (1980) 392.

[82] D. Bykov, "*The worldsheet low-energy limit of the AdS_4 x $CP\hat{3}$ superstring*", Nucl. Phys. B **838** (2010) 47 [arXiv:1003.2199 [hep-th]].

[83] H. Bethe, "*On the theory of metals. 1. Eigenvalues and eigenfunctions for the linear atomic chain*" (in German) Z. Phys. **71** (1931) 205.

[84] L. Faddeev, E. Sklyanin and L. Takhtajan, Theor. Math. Phys. 40 (1980) 688; N. Yu. Reshethikhin, L. Takhtajan and L.D. Faddeev, Len. Math. J. 1 (1990) 193.

L.D. Faddeev and L.A. Takhtajan, J. Sov. Math. 24 (1984) 241;
L.D. Faddeev and L.A. Takhtajan, Phys. Lett. 85A (1981) 375.
L.A. Takhtajian and L.D. Faddeev, Russ. Math. Sur. 34:5 (1979) 13.
V.E. Korepin, Theor. Math. Phys. 76 (1980) 165;
V.E. Korepin, G. Izergin and N.M. Bogoliubov, "*Quantum Inverse Scattering Method, Correlation Functions and Algebraic Bethe Ansatz*", (Cambridge University Press, 1993).
L.A. Takhtajan, "*Quamtum Groups, Introduction to Quantum Groups and Intergable Massive models of Quantum Field Theory*", eds, M.-L. Ge and B.-H. Zhao, Nankai Lectures on Mathematical Physics, World Scientific, 1990, p.p. 69.

[85] L.D. Faddeev, hep-th/9605187.
L.D. Faddeev, Int. J. Mod. Phys. A10 (1995) 1845, hep-th/9494013.
E.K. Sklyanin, hep-th/9211111.
R.I. Nepomechie, Int. J. Mod. Phys. B13 (1999) 2973, hep-th/9810032.
A. Klumper, Lect. Notes Phys. 645 (2004) 349, cond-mat/0502431.

[86] A. Doikou, S. Evangelisti, G. Feverati and N. Karaiskos, "*Introduction to Quantum Integrability*", Int. J. Mod. Phys. A **25** (2010) 3307 [arXiv:0912.3350 [math-ph]].

[87] A. B. Zamolodchikov, "*Thermodynamic Bethe Ansatz in Relativistic Models. Scaling Three State Potts and Lee-yang Models*", Nucl. Phys. B **342** (1990) 695.

[88] T. Klassen and E. Melzer, "*Purely elastic scattering theories and their ultraviolet limits*", Nucl. Phys B342, 695 (1990).

[89] T. Klassen and E. Melzer, "*The thermodynamics of purely elastic scattering theories and conformal perturbation theory*", Nucl. Phys. B350, 635 (1991).

[90] A. B. Zamolodchikov, "*On the thermodynamic Bethe ansatz equations for reflectionless ADE scattering theories*", Phys. Lett. B **253** (1991) 391.

[91] A. B. Zamolodchikov, "*Thermodynamic Bethe ansatz for RSOS scattering theories*", Nucl. Phys. B **358** (1991) 497.

[92] A. B. Zamolodchikov, "*From tricritical Ising to critical Ising by thermodynamic Bethe ansatz*", Nucl. Phys. B **358** (1991) 524.

[93] A. B. Zamolodchikov, "*TBA equations for integrable perturbed SU(2)-k x SU(2)-l / SU(2)-k+ l coset models*", Nucl. Phys. B **366** (1991) 122.

[94] A. B. Zamolodchikov, "*On the thermodynamic Bethe ansatz equation in sinh-Gordon model*", J. Phys. A **39** (2006) 12863 [hep-th/0005181].

[95] C. -N. Yang and T. D. Lee, "*Statistical theory of equations of state and phase transitions. 1. Theory of condensation*", Phys. Rev. **87** (1952) 404.

[96] M. E. Fisher, "*Yang-Lee Edge Singularity and phi**3 Field Theory*", Phys. Rev. Lett. **40** (1978) 1610.

[97] H.A. Bethe, *Z.Physik.* **71** (1998) 588.

[98] J. L. Cardy, "*Conformal invariance and universality in finite-size scaling*", J. Phys. A **17** (1984) L385.

[99] V. A. Fateev, E. Onofri and A. B. Zamolodchikov, "*The Sausage model (integrable deformations of O(3) sigma model)*", Nucl. Phys. B **406** (1993) 521.

[100] Z. Bajnok, C. Rim and A. .Zamolodchikov, "*Sinh-Gordon boundary TBA and boundary Liouville reflection amplitude*", Nucl. Phys. B **796** (2008) 622 [arXiv:0710.4789 [hep-th]].

[101] F. Ravanini, R. Tateo and A. Valleriani, "*Dynkin TBAs*" Int. J. Mod. Phys. A **8** (1993) 1707 [hep-th/9207040].

[102] A.N. Kirillov, "*Dilogarithm identities*", Prog. Theor. Phys. Suppl. 118, 61, (1995),[hep-th/9408113].

[103] P. Fendley, "*Sigma models as perturbed conformal field theories*", Phys. Rev. Lett. **83** (1999) 4468 [hep-th/9906036].

[104] P. Fendley, "*Integrable sigma models and perturbed coset models*", JHEP **0105** (2001) 050 [hep-th/0101034].

[105] F. Ravanini, "*Thermodynamic Bethe ansatz for G(k) x G(l) / G(k+l) coset models perturbed by their phi(1,1,Adj) operator*", Phys. Lett. B **282** (1992) 73 [hep-th/9202020].

[106] F. Ravanini, "*Finite size effects in integrable quantum field theories*", hep-th/0102148.

[107] E. Quattrini, F. Ravanini and R. Tateo, "*Integrable QFT in two-dimensions encoded on products of Dynkin diagrams*", hep-th/9311116.

[108] F. Ravanini, R. Tateo and A. Valleriani, "*A New family of diagonal A-D-E related scattering theories*", Phys. Lett. B **293** (1992) 361 [hep-th/9207069].

[109] A. Kirillov, Newton inst. preprints, [hep-th/9211137] and [hep-th/9212150]

[110] C. Ahn, D. Bernard and A. LeClair, "*Fractional Supersymmetries in Perturbed Coset Cfts and Integrable Soliton Theory*", Nucl. Phys. B **346** (1990) 409.

[111] P. Di Francesco, H. Saleur and J. B. Zuber, "*Generalized Coulomb Gas Formalism for Two-dimensional Critical Models Based on SU(2) Coset Construction*", Nucl. Phys. B **300** (1988) 393.

[112] D. Bernard and A. Leclair, "*Residual Quantum Symmetries of the Restricted Sine-Gordon Theories*", Nucl. Phys. B **340** (1990) 721.

[113] P. Fendley and K. A. Intriligator, "*Scattering and thermodynamics of fractionally charged supersymmetric solitons*", Nucl. Phys. B **372** (1992) 533 [hep-th/9111014].

[114] V. A. Fateev and A. B. Zamolodchikov, "*Integrable perturbations of Z(N) parafermion models and O(3) sigma model*", Phys. Lett. B **271** (1991) 91.

[115] F. Ravanini, "*Thermodynamic Bethe ansatz for G(k) x G(l) / G(k+l) coset models perturbed by their phi(1,1,Adj) operator*", Phys. Lett. B **282** (1992) 73 [hep-th/9202020].

[116] F. Ravanini, "*On the possibility of Z(N) exotic supersymmetry in two-dimensional conformal field theory*", Int. J. Mod. Phys. A **7** (1992) 4949 [hep-th/9109057].

[117] V. A. Fateev and A. B. Zamolodchikov, "*Representations of the Algebra of 'Parafermion Currents' of Spin 4/3 in Two-dimensional Conformal Field Theory. Minimal Models and the Tricritical Potts Z(3) Model*", Theor. Math. Phys. **71** (1987) 451 [Teor. Mat. Fiz. **71** (1987) 163].

[118] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, (Oxford, 1989)

[119] A. Zamolodchikov, "*Relativistic Factorized S Matrix in Two-Dimensional Space-Time with Isotopic O(n) Symmetry*", Pisma Zh. Eksp. Teor. Fiz. **26** (1977) 608.

[120] A. B. Zamolodchikov and A. B. Zamolodchikov, "*Factorized s Matrices in Two-Dimensions as the Exact Solutions of Certain Relativistic Quantum Field Models*", Annals Phys. **120** (1979) 253.

[121] M. J. Martins, "*The Two-dimensional o(3) nonlinear sigma model at finite temperature*", Phys. Lett. B **278** (1992) 167 [hep-th/9202042].

[122] N. Gromov, V. Kazakov and P. Vieira, "*Finite Volume Spectrum of 2D Field Theories from Hirota Dynamics*", JHEP **0912** (2009) 060 [arXiv:0812.5091 [hep-th]].

[123] J. Balog and A. Hegedus, "*TBA Equations for excited states in the O(3) and O(4) nonlinear sigma model*", J. Phys. A **37** (2004) 1881 [hep-th/0309009].

[124] A. M. Polyakov and P. B. Wiegmann, "*Theory of Nonabelian Goldstone Bosons*", Phys. Lett. B **131** (1983) 121.

[125] P. B. Wiegmann, "*Exact Solution Of The O(3) Nonlinear Sigma Model*", Phys. Lett. B **152** (1985) 209.

[126] M. Takahashi and M. Suzuki, "*One-Dimensional Anisotropic Heisenberg Model at Finite Temperatures*", Prog. Theor. Phys. **48** (1972) 2187.

[127] G. Arutyunov and S. Frolov, "*String hypothesis for the AdS(5) x S(5) mirror*", JHEP **0903** (2009) 152 [arXiv:0901.1417 [hep-th]].

[128] J. Balog and A. Hegedus, "*Virial expansion and TBA in O(N) sigma models*", Phys. Lett. B **523** (2001) 211 [hep-th/0108071].

[129] J. Balog and A. Hegedus, "*TBA Equations for excited states in the O(3) and O(4) nonlinear sigma model*", J. Phys. A **37** (2004) 1881 [hep-th/0309009].

[130] J. Balog and A. Hegedus, "*TBA equations for the mass gap in the O(2r) non-linear sigma-models*", Nucl. Phys. B **725** (2005) 531 [hep-th/0504186].

[131] J. Balog and A. Hegedus, "*The finite size spectrum of the 2-dimensional O(3) nonlinear sigma-model*", Nucl. Phys. B **829** (2010) 425 [arXiv:0907.1759 [hep-th]].

[132] P. Hasenfratz, M. Maggiore and F. Niedermayer, "*The Exact mass gap of the O(3) and O(4) nonlinear sigma models in d = 2*", Phys. Lett. B **245** (1990) 522.

[133] P. Hasenfratz and F. Niedermayer, "*The Exact mass gap of the O(N) sigma model for arbitrary N is >= 3 in d = 2*", Phys. Lett. B **245** (1990) 529.

[134] D. Volin, "*From the mass gap in O(N) to the non-Borel-summability in O(3) and O(4) sigma-models*", Phys. Rev. D **81** (2010) 105008 [arXiv:0904.2744 [hep-th]].

[135] P. Forgacs, S. Naik and F. Niedermayer, "*The Exact mass gap of the chiral Gross-Neveu model*", Phys. Lett. B **283** (1992) 282.

[136] J. M. Evans and T. J. Hollowood, "*The Exact mass gap of the supersymmetric cp**(n-1) sigma model*", Phys. Lett. B **343** (1995) 198 [hep-th/9409142].

[137] A. B. Zamolodchikov, "*Mass scale in the sine-Gordon model and its reductions*", Int. J. Mod. Phys. A **10** (1995) 1125.

[138] M. J. Martins, "*Complex excitations in the thermodynamic Bethe ansatz approach*", Phys. Rev. Lett. **67** (1991) 419.

[139] V. Privman and M.E. Fisher, *Finite-size effects at first-order transitions*, J. Stat. Phys. 33, 385 (1983).
E. Brézin and J. Zinn-Justin, *Finite size effects in phase transitions*, Nucl. Phys. B257, 867 (1985).
G. Münster, *Tunneling amplitude and surface tension in $\phi^4$ -theory*, Nucl. Phys.B324, 630 (1989).
K. Jansen and Y. Shen, *Tunneling and energy splitting in Ising models*, Nucl. Phys. B393, 658 (1993).

[140] D. Bombardelli, D. Fioravanti and R. Tateo, "*TBA and Y-system for planar AdS(4)/CFT(3)*" Nucl. Phys. B **834** (2010) 543 [arXiv:0912.4715 [hep-th]].

[141] L. F. Alday, J. Maldacena, A. Sever and P. Vieira, "*Y-system for Scattering Amplitudes*", J. Phys. A **43** (2010) 485401 [arXiv:1002.2459 [hep-th]].

[142] N. Gromov and F. Levkovich-Maslyuk, "*Y-system, TBA and Quasi-Classical strings in AdS(4) x CP3*", JHEP **1006** (2010) 088 [arXiv:0912.4911 [hep-th]].

[143] L. F. Alday and J. M. Maldacena, "*Comments on operators with large spin*", JHEP **0711** (2007) 019 [arXiv:0708.0672 [hep-th]].

[144] P. Dorey, R. Tateo and K. E. Thompson, "*Massive and massless phases in selfdual Z(N) spin models: Some exact results from the thermodynamic Bethe ansatz*", Nucl. Phys. B **470** (1996) 317 [hep-th/9601123].

[145] F. Gliozzi and R. Tateo, "*ADE functional dilogarithm identities and integrable models*", Phys. Lett. B **348** (1995) 84 [hep-th/9411203].

[146] A. Fabbri, D. Fioravanti, S. Piscaglia and R. Tateo, "*Exact results for the low energy $AdS_4$ x $\mathbb{CP}^3$ string theory*", JHEP **1311** (2013) 073 [arXiv:1308.1861 [hep-th]].

[147] R. Tateo, "*The Sine-Gordon model as SO(n)-1 x SO(n)-1 / SO(n)-2 perturbed coset theory and generalizations*", Int. J. Mod. Phys. A **10** (1995) 1357 [hep-th/9405197].

[148] D. Bernard and A. Leclair, "*The Fractional supersymmetric Sine-Gordon models*", Phys. Lett. B **247** (1990) 309.

[149] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, "*Integrable quantum field theories in finite volume: Excited state energies*", Nucl. Phys. B **489** (1997) 487 [hep-th/9607099].

[150] A. Klumper, M. T. Batchelor and P. A. Pearce, "*Central charges of the 6- and 19- vertex models with twisted boundary conditions*", J. Phys. A **24** (1991) 3111.

[151] P. Fendley, "*Excited state thermodynamics*", Nucl. Phys. B **374** (1992) 667 [hep-th/9109021].

[152] P. Fendley, "*Excited state energies and supersymmetric indices*", Adv. Theor. Math. Phys. **1** (1998) 210 [hep-th/9706161].

[153] T. Nakanishi and R. Tateo, "*Dilogarithm identities for sine-Gordon and reduced sine-Gordon Y-systems*" SIGMA **6** (2010) 085 [arXiv:1005.4199 [math.QA]].

[154] A. Kuniba, T. Nakanishi and J. Suzuki, "*T-systems and Y-systems for quantum affinizations of quantum Kac-Moody algebras*", SIGMA **5** (2009) 108 [arXiv:0909.4618 [math.QA]].

A. Kuniba, T. Nakanishi and J. Suzuki, "*T-systems and Y-systems in integrable systems*", J. Phys. A **44** (2011) 103001 [arXiv:1010.1344 [hep-th]].

[155] A. Kuniba, T. Nakanishi and J. Suzuki, "*Functional relations in solvable lattice models. 1: Functional relations and representation theory*", Int. J. Mod. Phys. A **9** (1994) 5215 [hep-th/9309137].
A. Kuniba, T. Nakanishi and J. Suzuki, "*Functional relations in solvable lattice model 2: Applications*", Int. J. Mod. Phys. A **9** (1994) 5267 [hep-th/9310060].

[156] L. Rogers, "*On function sum theorems connected with the series* $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$", Proc. London Math. Soc., (1907) v.4, 169-189.

[157] L. Lewin, "*Structural properties of polylogarithms*", Math. Surveys and Monographs, 1991, v.37.

[158] G. Ray, "*Multivariable polylogarithm identities*", Chapter 7 in [157], pp. 123-169.

[159] V. Bazhanov and N. Reshetikhin, "*Restricted Solid on Solid Models Connected With Simply Based Algebras and Conformal Field Theory*", J. Phys. A **23** (1990) 1477.

[160] A. N. Kirillov, "*Dilogarithm identities and spectra in conformal field theory*", Preprint - Kirillov, A.N. (rec.Dec.92) 10 p.B e: LANL hep-th/9211137 [hep-th/9211137].

[161] A. N. Kirillov, "*Dilogarithm identities, partitions and spectra in conformal field theory. 1.*", hep-th/9212150.

[162] P. Dorey and R. Tateo, "*Excited states by analytic continuation of TBA equations*", Nucl. Phys. B **482** (1996) 639 [hep-th/9607167].

[163] C. Destri and H. J. de Vega, "*New thermodynamic Bethe ansatz equations without strings*", Phys. Rev. Lett. **69** (1992) 2313.

[164] C. Destri and H. J. De Vega, "*Unified approach to thermodynamic Bethe Ansatz and finite size corrections for lattice models and field theories*", Nucl. Phys. B **438** (1995) 413 [hep-th/9407117].

[165] D. Fioravanti, A. Mariottini, E. Quattrini and F. Ravanini, "*Excited state Destri-De Vega equation for Sine-Gordon and restricted Sine-Gordon models*", Phys. Lett. B **390** (1997) 243 [hep-th/9608091].