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A MOMENT SYMBOLIC REPRESENTATION OF LÉVY PROCESSES WITH APPLICATIONS

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Contents

Introduction

A family of polynomials $\{P(x,t)\}_{t\geq 0}$ is said to be *time-space harmonic* with respect to a stochastic process $\{X_t\}_{t\geq 0}$ if

$$E[P(X_t,t) \mid \mathfrak{F}_s] = P(X_s,s),$$

for all $s \leq t$, where $\mathfrak{F}_s = \sigma\left(X_\tau : \tau \leq s\right)$ is the natural filtration associated with $\{X_t\}_{t\geq 0}$. The usefulness of time-space harmonic polynomials with respect to Lévy processes is that the stochastic process $\{P(X_t,t)\}$ is a martingale, whereas $\{X_t\}$ does not necessarily have this property. Therefore to find polynomials such that it is a martingale the stochastic process obtained by replacing the indeterminate x with the Lévy process $\{X_t\}_{t\geq 0}$, could be particularly meaningful in several area and in particular in mathematical finance.

For random walks $\{X_n\}_{n\geq 0}$, the family of time-space harmonic polynomials has been characterized in the literature as the coefficients of the Taylor expansion

$$\frac{\exp\{zX_n\}}{E[\exp\{zX_n\}]} = \sum_{k\geq 0} R_k(X_n,n)\frac{z^k}{k!}$$

in some neighborhood of the origin [39]. If $\{X_n\}_{n\geq 0}$ is replaced by a Lévy process $\{X_t\}_{t\geq 0}$, the left-hand side of the previous equality is the so-called Wald's exponential martingale [34]. The Wald's exponential martingale is well defined only when the process admits moment generating function $E[\exp\{zX_t\}]$ in a suitable neighborhood of the origin. Different authors have tried to overcome this gap by using other tools. Sengupta [52] uses a discretization procedure to extend the results proved by Goswami and Sengupta in [32]. Solé and Utzet [54] use Ito's formula showing that time-space harmonic polynomials with respect to centred Lévy processes are linked to exponential complete Bell polynomials [19]. The Wald's exponential martingale has been recently reconsidered also in [53], but without this giving rise to a closed expression for these polynomials. The Teugels martingale

[51] has been employed referring to centered Lévy processes, for which the martingale property holds.

This manuscript introduces a symbolic theory of univariate and multivariate Lévy processes with applications to time-space harmonic polynomials, which includes and generalizes the exponential complete Bell polynomials [19]. This symbolic representation of Lévy processes is obtained by using a moment symbolic calculus, which essentially relies on the so-called *classical umbral calculus* introduced in [46].

The term *umbral calculus* appeared for the first time in the literature in 1861, in a series of papers by Rev. John Blissard [6] $\div$ [15], who referred to a set of mathematical tricks consisting in dealing with sequences of numbers, indexed by nonnegative integers, where the subscripts were treated as they were powers. Blissard extensively used these tricks, but he was not able to give them a rigorous proof. Afterwards, Bell [1] $\div$ [5] reviewed the whole subject in a series of papers, with the intent to give it a rigorous foundation, but he failed. In the first modern textbook of combinatorics [42], Riordan employed umbral methods, without giving any formal justification. We have to wait for 1964, when Rota published *The number of partitions of a set* [45], in which he disclosed "the umbral magic art" of lowering and raising exponents, bringing to the light the underlying linear functional.

The classical umbral calculus, in the version introduced by Rota and Taylor [46] in 1994 and rigorously completed and formalized by Di Nardo and Senato in a series of papers starting from 2001 [22, 29, 30], is a syntax consisting in an alphabet $\mathcal{A} = \{\alpha, \beta, \gamma, \ldots\}$ of symbols, called *umbrae*, and a suitable linear functional E, called *evaluation*. This operator resembles the expectation operator in probability theory and shares the so-called *uncorrelation property* with respect to the product of different symbols. Such property parallels the independence property of random variables with respect to the expectation. Therefore umbrae look like the framework of random variables, with no reference to any probability space [29].

The main devices of this method are essentially two. The first key point is to associate a unital number sequence $1, a_1, a_2, \ldots$ to a sequence $1, \alpha, \alpha^2, \ldots$ of powers of α by means of the evaluation functional E. The sequence $\{a_n\}_{n \geq 0}$ is called the sequence of *moments* of the umbra α, similarly to the sequence of real numbers, obtained by evaluating the powers of a random variable X by means of the expectation symbol.

The second key point is that distinct umbrae could represent the same sequence $\{a_n\}$, so they are called *similar*. This is what happens in probability theory, where random variables with the same sequence of moments have the same distribution and so they are said to be identically distributed.

One tool used to manage umbrae is represented by the *generating function*, which is a formal power series in the commutative integral domain $\mathbb{R}[[z]]$ under the summation and the product. The usefulness of handling formal power series is because they can be treated as algebraic objects not involving questions of convergence [58]. Then operations among generating functions correspond to operations among umbrae.

The symbolic handling of Lévy processes requires the introduction of a new symbol representing suitable operations among uncorrelated and similar umbrae, that is the *auxiliary umbra*. As example, the *dot-product* $n.\alpha$ of a nonnegative integer n and an umbra α represents the summation of n umbrae $\alpha', \alpha'', \ldots, \alpha'''$. Moreover, it is possible to replace the integer n with $t \in \mathbb{R}$ or $\gamma \in \mathcal{A}$, by obtaining the auxiliary umbrae $t.\alpha$ and $\gamma.\alpha$ [22]. In particular, we prove that a Lévy process is umbrally represented by the family of auxiliary umbrae $\{t.\alpha\}_{t\geq 0}$ and this is possible by taking into account their infinite divisibility property [48].

A special dot-product among umbrae, which plays a crucial role in the symbolic representation, is the *cumulant umbra* [30] whose sequence of moments is the sequence of cumulants associated to a random variable [41]. Thanks to the additivity property and the invariance property under translations, cumulants could be analyzed from an algebraic standpoint. For this reason, cumulants are introduced also in the boolean context, as well as in the free one [62].

By using the umbral parametrization of cumulants in terms of moments and vice versa, a very simple, efficient and fast algorithm is introduced. This algorithm is based on a unique formula for classical, boolean and free cumulants. Comparisons with other procedures existing in the literature show the usefulness of this algorithm, especially in the free context [23].

Cumulant umbrae also enter in the symbolic representation of Lévy processes. Hence, a connection between Lévy processes and classical, boolean and free cumulant umbrae is also studied [24].

The Lévy processes here considered are not necessarily centered and so they do not share the martingale property, in general. In particular, in the symbolic version here introduced, this property is related to the role played by cumulants and highlighted by means of the Lévy-Khintchine formula. Thanks to this representation, the martingale property is taken back to the enrollment in the cumulants of a special umbra, the singleton umbra, although it does not have a probabilistic counterpart.

Cumulants play the same role also in a different class of polynomials, named the *Kailath-Segall polynomials* [33] and not necessarily time-space harmonic, linking the variations of a Lévy process $\{X_t\}_{t\geq 0}$ to its iterated stochastic integrals. In particular the Kailath-Segall formula and its inversion turn out to be a suitable generalization of the well-known formulae giving elementary symmetric polynomials in terms of power sum symmetric polynomials [20].

The main tool to deal with time-space harmonic polynomials is the definition of a new operator on the alphabet $\mathcal{A}$ that acts like the well-known conditional expectation of random variables [26]. The family of time-space harmonic polynomials here introduced is defined in a very simple way as $Q_k(x,t) = E[(x - t.\alpha)^k]$ and relies on a very simple closed-form of the corresponding coefficients which can be easily implemented in any symbolic software [23]. As corollaries, many identities are given on the coefficients of this family of time-space harmonic polynomials with respect to random walks and Lévy processes.

The new family of polynomials $\{Q_k(x,t)\}$ includes Hermite polynomials, which are time-space harmonic with respect to Brownian motion, Poisson-Charlier polynomials, which are linear combinations of the introduced polynomials when they are time-space harmonic with respect to compensated Poisson processes, the actuarial polynomials and the Laguerre polynomials with respect to Gamma processes, the Meixner polynomials of the first kind with respect to Pascal processes, the Bernoulli, Euler and Krawtchuk polynomials with respect to suitable random walks. Also the so-called Lévy-Sheffer polynomials are linear combinations of the introduced polynomials. We are able also to define new families of time-space harmonic polynomials with respect to the Inverse Gaussian processes and the stable processes.

The expression of the family of time space-harmonic polynomials here introduced has some connections with the so-called moment representation of various families of multivariate polynomials that recently many authors have obtained [63, 64, 65]. For the multivariate Hermite polynomials $H_{\mathbf{v}}(\mathbf{x})$, with $\mathbf{v} = (v_1, \ldots, v_d) \in \mathbb{N}_0^d$ a multi-index, i.e. a vector of nonnegative integers, the moment representation is $H_{\mathbf{v}}(\mathbf{x}) = E[(\mathbf{x}\Sigma^{-1} + i\mathbf{Y})^{\mathbf{v}}]$ [64, 65], with E the expectation symbol, i the imaginary unit, $\mathbf{Y} \sim N(\mathbf{0}, \Sigma^{-1})$ and Σ a covariance matrix of full rank d. In this manuscript, such a moment representation has been studied for the first time in connection with the time-space harmonic property with respect to a suitable symbolic multivariate Lévy process. So the multivariate time-space harmonic property has been introduced by using a suitable multivariate conditional evaluation. For a multivariate account of the umbral calculus see [20, 21].

Multivariate Hermite polynomials and their properties have been studied in connection with a symbolic version of the multivariate Brownian motion. Multivariate Bernoulli and Euler polynomials are represented as powers of multivariate polynomials which are time-space harmonic with respect to suitable multivariate Lévy processes. Many properties of these polynomials can be straightforwardly recovered thanks to these representations. A very simple and new relation between these two types of multivariate polynomials is also provided [27].

Chapter 1

The classical umbral calculus

An *umbral calculus* is a syntax consisting of the following data:

(i) a set $\mathcal{A} = \{\alpha, \beta, \gamma, \ldots\}$ of objects called *umbrae*;

(ii) a linear functional $E : R[x][\mathcal{A}] \longrightarrow R[x]$, called *evaluation*, such that

- $E[1] = 1$;
- $E[x^n \, \alpha^i \, \beta^j \, \cdots \, \gamma^k] = x^n \, E[\alpha^i] \, E[\beta^j] \, \cdots \, E[\gamma^k]$ for all distinct umbrae $\alpha, \beta, \ldots, \gamma \in \mathcal{A}$ and for all nonnegative integers $n, i, j, \ldots k$ (*uncorrelation property*);

(iii) an element $\epsilon \in \mathcal{A}$, called *augmentation*, such that $E[\epsilon^k] = \delta_{0,k}$, for all nonnegative integers k, where $\delta_{0,k}$ is the *Kronecker symbol* with $\delta_{0,k} = 1$ if $k = 0$ and 0, otherwise;

(iv) an element $u \in \mathcal{A}$, called *unity umbra*, such that $E[u^k] = 1$, for all nonnegative integers k.

Remark 1.1. The symbol R represents a commutative integral domain, whose quotient field has characteristic zero. In the following, we will consider $R = \mathbb{R}$.

A unital sequence of real numbers $a_0 = 1, a_1, a_2, \ldots$ is said to be *umbrally represented* by an umbra α if

$$E[\alpha^k] = a_k, \qquad \text{for all } k \geq 0.$$

The k-th element of the sequence is called the k-th moment of the umbra α.

An umbra α is said to be a *scalar* umbra if its moments are in $\mathbb{R}$, while it is said to be a *polynomial* umbra if its moments are in $\mathbb{R}[x]$.

Two distinct umbrae α and γ are said to be *similar* if and only if they have the same sequence of moments, in symbols

$$\alpha \equiv \gamma \Leftrightarrow E[\alpha^k] = E[\gamma^k], \quad \text{for } k = 0, 1, 2, \ldots$$

Given an umbra α, its *k-th factorial moment* is

$$E[(\alpha)_k] = \begin{cases} 1, & \text{if } k = 0 \\ E[\alpha(\alpha-1)\cdots(\alpha-k+1)], & \text{if } k > 0. \end{cases}$$

We set $E[(\alpha)_k] = a_{(k)}$, for all nonnegative integers k.

A polynomial $p \in \mathbb{R}[\mathcal{A}]$ is called an *umbral polynomial*. The *support* of p is defined to be the set of all occurring umbrae of $\mathcal{A}$. Two umbral polynomials p and q are said to be *uncorrelated* when their supports are disjoint.

Two umbral polynomials p and q are said to be *umbrally equivalent* if and only if $E[p] = E[q]$, in symbols $p \simeq q$.

The notion of similarity among umbrae can be expressed in terms of umbral equivalence in the following way

$$\alpha \equiv \gamma \Leftrightarrow \alpha^k \simeq \gamma^k, \quad \text{for all } k \geq 0.$$

Remark 1.2. There are three equivalence relations among umbrae: equality, similarity and umbral equivalence. Equality implies similarity, which implies umbral equivalence, but the converse is false.

1.1 The generating function of an umbra

The formal power series

$$u + \sum_{k \geq 1} \alpha^k \frac{z^k}{k!} \in \mathbb{R}[\mathcal{A}][z] \tag{1.1}$$

is the *generating function* of the umbra α and it is denoted by $e^{\alpha z}$.

By extending coefficientwise the notion of similarity among umbrae, any exponential formal power series

$$f(z) = 1 + \sum_{k \geq 1} a_k \frac{z^k}{k!} \tag{1.2}$$

can be umbrally represented by a formal power series of form (1.1) and so, if the sequence $\{a_k\}_{k \geq 0}$ is umbrally represented by the umbra α, we have $f(z) = E[e^{\alpha z}]$. We denote the formal power series in (1.2) by $f(\alpha, z)$.

As example, the generating functions of the augmentation umbra ϵ and the unity umbra u are

$$f(\epsilon, z) = 1, \qquad f(u, z) = e^z. \tag{1.3}$$

Generating functions (1.2) are *exponential* formal power series, to distinguish from the *ordinary* formal power series $1 + \sum_{k\geq 1} a_k z^k$.

The degree of a non-zero formal power series $f(z)$, say $deg(f(z))$ is the least integer k such that $a_k \neq 0$.

The set of all formal power series with real coefficients a_k forms a commutative integral domain, denoted by $\mathbb{R}[[z]]$, under the summation and the product among generating functions:

$$\left(1 + \sum_{k\geq 1} a_k \frac{z^k}{k!}\right) + \left(1 + \sum_{k\geq 1} b_k \frac{z^k}{k!}\right) = 1 + \sum_{k\geq 1} (a_k + b_k) \frac{z^k}{k!}$$
$$\left(1 + \sum_{k\geq 1} a_k \frac{z^k}{k!}\right) \left(1 + \sum_{k\geq 1} b_k \frac{z^k}{k!}\right) = 1 + \sum_{k\geq 1} c_k \frac{z^k}{k!},$$

where $c_k = \sum_{i=1}^{k} a_i b_{k-i}$, for all nonnegative integers $k \geq 1$.

Note that these operations obey the ordinary laws of algebra, as associativity and commutativity of addition and multiplication, distributivity of multiplication over addition and cancellation of multiplication.

We could define other operations among generating functions, under suitable hypothesis. As example, if $f(z) = 1 + \sum_{k\geq 1} a_k \frac{z^k}{k!}$ and $g(z) = 1 + \sum_{k\geq 1} b_k \frac{z^k}{k!}$ are elements of $\mathbb{R}[[z]]$ such that $f(z)g(z) = 1$, then we write $f(z) = g(z)^{-1}$ and $g(z)^{-1}$ exists if and only if $a_1 \neq 0$.

What is more, if $f(z) = 1 + \sum_{k\geq 1} a_k \frac{z^k}{k!}$, the formal derivative $f'(z)$ is the exponential formal power series such that

$$f'(z) = \sum_{k\geq 1} a_k \frac{z^{k-1}}{(k-1)!}.$$

It is easy to check that all familiar laws of differentiation that are formally well-defined continue to be valid for formal power series.

We thus have a formal calculus for formal power series, for which we do not take into account any question of being the radius of convergence positive or the coefficients undefined. More precisely, we have identities involving power series which are valid when the power series are regarded as functions, so that the variables are sufficiently small real numbers. Then, these identities continue to remain valid when regarded as identities among formal power series, provided the operations defined in the formulae are well-defined for formal power series.

Actually, in spite of generating functions are not interpreted as analytic objects, we need to put some additional structure on $\mathbb{R}[[z]]$, namely, the notion of *convergence* [58].

As example, the identity

$$\sum_{k \geq 0} \frac{(z+1)^k}{k!} = e \sum_{k \geq 0} \frac{z^k}{k!}$$

is valid at the function-theoretic level, since it states that $e^{(z+1)} = e - e^z$, but does not make sense as a statement involving formal power series. There is no formal procedure for writing $\sum_{k \geq 0}((z+1)^k)/k!$ as a member of $\mathbb{R}[[z]]$. For instance, the constant term of $\sum_{k \geq 0}((z+1)^k)/k!$ is $\sum_{k \geq 0} 1/k!$, whose interpretation as a member of $\mathbb{R}[[z]]$ involves the consideration of convergence.

Definition 1.3. The infinite series $\sum_{j \geq 0} f_j(z)$ converges if and only if

$$\lim_{j \to \infty} deg(f_j(z)) = \infty.$$

The most important application of Definition 1.3 is to composition of power series. Given any two formal power series $f(z) = 1 + \sum_{k \geq 0} a_k \frac{z^k}{k!}$ and $g(z)$ such that $g(0) = 0$, define the composition $f(g(z))$ to be the infinite sum $1 + \sum_{k \geq 1} a_k \frac{(g(z))^k}{k!}$.

Remark 1.4. The algebraic structure of formal power series is isomorphic to that of sequences endowed with the convolution product, each series corresponding to the sequence of its coefficients. Thus, equality of two formal power series is interpreted as the equality of their corresponding coefficients.

If the coefficients are defined only up to some finite m, then one works with sequences of m elements only. This means that formal power series are replaced by polynomials of degree m and operations like summation and multiplication among formal power series can be repeated in terms of these polynomials [59].

The umbral notation allows us to represent operations among generating functions through symbolic operations among umbrae. For example, given two distinct umbrae α and γ with generating functions $f(\alpha, z)$ and $f(\gamma, z)$, respectively, we have

$$f(\alpha, z)f(\gamma, z) = E[e^{(\alpha+\gamma)z}] = f(\alpha + \gamma, z),$$

that is, the product of exponential generating functions is umbrally represented by the sum of the corresponding umbrae.

Summation and difference among generating functions are umbrally represented by the so-called *disjoint sum* and *disjoint difference.*

Definition 1.5. Given two distinct umbrae α and γ, their disjoint sum $\alpha\dot{+}\gamma$ and disjoint difference $\alpha\dot{-}\gamma$ are such that

$$(\alpha\dot{+}\gamma)^k \simeq \begin{cases} u, & \text{if } k = 0 \\ \alpha^k + \gamma^k, & \text{if } k > 0 \end{cases} \quad (\alpha\dot{-}\gamma)^k \simeq \begin{cases} u, & \text{if } k = 0 \\ \alpha^k - \gamma^k, & \text{if } k > 0 \end{cases}. \tag{1.4}$$

By (1.4) and by definition of generating function, we have

$$\begin{aligned} E[e^{(\alpha\dot{\pm}\gamma)z}] &= 1 + \sum_{k\geq 1} E[(\alpha\dot{\pm}\gamma)^k]\frac{z^k}{k!} \\ &= 1 + \sum_{k\geq 1} \left(E[\alpha^k] \pm E[\gamma^k]\right)\frac{z^k}{k!} \\ &= 1 + \sum_{k\geq 1} E[\alpha^k]\frac{z^k}{k!} \pm \sum_{k\geq 1} E[\gamma^k]\frac{z^k}{k!} \\ &= 1 + \sum_{k\geq 1} E[\alpha^k]\frac{z^k}{k!} \pm \left[1 + \sum_{k\geq 1} E[\gamma^k]\frac{z^k}{k!} - 1\right], \end{aligned}$$

that is,

$$E[e^{(\alpha\dot{\pm}\gamma)z}] = f(\alpha, z) \pm [f(\gamma, z) - 1]. \tag{1.5}$$

Remark 1.6. The key point of the theory in dealing with the umbral calculus is the idea to associating a sequence of numbers to a sequence of symbols by means of the evaluation functional E.

This idea is familiar in probability theory, where the k-th element of a number sequence is the k-th moment of a random variable, under suitable hypothesis. From this point of view, the evaluation functional E plays the role of the expectation $\mathbb{E}$ of a random variable, while an umbra looks like the framework of a random variable, with no reference to any probability space. What is more, uncorrelated umbrae correspond to independent random variables and similarity among umbrae is the umbral counterpart of the identical distribution property among random variables.

The way to recognize the umbra corresponding to a random variable is to characterize the sequence of moments $\{a_k\}_{k\geq 0}$. When the sequence exists, this can be done by comparing the moment generating function of the random variable with the generating function of the umbra.

As example, the augmentation umbra ϵ is the umbral counterpart of a random variable X such that $P(X=0)=1$, while the unity umbra u corresponds to the random variable X such that $P(X=1)=1$.

Some special umbrae, very useful in the following, are given in the following.

Singleton umbra. The *singleton umbra* χ is the umbra with moments

$$E[\chi^k] = \begin{cases} 1, & \text{if } k=1 \\ 0, & \text{if } k>1. \end{cases}$$

Hence, the generating function of the singleton umbra is

$$f(\chi, z) = 1 + z. \tag{1.6}$$

Remark 1.7. The singleton umbra is an umbra with no probabilistic counterpart. Indeed, suppose X to be the random variable whose sequence of moments is umbrally represented by moments of the singleton umbra χ. We have

$$\mathbb{E}[X^0] = E[\chi^0] = 1,\ \mathbb{E}[X] = E[\chi] = 1,\ \mathbb{E}[X^k] = E[\chi^k] = 0,$$

for all $k \geq 2$.

So, the expected value of X is $\mathbb{E}[X]=1$. This allows us to compute the variance of the random variable

$$\begin{aligned} Var(X) &= \mathbb{E}[(X-\mathbb{E}[X])^2] = \mathbb{E}[(X-1)^2] = \mathbb{E}[X^2+1-2X] \\ &= \mathbb{E}[X^2]+1-2\mathbb{E}[X] = 1-2 = -1. \end{aligned}$$

Therefore, the singleton umbra χ should be the umbral counterpart of a random variable X with negative variance, but this is not allowed, by definition of variance of a random variable.

Nevertheless, the singleton umbra χ plays a central role in the umbral setting, since it will appear in the treatment of cumulants and it will be fundamental in the symbolic representation of Lévy processes.

Bell umbra. The umbra β is said to be the Bell umbra if

$$E[(\beta)_k] = 1, \quad k = 1, 2, \ldots$$

Its moments are the Bell numbers, that is the number of partitions of a finite non-empty set with k elements or the k-th coefficient in the Taylor series expansion of the function $\exp\{e^z - 1\}$ [19].

Thus, the generating function of the Bell umbra β is

$$f(\beta, z) = \exp\{e^z - 1\}. \tag{1.7}$$

Remark 1.8. The Bell umbra can be viewed as a Poisson random variable with parameter 1, since the generating function in (1.7) is equal to the moment generating function of the Poisson random variable with parameter 1.

1.2 Auxiliary umbrae

1.2.1 The dot-product $n.\alpha$

Consider the set $\{\alpha', \alpha'', \ldots, \alpha'''\}$ of n distinct umbrae, similar to the umbra α but uncorrelated. Let us introduce the new symbol $n.\alpha$ representing the summation of these umbrae, that is,

$$n.\alpha = \alpha' + \alpha'' + \cdots + \alpha'''. \tag{1.8}$$

We call the umbra $n.\alpha$ the *dot-product* of the nonnegative integer n and the umbra α. Moreover, we assume that $0.\alpha \equiv \epsilon$.

Remark 1.9. The umbral calculus obtained by adding auxiliary umbrae to the set $\mathcal{A}$ is said to be *saturated* [46]. More precisely, a saturated umbral calculus with base alphabet $\mathcal{A}$ is an umbral calculus on an alphabet $\mathcal{A} \cup \mathcal{B}$, where the symbols in $\mathcal{B}$ are auxiliary umbrae.

From now on, we will always deal with a saturated umbral calculus.

Next proposition easily follows from the definition of the dot-product.

Proposition 1.10. *We have*

(i) $n.\alpha \equiv n.\gamma$ *for some integer* $n \neq 0$ *if and only if* $\alpha \equiv \gamma$;

(ii) if $c \in \mathbb{R}$, *then* $n.(c\alpha) \equiv c(n.\alpha)$, *for any nonnegative integer* n;

(iii) $n.(m.\alpha) \equiv (nm).\alpha \equiv m.(n.\alpha)$, *for any two nonnegative integers* n, m;

(iv) (distributive property of the summation with respect to the dot-product) $(n+m).\alpha \equiv n.\alpha + m.\alpha'$*, for any two nonnegative integers* n, m *and for any two similar umbrae* α *and* α'*.*

(v) (distributive property of the dot-product with respect to the summation of umbrae) $n.(\alpha+\gamma) \equiv n.\alpha + n.\gamma$*, for any nonnegative integer* n *and for any two distinct umbrae* α *and* γ*.*

Remark 1.11. Since $n.\alpha$ and $m.\alpha$ are two different symbols, they represent two uncorrelated auxiliary umbrae, so it is unnecessary to use the two distinct symbols α and α'. The use of two distinct symbols turns to be fundamental when $n = m$.

The generating function of the dot-product $n.\alpha$ is

$$f(n.\alpha, z) = [f(\alpha, z)]^n, \tag{1.9}$$

where $f(\alpha, z)$ is the generating function of the umbra α.

The moments of the dot-product $n.\alpha$ are (see [29] for further details) for $k \geq 1$

$$E[(n.\alpha)^k] = \sum_{i=1}^{k} (n)_i B_{k,i}(a_1, a_2, \ldots, a_{k-i+1}), \tag{1.10}$$

where $B_{k,i}(a_1, a_2, \ldots, a_{k-i+1})$ are the partial exponential Bell polynomials [19], $(n)_i = n(n-1)\cdots(n-i+1)$ is the lower factorial and $a_j = E[\alpha^j]$, for all $j = 1, \ldots, k-i+1$.

Remark 1.12. If we replace α with the Bell umbra β, the resulting umbra $n.\beta$ is the sum of n similar uncorrelated Bell umbrae, likewise in probability theory where a Poisson random variable with parameter n can be viewed as the sum of n independent and identically distributed Poisson random variables with parameter 1.

Remark 1.13. Moments of $n.\alpha$ can be expressed using the notions of integer partition and dot-power. As regards the former, recall that, given a nonnegative integer k, a partition λ of k, in symbols $\lambda \vdash k$, is a sequence of weakly decreasing positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ such that $\sum_{j=1}^m \lambda_j = k$. The integers λ_j are named *parts* of λ. The *length* of λ is the number of its parts and it is indicated by $l(\lambda)$.

A different notation is $\lambda = (1^{r_1}, 2^{r_2}, \ldots)$, where r_j is the numbers of parts of λ equal to j and $r_1 + r_2 + \cdots = l(\lambda)$.

As regards the latter, we introduce the auxiliary umbra $\alpha^{.n}$ denoting the product $\alpha'\alpha'' \cdots \alpha'''$ of n umbrae, similar to the umbra α but uncorrelated, that is,

$$\alpha^{.n} = \alpha'\alpha'' \cdots \alpha'''$$

and we call it the *dot-power* of n and the umbra α. Moments of $\alpha^{.n}$ can be easily recovered from its definition. Indeed, if the umbra α represents the sequence $1, a_1, a_2, \ldots$, then $E[(\alpha^{.n})^k] = a_k^n$ for all nonnegative integers k and n.

By using an umbral version of the well-known multinomial expansion theorem, we have

$$(n.\alpha)^k \simeq \sum_{\lambda \vdash k} (n)_{l(\lambda)} d_\lambda \alpha_\lambda, \tag{1.11}$$

for all $k \geq 0$, where the sum is over all partitions $\lambda = (1^{r_1}, 2^{r_2}, \ldots)$ of the integer k, $(n)_{l(\lambda)} = 0$ for $l(\lambda) > n$,

$$d_\lambda = \frac{1}{r_1! r_2! \cdots} \frac{k!}{(1!)^{r_1}(2!)^{r_2} \cdots} \quad \text{and} \quad \alpha_\lambda = [\alpha']^{.r_1}[\alpha'']^{.r_2} \cdots . \tag{1.12}$$

1.2.2 The dot-product $t.\alpha$

Equation (1.10) suggests a way to define new auxiliary umbrae depending on a real parameter t. Indeed, the k-th moment of the dot-product $n.\alpha$ is a polynomial, say $q_k(n)$, of degree k in the variable n. The integer n can be replaced by any $t \in \mathbb{R}$ so that, for all $k \geq 1$,

$$q_k(t) = \sum_{i=1}^{k} (t)_i B_{k,i}(a_1, a_2, \ldots, a_{k-i+1}) \tag{1.13}$$

still denotes polynomials of degree k in t.

We introduce the new symbol $t.\alpha$ and we call it the dot-product of the real number t and the umbra α as the auxiliary umbra with $E[(t.\alpha)^k] = q_k(t)$, for all nonnegative integers k. The generating function of $t.\alpha$ is

$$f(t.\alpha, z) = [f(\alpha, z)]^t, \tag{1.14}$$

where $f(\alpha, z)$ is the generating function of the umbra α.

Proposition 1.10 can be easily generalized to the dot-product $t.\alpha$.

Proposition 1.14. *We have*

(i) if $t.\alpha \equiv t.\gamma$ for some $t \in \mathbb{R}$ with $t \neq 0$, then $\alpha \equiv \gamma$;

(ii) if $c \in \mathbb{R}$*, then* $t.(c\alpha) \equiv c(t.\alpha)$*, for any* $t \in \mathbb{R}$*;*

(iii) $t.(s.\alpha) \equiv (ts).\alpha \equiv s.(t.\alpha)$*, for any two* $t, s \in \mathbb{R}$*;*

(iv) (distributive property of the summation with respect to the dot-product) $(t + s).\alpha \equiv t.\alpha + s.\alpha'$*, for any two* $t, s \in \mathbb{R}$ *and for any two similar umbrae* α *and* α'*.*

(v) (distributive property of the dot-product with respect to the summation of umbrae) $t.(\alpha+\gamma) \equiv t.\alpha+t.\gamma$*, for any* $t \in \mathbb{R}$ *and for any two distinct umbrae* α *and* γ*.*

As examples of dot-product, we can choose first of all the Bell umbra β instead of α. The umbra $t.\beta$ is the *Bell polynomial umbra*, such that $(t.\beta)_k \simeq t^k$, for all $k \geq 1$, . We have [29]

$$E[(t.\beta)^k] = \sum_{i=0}^{k} S(k,i)t^i = \Phi_k(t), \tag{1.15}$$

for all $k \geq 0$, where $\Phi_k(t)$ are umbral polynomials with a statistical origin, known in the literature as *exponential polynomials* or *Touchard polynomials* [60].

By equations (1.7) and (1.14), the generating function of the Bell polynomial umbra is

$$f(t.\beta, z) = \exp\{t[e^z - 1]\}. \tag{1.16}$$

Remark 1.15. The Bell polynomial umbra $t.\beta$ can be viewed as a Poisson random variable with parameter t. Indeed, the generating function of the Bell polynomial umbra is $\exp\{t(e^z - 1)\}$, so that $P(e^x) = \exp\{t(e^x - 1)\}$, where $P(x)$ is the probability generating function, and therefore $P(y) = \exp\{t(y - 1)\}$.

Afterwards, we can choose the singleton umbra χ instead of α. We obtain the auxiliary umbra $t.\chi$, such that

$$E[(t.\chi)^k] = \sum_{i=0}^{k} s(k,i)t^i = (t)_k, \text{ for all } k \geq 0,$$

where $s(k,i)$ are the Stirling numbers of first kind.

Remark 1.16. In equivalence (1.11) we can replace the nonnegative integer n by $t \in \mathbb{R}$, so we obtain an alternative way of representing moments of dot-products very useful in some applications

$$(t.\alpha)^k \simeq \sum_{\lambda \vdash k} (t)_{l(\lambda)} d_\lambda \alpha_\lambda. \tag{1.17}$$

The inverse of an umbra. Given an umbra α, the umbra $-1.\alpha$ such that

$$-1.\alpha + \alpha \equiv \epsilon \tag{1.18}$$

is called the *inverse* of α.

Let us observe that the inverse of an umbra α is the dot-product $t.\alpha$ with $t = -1$. Hence, by equivalence (1.9), if $f(\alpha, z)$ is the generating function of the umbra α, the generating function of the inverse umbra $-1.\alpha$ is

$$f(-1.\alpha, z) = \frac{1}{f(\alpha, z)}, \tag{1.19}$$

We can generalize the dot-product $t.\alpha$ by replacing the umbra α with its inverse $-1.\alpha$, so we have

$$t.(-1.\alpha) \equiv -t.\alpha, \tag{1.20}$$

thanks to Proposition 1.10-(iii).

By equivalence (1.11), moments of $-n.\alpha$ are

$$(-n.\alpha)^k \simeq \sum_{\lambda \vdash k} (-n)_{l(\lambda)} d_\lambda \alpha_\lambda \simeq \sum_{\lambda \vdash k} (-1)^{l(\lambda)} \langle n \rangle_{l(\lambda)} d_\lambda \alpha_\lambda, \tag{1.21}$$

for all $k \geq 0$, where $\langle n \rangle_{l(\lambda)}$ is the rising factorial of the integer n.

The following examples give an idea of the probabilistic meaning of the inverse of an umbra.

Gamma random variable. A Gamma random variable $X^{(a,c)}$ with shape parameter $a > 0$ and scale parameter $c > 0$ has moment generating function

$$M(z) = \frac{1}{(1 - cz)^a}.$$

The Gamma random variable is related to the inverse of the singleton umbra χ. Indeed,

$$\begin{aligned} \frac{1}{(1-cz)^a} &= (1+z)^{-a}\big|_{z=-cz} = f(-a.\chi, -cz) \\ &= f(-1.a.\chi, -cz) = f(-c(-1.a.\chi), z) \\ &= f(-1.a.(-c\chi), z). \end{aligned}$$

Bernoulli umbra. The Bernoulli umbra ι is the umbra whose moments are the Bernoulli numbers B_k, for all $k \geq 0$ [19]. Its generating function is

$$f(\iota, z) = \frac{z}{e^z - 1}. \tag{1.22}$$

The Bernoulli umbra was first introduced in [44] as the umbra satisfying the umbral equivalence $\iota + u \equiv -\iota$.

Remark 1.17. By comparing the generating function (1.22) and the moment generating function of a uniform random variable on $[0, 1]$, say $U([0, 1])$, we have

$$M(z) = \frac{e^z - 1}{z} = f(\iota, z)^{-1} = f(-1.\iota, z),$$

thanks to (1.19). Hence, we have obtained that a uniform random variable on $[0, 1]$ is umbrally represented by the inverse of the Bernoulli umbra.

More in general, a uniform random variable on the interval $[a, b]$ has moment generating function

$$M(z) = \frac{e^{z\,b} - e^{z\,a}}{z(b-a)} = e^{z\,a} \left[\frac{e^s - 1}{s}\right]_{s=t(b-a)}.$$

From an umbral point of view, $e^{az} = f(au, z)$ and

$$\left.\frac{e^s - 1}{s}\right|_{s=z(b-a)} = f(-1.\iota, z(b-a)) = f((b-a)(-1.\iota), z).$$

Therefore,

$$M(z) = f(au, z) f((b-a)(-1.\iota), z) = f(at + (b-a)(-1.\iota), z),$$

that is, a uniform random variable on $[a, b]$ is umbrally represented by the umbra $a + (b-a)(-1.\iota)$.

Euler umbra. The Euler umbra η is the umbra with moments the Euler numbers $\mathfrak{E}_k$, for all $k \geq 0$ [19] and generating function

$$f(\eta, z) = \frac{2e^z}{e^{2z} + 1}. \tag{1.23}$$

Remark 1.18. From a probabilistic point of view, let us consider the inverse of the Euler umbra $-1.\eta$, whose generating function is

$$f(-1.\eta, z) = \frac{e^{2z}+1}{2e^{z}} = \frac{e^{2z}+1}{2}\, e^{-z}.$$

Since $f(u, z) = e^{z}$, then

$$\frac{e^{2z}+1}{2} = f(-1.\eta, z)\, f(u, z) = f(-1.\eta + u, z). \tag{1.24}$$

If we replace z by $z/2$ in (1.24), we have

$$\frac{e^{2(z/2)}+1}{2} = f\left(-1.\eta + u, \frac{z}{2}\right) = f\left(\frac{1}{2}\Big[-1.\eta + u\Big], z\right),$$

that is,

$$f\left(\frac{1}{2}\Big[-1.\eta + u\Big], z\right) = \frac{e^{z}+1}{2}, \tag{1.25}$$

which is the moment generating function of a Bernoulli random variable with parameter $1/2$.

Hence, we have proved that a Bernoulli random variable of parameter $1/2$ is umbrally represented by the umbra $\frac{1}{2}\Big[-1.\eta + u\Big]$, that is, the umbra $-1.\eta$ is the umbral counterpart of the random variable $Y = 2X - 1$, with X a Bernoulli random variable with parameter $1/2$.

Once we have introduced the auxiliary umbra $t.\alpha$, we can consider the umbra $-t.\alpha$.

As done in (1.21), we can consider the auxiliary umbra which is the dot-product of the real parameter t and the inverse umbra $-1.\alpha$, in symbols, $t.(-1.\alpha) \equiv -t.\alpha$, by virtue of Proposition 1.14-(iii).

From equivalence (1.17), we have

$$(-t.\alpha)^k \simeq \sum_{\lambda \vdash k} (-t)_{l(\lambda)} d_\lambda \alpha_\lambda \simeq \sum_{\lambda \vdash k} (-1)^{l(\lambda)} \langle t \rangle_{l(\lambda)}\, d_\lambda \alpha_\lambda, \tag{1.26}$$

where $\langle t \rangle_{l(\lambda)}$ denotes the rising factorial.

1.2.3 The dot-product among umbrae

Similarly to what has been done for the auxiliary umbra $t.\alpha$, we define a dot-product among umbrae.

If the parameter t in (1.13) is replaced by any umbra γ, then we have

$$q_k(\gamma) = \sum_{i=1}^{k} (\gamma)_i \, B_{k,i}(a_1, a_2, \ldots, a_{k-i+1}). \tag{1.27}$$

The umbra $\gamma.\alpha$ such that $E[(\gamma.\alpha)^k] = E[q_k(\gamma)]$ is called the dot-product of the umbrae α and γ.

The generating function of the dot-product $\gamma.\alpha$ is [29]

$$f(\gamma.\alpha, z) = f(\gamma, \log[f(\alpha, z)]). \tag{1.28}$$

Remark 1.19. From a probabilistic point of view, the dot-product $\gamma.\alpha$ corresponds to a random sum $S_N = X_1 + \cdots + X_N$, where $X_1, \ldots, X_N$ are independent and identically distributed random variables and N is a discrete random variable. Therefore, if α umbrally represents the random variables X_i, for all $i = 1, \ldots, N$ and γ umbrally represents the random variable N, the random sum S_N is umbrally represented by the auxiliary umbra $\gamma.\alpha$.

In particular, if we replace the umbra α by the Bell umbra β, then the auxiliary umbra $\gamma.\beta$ represents a randomized Poisson random variable with random parameter Y, umbrally represented by the umbra γ.

Remark 1.20. In equivalence (1.11) we can replace the nonnegative integer n by $\gamma \in \mathcal{A}$, so we obtain an alternative way of representing moments of dot-products

$$(\gamma.\alpha)^k \simeq \sum_{\lambda \vdash k} (\gamma)_{l(\lambda)} d_\lambda \alpha_\lambda. \tag{1.29}$$

Proposition 1.21. *We have*

(i) $\xi.\alpha \equiv \xi.\gamma$ *if and only if* $\alpha \equiv \gamma$;

(ii) if $c \in \mathbb{R}$, *then* $\xi.(c\alpha) \equiv c(\xi.\alpha)$ *for any two distinct umbrae* ξ *and* α;

(iii) $\xi.(\gamma.\alpha) \equiv (\xi.\gamma).\alpha$.

There are some special auxiliary umbrae, which will play a central role in the following.

α-partition umbra. If we replace, in the dot-product $\gamma.\alpha$, the umbra γ with the Bell umbra β, we have the so-called *α-partition umbra* $\beta.\alpha$, that is, the umbra such that

$$E[(\beta.\alpha)^k] = Y_k(a_1, \ldots, a_k), \tag{1.30}$$

for $k = 1, 2, \ldots$, where Y_k are the complete exponential Bell polynomials [19]. By using (1.7) and (1.28), the generating function of the α-partition umbra is

$$f(\beta.\alpha, z) = \exp\{f(\alpha, z) - 1\}. \tag{1.31}$$

Proposition 1.22. *For $\alpha, \gamma \in \mathcal{A}$, we have*

$$\beta.\alpha + \beta.\gamma \equiv \beta.(\alpha \dot{+} \gamma),$$

where β is the Bell umbra.

Proof. Via generating function, we have

$$\begin{aligned} f(\beta.\alpha + \beta.\gamma, z) &= f(\beta.\alpha, z)\, f(\beta.\gamma, z) \\ &= \exp\{f(\alpha, z) - 1\} \exp\{f(\gamma, z) - 1\} \\ &= \exp\{f(\alpha, z) - 1 + f(\gamma, z) - 1\} \\ &= \exp\{(f(\alpha, z) + f(\gamma, z) - 1) - 1\} \\ &= \exp\{f(\alpha \dot{+} \gamma, z) - 1\} \\ &= f(\beta.(\alpha \dot{+} \gamma), z), \end{aligned}$$

where the last two equalities hold by virtue of (1.5) and (1.31), respectively. □

Remark 1.23. A compound Poisson random variable with parameter 1 is a random sum $S_N = X_1 + X_2 + \cdots + X_N$ where N has a Poisson distribution with parameter 1 and X_i are independent and identically distributed random variables, for $i = 1, \ldots, N$ [31].

The definition of α-partition umbra fits perfectly this probabilistic notion, indeed the parameter N is umbrally represented by the Bell umbra β and, if we denote by α the umbra representing the random variables X_i, for $i = 1, \ldots, N$, the random sum S_N is umbrally represented by the dot-product $\beta.\alpha$, thanks to definition of dot-product.

The name "partition umbra" has also a probabilistic ground. Indeed the parameter of a Poisson random variable is usually denoted by λt with t representing a time interval, so that when this interval is partitioned into non-overlapping ones, their contributions are stochastically independent and add to S_N. This last circumstance is umbrally expressed by the relation

$$(t + s).\beta.\alpha \equiv t.\beta.\alpha + s.\beta.\alpha,$$

that also assures the binomial property for the polynomial sequence umbrally represented by $t.\beta.\alpha$ [29].

An interesting example of partition umbra is given by replacing α with the singleton umbra χ. By virtue of (1.31) and (1.6), we have

$$f(\beta.\chi, z) = \exp\{(1+z)-1\} = e^z = f(u, z),$$

so $\beta.\chi \equiv u$.

If we consider now the dot-product $\gamma.\alpha$ with the umbrae γ and α replaced by the singleton umbra χ and the Bell umbra β, respectively, we have the auxiliary umbra $\chi.\beta$.

Via (1.28), we have

$$f(\chi.\beta, z) = f(\chi, \log(f(\beta, z))) = 1 + e^z - 1 = e^z = f(u, z)$$

so $\chi.\beta \equiv u$ and the connection between the singleton umbra χ and the Bell umbra β

$$\beta.\chi \equiv u \equiv \chi.\beta \tag{1.32}$$

is made clear.

Gaussian umbra. Let us introduce the umbra δ such that

$$f(\delta, z) = 1 + \frac{z^2}{2}. \tag{1.33}$$

Hence, the umbra δ is such that its only non-zero moment is the second one, that is

$$E[\delta^k] = \begin{cases} 1, & \text{if } k = 2 \\ 0, & \text{if } k \neq 2 \end{cases} \tag{1.34}$$

Definition 1.24. Given a real parameter m and a nonnegative real parameter s, the *Gaussian umbra* is the umbra $m + \beta.(s\delta)$ with generating function

$$f(m + \beta.(s\delta), z) = \exp\left\{mz + \frac{1}{2}s^2z^2\right\}. \tag{1.35}$$

The following proposition gives the expression of the k-th moment of the Gaussian umbra.

Proposition 1.25. *For k=1,2,... we have*

$$E\left[\{m + \beta.(s\delta)\}^k\right] = \sum_{i=0}^{\lfloor k/2 \rfloor} \left(\frac{s^2}{2}\right)^i \frac{(k)_{2i}}{i!} m^{k-2i}. \tag{1.36}$$

Proof. By using the binomial expansion and (1.29), we have

$$(m+\beta.(s\delta))^k \simeq \sum_{j=0}^{k}\binom{k}{j} m^{k-j}\,[\beta.(s\delta)]^j \simeq \sum_{j=0}^{k}\binom{k}{j} m^{k-j}\sum_{\lambda\vdash j} d_\lambda (s\delta)_\lambda. \tag{1.37}$$

Suppose $\lambda=(1^{r_1},2^{r_2},\ldots)$, then

$$(s\delta)_\lambda \equiv (s\delta')^{.r_1}[s^2(\delta'')^2]^{.r_2}\cdots \equiv s^{(r_1+r_2+\cdots)}\,(\delta')^{.r_1}(\delta'')^{.r_2}\cdots \equiv s^j\delta_\lambda.$$

On the other hand, due to the definition of the umbra δ, we have $E[\delta_\lambda]\neq 0$ if and only if the partition λ is of type (2^{r_2}).

The result depends on the parity of the integer j. If $j=2i+1$, that is, j is odd, then there does not exist any partition of this type and so $\big[\beta.(s\delta)\big]^j \simeq 0$.

If j is even, say $j=2i$, there exists a unique partition of type (2^{r_2}) which corresponds to $r_2=i$. For this partition, we have $d_\lambda=(2i)!/((2!)^i\, i!)$ so that

$$\sum_{\lambda\vdash 2i} d_\lambda (s\delta)_\lambda \simeq (2i)!\,\frac{s^{2i}}{(2!)^i\, i!}. \tag{1.38}$$

Hence, we put $j=2i$ to wit $i=j/2$, so that, when j ranges over $\{0,\ldots,k\}$, the new variable i ranges over $\{0,\ldots,\lfloor k/2\rfloor$ and replace (1.38) in (1.37). We have

$$\begin{aligned}(m+\beta.(s\delta))^k &\simeq \sum_{i=0}^{\lfloor k/2\rfloor}\frac{k!}{(2i!)(k-2i)!}m^{k-2i}\frac{(2i!)s^{2i}}{(2!)^i i!}\\ &\simeq \sum_{i=0}^{\lfloor k/2\rfloor}\frac{k!}{(k-2i)!i!}m^{k-2i}\left(\frac{s^2}{2}\right)^i\\ &\simeq \sum_{i=0}^{\lfloor k/2\rfloor}\left(\frac{s^2}{2}\right)^i\frac{(k)_{2i}}{i!}m^{k-2i}.\end{aligned}$$

□

Remark 1.26. In the Proposition above we have proved by umbral tools that moments of the Gaussian umbra can be expressed by using a particular sequence of orthogonal polynomials, the *Hermite polynomials.* We will return on this family of polynomials later on.

Remark 1.27. Let us observe that, by virtue of first equivalence in (1.32), we have

$$m + \beta.(s\delta) \equiv u.m + \beta.(s\delta) \equiv \beta.\chi.m + \beta.(s\delta) \equiv \beta.(m\chi \dot{+} s\delta),$$

where the last equivalence results thanks to Proposition 1.22.

Hence, the Gaussian umbra can be considered as an example of α-partition umbra, with $\alpha \equiv m\chi \dot{+} s\delta$.

Remark 1.28. The Gaussian umbra $m+\beta.(s\delta)$ owes its name to the relation between its generating function (1.35) and the moment generating function of a Gaussian random variable $\mathcal{N}(m, s^2)$ with mean m and variance s^2. Since these two expressions are equal, then the Gaussian umbra is the umbral counterpart of $\mathcal{N}(m, s^2)$.

Composition umbra. The dot-product can be generalized. If in the dot-product $\gamma.\alpha$ we replace γ by the polynomial Bell umbra $t.\beta$, then we have the *polynomial partition umbra*

$$(t.\beta).\alpha \equiv t.\beta.\alpha.$$

The previous equivalence holds thanks to Proposition 1.21 - (iii), with the umbra ξ replaced by $t.u$, so $(t.u).(\beta.\alpha) \equiv (t.u.\beta).\alpha$. The result follows, since $u.\beta \equiv \beta.u \equiv \beta$.

By equivalence (1.31), we have

$$f(t.\beta.\alpha, z) = \exp\{t[f(\alpha, z) - 1]\}. \qquad (1.39)$$

Remark 1.29. From a probabilistic point of view, the polynomial partition umbra corresponds to a compound Poisson random variable of parameter t.

If the umbra γ is replaced by the dot-product $\gamma.\beta$, then we have the *composition umbra* of α and γ, represented by $\gamma.\beta.\alpha$. Its generating function is the composition of $f(\alpha, z)$ and $f(\gamma, z)$, that is,

$$f(\gamma.\beta.\alpha, z) = f(\gamma, f(\alpha, z) - 1). \qquad (1.40)$$

The moments are, for all $k \geq 0$, [29]

$$E[(\gamma.\beta.\alpha)^k] = \sum_{i=1}^{k} E[\gamma^i] B_{k,i}(a_1, a_2, \ldots, a_{k-i+1}). \qquad (1.41)$$

Remark 1.30. The umbra $\gamma.\beta.\alpha$ generalizes the concept of a random sum of independent and identically distributed compound Poisson random variables with parameter 1, indexed by an integer random variable Y, that is, $S_N = X_1 + \cdots + X_N$ is a randomized compound Poisson random variable with random parameter Y.

More precisely, let α and γ be the umbrae representing the random variable X_i, for all $i = 1, \ldots, N$ and the random variable Y, respectively. Then, the Poisson random variable N is umbrally represented by the umbra $\gamma.\beta$ and so the random sum S_N is umbrally represented by $(\gamma.\beta).\alpha \equiv \gamma.\beta.\alpha$.

As example of composition umbra, the *compositional inverse umbra* $\alpha^{<-1>}$ of an umbra α is such that

$$\alpha^{<-1>}.\beta.\alpha \equiv \chi \equiv \alpha.\beta.\alpha^{<-1>}. \tag{1.42}$$

In particular, we have $f(\alpha^{<-1>}, z) = f^{<-1>}(\alpha, z)$, where $f^{<-1>}$ denotes the compositional inverse of $f(\alpha, z)$, which satisfies

$$f(\alpha^{<-1>}, f(\alpha, z) - 1) = 1 + z = f(\alpha, f(\alpha^{<-1>}, z) - 1). \tag{1.43}$$

If we choose in particular $\alpha \equiv u$, we have

$$u.\beta.u^{<-1>} \equiv \chi \equiv u^{<-1>}.\beta.u. \tag{1.44}$$

On the other hand, $u.\beta \equiv \beta \equiv \beta.u$, so we have

$$\beta.u^{<-1>} \equiv \chi \equiv u^{<-1>}.\beta. \tag{1.45}$$

Chapter 2

Symbolic representation of univariate Lévy processes

2.1 Cumulants

Suppose that X is a real random variable with moment generating function

$$M(z) = \mathbb{E}[e^{zX}] = 1 + \sum_{k \geq 1} a_k \frac{z^k}{k!},$$

with $a_k = \mathbb{E}[X^k]$ the k-th moment of X, for all nonnegative integers k.

Consider the composition of the moment generating function of X and the logarithmic function $\log(z)$, which has a power series expansion. Recalling that the composition of functions with power series expansion is a function with power series expansion itself [19] and by observing that $M(0) = 1 \neq 0$, we have

$$\log(M(z)) = \sum_{k \geq 1} c_k \frac{z^k}{k!}. \tag{2.1}$$

We set

$$K(z) = \log(M(z)). \tag{2.2}$$

The coefficients $\{c_k\}_{k \geq 1}$ in (2.1) are called the *cumulants* of the random variable X and the expansion (2.1) is called the *cumulant generating function* of X.

Roughly speaking, just as the moment generating function $M(z)$ of a random variable X "generates" its moments, the logarithm of $M(z)$ "generates" its cumulants.

Remark 2.1. We give some examples of cumulants related to the most known distributions.

First of all, suppose $X \sim N(m, s^2)$ is the Gaussian random variable with mean m and variance s^2. Since $M(z) = \exp\{mz + s^2z^2/2\}$, then

$$K(z) = mz + \frac{s^2z^2}{2}$$

and so, by the uniqueness of the coefficients in (2.1),

$$c_1 = m, \ c_2 = s^2, \ c_k = 0, \ \text{for all } k \geq 3.$$

Then, suppose $X \sim Po(\lambda)$ is the Poisson random variable with intensity parameter $\lambda > 0$. Since $M(z) = \exp\{\lambda(e^z - 1)\}$, then

$$K(z) = \lambda(e^z - 1) = \sum_{k\geq 1} \lambda \frac{z^k}{k!}.$$

Evidently, by the uniqueness of the coefficients in (2.1), we have

$$c_k = \lambda, \ \text{for all } k \geq 1.$$

Finally, suppose $X \sim Gamma(1, r)$ is the Gamma random variable with scale parameter 1 and shape parameter $r > 0$. Since $M(z) = (1 - z)^{-r}$, we have

$$K(z) = \log[(1-z)^{-r}] = -r\log(1-z) = r\sum_{k\geq 1}\frac{z^k}{k} = r\sum_{k\geq 1}(k-1)!\frac{z^k}{k!}.$$

Hence, by the uniqueness of the coefficients in (2.1), we have

$$c_k = r(k-1)!, \ \text{for all } k \geq 1.$$

Cumulants share some interesting and useful algebraic properties, which can be summarized in the following theorem

Theorem 2.2. *For all $C \in \mathbb{R}$ and for all $k \geq 1$, we have*

(i) (Homogeneity property) $c_k(CX) = C^k c_k(X)$;

(ii) (Additivity property) if X and Y are two independent random variables, then $c_k(X + Y) = c_k(X) + c_k(Y)$;

(iii) (Semi-invariance property)

$$c_k(X+C) = \begin{cases} c_k(X)+C, & \text{if } k=1 \\ c_k(X), & \text{if } k>1. \end{cases}$$

Remark 2.3. Property (ii) in Theorem 2.2 is also called accumulate cumulants property. This explains the reason for the name *cumulants*.

It is well-known that there exists a close relationship between moments and cumulants of a random variable. In particular, for a random variable having moments $a_1, \ldots, a_k$ and cumulants $c_1, \ldots, c_k$, we have

$$c_k = \sum_{\lambda \vdash k} \tilde{d}_\lambda a_\lambda \text{ and } a_k = \sum_{\lambda \vdash k} d_\lambda c_\lambda, \tag{2.3}$$

where the sums are over all partitions $\lambda = (1^{r_1}, 2^{r_2}, \ldots,)$ of the integer k with $l(\lambda) = r_1 + r_2 + \cdots$ and

$$a_\lambda = \prod_{j \in \lambda} a_j, \quad c_\lambda = \prod_{j \in \lambda} c_j,$$
$$\tilde{d}_\lambda = d_\lambda (-1)^{l(\lambda)-1} (l(\lambda)-1)!$$

and d_λ defined in (1.12)

2.2 Symbolic treatment of univariate cumulants

Definition 2.4. Given an umbra α with generating function $f(\alpha, z)$, the α-cumulant umbra κ_α is defined by

$$\kappa_\alpha \equiv \chi.\alpha,$$

where χ is the singleton umbra.

Remark 2.5. The α-cumulant umbra is another example of auxiliary umbra, since it is defined as the dot-product of the singleton umbra χ and an umbra α. By virtue of (1.29), moments of the α-cumulant umbra are

$$(\chi.\alpha)^k \simeq \sum_{\lambda \vdash k} (-1)^{l(\lambda)-1} (l(\lambda)-1)! \, d_\lambda \, \alpha_\lambda, \tag{2.4}$$

since $E[(\chi)_k] = (-1)^k (k-1)!$, for all nonnegative integers $k \geq 1$.

Thanks to (1.28) and (1.6), the generating function of the α-cumulant umbra is

$$f(\kappa_\alpha, z) = f(\chi, \log(f(\alpha, z))) = 1 + \log(f(\alpha, z)). \tag{2.5}$$

Remark 2.6. The expression of the generating function of κ_α agrees with equation (2.1). This means that if the umbra α is the umbral counterpart of the random variable X, its cumulants are umbrally represented by κ_α.

Examples of cumulant umbrae If $\alpha \equiv \epsilon$, since $\chi.\epsilon \equiv \epsilon$, the umbra ϵ is the cumulant umbra of itself, that is, $\kappa_\epsilon \equiv \epsilon$.

If $\alpha \equiv u$, since $\chi.u \equiv \chi$, the umbra χ is the cumulant umbra of the unity umbra u, that is, $\kappa_u \equiv \chi$.

If $\alpha \equiv \beta$, since $\chi.\beta \equiv u$, the unity umbra u is the cumulant umbra of the Bell umbra β, that is, $\kappa_\beta \equiv u$.

If $\alpha \equiv \chi$, the cumulant umbra of the singleton umbra χ is given by $\kappa_\chi \equiv u^{<-1>}$, by virtue of the first equivalence in (1.45).

The algebraic properties in Theorem 2.2 can be easily recovered by umbral tools

Proposition 2.7. *Given two distinct umbrae $\alpha, \gamma \in \mathcal{A}$ and for any $C \in \mathbb{R}$, we have*

(i) (Homogeneity property) $\kappa_{C\alpha} \equiv C\kappa_\alpha$;

(ii) (Additivity property) $\kappa_{\alpha+\gamma} \equiv \kappa_\alpha \dot{+} \kappa_\gamma$;

(iii) (Semi-invariance property) $\kappa_{\alpha+C.u} \equiv \kappa_\alpha \dot{+} C\kappa_u$.

Proof. The homogeneity property follows by using Proposition 1.21-(ii), with the umbra ξ replaced by the singleton umbra χ.

The additivity property follows via generating functions, indeed we have

$$\begin{aligned} f(\chi.(\alpha+\gamma), z) &= 1 + \log(f(\alpha+\gamma, z)) \\ &= 1 + \log(f(\alpha, z)) + \log(f(\gamma, z)) \\ &= f(\chi.\alpha, z) + f(\chi.\gamma, z) - 1 \\ &= f(\chi.\alpha \dot{+} \chi.\gamma, z). \end{aligned}$$

The semi-invariance property follows from the additivity property, setting $\gamma \equiv C.u$, for all $C \in \mathbb{R}$. □

Now we want to show how the umbral calculus simplifies the expression in (2.3). Definition 2.4 is a closed form of the first equality in (2.3).

The umbral version of the second equality in (2.3) is given by the following Inversion theorem [30].

Theorem 2.8. *Let κ_α be the α-cumulant umbra, then*

$$\alpha \equiv \beta . \kappa_\alpha, \tag{2.6}$$

where β is the Bell umbra.

Proof. We have

$$\beta.\kappa_\alpha \equiv \beta.\chi.\alpha \equiv u.\alpha \equiv \alpha.$$

□

Remark 2.9. Theorem 2.8 allows us to express moments of an umbra α according to its cumulants. In fact, equivalence (2.6) simply means that any umbra α can be written as the partition umbra of its cumulant umbra.

By (1.30), we have

$$a_k = Y_k[(c_a)_1, \ldots, (c_a)_k], \tag{2.7}$$

where a_k is the k-th moment of the umbra α and $(c_a)_k$ is the k-th moment of the umbra κ_α.

The complete Bell polynomials in (2.7) are a polynomial sequence of binomial type. So, it is possible to prove a more general result: every polynomial sequence of binomial type is completely determined by its sequence of formal cumulants. Indeed, in [29] it is proved that any polynomial sequence of binomial type represents the moments of a polynomial umbra $t.\alpha$ and vice versa. So from the inversion theorem any polynomial sequence of binomial type represents the moments of a polynomial umbra $t.\beta.\kappa_\alpha$.

From Theorem 2.8 and by recalling $E[(\beta)_k] = 1$ for all nonnegative integers k and equivalence (1.29) we have

$$\alpha^k \simeq (\beta . \kappa_\alpha)^k \simeq \sum_{\lambda \vdash k} (\beta)_{l(\lambda)} d_\lambda (\kappa_\alpha)_\lambda \simeq \sum_{\lambda \vdash k} d_\lambda \, (\kappa_\alpha)_\lambda. \tag{2.8}$$

Theorem 2.10 (Parametrization)**.** *Let κ_α be the α-cumulant umbra. For $k = 1, 2, \ldots$ we have*

$$\alpha^k \simeq \kappa_\alpha (\kappa_\alpha + \beta.\kappa_\alpha)^{k-1} \quad \kappa_\alpha^k \simeq \alpha(\alpha - 1.\alpha)^{k-1}, \tag{2.9}$$

where $-1.\alpha$ denotes the inverse of the umbra α.

There are two more families of cumulants: the Boolean and the free cumulants. Let us describe them from an umbral point of view.

α-boolean cumulant umbra. The notion of boolean cumulants arises from considering the boolean convolution of probability measures [28].

Let $M(z)$ be the ordinary generating function of a random variable X, that is

$$M(z) = 1 + \sum_{k \geq 1} a_k z^k$$

where $a_k = E[X^k]$. We have

$$M(z) = \frac{1}{1 - H(z)}, \quad \text{where} \quad H(z) = \sum_{k \geq 1} h_k z^k,$$

and h_k are called *boolean cumulants* of X, [28].

In order to associate a sequence of boolean cumulants to an umbra, it is necessary to characterize umbrae whose generating functions are of ordinary type. This is possible by introducing the *boolean unity umbra*, that is, the umbra $\bar{u}$ such that, for all nonnegative integers k,

$$E[\bar{u}^k] = k!.$$

Therefore, we can consider the umbra $\bar{\alpha} \equiv \bar{u}\alpha$, such that

$$E[\bar{\alpha}^k] = k! E[\alpha^k] = k! a_k, \text{ for all } k \geq 0.$$

Finally, we have

$$f(\bar{\alpha}, z) = 1 + \sum_{k \geq 1} a_k z^k. \tag{2.10}$$

Definition 2.11. The *α-boolean cumulant umbra* η_α is the umbra such that

$$\bar{\eta}_\alpha \equiv \bar{u}^{<-1>}.\beta.\bar{\alpha}, \tag{2.11}$$

where $\bar{u}^{<-1>}$ is the compositional inverse of the boolean unity umbra $\bar{u}$ with generating function

$$f(\bar{u}^{<-1>}, z) = 1 + \log(1 + z). \tag{2.12}$$

The moments of the α-boolean cumulant umbra are $E[\eta_\alpha^k] = h_k$, for all nonnegative integers k.

Since the α-boolean cumulant umbra is defined as a composition umbra, by applying (1.40) we obtain that the generating function of the α-boolean cumulant umbra η_α is

$$f(\bar{\eta}_\alpha, z) = 2 - \frac{1}{f(\bar{\alpha}, z)}. \tag{2.13}$$

By equivalence (2.11), we get

$$\bar{\eta}_\alpha^k \simeq \sum_{\lambda \vdash k} (-1)^{l(\lambda)-1} l(\lambda)!\, d_\lambda\, \bar{\alpha}_\lambda. \tag{2.14}$$

The previous equivalence allows us to express boolean cumulants in terms of moments. In order to get the inverted expression, we need the following equivalence

$$\bar{\alpha} \equiv \bar{u}.\beta.\bar{\eta}_\alpha,$$

that has been proved in the Boolean Inversion Theorem (cfr. [23]). We have

$$\bar{\alpha}^k \simeq \sum_{\lambda \vdash k} l(\lambda)!\, d_\lambda\, (\bar{\eta}_\alpha)_\lambda.$$

Remark 2.12. Observe the analogy between the similarity $\bar{\eta}_\alpha \equiv \bar{u}^{<-1>}.\beta.\bar{\alpha}$, and the one characterizing the α-cumulant umbra $\kappa_\alpha \equiv \chi.\alpha \equiv u^{<-1>}.\beta.\alpha$.

Similarly to the case of the α- cumulant umbra, additivity property, homogeneity property and semi-invariance property still hold for α-boolean cumulant umbrae [28].

Theorem 2.13 (Parametrization)**.** *Let η_α be the α-boolean cumulant umbra. For $k = 1, 2, \ldots,$ we have*

$$\bar{\alpha}^k \simeq \bar{\eta}_\alpha(\bar{\eta}_\alpha + 2.\bar{u}.\beta.\bar{\eta}_\alpha)^{k-1} \quad \bar{\eta}_\alpha^k \simeq \bar{\alpha}(\bar{\alpha} - 2.\bar{\alpha})^{k-1}. \tag{2.15}$$

α-free cumulant umbra. Free probability theory is a non-commutative probability theory introduced by Voiculescu [62]. A new kind of independence is defined by replacing tensor product with free products.

The combinatorics underling this subject is based on the notion of non-crossing partition [28].

Let us consider a non-commutative random variable X, that is, an element of a unital non-commutative algebra $\mathbb{A}$. Suppose $\phi : \mathbb{A} \to \mathbb{C}$ is a unital linear functional. The k-th moment of X is the complex number $m_k = \phi(X^k)$ while its generating function is the formal power series

$$M(z) = 1 + \sum_{k \geq 1} m_k z^k.$$

The *non-crossing* (or *free*) cumulants of X are the coefficients r_k of the ordinary power series

$$R(z) = 1 + \sum_{k \geq 1} r_k z^k \text{ such that } M(z) = R[zM(z)]. \tag{2.16}$$

The umbral theory of free cumulants has been introduced in [23].

In order to give the definition of free cumulant umbrae, we have to introduce a new umbra, called the derivative umbra of an umbra α [28].

Given an umbra α, the *derivative umbra* α_D is the umbra whose moments are, for all $k \geq 1$,

$$(\alpha_D)^k \simeq \partial_k(\alpha^k) \simeq k\alpha^{k-1}. \tag{2.17}$$

The generating function of the derivative umbra is

$$f(\alpha_D, z) = 1 + zf(\alpha, z). \tag{2.18}$$

Definition 2.14. Given an umbra α, the *α-free cumulant umbra* $\mathfrak{K}_\alpha$ is the umbra such that

$$(-1.\mathfrak{K}_{\bar{\alpha}})_D \equiv \bar{\alpha}_D^{<-1>}, \tag{2.19}$$

where α_D is the derivative umbra.

The $\bar{\alpha}$-free cumulant $\mathfrak{K}_{\bar{\alpha}}$ has been characterized so that $E[\mathfrak{K}_{\bar{\alpha}}^k] = k!r_k$.

The moments of $\mathfrak{K}_\alpha$ are the *free cumulants* of the umbra α. This statement is justified by the fact that one can prove that

$$\bar{\alpha} \equiv \bar{\mathfrak{K}}_\alpha.\beta.\bar{\alpha}_D$$

and so, if we choose $M(z) = f(\bar{\alpha}, z)$ and $R(z) = f(\mathfrak{K}_\alpha, z)$, we obtain equation (2.16).

Remark 2.15. It is quite obvious to observe the difference between the previous equivalence and the ones characterizing both the α-classical and the α-boolean cumulant umbrae. This is why the computation of free cumulants is quite difficult compared with the classical and boolean ones.

Theorem 2.16 (Parametrization)**.** *Let $\mathfrak{K}_{\bar{\alpha}}$ be the $\bar{\alpha}$-free cumulant umbra. For $k = 1, 2, \ldots$ we have*

$$\bar{\alpha}^k \simeq \mathfrak{K}_{\bar{\alpha}}(\mathfrak{K}_{\bar{\alpha}} + k.\mathfrak{K}_{\bar{\alpha}})^{k-1} \quad \mathfrak{K}_{\bar{\alpha}}^k \simeq \bar{\alpha}(\bar{\alpha} - k.\bar{\alpha})^{k-1}. \tag{2.20}$$

Also in the free setting, one can prove that homogeneity property, additivity property and semi-invariance property still hold [28].

2.2.1 The umbral algorithm for classical, boolean and free cumulants

Theorems 2.10, 2.13 and 2.16 give parametrization formulae, expressing moments in terms of classical, boolean and free cumulants and vice versa.

The umbral polynomials appearing in these theorems can be viewed as special cases of a more general umbral polynomial, with a structure very similar to Abel polynomials.

Recall that the umbral Abel polynomials are [28]

$$A_k(x,\alpha) \simeq \begin{cases} u, & \text{if } k=0 \\ x(x-k.\alpha)^{k-1}, & \text{if } k \geq 1 \end{cases} \tag{2.21}$$

If the umbra α in (2.21) is replaced by $a.u$, where $a \in \mathbb{R}$, then $E[A_k(x,a.u)] = A_k(x,a)$, where $A_k(x,a) = x(x-ka)^{k-1}$ are the Abel polynomials.

Proposition 2.17. *If $\xi, \theta \in \mathcal{A}$, then*

$$\theta(\theta+\xi.\theta)^{k-1} \simeq \sum_{\lambda \vdash k} (\xi)_{l(\lambda)-1}\, d_\lambda\, \theta_\lambda, \tag{2.22}$$

where the sum ranges over all the integer partitions $\lambda = (1^{r_1}, 2^{r_2}, \ldots)$ of the integer k, $l(\lambda)$ is the length of λ, $\theta_\lambda = \theta_1^{.r_1}(\theta_2^2)^{.r_2}\cdots$, with $\theta_1, \theta_2, \ldots$ uncorrelated umbrae, similar to θ and d_λ is the number of set partitions of type λ, given in (1.12).

Proof. By using the binomial expansion and equivalence (1.11), we have

$$\theta(\theta+\xi.\theta)^{k-1} \simeq \sum_{s=1}^{k} \binom{k-1}{s-1} \theta^s \sum_{\lambda \vdash k-s} (\xi)_{l(\lambda)} d_\lambda \theta_\lambda. \tag{2.23}$$

Suppose we consider the partition μ of the integer k, obtained by adding the integer s to the partition λ. Then we have $\theta^s \theta_\lambda \equiv \theta_\mu$ and $l(\lambda) = l(\mu) - 1$.

If c_s denotes the multiplicity of s in λ and m_s denotes the multiplicity of s in μ, then $m_s = c_s + 1$. Therefore, we have

$$\binom{k-1}{s-1} d_\lambda = \frac{s}{k}\, \frac{k!}{(1!)^{c_1}\cdots(s!)^{c_s+1}\cdots\; c_1!\cdots c_s!\cdots} = s\, m_s\, \frac{d_\mu}{k},$$

where the last equality comes by multiplying numerator and denominator for $m_s > 0$. Recalling that $\sum s\, m_s = k$, equivalence (2.22) follows. □

We propose an algorithm which relies on the efficient expansion of the umbral polynomial $\theta(\theta + \xi.\theta)^{k-1}$ given in Proposition 2.17.

In order to evaluate $\theta(\theta + \xi.\theta)^{k-1}$ via (2.22), we need the factorial moments of the umbra ξ and the moments of the umbra θ. Recall that, if we just have information on moments $E[\xi^k]$, the factorial moments can be recovered by using the well-known change of bases:

$$(\xi)_k \simeq \sum_{i=1}^{k} s(k,i)\xi^i, \tag{2.24}$$

where $\{s(k,i)\}$ are the Stirling numbers of the first kind.

In particular, equivalence (2.22) allows us to give any expression of classical, boolean and free cumulants in terms of moments and vice versa.

i) For classical cumulants in terms of moments, due to the latter of (2.9), we set $\xi = -1.u$ and $\theta = \alpha$. By using (2.24), we find $E[(-1.u)_k] = (-1)_k = (-1)^k k!$.

ii) For moments in terms of classical cumulants, due to the first of (2.9), we set $\xi = \beta$ and $\theta = \kappa_\alpha$. We already know that $E[(\beta)_k] = 1$.

iii) For boolean cumulants in terms of moments, due to the latter of (2.15), we set $\xi = -2.u$ and $\theta = \bar{\alpha}$. Here we find $E[(-2.u)_k] = (-1)^k(k+1)!$.

iv) For moments in terms of boolean cumulants, due to the first of (2.15), we set $\xi = 2.\bar{u}.\beta$ and $\theta = \bar{\eta}_\alpha$. Here we have $E[(2.\bar{u}.\beta)_k] = E[(2.\bar{u}.\beta.\chi)^k] = E[(2.\bar{u})^k] = E[(\bar{u} + \bar{u}')^k] = (k+1)!$.

v) For free cumulants in terms of moments, due to the latter of (2.20), we set $\xi = -n.u$ and $\theta = \bar{\alpha}$. Here we have $E[(-n.u)_k] = (-n)_k$.

vi) Finally, for moments in terms of free cumulants, due to the first of (2.20), we set $\xi = n.u$ and $\theta = \mathfrak{K}_{\bar{\alpha}}$. Here we have $E[(n.u)_k] = (n)_k$.

The umbral algorithm in `MAPLE` is the following:

```
y:=combinat['partition'](i):
umbralg := proc(i,fm,y)
            i! * add(fm[j] *
                mul(((g[x[1]]/x[1]!)^x[2])/x[2]!,
                   x=convert(y[j],multiset)),
            j=1..nops(y));
 end:
```

Remark 2.18. In the `MAPLE` procedure, the factorial moments $E[(\xi)_{j-1}]$ are referred by the vector `fm[j]` and the moments $E[\theta^k]$ are referred by the vector `g[k]`.

Remark 2.19. We can compare the `MAPLE` algorithm with other procedures given in literature, see e.g. Bryc in [17].

However, such procedures are based on different approaches.

Table 1 refers to computational times (in seconds) reached by using the umbral algorithm and the Bryc's procedure [17], both implemented in `MAPLE`, release 7, when we need free cumulants in terms of moments [1].

k	`MAPLE` (umbral)	`MAPLE` (Bryc)
15	0.015	0.016
18	0.031	0.062
21	0.078	0.141
24	0.172	0.266
27	0.375	0.703

Table 2.1: Comparisons of computational times needed to compute free cumulants in terms of moments. Tasks performed on Intel (R) Pentium (R),CPU 3.00 GHz, 512 MB RAM.

Comparisons confirm the competitiveness of the umbral algorithm.

2.2.2 Computing cumulants of some known laws through the umbral algorithm

We take up some examples of the most known probability distributions, showing how they can be recovered through the umbral algorithm by a suitable characterization of the involved umbrae.

Poisson distribution. We have already showed in Remark 1.15 that a Poisson random variable of parameter λ is umbrally represented by the Bell polynomial umbra $\lambda.\beta$, whose moments are, thanks to equation (1.15) $E[(\lambda.\beta)^k] = \sum_{i=1}^{k} S(k,i)\lambda^i$, where $S(i,k)$ are the Stirling numbers of second kind.

Then, cumulants of a Poisson random variable can be computed via the umbral algorithm, taking as input the sequence of moments $E[(\lambda.\beta)^k]$.

If the input is the sequence of factorial moments $\{(-1)^k k!\}$, we get classical cumulants; if the input is the sequence of factorial moments $\{(-1)^k(k+1)!\}$, we get boolean cumulants; if the input is the sequence of factorial moments $\{(-n)_k\}$ we get free cumulants (cfr. Tables 3 and 4 in [17]).

Exponential distribution. Given a nonnegative real parameter λ, the

[1] The output is in the same form of the one given by Bryc's procedure.

umbral counterpart of an exponential random variable is the umbra

$$\frac{\bar{u}}{\lambda} \equiv \bar{u}.\frac{1}{\lambda},$$

where $\bar{u}$ is the boolean unity umbra. Indeed, its generating function is

$$f\left(\frac{\bar{u}}{\lambda}, z\right) = \frac{1}{1 - \frac{z}{\lambda}},$$

which is the moment generating function of an exponential random variable with parameter $\lambda > 0$. Moments are

$$E\left[\left(\frac{\bar{u}}{\lambda}\right)^k\right] = \frac{k!}{\lambda}.$$

In order to obtain the second column of Table 3 in [17], choose in (2.22) $\lambda = 1$ and $\{(-n)_k\}$ as factorial moments.

Uniform distribution. Let ι be the Bernoulli umbra. Recalling that $E[(-1.\iota)^k] = 1/(k+1)$, for all nonnegative integers k, we get

$$E\left\{[a + (b-a)(-1.\iota)]^k\right\} = \sum_{j=0}^{k} \binom{k}{j} a^{k-j} \frac{(b-a)^j}{j+1}.$$

In order to obtain the last column of Table 3 in [17], choose in (2.22) $a = -1, b = 1$ and $\{(-n)_k\}$ as factorial moments.

Binomial distribution. A Bernoulli random variable of parameter $p \in (0,1)$ has moment generating function

$$M(z) = q + pe^z = 1 + p(e^z - 1).$$

The umbra with generating function $M(z)$ is $\chi.p.\beta$ whose moments are all equal to p. A binomial random variable of parameters n, a positive integer, and $p \in (0,1)$ is a sum of n independent and identically distributed Bernoulli random variables.

A binomial random variable is umbrally represented by $n.(\chi.p.\beta)$. Due to equivalence (1.11) with α replaced by $\chi.p.\beta$, we have

$$E\{[n.(\chi.p.\beta)]^k\} = \sum_{\lambda \vdash k} (n)_{l(\lambda)} d_\lambda p^{l(\lambda)}.$$

In order to obtain the second column of Table 4 in [17], choose $\{(-n)_k\}$ as factorial moments in (2.22).

Gaussian distribution. A Gaussian random variable $\mathcal{N}(m, s^2)$ with mean m and variance s^2 is umbrally represented by $m + \beta.(s\delta)$. Its moments are The Hermite polynomials, as we have already proved in Proposition 1.25.

The first column Table 3 in [17] can be recovered from the umbral algorithm by using $\{(-n)_k\}$ as factorial moments in (2.22).

Wigner semicircle distribution. In free probability, the Wigner semicircle distribution is analogous to the Gaussian random variable in the classical probability. Indeed, free cumulants of degree higher than 2 of the Wigner semicircle random variable are zero. The first column in Table 2 shows moments of the Wigner semicircle random variable X, computed by the umbral algorithm. They are compared with Catalan numbers C_k (second column), since it is well-known that $E[X^{2k}] = C_k$ and $E[X^{2k+1}] = 0$. By using equivalence (1.41) it is straightforward to prove that the umbra corresponding to the Wigner semicircle distribution is $\bar{\varsigma}.\beta.\bar{\delta}$, where ς is the umbra whose moments are the Catalan numbers [28] and δ is an umbra appearing in the expression of the Gaussian umbra.

Marchenko-Pastur distribution. In free probability, the Marchenko-Pastur distribution is analogous to the Poisson random variable in the classical probability. Indeed, the free cumulants are all equal to a parameter λ. The last column in Table 2 shows moments of the Marchenko-Pastur distribution computed by the umbral algorithm.

k	Wigner random variable	Catalan numbers	Marchenko-Pastur random variable
2	1	2	$\lambda^2 + \lambda$
3	0	5	$\lambda^3 + 3\lambda^2 + \lambda$
4	2	14	$\lambda^4 + 6\lambda^3 + 6\lambda^2 + \lambda$
6	5	132	$\lambda^6 + 15\lambda^5 + 50\lambda^4 + 50\lambda^3 + 15\lambda^2 + \lambda$
8	14	1430	$\lambda^8 + 28\lambda^7 + 196\lambda^6 + 490\lambda^5 + 490\lambda^4$ $+196\lambda^3 + 28\lambda^2 + \lambda$

Table 2.2: Moments of some special free distributions.

2.3 Symbolic representation of Lévy processes

2.3.1 Probabilistic background

In this section we give basic definitions and main results on stochastic processes and, in particular, on Lévy processes. We skip any proof. See [48] for further details.

Definition 2.20. A *probability space* $(\Omega, \mathcal{F}, P)$ is a triplet of a set Ω, a family $\mathcal{F}$ of subset of Ω and a mapping P from $\mathcal{F}$ into $\mathbb{R}$ satisfying the following conditions:

(i) $\Omega \in \mathcal{F}$, $\emptyset \in \mathcal{F}$;

(ii) if $A_n \in \mathcal{F}$ for $n = 1, 2, \dots$, then $\bigcup_{n=1}^{\infty} A_n$ and $\bigcap_{n=1}^{\infty} A_n$ are in $\mathcal{F}$;

(iii) if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$;

(iv) $0 \leq P(A) \leq 1$, $P(\Omega) = 1$ and $P(\emptyset) = 0$;

(v) if $A_n \in \mathcal{F}$ for $n = 1, 2, \dots$ and $A_i \cap A_j \neq \emptyset$, for $i \neq j$, then $P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$.

Remark 2.21. Recall that $\mathcal{F}$ satisfying properties (i), (ii) and (iii) is said to be a *σ-algebra* on Ω. A mapping P from $\mathcal{F}$ into Ω satisfying properties (iv) and (v) is a *probability measure.*

Definition 2.22. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A mapping X from Ω into $\mathbb{R}$ is a *real-valued random variable* if it is $\mathcal{F}$-measurable, that is,

$$\text{for all } A \in \mathcal{F},\ X^{-1}(A) = \{\omega \in \Omega \,|\, X(\omega) \in A\} \in \mathcal{F}. \tag{2.25}$$

We denote by $P(X^{-1}(A)) = P^X(A)$ the *distribution* of the random variable X.

Definition 2.23. A family $\{X_t\}_{t \geq 0}$ of random variables on $\mathbb{R}$, defined on a common probability space and depending on a parameter $t \in \mathbb{R}^+$, is called a *stochastic process* on $\mathbb{R}$.

Definition 2.24. A stochastic process $X = \{X_t\}_{t \geq 0}$ on $\mathbb{R}$ is a *Lévy process* if

(i) $X_0 = 0$ a.s.

(ii) For all $n \geq 1$ and for all $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n < \infty$, the random variables $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \dots, X_{t_{n-1}} - X_{t_n}$ are independent.

(iii) For all $s \leq t$, $X_{t+s} - X_s \overset{d}{=} X_t$.

(iv) For all $\varepsilon > 0$, $\lim_{h \to 0} P(|X_{t+h} - X_t| > \varepsilon) = 0$.

(v) $t \mapsto X_t(\omega)$ are cádlág [2], for all $\omega \in \Omega$.

Remark 2.25. Let us better explain Definition 2.24.

Properties (i) and (ii) are obvious.

In property (iii) the symbol $\stackrel{d}{=}$ means *equal in distribution*, that is, the distribution of the increment $X_{t+s} - X_s$ does not depend on s.

Property (iv) states the so-called *stochastic continuity* or *continuity in probability.*

Recall that a stochastic process on $(\Omega, \mathcal{F}, P)$ is a map $X : [0,T]\times\Omega \to \mathbb{R}$ such that $(t,\omega) \mapsto X(t,\omega) = X_t(\omega)$. If we fix $t \in [0,T]$, the map $\omega \mapsto X_t(\omega)$ is a random variable, according to Definition 2.22. If we fix $\omega \in \Omega$, the map $t \mapsto X_t(\omega)$ is a deterministic function, called a trajectory of the process. Therefore, Property (v) means that the trajectories of a Lévy process are right-continuous functions and have left limits in $t \geq 0$.

Let us recall a very important property shared by Lévy processes, the *infinite divisibility property.*

Definition 2.26. A random variable X is said to be having the *infinite divisibility property* if, for any $n \geq 1$, there exist a sequence of independent and identically distributed random variables $X_{1,n}, \dots, X_{n,n}$ such that

$$X \stackrel{d}{=} X_{1,n} + \cdots + X_{n,n},$$

where the symbol $\stackrel{d}{=}$ means *equal in distribution.*

The following definition is equivalent to the previous one.

Definition 2.27. A probability measure μ on $\mathbb{R}$ is *infinitely divisible* if, for any positive integer n, there exists a probability measure μ_n such that

$$\mu = \underbrace{\mu_n * \cdots * \mu_n}_{n},$$

that is, μ is the n-fold convolution of the probability measure μ_n with itself.

The following theorem proves that the collection of all infinitely divisible distributions is in one-to-one correspondence with the collection of all Lévy processes. The proof is in [48].

[2]Recall that a cádlág function is a function defined on the real numbers (or a subset of them) that is everywhere right-continuous and has left limits everywhere, see next Remark for further details.

Theorem 2.28. *A stochastic process $\{X_t\}_{t\geq 0}$ with distribution P^{X_t} is a Lévy process if and only if the distribution P^{X_t} is infinitely divisible and $P^{X_t} = (P^{X_1})^t$.*

In the theorem above, P^{X_1} is said to be the infinitely divisible distribution corresponding to the Lévy process $\{X_t\}_{t\geq 0}$. Conversely, $\{X_t\}_{t\geq 0}$ is said to be the Lévy process corresponding to the infinitely distribution P^{X_1}.

We introduce now a tool very useful in identifying the distribution of random variables.

Definition 2.29. The *moment generating function* of a random variable X is the function

$$\varphi_X : \mathbb{R} \to \mathbb{R}, \quad \varphi_X(z) = \mathbb{E}[e^{Xz}],$$

where $\mathbb{E}$ is the expectation functional.

The moment generating functions of infinitely divisible distributions are completely characterized by the following result, known in the literature as the *Lévy-Khintchine formula*, [48].

Theorem 2.30. *$X = \{X_t\}_{t\geq 0}$ is a Lévy process if and only if there exists $m \in \mathbb{R}$, $s > 0$ and a measure ν on $\mathbb{R}$ with*

$$\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}} (|x|^2 \wedge 1)\nu(dx) < \infty$$

such that

$$\varphi_X(z) = \exp\left\{ t\left(zm + \frac{1}{2}s^2z^2 + \int_{\mathbb{R}} (e^{zx} - 1 - zx\mathbf{1}_{|x|\leq 1})\nu(dx) \right) \right\}. \quad (2.26)$$

What is more, the representation of $\varphi_X(z)$ in (2.26) by m, s and ν is unique.

Remark 2.31. We call (m, s^2, ν) in Theorem 2.30 the *Lévy triplet* or the *characteristic triplet*, where m is the drift of the process, s^2 is its Gaussian variance and ν is the *Lévy measure.*

2.3.2 Lévy processes via umbral calculus

Now we focus our attention on the family of auxiliary umbrae $\{t.\alpha\}_{t\geq 0}$ introduced in Section 1.2. If the moments of the umbra α are all finite, this family is the umbral counterpart of a stochastic process $\{X_t\}_{t\geq 0}$ such that $E[X_t^k] = E[(t.\alpha)^k]$ given in (1.13), for all nonnegative integers k. This stochastic process is a Lévy process.

Theorem 2.32. *Let $\{X_t\}_{t\geq 0}$ be a Lévy process and let α be the umbra such that $f(\alpha, z) = E[e^{zX_1}]$. Then, the Lévy process $\{X_t\}_{t\geq 0}$ is umbrally represented by the family of auxiliary umbrae $\{t.\alpha\}_{t\geq 0}$.*

Proof. We have to show that the random variables X_t, with $t \geq 0$, which form the process, have the same moment generating function of the auxiliary umbra $t.\alpha$.

Let $\{X_t\}_{t\geq 0}$ be a Lévy process with moment generating function $E[e^{zX_t}]$ and let $E[e^{zX_1}]$ be the moment generating function of X_1.

Theorem 2.28 guarantees that $\{X_t\}_{t\geq 0}$ has infinitely divisible distributions, that is,

$$E[e^{zX_t}] = \left(E[e^{zX_1}]\right)^t.$$

On the other hand, by definition, α is an umbra such that $f(\alpha, z) = E[e^{zX_1}]$, so the result follows, since

$$E\left[e^{zX_t}\right] = (f(\alpha, z))^t = f(t.\alpha, z),$$

where the last equality holds by virtue of (1.14). □

Remark 2.33. In [45], Rota wrote that the classical umbral calculus can be systematically interpreted as a calculus of measures on Poisson algebras, generalizing compound Poisson processes. This statement is proved in the Inversion Theorem 2.8, which states that $\alpha \equiv \beta.\kappa_\alpha$ and $f(\alpha, z) = \exp\{f(\alpha, z) - 1\}$. Therefore, the auxiliary umbra $\beta.\kappa_\alpha$ is the umbral counterpart of a compound Poisson random variable with parameter 1.

More in general, compound Poisson random variables of parameter t are represented by the auxiliary umbrae $t.\beta.\kappa_\alpha$ with generating function

$$f(t.\beta.\kappa_\alpha, z) = \exp\{t[f(\kappa_\alpha, z) - 1]\}. \tag{2.27}$$

The umbra $t.\alpha \equiv t.\beta.\kappa_\alpha$ corresponds to a Lévy process. Indeed, if we denote by $\phi(z, t)$, $\psi(z)$ and $k(z)$ the moment generating function of the increment $X_{t+s} - X_t$ of the process, the moment generating function of X_1 and the cumulant generating function of X_1, respectively, Theorem 2.32 guarantees that

$$f(t.\alpha, z) = \phi(z, t) = \exp\{\log(\psi(z))\} = \exp\{tk(z)\}, \tag{2.28}$$

with $k(0) = 0$.

Comparing $\exp\{tk(z)\}$ and $\exp\{t[f(\kappa_\alpha, z) - 1]\}$, the correspondence between $t.\alpha$ and the Lévy process is immediate. This is why we call $t.\alpha$ the *Lévy umbra* associated to the umbra α.

Remark 2.34. The umbral representation of Lévy processes given in Theorem 2.32 allows us to recover their classical properties.

Indeed, the homogeneity property $t.(C\alpha) \equiv C(t.\alpha)$ is proved by virtue of Proposition 1.14 -(ii), the nesting property $t.(s.\alpha) \equiv s.(t.\alpha) \equiv ts.\alpha$ is proved by virtue of Proposition 1.14-(iii), the binomial property $(t+s).\alpha \equiv t.\alpha + s.\alpha$ is proved by virtue of Proposition 1.14-(iv) and the additivity property $t.(\alpha+\gamma) \equiv t.\alpha + t.\gamma$ is proved by virtue of Proposition 1.14 - (v).

From the Lévy-Khintchine formula (2.26), we obtain the moment generating function of the process $\{X_t\}_{t\geq 0}$ of form

$$E[e^{zX_t}] = \exp\left\{t\left[zm + \frac{1}{2}s^2z^2 + \int_{\mathbb{R}}(e^{zx} - 1 - zx\mathbf{1}_{|x|\leq 1})\nu(dx)\right]\right\}. \quad (2.29)$$

If ν is a measure admitting all moments and if we set

$$c_0 = m + \int_{\{|x|>1\}} x\nu(dx) < \infty,$$

then the moment generating function (2.29) becomes

$$\begin{aligned} E[e^{zX_t}] &= \exp\left\{t\left[zm + \frac{1}{2}s^2z^2 + \int_{\mathbb{R}}(e^{zx} - 1 - zx\mathbf{1}_{|x|\leq 1})\nu(dx)\right.\right. \\ &\quad \left.\left. - \int_{\mathbb{R}} zx\nu(dx) + \int_{\mathbb{R}} zx\nu(dx)\right]\right\} \\ &= \exp\left\{t\left[z\left(m + \int_{\mathbb{R}} x\nu(dx)\right) + \frac{1}{2}s^2z^2 + \int_{\mathbb{R}}(e^{zx} - 1 - zx)\nu(dx)\right]\right\} \\ &= \exp\left\{t\left[zc_0 + \frac{1}{2}s^2z^2 + \int_{\mathbb{R}}(e^{zx} - 1 - zx)\nu(dx)\right]\right\}. \end{aligned}$$

Finally, we have

$$E[e^{zX_t}] = \exp\left\{t\left[zc_0 + \frac{1}{2}s^2z^2\right]\right\} \exp\left\{t\int_{\mathbb{R}}(e^{zx} - 1 - zx)\nu(dx)\right\}. \quad (2.30)$$

Now we are able to give the umbral version of a Lévy process, according to the Lévy-Khintchine formula (2.30).

Definition 2.35. The umbra γ whose moments $m_k = E[\gamma^k]$, for all $k \geq 0$, are such that

$$m_0 = 1, \quad m_1 = 0, \quad m_k = \int_{\mathbb{R}} x^k \nu(dx), \text{ for all } k \geq 2$$

is called the umbra *associated* to the Lévy measure ν.

We can construct the generating function of the umbra γ. Indeed, by definition of generating function, we have

$$f(\gamma, z) = \sum_{k \geq 0} m_k \frac{z^k}{k!} = 1 + \sum_{k \geq 2} m_k \frac{z^k}{k!} = 1 + \sum_{k \geq 2} \left(\int_{\mathbb{R}} x^k \nu(dx) \right) \frac{z^k}{k!}.$$

By exchanging the series and the integral, we have

$$\begin{aligned} f(\gamma, z) &= 1 + \int_{\mathbb{R}} \left(\sum_{k \geq 2} x^k \frac{z^k}{k!} \right) \nu(dx) \\ &= 1 + \int_{\mathbb{R}} \left(\sum_{k \geq 2} \frac{(xz)^k}{k!} \right) \nu(dx) \\ &= 1 + \int_{\mathbb{R}} \left(\sum_{k \geq 0} \frac{(xz)^k}{k!} - 1 - xz \right) \nu(dx). \end{aligned}$$

So, we obtain that the generating function of the umbra γ associated to the Lévy measure is

$$f(\gamma, z) = 1 + \int_{\mathbb{R}} (e^{xz} - 1 - xz) \nu(dx). \tag{2.31}$$

Remark 2.36. By definition, the umbra γ has first moment equal to 0. This means that the corresponding generating function $f(\gamma, z)$ does not admit compositional inverse.

Theorem 2.37. *A Lévy process $\{X_t\}_{t \geq 0}$ is umbrally represented by the family of auxiliary umbrae*

$$\{t.\beta.[c_0\chi \dot{+} s\delta \dot{+} \gamma]\}_{t \geq 0}, \tag{2.32}$$

where $c_0 = m + \int_{\{|x|>1\}} x\nu(dx) < \infty$, χ is the singleton umbra, $s \in \mathbb{R}^+$, δ is the umbral counterpart of a standard Gaussian random variable and γ is the umbra associated to the Lévy measure.

Proof. Let $\{X_t\}_{t \geq 0}$ be a Lévy process with moment generating function (2.30) and consider its two terms separately. As regards the first one, recall that the singleton umbra χ is such that $f(\chi, z) = 1 + z$, so we have

$$1 + c_0 z = f(\chi, c_0 z) = E[e^{(c_0 z)\chi}] = E[e^{(c_0 \chi) z}] = f(c_0\chi, z),$$

so that

$$c_0 z = f(c_0\chi, z) - 1. \tag{2.33}$$

On the other hand, the umbra δ is such that $f(\delta, z) = 1 + z^2/2$, hence

$$1 + \frac{1}{2}s^2 z^2 = f(\delta, sz) = E[e^{(sz)\delta}] = E[e^{(s\delta)z}] = f(s\delta, z)$$

so that

$$\frac{1}{2}s^2 z^2 = f(s\delta, z) - 1. \tag{2.34}$$

By adding (2.33) together with (2.34), we have

$$\begin{aligned} c_0 z + \frac{1}{2}s^2 z^2 &= f(c_0\chi, z) - 1 + f(s\delta, z) - 1 \\ &= (f(c_0\chi, z) + f(s\delta, z) - 1) - 1 \\ &= f(c_0\chi \dot{+} s\delta, z) - 1, \end{aligned}$$

where the last equality holds by virtue of definition of disjoint sum of the umbrae $c_0\chi$ and $s\delta$.

Finally, thanks to equality (1.39), we have

$$\exp\left\{t\left[zc_0 + \frac{1}{2}s^2 z^2\right]\right\} = \exp\{t[f(c_0\chi \dot{+} s\delta, z) - 1]\} = f(t.\beta.(c_0\chi \dot{+} s\delta), z). \tag{2.35}$$

As regards the second term in (2.30), we have

$$\begin{aligned} \exp\left\{t\int_{\mathbb{R}}(e^{zx} - 1 - zx\nu(dx))\right\} &= \exp\left\{t\left[1 + \int_{\mathbb{R}}(e^{zx} - 1 - zx)\nu(dx) - 1\right]\right\} \\ &= \exp\{t[f(\gamma, z) - 1]\} = f(t.\beta.\gamma, z), \quad (2.36) \end{aligned}$$

by virtue of equalities (1.39) and (2.31).

Now, we compare (2.30), (2.35) and (2.36) and apply Proposition 1.22, so that

$$\begin{aligned} E[e^{zX_t}] &= f(t.\beta.(c_0\chi \dot{+} s\delta), z)\, f(t.\beta.\gamma, z) \\ &= f(t.\beta.(c_0\chi \dot{+} s\delta) + t.\beta.\gamma, z) \\ &= f(t.\beta.(c_0\chi \dot{+} s\delta \dot{+} \gamma), z). \end{aligned}$$

□

Chapter 3

Time-space harmonic polynomials

3.1 Umbral time-space harmonic polynomials

It is well-known that there exists a correspondence between stochastic processes and some classes of polynomials [50]. In this Chapter we deal with a family of polynomials, called *time-space harmonic polynomials*, related to Lévy processes, which take advantage of the umbral notation.

In [39], Neveu characterizes the family of time-space harmonic polynomials with respect to a random walk $\{X_n\}_{n\geq 0}$ as the coefficients of the Taylor expansion

$$\frac{\exp\{zX_n\}}{E[\exp\{zX_n\}]} = \sum_{k\geq 0} R_k(X_n, n)\frac{z^k}{k!} \tag{3.1}$$

in some neighborhood of the origin.

If we replace the random walk $\{X_n\}_{n\geq 0}$ with a Lévy process $\{X_t\}_{t\geq 0}$, then (3.1) is the so-called *Wald's exponential martingale*.

Remark 3.1. Let us recall that a martingale is a stochastic process $\{M_t\}_{t\geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ such that

$$E[M_t | \, \mathcal{F}_s] = M_s,$$

for all $0 \leq s \leq t$.

Definition 3.2. A family of polynomials $\{P(x,t)\}_{t\geq 0}$ is said to be *time-space harmonic* with respect to a stochastic process $\{X_t\}_{t\geq 0}$ if

$$E[P(X_t, t) \mid \mathfrak{F}_s] = P(X_s, s), \text{ for all } s \leq t, \tag{3.2}$$

where $\mathfrak{F}_s = \sigma(X_\tau : \tau \leq s)$ is the natural filtration associated with $\{X_t\}_{t\geq 0}$.

Roughly speaking, the time-space harmonic polynomials are polynomials in the variable x, depending on a nonnegative real parameter t, such that, when we replace the variable x with the stochastic process X_t, we obtain a martingale [54].

In order to define a new family of time-space harmonic polynomials with respect to a Lévy process by using umbral tools, we have to introduce a new linear functional, which recalls the conditional expectation appearing in Definition 3.2.

Denote by $\mathcal{X}$ the set $\mathcal{X} = \{\alpha\}$.

Definition 3.3. The linear operator $E(\,\cdot \,|\, \alpha) : \mathbb{R}[x][\mathcal{A}] \longrightarrow \mathbb{R}[\mathcal{X}]$ such that

i) $E(1 \,|\, \alpha) = 1$;

ii) $E(x^m \alpha^n \gamma^i \delta^j \cdots \,|\, \alpha) = x^m \alpha^n E[\gamma^i] E[\delta^j] \cdots$ for uncorrelated umbrae $\alpha, \gamma, \delta, \ldots$ and for nonnegative integers $m, n, i, j, \ldots$

is called *conditional evaluation* with respect to the umbra α.

In other words, Definition 3.3 says that the conditional evaluation with respect to α handles the umbra α as it was an indeterminate.

Proposition 3.4. *If $\alpha \in \mathcal{A}$ and $p \in \mathbb{R}[x][\mathcal{A}]$ is an umbral polynomial with $\alpha \notin supp(p)$, then $E(p \,|\, \alpha) = E[p]$.*

Proof. Let p be an umbral monomial whose support does not contain the umbra α. For the sake of simplicity, suppose $p = x^i \xi_1^j \xi_2^k \cdots \xi_n^h$, for all nonnegative integers $i, j, k, \ldots, h$. Thanks to Definition 3.3, we have

$$E(p \,\big|\, \alpha) = E(x^i \xi_1^j \xi_2^k \cdots \xi_n^h \,\big|\, \alpha) = x^i E(\xi_1^j) E(\xi_2^k) \cdots E(\xi_n^h) = E(p).$$

The proof for umbral polynomials follows by linearity. □

The proof of the following Corollary is straightforward taking into account Definition 3.3 and Proposition 3.4.

Corollary 3.5. *If $\alpha \in \mathcal{A}$ and $p \in \mathbb{R}[x][\mathcal{A}]$, then $E[E(p \,|\, \alpha)] = E[p]$.*

This last corollary brings to light the parallelism between the conditional evaluation $E(\,\cdot \,|\, \alpha)$ and the well-known conditional expectation in probability theory [31]. As it happens in probability theory, the conditional evaluation is an element of $\mathbb{R}[x][\mathcal{A}]$ and, if we take the overall evaluation of $E(p \,|\, \alpha)$, this gives $E[p]$.

Definition 3.6. Let $\{P(x,t)\} \in \mathbb{R}[x]$ be a family of polynomials indexed by $t \geq 0$. $P(x,t)$ is said to be a *time-space harmonic polynomial* with respect to the family of umbral polynomials $\{q(t)\}_{t \geq 0}$ if and only if

$$E\left(P(q(t),t) \mid q(s)\right) = P(q(s),s), \quad \text{for all } s \leq t. \tag{3.3}$$

Since the property of being time-space harmonic is given with respect to a Lévy process and since we have thoroughly explained in Chapter 2 that a Lévy process is umbrally represented by a family of auxiliary umbrae, it is natural to consider the case of auxiliary umbrae in Definition 3.6. In particular, for the conditional evaluation with respect to the dot-product $n.\alpha$, we assume

$$E[(n+1).\alpha \mid n.\alpha] = E[n.\alpha + \alpha' \mid n.\alpha],$$

so that $= E[(n+1).\alpha \mid n.\alpha] = n.\alpha + E[\alpha']$.

More in general, for all $n, m \in \mathbb{Z}$ and for all nonnegative integer k, we will assume

$$E[\{(n+m).\alpha\}^k \mid n.\alpha] = E[\{n.\alpha + m.\alpha'\}^k \mid n.\alpha],$$

so that

$$\begin{aligned} E\left[\{(n+m).\alpha\}^k \,\middle|\, n.\alpha\right] &= E\left[\sum_{j=0}^{k} \binom{k}{j} (n.\alpha)^j (m.\alpha')^{k-j} \,\middle|\, n.\alpha\right] \\ &= \sum_{j=0}^{k} \binom{k}{j} E\left[(n.\alpha)^j (m.\alpha')^{k-j} \,\middle|\, n.\alpha\right] \\ &= \sum_{j=0}^{k} \binom{k}{j} (n.\alpha)^j E[(m.\alpha')^{k-j}]. \end{aligned} \tag{3.4}$$

On the other hand, Corollary 3.5 gives

$$E[E[\{(n+m).\alpha\}^k \mid n.\alpha]] = E[\{(n+m).\alpha\}^k],$$

hence, we recover the binomial property

$$E[\{(n+m).\alpha\}^k] = \sum_{j=0}^{k} \binom{k}{j} E[(n.\alpha)^j] E[(m.\alpha')^{k-j}].$$

By analogy with (3.4), for $t \geq 0$, we set

$$E\left[(t.\alpha)^k \,\middle|\, s.\alpha\right] = \sum_{j=0}^{k} \binom{k}{j} (s.\alpha)^j E[\{(t-s).\alpha'\}^{k-j}]. \tag{3.5}$$

Theorem 3.7. *For all nonnegative integers k, the family of polynomials*

$$Q_k(x,t) = E[(x - t.\alpha)^k] \in \mathbb{R}[x] \tag{3.6}$$

is time-space harmonic[1] *with respect to $\{t.\alpha\}_{t\geq 0}$.*

Proof. We have to prove that equality (3.3) holds, with the umbral polynomials given by the family of auxiliary umbrae $\{t.\alpha\}_{t\geq 0}$.

From (3.6), by applying the linearity property of the evaluation E, we have

$$Q_k(x,t) = \sum_{j=0}^{k} \binom{k}{j} x^{k-j} E[(-t.\alpha)^j] \tag{3.7}$$

for all nonnegative integers k. If we replace x with $t.\alpha$ in (3.7) and we take the conditional evaluation $E[\,\cdot \mid s.\alpha]$, we have

$$\begin{aligned}
E\left[Q_k(t.\alpha,t) \mid s.\alpha\right] &= E\left[\sum_{j=0}^{k}\binom{k}{j}(t.\alpha)^{k-j}E[(-t.\alpha)^j] \,\middle|\, s.\alpha\right]\\
&= \sum_{j=0}^{k}\binom{k}{j}E\left[(t.\alpha)^{k-j}E[(-t.\alpha)^j] \,\middle|\, s.\alpha\right]\\
&= \sum_{j=0}^{k}\binom{k}{j}E\left[(t.\alpha)^{k-j} \,\middle|\, s.\alpha\right]E[(-t.\alpha)^j]\\
&= \sum_{j=0}^{k}\binom{k}{j}E[(-t.\alpha)^j]\left\{\sum_{i=0}^{k-j}\binom{k-j}{i}(s.\alpha)^i E\left([(t-s).\alpha']^{k-j-i}\right)\right\},
\end{aligned}$$

where the last equality hold by virtue of (3.5). We claim that

$$\begin{aligned}
&\sum_{j=0}^{k}\binom{k}{j}E[(-t.\alpha)^j]\left\{\sum_{i=0}^{k-j}\binom{k-j}{i}(s.\alpha)^i E\left([(t-s).\alpha']^{k-j-i}\right)\right\}\\
&= \sum_{j=0}^{k}\binom{k}{j}(s.\alpha)^j\left\{\sum_{i=0}^{k-j}\binom{k-j}{i}E\left(\{(t-s).\alpha\}^{k-j-i}\right)E[(-t.\alpha)^i]\right\},
\end{aligned} \tag{3.8}$$

so that

[1] When no confusion occurs, we will use the notation $x - t.\alpha$ to denote the polynomial umbra $-t.\alpha + x = x + (-t).\alpha$.

$$\sum_{j=0}^{k} \binom{k}{j} (s.\alpha)^j \left\{ \sum_{i=0}^{k-j} \binom{k-j}{i} E\left(\{(t-s).\alpha\}^{k-j-i}\right) E[(-t.\alpha)^i] \right\}$$
$$= \sum_{j=0}^{k} \binom{k}{j} (s.\alpha)^j E\left[\{-t.\alpha + (t-s).\alpha\}^{k-j}\right]$$
$$= \sum_{j=0}^{k} \binom{k}{j} (s.\alpha)^j E[(-s.\alpha)^{k-j}],$$

that is,

$$E\left[Q_k(t.\alpha, t) \mid s.\alpha\right] = Q_k(s.\alpha, s),$$

again by virtue of Proposition 3.5.

Remain to prove equality (3.8). We have

$$\sum_{j=0}^{k} \binom{k}{j} E[(-t.\alpha)^j] \left\{ \sum_{i=0}^{k-j} \binom{k-j}{i} (s.\alpha)^i E\left([(t-s).\alpha']^{k-j-i}\right) \right.$$
$$= \sum_{j=0}^{k} \sum_{i=0}^{k-j} \binom{k}{j} \binom{k-j}{i} (s.\alpha)^i E[(-t.\alpha)^j] E[((t-s).\alpha')^{k-j-i}]$$
$$= (s.\alpha)^0 \sum_{j=0}^{k} \binom{k}{j} \binom{k-j}{0} E[(-t.\alpha)^j] E([(t-s).\alpha']^{k-j})$$
$$+ (s.\alpha) \sum_{j=0}^{k-1} \binom{k}{j} \binom{k-j}{1} E[(-t.\alpha)^j] E([(t-s).\alpha']^{k-1-j})$$
$$+ (s.\alpha)^2 \sum_{j=0}^{k-2} \binom{k}{j} \binom{k-j}{1} E[(-t.\alpha)^j] E([(t-s).\alpha']^{k-2-j})$$
$$+ \cdots + (s.\alpha)^{k-1} \sum_{j=0}^{1} \binom{k}{j} \binom{k-j}{k-1} E[(-t.\alpha)^j] E([(t-s).\alpha']^{1-j})$$
$$+ (s.\alpha)^k \sum_{j=0}^{0} \binom{k}{j} \binom{0}{k} E[(-t.\alpha)^j] E([(t-s).\alpha']^{-j})$$
$$= \sum_{i=0}^{k} (s.\alpha)^i \sum_{j=0}^{k-j} \binom{k}{j} \binom{k-j}{i} E[(-t.\alpha)^j] E([(t-s).\alpha']^{k-j-i}).$$

On the other hand, we have

$$\binom{k}{j}\binom{k-j}{i} = \binom{k}{i}\binom{k-i}{j},$$

therefore,

$$\sum_{j=0}^{k}\sum_{i=0}^{k-i}\binom{k}{i}\binom{k-i}{j}E[(-t.\alpha)^i]E([(t-s).\alpha']^{k-i-j})$$
$$= \sum_{j=0}^{k}(s.\alpha)^j\sum_{i=0}^{k-i}\binom{k}{j}\binom{k-j}{i}E[(-t.\alpha)^i]E([(t-s).\alpha']^{k-i-j})$$
$$= \sum_{j=0}^{k}\binom{k}{j}(s.\alpha)^j\sum_{i=0}^{k-i}\binom{k-j}{i}E[(-t.\alpha)^i]E([(t-s).\alpha']^{k-i-j}).$$

□

Definition 3.8. The polynomial umbra $x - t.\alpha$ is called the *time-space harmonic polynomial umbra* with respect to the family of auxiliary umbrae $\{t.\alpha\}_{t\geq 0}$.

It is straightforward to verify that the generating function of $x - t.\alpha$ is

$$f(x - t.\alpha, z) = \frac{\exp\{xz\}}{f(\alpha, z)^t} = \sum_{k\geq 0} Q_k(x,t)\frac{z^k}{k!}. \tag{3.9}$$

Remark 3.9. By replacing x with $t.\alpha$ in (3.9), we recover the Wald's exponential martingale (3.1). Equality of two formal power series is interpreted as the equality of their coefficients, so that $E[R_k(X_t,t)] = E[Q_k(t.\alpha,t)]$.

Also the Wald's identity $\sum_{k\geq 0} E[R_k(X_t,t)]z^k/k! = 1$ follows from (3.1). Therefore, the sequence $\{E[R_k(X_t,t)]\}_{k\geq 0} \in \mathbb{R}$ is umbrally represented by the augmentation umbra ϵ, since $f(\epsilon, z) = 1$, for all $t \geq 0$. But this is exactly what it happens when in the polynomial umbra $x - t.\alpha$ we replace x with $t.\alpha$.

The following corollary specifies the dependence of the coefficients of $Q_k(x,t)$ in (3.6) on the auxiliary umbra $\{t.\alpha\}_{t\geq 0}$.

Corollary 3.10. *If we set*

$$Q_k(x,t) = \sum_{j=0}^{k} q_j^{(k)}(t)\, x^j, \tag{3.10}$$

then

(i) $q_j^{(k)}(t) = \binom{k}{j} E[(-t.\alpha)^{k-j}]$, *for all* $j = 0, \ldots, k$;

(ii) $q_j^{(k)}(0) = 0$, *for all* $j = 0, \ldots, k-1$.

Proof. **(i)** We apply the binomial theorem and the linearity of the evaluation functional E, so that

$$Q_k(x,t) = E[(x - t.\alpha)^k] = \sum_{j=0}^{k} \binom{k}{j} E[(t.\alpha)^{k-j}] x^j. \tag{3.11}$$

The result follows by comparing (3.10) and (3.11).

(ii) From (i), we have

$$q_j^{(k)}(0) = \binom{k}{j} E[(0.\alpha)^{k-j}].$$

When we have introduced the auxiliary umbrae, we said that we set $0.\alpha \equiv \epsilon$, so that $E[(0.\alpha)^{k-j}] = 0$, when $j \neq k$, that is, $q_j^{(k)}(0) = 0$, for all $j < k$.

□

The following corollary specifies the dependence of the coefficients of $Q_k(x,t)$ in (3.6) on the umbra α.

Corollary 3.11. *If* $Q_k(x,t) = \sum_{j,i=0}^{k} c_{i,j}^{(k)}\, t^i\, x^j$, *then*

$$c_{i,j}^{(k)} \simeq \binom{k}{j} \sum_{\lambda \vdash k-j} \mathrm{d}_\lambda \alpha_\lambda (-1)^{2l(\lambda)+i}\, s(l(\lambda), i) \tag{3.12}$$

where $s(l(\lambda), i)$ *are the Stirling numbers of the first kind.*

Proof. Corollary 3.10 - (i) states

$$Q_k(x,t) \simeq \sum_{j=0}^{k} \binom{k}{j} (-t.\alpha)^{k-j} x^j. \tag{3.13}$$

On the other hand, equivalence (1.21) guarantees that

$$\begin{aligned} (-t.\alpha)^{k-j} &\simeq \sum_{\lambda \vdash k-j} a_\lambda \mathrm{d}_\lambda (-1)^{l(\lambda)} \langle t \rangle_{l(\lambda)} \\ &\simeq \sum_{\lambda \vdash k-j} a_\lambda \mathrm{d}_\lambda \sum_{i=0}^{l(\lambda)} (-1)^{i+2l(\lambda)} s(l(\lambda), i) t^i, \end{aligned} \tag{3.14}$$

since we have

$$\begin{aligned}\langle t\rangle_{l(\lambda)} = (-1)^{l(\lambda)}(-t)_{l(\lambda)} &= (-1)^{l(\lambda)}\sum_{i=0}^{l(\lambda)} s(l(\lambda),i)(-t)^i \\ &= (-1)^{l(\lambda)}\sum_{i=0}^{l(\lambda)} s(l(\lambda),i)(-1)^i t^i \\ &= \sum_{i=0}^{l(\lambda)}(-1)^{i+l(\lambda)} s(l(\lambda),i)t^i,\end{aligned}$$

where $s(l(\lambda),i)$ is the i-th Stirling number of the first kind.

By replacing (3.14) in (3.13), we have

$$Q_k(x,t) \simeq \sum_{j=0}^{k}\binom{k}{j}\sum_{\lambda\vdash k-j} a_\lambda \mathrm{d}_\lambda \sum_{i=0}^{l(\lambda)}(-1)^{i+l(\lambda)} s(l(\lambda),i)t^i x^j. \tag{3.15}$$

Observe that $s(l(\lambda),i)=0$, for all $l(\lambda)<i\leq k$, then we can take the summation in (3.15) ranging over the set $\{0,\ldots,k\}$ instead of over the set $\{0,\ldots,l(\lambda)\}$, hence we have

$$\begin{aligned}Q_k(x,t) &\simeq \sum_{j=0}^{k}\binom{k}{j}\sum_{\lambda\vdash k-j} a_\lambda \mathrm{d}_\lambda \sum_{i=0}^{k}(-1)^{i+l(\lambda)} s(l(\lambda),i)t^i x^j \\ &\simeq \sum_{i,j=0}^{k}\left[\binom{k}{j}\sum_{\lambda\vdash k-j} a_\lambda \mathrm{d}_\lambda (-1)^{i+l(\lambda)} s(l(\lambda),i)t^i\right] x^j\end{aligned}$$

and the result follows. □

Remark 3.12. Since $Q_k(x,t)$ is of degree k, for any k, every linear combination of time-space harmonic polynomials $\{Q_k(x,t)\}_{k\geq 1}$ is a time-space harmonic polynomial with respect to $\{t.\alpha\}_{t\geq 0}$.

Theorem 3.13. *A polynomial $P(x,t)=\sum_{j=0}^{k} p_j(t)\,x^j$, of degree k for all $t\geq 0$ is a time-space harmonic polynomial with respect to $\{t.\alpha\}_{t\geq 0}$ if and only if*

$$p_j(t) = \sum_{i=j}^{k}\binom{i}{j} p_i(0)\, E[(-t.\alpha)^{i-j}], \quad \text{for } j=0,\ldots,k. \tag{3.16}$$

Proof. Assume $P(x,t) = \sum_{j=0}^{k} p_j(t)\, x^j$ a polynomial whose coefficients satisfy equivalence (3.16). Then we have

$$
\begin{aligned}
P(x,t) &\simeq \sum_{j=0}^{k}\sum_{i=j}^{k}\binom{i}{j} p_i(0)\,(-t.\alpha)^{i-j}x^j \\
&\simeq \sum_{i=0}^{k}\binom{i}{0} p_i(0)\,(-t.\alpha)^{i}x^0 + \sum_{i=1}^{k}\binom{i}{1} p_i(0)\,(-t.\alpha)^{i-1}x \\
&+ \sum_{i=2}^{k}\binom{i}{2} p_i(0)\,(-t.\alpha)^{i-2}x^2 + \cdots + \\
&+ \sum_{i=k-1}^{k}\binom{i}{k-1} p_i(0)\,(-t.\alpha)^{i-k+1}x^{k-1} + \sum_{i=k}^{k}\binom{i}{j} p_i(0)\,(-t.\alpha)^{i-k}x^k \\
&= \binom{0}{0}p_0(0)(-t.\alpha)^0x^0 + \binom{1}{0}p_1(0)(-t.\alpha)x^0 + \binom{2}{0}p_2(0)(-t.\alpha)^2x^0 \\
&= \binom{0}{0}p_0(0)(-t.\alpha)^0x^0 + p_1(0)\left[\binom{1}{0}(-t.\alpha)x^0 + \binom{1}{1}(-t.\alpha)^0x\right] \\
&+ p_2(0)\left[\binom{2}{0}(-t.\alpha)^2x^0 + \binom{2}{1}(-t.\alpha)x + \binom{2}{2}(-t.\alpha)^0x^2\right] \\
&+ \cdots + p_k(0)\left[\binom{k}{0}(-t.\alpha)^kx^0 + \binom{k}{1}(-t.\alpha)^{k-1}x + \binom{k}{2}(-t.\alpha)^{k-2}x^2\right. \\
&\left. + \cdots + \binom{k}{k-1}(-t.\alpha)x^{k-1} + \binom{k}{k}(-t.\alpha)^0x^k\right] \\
&= \sum_{j=0}^{k} p_j(0)\sum_{i=0}^{j}\binom{j}{i}(-t.\alpha)^{j-i}x^i = \sum_{j=0}^{k} p_j(0)(x-t.\alpha)^j.
\end{aligned}
$$

Since $P(x,t)$ results to be a linear combination of $\{Q_k(x,t)\}_{k\geq 1}$, then $P(x,t)$ is a time-space harmonic polynomial with respect to $\{t.\alpha\}_{t\geq 0}$, thanks to Remark 3.12.

Vice versa, if $P(x,t) = \sum_{j=0}^{k} p_j(t)\, x^j$ is a time-space harmonic polynomial with respect to $\{t.\alpha\}_{t\geq 0}$, then we can think of it as a linear combination of suitable time-space harmonic polynomials, that is,

$$P(x,t) = \sum_{i=0}^{k} c_i E[(x-t.\alpha)^i],$$

with $\{c_i\} \in \mathbb{R}$. Therefore, by retracing the steps above, for $j = 0, \ldots, k$ we have

$$
\begin{aligned}
P(x,t) &\simeq \sum_{i=0}^{k} c_i \sum_{j=0}^{i} \binom{i}{j} (-t.\alpha)^{i-j} x^j \\
&\simeq c_0 \binom{0}{0} (-t.\alpha)^0 x^0 + c_1 \left[\binom{1}{0} (-t.\alpha) x^0 + \binom{1}{1} (-t.\alpha)^0 x \right] \\
&+ c_2 \left[\binom{2}{0} (-t.\alpha)^2 x^0 + \binom{2}{1} (-t.\alpha) x + \binom{2}{2} (-t.\alpha)^0 x^2 \right] \\
&+ \cdots + c_k \left[\binom{k}{0} (-t.\alpha)^k x^0 + \binom{k}{1} (-t.\alpha)^{k-1} x + \binom{k}{2} (-t.\alpha)^{k-2} x^2 \right. \\
&\left. + \cdots + \binom{k}{k} (-t.\alpha)^0 x^k \right] \\
&\simeq x^0 \left[c_0 \binom{0}{0} (-t.\alpha)^0 + c_1 \binom{1}{0} (-t.\alpha) + c_2 \binom{2}{0} (-t.\alpha)^2 \right. \\
&\left. + \cdots + c_k \binom{k}{0} (-t.\alpha)^k \right] + x \left[c_1 \binom{1}{1} (-t.\alpha)^0 + c_1 \binom{2}{1} (-t.\alpha) \right. \\
&\left. + \cdots + c_k \binom{k}{1} (-t.\alpha)^{k-1} \right] + \cdots + x^k c_k \binom{k}{k} (-t.\alpha)^0 \\
&\simeq x^0 \sum_{i=0}^{k} c_i \binom{i}{0} (-t.\alpha)^i + x \sum_{i=1}^{k} c_i \binom{i}{1} (-t.\alpha)^{i-1} \\
&+ \cdots + x^k x^0 \sum_{i=k}^{k} c_i \binom{i}{k} (-t.\alpha)^{k-i} \simeq \sum_{j=0}^{k} \sum_{i=j}^{k} c_i \binom{i}{j} (-t.\alpha)^{i-j} x^j.
\end{aligned}
$$

Hence, for all $j = 0, \ldots, k$, we take the evaluation of both sides and we have

$$q_j(t) = \sum_{i=j}^{k} c_i \binom{i}{j} E[(-t.\alpha)^{i-j}].$$

To complete the proof, we have to verify that $c_i = p_i(0)$, for all $i \leq k$. To this aim, set $t = 0$, then

$$q_j(0) = \sum_{i=j}^{k} \binom{i}{j} c_i \, E[(-0.\alpha)^{i-j}] = \sum_{i=j}^{k} \binom{i}{j} c_i \, E[\epsilon^{i-j}].$$

Since $\epsilon^{i-j} \simeq \delta_{ij}$, where δ_{ij} is the Kronecker symbol, we obtain that all the terms with $j \neq 1$ vanish, then

$$q_j(0) = \binom{j}{j} c_j = c_j.$$

□

Corollary 3.14. *If $P(x,t) = \sum_{j=0}^{k} p_j(t)\, x^j$ is a polynomial of degree k for all $t \geq 0$, then there exists an umbra α such that $P(x,t)$ is a time-space harmonic polynomial with respect to $\{t.\alpha\}_{t\geq 0}$.*

Proof. By Theorem 3.13, the polynomial $P(x,t)$ is time-space harmonic with respect to the Lévy process $\{t.\alpha\}_{t\geq 0}$ if and only if equation (3.16) holds, for all $j = 0, \ldots, k$. This is equivalent to consider the following system

$$\begin{cases} p_0(t) = \sum_{i=0}^{k} \binom{i}{0}\, p_i(0)\, E[(-t.\alpha)^i] \\ p_1(t) = \sum_{i=1}^{k} \binom{i}{1}\, p_i(0)\, E[(-t.\alpha)^{i-1}] \\ \qquad \vdots \\ p_{k-1}(t) = \sum_{i=k-1}^{k} \binom{i}{k-1}\, p_i(0)\, E[(-t.\alpha)^{i-k+1}] \\ p_k(t) = \sum_{i=k}^{k} \binom{i}{k}\, p_i(0)\, E[(-t.\alpha)^{i-k}] \end{cases} \tag{3.17}$$

We solve the system (3.17) starting from the bottom and we obtain

$$p_k(t) = \binom{k}{k} p_k(0)\, E[(-t.\alpha)^0] = p_k(0).$$

We replace this last result in the $(k-1)$-th equation of the system, then

$$p_{k-1}(t) = \binom{k-1}{k-1} p_{k-1}(0)\, E[(-t.\alpha)^{k-1-k+1}] + \binom{k}{k-1} p_k(0)\, E[(-t.\alpha)^{k-k+1}]$$
$$\Rightarrow\ E[(-t.\alpha)] = \frac{p_{k-1}(t) - p_k(0)}{k p_k(0)}.$$

By repeating this trick for the whole system, we have the expression of the moments of the umbra $t.\alpha$. □

Remark 3.15. Up to now, we have considered polynomials of form $E[(x - t.\alpha)^k]$ and we have seen that they are time-space harmonic with respect to Lévy processes, umbrally represented by the family of auxiliary umbrae $\{t.\alpha\}_{t\geq 0}$. Now, we could ask what happens if we consider polynomials of form $E[(x + t.\alpha)^k]$.

By using the same tools, one can prove that polynomials of form $E[(x + t.\alpha)^k]$ are time-space harmonic with respect to the family of auxiliary umbrae $\{-t.\alpha\}_{t\geq 0}$, since we have

$$x + t.\alpha \equiv x + (-1).(-1).t.\alpha \equiv x - (-t.\alpha).$$

On the other hand, in Chapter 1 we have already introduced the auxiliary umbra $-t.\alpha$ and we have shown in equivalence (1.20) that $-t.\alpha \equiv t.(-1.\alpha)$, so that the family of umbrae $\{-t.\alpha\}_{t\geq 0}$ is still the umbral counterpart of a Lévy process.

3.2 Connection with classical, boolean and free cumulants

Proposition 3.16. *For the sequence of polynomials $\{Q_k(x,t)\}$ umbrally represented by the time-space harmonic polynomial umbra $x - t.\alpha$, we have*

$$Q_k(x,t) = Y_k(h_1,\ldots,h_k), \tag{3.18}$$

with Y_k the complete exponential Bell polynomials and $\{h_n\}$ the sequence of cumulants of the umbra $-t.\alpha$.

Proof. By equation (1.30), we have

$$E[(\beta.\xi)^k] = Y_k(g_1,\ldots,g_k),$$

with $g_n = E[\xi^n]$, for all nonnegative integers n. Therefore, we will prove equation (3.18) if we show that $x + t.(-1.\alpha) \equiv \beta.\xi$, for some umbra ξ.

Choose as umbra ξ the umbra $\kappa_{x.u} \dot{+} \kappa_{(-t.\alpha)}$, where $\kappa_{x.u}$ is the cumulant umbra of $x.u$ and $\kappa_{(-t.\alpha)}$ is the cumulant umbra of $-t.\alpha$.

Thanks to the first equivalence in (1.4), we have

$$E[(\kappa_{x.u} \dot{+} \kappa_{(-t.\alpha)})^k] = \begin{cases} x + h_1, & \text{if } k = 1 \\ h_k, & \text{if } k > 1. \end{cases}$$

The result follows by applying Proposition 1.22, since

$$x + t(-1.\alpha) \equiv \beta.\kappa_{x.u} + \beta.\kappa_{(-t.\alpha)} \equiv \beta.(\kappa_{x.u} \dot{+} \kappa_{(-t.\alpha)}).$$

□

In Chapter 1 we have proved that every umbra α is the partition umbra of its cumulant umbra, that is, $\alpha \equiv \beta.\kappa_\alpha$. The following theorem explicitly states the connection between time-space harmonic polynomials with respect to the Lévy process $\{X_t\}_{t\geq 0}$ and the sequence of cumulants of X_1.

Theorem 3.17. *For all nonnegative integers k, the family of polynomials*

$$Q_k(x,t) = E[(x - t.\beta.\kappa_\alpha)^k] \in \mathbb{R}[x] \tag{3.19}$$

is time-space harmonic with respect to $\{t.\alpha\}_{t\geq 0}$, where κ_α is the α- cumulant umbra.

Proof. The result follows since in (3.7) $E[(-t.\alpha)^j] = E[(-t.\beta.\kappa_\alpha)^j]$, for all nonnegative integers j. □

Remark 3.18. Again in this case, every linear combination of time-space harmonic polynomials with respect to the Lévy process $\{t.\beta.\kappa_\alpha\}_{t\geq 0}$ is a time-space harmonic polynomial with respect to $\{t.\alpha\}_{t\geq 0}$.

Sheffer umbra and Appell umbra. Given two distinct umbrae α and γ with $E[\gamma] \neq 0$, we say that a polynomial umbra σ_x is a *Sheffer umbra* for (α, γ) if and only if

$$\sigma_x \equiv (-1.\alpha + x.u).\gamma^*, \tag{3.20}$$

where $\gamma^* \equiv \beta.\gamma^{<-1>}$ is the *adjoint umbra* of γ [22].

The generating function of a Sheffer umbra is [22]

$$f(\sigma_x, z) = \frac{1}{f(\alpha, f^{<-1>}(\gamma, z) - 1)} e^{x(f^{<-1>}(\alpha,z)-1)}. \tag{3.21}$$

If $\gamma \equiv \chi$ in equivalence (3.20), the umbra

$$-1.\alpha + x.u \tag{3.22}$$

is the *Appell umbra* of α, with generating function

$$f(-1.\alpha + x.u, z) = \frac{1}{f(\alpha, z)} e^{xz}. \tag{3.23}$$

Theorem 3.19 (The Appell identity)**.** *The polynomial umbra σ_x is an Appell umbra for some umbra α if and only if*

$$\sigma_{x+y} \equiv \sigma_x + y.$$

Corollary 3.20. *The polynomial umbra σ_x is an Appell umbra for some umbra α if and only if*

$$\sigma^k_{\chi+x.u} \simeq \sigma^k_x + k\sigma^{k-1}_x. \tag{3.24}$$

Corollary 3.20 says that, when we replace x with $\chi + x.u$ in the Appell umbra, the singleton umbra χ acts as a derivative operator, that is, [22]

$$\frac{d}{dx}(x+\alpha)^k \simeq (x+\chi+\alpha)^k - (x+\alpha)^k \simeq k(x+\alpha)^{k-1}. \tag{3.25}$$

Proposition 3.21. *The umbra $x + t.\alpha$ is the Appell umbra of $-t.\alpha$.*

Proof. The result follows by applying (3.25) with the umbra α replaced by the auxiliary umbra $-t.\alpha$ and the derivative operator replaced by the partial derivative operator. Indeed, if we set $Q_k(x,t) = E[(x+t.\alpha)^k]$ for all nonnegative integers k, we have

$$\begin{aligned}
\frac{\partial}{\partial x}Q_k(x,t) &= \frac{\partial}{\partial x}E[(x+t.\alpha)^k] = E\left[\frac{\partial}{\partial x}(x+t.\alpha)^k\right] \\
&= E[(x+\chi+t.\alpha)^k] - E[(x+t.\alpha)^k] \\
&= E\left[\sum_{j=0}^{k}\binom{k}{j}(x+t.\alpha)^j\chi^{k-j}\right] \\
&= \sum_{j=0}^{k}\binom{k}{j}E[(x+t.\alpha)^j]E[\chi^{k-j}] \\
&= \binom{k}{k-1}E[(x+t.\alpha)^{k-1}],
\end{aligned}$$

where the last equality holds, due to the moments of the singleton umbra.

Hence, by taking the evaluation of both sides, we have

$$\frac{\partial}{\partial x}Q_k(x,t) = kQ_{k-1}(x,t). \tag{3.26}$$

□

Thanks to (3.26), the sequence of polynomials $\{Q_k(x,t)\}$ is an Appell sequence with respect to x.

By virtue of Remark 3.15, we can also consider polynomials of form $Q_k(x,t) = E[(x+t.\beta.\kappa_\alpha)^k]$, which are time-space harmonic with respect to $\{-t.\beta.\kappa_\alpha\}_{t\geq 0}$.

For the sake of simplicity, in the following results we will refer to this last family of time-space harmonic polynomials.

Proposition 3.22. *If $Q_k(x,t) = E[(x+t.\beta.\kappa_\alpha)^k]$ for all nonnegative integers k, then* *(i) (Sheffer property)*

$$Q_k(x, t+s) = \sum_{j=0}^{k} \binom{k}{j} P_j(s) Q_{k-j}(x, t), \tag{3.27}$$

where $P_j(s) = Q_j(0, s)$, *for all* $j = 0, \ldots, k$;

(ii) (Uniqueness property)

$$Q_k(x, 0) = x^k; \tag{3.28}$$

(iii) (Homogeneity property) for all $c \in \mathbb{R}$, *if we denote by* $Q_k^{(\alpha)}(x, t) = E[(x + t.\beta.\kappa_\alpha)^k]$, *then*

$$Q_k^{(\alpha)}(x, ct) = Q_k^{(c\alpha)}(x, t). \tag{3.29}$$

Proof. **(i)** Thanks to the distributive property given in Proposition 1.14-(iv), we have

$$\begin{aligned} Q_k(x, t+s) &= E[(x + (t+s).\beta.\kappa_\alpha)^k] = E[(x + t.\beta.\kappa_\alpha + s.\beta.\kappa_\alpha)^k] \\ &= E\left[\sum_{j=0}^{k} \binom{k}{j} (s.\beta.\kappa_\alpha)^j (x + t.\beta.\kappa_\alpha)^{k-j}\right] \\ &= \sum_{j=0}^{k} \binom{k}{j} E[(s.\beta.\kappa_\alpha)^j] E[(x + t.\beta.\kappa_\alpha)^{k-j}]. \end{aligned}$$

Let us observe that

$$(s.\beta.\kappa_\alpha)^j \simeq (0 + s.\beta.\kappa_\alpha)^j \simeq Q_j(0, s) = P_j(s),$$

so that, by taking the evaluation of both sides, equation (3.27) follows.

(ii) We have

$$\begin{aligned} Q_k(x, 0) &= E[(x + 0.\beta.\kappa_\alpha)^k] = E[(x + \epsilon)^k] \\ &= E\left[\sum_{j=0}^{k} \binom{k}{j} x^j \epsilon^{k-j}\right] = \sum_{j=0}^{k} \binom{k}{j} x^j E[\epsilon^{k-j}] = x^k, \end{aligned}$$

where the last equality holds since $E[\epsilon^{k-j}] = 1$ if $j = k$ and it is 0, otherwise.

(iii) Via generating functions, we have

$$\begin{aligned} f((ct).\beta.\kappa_\alpha, z) &= \exp\{(ct)[1+\log(f(\alpha,z))-1]\} \\ &= \exp\{(ct)\log(f(\alpha,z))\} \\ &= \exp\{\log(f(\alpha,z)^{ct})\} = f(\alpha,z)^{ct}. \end{aligned} \tag{3.30}$$

On the other hand, $t.\beta.\kappa_{c\alpha} \equiv t.\beta.(c\kappa_\alpha)$, thanks to Proposition 2.7-(i), so that

$$\begin{aligned} f(t.\beta.\kappa_{c\alpha}, z) &= \exp\{t[1+c\log(f(\alpha,z))-1]\} \\ &= \exp\{(ct)\log(f(\alpha,z))\} \\ &= \exp\{\log(f(\alpha,z))^{ct}\} = f(\alpha,z)^{ct}. \end{aligned} \tag{3.31}$$

By comparing (3.30) and (3.31), we obtain $(ct).\beta.\kappa_\alpha \equiv t.\beta.\kappa_{c\alpha}$, by which (3.29) follows.

□

Remark 3.23. Let us consider polynomials of the form

$$\tilde{Q}_k(x,t) = E[m(x+t.\beta.\kappa_\alpha)^k], \tag{3.32}$$

with $m \in \mathbb{R}$. Since $\tilde{Q}_k(x,t) = mQ_k(x,t)$, by applying equation (3.26), we have

$$\begin{aligned} \frac{\partial}{\partial x}\tilde{Q}_k(x,t) &= \frac{\partial}{\partial x}\left(mQ_k(x,t)\right) = m\frac{\partial}{\partial x}\left(Q_k(x,t)\right) \\ &= kmQ_{k-1}(x,t) = m\tilde{Q}_k(x,t), \end{aligned}$$

Hence, the Appel property holds.

On the other hand, Sheffer property does not hold, since

$$\begin{aligned} \tilde{Q}_k(x,t+s) &= E[d(x+(t+s).\beta.\kappa_\alpha)^k] = mQ_k(x,t+s) \\ &= m\sum_{j=0}^{k}\binom{k}{j}P_j(s)Q_{k-j}(x,t), \end{aligned}$$

which is different from

$$\sum_{j=0}^{k}\binom{k}{j}\tilde{P}_j(s)\tilde{Q}_{k-j}(x,t),$$

with $\tilde{P}_j(s) = \tilde{Q}_j(0,s)$.

As regards to the homogeneity property, we have

$$\begin{aligned}\tilde{Q}_k^{(\alpha)}(x, ct) &= E[m(x + ct.\beta.\kappa_\alpha)^k] \\ &= E[m(x + t.\beta.\kappa_{c\alpha})^k] = \tilde{Q}_k^{(c\alpha)}(x, t),\end{aligned}$$

thanks to Proposition 3.22-(iii).

Let us consider now polynomials of the form

$$\bar{Q}_k(x, t) = E[m^k(x + t.\beta.\kappa_\alpha)^k], \tag{3.33}$$

with $m \in \mathbb{R}$. We have $\bar{Q}_k(x, t) = m^k Q_k(x, t)$, so that

$$\begin{aligned}\frac{\partial}{\partial x}\bar{Q}_k(x, t) &= m^k \frac{\partial}{\partial x} Q_k(x, t) = m^k k Q_{k-1}(x, t) \\ &= k m^{k-1+1} Q_{k-1}(x, t) = k m m^{k-1} Q_{k-1}(x, t) \\ &= k m \bar{Q}_{k-1}(x, t),\end{aligned}$$

which is different from $k\bar{Q}_k(x, t)$, therefore the Appell property does not hold.

The Sheffer property holds, in fact, by applying Proposition 3.22-(i), we have

$$\begin{aligned}\bar{Q}_k(x, t + s) &= E[m^k(x + t.\beta.\kappa_\alpha)^k] = m^k Q_k(x, t + s) \\ &= m^k \sum_{j=0}^{k} \binom{k}{j} P_j(s) Q_{k-j}(x, t) \\ &= \sum_{j=0}^{k} \binom{k}{j} m^{k-j+j} P_j(s) Q_{k-j}(x, t) \\ &= \sum_{j=0}^{k} \binom{k}{j} m^j P_j(s) m^{k-j} Q_{k-j}(x, t) \\ &= \sum_{j=0}^{k} \binom{k}{j} \bar{P}_j(s) \bar{Q}_{k-j}(x, t).\end{aligned}$$

Finally, let us observe that

$$\bar{Q}_k^{(\alpha)}(x, ct) = m^k Q_k^{(\alpha)}(x, ct) = E[m^k(x + t.\beta.\kappa_{c\alpha})^k] = \bar{Q}_k^{(c\alpha)}(x, t)$$

by using Proposition 3.22-(iii), so the homogeneity property holds.

Summarizing, we have:

- Appell property and homogeneity property hold for families of polynomials of form $\{\tilde{Q}_k(x,t)\}$ in (3.32).
- Sheffer property and homogeneity property hold for families of polynomials of form $\{\bar{Q}_k(x,t)\}$ in (3.33).

Let us observe that $\tilde{Q}_k(x,t)$ and $\bar{Q}_k(x,t)$ are more general families of time-space harmonic polynomials than the basis $\{Q_k(x,t)\}$.

Corollary 3.24. *For the coefficients* $\{q_j^{(k)}(t)\}$ *of the sequence of time-space harmonic polynomials* $Q_k(x,t) = E[(x + t.\beta.\kappa_\alpha)^k]$, *we have*

(i) for all $j = 0, \ldots, k-1$, $q_j^{(k)}(0) = 0$;

(ii) for all $j = 0, \ldots, k$, $j q_j^{(k)}(t) = k q_{j-1}^{(k-1)}(t)$;

(iii) for all $j = 0, \ldots, k$,

$$\frac{\mathrm{d}}{\mathrm{dt}} q_j^{(k)}(t) = \sum_{i=1}^{k-j} \binom{k}{j} h_i q_j^{(k-j)}(t),$$

where $\{h_i\}$ *is the sequence of cumulants of the umbra* α.

Proof. **(i)** By applying the binomial theorem, we have

$$Q_k(x,t) \simeq \sum_{j=0}^{k} \binom{k}{j} (t.\beta.\kappa_\alpha)^{k-j} x^j,$$

and so $q_j^{(k)}(0) \simeq \binom{k}{j}(0.\beta.\kappa_\alpha)^{k-j} \simeq \binom{k}{j}\epsilon^{k-j} \simeq 0$, for all $j < k$.
The result follows by taking the evaluation of both sides.

(ii) For all $j = 0, \ldots, k$, we have

$$\begin{aligned} j q_j^{(k)}(t) &\simeq j \binom{k}{j} (t.\beta.\kappa_\alpha)^{k-j} \simeq \frac{k(k-1)!}{(j-1)!(k-j)!} (t.\beta.\kappa_\alpha)^{k-j} \\ &\simeq k \binom{k-1}{j-1} (t.\beta.\kappa_\alpha)^{(k-1)-(j-1)} \simeq k q_{j-1}^{(k-1)}(t) \end{aligned}$$

and the result follows by taking the evaluation E of both sides.

(ii) For all $j = 0, \ldots, k$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}} q_j^{(k)}(t) &\simeq \frac{\mathrm{d}}{\mathrm{dt}} \left[\binom{k}{j} (t.\beta.\kappa_\alpha)^{k-j} \right] \\
&\simeq \binom{k}{j} \left\{ [(t+\chi).\beta.\kappa_\alpha]^{k-j} - [t.\beta.\kappa_\alpha]^{k-j} \right\} \\
&\simeq \binom{k}{j} \{ (t.\beta.\kappa_\alpha + \kappa_\alpha)^{k-j} - (t.\beta.\kappa_\alpha)^{k-j} \} \\
&\simeq \binom{k}{j} \left\{ \sum_{i=0}^{k-j} \binom{k-j}{i} \kappa_\alpha^i (t.\beta.\kappa_\alpha)^{k-j-i} - (t.\beta.\kappa_\alpha)^{k-j} \right\} \\
&\simeq \binom{k}{j} \sum_{i=1}^{k-j} \binom{k-j}{i} \kappa_\alpha^i (t.\beta.\kappa_\alpha)^{k-j-i} \\
&\simeq \sum_{i=1}^{k-j} \binom{k}{i} \binom{k-i}{j} \kappa_\alpha^i (t.\beta.\kappa_\alpha)^{k-j-i}.
\end{aligned}$$

Hence, by setting $E[\kappa_\alpha^i] = h_i$ and by applying the evaluation functional E, we have

$$\frac{\mathrm{d}}{\mathrm{dt}} q_j^{(k)}(t) = \sum_{i=1}^{k-j} \binom{k}{i} h_i q_j^{(k-i)}(t).$$

□

Connection with boolean and free cumulants. Let us consider the α-boolean cumulant umbra η_α given by equivalence (2.11). Thanks to Theorem 3.7, the polynomials

$$Q_k(x,t) = E[(x - t.\bar{u}.\beta.\bar{\eta}_\alpha)^k]$$

are time-space harmonic polynomials with respect to the family $\{t.\bar{\alpha}\}_{t\geq 0}$.

Consider now the α-free cumulant umbra $\mathfrak{K}_\alpha$ given by equivalence (2.19). This umbra allows us a different parametrization of $Q_k(x,t) = E[(x - t.\bar{\alpha})^k]$.

Indeed, the polynomials

$$Q_k(x,t) = E[(x + t.(-1.\bar{\mathfrak{K}}_\alpha).\beta.(-1.\bar{\mathfrak{K}}_\alpha)_D^{<-1>})^k]$$

are time-space harmonic polynomials with respect to the family $\{t.\bar{\alpha}\}_{t\geq 0}$.

Let us remark that the computations of the coefficients of these polynomials could be easily performed by using the algorithms proposed in Subsection 2.2.1.

3.2.1 Discrete case

In the following, assume $Q_k(x,t)$ as in (3.10) and denote by $\{a_n\}$ the sequence of moments umbrally represented by the umbra α in (3.6).

We start to consider the case in which the parameter t is replaced by a nonnegative integer n, so the coefficients $q_j^{(k)}(n)$ of time-space harmonic polynomials satisfy further properties thanks to the umbral representation (3.6). Some of these properties have been used in [32] in order to prove the existence of time-space harmonic polynomials. Differently from [32], our proofs do not rely on induction and result to be simplified.

Note that all these results here are recovered as corollaries, whereas in [32] are necessary to build the whole theory.

Proposition 3.25. *For all $j = 1, \ldots, k$, we have*

$$q_j^{(k)}(n-1) = \sum_{i=j}^{k} \binom{i}{j} q_i^{(k)}(n) a_{i-j}. \tag{3.34}$$

Proof. The result follows from Corollary 3.10-(i), indeed

$$\begin{aligned} q_j^{(k)}(n-1) &\simeq \binom{k}{j}[-(n-1).\alpha]^{k-j} \simeq \binom{k}{j}(-n.\alpha+\alpha)^{k-j} \\ &\simeq \binom{k}{j}\sum_{s=0}^{k-j}\binom{k-j}{s}(-n.\alpha)^{k-j-s}\alpha^s \\ &\simeq \sum_{s=0}^{k-j}\frac{k!}{s!j!(k-j-s)!}(-n.\alpha)^{k-j-s}\alpha^s. \end{aligned}$$

We make a change of variable, by setting $j+s=i$, so that $s=i-j$ and $j=i-s$. Furthermore,when the “old” variable s ranges over the set $\{0,\ldots,k-j\}$, the “new” variable i ranges over the set $\{j,\ldots,k\}$. Then, we have

$$\begin{aligned} q_j^{(k)}(n-1) &\simeq \sum_{i=j}^{k}\frac{k!}{j!}\frac{1}{(i-j)!(k-i)!}(-n.\alpha)^{k-i}\alpha^{i-j} \\ &\simeq \sum_{i=j}^{k}\binom{k}{i}\binom{i}{j}(-n.\alpha)^{k-i}\alpha^{i-j} \\ &\simeq \sum_{i=j}^{k}\binom{i}{j}\alpha^{i-j}\binom{k}{i}(-n.\alpha)^{k-i} \end{aligned}$$

$$\simeq \sum_{i=j}^{k} \binom{i}{j} \alpha^{i-j} q_i^{(k)}(n),$$

so we have

$$q_j^{(k)}(n-1) \simeq \sum_{i=j}^{k} \binom{i}{j} \alpha^{i-j} q_i^{(k)}(n).$$

The result follows by taking the evaluation of both sides in the previous equivalence. □

Corollary 3.26. *We have*

$$a_k q_k^{(k)}(n) = q_0^{(k)}(n-1) - \sum_{j=0}^{k-1} a_j q_j^{(k)}(n).$$

Proof. By using Proposition 3.25, we have

$$q_0^{(k)}(n-1) = \sum_{i=0}^{k} \binom{i}{0} q_i^{(k)}(n) a_i = \sum_{i=0}^{k} q_i^{(k)}(n) a_i = a_k q_k^{(k)}(n) \sum_{i=0}^{k-1} q_i^{(k)}(n) a_i.$$

The result follows because

$$q_0^{(k)}(n-1) - \sum_{j=0}^{k-1} a_j q_j^{(k)}(n) = \alpha^k q_k^{(k)}(n) + \sum_{j=0}^{k-1} a_j q_j^{(k)}(n)$$
$$- \sum_{j=0}^{k-1} a_j q_j^{(k)}(t) = a_k q_k^{(k)}(n).$$

□

Proposition 3.27. *We have*

$$a_k = q_0^{(k)}(n-1) - \sum_{j=0}^{k-1} a_j q_j^{(k)}(n).$$

Proof. We have

$$\sum_{j=0}^{k-1} \alpha^j q_j^{(k)}(n) \simeq \sum_{j=0}^{k-1} \alpha^j \binom{k}{j} (-n.\alpha)^{k-j} \sum_{j=0}^{k} \alpha^j \binom{k}{j} (-n.\alpha)^{k-j} - \alpha^k$$
$$\simeq (\alpha - n.\alpha)^k - \alpha^k \simeq (-(n-1).\alpha)^k - \alpha^k.$$

From Corollary 3.10, we have $q_0^{(k)}(n-1) \simeq \binom{k}{0}(-(n-1).\alpha)^k$, therefore

$$q_0^{(k)}(n-1) - \sum_{j=0}^{k-1} \alpha^j q_j^{(k)}(t) \simeq \binom{k}{0}(-(n-1).\alpha)^k - (-(n-1).\alpha)^k + \alpha^k \simeq \alpha^k$$

an the result follows by taking the evaluation of both sides in the previous equivalence. □

Proposition 3.28. *We have*

$$q_0^{(k)}(n) + \sum_{l=1}^{n}\sum_{j=1}^{k} a_j q_j^{(k)}(l) = 0.$$

Proof. From Corollary 3.10, we have

$$\begin{aligned}
\sum_{l=1}^{n}\sum_{j=1}^{k}\alpha^j q_j^{(k)}(l) &\simeq \sum_{j=1}^{k}\alpha^j\binom{k}{j}(-1.\alpha)^{k-j} + \sum_{j=1}^{k}\alpha^j\binom{k}{j}(-2.\alpha)^{k-j} \\
&\quad + \cdots + \sum_{j=1}^{k}\alpha^j\binom{k}{j}(-n.\alpha)^{k-j} \\
&\simeq (\alpha - 1.\alpha)^k - (-1.\alpha)^k + (\alpha - 2.\alpha)^k - (-2.\alpha)^k \\
&\quad + \cdots + (\alpha - (n-1).\alpha)^k - (-(n-1).\alpha)^k + (\alpha - n.\alpha)^k - (-n.\alpha)^k.
\end{aligned}$$

By expanding the powers, we have

$$\begin{aligned}
&\epsilon^k - (-1.\alpha)^k + (-1.\alpha)^k - (-2.\alpha)^k + (-2.\alpha)^k \\
&\quad + \cdots + (-(n-2).\alpha)^k - (-(n-1).\alpha)^k + (-(n-1).\alpha)^k - (-n.\alpha)^k \\
&\quad \simeq \epsilon^k - (-n.\alpha)^k \simeq -(-n.\alpha)^k.
\end{aligned}$$

Therefore,

$$\begin{aligned}
q_0^{(k)}(n) + \sum_{l=1}^{n}\sum_{j=1}^{k} a_j q_j^{(k)}(l) &\simeq \binom{k}{0}(-n.\alpha)^k - (-n.\alpha)^k \\
&\simeq (-n.\alpha)^k - (-n.\alpha)^k \simeq 0
\end{aligned}$$

and the result follows by taking the evaluation of both sides. □

Proposition 3.29. *For all $j = 0, \dots, k-2$, we have*

$$q_j^{(k)}(n) + \sum_{i=j+1}^{k} \binom{i}{j} \sum_{l=1}^{n} a_{i-j} q_i^{(k)}(l) = 0.$$

Proof. From Corollary 3.10, we have

$$\sum_{i=j+1}^{k} \binom{i}{j} \sum_{l=1}^{n} \alpha^{i-j} q_i^{(k)}(l) \simeq \sum_{l=1}^{n} \sum_{i=j+1}^{k} \binom{i}{j} \alpha^{i-j} \binom{k}{j} (-l.\alpha)^{k-i}. \tag{3.35}$$

In (3.35) we have

$$\begin{aligned}
\binom{i}{j} \sum_{i=j+1}^{k} \alpha^{i-j} \binom{k}{j} (-l.\alpha)^{k-i} &\simeq \sum_{l=1}^{n} \alpha^{i-j} \binom{i}{j} \binom{k}{j} (-l.\alpha)^{k-i} \\
&\simeq \sum_{i=j+1}^{k} \frac{i!}{j!(i-j)!} \frac{k!}{i!(k-i)!} \alpha^{i-j} (-l.\alpha)^{k-i} \\
&\simeq \sum_{i=j+1}^{k} \frac{k!}{j!} \frac{1}{(i-j)!(k-i)!} \alpha^{i-j} (-l.\alpha)^{k-i}.
\end{aligned}$$

We make a change of variable, by setting $i - j = s$, so that $i = s + j$. Furthermore,when the “old ”variable i ranges over the set $\{j+1, \dots, k\}$, the “new ”variable s ranges over the set $\{1, \dots, k-j\}$. Then, we have

$$\begin{aligned}
&\sum_{s=1}^{k-j} \frac{k!}{j!} \frac{1}{s!(k-s-j)!} \alpha^{s} (-l.\alpha)^{k-s-j} \\
&\quad \simeq \sum_{s=1}^{k-j} \binom{k}{j} \binom{k-j}{s} \alpha^{s} (-l.\alpha)^{(k-s)-j} \\
&\quad \simeq \binom{k}{j} \left[\sum_{s=0}^{k-j} \binom{k-j}{s} \alpha^{s} (-l.\alpha)^{(k-s)-j} - (-l.\alpha)^{k-j} \right] \\
&\quad \simeq \binom{k}{j} \left[(\alpha - l.\alpha)^{k-j} - (-l.\alpha)^{k-j} \right] \\
&\quad \simeq \binom{k}{j} (-((l-1).\alpha)^{k-j} - \binom{k}{j} (-l.\alpha)^{k-j}) \\
&\quad \simeq q_j^{(k)}(l-1) - \simeq q_j^{(k)}(l),
\end{aligned}$$

where the last equivalence holds by virtue of Corollary 3.10-(i).

Now, we can sum over $l = 1, \dots, n$ and we have

$$\sum_{l=1}^{n} \left(q_j^{(k)}(l-1) - \simeq q_j^{(k)}(l) \right) \simeq q_j^{(k)}(0) - q_j^{(k)}(1) + \cdots + q_j^{(k)}(n-1) + q_j^{(k)}(n).$$

On the other hand, Corollary 3.10-(ii) guarantees that $q_j^{(k)}(0) \simeq 0$, for all $j < k$. Finally, we have

$$q_j^{(k)}(n) + \sum_{i=j+1}^{k} \binom{i}{j} \sum_{l=1}^{n} a_{i-j} q_i^{(k)}(l) \simeq q_j^{(k)}(n) - q_j^{(k)}(n) \simeq 0$$

and the result follows by taking the evaluation of both sides in the previous equivalences. □

Next Proposition follows from Theorem 3.13, after replacing t by n.

Proposition 3.30. *For all $j = 0, \dots, k$, we have*

$$q_j(n) - q_j(0) + \sum_{i=j+1}^{k} \binom{i}{j} \sum_{l=1}^{n} a_{i-j} q_i(l) = 0.$$

Proof. Let us observe that

$$\begin{aligned} \sum_{i=j+1}^{k} \binom{i}{j} \sum_{l=1}^{n} q_i(l) &\simeq \sum_{i=j}^{k} \binom{i}{j} \sum_{l=1}^{n} q_i(l) - \sum_{l=1}^{n} q_j(l) \\ &\simeq \sum_{i=j}^{k} \binom{i}{j} \sum_{l=1}^{n} q_i(l) - q_j(n) - \sum_{l=1}^{n-1} q_j(l), \end{aligned}$$

then we have

$$\begin{aligned} & q_j(n) - q_j(0) + \sum_{i=j+1}^{k} \binom{i}{j} \sum_{l=1}^{n} \alpha^{i-j} q_i(l) \\ &\quad \simeq q_j(n) - q_j(0) + \sum_{i=j}^{k} \binom{i}{j} \sum_{l=1}^{n} \alpha^{i-j} q_i(l) - q_j(n) - \sum_{l=1}^{n-1} q_j(l) \\ &\quad \simeq \sum_{i=j}^{k} \binom{i}{j} \sum_{l=1}^{n} \alpha^{i-j} q_i(l) - q_j(n) - \sum_{l=0}^{n-1} q_j(l) \\ &\quad \simeq \sum_{l=1}^{n} \sum_{i=j}^{k} \binom{i}{j} \alpha^{i-j} \sum_{s=i}^{k} q_(s)(0) q_i^{(s)}(l) - \sum_{l=0}^{n-1} \sum_{s=j}^{k} q_s(0) q_j^{(s)}(l), \end{aligned}$$

where the last equivalence holds by virtue of Theorem 3.13 and Corollary 3.10-(i).

We make a change of variable in the second term, by setting $h = l + 1$, so that $l = h - 1$. Furthermore,when the "old "variable l ranges over the set $\{0, \dots, n-1\}$, the "new "variable h ranges over the set $\{1, \dots, n\}$. For convenience, we rename the new variable h with l. We have

$$\begin{aligned} & q_j(n) - q_j(0) + \sum_{i=j+1}^{k} \binom{i}{j} \sum_{l=1}^{n} \alpha^{i-j} q_i(l) \\ & \quad \simeq \sum_{l=1}^{n} \sum_{i=j}^{k} \binom{i}{j} \alpha^{i-j} \sum_{s=i}^{k} q_(s)(0) q_i^{(s)}(l) - \sum_{l=1}^{n} \sum_{s=j}^{k} q_s(0) q_j^{(s)}(l-1) \\ & \quad \simeq \sum_{l=1}^{n} \sum_{i=j}^{k} \binom{i}{j} \alpha^{i-j} \sum_{s=i}^{k} q_(s)(0) q_i^{(s)}(l) - \sum_{l=1}^{n} \sum_{s=j}^{k} q_s(0) \sum_{i=j}^{s} \binom{i}{j} \alpha^{i-j} q_i^{(s)}(l), \end{aligned} \tag{3.36}$$

thanks to Proposition 3.25.

Look at the second term in the previous summation and consider its general term. We have

$$\begin{aligned} & \sum_{s=j}^{k} q_s(0) \sum_{i=j}^{s} \binom{i}{j} \alpha^{i-j} q_i^{(s)}(l) \\ & \simeq q_j(0) \sum_{i=j}^{j} \binom{i}{j} \alpha^{i-j} q_i^{(j)}(l) + q_{j+1}(0) \sum_{i=j}^{j+1} \binom{i}{j} \alpha^{i-j} q_i^{(j+1)}(l) \\ & \simeq q_{j+2}(0) \sum_{i=j}^{j+2} \binom{i}{j} \alpha^{i-j} q_i^{(j+2)}(l) + q_{j+3}(0) \sum_{i=j}^{j+3} \binom{i}{j} \alpha^{i-j} q_i^{(j+3)}(l) \\ & + \cdots + q_{k-1}(0) \sum_{i=j}^{k-1} \binom{i}{j} \alpha^{i-j} q_i^{(k-1)}(l) + q_k(0) \sum_{i=j}^{k} \binom{i}{j} \alpha^{i-j} q_i^{(k)}(l) \\ & \simeq \binom{j}{j} \alpha^0 \sum_{s=j}^{k} q_s(0) q_j^{(s)}(l) + \binom{j+1}{j} \alpha \sum_{s=j+1}^{k} q_s(0) q_{j+1}^{(s)}(l) \\ & + \binom{j+2}{j} \alpha^2 \sum_{s=j+2}^{k} q_s(0) q_{j+2}^{(s)}(l) + \binom{j+3}{j} \alpha^3 \sum_{s=j+3}^{k} q_s(0) q_{j+3}^{(s)}(l) \\ & + \cdots + \binom{k-1}{j} \alpha^{k-j-1} \sum_{s=k-1}^{k} q_s(0) q_{k-1}^{(s)}(l) + \binom{k}{j} \alpha^{k-j} \sum_{s=k}^{k} q_s(0) q_k^{(s)}(l) \end{aligned}$$

$$\simeq \sum_{m=0}^{k-j} \binom{j+m}{j} \alpha^j \sum_{s=j+m}^{k} q_s(0) q_{j+m}^{(s)}(l).$$

If we set $j+m=i$, we have $m=i-j$, with $i=j,\ldots,k$, hence

$$\sum_{s=j}^{k} q_s(0) \sum_{i=j}^{s} \binom{i}{j} \alpha^{i-j} q_i^{(s)}(l) \simeq \sum_{i=j}^{k} \binom{i}{j} \alpha^{i-j} \sum_{s=i}^{k} q_s(0) q_i^{(s)}(l).$$

The result follows by replacing this last equivalence in (3.36) and by taking the evaluation of both sides. □

Remark 3.31. Let us observe that the family of umbrae $\{n.\alpha\}_{n\geq 0}$ corresponds to a discrete martingale $\{X_n\}_{n\geq 0}$ with $X_0=0$ and independent and identically distributed difference sequence with zero mean. Recall that the difference sequence associated to $\{X_n\}_{n\geq 0}$ is a sequence of random variables $\{M_n\}_{n\geq 0}$ such that $M_0=X_0=0$ and $M_n=X_n-X_{n-1}$, for all nonnegative integers n. The umbra $n.\alpha$ generalizes $X_n=M_1+M_2+\ldots+M_n$.

Suppose to remove the identical distribution hypothesis on $\{M_n\}_{n\geq 0}$: in umbral terms, the martingale $\{X_n\}_{n\geq 0}$ corresponds to the umbra $\alpha_1+\alpha_2+\cdots+\alpha_n$. The time-space harmonic polynomials $E[(x-n.\alpha)^k]$ need to be replaced by $E[(x-1.(\alpha_1+\alpha_2+\cdots+\alpha_n))^k]$. The previous properties can be recovered by similar arguments.

3.3 Examples of time-space harmonic polynomials

In the following we will introduce some families of time-space harmonic polynomials with respect to some special Lévy processes.

3.3.1 Hermite polynomials

A standard Brownian motion $\{W_t\}_{t\geq 0}$, also known as Wiener process with 0 drift, is a Lévy process whose increments are distributed according to the standard Gaussian distribution [49].

Thanks to the Lévy-Khintchine formula, the Lévy triplet is such that $c_0=0$, $s=1$ and $\nu=0$, so $E[\gamma^i]=\delta_{0,i}$, which means that $\gamma \equiv \epsilon$.

By applying (2.32) and Theorem 2.37, a standard Brownian motion results to be umbrally represented by the family of auxiliary umbrae

$$\{t.\beta.\delta\}_{t\geq 0}, \tag{3.37}$$

with β the Bell umbra and δ the umbral counterpart of a standard Gaussian random variable.

A Brownian motion $\{B_t\}_{t\geq 0}$ is a Lévy process whose increments are distributed as a Gaussian random variable with zero mean and variance $s^2 \neq 1$.

The Lévy triplet in this case is $(0, s^2, 0)$, then, again by virtue of (2.32) in Theorem 2.37, a Brownian motion is umbrally represented by

$$\{t.\beta.(s\delta)\}_{t\geq 0}. \tag{3.38}$$

Thanks to Theorem 3.17, we have that

$$Q_k(x,t) = E[(x - t.\beta.(s\delta))^k] \tag{3.39}$$

are time-space harmonic polynomials with respect to the Brownian motion umbrally represented by the family of umbrae in (3.38).

Proposition 3.32. *For all nonnegative integers $k \geq 1$, we have*

$$Q_k(x,t) = H_k^{(s^2 t)}(x),$$

where $\{H_k^{(s^2 t)}(x)\}$ are the generalized Hermite polynomials.

Proof. let $\{H_k^{(s^2)}(x)\}$ be the family of generalized Hermite polynomials, whose generated function id given by [43]

$$\sum_{k\geq 0} H_k^{(s^2)}(x) \frac{z^k}{k!} = \exp\left\{xz - \frac{s^2 z^2}{2}\right\}. \tag{3.40}$$

On the other hand, equation (3.40) can be seen as the generating function of a suitable polynomial umbra, indeed

$$\exp\left\{xz - \frac{s^2 z^2}{2}\right\} = \exp\{xz\} \exp\left\{-\frac{s^2 z^2}{2}\right\},$$

where $e^{xz} = f(u,z)$ whereas

$$\begin{aligned}\exp\left\{-\frac{s^2 z^2}{2}\right\} &= \exp\left\{-\left[1 + \frac{s^2 z^2}{2} - 1\right]\right\} = \exp\{-[f(\delta, sz) - 1]\} \\ &= \exp\{-[f(s\delta, z) - 1]\} = f(-1.\beta.(s\delta), z).\end{aligned}$$

Hence,

$$\exp\left\{xz - \frac{s^2 z^2}{2}\right\} = f(x - 1.\beta.(s\delta), z),$$

that is,

$$H_k^{(s^2)}(x) = E[(x - 1.\beta.(s\delta))^k]. \tag{3.41}$$

If we replace s^2 with $s^2 t$ in (3.41), we obtain

$$H_k^{(s^2 t)}(x) = E[(x - 1.\beta.(\sqrt{t}s\delta))^k]. \tag{3.42}$$

Finally, let us observe that, via generating functions, we have

$$\begin{aligned} f(-1.\beta.(\sqrt{t}s\delta), z) &= \frac{1}{f(\beta.(\sqrt{t}s\delta), z)} = \frac{1}{\exp\{f(\sqrt{t}s\delta, z) - 1\}} \\ &= \frac{1}{\exp\left\{1 + \frac{ts^2z^2}{2} - 1\right\}} = \exp\left\{-\frac{ts^2z^2}{2}\right\} \\ &= \exp\left\{-t\left[1 + \frac{s^2z^2}{2} - 1\right]\right\} = \exp\{-t[f(s\delta, z) - 1]\} \\ &= f(-t.\beta.(s\delta), z), \end{aligned}$$

that is,

$$-1.\beta.(\sqrt{t}s\delta) \equiv -t.\beta.(s\delta).$$

The result follows by replacing this last equivalence in (3.42). □

Here we present an umbral version of the recurrence formula for the Hermite polynomials.

Theorem 3.33. *We have*

$$H_{k+1}^{(t)}(x) = xH_k^{(t)}(x) - tkH_{k-1}^{(t)}(x). \tag{3.43}$$

Proof. Recall that $kH_{k-1}^{(t)}(x) = D_x H_k^{(t)}(x)$. Then the recurrence formula (3.43) becomes

$$(x + -t.\beta.\delta)^{k+1} \simeq x(x + -t.\beta.\delta)^k - tD_x[(x + -t.\beta.\delta)^k],$$

which is equivalent to prove that

$$\begin{aligned} \sum_{j=0}^{k+1} \binom{k+1}{j} x^j (-t.\beta.\delta)^{k+1-j} &\simeq \sum_{j=0}^{k} \binom{k}{j} x^{j+1} (-t.\beta.\delta)^{n-j} \\ &\quad - \sum_{i=1}^{k} \binom{k}{j} i x^{i-1} t(-t.\beta.\delta)^{k-i}, \end{aligned} \tag{3.44}$$

by applying the binomial theorem.

Look at the left-hand side of (3.44). We have

$$\sum_{j=0}^{k+1} \binom{k+1}{j} x^j (-t.\beta.\delta)^{k+1-j} \simeq x^{k+1} + \binom{k+1}{k} x^k (-t.\beta.\delta) + \sum_{j=1}^{k-1} \binom{k+1}{j} x^j (-t.\beta.\delta)^{k+1-j}. \tag{3.45}$$

As regards the right-hand side of (3.44), we make a change of variable in the second summation, by setting $s = i - 1$ so that $i = s + 1$, hence, when i ranges in $\{1, \ldots, k\}$, s ranges in $\{0, \ldots, k-1\}$. We have

$$\begin{aligned}
&\sum_{j=0}^{k} \binom{k}{j} x^{j+1} (-t.\beta.\delta)^{k-j} - \sum_{i=1}^{i} \binom{k}{j} i x^{i-1} t (-t.\beta.\delta)^{k-i} \\
&\simeq \sum_{j=0}^{k} \binom{k}{j} x^{j+1} (-t.\beta.\delta)^{k-j} - \sum_{s=0}^{k-1} \binom{k}{s+1} (s+1) x^s t (-t.\beta.\delta)^{k-s-1} \\
&\simeq x^{k+1} + \sum_{j=0}^{k-1} \binom{k}{j} x^{j+1} (-t.\beta.\delta)^{k-j} - \sum_{s=0}^{k-1} \binom{k}{s+1} (s+1) x^s t (-t.\beta.\delta)^{k-s-1} \\
&\simeq x^{k+1} + \binom{k}{k-1} x^k (-t.\beta.\delta) + \sum_{j=0}^{k-2} \binom{k}{j} x^{j+1} (-t.\beta.\delta)^{k-j} \\
&- \sum_{s=0}^{k-1} \binom{k}{s+1} (s+1) x^s t (-t.\beta.\delta)^{k-s-1}.
\end{aligned} \tag{3.46}$$

Replace equivalences (3.45) and (3.46) in (3.44) and compare the two sides. The result follows if we prove that

$$\begin{aligned}
\sum_{j=0}^{k-1} \binom{k+1}{j} x^j (-t.\beta.\delta)^{k+1-j} &\simeq \sum_{j=0}^{k-2} \binom{k}{j} x^{j+1} (-t.\beta.\delta)^{k-j} \\
&- \sum_{j=0}^{k-1} \binom{k}{j+1} (j+1) x^j t (-t.\beta.\delta)^{k-j-1} \\
&\simeq \sum_{j=1}^{k-1} \binom{n}{j-1} x^j (-t.\beta.\delta)^{k-j+1} - \sum_{j=0}^{k-1} \binom{k}{j+1} (j+1) x^j t (-t.\beta.\delta)^{k-j-1},
\end{aligned} \tag{3.47}$$

after a change of variable in the first summation. In particular, we have to prove that

$$(-t.\beta.\delta)^{k+1} \simeq -\binom{k}{1} t(-t.\beta.\delta)^{k-1} \quad \text{for } j=0;$$
$$\binom{k+1}{j}(-t.\beta.\delta)^{k+1-j} \simeq \binom{k}{j-1}(-t.\beta.\delta)^{k-j+1} \quad \text{for } j=1,\ldots,k-1.$$

We apply the *transfer formula* [22]

$$(x.\gamma^*)^{k+1} \simeq x\gamma^{<-1>}[(x+\chi).\gamma^*]^k \tag{3.48}$$

with $x = t$ and $\gamma \equiv \delta$, so that $\gamma^* \equiv \beta.\gamma^{<-1>}$.
Hence, the recurrence formula (3.43) for $j = 0$ gives

$$\begin{aligned}(-t.\beta.\delta)^{k+1} &\simeq -t\delta[(-t+\chi).\beta.\delta]^k \simeq -t\delta[-t.\beta.\delta+\delta]^k \\ &\simeq \sum_{j=0}^{k}\binom{n}{j}\delta^{j+1} - t(-t.\beta.\delta)^{k-j} \simeq \binom{k}{1} - t(t.\beta.\delta)^{k-1},\end{aligned}$$

where the last equivalence holds by equality (1.34) on the moments of the umbra δ.

When $j = 1, \ldots, k-1$, recalling that [19]

$$\binom{k+1}{j} = \binom{k}{j-1} + \binom{k}{j}$$

and by applying the transfer formula (3.48) again, we have

$$\begin{aligned}&\left[\binom{k+1}{j} - \binom{k}{j-1}\right](-t.\beta.\delta)^{k-j+1} \simeq \binom{k}{j}(-t.\beta.\delta)^{k-j+1} \\ &\simeq \binom{k}{j}(-t)\delta[(-t+\chi).\beta.\delta]^{k-j} \simeq \binom{k}{j}(-t)\delta[-t.\beta.\delta+\delta]^{k-j} \\ &\simeq \binom{k}{j}(-t)\delta\sum_{i=0}^{k-j}\binom{k-j}{i}\delta^i(-t.\beta.\delta)^{k-j-i} \\ &\simeq \sum_{i=0}^{k-j}\binom{k}{j}\binom{k-j}{i}\delta^{i+1}(-t)(-t.\beta.\delta)^{k-j-i} \\ &\simeq \binom{k}{j}\binom{k-j}{1}(-t)(-t.\beta.\delta)^{k-j-1},\end{aligned}$$

where the last equivalence holds by equality (1.34) on the moments of the umbra δ.

On the other hand,

$$\binom{k}{j}\binom{k-j}{1} = \binom{k}{j+1}(j+1),$$

thus

$$\binom{k}{j}(-t.\beta.\delta)^{k-j+1} \simeq \binom{k}{j+1}(j+1)(-t)(-t.\beta.\delta)^{k-j-1}.$$

□

??????????????????????????????

3.3.2 Poisson-Charlier polynomials

A Poisson process $\{N_t\}_{t\geq 0}$ is a pure jumps Lévy process, whose increments follow a Poisson distribution with intensity parameter $\lambda > 0$. The moment generating function of a Poisson process is [50]

$$E[e^{zN_t}] = \exp\{t[\lambda(e^z - 1)]\}.$$

If we compare this formula with (2.30), we have $c_0 = 0$, $s = 0$ and $\nu = \delta(1)$, that is, the Delta measure concentrated in 1, so that $E[\gamma^i] = \int_{\mathbb{R}} x^i \delta(1) = 1$, by definition of Delta measure. Then we have $\gamma \equiv u$.

Hence, by applying (2.32) in Theorem 2.37, the umbral version of the Poisson process is

$$\{(\lambda t).\beta.u\}_{t\geq 0} = \{(\lambda t).\beta\}_{t\geq 0}. \tag{3.49}$$

Remark 3.34. The result in (3.49) is in agreement with those in Chapter 2, where we have described the auxiliary umbrae $\{t.\alpha\}_{t\geq 0}$ as the umbral counterpart of Poisson random variables.

By virtue of Theorem 3.17, the polynomials

$$Q_k(x, \lambda t) = E[(x - t.\lambda.\beta)^k]$$

are time-space harmonic with respect to the Poisson process umbrally represented by (3.49).

Let us consider now the Poisson-Charlier polynomials $\{\tilde{C}_k(x, \lambda t)\}$, whose moment generating function is [43]

$$\sum_{k\geq 0} \tilde{C}_k(x, \lambda t)\frac{z^k}{k!} = e^{-\lambda t z}(1+z)^x. \tag{3.50}$$

Next proposition states that the Poisson-Charlier polynomials are time-space harmonic with respect to the Poisson process $\{N_t\}_{t\geq 0}$.

Proposition 3.35. *For all nonnegative integers $k \geq 1$, we have*

$$\tilde{C}_k(x, \lambda t) = \sum_{j=1}^{k} s(k,j) Q_j(x, \lambda t),$$

where $\{s(k,j)\}$ are the Stirling numbers of the first kind.

Proof. From (3.50), we have

$$\tilde{C}_k(x, \lambda t) \simeq (x.\chi - (\lambda t).u)^k \simeq (x.\chi - \lambda.t.u)^k.$$

Observe that, by applying the second equivalence in (1.32) and the distributive property of the summation of umbrae with respect to the dot-product, we have

$$x.\chi - \lambda.t.u \equiv x.\chi - \lambda.t.\beta.\chi \equiv (x - \lambda.t.\beta).\chi, \tag{3.51}$$

then

$$(x.\chi - \lambda.t.u)^k \simeq [(x - \lambda.t.\beta).\chi]^k.$$

Note that the umbra in (3.51) is of the form $\alpha.\chi$, with $\alpha \equiv x - \lambda.t.\beta$, i.e., it is the α- factorial umbra [30], for which we have $E[(\alpha.\chi)^k] = (a)_k$, where $(a)_k$ is the k-th factorial moment of the umbra α introduced at the beginning of Chapter 1.

Hence, in this case we have

$$(x.\chi - \lambda.t.u)^k \simeq (x - \lambda.t.\beta)_k \simeq \sum_{j=0}^{k} s(k,j)(x - \lambda.t.\beta)^j,$$

thanks to the properties of Stirling numbers of the first kind $s(k,j)$ [19], by which the result follows. □

The Poisson-Charlier polynomials are linear combination of the time-space harmonic polynomials with respect to the Poisson process $\{\lambda.t.\beta\}_{t\geq 0}$, hence they are time-space harmonic with respect to the same stochastic process, in turn, thanks to Remark 3.12.

3.3.3 Lévy-Sheffer systems

A sequence of polynomials $\{V_k(x,t)\}_{t\geq 0}$ is a *Lévy-Sheffer system* if it is defined by the following generating function [51]

$$\sum_{k\geq 0} V_k(x,t)\frac{z^k}{k!} = (g(z))^t \exp\{xu(z)\}, \tag{3.52}$$

where $g(z)$ and $u(z)$ are analytic in a neighborhood of $z=0$, $u(0)=0$, $g(0)=1$, $u'(0)\neq 0$ and $1/g(\tau(z))$ is a moment generating function associated to an infinitely divisible distribution, with $\tau(z)$ such that $\tau(u(z))=z$. Schoutens in [51] states that the basic link between these polynomials and Lévy processes is a *martingale equality* (cf. page. 337 eq. (6)), which is equivalent to ask that these polynomials are time-space harmonic with respect to Lévy processes.

In the following, we show that $V_k(x,t)$ is a linear combination of suitable time-space harmonic polynomials $Q_k(x,t)$ and therefore, they share the same property.

Theorem 3.36. *We have*

$$V_k(x,t) = \sum_{i=0}^{k} E[(x+t.\beta.\kappa)^i] B_{k,i}(g_1,\ldots,g_{k-i+1}), \tag{3.53}$$

where $g_j = E[\gamma^j]$, for all nonnegative j and κ is the cumulant umbra of $\alpha.\beta.\gamma^{<-1>}$, with $\gamma^{<-1>}$ the compositional inverse of the umbra γ.

Proof. Let α and γ be two umbrae such that $f(\alpha,z)=g(z)$ and $f(\gamma,z)=1+u(z)$, with $E[\gamma]=u'(0)\neq 0$. Hence, $u(z)=f(\gamma,z)-1$.

Due to the form of the generating function in (3.52), we have

$$(g(z))^t \exp\{xu(z)\} = (f(\alpha,z))^t \exp\{x[f(\gamma,z)-1]\}.$$

Hence, by applying (1.14) and (1.39), the polynomials $\{V_k(x,t)\}_{k\geq 0}$ are umbrally represented by the polynomial umbra $t.\alpha + x.\beta.\gamma$, in symbols

$$V_k(x,t) = E[(t.\alpha + x.\beta.\gamma)^k,] \quad \text{for all } k\geq 1. \tag{3.54}$$

What is more, thanks to (1.32), (1.42) and the distributive property, we have

$$\begin{aligned} x.\beta.\gamma + t.\alpha.u &\equiv x.\beta.\gamma + t.\alpha.\beta.\chi \equiv x.\beta.\gamma + t.\alpha.\beta.\gamma^{<-1>}.\beta.\gamma \\ &\equiv (x + t.\alpha.\beta.\gamma^{<-1>}).\beta.\gamma. \end{aligned} \tag{3.55}$$

The result follows thanks to (1.41), by replacing (3.55) in (3.54), that is,

$$V_k(x,t) = E[((x + t.\alpha.\beta.\gamma^{<-1>}).\beta.\gamma)^k]$$
$$= \sum_{i=0}^{k} E[(x + t.\alpha.\beta.\gamma^{<-1>})^i] B_{k,i}(g_1, \dots, g_{k-i+1}).$$

□

Corollary 3.37. *The Lévy-Sheffer polynomials* $\{V_k(x,t)\}_{t\geq 0}$ *are time-space harmonic with respect to Lévy processes umbrally represented by the family of auxiliary umbrae* $\{-t.\alpha.\beta.\gamma^{<-1>}\}_{t\geq 0}$.

Proof. By Theorem 3.7, the time-space harmonic polynomials with respect to $\{-t.\alpha.\beta.\gamma^{<-1>}\}_{t\geq 0}$ are

$$Q_k(x,t) = E[(x - t.\alpha.\beta.\gamma^{<-1>})^k],$$

for all $k \geq 1$.

Thus, Theorem 3.36 states that every Lévy-Sheffer system is a linear combination of time-space harmonic polynomials with respect to the Lévy process umbrally represented by $\{-t.\alpha.\beta.\gamma^{<-1>}\}_{t\geq 0}$.

The result follows, thanks to Remark 3.12. □

3.3.4 Laguerre polynomials and actuarial polynomials

A Gamma process $\{G_t(\lambda, b)\}_{t\geq 0}$ with scale parameter $\lambda > 0$ and shape parameter $b > 0$ is a Lévy process with stationary, independent and Gamma-distributed increments.

If we set $b = 1$, then the moment generating function of the Gamma process is [50]

$$E[e^{zG_t(\lambda,1)}] = [(1-z)^{-\lambda}]^t.$$

The Lévy triplet of the Gamma process is $(0, 0, \nu)$, [50] where the lévy measure is given by $\nu = \lambda e^{-x} x^{-1} \mathbf{1}_{x\geq 0}(x)dx$, hence

$$E[\gamma^i] = \int_{\mathbb{R}} x^i \nu(dx) = \lambda \int_{\mathbb{R}} x^{i-1} e^{-x} \mathbf{1}_{x\geq 0}(x)dx = \lambda \int_0^{+\infty} x^{i-1} e^{-x} dx.$$

By iteration of integration by parts, we have, for $i = 0, 1, \dots,$

$$E[\gamma^i] = -\lambda \left[x^{i-i} e^{-x} \Big|_0^{+\infty} - \int_0^{+\infty} (i-1) x^{i-2} e^{-x} dx \right]$$

$$
\begin{aligned}
&= -\lambda \int_0^{+\infty} (i-1)x^{i-2}(-e^{-x})dx \\
&= -\lambda(i-1)\left[x^{i-2}e^{-x}\Big|_0^{+\infty} - \int_0^{+\infty} (i-2)x^{i-3}e^{-x}dx \right] \\
&= \ldots = \lambda(i-1)(i-2)\cdots(i-i+1)\int_0^{+\infty}(-e^{-x})dx \\
&= \lambda(i-1)!e^{-x}\Big|_0^{+\infty} = \lambda(i-1)!
\end{aligned}
$$

Therefore, since

$$1 + \log\left(\frac{1}{1-z}\right) = 1 - \log(1-z) = 1 + \sum_{k\geq 1}\frac{z^k}{k} = 1 + \sum_{k\geq 1}(k-1)!\frac{z^k}{k!},$$

we have, by virtue of (2.5),

$$f(\gamma, z) = 1 + \lambda\sum_{k\geq 1}(k-1)!\frac{z^k}{k!} = 1 + \log\left(\left(\frac{1}{1-z}\right)^{\lambda}\right) = f(\chi.\lambda.\bar{u}, z),$$

where $\bar{u}$ is the boolean unity umbra.

Hence, by virtue of Theorem 2.37 and by recalling that $\beta.\chi \equiv u$, the umbral version of the Gamma process is

$$\{t.\lambda.\bar{u}\}_{t\geq 0}. \tag{3.56}$$

Due to Theorem 3.7, the time-space harmonic polynomials with respect to the Gamma process $\{G_t(\lambda, 1)\}$ are

$$Q_k(x,t) = E[(x - t.\lambda.\bar{u})^k], \quad \text{for all } k \geq 1. \tag{3.57}$$

There are two families of polynomials which are time-space harmonic with respect to Gamma processes, according to the value of the scale parameter λ.

If we set $\lambda = 1$, we have the *Laguerre polynomials* $\{\mathcal{L}_k^{(t-k)}(x)\}$, with generating function [43]

$$\sum_{k\geq 0}(-1)^k\mathcal{L}_k^{(t-k)}(x)z^k = (1-z)^t e^{xz}.$$

Since we work with exponential generating functions, set

$$\sum_{k\geq 0}(-1)^k k!\mathcal{L}_k^{(t-k)}(x)\frac{z^k}{k!} = (1-z)^t e^{xz}. \tag{3.58}$$

Hence, the umbral expression of Laguerre polynomials is given by observing that

$$(1-z)^t e^{xz} = f(t.(-\chi), z) f(x.u, z) = f(x + t.(-\chi), z),$$

by using (1.14) and by recalling that $f(\chi, z) = 1 + z$.
Therefore, we have

$$(-1)^k k! \mathcal{L}_k^{t-k}(x) = E[(x + t.(-\chi))^k], \qquad k = 1, 2, \ldots \tag{3.59}$$

Theorem 3.38. *The Laguerre polynomials $\{\mathcal{L}_k^{t-k}(x)\}_{t\geq 0}$ are time-space harmonic with respect to the Gamma process $\{G_t(1,1)\}_{t\geq 0}$.*

Proof. We want to bring us back to equation (3.57) with $\lambda = 1$. Via generating functions, we have $-1.\bar{u} \equiv -\chi$, so that $-t.\bar{u} \equiv t.(-1.\bar{u}) \equiv t.(-\chi)$, then

$$x - t.\bar{u} \equiv x + t.(-\chi).$$

The result follows by comparing this last result with (3.59). □

When $\lambda \neq 1$, Roman [43] defines the class of the actuarial polynomials as the sequence of polynomials with generating function

$$\sum_{k\geq 0} g_k(x, \lambda t) \frac{z^k}{k!} = \exp\{\lambda t z + x(1-e^z)\} = \exp\{\lambda t z\} \exp\{x(1-e^z)\}. \tag{3.60}$$

To get the umbral expression of $g_k(x, \lambda t)$ we use the umbral Lévy-Sheffer systems introduced in Section 3.3.3. By comparing (3.60) and (3.52), we have

$$f(z) = e^{\lambda t z}, \text{ so that } \alpha \equiv (\lambda t).u$$

$$u(z) = g(z) - 1 = 1 - e^z, \text{ so that } g(z) = 2 - e^z$$

We want to obtain an explicit expression of the umbra γ. Since its generating function is not referable to any known umbra, we can suppose that $g(z)$ is the generating function of the compositional inverse of some umbra we want to determine. Let ϑ be such umbra. By definition of compositional inverse, we have

$$f(\vartheta^{<-1>}, f(\vartheta, z) - 1) = 1 + z \Rightarrow 2 - e^z \Big|_{z = f(\vartheta, z) - 1} = 1 + z$$

$$\begin{aligned}
&2 - e^{f(\vartheta,z)-1} = 1 + z \Rightarrow e^{f(\vartheta,z)-1} = 1 - z \\
&f(\vartheta, z) - 1 = \log(1 - z) \Rightarrow f(\vartheta, z) = 1 + \log(1 - z) \\
&= 1 + \log(f(-\chi, z)) = f(\chi.(-\chi), z)
\end{aligned}$$

Then, $\vartheta \equiv \chi.(-\chi)$ and so $\gamma \equiv \vartheta^{<-1>} \equiv (\chi.(-\chi))^{<-1>}$. Hence, the umbra having (3.60) as generating function is $\lambda t + x.\beta.(\chi.(-\chi))^{<-1>}$ and its moments are

$$g_k(x, \lambda t) = E\{[\lambda t + x.\beta.(\chi.(-\chi))^{<-1>}]^k\}, \quad \text{for all } k \geq 0. \tag{3.61}$$

Theorem 3.39. *The actuarial polynomials* $\{g_k(x, \lambda t)\}_{t\geq 0}$ *are time-space harmonic with respect to the Gamma process* $\{G_t(\lambda, 1)\}_{t\geq 0}$.

Proof. We have

$$\begin{aligned}
\lambda t + x.\beta.(\chi.(-\chi))^{<-1>} &\equiv x.\beta.(\chi.(-\chi))^{<-1>} + (\lambda t).u \\
&\equiv (x + t.\lambda.u.\beta.\chi.(-\chi)).\beta.(\chi.(-\chi))^{<-1>} \\
&\equiv (x + t.\lambda.u.(-\chi)).\beta.(\chi.(-\chi))^{<-1>}.
\end{aligned}$$

Therefore, thanks to equation (3.61),

$$\begin{aligned}
g_k(x, \lambda t) &= E\{[(x + t.\lambda.u.(-\chi)).\beta.(\chi.(-\chi))^{<-1>}]^k\} \\
&= \sum_{j=0}^{k} E[(x + t.\lambda.u.(-\chi))^k] B_{k,j}(m_1, \ldots, m_{k-j-1}),
\end{aligned} \tag{3.62}$$

where $m_i = E[(\chi.(-\chi))^i]$, for all $i = 1, \ldots, k$.

Let us observe that

$$\begin{aligned}
x + t.\lambda.u.(-\chi) &\equiv x + (t\lambda).u.(-\chi) \equiv x + (t\lambda).(-1).(-1).(-\chi) \\
&\equiv x - (t\lambda).(-1.(-\chi)) \equiv x - (\lambda t).\bar{u}.
\end{aligned}$$

Finally, by using (3.57) and (3.62), we have

$$\begin{aligned}
g_k(x, \lambda t) &= \sum_{j=1}^{k} E[(x - (\lambda t).\bar{u})^j] B_{k,j}(m_1, \ldots, m_{k-j-1}) \\
&= \sum_{j=1}^{k} Q_k(x, t) B_{k,j}(m_1, \ldots, m_{k-j-1}).
\end{aligned}$$

So, we have proved that the actuarial polynomials $\{g_k(x, \lambda t)\}$ are a linear combination of time-space harmonic polynomials with respect to the Gamma process and so they are time-space harmonic with respect to the same process, in turn, thanks to Remark 3.12. $\square$

3.3.5 Meixner polynomials of first kind

Let $\{Pa(t,p)\}_{t\geq 0}$ be a Pascal process, that is, a Lévy process whose increments have binomial distribution with parameter p. The moment generating function of the Pascal process is

$$E[e^{zPa(t,p)}] = \left(\frac{1-p}{1-pe^z}\right)^t.$$

If we put $1+p=q$ and $d=p/q$, we have

$$\begin{aligned}\left(\frac{1-p}{1-pe^z}\right)^t &= \left(\frac{q}{p+q-pe^z}\right)^t = \left(\frac{q}{q+p(1-e^z)}\right)^t \\ &= \left(\frac{q}{q[1+\frac{p}{q}(1-e^z)]}\right)^t = \left(\frac{1}{1+\frac{p}{q}(1-e^z)}\right)^t \\ &= \left(\frac{1}{1-\frac{p}{q}(e^z-1)}\right)^t = \left(\frac{1}{1-d(e^z-1)}\right)^t \\ &= \left(\frac{1}{1-dz}\right)^t\Bigg|_{z=e^z-1} = (f(d\bar{u}, e^z-1))^t \\ &= (f(\bar{u}.d.\beta, z))^t = f(t.\bar{u}.d.\beta, z).\end{aligned}$$

Hence, by applying (2.32) in Theorem 2.37, the umbral version of the Pascal process is

$$\{t.\bar{u}.d.\beta\}_{t\geq 0}. \tag{3.63}$$

Consider the family of Meixner polynomials of the first kind $\{M_k(x,t,p)\}$ such that [49]

$$\sum_{k\geq 0}(-1)^k (t)_k M_k(x,t,p)\frac{z^k}{k!} = \left(1+\frac{z}{p}\right)^x (1+z)^{-x-t}. \tag{3.64}$$

The umbral expression of the Meixner polynomials of the first kind is

$$(-1)^k (t)_k M_k(x,t,p) = E\left\{\left[x.\left(-1.\chi+\frac{\chi}{p}\right) - t.\chi\right]^k\right\}. \tag{3.65}$$

Indeed we have

$$\left(1+\frac{z}{p}\right)^x (1+z)^{-x-t} = \left(1+\frac{z}{p}\right)^x (1+z)^{-x}(1+z)^{-t}$$

$$= \left(\frac{1+\frac{z}{p}}{1+z}\right)^x (1+z)^{-t} = \exp\left\{\log\left(\frac{1+\frac{z}{p}}{1+z}\right)^x\right\}(1+z)^{-t}$$
$$= \exp\left\{x\left[1+\log\left(\frac{1+\frac{z}{p}}{1+z}\right)-1\right]\right\}(1+z)^{-t}$$
$$= \exp\left\{x\left[1+\log\left(\frac{1+\frac{z}{p}}{1+z}\right)-1\right]\right\}\left(\frac{1}{1+z}\right)^t.$$

Let us observe that

$$\frac{1+\frac{z}{p}}{1+z} = f\left(\frac{\chi}{p}, z\right) f(-1.\chi, z) = f(-1.\chi + \frac{\chi}{p}, z)$$
$$\Rightarrow\ 1+\log\left(\frac{1+\frac{z}{p}}{1+z}\right) = f\left(\chi.\left(-1.\chi+\frac{\chi}{p}\right), z\right).$$

Then, equality (3.65) holds, since

$$\left(1+\frac{z}{p}\right)^x (1+z)^{-x-t} = f\left(t.(-1.\chi) + x.\beta.\chi.\left(-1.\chi+\frac{\chi}{p}\right)\right)$$
$$= f\left(-t.\chi + x.\left(-1.\chi+\frac{\chi}{p}\right)\right).$$

Theorem 3.40. *The Meixner polynomials of the first kind are time-space harmonic with respect to the Pascal process $\{Pa(p,t)\}_{t\geq 0}$.*

Proof. The Meixner polynomials of the first kind form a Lévy-Sheffer system, so they are represented by the polynomial umbra $x.\beta.(\chi.(-1.\chi+\chi/p)) + t.(-1.\chi)$. First of all, we have to prove that the first moment of the umbra $\chi.\left(-1.\chi+\frac{\chi}{p}\right)$ is non-zero. In fact,

$$f\left(\chi.\left(-1.\chi+\frac{\chi}{p}\right), z\right) = 1+\log\left(\chi.\left(-1.\chi+\frac{\chi}{p}\right)\right) = 1+\log\left(\frac{1+\frac{z}{p}}{1+z}\right).$$

By developing in formal power series and by setting $\gamma \equiv \chi.\left(-1.\chi+\frac{\chi}{p}\right)$, we have

$$f(\gamma, z) = 1+\log\left(1+\frac{z}{p}\right) - \log(1+z) = 1 + \sum_{k\geq 1}(-1)^{k+1}\frac{(z/p)^k}{k} + \sum_{k\geq 1}\frac{z^k}{k}$$
$$= 1 + \sum_{k\geq 1}\frac{(-1)^{k+1}(k-1)!}{p^k}\frac{(z/p)^k}{k!} + \sum_{k\geq 1}\frac{k!}{k}\frac{z^k}{k!}$$

$$= 1 + \sum_{k \geq 1} \frac{(-1)^{k+1}(k-1)!}{p^k} \frac{(z/p)^k}{k!} + \sum_{k \geq 1} (k-1)! \frac{z^k}{k!}$$
$$= 1 + \sum_{k \geq 1} \left[(k-1)! \left(\frac{(-1)^{k+1}}{p^k} + 1 \right) \right] \frac{z^k}{k!} = 1 + \sum_{k \geq 1} E[\gamma^k] \frac{z^k}{k!}.$$

Then, $E[\gamma] = \frac{1}{p} + 1 \neq 0$, so we are sure that the umbra $\gamma^{<-1>}$ does exist. Now we write

$$x.\left(-1.\chi + \frac{\chi}{p}\right) - t.\chi \equiv x.\beta.\left(\chi.\left(-1.\chi + \frac{\chi}{p}\right)\right) + t.(-1.\chi)$$
$$\equiv \left(x + t.(-1.\chi).\beta.\left(\chi.\left(-1.\chi + \frac{\chi}{p}\right)\right)^{<-1>}\right).\beta.\left(\chi.\left(-1.\chi + \frac{\chi}{p}\right)\right).$$

We are looking for an explicit expression of $\gamma^{<-1>} \equiv \chi.\left(-1.\chi + \frac{\chi}{p}\right)^{<-1>}$. By recalling that $f(\gamma, f(\gamma^{<-1>}, z) - 1) = 1 + z$, we have

$$1 + \log\left(\frac{1 + z/p}{1 + z}\right)\Bigg|_{z = f(\gamma^{<-1>}, z) - 1} = 1 + z$$
$$\Rightarrow \log\left(\frac{1 + (f(\gamma^{<-1>}, z) - 1)/p}{1 + f(\gamma^{<-1>}, z) - 1}\right) = z$$
$$\Rightarrow \frac{1 + (f(\gamma^{<-1>}, z) - 1)/p}{f(\gamma^{<-1>}, z)} = e^z \Rightarrow \frac{p + f(\gamma^{<-1>}, z) - 1}{pf(\gamma^{<-1>}, z)} = e^z$$
$$\Rightarrow p - 1 + f(\gamma^{<-1>}, z) = pe^z f(\gamma^{<-1>}, z) \Rightarrow f(\gamma^{<-1>}, z)(1 - pe^z) = 1 - p$$
$$\Rightarrow f(\gamma^{<-1>}, z) = \frac{1 - p}{1 - pe^z}.$$

On the other hand, taking into account that $p + q = 1$ and $\frac{p}{q} = d$, we have

$$\frac{1-p}{1-pe^z} = \frac{q}{1-pe^z} = \frac{q}{p + q - p(e^z - 1 + 1)} = \frac{q}{q - p(e^z - 1)}$$
$$= \frac{1}{1 - \frac{p}{q}(e^z - 1)} = \frac{1}{1 - d(e^z - 1)} = \frac{1}{1 - dz}\Bigg|_{z = e^z - 1}$$
$$= f(d\bar{u}, e^z - 1) = f(d\bar{u}, f(u, z) - 1) = f(d\bar{u}.\beta.u, z).$$

Thus,

$$\gamma^{<-1>} \equiv d\bar{u}.\beta.u \equiv \bar{u}.d.\beta. \tag{3.66}$$

Equivalence (3.66) gives

$$x.\left(-1.\chi + \frac{\chi}{p}\right) - t.\chi \equiv (x + t.(-1.\chi).\beta.\bar{u}.d.\beta).\beta.\left(\chi.\left(-1.\chi + \frac{\chi}{p}\right)\right).$$

The Meixner polynomials of the first kind can be written in the following way

$$(-1)^k (t)_k M_k(x,t,p) = \sum_{j=1}^{k} E[(x - t.\bar{u}.d.\beta)^j] B_{k,j}(m_1, \ldots, m_{k-j-1}),$$

where $m_i = E\left[\left(\chi.\left(-1.\chi + \frac{\chi}{p}\right)\right)^i\right]$. The result follows by recalling Remark 3.12. □

3.3.6 Random walks

The results in the literature involving the polynomials we are going to introduce refer to an integer parameter n. In order to highlight their time-space harmonic property, we can consider the discrete version of a Lévy process, that is a random walk $S_n = X_1 + X_2 + \cdots + X_n$, with $\{X_i\}$ independent and identically distributed random variables.

Recall that a random walk is the discrete counterpart of a Lévy process. On the other hand, random walks are not infinitely divisible, whereas one can prove that a stochastic process is a Lévy process if and only if its distribution shares the infinite divisibility property (see Lemma ?? [48]).

Nevertheless, for the symbolic representation of a Lévy process we have dealt with, a random walk is umbrally represented by $n.\alpha$, where n is a nonnegative integer.

This is a further confirmation that the umbral tools result to be a more general approach to the theory of Lévy processes.

The generality of the symbolic approach shows that if the parameter n is replaced by t, that is if the random walk is replaced by a Lévy process, more general classes of polynomials can be recovered for which many of the properties here introduced still hold.

Bernoulli polynomials. The Bernoulli polynomials $\{B_k(x,n)\}$ are defined by the generating function [43]

$$\sum_{k \geq 0} B_k(x,n) \frac{z^k}{k!} = \left(\frac{z}{e^z - 1}\right)^n e^{zx}. \tag{3.67}$$

By recalling the Bernoulli umbra defined in (1.22), the umbra whose generating function is given by (3.67) is $x + n.\iota$. Therefore, the Bernoulli polynomials are

$$B_k(x, n) = E[(x + n.\iota)^k], \tag{3.68}$$

for all nonnegative integers $k \geq 1$.

Theorem 3.41. *The Bernoulli polynomials $\{B_k(x, n)\}$ are time-space harmonic with respect to the random walk $\{n.(-1.\iota)\}_{n\geq 0}$.*

Proof. Let us consider the random walk $S_n = X_1 + X_2 + \cdots + X_n$ such that $X_1, X_2, \ldots, X_n$ are n independent and identically distributed random variables with uniform distribution on the interval $[0, 1]$.

Since X_i is umbrally represented by the umbra $-1.\iota$, for all $i = 1, \ldots, n$, as we have already seen in Remark 1.17, the random walk S_n is umbrally represented by the family of auxiliary umbrae $\{n.(-1.\iota)\}_{n\geq 0}$.

Theorem 3.7 ensures that the polynomials

$$Q_k(x, n) = E\left[(x - n.(-1.\iota))^k\right] \tag{3.69}$$

are time-space harmonic with respect to S_n, for all $k \geq 0$.

On the other hand, $n.(-1.\iota) \equiv -n.\iota$, hence $E[(x - n.(-1.\iota))^k] = E[(x + n.\iota)^k]$, that is,

$$B_k(x, n) = Q_k(x, n).$$

□

Euler polynomials. The Euler polynomials $\{\mathcal{E}_k(x, n)\}$ are defined by the generating function [43]

$$\sum_{k\geq 0} \mathcal{E}_k(x, n) \frac{z^k}{k!} = \left(\frac{2}{e^z + 1}\right)^n e^{zx}. \tag{3.70}$$

From (3.70) and (1.25), we have

$$\begin{aligned} \left(\frac{2}{e^z + 1}\right)^n e^{zx} &= f(u.x, z)\, f\left(n.\frac{1}{2}[\eta + (-1.u)], z\right) \\ &= f\left(x + n.\frac{1}{2}[\eta + (-1.u)], z\right) \end{aligned}$$

Therefore we have $\mathcal{E}_k(x, n) = E[\left(x + n.\left[\frac{1}{2}(-1.u + \eta)\right]\right)^k]$ for all nonnegative integers k.

Theorem 3.42. *The Euler polynomials* $\{\mathcal{E}_k(x,n)\}$ *are time-space harmonic with respect to the random walk* $\{n.\left[\frac{1}{2}\left(-1.\eta+u\right)\right]\}_{n\geq 0}$.

Proof. Let us consider the random walk $S_n = X_1 + X_2 + \cdots + X_n$ such that $X_1, X_2, \ldots, X_n$ are n independent and identically distributed Bernoulli random variables with parameter 1/2. Since X_i is umbrally represented by the umbra $\frac{1}{2}\left(-1.\eta+u\right)$, for all $i = 1, \ldots, n$, as we have already seen in Remark 1.18, the random walk S_n is umbrally represented by the family of auxiliary umbrae $\{n.\left[\frac{1}{2}\left(-1.\eta+u\right)\right]\}_{n\geq 0}$.

Theorem 3.7 ensures that the polynomials

$$Q_k(x,n) = E\left[\left(x - n.\left[\frac{1}{2}\left(-1.\eta+u\right)\right]\right)^k\right] \tag{3.71}$$

are time-space harmonic with respect to S_n, for all $k \geq 0$.

Hence,

$$\mathcal{E}_k(x,n) = Q_k(x,n).$$

□

Krawtchouk polynomials. The Krawtchouk polynomials $\{\mathcal{K}_k(x,p,n)\}$ are defined by the generating function [43]

$$\sum_{k\geq 0}\binom{n}{k}\mathcal{K}_k(x,p,n)z^k = \left(1 - \frac{1-p}{p}z\right)^x (1+z)^{n-x}.$$

We can rewrite this generating function in such a way it becomes of exponential type, that is

$$\begin{aligned}\sum_{k\geq 0}\binom{n}{k}\mathcal{K}_k(x,p,n)z^k &= \sum_{k\geq 0} k!\binom{n}{k}\mathcal{K}_k(x,p,n)\frac{z^k}{k!} \\ &= \left(1 - \frac{1-p}{p}z\right)^x (1+z)^{n-x}. \end{aligned} \tag{3.72}$$

Observe that, if we set $p+q=1$ and $p/q=d$, we have

$$\left(1 - \frac{1-p}{p}z\right)^x (1+z)^{n-x} = \left(1 - \frac{z}{d}\right)^x (1+z)^{n-x},$$

thus, the umbra with generating function (3.72) is

$$x.\left(-\frac{\chi}{d}\right) + (n-x).\chi \equiv x.\left(-\frac{\chi}{d}\right) + n.\chi - x.\chi \equiv n.\chi + x.\left(-1.\chi - \frac{\chi}{d}\right).$$

Then, for all nonnegative integers k we have

$$\frac{n!}{(n-k)!}\mathcal{K}_k(x,p,n) = E\left\{\left[n.\chi + x.\left(-1.\chi - \frac{\chi}{d}\right)\right]^k\right\}. \tag{3.73}$$

Theorem 3.43. *The Krawtchouk polynomials are time-space harmonic with respect to the random walk $\{n.(-1.\mu)\}_{n\geq 0}$, where $-1.\mu$ is the umbral counterpart of a Bernoulli random variable with parameter p.*

Proof. For $i = 1,\ldots,n$, let X_i be a random variable with Bernoulli distribution of parameter p. Let $\mu \equiv -1.\chi.p.\beta$ be the umbra such that $f(\mu,z) = 1/(pe^z + (1-p))$, so the random walk $S_n = X_1 + X_2 + \cdots + X_n$ is umbrally represented by the family of auxiliary umbrae $\{n.(-1.\mu)\}_{n\geq 0}$.

From Theorem 3.7, the polynomials $Q_k(x,n) = E[(x - n.(-1.\mu))^k] = E[(x+n.\mu)^k]$ are time-space harmonic with respect to S_n for all nonnegative integers k. In order to obtain a Lévy-Sheffer system, we have

$$\begin{aligned}
n.\chi + x.\left(-1.\chi - \frac{\chi}{d}\right) &\equiv n.\chi + x.\beta.\left(\chi.\left(-1.\chi - \frac{\chi}{d}\right)\right)\\
&\equiv \left(x + n.\chi.\beta.\left(\chi.\left(-1.\chi - \frac{\chi}{d}\right)\right)^{<-1>}\right).\beta.\left(\chi.\left(-1.\chi - \frac{\chi}{d}\right)\right)\\
&\equiv \left(x + n.\left(\chi.\left(-1.\chi - \frac{\chi}{d}\right)\right)^{<-1>}\right).\beta.\left(\chi.\left(-1.\chi - \frac{\chi}{d}\right)\right).
\end{aligned}$$

By applying (3.73), we have

$$\begin{aligned}
\frac{n!}{(n-k)!}\mathcal{K}_k(x,p,n) &= E\left[\left(n.\chi + x.\left(-1.\chi - \frac{\chi}{d}\right)\right)^k\right]\\
&= E\left\{\left[\left(x + n.\left(\chi.\left(-1.\chi - \frac{\chi}{d}\right)\right)^{<-1>}\right).\beta.\left(\chi.\left(-1.\chi - \frac{\chi}{d}\right)\right)\right]^k\right\}\\
&= \sum_{j=1}^{k} E\left[\left\{x + n.\left[\chi.\left(-1.\chi - \frac{\chi}{d}\right)\right]^{<-1>}\right\}^j\right] B_{k,j}(m_1,\ldots,m_{k-j+1}),
\end{aligned} \tag{3.74}$$

where $m_i = E[(\chi.(-1.\chi - \chi/d))^i]$.

Let us consider now $E\left[\left\{x + n.\left[\chi.\left(-1.\chi - \frac{\chi}{d}\right)\right]^{<-1>}\right\}^j\right]$. We have

$$\begin{aligned}
x + n.\left[\chi.\left(-1.\chi - \frac{\chi}{d}\right)\right]^{<-1>} &\equiv x + n.(-1).(-1).\left[\chi.\left(-1.\chi - \frac{\chi}{d}\right)\right]^{<-1>}\\
&\equiv x - n.\left(-1.\left[\chi.\left(-1.\chi - \frac{\chi}{d}\right)\right]^{<-1>}\right).
\end{aligned}$$

We want to prove that this last umbra is similar to $x - n.\mu$. In particular, it will suffice to prove that $\mu \equiv -1.\left[\chi.\left(-1.\chi - \frac{\chi}{d}\right)\right]^{<-1>}$ via generating functions.

First of all, observe that

$$f\left(-1.(\chi.(-1.\chi - \chi/d))^{<-1>}, z\right) = \frac{1}{f(\chi.(-1.\chi - \chi/d))^{<-1>}, z)}.$$

On the other hand, if for the sake of simplicity we set $\vartheta \equiv \chi.(-1.\chi - \chi/d))^{<-1>}$, we have

$$\begin{aligned}
& f(\vartheta, f(\vartheta^{<-1>}, z) - 1) = 1 + z \;\Rightarrow\; 1 + \log\left(\frac{1 - z/d}{1+z}\right)\Bigg|_{f(\vartheta^{<-1>}, z)-1} = 1 + z \\
& \Rightarrow \log\left(\frac{1 - \frac{f(\vartheta^{<-1>}, z)-1}{d}}{f(\vartheta^{<-1>}, z) - 1}\right) = z \;\Rightarrow\; \frac{1 - \frac{f(\vartheta^{<-1>}, z)-1}{d}}{f(\vartheta^{<-1>}, z) - 1} = e^z \\
& \Rightarrow \frac{d - f(\vartheta^{<-1>}, z) + 1}{df(\vartheta^{<-1>}, z) + 1} = e^z \;\Rightarrow\; d + 1 - f(\vartheta^{<-1>}, z) = de^z f(\vartheta^{<-1>}, z) \\
& \Rightarrow f(\vartheta^{<-1>}, z) + de^z f(\vartheta^{<-1>}, z) = d + 1 \;\Rightarrow\; f(\vartheta^{<-1>}, z)(1 + de^z) = d + 1 \\
& \Rightarrow f(\vartheta^{<-1>}, z) = \frac{d+1}{1 + de^z} = \frac{\frac{p}{q} + 1}{1 + \frac{p}{q} e^z} = \frac{1}{q + pe^z} = f(-1.\mu, z).
\end{aligned}$$

Hence,

$$f\left(-1.(\chi.(-1.\chi - \chi/d))^{<-1>}, z\right) = f(-1.\vartheta^{<-1>}, z) f(\mu, z),$$

that is, $\mu \equiv -1.(\chi.(-1.\chi - \chi/d))^{<-1>}$. Therefore,

$$E\left[\{x + n.(\chi.(-1.\chi - \chi/d))^{<-1>}\}^j\right] = E[(x + n.\mu)^j] = Q_j(x, n).$$

By replacing this result in (3.74), we have

$$\frac{n!}{(n-k)!} \mathcal{K}_k(x, p, n) = \sum_{j=1}^{k} Q_j(x, n) B_{k,j}(m_1, \ldots, m_{k-j+1}),$$

and the result follows, thanks to Remark 3.12. □

Pseudo-Narumi polynomials. The family of pseudo-Narumi polynomials $\{N_k(x, an)\}$, $a \in \mathbb{N}$, is the sequence of the coefficients with the following power series [43]

$$\sum_{k\geq 0} N_k(x,an) z^k = \sum_{k\geq 0} k! N_k(x,an) \frac{z^k}{k!} = \left(\frac{\log(1+z)}{z} \right)^{an} (1+z)^x. \quad (3.75)$$

In order to give the umbral representation of pseudo-narumi polynomials, we have to introduce the *α-primitive umbra* α_P, that is the umbra whose moments are, for all $k \geq 1$,

$$(\alpha_P)^k \simeq \frac{\alpha^{k+1}}{k+1}. \quad (3.76)$$

Observe that

$$\begin{aligned} f(\alpha_P, z) &\simeq \sum_{k\geq 0} \frac{\alpha^{k+1}}{k+1} \frac{z^k}{k!} \simeq \sum_{k\geq 0} \alpha^{k+1} \frac{z^k}{(k+1)!} \\ &\simeq \alpha + \sum_{k\geq 1} \alpha^{k+1} \frac{z^k}{(k+1)!} \simeq \alpha + \sum_{k\geq 2} \alpha^k \frac{z^{k-1}}{k!} \\ &\simeq \sum_{k\geq 1} \alpha^k \frac{z^{k-1}}{k!} \simeq \frac{1}{z} \sum_{k\geq 1} \alpha^k \frac{z^k}{k!} \\ &\simeq \frac{1}{z} \left(\sum_{k\geq 0} \alpha^k \frac{z^k}{k!} - 1 \right) \simeq \frac{(f(\alpha,z)-1)}{z}. \end{aligned}$$

Thus, the generating function of the primitive umbra is

$$f(\alpha_P, z) = \frac{f(\alpha,z)-1}{z}. \quad (3.77)$$

From (3.75), the pseudo-Narumi polynomials result to be the moments of the umbra $x.\chi + (an).u_P^{<-1>}$, where $u_P^{<-1>}$ is the primitive umbra of the compositional inverse $u^{<-1>}$. For all nonnegative integers k, we have

$$k! N_k(x,an) = E\{[(an).u_P^{<-1>} + x.\chi]^k\}. \quad (3.78)$$

Theorem 3.44. *The pseudo-Narumi polynomials are time-space harmonic with respect to the random walk* $\{(an).(-1.\iota)\}_{n\geq 0}$.

Proof. Consider the random walk $S_n = X_1 + X_2 + \cdots + X_n$, where, for $i = 1, \ldots, n$, X_i is a sum of $a \in \mathbb{N}$ random variables with uniform distribution on the interval $[0,1]$. Therefore, for $i = 1, \ldots, n$, X_i is umbrally represented by $a.(-1.\iota)$ and S_n is umbrally represented by $\{n.a.(-1.\iota)\}_{n\geq 0}$.

By applying Theorem 3.7, it is straightforward to prove that $Q_k(x,n) = E[(x-(an).(-1.\iota))^k]$ are time-space harmonic with respect to S_n.

On the other hand,

$$\begin{aligned} x.\chi + (an).u_P^{<-1>} &\equiv x.\beta.(\chi.\chi) + (an).u_P^{<-1>} \\ &\equiv x.\beta.u^{<-1>} + (an).u_P^{<-1>} \equiv (x + (an).u_P^{<-1>}.\beta.u).\beta.u^{<-1>} \end{aligned}$$

and then, from (3.78),

$$\begin{aligned} k!N_k(x,an) =& E[((x + (an).u_P^{<-1>}.\beta.u).\beta.u^{<-1>})^k] \\ &= \sum_{j=1}^{k} E[(x + (an).u_P^{<-1>}.\beta)^j] B_{k,j}(m_1,\ldots,m_{k-j+1}), \end{aligned}$$

where $m_i = E[(u^{<-1>})^i]$. To prove the result, it is sufficient to show that $E[(x+(an).u_P^{<-1>}.\beta)^j]$ is the j-th time-space harmonic polynomial $Q_j(x,n)$. Via generating function, we have

$$\begin{aligned} &f(u_P^{<-1>}.\beta, z) = f(u_P^{<-1>}, e^z - 1) \\ &= \left.\frac{f(u^{<-1>}, z) - 1}{z}\right|_{z=e^z-1} = \frac{\log(1 + e^z - 1)}{e^z - 1} \\ &= \frac{z}{e^z - 1} = f(\iota, z), \end{aligned}$$

which gives

$$k!N_k(x,an) = \sum_{j=1}^{k} Q_j(x,n) B_{k,j}(m_1,\ldots,m_{k-j+1}),$$

and the result follows, thanks to Remark 3.12. □

3.3.7 Further examples

Sum of two independent Lévy processes Let us consider two independent Lévy processes $W = \{W_t\}_{t\geq 0}$ and $Z = \{Z_t\}_{t\geq 0}$, umbrally represented by $\{t.\alpha\}_{t\geq 0}$ and $\{t.\gamma\}_{t\geq 0}$, respectively. Due to the distributive property (1.14)-(v), the process $X = W + Z$ is umbrally represented by $t.(\alpha+\gamma) \equiv t.\alpha + t.\gamma$. If we replace $\mathbb{R}[x]$ with $\mathbb{R}[x,w,z]$ [20], and denote by $\{Q_k(x,t)\}_{k\geq 0}$, $\{Q'_k(x,t)\}_{k\geq 0}$ and $\{Q''_k(x,t)\}_{k\geq 0}$ the time-space harmonic polynomials with respect to $\{X_t\}_{t\geq 0}$, $\{W_t\}_{t\geq 0}$ and $\{Z_t\}_{t\geq 0}$, respectively, we have if $x = w + z$,

$$Q_k(x,t) = \sum_{j=0}^{k} \binom{k}{j} Q'_j(w,t) Q''_{k-j}(z,t).$$

Inverse Gaussian process An Inverse Gaussian process $\{X_t^{(IG)}\}_{t\geq 0}$ is a Lévy process with independent, stationary and Inverse Gaussian distributed increments, that is, $X_t^{(IG)} \sim IG(a,b)$.

The moment generating function of this kind of process is [18]

$$E\left[e^{zX_t^{(IG)}}\right] = \exp\left\{\frac{b}{a}\left[1-\left(\sqrt{1-\frac{2a^2t}{b}}\right)\right]\right\}. \tag{3.79}$$

The name *inverse Gaussian* depends on the fact that, as Tweedie observed in [61], there exists an inverse relationship between the cumulant generating functions of this distribution of probability and the cumulant generating function of gaussian random variables. More precisely, if X is an inverse gaussian random variable and Y is a Gaussian random variable, we have

$$\log(E[e^{-zX}]) = (\log(E[e^{-zY}]))^{<-1>}. \tag{3.80}$$

We use this property to introduce the *inverse Gaussian umbra.*

Definition 3.45. An umbra α is said to be the *inverse Gaussian umbra* if

$$-\alpha \equiv \beta.(-m\chi\dot{+}s\delta)^{<-1>}. \tag{3.81}$$

The generating function of the inverse Gaussian umbra is given in (3.79). In fact, we have

$$\begin{aligned}
&\chi.(-\alpha) \equiv [\chi.(-m+\beta.(s\delta))]^{<-1>}\\
&\quad\Rightarrow\ -\alpha \equiv \beta.[\chi.(-m+\beta.(s\delta))]^{<-1>}\\
&\quad\Rightarrow\ -\alpha \equiv \beta.[\chi.(\beta.(-m\chi\dot{+}s\delta))]^{<-1>}\\
&\quad\Rightarrow\ -\alpha \equiv \beta.(-m\chi\dot{+}s\delta)^{<-1>}.
\end{aligned}$$

By recalling (1.31), we have

$$f(-\alpha,z) = f(\beta.(-m\chi\dot{+}s\delta)^{<-1>},z) = \exp\{f((-m\chi\dot{+}s\delta)^{<-1>},z)-1\}.$$

On the other hand, thanks to (1.43) and (1.5) and by setting, for the sake of simplicity, $f^{<-1>}(-m\chi\dot{+}s\delta,z) = \bar{f}$,

$$\begin{aligned}
&f(-m\chi\dot{+}s\delta,\bar{f}-1) = 1+z\\
&\Rightarrow\ 1-mz+\frac{1}{2}s^2z^2-1\Big|_{z=\bar{f}-1} = 1+z\\
&\Rightarrow\ -mz+\frac{1}{2}s^2z^2\Big|_{\bar{f}-1} = z
\end{aligned}$$

$$\Rightarrow -m(\bar{f}-1)+\frac{1}{2}s^2(\bar{f}-1)^2 = z$$
$$\Rightarrow -m\bar{f}+m+\frac{1}{2}s^2(\bar{f}^2+1-2\bar{f}) = z$$
$$\Rightarrow -m\bar{f}+m+\frac{1}{2}s^2\bar{f}^2+\frac{1}{2}s^2-s^2\bar{f}-z = 0$$
$$\Rightarrow -2m\bar{f}+2m+s^2\bar{f}^2+s^2-2s^2\bar{f}-2z = 0$$
$$\Rightarrow s^2\bar{f}^2-2\bar{f}(m+s^2)+2m+s^2-2z = 0$$

We solve the previous quadratic equation:

$$\begin{aligned}
\bar{f} &= \frac{m+s^2 \pm \sqrt{m^2+s^4+2ms^2-2ms^2-s^4+2zs^2}}{s^2} \\
&= \frac{m+s^2 \pm \sqrt{m^2+2zs^2}}{s^2} \\
&= \frac{m}{s^2}+1 \pm \sqrt{\frac{m^2+2zs^2}{s^4}} \\
&= 1+\frac{m}{s^2} \pm \sqrt{\frac{m^2}{s^4}+\frac{2z}{s^2}} \\
&= 1+\frac{m}{s^2} \pm \sqrt{\frac{m^2}{s^4}\left(1+\frac{2z}{s^2}\frac{s^4}{m^2}\right)} \\
&= 1+\frac{m}{s^2} \pm \frac{m}{s^2}\sqrt{1+\frac{2zs^2}{m^2}} \\
&= 1+\frac{m}{s^2}\left[1 \pm \sqrt{1+\frac{2zs^2}{m^2}}\right].
\end{aligned}$$

We choose the smaller solution of the quadratic equation, since we are considering a generating function, which has to start from 1. Therefore, we have

$$f((-m\chi\dot{+}s\delta)^{<-1>}, z) = 1+\frac{m}{s^2}\left[1-\sqrt{1+\frac{2zs^2}{m^2}}\right]$$
$$\Rightarrow f(\beta.(-m\chi\dot{+}s\delta)^{<-1>}, z) = \exp\left\{1+\frac{m}{s^2}\left[1-\sqrt{1+\frac{2zs^2}{m^2}}\right]-1\right\}$$
$$= \exp\left\{\frac{m}{s^2}\left[1-\sqrt{1+\frac{2zs^2}{m^2}}\right]\right\}.$$

If we set $\frac{b}{a}=\frac{m}{s^2}$, we obtain the generating function (3.79).

Remark 3.46. Thanks to Theorem 2.37, the umbral version of an Inverse Gaussian process is
$$\{t.\beta.(-m\chi\dot{+}s\delta)^{<-1>}\}_{t\geq 0}. \tag{3.82}$$

By virtue of Theorem 3.7, it is straightforward to prove that polynomials of form
$$Q_k(x,t) = E[(x - t.\beta.(-m\chi\dot{+}s\delta)^{<-1>})^k]$$
are time-space harmonic with respect to the Inverse Gaussian process umbrally represented by the family of auxiliary umbrae given in (3.82).

Stable process

Definition 3.47. An umbra $\alpha \in \mathcal{A}$ is said to be *stable* with *stability parameter* $m \in [0,2)$ if there exist b_n, $c_n \in \mathbb{R}$ such that
$$n.\alpha \equiv c_n\alpha + b_n \tag{3.83}$$
or, equivalently,
$$\alpha \equiv n.\left[\frac{1}{c_n}\left(\alpha - \frac{b_n}{n}\right)\right]. \tag{3.84}$$

Remark 3.48. The definition of stable umbrae parallels the definition of stable random variables given in probability theory [35, 36].

As it happens in probability theory, one can prove that $c_n = n^{1/m}$ (see [36] for further details).

We characterize stable umbrae through their moments. By definition, we have, for all nonnegative integers $k \geq 1$,
$$E[(n.\alpha)^k] = E[(c_n\alpha + b_n)^k]. \tag{3.85}$$
If $k = 1$, we have
$$E[n.\alpha] = E[c_n\alpha + b_n] = c_nE[\alpha] + b_n = c_na_1 + b_n,$$
where we set $E[\alpha^k] = a_k$, for all $k \geq 1$.

On the other hand, the expression of the moments of the auxiliary umbra $n.\alpha$ given in (1.10) guarantees that
$$E[(n.\alpha)] = na_1.$$

By comparing these last two results and by recalling the second part of Remark 3.48, we have
$$na_1 = c_1a_1 + b_1 \;\Rightarrow\; a_1(n - c_n) = b_1$$

$$\Rightarrow a_1 = \frac{b_n}{n - c_n} = \frac{b_n}{n - n^{1/m}} \tag{3.86}$$

If $k = 2$, we have, thanks to equation (1.10),

$$\begin{aligned} E[(n.\alpha)^2] &= \sum_{j=1}^{2} (n)_j B_{2,j}(a_1, a_2) = nB_{2,1}(a_1) + n(n-1)B_{2,2}(a_1, a_2) \\ &= na_2 + n^2 a_1^2 - na_1^2. \end{aligned} \tag{3.87}$$

On the other hand, (3.85) implies that $E[(n.\alpha)^2] = c_n^2 a_2 + b_n^2 + 2c_n b_n a_1$. By comparing with (3.87), we have

$$\begin{aligned} &na_2 + n^2 a_1^2 - na_1^2 = c_n^2 a_2 + b_n^2 + 2c_n b_n a_1 \\ &\quad \Rightarrow (n - c_n^2)a_2 = b_n^2 + 2c_n b_n a_1 - (n^2 - n)a_1^2. \end{aligned}$$

Therefore, by replacing (3.86),

$$\begin{aligned} a_2 &= \frac{b_n^2 + 2c_n b_n a_1 - (n^2 - n)a_1^2}{(n - c_n^2)} \\ &= \frac{b_n^2 + 2\frac{n^{1/m} b_n^2}{n - n^{1/m}} - (n^2 - n)\left(\frac{b_n}{n - n^{1/m}}\right)^2}{(n - n^{2/m})} \\ &= \frac{b_n^2 \left[1 + 2\frac{n^{1/m}}{n - n^{1/m}} + \frac{(n - n^2)}{(n - n^{1/m})^2}\right]}{(n - n^{2/m})} \\ &= \frac{b_n^2}{n - n^{2/m}} \left[\frac{(n - n^{1/m})^2 + 2n^{1/m}(n - n^{1/m}) + n - n^2}{(n - n^{1/m})^2}\right] \\ &= \frac{b_n^2 (n^2 - 2n^{1+1/m} + n^{2/m} + 2n^{1+1/m} - 2n^{2/m} + n - n^2)}{(n - n^{2/m})(n - n^{1/m})^2} \\ &= \frac{b_n^2 (n - n^{2/m})}{(n - n^{2/m})(n - n^{1/m})^2} = \frac{b_n^2}{(n - n^{1/m})^2}. \end{aligned}$$

Finally, we obtain

$$a_2 = \frac{b_n^2}{(n - n^{1/m})^2}. \tag{3.88}$$

With similar arguments, the k-th moment of a stable umbra is

$$a_k = \frac{b_n^k}{(n - n^{1/m})^k}, \quad \text{for all } k \geq 1. \tag{3.89}$$

A *stable* process is a Lévy process whose increments are independent, stationary and stable-distributed [36]. From an umbral point of view, a stable process in umbrally represented by the family of auxiliary umbrae $\{t.\alpha\}_{t\geq 0}$, where α is a stable umbra.

By applying Theorem 3.7, it is straightforward to prove that the time-space harmonic polynomials with respect to stable processes are

$$Q_k(x,t) = E[(x - t.\alpha)^k],$$

with α stable umbra.

3.4 The Kailath-Segall polynomials

Let $\{X_t\}_{t\geq 0}$ be a centered Lévy process with moments of all orders and let $\{X_t^{(n)}\}_{t\geq 0}$ be the variations

$$X_t^{(1)} = X_t, \;\; X_t^{(2)} = [X,X]_t, \;\; X_t^{(k)} = \sum_{s\geq t} (\Delta X_s)^k, \;\; k \geq 3, \tag{3.90}$$

of the process. The iterated stochastic integrals

$$P_t^{(0)} = 1, \;\; P_t^{(1)} = X_t, \;\; P_t^{(k)} = \int_0^t P_{s-}^{(k-1)} \mathrm{d}X_s, \;\; k \geq 2 \tag{3.91}$$

are related to the variations $\{X_t^{(k)}\}_{t\geq 0}$ by the Kailath-Segall formula [33]

$$P_t^{(k)} = \frac{1}{k}\left(P_t^{(k-1)} X_t^{(1)} - P_t^{(k-2)} X_t^{(2)} + \cdots + (-1)^{k+1} P_t^{(0)} X_t^{(k)}\right). \tag{3.92}$$

Then, $P_t^{(k)} = P_k\left(X_t^{(1)}, \ldots, X_t^{(k)}\right)$ is a polynomial in $X_t^{(1)}, X_t^{(2)}, \ldots, X_t^{(k)}$, called the k-th *Kailath-Segall polynomial*.

Let us introduce the families of umbrae $\{\Upsilon_t\}_{t\geq 0}$ and $\{\sigma_t\}_{t\geq 0}$ such that $E[\Upsilon_t^k] = k! E\left[P_t^{(k)}\right]$ and $E[\sigma_t^k] = E[X_t^{(k)}]$, respectively, for all nonnegative integers k. The following theorem states the umbral version of the Kailath-Segall formula and its inversion.

Theorem 3.49. *We have*

$$\Upsilon_t \equiv \beta.[(\chi.\chi)\sigma_t] \;\; \textit{and} \;\; (\chi.\chi)\sigma_t \equiv \chi.\Upsilon_t. \tag{3.93}$$

Proof. Assume $\psi_t \equiv (\chi.\chi)\sigma_t$ where $E[(\chi.\chi)^k] = (-1)^{k-1}(k-1)!$ [30]. The recurrence relation (3.92) is equivalent to $E[\Upsilon_t^k] = E[\psi_t(\Upsilon_t + \psi_t)^{k-1}]$, for all $k \geq 1$. Indeed, by definition of umbrae ψ_t and Υ_t, we have

$$E[\Upsilon_t^k] = k!\,\frac{1}{k}\left\{\frac{E\left[\Upsilon_t^{k-1}\right]E\left[\psi_t\right]}{(k-1)!} + \frac{E\left[\Upsilon_t^{k-2}\right]E\left[\psi_t^2\right]}{(k-2)!} + \cdots + \frac{E\left[\psi_t^k\right]}{(k-1)!}\right\}$$
$$= \sum_{j=0}^{k-1}\binom{k-1}{j}E\left[\Upsilon_t^{k-1-j}\right]E\left[\psi_t^{j+1}\right] = E\left[\psi_t(\Upsilon_t + \psi_t)^{k-1}\right].$$

By using the first equivalence of Theorem 2.10 given in Chapter 2, we have the inversion of the Kailath-Segall formula

$$\psi_t \equiv (\chi.\chi)\sigma_t \equiv \chi.\Upsilon_t.$$

The second equivalence follows by observing that $\psi_t \equiv \chi.\Upsilon_t \Leftrightarrow \beta.\psi_t \equiv \beta.\chi.\Upsilon_t$ and $\beta.\chi \equiv u$. □

By recalling that the moments of $\beta.\alpha$ are the (exponential) complete Bell polynomials [19] in the moments of α, then the *Kailath-Segall polynomials* are complete Bell exponential polynomials in $\{(-1)^{k-1}(n-1)!E[X_t^{(k)}]\}$.

Corollary 3.50. *If* $c_j = j!E\left[P_t^{(j)}\right]$ *for* $j = 1, \ldots, k,$ *then*

$$E\left[X_t^{(k)}\right] = \sum_{j=1}^{k}\frac{(-1)^{k-j}}{(k-1)_{k-j}}B_{k,j}(c_1, c_2, \ldots, c_{k-j+1}).$$

Proof. From the latter of (3.93), we have

$$((\chi.\chi)\sigma_t)^k \simeq (\chi.\Upsilon_t)^k \simeq \sum_{j=1}^{k}(\chi)_j B_{k,j}(c_1, \ldots, c_{k-j+1}),$$

where the last equivalence holds by virtue of equality (1.27), with γ and α replaced by χ and Υ_t, respectively.

Furthermore, let us recall that $(\chi)_j \simeq (-1)^{j-1}(j-1)!$, for all $j = 1, \ldots, k$ and $E[((\chi.\chi)\sigma_t)^k] = E[(\chi.\chi)^k\sigma_t^k] = (-1)^{k-1}(k-1)!E[\sigma_t^k]$, so that

$$(-1)^{k-1}(k-1)!\sigma_t^k \simeq \sum_{j=1}^{k}(-1)^{j-1}(j-1)!B_{k,j}(c_1, \ldots, c_{k-j+1}).$$

By definition, $\sigma_t^k \simeq X_t^{(k)}$, then

$$(-1)^{k-1}(k-1)!X_t^{(k)} \simeq \sum_{j=1}^{k}(-1)^{j-1}(j-1)!B_{k,j}(c_1,\ldots,c_{k-j+1})$$

$$\Rightarrow X_t^{(k)} \simeq \sum_{j=1}^{k}(-1)^{-k+j}\frac{(j-1)!}{(k-1)!}B_{k,j}(c_1,\ldots,c_{k-j+1})$$

$$= \sum_{j=1}^{k}(-1)^{k-j}\frac{(1}{(k-1)_{k-j}}B_{k,j}(c_1,\ldots,c_{k-j+1}).$$

The result follows by taking the evaluation of both sides. □

Remark 3.51. The inversion of the Kailath-Segall formula in Theorem 3.49 is a generalization of formula (3.2) in [20] which gives the elementary symmetric polynomials in terms of power sum symmetric polynomials. That is, if we replace the jumps $\{\Delta X_s\}$ in $X_t^{(n)}$ with suitable indeterminates $\{x_s\}$, then the Kailath-Segall polynomials reduce to the polynomials given in [59].

Chapter 4

Multivariate time-space harmonic polynomials

4.1 Multivariate umbral calculus

In the univariate classical umbral calculus, the main device is to replace a_n with α^n via the linear evaluation E. Similarly, in the multivariate case, the main device is to replace sequences like $\{g_{i_1,i_2,\ldots,i_d}\}$ with a product of powers $\mu_1^{i_1}\mu_2^{i_2}\ldots\mu_d^{i_d}$, where $\{\mu_1,\mu_2,\ldots,\mu_d\}$ are umbral monomials and $i_1,\ldots,i_d$ are nonnegative integers. Note that the supports of the umbral monomials in $\{\mu_1,\mu_2,\ldots,\mu_d\}$ are not necessarily disjoint.

In order to manage a product like $\mu_1^{i_1}\mu_2^{i_2}\ldots\mu_d^{i_d}$ as a power of an umbra, we will use a multi-index notation. Let us recall that a d-dimensional multi-index is a d-tuple $\mathbf{v}=(v_1,\ldots,v_d)$ of nonnegative integers, such that the following properties hold

Componentwise sum and difference: $\mathbf{v}\pm\mathbf{w}=(v_1\pm w_1,\ldots,v_d\pm w_d)$

Partial order: $\mathbf{v}\leq\mathbf{w} \;\Leftrightarrow\; v_i\leq w_i,$ for all $i=1,\ldots,d$

Sum of components: $|\mathbf{v}|=v_1+\cdots+v_d$

Factorial: $\mathbf{v}!=v_1!\cdots v_d!$

Binomial coefficient: $\binom{\mathbf{v}}{\mathbf{w}}=\binom{v_1}{w_1}\cdots\binom{v_d}{w_d}$

Power: $\mathbf{x}^{\mathbf{v}}=x_1^{v_1}\cdots x_d^{v_d}$, for all $\mathbf{x}=(x_1,\ldots,x_d)\in\mathbb{R}^d$.

Definition 4.1. A partition $\boldsymbol{\lambda}$ of a multi-index $\mathbf{v}$, in symbols $\boldsymbol{\lambda} \vdash \mathbf{v}$, is a matrix $\boldsymbol{\lambda} = (\lambda_{ij})$ of nonnegative integers and with no zero columns in lexicographic order such that $\lambda_{r_1} + \lambda_{r_2} + \cdots + \lambda_{r_k} = v_r$ for $r = 1, 2, \ldots, d$. The number of columns of $\boldsymbol{\lambda}$ is denoted by $l(\boldsymbol{\lambda})$. The notation $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1^{r_1}, \boldsymbol{\lambda}_2^{r_2}, \ldots)$ means that in the matrix $\boldsymbol{\lambda}$ there are r_1 columns equal to $\boldsymbol{\lambda}_1$, r_2 columns equal to $\boldsymbol{\lambda}_2$ and so on, with $\boldsymbol{\lambda}_1 \prec \boldsymbol{\lambda}_2 \prec \ldots$. We set $\mathfrak{m}(\boldsymbol{\lambda}) = (r_1, r_2, \ldots)$, $\mathfrak{m}(\boldsymbol{\lambda})! = r_1! r_2! \ldots$ and $\boldsymbol{\lambda}! = \boldsymbol{\lambda}_1! \boldsymbol{\lambda}_2! \cdots$.

A multivariate version of the classical umbral calculus has been given for the first time in [21].

A sequence $\{g_{\mathbf{v}}\}_{\mathbf{v} \in \mathbf{N}_0^d}$ with $g_{\mathbf{v}} = g_{v_1, \ldots, v_d}$ and $g_{\mathbf{0}} = 1$ is umbrally represented by the d-tuple $\boldsymbol{\mu}$ if

$$E[\boldsymbol{\mu}^{\mathbf{v}}] = g_{\mathbf{v}},$$

for all $\mathbf{v} \in \mathbb{N}_0^d$. Then $g_{\mathbf{v}}$ is called the *multivariate moment* of $\boldsymbol{\mu}$.

Two d-tuples $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ of umbral monomials are said to be similar if they represent the same sequence of multivariate moments, in symbols

$$E[\boldsymbol{\mu}^{\mathbf{v}}] = E[\boldsymbol{\nu}^{\mathbf{v}}], \text{ for all } \mathbf{v} \in \mathbb{N}_0^d.$$

We introduce the notion of generating function of the d-tuple $\boldsymbol{\mu}$. The exponential multivariate formal power series

$$e^{\boldsymbol{\mu}\mathbf{z}^T} = \boldsymbol{u} + \sum_{k \geq 1} \sum_{\substack{\mathbf{v} \in \mathbb{N}_0^d \\ |\mathbf{v}| = k}} \boldsymbol{\mu}^{\mathbf{v}} \frac{\mathbf{z}^{\mathbf{v}}}{\mathbf{v}!} \tag{4.1}$$

is said to be the *generating function* of the d-tuple $\boldsymbol{\mu}$. Now, assume $\{g_{\mathbf{v}}\}_{\mathbf{v} \in \mathbb{N}_0^d}$ is umbrally represented by the d-tuple $\boldsymbol{\mu}$. If the sequence $\{g_{\mathbf{v}}\}_{\mathbf{v} \in \mathbb{N}_0^d}$ has exponential multivariate generating function

$$f(\boldsymbol{\mu}, \mathbf{z}) = 1 + \sum_{k \geq 1} \sum_{\substack{\mathbf{v} \in \mathbb{N}_0^d \\ |\mathbf{v}| = k}} g_{\mathbf{v}} \frac{\mathbf{z}^{\mathbf{v}}}{\mathbf{v}!},$$

by suitably extending the action of E coefficientwise to generating function (4.1), we have $E[e^{\boldsymbol{\mu}\mathbf{z}^T}] = f(\boldsymbol{\mu}, \mathbf{z})$. Henceforth, when no confusion occurs, we refer to $f(\boldsymbol{\mu}, \mathbf{z})$ as the generating function of the d-tuple $\boldsymbol{\mu}$.

Some examples of umbral d-tuples are

Multivariate augmentation umbra. The *multivariate augmentation umbra* is the umbral d-tuple $\boldsymbol{\epsilon} = (\epsilon, \dots, \epsilon)$ such that

$$f(\boldsymbol{\epsilon}, \mathbf{z}) = 1. \tag{4.2}$$

Then $E[\boldsymbol{\epsilon}^{\mathbf{v}}] = 1$, when $\mathbf{v} = 0$, and 0 otherwise.

Multivariate unity umbra. The *multivariate unity umbra* is the umbral d-tuple $\boldsymbol{u} = (u, \dots, u)$ such that

$$f(\boldsymbol{u}, \mathbf{z}) = \exp\{z_1 + \dots + z_d\}. \tag{4.3}$$

Multivariate disjoint sum and difference. The *multivariate disjoint sum and difference* are the auxiliary umbrae $\boldsymbol{\mu}\dot{+}\boldsymbol{\nu}$ and $\boldsymbol{\mu}\dot{-}\boldsymbol{\nu}$ with generating functions respectively

$$f(\boldsymbol{\mu}\dot{+}\boldsymbol{\nu}, \mathbf{z}) = f(\boldsymbol{\mu}, \mathbf{z}) + f(\boldsymbol{\nu}, \mathbf{z}) - 1 \quad f(\boldsymbol{\mu}\dot{-}\boldsymbol{\nu}, \mathbf{z}) = f(\boldsymbol{\mu}, \mathbf{z}) - f(\boldsymbol{\nu}, \mathbf{z}) + 1. \tag{4.4}$$

In the univariate case we have defined the auxiliary umbra $n.\alpha$ and the composition umbra $\gamma.\beta.\alpha$. We follow the same steps in the multivariate case.

Let $\{\boldsymbol{\mu}', \boldsymbol{\mu}'', \dots, \boldsymbol{\mu}'''\}$ be a set of m uncorrelated d-tuples similar to $\boldsymbol{\mu}$. Define the *dot-product* of m and $\boldsymbol{\mu}$ and the m-th dot-power of $\boldsymbol{\mu}$ as the auxiliary umbrae

$$m.\boldsymbol{\mu} = \boldsymbol{\mu}' + \boldsymbol{\mu}'' + \ldots + \boldsymbol{\mu}''' \quad \boldsymbol{\mu}^{.m} = \boldsymbol{\mu}'\boldsymbol{\mu}'' \ldots \boldsymbol{\mu}'''. \tag{4.5}$$

One can prove that [21]

$$(m.\boldsymbol{\mu})^{\mathbf{v}} \simeq \sum_{\boldsymbol{\lambda} \vdash \mathbf{v}} \frac{\mathbf{v}!}{\mathfrak{m}(\boldsymbol{\lambda})\boldsymbol{\lambda}!} (m.\chi)^{l(\boldsymbol{\lambda})} \boldsymbol{\mu}_{\boldsymbol{\lambda}},$$

where $\boldsymbol{\mu}_{\boldsymbol{\lambda}} = (\boldsymbol{\mu}'^{\boldsymbol{\lambda}_1}_{\boldsymbol{\lambda}})^{.r_1} \boldsymbol{\mu}''^{\boldsymbol{\lambda}_2}_{\boldsymbol{\lambda}})^{.r_2} \dots$, with $\boldsymbol{\mu}', \boldsymbol{\mu}'', \dots$ uncorrelated d-tuple similar to $\boldsymbol{\mu}$ (see [21]).

As it happens in the univariate case, we can replace m with $t \in \mathbb{R}$, so we can consider $t.\boldsymbol{\mu}$ as an umbra such that

$$(t.\boldsymbol{\mu})^{\mathbf{v}} \simeq \sum_{\boldsymbol{\lambda} \vdash \mathbf{v}} \frac{\mathbf{v}!}{\mathfrak{m}(\boldsymbol{\lambda})\boldsymbol{\lambda}!} (t.\chi)^{l(\boldsymbol{\lambda})} \boldsymbol{\mu}_{\boldsymbol{\lambda}}.$$

The generating function of $t.\boldsymbol{\mu}$ is

$$f(t.\boldsymbol{\mu}, \mathbf{z}) = [f(\boldsymbol{\mu}, \mathbf{z})]^t. \tag{4.6}$$

Remark 4.2. The *distributive property* of the dot-product over the summation can be easily generalized to the multivariate case, so we have

$$t.(\boldsymbol{\mu} + \boldsymbol{\nu}) \equiv t.\boldsymbol{\mu} + t.\boldsymbol{\nu}. \tag{4.7}$$

We define the auxiliary umbrae $t.\beta.\boldsymbol{\mu}$ and $\gamma.\beta.\boldsymbol{\mu}$ as respectively

$$(\gamma.\beta.\boldsymbol{\mu})^{\mathbf{v}} \simeq \sum_{\boldsymbol{\lambda} \vdash \mathbf{v}} \frac{\mathbf{v}!}{\mathfrak{m}(\boldsymbol{\lambda})\boldsymbol{\lambda}!} (\gamma)^{l(\boldsymbol{\lambda})} \boldsymbol{\mu}_{\boldsymbol{\lambda}}, \quad (t.\beta.\boldsymbol{\mu})^{\mathbf{v}} \simeq \sum_{\boldsymbol{\lambda} \vdash \mathbf{v}} \frac{\mathbf{v}!}{\mathfrak{m}(\boldsymbol{\lambda})\boldsymbol{\lambda}!} t^{l(\boldsymbol{\lambda})} \boldsymbol{\mu}_{\boldsymbol{\lambda}}, \tag{4.8}$$

where $\boldsymbol{\mu}_{\boldsymbol{\lambda}} = (\boldsymbol{\mu}'^{\boldsymbol{\lambda}_1}_{\boldsymbol{\lambda}})^{.r_1} \boldsymbol{\mu}''^{\boldsymbol{\lambda}_2}_{\boldsymbol{\lambda}})^{.r_2} \ldots$, with $\boldsymbol{\mu}', \boldsymbol{\mu}'', \ldots$ uncorrelated d-tuple similar to $\boldsymbol{\mu}$.

Proposition 4.3. *Let $\boldsymbol{\mu}$ be a d-tuple of umbral monomials with generating function $f(\boldsymbol{\mu}, \mathbf{z})$. The auxiliary umbrae $\gamma.\beta.\boldsymbol{\mu}$ and $t.\beta.\boldsymbol{\mu}$ have generating function respectively*

$$f(\gamma.\beta.\boldsymbol{\mu}, \mathbf{z}) = f(\gamma, f(\boldsymbol{\mu}, \mathbf{z}) - 1) \qquad f(t.\beta.\boldsymbol{\mu}, \mathbf{z}) = f(t, f(\boldsymbol{\mu}, \mathbf{z}) - 1).$$

In Proposition 4.3 we can replace the umbra γ with the umbra $\chi.\chi$ with generating function $f(\chi.\chi, z) = 1 + \log(1 + z)$ and, since $\chi.\beta \equiv u$, as we have already proved in Chapter 1 we obtain

$$f(\chi.\chi.\beta.\boldsymbol{\mu}, \mathbf{z}) = f(\chi.\boldsymbol{\mu}, \mathbf{z}) = 1 + \log(f(\boldsymbol{\mu}, \mathbf{z})). \tag{4.9}$$

The moments of the auxiliary umbra $\chi.\boldsymbol{\mu}$ with generating function (4.9) are called *multivariate cumulants* of $\boldsymbol{\mu}$.

The equations giving multivariate cumulants in terms of multivariate moments and vice versa, corresponding to Theorems 2.9, 2.15 and 2.20, are considered in [38].

4.2 Symbolic representation of multidimensional Lévy processes

Definition 4.4. A stochastic process $\mathbf{X} = \{\mathbf{X}_t\}_{t \geq 0}$ on $\mathbb{R}^d$ is a *multidimensional Lévy process* if

(i) $\mathbf{X}_0 = \mathbf{0}$ a.s.

(ii) For all $n \geq 1$ and for all $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n < \infty$, the random variables $\mathbf{X}_{t_2} - \mathbf{X}_{t_1}, \mathbf{X}_{t_3} - \mathbf{X}_{t_2}, \ldots$ are independent.

(iii) For all $s \leq t$, $\mathbf{X}_{t+s} - \mathbf{X}_s \stackrel{d}{=} \mathbf{X}_t$.

(iv) For all $\varepsilon > 0$, $\lim_{h \to 0} P(|\mathbf{X}_{t+h} - \mathbf{X}_t| > \varepsilon) = 0$.

(v) $t \mapsto \mathbf{X}_t(\omega)$ are cádlág, for all $\omega \in \Omega$.

Remark 4.5. Definition 4.4 says that, since we have $\mathbf{X}_t = \left(X_1^{(t)}, \ldots, X_d^{(t)}\right)$, the random variables which form the stochastic process are d-dimensional vectors, whose components $X_j^{(t)}$ are random variable in turn, for all $j = 1, \ldots, d$. In particular, we have $\mathbf{X}_1 = \left(X_1^{(1)}, \ldots, X_d^{(1)}\right)$, so the corresponding moment generating function is

$$\varphi_{\mathbf{X}_1}(\mathbf{z}) = E\left[e^{\mathbf{z}\mathbf{X}_1^T}\right], \tag{4.10}$$

with $\mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{R}^d$.

The whole theory of Lévy processes discussed in Chapter 2 can be generalized to the multidimensional case. In particular, in Chapter 2 we have proved that every Lévy process $X = \{X_t\}_{t \geq 0}$ on $\mathbb{R}$ is umbrally represented by the family of auxiliary umbrae $\{t.\alpha\}_{t \geq 0}$ such that $E[(t.\alpha)^k] = E[X_t^k]$, for all nonnegative integers k. Now, we generalize this theorem to the multivariate case.

Theorem 4.6. *A Lévy process $\mathbf{X} = \{\mathbf{X}_t\}_{t \geq 0}$ in $\mathbb{R}^d$ is umbrally represented by $\{t.\boldsymbol{\mu}\}_{t \geq 0}$, where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d)$ is a d-tuple of umbral monomials such that $E[\boldsymbol{\mu}^{\mathbf{v}}] = E\left[\{X_1^{(t)}\}^{v_1} \cdots \{X_d^{(t)}\}^{v_d}\right]$, for all $\mathbf{v} \in \mathbf{N}_0^d$.*

Proof. Denote by $\varphi_{\mathbf{X}}(\mathbf{z})$ the moment generating function of $\{\mathbf{X}_t\}_{t \geq 0}$ and by $\varphi_{\mathbf{X}_1}(\mathbf{z})$ the moment generating function of $\mathbf{X}_1$. Thanks to the infinite divisibility property given in Definition 2.27, we have $\varphi_{\mathbf{X}}(\mathbf{z}) = [\varphi_{\mathbf{X}_1}(\mathbf{z})]^t$, for all $t \geq 0$.

Let $\boldsymbol{\mu}$ be the umbral d-tuple such that $f(\boldsymbol{\mu}, \mathbf{z}) = \varphi_{\mathbf{X}_1}(\mathbf{z})$. Then

$$\varphi_{\mathbf{X}}(\mathbf{z}) = [\varphi_{\mathbf{X}_1}(\mathbf{z})]^t = \exp\{t \log(f(\boldsymbol{\mu}, \mathbf{z}))\} = \exp\{t[f(\chi.\boldsymbol{\mu}, \mathbf{z}) - 1]\},$$

due to equation (4.9). Thanks to Proposition 4.3, we have

$$\varphi_{\mathbf{X}}(\mathbf{z}) = f(t.\beta.\chi.\boldsymbol{\mu}, \mathbf{z}) = f(t.\boldsymbol{\mu}, \mathbf{z}),$$

as $f(\beta.\chi, z) = e^z$, by which the result follows. $\square$

As example, recall that a Gamma process $\{G_t\}_{t\geq 0}$ on $\mathbb{R}$ with both shape parameter and scale parameter equal to 1 is umbrally represented by the family of auxiliary umbrae $\{t.\bar{u}\}_{t\geq 0}$, where $\bar{u}$ is the boolean unity umbra.

If we consider the *multivariate boolean unity* $\bar{\boldsymbol{u}} = (\bar{u}, \bar{u}, \ldots, \bar{u})$ with generating function

$$f(\bar{\boldsymbol{u}}, \mathbf{z}) = \frac{1}{[1 - (z_1 + \cdots + z_d)]}, \tag{4.11}$$

the family $\{t.\bar{\boldsymbol{u}}\}_{t\geq 0}$ is an example of Lévy process, having generating function

$$f(t.\bar{\boldsymbol{u}}, \mathbf{z}) = \frac{1}{[1 - (z_1 + \cdots + z_d)]^t}. \tag{4.12}$$

The moment generating function of a Lévy process in $\mathbb{R}^d$ is given by the Lévy-Khintchine formula [48].

Theorem 4.7. *A stochastic process $\mathbf{X} = \{\mathbf{X}_t\}_{t\geq 0}$ is a Lévy process if and only if there exists $\mathbf{m}_1 \in \mathbb{R}^d$, a symmetric, positive defined, $d \times d$ matrix $\Sigma > 0$ and a measure ν on $\mathbb{R}^d$ with*

$$\nu(\{0\}) = 0 \ \textit{and} \ \int_{\mathbb{R}} (|\mathbf{x}|^2 \wedge 1)\nu(d\mathbf{x}) < \infty$$

such that

$$\varphi_{\mathbf{X}}(\mathbf{z}) = \exp\left\{t\left[\frac{1}{2}\mathbf{z}\Sigma\mathbf{z}^T + \mathbf{m}_1\mathbf{z}^T + \int_{\mathbb{R}^d}(e^{\mathbf{x}\mathbf{z}^T} - 1 - \mathbf{x}\mathbf{z}^T\mathbf{1}_{\{|\mathbf{x}|\leq 1\}}(\mathbf{x}))\nu(\mathrm{d}\mathbf{x})\right]\right\} \tag{4.13}$$

and the representation of $\varphi_{\mathbf{X}}(\mathbf{z})$ in (4.13) by $\boldsymbol{m_1}$, Σ and ν is unique.

Remark 4.8. We can rewrite (4.13) in the following way:

$$\begin{aligned}
\varphi_{\mathbf{X}}(\mathbf{z}) &= \exp\left\{\left[\frac{1}{2}\mathbf{z}\Sigma\mathbf{z}^T + \mathbf{m}_1\mathbf{z}^T + \int_{\mathbb{R}^d}(e^{\mathbf{z}\mathbf{x}^T} - 1 - \mathbf{z}\mathbf{x}^T\mathbf{1}_{\{|\mathbf{x}|\leq 1\}}(\mathbf{x}))\nu(d\mathbf{x})\right]\right\} \\
&= \exp\left\{t\left[\frac{1}{2}\mathbf{z}\Sigma\mathbf{z}^T + \mathbf{m}_1\mathbf{z}^T + \int_{\mathbb{R}^d}(e^{\mathbf{z}\mathbf{x}^T} - 1 - \mathbf{z}\mathbf{x}^T\mathbf{1}_{\{|\mathbf{x}|\leq 1\}}(\mathbf{x}))\nu(d\mathbf{x})\right.\right. \\
&+ \left.\left.\int_{\mathbb{R}^d}\mathbf{z}\mathbf{x}^T\mathbf{1}_{\{|\mathbf{x}|>1\}}(\mathbf{x})\nu(d\mathbf{x}) - \int_{\mathbb{R}^d}\mathbf{z}\mathbf{x}^T\mathbf{1}_{\{|\mathbf{x}|>1\}}(\mathbf{x})\nu(d\mathbf{x})\right]\right\} \\
&= \exp\left\{t\left[\frac{1}{2}\mathbf{z}\Sigma\mathbf{z}^T + \mathbf{m}_1\mathbf{z}^T + \int_{\mathbb{R}^d}\mathbf{z}\mathbf{x}^T\mathbf{1}_{\{|\mathbf{x}|>1\}}(\mathbf{x})\nu(d\mathbf{x})\right.\right. \\
&+ \left.\left.\int_{\mathbb{R}^d}(e^{\mathbf{z}\mathbf{x}^T} - 1 - \mathbf{z}\mathbf{x}^T)\nu(d\mathbf{x})\right]\right\}.
\end{aligned}$$

If we put $\mathbf{m}_2\mathbf{z}^T = \int_{\mathbb{R}^d} \mathbf{z}\mathbf{x}^T \mathbf{1}_{\{|\mathbf{x}|>1\}}(\mathbf{x})\nu(d\mathbf{x})$ and $\mathbf{m} = \mathbf{m}_1 + \mathbf{m}_2$, we have

$$\varphi_{\mathbf{X}}(\mathbf{z}) = \exp\left\{t\left[\frac{1}{2}\mathbf{z}\Sigma\mathbf{z}^T + \mathbf{m}\mathbf{z}^T + \int_{\mathbb{R}^d}(e^{\mathbf{z}\mathbf{x}^T} - 1 - \mathbf{z}\mathbf{x}^T)\nu(d\mathbf{x})\right]\right\},$$

that is,

$$\varphi_{\mathbf{X}}(\mathbf{z}) = \exp\left\{t\left[\frac{1}{2}\mathbf{z}\Sigma\mathbf{z}^T + \mathbf{m}\mathbf{z}^T\right]\right\}\exp\left\{t\left[\int_{\mathbb{R}^d}(e^{\mathbf{z}\mathbf{x}^T} - 1 - \mathbf{z}\mathbf{x}^T)\nu(d\mathbf{x})\right]\right\}. \tag{4.14}$$

Definition 4.9. Consider the umbral d-tuple $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_d)$, with $\delta_1, \ldots, \delta_d$ uncorrelated umbrae similar to the umbra δ, with generating function

$$f(\boldsymbol{\delta}, \mathbf{z}) = 1 + \frac{1}{2}(z_1^2 + \ldots + z_d^2) = 1 + \frac{\mathbf{z}\mathbf{z}^T}{2}. \tag{4.15}$$

The umbral d-tuple $\mathbf{m}+\beta.(\boldsymbol{\delta}C^T)$ is the *multivariate gaussian umbra* with generating function

$$f(\mathbf{m} + \beta.(\boldsymbol{\delta}C^T), \mathbf{z}) = \exp\left\{\mathbf{m}\mathbf{z}^T + \frac{1}{2}\mathbf{z}\Sigma\mathbf{z}^T\right\}, \tag{4.16}$$

where C is the square root of Σ, that is, $\Sigma = CC^T$.

Remark 4.10. The umbral d-tuple $\mathbf{m}+\beta.(\boldsymbol{\delta}C^T)$ is the umbral counterpart of a multivariate gaussian random variable $X \sim N(\mathbf{m}, \Sigma)$, whose generating function is $\exp\{\mathbf{m}\mathbf{z}^T + \mathbf{z}\Sigma\mathbf{z}^T/2\}$.

Obviously, a multivariate standard random variable $X \sim N(\mathbf{0}, I)$ is umbrally represented by $\beta.(\boldsymbol{\delta}I)$, while the auxiliary umbra $\beta.(\boldsymbol{\delta}C^T)$ is the umbral counterpart of the gaussian random variable $X \sim N(\mathbf{0}, \Sigma)$.

Proposition 4.11. *If we choose two d-tuple $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ of umbral monomials, we have*

$$\beta.(\boldsymbol{\mu}\dot{+}\boldsymbol{\nu}) \equiv \beta.\boldsymbol{\mu} + \beta.\boldsymbol{\nu}. \tag{4.17}$$

Proof. Via generating function, we have

$$\begin{aligned}f(\beta.(\boldsymbol{\mu}\dot{+}\boldsymbol{\nu}), \mathbf{z}) &= \exp\{f(\boldsymbol{\mu}\dot{+}\boldsymbol{\nu}, \mathbf{z}) - 1\} = \exp\{f(\boldsymbol{\mu}, \mathbf{z}) + f(\boldsymbol{\nu}, \mathbf{z}) - 2\}\\ &= \exp\{f(\boldsymbol{\mu}, \mathbf{z}) - 1 + f(\boldsymbol{\nu}, \mathbf{z}) - 1\} = \exp\{f(\boldsymbol{\mu}, \mathbf{z}) - 1\}\exp\{f(\boldsymbol{\nu}, \mathbf{z}) - 1\}\\ &= f(\beta.\boldsymbol{\mu}, \mathbf{z})f(\beta.\boldsymbol{\nu}, \mathbf{z}) = f(\beta.\boldsymbol{\mu} + \beta.\boldsymbol{\nu}, \mathbf{z}).\end{aligned}$$

□

By applying Proposition 4.11, we have

$$\mathbf{m} + \beta.(\boldsymbol{\delta} C^T) \equiv \beta.\chi.\mathbf{m} + \beta.(\boldsymbol{\delta} C^T) \equiv \beta.(\chi.\mathbf{m} \dot{+} (\boldsymbol{\delta} C^T)).$$

Hence, also in the multivariate case the gaussian umbra preserves the structure of dot-product with the Bell umbra β.

Definition 4.12. The umbral d-tuple $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_d)$, with $\gamma_1, \ldots, \gamma_d$ umbrae similar to the umbra γ associated to the Lévy measure, introduced in Definition 2.35, such that

$$E[\boldsymbol{\gamma}^{\mathbf{i}}] = \begin{cases} \int_{\mathbb{R}^d} \mathbf{x}^{\mathbf{i}} \nu(d\mathbf{x}) & \mathbf{i} \geq 2 \\ 0, & \text{otherwise} \end{cases}$$

is said to be *associated* to the Lévy measure.

Thanks to Definition 4.12, the generating function of $\boldsymbol{\gamma}$ is

$$\begin{aligned} f(\boldsymbol{\gamma}, \mathbf{z}) &= 1 + \sum_{k \geq 1} \sum_{\substack{\mathbf{i} \in \mathbb{N}_0^d \\ |\mathbf{i}| = k}} E[\boldsymbol{\gamma}^{\mathbf{i}}] \frac{\mathbf{z}^{\mathbf{i}}}{\mathbf{i}!} = 1 + \sum_{k \geq 2} \sum_{\substack{\mathbf{i} \in \mathbb{N}_0^d \\ |\mathbf{i}| = k}} \int_{\mathbb{R}^d} \mathbf{x}^{\mathbf{i}} \nu(d\mathbf{x}) \frac{\mathbf{z}^{\mathbf{i}}}{\mathbf{i}!} \\ &= 1 + \int_{\mathbb{R}^d} \sum_{k \geq 2} \sum_{\substack{\mathbf{i} \in \mathbb{N}_0^d \\ |\mathbf{i}| = k}} \frac{\mathbf{x}^{\mathbf{i}} \mathbf{z}^{\mathbf{i}}}{\mathbf{i}!} \nu(d\mathbf{x}) = 1 + \int_{\mathbb{R}^d} (e^{\mathbf{z}\mathbf{x}^T} - 1 - \mathbf{z}\mathbf{x}^T) \nu(d\mathbf{x}). \end{aligned}$$

Therefore,

$$\exp\left\{ \int_{\mathbb{R}^d} (e^{\mathbf{z}\mathbf{x}^T} - 1 - \mathbf{z}\mathbf{x}^T) \nu(d\mathbf{x}) \right\} = f(\beta.\boldsymbol{\gamma}, \mathbf{z})$$

and then

$$f(t.\beta.\boldsymbol{\gamma}, \mathbf{z}) = \exp\left\{ t \int_{\mathbb{R}^d} (e^{\mathbf{z}\mathbf{x}^T} - 1 - \mathbf{z}\mathbf{x}^T) \nu(d\mathbf{x}) \right\}.$$

Remark 4.13. The exchange order of integration and summation, given in the third equality in the previous passages, is correct, thanks to the Fubini's theorem, if we take into account that a sum is a particular case of an integral, with respect to the counting measure.

Now, we are able to get the symbolic version of a multivariate Lévy process.

Theorem 4.14. *Every Lévy process* $\{\mathbf{X}_t\}_{t\geq 0}$ *on* $\mathbb{R}^d$ *is umbrally represented by the family of auxiliary umbrae*

$$\{t.\beta.(\chi.\mathbf{m}\dot{+}\boldsymbol{\delta}C^T\dot{+}\boldsymbol{\gamma})\}_{t\geq 0}, \tag{4.18}$$

where β *is the Bell umbra,* $\mathbf{m} \in \mathbb{R}^d$*,* $\boldsymbol{\delta}$ *is the multivariate umbral counterpart of a standard gaussian random variable,* C *is the square root of the covariance matrix* Σ *and* $\boldsymbol{\gamma}$ *is the multivariate umbra associated to the Lévy measure.*

Proof. Thanks to Proposition 4.3, we have

$$f[t.\beta.(\chi.\mathbf{m}\dot{+}\boldsymbol{\delta}C^T\dot{+}\boldsymbol{\gamma}),\mathbf{z}] = \exp\{t[f(\boldsymbol{\delta}C^T\dot{+}\chi.\mathbf{m}\dot{+}\boldsymbol{\gamma},\mathbf{z}) - 1]\}, \tag{4.19}$$

where

$$f(\chi.\mathbf{m}\dot{+}\boldsymbol{\delta}C^T\dot{+}\boldsymbol{\gamma},\mathbf{z}) = 1 + [f(\chi.\mathbf{m},\mathbf{z}) - 1] + [f(\boldsymbol{\delta}C^T,\mathbf{z}) - 1] + [f(\boldsymbol{\gamma},\mathbf{z}) - 1].$$

On the other hand,

$$f(\chi.\mathbf{m},\mathbf{z}) = 1 + \log(f(\mathbf{m},\mathbf{z})) = 1 + \mathbf{m}\mathbf{z}^T$$
$$f(\boldsymbol{\delta}C^T,\mathbf{z}) = 1 + \frac{(\mathbf{z}C)(\mathbf{z}C)^T}{2} = 1 + \frac{\mathbf{z}CC^T\mathbf{z}}{2} = 1 + \frac{\mathbf{z}\Sigma\mathbf{z}^T}{2}$$

by virtue of (4.9) and (4.15), respectively. Hence,

$$f(\chi.\mathbf{m}\dot{+}\boldsymbol{\delta}C^T\dot{+}\boldsymbol{\gamma},\mathbf{z}) = 1 + \mathbf{m}\mathbf{z}^T + \frac{\mathbf{z}\Sigma\mathbf{z}^T}{2} + \int_{\mathbb{R}^d}(e^{\mathbf{z}\mathbf{x}^T} - 1 - \mathbf{z}\mathbf{x}^T)\nu(d\mathbf{x}).$$

The result follows by replacing this last equation in (4.19) and by comparing with (4.14). □

Remark 4.15. Every auxiliary umbra $t.\beta.\boldsymbol{\nu}$ is a symbolic version of a multivariate compound Poisson random variable of parameter t, that is a random sum $S_N = \mathbf{Y}_1 + \cdots + \mathbf{Y}_N$ of independent and identically distributed random vectors $\{\mathbf{Y}_i\}$, whose index N is a Poisson random variable of parameter t, see [21]. Therefore, by the symbolic method, we have proved that a multivariate Lévy process is a multivariate compound Poisson random variable of parameter t and $(\boldsymbol{\delta}C^T\dot{+}\chi.\mathbf{m}\dot{+}\boldsymbol{\gamma})$ represents any of the random vectors $\{\mathbf{Y}_i\}$. We observe that $\chi.\mathbf{m}$ has not a probabilistic counterpart. If $\mathbf{m}$ is not equal to the zero vector, this parallels the well-known difficulty to interpret the Lévy measure as a probability measure.

Remark 4.16. At a first glance, the results of Theorems 4.6 and 4.14 seem to be different by a symbolic point of view. But by using definition of multivariate cumulants and by applying equivalence (1.32), it is straightforward to prove that, for all umbral d-tuple $\boldsymbol{\mu}$, there exists an umbral d-tuple $\boldsymbol{\nu}$ such that

$$f(\boldsymbol{\mu}, \mathbf{z}) = f(\beta.\boldsymbol{\nu}, \mathbf{z}).$$

This equality represents a generalization to the multivariate case of the Inversion Theorem 2.8, if we denote by $\boldsymbol{\nu}$ the umbral d-tuple whose moments are the multivariate cumulants of $\boldsymbol{\mu}$.

Thus, the relation between Theorems 4.6 and 4.14 specifies the d-tuple $\boldsymbol{\nu}$ associated to the umbral d-tuple $\boldsymbol{\mu}$, when instead of $t.\boldsymbol{\mu}$ we write $t.\beta.\boldsymbol{\nu}$.

Corollary 4.17. *If $\mathbf{X} = \{\mathbf{X}_t\}_{t\geq 0}$ is Lévy process in $\mathbb{R}^d$ then*

$$E[\mathbf{X}_t^{\mathbf{v}}] = \sum_{\boldsymbol{\lambda}\vdash\mathbf{v}} \frac{\mathbf{v}!}{m(\boldsymbol{\lambda})!\boldsymbol{\lambda}!} t^{l(\boldsymbol{\lambda})} E[(\boldsymbol{\delta}C^T \dot{+} \chi.\mathbf{m} \dot{+} \boldsymbol{\gamma})_{\boldsymbol{\lambda}}] \qquad \text{for all } \mathbf{v} \in \mathbf{N}_0^d \tag{4.20}$$

where the sum is over all partitions $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1^{r_1}, \boldsymbol{\lambda}_2^{r_2}, \ldots)$ of the multi-index $\mathbf{v}$ and for $\boldsymbol{\mu} \equiv \boldsymbol{\delta}C^T \dot{+} \chi.\mathbf{m} \dot{+} \boldsymbol{\gamma}$ we have $E[\boldsymbol{\mu}_{\boldsymbol{\lambda}}] = \left(E[\boldsymbol{\mu}^{\boldsymbol{\lambda}_1}]\right)^{r_1} \left(E[\boldsymbol{\mu}^{\boldsymbol{\lambda}_2}]\right)^{r_2} \cdots$.

Proof. Thanks to Theorem 4.14, the moments of a multivariate Lévy process are

$$E[\mathbf{X}_t^{\mathbf{v}}] = E[\{t.\beta.(\boldsymbol{\delta}C^T \dot{+} \chi.\mathbf{m} \dot{+} \boldsymbol{\gamma})\}^{\mathbf{v}}],$$

for all multi-index $\mathbf{v} \in \mathbb{N}_0^d$.

On the other hand, thanks to equivalence (4.8), we have

$$(t.\beta.(\boldsymbol{\delta}C^T \dot{+} \chi.\mathbf{m} \dot{+} \boldsymbol{\gamma}))^{\mathbf{v}} \simeq \sum_{\boldsymbol{\lambda}\vdash\mathbf{v}} \frac{\mathbf{v}!}{\mathfrak{m}(\boldsymbol{\lambda})\boldsymbol{\lambda}!} (\boldsymbol{\delta}C^T \dot{+} \chi.\mathbf{m} \dot{+} \boldsymbol{\gamma})_{\boldsymbol{\lambda}}.$$

The result follows by taking the evaluation of both sides. □

Corollary 4.18. *Every multivariate Brownian motion with covariance matrix CC^T and mean $\mathbf{m}$ is umbrally represented by the family of auxiliary umbrae $\{t.\beta.(\chi.\mathbf{m} \dot{+} \boldsymbol{\delta}C^T)\}_{t\geq 0}$.*

Proof. Let us consider $\mathbf{X} = \{\mathbf{X}_t\}_{t\geq 0}$ such that $\mathbf{X}_t = \mathbf{m}t + C\mathbf{B}_t$, where $\mathbf{m} \in \mathbb{R}^d$, C is a $d \times d$ matrix, whose determinant is not zero, and $\{\mathbf{B}_t\}_{t\geq 0}$ is the multivariate standard brownian motion in $\mathbb{R}^d$. The moment generating function of $\mathbf{X}$ is

$$\varphi_{\mathbf{X}}(\mathbf{z}) = \exp\left\{ t\left(\mathbf{m}\mathbf{z}^T + \frac{1}{2}\mathbf{z}\Sigma\mathbf{z}^T \right) \right\}, \tag{4.21}$$

The result follows from Theorem 4.14, by comparing equation (4.21) with equation (4.14). □

Corollary 4.19. *Every multivariate compound Poisson process with intensity parameter $\lambda > 0$ is umbrally represented by the family of auxiliary umbrae $\{(\lambda t).\beta.\boldsymbol{\gamma}\}_{t \geq 0}$ where $\boldsymbol{\gamma}$ is the d-tuple associated with the Lévy measure.*

Proof. A stochastic process $\mathbf{X} = \{\mathbf{X}_t\}_{t\geq 0}$ in $\mathbb{R}^d$ is a *multivariate compound Poisson process* with intensity parameter $\lambda > 0$ if $\mathbf{X}_t = \mathbf{Z}_1 + \cdots + \mathbf{Z}_{N_t}$ for all $t \geq 0$, with $\{\mathbf{Z}_j\}$ independent and identically distributed random vectors and N_t a Poisson process of intensity parameter λ. As proved in [48], these processes are actually Lévy processes, with moment generating function

$$\varphi_{\mathbf{X}}(\mathbf{z}) = \exp\left\{(\lambda t)\int_{\mathbb{R}^d}(e^{\mathbf{x}\mathbf{z}^T} - 1)\nu(\mathrm{d}\mathbf{x})\right\}. \tag{4.22}$$

The result follows from Theorem 4.14, by comparing equation (4.22) with equation (4.14). □

4.3 Multivariate time-space harmonic polynomials

Definition 4.20. Let $\mathcal{X} = \{\mu_1, \mu_2, \ldots, \mu_d\}$. The linear operator

$$E(\,\cdot \,|\, \boldsymbol{\mu}) : \mathbb{R}[x_1, \ldots, x_d][A] \longrightarrow \mathbb{R}[\mathcal{X}]$$

such that

i) $E(1 \,|\, \boldsymbol{\mu}) = 1;$

ii) $E(x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d} \boldsymbol{\mu}^{\mathbf{i}} \boldsymbol{\nu}^{\mathbf{j}} \boldsymbol{\gamma}^{\mathbf{k}} \cdots \,\big|\, \boldsymbol{\mu}) = x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d} \boldsymbol{\mu}^{\mathbf{i}} E[\boldsymbol{\nu}^{\mathbf{j}}] E[\boldsymbol{\gamma}^{\mathbf{k}}] \cdots$
for uncorrelated d-tuples $\boldsymbol{\mu}, \boldsymbol{\nu}, \boldsymbol{\gamma} \ldots$, for all $\mathbf{m}, \mathbf{i}, \mathbf{j}, \ldots \in \mathbb{N}_0^d$ and k_i nonnegative integers, for $i = 1, \ldots, d$.

is called *multivariate conditional evaluation* with respect to the umbral d-tuple $\boldsymbol{\mu}$.

For the sake of brevity we will still use $\mathbf{x}^{\mathbf{m}}$ to denote the product $x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d}$.

Definition 4.21. Let $\{P(\mathbf{x}, t)\} \in \mathbb{R}[x_1, \ldots, x_d]$ be a family of polynomials indexed by $t \geq 0$. $P(\mathbf{x}, t)$ is said to be a *multivariate time-space harmonic polynomial* with respect to the family of auxiliary umbrae $\{t.\boldsymbol{\mu}\}_{t\geq 0}$ if and only if

$$E\left(P(t.\boldsymbol{\mu}, t) \mid s.\boldsymbol{\mu}\right) = P(s.\boldsymbol{\mu}, s), \quad \text{for all } s \leq t. \tag{4.23}$$

For all $n, m \in \mathbb{Z}$ and for all multi-index $\mathbf{v} \in \mathbb{N}_0^d$, we will assume

$$E[\{(n+m).\boldsymbol{\mu}\}^{\mathbf{v}} \mid n.\boldsymbol{\mu}] = E[\{n.\boldsymbol{\mu} + m.\boldsymbol{\mu}'\}^{\mathbf{v}} \mid n.\boldsymbol{\mu}],$$

so that

$$\begin{aligned} E\left[\{(n+m).\boldsymbol{\mu}\}^{\mathbf{v}} \mid n.\boldsymbol{\mu}\right] &= E\left[\sum_{\mathbf{k}\leq\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} (n.\boldsymbol{\mu})^{\mathbf{k}} (m.\boldsymbol{\mu}')^{\mathbf{v}-\mathbf{k}} \,\middle|\, n.\boldsymbol{\mu}\right] \\ &= \sum_{\mathbf{k}\leq\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} (n.\boldsymbol{\mu})^{\mathbf{k}} E[(m.\boldsymbol{\mu}')^{\mathbf{v}-\mathbf{k}}]. \end{aligned} \tag{4.24}$$

By analogy with (4.24), for $t \geq 0$, we set

$$E\left[(t.\boldsymbol{\mu})^{\mathbf{v}} \mid s.\boldsymbol{\mu}\right] = \sum_{\mathbf{k}\leq\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} (s.\boldsymbol{\mu})^{\mathbf{k}} E[\{(t-s).\boldsymbol{\mu}'\}^{\mathbf{v}-\mathbf{k}}]. \tag{4.25}$$

Next theorem allows us to introduce the class of multivariate time-space harmonic polynomials with respect to a d-dimensional Lévy process.

Theorem 4.22. *For all multi-index* $\mathbf{v}$, *the family of polynomials*

$$Q_{\mathbf{v}}(\mathbf{x}, t) = E[(\mathbf{x} - t.\boldsymbol{\mu})^{\mathbf{v}}] \in \mathbb{R}[x_1, \ldots, x_d] \tag{4.26}$$

is time-space harmonic with respect to $\{t.\boldsymbol{\mu}\}_{t\geq 0}$.

Proof. We have to prove that equality (4.23) holds. Observe that

$$Q_{\mathbf{v}}(\mathbf{x}, t) = E\left[\sum_{\mathbf{k}\leq\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} \mathbf{x}^{\mathbf{v}-\mathbf{k}} (-t.\boldsymbol{\mu})^{\mathbf{k}}\right] = \sum_{\mathbf{k}\leq\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} \mathbf{x}^{\mathbf{v}-\mathbf{k}} E[(-t.\boldsymbol{\mu})^{\mathbf{k}}].$$

Thus,

$$Q_{\mathbf{v}}(t.\boldsymbol{\mu}, t) = \sum_{\mathbf{k}\leq\mathbf{v}} \binom{\mathbf{k}}{\mathbf{j}} (t.\boldsymbol{\mu})^{\mathbf{v}-\mathbf{k}} E[(-t.\boldsymbol{\mu})^{\mathbf{k}}]. \tag{4.27}$$

By applying (4.25) and (4.27), we have

$$\begin{aligned} E\left[Q_{\mathbf{v}}(t.\boldsymbol{\mu}, t) \mid s.\boldsymbol{\mu}\right] &= \sum_{\mathbf{k}\le\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} E\left[(t.\boldsymbol{\mu})^{\mathbf{v}-\mathbf{k}} E[(-t.\boldsymbol{\mu})^{\mathbf{k}}] \,\middle|\, s.\boldsymbol{\mu}\right] \\ &= \sum_{\mathbf{k}\le\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} E\left[(t.\boldsymbol{\mu})^{\mathbf{v}-\mathbf{k}} \,\middle|\, s.\boldsymbol{\mu}\right] E[(-t.\boldsymbol{\mu})^{\mathbf{k}}] \\ &= \sum_{\mathbf{k}\le\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} \left\{ \sum_{\mathbf{j}\le\mathbf{v}-\mathbf{k}} \binom{\mathbf{v}-\mathbf{k}}{\mathbf{j}} (s.\boldsymbol{\mu})^{\mathbf{j}} E[\{(t-s).\boldsymbol{\mu}'\}^{\mathbf{v}-\mathbf{k}-\mathbf{j}}] \right\} E[(-t.\boldsymbol{\mu})^{\mathbf{k}}] \\ &= \sum_{\mathbf{k}\le\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} (s.\boldsymbol{\mu})^{\mathbf{k}} E[\{(t-s).\boldsymbol{\mu}' + (-t.\boldsymbol{\mu})\}^{\mathbf{v}-\mathbf{k}}] \\ &= \sum_{\mathbf{k}\le\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} (s.\boldsymbol{\mu})^{\mathbf{k}} E[(-s.\boldsymbol{\mu})^{\mathbf{v}-\mathbf{k}}] = Q_{\mathbf{v}}(s.\boldsymbol{\mu}, s), \end{aligned}$$

where the last third equality is justified by the fact that, since the multi-index notation is used, the summation is actually a product of summations of univariate terms, for which the proof of Theorem 3.7 in Chapter 3 and a suitable rearrangement of terms has been used. □

Corollary 4.23. *If we set*

$$Q_{\mathbf{v}}(\mathbf{x}, t) = \sum_{\mathbf{k}\le\mathbf{v}} q_{\mathbf{k}}(t)\, \mathbf{x}^{\mathbf{k}}, \tag{4.28}$$

then

(i) $q_{\mathbf{k}}(t) = \binom{\mathbf{v}}{\mathbf{k}} E[(-t.\boldsymbol{\mu})^{\mathbf{v}-\mathbf{k}}]$, *for all* $\mathbf{k} \le \mathbf{v}$

(ii) $q_{\mathbf{k}}(0) = 0$, *for all* $\mathbf{k} < \mathbf{v}$.

Proof. **(i)** From Theorem 4.22 and by applying the multinomial theorem, we have

$$Q_{\mathbf{v}}(\mathbf{x}, t) = E\left[\sum_{\mathbf{k}\le\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} (-t.\boldsymbol{\mu})^{\mathbf{v}-\mathbf{k}} \mathbf{x}^{\mathbf{k}}\right] = \sum_{\mathbf{k}\le\mathbf{v}} E\left[\binom{\mathbf{v}}{\mathbf{k}} (-t.\boldsymbol{\mu})^{\mathbf{v}-\mathbf{k}}\right] \mathbf{x}^{\mathbf{k}}. \tag{4.29}$$

The result follows by comparing (4.28) and (4.29).

(ii) If $t = 0$, then $-t.\boldsymbol{\mu} \equiv \boldsymbol{\epsilon}$, so that

$$E[(-t.\boldsymbol{\mu})^{\mathbf{v}-\mathbf{k}}] = E[\boldsymbol{\epsilon}^{\mathbf{v}-\mathbf{k}}] = \begin{cases} 1, & \text{if } \mathbf{k} = \mathbf{v} \\ 0, & \text{otherwise.} \end{cases}$$

□

In the following, set $g_{\mathbf{v}} = E[\boldsymbol{\mu}^{\mathbf{v}}]$, for all $\mathbf{v} \in \mathbb{N}_0^d$.

Proposition 4.24. *For all* $\mathbf{k} < \mathbf{v}$ *we have*

$$q_{\mathbf{k}}(t-1) = \sum_{\mathbf{k} \le \mathbf{j} \le \mathbf{v}} \binom{\mathbf{j}}{\mathbf{k}} g_{\mathbf{j}}\, q_{\mathbf{j}}(t). \tag{4.30}$$

Proof. We have

$$\begin{aligned} q_{\mathbf{k}}(t-1) &= \binom{\mathbf{v}}{\mathbf{k}} E[(-(t-1).\boldsymbol{\mu})^{\mathbf{v}-\mathbf{k}}] = \binom{\mathbf{v}}{\mathbf{k}} E[(-t.\boldsymbol{\mu} + \boldsymbol{\mu}')^{\mathbf{v}-\mathbf{k}}] \\ &= \binom{\mathbf{v}}{\mathbf{k}} E\left[\sum_{\mathbf{j} \le \mathbf{v}-\mathbf{k}} \binom{\mathbf{v}-\mathbf{k}}{\mathbf{j}} (-t.\boldsymbol{\mu})^{\mathbf{v}-\mathbf{k}-\mathbf{j}} (\boldsymbol{\mu}')^{\mathbf{j}}\right] \\ &= \binom{\mathbf{v}}{\mathbf{k}} \sum_{\mathbf{j} \le \mathbf{v}-\mathbf{k}} \binom{\mathbf{v}-\mathbf{k}}{\mathbf{j}} E\left[(-t.\boldsymbol{\mu})^{\mathbf{v}-\mathbf{k}-\mathbf{j}}\right] g_{\mathbf{j}} \\ &= \sum_{\mathbf{j} \le \mathbf{v}-\mathbf{k}} \binom{\mathbf{v}}{\mathbf{k}} \binom{\mathbf{v}-\mathbf{k}}{\mathbf{j}} E\left[(-t.\boldsymbol{\mu})^{\mathbf{v}-\mathbf{k}-\mathbf{j}}\right] g_{\mathbf{j}} \\ &= \sum_{\mathbf{k} \le \mathbf{i} \le \mathbf{v}} \binom{\mathbf{i}}{\mathbf{k}} g_{\mathbf{i}-\mathbf{k}} E\left[\binom{\mathbf{v}}{\mathbf{i}} (-t.\boldsymbol{\mu})^{\mathbf{v}-\mathbf{i}}\right]. \end{aligned}$$

The result follows by applying Corollary 4.23-(i). □

Corollary 4.25. *We have*

$$g_{\mathbf{v}}\, q_{\mathbf{k}}(t) = q_{\mathbf{0}}(t-1) - \sum_{\mathbf{j} \le \mathbf{v}-1} g_{\mathbf{j}}\, q_{\mathbf{k}}(t).$$

Proof. Thanks to Proposition 4.24, we have

$$q_{\mathbf{0}}(t-1) = \sum_{\mathbf{j} \le \mathbf{v}} \binom{\mathbf{j}}{\mathbf{0}} q_{\mathbf{j}}(t)\, g_{\mathbf{j}} = \sum_{\mathbf{j} \le \mathbf{v}} q_{\mathbf{j}}(t)\, g_{\mathbf{j}} = q_{\mathbf{v}}(t) g_{\mathbf{v}} + \sum_{\mathbf{j} \le \mathbf{v}-1} q_{\mathbf{j}}(t)\, g_{\mathbf{j}}.$$

□

Proposition 4.26. *We have*

$$g_{\mathbf{v}} = q_{\mathbf{0}}(t-1) - \sum_{\mathbf{k} \leq \mathbf{v}-1} q_{\mathbf{k}}(t)\, g_{\mathbf{k}}.$$

Proof. We have

$$\begin{aligned}\sum_{\mathbf{k} \leq \mathbf{v}-1} q_{\mathbf{k}}(t)\, g_{\mathbf{k}} &= \sum_{\mathbf{k} \leq \mathbf{v}-1} \binom{\mathbf{v}}{\mathbf{k}} E[(-t.\boldsymbol{\mu})^{\mathbf{v}-\mathbf{k}}]\, g_{\mathbf{k}} \\ &= \sum_{\mathbf{k} \leq \mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} E[(-t.\boldsymbol{\mu})^{\mathbf{v}-\mathbf{k}}]\, g_{\mathbf{k}} - g_{\mathbf{v}} \\ &= E[(-t.\boldsymbol{\mu} + \boldsymbol{\mu}')^{\mathbf{v}}] - g_{\mathbf{v}} = E[(-(t-1).\boldsymbol{\mu})^{\mathbf{v}}] - g_{\mathbf{v}}.\end{aligned}$$

Hence, thanks to Corollary 4.23-(i), we obtain

$$q_{\mathbf{0}}(t-1) - \sum_{\mathbf{k} \leq \mathbf{v}-1} q_{\mathbf{k}}(t) g_{\mathbf{k}} = g_{\mathbf{v}}.$$

□

Theorem 4.27. *A polynomial*

$$P(\mathbf{x}, t) = \sum_{\mathbf{k} \leq \mathbf{v}} p_{\mathbf{k}}(t)\, \mathbf{x}^{\mathbf{k}} \tag{4.31}$$

is a time-space harmonic polynomial with respect to $\{t.\boldsymbol{\mu}\}_{t \geq 0}$ *if and only if*

$$p_{\mathbf{k}}(t) = \sum_{\mathbf{k} \leq \mathbf{i} \leq \mathbf{v}} \binom{\mathbf{i}}{\mathbf{k}} p_{\mathbf{k}}(0)\, E[(-t.\boldsymbol{\mu})^{\mathbf{i}-\mathbf{k}}], \quad \textit{for } \mathbf{k} \leq \mathbf{v}. \tag{4.32}$$

Proof. Suppose $P(\mathbf{x}, t)$ of form (4.31) such that its coefficients satisfy equation (4.32). By suitably rearranging the terms in the summations, we have

$$\begin{aligned}P(\mathbf{x}, t) &= \sum_{\mathbf{k} \leq \mathbf{v}} \left(\sum_{\mathbf{k} \leq \mathbf{i} \leq \mathbf{v}} \binom{\mathbf{i}}{\mathbf{k}} p_{\mathbf{k}}(0)\, E[(-t.\boldsymbol{\mu})^{\mathbf{i}-\mathbf{k}}] \right) \mathbf{x}^{\mathbf{k}} \\ &= \sum_{\mathbf{k} \leq \mathbf{v}} p_{\mathbf{k}}(0) \sum_{\mathbf{i} \leq \mathbf{k}} \binom{\mathbf{k}}{\mathbf{i}} E[(-t.\boldsymbol{\mu})^{\mathbf{k}-\mathbf{i}}] \mathbf{x}^{\mathbf{i}} \\ &= \sum_{\mathbf{k} \leq \mathbf{v}} p_{\mathbf{k}}(0) E[(\mathbf{x} - t.\boldsymbol{\mu})^{\mathbf{k}}] = \sum_{\mathbf{k} \leq \mathbf{v}} p_{\mathbf{k}}(0) Q_{\mathbf{k}}(\mathbf{x}, t).\end{aligned}$$

The result follows, since $Q_{\mathbf{k}}(\mathbf{x},t)$ is time-space harmonic, by virtue of Theorem 4.22 and taking into account that a linear combination of time-space harmonic polynomials is time-space harmonic in turn.

Vice versa, suppose $P(\mathbf{x},t) = \sum_{\mathbf{k}\leq \mathbf{v}} p_{\mathbf{k}}(t)\, \mathbf{x}^{\mathbf{k}}$ time-space harmonic with respect to the family of auxiliary umbrae $\{t.\boldsymbol{\mu}\}_{t\geq 0}$, then

$$P(\mathbf{x},t) = \sum_{\mathbf{k}\leq \mathbf{v}} c_{\mathbf{k}} E[(\mathbf{x}-t.\boldsymbol{\mu})^{\mathbf{k}}] = \sum_{\mathbf{k}\leq \mathbf{v}} c_{\mathbf{k}} \sum_{\mathbf{j}\leq \mathbf{k}} \binom{\mathbf{k}}{\mathbf{j}} E[(-t.\boldsymbol{\mu})^{\mathbf{k}-\mathbf{j}}]\mathbf{x}^{\mathbf{j}}$$
$$= \sum_{\mathbf{k}\leq \mathbf{v}} \left(\sum_{\mathbf{k}\leq \mathbf{j}\leq \mathbf{v}} \binom{\mathbf{j}}{\mathbf{k}} c_{\mathbf{k}} E[(-t.\boldsymbol{\mu})^{\mathbf{j}-\mathbf{k}}] \right) \mathbf{x}^{\mathbf{k}}.$$

We have
$$p_{\mathbf{k}}(t) = \sum_{\mathbf{k}\leq \mathbf{j}\leq \mathbf{v}} \binom{\mathbf{j}}{\mathbf{k}} p_{\mathbf{k}}(0)\, E[(-t.\boldsymbol{\mu})^{\mathbf{j}-\mathbf{k}}].$$

If $t=0$, then $E[(0.\boldsymbol{\mu})^{\mathbf{j}-\mathbf{k}}] = E[\boldsymbol{\epsilon}^{\mathbf{j}-\mathbf{k}}] = 1$, when $\mathbf{j}=\mathbf{k}$ and 0, otherwise, so that

$$p_{\mathbf{k}}(0) = \sum_{\mathbf{k}\leq \mathbf{j}\leq \mathbf{v}} \binom{\mathbf{j}}{\mathbf{k}} p_{\mathbf{k}}(0)\, E[\boldsymbol{\epsilon}^{\mathbf{j}-\mathbf{k}}] = c_{\mathbf{k}}.$$

Thus, we have

$$P(\mathbf{x},t) = \sum_{\mathbf{k}\leq \mathbf{j}\leq \mathbf{v}} \left(\binom{\mathbf{j}}{\mathbf{k}} p_{\mathbf{k}}(0)\, E[(-t.\boldsymbol{\mu})^{\mathbf{j}-\mathbf{k}}] \right) \mathbf{x}^{\mathbf{j}}.$$

□

4.4 Multivariate Bernoulli, Euler, Hermite polynomials

4.4.1 Multivariate Bernoulli numbers

Definition 4.28. The *multivariate Bernoulli umbra* $\boldsymbol{\iota}$ is the d-tuple $(\iota,\ldots,\iota)$ with all elements equal to the Bernoulli umbra ι.

The generating function of the multivariate Bernoulli umbra is

$$f(\boldsymbol{\iota},\mathbf{z}) = E[e^{\iota z_1+\cdots \iota z_d}] = f(\iota, z_1+\cdots+z_d) = \frac{z_1+\cdots+z_d}{e^{z_1+\cdots+z_d}-1}. \quad (4.33)$$

Lemma 4.29. *We have*

$$(\boldsymbol{u}+\boldsymbol{\iota})\dot{-}\boldsymbol{\iota} \equiv \chi.\boldsymbol{u}.$$

Proof. We prove this result via generating functions. By recalling (4.3) and by applying the latter of (4.4) and (4.33), we have

$$\begin{aligned} f((\boldsymbol{u}+\boldsymbol{\iota})\dot{+}\boldsymbol{\iota},\mathbf{z}) &= f(\boldsymbol{u}+\boldsymbol{\iota},\mathbf{z}) - f(\boldsymbol{\iota},\mathbf{z}) + 1 \\ &= f(\boldsymbol{u},\mathbf{z})f(\boldsymbol{\iota},\mathbf{z}) - f(\boldsymbol{\iota},\mathbf{z}) + 1 \\ &= f(\boldsymbol{\iota},\mathbf{z})(f(\boldsymbol{u},\mathbf{z})-1)+1 \\ &= z_1 + \cdots + z_d + 1. \end{aligned} \tag{4.34}$$

On the other hand, equation (4.9) guarantees that

$$f(\chi.\boldsymbol{u},\mathbf{z}) = 1+\log(f(\boldsymbol{u},\mathbf{z})) = 1+\log(\exp\{z_1+\cdots+z_d\}) = 1+z_1+\cdots+z_d. \tag{4.35}$$

By comparing (4.34) and (4.35), we have

$$f((\boldsymbol{u}+\boldsymbol{\iota})\dot{+}\boldsymbol{\iota},\mathbf{z}) = f(\chi.\boldsymbol{u},\mathbf{z}).$$

□

Corollary 4.30. *For all multi-index* $\mathbf{v}=(v_1,\ldots,v_d)\in\mathbb{N}_0^d$ *such that* $|\mathbf{v}|>1$, *we have*

$$(\boldsymbol{u}+\boldsymbol{\iota})^{\mathbf{v}} \simeq \boldsymbol{\iota}^{\mathbf{v}}.$$

Proof. Lemma 4.29 implies that, for all multi-index $\mathbf{v}=(v_1,\ldots,v_d)\in\mathbb{N}_0^d$,

$$(\chi.\boldsymbol{u})^{\mathbf{v}} \simeq (\boldsymbol{u}+\boldsymbol{\iota})\dot{-}\boldsymbol{\iota} \simeq (\boldsymbol{u}+\boldsymbol{\iota})^{\mathbf{v}} - \boldsymbol{\iota}^{\mathbf{v}}.$$

Since $f(\chi.\boldsymbol{u},\boldsymbol{z}) = 1+z_1+z_2+\cdots+z_d$ then for all $|\mathbf{v}|>1$, we have $(\chi.\boldsymbol{u})^{\mathbf{v}}\simeq 0$. □

Definition 4.31. The *first-order multivariate Bernoulli numbers* $\{B^{(1)}_{\mathbf{v}}\}_{\mathbf{v}\in\mathbb{N}_0^d}$ are the coefficients of the generating function (4.33), that is

$$B^{(1)}_{\mathbf{v}} = E[\boldsymbol{\iota}^{\mathbf{v}}] = E[\iota^{|\mathbf{v}|}].$$

Definition 4.31 generalizes the definition of multivariate Bernoulli numbers given in [37]. In particular, we have $B^{(0)}_{\mathbf{0}} = E[(0.\boldsymbol{\iota})^{\mathbf{0}}]=1$ and $B^{(0)}_{\mathbf{v}} = E[(0.\boldsymbol{\iota})^{\mathbf{v}}]=0$, if $|\mathbf{v}|>0$.

Proposition 4.32. $B^{(1)}_{\mathbf{v}} = \sum_{\mathbf{k}\le\mathbf{v}}\binom{\mathbf{v}}{\mathbf{k}}B^{(1)}_{\mathbf{k}}$ *for all* $\mathbf{v}\in\mathbb{N}_0^d$ *such that* $|\mathbf{v}|>1$.

Proof. From Definition 4.31, we have

$$\sum_{\mathbf{k}\leq\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} B_{\mathbf{k}}^{(1)} = \sum_{\mathbf{k}\leq\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} E[\boldsymbol{\iota}^{\mathbf{k}}] = \sum_{\mathbf{k}\leq\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} E[\boldsymbol{\iota}^{\mathbf{k}}]E[\boldsymbol{u}^{\mathbf{v}-\mathbf{k}}]$$

$$= E\left[\sum_{\mathbf{k}\leq\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} \boldsymbol{\iota}^{\mathbf{k}}\boldsymbol{u}^{\mathbf{v}-\mathbf{k}}\right] = E[(\boldsymbol{u}+\boldsymbol{\iota})^{\mathbf{v}}] = E[\boldsymbol{\iota}^{\mathbf{v}}],$$

where the last equality holds by taking the evaluation of both sides of the equivalence in Corollary 4.30. □

Proposition 4.33. $f(t.\boldsymbol{\iota},\mathbf{z}) = \left(\dfrac{z_1+\cdots+z_d}{e^{z_1+\cdots+z_d}-1}\right)^t.$

Proof. The result follows by applying equation (4.6), with $\boldsymbol{\mu}$ replaced by the multivariate Bernoulli umbra $\boldsymbol{\iota}$. □

If we set $t=-1$ in Proposition 4.33, we get the inverse $-1.\boldsymbol{\iota}$ of the d-tuple $\boldsymbol{\iota}$, such that

$$f(-1.\boldsymbol{\iota},\mathbf{z}) = \frac{1}{f(\boldsymbol{\iota},\mathbf{z})} = \frac{e^{z_1+\cdots+z_d}-1}{z_1+\cdots+z_d}. \tag{4.36}$$

Proposition 4.34. *The inverse $-1.\boldsymbol{\iota}$ of the multivariate Bernoulli umbra is the umbral counterpart of a d-tuple identically distributed to $(U,\ldots,U)$, where U is a uniform random variable on the interval $(0,1)$.*

Proof. The result follows by comparing the generating function (4.36) with the moments generating function of a multivariate uniform random variable on the interval $(0,1)$. □

Definition 4.35. The t-th-order multivariate Bernoulli numbers $\{B_{\mathbf{v}}^{(t)}\}_{\mathbf{v}\in\mathbb{N}_0^d}$ are the moments of the multivariate umbra $t.\boldsymbol{\iota}$, that is $B_{\mathbf{v}}^{(t)} = E[(t.\boldsymbol{\iota})^{\mathbf{v}}]$.

Proposition 4.36. $B_{\mathbf{v}}^{(t)} = \sum_{\mathbf{k}\leq\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} B_{\mathbf{k}}^{(s)} B_{\mathbf{v}-\mathbf{k}}^{(t-s)}$, *for all $s,t\in\mathbb{R}$ and $\mathbf{v}\in\mathbb{N}_0^d$.*

Proof. The proof is straightforward if $s=t$. For all $s,t\in\mathbb{R}$ such that $s\neq t$, the result follows from the linearity of E, in fact

$$B_{\mathbf{v}}^{(t)} = E[(t.\boldsymbol{\iota})^{\mathbf{v}}] = E\left\{[(t-s).\boldsymbol{\iota}+s.\boldsymbol{\iota}]^{\mathbf{v}}\right\}$$

$$= \sum_{\mathbf{k}\leq\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} E[(t-s).\boldsymbol{\iota}]^{\mathbf{v}-\mathbf{k}}]E[(s.\boldsymbol{\iota})^{\mathbf{k}}]$$

$$= \sum_{\mathbf{k} \le \mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} B_{\mathbf{k}}^{(s)} B_{\mathbf{v}-\mathbf{k}}^{(t-s)}.$$

□

In Proposition 4.36, set $t = 0$. The following corollary follows.

Corollary 4.37. *For all $s \in \mathbb{R}$ we have $\sum_{\mathbf{k} \le \mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} B_{\mathbf{k}}^{(s)} B_{\mathbf{v}-\mathbf{k}}^{(-s)} = 1$ if $\mathbf{v} = \mathbf{0}$ otherwise being 0.*

4.4.2 Multivariate Euler numbers

Definition 4.38. The *multivariate Euler umbra* $\boldsymbol{\eta}$ is the d-tuple $(\eta, \ldots, \eta)$ with all elements equal to the Euler umbra η.

The generating function of the multivariate Euler umbra is

$$f(\boldsymbol{\eta}, \mathbf{z}) = E[e^{\eta z_1 + \cdots \eta z_d}] = f(\eta, z_1 + \cdots + z_d) = \frac{2e^{z_1 + \cdots + z_d}}{e^{2(z_1 + \cdots + z_d)} + 1}. \tag{4.37}$$

Definition 4.39. The *multivariate Euler numbers* $\{\mathfrak{E}_{\mathbf{v}}^{(1)}\}_{\mathbf{v} \in \mathbb{N}_0^d}$ are the coefficients of the generating function (4.37), that is, $\mathfrak{E}_{\mathbf{v}}^{(1)} = E[\boldsymbol{\eta}^{\mathbf{v}}] = E[\eta^{|\mathbf{v}|}]$.

Definition 4.40. The t-th-order multivariate Euler numbers $\mathfrak{E}_{\mathbf{v}}^{(t)}$ are the moments of the multivariate umbra $t.\boldsymbol{\eta}$, that is $\mathfrak{E}_{\mathbf{v}}^{(t)} = E[(t.\boldsymbol{\eta})^{\mathbf{v}}]$.

Proposition 4.41. $f(t.\boldsymbol{\eta}, \mathbf{z}) = \left(\dfrac{2\, e^{z_1 + \cdots + z_d}}{e^{2(z_1 + \cdots + z_d)} + 1} \right)^t$.

Proof. The result follows by applying equation (4.6), with $\boldsymbol{\mu}$ replaced by the multivariate Euler umbra $\boldsymbol{\eta}$. □

Proposition 4.42. *The inverse $-1.\boldsymbol{\eta}$ of the multivariate Euler umbra is the umbral counterpart of a d-tuple, identically distributed to $(X, \ldots, X)$, where $X = Y - 1$ with Y a Bernoulli random variable with parameter $1/2$.*

Proof. The result follows by comparing the generating function in Proposition 4.41 with the moment generating function of a multivariate Bernoulli random variable.

□

Proposition 4.43. *For all $s, t \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{N}_0^d$, we have*

$$\mathfrak{E}_{\mathbf{v}}^{(t)} = \sum_{\mathbf{k} \le \mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} \mathfrak{E}_{\mathbf{k}}^{(s)} \mathfrak{E}_{\mathbf{v}-\mathbf{k}}^{(t-s)}.$$

Proof. The proof is straightforward if $s = t$. For all $s, t \in \mathbb{R}$ such that $s \neq t$, the result follows from the linearity of E, that is

$$\begin{aligned}\mathfrak{E}_{\mathbf{v}}^{(t)} &= E[(t.\boldsymbol{\eta})^{\mathbf{v}}] = E[((t-s).\boldsymbol{\eta} + s.\boldsymbol{\eta})^{\mathbf{v}}] \\ &= \sum_{\mathbf{k} \leq \mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} E[((t-s).\boldsymbol{\eta})^{\mathbf{v}-\mathbf{k}}] E[(s.\boldsymbol{\eta})^{\mathbf{k}}] \\ &= \sum_{\mathbf{k} \leq \mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} \mathfrak{E}_{\mathbf{v}-\mathbf{k}}^{(t-s)} \mathfrak{E}_{\mathbf{k}}^{(s)}.\end{aligned}$$

□

In Proposition 4.43 set $t = 0$. We have the following corollary.

Corollary 4.44. *For all* $s \in \mathbb{R}$, *we have* $\sum_{\mathbf{k} \leq \mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} \mathfrak{E}_{\mathbf{k}}^{(s)} \mathfrak{E}_{\mathbf{v}-\mathbf{k}}^{(-s)} = 1$ *if* $\mathbf{v} = 0$ *otherwise being* 0.

4.4.3 Generalized multivariate Bernoulli and Euler polynomials

Definition 4.45. The t-th-order d-variable Bernoulli $\mathcal{B}_{\mathbf{v}}^{(t)}(\boldsymbol{x})$ and Euler $\mathcal{E}_{\mathbf{v}}^{(t)}(\boldsymbol{x})$ polynomials are respectively the coefficients of the following formal power series expansions:

$$\sum_{k \geq 0} \sum_{\substack{\mathbf{v} \in \mathbb{N}_0^d \\ |\mathbf{v}|=k}} \mathcal{B}_{\mathbf{v}}^{(t)}(\mathbf{x}) \frac{\mathbf{z}^{\mathbf{v}}}{\mathbf{v}!} = e^{\mathbf{x}\mathbf{z}^T} \left(\frac{z_1 + \cdots + z_d}{e^{z_1 + \cdots + z_d} - 1} \right)^t \tag{4.38}$$

$$\sum_{k \geq 0} \sum_{\substack{\mathbf{v} \in \mathbb{N}_0^d \\ |\mathbf{v}|=k}} \mathcal{E}_{\mathbf{v}}^{(t)}(\mathbf{x}) \frac{\mathbf{z}^{\mathbf{v}}}{\mathbf{v}!} = e^{\mathbf{x}\mathbf{z}^T} \left(\frac{2}{e^{z_1 + \cdots + z_d} + 1} \right)^t. \tag{4.39}$$

Theorem 4.46. *For all* $\mathbf{v} \in \mathbb{N}_0^d$, *we have*

$$B_{\mathbf{v}}^{(t)}(\mathbf{x}) = E[(\mathbf{x} + t.\boldsymbol{\iota})^{\mathbf{v}}] \qquad \mathcal{E}_{\mathbf{v}}^{(t)}(\mathbf{x}) = E\left[\left(\mathbf{x} + \frac{1}{2}(t.(\boldsymbol{\eta} - 1.\boldsymbol{u})) \right)^{\mathbf{v}} \right].$$

Proof. First of all, observe that $e^{\mathbf{x}\mathbf{z}^T} = f(\mathbf{x}, \mathbf{z})$. The former equation follows from (4.38) and from Proposition 4.33.

As regards the latter equation, from (4.39) we have

$$\frac{2}{e^{z_1 + \cdots + z_d} + 1} = \frac{2e^{(z_1 + \cdots + z_d)/2}}{e^{2(z_1 + \cdots + z_d)/2 + 1}} \frac{1}{e^{(z_1 + \cdots + z_d)/2}}$$

$$= f\left(\boldsymbol{\eta}, \frac{\mathbf{z}}{2}\right) f\left(-1.\boldsymbol{u}, \frac{\mathbf{z}}{2}\right) = f\left(\boldsymbol{\eta} - 1.\boldsymbol{u}, \frac{\mathbf{z}}{2}\right)$$
$$= f\left(\frac{1}{2}(\boldsymbol{\eta} - 1.\boldsymbol{u}), \mathbf{z}\right).$$

The result follows by replacing this last result in (4.39). □

Corollary 4.47. *For all $t \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{N}_0^d$ we have*

$$\mathcal{B}_{\mathbf{v}}^{(t)}(\mathbf{x}) = \sum_{\mathbf{k}\leq\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} \mathbf{x}^{\mathbf{v}-\mathbf{k}} B_{\mathbf{k}}^{(t)} \qquad 2^{|\mathbf{v}|}\mathcal{E}_{\mathbf{v}}^{(t)}\left(\frac{1}{2}\mathbf{x} + \frac{t}{2}\mathbf{1}\right) = \sum_{\mathbf{k}\leq\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} \mathbf{x}^{\mathbf{v}-\mathbf{k}} \mathfrak{E}_{\mathbf{k}}^{(t)}, \tag{4.40}$$

where $\mathbf{1}$ is the d-tuple with all elements equal to 1 and $B_{\mathbf{k}}^{(t)}, \mathfrak{E}_{\mathbf{k}}^{(t)}$ are the t-th-order multivariate Bernoulli and Euler numbers, respectively.

Proof. From the moment representation of multivariate Bernoulli polynomials (Proposition 4.46) we have $\mathcal{B}_{\mathbf{v}}^{(t)}(\mathbf{x}) = \sum_{\mathbf{k}\leq\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} \mathbf{x}^{\mathbf{v}-\mathbf{k}} E[(t.\boldsymbol{\iota})^{\mathbf{k}}]$, by which the former equality in (4.40) follows immediately.

For the latter equality, observe that

$$E\left[\left(\frac{1}{2}\mathbf{x} + \frac{t}{2}\mathbf{1} + \frac{1}{2}[t.(\boldsymbol{\eta} - \boldsymbol{u})]\right)\right] = \sum_{\mathbf{k}\leq\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} E\left[\left\{\frac{t}{2}\mathbf{1} - \frac{1}{2}(t.\boldsymbol{u})\right\}^{\mathbf{k}}\right] \frac{E[(\mathbf{x} + t.\boldsymbol{\eta})^{\mathbf{v}-\mathbf{k}}]}{2^{|\mathbf{v}-\mathbf{k}|}}.$$

Since $E\left[\left\{\frac{t}{2}\mathbf{1} - \frac{1}{2}(t.\boldsymbol{u})\right\}^{\mathbf{k}}\right] = 0$ for all $\mathbf{k}$, except when $\mathbf{k} = \mathbf{0}$ which gives 1, we have

$$2^{|\mathbf{v}|}\mathcal{E}_{\mathbf{v}}^{(t)}\left(\frac{1}{2}\mathbf{x} + \frac{t}{2}\mathbf{1}\right) = E[(\mathbf{x} + t.\boldsymbol{\eta})^{\mathbf{v}}] = \sum_{\mathbf{k}\leq\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} \mathbf{x}^{\mathbf{v}-\mathbf{k}} \mathfrak{E}_{\mathbf{k}}^{(t)}.$$

□

Next Corollary states the relationship Bernoulli numbers-Bernoulli polynomials and Euler numbers-Euler polynomials.

Corollary 4.48. *If we set $\mathbf{0}, \mathbf{1}$ the d-tuples with all elements equal to 0 and 1, respectively, then*

$$B_{\mathbf{v}}^{(t)} = \mathcal{B}_{\mathbf{v}}^{(t)}(\mathbf{0}) \quad \mathfrak{E}_{\mathbf{k}}^{(t)} = 2^{|\mathbf{v}|}\mathcal{E}_{\mathbf{v}}^{(t)}\left(\frac{t}{2}\mathbf{1}\right).$$

Proof. If we set $\mathbf{x} = \mathbf{0}$ in the moment representation of multivariate Bernoulli polynomials, then we obtain

$$\mathcal{B}_{\mathbf{v}}^{(t)}(\mathbf{0} = E[(t.\boldsymbol{\iota})^{\mathbf{v}}] = B_{\mathbf{v}}^{(t)},$$

due to Definition 4.35.

In the same way, we set $\mathbf{x} = \frac{t}{2} - \mathbf{1}$ in the moment representation of multivariate Euler polynomials, then

$$\begin{aligned}
\mathcal{E}_{\mathbf{k}}^{(t)} &= E\left[\left(\mathbf{x} + \frac{1}{2}(t.\boldsymbol{\eta} - t.\boldsymbol{u})\right)^{\mathbf{v}}\right] \\
&= E\left[\left(\mathbf{x} + \frac{1}{2}(t.\boldsymbol{\eta}) + \frac{1}{2}(-t.\boldsymbol{u})\right)^{\mathbf{v}}\right] \\
&= 2^{|\mathbf{v}|}E[(2\mathbf{x} - t.\boldsymbol{u} + t.\boldsymbol{\eta})^{\mathbf{v}}] \\
&= 2^{|\mathbf{v}|}E\left[\sum_{\mathbf{k}\leq\mathbf{v}}\binom{\mathbf{v}}{\mathbf{k}}(2\mathbf{x} - t.\boldsymbol{u})^{\mathbf{v}-\mathbf{k}}(t.\boldsymbol{\eta})^{\mathbf{k}}\right] \\
&= 2^{|\mathbf{v}|}\sum_{\mathbf{k}\leq\mathbf{v}}\binom{\mathbf{v}}{\mathbf{k}}E[(2\mathbf{x} - t.\boldsymbol{u})^{\mathbf{v}-\mathbf{k}}]E[(t.\boldsymbol{\eta})^{\mathbf{k}}].
\end{aligned}$$

If $\mathbf{x} = \frac{t}{2}\mathbf{1}$, then $\mathcal{E}_{\mathbf{k}}^{(t)}(\mathbf{x}) = 2^{|\mathbf{v}|}\mathfrak{E}_{\mathbf{k}}^{(t)}$. □

Remark 4.49. The definition of multivariate Euler polynomial is given according to the terminology first introduced by Nörlund [40]. This definition does not allow us a connection between multivariate Euler numbers and multivariate Euler polynomials as simple as the same between multivariate Bernoulli numbers and multivariate Bernoulli polynomials. This explains why the moment representation of multivariate Euler polynomials is less simple than the one given for multivariate Bernoulli polynomials.

Corollary 4.50. *For all $s, t \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{N}_0^d$ we have*

$$\mathcal{B}_{\mathbf{v}}^{(t+s)}(\mathbf{x} + \boldsymbol{y}) = \sum_{\mathbf{k}\leq\mathbf{v}}\binom{\mathbf{v}}{\mathbf{k}}\mathcal{B}_{\mathbf{k}}^{(t)}(\mathbf{x})\mathcal{B}_{\mathbf{v}-\mathbf{k}}^{(s)}(\boldsymbol{y})$$

$$\mathcal{E}_{\mathbf{v}}^{(t+s)}(\mathbf{x} + \boldsymbol{y}) = \sum_{\mathbf{k}\leq\mathbf{v}}\binom{\mathbf{v}}{\mathbf{k}}\mathcal{E}_{\mathbf{k}}^{(t)}(\mathbf{x})\mathcal{E}_{\mathbf{v}-\mathbf{k}}^{(s)}(\boldsymbol{y})$$

Proof. We replace $\mathbb{R}[x_1, \ldots, x_d]$ with $\mathbb{R}[x_1, \ldots, x_d, y_1, \ldots, y_d]$ and repeat the same remarks made in the proof of Theorem 4.46, setting $\mathbf{x} = (x_1, \ldots, x_d)$

and $\mathbf{y} = (y_1, \ldots, y_d)$. Therefore the former result follows since $\mathcal{B}_{\mathbf{v}}^{(s+t)}(\mathbf{x} + \mathbf{y}) = E[(\mathbf{x} + s.\boldsymbol{\iota} + \mathbf{y} + t.\boldsymbol{\iota})^{\mathbf{v}}] = \sum_{\mathbf{k}\leq\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} E[(\mathbf{x} + s.\boldsymbol{\iota})^{\mathbf{k}}]E[(\boldsymbol{y} + t.\boldsymbol{\iota})^{\mathbf{v}-\mathbf{k}}]$. The latter result can be proved following the same arguments. □

Corollary 4.51. *For all $t \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{N}_0^d$, we have*

$$\sum_{\mathbf{k}\leq\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} \mathcal{E}_{\mathbf{k}}^{(t)}(\mathbf{x})\mathcal{E}_{\mathbf{v}-\mathbf{k}}^{(-t)}(\mathbf{x}) = \sum_{\mathbf{k}\leq\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} \mathcal{B}_{\mathbf{k}}^{(t)}(\mathbf{x})\mathcal{B}_{\mathbf{v}-\mathbf{k}}^{(-t)}(\mathbf{x}) = 2^{|\mathbf{v}|}\mathbf{x}^{\mathbf{v}}.$$

Proof. In Corollary 4.50, set $s = -t$. The result follows by observing that $\mathcal{E}_{\mathbf{k}}^{(0)}(2\mathbf{x}) = \mathcal{B}_{\mathbf{k}}^{(0)}(2\mathbf{x}) = E[(2\mathbf{x})^{\mathbf{v}}] = 2^{|\mathbf{v}|}\mathbf{x}^{\mathbf{v}}$. □

Proposition 4.52. *For all $\mathbf{v} \in \mathbb{N}_0^d$, set $\tilde{\mathbf{v}}_j = (v_1, \ldots, v_{j-1}, v_j - 1, v_{j+1}, \ldots, v_d)$. We have*

$$\frac{\partial}{\partial x_j}\mathcal{E}_{\mathbf{v}}^{(t)}(\mathbf{x}) = v_j \mathcal{E}_{\tilde{\mathbf{v}}}^{(t)}(\mathbf{x}) \qquad \frac{\partial}{\partial x_j}\mathcal{B}_{\mathbf{v}}^{(t)}(\mathbf{x}) = v_j \mathcal{B}_{\tilde{\mathbf{v}}}^{(t)}(\mathbf{x}).$$

Proof. We have

$$\begin{aligned}
\frac{\partial}{\partial x_j}\mathcal{B}_{\mathbf{v}}^{(t)}(\mathbf{x}) = \frac{\partial}{\partial x_j}E[(\mathbf{x} + t.\boldsymbol{\iota})^{\mathbf{v}}] &= \frac{\partial}{\partial x_j}E[(x_1 + t.\iota)^{v_1} \ldots (x_d + t.\iota)^{v_d}] \\
&= E\left[(x_1 + t.\iota)^{v_1}\frac{\partial}{\partial x_j}(x_j + t.\iota)^{v_j} \ldots (x_d + t.\iota)^{v_d}\right] \\
&= E[(x_1 + t.\iota)^{v_1} \ldots v_j(x_j + t.\iota)^{v_j - 1} \ldots (x_d + t.\iota)^{v_d}] \\
&= v_j E[(x_1 + t.\iota)^{v_1} \ldots (x_j + t.\iota)^{v_j - 1} \ldots (x_d + t.\iota)^{v_d}] \\
&= v_j \mathcal{B}_{\tilde{\mathbf{v}}}^{(t)}(\mathbf{x}).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\frac{\partial}{\partial x_j}\mathcal{E}_{\mathbf{v}}^{(t)}(\mathbf{x}) = \frac{\partial}{\partial x_j}E\left[\left(\mathbf{x} + \frac{1}{2}\left(t.\left(\boldsymbol{\eta} - 1.\boldsymbol{u}\right)\right)\right)^{\mathbf{v}}\right] \\
&= \frac{\partial}{\partial x_j}E\left[\left(x_1 + \frac{1}{2}\left(t.\left(\eta - 1.u\right)\right)\right)^{v_1} \ldots \left(x_d + \frac{1}{2}\left(t.\left(\eta - 1.u\right)\right)\right)^{v_d}\right] \\
&= E\Bigg[\left(x_1 + \frac{1}{2}\left(t.\left(\eta - 1.u\right)\right)\right)^{v_1} \frac{\partial}{\partial x_j}\left(x_j + \frac{1}{2}\left(t.\left(\eta - 1.u\right)\right)\right)^{v_j} \ldots \\
&\left(x_d + \frac{1}{2}\left(t.\left(\eta - 1.u\right)\right)\right)^{v_d}\Bigg] \\
&= E\Bigg[\left(x_1 + \frac{1}{2}\left(t.\left(\eta - 1.u\right)\right)\right)^{v_1} \ldots v_j\left(x_j + \frac{1}{2}\left(t.\left(\eta - 1.u\right)\right)\right)^{v_j - 1} \ldots
\end{aligned}$$

$$\left(x_d + \frac{1}{2}\left(t.\left(\eta - 1.u\right)\right)\right)^{v_d}\Bigg]$$
$$= v_j E\Bigg[\left(x_1 + \frac{1}{2}\left(t.\left(\eta - 1.u\right)\right)\right)^{v_1} \ldots \left(x_j + \frac{1}{2}\left(t.\left(\eta - 1.u\right)\right)\right)^{v_j - 1} \ldots$$
$$\left(x_d + \frac{1}{2}\left(t.\left(\eta - 1.u\right)\right)\right)^{v_d}\Bigg] = v_j \mathcal{E}^{(t)}_{\tilde{\mathbf{v}}}\left(\mathbf{x}\right).$$

□

Proposition 4.53. *For all* $\mathbf{v} \in \mathbb{N}_0^d$, *set* $\tilde{\mathbf{v}}_j = (v_1, \ldots, v_{j-1}, v_j - 1, v_{j+1}, \ldots, v_d)$. *We have*

$$\mathcal{B}^{(t)}_{\mathbf{v}}(\mathbf{x}+\mathbf{1}) - \mathcal{B}^{(t)}_{\mathbf{v}}(\mathbf{x}) = \sum_{j=1}^{d} v_j \mathcal{B}^{(t)}_{\tilde{\mathbf{v}}}(\mathbf{x})$$
$$\mathcal{E}^{(t)}_{\mathbf{v}}(\mathbf{x}+\mathbf{1}) + \mathcal{E}^{(t)}_{\mathbf{v}}(\mathbf{x}) = 2\mathcal{E}^{(t-1)}_{\mathbf{v}}(\mathbf{x})$$

Proof. By definition, we have $B^{(t)}_{\mathbf{v}}(\mathbf{x}) = E[(\mathbf{x} + t.\boldsymbol{\iota})^{\mathbf{v}}]$,

$$\Rightarrow B^{(t)}_{\mathbf{v}}(\mathbf{1}+\mathbf{x}) - B^{(t)}_{\mathbf{v}}(\mathbf{x}) = E[(\mathbf{1}+\mathbf{x}+t.\boldsymbol{\iota})^{\mathbf{v}}] - E[(\mathbf{x}+t.\boldsymbol{\iota})^{\mathbf{v}}]$$
$$= E[(\boldsymbol{u}+\mathbf{x}+(t-1+1).\boldsymbol{\iota})^{\mathbf{v}}] - E[(\mathbf{x}+(t-1+1).\boldsymbol{\iota})^{\mathbf{v}}]$$
$$= E[(\boldsymbol{u}+\mathbf{x}+(t-1).\boldsymbol{\iota}+\boldsymbol{\iota})^{\mathbf{v}}] - E[(\mathbf{x}+(t-1).\boldsymbol{\iota}+\boldsymbol{\iota})^{\mathbf{v}}]$$
$$= E\left[\sum_{\mathbf{k}\le\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} (\mathbf{x}+(t-1).\boldsymbol{\iota})^{\mathbf{k}} (\boldsymbol{u}+\boldsymbol{\iota})^{\mathbf{v}-\mathbf{k}}\right]$$
$$- E\left[\sum_{\mathbf{k}\le\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} (\mathbf{x}+(t-1).\boldsymbol{\iota})^{\mathbf{k}} (\boldsymbol{\iota})^{\mathbf{v}-\mathbf{k}}\right]$$
$$= E\left[\sum_{\mathbf{k}\le\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} (\mathbf{x}+(t-1).\boldsymbol{\iota})^{\mathbf{k}} \{(\boldsymbol{u}+\boldsymbol{\iota})^{\mathbf{v}-\mathbf{k}} - (\boldsymbol{\iota})^{\mathbf{v}-\mathbf{k}}\}\right].$$

If $\mathbf{k} = \mathbf{v}$, that is, $\mathbf{v} - \mathbf{k} = \mathbf{0}$, then $(\boldsymbol{u}+\boldsymbol{\iota})^{\mathbf{v}-\mathbf{k}} - (\boldsymbol{\iota})^{\mathbf{v}-\mathbf{k}} \simeq (\boldsymbol{u}+\boldsymbol{\iota})^{\mathbf{0}} - (\boldsymbol{\iota})^{\mathbf{0}} \simeq 1 - 1 \simeq 0$.

If $|\mathbf{v} - \mathbf{k}| > 1$, then, by virtue of Corollary 4.30, we have $(\boldsymbol{u}+\boldsymbol{\iota})^{\mathbf{v}-\mathbf{k}} - (\boldsymbol{\iota})^{\mathbf{v}-\mathbf{k}} \simeq 0$.

If $|\mathbf{v} - \mathbf{k}| = 1$, then by virtue of Corollary 4.30, we have $(\boldsymbol{u}+\boldsymbol{\iota}) - (\boldsymbol{\iota}) \simeq \boldsymbol{u}$. Note that $|\mathbf{v} - \mathbf{k}| = 1$ is equivalent to $(v_1 - k_1) + \cdots + (v_d - k_d) = 1$ and this is possible if and only if there exists an index j, $j = 1, \ldots, d$, such that $v_j - k_j = 1$ and $v_h - k_h = 0$, for all $h \neq j$, that is, there exists an index

j, $j = 1, \ldots, d$, such that $k_j = v_j - 1$ and $v_h = k_h$, for all $h \neq j$. If we put $\tilde{\mathbf{v}}_j = (v_1, \ldots, v_{j-1}, v_j - 1, v_{j+1}, \ldots, v_d)$, we have

$$E\left[\sum_{\mathbf{k}\leq\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} (\mathbf{x} + (t-1).\boldsymbol{\iota})^{\mathbf{k}} \{(\boldsymbol{u} + \boldsymbol{\iota})^{\mathbf{v}-\mathbf{k}} - (\boldsymbol{\iota})^{\mathbf{v}-\mathbf{k}}\}\right]$$
$$= E[v_1(\mathbf{x} + (t-1).\boldsymbol{\iota})^{\tilde{\mathbf{v}}_1} + \cdots + v_d(\mathbf{x} + (t-1).\boldsymbol{\iota})^{\tilde{\mathbf{v}}_d}]$$
$$= E\left[\sum_{j=1}^{d} v_j (\mathbf{x} + (t-1).\boldsymbol{\iota})^{\tilde{\mathbf{v}}_j}\right] = \sum_{j=1}^{d} v_j E[(\mathbf{x} + (t-1).\boldsymbol{\iota})^{\tilde{\mathbf{v}}_j}]$$
$$= \sum_{j=1}^{d} v_j B_{\tilde{\mathbf{v}}_j}^{(t-1)}(\mathbf{x}).$$

Similarly, we have

$$\mathcal{E}_{\mathbf{v}}^{(t)}(\mathbf{x} + \mathbf{1}) + \mathcal{E}_{\mathbf{v}}^{(t)}(\mathbf{x}) = E\left[\left(\mathbf{x} + \mathbf{1} + \frac{1}{2}(t.(\boldsymbol{\eta} - 1.\boldsymbol{u}))\right)^{\mathbf{v}}\right]$$
$$+ E\left[\left(\mathbf{x} + \frac{1}{2}(t.(\boldsymbol{\eta} - 1.\boldsymbol{u}))\right)^{\mathbf{v}}\right]$$
$$= E\left[\left(\mathbf{x} + \boldsymbol{u} + \frac{1}{2}((t-1+1).(\boldsymbol{\eta} - 1.\boldsymbol{u}))\right)^{\mathbf{v}}\right]$$
$$+ E\left[\left(\mathbf{x} + \frac{1}{2}((t-1+1).(\boldsymbol{\eta} - 1.\boldsymbol{u}))\right)^{\mathbf{v}}\right]$$
$$= E\left[\left(\mathbf{x} + \boldsymbol{u} + \frac{1}{2}((t-1).(\boldsymbol{\eta} - 1.\boldsymbol{u})) + \frac{1}{2}(\boldsymbol{\eta} - 1.\boldsymbol{u})\right)^{\mathbf{v}}\right]$$
$$+ E\left[\left(\mathbf{x} + \frac{1}{2}((t-1).\boldsymbol{\eta} - 1.\boldsymbol{u})\right) + \frac{1}{2}.(\boldsymbol{\eta} - 1.\boldsymbol{u}))^{\mathbf{v}}\right]$$
$$= E\left[\sum_{\mathbf{k}\leq\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} (\mathbf{x} + \frac{1}{2}((t-1).\boldsymbol{\eta} - 1.\boldsymbol{u}))^{\mathbf{k}} (\boldsymbol{u} + \frac{1}{2}.(\boldsymbol{\eta} - 1.\boldsymbol{u}))^{\mathbf{v}-\mathbf{k}}\right]$$
$$+ E\left[\sum_{\mathbf{k}\leq\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} (\mathbf{x} + (t-1).\frac{1}{2}.(\boldsymbol{\eta} - 1.\boldsymbol{u}))^{\mathbf{k}} (\frac{1}{2}.(\boldsymbol{\eta} - 1.\boldsymbol{u}))^{\mathbf{v}-\mathbf{k}}\right]$$
$$= E\Bigg[\sum_{\mathbf{k}\leq\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} (\mathbf{x} + (t-1).\frac{1}{2}.(\boldsymbol{\eta} - 1.\boldsymbol{u}))^{\mathbf{k}} \{(\boldsymbol{u} + \frac{1}{2}.(\boldsymbol{\eta} - 1.\boldsymbol{u}))^{\mathbf{v}-\mathbf{k}} +$$
$$+ (\frac{1}{2}.(\boldsymbol{\eta} - 1.\boldsymbol{u}))^{\mathbf{v}-\mathbf{k}}\}\Bigg].$$

Since

$$(\boldsymbol{u}+\frac{1}{2}.(\boldsymbol{\eta}-1.\boldsymbol{u}))^{\mathbf{v}-\mathbf{k}}+(\frac{1}{2}.(\boldsymbol{\eta}-1.\boldsymbol{u}))^{\mathbf{v}-\mathbf{k}} \simeq \begin{cases} 2, & \text{if } \mathbf{v}-\mathbf{k}=\mathbf{0} \\ 0, & \text{if } |\mathbf{v}-\mathbf{k}|>0, \end{cases}$$

then $\mathcal{E}_{\mathbf{v}}^{(t)}(\mathbf{1}+\mathbf{x})+\mathcal{E}_{\mathbf{v}}^{(t)}(\mathbf{x}) = 2E[(\mathbf{x}+(t-1).\frac{1}{2}.(\boldsymbol{\eta}-1.\boldsymbol{u}))^{\mathbf{v}}] = 2\mathcal{E}_{\mathbf{v}}^{(t-1)}(\mathbf{x})$. □

Proposition 4.54. *For all* $\mathbf{v} \in \mathbb{N}_0^d$, *we have*

$$\mathcal{B}_{\mathbf{v}}^{(t)}(t\mathbf{1}-\mathbf{x}) = (-1)^{|\mathbf{v}|}\mathcal{B}_{\mathbf{v}}^{(t)}(\mathbf{x}) \quad \mathcal{E}_{\mathbf{v}}^{(t)}(t\mathbf{1}-\mathbf{x}) = (-1)^{|\mathbf{v}|}\mathcal{E}_{\mathbf{v}}^{(t)}(\mathbf{x}).$$

Proof. The former equality follows by observing that

$$\begin{aligned}\mathcal{B}_{\mathbf{v}}^{(t)}(t\mathbf{1}-\mathbf{x}) &= E[(t.\boldsymbol{u}-\mathbf{x}+t.\boldsymbol{\iota})^{\mathbf{v}}] = (-1)^{|\mathbf{v}|}E[(t.(-\boldsymbol{u})+\mathbf{x}+t.(-\boldsymbol{\iota}))^{\mathbf{v}}] \\ &= (-1)^{|\mathbf{v}|}E\left[(\mathbf{x}+t.(-(\boldsymbol{u}+\boldsymbol{\iota})))^{\mathbf{v}}\right].\end{aligned}$$

Since $E[(\iota+u)^k] = (-1)^k E[\iota^k]$ for all nonnegative integers k, then $E[(-(\boldsymbol{\iota}+\boldsymbol{u}))^{\mathbf{v}}] = (-1)^{|\mathbf{v}|}[(\iota+u)^{|\mathbf{v}|}] = E[\iota^{|\mathbf{v}|}] = E[\boldsymbol{\iota}^{\mathbf{v}}]$. Then we have

$$-(\boldsymbol{\iota}+\boldsymbol{u}) \equiv \boldsymbol{\iota} \quad t.(-(\boldsymbol{\iota}+\boldsymbol{u})) \equiv t.\boldsymbol{\iota},$$

by which the result follows.

Similarly, we have

$$\mathcal{E}_{\mathbf{v}}^{(t)}(t\mathbf{1}-\mathbf{x}) = E\left[\left(t\mathbf{1}-\mathbf{x}+t.\frac{\boldsymbol{\eta}}{2}-\frac{t}{2}\mathbf{1}\right)^{\mathbf{v}}\right] = (-1)^{|\mathbf{v}|}E\left[\left(\mathbf{x}+t.\frac{-\boldsymbol{\eta}}{2}+t.\frac{\boldsymbol{u}}{2}\right)^{\mathbf{v}}\right].$$

Since $f\left(-\frac{\boldsymbol{\eta}}{2},\mathbf{z}\right) = f\left(\frac{\boldsymbol{\eta}}{2},\mathbf{z}\right)$, the latter result follows. □

Up to now, it is clear that multivariate Bernoulli and Euler polynomials share many properties. Undoubtedly, this is due to the similar moment representation they have, but it is also reasonable to ask for a deeper connection between these two families of polynomials, that is a connection between the multivariate umbrae they are related to.

Lemma 4.55. *If* $\boldsymbol{\iota}$ *is the multivariate Bernoulli umbra and* $\boldsymbol{\eta}$ *is the multivariate Euler umbra, then*

$$2\boldsymbol{\iota} \equiv \frac{1}{2}(\boldsymbol{\eta}-1.\boldsymbol{u})+\boldsymbol{\iota}.$$

Proof. Via generating function, we have

$$f(2\boldsymbol{\iota}, \mathbf{z}) = f(\boldsymbol{\iota}, 2\mathbf{z}) = \frac{2(z_1 + \cdots + z_d)}{e^{2(z_1+\cdots+z_d)} - 1} = f(\boldsymbol{\eta}, \mathbf{z}) f(\boldsymbol{\iota}, \mathbf{z}) \frac{1}{f(\boldsymbol{u}, \mathbf{z})},$$

by using equations (4.3) (4.33) and (4.37). □

Theorem 4.56. *For all $t \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{N}_0^d$ we have*

$$2^{|\mathbf{v}|} \mathcal{B}_{\mathbf{v}}^{(t)} \left(\frac{\mathbf{x}}{2} \right) = \mathcal{E}_{\mathbf{v}}^{(t)} (\mathbf{x} + t.\boldsymbol{\iota}).$$

Proof. From Lemma 4.55, we have

$$2^{|\mathbf{v}|} \mathcal{B}_{\mathbf{v}}^{(t)} \left(\frac{\mathbf{x}}{2} \right) = E[(\mathbf{x} + t.(2\boldsymbol{\iota}))^{\mathbf{v}}] = E\left[\left(\mathbf{x} + t.\boldsymbol{\iota} + \frac{1}{2} t.(\boldsymbol{\eta} - 1.\boldsymbol{u}) \right)^{\mathbf{v}} \right].$$

□

A more complex relation between multivariate Bernoulli and Euler polynomials is given in [37]. For completeness, we add its generalization and umbral proof.

Proposition 4.57. *For all $\mathbf{v} \in \mathbb{N}_0^d$, set $\tilde{\mathbf{v}}_j = (v_1, \ldots, v_{j-1}, v_j - 1, v_{j+1}, \ldots, v_d)$. We have*

$$\sum_{j=1}^{d} v_j \mathcal{E}_{\tilde{\mathbf{v}}_j}^{(t)} (\mathbf{x}) = \sum_{\mathbf{k} \le \mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} 2^{|\mathbf{k}|} \left[\mathcal{B}_{\mathbf{k}}^{(t)} \left(\frac{\mathbf{x} + \mathbf{1}}{2} \right) - \mathcal{B}_{\mathbf{k}}^{(t)} \left(\frac{\mathbf{x}}{2} \right) \right] B_{\mathbf{v}-\mathbf{k}}^{(1-t)},$$

for all $t \in \mathbb{R}$ with $\{B_{\mathbf{v}}^{(t)}\}$ the multivariate Bernoulli numbers.

Proof. From Theorem 4.56, we have

$$\begin{aligned} 2^{|\mathbf{k}|} \mathcal{B}_{\mathbf{k}}^{(t)} \left(\frac{\mathbf{x} + \mathbf{1}}{2} \right) &= \mathcal{E}_{\mathbf{k}}^{(t)} (\mathbf{x} + \boldsymbol{u} + t.\boldsymbol{\iota}) = E[(\mathbf{x} + \boldsymbol{u} + t.\boldsymbol{\iota} + \frac{1}{2}(t.(\boldsymbol{\eta} - 1.\boldsymbol{u})))^{\mathbf{k}}] \\ &= E[(\mathbf{x} + t.\boldsymbol{\iota} + \frac{1}{2}(t.\boldsymbol{\eta}) + (1 \frac{t}{2}).\boldsymbol{u})^{\mathbf{k}}]. \end{aligned}$$

Then,

$$\begin{aligned} \sum_{\mathbf{k} \le \mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} 2^{|\mathbf{k}|} \mathcal{B}_{\mathbf{k}}^{(t)} \left(\frac{\mathbf{x} + \mathbf{1}}{2} \right) B_{\mathbf{v}-\mathbf{k}}^{(1-t)} &= \sum_{\mathbf{k} \le \mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} E[(\mathbf{x} + t.\boldsymbol{\iota} + \frac{1}{2}(t.\boldsymbol{\eta}) \\ &\quad + (1 \frac{t}{2}).\boldsymbol{u})^{\mathbf{k}}] \, E[((1-t).\boldsymbol{\iota})^{\mathbf{v}-\mathbf{k}}] \end{aligned}$$

$$= E[(\mathbf{x}\frac{1}{2}(t.\boldsymbol{\eta}) + \boldsymbol{u} - \frac{1}{2}(t.\boldsymbol{u}) + (1-t).\boldsymbol{\iota})^{\mathbf{v}}]$$
$$= E[(\mathbf{x} + \boldsymbol{u} + \boldsymbol{\iota} + \frac{1}{2}(t.(\boldsymbol{\eta} - 1.\boldsymbol{u})))^{\mathbf{v}}].$$

Similarly, we have

$$\sum_{\mathbf{k}\leq\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} 2^{|\mathbf{k}|} \mathcal{B}_{\mathbf{k}}^{(t)}\left(\frac{\mathbf{x}}{2}\right) B_{\mathbf{v}-\mathbf{k}}^{(1-t)} = \sum_{\mathbf{k}\leq\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} \mathcal{E}_{\mathbf{k}}^{(t)}(\mathbf{x} + t.\boldsymbol{\iota})\, B_{\mathbf{v}-\mathbf{k}}^{(1-t)}$$
$$= \sum_{\mathbf{k}\leq\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} E[(\mathbf{x} + t.\boldsymbol{\iota} + \frac{1}{2}t.(\boldsymbol{\eta} - 1.\boldsymbol{u}))^{\mathbf{k}}] E[((1-t).\boldsymbol{\iota})^{\mathbf{v}-\mathbf{k}}]$$
$$= E[(\mathbf{x} + \frac{1}{2}(t.(\boldsymbol{\eta} - 1.\boldsymbol{u})) + (1-t).\boldsymbol{\iota})^{\mathbf{v}}].$$

By expanding the powers, we have

$$E\left[\sum_{\mathbf{k}\leq\mathbf{v}} \binom{\mathbf{v}}{\mathbf{k}} (\mathbf{x} + t.\boldsymbol{\eta} - t.\boldsymbol{u})^{\mathbf{k}} \{(\boldsymbol{u} + \boldsymbol{\iota})^{\mathbf{v}-\mathbf{k}} - (\boldsymbol{\iota})^{\mathbf{v}-\mathbf{k}}\}\right],$$

by which the result follows, thanks to Corollary 4.30. □

Theorem 4.58. *The generalized multivariate Bernoulli and Euler polynomials are time-space harmonic with respect to the Lévy processes umbrally represented by* $\{t.(-1.\boldsymbol{\iota})\}_{t\geq 0}$ *and* $\{t.(-1.\boldsymbol{\eta})\}_{t\geq 0}$, *respectively.*

Proof. The result follows by comparing the moment representation of multivariate Bernoulli and Euler polynomials (Theorem 4.46) with Theorem 4.22. □

The property of being time-space harmonic polynomials, shared by Bernoulli and Euler polynomials, is also confirmed by the next Corollary, which states that their overall evaluation is zero.

Corollary 4.59. $E[\mathcal{B}_{\mathbf{v}}^{(t)}(-t.\boldsymbol{\iota})] = E[\mathcal{B}_{\mathbf{v}}^{(t)}(t.(-1.\boldsymbol{\iota}))] = 0$ *and* $E[\mathcal{E}_{\mathbf{v}}^{(t)}(\frac{1}{2}[t.(\boldsymbol{u} - 1.\boldsymbol{\eta})])] = 0$.

Proof. The result follows by replacing $\mathbf{x}$ with $-t.\boldsymbol{\iota}$ and $\frac{1}{2}t.(\boldsymbol{u} - 1.\boldsymbol{\eta})$ in Theorem 4.46. □

4.4.4 Multivariate Hermite polynomials

Recall that a multivariate Brownian motion is umbrally represented by the family of auxiliary umbrae $\{t.\beta.(\chi.\mathbf{m}\dot{+}\boldsymbol{\delta}C^T)\}_{t\geq 0}$.

In [64], Withers gives the following moment representation of multivariate Hermite polynomials

$$H_{\mathbf{v}}(\mathbf{x},\Sigma) = E[(\mathbf{x}\Sigma^{-1}+iY)^{\mathbf{v}}]$$
$$\tilde{H}_{\mathbf{v}}(\mathbf{x},\Sigma) = E[(\mathbf{x}+iZ)^{\mathbf{v}}]$$

where $Y \sim N(\mathbf{0},\Sigma)$ and $Z \sim N(\mathbf{0},\Sigma)^{-1}$.

We will consider the family of polynomials $\{\tilde{H}_{\mathbf{v}}(\mathbf{x},\Sigma)\}$, since they are monic and orthogonal with respect to the multivariate gaussian density function.

Theorem 4.60. *The family $\{\tilde{H}^{(t)}_{\mathbf{v}}(\mathbf{x},\Sigma)\}_{t\geq 0}$ of generalized multivariate Hermite polynomials is time-space harmonic with respect to the multivariate Brownian motion without drift $\{t.\beta.(\boldsymbol{\delta}C^T)\}$.*

Proof. The moment generating function of $\{\tilde{H}^{(t)}_{\mathbf{v}}(\mathbf{x},\Sigma)\}_{t\geq 0}$ is

$$1+\sum_{k\geq 1}\sum_{|\mathbf{v}|=k}\tilde{H}^{(t)}_{\mathbf{v}}(\mathbf{x},\Sigma)\frac{\mathbf{x}^{\mathbf{v}}}{\mathbf{v}!} = \exp\left\{\mathbf{x}\mathbf{z}^T - \frac{t}{2}\mathbf{z}\Sigma\mathbf{z}^T\right\}. \tag{4.41}$$

Equation (4.41) can be written in the following way

$$\exp\left\{\mathbf{x}\mathbf{z}^T - \frac{t}{2}\mathbf{z}\Sigma\mathbf{z}^T\right\} = \exp\left\{\mathbf{x}\mathbf{z}^T\right\}\,\exp\left\{-\frac{t}{2}\mathbf{z}\Sigma\mathbf{z}^T\right\}.$$

Thanks to equation (4.3), we have

$$\exp\left\{\mathbf{x}\mathbf{z}^T\right\} = f(\mathbf{x},\mathbf{z}).$$

Since $\Sigma = CC^T$, we have

$$\begin{aligned}
\exp\left\{-\frac{t}{2}\mathbf{z}\Sigma\mathbf{z}^T\right\} &= \exp\left\{-\frac{t}{2}\mathbf{z}CC^T\mathbf{z}^T\right\}\\
&= \exp\left\{-\left[1+\frac{t}{2}\mathbf{z}CC^T\mathbf{z}^T-1\right]\right\}\\
&= \exp\left\{-\left[1+\frac{1}{2}[\mathbf{z}(t^{1/2}C)][(t^{1/2}C^T)\mathbf{z}^T]-1\right]\right\}\\
&= \exp\left\{-\left[1+\frac{1}{2}[\mathbf{z}(t^{1/2}C)][(t^{1/2}C)\mathbf{z}]^T-1\right]\right\}
\end{aligned}$$

$$= \left(\exp\left\{ \left[1 + \frac{1}{2}[\mathbf{z}(t^{1/2}C)][(t^{1/2}C)\mathbf{z}]^T - 1 \right] \right\} \right)^{-1}$$
$$= \left(\exp\left\{ f(\boldsymbol{\delta}, \mathbf{z}(t^{1/2}C)) - 1 \right\} \right)^{-1}.$$

Observe that, for all umbral d-tuple $\boldsymbol{\mu}$ and for all $d \times d$ matrix A, we have

$$f(\boldsymbol{\mu}, \mathbf{z}A) = E\left[e^{\boldsymbol{\mu}(\mathbf{z}A)^T} \right] = E\left[e^{\boldsymbol{\mu}A^T} \mathbf{z}^T \right]$$
$$= E\left[e^{(\boldsymbol{\mu}A^T)\mathbf{z}^T} \right] = f(\boldsymbol{\mu}A^T, \mathbf{z}),$$

then, setting $\boldsymbol{\mu} = \boldsymbol{\delta}$ and $A = t^{1/2}C$, we have

$$f(\boldsymbol{\delta}, \mathbf{z}(t^{1/2}C)) = f(\boldsymbol{\delta}(t^{1/2}C)^T, \mathbf{z}) = f(\boldsymbol{\delta}(t^{1/2}C^T), \mathbf{z}).$$

Therefore,

$$\exp\left\{ -\frac{t}{2}\mathbf{z}\Sigma\mathbf{z}^T \right\} = \left(\exp\left\{ f(\boldsymbol{\delta}, \mathbf{z}(t^{1/2}C)) - 1 \right\} \right)^{-1}$$
$$= \left(\exp\left\{ f(\boldsymbol{\delta}(t^{1/2}C^T), \mathbf{z}) - 1 \right\} \right)^{-1}$$
$$= \left(f(\beta.[\boldsymbol{\delta}(t^{1/2}C^T)], \mathbf{z}) \right)^{-1} = f(-1.\beta.[\boldsymbol{\delta}(t^{1/2}C^T)], \mathbf{z}).$$

Thus,

$$\exp\left\{ \mathbf{x}\mathbf{z}^T - \frac{t}{2}\mathbf{z}\Sigma\mathbf{z}^T \right\} = f(\mathbf{x}, \mathbf{z})\, f(-1.\beta.[\boldsymbol{\delta}(t^{1/2}C^T)], \mathbf{z})$$
$$= f(\mathbf{x} - 1.\beta.[\boldsymbol{\delta}(t^{1/2}C^T)], \mathbf{z}).$$

On the other hand, we have

$$f(-t.\beta.(\boldsymbol{\delta}C^T), \mathbf{z}) = [f(\beta.(\boldsymbol{\delta}C^T), \mathbf{z})]^{-t} = [\exp\{f(\boldsymbol{\delta}C^T, \mathbf{z}) - 1\}]^{-t}$$
$$= \exp\{-t[f(\boldsymbol{\delta}C^T, \mathbf{z}) - 1]\} = \exp\left\{ -t\left[\frac{1}{2}\mathbf{z}\Sigma\mathbf{z}^T \right] \right\}$$
$$= \exp\left\{ \left[-\frac{t}{2}\mathbf{z}\Sigma\mathbf{z}^T \right] \right\},$$

that is, via generating function we have proved that

$$-t.\beta.(\boldsymbol{\delta}C^T) \equiv -1.\beta[\boldsymbol{\delta}(t^{1/2}C^T)].$$

Hence,

$$\exp\left\{\mathbf{x}\mathbf{z}^T - \frac{t}{2}\mathbf{z}\Sigma\mathbf{z}^T\right\} = f(\mathbf{x} - t.\beta.(\boldsymbol{\delta}C^T), \mathbf{z})$$

and

$$\tilde{H}_{\mathbf{v}}^{(t)}(\mathbf{x}, \Sigma) = E[(\mathbf{x} - t.\beta.(\boldsymbol{\delta}C^T))^{\mathbf{v}}].$$

□

Conclusions

In this manuscript a symbolic representation of a specific family of stochastic processes, known as Lévy processes, is presented.

Throughout the research time, we came upon two classes of polynomials related to Lévy processes: the Kailath-Segall polynomials and the time-space harmonic polynomials. In particular, the latter have shown to share several properties which well fit in the umbral syntax, in the sense that the symbolic techniques described and applied in the manuscript allow us to streamline their proofs.

However, it is not a case of making a pure rewriting of known tools, already dealt in the literature, by means of a new language. This is rather a different approach, which proceeds in an opposite direction to that described in the literature by several authors. As a matter of facts, all the main properties of time-space harmonic polynomials and their coefficients, which are the building blocks of the theory in the classical case, are obtained in this manuscript as quite simple consequence of a unique closed-form formula.

What is more, we have generalized such expressions to the multivariate case, this allowing us to create a theory of multivariate time-space harmonic polynomials which is general, complete, rigorous and original. All these symbolic representations could be quickly implemented by means of suitable symbolic software, as it happens with the parametrization formulae between moments and cumulants.

A future direction of research would be the combinatorial interpretation of stochastic integrals, whose differential involves Lévy processes. Indeed in 1997, Rota and Wallstrom conceived a combinatorial definition of stochastic integration in the setting of random measures. The starting point was the Kailath-Segall formula interpreted in combinatorial terms and applied to derive recursion relations for some classes of orthogonal polynomials. Such approach was a real revolution, since, up to that moment, advanced techniques of functional analysis were the only tools used to handle stochastic integrals. We believe that the symbolic theory of Lévy processes could help in approaching stochastic integration by allowing us also new algorithms for their computations.

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