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# STRONG SEMICLASSICAL GRAVITY IN THE UNIVERSE AND THE LABORATORY

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# Introduction

Despite gravity was the first fundamental force to be recognized, three centuries ago, it will be the last one to be completely understood. One century after the formulation of general relativity by Albert Einstein we do not have yet any quantum theory of gravitation and the unification of gravity with other fundamental forces is probably the most important and burdensome problem in physics nowadays.

The need for a quantization of gravity comes from different aspects of the theory itself. Just to give an example we remind that the source of gravity could be any other quantized field (e.g. quantum harmonic oscillators), not to mention all the problems related to divergences, singularities and final stages of black hole collapse and evaporation. Another trouble with gravity is the "hierarchy problem": by this name one intends the great difference between the scales of energy on which gravity "works" and the scales of other fundamental forces. It seems in fact bizarre that gravity is by far the weakest force in Nature: $10^{25}$ times feebler than the weak force, $10^{36}$ times feebler than the electromagnetic force and $10^{38}$ times feebler than the strong force. Despite this, the whole universe on large scale is ruled by gravitational interaction.

In order to quantize gravity, two main roads are followed today: *string theory* (stemming from particle physics) and *loop quantum gravity* (deriving from quantum geometry). String theory is a candidate "Theory of Everything" which describes particles in terms of vibrating strings. This theory presents itself in a variety of sub-theories, each characterized by its pro's and con's. One peculiarity of string theory is the need for extra-spatial dimensions in order to have a consistent quantum theory. By assuming supersymmetry, the number of dimensions would be 10+1 in the M-theory, eventually reducing to 9+1 in other classes ("types") of theory. By the way, everyday experience tells that we are living in a 3+1 spacetime. To justify this discrepancy between theoretical needs and empirical evidence, different scenarios have been proposed. One in particular assumes that the 6 or 7 extra dimensions are compactified on very small scales. Another choice is the *brane world scenario* according to which we inhabit a 3+1 subspace embedded in a 4+1 "bulk". The extra dimension is accessible only by gravity. We will be back on this specific topic in chapter 3, where we present some possible experimental consequences of this assumption.

The other candidate for a quantum gravity theory is loop quantum gravity (LQG). LQG does not make use of any extra dimensions nor vibrating strings. Rather, it describes spacetime as a discretized network of loops. Spacetime is no longer fundamental, but is the natural outcome of these *spin networks*, from which surfaces can be constructed together with their related properties (curvature, volume, area...).

The main problem with string theory and loop quantum gravity is that they are not experimentally testable. String theory predicts unification of all fundamental forces on unachievable energy scales and due to its perturbative approach it is not background-independent. Also, string theory has a huge number of solutions, called string vacua, constituting the *landscape* (on which we will spend few more words in chapter 1). This causes the theory to predict almost everything, and to a physicist this is like predicting nothing. On the other side loop quantum gravity has no semiclassical limit, therefore does not recover general relativity at low energies.

Waiting for a trustable theory of quantum gravity, we have to rely on a compromise between general relativity and quantum mechanics. *Semiclassical gravity* is, by now, the only approach that can give experimentally testable predictions. As already observed by Planck in 1899, one can use the fundamental constants of nature to construct a new fundamental unit of length $\ell_{\rm P} = \sqrt{G\hbar/c^3} = 1.616 \times 10^{-33}$ cm, and a new fundamental unit of time $t_{\rm P} = \sqrt{G\hbar/c^5} = 5.39 \times 10^{-44}$ s. These incredibly small values outline the borders of quantum gravity and easily give an idea of the great effort one has to take to experimentally reach them. It is curious on the other hand how big is the new fundamental mass unit, the Planck mass $M_{\rm P} = \sqrt{\hbar c/G} = 2.176 \times 10^{-8}$ Kg, more or less the same mass as a flea egg. It is expected that on scales (energy, time, space...) much larger than $\ell_{\rm P}$, $t_{\rm P}$ and smaller than $M_{\rm P}$ one could approximate quantum gravity by assuming that all matter fields are quantized, whereas gravity preserves its classical description. One has then to use a theory for quantum fields on curved spacetimes which presents even more ambiguities than regular quantum field theory, mostly related to the lack of global Lorentz invariance (this will be discussed more deeply in chapter 2). Using other words, semiclassical gravity corresponds to summing all Feynman diagrams which do not have loops of gravitons (gravitational quanta) but can have any number of matter loops.

Contrary to string theory and loop quantum gravity, semiclassical gravity already gives us some hints on what we should expect to see in forthcoming experiments and astronomical observations. Thanks to the work developed from the end of 1960's by various brilliant physicists, we know what we should be looking for when searching the sky with telescopes or probing the "infinitely small" by means of particle accelerators. The most notable semiclassical effects happen in the presence of event horizons: for this reason most efforts, both theoretical and experimental, are directed towards the study of black holes. However, this approach is not free of problems.

With the beginning of the third millennium, Physics reached a critical point in its history. Probably for the first time since Galileo Galilei, experiments can't keep pace with theory. Even the most advanced (and expensive) technology seems unable to investigate as deep as needed the unexplored folds of Nature, and no sufficient technological advancements are expected in the next decades to survey the most fundamental theoretical models. The Large Hadron Collider (LHC) at CERN is the biggest and most "extreme" machine ever built by Mankind. It is expected that in the next years LHC will be able to collide particles at 14 TeV, an energy scale never probed before. Despite this, the Physics we are looking for might be out of its reach. If LHC fails in revealing new Physics, then we have to wait longtime before we develop the technology needed to investigate the "infinitely small". The only alternative is to look the other way, to the "infinitely large". Sky is crowded with stars and galaxies: collisions between celestial objects and stellar collapses involve enough energy to test Physics beyond our most unattainable requests, but they can't be reproduced on demand and they always happen far away, where we can barely see them. Astronomical incredibly high-energy events occur for free, but as in every free show, we can only hope to be lucky and find a good seat to enjoy the spectacle. While collecting astronomical data, the scientific community is focusing on repeatable experiments performed with particle accelerators. Waiting for some concrete results, a lot of hypothesis have been formulated on the possible outcome of such extreme scenarios. Every new fundamental theory must be experimentally falsifiable, and falsification is achieved by testing simple models to see whether they are consistent with observations.

The scope of this Ph.D. thesis is to collect some theoretical models which we elaborated relying on different popular theoretical backgrounds. Three fundamental aspects of semiclassical gravity are discussed.

Firstly, we apply the simple notion of quantum tunneling to the primordial universe. Application of quantum mechanical concepts to cosmology led in the 70's to one of the first important achievements of semiclassical gravity. It is supposed that our universe could have nucleated during a phase transition. Such nucleation more likely occurs in spherical symmetry, giving rise to a new phase bubble that expands in the old medium. This model can be applied to a variety of contexts. The growth or collapse of such a bubble is not obvious at all in dynamical backgrounds: for this reason we worked out a perturbative method to study its evolution.

Secondarily we investigate one of the most important phenomena in semiclassical gravity, the Unruh effect. Actually, Unruh effect is not gravitational since it does not regard the bending of spacetime due to mass and energy, strictly speaking. Despite this, there is a strong connection with general relativity via the equivalence principle which states that the local effects of motion in a curved space (gravitation) are indistinguishable from those of an accelerated observer in flat space. Because of this, one expects an accelerated observer to measure some thermal radiation emitted from an horizon the same way a static observer does when watching black hole. Anyway there are some problems connected with the definition of the horizon temperature in some cases. We are going to review these problems and study one possible solution.

Finally we discuss a subject that is of great importance for the semiclassical theory: the possibility of observing black holes formation at LHC. If extra dimensions are present it is possible that microscopic black holes are created in the laboratory. The detection of the emitted Hawking radiation would confirm the validity of the semiclassical approach but, as we will show, there might be more accidents than expected.

# Notation

In the present thesis we will mostly use the following notation, unless otherwise specified:

- $c = \hbar = 1$
- $G_N$ is the Newton constant
- $\kappa = 8\pi G_N/3$
- $\dot{a}$ is the derivative of the quantity $a$ with respect to some specified timelike coordinate
- $a'$ is the derivative of the quantity $a$ with respect to some specified spacelike coordinate (usually the radius)

# Contents

# Chapter 1

# New phase bubbles

In this chapter we discuss the bubble formation and expansion as a result of a phase transition. Results have been published in [1].

In Section 1.1 we briefly introduce the main reasons for which physicists have paid so much attention to new phase bubbles in the 80s and 90s and why this topic is still discussed today.

In Section 1.2 one of the pioneering researches concerning new phase bubbles is resumed.

In Section 1.3 we review the fundamental equations and constraints that describe general timelike bubble dynamics.

In Sections 1.4, 1.5 and 1.6, we work out the explicit case of a radiation bubble of constant surface density nucleated inside collapsing or expanding dust, for which we obtain the initial minimum radius in terms of the inner and outer energy densities.

Finally, in Section 1.7 we make some considerations about our findings and possible future generalizations.

## 1.1 Why bubbles?

In physics jargon, *bubbles* are spherically symmetric objects whose interior is separated from the exterior by a thin *shell*. In general relativity bubbles refer to a spherical portion of spacetime in which physical parameters such as energy density, equation of state and cosmological constant are different from the exterior. Bubbles have been frequently used in different cosmological scenarios, with a peak of interest during the 80's, following Guth's and Linde's inflationary models [2, 3, 4]. In such models the inflating universe finds itself in a false metastable vacuum and there are possibilities that a portion of space undergoes phase transition to a stable vacuum (the true vacuum) at random times. The new vacuum is assumed to have spherical symmetry both for simplicity and energetic “convenience”. Its interior can be regarded as a new universe and this “bubbling” process is frequently invoked in the *eternal inflation scenario* [3], where the rate of bubble formation is outpaced by the accelerated expansion of the inflating false vacuum, and therefore inflation does not end everywhere (for a review on this subject see [5]).

Attention towards this subject rose again thanks to string theory. String theory is not yet completely understood: what is clear is that there is a continuum of solutions to some *Master theory*, or simply *M-theory*: the space of this solutions is called *the moduli space of supersymmetric vacua*, or briefly the *supermoduli-space* [6, 7]. Moduli are like fields carrying information about the different quantities that characterize a given universe. Unluckily the supermoduli space only provides solutions for a supersymmetric universe with zero cosmological constant, and this is clearly not our case. To justify the existence of our universe we assume that there is a *landscape* [8, 9] consisting of many metastable de Sitter vacua, populated by eternal inflation. Quantum tunneling between vacua produces bubbles of new vacuum that look like infinite open Friedmann universes to observers inside. Some of this vacua might have anthropically favorable conditions and one among them is the universe we live in.

There are other cases of particular physical interest that might be studied by means of “bubbles”. For example, one can conceive the density inside a collapsing astrophysical object might be large enough to allow for the creation of supersymmetric matter which, in turn, would then annihilate regular matter and produce a ball of radiation [10, 11]. Also, the mathematical formalism that governs bubble dynamics has been used to model gravitational vacuum stars (*gravastars*), an alternative solution to black holes as a result of gravitational collapse [12]. Gravastars have an internal de Sitter core that is matched to the exterior Schwarzschild metric by means of an intermediate thin layer of matter with different equation of state.

In the next section we review the quantum mechanics behind the nucleation of new phase bubbles, and the mathematical tools needed to face this subject are synthetically resumed in the following chapter, with references to a complete and exhaustive bibliography.

## 1.2 Bubble nucleation: a summary

Coleman [13] in 1977 developed a theory on the decay from false vacuum following the tracks of Voloshin, Kobzarev and Okun [14], and he did this before any inflationary theory was formulated. Three years later Coleman and De Luccia proposed a more general theory that included the effects of gravity on the nucleation of the bubble [15]. Here we present a synthesis of [15] where the effects of gravity are considered. We also recommend to carefully read a recent paper by Copsey [16] where many subtleties are considered and new light is shed over this topic. By now, for simplicity, we just present the original research.

Assume we have a potential $U(\phi)$ with two minima $\phi_+$ (relative minimum) and $\phi_-$ (absolute minimum), see figure (1.1).

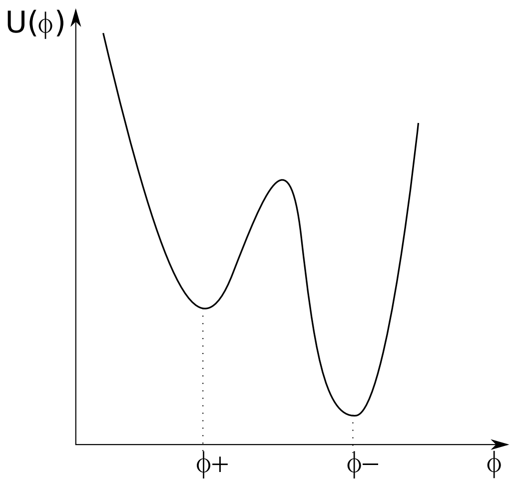


Figure 1.1: Tunneling through the potential barrier

Transition can occur between the two minima with relative probability $\Gamma/V$ per unit time and volume. In a semiclassical approximation this reads

$$\Gamma/V = Ae^{-B/\hbar}[1 + O(\hbar)] \tag{1.1}$$

with

$$B = S_E(\phi) - S_E(\phi_+) \ , \tag{1.2}$$

where $S_E$ is the euclidean action and A an undefined coefficient. Despite the small (quantum) scale of the problem, gravity has its role in the nucleation. In flat spacetime adding a constant term to a Lagrangian has no effect. On the contrary when including gravitation, the square root of the metric $\sqrt{-g}$ will multiply the constant term:

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - U(\phi) - \frac{R}{16\pi G_N}\right) \ , \tag{1.3}$$

i.e. introducing a potential $U$ is like introducing a cosmological constant. Once the vacuum decays and the bubble forms, the cosmological constant may be different.

First of all we write the Euclidean action $S_E$ by means of Wick rotation; then we find $\phi$, the solution of the Euler-Lagrange equation with euclidean signature $(++++)$. We demand that at euclidean infinity $\phi$ coincide with the false vacuum field $\phi_+$, that $\phi$ be not constant and that $\phi$ minimize the action. Coleman calls $\phi$ the *bounce*.

If the whole (Euclidean) spacetime is $O(4)$ symmetric (i.e. there is rotational invariance) the metric reads

$$ds^2 = d\xi^2 + \rho(\xi)^2 d\Omega^2 \tag{1.4}$$

where $d\Omega$ is the element of distance on a unit three-sphere and $\rho$ gives the radius of curvature of each three-sphere. One can then find the Euclidean equations of motion for $\phi$:

$$\phi'' + \frac{3\rho'}{\rho}\phi' = \frac{dU}{d\phi} \tag{1.5}$$

where a prime denotes derivation by $\xi$, whereas Einstein equation

$$G_{\xi\xi} = 3\kappa T_{\xi\xi} \tag{1.6}$$

with $\kappa = 8\pi G_N/3$ for later convenience, becomes

$$\rho'^2 = 1 + \kappa\rho^2 \left(\frac{1}{2}\phi'^2 - U\right) \ . \tag{1.7}$$

The other Einstein equations are identities of this one or at most simple consequences. Evaluation of the angular components of the Euclidean action gives:

$$S_E = 2\pi^2 \int d\xi \left[\rho^3 \left(\frac{1}{2}\phi'^2 + U\right) + \frac{1}{\kappa}\left(\rho^2\rho'' + \rho\rho'^2 - \rho\right)\right] \ . \tag{1.8}$$

In the thin-wall approximation one can demonstrate that $\rho'$ is negligible[1]. Also, we define a very small $\epsilon$ by

$$\epsilon = U(\phi_+) - U(\phi_-) \tag{1.9}$$

and write $U$ as

$$U(\phi) = U_0(\phi) + \mathcal{O}(\epsilon) \tag{1.10}$$

[1] See Copsey [16] for discussion on the validity of this approximation.

where $U_0$ is a function such that $U_0(\phi_-) = U_0(\phi_+)$ and such that $dU_0/d\phi$ vanishes at both $\phi_\pm$.

With this approximation we can easily integrate the equations of motion to find that:

$$\frac{1}{2}(\phi')^2 - U_0(\phi) = -U_0(\phi_\pm) \ , \tag{1.11}$$

in which we have considered that $\phi(\infty) = \phi_+$. This equation determines $\phi$ up to an integration constant $\bar{\xi}$ that we choose to be the point at which $\phi$ is the average of its two extreme values:

$$\int_{(\phi_+ + \phi_-)/2}^{\phi} d\phi \left[2\left(U_0 - U_0(\phi_\pm)\right)\right]^{-1/2} = \int_{\bar{\xi}}^{\xi} d\xi = \xi - \bar{\xi} \ . \tag{1.12}$$

Then $\bar{\rho} \equiv \rho(\bar{\xi})$ is the average curvature radius of the bubble, obtained by solving (1.7). To find $\bar{\rho}$ one has to compute $B$, the difference in action between the bounce and the false vacuum. Then, by demanding stationarity of $B$, we get $\bar{\rho}$. After integration by parts and discarding boundary terms, one has

$$S_E = 4\pi^2 \int d\xi \left[\rho^3 \left(\frac{1}{2}\phi'^2 + U\right) - \frac{1}{\kappa}\left(\rho\rho'^2 + \rho\right)\right] \ . \tag{1.13}$$

We substitute $\rho'$ using (1.7) and get

$$S_E = 4\pi^2 \int d\xi \left(\rho^3 U - \frac{1}{k}\rho\right) \ . \tag{1.14}$$

We divide the integration interval in three parts: "outside" the bubble the bounce and the false vacuum are identical:

$$B_{out} = S_E(\phi_+) - S_E(\phi_+) = 0 \ . \tag{1.15}$$

Within the wall we can set $\rho = \bar{\rho}$ and $U = U_0$:

$$B_{wall} = 2\pi^2 \bar{\rho}^3 S_1 \tag{1.16}$$

where $S_1$ is

$$S_1 = \int_{\phi_-}^{\phi_+} d\phi \left[2\left(U_0(\phi) - U_0(\phi_+)\right)\right]^{1/2} = 2\int d\xi \left[U_0(\phi) - U_0(\phi_+)\right] \tag{1.17}$$

and can be regarded to as the surface energy density of the bubble. Inside the wall $\phi$ is constant, and by using the radial component of the Friedmann-Robertson-Walker line element

$$d\xi = \frac{d\rho}{\sqrt{1 - \kappa\rho^2 U}} \tag{1.18}$$

we define $\xi$ to be the "physical length". Hence:

$$B_{in} = \frac{4\pi^2}{3\kappa^2}\left\{U(\phi_-)^{-1}\left[\left(1-\kappa\bar{\rho}^2U(\phi_-)\right)^{3/2}-1\right]-(\phi_-\mapsto\phi_+)\right\}\ . \quad (1.19)$$

Let's make some considerations of physical nature. If we consider transitions from higher to lower energy densities then $B$ is stationary at

$$\bar{\rho} = \frac{12S_1}{4\epsilon \pm 9\kappa S_1^2} = \frac{\bar{\rho}_0}{1\pm(\bar{\rho}_0/2\Lambda)^2} \quad (1.20)$$

where $\bar{\rho}_0 = 3S_1/\epsilon$ is the bubble radius in the absence of gravity and $\Lambda = (\kappa\epsilon)^{-1/2}$ is the Schwarzschild radius of a sphere that exactly matches its own Schwarzschild radius (this outlines how gravitation enters the problem and how much important it is). The $\pm$ sign changes according to the content of the phases: it is $+$ when transition occurs from a phase with positive energy density to one with zero energy density ($U(\phi_+) = \epsilon,\ \ U(\phi_-) = 0$), whereas it is $-$ when going from zero energy density to negative energy density ($U(\phi_+) = 0,\ \ U(\phi_-) = -\epsilon$). Hence:

$$B = \frac{B_0}{[1\pm(\bar{\rho}_0/2\Lambda)^2]^2} \qquad B_0 = 27\pi^2S_1^4/2\epsilon^3 \quad (1.21)$$

where $B_0$ is the decay coefficient in the absence of gravity.

One can see then in the first case (decay from positive to zero energy density) gravitation makes the nucleation more probable and reduces the initial radius of the bubble. In the second case (decay from zero to negative energy density) things go just the opposite way. We can understand this if we keep in mind that during the phase transition energy must be conserved. Assume that we are in a universe with positive energy density $\epsilon$ and a spherical portion of this old phase, with radius $\bar{\rho}$ is converted into a new phase bubble whose energy density is zero. The energy balance require that what was inside the bubble ($E = 4\pi\bar{\rho}^3\epsilon/3$) must be totally transferred to the walls ($E = 4\pi\bar{\rho}^2S$, where $S$ is the surface energy density of the bubble). Nothing is left in the inside. In a flat geometry this is clear, but when introducing gravity the presence of matter/energy curves the spacetime. Therefore the volume of the bubble is not the same as before. One can see that positive energy densities cause the volume/surface ratio to increase, whereas negative energy densities make the same ratio smaller. So, if the transition occurs from a positive to a zero energy density phase there is more energy stored inside a bubble of given radius: otherwise we can say that a smaller radius is required to nucleate a bubble with given initial energy density $\epsilon$ and final surface density $S$. The opposite is true for transition from a zero to a negative energy density phase. In this latter case it is then possible that, for sufficiently small $\epsilon$, no bubble could be created.

## 1.3 Mathematical formalism and bubble dynamics

Bubbles, intended as spherically symmetric objects, have been studied in General Relativity to answer the question: "What happens to the geometry of spacetime when there is a discontinuity in the stress-energy tensor along a surface?". Originally there was no corellation with the study of the origin of our universe: this only became clear later. An interesting approach was formulated by Israel in 1965 [17].

Let's take an infinitely thin, spherically symmetric shell $\Sigma$ separating two spacetime regions $\Omega_\pm$ with given metrics. Given the symmetry of the system, we can use spherical coordinates $y^\mu_\pm = \{t_\pm, r_\pm, \theta, \phi\}$ in $\Omega_\pm$, respectively, where the angular coordinates are the same in both patches, and $0 \leq r_- < r^{\mathrm{s}}_-$, $r^{\mathrm{s}}_+ < r_+$, with $r^{\mathrm{s}}_\pm = r^{\mathrm{s}}_\pm(t_\pm)$ the (in general time-dependent) radial coordinates of the shell in $\Omega_\pm$. One then takes specific solutions $g^\pm_{\mu\nu}$ of the Einstein equations inside $\Omega_\pm$ and imposes suitable junction conditions across $\Sigma$, namely the metric is required to be continuous across the shell:

$$g^+_{\mu\nu}\big|_\Sigma = g^-_{\mu\nu}\big|_\Sigma \ . \tag{1.22}$$

One can do this by defining the shell's hypersurface on both sides by the equation $F^\pm(y^\pm) = 0$ and introducing new coordinates $(n, x)$ in $\Omega_\pm$ in such a way that $n_\pm = 0$ coincides with $F^\pm(y^\pm) = 0$. Then the metric in $\Omega_\pm$ can be written as:

$$ds^2 = \eta\ dn_+{}^2 + \gamma^+_{ij}(n_+, x_+) dx_+{}^i dx_+{}^j \tag{1.23}$$

$$ds^2 = \eta\ dn_-{}^2 + \gamma^-_{ij}(n_-, x_-) dx_-{}^i dx_-{}^j \tag{1.24}$$

where $\eta = \pm 1$ according to whether the shell is spacelike $(-)$ or timelike $(+)$, $x^i$ are coordinates on the shell and italic indices run on the shell's three-dimensional world-sheet. Then one has to find the *first junction*

$$\gamma^+_{ij}(0, x_+) = \gamma^-_{ij}(0, x_-) \frac{\partial x_-{}^i}{\partial x_+{}^k} \frac{dx_-{}^j}{\partial x_+{}^l} \tag{1.25}$$

and once the junction is carried out we can cover the whole manifold by Gaussian coordiantes in which $n = 0$ is the equation of the hypersurface $\Sigma$:

$$ds^2 = \eta dn^2 + \gamma_{ij}(x, n) dx^i dx^j \ . \tag{1.26}$$

We can use coordinates (1.26) to define a discontinuous stress-energy tensor over the whole spacetime, composed by the interior and exterior sources $T$ and the shell's surface density $\sigma$

$$T_{\mu\nu} = T^+_{\mu\nu}(y)\Theta(n) + T^-_{\mu\nu}(y)\Theta(-n) + \sigma_{\mu\nu}(y)\delta(n) \tag{1.27}$$

where $\Theta$ is Heaviside's step function and $\delta$ is the Dirac delta, and then find Einstein equations on the shell:

$$(\Gamma^n_{ij} - \gamma_{ij}\Gamma^n_{kl}\gamma^{kl})^+ - (...)^- = 8\pi\kappa\sigma_{ij} \ . \tag{1.28}$$

It is more convenient to write this in terms of the extrinsic curvature $K_{ij}$ of $\Sigma$. One then sees that the extrinsic curvature is allowed to have a jump proportional to the surface stress-energy tensor of the time-like shell $\sigma_{ij}$,

$$K_{\mu\nu} = \eta n_\alpha \Gamma^\alpha_{\mu\nu} \tag{1.29}$$

$$\left[K_i^{\ j}\right] - \delta_i^{\ j}\left[K_l^{\ l}\right] = \kappa\, \sigma_i^{\ j} \quad , \tag{1.30}$$

in which $n_\alpha$ is the unit vector normal to the shell and $[K_i^{\ j}] \equiv K_i^{\ j}|_+ - K_i^{\ j}|_-$ denotes the difference between extrinsic curvatures on the two sides of the shell. Note that we set $c = 1$ and $\kappa = 8\,\pi\, G_{\rm N}/3 = \ell_{\rm P}/M_{\rm P}$ (= 1 when convenient), where $G_{\rm N}$ is Newton's constant and $\ell_{\rm P}$ ($M_{\rm P}$) the Planck length (mass). Although the classical evolution equation (1.30) may appear simple, it has been solved only in a few special cases, most notably for the vacuum or with cosmological constants in $\Omega_\pm$ (for an extensive bibliography see Ref. [18]).

For our analysis, we will mostly follow the notation of Ref. [19], where the metric in each portion $\Omega_\pm$ of spacetime is given by

$$ds^2 = e^{\nu(r,t)}\, dt^2 - e^{\lambda(r,t)}\, dr^2 - R^2(r,t)\, d\Omega^2 \ , \tag{1.31}$$

where $t = t_\pm$ are the time coordinates inside the corresponding patches, and likewise for the three spatial coordinates. On the shell time-like surface $\Sigma$, one has the line element

$$ds^2\big|_\Sigma = d\tau^2 - \rho^2(\tau)\, d\Omega^2 \ , \tag{1.32}$$

in which $\tau$ is the proper time as measured by an observer at rest with the shell of areal radius $\rho = R_\pm(r^{\rm s}_\pm(t_\pm), t_\pm)$. The relation between $\tau$ and $t_\pm$ is obtained from the equation of continuity of the metric, Eq. (1.22), and is displayed below in Eq. (1.53) for the cases of interest. On solving Eq. (1.30) in terms of $\rho$ and $\dot\rho = {\rm d}\rho/{\rm d}\tau$, one gets the dynamical equation

$$\dot\rho^2(\tau) = B^2(\tau)\, \rho^2(\tau) - 1 \ , \tag{1.33}$$

where

$$B^2 = \frac{\left(\epsilon_+ + \epsilon_- + 9\,\kappa\,\sigma^2/4\right)^2 - 4\,\epsilon_-\,\epsilon_+}{9\,\sigma^2} \ , \tag{1.34}$$

with $\sigma = {\sigma^0}_0(\tau)$ the shell's surface density and $\epsilon_\pm = \epsilon_\pm(t_\pm)$ the time-dependent energy densities in $\Omega_\pm$ respectively. It is important to recall that metric junctions can involve different topologies for $\Omega_\pm$, but we are here considering only the so-called "black hole" type, in which both portions of spacetime have increasing area radii $R_\pm$ in the outward direction (of increasing $r_\pm$). Assuming the surface density of the shell is positive, one must then have [2]

$$\epsilon_+(\tau) - \epsilon_-(\tau) > \frac{9}{4}\,\kappa\,\sigma^2(\tau) \ , \tag{1.35}$$

at all times, in order to preserve the chosen spacetime topology [19, 20].

In the pure vacuum case, $\epsilon_\pm$ are constant and for constant $\sigma$ the solution is straightforwardly given by

$$\rho(\tau) = B^{-1}\,\cosh(B\,\tau) \ , \tag{1.36}$$

where $B = B(\epsilon_\pm, \sigma)$ from Eq. (1.34) is also constant [4].

In the non-vacuum cases, finding a solution is however significantly more involved. Regardless of the matter content of $\Omega_\pm$, it is nonetheless possible to derive a few general results for a bubble which nucleates at a time $\tau = \tau_0$, that is a shell that expands from rest,

$$\dot\rho_0 = 0 \ , \tag{1.37}$$

with initial finite (turning) radius ($\rho_0 > 0$), where the subscript 0 will always indicate quantities evaluated at the time $\tau = \tau_0$. First of all, from Eq. (1.33), the initial radius must be given by

$$\rho_0 = \left|B_0^{-1}\right| \ , \tag{1.38}$$

which requires $B_0$ real, or

$$\left(\epsilon_{0+} + \epsilon_{0-} + 9\,\kappa\,\sigma_0^2/4\right)^2 > 4\,\epsilon_{0-}\,\epsilon_{0+} \ . \tag{1.39}$$

This condition is always satisfied if $\epsilon_0^{\rm dust}$ and $\epsilon_0^{\rm rad}$ are both positive and will therefore be of no relevance in the present case, but must be carefully considered when allowing for negative energy densities (and non-vanishing spatial curvature). Further, upon deriving Eq. (1.33) with respect to $\tau$ (always denoted by a dot)

$$2\,\dot\rho\,\ddot\rho = 2\left(B\,\dot B\,\rho^2 + B^2\,\rho\,\dot\rho\right) \ , \tag{1.40}$$

[2] This also implies that $\epsilon_{0+} > \epsilon_{0-}$ and, from the Friedmann equation (1.44) given below, $H_+^2 > H_-^2$.

and using Eq. (1.37), one also obtains

$$\dot{B}_0 = 0 \ , \tag{1.41}$$

assuming $\ddot{\rho}_0$ is not singular. The constraint in Eq. (1.35) at $\tau = \tau_0$,

$$\epsilon_{0+} - \epsilon_{0-} > \frac{9}{4}\,\kappa\,\sigma_0^2 \ , \tag{1.42}$$

and the conditions in Eqs. (1.37) and (1.41) will play a crucial role in the following.

## 1.4 Flat FRW regions

The approach we developed could turn out to be suitable to describe the decay into radiation. We have focused our interest in presenting an analytical (perturbative in time) approach to study a time-like shell's dynamics and to obtain analytical conditions for the existence of expanding bubbles in terms of the energy densities inside and outside the shell, when such regions contain homogeneous dust or radiation. This problem is made technically cumbersome because of the occurrence of algebraic constraints (to ensure the arguments of proliferating square roots are positive). We shall therefore consider in details only the specific case of nucleation of a spatially flat radiation bubble inside (spatially flat) collapsing or expanding dust, in order to keep the presentation of our method more streamlined. Since we wish to study the particular case of a shell $\Sigma$ separating two regions $\Omega_\pm$ filled with homogeneous fluids, the metrics in $\Omega_\pm$ will be taken to be the usual FRW expressions. Moreover, we already assumed $\Omega_-$ has finite initial extension and, by definition, represents the interior of the shell. As a further simplification, we shall only consider flat spatial curvature and set the cosmological constant $\Lambda = 0$ everywhere. One can use such a model to approximate the formation of radiation from a decaying scalar field during reheating after inflation. It is known that for an inflaton with a quadratic potential, the time averaged dynamics of the final oscillation phase mimics that of matter. In the end, knowing the correct evolution of such a bubble would be of great help in understanding how defects formed during phase transitions are "ironed out" by the expansion of the new phase.

The metrics (1.31) in the inner and outer regions are therefore given by

$$ds^2 = dt^2 - a^2(t)\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] \ , \tag{1.43}$$

where $a(t)$ is the scale factor which evolves according to the Friedmann equations

$$H^2 = \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \kappa\,\epsilon \tag{1.44}$$

$$2\,\frac{1}{a}\,\frac{\mathrm{d}^2 a}{dt^2} + \left(\frac{1}{a}\frac{da}{dt}\right)^2 = -3\,\kappa\,p\ . \tag{1.45}$$

We assume the energy density $\epsilon$ and pressure $p$ of the fluids obey barotropic equations of state,

$$p = w\,\epsilon(t)\ , \tag{1.46}$$

and recover the well-known behaviors

$$\epsilon(t)\,\left(\frac{a(t)}{a_0}\right)^{3\,(w+1)} = \epsilon_0\ , \tag{1.47}$$

in which $\epsilon_0$ is the density evaluated at a reference instant of time $t = t_0$ and $a_0 = a(t_0)$. For dust, $w = 0$ ($p = 0$), whereas for radiation $w = 1/3$, so that

$$\begin{aligned} \epsilon^{\mathrm{dust}}(t) &= \frac{\epsilon_0\,a_0^3}{a^3(t)} \\ \epsilon^{\mathrm{rad}}(t) &= \frac{\epsilon_0\,a_0^4}{a^4(t)}\ . \end{aligned} \tag{1.48}$$

The evolution of scale factors in cosmic time for expanding ($\uparrow$) and contracting ($\downarrow$) solutions are finally given by

$$\begin{aligned} a^{\mathrm{rad}}_{\uparrow\downarrow}(t) &= \left(\gamma \pm 2\,\sqrt{M^{\mathrm{rad}}}\,t\right)^{1/2} \\ \frac{da^{\mathrm{rad}}_{\uparrow\downarrow}}{dt} &= \pm\frac{\sqrt{M^{\mathrm{rad}}}}{a^{\mathrm{rad}}(t)} \end{aligned} \tag{1.49}$$

and

$$\begin{aligned} a^{\mathrm{dust}}_{\uparrow\downarrow}(t) &= \left(\delta \pm \tfrac{3}{2}\,\sqrt{M^{\mathrm{dust}}}\,t\right)^{2/3} \\ \frac{da^{\mathrm{dust}}_{\uparrow\downarrow}}{dt} &= \pm\,\sqrt{\frac{M^{\mathrm{dust}}}{a^{\mathrm{dust}}(t)}}\ , \end{aligned} \tag{1.50}$$

where, in the above r.h.s., the $+$ signs are for expansion and $-$ signs for contraction,

$$\begin{aligned} M^{\rm rad} &= \kappa\, a_0^4\, \epsilon_0^{\rm rad} \\ M^{\rm dust} &= \kappa\, a_0^3\, \epsilon_0^{\rm dust}\ , \end{aligned} \tag{1.51}$$

and $\gamma$ and $\delta$ integration constants that determine the size of the scale factors at $t = 0$. Later, for convenience, we will set $\gamma = \delta = 1$ at $t = 0$, so that $\epsilon(0) = \epsilon_0$ and $a(0) = a_0 = 1$.

Let us now consider the time-like shell $\Sigma$ at $r_\pm = r^{\rm s}_\pm(t_\pm)$ separating the two regions $\Omega_\pm$. Clearly, metric continuity implies

$$\rho = a_\pm(t_\pm)\, r^{\rm s}_\pm(t_\pm)\ . \tag{1.52}$$

The inner and outer spaces are characterized by different physical parameters. In particular, as one can see from Eq. (1.34), the shell's dynamics are determined by:

1. The type of fluid inside the shell (its equation of state $w_-$);
2. The initial values $a_{0-}$ and $\epsilon_{0-}$;
3. The type of fluid outside the shell (its equation of state $w_+$);
4. The initial values $a_{0+}$ and $\epsilon_{0+}$;
5. The shell surface density $\sigma$ (as a function of the radius $\rho$).

A given configuration of dust, radiation and surface density is admissible only if the corresponding initial conditions are such that Eqs. (1.37), (1.41) and (1.42) are satisfied.

## 1.5 Time transformations and expansion

The densities $\epsilon_\pm$ in Eq. (1.48) are given in terms of coordinate times $t_\pm$. However, it is the time $\tau$ measured by an observer on the shell which appears in the evolution equation (1.33). Hence we need to find the transformation from $t_\pm$ to $\tau$. Following Ref [19], we recall that metric continuity (1.22) implies [3]

$$\left.\frac{dt_\pm}{d\tau}\right|_\Sigma = \left\{ \frac{H\,\rho\,\dot\rho}{\Delta} \left[ 1 \pm \sqrt{1 + \frac{\Delta^2 - \Delta\,(\dot\rho^2 + H^2\rho^2)}{(H\,\rho\,\dot\rho)^2}} \right] \right\}_\pm , \tag{1.53}$$

[3] There is a typo in Eq. (B10) of Ref [19]: $\rho^2$ is missing in the last term in the square root.

in which $H$ is again the Hubble "constant", $\Delta = \kappa\,\epsilon\,\rho^2 - 1$, and the expression within braces must be estimated on the two sides of the shell [4]. Now, the above two equations should be solved along with Eq. (1.33), which makes it clear why it is impossible to obtain general analytic solutions.

An important result can be obtained by considering the time when the bubble is at rest, that is $t_\pm = t_{0\pm}$ and $\tau = \tau_0$, with $\dot\rho(\tau_0) = 0$ and $H_0 \equiv H(t_0)$, namely

$$\left.\frac{dt_\pm}{d\tau}\right|_{\Sigma,0} = \pm\sqrt{1 - \frac{H_{0\pm}^2\,\rho_0^2}{\Delta_{0\pm}}}\ . \tag{1.54}$$

From Eq. (1.44) we see that $\Delta = H^2\rho^2 - 1$, therefore

$$\begin{aligned}\left.\frac{dt_\pm}{d\tau}\right|_{\Sigma,0} &= \frac{\pm 1}{\sqrt{1 - H_{0\pm}^2\,\rho_0^2}} = \frac{\pm 1}{\sqrt{1 - \kappa\,\epsilon_{0\pm}\,\rho_0^2}} \\ &= \frac{\pm 1}{\sqrt{-\Delta_{0\pm}}}\ ,\end{aligned} \tag{1.55}$$

with the signs in the numerator simply reflecting the directions $t_\pm$ flow relative to $\tau$. It is clear that real solutions to Eq. (1.55) exist only if

$$\Delta_{0\pm} < 0\ . \tag{1.56}$$

Remarkably, this is the same as stating that the energy density inside the radius $\rho = \rho_0$ must not generate a black hole, as one can easily check by considering the Schwarzschild radius $r_{\rm S} = 2\,G_{\rm N}\,M$ with $M = (4\,\pi/3)\,\epsilon_0\,\rho_0^3$. This is manifest when considering $\epsilon_{0-}$ inside the bubble, but it must also hold for the energy density $\epsilon_{0+}$ outside the bubble. For the outer region, this means the bubble must lie inside the Hubble radius, $\rho_0 < H_0^{-1}$. Putting together these conditions tells us that the temporal coordinates are properly transformed only within a causal region of the spacetime.

In order to study how the bubble grows after nucleation, we can expand $t = t(\tau)$ for short times about $t_{0\pm}$ and $\tau_0$. Further, we want all times directed the same way, so we choose the + sign in the above expression and obtain, to linear order,

$$t_\pm \simeq t_{0\pm} + \frac{\tau - \tau_0}{\sqrt{-\Delta_{0\pm}}}\ , \tag{1.57}$$

where $t_{0\pm}$ are integration constants.

[4] Note the sign ambiguity $\pm$ in front of the square root just reflects the double root of a second degree equation and is *not* associated with the interior or exterior regions.

Unfortunately, a first order expansion is not sufficient to study the evolution of the bubble radius. Since $\dot{\rho}_0 = 0$, we need at least second order terms in $\tau$ to get significant results, which makes all expressions very cumbersome. We shall therefore just consider a few specific cases, generalizing the exact result (1.36) for a shell of constant surface energy in vacuum. For such cases, our perturbative approach will yield exact conditions for the bubble's existence, which we see as a clear advantage with respect to purely numerical solutions. Other advantages would be that having analytical expressions is a necessary ingredient for quantum mechanical (or semiclassical) studies of these systems. Moreover, adapting our procedure to all possible combinations of fluids in $\Omega_\pm$, and for more general shell surface density, should be rather straightforward.

## 1.6 Radiation bubble inside dust

The main idea in our approach stems from the observation that the (three) fundamental (first order differential) equations (1.33) and (1.53) contain six functions of the proper time $\tau$: the shell radius $\rho$, its surface density $\sigma$, the two times $t_\pm$ and the two Hubble functions $H_\pm$. Once we choose the matter content inside $\Omega_\pm$ and on the shell, the Hubble functions and surface density are uniquely fixed and we are left with the three unknowns $\rho$ and $t_\pm$ (and a set of constraints for the initial conditions). To determine these unknowns, we find it convenient to formally expand the shell radius $\rho$ and Hubble functions $H_\pm$ for short (proper) time "after the nucleation of the bubble" (when $\dot{\rho}_0 = 0$), and solve Eqs. (1.33) and (1.53) order by order.

Since expressions rapidly become involved, and a general treatment for all combinations of matter content in $\Omega_\pm$ and shell surface density would be hardly readable, we shall only consider the specific case of a radiation bubble ($w_- = 1/3$) inside a region filled with dust ($w_+ = 0$). We further assume the shell's surface density

$$\sigma(\tau) = \sigma_0 > 0 \tag{1.58}$$

is constant and positive. Since one would expect the shell's density decreases as the shell's surface grows, this assumption might appear rather strong. However, it is the simplest way to ensure the junction remains of the "black hole" type, and a more thorough discussion of this point can be found in Ref. [19]. In order to keep the presentation uncluttered, we also set $\kappa = 1$ from now on and regard all quantities as dimensionless (tantamount to assuming they are rescaled by suitable powers of $\kappa = \ell_\text{P}/M_\text{P}$). This means that densities will be measured in Planck units, that is $\epsilon = 1$ corresponds to the Planck density $\epsilon_{\rm P} = M_{\rm P}/\ell_{\rm P}^3 = \ell_{\rm P}^{-2}$ and $\sigma = 1$ to $\sigma_{\rm P} = M_{\rm P}/\ell_{\rm P}^2 = \ell_{\rm P}^{-1}$. Likewise, $\rho = 1$ is the Planck length $\ell_{\rm P}$. We also express the shell surface density and radiation energy density as fractions of $\epsilon_0^{\rm dust} > 0$,

$$\epsilon_0^{\rm rad} = \epsilon_0^{\rm dust}\, x \ , \qquad \sigma_0 = \sqrt{\epsilon_0^{\rm dust}}\, y \ , \tag{1.59}$$

with $0 \le x \le 1$ and $y \ge 0$.

It is natural to choose $\tau_0 = 0$, and then proceed to analyze Eqs. (1.33) and (1.53) by formally expanding all relevant time-dependent functions in powers of $\tau - \tau_0 = \tau$:

**Step 1)** since $\dot\rho_0 = 0$ [see Eq. (1.37)], we can formally write the bubble radius as

$$\rho(\tau) = \rho_0 + \frac{1}{2}\,\ddot\rho_0\,\tau^2 + \mathcal{O}(\tau^3) \ , \tag{1.60}$$

where $\rho_0$ and $\ddot\rho_0$ are parameters to be determined. In particular, from Eqs. (1.34) and (1.38), we obtain the (not yet final) expression

$$\begin{aligned} \rho_0 &= \frac{3\,\sigma_0}{\sqrt{\left(\epsilon_{0+} + \epsilon_{0-} + 9\,\kappa\,\sigma_0^2/4\right)^2 - 4\,\epsilon_{0-}\,\epsilon_{0+}}} \\ &= \frac{3\,(\epsilon_0^{\rm dust})^{-1/2}\,y}{\sqrt{\left(1 + x + 9\,y^2/4\right)^2 - 4\,x}} \ , \end{aligned} \tag{1.61}$$

which only depends on $\epsilon_{0\pm}$ and $\sigma_0$. More precisely, $\epsilon_0^{\rm dust}$ sets the overall scale of the shell radius and the fractions $x$ and $y$ defined in Eq. (1.59) the detailed form.

We next obtain $t_\pm = t_\pm(\tau)$ by solving Eq. (1.53). However, for this purpose we need the Hubble parameters $H_\pm$ as functions of $\tau$, whereas they explicitly depend on $t_\pm$:

**Step 2)** we replace $H$ in Eq. (1.53) with the formal expansion

$$\mathcal{H} = \mathcal{H}_0 + \dot{\mathcal{H}}_0\,\tau + \mathcal{O}(\tau^2) \ , \tag{1.62}$$

where $\mathcal{H}_0$ and $\dot{\mathcal{H}}_0$ are unknown constant quantities to be determined by consistency. By expanding the right hand side of Eq. (1.53) to first order in $\tau$ and then integrating, we obtain $t$ to second order in $\tau$,

$$t_\pm \simeq t_{0\pm} + \frac{\tau}{\sqrt{-\Delta_{0\pm}}} + \frac{\rho_0\,\mathcal{H}_{0\pm}}{2\,\Delta_{0\pm}} \left(\ddot\rho_0 - \frac{\rho_0\,\dot{\mathcal{H}}_{0\pm}}{\sqrt{-\Delta_{0\pm}}}\right)\tau^2, \tag{1.63}$$

where $\rho_0$ must now be understood as the expression given in Eq. (1.61) and $t_{0\pm}$ are integration constants we can set to zero without loss of generality. In fact, let us assume we are at rest in an expanding (or contracting) universe, corresponding to the old exterior phase (with parameters $\epsilon_{0+}$ and $H_{0+}$), and measure a time $t'_+$ from the "Big Bang" (or beginning of collapse) of this exterior universe $[a_+(t'_+ = 0) = 0$ or $a_+(t'_+ = 0) > a(t'_+)$ for $t'_+ > 0$, respectively]. If, for instance, a new phase bubble arises at rest at the instant $t'_+ = t'_{0+}$, we can define $t_+ = t'_+ - t'_{0+}$, so that the bubble is created at $t_+ = 0$, and also set $t_{0-} = 0$, since an "inner time" is meaningless before any "inner part" exists. For different pictures, similar arguments can likewise be formulated.

**Step 3)** From Eq. (1.49) and (1.50), we choose an expanding radiation interior and contracting or expanding dust exterior,

$$a_- = a_\uparrow^{\rm rad}(t_-) \ , \qquad a_+ = a_{\downarrow\uparrow}^{\rm dust}(t_+) \ , \tag{1.64}$$

set $a_{0\pm} = 1$ and express $t_\pm$ according to Eq. (1.63). In so doing, $a_\pm$ and $da_\pm/dt_\pm$ become explicit functions of $\tau$ containing $\rho_0$, $\mathcal{H}_{0\pm}$ and $\dot{\mathcal{H}}_{0\pm}$. For consistency with Eq. (1.62), we must therefore require

$$H(\tau) = \frac{1}{a}\frac{da}{dt} = \mathcal{H}(\tau) \ , \tag{1.65}$$

with $H_- = \mathcal{H}_- > 0$ and $H_+ = \mathcal{H}_+ < 0$ for collapsing dust, or $H_+ = \mathcal{H}_+ > 0$ for expanding dust.

**Step 4)** To zero order in $\tau$, Eq. (1.65) gives rise to a first-order equation for $\mathcal{H}_0$,

$$H_0 = \frac{1}{a_0}\left.\frac{da}{dt}\right|_{\tau=0} = \left.\frac{da}{dt}\right|_{\tau=0} = \mathcal{H}_0 \ . \tag{1.66}$$

The solutions are uniquely given by

$$\mathcal{H}_0^{\uparrow\downarrow} = \pm\sqrt{\epsilon_0} \ , \tag{1.67}$$

in which the $\uparrow$ and $+$ sign (respectively $\downarrow$ and $-$ sign) refer to expanding (contracting) solutions, i.e. solutions with increasing (decreasing) scale factor, as before [5]. Note this result also follows directly from the Friedmann equation (1.44) for $\tau = t = 0$.

To first order in $\tau$, one analogously obtains

$$\dot{\mathcal{H}}_0^{\uparrow\downarrow} = -\frac{n\,\epsilon_0}{\sqrt{1-\rho_0^2\,\epsilon_0}} \ , \tag{1.68}$$

[5]Note that, for example, the Hubble parameter for the expanding interior phase will carry a second subscript sign and will then be denoted as $\mathcal{H}_-^\uparrow$, where the subscript $-$ indicates the inner region and the apex $\uparrow$ stands for expansion.

with $n = 2$ for radiation and $n = 3/2$ for dust, and $\rho_0$ must again be understood as the expression given in Eq. (1.61).

**Step 5)** Replace $\rho_0$ from Eq. (1.61) and the chosen combination of Hubble parameters (1.67) inside $\dot{B}_0$, which must then satisfy Eq. (1.41). This equation will only contain $\epsilon_{0-} = \epsilon_0^{\rm rad}$, $\epsilon_{0+} = \epsilon_0^{\rm dust}$ and $\sigma_0$, so that it can be used to determine $\sigma_0 = \sigma_0(\epsilon_{0-}, \epsilon_{0+})$. In particular, introducing the fractions in Eq. (1.59), we obtain

$$\dot{B}_0 = \epsilon_0^{\rm dust}\, \dot{b}_0(x, y) \ , \tag{1.69}$$

from which it appears that the dust energy density just sets the overall scale like in Eq. (1.61). For any given values of $\epsilon_0^{\rm dust}$, the shell surface density is instead determined by the radiation energy density according to

$$\dot{b}_0(x, y) = 0 \ , \tag{1.70}$$

which, for the cases of interest, is a fourth-order algebraic equation for $y$. Analytic solutions can be found (in suitable ranges of $x$), which we denote as $\bar{y} = \bar{y}(x)$, so that the allowed surface densities are given by

$$\bar{\sigma}_0 = \sqrt{\epsilon_0^{\rm dust}}\, \bar{y} \ . \tag{1.71}$$

**Step 6)** Replace the above surface density $\bar{\sigma}_0$ into the initial radius (1.61) and obtain its final expression,

$$\bar{\rho}_0 = \frac{3\,(\epsilon_0^{\rm dust})^{-1/2}\, \bar{y}}{\sqrt{(1 + x + 9\,\bar{y}^2/4)^2 - 4\,x}} \ , \tag{1.72}$$

which can then be used to determine the final forms of $\dot{\mathcal{H}}_{0\pm}$ and the scale factors $a_\pm$ to first order in $\tau$.

**Step 7)** One must now check that $\bar{\sigma}_0$ and $\bar{\rho}_0$ satisfy all of the initial constraints and lead to valid time transformations (1.53), at least for some values of $\epsilon_0^{\rm rad}$ and $\epsilon_0^{\rm dust}$. If not all of these conditions are met, one must conclude the corresponding physical system may not exist. Moreover, we note the condition (1.56) requires $\epsilon_0 \lesssim \epsilon_{\rm P}$ in order to have a (semi)classical bubble with $\rho_0 \gtrsim \ell_{\rm P}$. In the following Section, one should therefore consider only dust and radiation energy densities $\epsilon_0 \ll \epsilon_{\rm P}$ and look at the limiting case $\epsilon_0 \simeq \epsilon_{\rm P}$ as a glimpse into the quantum gravity regime.

If a consistent solution for $\bar{\sigma}_0$ and $\bar{\rho}_0$ exists, one can proceed to determine higher orders terms (in $\tau$). However, due to the increasing degree of complexity of the resulting expressions, we shall not go any further here. We instead present our findings for the two cases of interest separately.

### 1.6.1 Collapsing dust

This case is defined by choosing the scale factors

$$a_- = a_\uparrow^{\rm rad}(t_-) \ , \qquad a_+ = a_\downarrow^{\rm dust}(t_+) \ , \tag{1.73}$$

and proceeding as described above. We can then prove that this case does not admit solutions, in general, by simply analyzing the constraint (1.70),

$$\begin{aligned} \dot{b}_0 \ \propto \ & 16\,x^2\big(3+4\,x^{3/2}\big)+4\,x^{3/2}\big(4-9\,y^2\big)\big(4-8\,x-9\,y^2\big) \\ & +3\left(4+9\,y^2\right)\left(4-8\,x+9\,y^2\right)=0 \ , \end{aligned} \tag{1.74}$$

which admits the four solutions

$$\bar{y}_{\pm\pm} = \pm\frac{2}{3}\sqrt{(1-x)\,\frac{2\,x^{3/4}\pm i\,\sqrt{3}}{2\,x^{3/4}\mp i\,\sqrt{3}}} \ . \tag{1.75}$$

However, for $x \neq 1$, all the $\bar{y}_{\pm\pm}$ are complex and a complex surface density is obviously unphysical. One is then apparently left with the only trivial case $x = 1$, corresponding to $\bar{y} = 0$ and

$$\bar{\rho}_0 = (\epsilon_0^{\rm dust})^{-1/2}\sqrt{\frac{3+4\,x^{3/2}}{3+4\,x^{1/2}}} = 1 \ . \tag{1.76}$$

This case however does not satisfy all the required constraints. For example, Eq. (1.42) for $\sigma_0 = 0$ yields

$$\epsilon_0^{\rm dust} > \epsilon_0^{\rm rad} = x\,\epsilon_0^{\rm dust} \ , \tag{1.77}$$

which clearly contradicts $x = 1$. Correspondingly, the time transformations (1.53) are not well-defined, because $\dot{t}_{0+} = \dot{t}_+(\tau = 0; x)$ is complex for $0 < x < 1$ and both $\dot{t}_{0\pm}$ diverge for $x \to 1$.

The overall conclusion is thus that expanding radiation bubbles with a turning point of minimum radius cannot be matched with a collapsing dust exterior.

### 1.6.2 Expanding dust

This case is defined by choosing the scale factors

$$a_- = a_\uparrow^{\rm rad}(t_-) \ , \qquad a_+ = a_\uparrow^{\rm dust}(t_+) \ . \tag{1.78}$$

The crucial task is again to solve the constraint in Eq. (1.70), namely

$$\begin{aligned} \dot{b}_0 \ \propto \ & 16\,x^2\big(3-4\,x^{3/2}\big)-4\,x^{3/2}\big(4-9\,y^2\big)\big(4-8\,x-9\,y^2\big) \\ & -3\left(4+9\,y^2\right)\left(4-8\,x+9\,y^2\right)=0 \ , \end{aligned} \tag{1.79}$$

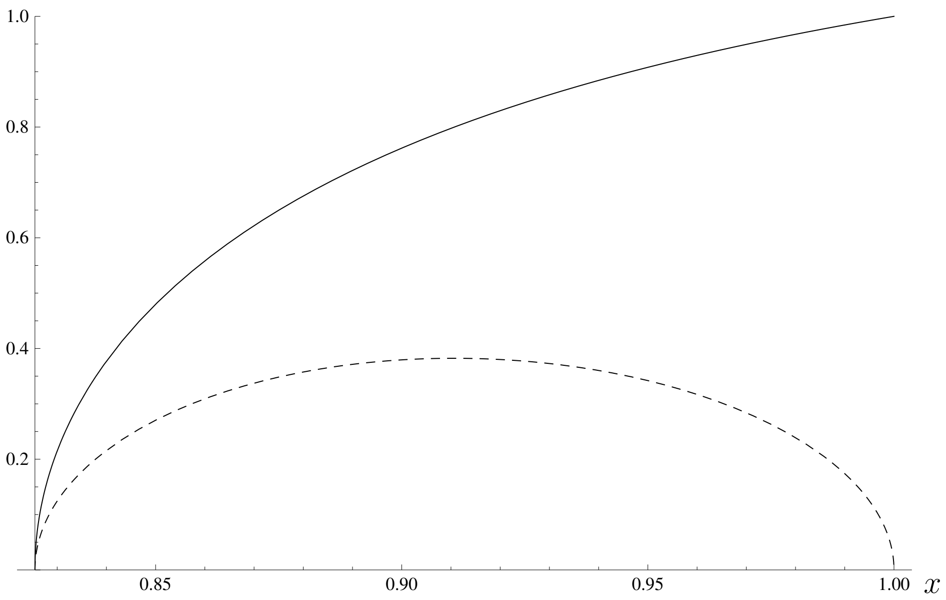


Figure 1.2: Plot of $\bar{\sigma}_0/\sqrt{\epsilon_0^{\rm dust}} = \bar{y}_{+-}(x)$ (magnified by a factor of 10 for convenience, dashed line) and corresponding $\bar{\rho}_0/(\epsilon_0^{\rm dust})^{-1/2}$ (solid line) in the range (1.81).

admitting the four solutions

$$\bar{y}_{\pm\pm} = \pm\frac{2}{3}\sqrt{(1-x)\,\frac{2\,x^{3/4}\pm\sqrt{3}}{2\,x^{3/4}\mp\sqrt{3}}}\ , \tag{1.80}$$

which are real for

$$x_{\rm min} = \left(\frac{3}{4}\right)^{2/3} < x < 1\ . \tag{1.81}$$

We discard the negative solutions $\bar{y}_{-\pm}$ associated to negative surface densities and just analyze the positive cases $\bar{y}_{+\pm}$. It is then easy to see that $\bar{y}_{++}$ leads to a surface density that diverges for $x \to x_{\rm min}$, and is always too large to satisfy the condition (1.42), since

$$1 - x - \frac{9}{4}\,\bar{y}_{++}^2 = \frac{\sqrt{3}\,(x-1)}{2\,x^{3/4}-\sqrt{3}} < 0\ , \tag{1.82}$$

in the range (1.81). In the limit for $x \to 1$, $\bar{y}_{++} \to 0$, however, Eq. (1.42) is still violated in the strict sense and one can in fact show that $\dot{t}_{0+}$ diverges.

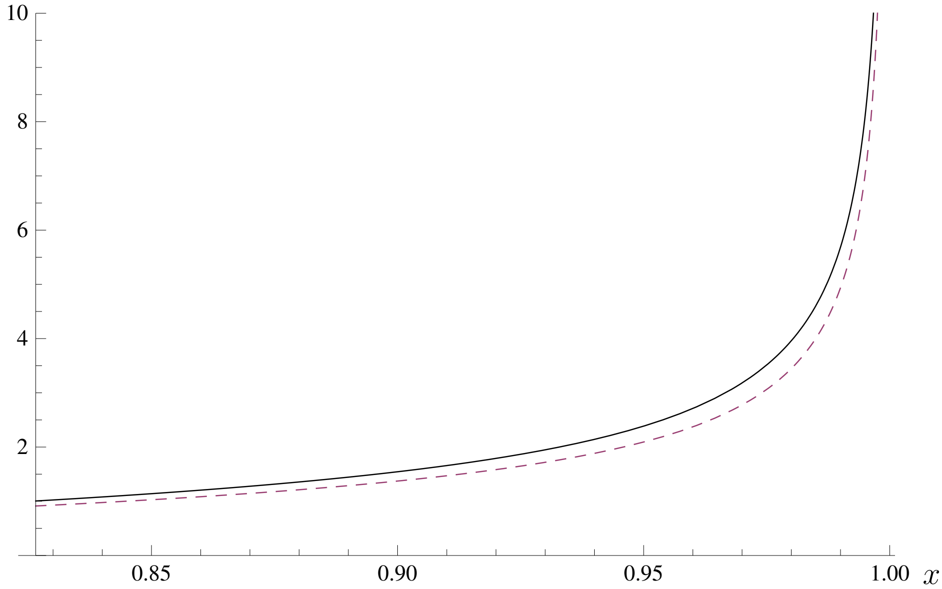


Figure 1.3: Plot of $\dot{t}_{0+}$ (solid line) and $\dot{t}_{0-}$ (dashed line) for $y = \bar{y}_{+-}(x)$ in the range (1.81).

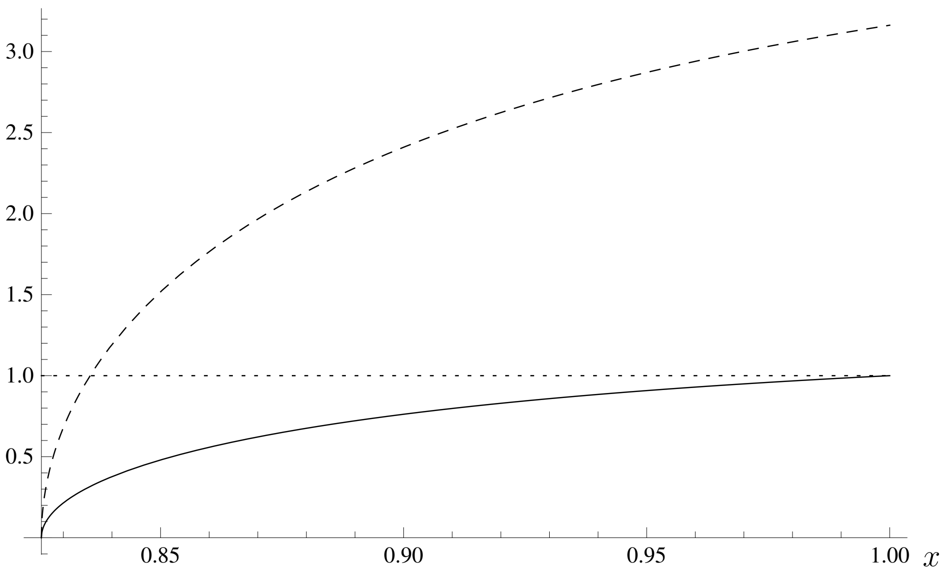


Figure 1.4: Plot of $\bar{\rho}_0$ for $y = \bar{y}_{+-}(x)$ with $\epsilon_0^{\rm dust} = \epsilon_{\rm P}/10$ (dashed line) and $\epsilon_0^{\rm dust} = \epsilon_{\rm P}$ (solid line) in the range (1.81). Only values above $\ell_{\rm P} = 1$ represent acceptable semiclassical radii.

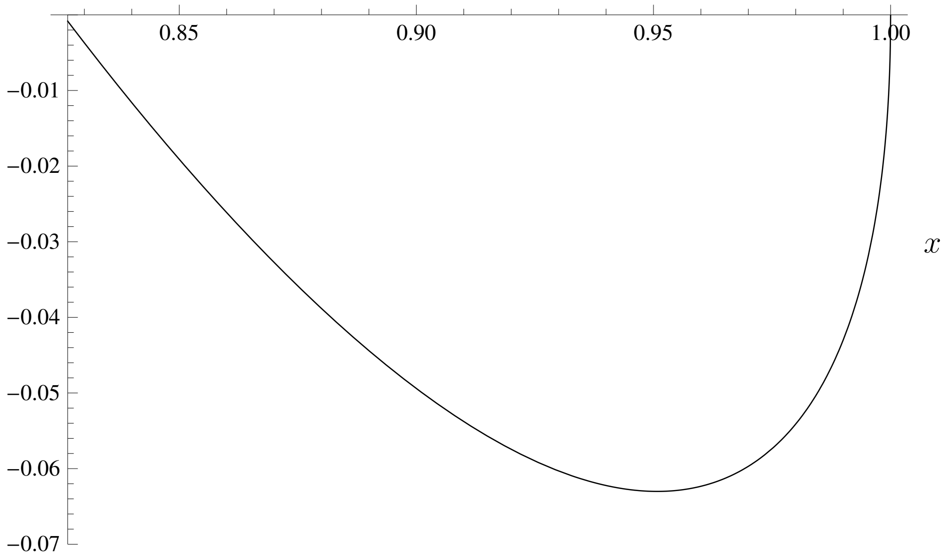


Figure 1.5: Plot of $\bar{C}_0/\bar{M}_0^{\rm dust}$ for $\bar{\sigma}_0 > 0$ and $y = \bar{y}_{+-}(x)$ in the range (1.81).

The only solution which appears consistent is therefore

$$\begin{aligned}\bar{\sigma}_0 &= \sqrt{\epsilon_0^{\rm dust}}\,\bar{y}_{+-} \\ &= \frac{2}{3}\sqrt{\epsilon_0^{\rm dust}}\sqrt{(1-x)\,\frac{2\,x^{3/4}-\sqrt{3}}{2\,x^{3/4}+\sqrt{3}}}\ , \end{aligned} \tag{1.83}$$

with $x$ again in the range (1.81). This expression yields a vanishing surface density for the limiting values $x \to 1$ and $x \to x_{\rm min}$ (see Fig. 1.2) and further satisfies the condition (1.42),

$$1 - x - \frac{9}{4}\,\bar{y}_{+-}^2 = \frac{\sqrt{3}\,(1-x)}{2\,x^{3/4}+\sqrt{3}} > 0\ . \tag{1.84}$$

The corresponding initial bubble radius is an increasing function of $x$ (see Fig.1.2),

$$\bar{\rho}_0 = (\epsilon_0^{\rm dust})^{-1/2}\sqrt{\frac{4\,x^{3/2}-3}{x\,(4\,x^{1/2}-3)}} < (\epsilon_0^{\rm dust})^{-1/2}\ , \tag{1.85}$$

with $\bar{\rho}_0(x \to 1) = (\epsilon_0^{\rm dust})^{-1/2}$. Further, the products

$$\epsilon_0^{\rm dust}\,\bar{\rho}_0^2 < 1 \quad \text{and} \quad \epsilon_0^{\rm rad}\,\bar{\rho}_0^2 < 1\ , \tag{1.86}$$

for $x_{\rm min} < x < 1$, as required by the condition (1.56). In fact the initial time derivatives $\dot{t}_{0\pm}$ are well defined in this range (see Fig. 1.3) and only diverge for $x \to 1$. Note the above initial radius can be larger than $\ell_{\rm P}$ only if $\epsilon_0^{\rm dust} < \epsilon_{\rm P}$ and for sufficiently large $x$, since $\bar{\rho}_0 \to 0$ for $x \to x_{\rm min}$ (see, for example, Fig. 1.4).

Finally, let us check if one can use the process of bubble nucleation to describe a phase transition from dust to radiation for the matter inside the sphere of radius $\bar{\rho}_0$, accompanied by the creation of a layer of non-vanishing surface density $\bar{\sigma}_0$. From Eqs. (1.83) and (1.85), one has

$$\bar{C}_0 \equiv \bar{M}_0^{\rm dust} - \bar{M}_0^{\rm rad} - \bar{E}_0^{\Sigma} < 0 \ , \tag{1.87}$$

which means the dust energy inside the sphere of radius $\bar{\rho}_0$ at time of bubble formation, $\bar{M}_0^{\rm dust} = (4\,\pi/3)\,\bar{\rho}_0^3\,\epsilon_0^{\rm dust}$, is not sufficient to produce the radiation energy $\bar{M}_0^{\rm rad} = (4\,\pi/3)\,\bar{\rho}_0^3\,\epsilon_0^{\rm rad}$ and surface energy $\bar{E}_0^{\Sigma} = 4\,\pi\,\bar{\sigma}_0\,\rho_0^2$. An extra source is thus needed to provide the energy $-\bar{C}_0 > 0$. The reverse process of collapsing radiation reaching a minimum size $\rho = \bar{\rho}_0$ and then turning into collapsing dust would instead be energetically favored, with the amount of energy $-\bar{C}_0$ now being released. Of course, in order to support this kind of argument, the extra contribution should be a small perturbation on the given background,

$$|\bar{C}_0| \ll \bar{M}_0^{\rm dust} \ , \tag{1.88}$$

since it was not included in the dynamical equations. From Fig. (1.5), we expect this is indeed a very good approximation since $0 < -\bar{C}_0 \lesssim 0.06\,\bar{M}_0^{\rm dust}$.

## 1.7 Conclusions and outlook

We have analyzed bubbles of radiation whose timelike surface starts to expand inside collapsing or expanding dust with vanishing initial rate, and with the further (simplifying) assumptions that the bubble's surface density is constant and positive, and both interior and exterior are spatially flat. These bubbles generalize the simplest self-gravitating case of a shell with constant surface density expanding in vacuum, for which the exact trajectories are known [17, 18]. These generalizations are of potential interest both for the physics of the early universe and the description of astrophysical processes. However, although the general formalism was already developed a long time ago [17], and the dynamics are ruled by apparently simple equations [19], finding explicit solutions is not straightforward.

By developing an approach to obtain analytical expressions for the evolution of the bubble radius in the shell's proper time, $\rho = \rho(\tau)$ with $\dot{\rho}(\tau =$

0) = 0, we determined the conditions which allow for the existence of such configurations. Although our approach is perturbative (with an expansion for short times after nucleation), the conditions for the bubble's existence are exact, which is a clear advantage with respect to purely numerical solutions.

We then found that:

- *expanding radiation bubbles of constant surface density may not be matched to a collapsing dust exterior*. More precisely, we found that inside collapsing dust there may not be a bubble of radiation whose surface ever reaches vanishing speed of expansion at finite radius.

- Bubbles whose radius admits a turning point are instead *allowed inside an expanding dust-dominated universe.*

In this latter case, bubbles can further be used to model a phase transition from radiation to dust if an external source provides part of the the energy required to build the shell, or the converse process with release of energy (albeit, of an amount small enough to leave the background configuration unaffected).

Let us clarify this point about energy conservation. The fundamental Eqs. (1.33) and (1.53) are just a different form of the junction equations (1.22) and (1.30) which, in turn, follow from the Einstein equations. Conservation of the energy-momentum in a given system is therefore guaranteed. However, when we use bubbles to model a phase transition, we are considering the possibility that a region of space filled with dust be replaced by radiation enclosed inside an expanding shell, or the reverse process. Technically, we are therefore considering two different systems: one with dust and one with a bubble of radiation within a shell of positive surface density whose radius evolves along a trajectory with a turning point (zero speed at finite minimum radius). The total energy in the two configurations differ by the amount $\bar{C}_0$ defined in Eq. (1.87), and a (quantum) transition between them would therefore violate energy conservation and be suppressed in the semiclassical regime. By looking at Fig. 1.5, we however see that $|\bar{C}_0| \ll \bar{M}_0^{\rm dust}$. One may thus argue the unspecified matter contribution carrying the energy $|\bar{C}_0|$ should be well approximated as a perturbation with respect to the dust and radiation, with its backreaction on the chosen configuration consistently negligible. If so, one can further speculate if the extra energy required to nucleate radiation could be provided by pressure in the initial cloud or by the decay of a region of false vacuum (with vacuum energy or cosmological constant $\Lambda_+$) to true vacuum (with cosmological constant $\Lambda_- < \Lambda_+$), like in the seminal Refs. [13].

The fact that no consistent solution was found inside collapsing dust does not mean that expanding radiation bubbles may not be produced at all in this context, which includes, for example, the collapsing core of a supernova or other astrophysical processes leading to black hole formation. In fact, the situation might change if one, for instance (and more realistically), includes matter pressure or a radius-dependent surface density, $\sigma = \sigma(\rho)$. This observation thus brings us to briefly comment on the possible generalizations and extensions of the present work, which include the just mentioned non-constant $\sigma$, as well as different combinations of matter inside and outside the shell, and cosmological constant(s) $\Lambda_\pm$. Moreover, one might like to consider the vacuum inside the shell and radiation outside (with or without $\Lambda_\pm$) and apply the corresponding results to the thick shell model previously studied in Refs. [21, 22, 23, 24].

Finally, our analysis is entirely based on classical General Relativity and no attempt was made to compute the quantum mechanical "tunneling" probability for radiation bubbles to come into existence (or convert to dust). Such an analysis requires the (effective Euclidean) action to be integrated along the (classically forbidden) trajectory for the bubble radius $\rho$ to go from 0 to $\rho_0$ [25], whose construction is clearly no easy task, given the classical trajectories are so difficult to determine. Nonetheless, another advantage of our approach is that it provides analytical (albeit perturbative) expressions, which is a property one needs for any quantum mechanical (or semiclassical) studies of these systems. Of course, energy densities above the Planck scale would not be meaningful in this context, since one then has no guarantee the dynamical equations derived from General Relativity can be trusted at all.

# Chapter 2

# The Kodama vector field

Here is presented another original research developed jointly with a group at Trento University [26]. In very recent years some attention has been paid to the Kodama vector field [27] as a tool for solving dynamical problems in general relativity. We decided to investigate the physical meaning of such a vector field on the slippery ground of radiating horizons, by studying Unruh radiation as seen by a finite size particle detector accelerating along a Kodama trajectory in a de Sitter universe.

In Section 2.1 we resume the most important notions about Hawking radiation, underlining the role that Killing vectors play.

In Section 2.2 we introduce the Kodama vector field and discuss its geometrical properties.

In Section 2.3 the Unruh effect and the particle concept are explained.

In Section 2.4 the original content is presented: the amount and peculiarities of Unruh radiation as measured by a finite size, two-states detector moving along a Kodama trajectory.

## 2.1 Particle production in curved spacetime

Since Hawking's first papers on evaporating black holes [28, 29] and subsequent discovery by Bekenstein that black holes have an associated entropy [30], physicists devoted a lot of attention to black hole thermodynamics. Black hole thermodynamics is the perfect ground for studying a semiclassical approximation of quantum gravity, whereas a full unified theory is far from being achieved. Starting from black holes, investigation has broadened to include every kind of horizon allowed by general relativity.

We will now show the role played by Killing vectors in the black hole evaporation process by considering a Klein-Gordon field in a generic spacetime, obeying

$$\Box\Phi + m^2\Phi = 0 \ , \tag{2.1}$$

whose (classical) solutions are plane waves of the kind $f_\omega \sim e^{-ik_\mu x^\mu} \sim e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})}$. The frequency $\omega$ must satisfy the dispersion relation

$$\omega^2 = \mathbf{k}^2 + m^2 \tag{2.2}$$

and solutions will therefore be parametrized by $\mathbf{k}$ and the sign of $\omega$. To give the most general solution one must build a complete, orthonormal set of modes by introducing a suitable inner product $(\Phi_1, \Phi_2)$. One general way to define a pseudo-inner product is to use the *particle number current* $j^\mu[\Phi_1, \Phi_2]$:

$$\begin{aligned} j^\mu[\Phi_1, \Phi_2] &= \Phi_1 \nabla^\mu \Phi_2^* - \Phi_2^* \nabla^\mu \Phi_1 && (2.3)\\ (\Phi_1, \Phi_2) &= -i \int_\Sigma d^3x \sqrt{h}\ n^\mu j_\mu[\Phi_1, \Phi_2] && (2.4)\\ &= -i \int_\Sigma d^3x \sqrt{h}\ n^\mu \left(\Phi_1 \nabla_\mu \Phi_2^* - \Phi_2^* \nabla_\mu \Phi_1\right) \ , && (2.5) \end{aligned}$$

where the integration is independent of the spacelike hypersurface $\Sigma$ with induced metric $h$ and unit normal vector $n^\mu$ [31, 32]. We call this a *pseudo*-inner product since it is not positive definite. As an example, by multiplying two plane waves with different frequencies $\omega_1$ and $\omega_2$, and different wave vectors $\mathbf{k_1}$ and $\mathbf{k_2}$, one gets *in flat spacetime*

$$(e^{-ik_1^\mu x_\mu}, e^{ik_1^\mu x_\mu}) = (2\pi)^3(\omega_1 + \omega_2)e^{-i(\omega_1-\omega_2)t}\delta^3(\mathbf{k_1} - \mathbf{k_2}) \ , \tag{2.6}$$

and clearly this can be negative, depending on the sign of $\omega$. In order to have a real inner product we must select only positive $\omega$. This is where things start differing from flat to curved spacetime.

In Minkowski spacetime we can find a positive definite inner product by integrating over constant time hypersurfaces $\Sigma_t$ (the choice of the particular spacelike hypersurface does not affect the result in virtue of Stoke's theorem)

$$(\Phi_1, \Phi_2) = -i \int_{\Sigma_t} \left(\Phi_1 \partial_t \Phi_2^* - \Phi_2^* \partial_t \Phi_1\right) d^3x \ . \tag{2.7}$$

From (2.2) we can impose $\omega > 0$ by defining $\omega = |\mathbf{k}|$, and then the correct normalization of the plane waves can be found

$$f_\mathbf{k}(x^\mu) = \frac{e^{-ik_\mu x^\mu}}{\sqrt{(2\pi)^3 2\omega}} \ . \tag{2.8}$$

The $f_{\mathbf{k}}$ and $f^*_{\mathbf{k}}$ are the normal modes we need to expand the field solution $\Phi$. The $f_{\mathbf{k}}$ are said to be positive-frequency and satisfy

$$\partial_t f_{\mathbf{k}} = -i\omega f_{\mathbf{k}} \qquad (\omega > 0) \tag{2.9}$$

whereas $f^*_{\mathbf{k}}$ modes are said to be negative-frequency and satisfy

$$\partial_t f_{\mathbf{k}} = +i\omega f^*_{\mathbf{k}} \qquad (\omega > 0)\ . \tag{2.10}$$

This neat distinction between positive and negative frequency modes is of fundamental importance and let us build all the Hilbert space on which quantum mechanics rely on. A quantum field can always be expanded by means of a complete basis of solutions $f_i(x^\mu), f^*_i(x^\mu)$ (we now use discrete indices to simplify calculations)

$$\Phi = \sum_{\mathbf{k}_i} \left( \hat{a}_{\boldsymbol{k}i} f_{\mathbf{k}_i} + \hat{a}^\dagger_{\boldsymbol{k}i} f^*_{\mathbf{k}_i} \right)\ , \tag{2.11}$$

with coefficients given by the usual annihilation and creation operators $\hat{a}_{\boldsymbol{k}_i}$, $\hat{a}^\dagger_{\boldsymbol{k}_i}$ such that

$$[\hat{a}_{\boldsymbol{k}_i}, \hat{a}_{\boldsymbol{k}_j}] = [\hat{a}^\dagger_{\boldsymbol{k}_i}, \hat{a}^\dagger_{\boldsymbol{k}_j}] = 0\ , \qquad [\hat{a}_{\boldsymbol{k}_i}, \hat{a}^\dagger_{\boldsymbol{k}_j}] = \delta_{ij} \tag{2.12}$$

and the vacuum state $|\, 0_f \,\rangle$ for the basis $f, f^*$ defined as

$$\hat{a}_{\boldsymbol{k}_i} |\, 0_f \,\rangle = 0 \quad \forall i\ . \tag{2.13}$$

A state with $n_i$ excitations of different momenta $\mathbf{k}_i$ is created by repeated action of $\hat{a}^\dagger_{\boldsymbol{k}_i}$:

$$|\, n_1, n_2, ..., n_j \,\rangle_f = \frac{1}{\sqrt{n_1! n_2! ... n_j!}} \left(\hat{a}^\dagger_{\boldsymbol{k}_1}\right)^{n_{\boldsymbol{k}_1}} \left(\hat{a}^\dagger_{\boldsymbol{k}_2}\right)^{n_{\boldsymbol{k}_2}} ... \left(\hat{a}^\dagger_{\boldsymbol{k}_j}\right)^{n_{\mathbf{k}_j}} |\, 0 \,\rangle_f\ , \tag{2.14}$$

and a number operator can be defined for each wave vector

$$\hat{n}^a_{\mathbf{k}_i} = \hat{a}^\dagger_{\boldsymbol{k}_i} \hat{a}_{\boldsymbol{k}_i}\ . \tag{2.15}$$

We can of course consider an alternative set of modes $g_i(x^\mu)$ and coefficients $\hat{b}_{\mathbf{k}_i}, \hat{b}^\dagger_{\mathbf{k}_i}$ satisfying the same properties as before

$$\Phi = \sum_{\mathbf{k}_i} \left( \hat{b}_{\mathbf{k}_i} f_{\mathbf{k}_i} + \hat{b}^\dagger_{\mathbf{k}_i} f^*_{\mathbf{k}_i} \right)\ , \tag{2.16}$$

but with states built from another vacuum state that is generally different:

$$\hat{b}_{\mathbf{k}_i} |\, 0_g \,\rangle = 0 \quad \forall i \tag{2.17}$$
$$\hat{n}^g_{\mathbf{k}_i} = \hat{b}^\dagger_{\mathbf{k}_i} \hat{b}_{\mathbf{k}_i} \ . \tag{2.18}$$

Since basis sets are complete, one can expand one set in terms of the other by using Bogoliubov transformations (we omit index **k** for brevity):

$$g_i = \sum_j \left(\alpha_{ij} f_j + \beta_{ij} f^*_j\right) \tag{2.19}$$
$$f_i = \sum_j \left(\alpha^*_{ij} g_j - \beta_{ij} g^*_j\right) \tag{2.20}$$

where Bogoliubov coefficients can be found using the orthonormality of the mode functions

$$\alpha_{ij} = (g_i, f_j) \tag{2.21}$$
$$\beta_{ij} = -\left(g_i, f^*_j\right) \tag{2.22}$$

and they obey their own normalization conditions:

$$\sum_j \left(\alpha_{ik} \alpha^*_{jk} - \beta_{ik} \beta^*_{jk}\right) = \delta_{ij} \tag{2.23}$$
$$\sum_j \left(\alpha_{ik} \beta_{jk} - \beta_{ik} \alpha_{jk}\right) = 0 \ . \tag{2.24}$$

Then, annihilation (and creation operators can be rewritten in terms of the same operators in the other basis

$$\hat{a}_i = \sum_j \left(\alpha_{ji} \hat{b}_j + \beta^*_{ji} \hat{b}^\dagger_j\right) \tag{2.25}$$
$$\hat{b}_i = \sum_j \left(\alpha^*_{ji} \hat{a}_j - \beta^*_{ij} \hat{a}^\dagger_j\right) \ . \tag{2.26}$$

The vacuum defined by $f$-modes might be a non-vacuum state when expressed in $g$-modes. To see this, one uses the number operator of one basis on the states of the other basis:

$$\langle\, 0_f \,| \hat{n}^g_i |\, 0_f \,\rangle = \langle\, 0_f \,| \hat{b}^\dagger_i \hat{b}_i |\, 0_f \,\rangle = \sum_j |\beta_{ij}|^2 \ . \tag{2.27}$$

If $\beta_{ij} = 0$ then the f-vacuum is perceived as a zero-particles state also in the g-modes basis, whereas $\beta_{ij} \neq 0$ implies that in the g-basis one would actually see some particles. The meaning of $\beta_{ij}$ can be understood from (2.22): it represents the orthogonality relation between particles of the g-basis and antiparticles of the f-basis, i.e. vacuum in one basis becomes populated when one can mix positive and negative frequency modes of the other basis. This mixing can occur only if Lorentz invariance is violated. In flat spacetime we can perform Lorentz transformations

$$t' = \gamma t - \gamma \mathbf{v} \cdot \mathbf{x} \qquad \mathbf{x}' = \gamma \mathbf{x} - \gamma \mathbf{v} t \tag{2.28}$$

on the time derivative $\partial_t$ so that we get

$$\partial_{t'} f_{\mathbf{k}} = i\omega' f_{\mathbf{k}} \tag{2.29}$$

$$\omega' = \gamma\omega - \gamma \mathbf{v} \cdot \mathbf{k} \ . \tag{2.30}$$

As one can see, all frequencies are boosted in the same exact way. Therefore there is a 1-1 correspondence between different states in the two different reference frames. Particles of a given positive frequency might be "redshifted" or "blueshifted" but their frequency can't become negative. The same occurs for antiparticles, whose frequency can change but will always be negative. Therefore one has no mixing and $\beta_{ij} = 0$, so the vacuum state is the same for every observer.

Things change when going to curved spacetime. There, global Lorentz invariance does not hold anymore and one can find transformations that mix up positive and negative frequencies. Anyway, the signature of the frequencies is strictly related to the existence of a timelike Killing vector $K$. Relations (2.9) and (2.10) can be recast in a coordinate-invariant form as

$$\mathcal{L}_K f_\omega = K^\mu \partial_\mu f_\omega = i\omega f_\omega, \qquad \omega > 0 \ , \tag{2.31}$$

where $\mathcal{L}_K$ denotes the Lie derivative along $K$. In static spacetime it is always possible to find such a $K$ along which the vacuum state keeps being "empty". Dynamical spacetimes on the other side lack any asymptotically timelike Killing vector that could help us defining positive and negative frequencies.

To show the effects of a dynamical spacetime on the propagation of modes consider a spherical ball of matter, centered in the origin of our coordinate system and surrounded by empy space. Initially the ball of matter is extended to the infinity and has infinitesimal density, so we can construct the standard Minkowski vacuum. After the matter starts collapsing, on its outside the metric will be the usual Schwarzschild metric

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \ , \tag{2.32}$$

whereas the interior metric is of no practical importance. We then consider a massless scalar field obeying $\Box\Phi = 0$ and, as shown before, we decompose $\Phi$ into a complete set of positive frequency modes $f_\omega$ (indices referring to observables associated with spherical simmetry are omitted)

$$\Phi = \int d\omega \left( a_\omega f + a_\omega^\dagger f^* \right) \ , \tag{2.33}$$

where the quantum vacuum is defined at the past infinite $\mathcal{I}^-$ by $a_\omega | \, 0 \, \rangle = 0$. Since the field is massless it will propagate in spacetime at the speed of light, therefore passing through the collapsing matter shell before the collapse is over. Because of this we assume that the field does not interact with the matter and introduce the null coordinates

$$u = t - r^* + R_0^* = t - r - 2m \ln |r/2m - 1| + R_0^* \tag{2.34}$$
$$\tag{2.35}$$
$$v = t + r^* - R_0^* = t + r + 2m \ln |r/2m - 1| - R_0^* \tag{2.36}$$

where $R_0^*$ is a constant. Outside the ball of matter the line element reads

$$ds^2 = C(r) du dv \tag{2.37}$$
$$r^* = \int \frac{dr}{C(r)} \ , \tag{2.38}$$

where $C \to 1$ and $\partial C / \partial r \to 0$ as $r \to \infty$. To describe the interior part, we define other null coordinates

$$U = \tau - r + R_0 \tag{2.39}$$
$$\tag{2.40}$$
$$V = \tau + r - R_0 \tag{2.41}$$

and we take the general line element

$$ds^2 = A(U, V) dU dV \tag{2.42}$$

where $A(U, V)$ is arbitrary but smooth and non-singular. We assume that collapse starts at $t = \tau = 0$ and the ball has its surface at $u = U = v = V = 0$, whereas it follows $r = R(\tau)$ when shrinking.

By fixing these initial conditions, the mode solutions of the massless scalar field is

$$f_\omega = \frac{i}{\sqrt{4\pi\omega}} \left( e^{-i\omega v} - e^{+i\omega v} \right) \tag{2.43}$$

with

$$U = \alpha u \qquad v = \beta(V) \qquad V = U - 2R_0 \tag{2.44}$$
$$\Rightarrow v = \beta[\alpha(u) - 2R_0] \ . \tag{2.45}$$

On the event horizon ($C = 0$) one can approximate

$$\frac{dU}{du} \sim C(R)\frac{\dot{R} - 1}{2\dot{R}} \tag{2.46}$$
$$\frac{dv}{dV} \sim A\frac{(1 - \dot{R})}{2\dot{R}} \tag{2.47}$$
$$R(\tau) \sim R_h - \dot{R}(\tau_h)(\tau_h - t) \tag{2.48}$$

where the subscript $h$ means that the variable is evaluated on the horizon. By integrating one gets

$$\kappa u = -\ln|U + R_h - R_0 - \tau_h| + \text{constant} \tag{2.49}$$
$$\kappa = \frac{1}{2}\frac{\partial C}{\partial r}\bigg|_{r=R_h} \tag{2.50}$$

where $\kappa$ is the surface gravity on the horizon. We also integrate (2.47) by considering $A$ constant so that we get $v$ and the late time asymptotic modes:

$$v \sim \text{constant} + AV\frac{(1 - \dot{R})}{2\dot{R}} \tag{2.51}$$
$$f_\omega = \frac{i}{\sqrt{4\pi\omega}}\left(e^{-i\omega v} - e^{+i\omega(ce^{-\kappa u + d})}\right) \ , \tag{2.52}$$

with $c$ and $d$ constants. Because of this complicated expression for the modes at $\mathcal{I}^+$, the Bogoliubov coefficient $\beta_{ij}$ in (2.22) will be non-zero:

$$|\beta_{\omega'\omega}|^2 = \frac{1}{2\pi\kappa\omega'}\frac{1}{e^{2\pi\omega/\kappa} - 1} \ . \tag{2.53}$$

By analogy with statistical mechanics one can therefore associate a temperature to the horizon:

$$T = \kappa/(2\pi k_B) \tag{2.54}$$

where $k_B$ is the Boltzmann constant. As it can be seen from (2.53) the absence of any horizon (i.e. $\kappa = 0$) leads to a vanishing $\beta$ and the thermal spectrum disappears: no particles are detected.

## 2.2 The Kodama vector field

As we've just seen, a timelike Killing vector field is need in order to define positive and negative frequency modes since it corresponds to the operator whose eigenvalue is associated to the energy. The lack of such a vector field leave us no means to solve the problem of defining a common vacuum state on which to define particles and antiparticles. In dynamical spacetimes there are no asymptotically timelike Killing vectors in general. Because of this we can't define the temperature for a dynamical horizon via $\beta_{ij}$ (see (2.22)), and in general we can not define other *conserved charges* such as energy or momentum.

The reason why *timelike* Killing vectors are required to find a conserved *charge* is that we have to integrate over a *spacelike* hypersurface $\Sigma$ to get the total energy [32]:

$$E_T = \int_\Sigma \sqrt{h}\ J^\mu_T\ n_\mu\ \mathrm{d}^3x\ , \tag{2.55}$$

in which $h$ is the induced metric on $\Sigma$, $n_\mu$ is the unit vector normal to $\Sigma$ and $J^\mu_T = T^{\mu\nu}X_\nu$ is the conserved *current* as determined by means of Killing vectors $X^\nu$. The spacelike/timelike issue arises when trying to integrate such expression.

In 1980 Kodama published a paper [27] in which he introduced a particular vector that can determine a conserved current in spherical symmetry even if there is no suitable Killing vector field. In [34] Hayward reintroduced this vector field referring to it as the Kodama vector, aiming to find a more general definition of energy in General Relativity that could match the usual definitions as given by Arnowitt-De Witt-Meser (ADM), Misner-Sharp (MS) and Bondi-Sachs (BS) in the appropriate limits.

According to Hayward's more formal definition, one can define the Kodama vector by:

$$k = \mathrm{curl}\ r\ , \tag{2.56}$$

where the curl is taken in the 1+1 dimensional space orthogonal to the spheres of symmetry and $r$ is the radius that gives the invariant area $A = 4\pi r^2$. Another definition can be given by means of the Hodge dual $\star$:

$$k = g^{-1}(\star dr)\ . \tag{2.57}$$

The *Kodama current* $j$ satisfies

$$g(j) = -T(k) \tag{2.58}$$

where $k$ is the Kodama vector, $g$ is the metric tensor and $T$ the stress-energy tensor. In other words:

$$j^a = T^{ab} k_b \tag{2.59}$$
$$\nabla_a j^a = 0 \tag{2.60}$$
$$\nabla_a k^a = 0 \; . \tag{2.61}$$

In [35] the same expression as (2.55) is used to define energy with Kodama vectors (we report it as it is found in the original paper):

$$E = -\int_\Sigma \star u^\flat \cdot j = \int_\Sigma \star u \cdot T^\flat \cdot k \tag{2.62}$$

where $\star$ denotes the volume form and $u$ the unit future normal vector of an arbitrary spatial hypersurface $\Sigma$ with regular center. It follows that the Kodama vector can define energy accordingly to Misner-Sharp [36] for a Schwarzschild metric:

$$g(k,k) = \frac{2E}{r} - 1 \; . \tag{2.63}$$

It seems like the Kodama vector can take the place of the Killing vector in defining quantities such as surface gravity in spherical symmetric dynamical scenarios. The Kodama vector seems to be preferable to the Killing vector because:

1. It provides a conserved current (the energy) even in dynamical scenario, whereas the Killing vector cannot be used to determine the total energy since it might not be asymptotically timelike and therefore no integration can be performed to get the right value of the energy;

2. in some static spherically symmetric cases it agrees with the Killing vector.

Using the general spherically symmetric metric

$$ds^2 = g_{00}(dx^0)^2 + 2g_{01}dx^0dx^1 + g_{11}(dx^1)^2 + r^2_{(0,1)}d\Omega^2 \tag{2.64}$$

we write explicitly the most general Kodama vector:

$$k^0 = \sqrt{\gamma}(g^{00}g^{11} - g^{01}g^{10})\partial_1 r = \tfrac{1}{\sqrt{\gamma}}\partial_1 r \tag{2.65}$$
$$k^1 = \sqrt{\gamma}(-g^{00}g^{11} + g^{01}g^{10})\partial_0 r = -\tfrac{1}{\sqrt{\gamma}}\partial_0 r \tag{2.66}$$
$$k^\theta = k^\psi = 0 \tag{2.67}$$

in which $r$ is again the radius that gives the invariant area and $\gamma$ is the determinant of the 2-metric associated to coordiantes $x^0, x^1$ normal to the spheres of symmetry. Coordinates 0 and 1 can be chosen accordingly to our preferences (temporal-radial, null, double null and so on).

Substituting the general expression for the Kodama vector into the Killing equation leads to:

$$\begin{aligned} & g_{\nu\alpha}\nabla_\mu k^\alpha + g_{\mu\beta}\nabla_\nu k^\beta = \\ & g_{\nu\alpha}\nabla_\mu\left(\epsilon^{\alpha\sigma}\nabla_\sigma r\right) + g_{\mu\beta}\nabla_\nu\left(\epsilon^{\beta\sigma}\nabla_\sigma r\right) = \\ & g_{\nu\alpha}\epsilon^{\alpha\sigma}\nabla_\mu\nabla_\sigma r + g_{\mu\beta}\epsilon^{\beta\sigma}\nabla_\nu\nabla_\sigma r = 0 \ . \end{aligned} \tag{2.68}$$

Since all the indices are 0 or 1, one has the conditions

$$\begin{aligned} & g_{00}\Gamma^\lambda_{01}\partial_\lambda r = g_{01}\Gamma^\lambda_{01}\partial_\lambda r \\ & g_{10}\Gamma^\lambda_{01}\partial_\lambda r = \left(g_{11}\Gamma^\lambda_{00} + g_{00}\Gamma^\lambda_{11}\right)\partial_\lambda r \\ & g_{10}\Gamma^\lambda_{11}\partial_\lambda r = g_{11}\Gamma^\lambda_{10}\partial_\lambda r \ . \end{aligned} \tag{2.69}$$

One can see that using the so-called *dirty Schwarzschild metric*

$$ds^2 = e^{2\psi}Adt^2 - \frac{1}{A}dr^2 - r^2 d\Omega^2 \tag{2.70}$$

$$A = (1 - 2m/r) \ , \tag{2.71}$$

Killing and Kodama vectors only coincide if we set $\psi = constant$ in the metric. Solving for the *clean* ($\psi = 0$) Schwarzschild solution in the Lemaitre-Rylov gauge

$$ds^2 = dt^2 - \tfrac{1}{B}dr^2 - B^2 d\Omega^2 \tag{2.72}$$

$$B = \left[\tfrac{3}{2}(r - t)\right]^{3/2} \tag{2.73}$$

shows that in this case too[1] the Kodama vector is a Killing vector.

For a Schwarzschild-de Sitter metric the Kodama vector is $k^\mu = (1, 0, 0, 0)$ and also in this case the above criterion shows that this is a Killing vector too.

Solving this equation in a flat Friedmann-Robertson-Walker (FRW) universe with metric given by

$$ds^2 = dt^2 - a^2(t)\left[dr^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] \tag{2.74}$$

one finds out that the Kodama vector is a Killing vector only if $a\ddot{a} = a\dot{a}$ ($a$ is the scale factor), i.e. when the universe is de Sitter.

[1] This gauge is somehow unusual since the spherical radius is a function of time.

### 2.2.1 Which surface gravity?

In [35] Hayward provides a new definition of *surface gravity*:

$$\kappa_H = \frac{1}{2}\nabla_\mu \nabla^\mu|_\gamma r = \frac{1}{2}\Box|_\gamma r \ , \tag{2.75}$$

where the d'Alembertian operator is taken in the 2-metric (time-radius) and $r$ is the area radius. According to the author this "gives a new definition of surface gravity for spherically symmetric black holes, with the same form as the usual stationary definitions involving the Killing vector on the Killing horizon, but instead using the Kodama vector on the trapping horizon [...] A trapping horizon is said to be *outer*, *degenerate* or *inner* as $\nabla^2 r > 0$, $\nabla^2 r = 0$ or $\nabla^2 r < 0$ respectively". We just remind the definition of surface gravity $\kappa_X$ as can be found in [32] in terms of Killing vectors $X^\mu$:

$$\kappa_X = \sqrt{\nabla_\mu V \nabla^\mu V} \ , \tag{2.76}$$

$$V := \sqrt{-X^\mu X_\mu} \ . \tag{2.77}$$

At this point it might be useful to gather all the ingredients to make a comparison that shows the main differences between the two definitions of surface gravity. We "empirically" approach the problem of evaluating the surface gravity by using:

- 2 kind of spherically symmetric metrics: stationary or dynamical, keeping in mind there are dynamical metrics that can be written in a static way taking "pictures" of the universe at different times, such as it happens in FRW metrics

- 2 formulas for the surface gravity: the usual one and the new definition given by Hayward ($\kappa_H = \frac{1}{2}\Box r$)

- 2 vector fields: Killing and Kodama

Of course we could use a very general metric, compute everything and then simplify according to whether it is static, dynamical or whatever. Despite being more general, this risks to distract from the main issue. Also consider that calculating the surface gravity by means of Hayward's expression does not require any particular vector, so this shifts the question from "which vector do we need?" to "is Hayward's surface gravity correct?" We will concentrate on two spherically symmetric metrics: the dirty metric (2.70) and the geometrically flat Friedmann-Robertson-Walker (FRW) metric in its "dynamical" version (2.74). Now we directly evaluate the 6 possible combinations for the surface gravity. We will define

$$\kappa_H \equiv \frac{1}{2}\Box r \tag{2.78}$$

where $r$ is the physical radius of the invariant area, and

$$\kappa_Y \equiv \sqrt{\partial_\mu V \partial^\mu V} \tag{2.79}$$

with $V = \sqrt{-Y_\mu Y^\mu}$ and $Y$ can be the Kodama vector $K$ or the Killing vector $X$. Here are the results.

**Static dirty metric**

- $\kappa_H = \frac{\psi'}{2}A + \frac{1}{2}A'$
- the Kodama vector is $(e^{-\psi}, 0, 0, 0)$ and
  $\kappa_K = \frac{1}{2}A'$
- the Killing vector is $(1, 0, 0, 0)$ and
  $\kappa_X = \frac{1}{2}e^{\psi}A' + \psi' e^{\psi} A$

**Dynamical FRW metric**

- $\kappa_H = -\frac{1}{2}(H + \frac{\dot{H}}{H} + 1)$
- the Kodama vector is $(1, -Hr, 0, 0)$:
  $\kappa_K = (H^2 r^2 a^2 - 1)^{-1/2}\sqrt{-\left(\frac{\dot{H}}{H}\right)^2 - 2\dot{H}}$, and this diverges in the limit $r \to \frac{1}{aH}$
- $\kappa_X$ with a test vector $(1, 0, 0, 0)$: $\kappa_X = 0$

It is then clear by comparison that Hayward's surface gravity only matches the usual definition of surface gavity when the metric is static and *clean*, i.e. when the Kodama vector reduces to the Killing vector. In dynamical scenarios, when the Kodama vector should be useful, it seems that we have two different choices for the surface gravity. Of course, Hayward's surface gravity gives the correct outcome only in a very special case, and there is no way to grant that it is the right extension of the usual surface gravity, since it does not match the well-know expression for $\kappa$ in the dirty case. Because of this discrepancies, we'll try to better understand the meaning of the Kodama vector in another scenario: the particle detection by an accelerated observer.

## 2.3 Unruh radiation

A key concepts of Physics – no matter if classical or quantum – is the concept of *particle*. The question "What is a particle?" is so important in our atomistic viewpoint that even in Quantum Field Theory (QFT) we feel necessary to find an answer.

As far as gravitation is not considered (an approximation which turns out to be often natural since gravity is the weakest known interaction), the answer runs straight without many accidents: in this case, in fact, the spacetime where quantum fields propagate is Minkowskian. It is well known [39] that Minkowski spacetime possesses a rectangular coordinate system $(t, x, y, z)$ naturally associated with the Poincaré group, whose action leaves the Minkowski line element unchanged. The vector field $\partial_t$ is a Killing vector of Minkowski spacetime orthogonal to the space-like hypersurfaces $t =$ constant and the wave modes $u_{\mathbf{k}} \propto \exp(i\,\omega_{\mathbf{k}}\,t - \mathbf{k}\cdot\mathbf{x})$ are eigenfunctions of this Killing vector with eigenvalues $i\,\omega$. Thus, a particle mode solution in Minkowski space is one which is positive frequency $(\omega > 0)$ with respect to the time coordinate $t$. Under Poincaré transformations, positive frequency solutions transform to positive frequency solutions and the concept of particle is the same for every inertial observer, in the sense that all inertial observers agree on the number of particles present. Further, the Minkowski vacuum state, defined as the state with no particles present, is invariant under the Poincaré group.

Problems arise as one turns gravity on or in the presence of an accelerated observer. Equivalence principle equates (locally) a gravitational field with a uniformly accelerating reference frame, and as shown by Rindler [33] an accelerated rod in flat spacetime perceives the same physics as a static observer in Schwarzschild spacetime, being the two cases formally equivalent.

In curved spacetime the Poincaré group is no longer a symmetry group. In general, there are no Killing vectors with which to define positive frequency modes and no coordinate choice is available to make the field decomposition in some modes more natural than others. This, of course, is not just an accident but is rooted in the same guiding principle of general relativity: that coordinate systems are physically irrelevant.

As a possible way out, DeWitt and others suggested an operational definition of a particle: "a particle is something that can be detected by a particle detector". The particle detector proposed by Unruh [40] and later, in simplified version, by DeWitt [41] can be described as a quantum mechanical particle with many energy levels linearly coupled to a massless scalar field via a monopole moment operator. The particle is accelerated in Minkowski vacuum and plays the role of the Rindler observer [33]. In Minkwoski coordinates $\{t, x\}$, the trajectory of the accelerated particle/observer/detector is given by the coordinate transformations

$$\begin{cases} t = \dfrac{e^{a\xi}}{a} \sinh(aT) \\ \\ x = \dfrac{e^{a\xi}}{a} \cosh(aT) \end{cases} , \tag{2.80}$$

where $ae^{-a\xi}$ is the acceleration as measured at coordinate $\xi$ (in the accelerated detector's rest frame one has $\xi = 0$). This leads to

$$\begin{cases} \dfrac{\partial t}{\partial T} \equiv \tilde{u}^t = e^{a\xi} \cosh(aT) \\ \\ \dfrac{\partial x}{\partial T} \equiv \tilde{u}^x = e^{a\xi} \sinh(aT) \end{cases} \tag{2.81}$$

with $(\tilde{u}^t)^2 - (\tilde{u}^x)^2 = e^{2a\xi}$. The Minkowksi line element in the coordinates $\{T, \xi\}$ is

$$ds^2 = e^{2a\xi} \left( dT^2 - \mathrm{d}\xi^2 \right) , \tag{2.82}$$

and the *proper time* of the Rindler observer

$$\tau = e^{a\xi} T . \tag{2.83}$$

It is then easy to derive the components of the four-velocity and four-acceleration,

$$\begin{cases} u^t = \cosh(ae^{-a\xi}\tau) \\ \\ u^x = \sinh(ae^{-a\xi}\tau) \end{cases} \qquad \begin{cases} a^t = ae^{-a\xi} \sinh(ae^{-a\xi}\tau) \\ \\ a^x = ae^{-a\xi} \cosh(ae^{-a\xi}\tau) . \end{cases} \tag{2.84}$$

Therefore

$$(a^t)^2 - (a^x)^2 = -a^2 e^{-2a\xi} , \qquad a^\mu \, u_\mu = 0 , \tag{2.85}$$

and we can check that the acceleration is $ae^{-a\xi}$.

Note that, by choosing $\xi = 0$, the parameter $a$ equals the proper acceleration and the Rindler trajectory reduces to the simpler form

$$\begin{cases} t = \dfrac{1}{a} \sinh(a\tau) \\ \\ x = \dfrac{1}{a} \cosh(a\tau) , \end{cases}$$

without loss of generality.

Now let us consider a massless scalar field $\phi$ with Hamiltonian $\hat{H}_\phi$ obeying the massless Klein-Gordon equation. The free field operator is expanded in terms of a complete orthonormal set of solutions to the field equation as

$$\hat{\phi}(t,\mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \frac{d\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}\left(\hat{a}_{\mathbf{k}}e^{-i(\omega_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})} + \hat{a}^\dagger_{\mathbf{k}}e^{i(\omega_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})}\right) , \tag{2.86}$$

where, in the massless case, $\omega_{\mathbf{k}} = |\mathbf{k}|$. Field quantization is realized by imposing the usual commutation relations on the creation and annihilation operators,

$$[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = \delta^3(\mathbf{k}-\mathbf{k}') , \qquad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = 0 = [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] . \tag{2.87}$$

The Minkowski vacuum is the state $|\,0\,\rangle$ annihilated by $\hat{a}_{\mathbf{k}}$, for all $\mathbf{k}$.

The detector is a quantum mechanical system with a set of energy eigenstates $\{|\,0\,\rangle_d, |\,E_i\,\rangle\}$ which moves along a prescribed classical trajectory $t = t(\tau),\ \mathbf{x} = \mathbf{x}(\tau)$, where $\tau$ is the detector's proper time. The detector is coupled to the scalar field $\phi$ via the interaction Hamiltonian

$$\hat{H}_{int} = \lambda\,\hat{M}(\tau)\hat{\phi}(\tau) . \tag{2.88}$$

$\lambda$ is treated here as a small parameter while $\hat{M}(\tau)$ is the detector's monopole moment operator whose evolution is provided by

$$\hat{M}(\tau) = e^{i\hat{H}_d\tau}\,\hat{M}(0)\,e^{-i\hat{H}_d\tau} , \tag{2.89}$$

$\hat{H}_d$ being the detector's Hamiltonian. This model is also known as the *point-like detector* since the interaction takes place at a point along the given trajectory at any given time.

Suppose that at time $\tau_0$ the detector and the field are in the product state $|\,0, E_0\,\rangle = |\,0\,\rangle|\,E_0\,\rangle$, where $|\,E_0\,\rangle$ is a detector state with energy $E_0$ [42]. We want to know the probability that at a later time $\tau_1 > \tau_0$ the detector is found in the state $|\,E_1\,\rangle$ with energy $E_1 \gtrless E_0$, no matter what is the final state of the field $\phi$. The answer is provided by the so-called interaction picture where we make both operators and states to evolve in time according to the following understandings: operators evolution is governed by free Hamiltonians; states evolution is governed by the Schrödinger equation depending on the interaction Hamiltonian, that is

$$i\,\frac{d}{d\tau}|\,\varphi(\tau)\,\rangle = \hat{H}_{int}|\,\varphi(\tau)\,\rangle . \tag{2.90}$$

The amplitude for the transition from the state $|\,0,E_0\,\rangle$ at time $\tau=\tau_0$ to the state $|\,\varphi,E_1\,\rangle$ at time $\tau=\tau_1$ is then provided by

$$\langle\,\varphi,E_1\mid 0,E_0\,\rangle=\langle\,\varphi,E_1\,|\hat{T}\exp\left(-i\int_{\tau_0}^{\tau_1}d\tau\,\hat{H}_{int}(\tau)\right)|\,0,E_0\,\rangle\;, \qquad (2.91)$$

where $\hat{T}$ is the time-ordering operator. To first order in perturbation theory, we get

$$\begin{aligned}\langle\,\varphi,E_1\mid 0,E_0\,\rangle &= \langle\,\varphi,E_1\,|\hat{\mathbb{I}}|\,0,E_0\,\rangle- && (2.92)\\ &\quad -i\,\lambda\,\langle\,\varphi,E_1\,|\int_{\tau_0}^{\tau_1}d\tau\,e^{i\hat{H}_d\tau}\,\hat{M}(0)\,e^{-i\hat{H}_d\tau}\hat{\phi}(\tau)|\,0,E_0\,\rangle+\ldots\\ &= -i\,\lambda\,\langle\,E_1\,|\hat{M}(0)|\,E_0\,\rangle\,\cdot && (2.93)\\ &\quad \cdot\int_{\tau_0}^{\tau_1}d\tau\,e^{i\tau(E_1-E_0)}\langle\,\varphi\,|\hat{\phi}(\tau)|\,0\,\rangle+\ldots && (2.94)\end{aligned}$$

The transition probability to all possible finale states of the field $\phi$ is given by squaring (2.94) and summing over the complete set $\{|\,\varphi\,\rangle\}$ of final unobserved field states,

$$\begin{aligned}\sum_{\varphi}|\langle\,\varphi,E_1\mid 0,E_0\,\rangle|^2 &= \lambda^2|\langle\,E_1\,|\hat{M}(0)|\,E_0\,\rangle|^2\,\cdot && (2.95)\\ &\quad \cdot\int_{\tau_0}^{\tau_1}d\tau\int_{\tau_0}^{\tau_1}d\tau'\,e^{-i(E_1-E_0)(\tau-\tau')}\,\cdot && (2.96)\\ &\quad \cdot\langle\,0\,|\hat{\phi}(\tau)\hat{\phi}(\tau')|\,0\,\rangle\;. && (2.97)\end{aligned}$$

This expression has two parts: the pre-factor $\lambda^2|\langle\,E_1\,|\hat{M}(0)|\,E_0\,\rangle|^2$ which depends only on the peculiar details of the detector and the *response function*

$$R_{\tau_0,\tau_1}(\Delta E)=\int_{\tau_0}^{\tau_1}d\tau\int_{\tau_0}^{\tau_1}d\tau'\,e^{-i\Delta E(\tau-\tau')}\langle\,0\,|\hat{\phi}(\tau)\hat{\phi}(\tau')|\,0\,\rangle\;, \qquad (2.98)$$

which is insensitive to the internal structure of the detector and is thus the same for all possible detectors. Here, we have set the energy gap $\Delta E\equiv E_1-E_0\gtrless 0$ for excitations or decay, respectively. ¿From now on, we will only consider the model-independent response function.

Introducing new coordinates $u:=\tau$, $s:=\tau-\tau'$ for $\tau>\tau'$ and $u:=\tau'$, $s:=\tau'-\tau$ for $\tau'>\tau$, the response function can be re-written as

$$R_{\tau_0,\tau_1}(\Delta E)=2\int_{\tau_0}^{\tau_1}du\int_0^{u-\tau_0}ds\;\mathrm{Re}\left(e^{-i\Delta E s}\langle\,0\,|\hat{\phi}(u)\hat{\phi}(u-s)|\,0\,\rangle\right)\;, \qquad (2.99)$$

having used $\langle 0 |\hat{\phi}(\tau')\hat{\phi}(\tau)| 0 \rangle = \langle 0 |\hat{\phi}(\tau)\hat{\phi}(\tau')| 0 \rangle^*$, since $\hat{\phi}$ is a self-adjoint operator. Eq. (2.99) can be differentiated with respect to $\tau_1$ in order to obtain the *transition rate*

$$\dot{R}_{\tau_0,\tau}(\Delta E) = 2\int_0^{\tau-\tau_0} ds \,\mathrm{Re}\left(e^{-i\Delta E s}\langle 0 |\hat{\phi}(\tau)\hat{\phi}(\tau-s)| 0 \rangle\right) \;, \tag{2.100}$$

where we set $\tau_1 \equiv \tau$. If the correlation function $\langle 0 |\hat{\phi}(\tau)\hat{\phi}(\tau-s)| 0 \rangle$ is invariant under $\tau$-translations, (2.100) can be further simplified,

$$\dot{R}_{\tau_0,\tau}(\Delta E) = \int_{-(\tau-\tau_0)}^{\tau-\tau_0} ds\; e^{-i\Delta E s}\langle 0 |\hat{\phi}(s)\hat{\phi}(0)| 0 \rangle \;. \tag{2.101}$$

The correlation function $\langle 0 |\hat{\phi}(x)\hat{\phi}(x')| 0 \rangle$ which appears in these expressions is the positive frequency Wightman function that can be obtained from (2.86),

$$\langle 0 |\hat{\phi}(x)\hat{\phi}(x')| 0 \rangle = \frac{1}{(2\pi)^3}\int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}} e^{-i\omega_{\mathbf{k}}(t-t')+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \;. \tag{2.102}$$

The integral in $|\mathbf{k}|$ contains UV divergences and can be regularized [39] by introducing the exponential cut-off $e^{-\epsilon|\mathbf{k}|}$, with $\epsilon > 0$ and small, in the high frequency modes. The resulting expression is

$$\begin{aligned}\langle 0 |\hat{\phi}(x(\tau))\hat{\phi}(x(\tau'))| 0 \rangle \;&=\; \frac{1/4\pi^2}{|\mathbf{x}(\tau)-\mathbf{x}(\tau')|^2 - [t(\tau)-t(\tau')-i\epsilon]^2} \\ &\equiv W_\epsilon(x(\tau),x(\tau')) \;,\end{aligned} \tag{2.103}$$

so that, we finally have

$$\dot{R}_{\tau_0,\tau}(\Delta E) = \lim_{\epsilon\to 0^+}\int_{-(\tau-\tau_0)}^{\tau-\tau_0} ds\; e^{-i\Delta E s}\, W_\epsilon(x(s),x(0)) \;. \tag{2.104}$$

## 2.4 Unruh effect along the Kodama trajectory

Our aim is to study the Unruh radiation as seen by an observer accelerated along a Kodama trajectory in de Sitter spacetime. For an observer comoving with the cosmic fluid, the de Sitter metric reads [2]

$$ds^2 = dt^2 - e^{2Ht}\,dr^2 \;. \tag{2.105}$$

[2]We consider a two-dimensional spacetime for simplicity. Alternatively, $r = x^1$ can be viewed as a cartesian coordinate on the sub-manifold $x^2 = x^3 = 0$.

We are here interested in a detector moving along the trajectory

$$r\,e^{Ht} = K \ , \tag{2.106}$$

where $K$ is constant. Its peculiar velocity is then given by

$$v = \frac{dr}{dt} = K\frac{de^{-Ht}}{dt} = -HKe^{-Ht} = -Hr \ . \tag{2.107}$$

From $u^\mu\,u_\mu = 1$, we obtain the dilatation factor

$$\gamma = \frac{1}{\sqrt{1-v^2e^{2Ht}}} = \frac{1}{\sqrt{1-H^2r^2e^{2Ht}}} = \frac{1}{\sqrt{1-K^2H^2}} \tag{2.108}$$

and the four-velocity of the detector in the de Sitter frame is

$$u^\mu = (\gamma, \gamma\,v) = \left(\frac{1}{\sqrt{1-H^2K^2}}, -\frac{Hr}{\sqrt{1-H^2K^2}}\right) \ . \tag{2.109}$$

Analogously, the four-acceleration is

$$\begin{aligned} a^t &= u^t\partial_t u^t + u^r\partial_r u^t + \Gamma^t_{rr}u^r u^r \\ &= \frac{H^3r^2e^{2Ht}}{1-H^2K^2} = \frac{Hv^2e^{2Ht}}{1-H^2K^2} = \frac{H^3K^2}{1-H^2K^2} \end{aligned} \tag{2.110}$$

$$\begin{aligned} a^r &= u^t\partial_t u^r + u^r\partial_r u^r + 2\Gamma^r_{rt}u^t u^r \\ &= \gamma^4 Hv^3e^{2Ht} - \gamma^4 Hv^3e^{2Ht} - \gamma^2 Hv + 2Hu^0u^1 \\ &= -\gamma^2 Hv + 2\gamma^2 Hv \\ &= \frac{Hv}{1-H^2K^2} = -\frac{H^2r}{1-H^2K^2} \end{aligned} \tag{2.111}$$

$$a^\mu\,a_\mu = \gamma^4H^4K^2(H^2K^2-1) = -\frac{H^4K^2}{1-H^2K^2} \ . \tag{2.112}$$

In the following we shall need to relate a change in acceleration $(\delta a)$ to a change in the trajectory $(\delta r)$ at fixed time, namely

$$\begin{cases} \delta a^t = \dfrac{\delta a^t}{\delta r}\delta r = \dfrac{2H^3re^{2Ht}}{(1-H^2K^2)^2}\delta r \\[2ex] \delta a^r = \dfrac{\delta a^r}{\delta r}\delta r = -H^2\dfrac{1+H^2K^2}{1-H^2K^2}\delta r \ . \end{cases} \tag{2.113}$$

### 2.4.1 Pointlike detector

Let us apply the above construction to Kodama detectors in de Sitter spacetime. We first recall that de Sitter metric in the static patch is

$$ds^2 = \left(1 - H^2\bar{r}^2\right) dt^2 - \left(1 - H^2\bar{r}^2\right)^{-1} \mathrm{d}\bar{r}^2 - \bar{r}^2 \mathrm{d}\Omega^2 \ , \tag{2.114}$$

and in the cosmological global system

$$ds^2 = \mathrm{d}\tau^2 - H^{-2}\cosh^2(H\tau)\, \mathrm{d}\Omega_3^2 \ . \tag{2.115}$$

An easy calculation then gives the equivalent definitions of the Kodama trajectory (2.106)

$$r = K\, e^{-H\,t} \quad \Longleftrightarrow \quad \bar{r} = K \quad \Longleftrightarrow \quad \sin\chi = \frac{K H}{\cosh(H\tau)} \ . \tag{2.116}$$

Conceptually, the relevant equality is the second one: it means that the Kodama observers are just the stationary de Sitter observers at constant distance from their cosmological horizon. We know that these observers will perceive a thermal bath at de Sitter temperature $T = H/2\pi$, so this must be true in the inflationary patch as well.

We can confirm this expectation rewriting Eq. (2.103) by making use of the following relations for de Sitter space

$$\eta = -H^{-1}e^{-H\,t}, \quad r = K\, e^{-Ht} = -K\, H\, \eta, \quad \tau = t\sqrt{1 - K^2\, H^2} \ . \tag{2.117}$$

Then, provided $1 - K^2H^2 > 0$, the Wightman function (2.103) becomes

$$W_\epsilon(x, x') = -\frac{1}{4\pi^2}\frac{H^2}{e^{H(t+t')}}\frac{1}{(1 - K^2H^2)(e^{-Ht} - e^{-Ht'} - i\epsilon)^2} \ . \tag{2.118}$$

and, in the limit of $\tau_0 \to -\infty$, the detector transition rate (2.104) becomes

$$\dot{R}(\Delta E) = -\frac{H^2}{4\pi^2}\lim_{\epsilon\to 0^+}\int_{-\infty}^{+\infty} ds\, \frac{\exp(-i\Delta E\sqrt{1 - K^2H^2}\, s)}{(e^{Hs/2} - e^{-Hs/2} - i\epsilon)^2} \ . \tag{2.119}$$

As long as $H > 0$ (expanding universe) we can write the denominator as

$$\left(e^{H\,s/2} - e^{-H\,s/2} - i\epsilon\right)^2 = 4\, \sinh^2\left[H\,(s - i\epsilon)/2\right] \ . \tag{2.120}$$

This function has infinitely many double poles in the complex $s$-plane, namely for $s = s_j$ with

$$s_j = \frac{2\,\pi\, i}{H} j \ , \quad j \in \mathbb{Z} \ . \tag{2.121}$$

Since we are interested in the case of $\Delta E > 0$, we can close the contour of integration in the lower half plane, summing over the residues of all the double poles in the lower complex $s$-plane, with the exception of the $s = 0$ pole which has been slightly displaced above the integration path by the $i\epsilon$-prescription. The well known result turns out to be confirmed, namely

$$\dot{R}(\Delta E) = \frac{\Delta E\,\sqrt{1-K^2H^2}\,\exp\left(\frac{2\,\pi}{H}\,\Delta E\,\sqrt{1-K^2\,H^2}\right)}{\exp\left(\frac{2\,\pi\,\Delta E\,\sqrt{1-K^2\,H^2}}{H}\right)-1}\ , \tag{2.122}$$

showing the presence of a cosmological Unruh effect with de Sitter temperature

$$T = \frac{H}{2\,\pi}\ , \tag{2.123}$$

red-shifted by the Tolman factor $\sqrt{1-K^2\,H^2}$, here appearing as a Doppler shift due to the proper motion of the detector. In fact we remember after Eq (2.116) that $K$ is also the value of the static coordinate $\bar{r}$ owned by the detectors relative to the static patch. Another intriguing formula can be written if we recall the value (2.112) of the acceleration: one easily sees that

$$\frac{T}{\sqrt{1-H^2K^2}} = \frac{1}{2\pi}\,\sqrt{A^2+H^2} \tag{2.124}$$

One can interpret this formula by saying that, actually, the temperature is due to a mixing of a pure Unruh effect (the acceleration term) plus a cosmological expansion term (the $H$ term) and it is the de Sitter version [43]of a formula discovered by Deser & Levine for detectors in anti-de Sitter space[44]. Alternatively, we can understand this effect as the transition from cosmological energy $\Delta E$, conjugated to cosmic de Sitter time, to the energy

$$\Delta E\,\sqrt{1-K^2H^2} \tag{2.125}$$

as measured locally by Kodama's observers.

### 2.4.2 Extended detector and backreaction

We now repeat the previous analysis assuming the detector's size is not *a priori* negligible. In order to simplify the computation, we still assume spherical symmetry and the detector therefore only extends along one dimension. We will not display all the details but focus on the main differences.

It is easy to see that a detector moving along a Kodama trajectory (2.106) in an expanding de Sitter universe, has the same dynamics of a particle which moves along the separatrix in the potential of an inverted harmonic oscillator.

From the equation of motion (2.106), we can therefore introduce the effective Lagrangian

$$\mathcal{L}_{\rm iho} = m \left( \frac{{\rm d}^2 r}{dt^2} + H^2 r^2 \right) , \tag{2.126}$$

where $m$ is a parameter with mass dimensions whose physical meaning will be clarified later. Our detector is initially (at $t = 0$) represented by a Gaussian wave packet of size $b$ peaked around $r = K$,

$$\psi(t, r) = \frac{\exp\left[ -i \frac{H\, K\, m\, r}{\hbar} - \frac{(r-K)^2}{2\, b^2} \right]}{\sqrt{b\,\sqrt{\pi}}} , \tag{2.127}$$

which is then propagated to later times by the propagator [37]

$$G(t, r; 0, r') = \sqrt{\frac{i\, H\, m}{2\, \pi\, \hbar\, \sinh(H\, t)}} \exp\left[ i\, \frac{H\, m\, [(r^2 - r'^2)\, \cosh(H\, t) - 2\, r\, r']}{2\, \hbar\, \sinh(H\, t)} \right] .$$

The complete expression of the detector's propagated wavefunction is rather cumbersome, however we notice that its square modulus yields

$$|\psi(t, r)|^2 \sim \exp\left[ - \frac{2\, b^2\, H^2\, m^2\, (r - K\, e^{-H\, t})^2}{b^4\, H^2\, m^2 - \hbar^2 + (\hbar^2 + b^4\, H^2\, m^2)\, \cosh(H\, t)} \right] , \tag{2.128}$$

and the classical behavior is properly recovered in the limit $\hbar \to 0$ followed by $b \to 0$ [37], in which the detector's wavefunction $\psi(r, t)$ reproduces the usual delta-function peaked on the classical trajectory employed previously.

However, in order to study the probability for the detector to absorb a scalar quantum and make a transition between two different trajectories (parameterized by different $m_i$ and $K_i$), one needs to compute the transition amplitude for finite $b$ and $\hbar$ (otherwise the result would automatically vanish). The detector now interacts with the quantized scalar field $\varphi = \varphi(t, r)$ according to

$$\mathcal{L}_{\rm int} = \frac{1}{2}\, Q\, (\psi_2^*\, \psi_1 + \psi_2\, \psi_1^*)\, \varphi , \tag{2.129}$$

where $Q$ is a coupling constant and $\psi_i = \psi_i(t, r)$ two possible states of the detector corresponding to different trajectories $r_i = K_i\, e^{-H\, t}$ and mass parameters $m_i$ [3]. We assume the difference between the two states is small,

$$\begin{cases} K_1 = K - \frac{1}{2}\,\delta K \\ \\ K_2 = K + \frac{1}{2}\,\delta K \end{cases} \qquad \begin{cases} m_1 = m - \frac{1}{2}\,\delta m \\ \\ m_2 = m + \frac{1}{2}\,\delta m \ , \end{cases} \tag{2.130}$$

and expand to leading order in $\delta K$ and $\delta m$ and, subsequently, for short times ($H\,t \sim H\,t' \ll 1$). In particular, one obtains

$$\psi_2^*\,\psi_1(t)\,\psi_1^*\,\psi_2(t') \sim \exp\left[-i\,\frac{H^2\,K^2}{\hbar}\,\delta m\,(t-t')\right] \ , \tag{2.131}$$

in which we have evaluated the phase for $r$ along the average trajectory between $r_1$ and $r_2$. Upon comparing with the result obtained for the point-like case, we immediately recognize that

$$H^2\,K^2\,m = E\,\sqrt{1-H^2\,K^2} \ , \tag{2.132}$$

where $E$ is the detector's proper energy and

$$\psi_2^*\,\psi_1(t)\,\psi_1^*\,\psi_2(t') \quad \sim \quad \exp\left[-\frac{i}{\hbar}\,\delta E\,\sqrt{1-H^2\,K^2}\,(t-t')\right] \tag{2.133}$$

$$+\exp\left[+i\,\frac{2-H^2\,K^2}{\hbar\,K^3\,H^2}\,E\,\delta K\,(t-t')\right] \ . \tag{2.134}$$

We recall from Ref. [38] that, requiring the analogue of the second factor above equaled one, leads to the equation of motion for a uniformly accelerated detector in Minkowski spacetime, namely $m\,a = f$ and constant. Following the same line of reasoning, we now obtain the equation of motion

$$\delta K = 0 \ . \tag{2.135}$$

This can be interpreted as meaning the Kodama trajectory is stable against thermal emission of scalar quanta in the de Sitter background.

The transition probability per unit de Sitter time can finally be computed by taking the classical limit, in which one recovers the same result as in the previous Section, namely Eq. (2.122) with $\Delta E = \delta E$.

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[3] A fundamental difference with respect to the Unruh effect analyzed in Ref. [38] is that the acceleration parameter $H$ is not varied here, since it is a property of the background spacetime. A change $\delta K$ implies a change in the detector's acceleration according to Eq. (2.112).

## 2.5 Conclusions and outlook

After 30 years the factual utility of Kodama vector is still uncertain. When a Killing vector field is present, the Kodama vector seems to mimic its properties. On the other side, when there is no Killing vector, we have no comparison terms to decide whether this "new" vector field is meanin or not. The Schwarzschild dirty metric (2.70) is the only proving ground on which one has two distinct choices but there is no known criterion to decide which is the right one.

When introducing the Kodama vector in the study of the Unruh effect, one gathers few more informations. The Kodama trajectory is not a free-falling trajectory and this adds a redshift factor to the horizon thermal energy measured by the detector, as one expects. What is more interesting is the absence of any backreaction on the detector. In first approximation it seems like the detector does not recoil (actually: "jumps outward") but rather stays stable on its trajectory. To clarify this one needs to investigate the backreaction at higher orders of approximation to be sure that this curious effect is not due to the lack of precision in the computation.

# Chapter 3

# Light Black Holes

The possibility of detecting microscopic black holes is presented in this last chapter. In recent times there has been great interest in this subject, due to the birth of new theoretical scenarios and the attainment of new records in high energy physics experiments.

In Section 3.1 we present a synthetic resumée of the main concepts behind theories of gravitation with extra-dimensions.

In Section 3.2 we focus on the Randall-Sundrum (RS) model [48]

In Section 3.3 we present our research [45] on the possibility that tidally charged black holes could be created or not at the LHC.

## 3.1 Gravity with extra dimensions

At very high energies, Einstein's theory of general relativity is not able to give an exact description of Nature, and quantum gravity must be introduced. There is not yet any accepted quantum gravity theory. Two main candidates are string theory and loop quantum gravity, both still presenting major flaws. One of the fundamental aspects of string theory is the need for extra spatial dimensions. The first theory with extra dimensions is the Kaluza-Klein theory that dates back to the 1920's and whose aim was to get the photon field from a fifth component of the metric tensor. In the 1980's the wide-spreading String Theory attracted more and more interest on extra dimensions. To match the empirical evidence that our world only 1+3 dimensions, one can assume that the extra dimensions are very small and compactified: therefore they can be "seen" only at very high energy scales. The observable universe is then a 1+3-surface (called "brane") embedded in a 1+3+d - dimensional spacetime (called "bulk"), and Standard Model particles and fields would live on the brane while gravitational interaction is located in the whole bulk. p-branes are branes with $p$ spatial dimensions (then a 2-brane is a common “membrane” and a 1-brane is a string), whereas D-branes (Dirichlet-branes) are surfaces where open string can end on. Opens strings are the source of fields that generate the Standard Model.

A most exciting feature of models with large extra dimensions [46, 48] is that the fundamental scale of gravity $M_{\rm G}$ could be much smaller than the Planck mass $M_{\rm P}$ and as low as the electro-weak scale ($M_{\rm G} \simeq 1\,$TeV), hence solving the hierarchy problem. This can be seen by considering the action $S$ of the extra-dimensional theory (with $d$ extra dimensions):

$$S_{4+d} = -M_{\rm G}^{2+d} \int d^{4+d}x \sqrt{g^{(4+d)}}\, R^{(4+d)} \ . \tag{3.1}$$

Here $g^{(4+d)}$ and $R^{(4+d)}$ are the determinant of the metric and the Ricci scalar in $4+d$ dimensions, whereas $M_{\rm G}$ is the new scale of gravity, needed to make the action dimensionless. For $d = 0$ one recover the usual Planck mass $M_{\rm G}^2 = M_{\rm P}^2 \sim 10^{19}$GeV.

The static weak field limit of the field equations gives the Poisson equation in $4 + d$ dimensions, whose solution is the gravitational potential

$$V(r) \sim \frac{8\pi}{M_{\rm G}^{2+d} r^{1+d}} \ . \tag{3.2}$$

Given $L$ the length scale of the extra dimensions, on large scales ($r > L$) the potential $V(r)$ reduces to the usual form $1/r$. By contrast on small scales ($r < L$) all dimensions contribute and give the $4 + d$ dimensional potential $V(r) \sim 1/r^{1+d}$. In virtue of (3.2) one can also write

$$M_{\rm P}^2 = M_{\rm G}^{2+d} \mathcal{V} \tag{3.3}$$

where $\mathcal{V} \sim r^d$ is the volume of the extra dimensions. The dilution of gravity through the other dimensions would explain its relative weakness. How “large” are these extra dimensions? Newtonian gravity has been tested down to scales of one tenth of a millimeter. There, molecular forces are too strong and prevent better measurements of the gravitational attraction. According to (3.3) one has

$$r \sim \frac{1}{M_{\rm G}} \left( \frac{M}{M_{\rm G}} \right)^{2/d} \ . \tag{3.4}$$

Using this expression one can clearly see that if $M_{\rm G} \sim 1$ TeV and $d = 1$, the extra (flat) dimension would be of astronomical scales, then it would have already been “detected” since longtime. For $d = 2$ anyway at the scale goes down to millimeters. Therefore if there are two extra dimensions they might be detected in present experiments at the TeV energy. For $n > 2$ the size of the extra dimensions is less than $10^{-6}$ cm and $M_{\mathrm{G}}$ could really be found at the TeV scale without disagreeing with experimental evidences.

## 3.2 The Randall-Sundrum model

There exist actually different string theories, but it is supposed that they are just limits of a single theory, the M theory. At low energies, M theory can be approximated by 1+10 dimensional supergravity. In the Horava-Witten solution gauge fields of the standard model are confined on two 1+9 branes found at the end points of an $S^1/\mathbb{Z}_2$ orbifold, i.e. a circle whose points are identified two-by-two across a diameter (see figure 3.1). Of the

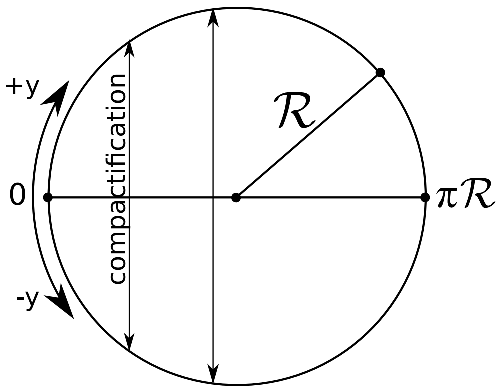


Figure 3.1: Representation of the $S^1/\mathbb{Z}_2$ orbifold

9 spatial dimensions of the branes, 3 are the ordinary ones, whereas the other 6 are compactified on very small scales. The Randall-Sundrum (RS) model constitutes an effective 5-dimensional solution of Horava-Witten's, the difference being that in RS there is a "preferred" extra dimension, while at the same time the others are negligible. The metric is obtained as follow. Let's denote by $y$ the coordinate labeling the extra dimension: we want it to be compactified with period $\pi\mathcal{R} \equiv L$ (here $\mathcal{R}$ is the compactification radius). The extrema of this compactified circle are of course in $y = 0$ and $y = L$; there are located the 3 branes where the gauge fields "live". Then we have two branes, at a distance L one from the other, enclosing a 5D bulk. We start assuming that the action is the sum of the Einstein-Hilbert action $S_H$

and a matter part $S_M$:

$$S = S_H + S_M = \int d^4x \int_{-L}^{+L} dy \sqrt{-g}(M_{\rm G}^3 R - \Lambda) \ , \tag{3.5}$$

where $M_{\rm G}$ is the fundamental scale of gravity, R the Ricci curvature scalar and $g$ the determinant of the 5D metric. The metric must preserve Poincaré invariance on the brane; therefore we make the ansatz

$$ds^2 = e^{-2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2 \tag{3.6}$$

where greek indices run in the usual 4D spacetime. The *warp factor* $e^{-2A(y)}$ is written as an exponential for convenience and its dependance on the extra dimension $y$ makes it non-factorizable (i.e.: the metric is *not* the product of Minkowski spacetime with a sphere $S^1$).

By solving the 5D Einstein's equations

$$G_{MN} = R_{MN} - \frac{1}{2} g_{MN} R = \kappa^2 T_{MN} \tag{3.7}$$

with $M, N = 0, 1, 2, 3, 4$ we find the 5-dimensional Newton constant

$$\kappa^2 \equiv \frac{1}{2M_{\rm G}^3} \tag{3.8}$$

and in particular the new (fifth) component of Einstein's equations gives

$$G_{55} = 6A'^2 = -\frac{\Lambda}{2M_{\rm G}^3} \Rightarrow A'^2 = -\frac{\Lambda}{12M_{\rm G}^3} \equiv k^2 \ . \tag{3.9}$$

This requires the bulk cosmological constant to be negative and defines $A(y)$. Requiring invariance under $y \mapsto -y$ (the orbifold symmetry) one finally finds:

$$A(y) = k|y| \tag{3.10}$$

from which the correct metric follows:

$$ds^2 = e^{-2k|y|} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2 \ . \tag{3.11}$$

Anyway we still need some corrections to the original action. We can see that the $G_{\mu\nu}$ components of Einstein's equations contain first and second derivatives of $A(y)$:

$$G_{\mu\nu} = (6A'^2 - 3A''^2) g_{\mu\nu} \tag{3.12}$$

$$A' = \text{sgn}(y) k \tag{3.13}$$

$$A'' = 2k(\delta(y) - \delta(y - L)) \tag{3.14}$$

where we have kept account of the discontinuities of $A$ in $y = 0$ and $y = L$. These two delta terms found in Einstein's equations need a counterpart in the action, or their presence would be unexplained. For this reason we add to the action (3.5) one term for each brane, corresponding to the brane tensions $\lambda_1$ and $\lambda_2$:

$$S_1 = -\int d^4x\sqrt{-g_1}\lambda_1 = -\int d^4x dy\sqrt{-g_1}\lambda_1\delta(y) \tag{3.15}$$

$$S_2 = -\int d^4x\sqrt{-g_2}\lambda_2 = -\int d^4x dy\sqrt{-g_2}\lambda_2\delta(y) \tag{3.16}$$

whre $g_1$ and $g_1$ is the determinant of the metric induced on the branes. To satisfy Einstein's equations one must have

$$\lambda_1 = -\lambda_2 = 12kM_{\rm G}^3 \tag{3.17}$$

and because of the definition of $k$ we gets

$$\Lambda = -\frac{\lambda_1^2}{12M_{\rm G}^3} \; . \tag{3.18}$$

The positive tension brane is said to be the *hidden brane*, whereas the

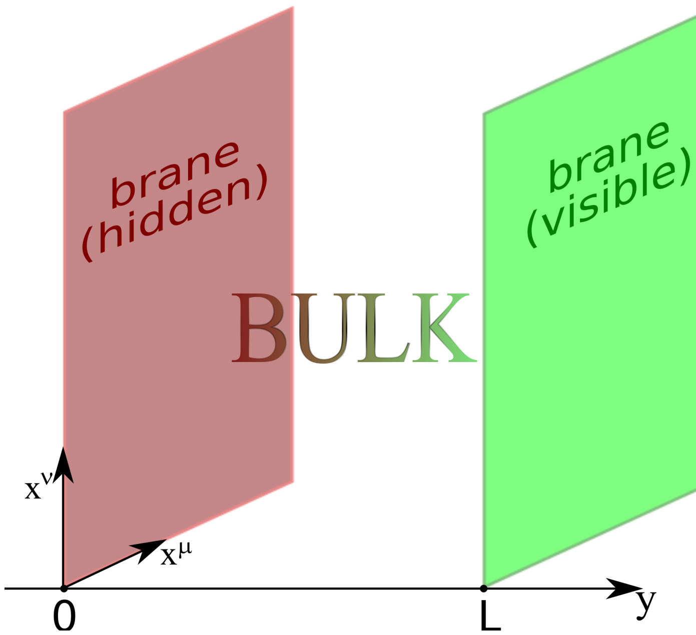


Figure 3.2: Representation of the two branes

negative tension brane is the *visible brane* (see figure 3.2). Because of the warp factor, the effective Plank scale $M_{\rm G}$ on the visible brane at $y = L$ is given by

$$M_{\rm P}^2 = \frac{M_{\rm G}^3}{k}\left(1 - e^{-2Lk}\right) \tag{3.19}$$

All this makes the 4D universe flat and static, since the four-dimensional brane sources are balanced by the 5D bulk cosmological constant $\Lambda$ so that the effective 4D cosmological constant is vanishing. This process of "flattening" the bulk by transferring the cosmological constant from the bulk to the brane is called "off-loading".

### 3.2.1 The tidally charged black hole

Finding nontrivial analytical solutions in the RS model is not an easy task. There exists anyway an exact solution for a static black hole localized on a 3-brane in 5D gravity [78]. The field equations in the bulk are given by

$$\tilde{G}_{AB} = \tilde{\kappa}^2\left[-\tilde{\Lambda}\tilde{g}_{AB} + \delta(y)(-\lambda gAB + T_{AB})\right] \tag{3.20}$$

where the tildes denote bulk quantities and capital indices run from 0 to 4, with $4 \equiv y$ the extra dimension. The 5D Planck mass $M_{\rm G}$ enters via $\tilde{\kappa}^2 = 8\pi/M_{\rm G}^3$. The brane tension is denoted by $\lambda$, whereas $\tilde{\Lambda}$ is the bulk cosmological constant. The brane is found at $y = 0$. The field equations induced on the brane can be derived from (3.20) and present themselves as a modification of the standard Einstein equations [77]:

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa^2 T_{\mu\nu} + \tilde{\kappa}^4 S_{\mu\nu} - \mathcal{E}_{\mu\nu} \tag{3.21}$$

where $\kappa^2 = 8\pi/M_{\rm P}^2$. One also finds the useful relations:

$$M_{\rm P} = \sqrt{\frac{3}{4\pi}}\left(\frac{M_{\rm G}^2}{\sqrt{\lambda}}\right)M_{\rm G}\ , \qquad \Lambda = \frac{4\pi}{M_{\rm G}^3}\left[\tilde{\Lambda} + \left(\frac{4\pi}{3M_{\rm G}^3}\right)\lambda^2\right]. \tag{3.22}$$

In vacuum one has $T_{\mu\nu} = S_{\mu\nu} = 0$, and in order to have the brane cosmological constant $\Lambda = 0$ we set the bulk cosmological constant to $\tilde{\Lambda} = -4\pi\lambda^2/3M_{\rm G}^3$. Doing so, eq. (3.21) reduces to

$$R_{\mu\nu} = -\mathcal{E}_{\mu\nu}\ , \qquad R_{\mu}^{\ \mu} = \mathcal{E}_{\mu}^{\ \mu} \tag{3.23}$$

where $\mathcal{E}_{\mu\nu}$ is the bulk Weyl tensor projeted on the brane. Together with this latter expression, a vacuum solution outside a mass localized on the brane must satisfy $\nabla^{\mu}\mathcal{E}_{\mu\nu} = 0$. In general we can decompose $\mathcal{E}_{\mu\nu}$ as

$$\mathcal{E}_{\mu\nu} = -\left(\frac{\tilde{\kappa}}{\kappa}\right)^4\left[\mathcal{U}(u_\mu u_\nu + \frac{1}{3}h_{\mu\nu}) + \mathcal{P}_{\mu\nu} + 2\mathcal{Q}_{(\mu}u_{\nu)}\right]\ . \tag{3.24}$$

Here $u_\mu$ is a chosen 4-velocity vector field, $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ and round brackets denote symmetrization. We also defined the energy density on the brane by

$$\mathcal{U} = -\left(\frac{\kappa}{\tilde{\kappa}}\right)^4 \mathcal{E}_{\mu\nu} u^\mu u^\nu \; . \tag{3.25}$$

This energy density arises on the brane from the free gravitational field in the bulk, and can be bot positive or negative. In the same way, there is an effective anisotropic stress on the brane:

$$\mathcal{P}_{\mu\nu} = -\left(\frac{\kappa}{\tilde{\kappa}}\right)^4 [h_{(\mu}{}^\alpha h_{\nu)}{}^\beta - \frac{1}{3} h_{\mu\nu} h^{\alpha\beta}] \mathcal{E}_{\alpha\beta} \; . \tag{3.26}$$

The energy flux from the free gravitational field in the bulk is represented by

$$\mathcal{Q}_\mu = \left(\frac{\kappa}{\tilde{\kappa}}\right)^4 h_\mu{}^\alpha \mathcal{E}_{\alpha\beta} u^\beta \; . \tag{3.27}$$

If $u^\mu$ is oriented along a Killing vector field, the vacuum is static and one has $\mathcal{Q}_\mu = 0$. We then define the 4-acceleration $A_\mu = u^\nu \nabla_\nu u_\mu$. If the spacetime is spherically symmetric and static, the one has

$$A_\mu = A(r) \hat{r}_\mu \; , \quad \mathcal{P}_{\mu\nu} = \mathcal{P}(r) [\hat{r}_\mu \hat{r}_\nu - \frac{1}{3} h_{\mu\nu}] \tag{3.28}$$

where $\hat{r}_\mu$ is a unit radial vector and $r$ is the areal radius. Using the conservation law for the Weyl tensor and the given decomposition, one finds

$$\mathcal{U} = \left(\frac{\kappa}{\tilde{\kappa}}\right)^4 \frac{Q}{r^4} = -\frac{1}{2} \mathcal{P} \; , \tag{3.29}$$

with $Q$ is the tidal charge to be determined. We now recall the most general spherically symmetric line element

$$ds^2 = -A(r) dt^2 + B(r) dr^2 + r^2 d\Omega^2 \tag{3.30}$$

and the modified Schwarzschild potential $\Phi$

$$\Phi = -\frac{M}{M_{\mathrm{P}}^2 r} + \frac{Q}{2r^2} \; . \tag{3.31}$$

Collecting all the above relations one arrives at

$$A = B^{-1} = 1 + \frac{\alpha}{r} + \frac{\beta}{r^2} \tag{3.32}$$

$$\mathcal{E}_t{}^t = \mathcal{E}_r{}^r = -\mathcal{E}_\theta{}^\theta = -\mathcal{E}_\phi{}^\phi = \frac{\beta}{r^4} \tag{3.33}$$

where $\alpha$ can be determined in the far-field Newtonian limit $\alpha = -2M/M_{\rm P}^2$ and $\beta$ is given by (3.24) and (3.29): $\beta = Q$. Finally, the metric of the tidally charged black hole reads:

$$ds^2 = -A(r)dt^2 + \frac{1}{A(r)}dr^2 + r^2 d\Omega^2 \tag{3.34}$$

$$A(r) = 1 - \left(\frac{2M}{M_{\rm P}^2}\right)\frac{1}{r} + \left(\frac{q}{M_{\rm G}^2}\right)\frac{1}{r^2} \tag{3.35}$$

where $q = QM_{\rm G}^2$ is a dimensionless tidal charge parameter.

The 4D horizons of this solution depends on the sign of $q$. For $q > 0$ there is a striking resemblance to the Reissner-Nördstrom solution for electrically charged black holes:

$$r_\pm = \frac{M}{M_{\rm P}^2}\left[1 \pm \sqrt{1 - q\frac{M_{\rm P}^4}{M^2 M_{\rm G}^2}}\right] . \tag{3.36}$$

Both horizons lie inside the Schwarzschild horizon: $0 \leq r_- \leq r_+ \leq r_S = 2M/M_{\rm P}^2$. The square root gives an upper limit for $q$:

$$q \leq q_{max} = \left(\frac{M_{\rm G}}{M_{\rm P}}\right)\left(\frac{M}{M_{\rm P}}\right)^2 . \tag{3.37}$$

The case qith $q < 0$ is not contemplated in 4D general relativity, but in this case there is one, with just one horizon:

$$r_+ = \frac{M}{M_{\rm P}^2}\left[1 + \sqrt{1 - q\frac{M_{\rm P}^4}{M^2 M_{\rm G}^2}}\right] > r_S . \tag{3.38}$$

The negative tidal charge then increase the horizon area and entropy, while reducing the black hole temperature. One important difference is that when $q < 0$ the singularity in the origin is spacelike just like in the usual Schwarzschild case, whereas for $q > 0$ the singularity is timelike.

## 3.3 Light black holes

Due to the lowering of the effective scale of gravity, microscopic black holes may be created in our accelerators [47, 49, 50, 51, 52] with a production cross section given, according to the hoop conjecture [53], by $\sigma \sim R_{\rm H}^2$ [49, 51], where $R_{\rm H}$ is the radius of the forming horizon and is bounded below by the wavelength of typical quantum fluctuations [49, 54, 55]. After the black hole has formed (and possible transients), the Hawking radiation [56] is expected to set off, with the most common description based on the canonical Planckian distribution for the emitted particles and consequent instantaneous decay [51]. This standard picture and a variety of refinements have been implemented in the most recent Monte Carlo codes [57] and the outcome is being confronted with Large Hadron Collider (LHC) data [58, 59]. One problem with the canonical description is that the black hole specific heat is in general negative and one should therefore use the more consistent microcanonical description [60, 61], which however requires an explicit counting of the black hole microscopic degrees of freedom (or degeneracy).

For this counting, one may appeal to the *area law* [62], from which it can be inferred the horizon area describes the black hole degeneracy [63, 64]. The area–entropy correspondence has inspired the holographic principle [65] in order to solve the black hole information paradox [66]. This principle has widely been developed [67], and a theoretical support was found in the AdS/CFT correspondence [68] that conjectures the equivalence of a string theory with gravity in anti-de Sitter space with a quantum field theory without gravity on the boundary. In this letter we shall analyze the interplay between the classicality condition that must be met in black hole formation and the horizon area as a measure of the entropy of black holes in the braneworld [48]. Our results allow for the existence of "lightweight" microscopic black holes (LBH) with mass below $M_{\rm G}$, or could explain the non existence of black holes within the reach of LHC experiments [58, 59] even if $M_{\rm G} \simeq 1$ TeV.

The issue concerning the lowering of the minimum mass scale to values that would make microscopic black hole production possible in our accelerators has been already considered from different points of view, for example, in Refs. [49, 69]. In particular, warping of the extra spatial dimension was identified in Ref. [69] as the possible cause of such lowering, which is precisely the Randall-Sundrum scenario [48] considered in the present work. Successful black hole production in that context was however shown to occur if particles collide with their own orbifold images, and the generality, or phenomenological relevance, of that argument therefore remains to be investigated. We must therefore point out that our approach here is different, in that we are trying to derive a minimum black hole mass from the geometrical properties of a candidate black hole metric discovered in Ref. [78], and we do so without modeling collision processes (at the present stage). Eventually, such processes will have to be investigated in more details, for example, by supplementing the arguments of Refs. [55, 69] with the conclusions drawn from the metric employed here, or by adapting the study of gravitational collapse previously considered in Ref. [70] to the case of microscopic black holes. In fact, only a fully dynamical description of black hole formation, a rather difficult task already at the purely classical level in any space-time dimensions, and theory of gravity, will provide the much sought-after final theoretical answer.

### 3.3.1 Compton classicality

A black hole is a classical space-time configuration and its production in a collider is therefore a "classicalization" process in which quantum mechanical particles are trapped by gravitational self-interaction within the horizon [49, 54, 55]. Consequently, quantum fluctuations should be negligible for the final state (of total energy $M$) which sources such a metric. A widely accepted condition of classicality is then expressed by assuming the Compton wavelength $\lambda_{\rm C} \simeq \hbar/M = \ell_{\rm P}\, M_{\rm P}/M$ [1] of the black hole, viewed as *one particle*, is the lower bound for the "would-be horizon radius" $R_{\rm H}$, that is

$$R_{\rm H} \gtrsim \lambda_{\rm C} \ , \tag{3.39}$$

where $R_{\rm H} = R_{\rm H}(M)$ depends on the specific black hole metric. In four dimensions, using the Schwarzschild metric, one obtains

$$R_{\rm H} = 2\,\ell_{\rm P}\,\frac{M}{M_{\rm P}} \gtrsim \ell_{\rm P}\,\frac{M_{\rm P}}{M} \quad \Rightarrow \quad M \gtrsim M_{\rm C} \simeq M_{\rm P} \ , \tag{3.40}$$

which is supported by perturbative calculations of scattering amplitudes for particles with centre-mass energy $M$ [55]. The above derivation does not make full use of the space-time geometry and, in particular, neglects that, for $M$ approaching the scale $M_{\rm G}$, quantum fields should be affected by extra-spatial dimensions (if they exist).

### 3.3.2 Entropic classicality

Another classicality argument can be given, which does not involve black hole wavefunctions, but relies on Bekenstein's conjectured correspondence between the entropy of thermodynamical systems and the area of black hole horizons [63]. Christoudolou [71] first pointed out that the irreducible mass $M_{\rm ir}$ of a Kerr black hole, *i.e.* the amount of energy that cannot be converted into work by means of the Penrose process [72], is related to the horizon area $A$ as $M_{\rm ir} = M_{\rm P}\,\sqrt{A/16\,\pi\,\ell_{\rm P}^2} \equiv \sqrt{A_{\rm ir}}$. Now, in thermodynamics, an increase in entropy is associated with a degradation of energy because the work we can extract from the system is reduced. The similarity is clear,

[1] We shall mostly use units with the Boltzmann constant $k_{\rm B} = c = 1$, $G_{\rm N} = \ell_{\rm P}/M_{\rm P}$ and $\hbar = \ell_{\rm P}\, M_{\rm P} = \ell_{\rm G}\, M_{\rm G}$.

but goes beyond this simple statement. For a Schwarzschild black hole, $M_{\rm ir} = M$ and no energy at all can be extracted. Nonetheless, we can take a collection of fully degraded subsystems (Schwarzschild black holes) and still get some work out of them. In fact, if we merge two or more black holes, the total horizon area must equal at least the sum of all their original areas [62]. Denoting by $M_i$ and $A_i$ the initial irreducible masses and areas, and by $M_F$ and $A_F$ the final irreducible mass and horizon area, we see that $M_F = \sqrt{A_F} = \sqrt{\sum_i A_i} < \sum_i \sqrt{A_i} = \sum_i M_i$. The final irreducible mass is then less than the sum of all the initial irreducible masses: some more work can be extracted by merging fully degraded black holes. The same occurs by collecting thermodynamical systems that – individually – are fully degraded but together can still provide work. Bekenstein [63] remarked how we can clarify these similarities by invoking Shannon's entropy

$$S = -\sum_n p_n \ln p_n \ , \tag{3.41}$$

where $p_n$ is the probability for a thermodynamical system to be found in the $n$-th state. A thermodynamical system is described in terms of a few macroscopical variables (like energy, temperature and pressure). Once these variables are fixed, the system can nevertheless be described by a huge amount of *microscopically inequivalent* states. Hence, entropy can be seen as the lack of information about the actual internal structure of the system. Analogously, any four-dimensional black hole can be described in terms of three macroscopic variables: mass, angular momentum and charge. All information about the matter which formed the black hole is lost beyond the horizon. Because of properties shared by thermodynamical entropy and horizon area, Bekenstein found the simplest expression (with dimensions of $\hbar$) which satisfies the conditions on the irreducible mass is

$$S_{\rm BH} = \frac{M_{\rm P}\, A}{16\,\pi\,\ell_{\rm P}} \ . \tag{3.42}$$

Using a *gedanken experiment*, Bekenstein [73] further obtained the so-called entropy bound $S_{\rm BH} \leq 2\,\pi\,R_{\rm H}\,M$, and this topic has by now been extended to more general scenarios (for a review, see Ref. [74]).

From Eq. (3.42), we can now infer an *entropic condition* for black hole classicality: a four-dimensional classical black hole should have a large degeneracy (in units of the Planck scale), that is

$$\tilde{S}^{\rm E}_{(4)} \equiv \frac{S_{\rm BH}}{\ell_{\rm P}\, M_{\rm P}} = \frac{4\,\pi\,R_{\rm H}^2}{16\,\pi\,\ell_{\rm P}^2} \simeq \left(\frac{M}{M_{\rm eff}}\right)^2 \gtrsim 1 \ , \tag{3.43}$$

which, for the Schwarzschild metric, leads to $M \gtrsim M_{\rm eff} \simeq M_{\rm P}$. This conclusion is also supported by perturbative calculations of scattering amplitudes, since the entropy (3.42) can be reproduced by assuming the final classical black holes are composed of quanta with wavelength $\lambda \sim R_{\rm H}$ [55]. Note, however, that the physical meaning of the two scales is not quite the same: $M_{\rm eff}$ is the natural unit for measuring black hole internal degrees of freedom (like the gap between energy levels of the harmonic oscillator), whereas $M_{\rm C}$ is the minimum mass below which black holes do not exist (like the threshold in massive particle production). That $M_{\rm eff} \simeq M_{\rm C} \simeq M_{\rm P}$ is expected – because gravity in four dimensions entails one scale – but is till a remarkable evidence that black holes hide most information about forming matter.

### 3.3.3 ADD black holes

Both classicality conditions (3.40) and (3.43) can be straightforwardly generalized to models with extra-spatial dimensions by replacing $M_{\rm P}$ and $\ell_{\rm P}$ with $M_{\rm G}$ and $\ell_{\rm G}$, and using the appropriate expressions for the horizon radius. For example, in the ADD scenario of Refs. [46], the brane tension is neglected and one can therefore consider vacuum solutions to the Einstein equations in $4+d$ dimensions to derive the following relation between the mass and horizon radius [51],

$$R_{\rm H} = \frac{\ell_{\rm G}}{\sqrt{\pi}} \left(\frac{M}{M_{\rm G}}\right)^{\frac{1}{1+d}} \left(\frac{8\,\Gamma\left(\frac{d+3}{2}\right)}{2+d}\right)^{\frac{1}{1+d}} , \tag{3.44}$$

where $\Gamma$ is the usual Gamma function. Inserting the above into Eq. (3.39) yields [2]

$$R_{\rm H} \gtrsim \ell_{\rm G}\,\frac{M_{\rm G}}{M} \qquad \Rightarrow \qquad M \gtrsim M_{\rm C} \simeq M_{\rm G} \ , \tag{3.45}$$

as one would naively expect. Moreover, the same result is again obtained by generalizing the entropic argument to $4+d$ dimensions, namely

$$\tilde{S}^{\rm E}_{(4+d)} \simeq \left(\frac{R_{\rm H}}{\ell_{\rm G}}\right)^{2+d} \sim \left(\frac{M}{M_{\rm eff}}\right)^{\frac{2+d}{1+d}} \gtrsim 1 \ , \tag{3.46}$$

where $M_{\rm eff} \simeq M_{\rm G}$. One therefore concludes that even in the ADD scenario, gravity enters black hole physics with one scale, $M_{\rm G}$, like in four dimensions.

[2]This is the kind of condition employed in all Monte Carlo studies of black hole production at the LHC [57].

### 3.3.4 Brane-world black holes

The situation appears more involved in the brane-world (RS) scenario [48], in which the brane tension is not ignored and the bulk is consequently warped. This has made it very hard to describe black holes [75] [3], and only a few analytical candidates are known which solve the effective four-dimensional vacuum Einstein equations [77]. We just remind the tidally charged metric we worked out before:

$$ds^2 = -A\,dt^2 + A^{-1}\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right) \ , \tag{3.47}$$

with

$$A = 1 - \frac{2\,\ell_{\rm P}\,M}{M_{\rm P}\,r} - q\,\frac{\ell_{\rm G}^2}{r^2} \ , \tag{3.48}$$

which has been extensively studied in Refs. [61, 79, 80, 81]. In the above and what follows, the tidal charge $q$ and $M$ are treated as independent quantities, although one expects $q$ vanishes when the black hole mass $M = 0$ (that is, when there is no black hole). Further, we only consider the case $q > 0$ since negative tidal charge would yield anti-gravity effects [80]. A relation $q = q(M)$ should be obtained by solving the complete five-dimensional Einstein equations [81] (or by means of supplementary arguments [82]). Nevertheless, since we are here interested in black holes near their minimum possible mass $M_{\rm C} \sim M_{\rm G} \ll M_{\rm P}$, we can approximate $q \simeq q(M_{\rm C})$ and constant, and expand all final expressions for $M \sim M_{\rm G} \ll M_{\rm P}$.

We can first apply the usual classicality argument (3.39), with the horizon radius

$$R_{\rm H} = \ell_{\rm P}\left(\frac{M}{M_{\rm P}} + \sqrt{\frac{M^2}{M_{\rm P}^2} + q\,\frac{M_{\rm P}^2}{M_{\rm G}^2}}\right) \ , \tag{3.49}$$

and obtain $M \gtrsim M_{\rm C}$, where the minimum mass

$$M_{\rm C} \simeq \frac{M_{\rm G}}{\sqrt{q}} \ , \tag{3.50}$$

for $M \sim M_{\rm G} \ll M_{\rm P}$.

We can also repeat the entropic argument by employing the effective four-dimensional action, namely

$$\tilde{S}^{\rm E}_{\rm eff} \simeq \frac{4\,\pi\,R_{\rm H}^2}{16\,\pi\,\ell_{\rm P}^2} \ , \tag{3.51}$$

[3] Arguments have been formulated against the existence of static brane-world black hole metrics [76]. Given the Hawking radiation is likely a strong effect for microscopic black holes, their instability is here taken as granted.

which, however, is not at a minimum for $M \simeq M_\mathrm{C}$, as one would instead expect from previous cases. This discrepancy can be cured by recalling the Euclidean action (as well as the thermodynamical entropy) is defined modulo constant terms, which, for example, do not affect the value of the Hawking temperature nor the microcanonical description of the Hawking radiation [60]. By subtracting from Eq. (3.51) a suitable constant, namely

$$\tilde{S}^\mathrm{E}_\mathrm{sub} = \tilde{S}^\mathrm{E}_\mathrm{eff}(M) - \tilde{S}^\mathrm{E}_\mathrm{eff}(M_\mathrm{C}) \ , \tag{3.52}$$

and expanding for $M \sim M_\mathrm{G} \ll M_\mathrm{P}$, we finally obtain

$$\tilde{S}^\mathrm{E}_\mathrm{sub} \simeq \frac{M}{M_\mathrm{eff}} \ , \tag{3.53}$$

where there now appears the effective degeneracy scale

$$M_\mathrm{eff} \simeq \frac{M_\mathrm{G}}{\sqrt{q}} \ . \tag{3.54}$$

It is again remarkable that $M_\mathrm{eff} \simeq M_\mathrm{C}$ and brane-world black holes are also described by one scale [recall that $q \simeq q(M_\mathrm{C})$ is not truly independent]. The "natural" choice would now be $q \simeq 1$, so that $M_\mathrm{eff} \simeq M_\mathrm{C} \simeq M_\mathrm{G}$, but the effective scale $M_\mathrm{eff} \simeq M_\mathrm{C}$ could also be either larger ($q \ll 1$) or smaller ($q \gg 1$) than $M_\mathrm{G}$.

## 3.4 Conclusions and outlook

Detection of black holes at the LHC would be a clear signal that we are embedded in a higher-dimensional space-time and the fundamental scale of gravity $M_\mathrm{G} \simeq 1\,\mathrm{TeV}$. The existence of extra spatial dimensions could also be uncovered by means of particle processes which involve the exchange of bulk gravitons [84]. Such processes are perturbatively described by operators suppressed by inverse powers of $M_\mathrm{G}$, and might not be detectable if the latter is larger than a few TeV [85]. We have shown that both classicality conditions, from quantum mechanics and the entropic counting of internal degrees of freedom, allow for brane-world black holes with minimum mass (3.50). The latter might be different from the fundamental scale $M_\mathrm{G}$, if $q$ departs significantly from 1 (which should be related to details of the mechanism confining standard model particles and four-dimensional modes of gravity on the brane). This introduces two alternative scenarios:

i) for $q \gg 1$, "lightweight black holes" (LBH) with $M_\mathrm{C} \lesssim M \lesssim M_\mathrm{G}$ may exist and be produced at the LHC even if $M_\mathrm{G} \gtrsim 10\,\mathrm{TeV}$. In this case, the effects due to bulk graviton exchanges would remain undetected;

ii) if $q \ll 1$, black holes do not exist with $M \simeq M_{\rm G}$, even if $M_{\rm G} \simeq 1\,$TeV, and processes involving bulk gravitons are going to be the only available signature of extra-spatial dimensions.

The former scenario might have important phenomenological implications both for accelerator physics and in astrophysics. Although recent LHC data at 7 TeV center mass energy seem to exclude the production of microscopic black holes [58, 59], there is still the possibility that future runs at 14 TeV will achieve this goal. Further, LHB might play a role in cosmological models as primordial black holes produced in the early universe, and in astrophysics as the outcome of high energy cosmic rays colliding against dense stars [86]. The case of $q \ll 1$ might instead explain why there is no evidence of black holes at the LHC [58, 59], even if extra spatial dimensions exist.

While only data collected by the LHC or cosmic rays detectors will finally assess which of these scenarios is correct (if any), there clearly remains a fair amount of theoretical work ahead of us. For example, one may hope to estimate the dependence of the tidal charge $q$ on other physical parameters, such as the mass, by studying stars and black holes of astrophysical size [82], and then devise some mathematically sensible limiting procedure to reduce the mass down to the TeV range. And, eventually, one will have to tackle the problem of black hole formation by collision or collapse in the brane-world [69, 70], using an approach which keeps the size of quantum fluctuations under control and is still mathematically manageable. A problem which, as we recalled in the Introduction, appears rather severe in any (classical) theory of gravity.

# Chapter 4

# Final remarks

This doctoral thesis is a collection of three original researches on various subjects.

The first chapter is more "cosmology-oriented" and the original results in it are purely relativistic (i.e. geometrical) in nature. Anyway, the shell model is quite adaptable to a variety of cases, even black hole evaporation if we assume that the final stages of emission consist in a massive spherically symmetric outburst of particles, forming a shell. We successfully found a perturbative solution to a problem that has never been treated because considered too complicated. Present day applications mostly use the thin shell formalism to describe gravitational collapse or wormhole formation, but they always restrict to the simple vacuum case. Our hope is that the procedure we outlined will be of some help in facing more realistic, non-vacuum toy models in the future.

In the second chapter we have used a typically semiclassical effect (the Unruh effect) to investigate the qualities of a geometrical object, the Kodama vector. By simply considering its geometrical properties, not much can be said on the nature of the Kodama vector and its possible uses. The only way to better understand the Kodama vector is a face to face comparison with the Killing vector on different proving grounds. In the semiclassical framework, in some cases, it can provide different temperature values for an horizon: the temperature is redshifted by some function with respect to the original results. In the meanwhile, more research has been done on this vector field and results are going to be resumed and published.

The third chapter is more experimentally oriented and casts predictions on the possible detection of microscopic black holes in particle accelerators. The tidally charged black hole is the simplest analytical solution of the Randall-Sundrum model but is still not perfectly clear. The tidal charge has a very important role in the horizon formation and at the same time is not well understood.

By one side it was certainly not easy to frame such different topics in one unique work. On the other hand this gives an idea of the powerful tools provided by the semiclassical theory of gravity: its relative simplicity (both conceptual and technical) make us able to investigate the most different features of borderline physics. We still have so much to learn (and verify) from semiclassical gravity that any deeper inquiry might seem premature.

Our hope is to detect the first signs of black hole formation and evaporation at LHC in the next years. This might never happen of course, and in this case we will have to face very important issues:

- what if the scale of gravity is really that far from the TeV scale?
- what if the scale of gravity is just few TeV out of reach? Is it worth to build new accelerators?
- what if the scale of gravity is actually reachable but no semiclassical effects are produced or recognized?

Different scenarios would open in each case, everyone carrying with itself a whole “bouqet” of new theories and predictions. If nothing new is going be observed in the next years, scientists will not be discouraged anyway. Physics research needs affirmative results to achieve victory, but it is not success that pushes physicists. It is challenge, indeed.

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