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WEIGHTED INEQUALITIES AND LIPSCHITZ SPACES

Presentata da: MARIA ROSARIA TUPPUTI

Coordinatore Dottorato:
Chiar.mo Prof.
ALBERTO PARMEGGIANI

Relatore:
Chiar.mo Prof.
NICOLA ARCOZZI

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Chapter 1

Introduction

Trace inequalities are an important and very large topic of mathematical analysis. They are used in many problems of partial differential equations, potential theory, harmonic and complex anlysis, just to name a few. In the most general and abstract sense, given B, a space of functions defined on a metric space X, we say that a Borel measure μ satisfies a *trace inequality for B with exponent* $p > 0$, if the immersion

$$(\star) \quad i : B \to L^p(\mu)$$

is bounded. In this case we say that μ is a *trace measure for B*.

1.1 Main Results

In this thesis we have characterized trace measures for some potential spaces living on $\mathbb{H}$, the upper half-space of $\mathbb{R}^n$. These spaces are described by integral operators with potential kernels $K_{a,b}$ can be thought as Bessel kernels on $\mathbb{R}^n$ mollified on $\mathbb{H}$ in the sense of following definition

$$K_{a,b}(x,t) \approx \begin{cases} e^{-\varepsilon(\|x\|+t)} & \|x\|+t \geq 1 \\ \frac{t^a}{(\|x\|+t)^b} & \|x\|+t<1 \end{cases}$$

where $a \geq 0$, $b > 0$ and $0 < \varepsilon < 1$. Fixed $c \in \mathbb{R}$, the corresponding integral operators $I^c_{a,b}$ act on the functions $u : \mathbb{H} \to \mathbb{R}$ similarly to a convolution operator. $I^c_{a,b}$ is defined by:

$$I^c_{a,b}u(x,t) := \int_{\mathbb{H}} K_{a,b}(x-y, t+s)u(y,s)s^c ds dy,$$

for all functions $u : \mathbb{H} \to \mathbb{R}$ for which the above integral exists almost everywhere. We can arbitrarily call these spaces $I^c_{a,b}$ spaces. Given $1 < p, p' < \infty$

such that $1/p' + 1/p = 1$ and denoting by m_c the weighted Lebesgue measure $t^c dtdx$, we say that a Borel measure μ on $\mathbb{H}$ is a trace measure for $I^c_{a,b}$ space if $I^c_{a,b} : L^p(\mathbb{H}, m_c) \to L^p(\mathbb{H}, \mu)$ is bounded, or, using the duality argument, if the dual operator $I^*_{a,b} : L^{p'}(\mathbb{H}, \mu) \to L^{p'}(\mathbb{H}, m_c)$ (see section 1.4 for more details and definitions) is bounded, then there exists a constant $C(\mu) > 0$ such that for any $u \in L^{p'}(\mathbb{H}, \mu)$:

$$\int_{\mathbb{H}} \left(\int_{\mathbb{H}} K_{a,b}(x-y, t+s) u(y,s) d\mu(y,s) \right)^{p'} t^c dtdx \\ \leq C(\mu) \int_{\mathbb{H}} |u(x,t)|^{p'} d\mu(x,t). \tag{1.1}$$

Theorem 1. *Given $c > -1$, if $c > b - n - 1$ we have that a bounded positive Borel measure μ on $\mathbb{H}$ is a trace measure for $I^c_{a,b}$ space if and only if there exists a constant $C(\mu) > 0$ such that for any $S \subset \mathcal{S}(\mathcal{Q})$:*

$$\int_S \left(\int_S K_{a,b}(x-y, t+s) d\mu(y,s) \right)^{p'} t^c dtdx \leq C(\mu)\mu(S), \tag{1.2}$$

where $\mathcal{S}(\mathcal{Q})$ is the set of all dyadic Carleson boxes.

A set $S \subset \mathbb{H}$ is a dyadic Carleson box if there exists a dyadic cube $\mathbb{R}^n$ with lenghtside $\ell(Q)$ such that $(x,t) \in S$ iff $x \in Q$ and $t < \ell(Q)$. We write also $S(Q) := S$ for any dyadic Carleson box S. To prove the inequality (1.2) as sufficient condition for the (1.1) we have generalized the Wolff (or Muckenhoupt-Wheeden) inequality to Borel measures supported on $\mathbb{H}$ and potential kernels of Riesz type.

Theorem 2. *If $c > b - n - 1$ and μ is a Borel measure on $\mathbb{H}$, then:*

$$\int_{S_0} \left(\sum_{S(Q) \subseteq S_0} \ell(S(Q))^{a-b} \mu(S(Q)^*) \chi_{S(Q)}(x,t) \right)^{p'} t^c dtdx \\ \leq C \int_{S_0} \sum_{S(Q) \subseteq S_0} \ell(S(Q))^{p'(a-b)} \mu(S(Q)^*)^{p'} \chi_{S(Q)}(x,t) t^c dtdx, \tag{1.3}$$

where S_0 is a dyadic Carleson box such that for any $(x,t), (y,s) \in S_0$ we have $\|x - y\| + t + s < 1$, and $S(Q)^*$ is a suitable expansion of $S(Q)$.

1.2 Historical Remarks on Trace Inequalities

The Sobolev spaces were among the first spaces where the study of trace measures began. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ belongs to Sobolev space $W^{\alpha,p}$, where $q > 1$ and $0 < \alpha < n$ is an integer, if for any multi-index $\beta = (\beta_1, ..., \beta_n)$ such that $|\beta| := \beta_1 + ... + \beta_n \leq \alpha$, we have $D^\beta f \in L^q$. $D^\beta f$ is the weak partial derivative with exponent β of f. A norm on $W^{\alpha,q}$ is the sum of L^q-norm of all derivatives $D^\beta f$. If $B = W^{\alpha,q}$ and μ is the Lebesgue measure, $(\star)$ represents, if $\alpha p < n$, the classical Sobolev embeddings $W^{\alpha,q} \hookrightarrow L^p$, with $q = pn/(n - p\alpha p)$, since $\|f\|_{L^p} \leq C\|f\|_{W^{\alpha,q}}$ for any $f \in W^{\alpha,q}$. If we replace in the Sobolev embeddings the Lebesgue measure with a generic Borel measure μ, we have the problem to characterize the measures μ for which $W^{\alpha,q} \hookrightarrow L^p(\mu)$ is an embedding. These measures are called trace measures for Sobolev spaces.

In the early 60's the first characterizations of trace measures for Sobolev spaces were given by means of capacitary conditions by Maz'ya and Adams. The Sobolev spaces $W^{\alpha,p}$ are related to Bessel potential spaces $L^{\alpha,p}$. For $0 < \alpha < n$, denoted by G_α the Bessel kernel

$$G_\alpha(x) \approx \begin{cases} \|x\|^{\alpha-n} & \text{if } \|x\| < 1, \\ e^{-\frac{1}{2}\|x\|} & \text{if } \|x\| \geq 1, \end{cases}$$

the functions in $L^{\alpha,p}$ are those having the form $G_\alpha * f$ with $f \in L^p$. A norm on Bessel potential space is $\|G_\alpha * f\|_{L^{\alpha,p}} := \|f\|_{L^p}$. Calderón in [18] proved that if α is a positive integer then $W^{\alpha,p} = L^{\alpha,p}$ with equivalence of norm. In this case a Borel measure μ is a trace measure for $W^{\alpha,p}$ if and only if is a trace measure for $L^{\alpha,p}$. The immersion $i : L^{\alpha,p} \to L^p(\mu)$ is bounded if there exists a constant $C(\mu) > 0$ such that for any $f \in L^p$:

$$\int_{\mathbb{R}^n} |G_\alpha * f|^p d\mu \leq C(\mu) \int_{\mathbb{R}^n} |f|^p dx.$$

Let $I_\alpha := G_\alpha * f, f \in L^p$, be its Bessel potential. By definition μ is a trace measure for $L^{\alpha,p}$ if and only if $I_\alpha : L^p \to L^p(\mu)$ is bounded. The norms L^p are simpler to deal with than the norms $W^{\alpha,p}$, so the identification between $W^{\alpha,p}$ and $L^{\alpha,p}$ is convenient. The classical Bessel capacity is defined on compact sets $K \subset \mathbb{R}^n$, for $1 < p < n/\alpha$, as following:

$$C_{\alpha,p}(K) := \inf \left\{ \|f\|^p_{L^p} \,|\, f \in L^p,\, I_\alpha f \geq 1 \,\text{on}\, K \right\}.$$

The (α, p)-Capacity can be extended for any $E \subset \mathbb{R}^n$ (see § 2 in [4] for more details). Since the (α, p)-Capacity is the infimum of $L^{\alpha,p}$ norms, it seems natural to use for the characterization of the boundedness of I_α. In the early 60's Maz'ya gave in [32], [33] and [34] the first characterization of trace measures for Bessel spaces via capacity. He discovered that a sufficient and necessary condition for the imbedding of $L^{\alpha,p}$ in $L^p(\mu)$ is that there must be a constant $C(\mu) > 0$ such that for any $E \subset \mathbb{R}^n$:

$$\mu(E) \leq C(\mu)C_{\alpha,p}(E), \tag{1.4}$$

with $1 < p < n/\alpha$. This kind of inequality is called by Maz'ya *isocapacitary inequality* and it is connected with the following *capacity strong type inequality* proved by Maz'ya in [35]:

$$\int_0^{+\infty} C_{\alpha,p}(\{I_\alpha f > t\})dt^p \leq C\|f\|^p_{L^p}. \tag{1.5}$$

In the well known Fefferman-Phong inequality [24] we find a box condition which is a sufficient condition for (1.4), but not necessary condition.
Another renowned inequality in potential theory is *Wolff's inequality*

$$\|I_\alpha \mu\|^{p'}_{L^{p'}} \leq C \int_{\mathbb{R}^n} W^\mu_{\alpha,p} d\mu, \tag{1.6}$$

where for $\alpha p < n$ the function

$$W^\mu_{\alpha,p}(x) = \int_0^{+\infty} [r^{\alpha p - n}\mu(B_r(x))]^{p'-1}\frac{dr}{r}$$

is called *Wolff potential.* The boundedness of potential $W^\mu_{\alpha,p}$ is a sufficient but not necessary condition for (1.4). For any compact set $K \subset \mathbb{R}^n$ if μ is a Borel measure supported on K we have that for $f \in L^{\alpha,p}$, using the Hölder inequality:

$$\begin{aligned} \mu(K) &\leq \int_{\mathbb{R}^n} I_\alpha f d\mu \leq \|f\|_{L^p}\|I_\alpha \mu\|_{L^{p'}} \\ &\leq C\|f\|_{L^p}\|W^\mu_{\alpha,p}\|^{\frac{1}{p'}}_{L^\infty}\mu(K)^{\frac{1}{p'}}, \end{aligned}$$

so if we suppose $\|W^\mu_{\alpha,p}\|_{L^\infty} < \infty$ then (1.4) follows. On the other hand it is clear that from (1.4) does not follow the boundedness of $W^\mu_{\alpha,p}$.
In 1985 Kerman and Sawyer found a characterization of trace measures for more general potential spaces by testing on balls (or equivalently on the dyadic cubes) of $\mathbb{R}^n$. In [27] Kerman and Sawyer studied the trace inequalities' conditions for potential operators T_ϕ defined as convolution operators with kernel ϕ on L^p functions. The kernel ϕ is assumed to be a function locally integrable on $\mathbb{R}^n$, nonnegative and radially decreasing. It is clear that the Bessel potential I_α is included in this family of potential operators. In this potential spaces a Borel measure μ is a trace measure if $T_\phi : L^p \to L^p(\mu)$

is bounded, similarly to the case of Bessel spaces. The Theorem 2.3 in [27] states that for a positive Borel locally finite measure μ, a sufficient and necessary condition for the boundedness of T_ϕ is that there must be $C(\mu) > 0$ such that for any dyadic cube (or balls) $Q \subset \mathbb{R}^n$:

$$\int_{\mathbb{R}^n} \left[\int_Q \phi(x-y) d\mu(y) \right]^{p'} dx \leq C(\mu)\mu(Q), \tag{1.7}$$

where p' is such that $pp' = p + p'$. The inequality (1.7) means that the dual operator T_ϕ^* is bounded on the characteristic functions of dyadic cubes. We can see with a simple calculation that for $f \in L^{p'}(\mu)$ and $x \in \mathbb{R}^n$:

$$T_\phi^*(f)(x) = \int_{\mathbb{R}^n} \phi(x-y) f(y) d\mu(y).$$

$T_\phi^* : L^{p'}(\mu) \to L^{p'}$ is bounded if for any $f \in L^{p'}(\mu)$:

$$\int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \phi(x-y) f(y) d\mu(y) \right]^{p'} dx \leq C \int_{\mathbb{R}^n} |f(x)|^{p'} d\mu(x), \tag{1.8}$$

so for $f = \chi_Q$ we find the inequality (1.7). However, Hansson in [29], and also Dahlberg in [23], gave, generalizing the results of Maz'ya and Adams, a capacity condition of type (1.4) for the potential spaces defined by operators T_ϕ.

1.3 Historical Remarks on Carleson Measures

Still in the early 60's a problem similar to trace measures characterization for Sobolev spaces was solved in complex analysis by Carleson in research related to his proof of the Corona theorem in 1962. He was interested in studying which measures μ supported on unit complex disc $\mathbb{D}$ satisfied the inequality

$$\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \leq C(\mu) \|f\|_{H^2}^2, \tag{1.9}$$

where H^2 is the Hardy space, that is the space of all holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{H^2}^2 := \sup_{0 \leq r < 1} \int_{\partial \mathbb{D}} |f(re^{it})|^2 dt < \infty.$$

In [19] Carleson found that a measure μ satisfies (1.9) if and only if for any arc $I \subset \partial\mathbb{D}$:

$$\mu(S(I)) \leq C(\mu)|I|, \tag{1.10}$$

where $S(I) := \{z \in \mathbb{D} \,|\, z/|z| \in I,\, 1 - |I|/2\pi < |z|\}$ is the *Carleson box with base I.* This type of measures was called, for obvious reasons, *Carleson measures.* Remembering that

$$\|f\|^2_{H^2} \sim \|f\|^2_{\mathcal{D}_1},$$

where $\mathcal{D}_1$ is the space of holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ with norm

$$\|f\|^2_{\mathcal{D}_1} := \int_{\mathbb{D}} |f'(z)|(1 - |z|)dA(z) + |f(0)| < \infty,$$

we observe that (1.9) is an weighted Sobolev-Poincaré embedding for holomorphic functions. We also observe that the ball conditions seen in the Sobolev spaces here are replaced by conditions on the border $\partial\mathbb{D}$. This is due to the fact that the holomorphic functions may exhibit irregularities in near of the border $\partial\mathbb{D}$; unlike to Sobolev functions which may have them everywhere. Carleson's result opened the problem of characterizing Carleson measures for weighted analytic Dirichlet spaces $\mathcal{D}_a$ and analytic Besov spaces $\mathcal{B}_p$.

For $0 \leq \alpha < 1$, the functions of $\mathcal{D}_a$ are holomorphic functions with norm

$$\|f\|^2_{\mathcal{D}_a} := \int_{\mathbb{D}} |f'(z)|^2(1 - |z|^2)^a dA(z) + |f(0)|^2 < \infty,$$

For $a = 0$ we found the classical Dirichlet space $\mathcal{D}$. A Carleson measure for $\mathcal{D}_a$ is a measure for which, replacing the H^2-norm with $\mathcal{D}_a$-norm, inequality (1.9) holds. The first to characterize Carleson measures for the spaces $\mathcal{D}_a$ was Stegenga in 1980 which generalized the Carleson's characterization by combining the Carleson boxes condition with a Maz'ya type capacitary condition. He proved in [41] that for $0 < a \leq 1/2$ a sufficient and necessary conditions for the Carleson measures μ in $\mathcal{D}_a$ is that for any finite sequence of arcs $I_1, ..., I_n$ in $\partial\mathbb{D}$:

$$\mu\left(\bigcup_{k=1}^{n} S(I_k)\right) \leq C(\mu)C_{a,2}\left(\bigcup_{k=1}^{n} I_k\right)$$

where $C_{a,2}$ is the Bessel capacity. Unlike the condition (1.10) Stegenga's characterization is not a one-box condition. The results of Stegenga have been generalize to analytic Dirichlet and Besov spaces by Cascante and Ortega in [21] and [22], Verbitsky in [43] and Wu in [44] using again test conditions of capacity type. A test condition without capacity is given by Kerman and Sawyer in [28] for some particular weighted Dirichlet spaces. A simple one-box condition was found by Arcozzi, Rochberg and Sawyer for weighted Besov spaces $\mathcal{B}_p(\rho)$. This condition type seems much easier to verify than the capacitary conditions. Now, we remember that, an holomorphic function $f : \mathbb{D} \to \mathbb{C}$ belongs to $\mathcal{B}_p(\rho)$ if

$$\|f\|^p_{\mathcal{B}_p(\rho)} := \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} \rho(z) dA(z) + |f(0)|^p < \infty,$$

where ρ is a weight belonging to a special class of functions said p-admissible. The Besov spaces, as it can see from their definition, generalize the Dirichlet space to exponent p other than 2. A Borel measure μ is a Carleson for $\mathcal{B}_p(\rho)$ if there exists a constant $C(\mu) > 0$ such that for any $f \in \mathcal{B}_p(\rho)$ we have

$$\int_{\mathbb{D}} |f(z)|^p d\mu(z) \leq C(\mu) \|f\|^p_{\mathcal{B}_p(\rho)}. \tag{1.11}$$

In [8] and then more simply in [11], Arcozzi, Rochberg and Sawyer proved that the Carleson measures characterization's problem for $\mathcal{B}_p(\rho)$ could be reduced to an equivalent discrete problem on a tree T via Whitney decomposition of the complex unit disk. They built a correspondence between the Besov space $\mathcal{B}_p(\rho)$ and its discrete model $B_p^T(\rho)$ by matching a vertex α in T to each Whitney box S in $\mathbb{D}$. According to this model, the discrete to (1.11) corresponding is

$$\sum_{\alpha \in T} |\varphi(\alpha)|^p \mu(\alpha) \leq C(\mu) \sum_{\alpha \in T} |\Delta\varphi(\alpha)|^p \rho(\alpha), \tag{1.12}$$

where φ is a function on T to f corresponding and $\Delta\varphi$ is the "discrete derivative" (difference operator which will be defined in Chapter 3) corresponding to f'. The measures μ and ρ on $\mathbb{D}$, under the correspondence $\alpha \in T \leftrightarrow S(\alpha) \subseteq \mathbb{D}$ remains the same. The approach used to characterize the Carleson measures for $\mathcal{B}_p^T(\rho)$ follows the reasoning of Kerman and Sawyer which we have seen in the characterization of trace measures for potentials T_ϕ. The test condition in fact is given on the equivalent dual inequality of (1.12). A sufficient and necessary condition for Carleson measures μ for $\mathcal{B}_p^T(\rho)$ is that

$$\sum_{\beta \geq \alpha} \mu(\bar{S}(\beta))^{p'} \rho^{1-p'}(\beta) \leq C(\mu)\mu(\bar{S}(\alpha)), \tag{1.13}$$

where $\beta \geq \alpha$ if and only if $S(\beta) \subseteq S(\alpha)$.

1.4 Some Words on the Proofs

In this thesis has been given a trace measures characterization of type ball condition similar to that just seen of Arcozzi, Rochberg and Sawyer. We observe infact that the geometry of our characterization's problem is similar to that which governs the Carleson measures in unit complex disk. Following the procedure of Kerman and Sawyer seen for the T_ϕ-potential spaces, we have that the box condition (1.2) follows from the boundedness of adjoint operator $I^*_{a,b} : L^{p'}(\mathbb{H}, \mu) \to L^{p'}(\mathbb{H}, m_c)$. The inequality (1.1) is equivalent infact to the boundedness of $I^*_{a,b}$. $I^c_{a,b}$ is a symmetric operator, so for any $u \in L^{p'}(\mathbb{H}, \mu)$ and $(x, t) \in \mathbb{H}$:

$$I^*_{a,b}u(x, t) = \int_{\mathbb{H}} K_{a,b}(x - y, t + s)u(y, s)d\mu(y, s).$$

Then if we suppose $I^*_{a,b}$ bounded only on characteristic functions χ_S, where $S \in \mathcal{S}(\mathcal{Q})$, we find (1.2). We note that any box $S(Q)$ is similar to a Carleson box that is built on a base belonging to the boundary $\partial\mathbb{D}$. The assumption on c is used to control the convergence of the kernel $K_{a,b}$ approximations on the elementary boxes $S \in \mathcal{S}(\mathcal{Q})$ close the singularities on the boundary $\mathbb{R}^n$. This, as we shall see, is crucial in order to generalize Wolff's inequality to our kernel. From the assumption that μ is bounded we can prove that the $K_{a,b}$ exponential error

$$I^\mu_e u(x, t) := \int_{\mathbb{H}} e^{-\varepsilon(\|x-y\|+t+s)}u(y, s)d\mu(y, s)$$

is a bounded operator from $L^{p'}(\mathbb{H}, \mu)$ to $L^{p'}(\mathbb{H}, m_c)$. Therefore, the test on the boxes must be check on only with respect to the $K_{a,b}$ principal part given by the integral operator

$$I^\mu_{pr}u(x, t) := \int_{S_0} \frac{(t + s)^a}{(\|x - y\| + t + s)^b}u(y, s)d\mu(y, s),$$

At this point the problem of boundedness of I^μ_{pr} can be reduced in an equivalent discrete problem. S_0 can be modeled by a dyadic type tree T. We identify S_0 with the root o of T and each vertex $\alpha \in T$ with the top-half of $S(Q_\alpha) \in S_0$. The characterization (1.2) follows from the approximation

$$\int_{S_0} |I^\mu_{pr}u(x, t)|^{p'} t^c dtdx \approx \sum_{\alpha\in T} |I^*_\mu u(\alpha)|^{p'} \rho(\alpha), \tag{1.14}$$

where ρ is an appropriate weight measure on T and I^*_μ is the dual discrete operator on T corresponding to $I^*_{a,b}$. In [14] it is proved that a sufficient and necessary condition for the boundedness of $I^*_\mu : L^{p'}(T,\mu) \to L^{p'}(T,\rho)$ is the boundedness of I^*_μ on characteristic functions $\chi_{S(\alpha)}$ only, where $S(\alpha)$ is a box of T. The approximation (1.14) is not triavial. As mentioned at the beginning, to prove (1.14) it was necessary to generalize to our case the Muckenhoupt-Wheeden [31] and Wolff's inequality [30]. In these inequalities it is possible to reverse on average the natural inclusion $\ell^1 \subseteq \ell^{p'}$. Since the kernel $K_{a,b}$ can be approximated by $\ell(S(Q_\alpha))^{a-b}$ on each box $S(Q_\alpha)$ it follows that

$$\int_{S_0} |I^\mu_{pr} u(x,t)|^{p'} t^c dt dx$$
$$\approx \int_{S_0} \left| \sum_{\alpha\in T} \ell(S(Q_\alpha))^{a-b} I^*_\mu u(\alpha) \chi_{S(Q_\alpha)}(x,t) \right|^{p'} t^c dt dx. \tag{1.15}$$

If u is a positive function we have that $I^*_\mu u$ can be seen as a Borel measure. It is not restrictive to consider only positive functions. Appliyng to $I^*_\mu u$ the generalized Wolff (or Muckenhoupt-Wheeden) inequality (1.3) we have:

$$\int_{S_0} \left| \sum_{\alpha\in T} \ell(S(Q_\alpha))^{a-b} I^*_\mu u(\alpha) \chi_{S(Q_\alpha)}(x,t) \right|^{p'} t^c dt dx$$
$$\leq C \int_{S_0} \sum_{\alpha\in T} \ell(S(Q_\alpha))^{p'(a-b)} |I^*_\mu u(\alpha)|^{p'} \chi_{S_(Q_\alpha)} t^c dt dx, \tag{1.16}$$

so, if $\rho(\alpha) := \ell(S(Q_\alpha))^{p'(a-b)+c+n+1}$, using for the other direction of the inequality (1.16) the inclusion $\ell^1 \subseteq \ell^{p'}$, we find (1.14). We observe that in (1.16) the p' power has been moved inside the sum, "as if" $\ell^{p'} \subseteq \ell^1$.

1.5 An Application to Lipschitz-Besov Spaces

What we have found for the spaces with potentials $I^c_{a,b}$ follows by intention to give a necessary and sufficient box condition for trace measures for the harmonic extension of Lipschitz-Besov spaces $\Lambda^{p,q}_\delta$, where $0 < \delta < 1$ and $1 < p, q < +\infty$ (see section 2.3 for a definition of $\Lambda^{p,q}_\delta$). We denote by $H^p_{\alpha,\beta}$ the space of harmonic functions $u : \mathbb{H} \to \mathbb{R}$ such that

$$\|u\|^p_{H^p_{\alpha,\beta}} := \int_0^{+\infty} t^{p(1-\delta)} \|\partial_t u\|^p_p \frac{dt}{t} dx + \int_{\mathbb{R}^n} |u(x,0)|^p dx < \infty,$$

where $\delta := \alpha - (\beta+1)/2$. We can show that $H^p_{\alpha,\beta}$ is the harmonic extension of $\Lambda^{p,p}_\delta$ (see § 5 in [40] for more details). A Borel measure μ on $\mathbb{H}$ is a trace measure for $H^p_{\alpha,\beta}$ if there exists a constant $C(\mu) > 0$ such that for any $u \in H^p_{\alpha,\beta}$ we have

$$\|u\|_{L^p(\mathbb{H},\mu)} \le C(\mu)\|u\|_{H^p_{\alpha,\beta}}.$$

It should be possible to prove that μ is a Carleson trace measure for $H^2_{\alpha,\beta}$ if and only if the integral operator $\Theta^*_\mu : L^2(\mathbb{H},\mu) \to L^2(\mathbb{H}, m_{1-2\delta})$ is bounded, where for any $u \in L^2(\mathbb{H},\mu)$ and $(x,t) \in \mathbb{H}$

$$\begin{aligned}\Theta^*_\mu u(x,t) \approx &\int_{\|x-y\|+t+s<1} \frac{\|x-y\|^2 - n(t+s)}{(\|x-y\|^2+(t+s)^2)^{\frac{n+3}{2}}} u(y,s)d\mu(y,s)\\ &+ \int_{\|x-y\|+t+s\ge 1} e^{-\frac{1}{2}(\|x-y\|+t+s)} u(y,s)d\mu(y,s).\end{aligned}$$

We define for any $u \in L^2(\mathbb{H},\mu)$ and $(x,t) \in \mathbb{H}$

$$\begin{aligned}\Theta^{*+}_\mu u(x,t) := &\int_{\|x-y\|+t+s<1} \frac{1}{(\|x-y\|^2+(t+s)^2)^{\frac{n+1}{2}}} u(y,s)d\mu(y,s)\\ &+ \int_{\|x-y\|+t+s\ge 1} e^{-\frac{1}{2}(\|x-y\|+t+s)} u(y,s)d\mu(y,s).\end{aligned}$$

It is clear that

$$|\Theta^*_\mu u| \le \Theta^{*+}_\mu |u| \approx I^*_{0,n+1}|u|,$$

so the test condition (1.2) is sufficient for the boundedness of Θ^*_μ, but not necessary. If we could prove that $|\Theta^*_\mu u| \approx \Theta^{*+}_\mu u$, then the box condition (1.2) would be a necessary and sufficient condition for the trace measures on $H^2_{\alpha,\beta}$. I do not expect $\Theta^{*+}_\mu u$ and $\Theta^*_\mu u$ to be comparable for each given $u \ge 0$, but I do expect Θ^{*+}_μ to be bounded from $L^2(\mathbb{H},\mu)$ to $L^2(\mathbb{H}, m_{1-2\delta})$ iff Θ^*_μ is. At present I do have a proof of this for a discrete version of the originally problem only, as we will show in Chapter 3.
A characterization of trace measures for Lipschitz-Besov spaces $\Lambda^{p,q}_\delta$ has recently been given by Guliyev and Wu in [25] via Maz'ya type capacitary condition. They have found a capacity strong type inequality as in (1.5)

$$\int_0^{+\infty} cap_{\Lambda^{p,q}_\delta}(\{|f| > t\}dt^q \le C\|f\|^q_{\Lambda^{p,q}_\delta},$$

where for any open set $O \subset \mathbb{R}^n$ they define

$$cap_{\Lambda^{p,q}_\delta}(O) := \inf\{\|f\|^q_{\Lambda^{p,q}_\delta} \mid f \ge 1 \text{ in } O\}.$$

A nonnegative Borel measure μ on $\mathbb{H}$ satisfies a trace measures for $\Lambda^{p,q}_\delta$, if there exist a constant $C(\mu) > 0$ such that for any $f \in \Lambda^{p,q}_\delta$:

$$\int_{\mathbb{H}} |P_t * \ f(x)|^q d\mu(x,t) \le C(\mu)\|f\|^q_{\Lambda^{p,q}_\delta}, \tag{1.17}$$

where $P * f$ is the harmonic extension of f by Poisson kernel. A sufficient and necessary condition for the (1.17) is that there exists a constant $C(\mu) > 0$ such that for any bounded open set $O \subset R^n$, denoted by $T(O)$ the Carleson box on $\mathbb{H}$ with basis O:

$$\mu(T(O)) \leq C(\mu) cap_{\Lambda_\delta^{p,q}}(O), \tag{1.18}$$

according to the Maz'ya condition (1.4). Indeed, condition (1.18) seems difficult to check for concrete measures μ.

1.6 A Recent Result by Alemann, Pott and Reguera

I have recently become aware of yet unwritten research work by Aleman, Pott and Reguera on topics close to those of the present thesis (see [5]). They consider in the recent work the Bergman projection $P_B : L^p(\mathbb{D}) \to A^p(\mathbb{D})$, where $p > 1$. We recall that $A^p(\mathbb{D}) = L^p(\mathbb{D}) \cap Hol(\mathbb{D})$ is the Bergman space and for any $f \in L^p(\mathbb{D})$:

$$P_B f(z) = \int_{\mathbb{D}} \frac{f(\xi)}{(1 - z\bar{\xi})^2} dA(\xi),$$

is the Bergman projection, dA is area measure. The Bergman projection has a geometry similar to the principal part of Θ_μ^* operator. Aleman, Pott and Reguera are interested in studying for which couples of weights u and v on $\mathbb{D}$ the projection $P_B : L_u^2(\mathbb{D}) \to L_v^2(\mathbb{D})$ is bounded, where $f \in L_u^2(\mathbb{D})$ if

$$\int_{\mathbb{D}} |f(z)|^2 u(z) dA(z) < \infty,$$

and similarly $g \in L_v^2(\mathbb{D})$ if

$$\int_{\mathbb{D}} |g(z)|^2 v(z) dA(z) < \infty.$$

Since P_B may have cancellations, they considered the simpler two weight inequality for the positive operator

$$P_B^+ f(z) = \int_{\mathbb{D}} \frac{f(\xi)}{|1 - z\bar{\xi}|} dA(\xi).$$

They found that $P_B^+ : L^2_u(\mathbb{D}) \to L^2_v(\mathbb{D})$ is bounded if and only if there exist K_1, K_2, K_3 such that:

1. $\sup_{I\subset\partial\mathbb{D} \text{ interval}} \left(\frac{1}{|Q_I|}\int_{Q_I} v(z)dA(z)\right)\left(\frac{1}{|Q_I|}\int_{Q_I} u^{-1}(z)dA(z)\right) \le K_1$
2. $\|P_B^+ u^{-1}\chi_{Q_I}\|_{L^2_v} \le K_2\|\chi_{Q_I}\|_{L^2_{u^{-1}}}$
3. $\|P_B^+ v\chi_{Q_I}\|_{L^2_{u^{-1}}} \le K_3\|\chi_{Q_I}\|_{L^2_v}$

where Q_I is the Carleson box with basis $I \subset \partial\mathbb{D}$.
We note that the characterization of boundedness of P_B^+ has a Carleson box condition. The test on Carleson boxes regards the adjoint operators respectively to two weights u and v. Our trace inequality is a multidimensional, real variable version of Aleman, Pott and Reguera with the weight u belonging to a special class. Our result on discrete martingales could be seen as a version of the original Bergman projection (which is still open) for a special class of weights.

1.7 Guide to the Thesis

This thesis is structured in two chapters. In first chapter we give a characterization theorem for trace measures for potential spaces with integral potentials $I^c_{a,b}$ and the generalized Wolff inequality used to prove characterization theorem. At the end of first chapter we show as our characterization theorem can be partially applied to Lipschitz-Besov spaces and the cancellation's problem associated. In second chapter we solve cancellation's problem on discrete Besov spaces of martingales on an homogeneous tree (this space represents a discrete model for the harmonic extension of Lipschitz-Besov spaces).

Chapter 2

Weighted Inequalities

Introduction

Chapter 2 is divided in three sections. In the first section we prove the generalized Wolff (or Muckenhoupt-Wheeden) inequalities for potential integral operators with Riesz type kernels and for Borel measures supported on $\mathbb{H}$. In the second section we deal with the problem of the characterization of trace measures for weighted potential spaces defined by integral operators with Bessel type kernels. We give a characterization theorem which provides a box test as a necessary and sufficient condition for trace measures. We note that the generalized Wolff inequality has a key role in the proof of the box test as a sufficient condition for trace measures in our potential spaces. In the last section we show the possible applications of characterization theorem to the harmonic extension of Lipschitz-Besov spaces.

2.1 Wolff type Inequalities

The Wolff inequality is described by Theorem 1 in [30] which states that, given a Borel measure μ on $\mathbb{R}^n$, $\alpha > 0$ and $1 < q \leq n/\alpha$, there exists $A > 0$ such that:

$$\int_{\mathbb{R}^n} (G_\alpha * \mu)^p dx \leq A \int_{\mathbb{R}^n} W^\mu_{\alpha,q} d\mu, \tag{2.1}$$

where p is the conjugate of q, G_α is the Bessel kernel and $W^\mu_{\alpha,q}$ is a specific function named *Wolff potential* . The key point of the proof of Theorem 1 is in fact proving

$$\int_{I_0}\left(\sum_{\ell(Q)\le 1}\ell(Q)^{\alpha-n}\mu(\tilde{Q})\chi_Q(x)\right)^p dx$$
$$\le A\int_{I_0}\sum_{\ell(Q)\le 1}(\ell(Q)^{\alpha-n}\mu(Q))^p\chi_Q(x)dx; \tag{2.2}$$

where I_0 is the unit cube in $\mathbb{R}^n$, $Q \subseteq I_0$ is a generic cube of the dyadic subdivision of $\mathbb{R}^n$ with side-lenght $\ell(Q)$ and $\tilde{Q}$ the cube concentric to Q with side-lenght $3\ell(Q)$. The inequality (2.2) ensures that for the functions of type $g^\mu : x \in \mathbb{R}\times Q \subseteq I_0 \to \mathbb{R}$ such that $g^\mu(x,Q) := \ell(Q)^{\alpha-n}\mu(Q)\chi_Q(x)$ it is true that:

$$\int_{I_0}\|g^\mu_x\|^p_{\ell^1}dx \approx \int_{I_0}\|g^\mu_x\|^p_{\ell^p};$$

It is elementary in fact that

$$\sum_{\ell(Q)\le 1}(\ell(Q)^{\alpha-n}\mu(Q))^p\chi_Q(x)$$
$$\le \left(\sum_{\ell(Q)\le 1}\ell(Q)^{\alpha-n}\mu(\tilde{Q})\chi_Q(x)\right)^p dx.$$

The possibility to invert on average the natural inclusion $\ell^1 \subset \ell^p$ may be useful in many contexs. In our case, to prove the Characterization Theorem in section 2.2, it was necessary to generalize the inequality (2.2) for the spaces $\Pi^p_{K_{a,b}}(\mathbb{H}, m_c)$ (see Definition 2.2.1), which represent the extension in the upper half-space of $\mathbb{R}^n$ of the potential Bessel spaces.
The merit of Wolff is to have found an ingenious algebraic-geometric method which makes possible (2.2). In our case this is possible only for particular spaces $\Pi^p_{K_{a,b}}(\mathbb{H}, m_c)$, i.e. only for those with weighted measure m_c where c is large enough to control the singularity of the kernel $K_{a,b}$.

The generalization of the Wolff inequality requires that the upper half-space $\mathbb{H}$ be subdivided into elementary sets constructed from the dyadic subdivision of $\mathbb{R}^n$. We start by giving some preliminary definitions.

Definition 2.1.1. *A set S is a box of $\mathbb{H}$ if there exists $(x,t) \in \mathbb{H}$, where $x \in \mathbb{R}^n$, $t \ge 0$, such that:*

$$(y,s) \in S \Leftrightarrow s < 2t \text{ and } \sup_{1\le i\le n}|x_i - y_i| < t.$$

In this case we can write $S_{x,t} := S$.

Remark 2.1.1. Given a box $S \subset \mathbb{H}$, we denote by $\ell(S)$ its side-lenght, then from Definition 2.1.1 there is $t \geq 0$ such that $\ell(S) = 2t$.

Definition 2.1.2. *For any $k \in \mathbb{N}$, we define on $\mathbb{R}^n$ the set of all dyadic cubes at level k*

$$\mathcal{Q}_k := \left\{ \left[\frac{a_i}{2^k}, \frac{a_i+1}{2^k}\right)^n \,\Big|\, a_i \in \mathbb{Z} \right\};$$

and then the set all dyadic cubes of $\mathbb{R}^n$:

$$\mathcal{Q} := \{Q \,|\, Q \in \mathcal{Q}_k,\, k \in \mathbb{N}\}.$$

Definition 2.1.3 (Carleson type box on the real half-upper space). *For any $Q \in \mathcal{Q}$ we define $S(Q) \subset \mathbb{H}$ as the box such that*

$$(x,t) \in S(Q) \Leftrightarrow x \in Q, t < \ell(Q).$$

Then we can define:

$$\mathcal{S}(\mathcal{Q}) := \{S(Q) \,|\, Q \in \mathcal{Q}\}.$$

Remark 2.1.2. From the definitions 2.1.1 and 2.1.3 is clear that for any $S(Q) \in \mathcal{S}(\mathcal{Q})$ we have $\ell(S(Q)) = \ell(Q)$.

Definition 2.1.4. *Considered $\mathcal{Q}_0 := \{Q \in \mathcal{Q} \,|\, \ell(Q) \leq 1/4\}$, we put:*

$$\mathcal{S}(\mathcal{Q}_0) := \{S(Q) \,|\, Q \in \mathcal{Q}_0\}.$$

For any box $S \subset \mathbb{H}$, writing $S \subseteq S_0$ we intend $S(Q) \in \mathcal{S}(\mathcal{Q}_0)$.

Remark 2.1.3. Given $S \subseteq S_0$, for any $(x,t),(y,s) \in S$ we have

$$\|x-y\| + t + s < 1.$$

If in fact $S \subseteq S_0$ there is $Q \in \mathcal{Q}_0$ such that $S = S(Q)$, then $x, y \in Q$ and $t, s < \ell(Q)$, so:

$$\|x-y\| + t + s \leq \|x\| + \|y\| + t + s < 4\ell(Q) \leq 1.$$

Definition 2.1.5. *Let $Q \in \mathcal{Q}$. We define $S(Q)^*$ the box of $\mathbb{H}$ such that:*

1. *$S(Q) \subset S(Q)^*$,*

2. *$\ell(S(Q)^*) = 3\ell(Q)$,*

3. *$d(\partial S(Q), \partial S(Q)^*) = \ell(Q)$.*

Remark 2.1.4. If $x \in Q$ then $S_{x,\ell(Q)} \subset S(Q)^*$. We suppose that there is $(y,s) \in S_{x,\ell(Q)}$ such that $(y,s) \notin S(Q)^*$, then from Definition 2.1.5 we have:

$$\|x-y\| > \sup_{z\in S(Q)^*} \|x-z\| > d(\partial S(Q), \partial S(Q)^*) = \ell(Q)$$

but this is an absurd because, according to Definition 2.1.1, it must be $\|x-y\| < \ell(Q)$.

Remark 2.1.5. Let $x \in \mathbb{R}^n$ and $t \geq 0$, then for any $(y,s) \in S_{x,t}$ we have $\|x-y\| + s + t \approx t$. In fact if $(y,s) \in S_{x,t}$, then from Definition 2.1.1 it is follows that $s < 2t$ and $\|x-y\| \approx \sup_{1\leq i\leq n} |x_i - y_i| < t$, so we have $t \leq \|x-y\| + t + s < 4t$.

The following lemma extends to our case Lemma 1 in [30]. In fact we can see in the proof of Theorem 2.1.1 that the inequality (2.3) is the main property which leads to (2.8).

Lemma 2.1.1. *Let μ be a Borel measure on $\mathbb{H}$. Given $b > 0$ for any box $S \subseteq S_0$ we set $d(S) := \ell(S)^{-b}\mu(S)$. Fixed $\tilde{S} \subseteq S_0$, if $c > b - n - 1$, for any $s > 0$ and $r \geq 1$ there is a constant $C > 0$ such that:*

$$\sum_{S'\subseteq \tilde{S}} d(S^*)^s \sum_{S'\subseteq S} d(S'^*)^r \ell(S')^c |S'| \leq C \sum_{S\subseteq \tilde{S}} d(S^*)^{r+s} \ell(S)^c |S|, \tag{2.3}$$

where $|S| = \ell(S)^{n+1}$ is the Lebesgue measure of box S.

Proof. If $r = 1$, for $S \subseteq \tilde{S}$ there is $k \geq 0$ such that $\ell(S) = 2^{-k}$; so

$$\begin{aligned}
\sum_{S'\subseteq S} d(S'^*)\ell(S')^c|S'| &\leq C \sum_{S'\subseteq S} \ell(S')^{-b+c+n+1}\mu(S'^*) \\
&\leq C \int_{\mathbb{H}} \sum_{S'\subseteq S} \ell(S')^{-b+c+n+1}\chi_{S'^*}(x,t)d\mu(x,t) \\
&\leq C\mu(S^*) \sum_{j=k}^{+\infty} 2^{-j(-b+c+n+1)} \\
&\leq C\mu(S^*)2^{-k(-b+c+n+1)} = Cd(S^*)\ell(S)^c|S|,
\end{aligned}$$

from which

$$\sum_{S\subseteq \tilde{S}} d(S^*)^s \sum_{S'\subseteq S} d(S'^*)\ell(S')^c|S'| \leq C \sum_{S\subseteq \tilde{S}} d(S^*)^{1+s}\ell(S)^c|S|.$$

If $r > 1$, first we choose $S = S_0$, then, by Hölder's inequality, we have

$$\begin{aligned}
\sum_{S'\subseteq S_0} d(S'^*)^r \ell(S')^c |S'| &= C \sum_{S'\subseteq S_0} d(S'^*)^{r-1} \ell(S')^{-b+c} |S'| \mu(S'^*) \\
&\leq C \left(\sum_{S'\subseteq S_0} d(S'^*)^{r-1+s} \ell(S')^{-b+c+\varepsilon} |S'| \mu(S'^*) \right)^{\frac{r-1}{r-1+s}} \\
&\quad \cdot \left(\sum_{S'\subseteq S_0} \ell(S')^{-b+c-\varepsilon'} |S'| \mu(S'^*) \right)^{\frac{s}{r-1+s}} \\
&\leq C \left(\sum_{S'\subseteq S_0} d(S'^*)^{r+s} \ell(S')^{c+\varepsilon} |S'| \right)^{\frac{r-1}{r-1+s}} \\
&\quad \cdot \left(\sum_{S'\subseteq S_0} \ell(S')^{-b+c-\varepsilon'} |S'| \mu(S'^*) \right)^{\frac{s}{r-1+s}} ;
\end{aligned}$$

where $\varepsilon := \nu(r-1)/(r-1+s)$ and $\varepsilon' := \nu s/(r-1+s)$ with $\nu > 0$ such that $\varepsilon' < c - b$. Observing that

$$\begin{aligned}
\sum_{S'\subseteq S_0} \ell(S')^{-b+c-\varepsilon'} |S'| \mu(S'^*) &= \int_{\mathbb{H}} \sum_{S\subseteq S_0} \ell(S')^{-b+c-\varepsilon'+n+1} \chi_{S'^*}(x,t) d\mu(x,t) \\
&\leq C\mu(S_0^*) \sum_{j=0}^{+\infty} 2^{-j(-b+c-\varepsilon'+n+1)} \leq C\mu(S_0^*),
\end{aligned}$$

our Hölder's inequality becomes

$$\sum_{S'\subseteq S_0} d(S'^*)^r \ell(S')^c |S'| \leq C\mu(S_0^*)^{\frac{s}{r-1+s}} \left(\sum_{S'\subseteq S_0} d(S'*)^{r+s} \ell(S')^{\varepsilon+c} |S'| \right)^{\frac{r-1}{r-1+s}} . \tag{2.4}$$

Trivially:

$$\mu(S_0^*)^{r+s} = Cd(S_0^*)^{r+s} \ell(S_0)^{c+\varepsilon} |S_0| \leq C \sum_{S'\subseteq S_0} d(S'^*)^{r+s} \ell(S')^{c+\varepsilon} |S'|;$$

so multiplying (2.4) by $\mu(S_0^*)^s$ we obtain:

$$\begin{aligned}
\mu(S_0^*)^s \sum_{S'\subseteq S_0} d(S'^*)^r \ell(S')^c |S'| &\leq C\mu(S_0^*)^{\frac{r(r+s)}{r-1+s}} \left(\sum_{S'\subseteq S_0} d(S'^*)^{r+s} \ell(S')^{c+\varepsilon} |S'| \right)^{\frac{r-1}{r-1+s}} \\
&\leq C \sum_{S'\subseteq S_0} d(S'^*)^{r+s} \ell(S')^{c+\varepsilon} |S'|. \qquad (2.5)
\end{aligned}$$

For each $S \subseteq S_0$, by homogeneity, the inequality (2.5) becomes

$$\ell(S^*)^s \mu(S^*)^s \ell(S)^\varepsilon \sum_{S' \subseteq S} d(S'^*)^r \ell(S')^c |S'| \leq C \sum_{S' \subseteq S} d(S'^*)^{r+s} \ell(S')^{c+\varepsilon} |S'|,$$

therefore, dividing by $\ell(S)^\varepsilon$ and summing over $S \subseteq \tilde{S}$, it follows that:

$$\begin{aligned}
\sum_{S \subseteq \tilde{S}} d(S^*)^s \sum_{S' \subseteq S} d(S'^*)^r \ell(S')^c |S'| &\leq C \sum_{S \subseteq \tilde{S}} \sum_{S' \subset S} d(S'^*)^{r+s} \ell(S')^{c+\varepsilon} |S'| l(S)^{-\varepsilon} \\
&= C \sum_{S \subseteq \tilde{S}} d(S^*)^{r+s} \ell(S)^{c+\varepsilon} |S| \sum_{S \subseteq S' \subset \tilde{S}} \ell(S')^{-\varepsilon} \\
&\leq C \sum_{S \subseteq \tilde{S}} d(S^*)^{r+s} \ell(S)^{c+\varepsilon} |S| l(S)^{-\varepsilon} \\
&= C \sum_{S \subseteq \tilde{S}} d(S^*)^{r+s} \ell(S)^c |S|.
\end{aligned}$$

□

Theorem 2.1.1 (Generalized Wolff Inequality). *Let μ be a Borel measure on $\mathbb{H}$. Given $p > 1$ and $b > 0$ if $c > b - n - 1$ there exist $C > 0$ such that:*

$$\int_{S_0} \left(\sum_{S \subseteq S_0} \ell(S)^{-b} \mu(S^*) \chi_S(x,t) \right)^p t^c dt dx$$
$$\leq C \sum_{S \subseteq S_0} \ell(S)^{c-bp} \mu(S^*)^p |S|. \tag{2.6}$$

Proof. For $p > 1$ there exists a positive integer $n \geq 0$ and $0 \leq s < 1$ such that $p = n + s$, so using the notation of Lemma 2.1.1 we can do the following expansion:

$$\begin{aligned}
&\int_{S_0} \left(\sum_{S \subseteq S_0} \ell(S)^{-b} \mu(S^*) \chi_S(x,t) \right)^p t^c dt dx \\
\leq\ & C \int_{S(Q_0)} \sum_{S_1 \subseteq \ldots \subseteq S_n \ldots \subseteq S_0} d(S_1^*) \chi_{S_1}(x,t) \cdot \ldots \cdot d(S_n^*) \chi_{S_n}(x,t) \\
&\cdot \left(\int_{S \subseteq S_0} d(S^*) \chi_S(x,t) \right)^s t^c dt dx. \qquad (2.7)
\end{aligned}$$

For fixed $S_1 \subseteq \ldots \subseteq S_n \subseteq S_0$ we have different cases depending on $S \subseteq S_0$.
Case 1. There is $k \in \{1, \ldots, n-1\}$ such that $S_k \subseteq S \subseteq S_{k+1}$, then applying

to (2.7) the inequality (2.3) we have:

$$\begin{aligned}
& \int_{S_0} \sum_{S_1\subseteq\ldots\subseteq S_n\subseteq S_0} d(S_1^*)\chi_{S_1}(x,t)\cdot\ldots\cdot d(S_n^*)\chi_{S_n}(x,t) \\
& \cdot\left(\int_{S\subseteq S_0} d(S^*)\chi_S(x,t)\right)^s t^c dt dx \\
\leq\quad & C \sum_{S_{k+1}\subseteq\ldots\subseteq S_n\subseteq S_0} d(S_{k+1}^*)\cdot\ldots\cdot d(S_n^*) \sum_{S\subseteq S_{k+1}} d(S^*)^s \\
& \cdot \sum_{S_3\subseteq\ldots\subseteq S_k\subseteq S} d(S_3^*)\cdot\ldots\cdot d(S_k^*) \sum_{S_2\subseteq S_3} d(S_2^*) \sum_{S_1\subseteq S_2} d(S_1^*)\ell(S_1)^c|S_1| \\
\leq\quad & C \sum_{S_{k+1}\subseteq\ldots\subseteq S_n\subseteq S_0} d(S_{k+1}^*)\cdot\ldots\cdot d(S_n^*) \sum_{S\subseteq S_{k+1}} d(S^*)^c \\
& \cdot \sum_{S_3\subseteq\ldots\subseteq S_k\subseteq S} d(S_3^*)\cdot\ldots\cdot d(S_k^*) \sum_{S_2\subseteq S_3} d(S_2^*)^2\ell(S_2)^c|S_2|
\end{aligned}$$

proceeding by iterations

$$\begin{aligned}
\leq\ldots\leq\quad & C \sum_{S_{k+1}\subseteq\ldots\subseteq S_n\subseteq S_0} d(S_{k+1}^*)\cdot\ldots\cdot d(S_n^*) \sum_{S\subseteq S_{k+1}} d(S^*)^s \\
& \cdot \sum_{S_k\subseteq S} d(S_k^*)^k\ell(S_k)^c|S_k| \\
\leq\quad & C \sum_{S_{k+1}\subseteq\ldots\subseteq S_n\subseteq S_0} d(S_{k+1}^*)\cdot\ldots\cdot d(S_n^*) \sum_{S\subseteq S_{k+1}} d(S^*)^{k+s}\ell(S)^c|S| \\
\leq\ldots\leq\quad & C \sum_{S\subseteq S_0} d(S^*)^{n+s}\ell(S)^c|S| \\
=\quad & C \sum_{S\subseteq S_0} \ell(S)^{c-pb}\mu(S^*)^p|S|.
\end{aligned}$$

Case 2. If $S_n \subseteq S$ then, using the Lemma 2.1.1 again:

$$
\begin{aligned}
& \int_{S(Q_0)} \sum_{S_1 \subseteq \ldots \subseteq S_n \subseteq S_0} d(S_1^*)\chi_{S_1}(x,t) \cdot \ldots \cdot d(S_n^*)\chi_{S_n}(x,t) \\
& \cdot \left(\int_{S \subseteq S_0} d(S^*)\chi_S(x,t) \right)^s t^c dt dx \\
\leq \quad & C \sum_{S \subseteq S_0} d(S^*)^s \sum_{S_3 \subseteq \ldots \subseteq S_n \subseteq S} d(S_3^*) \cdot \ldots \cdot d(S_n^*) \\
& \cdot \sum_{S_2 \subseteq S_3} d(S_2^*) \sum_{S_1 \subseteq S_2} d(S_1^*)\ell(S_1)^c |S_1| \\
\leq \ldots \leq \quad & C \sum_{S \subseteq S_0} d(S^*)^s \sum_{S_n \subseteq S} d(S_n^*)^n \ell(S_n)^c |S_n| \\
\leq \quad & C \sum_{S \subseteq S_0} d(S^*)^{s+n} \ell(S)^c |S| \\
= \quad & C \sum_{S \subseteq S_0} \ell(S)^{c-pb} \mu(S^*)^p |S|.
\end{aligned}
$$

Case 3. If $S \subseteq S_1$, let $k \geq 1$ such that $\ell(S_1) = 2^{-k}$. By Hölder inequality we have:

$$
\begin{aligned}
& \int_{S_1} \left(\sum_{S \subseteq S_1} d(S^*)\chi_S(x,t) \right)^s t^c dt dx \\
& \leq \left(\ell(S_1)^c |S_1| \right)^{1-s} \left(\int_{S_1} \sum_{S \subseteq S_1} d(S^*)\chi_S(x,t) t^c dt dx \right)^s \\
& \leq C \left(\ell(S_1)^c |S_1| \right)^{1-s} \mu(S_1^*)^s \left(\sum_{j=k}^{+\infty} 2^{-j(c-b+n+1)} \right)^s \\
& \leq C \left(\ell(S_1)^c |S_1| \right)^{1-s} \mu(S_1^*)^s 2^{-ks(c-b+n+1)} = C d(S_1^*)^s \ell(S_1)^c |S_1|,
\end{aligned}
$$

so from Lemma 2.1.1 it follows that:

$$\begin{aligned}
&\int_{S_0} \sum_{S_1\subseteq \ldots \subseteq S_n \subseteq S_0} d(S_1^*)\chi_{S_1}(x,t)\cdot \ldots \cdot d(S_n^*)\chi_{S_n}(x,t) \\
&\cdot \left(\sum_{S\subseteq S_0} d(S^*)\chi_S(x,t)\right)^s t^c dtdx \\
\leq & C \sum_{S_1\subseteq \ldots \subseteq S_n \subseteq S_0} d(S_1^*)\cdot \ldots \cdot d(S_n^*) \int_{S_1} \left(\sum_{S\subseteq S_1} d(S^*)\chi_S(x,t)\right)^s t^c dtdx \\
\leq & C \sum_{S_1\subseteq \ldots \subseteq S_n \subseteq S_0} d(S_n^*)\cdot \ldots \cdot d(S_1^*)^{s+1}\ell(S_1)^c|S_1| \\
= & C \sum_{S_3\subseteq \ldots \subseteq S_n} d(S_n^*)\cdot \ldots \cdot d(S_3^*) \sum_{S_2\subseteq S_3} d(S_2^*) \sum_{S_1\subseteq S_2} d(S_1^*)^{s+1}\ell(S_1)^c|S_1| \\
\leq \ldots \leq & C \sum_{S\subseteq S_0} d(S^*)^{n+s}\ell(S)^c|S| \\
= & C \sum_{S\subseteq S_0} \ell(S)^{c-pb}\mu(S^*)^p|S|.
\end{aligned}$$

Combining the cases 1,2, and 3 in the (2.7) the inequality (2.8) follows. $\square$

Corollary 2.1.1. *Let μ be a Borel measure on $\mathbb{H}$. Given $p > 1$ and $a \geq 0$, $b > 0$ we have that if $c > b - n - 1$ there exists $C > 0$ such that:*

$$\int_{S_0} \left(\sum_{S\subseteq S_0} \ell(S)^{a-b}\mu(S^*)\chi_S(x,t)\right)^p t^c dtdx \leq C \sum_{S\subseteq S_0} \ell(S)^{c+p(a-b)}\mu(S^*)^p|S|. \tag{2.8}$$

Proof. Let $S \in \mathcal{S}(\mathcal{Q})$, then we define $\nu(S) := \ell(S)^a\mu(S)$ if $S \subseteq S_0$, otherwise $\nu(S) = 0$. ν is a borelian measure on $\mathbb{H}$, so from the Theorem 2.1.1 it follows that:

$$\begin{aligned}
&\int_{S_0} \left(\sum_{S\subseteq S_0} \ell(S)^{a-b}\mu(S^*)\chi_S(x,t)\right)^p t^c dtdx \\
= &\int_{S_0} \left(\sum_{S\subseteq S_0} \ell(S)^{-b}\nu(S^*)\chi_S(x,t)\right)^p t^c dtdx \\
\leq & C \sum_{S\subseteq S_0} \ell(S)^{c-bp}\nu(S)^p|S| \\
= & C \sum_{S\subseteq S_0} \ell(S)^{c+p(a-b)}\mu(S^*)^p|S|.
\end{aligned}$$

$\square$

Corollary 2.1.2. *Let μ be a Borel measure on $\mathbb{H}$. Given $p > 1, a \geq 0$ and $b > 0$, if $c > b - n - 1$ there exist $C > 0$ such that:*

$$\int_{S_0} \left(\int_{S_0} \frac{(t+s)^a}{(\|x-y\| + (t+s))^b} d\mu(y,s) \right)^p t^c dt dx$$
$$\leq C \sum_{S \subseteq S_0} \ell(S)^{c+p(a-b)} \mu(S^*)^p |S|. \tag{2.9}$$

Proof. For each $(x,t) \in S_0$ with $t > 0$ there exists $k(t) \geq 0$ such that $k(t) \approx \log_2(1/t) + o(1)$, so using the proprieties described by remarks 2.1.4 and 2.1.5 we have:

$$\begin{aligned}
\int_{S_0} \frac{(t+s)^a}{(\|x-y\| + (s+t))^b} d\mu(y,s) &\leq \sum_{k=1}^{k(t)} \int_{S_{x,2^k t}} \frac{(t+s)^a}{(\|x-y\| + (s+t))^b} d\mu(y,s) \\
&\approx \sum_{k=1}^{k(t)} 2^{-(a-b)(k(t)-k)} \mu(S_{x,2^{-(k(t)-k)}}) \\
&\leq \sum_{x \in Q \in \mathcal{Q}_0, \ell(Q) \geq t} \ell(Q)^{a-b} \mu(S(Q)^*) \\
&= \sum_{S \in \mathcal{S}_0} \ell(S)^{a-b} \mu(S^*) \chi_S(x,t),
\end{aligned} \tag{2.10}$$

From the Corollary 2.1.1 it follows:

$$\begin{aligned}
&\int_{S_0} \left(\int_{S_0} \frac{(t+s)^a}{(\|x-y\| + (t+s))^b} d\mu(y,s) \right)^p t^c dt dx \\
&\leq C \int_{S_0} \left(\sum_{S \subseteq S_0} \ell(S)^{a-b} \mu(S^*) \chi_S(x,t) \right)^p t^c dt dx \\
&\leq C \sum_{S \subseteq S_0} \ell(S)^{c+p(a-b)} \mu(S^*)^p |S|.
\end{aligned}$$

$\square$

Remark 2.1.6. The Corollary 2.1.2 extends the inequality (2.2) to all potential spaces on $\mathbb{H}$ with potential kernels K which behave in the unit box as a Riesz potential, i.e. for $(x,t), (y,s) \in S_0$:

$$K_{x,t}(y,s) \approx \frac{(t+s)^a}{(\|x-y\| + t + s)^b}.$$

2.2 Trace Inequalities

In this section we characterize trace measures for particular weighted potential spaces on $\mathbb{H}$. Let m a weighted-measure on $\mathbb{H}$, then our potential space is described by a potential operator $T_K : \mathbb{H} \to \mathbb{R}$, with kernel $K : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$, such that for any $u \in L^p(\mathbb{H}, m)$ and for almost everywhere $(x,t) \in \mathbb{H}$:

$$T_K(u)(x,t) := \int_{\mathbb{H}} K_{x,t}(y,s)u(y,s)dm(y,s) < \infty.$$

To characterize a trace measure μ for this type of space means to characterizing all Borel measures μ on $\mathbb{H}$ for which

$$T_K : L^p(\mathbb{H}, m) \to L^p(\mathbb{H}, \mu)$$

is a bounded operator. In particular, following Kerman and Sawyer, we are interested in characterizing, in equivalent manner, the measures μ for which the dual operator

$$T^*_{\mu,K} : L^{p'}(\mathbb{H}, \mu) \to L^{p'}(\mathbb{H}, m)$$

is bounded, where p' is the conjugate of p.
Let $u \in L^p(\mathbb{H}, m)$, $v \in L^{p'}(\mathbb{H}, \mu)$, then we have:

$$\langle T_K u, v\rangle_{L^2(\mathbb{H},\mu)} = \langle u, T^*_{\mu,K}\rangle_{L^2(\mathbb{H},m)}. \tag{2.11}$$

We compute:

$$\begin{aligned}
\langle T_K u, v\rangle_{L^2(\mathbb{H},\mu)} &= \int_{\mathbb{H}} T_K u(x,t)v(x,t)d\mu(x,t) \\
&= \int_{\mathbb{H}} \left(\int_{\mathbb{H}} K_{x,t}(y,s)u(y,s)dm(y,s) \right) v(x,t)d\mu(x,t) \\
&\quad \text{by Fubini-Tonelli Theorem} \\
&= \int_{\mathbb{H}} \left(\int_{\mathbb{H}} K_{x,y}(y,s)v(x,t)d\mu(x,t) \right) u(y,s)dm(y,s),
\end{aligned}$$

Then from (2.11) it follows that

$$T^*_{\mu,K}(v)(x,t) = \int_{\mathbb{H}} K^*_{x,t}(y,s)v(y,s)d\mu(y,s),$$

where $K^*_{x,t}(y,s) := K_{y,s}(x,t)$. It is clear that if K is a symmetric kernel then $K^* = K$.

These remarks motivate the following definitions.

Definition 2.2.1. *Given $p > 1$ and m a measure on $\mathbb{H}$, let $K : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ be a potential kernel and T_K the integral operator with kernel K. We say that a function u belongs to $\Pi^p_K(\mathbb{H}, m)$ if $u \in L^p(\mathbb{H}, m)$ and $T_K(u) < \infty$ almost everywhere.*

Definition 2.2.2. *A Borel measure μ on $\mathbb{H}$ is a trace measure for $\Pi^p_K(\mathbb{H}, m)$ if there exists a constant $C(\mu) > 0$ such that for any $u \in L^{p'}(\mathbb{H}, \mu)$:*

$$\int_{\mathbb{H}} \left(\int_{\mathbb{H}} K^*_{x,t}(y, s) u(y, s) d\mu(y, s) \right)^{p'} dm(x, t) \\ \leq C(\mu) \int_{\mathbb{H}} |u(x, t)|^{p'} d\mu(x, t). \tag{2.12}$$

Remark 2.2.1. The inequality (2.12) proves the boundedness of the operator $T^*_{\mu,K}$. Since $T^*_{\mu,K}$ is boundedness if and only if T_K is boundedness, we can also say that μ is a trace measure for $\Pi^p_K(\mathbb{H}, m)$ if there exists a constant $C(\mu) > 0$ such that for any $u \in L^p(\mathbb{H}, m)$:

$$\int_{\mathbb{H}} \left(\int_{\mathbb{H}} K_{x,t}(y, s) u(y, s) dm(y, s) \right)^{p} d\mu(x, t) \\ \leq C(\mu) \int_{\mathbb{H}} |u(x, t)|^{p} dm(x, t). \tag{2.13}$$

Our characterization of trace measures is directed to a particular family of spaces $\Pi^p_{K_{a,b}}(\mathbb{H}, m_c)$.
For $c \in \mathbb{R}$, the measure m_c is defined as $dm_c(x, t) := t^c dtdx$, where dtdx is the classical Lebesgue measure, and, with parameters $a \geq 0$ and $b > 0$, the kernel $K_{a,b} : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ is such that

$$\begin{aligned} K_{a,b}(x - y, t + s) &\approx \frac{(t + s)^a}{(\|x - y\| + t + s)^b} && \text{if } \|x - y\| + t + s < 1, \\ K_{a,b}(x - y, t + s) &\approx e^{-\varepsilon(\|x-y\|+t+s)} && \text{if } \|x - y\| + t + s \geq 1, \end{aligned}$$

where $0 < \varepsilon < 1$ is fixed.

Remark 2.2.2. For $0 < \delta < n$, the kernel $K_0, n - \delta$ is the extension on $\mathbb{H}$ of the Bessel kernel $G_\delta : \mathbb{R}^n \to \mathbb{R}$. It is well known infact that $G_\delta(x) \approx \|x\|^{\delta-n}$ if $\|x\| \leq 1$, otherwise $G_\delta \approx e^{-\frac{1}{2}\|x\|}$. Fixed $0 < t < 1$, we observe also that for $\|x\| < 1 - t$ the kernel $K_{0,n-\delta}$ acts as a mollifier for Riesz kernel at step t.

Remark 2.2.3. Since $K_{a,b}$ is a symmetric kernel $K^*_{a,b} = K_{a,b}$. Then μ is a trace measure for $\Pi^p_{K_{a,b}}(\mathbb{H}, m_c)$ if there is a constant $C(\mu) > 0$ such that for any $u \in L^{p'}(\mathbb{H}, \mu)$:

$$\int_{\mathbb{H}} \left(\int_{\mathbb{H}} K_{a,b}(x-y, t+s) u(y,s) d\mu(y,s) \right)^{p'} dm(x,t) \\ \leq C(\mu) \int_{\mathbb{H}} |u(x,t)|^{p'} d\mu(x,t). \tag{2.14}$$

Lemma 2.2.1. *Given $c > -1$ and $p > 1$, we can define, fixed $0 < \varepsilon < 1$, the integral operator $T_e : L^p(\mathbb{H}, m_c) \to L^p(\mathbb{H}, \mu)$ such that for any $u \in L^p(\mathbb{H}, m_c)$ and $(x,t) \in \mathbb{H}$:*

$$T_e(u)(x,t) := \int_{\mathbb{H}} e^{-\varepsilon(\|x-y\|+t+s)} u(y,s) dm_c(y,s).$$

Given a bounded Borel measure μ on $\mathbb{H}$, there is a constant $C(\mu) > 0$ such that for any $u \in L^{p'}(\mathbb{H}, \mu)$:

$$\int_{\mathbb{H}} |T^*_{\mu,e}(u)(x,t)|^{p'} dm_c(x,t) \leq C(\mu) \int_{\mathbb{H}} |u(x,t)|^{p'} d\mu(x,t), \tag{2.15}$$

where $p + p' = p'p$.

Proof. The kernel K_e is a symmetric function, so $K_e = K_e^*$, then for any $u \in L^{p'}(\mathbb{H}, \mu)$

$$\int_{\mathbb{H}} |T^*_{\mu,e}(u)(x,t)|^{p'} dm_c(x,t)$$
$$= \int_{\mathbb{H}} \left(\int_{\mathbb{H}} e^{-\varepsilon(\|x-y\|+t+s)} u(y,s) d\mu(y,s) \right)^{p'} dm_c(x,t)$$
from Hölder's inequality
$$\leq \|u\|^{p'}_{L^{p'}(\mathbb{H},\mu)} \int_{\mathbb{H}} \left(\int_{\mathbb{H}} e^{-p\varepsilon(\|x-y\|+t+s)} d\mu(y,s) \right)^{\frac{p'}{p}} dm_c(x,t). \tag{2.16}$$

We study:

$$\begin{aligned}
R(\mu) &:= \int_{\mathbb{H}} \left(\int_{\mathbb{H}} e^{-p\varepsilon(\|x-y\|+t+s)} d\mu(y,s) \right)^{\frac{p\prime}{p}} dm_c(x,t) \\
&\leq \int_{\mathbb{H}} \left(\sum_{k=1}^{+\infty} \int_{S_{x,k+\|x\|+t}} e^{-p\varepsilon(\|x-y\|+t+s)} d\mu(y,s) \right)^{\frac{p\prime}{p}} dm_c(x,t) \\
&\quad \text{for the remark 2.1.5} \\
&\approx \int_{\mathbb{H}} \left(\sum_{k=1}^{+\infty} e^{-p\varepsilon(k+\|x\|+t)} \mu(S_{x,k+\|x\|+t}) \right)^{\frac{p\prime}{p}} dm_c(x,t) \\
&\leq \left(\mu(\mathbb{H}) \sum_{k=1}^{+\infty} e^{-kp\varepsilon} \right)^{\frac{p\prime}{p}} \int_{\mathbb{H}} e^{-p\prime\varepsilon(\|x\|+t)} dm_c(x,t). \qquad (2.17)
\end{aligned}$$

Recalling that $\mu(\mathbb{H}) < \infty$ and $c > -1$ we have:

$$R(\mu) < \infty,$$

so there exists $C(\mu) > 0$ such that:

$$\int_{\mathbb{H}} |T^*_{\mu,e}(u)(x,t)|^{p\prime} dm_c(x,t) \leq C(\mu) \|u\|^{p\prime}_{L^{p\prime}(\mathbb{H},\mu)}.$$

□

Now, we give a Characterization Theorem for the trace measures on $\Pi^p_{K_{a,b}}(\mathbb{H}, m_c)$. The theorem states that the boundedness of the operator $T^*_{\mu,K_{a,b}}$ on the characteristic functions of Carleson boxes of $\mathbb{H}$ is a necessary and sufficient condition for its overall boundedness. The hypothesis of the theorem require that the exponent c of the weighted-measure m_c is great enough to control, if we replace in (2.12) m with m_c, the convergence of the left integral in (2.12) in the singular points of $K_{a,b}$. This hypothesis is the same of Theorem 2.1.1.

Finally the theorem gives a characterization only for boundedness Borel measures on $\mathbb{H}$. This condition, together with the assumption $c > -1$, is used to deal with the exponential error as we have seen in Lemma 2.2.1.

Theorem 2.2.1 (**Characterization Theorem**). *Let $p, p' > 1$ be conjugate and $c > -1$. If $c > b-n-1$ we have that a positive bounded Borel measure μ on $\mathbb{H}$ is a trace measure for $\Pi^p_{K_{a,b}}(\mathbb{H}, m_c)$ if and only if there exists a constant $C(\mu) > 0$ such that for any $S \subseteq S_0$:*

$$\int_{S_0} \left(\int_S K_{a,b}(x-y, t+s) d\mu(y,s) \right)^{p'} dm_c(x,t) \leq C(\mu)\mu(S). \tag{2.18}$$

Proof. We first note that it is sufficient to prove the Theorem 2.2.1 only for functions $u \geq 0$. If u is a not positive function, trivially:

$$\begin{aligned} &\int_{\mathbb{H}} \left(\int_{\mathbb{H}} K_{a,b}(x-y, t+s))u(y,s) d\mu(y,s) \right)^{p'} dm_c(x,t) \\ \leq &\int_{\mathbb{H}} \left(\int_{\mathbb{H}} K_{a,b}(x-y, t+s)|u(y,s)| d\mu(y,s) \right)^{p'} dm_c(x,t), \end{aligned}$$

so, if $|u|$ satisfies the trace inequality (2.14), then also u does.
From the definition of kernel $K_{a,b}$, it is clear that:

$$\begin{aligned} \int_{\mathbb{H}} |T^*_{\mu,K_{a,b}}(u)(x,t)|^{p'} dm_c(x,t) \approx & \int_{\mathbb{H}\setminus S_0} |T^*_{\mu,e}(u)(x,t)|^{p'} dm_c(x,t) \\ & + \int_{S_0} |T^*_{\mu,Pr}(u)(x,t)|^{p'} dm_c(x,t), \end{aligned} \tag{2.19}$$

where we set for $u \in L^{p'}(\mathbb{H}, \mu)$ and $(x,t) \in \mathbb{H}$:

$$T^*_{\mu,Pr}(u)(x,t) := \int_{S_0} \frac{(t+s)^a}{(\|x-y\| + t + s)^b} u(y,s) d\mu(y,s).$$

From Lemma 2.2.1 and (2.19) it follows that it is sufficient to prove that the operator $T^*_{\mu,Pr} : L^{p'}(\mathbb{H}, \mu) \to L^{p'}(\mathbb{H}, m_c)$ is bounded if and only if there is $C(\mu) > 0$ such that for any $S \subseteq S_0$ we have

$$\int_{S_0} \left(\int_S \frac{(t+s)^a}{(\|x-y\| + t + s)^b} d\mu(y,s) \right)^{p'} dm_c(x,t) \leq C(\mu)\mu(S). \tag{2.20}$$

In fact if $T^*_{\mu,K_{a,b}}$ is bounded then also $T^*_{\mu,Pr}$ is bounded because:

$$\begin{aligned} \int_{S_0} |T^*_{\mu,Pr}(u)(x,t)|^{p'} dm_c(x,t) \leq & \int_{\mathbb{H}} |T^*_{\mu,K_{a,b}}(u)(x,t)|^{p'} dm_c(x,t) \\ & + \int_{\mathbb{H}\setminus S_0} |T^*_{\mu,e}(u)(x,t)|^{p'} dm_c(x,t); \end{aligned}$$

and, on the other hand, if $T^*_{\mu,Pr}$ is bounded then from Lemma 2.2.1 and (2.19) we have that $T^*_{\mu,K_{a,b}}$ is bounded. Let $u \in L^{p'}(\mathbb{H},\mu)$ be a positive function, then it is always possible to define a positive Borel measure ν on $\mathbb{H}$ setting for any Borel set $E \subseteq \mathbb{H}$ (see for more details [39]):

$$\nu(E) := \int_E u(y,s)d\mu(y,s).$$

We observe that:

$$\int_{S_0} |T^*_{\mu,Pr}(u)(x,t)|^{p'} dm_c(x,t)$$
$$= \int_{S_0} \left(\int_{S_0} \frac{(t+s)^a u(y,s)}{(\|x-y\|+t+s)^b} d\mu(y,s) \right)^{p'} dm_c(x,t)$$

putting $k(t) \approx \log_2(1/t) + o(1)$

$$\approx \int_{S_0} \left(\sum_{k=0}^{k(t)} \int_{S_{x,2^k t}} \frac{(t+s)^a u(y,s)}{(\|x-y\|+t+s)^b} d\mu(y,s) \right)^{p'} dm_c(x,t)$$

for the remark 2.1.5

$$\approx \int_{S_0} \left(\sum_{k=0}^{k(t)} 2^{(a-b)(k-k(t))} \nu(S_{x,2^k t}) \right)^{p'} dm_c(x,t)$$
$$\approx \int_{S_0} \left(\sum_{x\in Q\in \mathcal{Q}_0, \ell(Q)\geq t} \ell(Q)^{a-b} \nu(S(Q)^*) \right)^{p'} dm_c(x,t)$$
$$= \int_{S_0} \left(\sum_{S\subseteq S_0} \ell(S)^{a-b} \nu(S^*) \chi_S(x,t) \right)^{p'} dm_c(x,t) \tag{2.21}$$

We set for any $S \subset S_0$

$$\rho(S) := \ell(S)^{c+n+1+p'(a-b)}.$$

Applying to the (2.21) the Corollary 2.1.1 we find:

$$\begin{aligned} \int_{S_0} |T^*_{\mu,Pr}(u)(x,t)|^{p'} dm_c(x,t) &\leq C \sum_{S\subseteq S_0} \ell(S)^{c+p'(a-b)} \nu(S)^{p'} |S| \\ &= \sum_{S\subseteq S_0} \left(\int_S u(y,s) d\mu(y,s) \right)^{p'} \rho(S). \end{aligned} \tag{2.22}$$

S_0 can be modeled by a tree T. We identify the root o of T with S_0 and we match the top-half of $S(Q_\alpha)$, denoted by $S(Q_\alpha)'$, to each vertex $\alpha \in T$. For any $v \in L^{p'}(T,\mu)$ we consider $I^*_\mu v : T \to \mathbb{R}$ such that for any $\alpha \in T$:

$$I^*_\mu v(\alpha) \quad := \quad \sum_{\beta \geq \alpha} v(\beta)\mu(\beta)$$

where $\beta \geq \alpha$ iff $Q_\beta \subseteq Q_\alpha$ and by $\mu(\beta)$ we intend $\mu(S(Q_\beta)')$. We have indicated by $L^{p'}(T,\mu)$ the space of functions $\varphi : T \to \mathbb{R}$ with norm

$$\|\varphi\|^{p\prime}_{L^{p\prime}(T,\mu)} := \sum_{\alpha \in T} |\varphi(\alpha)|^{p\prime}\mu(\alpha) < \infty.$$

where $\mu(\alpha) = \mu(S(Q_\alpha)')$. Let be $u \in L^{p'}(\mathbb{H},\mu)$, then we define the function $v_u : T \to \mathbb{R}$ such that for any $\beta \in T$:

$$v_u(\beta) := \frac{1}{\mu(S(Q_\beta)')} \int_{S(Q_\beta)'} u(y,s)d\mu(y,s),$$

We observe that:

$$\begin{aligned} I^*_\mu v_u(\alpha) &= \sum_{\beta \geq \alpha} v_u(\beta)\mu(\beta) = \sum_{\beta \geq \alpha} \int_{S(Q_\beta)'} u(y,s)d\mu(y,s) \\ &= \int_{S(Q_\alpha)} u(y,s)d\mu(y,s); \end{aligned}$$

and, using the Jensen's inequality, that

$$\begin{aligned} \|v_u\|^{p'}_{L^{p'}(T,\mu)} &\leq \sum_{\alpha \in T} \int_{S(Q_\alpha)'} |u(y,s)|^{p'} d\mu(y,s) \\ &= \|u\|^{p'}_{L^{p'}(\mathbb{H},\mu)}. \end{aligned} \tag{2.23}$$

By the correspondence $\alpha \leftrightarrow S(Q_\alpha)$ we have:

$$\begin{aligned} \sum_{S \subseteq S_0} \left(\int_S u(y,s)d\mu(y,s) \right)^{p'} \rho(S) &= \sum_{\alpha \in T} \left(\int_{S(Q_\alpha)} u(y,s)d\mu(y,s) \right)^{p'} \rho(\alpha) \\ &= \|I^*_\mu v_u\|^{p\prime}_{L^{p\prime}(T,\rho)}. \end{aligned}$$

From the (2.22), according to these new definitions, we can write:

$$\int_{S_0} |T^*_{\mu,Pr}(u)(x,t)|^{p\prime} dm_c(x,t) \leq C \|I^*_\mu v_u\|^{p\prime}_{L^{p\prime}(T,\rho)}. \tag{2.24}$$

On the other hand, once again from the (2.21):

$$
\begin{aligned}
&\int_{S_0} |T^*_{\mu,Pr}(u)(x,t)|^{p'} dm_c(x,t) \\
&\approx \int_{S_0} \left(\sum_{S \subseteq S_0} \ell(S)^{a-b} \nu(S^*) \chi_S(x,t) \right)^{p'} dm_c(x,t) \\
&\text{from the inclusion } \ell^1 \subseteq \ell^{p'} \\
&\geq C \sum_{S \subseteq S_0} \ell(S)^{c+p'(a-b)} \nu(S^*)^{p'} |S| \qquad (2.25)
\end{aligned}
$$

From the (2.24) and the (2.25) we obtain:

$$\int_{S_0} |T^*_{\mu,Pr}(u)(x,t)|^{p'} dm_c(x,t) \approx \|I^*_\mu v_u\|^{p'}_{L^{p'}(T,\rho)}. \qquad (2.26)$$

The operator $I^*_\mu : L^{p'}(T,\mu) \to L^{p'}(T,\rho)$ is a particular discrete dual operator. In [8], [11] and more in general contexts in [14] it can be seen that if μ is a positive bounded borel measure, then a sufficient and necessary condition for the boundedness of operator I^*_μ is that there exists a constant $C(\mu) > 0$ such that for any $S(\alpha) \subseteq T$:

$$\|I^*_\mu \chi_{S(\alpha)}\|^{p'}_{L^{p'}(T,\rho)} \leq C(\mu)\mu(S(\alpha)), \qquad (2.27)$$

where $S(\alpha) := \{\beta \geq \alpha \,|\, \beta \in T\}$. It is clear that $S(\alpha) \sim S(Q_\alpha)$.
If (2.27) holds then I^*_μ is bounded, and, from (2.23) and (2.26), it follows that:

$$
\begin{aligned}
\int_{S_0} |T^*_{\mu,Pr}(u)(x,t)|^{p'} dm_c(x,t) &\approx \|I^*_\mu v_u\|^{p'}_{L^{p'}(T,\rho)} \\
&\leq C(\mu)\|v_u\|^{p'}_{L^{p'}(T,\mu)} \\
&\leq C(\mu)\|u\|^{p'}_{L^{p'}(\mathbb{H},\mu)}, \qquad (2.28)
\end{aligned}
$$

so $T^*_{\mu,Pr}$ is bounded.
Finally by the correspondence $S(Q_\alpha) \sim S(\alpha)$ it is easy to see also that (2.20) implies (2.27), so the Theorem 2.2.1 is proved. $\square$

2.3 Ideas for Further Developments

Given $f \in L^2(\mathbb{R}^n)$ we denote by $\hat{f}$ or by $\mathcal{F}f$ the Fourier transform of f. Let $\mathcal{F}^{-1}$ be the anti-Fourier transform. For any $\alpha > 0$ the fractional laplacian $\Delta_x^{\alpha/2}$ is defined as the operator such that $-\Delta_x^{\alpha/2} f := \mathcal{F}^{-1}(\|x\|^\alpha \hat{f})$.
If $u : \mathbb{H} \to \mathbb{R}$ is an harmonic function we have $\partial_t^2 u = -\Delta_x u$, so we can define $\partial_t^\alpha u := -\Delta_x^{\alpha/2} u$. We recall that if u is an harmonic function we have $\hat{u}(x,t) = e^{-t\|x\|}\hat{f}(x)$, where $f(x) := u(x,0)$. Given $\beta \in \mathbb{R}$, we calculate:

$$\begin{aligned}
\int_{\mathbb{H}} |\partial_t^\alpha u|^2 t^\beta dtdx &= \int_0^{+\infty} t^\beta \int_{\mathbb{R}^n} |-\Delta_x^{\alpha/2} u|^2 dxdt \\
&= \int_0^{+\infty} t^\beta \int_{\mathbb{R}^n} |\mathcal{F}(-\Delta_x^{\alpha/2} u)|^2 dxdt \\
&= \int_{\mathbb{R}^n} \|x\|^{2\alpha} |\hat{f}(x)|^2 \int_0^{+\infty} t^\beta e^{-2t\|x\|} dtdx \\
&\quad \text{integrating by part respect to variable } t \\
&= C_1 \int_{\mathbb{R}^n} \|x\|^{2\alpha-\beta-1} |\hat{f}(x)|^2 dx. \qquad (2.29)
\end{aligned}$$

On the other hand for $0 < \delta < 1$ similarly we compute:

$$\begin{aligned}
\int_0^{+\infty} t^{2(1-\delta)} \|\partial_t u\|_2^2 \frac{dt}{t} &= \int_0^{+\infty} t^{1-2\delta} \int_{\mathbb{R}^n} |-\Delta^{1/2} u|^2 dxdt \\
&= \int_0^{+\infty} t^{1-2\delta} \int_{\mathbb{R}^n} |\mathcal{F}(-\Delta^{1/2} u)|^2 dxdt \\
&= \int_{\mathbb{R}^n} \|x\|^2 |\hat{f}(x)|^2 \int_0^{+\infty} t^{1-2\delta} e^{-2t\|x\|} dtdx \\
&= C_2 \int_{\mathbb{R}^n} \|x\|^{2\delta} |\hat{f}(x)|^2 dx. \qquad (2.30)
\end{aligned}$$

If we put $\delta := \alpha - (\beta+1)/2$, from (2.29) and (2.30) it follows that:

$$\int_{\mathbb{H}} |\partial_t^\alpha u|^2 t^\beta dtdx = C \int_0^{+\infty} t^{2(1-\delta)} \|\partial_t u\|_2^2 \frac{dt}{t}. \qquad (2.31)$$

Equality (2.31) proves that if u is an harmonic function such that

$$\int_{\mathbb{H}} |\partial_t^\alpha u|^2 t^\beta dtdx + \int_{\mathbb{R}^n} |f|^2 dx < \infty, \qquad (2.32)$$

then $f \in \Lambda_\delta^{2,2}(\mathbb{R}^n) = H^\delta(\mathbb{R}^n)$. Following the notation of Stein in [40], $\Lambda_\alpha^{p,q}(\mathbb{R}^n)$ is a Lipschitz-Besov spaces, i.e. the space of functions $f \in L^p(\mathbb{R}^n)$ such that:

$$\|f\|_{\Lambda_\alpha^{p,q}} := \|f\|_p + \left(\int_{\mathbb{R}^n} \frac{\|f(x+y) - f(x)\|_p^q}{\|y\|^{n+\alpha q}} dy\right)^{\frac{1}{q}} < \infty.$$

where $1 \leq p, q < \infty$ and $0 < \alpha < 1$.
We consider the Poisson kernel on the upper half-space of $\mathbb{R}^n$:

$$P_t(y) = \frac{c_n t}{(\|x\|^2 + t^2)^{\frac{n+1}{2}}},$$

where $x \in \mathbb{R}^n, t > 0$.
Proposition 7' in § 5 of [40] states that for any $f \in L^p(\mathbb{R}^n)$:

$$f \in \Lambda_\alpha^{p,q}(\mathbb{R}^n) \Leftrightarrow \int_0^{+\infty} t^{q(1-\alpha)} \|\partial_t(P_t * \ f)\|_p^q \frac{dt}{t} < \infty, \tag{2.33}$$

and

$$\int_{\mathbb{R}^n} \frac{\|f(x+y) - f(x)\|_p^q}{\|y\|^{n+\alpha q}} dy \approx \int_0^{+\infty} t^{q(1-\alpha)} \|\partial_t(P_t * \ f)\|_p^q \frac{dt}{t} < \infty; \tag{2.34}$$

so the square root of (2.32) is a norm on harmonic extension of $\Lambda_\delta^{2,2}(\mathbb{R}^n)$. We denote by $\mathcal{H}_{\alpha,\beta}^2$ the harmonic extension of $\Lambda_\delta^{2,2}(\mathbb{R}^n)$ and by $\|.\|_{\mathcal{H}_{\alpha,\beta}^2}$ the square root of (2.32). $\mathcal{H}_{\alpha,\beta}^2$ is an Hilbert space whit reproducing kernel:

$$K_{x,t}(y,s) = G_\delta * \ P_{x,t+s}(y), \tag{2.35}$$

where G_δ is the Bessel kernel with exponent δ. We recall that $K : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ is the reproducing kernel of $\mathcal{H}_{\alpha,\beta}^2$ if:

1. $K_{x,t} \in \mathcal{H}_{\alpha,\beta}^2$ for any fixed $(x,t) \in \mathbb{H}$,
2. $\forall u \in \mathcal{H}_{\alpha,\beta}^2, \forall (x,t) \in \mathbb{H}$: $\quad u(x,t) = \langle K_{x,t}, u \rangle_{\mathcal{H}_{\alpha,\beta}^2}$.

If K is the reproducing kernel of $\mathcal{H}_{\alpha,\beta}^2$, then for $u \in \mathcal{H}_{\alpha,\beta}^2$ and $(x,t) \in \mathbb{H}$ we have

$$\begin{aligned}
& u(x,t) = \langle K_{x,t}, u \rangle_{H_{\alpha,\beta}^2} \\
= \quad & \int_{\mathbb{H}} \Delta_y^{\alpha/2} K_{x,t}(y,s) \Delta_y^{\alpha/2} u(y,s) s^\beta ds dy + \int_{\mathbb{R}^n} K_{x,t}(y,0) f(y) dy \\
& \text{using the Fourier transform and integrating by part respect to variable } t \\
= \quad & \int_{\mathbb{R}^n} (\|y\|^{2\alpha-\beta-1} + 1) \mathcal{F}K_{x,t}(.,0)(y) \mathcal{F}f(y) dy \\
\approx \quad & \int_{\mathbb{R}^n} (\|y\|^2 + 1)^{\delta/2} \mathcal{F}K_{x,t}(.,0)(y) \mathcal{F}f(y) dy.
\end{aligned} \tag{2.36}$$

On the other hand since u is an harmonic function:

$$\begin{aligned}
u(x,t) &= \langle P_{x,t}, f \rangle_2 \\
&= \int_{\mathbb{R}^n} P_{x,t}(y) f(y) dy = \int_{\mathbb{R}^n} \mathcal{F}P_{x,t}(y) \mathcal{F}f(y) dy.
\end{aligned} \tag{2.37}$$

Recalling that $G_\delta = \mathcal{F}^{-1}((\|x\|^2+1)^{-\delta/2}$, from (2.36) and (2.37) we obtain:

$$\begin{aligned}
\mathcal{F}K_{x,t}(.,0)(y) &= (\|y\|^2+1)^{-\delta/2}\mathcal{F}P_{x,t}(y) \\
&= \mathcal{F}(\mathcal{F}^{-1}(\|.\|^2+1)^{-\delta/2})(y)\mathcal{F}P_{x,t}(y) \\
&\quad \text{for the Fourier transform property on convolution product} \\
&= \mathcal{F}(G_\delta * \ P_{x,t})(y),
\end{aligned}$$

so applying to this equality the anti-Fourier transforms we have (2.35).
It is easy to verify that $\Delta K_{x,t} = 0$. We want to verify if

$$\int_0^{+\infty} s^{2(1-\delta)-1} \int_{\mathbb{R}^n} |\partial_t K_{x,t}(y,s)|^2 dyds < \infty. \tag{2.38}$$

To prove (2.38) is equivalent to prove that

$$\int_{\mathbb{R}^n} (\|y\|^2+1)^{\delta/2} |\mathcal{F}K_{x,t}(y)|^2 dy < \infty. \tag{2.39}$$

We observe:

$$\begin{aligned}
\int_{\mathbb{R}^n} (\|y\|^2+1)^{\delta/2}|\mathcal{F}K_{x,t}(y)|^2 dy &= \int_{\mathbb{R}^n} (\|y\|^2+1)^{-\delta/2}|\mathcal{F}P_{x,t}(y)|^2 dy \\
&= \int_{\mathbb{R}^n} (\|y\|^2+1)^{-\delta/2} e^{-2t\|x-y\|} dy < \infty,
\end{aligned} \tag{2.40}$$

so K is the reproducing kernel of $\mathcal{H}^2_{\alpha,\beta}$. Using the properties of Fourier transform and reproducing kernels it should be possible to consider:

$$\partial_t K_{x,t}(y,s) \le \partial_t P_{x,t}(y,s)\chi_{\{\|x-y\|+t+s<1\}} + e^{-\frac{1}{2}(\|x-y\|+t+s)}\chi_{\{\|x-y\|+t+s\ge 1\}}. \tag{2.41}$$

It is easy to compute:

$$\partial_t P_{x,t}(y,s) = \frac{(\|x-y\|^2 - n(t+s)^2)}{(\|x-y\|^2+(t+s)^2)^{\frac{n+3}{2}}}, \tag{2.42}$$

so if (2.41) is true, we have:

$$\partial_t K_{x,t}(y,s) \le K_{0,n+1}(x-y, t+s)$$

in this case it is clear that Characterization Theorem holds as a sufficient condition for trace measures of $\mathcal{H}^2_{\alpha,\beta}$. Given a bounded Borel measures μ on

$\mathbb{H}$ we have in fact for any positive function $u \in L^2(\mathbb{H}, t^{2(1-\delta)}dtdx)$:

$$\int_{\mathbb{H}} \left(\int_{\mathbb{H}} \partial_t K_{x,t}(y,s) u(y,s) d\mu(y,s) \right)^2 t^{2(1-\delta)} dtdx$$
$$\leq \int_{\mathbb{H}} \left(\int_{\mathbb{H}} K_{0,n+1}(x-y, t+s) u(y,s) d\mu(y,s) \right)^2 t^{2(1-\delta)} dtdx; \quad (2.43)$$

in particular for any Carleson box $S \in \mathbb{H}$ we have:

$$\int_{\mathbb{H}} \left(\int_{S} \partial_t K_{x,t}(y,s) d\mu(y,s) \right)^2 t^{2(1-\delta)} dtdx$$
$$\leq \int_{\mathbb{H}} \left(\int_{S} K_{0,n+1}(x-y, t+s) d\mu(y,s) \right)^2 t^{2(1-\delta)} dtdx. \quad (2.44)$$

$\partial_t P_{x,t}$ could have many cancellations under integral. The eventual resolution of this cancellation's problem is the key point to apply the Characterization Theorem also as a necessary condition for trace measures of $\mathcal{H}^2_{\alpha,\beta}$. We would like to prove that:

$$\int_{S_0} \left(\int_{S_0} \partial_t P_{x,t}(y,s) |u(y,s)| d\mu(y,s) \right)^2 t^{2(1-\delta)} dtdx \leq C(\mu) \|u\|^2_{\mathcal{H}^2_{\alpha,\beta}}$$
$$\Leftrightarrow \int_{S_0} \left(\int_{S_0} \frac{|u(y,s)| d\mu(y,s)}{(\|x-y\| + t + s)^{n+1}} \right)^2 t^{2(1-\delta)} dtdx \leq C(\mu) \|u\|^2_{\mathcal{H}^2_{\alpha,\beta}}, \quad (2.45)$$

in this case the test on the Carleson box of Characterization Theorem holds as a necessary and sufficient condition for trace measures of $\mathcal{H}^2_{\alpha,\beta}$, but it seems very difficult to prove the equivalence (2.45).

Finally we would like generalize the space $\mathcal{H}^2_{\alpha,\beta}$ to exponents $p \neq 2$. For example, one difficulty we have is that, if we replace the weight $s^{2(1-\delta)}$ with $s^{p(1-\delta)}$, we have:

$$\int_{S_0} s^{p(1-\delta)} |\partial_s K_{t,x}(y,s)|^p \frac{ds}{s} dy = +\infty. \quad (2.46)$$

Chapter 3

Carleson Measures for Besov Spaces of Discrete Martingales

Introduction

In this chapter we characterize the Carleson measures for Besov spaces of discrete martingales defined on homogeneous trees. Let T be a k-homogeneous tree with root o (see below for these and other definitions). Given a martingale $\varphi : T \to \mathbb{R}$, consider the *martingale difference function* $D\varphi : T \to \mathbb{R}$ defined by

$$D\varphi(\alpha) = \begin{cases} \varphi(\alpha) - \varphi(\alpha^{-1}) \text{ if } \alpha \in T \setminus \{o\}, \\ \varphi(o) \text{ if } \alpha = o. \end{cases}$$

Here α^{-1} is the predecessor of α in T. On the level of metaphor, we think of the tree as a model for the unit disc in the complex plane, of martingales as harmonic functions and of martingale differences as gradients of harmonic functions. This viewpoint is well known. It has its roots in the seminal work of Cartier (with harmonic functions instead of martingales) [20] and in the influential article [26].

The characterization we have is in terms of "testing conditions" similar to those Kerman and Saywer [27] found in their work on weighted trace inequalities. It will be shown how in this discrete case it is possible to solve a cancellation problem using the stopping time argument, already used in [7] to solve the same type of problem for analytic Besov spaces on dyadic trees. Our cancellations problem, as we will see, is caused by the non constant sign of the discrete integral operator $\Theta : \mathcal{B}_p^T \to L^{p'}(\mu)$ of which boundedness is equivalent to Carleson measures inequality. Unfortunately, the results found in the discrete case can not reapplied to the continuous case where the cancellations problem is still open.

3.1 Besov Spaces of Discrete Martingales

An analytic Besov space can be reduced to a corresponding discrete model. We give some preliminary definitions about the tree model.

Definition 3.1.1. *A k-homogeneous tree T, where $k \geq 2$ is an integer, is defined by the set*

$$T := \{(n, j) \in \mathbb{N} \times \mathbb{N} : 1 \leq j \leq k^n\}.$$

A generic $\alpha = (n, j) \in T$ is called vertex of T.

In our case we identify the interval $[0, 1]$ with the root $o = (0, 1) \in T$. At each vertex $\alpha = (n, j) \in T$ we match the interval $I(\alpha) := [k^{-n}(j-1), k^{-n}j]$, so $|I(\alpha)| = k^{-n}$ is the length of $I(\alpha)$. On T it possible to define a partial ordering as in the following definition.

Definition 3.1.2. *Given $\alpha, \beta \in T$*

$$\alpha < \beta \Leftrightarrow I(\beta) \subset I(\alpha)$$

If $\alpha < \beta$ or $\alpha > \beta$ and $|I(\alpha)| \cdot |I(\beta)|^{-1} \in \{k, 1/k\}$ we say that there is an *edge* between α and β. We write $d(\alpha, \beta)$ to indicate the minimum number of edges between α and β, in particular we set $d(\alpha) := d(\alpha, o)$, the distance in terms of edges from a generic element α to the root.

Definition 3.1.3. *Let α a vertex of T, then we define as predecessor of α the vertex $\alpha^{-1} \in T$ such that*

1. $\alpha > \alpha^{-1}$
2. $d(\alpha^{-1}) = d(\alpha) - 1$

Definition 3.1.4. *Let α a vertex of T, then we define as successor of α the vertex β such that*

1. $\beta > \alpha$
2. $d(\beta) = d(\alpha) + 1$

Remark 3.1.1. We observe that any vertex α of a k-homogeneous tree T has k successors which we indicate always by $\alpha_1, \alpha_2, ..., \alpha_k$.

In the following definitions we build an appropriate functions on T which discretize all continuous functions and operators describing a Besov space of harmonic function. An harmonic function can be discretized by a martingale, the derivative function by the operator D and finally the integral function by the operator I.

Definition 3.1.5. *A function $\varphi : T \to \mathbb{R}$ is a martingale if for all $\alpha \in T$:*

$$\varphi(\alpha) = \frac{1}{k} \sum_{j=1}^{k} \varphi(\alpha_j). \tag{3.1}$$

Remark 3.1.2. The definition of martingale corresponds to discrete version of mean value property characterizing the harmonic function.

Definition 3.1.6. *Given $\varphi : T \to \mathbb{R}$ the discrete derivative $D\varphi : T \to \mathbb{R}$ is the function:*

$$D\varphi(\alpha) = \begin{cases} \varphi(\alpha) - \varphi(\alpha^{-1}) & \text{if } \alpha \in T \setminus \{o\}, \\ \varphi(o) & \text{if } \alpha = o. \end{cases}$$

Definition 3.1.7. *Given $\varphi : T \to \mathbb{R}$ the discrete primitive $I\varphi : T \to \mathbb{R}$ is the function:*

$$I\varphi(\alpha) := \sum_{o \le \beta \le \alpha} \varphi(\beta)$$

Remark 3.1.3. It is straightforward to verify that $D \circ I = I \circ D = Id$.

Definition 3.1.8. *Given $1 < p < \infty$ we denote by $\mathcal{B}_p^T$ the Besov space of martinagales $\varphi : T \to \mathbb{R}$ such that:*

$$\|\varphi\|_{\mathcal{B}_p}^p = \sum_{\alpha \in T} |D\varphi(\alpha)|^p < \infty. \tag{3.2}$$

Remark 3.1.4. For $p = 2$ we have the Dirichlet space $\mathcal{B}_2^T$, which is an Hilbert space with inner product:

$$\langle \varphi, \psi \rangle_{\mathcal{B}_2^T} := \sum_{\alpha \in T} D\varphi(\alpha) \overline{D\psi(\alpha)}. \tag{3.3}$$

Definition 3.1.9. *$K : T \times T \to \mathbb{R}$ is a reproducing kernel for $\mathcal{B}_2^T$ if for all martingales $\varphi \in \mathcal{B}_2$ and for all $\alpha \in T$:*

$$\varphi(\alpha) = \langle K_\alpha, \varphi \rangle_{\mathcal{B}_2^T} = \sum_{\beta \in T} DK_\alpha(\beta) \overline{D\varphi(\beta)}. \tag{3.4}$$

Proposition 3.1.1. *The reproducing kernel of $\mathcal{B}_2^T$ is the function:*

$$K_\alpha(\beta) \quad = \quad \begin{cases} \frac{k-1}{k} d(\alpha) & \text{if } \alpha \le \beta, \\ \frac{k-1}{k}(d(\beta) + 1) & \text{if } \alpha > \beta. \end{cases}$$

Proof. If $\{\varphi_r\}_r$ is an orthonormal basis of $\mathcal{B}_2^T$ then it is known that for all $\alpha, \beta \in T$:

$$K_\alpha(\beta) = \sum_r \varphi_r(\alpha)\overline{\varphi_r(\beta)}. \tag{3.5}$$

We denote by D_α and D_β the corresponding derivatives of K from the variables α and β, so from (3.5) we have:

$$\begin{aligned} D_\beta K_\alpha(\beta) &= K_\alpha(\beta) - K_\alpha(\beta^{-1}) \\ &= \sum_r \varphi_r(\alpha)\overline{\varphi_r(\beta)} - \sum_r \varphi_r(\alpha)\overline{\varphi_r(\beta^{-1})} = \sum_r \varphi_r(\alpha)\overline{D\varphi_r(\beta)}; \end{aligned}$$

from which we get:

$$\begin{aligned} D_\alpha D_\beta K_\alpha(\beta) &= D_\beta K_\alpha(\beta) - D_\beta K_{\alpha^{-1}}(\beta) \\ &= \sum_r \varphi_r(\alpha)\overline{D\varphi_r(\beta)} - \sum_r \varphi_r(\alpha^{-1})\overline{D\varphi_r(\beta)} \\ &= \sum_r D\varphi_r(\alpha)\overline{D\varphi_r(\beta)}. \end{aligned} \tag{3.6}$$

Let $\mathcal{DB}_2^T$ be the space of functions ψ for which there is a martingale $\varphi \in \mathcal{B}_2^T$ such that $\psi = D\varphi$, then (3.1) is equivalent to:

$$\sum_{j=1}^k \psi(\alpha_j) = 0, \tag{3.7}$$

where $\alpha_1, \alpha_2, ..., \alpha_k$ are the successors of an element $\alpha \in T$. If we consider on $\mathcal{DB}_2^T$ the ℓ^2-norm we note that:

$$\|\psi\|^2_{\ell^2(T)} = \sum_{\alpha\in T} |\psi(\alpha)|^2 = \sum_{\alpha\in T} |D(I\psi)(\alpha)|^2 = \|I\psi\|^2_{\mathcal{B}_2^T} = \|\varphi\|^2_{\mathcal{B}_2^T}; \tag{3.8}$$

therefore it is obvious that $\{\varphi_r\}$ is an orthonormal basis in $\mathcal{B}_2^T$ if and only if $\{\psi_r\}$ is an orthonormal basis in $\mathcal{DB}_2^T$. We can choose as orthonormal basis of $\mathcal{DB}_2^T$ the normalized k-th roots of unity:

$$\psi_{\alpha,r}(\alpha_l) := \frac{1}{\sqrt{k}} e^{2\pi i \frac{rl}{k}}, \quad l \in \{1, 2, ..., k\}$$

where $r \in \{0, 1, ..., k-1\}$, α is an element of T and $\alpha_1, \alpha_2, ..., \alpha_k$ its successors. Each function $\psi_{\alpha,r}$ is zero on all other elements of T different from the successors of α. For all $\alpha \in T$ and $r \in \{1, 2, ..., k-1\}$ condition (3.7) is verified and clearly $\{\psi_{\alpha,j}\,|\,\alpha \in T, 0 \leq j \leq k-1\}$ is an orthonormal basis of $\mathcal{DB}_2^T$. By (3.6) we have:

$$D_\alpha D_\beta K_\alpha(\beta) \;=\; \sum_{\delta\in T}\sum_{r=1}^{k-1} \psi_{\delta,r}(\alpha)\overline{\psi_{\delta,r}(\beta)}.$$

The last expression is non-zero only if there is $\delta \in T$ such that $\alpha, \beta \in \{\delta_1, \delta_2, ..., \delta_k\}$, in which case there are $m, n \in \{1, 2, ..., k\}$ such that $\alpha = \delta_m$ and $\beta = \delta_n$, so:

$$\begin{aligned} D_\alpha D_\beta K_\alpha(\beta) &= \sum_{r=1}^{k-1} \psi_{\delta,r}(\delta_m)\overline{\psi_{\delta,r}(\delta_n)} \\ &= \frac{1}{k}\sum_{r=1}^{k-1} e^{2\pi i \frac{r(m-n)}{k}} = \begin{cases} \frac{k-1}{k} & \text{if } m = n, \\ -\frac{1}{k} & \text{if } m \neq n. \end{cases} \end{aligned} \tag{3.9}$$

Integrating with respect to the variable β, we obtain:

$$\begin{aligned} D_\alpha K_\alpha(\beta) &= I_\beta(D_\alpha D_\beta K_\alpha(\beta)) \\ &= \sum_{\delta=o}^{\beta} D_\alpha D_\beta K_\alpha(\delta) \\ &= \begin{cases} \frac{k-1}{k} & \text{if } \alpha \in [o, \beta], \\ -\frac{1}{k} & \text{if } \alpha \notin [o, \beta], d(\alpha, \alpha \wedge \beta) = 1, \end{cases} \end{aligned} \tag{3.10}$$

where $\alpha \in [o, \beta]$ means that $o \leq \alpha \leq \beta$ and we denote by $\alpha \wedge \beta$ *the confluent of* α *and* β, i.e. the only point of intersection between the two geodesics starting at o and containing α and β respectively. We recall that a geodesic of T is a sequence $\{z_n\}_{n\geq 0} \subseteq T$ such that $z_n > z_{n-1}$ and $d(z_n, z_{n-1}) = 1$ for each $n \geq 1$. Integrating respect to the variable α, we have:

$$\begin{aligned} K_\alpha(\beta) &= I_\alpha(D_\alpha K_\alpha(\beta)) \\ &= \sum_{\delta=0}^{\alpha} D_\alpha K_\delta(\beta) = \begin{cases} \frac{k-1}{k} d(\alpha) & \text{if } \alpha \leq \beta, \\ \frac{k-1}{k}(d(\beta)+1) & \text{if } \alpha > \beta. \end{cases} \end{aligned} \tag{3.11}$$

$\square$

3.2 Carleson Measures for $\mathcal{B}_p^T$ Spaces

Definition 3.2.1. *A positive measure μ on T is a Carleson measure for $\mathcal{B}_p^T$, $1 < p < \infty$, if there exists a constant $C(\mu) > 0$ such that for each $\varphi \in \mathcal{B}_p^T$ the following inequality holds:*

$$\sum_{\alpha \in T} |\varphi(\alpha)|^p \mu(\alpha) \leq C(\mu) \sum_{\alpha \in T} |D\varphi(\alpha)|^p \tag{3.12}$$

Remark 3.2.1. Equivalently the inequality (3.12) can be rewritten respect to I operator. We can say that a positive measure μ on T is a Carleson measure for $\mathcal{B}_p^T$ if there is a constant $C(\mu) > 0$ such that for any $\varphi \in \mathcal{B}_p^T$:

$$\sum_{\alpha \in T} |I\varphi(\alpha)|^p \mu(\alpha) \leq C(\mu) \sum_{\alpha \in T} |\varphi(\alpha)|^p. \tag{3.13}$$

If μ is a Carleson measure then $I : \mathcal{B}_p^T \to L^p(T, \mu)$ is a bounded operator. Let $\Theta : L^{p'}(T, \mu) \to \mathcal{B}_{p'}^T$ be the dual operator of I. Θ is bounded if and only if μ is a Carleson measure. We can say also that μ is a Carleson measure if there is a constant $C(\mu) > 0$ such that for any $\varphi \in L^{p'}(T, \mu)$:

$$\sum_{\alpha \in T} |D\Theta\varphi(\alpha)|^{p'} \leq C(\mu) \sum_{\alpha \in T} |\varphi(\alpha)|^{p'}. \tag{3.14}$$

Lemma 3.2.1. *For any $\varphi \in L^{p'}(\mu)$ we have:*

$$\begin{aligned} \|\Theta\varphi\|_{\mathcal{B}_{p'}^T}^{p'} = \sum_{x \in T} \left| \frac{k-1}{k} \sum_{y \in S(x)} \varphi(y)\mu(y) - \frac{1}{k} \sum_{y \sim x} \sum_{z \in S(y)} \varphi(z)\mu(z) \right|^{p'} \\ + \left| \frac{k-1}{k} \sum_{x \in T} \varphi(y)\mu(y) \right|^{p'}. \end{aligned} \tag{3.15}$$

Proof. $\Theta\varphi \in \mathcal{B}_{p'}^T$, so for the Definition 3.1.9 we have:

$$\begin{aligned} \Theta\varphi(x) &= \langle K_x, \Theta\varphi \rangle_{\mathcal{B}_2^T} \\ &= \langle K_x, \varphi \rangle_{\ell^2(\mu)} = \sum_{y \in T} K_x(y)\varphi(y)\mu(y), \end{aligned}$$

so

$$D_x\Theta\varphi(x) = \sum_{y \in T} D_x K_x(y)\varphi(y)\mu(y),$$

From the (3.10) of Proposition 3.1.1 we have that for any $x \in T$

$$\begin{aligned} D_x \Theta\varphi(x) &= \sum_{y \in T} D_x K_x(y)\varphi(y)\mu(y) \\ &= \frac{k-1}{k} \sum_{y \geq x} \varphi(y)\mu(y) - \frac{1}{k} \sum_{y \sim x} \sum_{z > y} \varphi(z)\mu(z) \\ &= \frac{k-1}{k} \sum_{y \in S(x)} \varphi(y)\mu(y) - \frac{1}{k} \sum_{y \sim x} \sum_{z \in S(y)} \varphi(y)\mu(y) \\ &\quad + \frac{k-1}{k}\varphi(x)\mu(x) \end{aligned}$$

where we denote by $y \sim x$ the successors of x^{-1} such that $y \neq x$. Then summing on $x \in T$ the (3.15) follows.

□

Theorem 3.2.1. *Given $p > 1$, let p' the conjugate of p. A positive and bounded measure μ on T is a Carleson measure for $\mathcal{B}_p^T$ if and only if for all $\alpha \in T$:*

$$\sum_{x \geq \alpha} \mu(S(x))^{p'} \leq C(\mu)\mu(S(\alpha)); \tag{3.16}$$

where for each $\alpha \in T$ we define $S(\alpha) := \{\beta \,|\, \beta > \alpha\}$ as the box of vertex α.

In Theorem 3.2.1 Carleson measures are characterized by testing inequality (3.12) on characteristic functions of special sets. The result is similar to those obtained in a different context by Kerman and Sawyer. Let me say a few words about the proof. The inequality (3.12) means that the operator $Id : \mathcal{B}_p^T \to L^p(\mu)$ is bounded. This is equivalent to the boundedness of the adjoint operator $\Theta : L^{p'}(\mu) \to \mathcal{B}_{p'}^T$. Now, Θ is an integral operator with a kernel having many cancellations. It was proved in [8] that the hypothesis of Theorem 3.2.1 is equivalent to the boundedness of the operator $|\Theta|$ which is obtained from Θ by replacing the signed kernel with its absolute value. We are left, then, with the task of proving that the boundedness of Θ, the signed kernel, implies condition (3.16). The same problem arises in the dyadic case, which was considered in [7]. To get around the problem of cancellations we use a stopping time argument. In the continuos case, a result analogous to Theorem 1 for the corresponding Besov spaces of harmonic functions on $\mathbb{R}^n_+$ seems to be still open. The cancellations in the kernel are of the same kind, but the more involved geometry of the upper-half space makes it difficult to efficiently run a stopping time argument.

Proof. For $x \in T$ we set $M_x := \mu(S(x))$. If $\varphi = \chi_{S(\alpha)}$, from (3.15) we have

$$\|\Theta\varphi\|^{p'}_{\mathcal{B}^T_{p'}} = \sum_{x>\alpha} \left| \frac{k-1}{k} M_x - \frac{1}{k} \sum_{y \sim x} M_y \right|^{p'} + \left| \frac{k-1}{k} M_\alpha \right|^{p'}. \tag{3.17}$$

The Theorem 1 is proved if we show that:

$$\sum_{x>\alpha} \left| \frac{k-1}{k} M_x - \frac{1}{k} \sum_{y \sim x} M_y \right|^{p'} + \left| \frac{k-1}{k} M_\alpha \right|^{p'} \approx \sum_{x \geq \alpha} M_x^{p'}. \tag{3.18}$$

Infact if we consider $I^*_\mu : L^{p'}(\mu) \to L^{p'}$, the adjoint of $I : L^p \to L^p(\mu)$, we have that for $\varphi := \chi_{S(\alpha)}$ and $x \in T$

$$I^*_\mu(\varphi)(x) = \sum_{y \in S(x)} \varphi(y)\mu(y) = \begin{cases} M_x & \text{if } x > \alpha, \\ M_\alpha & \text{if } x \leq \alpha; \end{cases} \tag{3.19}$$

thus

$$\|I^*_\mu(\varphi)\|^{p'}_{L^{p'}} = \sum_{x \in T} |I^*_\mu(\varphi)(x)|^{p'} \geq \sum_{x \geq \alpha} M_x^{p'}. \tag{3.20}$$

If (3.18) is true, from (3.17) and (3.20) we obtain:

$$\|\Theta\varphi\|^{p'}_{\mathcal{B}^T_{p'}} \leq \|I^*_\mu(\varphi)\|^{p'}_{L^{p'}}. \tag{3.21}$$

In [8] it has been proved that I^*_μ is bounded if and only if it is bounded only on characteristic functions $\chi_{S(\alpha)}$. By (3.21) it is sufficient to test the boundedness of the operator Θ only on this type of functions.

We note immediately that:

$$\begin{aligned} \sum_{x>\alpha} \left| \frac{k-1}{k} M_x - \frac{1}{k} \sum_{y \sim x} M_y \right|^{p'} &\leq \sum_{x>\alpha} \left| \frac{k-1}{k} M_x + \frac{1}{k} \sum_{y \sim x} M_y \right|^{p'} \\ &\leq \sum_{x>\alpha} \left| \frac{k-1}{k} M_{x^{-1}} + \frac{k-1}{k} M_{x^{-1}} \right|^{p'} \\ &\leq C \sum_{x \geq \alpha} M_x^{p'}. \end{aligned} \tag{3.22}$$

The opposite inequality

$$\sum_{x \geq \alpha} M_x^{p'} \leq C \left| \frac{k-1}{k} M_\alpha \right|^{p'} + C \sum_{x>\alpha} \left| \frac{k-1}{k} M_x - \frac{1}{k} \sum_{y \sim x} M_y \right|^{p'} \tag{3.23}$$

is not trivial.

To prove (3.23) we need some preliminary definitions. Let $\varepsilon > 0$ to be specified later. Let $\alpha \in T$ and $\xi := \{z_n\}_{n\geq 0}$ a geodesic such that $z_0 = \alpha$. We define*m-stopping times* recursively:

$$t_m := t_m(\xi) := \inf\left\{t > t_{m-1} : M_{z_t} > \left(\frac{1+\varepsilon}{k}\right) M_{z_t^{-1}}\right\},$$

where $m > 0$ and $t_0 := 0$. A point $b \in T$ such that $b = z_{t_m}$ on some geodesic staring at α is called *m-stopping point* and let $A(b)$ to be as the set of the $(m+1)$-stopping points c such that $c > b$. Let $SP(\alpha)$ be the set of the stopping points, then we prove that:

$$\sum_{x\geq\alpha} M_x^{p'} \leq C \sum_{x\geq\alpha, x\in SP(\alpha)} M_x^{p'}. \tag{3.24}$$

This last inequality is proved if for all $m \geq 0$ and for each m-stopping point b:

$$\sum_{b\leq c<A(b)} M_c^{p'} \leq C M_b^{p'}, \tag{3.25}$$

where $c < A(b)$ means that $c < u$ there is no $u \in A(b), c \geq u$. Let n be a positive integer and $b \leq c < \mathcal{B}(b)$ such that $d(b,c) = n$. Let $c_1, c_2, ..., c_k$ be the successors of c^{-1}, then there is $j \in \{1, 2, ..., k\}$ such that $c = c_j$. If there exists $l \in \{1, 2, ..., k-1\}$ such that for all $1 \leq s \leq l$ we have $b < c_s < \mathcal{B}(b)$ then:

$$M_c^{p'} + \sum_{s=1}^{l} M_{c_s}^{p'} \leq (l+1)\left(\frac{1+\varepsilon}{k}\right)^{p'} M_{c^{-1}}^{p'} \leq k^{1-p'}(1+\varepsilon)^{p'} M_{c^{-1}}^{p'}. \tag{3.26}$$

If $c_1, ..., c_{j-1}, c_{j+1}, ..., c_k$ are not between b and $A(b)$, then:

$$M_c^{p'} \leq \left(\frac{1+\varepsilon}{k}\right)^{p'} M_{c^{-1}}^{p'} \leq k^{1-p'}(1+\varepsilon)^{p'} M_{c^{-1}}^{p'}. \tag{3.27}$$

We choose ε such that $k^{1-p'}(1+\varepsilon)^{p'} = 1-\delta < 1$, where $0 < \delta < 1$. If either (3.26) or (3.27) holds, we have, by iteration $d(b,c)$:

$$\begin{aligned}\sum_{b<c<A(b),d(b,c)=n} M_c^{p'} &\leq (1-\delta) \sum_{b<c<A(b),d(b,c)=n-1} M_c^{p'} \\ &\leq (1-\delta)^2 \sum_{b<c<A(b),d(b,c)=n-2} M_c^{p'} \\ &\leq ... \leq (1-\delta)^n M_b^{p'}. \end{aligned}\tag{3.28}$$

Summing on $n > 0$ in (3.28), the (3.25) is proved. In fact:

$$\sum_{b \le c < A(b)} M_c^{p'} \le M_b^{p'} \sum_{n=0}^{+\infty} (1-\delta)^n \le C M_b^{p'}.$$

Now let $b > \alpha$ be a stopping point. Since for $y \sim b$ the boxes $S(y) \subset S(b^{-1})$ are all mutually disjoint

$$\sum_{y \sim b} M_y \le M_{b^{-1}} - M_b.$$

Then

$$\begin{aligned} \frac{1}{k} \sum_{y \sim b} M_y &\le \frac{1}{k}(M_{b^{-1}} - M_b) \le \frac{1}{k}\left(\frac{k}{1+\varepsilon} - 1\right) M_b \\ &\le \left(\frac{1}{1+\varepsilon} - 1\right) M_b + \frac{k-1}{k} M_b. \end{aligned}$$

Hence:

$$\frac{k-1}{k} M_b - \frac{1}{k} \sum_{y \sim b} M_y \ge \varepsilon M_b. \tag{3.29}$$

Putting $C := \varepsilon^{-p'}$, we obtain:

$$M_b^{p'} \le C \left| \frac{k-1}{k} M_b - \frac{1}{k} \sum_{y \sim b} M_y \right|^{p'}. \tag{3.30}$$

Finally summing over b, we get:

$$\begin{aligned} \sum_{b > \alpha, b \in SP(\alpha)} M_b^{p'} &\le C \sum_{b > \alpha, b \in SP(\alpha)} \left| \frac{k-1}{k} M_b - \frac{1}{k} \sum_{y \sim b} M_y \right|^{p'} \\ &\le C \left| \frac{k-1}{k} M_\alpha \right|^{p'} + C \sum_{b > \alpha} \left| \frac{k-1}{k} M_b - \frac{1}{k} \sum_{y \sim b} M_y \right|^{p'}; \end{aligned}$$

by the definition of stopping point occurring:

$$\sum_{b \ge \alpha} M_b^{p'} \le \sum_{b > \alpha, b \in SP(\alpha)} M_b^{p'};$$

the (3.23) is fully proved. $\square$

Bibliography

[1] Adams D. R., *Trace of potentials arising from translation invariant operators*, Ann. Sc. Norm. Super. Pisa, **25**, (1971), 203-217

[2] Adams D. R., *A trace inequality for generalized potentials*, Studia Math., **48**, (1973), 99-105

[3] Adams D. R., *On the existence of capacitary strong type estimates in* $\mathbb{R}^n$, Ark. Mat., **14**, (1976), 125-140

[4] Adams D. R., Hedberg L. I., *Function spaces and potential Theory*, Springer, Grundlehren Math. Wiss. Bd. 314, Berlin (1999)

[5] Aleman A., Pott S., Reguera M. C., *A linear bound for weighted Bergman projection in terms of the Bekolle constant*, http://webpages.ull.es/users/hcaa2012/media/ficheros/pott.pdf

[6] Arcozzi N., *Carleson measures for analytic Besov spaces: the upper triangle case*, JIPAM. J. Inequal. Pure Appl. Math., **6**, No. 1, (2005), Article 13, 15 pp.

[7] Arcozzi N., Rochberg R., *Topics in dyadic Dirichlet spaces*, New York J. of Math., **10** (2004), 45-67

[8] Arcozzi N., Rochberg R., Sawyer E., *Carleson measures for analytic Besov spaces*, Rev. Mat. Iberoamericana, **18** (2002), 443-510

[9] Arcozzi N., Rochberg R., Sawyer E., *Carleson measures and interpolating sequences for Besov spaces on complex balls*, Mem. Amer. Math. Soc., **182**, No. 859, (2006), vi+163 pp.

[10] Arcozzi N., Rochberg R., Sawyer E., *Some problems on Carleson measures for Besov-Sobolev spaces*, Topics in Complex Analysis and Operator Theory, Proceeding of the Winter School held in Antequera, Malaga, Spain, (February 5-9-2006), 141-148

[11] Arcozzi N., Rochberg R., Sawyer E., *The characterization of the Carleson measures for analytic Besov spaces: a simple proof*, Complex and harmonic analysis, DEStech Publ., Inc., Lancaster, PA, (2007), 167-177

[12] Arcozzi N., Rochberg R., Sawyer E., *Capacity, Carleson measures, boundary convergence, and exceptional sets*, "Perspective in Harmonic Analysis and Applications" in honor of V. G. Maz'ya 70-th birthday, AMS Proceedings of Symposia in Pure and Applied Mathematics, (2008)

[13] Arcozzi N., Rochberg R., Sawyer E., Wick B., *The Dirichlet space: a survey*, New York Journal of Mathematics, **17a**, (2010)

[14] Arcozzi N., Rochberg R., Sawyer E., Wick B., *Potential theory on trees, graphs and Ahlfors regular metric spaces*, arXiv:1010.4788, (2010)

[15] Aronszajn N., *Theory of reproducing kernels*, Transactions of American Mathematical Society, **68**, No. 3, (1950), 337-404

[16] Aronszajn N,, Smith K. T., *Theory of Bessel potential. I*, Annales de l'institut Fourier, **11**, (1961), 385-475

[17] Axler S., Bourdon P., Ramey W., *Harmonic function theory*, Springer-Verlag, New York, 1962

[18] Calderón A. P., *Lebesgue spaces of differentiable functions and distributions*, Proc. Sympos. Pure Math., **4**, (1961), 33-49

[19] Carleson L., *Interpolation by bounded analytic functions and the corona problem*, Ann. Math., **76**, (1962), 547-559

[20] Cartier P., *Fonctions harmoniques sur un arbre*, Symp. Math., **9** (1972), 203-270, Academic Press, London, 1972.

[21] Cascante C., Ortega J. M., *Carleson measures on spaces of Hardy-Sobolev type*, Can. J. Math., **47**, (1995), 1177-1200

[22] Cascante C., Ortega J. M., *On q-Carleson measures for spaces of $\mathcal{M}$-harmonic functions*, Can. J. Math., **49**, (1997), 653-674

[23] Dahlberg B., *Regularity properties of Riesz potentials*, Ind. U. Math. J., (28), (1979), 257-268

[24] Fefferman C., Phong D. H., *On positivity of pseudo-differential operators*, Proc. Natl. Acad. Sci. USA, **75**, No. 10, (1978), 4673-4674

[25] Guliyev V., Wu S. Z., *Strong type estimate and Carlerson Measures for Weighted Besov-Lipschitz Spaces*, Further Progress in Analysis, 132-141

[26] Korányi A., Picardello M. A., Taibleson M. H., *Hardy spaces on non-homogeneous trees*, Symp. Math. **29** (1984), 205-265

[27] Kerman R., Sawyer E., *The trace inequality and eigenvalue estimates for Schrödinger operators*, Annales de l'institut Fourier, **36** No. 4, (1985), 207-228

[28] Kerman R., Sawyer E., *Carleson measures and multipliers of Dirichlet-type spaces*, Trans. Amer. Math. Soc., **309**, (1988), 87-98

[29] Hansson K., *Continuity and compactness of certain convolution operators*, Institut Mittage-Leffler, Report No. 9, (1982)

[30] Hedberg L. I, Wolff T. H., *Thin sets in non linear potential theory*, Annales de l'institut Fourier, **33**, No. 4, (1983), 161-187

[31] Hunt R., Muckenhoupt B., Wheeden R., *Weighted norm inequalities for the conjugate function and Hilber transform*, Trans. Amer. Math. Soc., **176**, (1973), 227-251

[32] Maz'ya V. G., *The negative spectrum of the n-dimensional Schrödinger operator*, Soviet Math Dokl., **3**, (1962), 808-810

[33] Maz'ya V. G., *The Dirichlet problem for elliptic equations of arbitrary order in unbounded regions*, Soviet Math. Dokl., **4**, (1963), 860-863

[34] Maz'ya V. G., *On the theory of the multidimensional Schrödinger operator*, Izv. Akad. Nauk SSSR, **28**, (1964), 1145-1172

[35] Maz'ya V. G., *On some integral inequalities for functions of many variables*, J. Soviet Math., **1**, (1973), 205-234

[36] Maz'ya V. G., *On capacitary estimates of the strong type for the fractional norm*, Zap. Sen. LOMI Leningrad, **70**, (1977), 161-168

[37] Maz'ya V. G., *Sobolev spaces*, Springer, Berlin, (1985)

[38] Muramatu T., *On the Dual of Besov Spaces*, Publ. RIMS, Kyoto Univ., **12**, (1976), 123-140

[39] Rudin W., *Real and complex analysis*, McGraw-Hill, (1970)

[40] Stein E. M., *Singular integrals and differentiability properties of functions*, Princeton University Press, (1970)

[41] Stegenga D. A., *Multipliers of the Dirichlet space*, Illinois Journal of Mathematics, 24, No. 1, (1980)

[42] Triebel H., *Theory of function spaces III*, Birkhäuser Verlag, 2006

[43] Verbitsky I. E:, *Multipliers in space with "fractional" norms, and inner functions*, Siber. J. of Mathematics, **26**, (1985), 198-216

[44] Wu Z., *Carleson measures and multipliers for Dirichlet spaces*, J. Funct. Anal., **169**, (1999), 148-163

[45] Zhu K., *Operator theory in function spaces*, Monographs and Textbooks in Pure and Applied Mathematics, 139. Marcel Dekker, Inc., New York, (1990)