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ASYMPTOTIC BEHAVIOUR OF ZERO MASS FIELDS WITH SPIN 1 OR 2 PROPAGATING ON CURVED BACKGROUND SPACETIMES

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Asymptotic Behaviour of Zero Mass Fields of spin 1 and 2 Propagating on Background Curved Spacetimes

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0.1 Introduction

The problem we are going to investigate in this work concerns the asymptotic behavior of the solutions of a particular class of equations, the zero-rest mass field equations with spin $s = 1, 2$. The reasons why we are interested on this problem are connected to some open problems associated to the General Relativity Einstein equations. Let us begin recalling some basic definitions and some important notions of the Einstein equations connected to the arguments discussed in this thesis.

Definition 0.1.1. *An Einstein spacetime, $(\mathcal{M}, g)$, is a 4-dimensional manifold $\mathcal{M}$ equipped with a Lorentz metric g which satisfies the Einstein's equations:*

$$G_{\mu\nu}(g) = 8\pi T_{\mu\nu} \ . \tag{0.1.1}$$

G is the Einstein tensor, defined from the Ricci tensor $R_{\mu\nu}$ and its trace part R, the scalar curvature of $\mathcal{M}$, through the following relation

$$G_{\mu\nu}(g) = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

and $T_{\mu\nu}$ is the energy-momentum tensor relative to the matter and to the electromagnetic fields present in the spacetime.[1]

The Einstein equations are a set of 10 partial differential equations, where the unknown functions are the components of the metric tensor g, as one can see choosing a coordinate system and expressing, in the coordinate basis, the components $R_{\mu\nu}$ of the Ricci tensor in terms of the metric tensor g. They connect the Ricci and the scalar part of the curvature tensor, associated to the metric g, to the matter and the electromagnetic field present in the spacetime. Due to their intrinsic tensorial nature, any possible solution of them is defined up to isometries, which correspond to every possible change of coordinates of the spacetime (see also [6], chapter 7).
The solutions of the set of equations

$$R_{\mu\nu} = 0$$

[1] In this work we adopt the Einstein index notation, where $\mu, \nu = 0, 1, 2, 3$ and the repeated high and low indices are thought as summed.

define the so called "Vacuum Einstein spacetimes", that is spacetimes where it is not present any matter nor electromagnetic field; in fact they correspond to the set

$$T_{\mu\nu} = 0.$$

Hereafter we will concentrate on the vacuum Einstein's equations; they are somewhat easier from a technical point of view, but they are, nevertheless physically very interesting, as they describe the nature of a spacetime region where no stellar bodies are present.
Even in the vacuum case, nevertheless, the Einstein equations are far to be easy. This is due to their intrinsic non linear nature which can be immediately recognized looking at the explicit expression of the Ricci tensor in terms of the metric components and their first and second partial derivatives. Once these equations are solved, assuming initial data have been assigned, from the metric g, solution of these equations, we can obtain the conformal part of the Riemann tensor $C_{\mu\nu\rho\sigma}$ which, in the vacuum case, coincides with the whole tensor, $C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}$, and satisfy the Bianchi equations:

$$D^{\mu} R_{\mu\nu\rho\sigma} = 0 \ . \tag{0.1.2}$$

In such a general form, Einstein's equations are very difficult to solve, in fact only a small number of exact solutions is known, all of them obtained imposing some particular symmetry conditions on the metric.
Let us recall some of them, associated to a single massive body:
The Minkowski spacetime, the spacetime of special relativity, whose metric is the Lorentzian metric with eigenvalues $(-1, 1, 1, 1)$. Here no matter is present nor any field energy, moreover it is invariant under the transformations of the extended Poincaré group, made by the rotations, the (spacetime) translations and the Lorentz transformations.
The Schwarzschild solution, discovered in 1916 ([17]), whose "external part" describes the spacetime of a static, spherically symmetric body, such as a star or as a black body with angular momentum equal zero.
The Kerr spacetime, discovered only in 1965 ([10]), a spacetime symmetric with respect to time translations and rotations with respect to a fixed axis. Physically it has a very important meaning, as it describes the spacetime geometry generated around a rotating body and, in particular, around a rotating black hole.
As these solutions are time independent, it is meaningless to talk for them of an evolution problem, but, viceversa, looking for more general solutions of

0.1.1, we have to solve an evolution problem with some suitable initial data. In order to define the Cauchy problem in General Relativity, let us make the following definitions:

Definition 0.1.2. *An initial data set is given by a set $\{\Sigma, \bar{g}, \bar{k}, \bar{\psi}\}$ where Σ is a three dimensional manifold, $\bar{\psi}$ is the prescribed matter field on it, $\bar{g}$ is a riemannian metric, and $\bar{k}$ is a covariant symmetric tensor field satisfying the constraint equations:*

$$\begin{aligned} \nabla^j \bar{k}_{ij} - \nabla_i tr\bar{k} &= 8\pi T_{0i}(\bar{\psi}) \\ \bar{R} - |\bar{k}|^2 + (tr\bar{k})^2 &= 16\pi T_{00}(\bar{\psi}), \end{aligned}$$

where $\bar{R}$ is the scalar curvature of $\bar{g}$.
Two initial data set are said to be equivalent if there exists a diffeomorphism in Σ which maps the first set made by the metric, the matter fields and the covariant symmetric tensor field $\bar{k}$ in the second set of initial data.

To solve the Einstein field equations with a given initial data set means finding a four-dimensional manifold $\mathcal{M}$, a Lorentz metric g and fields ψ satisfying the coupled Einstein equations as well as an imbedding

$$i : \Sigma \to \mathcal{M}$$

such that $i^*(g) = \bar{g}$, $i^*(k) = \bar{k}$, $i^*(\psi) = \bar{\psi}$, where g is the induced metric and k is the second fundamental form of the submanifold $i(\Sigma) \subset \mathcal{M}$. Two equivalent initial data sets are supposed to lead to equivalent solutions.
This definition of the Cauchy problem for the Einstein equations is "coordinate independent". If we choose a specific set of coordinates the Cauchy problem can be rephrased in the following way: Let us give, as initial data, $g_{\mu\nu}$ and $\frac{\partial g_{\mu\nu}}{\partial x_0}$ on a three dimensional surface defined by $x_0 = t$. The Einstein's equations written in this set of coordinates are a second order quasilinear system made by ten equations for the ten components of the metric g. The tensorial character of the Einstein equations imply, nevertheless, that four of the ten equations are "constraint equations" which do not depend on the second time derivatives of the metric g. It is well known that for any solutions of the remaining six (evolution) equations satisfying the constraint equations at $t = 0$, the constraint equations are automatically solved for all the remaining times. This has a twofold consequence: first the system

is in some sense underdetermined; this is not a drawback of the theory as due to the general covariance of the theory it is expected that a well defined spacetime corresponds to a whole family of solutions connected by diffeomorphisms which do not change the initial data. Second, the possibility of looking at the Einstein equations in a specific choice of coordinates allow to write them in a form suitable to use the mathematical knowledge and results about the quasilinear hyperbolic partial differential equations.
The existence of local (in time) solutions with generic initial data was proved first by Yvonne Chioquet-Bruhat in 1952 (see [1]), who studied Einstein's equations in a special set of coordinates, often said a specific gauge, the wavelike coordinates (or the harmonic gauge).
In these coordinates the Einstein equations have the form of a quasilinear hyperbolic system, then one can apply standard results of the theory of hyperbolic partial differential equations to show the existence and uniqueness of the solution.

In particular we know there exists a family of integral quantities written in terms of the first derivatives of the solution, called energy norms, and the existence proof is mainly based on proving the boundedness of these quantities. Roughly speaking, it can be shown that the solution of a quasilinear hyperbolic system does exist for all the times for which the energy norms are bounded (see [7], [18]).
As far as the global existence is concerned, it is very complicate to prove it, even if we require some smallness condition on initial data. In fact the first global solutions with generic small initial data were discovered only in 1993, by D. Christodoulou and S. Klainerman ([3]), who have shown the global existence of solutions with initial data "near" the Minkowski spacetime initial data. [2]

In the sequel of this discussion our attention will be concentrated on the asymptotically flat solutions: these are solutions whose initial data are given by a riemannian manifold equipped with a metric such that, outside a sufficiently large compact region (and in a determined coordinates set), it approaches asymptotically the flat metric, with a decay not faster than $\frac{1}{r}$.

The metric tensor of these spacetimes satisfies the Einstein vacuum equations and physically they can be interpreted as those generated by isolated system (without any symmetry condition imposed ab initio). To obtain them

[2]A previous work faced with different techniques (conformal compactification) is due to Friedrich, see [5].

one has to solve the vacuum Einstein equations globally. This means to prove global existence results for a complicated quasilinear set of equations of "hyperbolic type."[3] All the global existence results known up to now require "small initial data". Moreover once a global result is obtained then one can investigate the asymptotic structure of these spacetimes far from the isolated body. This is relevant as it is connected to the existence of gravitational waves propagating toward the (future) infinity.
Since the seventies there existed some conjectures and theorems due to R.Penrose and E.T.Newman (see [14], [15]) concerning the asymptotic behavior of the Riemann tensor for some well defined families of spacetimes. More precisely they introduced the notion of asymptotically simple spacetime, a generalization and a mathematically more precise formulation of the asymptotically flat spacetime. They are defined in the following way:

Definition 0.1.3. *Let $(\mathcal{M}, g)$ be a spacetime. It is said asymptotically simple if there exists an other manifold $\tilde{\mathcal{M}}$ with metric $\tilde{g}$ and a conformal isometry Ω such that:*
i) $\mathcal{M}$ is a submanifold of $\tilde{\mathcal{M}}$ with $\partial\mathcal{M}$ $\mathcal{C}^\infty$ boundary.
ii) $\Omega : \mathcal{M} \rightarrow \mathbb{R}$ has a smooth extension on $\bar{\mathcal{M}}$.
iii) Onto $\partial\mathcal{M}$, $\Omega = 0$ and $\nabla_a\Omega \neq 0$. iv) Every null geodesic in $\mathcal{M}$ has an initial and a final point in $\partial\mathcal{M}$.
v) $R_{ab} = 0$ in a neighborhood of $\partial\mathcal{M}$.

To analyze the behaviour at the (null) infinity of the asymptotically simple spacetimes is much easier because the whole "physical" spacetime is mapped into a finite region of a larger "unphysical" one. A consequence of this approach is the "Peeling theorem" (see [16]), which prescribes the asymptotic (null) behavior of zero-rest mass fields with any spin s propagating themselves in the asymptotically simple spacetimes along the directions of the null infinity. More specifically, a zero-rest mass fields of spin s is a tensor field $T_{\mu_1\mu_2....\mu_{2s}}$ which satisfies the equations:

$$D^{\mu_1}T_{\mu_1\mu_2...\mu_{2s}}.$$

We observe that, for $s = 2$, they correspond to the Bianchi equations satisfied by the Riemann tensor of a vacuum spacetime and for $s = 1$, together with the set of equations

$$D^{\mu *}F_{\mu\nu} = 0,$$

[3]The sense in which these equations are of hyperbolic type will be discussed later on.

(where ${}^*F_{\mu\nu}$ is its left Hodge dual (see (5.3.17)), they are just the Maxwell equations for an electromagnetic field propagating in the vacuum. The "Peeling theorem" tells us also how the various components of the Riemann tensor of an asymptotically spacetime have to decay at null infinity. Therefore if one has a global existence solution of the vacuum Einstein equations one also knows the asymptotic behaviour of the metric tensor toward the null infinity. Therefore as such behavior will depend on the asymptotic behavior of the initial data on the initial hypersurface $t = 0$, one has a necessary condition on the initial data of the Einstein equations to produce asymptotically simple spacetimes. These data in fact must be such that the global spacetime satisfies the "Peeling theorem". [4]

Only for Minkowski and Schwarzschild spacetimes it is explicitly known the conformal isometry Ω compactifying them, but recently S.Klainerman and F.Nicolò have shown under what decay and smallness hypothesis on the initial data the nonlinear perturbations of Minkowski and Schwarzschild spacetimes satisfy the Peeling theorem (see [12]).

As far as the Kerr spacetime is concerned, it is unknown if this spacetime is asymptotically simple, because one does not have the conformal isometry Ω, but an explicit calculation of the Riemann tensor shows that it satisfies the "peeling decay", therefore, at least the necessary condition for being asymptotically simple is satisfied.

A still open problem is to prove that some suitable nonlinear perturbations of Kerr spacetime satisfy the Peeling theorem. As said before this requires a global existence result, satisfying the peeling, for the vacuum Einstein equations with initial data near (in some appropriate norms) to the Kerr metric on the initial spacelike hypersurface. This result, even if we restrict ourselves to an external region far from the influence domain of a compact on Σ_0, is not at our disposal. More precisely the global existence result proved by S.Klainerman and F.Nicolò with initial data near the flat ones (see [11]) can be applied also to data near to the Kerr initial data only with very small angular momentum, but this result does not satisfy the peeling. The subsequent work by by S.Klainerman and F.Nicolò which proves the peeling under stronger asymptotic conditions for the initial data (see [12]) cannot be applied to initial data "near" to Kerr, the main difficulty being connected to

[4]The Newman-Penrose Peeling theorem is not a constructive result. In other words the definition of asymptotically simple spacetimes is given imposing some conditions on the "conformal" null infinity. Therefore to connect the Peeling asymptotic behaviour to the initial data requires a global existence proof for the Einstein vacuum equations.

the fact that the angular moment J of the Kerr spacetime is different from zero; in other words the Kerr spacetime is not spherical symmetric, but only axially symmetric.

The work of this thesis is connected to this problem and can be seen as an intermediate step toward its solution. In other words the results I am going to present, beside their intrinsic meaning, can be interpreted as a preliminary step toward the proof of peeling for spacetimes near to the Kerr spacetime. More precisely the following two results are proved:

1) The solutions of the massless spin-2 field equations, the Bianchi equations, with the Kerr spacetime as background spacetime, satisfy the decays prescribed by the Peeling theorem in the future null infinity direction.
2) The solutions of the Maxwell equations in the vacuum with the (external part of the) Schwarzschild spacetime as background spacetime, satisfy the Peeling theorem, without any strong condition on their initial data.

The strategy used in the proof follows the approach introduced by Klainerman and Nicolò in their proof of global stability of Minkowski spacetime [11] and in the subsequent paper [12]. Let us give a short summary of it. They do not use the wavelike (harmonic) gauge, because with them it is very difficult to get the needed apriori estimates for the energy norms at any time (see, nevertheless the recent results of H.Linblad and I.Rodniansky) (see [13]), rather it is based on a more geometrical approach. They foliate the spacetime with a double null foliation, the equivalent of the ingoing and outgoing null cones of Minkowski spacetime, and introduce a suitable set of coordinates and a null frame adapted to this foliation. The Riemann tensor, (let us indicate it with $\mathcal{R}$), is decomposed with respect to the null frame associated to the foliation and the connection coefficients relative to the null frame (we call them $\mathcal{O}$) are introduced. The Bianchi equations are written as equations for the Riemann components and the Einstein equations are written as a system of transport equations along the null directions for the connection coefficients $\mathcal{O}$ (equations which also depend on the Riemann null components $\mathcal{R}$). In this way the Einstein equations and the Bianchi equations appear as a coupled nonlinear system even if, in fact, the Einstein equations are equations for the metric components $g_{\mu\nu}$ and the Bianchi equations are automatically satisfied once the spacetime metric g is assigned. Let us indicate it, symbolically, in the following way:

$$\begin{cases} \frac{\partial}{\partial(u,\underline{u})}\mathcal{O} = \mathcal{F}(\mathcal{O}) + \mathcal{R} \\ \frac{\partial}{\partial(u,\underline{u})}\mathcal{R} = \mathcal{O}\mathcal{R} \ . \end{cases}$$

The global existence is proved by a bootstrap mechanism. One assumes that a maximal finite region exists where appropriate norms for $\mathcal{O}$ and $\mathcal{R}$ are bounded by a sufficiently small constant, then using the second set of equations one proves that the norms of $\mathcal{R}$ can be bounded in terms of the same norms written in terms of the initial data and, therefore, choosing the initial data sufficiently small, made even smaller; this at its turn allows to prove that, using the first set of equations, even the norms of $\mathcal{O}$ can be made smaller than the previously chosen constant. This allows to slightly enlarge the previous finite region which, therefore, is not the maximal one. By contradiction this implies that the region where the norms are bounded is in fact the whole "infinite" spacetime. The key ingredient to perform this bootstrap mechanism is, therefore, the possibility of expressing the norms of the Riemann null components $\mathcal{R}$ in terms of the same norms written for the initial data.
The fundamental step to get this result is the introduction of a family of integral norms, L^2 norms made along the null directions, which are a generalization of the classical energy norms. The main difference with the usual procedure for the non linear hyperbolic equations is that they are not expressed in terms of g and its first derivatives but in terms of the Bel-Robinson tensor (built in terms of the Riemann tensor). The crucial fact is, therefore, having them finite and small at the initial time $t = 0$ and then to prove their boundedness in the whole region.

Going back to this thesis work, it can be interpreted as a linearized part of this approach as it amounts to studying of Bianchi equations with respect to an assigned metric, the Kerr metric, that is in a assigned background spacetime, the (external part of the) Kerr spacetime. Therefore $\mathcal{R}$ become the null components of an external Weyl field, independent from the metric g, with null mass and spin 2. The structure equations are not needed now as the connection coefficients of the Kerr spacetime are known explicitly and we have only to investigate the Bianchi equation solutions.
Nevertheless, looking at the asymptotic behavior of a zero-rest mass field with spin 2 (and with spin 1 too, as we discuss later on), in this approach,

we discovered that four of the ten null components of Riemann tensor do not decay as suggested from the Peeling theorem, but in a lower way. From the previous considerations and the results of S.Klainerman and F.Nicolò for the perturbed Minkowski and Schwarzschild spacetimes (see [12]), it was reasonable to expect that, improving the techniques used in the proof, the bad asymptotic behaviors could be improved. Therefore we have modified the previously defined generalized energy norms by inserting a weight factor of the form $u^{5+\epsilon}$ (the reason for the choice of this factor will be discussed in the next chapters) , where u is the retarded optical function (the equivalent of $t-r$ in Minkowski spacetime). This weight factors should allow, with a long procedure discussed in detail later on, to get better decays for the Riemann components. Nevertheless with this norm modification a different problem arises; in fact as our spin 2 fields, W, should mimic in this linearized version, the behaviour of the Kerr Riemann tensor, its initial data should be assigned with the same (spatial) asymptotic behaviour, a behaviour which is not compatible with these new norms which become infinite. This is due to those terms of the Kerr metric tensor associated to the angular moment, which decay too slowly.
A way to exclude this term is based again on the underlined idea that the problem we are studying has to be connected to the problem of solving the Einstein equations for initial data near to the Kerr spacetime. Therefore we can look for a solution W of the following form

$$W = W^{(Kerr)} + \delta W. \tag{0.1.3}$$

where $W^{(Kerr)}$ is the Riemann tensor of the Kerr spacetime. In this way the Bianchi equations become:

$$D^{\mu}(\delta W)_{\mu\nu\rho\sigma} = 0,$$

as D is the covariant derivative with respect to the Kerr background metric and, therefore, $D^{\mu}W^{(Kerr)}{}_{\mu\nu\rho\sigma} = 0$. This does not seem very interesting due to the linearity of the problem and to the fact that we are free to impose an arbitrary decay on δW on the initial hypersurface. If we want that the problem we are considering, better mimics part of the associated Einstein problem, observing that in that case the Bianchi equations are not linear equations (the connection coefficients through the first transport equations depend on $\mathcal{R}$), it is more natural to look for a non linear spin 2-field equation, namely, recalling again that $D^{\mu}W^{(Kerr)}{}_{\mu\nu\rho\sigma} = 0$,

$$D^{\mu}(\delta W)_{\mu\nu\rho\sigma} = J(W^{Kerr} + \delta W)_{\nu\rho\sigma}$$

where the "current J is a non linear term for the whole field W.
If the non linear term is chosen sufficiently good we can expect to be able to prove, outside the "evolution region" of a compact ball of radius R, the global existence of the field W showing that the generalized energy norms are bounded for any t. To prove the peeling the same results should be proved for the analogous norms with the extra weight factors $u^{5+\epsilon}$. This can be investigated in a straightforward manner, but in this thesis a slightly different approach is chosen. In fact we study the non linear spin 2 equations for the tensor field $\mathcal{L}_{T_0}W$ instead than W, where $\mathcal{L}_{T_0}$ is the Lie derivative done with respect the Killing vector field of the Kerr spacetime generating the time symmetries. As the Kerr spacetime is static, two important facts follow:

$$\mathcal{L}_{T_0}W_{Kerr} = 0$$

and D commutes with $\mathcal{L}_{T_0}$,

$$[D, \mathcal{L}_{T_0}] = 0 \ .$$

The field equations for $\mathcal{L}_{T_0}W$ become, in this case,

$$D^{\mu}\mathcal{L}_{T_0}\delta W = H_{\nu\rho\sigma}(\mathcal{L}_{T_0}\delta W) \ .$$

In this thesis it is proved that any solution I can obtain globally, requiring suitable initial decays for the tensor $\mathcal{L}_{T_0}\delta W$, decays in a way consistent with the Peeling theorem and once we have $\mathcal{L}_{T_0}\delta W$ we obtain δW basically through an integration in the time and then again W adding to it $W^{(Kerr)}$. Therefore the final result is that for an appropriate class of solutions, depending on the initial data of δW, the asymptotic behaviour is in agreement with the Peeling theorem suggesting also that in the more complicated case of the Einstein equations the necessary condition for Kerr spacetime asymptotically simplicity could be verified.

In this thesis we have also investigated the asymptotic behavior, in the null directions, of the electromagnetic tensor $F_{\mu\nu}$ that satisfies vacuum Maxwell equations in Schwarzschild spacetime. Again we expect that its null components have to satisfy the Peeling theorem. Proceeding as before we realize that to obtain the expected behavior the energy norms with a weight factor $\underline{u}^{3+\epsilon}$ needed for the peeling seem to require, to be finite, the absence in the initial data of the electromagnetic field of the (time independent) dipole term. Even in this case it has been proved that looking at the Maxwell equations for the tensor field $\mathcal{L}_{T_0}W$ we can obtain the peeling without excluding the presence of a dipole term.

The plan of this work is the following:

In Chapter 1 we introduce all the important quantities, that we need to construct the energy norms, to prove their boundedness and the asymptotic behavior of W. We define in a precise way the energy norms we are going to use, indicate by $\mathcal{Q}$ (the one without the factor $u^{5+\epsilon}$). Then we state the main theorems we are going to prove, relative to the asymptotic behavior of the W null components using the energy norms, with or without the weight $u^{5+\epsilon}$. We also analyze the main results we will use in the remaining part of this work.
In Chapter 2 we examine the geometric and analytic properties of the Kerr spacetime in some details, we show how to introduce an appropriate foliation, we introduce some Killing and pseudo-Killing vector fields, their deformation tensors and we compute the asymptotic behavior of the most important geometric spacetime quantities, the connection coefficients.
In Chapter 3, the central part of the work, we estimate the $\mathcal{Q}$ norms relative to the double null foliation and we prove Theorem 1.2.1.
In Chapter 4 we show how modify this approach to obtain the asymptotic result, in particular we introduce the modified integral norms and we focus our attention on the tensor field $\hat{\mathcal{L}}_{T_0}W$, to obtain for the W null components a decay in agreement with the peeling theorem.
In Chapter 5 we show under which decay assumptions on the initial data of $\mathcal{L}_{T_0F}$, the electromagnetic tensor F satisfying the vacuum Maxwell equations and propagating in the (external) Schwarzschild spacetime (thought as a background spacetime) satisfies the Peeling theorem.

Chapter 1

The General Method Adapted to the Linear Case

In the first chapter, we provided a general motivation explaining why we are interested to the study of the solutions of zero-rest mass field equations, in fact to their asymptotic behavior, and a general discussion about the strategy to solve globally Einstein vacuum equations (with asymptotically flat initial data), following the methods introduced by **Ch-Kl and KL-Ni**.
The aim of this chapter will be to describe in more detail the technique of the generalized energy norms and to give a complete picture of the analytic tools necessary to prove our result, paying attention to explaining the logic which we tackle the problem with.
As we will see, the crucial difference treating the linearized version of the problem is that we already know the background spacetime, specifically the Kerr spacetime. As we have seen in the introduction, this shall produce many technical simplifications in obtaining our results.

1.1 Weyl fields and Bel-Robinson tensor

In order to prove the expected results about the asymptotic behavior of spin 2 zero-rest mass fields, we begin with some definitions.

Definition 1.1.1. *Given a spacetime* $(\mathcal{M}, g)$*, a Weyl field is a tensor field*

W which satisfies the following properties

$$\begin{aligned} &W_{\alpha\beta\gamma\delta} = W_{\gamma\delta\alpha\beta} = -W_{\beta\alpha\gamma\delta} = -W_{\alpha\beta\delta\gamma} \\ &W_{\alpha\beta\gamma\delta} + W_{\alpha\gamma\delta\beta} + W_{\alpha\delta\beta\gamma} = 0 \\ &g^{\alpha\gamma} W_{\alpha\beta\gamma\delta} = 0. \end{aligned} \tag{1.1.1}$$

Definition 1.1.2. *A Weyl tensor field W is a solution of the 2-spin and zero-rest mass field equations (called Bianchi equations too) in $(\mathcal{M}, g)$ if, relative to the Levi-Civita connection of g, it satisfies*

$$\mathbf{D}^\mu W_{\mu\nu\rho\sigma} = 0.$$

Remark 1.1.1. *If we decompose the Riemann tensor of the spacetime as*

$$R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + 2(g_{\alpha[\gamma} R_{\delta]\beta} - g_{\beta[\gamma} R_{\delta]\alpha}) - R g_{\alpha[\gamma} g_{\delta]\beta},$$

where C is called the conformal part and it is traceless, $R_{\alpha\gamma}$ is the Ricci tensor and R is the scalar curvature, we observe its conformal part, which is the part different from 0 at the infinity, is a Weyl tensor field.
Moreover, if $(\mathcal{M}, g)$ is a vacuum spacetime ($R_{\alpha\beta} = 0$), then its Riemann tensor is a Weyl field that satisfies the 0-rest mass and 2-spin field equations.

Definition 1.1.3. *Given a tensor field $W_{\alpha\beta\gamma\delta}$, we define the modified Lie derivative $\hat{\mathcal{L}}_X W$ in the following way*

$$\hat{\mathcal{L}}_X W = \mathcal{L}_X W - \frac{1}{2} {}^{(X)}[W] + \frac{3}{8} tr^{(X)}\pi W$$

where

$${}^{(X)}[W]_{\alpha\beta\gamma\delta} = {}^{(X)}\pi_\alpha^\lambda W_{\lambda\beta\gamma\delta} + {}^{(X)}\pi_\beta^\lambda W_{\alpha\lambda\gamma\delta} + {}^{(X)}\pi_\gamma^\lambda W_{\alpha\beta\lambda\delta} + {}^{(X)}\pi_\delta^\lambda W_{\alpha\beta\lambda\gamma}$$

being ${}^{(X)}\pi$ the deformation tensor relative to the vector field X.

It is useful to note that if W is a Weyl field, $\hat{\mathcal{L}}_X W$ is a Weyl field too.

Definition 1.1.4. *A double null foliation of $\mathcal{M}$ consists of the double family of lightlike hypersurface $\{C(u), \underline{C}(\underline{u})\}$ defined as the level hypersurface of the solutions of the eikonal equation*

$$g^{\mu\nu} \partial_\mu \partial_\nu v = 0.$$

$C(u)$ represent the outgoing hypersurfaces, while $\underline{C}(\underline{u})$ are the incoming null hypersurfaces. The intesection of $C(u)$ and $\underline{C}(\underline{u})$ is a two-sphere, which we indicate by $S(u, \underline{u})$.

Definition 1.1.5. *Given a 2-sphere $S(u, \underline{u})$, a null frame in $\mathcal{M}$ is a thetrad of vector fields $\{e_3, e_4, e_1, e_2\}$, where $\{e_1, e_2\}$ are vector fields tangent on the sphere, $\{e_3, e_4\}$ is a fixed null pair and they satisfy the following relations:*

$$\begin{aligned} g(e_3, e_4) &= -2 \\ g(e_a, e_3) &= g(e_a, e_4) = 0, \qquad \textit{for } a = 1, 2 \\ g(e_a, e_b) &= \delta_{ab}. \end{aligned}$$

Associated to the null frame there are some geometric quantities whose specific expression is related to the spacetime we are investigating, in particular:

Definition 1.1.6. *The connection coefficients of a spacetime $(\mathcal{M}, g)$ are the following quantities:*

$$\begin{aligned} &\chi_{ab} = g(D_{e_a} e_4, e_b), && \underline{\chi}_{ab} = g(D_{e_3} e_a, e_b) \\ &\xi_a = \frac{1}{2} g(D_{e_4} e_4, e_a), && \underline{\xi}_a \frac{1}{2} g(D_{e_3} e_3, e_a) \\ &\eta_a = -\frac{1}{2} g(D_{e_3} e_a, e_4), && \underline{\eta}_a = -\frac{1}{2} g(D_{e_4} e_a, e_e) \qquad (1.1.2) \\ &\omega = -\frac{1}{4} g(D_{e_4} e_3, e_4), && \underline{\omega} = -\frac{1}{4} g(D_{e_3} e_4, e_3). \qquad (1.1.3) \end{aligned}$$

Next we introduce the null decomposition of a Weyl tensor, i.e. we express W in terms of a null frame $\{e_3, e_4, e_1, e_2\}$ in the following way:

Definition 1.1.7. *Let e_3, e_4 be a null pair and W a Weyl field. At a given point $p \in \mathcal{M}$ we define the following tensors on the tangent space to the sphere $S(u, \underline{u})$ passing through p,*

$$\begin{aligned} &\alpha(W)(X, Y) = W(X, e_4, Y, e_4), && \underline{\alpha}(W)(X, Y) = W(X, e_3, Y, e_3) \\ &\beta(W)(X) = \frac{1}{2} W(X, e_4, e_3, e_4), && \underline{\beta}(W)(X) = \frac{1}{2} W(X, e_3, e_3, e_4) \qquad (1.1.4) \\ &\rho(W) = \frac{1}{4} W(e_3, e_4, e_3, e_4), && \sigma(W) = \frac{1}{4}{}^\star W(e_3, e_4, e_3, e_4), \end{aligned}$$

where ${}^\star W_{\alpha\beta\gamma\delta}$ is the left Hodge dual of W, defined in the following way:

$$ {}^\star W_{\alpha\beta\gamma\delta} = \frac{1}{2} \epsilon_{\alpha\beta\phi\psi} W^{\phi\psi}{}_{\gamma\delta}. $$

Proposition 1.1.1 (Bianchi Equations). *Expressed relatively to an adapted null frame, the Bianchi equations take the following form*

$$\begin{aligned}
\underline{\alpha}_4 &\equiv \not{D}_4\underline{\alpha} + \frac{1}{2}tr\chi\underline{\alpha} = -\not\nabla\hat{\otimes}\underline{\beta} + 4\omega\underline{\alpha} - 3(\underline{\hat{\chi}}\rho - {}^\star\underline{\hat{\chi}}\sigma) + (\zeta - 4\underline{\eta}\hat{\otimes})\underline{\beta} \\
\underline{\beta}_3 &\equiv \not{D}_3\underline{\beta} + 2tr\underline{\chi}\underline{\beta} = -\not{d}iv\underline{\alpha} - 2\underline{\omega}\underline{\beta} + (2\zeta - \eta)\cdot\underline{\alpha} \\
\underline{\beta}_4 &\equiv \not{D}_4\underline{\beta} + tr\chi\underline{\beta} = -\not\nabla\rho + 2\omega\underline{\beta} + 2\underline{\hat{\chi}}\cdot\beta + {}^\star\not\nabla\sigma - 3(\underline{\eta}\rho - {}^\star\underline{\eta}\sigma) \\
\rho_3 &\equiv \mathbf{D}_3\rho + \frac{3}{2}tr\underline{\chi}\rho = -\not{d}iv\underline{\beta} - \frac{1}{2}\hat{\chi}\cdot\underline{\alpha} + \zeta\cdot\underline{\beta} - 2\eta\cdot\underline{\beta} \\
\rho_4 &\equiv \mathbf{D}_4\rho + \frac{3}{2}tr\chi\rho = \not{d}iv\beta - \frac{1}{2}\underline{\hat{\chi}}\cdot\alpha + \zeta\cdot\beta + 2\underline{\eta}\cdot\beta \qquad (1.1.5) \\
\sigma_3 &\equiv \mathbf{D}_3\sigma + \frac{3}{2}tr\underline{\chi}\sigma = -\not{d}iv^\star\underline{\beta} + \frac{1}{2}\hat{\chi}\cdot^\star\underline{\alpha} - (\zeta + 2\eta)\cdot^\star\underline{\beta} \\
\sigma_4 &\equiv \mathbf{D}_4\sigma + \frac{3}{2}tr\chi\sigma = -\not{d}iv^\star\beta + \frac{1}{2}\underline{\hat{\chi}}\cdot^\star\alpha - (\zeta + 2\underline{\eta})\cdot^\star\beta \\
\beta_3 &\equiv \not{D}_3\beta + tr\underline{\chi}\beta = \not\nabla\rho +^\star \not\nabla\sigma + 2\underline{\omega}\beta + 2\hat{\chi}\cdot\underline{\beta} + 3(\eta\rho +^\star \eta\sigma) \\
\beta_4 &\equiv \not{D}_4\beta + 2tr\chi\beta = \not{d}iv\alpha - 2\omega\beta + (2\zeta + \underline{\eta})\alpha \\
\alpha_3 &\equiv \not{D}_3\alpha + \frac{1}{2}tr\underline{\chi}\alpha = \not\nabla\hat{\otimes}\beta + 4\underline{\omega}\alpha - 3(\hat{\chi}\rho +^\star \hat{\chi}\sigma) + (\zeta + 4\eta)\hat{\otimes}\beta,
\end{aligned}$$

where, here, $\not{D}_4$ *and* $\not{D}_3$ *are the projections on the tangent space to* $S(u,\underline{u})$ *of the covariant derivatives along* e_3, e_4*,* $\not{d}iv$ *and* $\not\nabla$ *are the projections on the tangent space to* $S(u,\underline{u})$ *of the divergence and the covariant derivative relative to* Σ_t*, and* $\hat{\otimes}$ *denotes twice the traceless part of the symmetric tensor product. The Hodge operator* * *indicates the dual of the tensor fields relative to the tangent space of* $S(u,\underline{u})$*, in particular*

Definition 1.1.8. *Given the 1-form* ψ *defined on* $S(u,\underline{u})$*, we define its Hodge dual:*

$${}^\star\psi_a = \epsilon_{ab}\psi_b,$$

where ϵ_{ab} *are the components of the area element of* $S(u,\underline{u})$ *relative to an orthonormal frame* $(e_a)_{a=1,2}$*.*
If ψ *is a symmetric traceless 2-tensor, we define the following left,* ${}^\star\psi$*, and right,*$\psi^\star$*, Hodge duals:*

$${}^\star\psi_{ab} = \epsilon_{ac}{\psi^c}_b, \psi^\star_{ab} = {\psi_a}^c\epsilon_{cb}.$$

Once introduced a Weyl field which satisfies the Bianchi equations, we are able to define the Bel-Robinson tensor associated to it, in the following way:

Definition 1.1.9. *The Bel-Robinson tensor field associated to the Weyl tensor W is the 4-covariant tensor field*

$$\begin{aligned} Q_{\alpha\beta\gamma\delta}[W] &= W_{\alpha\rho\gamma\sigma}W_{\beta}{}^{\rho}{}_{\delta}{}^{\sigma} + {}^{\star}W_{\alpha\rho\gamma\sigma}{}^{\star}W_{\beta}{}^{\rho}{}_{\delta}{}^{\sigma} \\ &= W_{\alpha\rho\gamma\sigma}W_{\beta}{}^{\rho}{}_{\delta\sigma} + W_{\alpha\rho\delta\sigma}W_{\beta}{}^{\rho}{}_{\gamma\sigma} - \frac{1}{8}g_{\alpha\beta}g_{\gamma\delta}W_{\rho\sigma\mu\nu}W^{\rho\sigma\mu\nu}. \end{aligned}$$

The Bel-Robinson tensor satisfies the following important

Proposition 1.1.2.
i) Q is symmetric and traceless relative to all pairs of indices.

ii) Q satisfies the following positivity condition: given any timelike vector fields X_μ, for $\mu = 1, ..., 4$

$$Q(X_1, X_2, X_3, X_4) > 0$$

unlike $W = 0$.
iii)If W is a solution of the Bianchi equations, it follows

$$D^\alpha Q_{\alpha\beta\gamma\delta} = 0.$$

For the proof, see [4].

Proposition 1.1.3. *Let $Q(W)$ be the Bel-Robinson tensor of a Weyl field W and X, Y, Z a triplet of vector fields in$\mathcal{M}$. We define the 1-form P associated at the triplet as*

$$P_\alpha = Q_{\alpha\beta\gamma\delta}X^\beta Y^\gamma Z^\delta. \tag{1.1.6}$$

Using all the symmetry properties of Q, we have:

$$\begin{aligned} \mathbf{Div}P &= \mathbf{Div}Q_{\beta\gamma\delta}X^\beta Y^\gamma Z^\delta \qquad (1.1.7) \\ &+ \frac{1}{2}Q_{\alpha\beta\gamma\delta}\big({}^{(X)}\pi^{\alpha\beta}Y^\gamma Z^\delta + {}^{(Y)}\pi^{\alpha\gamma}X^\beta Z^\delta + {}^{(Z)}\pi^{\alpha\delta}X^\beta Y^\gamma\big). \end{aligned}$$

Remark 1.1.2. *When X, Y, Z are Killing or conformal Killing vector fields and W satisfies Bianchi equations, it follows*

$$\mathbf{Div}P = 0$$

i.e. P is a conserved quantity.

From now on our attention will be focused upon Kerr spacetime, whose Killing vector fields are the generator of time translations

$$T_0 \equiv \frac{\partial}{\partial_t}$$

and the generator of rotation with respect to an axis

$$^{(1)}O \equiv \frac{\partial}{\partial \phi} = x_2 \frac{\partial}{\partial_{x_3}} - x_3 \frac{\partial}{\partial_{x_2}}.$$

The other vector fields which will be introduced in order to define a family of meaningful integral quantities are the following ones:

$$\begin{aligned} ^{(2)}O &= x_3 \frac{\partial}{\partial_{x_1}} - x_1 \frac{\partial}{\partial_{x_3}} \\ ^{(3)}O &= x_2 \frac{\partial}{\partial_{x_1}} - x_1 \frac{\partial}{\partial_{x_2}} \\ S &= \frac{\Omega}{2}(u e_e + \underline{u} e_4) \\ K_0 &= \frac{\Omega}{2}(u^2 e_3 + \underline{u}^2 e_4) \\ T &= \frac{1}{\Omega} T_0, \end{aligned} \tag{1.1.8}$$

the $^{(i)}O$ are the rotation vector fields, S corresponds to scaling transformations and K_0 to inverted time translations.

Theorem 1.1.1. *Let $\mathcal{M} = \Sigma \times \mathbb{R}$, where Σ is a three-dimensional spacelike surface, let W be a solution of Bianchi equations and let $X, Y, Z, V_1 ..., V_k$ be Killing or conformal Killing vector fields. Then*

*i)***Div**$P = 0$, *where P is defined in 1.1.6.*

ii) The integral $\int_{\Sigma_t} Q[W](X, Y, Z, T_0) d^3x$ is finite and constant for all t provided that they are finite at $t = 0$.

iii)The integral $\int_{\Sigma_t} Q[\hat{\mathcal{L}}_{V_1} \hat{\mathcal{L}}_{V_2} ... \hat{\mathcal{L}} V_k W](X, Y, Z, T_0) d^3x$ is finite and constant for all t provided that they are finite at $t = 0$.

This theorem shows the importance of assigning a spacetime foliation when we will have to estimate the norms constructed from the Bel-Robinson rensor, that we will call $\mathcal{Q}$ norms. In fact one can consider some different types of hypersurfaces foliating $\mathcal{M}$ (see for example [11]). We are adopting the approach of [11], by introducing a double null foliation (equivalent to outgoing and ingoing null cones of Minkowski spacetime).

1.2 The main Theorems

In this section we state the theorems we are going to prove in the subsequent chapters, which give the asymptotic behavior of the null components of a Weyl tensor propagating in the Kerr spacetime, according to their initial data, or to the initial data of their Lie derivative with respect to the time.

Theorem 1.2.1. *Let W be a Weyl field in a spacetime with assigned metric of Kerr, that satisfies the 2-spin and zero-rest mass field equations*

$$D^{\mu}W_{\mu\nu\rho\sigma} = 0.$$

Let us assume that the W null components for $t = 0$, decay like $r^{-(7/2+\epsilon)}$, all except $\rho(W)$, which behaves as r^{-3}, where r is the radial parameter of the sphere $S(u, \underline{u})$, that is

$$r^2(u, \underline{u}) = \frac{1}{4\pi}\int_{S(u,\underline{u})} d\sigma \quad .$$

Then the null components of W have the following asymptotic behavior along the null infinity

$$\begin{aligned}
&\sup_{\mathcal{K}} r^{\frac{7}{2}}|\alpha| \leq C_0, \quad \sup_{\mathcal{K}} r^{\frac{7}{2}}|\beta| \leq C_0 \\
&\sup_{\mathcal{K}} r^3|\rho| \leq C_0, \quad \sup r^3|u|^{\frac{1}{2}}|(\rho - \bar{\rho}, \sigma)| \leq C_0 \\
&\sup_{\mathcal{K}} r^2|u|^{\frac{3}{2}}\underline{\beta} \leq C_0, \quad \sup_{\mathcal{K}} r|u|^{\frac{5}{2}}|\underline{\alpha}| \leq C_0,
\end{aligned} \tag{1.2.9}$$

where C_0 is a constant that depends on the initial data.

Remark 1.2.1. *The decays we are assigning for the initial data of the null components are not completely arbitrary, but they are in agreement with the asymptotic spacelike behavior of the Kerr Riemann tensor.*

Theorem 1.2.2 (Peeling Theorem). *Let W be a Weyl field solution of the Bianchi equations in the Kerr spacetime.*
Let us assume that every null component of $\hat{\mathcal{L}}_T W$ on Σ_0 decay like $r^{-(6+\epsilon)}$. Then the W null components satisfy the following peeling decays:

$$\begin{aligned}
&\sup_{\mathcal{K}} r^5|u|^{\epsilon'}|\alpha| \leq C_0, \quad \sup_{\mathcal{K}} r^4|u|^{1+\epsilon'}|\beta| \leq C_0 \\
&\sup_{\mathcal{K}} r^3|\rho| \leq C_0, \quad \sup r^3|u|^{2+\epsilon'}|\sigma| \leq C_0 \\
&\sup_{\mathcal{K}} r^2|u|^{3+\epsilon'}\underline{\beta} \leq C_0, \quad \sup_{\mathcal{K}} r|u|^{4+\epsilon'}|\underline{\alpha}| \leq C_0,
\end{aligned} \tag{1.2.10}$$

with $\epsilon' = \frac{1}{2}\epsilon$ and C_0 is a constant depending on the initial data.

1.3 Analytic general results

The aim of this section is to present some analytic general results which will be needed to estimate the W null components, once introduced some suitable energy norms.
We start recalling the Gronwall inequality:

Proposition 1.3.1. *Let f,g :$[a,b) \to \mathbb{R}$ be continuous and nonnegative. Assume that*

$$f(t) \leq A + \int_a^t f(s)g(s)ds, \qquad A \geq 0.$$

Then

$$f(t) \leq A\exp\Big(\int_a^t g(s)ds\Big), \text{ for } t \in [a,b).$$

Gronwall inequality will be very useful in the study of the decay of the W null components. Another result needed in order to show their asymptotic behavior are the following Sobolev estimates:

Proposition 1.3.2. *Let F be a smooth S-tangent tensor field (it means at any point p, F is tangent to the 2-surface $S(u,\underline{u})$ passing through p). The following nondegenerate version of the global Sobolev inequality along $C(u)$*

holds true:

$$\begin{aligned}\sup_{S(u,\underline{u})}(r^{\frac{3}{2}}|F|) \;\leq\; & c\Big[(\int_{S(u,\underline{u}_0)} r^4|F|^4)^{\frac{1}{4}} + (\int_{S(u,\underline{u}_0)} r^4|r\nabla\!\!\!/ F|^4)^{\frac{1}{4}} \\ & +\Big(\int_{C(u)\cap V(u,\underline{u})} |F|^2 + r^2|\nabla\!\!\!/ F|^2 + r^2|\mathbf{D}\!\!\!\!/_4 F|^2 \\ & + r^4|\nabla\!\!\!/^2 F|^2 + r^4|\nabla\!\!\!/ \mathbf{D}\!\!\!\!/_4 F|^2\Big)^{\frac{1}{2}}\Big], \qquad (1.3.11)\end{aligned}$$

where $\tau_-^2 = (1+u^2)$*. We also have the degenerate version:*

$$\begin{aligned}\sup_{S(u,\underline{u})}(r\tau_-^{\frac{1}{2}}|F|) \;\leq\; & c\Big[(\int_{S(u,\underline{u}_0)} r^2\tau_-^2|F|^4)^{\frac{1}{4}} + (\int_{S(u,\underline{u}_0)} r^2\tau_-^2|r\nabla\!\!\!/ F|^4)^{\frac{1}{4}} \\ & +\Big(\int_{C(u)\cap V(u,\underline{u})} |F|^2 + r^2|\nabla\!\!\!/ F|^2 + \tau_-^2|\mathbf{D}\!\!\!\!/_4 F|^2 \\ & + r^4|\nabla\!\!\!/^2 F|^2 + r^2\tau_-^2|\nabla\!\!\!/ \mathbf{D}\!\!\!\!/_4 F|^2\Big)^{\frac{1}{2}}\Big]. \qquad (1.3.12)\end{aligned}$$

Analogous estimates are obtained along the null-incoming hypersurfaces $\underline{C}(\underline{u})$*:*

$$\begin{aligned}\sup_{S(u,\underline{u})}(r^{\frac{3}{2}}|F|) \;\leq\; & c\Big[(\int_{S(u_0,\underline{u})} r^4|F|^4)^{\frac{1}{4}} + (\int_{S(u_0,\underline{u})} r^4|r\nabla\!\!\!/ F|^4)^{\frac{1}{4}} \\ & +\Big(\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} |F|^2 + r^2|\nabla\!\!\!/ F|^2 + r^2|\mathbf{D}\!\!\!\!/_3 F|^2 \\ & + r^4|\nabla\!\!\!/^2 F|^2 + r^4|\nabla\!\!\!/ \mathbf{D}\!\!\!\!/_3 F|^2\Big)^{\frac{1}{2}}\Big] \qquad (1.3.13)\end{aligned}$$

and

$$\begin{aligned}\sup_{S(u,\underline{u})}(r\tau_-^{\frac{1}{2}}|F|) \;\leq\; & c\Big[(\int_{S(u_0,\underline{u})} r^2\tau_-^2|F|^4)^{\frac{1}{4}} + (\int_{S(u_0,\underline{u})} r^2\tau_-^2|r\nabla\!\!\!/ F|^4)^{\frac{1}{4}} \\ & +\Big(\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} |F|^2 + r^2|\nabla\!\!\!/ F|^2 + \tau_-^2|\mathbf{D}\!\!\!\!/_3 F|^2 \\ & + r^4|\nabla\!\!\!/^2 F|^2 + r^2\tau_-^2|\nabla\!\!\!/ \mathbf{D}\!\!\!\!/_3 F|^2\Big)^{\frac{1}{2}}\Big]. \qquad (1.3.14)\end{aligned}$$

The proofs of it is in appendix at [3], (see sections 5.1, 5.2).

In order to estimate null components of W in terms of the $\mathcal{Q}$ norms which we are going to introduce in the following section, as we will see in the next chapters, firstly we must to be able to estimate them and their tangential first derivatives in terms of their Lie derivative done with respect to the rotation vector fields ${}^{(i)}O$.

In fact the following result holds

Lemma 1.3.1. *The rotation vector fields ${}^{(i)}O$ satisfy the following properties:*
i) Given an S-tangent tensor field f on $\mathcal{M}$ there exists a constant c_0 such that

$$c_0^{-1}\int_{S(u,\underline{u})} r^2|\nabla\!\!\!/ f|^2 \le \int_{S(u,\underline{u})} |\mathcal{L}_O f|^2 \le c_0 \int_{S(u,\underline{u})} (|f|^2 + r^2|\nabla\!\!\!/ f|^2), \quad (1.3.15)$$

where $|\mathcal{L}_O f|^2 = \sum_{i=1}^3 |\mathcal{L}_{{}^{(i)}O} f|^2$.
ii) If f is a 1-form or a traceless symmetric 2-covariant tensor tangent to the surfaces $S(u,\underline{u})$, the following inequality holds:

$$c_0^{-1}\int_{S(u,\underline{u})} |f|^2 \le \int_{S(u,\underline{u})} |\mathcal{L}_O f|^2. \quad (1.3.16)$$

iii) If f is a scalar function and $\bar{f}$ is its mean value on the sphere then the Poincaré inequality holds:

$$\int_{S(u,\underline{u})} (f-\bar{f})^2 \le c_0 \int_{S(u,\underline{u})} |r\nabla\!\!\!/ f|^2. \quad (1.3.17)$$

Finally we will need the relations between the null components of the Weyl field $\hat{\mathcal{L}}_X W$ and the modified Lie derivative with respect to the vector field X of W null components, when $X = S, {}^{(i)}O, T$. First of all we introduce the following

Definition 1.3.1. *Let X be a vector field in the family $\{S, {}^{(i)}O, T\}$. We define the Lie coefficients of X through the following commutation relations: (see [11], prop. 7.3.1)*

$$\begin{aligned}
[X,e_3] &= {}^{(X)}\underline{P}_b e_b + {}^{(X)}\underline{M} e_3 + {}^{(X)}\underline{N} e_4 \\
[X,e_4] &= {}^{(X)}P_b e_b + {}^{(X)}N e_3 + {}^{(X)}M e_4 \\
[X,e_a] &= \Pi[X,e_a] + \frac{1}{2}\big({}^{(X)}Q_a e_3 + {}^{(X)}\underline{Q}_a\big)e_4,
\end{aligned} \quad (1.3.18)$$

where $\Pi[X,e_a]$ is the projection on $TS(u,\underline{u})$ of $[X,e_a]$.

The Lie coefficients of X appear when we commute $\hat{\mathcal{L}}_X$ with the null decomposition of a Weyl tensor. The result of this commutation is expressed in the following proposition (for the proof see [3], prop.7.3.1)

Proposition 1.3.3. *Let W an arbitrary Weyl tensor. Consider its null components as well as the null components of $\hat{\mathcal{L}}_X W$. Let $\hat{\not{\mathcal{L}}}_X$ be the projection on $S(u,\underline{u})$ of the Lie derivative $\mathcal{L}_X$, and let $\hat{\hat{\not{\mathcal{L}}}}_X\alpha, \hat{\hat{\not{\mathcal{L}}}}_X\underline{\alpha}$ be the traceless part of the tensors $\hat{\not{\mathcal{L}}}\alpha, \hat{\not{\mathcal{L}}}\underline{\alpha}$. Then the following relations hold:*

$$\begin{aligned}
\alpha(\hat{\mathcal{L}}_X W)_{ab} &= \hat{\not{\mathcal{L}}}_X\alpha(W)_{ab} + \Bigg[\Big(-({}^{(X)}M + {}^{(X)}\underline{M}) + \frac{1}{8}tr{}^{(X)}\pi\Big)\alpha(W)_{ab}\\
&- \big({}^{(X)}P_a + {}^{(X)}Q_a\big)\beta(W)_b - \big({}^{(X)}P_b + {}^{(X)}Q_b\big)\beta(W)_a\\
&+ \delta_{ab}\big({}^{(X)}P + {}^{(X)}Q\big)\cdot\beta(W)\Bigg]\\
\beta(\hat{\mathcal{L}}_X W)_a &= \hat{\not{\mathcal{L}}}_X\beta(W)_a + \Bigg[-\frac{1}{2}{}^{(X)}\hat{\pi}_{ab}\beta(W)_b - \Big({}^{(X)}M + \frac{1}{8}tr{}^{(X)}\pi\Big)\beta(W)_a\\
&- \frac{3}{4}\big({}^{(X)}P_a + {}^{(X)}Q_a\big)\rho(W) - \frac{3}{4}\epsilon_{ab}\big({}^{(X)}P_b + {}^{(X)}Q_b\big)\sigma(W)\\
&- \frac{1}{4}\big({}^{(X)}\underline{P}_b + {}^{(X)}\underline{Q}_b\big)\alpha(W)_{ab}\Bigg]
\end{aligned}$$

$$\begin{aligned}
\rho(\hat{\mathcal{L}}_X W) &= \mathcal{L}_X\rho(W) + \Bigg[-\frac{1}{8}tr{}^{(X)}\pi\rho(W) - \frac{1}{2}\big({}^{(X)}\underline{P}_a + {}^{(X)}\underline{Q}_a\big)\beta(W)_a\\
&+ \frac{1}{2}\big({}^{(X)}P_a + {}^{(X)}Q_a\big)\underline{\beta}(W)_a\Bigg]
\end{aligned}$$

$$\begin{aligned}
\sigma(\hat{\mathcal{L}}_X W) &= \mathcal{L}_X\sigma(W) + \Bigg[-\frac{1}{8}tr{}^{(X)}\pi\sigma(W) + \frac{1}{2}\big({}^{(X)}\underline{P}_a + {}^{(X)}\underline{Q}_a\big)^*\beta(W)_a\\
&+ \frac{1}{2}\big({}^{(X)}P_a + {}^{(X)}Q_a\big)^*\underline{\beta}(W)_a\Bigg]
\end{aligned}$$

$$
\begin{aligned}
\underline{\beta}(\hat{\mathcal{L}}_X W)_a &= \hat{\not{\mathcal{L}}}_X \underline{\beta}(W)_a + \Big[-\frac{1}{2} {}^{(X)}\hat{\pi}_{ab} \underline{\beta}(W)_b - \Big({}^{(X)}\underline{M} + \frac{1}{8} tr^{(X)}\pi \Big) \underline{\beta}(W)_a \\
&+ \frac{3}{4} \big({}^{(X)}\underline{P}_a + {}^{(X)}\underline{Q}_a \big) \rho(W) - \frac{3}{4} \epsilon_{ab} \big({}^{(X)}\underline{P}_b + {}^{(X)}\underline{Q}_b \big) \sigma(W) \\
&+ \frac{1}{4} \big({}^{(X)}P_b + {}^{(X)}Q_b \big) \underline{\alpha}(W)_{ab} \Big] \\
\underline{\alpha}(\hat{\mathcal{L}}_X W)_{ab} &= \hat{\not{\mathcal{L}}}_X \underline{\alpha}(W)_{ab} + \Big[\Big(-({}^{(X)}M + {}^{(X)}\underline{M}) + \frac{1}{8} tr^{(X)}\pi \Big) \underline{\alpha}(W)_{ab} \\
&+ \big({}^{(X)}\underline{P}_a + {}^{(X)}\underline{Q}_a \big) \underline{\beta}(W)_b + \big({}^{(X)}\underline{P}_b + {}^{(X)}\underline{Q}_b \big) \underline{\beta}(W)_a \\
&- \delta_{ab} \big({}^{(X)}\underline{P} + {}^{(X)}\underline{Q} \big) \cdot \underline{\beta}(W) \Big].
\end{aligned}
$$

As we will see in Chapter 4, the terms in square brackets behave better along the null infinity as far as the decay in r is concerned, so we shall treat them as any correction terms. Therefore this proposition shows us that the projection on $S(u, \underline{u})$ of the traceless part of Lie derivatives done with respect to $S, T, {}^{(i)}O$ of W null components, when $r \to 0$, behave as the null components of the Weyl field $\hat{\mathcal{L}}_X W$.

1.4 $\mathcal{Q}$ integral norms

In the previous section we have shown how to estimate the sup norms of a function f in terms of L_2 norms on the (incoming or outgoing) lightlike hypersurfaces of f and its derivatives up to the second order (see Proposition 1.3.2). Now we show which are the suitable integral norms to introduce on $C(u), \underline{C}(\underline{u})$ and on the initial hypersurface Σ_0 in order to study the asymptotic behavior of W null components. We denote these norms with $\mathcal{Q}[W]$. They have to satisfy two properties:
i) They have to estimate the W null components, up to the second order in L_2 norms.
ii) They have to be bounded.
Let us start by showing how to define the quantities which satisfy the first point.
As far as L_2 norms for f are concerned, we recall (1.3.16) and (1.3.17), which tell us how the W null components can be estimated by their Lie derivatives

done with respect to the rotation generators of the spacetime.[1] As far as the first derivatives are concerned, we note that (1.3.15) holds implying that DW can be estimated by $\mathcal{L}_O[W]$ too. But in the inequalities of the proposition 1.3.2 some terms of the type $D_4[W]$ or $D_3[W]$ appear too. Their control is more complicated but it is obtained using the Bianchi equations. Let us consider, for example, how to treat $\alpha(W)$. As far as $D_3\alpha$ is concerned , we use the Bianchi equation indicated by α_3, which expresses $D_3\alpha$ in terms of $\frac{1}{r}\alpha(W)$ and $\nabla\!\!\!/\beta(W)$. It follows $D_3\alpha$ is bounded by $\mathcal{L}_O\alpha(W)$.
As far as $D_4\alpha$ is concerned, we note that an evolution equation for α along outgoing null cones does not exist , but the following relation holds:

$$D_4\alpha = 2D_T\alpha - D_3\alpha$$

and $||D_T\alpha||_{L_2}$ is estimated by $||\mathcal{L}_T\alpha||_{L_2}$. Finally all the first derivatives of W null components will be bounded in terms of $\mathcal{L}_O[W]$ and $\mathcal{L}_T[W]$.
As far as second derivatives are concerned, we observe that $||\nabla\!\!\!/^2[W]||_{L_2}$ is estimated through $||\mathcal{L}^2_O W||_{L_2}$ and $||\nabla\!\!\!/ D_4W||_{L_2}$ will be bounded by $||\mathcal{L}^2_O W||_{L_2}$ and $||\mathcal{L}_O\mathcal{L}_T W||_{L_2}$.
In order to obtain any estimates in terms of the Bel-Robinson tensor $Q[W]$ related to a Weyl field, let us recall Proposition 1.3.3. Due to it the procedure to follow is the following one: first of all we have to introduce some suitable vector fields by which we saturate $Q[W]$. They must be:
i)Non-spacelike and future directed (in order to satisfy condition *ii*) of proposition 1.1.2),
ii) Normal to the foliation hypersurfaces and to hypersurfaces at $t = const$ (in order to apply the divergence theorem),
iii) They have to be the equivalent of Killing and conformal Killing vector fields of Minkowski spacetime (so we can suppose their deformation tensor to be small).
The suitable vector fields are just those already introduced, in particular they are the null pair, $\{e_3, e_4\}$, introduced in definition 1.1.5 , the vector field $T = \frac{\Omega}{2}T_0$[2] and $\bar{K} = K_0 + T$, where K_0 is defined in (1.1.8). The way we can saturate Bel-Robinson tensor Q with such vectors will be made clear in Chapter 4. After the considerations made up to now, let us introduce without further explanations the integral norms we will show to be bounded.

[1] Actually the problem for ρ and σ is not yet completely solved because proceeding in this way one estimates $\rho - \bar{\rho}$ and $\sigma - \bar{\sigma}$.
[2] Ω is the lapse function of the spacetime

Definition 1.4.1. *Using the vector fields $\bar{K}, S, T$ and ${}^{(i)}O$ and denoting*

$$V(u,\underline{u}) = J^{-}(S(u,\underline{u})),$$

we define the following energy-type norms:

$$\begin{aligned}\mathcal{Q}(u,\underline{u}) &= \mathcal{Q}_1(u,\underline{u}) + \mathcal{Q}_2(u,\underline{u})\\ \underline{\mathcal{Q}}(u,\underline{u}) &= \underline{\mathcal{Q}}_1(u,\underline{u}) + \underline{\mathcal{Q}}_2(u,\underline{u}),\end{aligned} \tag{1.4.19}$$

where

$$\begin{aligned}\mathcal{Q}_1(u,\underline{u}) &\equiv \int_{C(u)\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4)\\ &\quad + \int_{C(u)\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4)\\ \mathcal{Q}_2(u,\underline{u}) &\equiv \int_{C(u)\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4)\\ &\quad + \int_{C(u)\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_4)\\ &\quad + \int_{C(u)\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4)\end{aligned} \tag{1.4.20}$$

$$\begin{aligned}\underline{\mathcal{Q}}_1(u,\underline{u}) &\equiv \sup_{V(u,\underline{u})\cap\Sigma_0} |r^3\overline{\rho}|^2 + \int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_3)\\ &\quad + \int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_3),\end{aligned}$$

where $\bar{\rho}$ is the mean value of $\rho(W)$.

$$\begin{aligned}\underline{\mathcal{Q}}_2(u,\underline{u}) &\equiv \int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_3)\\ &\quad + \int_{\underline{C}(u)\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_3)\\ &\quad + \int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_3)\end{aligned} \tag{1.4.21}$$

and

$$\begin{aligned}
\mathcal{Q}_{1_{\Sigma_0\cap V(u,\underline{u})}} &\equiv \int_{\Sigma_0\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},T)\\
&+\int_{\Sigma_0\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,T)\\
&+\sup_{\Sigma_0\cap V(u,\underline{u})} |r^3\bar{\rho}|^2 \qquad (1.4.22)\\
\mathcal{Q}_{2_{\Sigma_0\cap V(u,\underline{u})}} &\equiv \int_{\Sigma_0\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},T)\\
&+\int_{\Sigma_0\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,T)\\
&+\int_{\Sigma_0\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},T). \qquad (1.4.23)
\end{aligned}$$

We introduce the following quantity too

$$\mathcal{Q}_{\mathcal{K}} \equiv \sup_{\{u,\underline{u}|S(u,\underline{u})\subseteq\mathcal{K}\}} \{\mathcal{Q}(u,\underline{u})+\underline{\mathcal{Q}}(u,\underline{u})\}. \qquad (1.4.24)$$

Moreover, on the initial spacelike hypersurface Σ_0 *we define*

$$\mathcal{Q}_{\Sigma_0\cap\mathcal{K}} = \sup_{\{u,\underline{u}|S(u,\underline{u})\subseteq\mathcal{K}\}} \{\mathcal{Q}_{1_{\Sigma_0\cap V(u,\underline{u})}}+\mathcal{Q}_{2_{\Sigma_0\cap V(u,\underline{u})}}\}. \qquad (1.4.25)$$

Now let us observe the Bel-Robinson tensor satisfies the following identities:

$$\begin{aligned}
Q(W)(e_3,e_3,e_3,e_3) &= 2|\underline{\alpha}|^2\\
Q(W)(e_4,e_4,e_4,e_4) &= 2|\alpha|^2\\
Q(W)(e_3,e_3,e_3,e_4) &= 4|\underline{\beta}|^2\\
Q(W)(e_3,e_4,e_4,e_4) &= 4|\beta|^2\\
Q(W)(e_3,e_3,e_4,e_4) &= 4(\rho^2+\sigma^2), \qquad (1.4.26)
\end{aligned}$$

and, using multilinearity properties,

$$\begin{aligned}
Q(W)(K_0,K_0,T,e_4) &= \frac{1}{4}\underline{u}^4|\alpha|^2+\frac{1}{2}(\underline{u}^4+2\underline{u}^2u^2)|\beta|^2+\frac{1}{2}(u^4+2\underline{u}^2u^2)(\rho^2+\sigma^2)\\
&\quad+\frac{1}{2}u^4|\underline{\beta}|^2\\
Q(W)(K_0,K_0,T,e_3) &= \frac{1}{4}u^4|\underline{\alpha}|^2+\frac{1}{2}(u^4+2\underline{u}^2u^2)|\underline{\beta}|^2+\frac{1}{2}(\underline{u}^4+2\underline{u}^2u^2)(\rho^2+\sigma^2)\\
&\quad+\frac{1}{2}\underline{u}^4|\beta|^2 \qquad (1.4.27)\\
Q(W)(K_0,K_0,K_0,e_4) &= \frac{1}{4}\underline{u}^6|\alpha|^2+\frac{3}{2}\underline{u}^4u^2|\beta|^2+\frac{3}{2}u^4\underline{u}^2(\rho^2+\sigma^2)+\frac{1}{2}u^6|\underline{\beta}|^2\\
Q(W)(K_0,K_0,K_0,e_3) &= \frac{1}{4}u^6|\underline{\alpha}|^2+\frac{3}{2}\underline{u}^2u^4|\underline{\beta}|^2+\frac{3}{2}u^2\underline{u}^4(\rho^2+\sigma^2)\\
&\quad+\frac{1}{2}\underline{u}^6|\beta|^2. \qquad (1.4.28)
\end{aligned}$$

We also have

$$\begin{aligned}
Q(W)(K_0,K_0,T,T) &= \frac{1}{8}\underline{u}^4|\alpha|^2+\frac{1}{8}u^4|\underline{\alpha}|^2+\frac{1}{2}(\underline{u}^4+\frac{1}{2}\underline{u}^2u^2)|\beta|^2\\
&\quad+\frac{1}{2}(u^4+4u^2\underline{u}^2+\underline{u}^4)(\rho^2+\sigma^2)+\frac{1}{2}(u^4+\frac{1}{2}u^2\underline{u}^2)|\underline{\beta}|^2\\
Q(W)(K_0,K_0,K_0,T) &= \frac{1}{8}\underline{u}^6|\alpha|^2+\frac{1}{8}u^6|\underline{\alpha}|^2+\frac{1}{4}u^4(u^2+3\underline{u}^2)|\underline{\beta}|^2 \qquad (1.4.29)\\
&\quad+\frac{3}{4}(\underline{u}^2+u^2)\underline{u}^2u^2(\rho^2+\sigma^2)+\frac{1}{4}\underline{u}^4(3u^2+\underline{u}^2)|\beta|^2.
\end{aligned}$$

So, once we prove the boundedness of $\mathcal{Q}_{\mathcal{K}}$ and $\underline{\mathcal{Q}}_{\mathcal{K}}$, we are able to obtain all the estimates for null components of the Weyl fields $\hat{\mathcal{L}}_O W, \hat{\mathcal{L}}_T W, \hat{\mathcal{L}}_0^2 W, \hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W$, and $\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W$.
To connect these quantities to null components of W, we will use Proposition (1.3.3), observing that the correction terms do not change their asymptotic behavior.
Let us discuss now the possibility of estimating the integral norms we have introduced. This is the most difficult point, because we shall have to exploit the intrinsic hyperbolicity of Bianchi equations to obtain an analogous of the energy estimates. To do it, Proposition (1.4.32) will be essential. In fact let us shortly illustrate our procedure, which will be developed in details in Chapter 4.

We start recalling that we have introduced a double null foliation, shaped by $\{C(u)\}, \{\underline{C}(\underline{u})\}$.
We want to obtain an estimate of the form:

$$\mathcal{Q}(u, \underline{u}) + \underline{\mathcal{Q}}(u, \underline{u}) \leq c\big(\mathcal{Q}_{\Sigma_0 \cap V(u,\underline{u})}\big). \qquad (1.4.30)$$

Then, denoted

$$\varepsilon(u, \underline{u}) \equiv \big(\mathcal{Q} + \underline{\mathcal{Q}} - \mathcal{Q}_{\Sigma_0 \cap V(u,\underline{u})}\big)(u, \underline{u}),$$

we wish to proof that ε is bounded for all $(u, \underline{u})$.
Actually in Chapter 4 we will show that in Kerr spacetime there exists a constant $c_1 = \frac{c_0}{r_0}$ such that

$$\varepsilon(u, \underline{u}) \leq c_1 \mathcal{Q}_{\mathcal{K}}$$

and, from this

$$\mathcal{Q}_{\mathcal{K}} \leq \frac{1}{1 - c_1} \mathcal{Q}_{\Sigma_0 \cap \mathcal{K}}. \qquad (1.4.31)$$

For r_0 sufficiently large, it means that, in the external Kerr spacetime, the $\mathcal{Q}_{\mathcal{K}}$ are bounded in terms of their initial values.
In order to prove it, Proposition (1.4.32) is fundamental together with Stokes theorem, from which we obtain the following equality:

$$\begin{aligned} &\int_{\underline{C}(\underline{u}) \cap V(u,\underline{u})} Q(W)(X, Y, Z, e_3) \;+\; \int_{C(u) \cap V(u,\underline{u})} Q(W)(X, Y, Z, e_4) \\ &- \int_{\Sigma_0 \cap V(u,\underline{u})} Q(W)(X, Y, Z, T) \\ = &\int_{V(u,\underline{u})} [\mathbf{Div} Q(W)_{\beta\gamma\delta} X^\beta Y^\gamma Z^\delta \;+\; \frac{1}{2} Q^{\alpha\beta\gamma\delta}(W)({}^{(X)}\pi_{\alpha\beta} Y_\gamma Z_\delta \\ &+ {}^{(Y)}\pi_{\alpha\beta} Y_\gamma Z_\delta + {}^{(Z)}\pi_{\alpha\beta} X_\gamma Y_\delta)] \quad , \end{aligned} \qquad (1.4.32)$$

where X, Y, Z are three vector fields on $\mathcal{M}$.
So, $\varepsilon(u, \underline{u})$ will be written in terms of integrals done on the whole spacetime region bounded by $C(u)$, $\underline{C}(\underline{u})$ and Σ_0. The most complicated part of our work will be the estimate of integrals of the form

$$\int_{V(u,\underline{u})} \mathbf{Div} Q(\hat{\mathcal{L}}_X W)(Y, Z, W), \quad \text{and} \int_{V(u,\underline{u})} \mathbf{Div} Q(\hat{\mathcal{L}}_{X_1} \hat{\mathcal{L}}_{X_2} W)(Y, Z, W).$$

Using the Bianchi equations, we are able to estimate the first integral with some terms of the form

$$\int_{V(u,\underline{u})} DW \cdot W \cdot {}^{(X)}\pi \text{ or } \int_{V(u,\underline{u})} DW \cdot DW \cdot {}^{(X)}\pi.$$

So we can substitute $\mathbf{Div}Q(\hat{\mathcal{L}}_X W)$ in which W derivatives up to the second order appear, with terms having only zero and first derivatives of W. A similar result holds as far as the terms of the form $\int_{V(u,\underline{u})} \mathbf{Div}Q(\hat{\mathcal{L}}_{X_1}\hat{\mathcal{L}}_{X_2}W)$ (Y, Z, W) are concerned. It remains a problem: In order to estimate terms of the form $DDW \cdot W$ with the $\mathcal{Q}$ norms, we need to control $Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)$ too; therefore this term has been inserted in the definition of the $\mathcal{Q}_2$ and $\underline{\mathcal{Q}}_2$ norms (see (4.2.4) and (4.2.5)).
In this thesis we will use this approach in order to prove Theorems 1.2.1 and 1.2.2.

In the next chapter we will examine the Kerr spacetime, the double null foliation we will use in the following, the adapted null frame associated to it and the other peculiar quantities that will appear in the error estimate (connection coefficients, Killing and quasi-Killing vector fields, deformation tensors...) with a particular attention to their decays along null infinity.

Chapter 2

Kerr Spacetime

From now on we focus our attention on Kerr spacetime, that is physically very important, because of it represents the spacetime generated outside a rotating body, in particular as it is reasonable to think black holes all rotating, Kerr metric describes the geometry spacetime around them. This is the situation that describes a nonspherical collapse, and in such a situation the spacetime geometry outside the collapsing body should vary with time, moreover large amounts of energy may be radiated away. But one would expect that at sufficiently late times, the spacetime geometry becomes stationary and all the matter present is rapidly swallowed up to the black hole, so the final state should be vacuum.

2.1 General structure of the most important quantities

Let us express the metric of Kerr spacetime the set of coordinates $\{t, r, \theta, \phi\}$, called Boyer-Lindquist coordinates. It assumes the following form

$$\begin{aligned} ds^2 &= -\frac{\Delta - a^2\sin^2\theta}{\Sigma}dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 - \frac{4Mar\sin^2\theta}{\Sigma}d\phi dt \\ &+ R^2\sin^2\theta d\phi^2, \end{aligned} \tag{2.1.1}$$

where:

$$\begin{aligned}\Delta &= r^2 + a^2 - 2Mr \\ \Sigma &= r^2 + a^2\cos^2\theta \\ R^2 &= \frac{1}{\Sigma}((r^2+a^2)^2 - \Delta a^2 \sin^2\theta),\end{aligned}$$

and we note the useful identities:

$$\begin{aligned}\Sigma R^2 = (r^2+a^2)^2 &- \Delta a^2\sin^2\theta, \\ g_{\phi\phi}g_{tt} - g_{\phi t}^2 = &- \Delta\sin^2\theta.\end{aligned}$$

Moreover,let us define the following quantities:

$$\begin{aligned}P^2(\theta,\lambda) &= a^2(\lambda - \sin^2\theta) \\ Q^2(r,\lambda,M) &= (r^2+a^2)^2 - a^2\lambda\Delta \\ K^2(r) &= r^2 + a^2,\end{aligned}$$

where λ is a constant.
This metric is stationary and axisymmetric, and it is asymptotically flat, because its components approach those of the Minkowski metric in spherical polar coordinates as $r \to \infty$.
Now, we are interesting in looking for ingoing and outgoing lightlike hypersurfaces of Kerr spacetime, like level hypersurfaces of the optical functions $u, \underline{u}$:

$$\begin{aligned}C(u) &= \{p \in \mathcal{M} | u(p) = u\} \\ \underline{C}(\underline{u}) &= \{p \in \mathcal{M} | \underline{u}(p) = \underline{u}\}. \qquad (2.1.2)\end{aligned}$$

Cause of the symmetries of Kerr spacetime, the three-dimensional surfaces we are going to find, have to be null and axisymmetric with respect to the axis of rotation, i.e. the two-dimensional surface

$$S(u,\underline{u}) = C(u) \cap \underline{C}(\underline{u})$$

have to be symmetric with respect to one of the two angle θ or ϕ. Let us observe that as Riemann tensor of Kerr spacetime in Petrov classification is of type D, it holds the following

Lemma 2.1.1 (Goldberg-Sachs theorem). *In a type D spacetime there exist two congruences of null shear-free geodesics, that means there exist two null vector fields $L, \underline{L}$, so that $\hat{\chi} = \underline{\hat{\chi}} = 0$.*

L and $\underline{L}$ have the following form:

$$\begin{aligned} L &= \frac{K^2}{\Delta}\partial_t + \partial_r + \frac{a}{\Delta}\partial_\phi \\ \underline{L} &= \frac{K^2}{\Delta}\partial_t - \partial_r + \frac{a}{\Delta}\partial_\phi . \end{aligned}$$

Adding to them the following two vector fields

$$\begin{aligned} \tilde{e}_\theta &= \frac{1}{\Sigma^{\frac{1}{2}}}\partial_\theta \\ \tilde{e}_\phi &= \frac{1}{\Sigma^{\frac{1}{2}}}\big(\frac{1}{\sin\theta}\partial_\phi + a\sin\theta\partial_t\big), \end{aligned}$$

and recalling the definition (1.1.5), it is easy to verify we obtain a moving frame. As we have seen in the precedent chapter, Killing vector fields of the Kerr spacetime are the generators of the symmetries with respect to the time t and to a fixed angle ϕ; let us indicate them with:

$$\begin{aligned} T_0 &:= \partial_t \\ {}^{(3)}O &:= \partial_\phi . \end{aligned} \tag{2.1.3}$$

We look for a null hypersurface that is obtained as an integral submanifold of $\mathcal{M}$ from $\{L, e_\theta, {}^{(3)}O\}$. As:

$$\begin{aligned} &[L, {}^{(3)}O] = [{}^{(3)}O, e_\theta] = 0 \\ &[L, e_\theta] = -\frac{1}{2}(\partial_r \log\Sigma)e_\theta, \end{aligned}$$

from Frobenius' theorem, it follows it is an integrable distribution, but it is not null, in fact

$$g(L, {}^{(3)}O) = \frac{a}{\Delta}g_{\phi\phi} + \frac{K^2}{\Delta}a\sin^2\theta g_{\theta\phi}$$

and so, L is not orthogonal to it.
Instead, whether we consider the distribution $\{L, e_\theta, e_\phi\}$, we observe it is not integrable because $[e_\theta, e_\phi]$ doesn't belong to the surface spanned from this triplet of vector fields .
So, by this way, we have not found any lightlike cones of Kerr spacetime, then we try again by studying the eikonal equation

$$g^{\alpha\beta}(\partial_\alpha v)(\partial_\beta v) = 0 \tag{2.1.4}$$

and we look for its solutions, because they will be the level functions of (2.1.2).
In Boyer-Lindquist coordinates, (2.1.4) is equivalent to

$$\Delta(\partial_r r_*)^2 + (\partial_\theta r_*)^2 = K^4\Delta - a^2 \sin^2\theta \tag{2.1.5}$$

and it is easy to obtain a particular solution of (2.1.5), depending on two arbitrary constants, by adding and subtracting an arbitrary separation quantity

$$a^2\lambda$$

on the right-hand side.
The solution is obtained integrating the exact differential

$$dr_* = (\frac{Q}{\Delta})dr + Pd\theta \tag{2.1.6}$$

at fixed λ and it is a function which depends on the variables r, θ and on the parameters λ, m:

$$r_* = \rho(r, \theta, \lambda, m).$$

But because of ρ depends on λ too, a more complete expression for its differential is

$$d\rho = (\frac{Q}{\Delta})dr + Pd\theta + (a^2/2)Fd\lambda, \tag{2.1.7}$$

where

$$(a^2/2)F(r, \theta, \lambda, m) = \partial_\lambda \rho(r, \theta, \lambda, m),$$

and its explicit form is

$$F(r, \theta, \lambda, m) = \int_0^\infty \frac{d\theta'}{P(\theta', \lambda)} + \int_0^\infty \frac{dr'}{Q(r', \lambda)} + f'(\lambda). \tag{2.1.8}$$

The equation (2.1.7)reduces to(2.1.5) in any two cases:

$$d\lambda = 0$$

that means $\lambda = c$, but also $\lambda = \lambda(r, \theta) \neq c$, provided that

$$F(r, \theta, \lambda, m) = 0.$$

Thus we have a general solution, depending upon an arbitrary function $f[\lambda(r,\theta)]$.
Then ρ assumes the following explicit form

$$\begin{aligned}\rho(r,\theta,\lambda) &= \int_r^\infty \frac{\bar{r}^2+a^2}{\Delta(\bar{r},M)}d\bar{r} + \int_r^\infty \frac{\bar{r}^2+a^2-Q(\bar{r},\lambda,M)}{\Delta(\bar{r},M)}d\bar{r} \\ &+ \int_0^\theta P(\bar{\theta},\lambda) + \frac{1}{2}a^2 f(\lambda). \end{aligned} \tag{2.1.9}$$

Remark 2.1.1. *The function $\rho(r,\theta,\lambda)$ is just the radial parameter of the Kerr metric, i.e. given a closed 2-dimensional surface S embedded in $(\mathcal{M},g)$, ρ is the radius of S,*

$$\rho(S) = \sqrt{\frac{1}{4\pi}|S|},$$

where $|S|$ is the area of S.

The functions w which satisfy the equation

$$w(t,r,\theta,\lambda) = t \pm \rho(r,\theta,\lambda) = const$$

represent an axisimmetric lightlike hypersurface (ingoing or outgoing according to the sign). As it is shown in [9], we can define an angle θ_* so that

$$\lambda = \sin^2\theta_*, \tag{2.1.10}$$

and

$$\theta_*(r=\infty,\theta) = \theta.$$

This choice fixes the arbitrary function $f'(\lambda)$ in (2.1.8), to be

$$f'(\lambda) = -\frac{2}{a}\frac{d}{d\lambda}\left(\int_0^{\theta_*}\sqrt{\lambda-\sin^2\theta'}d\theta'\right)$$

which implies the following expression for F holds:

$$F(r,\theta,\lambda,m) = \int_r^\infty \frac{dr'}{Q(r',\lambda,m)} - \int_\theta^{\theta_*}\frac{d\theta'}{P(\theta',\lambda)}. \tag{2.1.11}$$

The vector fields of the null frame adapted to Boyer-Lindquist coordinates have the following form:

$$\begin{aligned}
e_4 &= \frac{\sqrt{\Delta}}{R}\{\frac{1}{\Delta\sin^2\theta}[g_{\phi\phi}\partial_t - g_{t\phi}\partial_\phi] + \frac{1}{\Sigma}[Q\partial_r + P\partial_\theta]\}\\
e_3 &= \frac{\sqrt{\Delta}}{R}\{\frac{1}{\Delta\sin^2\theta}[g_{\phi\phi}\partial_t - g_{t\phi}\partial_\phi] - \frac{1}{\Sigma}[Q\partial_r + P\partial_\theta]\}\\
e_\theta &= \frac{1}{\Sigma R}(Q\partial_\theta - \Delta P\partial_r)\\
e_\phi &= \frac{1}{R\sin\theta}\partial_\phi. \qquad (2.1.12)
\end{aligned}$$

Let us observe $\frac{\Delta}{2R}$ is the lapse function Ω of Kerr spacetime.

Remark 2.1.2. *Because of the quantity λ is defined as a function that at the infinity tends to the quantity $\sin^2\theta$, this implies the following asymptotic relations hold true, when r tends to 0:*

$$\begin{aligned}
i)P^2 &= \partial_\theta(P^2) = 0\\
ii)Q^2 &= \Sigma R^2\\
iii)e_\theta &= \frac{1}{\sqrt{\Sigma}}\partial_\theta\\
iv)e_3 &= \frac{\sqrt{\Delta}}{R}\{\frac{1}{\Delta\sin^2\theta}[g_{\phi\phi}\partial_t - g_{t\phi}\partial\phi] - \frac{Q}{\Sigma}\partial_r\}\\
v)e_4 &= \frac{\sqrt{\Delta}}{R}\{\frac{1}{\Delta\sin^2\theta}[g_{\phi\phi}\partial_t - g_{t\phi}\partial_\phi] + \frac{Q}{\Sigma}\partial_r\}.
\end{aligned}$$

Lemma 2.1.2. *The new vector fields are related to those one of the precedent null frame in the following way*

$$\begin{aligned}
e_4 &= \frac{\sqrt{\Delta}}{R}[\frac{K^2+Q}{2\Sigma}L + \frac{K^2-Q}{2\Sigma}\underline{L} + \frac{P}{\sqrt{\Sigma}}\tilde{e}_\theta - \frac{a\sin\theta}{\sqrt{\Sigma}}\tilde{e}_\phi]\\
e_3 &= \frac{\sqrt{\Delta}}{R}[\frac{K^2-Q}{2\Sigma}L + \frac{K^2+Q}{2\Sigma}\underline{L} - \frac{P}{\sqrt{\Sigma}}\tilde{e}_\theta - \frac{a\sin\theta}{\sqrt{\Sigma}}\tilde{e}_\phi]\\
e_\theta &= \frac{Q}{\sqrt{\Sigma}R}\tilde{e}_\theta - \frac{\Delta P}{2\Sigma R}L + \frac{\Delta P}{2\Sigma R}\underline{L} \qquad (2.1.13)\\
e_\phi &= -\frac{\Delta a\sin\theta}{2\Sigma R}(L+\underline{L}) + \frac{K^2}{\sqrt{\Sigma}R}\tilde{e}_\phi,
\end{aligned}$$

or conversely, reversing these one, we can express the old vector fields in function of these news in the following way

$$\begin{aligned}
L &= \frac{K^2+Q}{2\sqrt{\Delta}R}e_4 + \frac{K^2-Q}{2\sqrt{\Delta}R}e_3 + \frac{a\sin\theta}{R}e_\phi \\
\underline{L} &= \frac{K^2-Q}{2\sqrt{\Delta}R}e_4 + \frac{K^2+Q}{2\sqrt{\Delta}R}e_3 + \frac{a\sin\theta}{R}e_\phi \\
\tilde{e}_\theta &= \frac{\sqrt{\Delta}P}{2\sqrt{\Sigma}R}e_4 - \frac{\sqrt{\Delta}P}{2\sqrt{\Sigma}R}e_3 + \frac{Q}{\sqrt{\Sigma}R}e_\theta \\
\tilde{e}_\phi &= \frac{K^2}{\sqrt{\Sigma}R}e_\phi + \frac{\sqrt{\Delta}a\sin\theta}{2R\sqrt{\Sigma}}e_4 + \frac{\sqrt{\Delta}a\sin\theta}{2R\sqrt{\Sigma}}e_3.
\end{aligned} \tag{2.1.14}$$

Proof. The proof of the first set of relations is a consequence of the explicit expression of the coordinate vector fields $\partial_t, \partial_r, \partial_\theta, \partial_\phi$ in function of $L, \underline{L}, \tilde{e}_a$. They are

$$\begin{aligned}
\partial_r &= \frac{1}{2}(L-\underline{L}) \\
\partial_\theta &= \sqrt{\Sigma}\tilde{e}_\theta \\
\partial_\phi &= -\frac{\Delta a\sin^2\theta}{2\Sigma}(L+\underline{L}) + \frac{K^2\sin\theta}{\sqrt{\Sigma}}\tilde{e}_\phi \\
\partial_t &= \frac{\Delta}{2\Sigma}(L+\underline{L}) - \frac{a\sin\theta}{\sqrt{\Sigma}}\tilde{e}_\phi.
\end{aligned}$$

The second set of equations derives from the expressions of $\partial_t, \partial_r, \partial_\theta, \partial_\phi$ written in terms of e_3, e_4, e_a:

$$\begin{aligned}
\partial_r &= \frac{Q}{2\sqrt{\Delta}R}(e_4-e_3) - \frac{P}{R}e_\theta \\
\partial_\theta &= \frac{\sqrt{\Delta}P}{2R}(e_4-e_3) + \frac{Q}{R}e_\theta \\
\partial_\phi &= R\sin\theta e_\phi \\
\partial_t &= \frac{\sqrt{\Delta}}{2R}(e_4+e_3) - \frac{2Mar\sin\theta}{\Sigma R}e_\phi.
\end{aligned}$$

Using them, from an explicit calculation, we find (2.1.13) and (2.1.14).

We note the useful identities we have used

$$
\begin{array}{rl}
i) & K^2 - a^2 \sin^2\theta = \Sigma \\
ii) & \Sigma R^2 - 2MrK^2 = \Sigma\Delta \\
iii) & \Sigma R^2 - 2Mra^2\sin^2\theta = \Sigma K^2.
\end{array}
$$

□

2.1.1 The function P

Now we want to describe the asymptotic behavior of

$$P = [a^2(\lambda - \sin^2\theta)]^{\frac{1}{2}}$$

with respect to r and to θ.
Thanks to (2.1.10), we can write

$$P = [a^2(\sin^2\theta_* - \sin^2\theta)]^{\frac{1}{2}}.$$

Let us suppose

$$\theta_* - \theta = O(\frac{1}{r^\alpha}),$$

we want to find α. First we observe that

$$
\begin{aligned}
\sin^2\theta_* &- \sin^2\theta = (\sin\theta_* - \sin\theta)(\sin\theta_* + \sin\theta) \\
&= 4\cos(\frac{\theta_*+\theta}{2})\sin(\frac{\theta_*-\theta}{2})\cos(\frac{\theta_*-\theta}{2})\sin(\frac{\theta_*+\theta}{2})
\end{aligned}
$$

and the quantity $\sin(\frac{\theta_*-\theta}{2})$ is the only part which tends to 0 for every $\theta \neq 0$.
Since at the first order

$$\sin(\theta_* - \theta) = \theta_* - \theta$$

it follows

$$\sqrt{\sin^2\theta_* - \sin^2\theta} = O(\sqrt{\sin(\theta_*-\theta)}) = O(\frac{1}{r^{\frac{\alpha}{2}}})$$

Then $P \to 0$ in the same way as $\theta_* \to \theta$ and so, given (2.1.11), it follows $F = 0$ if and only if the first integral is of the same order of the second, i.e.

$$\int_r^\infty \frac{dr'}{Q(r',\lambda)} \approx \int_\theta^{\theta_*} \frac{d\theta'}{P(\theta',\lambda)}.$$

Now

$$\int_r^\infty \frac{dr'}{Q(r',\lambda)} = O(\frac{1}{r^2})$$

and so it must be

$$\frac{1}{r^{\frac{\alpha}{2}}} = \frac{1}{r}$$

that means

$$\alpha = 2$$

and so P tends to 0 for θ_* tending to θ like $\frac{1}{r}$.

2.1.2 Kerr metric in Kruskal coordinates

We are enable to obtain another form of Kerr metric, starting from the function λ: since it is determined by the constraint $F = 0$, its partial derivatives can be computed from (2.1.8), by requiring $dF = 0$, i.e.

$$\mu d\lambda = -\frac{dr}{Q} + \frac{d\theta}{P}, \tag{2.1.15}$$

where $\mu = -\frac{\partial F}{\partial \lambda}$. It follows from it and from (2.1.6) that r_* and λ are orthogonal with respect to the intrinsic two-metric

$$\gamma^2 = \Sigma(\frac{dr^2}{\Delta} + d\theta^2)$$

of the spatial sections $(\phi, t) = const$ and further, since λ does not depend on ϕ and t, it follows that λ is constant along the lightlike generators, i.e.

$$l^\alpha \partial_\alpha \lambda = 0,$$

with

$$l_\alpha = -\partial_\alpha w = -\partial_\alpha (t \pm r_*)$$

If r_* and λ are adopted as any two new coordinates, in place of r and θ, defining the quantity $L = \mu PQ$, it follows the intrinsic two metric becomes:

$$\gamma^2 = R^{-2}(\Delta dr_*^2 + L^2 d\lambda^2)$$

and the Kerr metric assumes the form

$$ds^2 = \frac{\Delta}{R^2}(dr_*^2 - dt^2) + \frac{L^2}{R^2} d\lambda^2 + R^2 \sin^2\theta (d\phi - \omega_B dt)^2, \tag{2.1.16}$$

where

$$\omega_B = \frac{2Mar}{\Sigma R^2}.$$

Definition 2.1.1. *This new set of coordinates, composed by* $\{t, \phi, r_*, \lambda, \}$ *are called the Kruskal coordinates.*

2.2 The Minkowski spacetime

Let us suppose now the case $M = 0$. Then the Kerr line-element written in Boyer-Lindquist coordinates reduces itself to the metric of flat spacetime, expressed in oblates coordinates in fact it is enough to make the change of coordinates

$$\begin{aligned} x^2 + y^2 &= (r^2 + a^2)\sin^2\theta = K^2 \sin^2\theta \\ z &= r\cos\theta, \end{aligned}$$

that explicitly is

$$\begin{aligned} x &= K \sin\theta \cos\phi \\ y &= K \sin\theta \sin\phi \\ z &= r\cos\theta. \end{aligned}$$

It follows by these,

$$\begin{aligned} dx &= \frac{r}{K}\sin\theta\cos\phi dr + K\cos\theta\cos\phi d\theta - K\sin\theta\sin\phi d\phi \\ dy &= \frac{r}{K}\sin\theta\sin\phi dr + K\cos\theta\sin\phi d\theta + K\sin\theta\cos\phi d\phi \\ dz &= \cos\theta dr - r\sin\theta d\theta \end{aligned}$$

and finally Minkowski metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

assumes the form

$$ds^2 = -dt^2 + (\frac{r^2}{K^2}\sin^2\theta + \cos^2\theta)dr^2 + \Sigma d\theta^2 + K^2\sin^2\theta d\phi^2$$

that is just (2.1.1) with $M = 0$.

Definition 2.2.1. *We define the quasi-spherical light cones as null hypersurfaces which asymptotically approach Minkowski spherical light cones.*

Quasi-spherical light cones of the Kerr spacetime are the null hypersurface $C(u)$ and $\underline{C}(\underline{u})$ found in section 2.1.

Remark 2.2.1. *Studying the case $M = 0$, we note that the metric singularity $\Sigma = 0$ can not be a true singularity, but it is purely a coordinate singularity, located on the ring of radius a in the plane $z = 0$. Otherwise, testing the Kerr scalar curvature $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, it follows it blows up when $\Sigma = 0$ and $M \neq 0$, that means there is a true singularity, not just caused from a particular choice of the coordinates.*
Whit this in mind, we can give a good interpretation of such a singularity as a ring singularity, i.e. we define the Kerr metric on a manifold whose structure in a neighborhood of the set of points such that $\Sigma = 0$ has the topology of $\mathbb{R}^4$ with the set $S^1 \times \mathbb{R}$ removed.
As far as the other singularity is concerned, it is defined where $\Delta = 0$, that means $r_\pm = M \pm \sqrt{M^2 - a^2}$, possible only if $a \leq M$. Otherwise, as shown by Boyer and Lindquist the singularities in the metric components at $r = r_+$ and at $r = r_-$ are coordinate singularities of the same nature of the singularity $r = 2M$ in the Schwarzschild spacetime.

2.3 Kerr Christoffel symbols and connection coefficients

In the following, we write the Kerr Christoffel symbols with respect to the system of coordinates $\{t, r, \theta, \phi\}$, which are different from 0. Their explicit expression and their behavior at the highest order are:

$$
\begin{aligned}
\Gamma^t_{\theta t} &= -\frac{2Mra^2 \sin\theta\cos\theta}{\Sigma^2} = O(-\frac{2Ma^2\sin\theta\cos\theta}{r^3}) \\
\Gamma^t_{rt} &= -\frac{M}{\Delta\Sigma^2}(r^2+a^2)(\Sigma - 2r^2) = O(\frac{M}{r^2}) \\
\Gamma^t_{\theta\phi} &= \frac{2Mr(r^2+a^2)a^3\sin^3\theta\cos\theta}{\Sigma^3} = O(\frac{2Ma^3\sin^3\theta\cos\theta}{r^3}) \\
\Gamma^t_{r\phi} &= -\frac{Ma\sin^2\theta}{\Delta\Sigma^2}[4r^4 - (r-a)^2(\Sigma + 2ra)] = O(-\frac{3Ma\sin^2\ \theta}{r^2})
\end{aligned}
\qquad (2.3.17)
$$

$$
\begin{aligned}
\Gamma^r_{tt} &= \frac{M\Delta}{\Sigma^3}(\Sigma - 2r^2) = O(-\frac{M}{r^2}) \\
\Gamma^r_{\phi t} &= \frac{M\Delta}{\Sigma^3}(\Sigma - 2r^2)a\sin^2\theta = O(-\frac{Ma\sin^2\theta}{r^2}) \\
\Gamma^r_{r\theta} &= -\frac{a^2\sin\theta\cos\theta}{\Sigma} = O(-\frac{a^2\sin\theta\cos\theta}{r^2}) \\
\Gamma^r_{\theta\theta} &= -\frac{\Delta}{\Sigma}r = O(-r) \qquad (2.3.18) \\
\Gamma^r_{\phi\phi} &= -\frac{\Delta}{2\Sigma^2}[\sin^2\theta(4r(r^2+a^2) - 2(r-M)a^2\sin^2\theta) - 2rg_{\phi\phi}] = O(-r\sin^2\theta) \\
\Gamma^r_{rr} &= \frac{1}{\Sigma\Delta}(r\Delta - (r-M)\Sigma) = O(\frac{M}{r^2})
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^\theta_{tt} &= -\frac{2Ma^2r\sin\theta\cos\theta}{\Sigma^3} = O(-\frac{2Ma^2\sin\theta\cos\theta}{r^5}) \\
\Gamma^\theta_{\phi t} &= \frac{2Ma}{\Sigma^3}(r^2+a^2)r\sin\theta\cos\theta = O(\frac{2Ma\sin\theta\cos\theta}{r^3}) \\
\Gamma^\theta_{r\theta} &= \frac{r}{\Sigma} = O(\frac{1}{r}) \\
\Gamma^\theta_{\theta\theta} &= -\frac{a^2\sin\theta\cos\theta}{\Sigma} = O(-\frac{a^2\sin\theta\cos\theta}{r^2}) \qquad (2.3.19) \\
\Gamma^\theta_{\phi\phi} &= -\frac{1}{2\Sigma^3}(r^2+a^2)[(r^2+a^2)^2 - 2\Delta a^2\sin^2\theta]\sin\theta\cos\theta = O(-\sin\theta\cos\theta) \\
\Gamma^\theta_{rr} &= \frac{a^2\sin\theta\cos\theta}{\Sigma\Delta} = O(\frac{a^2\sin\theta\cos\theta}{r^4})
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^\phi_{\theta t} &= -\frac{2Mar\cot\theta}{\Sigma^2} = O(-\frac{2Ma\cot\theta}{r^3}) \\
\Gamma^\phi_{rt} &= \frac{Ma}{2\Delta\Sigma^3}\{2r[-2(r-M)\Sigma + 2r(\Delta - a^2\sin^2\theta)] + (\Delta - a^2\sin^2\theta)2(\Sigma - 2r^2)\} \\
&= O(-\frac{Ma}{r^4}) \qquad (2.3.20) \\
\Gamma^\phi_{r\phi} &= \frac{1}{2\Delta\Sigma^3}[4M^2a^2r\sin^2\theta(\Sigma - 2r^2) + (\Delta - a^2\sin^2\theta)[\Sigma(4r(r^2+a^2) \\
&- 2a^2\sin^2\theta(r-M)) - 2r((r^2+a^2)^2 - \Delta a^2\sin^2\theta)]] = O(\frac{1}{r}) \\
\Gamma^\phi_{\theta\phi} &= \frac{(r^2+a^2)\cot\theta}{\Delta\Sigma^3}[(\Delta - a^2\sin^2\theta)(r^2+a^2)^2 - 2(\Delta - a^2\sin^2\theta) \\
&\cdot\Delta a^2\sin^2\theta + 4M^2a^2r^2\sin^2\theta] = O(\cot\theta).
\end{aligned}
$$

Moreover, we have computed the connection coefficients, defined in (1.1.2) in terms of the null frame vector fields at the highest orders:

$$\begin{aligned}
\chi_{\theta\theta} &= \frac{1}{r} - \frac{M}{r^2} + \frac{P^2}{r^3} \\
\chi_{\theta\phi} &= 0 \\
\chi_{\phi\phi} &= \frac{1}{r} - \frac{M}{r^2} - \frac{P}{r^2}\cot\theta + \frac{a^2}{r^3}(2\sin^2\theta - \lambda) \\
\hat{\chi}_{\theta\theta} &= (\partial_\theta P - P\cot\theta)\frac{1}{r^2} + \frac{a^2}{r^3}(2\sin^2\theta - \lambda) \\
\hat{\chi}_{\phi\phi} &= -\hat{\chi}_{\theta\theta} \\
\hat{\chi}_{\theta\phi} &= 0 \\
\mathrm{tr}\chi &= \frac{2}{r} - \frac{2M}{r^2} - \frac{P\cot\theta}{r^2}, \qquad (2.3.21)
\end{aligned}$$

where

$$\begin{aligned}
&\mathrm{tr}\chi = g^{ab}\chi_{ab} = \chi_{\theta\theta} + \chi_{\phi\phi} \\
&\mathrm{tr}\underline{\chi} = g^{ab}\underline{\chi}_{ab} = \underline{\chi}_{\theta\theta} + \underline{\chi}_{\phi\phi} \\
&\hat{\chi}_{ab} = \chi_{ab} - \frac{1}{2}\delta^{ab}\chi_{ab} \\
&\underline{\hat{\chi}}_{ab} = \underline{\chi}_{ab} - \frac{1}{2}\delta^{ab}\underline{\chi}_{ab}.
\end{aligned}$$

Then:

$$\begin{aligned}
\zeta_\theta &= -\frac{1}{2r^3}(a^2\sin\theta\cos\theta + MP + P\partial_\theta P) \\
\zeta_\phi &= \frac{3Ma\sin\theta}{r^3} \\
\eta_\theta &= -\frac{1}{2r^3}(a^2\sin\theta\cos\theta + MP) \\
\eta_\phi &= \frac{3Ma\sin\theta}{r^3} \\
\omega &= -\frac{M}{2r^2} \\
\underline{\omega} &= \frac{M}{2r^2}. \qquad (2.3.22)
\end{aligned}$$

The following equalities are exact, i.e. they hold not only at the higher orders of decay, but for anyone:

$$
\begin{aligned}
\underline{\chi}_{\theta\theta} &= -\chi_{\theta\theta} \\
\underline{\chi}_{\theta\phi} &= \chi_{\theta\phi} \\
\underline{\chi}_{\phi\phi} &= -\chi_{\phi\phi} \\
\hat{\underline{\chi}}_{\theta\theta} &= -\chi_{\theta\theta} \\
\hat{\underline{\chi}}_{\phi\phi} &= -\chi_{\phi\phi} \qquad (2.3.23)\\
\hat{\underline{\chi}}_{\theta\phi} &= 0 \\
\mathrm{tr}\underline{\chi} &= -\mathrm{tr}\chi \qquad (2.3.24)\\
\underline{\eta}_{\theta} &= -\eta_{\theta} \\
\underline{\eta}_{\phi} &= -\eta_{\phi} \\
\omega &= -\underline{\omega} \qquad (2.3.25)
\end{aligned}
$$

We prove just one of it, in particular we show the expression for $\chi_{\theta\theta}$ is correct in the appendix.

2.4 Killing and pseudo-Killing vector fields

We have already introduced the Killing vector of Kerr spacetime in (2.1.3), let us recall them here:

$$
\begin{aligned}
T_0 &= \frac{\partial}{\partial_t} \\
{}^{(3)}O &= \frac{\partial}{\partial_\phi}
\end{aligned}
$$

they are the generators of the time translations and of the spatial symmetries.
The other two rotation vector fields (which are not Killing vector fields) written with respect to Boyer-Lindquist coordinates have the following expressions

$$
\begin{aligned}
{}^{(2)}O &= -\cos\phi\partial_\theta + \sin\phi\cot\theta\partial_\phi \\
{}^{(1)}O &= -\sin\phi\partial_\theta - \cos\phi\cot\theta\partial_\phi .
\end{aligned}
$$

Moreover we introduce the conformal Killing vector fields S, K_0 and $\bar{K}$ which will appear within the energy norms.

$$\begin{aligned} S &= \frac{1}{2}(\underline{u}e_4 + ue_3) \\ &= \frac{\sqrt{\Delta}}{R}\{(1+t^2+\rho^2)\frac{1}{\Delta\sin^2\theta}(g_{\phi\phi}\partial_t - g_{t\phi}\partial_\phi) + 2t\rho\frac{Q}{\Sigma}\partial_r\} \\ K_0 &= \frac{1}{2}(u^2 e_3 + \underline{u}^2 e_4) \\ \bar{K} &= \frac{1}{2}(\tau_+^2 e_4 + \tau_-^2 e_3) \qquad (2.4.26) \\ &= \frac{\sqrt{\Delta}}{R}\{\frac{t}{\Delta\sin^2\theta}(g_{\phi\phi}\partial_t - g_{t\phi}\partial_\phi) + \frac{\rho}{\Sigma}(Q\partial_r + P\partial_\theta)\}, \qquad (2.4.27) \end{aligned}$$

where ρ is defined in (5.6.43). Finally we define

$$T = \frac{1}{2}(e_3 + e_4) \qquad (2.4.28)$$

From the differential geometry, we recall the following

Definition 2.4.1. *Let X be a vector field, then the deformation tensor of X is defined as the quantity*

$$^{(X)}\pi_{\mu\nu} = \mathcal{L}_X g_{\mu\nu} = D_\mu X_\nu + D_\nu X_\mu$$

and its traceless part is

$$^{(X)}\hat{\pi}_{\mu\nu} = {}^{(X)}\pi_{\mu\nu} - \frac{1}{4}g_{\mu\nu}tr^{(X)}\pi.$$

Then, if X is a Killing vector field, it follows

$$^{(X)}\pi = 0.$$

Definition 2.4.2. *Given a Weyl tensor field W and a vector field X, we define the modified Lie derivative relative to X by*

$$\hat{\mathcal{L}}_X W = L_X W - \frac{1}{2}{}^{(X)}[W] + \frac{3}{8}tr^{(X)}\pi W, \qquad (2.4.29)$$

where

$$^{(X)}[W]_{\alpha\beta\gamma\delta} = {}^{(X)}\pi^\mu_\alpha W_{\mu\beta\gamma\delta} + {}^{(X)}\pi^\mu_\beta W_{\alpha\mu\gamma\delta} + {}^{(X)}\pi^\mu_\gamma W_{\alpha\beta\mu\delta} + {}^{(X)}\pi^\mu_\delta W_{\alpha\beta\gamma\mu}.$$

Remark 2.4.1. *The modified Lie derivative of a tensor field is a tensorial quantity too and the modified Lie derivative of a Weyl field is yet a Weyl field.*

In the following we give a decomposition of the deformation tensor of a vector X with respect to the null frame:

$$
\begin{aligned}
{}^{(X)}\pi_{ab} &= g(D_{e_a}X, e_b) + g(D_{e_b}X, e_a) \\
{}^{(X)}\pi_{a4} &= g(D_{e_a}X, e_4) + g(D_{e_4}X, e_a) \\
{}^{(X)}\pi_{a3} &= g(D_{e_a}X, e_3) + g(D_{e_3}X, e_a) \\
{}^{(X)}\pi_{34} &= g(D_{e_3}X, e_4) + g(D_{e_4}X, e_3) \\
{}^{(X)}\pi_{44} &= 2g(D_{e_4}X, e_4) \\
{}^{(X)}\pi_{33} &= 2g(D_{e_3}X, e_3),
\end{aligned}
\tag{2.4.30}
$$

and the traceless part of the various components is

$$
\begin{aligned}
{}^{(X)}\hat{\pi}_{ab} &= {}^{(X)}\pi_{ab} - \frac{1}{4}\delta_{ab}\mathrm{tr}^{(X)}\pi \\
{}^{(X)}\hat{\pi}_{a4} &= {}^{(X)}\pi_{a4} \\
{}^{(X)}\hat{\pi}_{a3} &= {}^{(X)}\pi_{a3} \\
{}^{(X)}\hat{\pi}_{34} &= {}^{(X)}\pi_{34} + \frac{1}{2}\mathrm{tr}^{(X)}\pi \\
{}^{(X)}\hat{\pi}_{44} &= {}^{(X)}\pi_{44} \\
{}^{(X)}\hat{\pi}_{33} &= {}^{(X)}\pi_{33}.
\end{aligned}
$$

Now let us introduce a new notation for the previous quantities:

$$
\begin{aligned}
{}^{(X)}i_{ab} = {}^{(X)}\hat{\pi}_{ab} \quad &, \quad {}^{(X)}j = {}^{(X)}\hat{\pi}_{34} \\
{}^{(X)}m_a = {}^{(X)}\hat{\pi}_{a4} \quad &, \quad {}^{(X)}\underline{m}_a = {}^{(X)}\hat{\pi}_{a3} \\
{}^{(X)}n = {}^{(X)}\hat{\pi}_{44} \quad &, \quad {}^{(X)}\underline{n} = {}^{(X)}\hat{\pi}_{33}.
\end{aligned}
\tag{2.4.31}
$$

2.5 Asymptotic behavior of deformation tensors

Proposition 2.5.1. *As* ${}^{(3)(O)}$ *is a Killing vector field, the deformation tensor related to* ${}^{(3)}O$ *is null. The null components of the two other rotation* *deformation tensors behave at the infinity in the following way:*

$$
\begin{aligned}
{}^{(2)(O)}i_{\theta\theta} &= -\frac{a}{r^2}(2a-3M)\sin\theta\cos\theta\cos\phi \\
{}^{(2)(O)}i_{\phi\phi} &= -\frac{3Ma}{r^2}\sin\theta\cos\theta\cos\phi \\
{}^{(2)(O)}i_{\theta\phi} &= -\frac{a^2}{r^2}\sin\theta\cos\theta\sin\phi \\
{}^{(2)(O)}j &= \frac{a}{r^2}(a-3M)\sin\theta\cos\theta\cos\phi \\
{}^{(2)(O)}m_{\theta} &= \frac{2Ma\sin\phi\cos^2\theta}{r^2}(1+\sin^2\theta) \\
{}^{(2)(O)}m_{\phi} &= 0 \\
{}^{(2)(O)}\underline{m}_{\theta} &= m_\theta + 2\frac{P^2}{r^2}\cos\phi \\
{}^{(2)(O)}\underline{m}_{\phi} &= -m_\phi = 0 \\
{}^{(2)(O)}n &= \frac{2a\sin\theta\cos\theta}{r^2}[a\cos\phi - M\sin\phi] \\
{}^{(2)(O)}\underline{n} &= \frac{2a\sin\theta\cos\theta}{r^2}[a\cos\phi + M\sin\phi]
\end{aligned}
\tag{2.5.32}
$$

and the behavior of ${}^{(1)}(O)\pi_{\mu\nu}$ *is very similar, in particular*

$$
\begin{aligned}
{}^{(1)(O)}i_{\theta\theta} &= -\frac{a}{r^2}(2a-3M)\sin\theta\cos\theta\sin\phi \\
{}^{(1)(O)}i_{\phi\phi} &= -\frac{3Ma}{r^2}\sin\theta\cos\theta\sin\phi \\
{}^{(1)(O)}i_{\theta\phi} &= \frac{a^2+M^2}{r^2}\sin\theta\cos\theta\cos\phi \\
{}^{(1)(O)}j &= \frac{a}{r^2}(a-3M)\sin\theta\cos\theta\sin\phi
\end{aligned}
\tag{2.5.33}
$$

$$
\begin{aligned}
{}^{(1)(O)}m_\theta &= -\frac{2Ma\cos\phi\cos^2\theta}{r^2}(1+\sin^2\theta) \\
{}^{(1)(O)}m_\phi = 0 & \\
{}^{(1)(O)}\underline{m}_\theta &= m_\theta + 2\frac{P^2}{r^2}\sin\phi \\
{}^{(1)(O)}\underline{m}_\phi &= -m_\phi \\
{}^{(1)(O)}n &= \frac{2a\sin\theta\cos\theta}{r^2}[a\sin\phi + M\cos\phi] \\
{}^{(1)(O)}\underline{n} &= \frac{2a\sin\theta\cos\theta}{r^2}[a\sin\phi - M\cos\phi]. \qquad (2.5.34)
\end{aligned}
$$

The proof is a straightforward direct calculation, starting from the definition of the null components of ${}^{(O^{(i)})}\pi$, and using the results found for the connection coefficients of Kerr spacetime. We have computed explicitly all this terms, but in order to not overload too much the work, we report only one of it at the appendix of the chapter.

Corollary 2.5.1. *As far as the O components are concerned, we have found in $\mathcal{K}$ the following inequalities:*

$$
|r^3 \nabla\!\!\!/ ({}^{(O)}i, {}^{(O)}j, {}^{(O)}m, {}^{(O)}\underline{m}, {}^{(O)}n, {}^{(O)}\underline{n})| \leq c \qquad (2.5.35)
$$

$$
|r^3 D\!\!\!\!/_4 ({}^{(O)}i, {}^{(O)}j, {}^{(O)}m, {}^{(O)}\underline{m}, {}^{(O)}n, {}^{(O)}\underline{n})| \leq c \qquad (2.5.36)
$$

$$
|r^3 D\!\!\!\!/_3 ({}^{(O)}i, {}^{(O)}j, {}^{(O)}m, {}^{(O)}\underline{m}, {}^{(O)}n, {}^{(O)}\underline{n})| \leq c. \qquad (2.5.37)
$$

Really, it holds the following

Proposition 2.5.2. *The first derivatives of the components of ${}^{(O)}\pi_{\mu\nu}$ satisfy the following L_p estimates on any $S \subset \mathcal{K}$, with $p \in [2,4]$:*

$$
\begin{aligned}
||r^{3-\frac{2}{p}} \nabla\!\!\!/ ({}^{(O)}i, {}^{(O)}j, {}^{(O)}m, {}^{(O)}\underline{m}, {}^{(O)}n, {}^{(O)}\underline{n})||_{p,S} &\leq c \\
||r^{3-\frac{2}{p}} D\!\!\!\!/_4 ({}^{(O)}i, {}^{(O)}j, {}^{(O)}m, {}^{(O)}\underline{m}, {}^{(O)}n, {}^{(O)}\underline{n})||_{p,S} &\leq c \\
||r^{3-\frac{2}{p}} D\!\!\!\!/_3 ({}^{(O)}i, {}^{(O)}j, {}^{(O)}m, {}^{(O)}\underline{m}, {}^{(O)}n, {}^{(O)}\underline{n})||_{p,S} &\leq c.
\end{aligned}
$$

Proof. This result and those following one relative to the derivatives of null components of the other deformation tensors are obtained observing the following relations hold:
i) when D_4 acts on a function f, it brings a term which is of the order of $\frac{1}{r}$, that means it improves its asymptotic behavior like r^{-1}.

ii)In general, when D_3 acts on f, it is substantially like $\frac{\partial}{\partial_u}$, then it brings a factor of the form u^{-1}, but whether f doesn't depend on t, but only on r, then the derivative with respect to e_3 produces a factor of the form $\frac{1}{r}$.
ii) When you make the tangential derivative $\nabla\!\!\!/$ on f, then if it depends on θ or ϕ, then $\nabla\!\!\!/ f$ will be of the order of $\frac{1}{r}f$; if it doesn't depend on any two angle, then $\nabla\!\!\!/ f = O(r^{-2})f$. (This last fact follows easily from the form of the vector fields e_θ, e_ϕ (see (2.1.12))). $\square$

The explicit expressions of the components of ${}^{(X)}\hat{\pi}_{\mu\nu}$, when $X = T, S, K_0$ are (see [3], (pg. 172-176)):

$$\begin{aligned}
{}^{(T)}i_{ab} &= \hat{\chi}_{ab} + \underline{\hat{\chi}}_{ab} + \frac{1}{2}\delta_{ab}(\frac{1}{2}(\mathrm{tr}\chi + \mathrm{tr}\underline{\chi}) + \omega + \underline{\omega}) \\
{}^{(T)}j &= \frac{1}{2}(\mathrm{tr}\chi + \mathrm{tr}\underline{\chi}) + (\omega + \underline{\omega}) \\
{}^{(T)}m_a &= \underline{\eta}_a - \zeta_a \\
{}^{(T)}\underline{m}_a &= \eta_a + \zeta_a \\
{}^{(T)}n &= -4\omega \\
{}^{(T)}\underline{n} &= -4\underline{\omega},
\end{aligned} \tag{2.5.38}$$

$$\begin{aligned}
{}^{(S)}i_{ab} &= \underline{u}\hat{\chi}_{ab} + u\underline{\hat{\chi}}_{ab} + \frac{1}{2}\delta_{ab}(\frac{1}{2}(\underline{u}\mathrm{tr}\chi + u\mathrm{tr}\underline{\chi}) + (\underline{u}\omega + u\underline{\omega}) - \frac{2R}{\sqrt{\Delta}}) \\
{}^{(S)}j &= \frac{1}{2}(\underline{u}\mathrm{tr}\chi + u\mathrm{tr}\underline{\chi}) + (\underline{u}\omega + u\underline{\omega}) - \frac{2R}{\sqrt{\Delta}} \\
{}^{(S)}m_a &= u\,{}^{(T)}m_a \\
{}^{(S)}\underline{m}_a &= \underline{u}\,{}^{(T)}\underline{m}_a \\
{}^{(S)}n &= u\,{}^{(T)}n \\
{}^{(S)}\underline{n} &= \underline{u}\,{}^{(T)}\underline{n},
\end{aligned} \tag{2.5.39}$$

$$\begin{aligned}
{}^{(K_0)}i_{ab} &= \underline{u}^2\hat{\chi}_{ab} + u^2\underline{\hat{\chi}}_{ab} + \frac{1}{2}\delta_{ab}(\frac{1}{2}(\underline{u}^2\mathrm{tr}\chi + u^2\mathrm{tr}\underline{\chi}) + (\underline{u}^2\omega + u^2\underline{\omega}) - (\underline{u} + u)\frac{2R}{\sqrt{\Delta}}) \\
{}^{(K_0)}j &= \frac{1}{2}(\underline{u}^2\mathrm{tr}\chi + u^2\mathrm{tr}\underline{\chi}) + (\underline{u}^2\omega + u^2\underline{\omega}) - (\underline{u} + u)\frac{2R}{\sqrt{\Delta}} \\
{}^{(K_0)}m_a &= u^2\,{}^{(T)}m_a \\
{}^{(K_0)}\underline{m}_a &= \underline{u}^2\,{}^{(T)}\underline{m}_a \\
{}^{(K_0)}n &= u^2\,{}^{(T)}n \\
{}^{(K_0)}\underline{n} &= \underline{u}^2\,{}^{(T)}\underline{n}.
\end{aligned} \tag{2.5.40}$$

Proposition 2.5.3. *Recalling the decays of the connection coefficients, we find the following asymptotic behavior for the components of the deformation tensor of the vector field* $T = \frac{2}{\Omega} T_0$*:*

$$
\begin{aligned}
{}^{(T)}i_{ab} &= 0 \\
{}^{(T)}j &= 0 \\
{}^{(T)}m_\theta &= \frac{1}{r^3}(a^2 \sin\theta\cos\theta + MP + \frac{P}{2}\partial_\theta P) \\
{}^{(T)}m_\phi &= -\frac{6Ma\sin\theta}{r^3} \\
{}^{(T)}\underline{m}_\theta &= -{}^{(T)}m_\theta \\
{}^{(T)}\underline{m}_\phi &= -{}^{(T)}m_\phi \\
{}^{(T)}n &= \frac{2M}{r^2} \\
{}^{(T)}\underline{n} &= -\frac{2M}{r^2}.
\end{aligned}
\tag{2.5.41}
$$

Their derivatives decay in the following way:

$$
\begin{aligned}
&\| \; r^{4-\frac{2}{p}} \nabla\!\!\!\!/\, {}^{(T)}m \; \|_{p,S} \leq c \\
&\| \; r^{4-\frac{2}{p}} \nabla\!\!\!\!/\, {}^{(T)}\underline{m} \; \|_{p,S} \leq c \\
&\| \; r^{4-\frac{2}{p}} \nabla\!\!\!\!/\, {}^{(T)}n \; \|_{p,S} \leq c \\
&\| \; r^{4-\frac{2}{p}} \nabla\!\!\!\!/\, {}^{(T)}\underline{n} \; \|_{p,S} \leq c.
\end{aligned}
\tag{2.5.42}
$$

$$
\begin{aligned}
&\| \; r^{4-\frac{2}{p}} \mathbf{D}\!\!\!\!/_4 {}^{(T)}m \; \|_{p,S} \leq c \\
&\| \; r^{4-\frac{2}{p}} \mathbf{D}\!\!\!\!/_4 {}^{(T)}\underline{m} \; \|_{p,S} \leq c \\
&\| \; r^{3-\frac{2}{p}} \mathbf{D}\!\!\!\!/_4 {}^{(T)}n \; \|_{p,S} \leq c \\
&\| \; r^{3-\frac{2}{p}} \mathbf{D}\!\!\!\!/_4 {}^{(T)}\underline{n} \; \|_{p,S} \leq c.
\end{aligned}
\tag{2.5.43}
$$

$$
\begin{aligned}
&\| \; r^{4-\frac{2}{p}} \mathbf{D}\!\!\!\!/_3 {}^{(T)}m \; \|_{p,S} \leq c \\
&\| \; r^{4-\frac{2}{p}} \mathbf{D}\!\!\!\!/_3 {}^{(T)}\underline{m} \; \|_{p,S} \leq c \\
&\| \; r^{3-\frac{2}{p}} \mathbf{D}\!\!\!\!/_3 {}^{(T)}n \; \|_{p,S} \leq c \\
&\| \; r^{3-\frac{2}{p}} \mathbf{D}\!\!\!\!/_3 {}^{(T)}\underline{n} \; \|_{p,S} \leq c.
\end{aligned}
\tag{2.5.44}
$$

Proposition 2.5.4. *The components of ${}^{(S)}\hat{\pi}_{\mu\nu}$ at the highest order have the following form:*

$$
\begin{aligned}
{}^{(S)}i_{\theta\theta} &= 2M\frac{\log r}{r} + \frac{5}{2}\frac{\partial_\theta P - P\cot\theta - M}{r}\\
{}^{(S)}i_{\phi\phi} &= \frac{2M\log r}{r} + \frac{3(P\cot\theta - \partial_\theta P) - 7M}{2r}\\
{}^{(S)}j &= 4M\frac{\log r}{r} - \frac{5M}{r}\\
{}^{(S)}m_\theta &= \frac{u}{2}\frac{2(a^2\sin\theta\cos\theta + MP) + P\partial_\theta P}{r^3}\\
{}^{(S)}m_\phi &= -6u\frac{Ma\sin\theta}{r^3}\\
{}^{(S)}\underline{m}_\theta &= -\frac{\underline{u}}{2}\frac{2(a^2\sin\theta\cos\theta + MP) + P\partial_\theta P}{r^3}\\
= &\quad -\frac{2(a^2\sin\theta\cos\theta + MP) + P\partial_\theta P}{r^2}\\
{}^{(S)}\underline{m}_\phi &= -\frac{\underline{u}}{2}\frac{2(a^2\sin\theta\cos\theta + MP) + P\partial_\theta P}{r^3}\\
= &\quad -\frac{2(a^2\sin\theta\cos\theta + MP) + P\partial_\theta P}{r^2}\\
{}^{(S)}n &= 2\frac{Mu}{r^2}\\
{}^{(S)}\underline{n} &= -2\frac{M\underline{u}}{r^2} = -4\frac{M}{r},
\end{aligned}
\tag{2.5.45}
$$

as:

$$
i)\frac{\tau_+}{r^2} \quad = \quad O(2).
$$

. Moreover, for their first derivatives, the following L_p estimates hold for any

$p \in [2,4]$ *and for any* $S \subset \mathcal{K}$:

$$\begin{aligned}
|| \quad r^{3-\frac{2}{p}} \nabla\!\!\!\!/^{(S)} i \quad &||_{p,S} \leq c \\
|| \quad r^{3-\frac{2}{p}} \nabla\!\!\!\!/^{(S)} j \quad &||_{p,S} \leq c \\
|| \quad \frac{r^{4-\frac{2}{p}}}{\tau_-} \nabla^{(S)} m \quad &||_{p,S} \leq c \\
|| \quad r^{3-\frac{2}{p}} \nabla\!\!\!\!/^{(S)} \underline{m} \quad &||_{p,S} \leq c \\
|| \quad r^{4-\frac{2}{p}} \frac{1}{\tau_-} \nabla\!\!\!\!/^{(S)} n \quad &||_{p,S} \leq c \\
|| \quad r^{3-\frac{2}{p}} \nabla\!\!\!\!/^{(S)} \underline{n} \quad &||_{p,S} \leq c.
\end{aligned} \tag{2.5.46}$$

$$\begin{aligned}
|| \quad r^{2-\frac{2}{p}} \mathbf{D}\!\!\!\!/_4{}^{(S)} i \quad &||_{p,S} \leq c \\
|| \quad r^{2-\frac{2}{p}} \mathbf{D}\!\!\!\!/_4{}^{(S)} j \quad &||_{p,S} \leq c \\
|| \quad r^{4-\frac{2}{p}} \frac{1}{\tau_-} \mathbf{D}\!\!\!\!/_4{}^{(S)} m \quad &||_{p,S} \leq c \\
|| \quad r^{3-\frac{2}{p}} \mathbf{D}\!\!\!\!/_4{}^{(S)} \underline{m} \quad &||_{p,S} \leq c \\
|| \quad r^{3-\frac{2}{p}} \frac{1}{\tau_-} \mathbf{D}\!\!\!\!/_4{}^{(S)} n \quad &||_{p,S} \leq c \\
|| \quad r^{2-\frac{2}{p}} \mathbf{D}\!\!\!\!/_4{}^{(S)} \underline{n} \quad &||_{p,S} \leq c.
\end{aligned} \tag{2.5.47}$$

$$\begin{aligned}
|| \quad r^{2-\frac{2}{p}} \mathbf{D}\!\!\!\!/_3{}^{(S)} i \quad &||_{p,S} \leq c \\
|| \quad r^{2-\frac{2}{p}} \mathbf{D}\!\!\!\!/_3{}^{(S)} j \quad &||_{p,S} \leq c \\
|| \quad r^{3-\frac{2}{p}} \mathbf{D}\!\!\!\!/_3{}^{(S)} m \quad &||_{p,S} \leq c \\
|| \quad r^{3-\frac{2}{p}} \mathbf{D}\!\!\!\!/_3{}^{(S)} \underline{m} \quad &||_{p,S} \leq c \\
|| \quad r^{2-\frac{2}{p}} \mathbf{D}\!\!\!\!/_3{}^{(S)} n \quad &||_{p,S} \leq c \\
|| \quad r^{2-\frac{2}{p}} \mathbf{D}\!\!\!\!/_3{}^{(S)} \underline{n} \quad &||_{p,S} \leq c.
\end{aligned} \tag{2.5.48}$$

At last we have computed the components of $\bar{K}$ deformation tensor at the first order in the same way and they result to be the following:

Proposition 2.5.5. *For any $S \subset \mathcal{K}$, the following estimates hold*

$$\begin{aligned}
{}^{(\bar K)}i_{\theta\theta} &= \frac{4Mt\log r}{r}\\
{}^{(\bar K)}i_{\phi\phi} &= \frac{4Mt\log r}{r}\\
{}^{(\bar K)}j &= 8Mt\frac{\log r}{r}\\
{}^{(\bar K)}m_\theta &= \tau_-^2\frac{1}{2}\frac{2(a^2\sin\theta\cos\theta + MP) + P\partial_\theta P}{r^3}\\
{}^{(\bar K)}m_\phi &= \tau_-^2(-6\frac{Ma\sin\theta}{r^3})\\
{}^{(\bar K)}\underline{m}_\theta &= \tau_+^2(-\frac{1}{2}\frac{2(a^2\sin\theta\cos\theta + MP) + P\partial_\theta P}{r^3})\\
= &\quad -2\frac{2(a^2\sin\theta\cos\theta + MP) + P\partial_\theta P}{r}\\
{}^{(\bar K)}\underline{m}_\phi &= \tau_+^2(-\frac{1}{2}\frac{2(a^2\sin\theta\cos\theta + MP) + P\partial_\theta P}{r^3})\\
= &\quad -2\frac{2(a^2\sin\theta\cos\theta + MP) + P\partial\theta P}{r}\\
{}^{(\bar K)}n &= 2\frac{M\tau_-^2}{r^2}\\
{}^{(\bar K)}\underline{n} &= -2\frac{M\tau_+^2}{r^2} = -8M.
\end{aligned} \tag{2.5.49}$$

Moreover, for every $p \in [2,4]$ the following inequalities hold:

$$\begin{aligned}
|| \quad \frac{r^{3-\frac{2}{p}}}{t}\nabla\!\!\!\!/\,{}^{(\bar K)}i \quad &||_{p,S} \le c\\
|| \quad \frac{r^{3-\frac{2}{p}}}{t}\nabla\!\!\!\!/\,{}^{(\bar K)}j \quad &||_{p,S} \le c\\
|| \quad r^{4-\frac{2}{p}}\frac{1}{\tau_-^2}\nabla\!\!\!\!/\,{}^{(\bar K)}m \quad &||_{p,S} \le c\\
|| \quad r^{2-\frac{2}{p}}\nabla\!\!\!\!/\,{}^{(\bar K)}\underline{m} \quad &||_{p,S} \le c\\
|| \quad r^{4-\frac{2}{p}}\frac{1}{\tau_-^2}\nabla\!\!\!\!/\,{}^{(\bar K)}n \quad &||_{p,S} \le c\\
|| \quad r^{4-\frac{2}{p}}\nabla\!\!\!\!/\,{}^{(\bar K)}\underline{n} \quad &||_{p,S} \le c.
\end{aligned} \tag{2.5.50}$$

The L_p estimates we have found are a consequence of a more general fact, easy to show:

Lemma 2.5.1. *If the L_∞ norm of a tensor field on $\mathcal{K}$ decays as $r^{-\alpha}$, then, on every S, its L_p norms decay along the null directions as $r^{-\alpha-\frac{2}{p}}$, for $p \in [2,4]$.*

2.6 Appendix

2.6.1 Estimate of $\chi_{\theta\theta}$

Let us calculate explicitly the connection coefficient $\chi_{\theta\theta} = g(D_{e_\theta} e_4, e_\theta)$:

$$\begin{aligned}
\chi_{\theta\theta} &= \frac{Q}{\Sigma R}\big[\partial_\theta\big(\frac{\sqrt{\Delta}P}{\Sigma R}\big)g(\partial_\theta, e_\theta) + \partial_\theta\big(\frac{\sqrt{\Delta}Q}{\Sigma R}g(\partial_r, e_\theta)\big]\\
&+\Gamma^\theta_{\rho\sigma}e_4^\rho e_\theta^\sigma g(\partial_\theta, e_\theta) - \frac{\Delta P}{\Sigma R}\big[\partial_r\big(\frac{\sqrt{\Delta}P}{\Sigma R}\big)g(\partial_\theta, e_\theta)\\
&+\partial_r\big(\frac{\sqrt{\Delta}Q}{\Sigma R}\big)g(\partial_r, e_\theta)\big] + \Gamma^r_{\rho\sigma}e_4^\rho e_\theta^\sigma g(\partial_r, e_\theta),
\end{aligned}$$

being the only terms different from 0, as ∂_θ and ∂_r the only coordinate vector fields not orthogonal to e_θ. Then:

$$\begin{aligned}
\chi_{\theta\theta} &= \frac{Q^2}{\Sigma R}O\big(\frac{\partial_\theta P}{r^2}\big) - \frac{\sqrt{\Delta}QP}{\Sigma R^3}\big[2a^2\sin\theta\cos\theta - \frac{1}{2}\partial_\theta\lambda\big]\frac{1}{r^2}\\
&+\big[\frac{r}{\Sigma}\big(\frac{\sqrt{\Delta}Q^2}{\Sigma^2R^2} - \frac{\Delta^{\frac{3}{2}}P^2}{\Sigma^2R^2}\big) - \frac{2a^2\sin\theta\cos\theta}{\Sigma}\frac{\sqrt{\Delta}PQ}{\Sigma^2R^2}\big]\frac{Q}{R}\\
&\frac{\Delta P}{\Sigma R}O\big(\frac{P}{r^3}\big)\frac{Q}{R} + \frac{M\Delta P^2 Q}{\Sigma^2R^2r^2} + \big[-\frac{a^2\sin\theta\cos\theta}{\Sigma}\\
&\big(\frac{\sqrt{\Delta}Q^2}{\Sigma^2R^2} - \frac{\Delta^{\frac{3}{2}}P^2}{\Sigma^2R^2}\big)\frac{\Delta}{\Sigma}r\frac{\sqrt{\Delta}PQ}{R^2\Sigma^2}\\
&+\frac{Mr^2}{\Sigma\Delta}\frac{\Delta^{\frac{3}{2}}QP}{\Sigma^2R^2}\big]\big(-\frac{P}{R}\big)\\
&= \frac{\partial_\theta P}{r^2} + O\big(\frac{2P(a^2\sin\theta\cos\theta - \frac{\partial_\theta\lambda}{2})}{r^4}\big) + O\big(\frac{1}{r}\big) + O\big(\frac{P^2}{r^3}\big)\\
&+O\big(\frac{-2Pa^2\sin\theta\cos\theta}{r^4}\big) + O\big(-\frac{P^2}{r^3}\big) + O\big(\frac{MP^2}{r^4}\big)\\
&+O\big(\frac{Pa^2\sin\theta\cos\theta}{r^4}\big) + O\big(\frac{a^2P^3}{r^6}\big) + O\big(\frac{P^2}{r^3}\big) + O\big(-\frac{P^2M}{r^4}\big),
\end{aligned}$$

where we have used the expressions in power series centered at $r = \infty$ for the following quantities:

$$\begin{aligned}
\frac{\sqrt{\Delta}}{R} &= 1 - \frac{M}{r} + \frac{a^2 M}{r^3}(\frac{1}{2} - \sin^2\theta) \\
\frac{Q}{\Sigma} &= 1 + \frac{a^2(\sin^2\theta - \frac{\lambda}{2})}{r^2} \\
\frac{\Delta}{\Sigma} &= 1 - \frac{M}{r} + \frac{a^2\sin^2\theta}{2r^2} \\
\frac{Q}{R^2} &= 1 - \frac{a^2\lambda}{2r^2}
\end{aligned}$$

The term $O\big(\frac{1}{r}\big)$ derives from the highest order of $\chi_{\theta\theta}$, that is $\frac{\sqrt{\Delta}Q^3 r}{\Sigma^3 R^3}$. Developing it as a series of powers, its first terms result to be:

$$\frac{1}{r} - \frac{M}{r^2} + \frac{3a^2}{r^3}\big(\sin^2\theta - \frac{\lambda}{2}\big),$$

and so at the higher orders, the component $\chi_{\theta\theta}$ assumes the following form:

$$\chi_{\theta\theta} = \frac{1}{r} - \frac{M}{r^2} + \frac{P^2}{r^3}. \tag{2.6.51}$$

2.6.2 Estimate of ${}^{(2)(O)}n$

$$
\begin{aligned}
{}^{(2)O}n \;=\; & 2g(D_{e_4}{}^{(2)}O, e_4) = \frac{4Mra}{\sqrt{\Delta}\Sigma R}\partial_\phi(-\cos\phi)g(\partial_\theta, e_4) + 2\Gamma^t_{\rho\sigma}{}^{(2)}O^\rho e_4^\sigma g(\partial_t, e_4) \\
& +2\Gamma^r_{\rho\sigma}{}^{(2)}O^\rho e_4^\sigma g(\partial_r, e_4) + 2\Gamma^\theta_{\rho\sigma}{}^{(2)}O^\rho e_4^\sigma g(\partial_\theta, e_4) \\
& = \frac{MPar\sin\phi}{\Sigma R^2} - 2\big[O\big(\frac{2a^3M\sin^3\theta\cos\theta}{r^3}\big)\big(-\frac{2Mra\cos\phi}{\sqrt{\Delta}\Sigma R} \\
& +\frac{\sqrt{\Delta}P\sin\phi\cos\theta}{\Sigma R\sin\theta}\big) + O\big(\frac{2Ma^2\sin\theta\cos\theta}{r^3}\big)\cos\phi\frac{R}{\sqrt{\Delta}}\big] \\
& +\frac{2Q}{\sqrt{\Delta}R}\big[O\big(-\frac{Ma\sin^2\theta}{r^2}\big)\sin\phi\cot\theta\frac{R}{\sqrt{\Delta}} + \frac{a^2\sin^2\theta\cos\theta}{\Sigma}\cos\phi\frac{\sqrt{\Delta}Q}{\Sigma R} \\
& +\frac{\Delta}{\Sigma}r\cos\phi\frac{\sqrt{\Delta}P}{\Sigma R} + O\big(-r\sin^2\theta\big)\sin\phi\cot\theta\frac{2Mra}{\sqrt{\Delta}\Sigma R}\big] + \frac{2\sqrt{\Delta}P}{R} \\
& \cdot\big[\frac{2Ma\sin\theta\cos\theta\sin\phi}{r^3}\cot\theta\frac{R}{\sqrt{\Delta}} - \frac{r}{\Sigma}\cos\phi\frac{Q}{\Sigma}\frac{\sqrt{\Delta}}{R} \\
& +\frac{a^2\sin\theta\cos\theta}{\Sigma}\cos\phi\frac{\sqrt{\Delta}P}{\Sigma R} - \frac{2Mra\sin\phi\cos^2\theta}{\sqrt{\Delta}\Sigma R}\big].
\end{aligned}
$$

The highest order terms came from:

$$g(\partial_r, e_4)\big(\Gamma^r_{t\phi}e_4^t{}^{(2)}O^\phi + \Gamma^r_{\theta r}{}^{(2)}O^\theta e_4^r.$$

So

$${}^{(2)O}n = O\big(\frac{2a\sin\theta\cos\theta}{r^2}(a\cos\phi - M\sin\phi)\big).$$

Chapter 3

The Error Estimate

3.1 Definitions and prerequisites

As far as we have claimed in the second chapter, we have to check the boundedness of the $\mathcal{Q}$ norms. More precisely, in this chapter we will prove the following

Theorem 3.1.1. *Let us suppose W be a Weyl tensor which propagates itself in Kerr spacetime and which satisfies the Bianchi equations.*
Then, the following inequality holds

$$\mathcal{Q}_{\mathcal{K}} \leq c\mathcal{Q}_{\Sigma_0 \cap \mathcal{K}}, \tag{3.1.1}$$

where $\mathcal{Q}_{\mathcal{K}}$ and $\mathcal{Q}_{\Sigma_0 \cap \mathcal{K}}$ norms are defined in definition 1.4.1.

Remark 3.1.1. *The proof will mimic the result obtained in [11], chapter 6). The necessary calculations using in the proof are many and complicate, then we will not report all, but, in order to render the exposition clear and to give some general ideas of the work, we do some of then explicitly in the next sections and we will report the other one in the appendix at the end of the chapter.*

Now we are trying to explicate basis concepts about the used techniques. In order to prove the result (3.1.6) we need to control the quantity that we call the *error term*, defined in the following way

$$\mathcal{E}(u, \underline{u}) \equiv (\mathcal{Q} + \underline{\mathcal{Q}})(u, \underline{u}) - \mathcal{Q}_{\Sigma_0 \cap V(u,\underline{u})}. \tag{3.1.2}$$

Recalling the definition of the 1-form P related to the Bel-Robinson tensor of W (see (1.1.7)) and using Stokes theorem we are able to prove the following equality holds:

$$\begin{aligned}
&\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} Q(W)(X,Y,Z,e_3) \;+\; \int_{C(u)\cap V(u,\underline{u})} Q(W)(X,Y,Z,e_4)\\
&-\int_{\Sigma_0\cap V(u,\underline{u})} Q(W)(X,Y,Z,T)\\
&=\int_{V(u,\underline{u})} [\mathbf{Div}Q(W)_{\beta\gamma\delta}X^\beta Y^\gamma Z^\delta \;+\; \frac{1}{2}Q^{\alpha\beta\gamma\delta}(W)({}^{(X)}\pi_{\alpha\beta}Y_\gamma Z_\delta\\
&+{}^{(Y)}\pi_{\alpha\beta}Y_\gamma Z_\delta + {}^{(Z)}\pi_{\alpha\beta}X_\gamma Y_\delta)] \quad , \qquad (3.1.3)
\end{aligned}$$

where X, Y, Z are three vector fields on $\mathcal{M}$.
Therefore we can decompose the error term in two parts, one of it related to $\mathcal{Q}_1$ norms and the other one associated to $\mathcal{Q}_2$, that means

$$\mathcal{E}(u,\underline{u}) \;\equiv\; \mathcal{E}_1(u,\underline{u}) + \mathcal{E}_2(u,\underline{u})$$

where

$$\begin{aligned}
\mathcal{E}_1(u,\underline{u}) \;=\; &\int_{V(u,\underline{u})} \mathbf{Div}Q(\hat{\mathcal{L}}_T W)_{\beta\gamma\delta}(\bar{K}^\beta\bar{K}^\gamma\bar{K}^\delta)\\
&+\int_{V(u,\underline{u})} \mathbf{Div}Q(\hat{\mathcal{L}}_O W)_{\beta\gamma\delta}(\bar{K}^\beta\bar{K}^\gamma T^\delta)\\
&+\frac{3}{2}\int_{V(u,\underline{u})} Q(\hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta) \qquad (3.1.4)\\
&+\int_{V(u,\underline{u})} Q(\hat{\mathcal{L}}_O W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma T^\delta)\\
&+\frac{1}{2}\int_{V(u,\underline{u})} Q(\hat{\mathcal{L}}_O W)_{\alpha\beta\gamma\delta}({}^{(T)}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta)
\end{aligned}$$

$$
\begin{aligned}
\mathcal{E}_2(u,\underline{u}) \quad = \quad & \int_{V(u,\underline{u})} \mathbf{Div} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)_{\beta\gamma\delta}(\bar{K}^\beta\bar{K}^\gamma\bar{K}^\delta) \\
& + \int_{V(u,\underline{u})} \mathbf{Div} Q(\hat{\mathcal{L}}_O^2 W)_{\beta\gamma\delta}(\bar{K}^\beta\bar{K}^\gamma T^\delta) \\
& + \int_{V(u,\underline{u})} \mathbf{Div} Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)_{\beta\gamma\delta}(\bar{K}^\beta\bar{K}^\gamma\bar{K}^\delta) \\
& + \frac{3}{2}\int_{V(u,\underline{u})} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta) \\
& + \frac{3}{2}\int_{V(u,\underline{u})} Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta) \qquad (3.1.5) \\
& + \int_{V(u,\underline{u})} Q(\hat{\mathcal{L}}_O^2 W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma T^\delta) \\
& + \frac{1}{2}\int_{V(u,\underline{u})} Q(\hat{\mathcal{L}}_O^2 W)_{\alpha\beta\gamma\delta}({}^{(T)}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta).
\end{aligned}
$$

This decomposition divides the terms depending only on the first derivatives of W, which appear in $\mathcal{E}_1$ from the terms that involve second derivatives, included in $\mathcal{E}_2$. We shall prove there exists a constant c_0 such that

$$\mathcal{E}(u,\underline{u}) \leq \frac{c_0}{r_0}\mathcal{Q}_{\mathcal{K}}, \qquad (3.1.6)$$

which implies

$$\mathcal{Q}_{\mathcal{K}} \leq \frac{1}{1-c_0/r_0}\mathcal{Q}_{\Sigma_0\cap\mathcal{K}} \qquad (3.1.7)$$

that, for $r \geq r_0$ sufficiently great, will conclude the proof of the theorem. This last consideration means that we have to consider only an outer region with respect to the domain of dependance of $B(0,r)$, ball of radius r contained in Σ_0. In the precedent chapter we have already seen that all the most sensitive quantities of Kerr spacetime, and in particular the connection coefficients, when $r \to 0$, tend to those one of Minkowski spacetime.
Even if quantities to control are similar to the one of [**?**], we had to make every calculation again, because main quantities of Kerr spacetime have a different asymptotic behavior from those one of a linear perturbation of Minkowski spacetime, so we must verify if these new behaviors allow boundedness and smallness of norms. More in details:
1) Connection coefficients decay as in the book case, except $\eta, \underline{\eta}, \zeta, \underline{\zeta}$, that

decay faster.
2) Our deformation tensors, have not null components equal to zero, except${}^{((3))O}\pi = 0$
3) Also some of the asymptotic behavior of ${}^{(T)}\hat{\pi}, {}^{(S)}\hat{\pi}$ are different, in which case they are better.
One could think the results easily follow from these better decays, but we have a problem regarding the smallness of $\epsilon(u, \underline{u})$, problem which we can solve only in an outer region, where $r \gg r_0$, so we need very good decays to show (3.1.7).
Before to estimate into details all these terms, we provide you a short sketch about how decompose the integrand for best using the previous estimates on the connection coefficients.
We first consider the term with involves $\mathbf{Div}(Q)$. Let us denote

$$D(X, W) \equiv \mathbf{Div}Q(\hat{\mathcal{L}}_X W).$$

We will see we are able to express it as a sum of products of type

$$W \cdot W \cdot {}^{(X)}\pi$$

where W is some of null components of the conformal part of Riemann tensor, or their first derivatives, ${}^{(X)}\pi$ represents a generic component of the deformation tensor of X or its first derivative (then it depends on the connection coefficients up to their second derivatives). Then

$$\int_{V(u,\underline{u})} D(X, W)$$

will be some thing as

$$\int_{V(u,\underline{u})} W \cdot W \cdot {}^{(X)}\pi$$

and so, applying Schwartz inequality,

$$|\int_{V(u,\underline{u})} W \cdot W \cdot {}^{(X)}\pi| \leq |\int_{V(u,\underline{u})} |W|^2|^{\frac{1}{2}} \cdot |\int_{V(u,\underline{u})} |W^{(X)}\pi|^2|^{\frac{1}{2}}.$$

Now, the first term will be just controlled by $c\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}$, while as far as the second term is concerned, we first observe

$$|\int_{V(u,\underline{u})} |W \cdot {}^{(X)}\pi|^2|^{\frac{1}{2}} = |\int_{u_0}^{u} du'(\int_{C(u';[\underline{u}_0,\underline{u}])} |W \cdot {}^{(X)}\pi|^2)|^{\frac{1}{2}},$$

then, whether ${}^{(X)}\pi(u,\underline{u})$ admits the right decay, we can estimate this term too with $c\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}$.

Remark 3.1.2. *Really, we know the decays in r, then first we will have to transform them in decays along u, or $\underline{u}$, noting that, on the outgoing light cones (where we are performing the integral), u is a constant and $\underline{u} \simeq r$.*

Remark 3.1.3. *Sometimes we can't directly estimate*

$$|\int_{V(u,\underline{u})} |W^{(X)}\pi|^2|^{\frac{1}{2}}$$

but we will resort to more sophisticated estimates, which will involve L_4 norms of W and ${}^{(X)}\pi$.

We give now an explicit example about the reduction of the various error terms to above mentioned integrals. As far as the terms with $\mathbf{Div}Q(\hat{\mathcal{L}}_X W)$ are concerned, we have to compute explicitly them, when $X = T, O$.
We start introducing some new quantities, defined in the following

Definition 3.1.1. *Given a Weyl tensor W and a vector field X, we define the Weyl current as*

$$J(X,W)_{\beta\gamma\delta} \equiv D^\alpha(\hat{\mathcal{L}}_X W)_{\alpha\beta\gamma\delta}.$$

J can be decomposed with respect to a null frame, as well as W has been decomposed, in the following way

$$\begin{aligned}
\Lambda(J) &= \frac{1}{4}J_{434}, \qquad \underline{\Lambda}(J) = \frac{1}{4}J_{343}, \qquad \Xi(J)_a = \frac{1}{2}J_{44a} \\
\underline{\Xi}(J)_a &= \frac{1}{2}J_{33a}, \qquad I(J)_a = \frac{1}{2}J_{34a}, \qquad \underline{I}(J)_a = \frac{1}{2}J_{43a} \\
K(J) &= \frac{1}{4}\epsilon^{ab}J_{4ab}, \qquad \underline{K}(J) = \frac{1}{4}\epsilon^{ab}J_{3ab} \qquad (3.1.8) \\
\Theta(J)_{ab} &= J_{a4b} + J_{b4a} - (\delta^{cd}J_{c4d})\delta_{ab}, \quad \underline{\Theta}(J)_{ab} = J_{a3b} + J_{b3a} - (\delta^{cd}J_{c3d})\delta_{ab}.
\end{aligned}$$

Let us consider the case $X = T$, and so let us analyze the term

$$D(T,W) = \mathbf{Div}Q(\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K}).$$

The first thing to do is expressing $\text{div}Q(\hat{\mathcal{L}}_T W)$ as a sum of products between the null components of deformation tensors and null components of Weyl tensor field. Let us project $\text{div}Q(\hat{\mathcal{L}}_T W)$ along the null frame:

$$\begin{aligned} D(T,W)(\bar{K},\bar{K},\bar{K}) &= \frac{1}{8}\tau_+^6 D(T,W)_{444} + \frac{3}{8}\tau_+^4\tau_-^2 D(T,W)_{344} \\ & \frac{3}{8}\tau_+^2\tau_-^4 D(T,W)_{334} + \frac{1}{8}\tau_-^6 D(T,W)_{333}, \end{aligned} \tag{3.1.9}$$

where

$$\begin{aligned} D(T,W)_{444} &= 4\alpha(\hat{\mathcal{L}}_T W)\cdot\Theta(T,W) - 8\beta(\hat{\mathcal{L}}_X W)\cdot\Xi(X,W) \\ D(T,W)_{443} &= 8\rho(\hat{\mathcal{L}}_X W)\cdot\Lambda(T,W) + 8\sigma(\hat{\mathcal{L}}_T W)K(\hat{\mathcal{L}}_T W) \\ &\quad +8\beta(\hat{\mathcal{L}}_T W)\cdot I(T,W) \\ D(T,W)_{334} &= 8\rho(\hat{\mathcal{L}}_T W)\underline{\Lambda}(T,W) - 8\sigma(\hat{\mathcal{L}}_T W)K(X,W) \\ &\quad -8\underline{\beta}(\hat{\mathcal{L}}_T W)\cdot\underline{I}(T,W) \\ D(T,W)_{333} &= 4\underline{\alpha}(\hat{\mathcal{L}}_T W)\cdot\underline{\Theta}(T,W) + 8\underline{\beta}(\hat{\mathcal{L}}_T W)\cdot\underline{\Xi}(T,W) \end{aligned} \tag{3.1.10}$$

Really the same expressions hold for $D(X,W)$, for every vector field X. In particular, when $X = O$, we are interesting in the estimate of the divergence of $Q(\hat{\mathcal{L}}_O W)$ saturated with $(\bar{K},\bar{K},T)$. So we find the following identity holds

$$\begin{aligned} D(O,W)(\bar{K},\bar{K},T) &= \frac{1}{8}\tau_+^4(D(O,W)_{444} + D(O,W)_{344}) \\ &+ \frac{1}{4}\tau_+^2\tau_-^2(D(O,W)_{344} + D(O,W)_{334}) \\ &+ \frac{1}{8}\tau_-^4(D(O,W)_{334} + D(O,W)_{333}). \end{aligned} \tag{3.1.11}$$

Let us consider now the second product term of $D(T,W)_{444}$:

$$4\beta(\hat{\mathcal{L}}_T W)\cdot\Xi(T,W). \tag{3.1.12}$$

In order to obtain the explicit dependance of $D(T,W)_{444}$ on the null components of W and their first derivatives and null components of ${}^{(T)}\pi$ up to the first derivatives, we decompose the null current $J(T,W)$ into three parts

$$J(T,W) = J^1(T,W) + J^2(T,W) + J^3(T,W)$$

where

$$J^1(T,W)_{\beta\gamma\delta} = \frac{1}{2}{}^{(T)}\hat{\pi}^{\mu\nu} D_\nu W_{\mu\beta\gamma\delta} \tag{3.1.13}$$

is such that it depends on ${}^{(T)}\hat{\pi}$ and on the derivatives of null components of Weyl tensor field up to the first order.

$$\begin{aligned} J^2(T,W)_{\beta\gamma\delta} &= \frac{1}{2}{}^{(T)}p_\lambda W^\lambda{}_{\beta\gamma\delta} \\ J^3(T,W)_{\beta\gamma\delta} &= \frac{1}{2}({}^{(T)}q_{\alpha\beta\lambda}W^{\alpha\lambda}{}_{\gamma\delta} + {}^{(T)}q_{\alpha\gamma\lambda}W^\alpha{}_{\beta\gamma}{}^\lambda) \end{aligned} \tag{3.1.14}$$

where

$$\begin{aligned} {}^{(T)}p_\lambda &= \mathbf{D}^\alpha(T)\hat{\pi}_{\alpha\lambda} \\ {}^{(T)}q_{\alpha\beta\gamma} &= \mathbf{D}^{\beta(T)}\hat{\pi}_{\alpha\gamma} - \mathbf{D}^{\gamma(T)}\hat{\pi}_{\alpha\beta} - \frac{1}{3}({}^{(T)}p_\gamma g_{\alpha\beta} - {}^{(T)}p_\beta g_{\alpha\gamma}). \end{aligned}$$

So J^2 and J^3 both depend on the first derivatives of ${}^{(T)}\hat{\pi}$, but in a different way, and on the null components of Weyl field not derived. As a consequence of this decomposition, also every null component of J results to be decomposed in three parts, in particular

$$\Xi(J(T,W)) = \Xi(J^1(T,W)) + \Xi(J^2(T,W)) + \Xi(J^3(T,W))$$

where $\Xi(J^1)$ is a sum of quadratic expressions between components of ${}^{(T)}\hat{\pi}$ along the null frame and null components of Weyl tensor or its first derivatives (plus lower order terms), explicitly (recalling ${}^{(T)}i = {}^{(T)}j = 0$)

$$\begin{aligned} \Xi(J^1(T,W)) &= Qr[{}^{(T)}\underline{m};\alpha_4] + Qr[{}^{(T)}m;\alpha_3] + Qr[{}^{(T)}m;\nabla\!\!\!/\beta] \\ &+ Qr[{}^{(T)}n;\beta_3] + \mathrm{tr}\chi\{Qr[{}^{(T)}\underline{m};\alpha]+\} \\ &+ \mathrm{tr}\underline{\chi}\{Qr[{}^{(T)}m;\alpha] + Qr[{}^{(T)}n;\beta]\} + l.o.t. \end{aligned} \tag{3.1.15}$$

For the explicit expressions of $\alpha_3, (\rho,\sigma)_{\{3,4\}}, \beta_{\{3,4\}}$ see the (1.1.5), while as far as the quantity which we denote α_4 is concerned, (recalling that it does not exist an evolution equation for α along null outgoing hypersurface, as well as there isn't the evolution equation of $\underline{\alpha}$ along incoming cones) it is obtained expressing it in terms of α_3 and $\mathbf{D}\!\!\!/_T\alpha$:

$$\alpha_4 = 2\mathbf{D}\!\!\!/_T\alpha + \alpha_3 + (\frac{5}{2}\mathrm{tr}\chi + \frac{1}{2}\mathrm{tr}\underline{\chi})\alpha \tag{3.1.16}$$

and analogously

$$\underline{\alpha}_3 = 2\mathbf{D}\!\!\!/_T\underline{\alpha} - \underline{\alpha}_4 + (\frac{5}{2}\mathrm{tr}\underline{\chi} + \frac{1}{2}\mathrm{tr}\underline{\chi})\underline{\alpha}.$$

Then

$$\begin{aligned}
\Xi(J^2(T,W)) &= Qr[{}^{(T)}\not{p}; \underline{\alpha}] + Qr[{}^{(T)}p_3; \underline{\beta}] \quad (3.1.17)\\
\Xi(J^3(T,W)) &= Qr[\alpha; (I, \underline{I})({}^{(T)}q)] + Qr[\beta; (K, \Lambda, \Theta)({}^{(T)}q)]\\
&+ Qr[(\rho,\sigma); \Xi({}^{(T)}q)]. \quad (3.1.18)
\end{aligned}$$

In order to not overload too much the error estimate, let us report the explicit expressions of the various functions of ${}^{(T)}q$ which appear in (3.1.18) and the expressions of all Weyl current components at the appendix of the chapter.

3.2 Error estimate into details

From now on, we shall follow the proceeding adopted in chapter 6 of [11]. First, let us write the explicit expressions for ${}^{(X)}p_\mu$; they are the following

$$\begin{aligned}
{}^{(X)}p_3 &= \not{d}\mathrm{iv}\,{}^{(X)}\underline{m} - \frac{1}{2}(\not{D}_4{}^{(X)}\underline{n} + \not{D}_3{}^{(X)}j) + (2\underline{\eta} + \eta - \zeta)\cdot{}^{(X)}\underline{m} \quad (3.2.19)\\
&- \hat{\chi}\cdot{}^{(X)}i - \frac{1}{2}\mathrm{tr}\chi(\mathrm{tr}{}^{(X)}i + {}^{(X)}j) - \frac{1}{2}\mathrm{tr}\underline{\chi}{}^{(X)}n - (\mathbf{D}_3\log\Omega){}^{(X)}n,
\end{aligned}$$

$$\begin{aligned}
{}^{(X)}p_4 &= \not{d}\mathrm{iv}\,{}^{(X)}m - \frac{1}{2}(\not{D}_3{}^{(X)}n + \not{D}_4{}^{(X)}j) + (\underline{\eta} + 2\eta + \zeta)\cdot{}^{(X)}m \quad (3.2.20)\\
&- \underline{\hat{\chi}}\cdot{}^{(X)}i - \frac{1}{2}\mathrm{tr}\underline{\chi}(\mathrm{tr}{}^{(X)}i + {}^{(X)}j) - \frac{1}{2}\mathrm{tr}\chi{}^{(X)}\underline{n} - (\mathbf{D}_4\log\Omega){}^{(X)}\underline{n},
\end{aligned}$$

$$\begin{aligned}
{}^{(X)}\not{p} &= \nabla_c{}^{(X)}i - \frac{1}{2}(\not{D}_4{}^{(X)}\underline{m} + \not{D}_3{}^{(X)}m) + \frac{1}{2}(\eta + \underline{\eta}){}^{(X)}j\\
&+ (\eta + \underline{\eta})\cdot{}^{(X)}i - \frac{1}{2}\hat{\chi}\cdot{}^{(X)}m - \frac{1}{2}\underline{\hat{\chi}}\cdot{}^{(X)}m - \frac{3}{4}\mathrm{tr}\chi{}^{(X)}\underline{m} - \frac{3}{4}\mathrm{tr}\underline{\chi}{}^{(X)}m\\
&- \frac{1}{2}(\mathbf{D}_4\log\Omega){}^{(X)}\underline{m} - \frac{1}{2}(\mathbf{D}_3\log\Omega){}^{(X)}m. \quad (3.2.21)
\end{aligned}$$

In the rest of the section we will need to know their decays when $X = T$, or $X = 0$. In particular, as we know the asymptotic behavior of connection coefficients of Kerr spacetime and that one of $T, {}^{(i)}O$ and S deformation tensors components, we are able to show the following asymptotic behaviors hold true:

Proposition 3.2.1. *Based on proposition (2.5.1) and on corollary (2.5.1), the following estimates for any $S \subset \mathcal{K}$ with $p \in [2,4]$ hold:*

$$||r^{3-\frac{2}{p}}({}^{(O)}p_3, {}^{(O)}p_4, {}^{(O)}\not{p}_a)||_{p,S} \leq c. \tag{3.2.22}$$

As far as ${}^{(T)}p_\mu$ are concerned, the following asymptotic inequalities hold

Proposition 3.2.2. *Based on proposition(2.5.3), we find the following estimates relative to ${}^{(T)}p_\lambda$ and relative to its derivatives for any $S \subset \mathcal{K}$ with $p \in [2,4]$:*

$$\begin{aligned} ||r^{3-\frac{2}{p}}\,{}^{(T)}p_3||_{p,S} &\leq c \\ ||r^{3-\frac{2}{p}}\,{}^{(T)}p_4||_{p,S} &\leq c \\ ||r^{3-\frac{2}{p}}\,{}^{(T)}\not{p}_a||_{p,S} &\leq c, \end{aligned}$$

$$\begin{aligned} ||r^{4-\frac{2}{p}}\not{\nabla}{}^{(T)}p_3||_{p,S} &\leq c \\ ||r^{4-\frac{2}{p}}\not{\nabla}{}^{(T)}p_4||_{p,S} &\leq c \\ ||r^{4-\frac{2}{p}}\,{}^{(T)}\not{p}_a||_{p,S} &\leq c. \end{aligned}$$

Proposition 3.2.3. *Making basic calculations, it follows the following estimates hold:*

$$\begin{aligned} ||r^{\frac{3}{2}}\,{}^{(T)}p_3||_{L_2(C\cap\mathcal{K})} &\leq c \\ ||r^{\frac{5}{2}}\not{\nabla}{}^{(T)}p_3||_{L_2(C\cap\mathcal{K})} &\leq c \\ ||r^{\frac{3}{2}}\mathcal{L}_S{}^{(T)}\not{p}_a||_{L_2(C\cap\mathcal{K})} &\leq c. \end{aligned}$$

Given a vector field X, we report the explicit expressions of any null components of the currents of W relative to X in the appendix at the chapter. In order to estimate the error term, it will be necessary to estimate their asymptotic behavior when $X = O, T$.
Now let us state the following propositions which prescribes their asymptotic decays

Proposition 3.2.4. *Given a Weyl field W propagating in the Kerr spacetime, the null components of the part of the current $J^3(O,W)$ satisfy the* *following estimates for any $S \subset \mathcal{K}$, with $p \in [2,4]$:*

$$\begin{aligned}
|r^{3-\frac{2}{p}}\Xi(O,W)|_{p,S} &\leq c \\
|r^{3-\frac{2}{p}}\Theta(O,W)|_{p,S} &\leq c \\
|r^{3-\frac{2}{p}}\Lambda(O,W)|_{p,S} &\leq c \\
|r^{3-\frac{2}{p}}K(O,W)|_{p,S} &\leq c \\
|r^{3-\frac{2}{p}}I(O,W)|_{p,S} &\leq c,
\end{aligned}$$

and

$$\begin{aligned}
|r^{3-\frac{2}{p}}\underline{\Xi}(O,W)|_{p,S} &\leq c \\
|r^{3-\frac{2}{p}}\underline{\Theta}(O,W)|_{p,S} &\leq c \\
|r^{3-\frac{2}{p}}\underline{\Lambda}(O,W)|_{p,S} &\leq c \\
|r^{3-\frac{2}{p}}\underline{K}(O,W)|_{p,S} &\leq c \\
|r^{3-\frac{2}{p}}\underline{I}(O,W)|_{p,S} &\leq c.
\end{aligned}$$

Proposition 3.2.5. *Let $T = \frac{2}{\Omega}\frac{\partial}{\partial_t}$ and W a Weyl tensor in a Kerr spacetime. Then the null components of the part $J^3(T,W)$ satisfy the following estimates for any $S \subset \mathcal{K}$, with $p \in [2,4]$:*

$$\begin{aligned}
|r^{4-\frac{2}{p}}\Xi(T,W)|_{p,S} &\leq c \\
|r^{4-\frac{2}{p}}\Theta(T,W)|_{p,S} &\leq c \\
|r^{3-\frac{2}{p}}\Lambda(T,W)|_{p,S} &\leq c \\
|r^{4-\frac{2}{p}}K(T,W)|_{p,S} &\leq c \\
|r^{3-\frac{2}{p}}I(T,W)|_{p,S} &\leq c,
\end{aligned}$$

and

$$\begin{aligned}
|r^{4-\frac{2}{p}}\underline{\Xi}(T,W)|_{p,S} &\leq c \\
|r^{4-\frac{2}{p}}\underline{\Theta}(T,W)|_{p,S} &\leq c \\
|r^{3-\frac{2}{p}}\underline{\Lambda}(T,W)|_{p,S} &\leq c \\
|r^{4-\frac{2}{p}}\underline{K}(T,W)|_{p,S} &\leq c \\
|r^{3-\frac{2}{p}}\underline{I}(T,W)|_{p,S} &\leq c.
\end{aligned}$$

At last we need the results of proposition 1.3.3 to estimate the modified Lie derivatives with respect to the vector field X of the null components of W in terms of the null components of the tensor field $\hat{\mathcal{L}}_X W$. We have yet to show that when $X = T, O$ the terms in the square brackets have a better asymptotic behavior, so we can disregard them. In particular, let us consider $X = T$. Then recalling the definition of the quantities P, Q...., given in (1.3.18), we have computed the asymptotic behaviors of the Lie coefficients of T, O and they are expressed in the following propositions.

Proposition 3.2.6. *The Lie coefficients of the vector field T have the following asymptotic behavior:*

$$
\begin{aligned}
{}^{(T)}P_a &= O(\frac{c}{r^3}) \\
{}^{(T)}\underline{P}_a &= O(\frac{c}{r^3}) \\
{}^{(T)}Q_a &= O(\frac{c}{r^3}) \\
{}^{(T)}\underline{Q}_a &= O(\frac{c}{r^3}) \\
{}^{(T)}M &= O(\frac{c}{r^2}) \\
{}^{(T)}\underline{M} &= O(\frac{c}{r^2}) \\
{}^{(T)}N &= O(\frac{c}{r^2}) \\
{}^{(T)}\underline{N} &= O(\frac{c}{r^2}).
\end{aligned}
$$

Then, comparing them with the expressions of the proposition 1.3.3, it follows at the highest order null components of $\hat{\mathcal{L}}_T W$ behave as the projection onto $S(u, \underline{u})$ of the modified lie derivative with respect to T of the corresponding null components of W.

Proposition 3.2.7. *The Lie coefficients of the vector field ${}^{(}2)O$ have the* *following asymptotic behavior:*

$$
\begin{aligned}
{}^{(2)(O)}P_\theta &= O(\sin\phi\cos\theta)\\
{}^{(2)(O)}P_\phi &= O(\cos\phi)\\
{}^{(2)(O)}\underline{P}_\theta &= O(-\sin\phi\cos\theta)\\
{}^{(2)(O)}\underline{P}_\phi &= O(-\cos\phi)\\
{}^{(2)(O)}Q_a &= O(-\sin\phi\cos\theta)\\
{}^{(2)(O)}\underline{Q}_a &= O(-\sin\phi\cos\theta)\\
{}^{(2)(O)}M &= O(\frac{c}{r^2})\\
{}^{(2)(O)}\underline{M} &= O(\frac{c}{r^2})\\
{}^{(2)(O)}N &= O(\frac{c}{r^2})\\
{}^{(2)(O)}\underline{N} &= O(\frac{c}{r^2}).
\end{aligned}
$$

Moreover, the following relations hold:

$$
\begin{aligned}
{}^{(2)(O)}P_\theta + {}^{(2)(O)}Q_\theta &= O(\frac{1}{r})\\
{}^{(2)(O)}P_\phi + {}^{(2)(O)}Q_\phi &= O(\frac{\partial P}{\partial_\theta}\frac{1}{r}\cos\phi).
\end{aligned}
$$

Then at the highest order null components of $\hat{\mathcal{L}}_{(2)O}W$ behave as the projection onto $S(u,\underline{u})$ of the modified lie derivative with respect to ${}^{(}2)O$ of the corresponding null components of W.

Remark 3.2.1. *The decays for the Lie coefficients of ${}^{(3)(O)}$ are very similar you have just to change $\sin\phi$ $\cos\phi$ and $\sin\theta$ with $-\cos\theta$.*

3.3 The error term ϵ_1

All the estimates present in this chapter are relative to the double null foliation and to the initial hypersurface Σ_0.

3.3.1 Estimate of $\int_{V(u,\bar{u})} \mathbf{Div} Q(\hat{\mathcal{L}}_T W)_{\beta\gamma\delta}(\bar{K}^\beta, \bar{K}^\gamma, \bar{K}^\delta)$

As we have seen before (see (3.1.9)),we have to estimate the following integrals:

$$\int_{V(u,\underline{u})} \tau_+^6 D(T,W)_{444}, \quad \int_{V(u,\underline{u})} \tau_+^4 \tau_-^2 D(T,W)_{344}$$
$$\int_{V(u,\underline{u})} \tau_+^2 \tau_-^4 D(T,W)_{334}, \quad \int_{V(u,\underline{u})} \tau_-^6 D(T,W)_{333}$$

. Let us control only the first integral, which has the highest weight factor in τ_+. From equation (3.1.10), we have to control the following integral:

$$\int_{V(u,\underline{u})} \tau_+^6 \beta(\hat{\mathcal{L}}_T W) \cdot \Xi(T,W)$$
$$\int_{V(u,\underline{u})} \tau_+^6 \alpha(\hat{\mathcal{L}}_T W) \cdot \Theta(T,W).$$

In fact, it holds the following

Proposition 3.3.1. *In Kerr spacetime, the following inequalities hold*

$$\begin{aligned} \Big|\int_{V(u,\underline{u})} \tau_+^6 \beta(\hat{\mathcal{L}}_T W) \cdot \Xi(T,W)\Big| &\leq \frac{c}{r_0} \mathcal{Q}_{\mathcal{K}} \\ \Big|\int_{V(u,\underline{u})} \tau_+^6 \alpha(\hat{\mathcal{L}}_T W) \cdot \Theta(T,W)\Big| &\leq \frac{c}{r_0} \mathcal{Q}_{\mathcal{K}}. \end{aligned}$$

Proof. We discuss into details the first integral, the estimate of the second one is similar. Using the coarea formula

$$\int_{V(u,\underline{u})} F = \int_{u_0}^{u} du' \int_{C(u') \cap V(u,\underline{u})} F,$$

with $u_0(\underline{u}) = u|_{\underline{C}(\underline{u}) \cap \Sigma_0}$, and the Schwartz inequality, it is majored by

$$\begin{aligned} \Big|\int_{V(u,\underline{u})} \tau_+^6 \beta(\hat{\mathcal{L}}_T W) \cdot \Xi(T,W)\Big| &\leq c \int_{u_0}^{u} du' \Big(\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^6 |\beta(\hat{\mathcal{L}}_T W)|^2\Big)^{\frac{1}{2}} \cdot \\ \Big(\underline{u}'^6 |\Xi(T,W)^2|\Big)^{\frac{1}{2}} &\leq c \mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}} \Big[\sum_{i=1}^{3} \int_{u_0}^{u} du' \cdot \qquad (3.3.23) \\ \Big(\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^6 \quad | \quad \Xi^{(i)}(T,W)|^2\Big)^{\frac{1}{2}}\Big]. \qquad (3.3.24) \end{aligned}$$

To complete the proof, we have to prove the following inequalities hold

$$\begin{aligned}
\big(\underline{u}'^6|\Xi^{(1)}(T,W)^2|\big)^{\frac{1}{2}} &\leq \frac{1}{|u'|^2}\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\\
\big(\underline{u}'^6|\Xi^{(2)}(T,W)^2|\big)^{\frac{1}{2}} &\leq \frac{1}{|u'|^2}\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\\
\big(\underline{u}'^6|\Xi^{(3)}(T,W)^2|\big)^{\frac{1}{2}} &\leq \frac{1}{|u'|^2}\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}. \qquad (3.3.25)
\end{aligned}$$

As far as the first integral is concerned, we have to estimate various terms (see the expression for $\Xi^{(1)}(T,W)$), which are all estimated in the same way. Because of ${}^{(T)}n$ is the ${}^{(T)}\hat{\pi}$ component that decay slowest, let us control only the terms which involve it, i.e.

$$\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^6|{}^{(T)}n|^2|\beta_3(W)|^2$$

$$\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^6|\mathrm{tr}\chi|^2|{}^{(T)}n|^2|\beta_3(W)|^2.$$

As far as the first integral is concerned, it can be estimated in the following way:

$$\begin{aligned}
\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^6|{}^{(T)}n|^2|\beta_3(W)|^2 &\leq c\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^6\frac{1}{r^4}\frac{1}{r^2}|\beta(W)|^2\\
&\leq c\frac{1}{u'^4}\int_{C(u';[\underline{u}_0,\underline{u}])}\frac{\underline{u}'^6}{r^2}|\beta(\mathcal{L}_O W)|^2
\end{aligned}$$

the first inequality following directly from the asymptotic behavior of ${}^{(T)}n$ and the second inequality being true, by holding (1.3.16). Then:

$$\big(\underline{u}'^6|\Xi^{(1)}(T,W)|^2\big)^{\frac{1}{2}} \leq \frac{1}{|u'|^2}\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}} \qquad (3.3.26)$$

that means

$$\begin{aligned}
\Big|\int_{V(u,\underline{u})}\tau_+^6\beta(\hat{\mathcal{L}}_T W)\cdot\Xi(T,W)\Big| &\leq \frac{c}{u'}\mathcal{Q}_{\mathcal{K}}\\
&\leq \frac{c}{r_0}\mathcal{Q}_{\mathcal{K}},
\end{aligned}$$

for r_0 sufficiently great.
To control the second integral of (3.3.25), recalling that

$$\Xi^{(2)}(T,W) = Qr[{}^{(T)}\not{p};\alpha] + Qr[{}^{(T)}p_4;\beta]$$

we have to estimate the integrals

$$\begin{aligned}&\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^6|{}^{(T)}\not{p}|^2|\alpha(W)|^2\\ &\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^6|{}^{(T)}p_4|^2|\beta(W)|^2. \end{aligned} \tag{3.3.27}$$

Let us study the second integral as an illustrative case. It is controlled in the following way

$$\begin{aligned}\left(\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^6|{}^{(T)}p_4|^2|\beta(W)|^2\right)^{\frac{1}{2}} &\leq c\left(\int_{C(u';[\underline{u}_0,\underline{u}])} \frac{\underline{u}^6}{r^6}|\beta(W)|^2\right)^{\frac{1}{2}}\\ &\leq \frac{c}{u'^2}\left(\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^4|\beta(\hat{\mathcal{L}}_O W)|^2\right)^{\frac{1}{2}}\\ &\leq \frac{c}{u'^2}\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}.\end{aligned}$$

To control the third integral of (3.3.25), let us recall the explicit expression for $\Xi^{(3)}(T,W)$:

$$\Xi^{(3)}(T,W) = Qr[\alpha;(I,\underline{I}),{}^{(T)}q] + Qr[\beta;(K,\Lambda,\Theta)^{(T)}q] + Qr[(\rho,\sigma);\Xi^{(T)}q]$$

Then we have to control the following integral terms:

$$\begin{aligned}&\int_{C(u';[\underline{u}_0,\underline{u}])} \tau_+^6|(I({}^{(T)}q),\underline{I}({}^{(T)}q))|^2|\alpha(W)|^2\\ &\int_{C(u';[\underline{u}_0,\underline{u}])} \tau_+^6|(K({}^{(T)}q),\Lambda({}^{(T)}q),\Theta({}^{(T)}q))|^2|\beta(W)|^2\\ &\int_{C(u';[\underline{u}_0,\underline{u}])} \tau_+^6|\Xi({}^{(T)}q)|^2|(\rho(W),\sigma(W))|^2.\end{aligned} \tag{3.3.28}$$

The terms having the worst asymptotic behavior are $I({}^{(T)}q), \underline{I}({}^{(T)}q), \Lambda({}^{(T)}q)$, that decay at null infinity like $\frac{c}{r^3}$, so let us estimate the first integral, in particular let us show we can control

$$\int_{C(u';[\underline{u}_0,\underline{u}])} \tau_+^6|(I({}^{(T)}q),\underline{I}({}^{(T)}q))|^2|\alpha(W)|^2.$$

It is majored by:

$$\frac{c}{u'^4}\int_{C(u';[\underline{u}_0,\underline{u}])}\frac{\tau_+^6}{r^6}\tau_+^4|\alpha(\hat{\mathcal{L}}_O W)|^2$$

and so

$$\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^6|(I({}^{(T)}q),\underline{I}({}^{(T)}q))|^2|\alpha(W)|^2\right)^{\frac{1}{2}}\leq\frac{c}{u'^2}\mathcal{Q}^{\frac{1}{2}}_{calK}.$$

□

3.3.2 Estimate of $\int_{V(u,\bar{u})}Q(\hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta)$

Proposition 3.3.2. *In Kerr spacetime the following inequalities holds:*

$$\int_{V(u,\underline{u})}|Q(\hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta)|\leq\frac{c}{r^0}\mathcal{Q}_{\mathcal{K}} \tag{3.3.29}$$

Proof. For the complete explicit expression of the integrand, see [11],(6.2.27)-(6.2.29).
All factors are cubic terms, quadratic in the null components of $\hat{\mathcal{L}}_T W$ and linear in ${}^{(\bar{K})}\pi$. Let us discuss the integral of terms that behave worst. They are those one involving ${}^{(\bar{K})}\underline{n}$ and $\rho(\hat{\mathcal{L}}_T W)$ and with the highest weight factor τ_+, exactly they are the integral relative to $\tau_+^4\,{}^{(\bar{K})}\underline{n}|\alpha(\hat{\mathcal{L}}_T W)|^2$ and the one relative to $\tau_+^4\,{}^{(\bar{K})}n|\rho(\hat{\mathcal{L}}_T W)|^2$. As far as the first integral is concerned, we obtain the following inequality:

$$\begin{aligned}&\int_{V(u,\underline{u})}\tau_+^4|\alpha(\hat{\mathcal{L}}_T W)|^2|\underline{n}|\leq c\int_{u_0}^{u}du'\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^6|\alpha(\hat{\mathcal{L}}_T W)|^2\frac{1}{r^2}|{}^{(\bar{K})}\underline{n}|\\ \leq\ & c\left(\sup_{\mathcal{K}}{}^{(\bar{K})}\underline{n}\right)\left(\sup_{\mathcal{K}}\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^6|\alpha(\hat{\mathcal{L}}_T W)|^2\right)\int_{u_0}^{u}du'\frac{1}{u'^2}\leq\frac{c}{r_0}\mathcal{Q}_{\mathcal{K}}.\end{aligned}$$

For the estimate of the other integral, we proceed in the same way,

$$\begin{aligned}&\int_{V(u,\underline{u})}\tau_+^4|\rho(\hat{\mathcal{L}}_T W)|^2|{}^{(\bar{K})}n|\leq c\int_{u_0}^{u}du'\frac{1}{u'^2}\int_{C(u';[\underline{u}_0,\underline{u}])}|\frac{r^2}{\tau_-^2}{}^{(\bar{K})}n|\tau_+^2\tau_-^4|\rho(\hat{\mathcal{L}}_T W)|^2\\ \leq\ & c\left(\sup_{\mathcal{K}}|\frac{r^2}{\tau_-^2}{}^{(\bar{K})}n|\right)\left(\sup_{\mathcal{K}}\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^2\tau_-^4|\rho(\hat{\mathcal{L}}_T W)|^2\right)\int_{u_0}^{u}du'\frac{1}{u'}\leq\frac{c}{r_0}\mathcal{Q}_{\mathcal{K}}.\end{aligned}$$

□

3.3.3 Estimate of $\int_{V(u,\bar{u})} \mathbf{Div} Q(\hat{\mathcal{L}}_O W)_{\beta\gamma\delta}(\bar{K}^\beta \bar{K}^\gamma \bar{K}^\delta)$

Recalling (3.1.11), it follows we have to control the following integrals:

$$\begin{aligned}
&\int_{V(u,\underline{u})} \tau_+^4 D(O,W)_{444}, \quad \int_{V(u,\underline{u})} \tau_+^4 D(O,W)_{344} \\
&\int_{V(u,\underline{u})} \tau_+^2 \tau_-^2 D(O,W)_{344}, \quad \int_{V(u,\underline{u})} \tau_+^2 \tau_-^2 D(O,W)_{334} \\
&\int_{V(u,\underline{u})} \tau_-^4 D(O,W)_{334}, \quad \int_{V(u,\underline{u})} \tau_-^4 D(O,W)_{333}.
\end{aligned} \tag{3.3.30}$$

The most sensitive terms are the one which contain the factor τ_+^4. Let us estimate into details the $\int_{V(u,\underline{u})} \tau_+^4 D(O,W)_{444}$, with the following

Proposition 3.3.3. *The following inequalities hold true:*

$$\begin{aligned}
\left| \int_{V(u,\underline{u})} \tau_+^4 D(O,W)_{444} \right| &\le \frac{c}{r_0} \mathcal{Q}_{\mathcal{K}} \\
\left| \int_{V(u,\underline{u})} \tau_+^4 D(O,W)_{444} \right| &\le \frac{c}{r_0} \mathcal{Q}_{\mathcal{K}}.
\end{aligned}$$

Proof:

$$\begin{aligned}
\int_{V(u,\underline{u})} \tau_+^4 D(O,W)_{444} &= 4 \int_{V(u,\underline{u})} \tau_+^4 \alpha(\hat{\mathcal{L}}_O W) \cdot \Theta(O,W) \\
&- 8 \int_{V(u,\underline{u})} \tau_+^4 \beta(\hat{\mathcal{L}}_O W) \cdot \Xi(O,W).
\end{aligned} \tag{3.3.31}$$

As far as the first integral is concerned, using Schwartz inequality, we write

$$\begin{aligned}
&| \int_{V(u,\underline{u})} \tau_+^4 \alpha(\hat{\mathcal{L}}_O W) \cdot \Theta(O,W)| \\
\le \ & \left(\sup \mathcal{K} \int_{C(u';[\underline{u}_0,\underline{u}])} \tau_+^4 |\alpha(\hat{\mathcal{L}}_O W)|^2 \right)^{\frac{1}{2}} \int_{u_0}^{u} du' \left(\int_{C(u';[\underline{u}_0,\underline{u}])} \tau_+^4 |\Theta(O,W)|^2 \right)^{\frac{1}{2}} \\
\le \ & c \mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}} \int_{u_0}^{u} du' \sum_{i=1}^{3} \left(\int_{C(u';[\underline{u}_0,\underline{u}])} \tau_+^4 |\Theta^{(i)}(O,W)|^2 \right)^{\frac{1}{2}}
\end{aligned} \tag{3.3.32}$$

and analogously for the second integral, we obtain the following estimate

$$|\int_{V(u,\underline{u})}\tau_+^4\beta(\hat{\mathcal{L}}_O W)\cdot\Xi(O,W)| \leq c\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\int_{u_0}^{u}du'\sum_{i=1}^{3}\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^4|\Xi^{(i)}(O,W)|^2\right)^{\frac{1}{2}}. \tag{3.3.33}$$

Let us discuss only the first term. The result is obtained by proving the next lemma

Lemma 3.3.1. *In Kerr spacetime the following inequalities hold*

$$\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^4|\Theta^{(1)}(T,W)|^2\right)^{\frac{1}{2}} \leq \frac{c}{r_0}\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\frac{1}{|u'|^2} \tag{3.3.34}$$

$$\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^4|\Theta^{(2)}(T,W)|^2\right)^{\frac{1}{2}} \leq \frac{c}{r_0}\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\frac{1}{|u'|^2}$$

$$\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^4|\Theta^{(3)}(T,W)|^2\right)^{\frac{1}{2}} \leq \frac{c}{r_0}\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\frac{1}{|u'|^2}. \tag{3.3.35}$$

Proof. We start by estimate the first integral, that is related to the J_1 part of the current. From the explicit expression of $\Theta^{(1)}(O,W)$ and from proposition 2.5.1, it follows

$$\begin{aligned}
|\Theta^{(1)}(O,W)|^2 &\leq c\left(\sup_{\mathcal{K}}|r^2({}^{(O)}i,{}^{(O)}j,{}^{(O)}m,{}^{(O)}\underline{m},{}^{(O)}n,{}^{(O)}\underline{n})|\right)^2\frac{1}{r^4}\\
&\cdot\ \big[(|\nabla\!\!\!/\alpha|^2+|\alpha_3|^2+|\alpha_4|^2+|\nabla\!\!\!/\beta|^2+|\beta_4|^2+|\beta_3|^2\\
&+\ |(\rho_4,\sigma_4)|^2+|\nabla\!\!\!/(\rho,\sigma)|^2+|(\rho_3,\sigma_3)|^2)+\frac{1}{r^2}(|\alpha|^2+|\beta|^2\\
&+\ |(\rho,\sigma)|^2)\big]+l.o.t.
\end{aligned}$$

Therefore

$$
\begin{aligned}
&\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^4|\Theta^{(1)}(O,W)|^2\right)^{\frac{1}{2}} \leq c\left(\sup_{\mathcal{K}}|r^2({}^{(O)}i,{}^{(O)}j,{}^{(O)}m,{}^{(O)}\underline{m},{}^{(O)}n,{}^{(O)}\underline{n})|\right)\\
&\quad\frac{1}{u'}\left[\int_{C(u';[\underline{u}_0,\underline{u}])}u'^2(|\nabla\!\!\!\!/\,\alpha|^2+|\alpha_3|^2+|\alpha_4|^2+|\nabla\!\!\!\!/\,\beta|^2+|\beta_4|^2\right.\\
+&\quad\left.|\beta_3|^2+|(\rho_4,\sigma_4)|^2+|\nabla\!\!\!\!/\,(\rho,\sigma)|^2+|(\rho_3,\sigma_3)|^2)+\frac{u'^2}{r^2}(|\alpha|^2+|\beta|^2+|(\rho,\sigma)|^2)\right]^{\frac{1}{2}}\\
\leq&\quad\frac{1}{u'^3}\left[\left(\sup_{\mathcal{K}}\int_{C(u';[\underline{u}_0,\underline{u}])}Q(\hat{\mathcal{L}}_OW)(\bar{K},\bar{K},T,e_4)\right)^{\frac{1}{2}}\right.\\
+&\quad\left.\left(\sup_{\mathcal{K}}\int_{C(\underline{u}';[u_0,u])}Q(\hat{\mathcal{L}}_OW)(\bar{K},\bar{K},T,e_3)\right)^{\frac{1}{2}}+\sup\mathcal{K}\cap\pm,|r^3(\bar{\rho},\bar{\sigma})|\right]\\
\leq&\quad\frac{c}{r_0}\frac{1}{|u'|^2}\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}.
\end{aligned}
\tag{3.3.36}
$$

As far as the term related to the J_2 current is concerned, as $\Theta^{(2)}(T,W)$ has the following form:

$$
\Theta^{(2)}(T,W)=Qr[{}^{(O)}p_3;\alpha]+Qr[{}^{(O)}p\!\!\!/;\beta]+Qr[{}^{(O)}p_4;(\rho,\sigma)],
$$

then we write it in the following way

$$
\begin{aligned}
&\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^4|\Theta^{(2)}(T,W)|^2\right)^{\frac{1}{2}}\\
\leq&\quad c\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^4(|{}^{(O)}p_3|^2\alpha(W)|^2+|{}^{(O)}p\!\!\!/|^2|\beta(W)|^2+|{}^{(O)}p_4|^2|(\rho,\sigma)(W)|^2|)\right)^{\frac{1}{2}}.
\end{aligned}
$$

All these terms have the same structure and they are estimated in an analogous way. Let us show the boundedness of the piece with $\rho(W)$. Recalling the proposition **??**, and applying the Poincaré inequality to $\rho-\bar{\rho}$, the following inequalities hold:

$$\begin{aligned}\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\right. & \left.\tau_+^4|^{(O)}p_4|^2|(\rho,\sigma)(W)|^2\right)^{\frac{1}{2}} \\ &\leq c\left[\int_{\underline{u}_0}^{\underline{u}} d\underline{u}'\frac{\tau_+^4}{r^3}\int_{S(u',\underline{u}')} r^2|\nabla\!\!\!/\rho|^2\right]+\sup_{\mathcal{K}} r^2|\bar{\rho}| \qquad (3.3.37)\\ &\leq c\frac{1}{|u'|^3}\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+\tau_-^3|\rho(\hat{\mathcal{L}}_O W)|^2\right)^{\frac{1}{2}}+\sup_{\mathcal{K}} r^2|\bar{\rho}|\\ &\leq c\frac{1}{|u'|^3}\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^2\tau_-^2|\rho(\hat{\mathcal{L}}_O W)|^2\right)^{\frac{1}{2}}\\ &\leq \frac{c}{r_0}\frac{1}{|u'|^3}\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}, \qquad (3.3.38)\end{aligned}$$

where we have also used the inequality (1.3.15) and the proposition 1.3.3 applied to $\nabla\!\!\!/\rho$.
The part depending on $\Theta^{(3)}$ has the following decomposition:

$$\begin{aligned}\Theta^{(3)}(O,W) &= Qr[\alpha;\underline{K}(^{(O)}q)]+Qr[\alpha;\underline{\Lambda}(^{(O)}q)]+Qr[\beta;(I,\underline{I})(^{(O)}q)]\\ &\quad Qr[(\rho,\sigma);\Theta(^{(O)}q)],\end{aligned}$$

from which

$$\begin{aligned}&\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^4|\Theta^{(3)}(O,W)|^2\right)^{\frac{1}{2}}\\ \leq\ & c\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^4|\underline{K}(^{(O)}q)|^2|\alpha(W)|^2+|\underline{\Lambda}(^{(O)}q)|^2|\alpha(W)|^2\right.\\ +\ & \left.|(I,\underline{I})(^{(O)}q)|^2|\beta(W)|^2+|\Theta(^{(O)}q)|^2|(\rho,\sigma)(W)|^2\right)^{\frac{1}{2}}.\end{aligned}$$

To estimate them, we recall the proposition 3.2.4 which tells us every J^3 null component, which we indicate now with $F(^{(O)}q)$ satisfies the following L_p estimate on $S(u,\underline{u})$, with $p\in[2,4]$:

$$\sup_{\mathcal{K}}|r^{3-\frac{2}{p}}F(^{(O)}q)|_{p,S}\leq c.$$

The estimate of the various terms can be done in the same way. In particular, is is easy to show they are majored by

$$\frac{c}{r_0}\frac{1}{|u'|^2}\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}},$$

so, also (3.3.35) is satisfied and it completes the proof of the boundedness of $\int_{V(u,\underline{u})}\mathbf{Div}Q(\hat{\mathcal{L}}_O W)_{\beta\gamma\delta}(\bar{K}^\beta\bar{K}^\gamma\bar{K}^\delta)$. □

3.3.4 Estimate of $\int_{V(u,\bar{u})} Q(\hat{\mathcal{L}}_O W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma T^\delta)$

Proposition 3.3.4. *In Kerr spacetime the following inequality holds*

$$\int_{V(u,\underline{u})} Q(\hat{\mathcal{L}}_O W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma T^\delta) \leq \frac{c}{r_0^{\frac{3}{2}}}\mathcal{Q}_{\mathcal{K}}. \tag{3.3.39}$$

Proof. The proof is a straightforward calculation, the result is obtained starting from the identity

$$\begin{aligned}{}^{(\bar{K})}\pi^{\alpha\beta}Q(\hat{\mathcal{L}}_O W)_{\alpha\beta\gamma\delta}\bar{K}^\gamma T^\delta &= {}^{(\bar{K})}\pi^{\alpha\beta}\big[\tau_+^2(Q(\hat{\mathcal{L}}_O W)_{\alpha\beta 44} + Q(\hat{\mathcal{L}}_O W)_{\alpha\beta 43})\\ &+ \tau_-^2(Q(\hat{\mathcal{L}}_O W)_{ab34} + Q(\hat{\mathcal{L}}_O W)_{ab33})\big],\end{aligned}$$

by writing explicitly the various term of the integrand (see [?], (6.2.40)-(6.2.43)). Let us discuss into details, one of these, in particular we want to check the boundedness of

$$\int_{V(u,\underline{u})}\tau_+^2|\rho(\hat{\mathcal{L}}_O W)||\alpha(\hat{\mathcal{L}}_O W)||{}^{(\bar{K})}i|.$$

Applying the Schwartz inequality, and recalling the decay of ${}^{(\bar{K})}i|$, we obtain the following estimate

$$
\begin{aligned}
&\int_{V(u,\underline{u})} \tau_+^2|\rho(\hat{\mathcal{L}}_O W)||\alpha(\hat{\mathcal{L}}_O W)||^{(\bar{K})}i| \\
&\leq \left(\int_{V(u,\underline{u})} |\rho(\hat{\mathcal{L}}_O W)|^2|^{(\bar{K})}i|^2\right)^{\frac{1}{2}} \left(\int_{V(u,\underline{u})} \tau_+^4|\alpha(\hat{\mathcal{L}}_O W)|^2\right)^{\frac{1}{2}} \\
\leq\ & c\left(\int_{u_0}^{u} du' \int_{C(u';[\underline{u}_o,\underline{u}])} |\frac{\log r}{r}\rho(\hat{\mathcal{L}}_O W)|^2|\frac{r}{\log r}{}^{(\bar{K})}i|^2\right)^{\frac{1}{2}} \\
\cdot\ & \left(\int_{u_0}^{u} du' \int_{C(u';[\underline{u}_o,\underline{u}])} \tau_+^4|\alpha(\hat{\mathcal{L}}_O W)|^2\right)^{\frac{1}{2}} \\
\leq\ & c\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\left(\sup_{\mathcal{K}}|\frac{r}{\log r}{}^{(\bar{K})}i|\right)\left(\sup_{\mathcal{K}}\int_{C(u';[\underline{u}_0,\underline{u}])} \tau_+^4|\rho(\hat{\mathcal{L}}_O W)|^2\right)\left(\int_{u_0}^{u} du'\frac{(\log r)^2}{r^6}\right)^{\frac{1}{2}} \\
\leq\ & c\mathcal{Q}_{\mathcal{K}}\left(\int_{u_0}^{u} du'\frac{1}{u'^4}\right)^{\frac{1}{2}} \\
\leq\ & \frac{c}{r_0^{\frac{3}{2}}}\mathcal{Q}_{\mathcal{K}}.
\end{aligned}
$$

□

3.3.5 Estimate of $\int_{V(u,\bar{u})} Q(\hat{\mathcal{L}}_O W)_{\alpha\beta\gamma\delta}({}^{(T)}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta)$

Proposition 3.3.5. *In Kerr spacetime the following estimate holds*

$$\int_{V(u,\underline{u})} |Q(\hat{\mathcal{L}}_O W)_{\alpha\beta\gamma\delta}({}^{(T)}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta)| \leq \frac{c}{r_0}\mathcal{Q}_{\mathcal{K}}. \tag{3.3.40}$$

The proof is very similar at the one precedent, with the employ of ${}^{(T)}\pi$ instead of ${}^{(\bar{K})}\pi$. Once decomposed the integrand in any various terms:

$$
\begin{aligned}
&Q(\hat{\mathcal{L}}_O W)_{\alpha\beta\gamma\delta}({}^{(T)}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta) = \\
&{}^{(T)}\pi^{\alpha\beta}\left\{Q(\hat{\mathcal{L}}_O W)_{\alpha\beta 44}\tau_+^4 + 2Q(\hat{\mathcal{L}}_O W)_{\alpha\beta 43}\tau_-^2\tau_+^2 * Q(\hat{\mathcal{L}}_O W)_{\alpha\beta 33}\tau_-^4\right\},
\end{aligned}
$$

the estimates follows in the same way from proposition 2.5.3.
So we have completed the estimate of $\mathcal{E}_1$, checking that it is bounded (in terms of the $\mathcal{Q}_{\mathcal{K}}$ norms) and it is small in the external region of Kerr spacetime, where $r > r_0$.

3.4 Estimate of ϵ_2

To estimate the part of the norms involving two Lie derivatives of the Weyl field W, we need some estimates about the behavior of ${}^S\pi$ (given in the proposition 5.2.1)and about the components of the currents $J(S,W)$. Then we give them in the following.

Proposition 3.4.1. *From the explicit expression of* ${}^{(S)}p_3$, ${}^{(S)}p_4$ *and* ${}^{(S)}\not{p}$, *we obtain the following estimates for any* $S \subset \mathcal{K}$ *with* $p \in [2,4]$*:*

$$
\begin{aligned}
||\tfrac{r^{2-\frac{2}{p}}}{\log r}{}^{(S)}p_3||_{p,S} &\leq c\\
||\tfrac{r^{2-\frac{2}{p}}}{\log r}{}^{(S)}p_4||_{p,S} &\leq c\\
||r^{3-\frac{2}{p}}{}^{(S)}\not{p}_a||_{p,S} &\leq c.
\end{aligned}
$$

Proposition 3.4.2. *The null components of the* S *current* J^3 *satisfy the following estimates for any* $S \subset \mathcal{K}$ *with* $p \in [2,4]$*:*

$$
\begin{aligned}
&|\frac{r^{2-\frac{2}{p}}}{\log r}\Lambda({}^{(S)}q)|_{p,S} \leq c\\
&|\frac{r^{3-\frac{2}{p}}}{\log r}K({}^{(S)}q)|_{p,S} \leq c\\
&|\frac{r^{4-\frac{2}{p}}}{\tau_-}\Xi({}^{(S)}q)|_{p,S} \leq c\\
&|r^{3-\frac{2}{p}}I({}^{(S)}q)|_{p,S} \leq c\\
&|\frac{r2-\frac{2}{p}}{\log r}\Theta({}^{(S)}q)|_{p,S} \leq c
\end{aligned}
$$

and

$$
\begin{aligned}
&|\frac{r^{2-\frac{2}{p}}}{\log r}\underline{\Lambda}({}^{(S)}q)|_{p,S} \leq c\\
&|\frac{r^{3-\frac{2}{p}}}{\log r}\underline{K}({}^{(S)}q)|_{p,S} \leq c\\
&|\frac{r^{4-\frac{2}{p}}}{\tau_-}\underline{\Xi}({}^{(S)}q)|_{p,S} \leq c\\
&|r^{3-\frac{2}{p}}\underline{I}({}^{(S)}q)|_{p,S} \leq c\\
&|\frac{r^{2-\frac{2}{p}}}{\log r}\underline{\Theta}({}^{(S)}q)|_{p,S} \leq c.
\end{aligned}
$$

Then we need some decays for the Lie derivatives of null components of the currents done with respect to O, in particular the following propositions hold:

Proposition 3.4.3. *Based on proposition (2.5.1) and on corollary (2.5.1), the following estimates hold:*

$$
\begin{aligned}
&||r^{\frac{3}{2}-\epsilon}\hat{\mathcal{L}}_O{}^{(O)}p_3|| \quad {}_{L_2(\underline{C}(\underline{u})\cap V(u,\underline{u}))} \leq c\\
&||r^{\frac{3}{2}-\epsilon}\hat{\mathcal{L}}_O{}^{(O)}p_4|| \quad {}_{L_2(\underline{C}(\underline{u})\cap V(u,\underline{u}))} \leq c\\
&||r^{\frac{3}{2}-\epsilon}\hat{\mathcal{L}}_O{}^{(O)}\not{p}_a|| \quad {}_{L_2(\underline{C}(\underline{u})\cap V(u,\underline{u}))} \leq c.
\end{aligned}
$$

Proposition 3.4.4. *The modified Lie derivative of the null components of $J^3(O,W)$ made with respect to the rotation vector fields O satisfy the following asymptotic estimates:*

$$
\begin{aligned}
&||r^{\frac{3}{2}-\epsilon}\hat{\mathcal{L}}_O\Theta({}^{(O)}q)||_{L_2(\underline{C}(\underline{u})\cap V(u,\underline{u}))} \leq c\\
&||r^{\frac{3}{2}-\epsilon}\hat{\mathcal{L}}_O\Lambda({}^{(O)}q)||_{L_2(\underline{C}(\underline{u})\cap V(u,\underline{u}))} \leq c\\
&||r^{\frac{3}{2}-\epsilon}\hat{\mathcal{L}}_O K({}^{(O)}q)||_{L_2(\underline{C}(\underline{u})\cap V(u,\underline{u}))} \leq c\\
&||r^{\frac{3}{2}-\epsilon}\hat{\mathcal{L}}_O I({}^{(O)}q)||_{L_2(\underline{C}(\underline{u})\cap V(u,\underline{u}))} \leq c\\
&||r^{\frac{3}{2}-\epsilon}\hat{\mathcal{L}}_O\Xi({}^{(O)}q)||_{L_2(\underline{C}(\underline{u})\cap V(u,\underline{u}))} \leq c
\end{aligned}
$$

.

For the underlined quantities hold the same inequalities.

3.5 The error term ϵ_2

$\mathcal{E}_2$ collects the error terms associated with the integrals $\mathcal{Q}_2$ and $\mathcal{Q}_2$, in particular:

$$\begin{aligned}
\mathcal{E}_2(u,\underline{u}) &= \int_{V(u,\underline{u})} \mathbf{Div} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)_{\beta\gamma\delta}(\bar{K}^\beta\bar{K}^\gamma\bar{K}^\delta) \\
&\quad + \int_{V(u,\underline{u})} \mathbf{Div} Q(\hat{\mathcal{L}}_O^2 W)_{\beta\gamma\delta}(\bar{K}^\beta\bar{K}^\gamma T^\delta) \\
&\quad + \int_{V(u,\underline{u})} \mathbf{Div} Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)_{\beta\gamma\delta}(\bar{K}^\beta\bar{K}^\gamma\bar{K}^\delta) \\
&\quad + \frac{3}{2}\int_{V(u,\underline{u})} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta) \\
&\quad + \frac{3}{2}\int_{V(u,\underline{u})} Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta) \\
&\quad + \int_{V(u,\underline{u})} Q(\hat{\mathcal{L}}_O^2 W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma T^\delta) \\
&\quad + \frac{1}{2}\int_{V(u,\underline{u})} Q(\hat{\mathcal{L}}_O^2 W)_{\alpha\beta\gamma\delta}({}^{(T)}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta).
\end{aligned}$$

Remark 3.5.1. *As far as the study of $\mathcal{E}_2$ is concerned, we note it is decomposed in many terms, but lots of them can be treated as like the corresponding ones in the previous section.*

First of all, given X, Y two vector fields on $T\mathcal{M}$, let us define the following quantity:

$$J(X,Y;W) = J^0(X,Y;W) + \frac{1}{2}\big(J^1(X,Y;W) + J^2(X,Y;W) + J^3(X,Y;W)\big), \tag{3.5.41}$$

where

$$\begin{aligned}
J^0(X,Y;W) &= \hat{\mathcal{L}}_X J(Y;W) \\
J^1(X,Y;W) &= J^1(X;\hat{\mathcal{L}}_Y W) \\
J^2(X,Y;W) &= J^2(X;\hat{\mathcal{L}}_Y W) \\
J^3(X,Y;W) &= J^3(X;\hat{\mathcal{L}}_Y W).
\end{aligned} \tag{3.5.42}$$

On a correspondence with (3.5.41), its null components $\Theta(X,Y;W)$, ..., $\Xi(X,Y;W)$ have the following structure:

$$F(X,Y;W) = F^0(X,Y;W)+\frac{1}{2}\big(F^1(X,Y;W)+F^2(X,Y;W)+F^3(X,Y;W)\big)$$

and

$$F^0(X,Y;W) = \frac{1}{2}\big[\hat{\mathcal{L}}_X F^1(Y;W) + \hat{\mathcal{L}}_X F^2(Y;W) + \hat{\mathcal{L}}_X F^3(Y,W)\big]. \quad (3.5.43)$$

After, by a straightforward calculation, the following decomposition for $\mathbf{Div}Q(\hat{\mathcal{L}}_X,\hat{\mathcal{L}}_Y W)$ results to be true (see [Ch-Kl], propositions 7.1.1, 7.1.2),

$$\begin{aligned}\mathbf{Div}Q(\hat{\mathcal{L}}_X\hat{\mathcal{L}}_Y W)_{\beta\gamma\delta} &= (\hat{\mathcal{L}}_X\hat{\mathcal{L}}_Y W)_{\beta\ \ \delta}^{\ \mu\ \nu} J(X,Y;W)_{\mu\gamma\nu} + (\hat{\mathcal{L}}_X\hat{\mathcal{L}}_Y W)_{\beta\ \ \gamma}^{\ \mu\ \nu} \\ &\cdot\ J(X,Y;W)_{\mu\delta\nu} + (\hat{\mathcal{L}}_X, Y;W)_{\beta\ \ \delta}^{\ \mu\ \nu} J(X,Y;W)^*{}_{\mu\gamma\nu} \\ &+\ (\hat{\mathcal{L}}_X\hat{\mathcal{L}}_Y)_{\beta\ \ \gamma}^{\ \mu\ \nu} J(X,Y;W)^*{}_{\mu\delta\nu}, \end{aligned} \quad (3.5.44)$$

where $J(X,Y;W)$ is defined by (3.5.41), (3.5.42). These new quantities shall intervene in the estimate of the terms involving the divergence, choosing X,Y between $\{O,T,S\}$ suitably.

3.5.1 Estimate of $\int_{V(u,\bar{u})}\mathbf{Div}Q(\hat{\mathcal{L}}_O^2 W)_{\beta\gamma\delta}(\bar{K}^\beta\bar{K}^\gamma T^\delta)$

Recalling the equation (3.1.9), it follows

$$\begin{aligned}\mathbf{Div}Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T) &= \frac{1}{8}\tau_+^4\big(D(O,O;W)_{444}+D(O,O;W)_{344}\big) \\ &+\ \frac{1}{4}\tau_+^2\tau_-^2\big(D(O,O;W)_{344}+D(O,O;W)_{334}\big) \\ &+\ \frac{1}{8}\tau_-^4\big(D(O,O;W)_{334}+D(O,O;W)_{333}\big),\end{aligned}$$

where these last ones are expressed in appendix (see (3.6.53)), with $X=Y=O$. Posing in (3.5.44) $X=Y=O$, it follows we have to control the following integrals:

$$\begin{aligned}&\int_{V(u,\underline{u})}\tau_+^4 D(O,O;W)_{444}, \quad \int_{V(u,\underline{u})}\tau_+^4 D(O,O;W)_{443} \\ &\int_{V(u,\underline{u})}\tau_+^2\tau_-^2 D(O,O;W)_{334}, \quad \int_{V(u,\underline{u})}\tau_-^4 D(O,O;W)_{333}.\end{aligned} \quad (3.5.45)$$

The most sensitive terms are the one containing the factor τ_+^4. Let us estimate the first one,

$$\int_{V(u,\underline{u})} \tau_+^4 D(O,O;W)_{444} = \frac{1}{2}\int_{V(u,\underline{u})} \tau_+^4 \alpha(\hat{\mathcal{L}}_O^2 W)\cdot\Theta(O,O;W) - \int_{V(u,\underline{u})} \tau_+^4 \beta(\hat{\mathcal{L}}_O^2 W)\cdot\Xi(O,O;W).$$

It holds the following

Proposition 3.5.1. *In Kerr spacetime, the following inequalities hold:*

$$\left|\int_{V(u,\underline{u})} \tau_+^4 \alpha(\hat{\mathcal{L}}_O^2 W)\cdot\Theta(O,O;W)\right| \leq \frac{c}{r_0}\mathcal{Q}_{\mathcal{K}}$$
$$\left|\int_{V(u,\underline{u})} \tau_+^4 \beta(\hat{\mathcal{L}}_O^2 W)\cdot\Xi(O,O;W)\right| \leq \frac{c}{r_0}\mathcal{Q}_{\mathcal{K}}.$$

Proof. As far as the first integral is concerned, let us observe the terms associated with $J^i(O,O;W)$ for $i=1,2,3$ are estimated exactly as the corresponding terms of proposition 3.3.3 by substituting W with $\hat{\mathcal{L}}_O W$.
We still have to control

$$\int_{V(u,\underline{u})} \tau_+^4 \alpha(\hat{\mathcal{L}}_O^2 W)\cdot\Theta^0(O,O;W) = \frac{1}{2}\sum_{i=1}^{3}\left(\int_{V(u,\underline{u})} \tau_+^4 \alpha\cdot\hat{\mathcal{L}}_O\Theta^i(O;W)\right).$$

$\hat{\mathcal{L}}_O\Theta^2(O;W)$ has the following expression:

$$\hat{\mathcal{L}}_O\Theta^2(O;W) = Qr[\hat{\mathcal{L}}_O{}^{(O)}p_3,\alpha(W)] + Qr[\hat{\mathcal{L}}_O{}^{(O)}\not{p};\beta(W)] + Qr[{}^{(O)}p_3;\hat{\mathcal{L}}_O\alpha(W)]$$
$$Qr[\hat{\mathcal{L}}_O p_4;(\rho,\sigma)(W)] + Qr[{}^{(O)}\not{p};\hat{\mathcal{L}}_O\beta(W)] + Qr[{}^{(O)}p_4;\hat{\mathcal{L}}_O(\rho,\sigma)(W)].$$

As far as the $\Theta^2(O;W)$ term is concerned, let us estimate the following terms

$$\int_{V(u,\underline{u})} \tau_+^4 \alpha(\hat{\mathcal{L}}_O^2 W) Qr[\hat{\mathcal{L}}_O{}^{(O)}p_3;\alpha(W)]$$
$$\int_{V(u,\underline{u})} \tau_+^4 \alpha(\hat{\mathcal{L}}_O^2 W) Qr[\hat{\mathcal{L}}_O{}^{(O)}\not{p};\beta(W)] \qquad (3.5.46)$$

that are the most complicated. We control the second one in the following way: by applying the Schwartz inequality,

$$\int_{V(u,\underline{u})} \tau_+^4 \alpha(\hat{\mathcal{L}}_O^2 W) Qr[\hat{\mathcal{L}}_O{}^{(O)}\not{p}; \beta(W)] \leq \int_{V(u,\underline{u})} \tau_+^4 |\alpha(\hat{\mathcal{L}}_O^2 W)||\hat{\mathcal{L}}_O{}^{(O)}\not{p}||\beta(W)|$$
$$\leq \left(\int_{V(u,\underline{u})} \tau_+^{2\gamma}|\alpha(\hat{\mathcal{L}}_O^2 W)|^2\right)^{\frac{1}{2}} \left(\int_{V(u,\underline{u})} \tau_+^{2\sigma}|\hat{\mathcal{L}}_O{}^{(O)}\not{p}|^2|\beta(W)|^2\right)^{\frac{1}{2}},$$

with $\gamma + \sigma = 4$. The first factor satisfies the inequality

$$\left(\int_{V(u,\underline{u})} \tau_+^{2\gamma}|\alpha(\hat{\mathcal{L}}_O^2 W)|^2\right)^{\frac{1}{2}} \leq c\mathcal{Q}_2$$

if and only if $2\gamma < 3$. In fact

$$\int_{V(u,\underline{u})} \tau_+^{2\gamma}|\alpha(\hat{\mathcal{L}}_O^2 W)|^2 = \int_{u_0}^{u} du' \int_{C(u';[\underline{u}_0,\underline{u}])} \tau_+^2\gamma|\alpha(\hat{\mathcal{L}}_O^2 W)|^2$$
$$\leq \int_{u_0}^{u} du' \frac{1}{u'4-2\gamma}\left(\sup_{\mathcal{K}} \int_{C(u';[\underline{u}_0,\underline{u}])} \tau_+^4|\alpha(\hat{\mathcal{L}}_O^2 W)|^2\right) \leq c\mathcal{Q}_2.$$

As far as the second factor is concerned, it holds the following inequality:

$$\begin{aligned}\int_{V(u,\underline{u})} \tau_+^2\sigma|\hat{\mathcal{L}}_O{}^{(O)}\not{p}|^2 &\leq \left(\sup_{\mathcal{K}} |r^{\frac{7}{2}}\beta(W)|^2\right) \int_{V(u,\underline{u})} \tau_+^{-(7-2\sigma)}||\hat{\mathcal{L}}_O{}^{(O)}\not{p}|^2 \\ &\leq c\mathcal{Q}_{\mathcal{K}} \int_{V(u,\underline{u})} \tau_+^{-(7-2\sigma)}|\hat{\mathcal{L}}_O{}^{(O)}\not{p}|^2 \\ &\leq c\mathcal{Q}_{\mathcal{K}} \int_{u_0}^{u} du' \frac{1}{u'^{1+\epsilon}}\left(\int_{C(u';[\underline{u}_0,\underline{u}])} \frac{\tau_+^{6-2\sigma+\epsilon}}{r^3}\right) \\ &\leq c\mathcal{Q}_{\mathcal{K}} \int_{\underline{u}_0}^{\underline{u}} d\underline{u}' \frac{1}{\underline{u}'^{1+\epsilon}}\left(\int_{\underline{C}(\underline{u}';[u_0,u])} \tau_+^{-(2-\epsilon)}\right) \leq \frac{c}{r_0},\end{aligned}$$

the last inequality due to the fact that $2\sigma > 5$. The first integral in (3.5.46) is controlled in the same way, so we do not report it here. The contribution of $Qr[\hat{\mathcal{L}}_O{}^{(O)}p_4; (\rho, \sigma)]$ is easier to treat, so we don't discuss it. Also the other terms present in the expression of $\hat{\mathcal{L}}_O\Theta^2(O, W)$ turn out to be easier to control. So we have only to estimate the terms associated with $\hat{\mathcal{L}}_O\Theta^3(O; W)$. From the expression of $\Theta^3(O, W)$ given in appendix (see (3.6.49)), we need the estimates of the null components of ${}^{(O)}q$, furnished by the proposition 3.4.4. Looking at it we note the $L_2(\underline{C}(\underline{u};[u_0,u]))$ norm of the null components of $\hat{\mathcal{L}}_O{}^{(O)}q$ decay in the same manner, that is $r^{-(\frac{3}{2}-\epsilon)}$, so let us check only the term with $\rho(W)$:

$$\int_{V(u,\underline{u})} \tau_+^4 \alpha(\hat{\mathcal{L}}_O^2 W) Qr[\hat{\mathcal{L}}_O I({}^{(O)}q);\rho(W)]$$

This integral is estimated in the same way as the previous one, i.e. posing $\sigma+\gamma=4$

$$\begin{aligned}
&\int_{V(u,\underline{u})} \tau_+^4 \alpha(\hat{\mathcal{L}}_O^2 W) Qr[\hat{\mathcal{L}}_O I({}^{(O)}q);\rho(W)] \\
\leq\; & \left(\int_{V(u,\underline{u})} \tau_+^{2\gamma} |\alpha(\hat{\mathcal{L}}_O^2 W)|^2\right)^{\frac{1}{2}} \left(\int_{V(u,\underline{u})} \tau_+^{2\sigma} |\rho(W)|^2 |\hat{\mathcal{L}}_O I({}^{(O)}q)|^2\right)^{\frac{1}{2}} \\
\leq\; & c\mathcal{Q}_2^{\frac{1}{2}} \big(\sup_{V(u,\underline{u})} |r^3(\rho-\bar{\rho})|^2 + |r^3\bar{\rho}|^2\big)^{\frac{1}{2}} \int_{\underline{u}_0}^{\underline{u}} d\underline{u}' \frac{1}{\underline{u}'^{1+\epsilon}} \left(\int_{\underline{C}(\underline{u}';[u_0,u])} \frac{\tau_+^{2\sigma-6+1+\epsilon}}{r^{3-\epsilon}}\right)^{\frac{1}{2}} \\
\leq\; & \frac{c}{r_0}\mathcal{Q}_{\mathcal{K}}
\end{aligned}$$

if and only if $5 < 2\sigma \leq 7-\epsilon$.
The terms involving Θ^1 have a better decay, so we do not treat them here. □

3.5.2 Estimate of $\int_{V(u,\bar{u})} \mathbf{Div} Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)_{\beta\gamma\delta}(\bar{K}^\beta \bar{K}^\gamma \bar{K}^\delta)$

Posing in (3.5.44) $X=O, Y=T$, and recalling (5.3.26), we note the terms associated with the current $J^1(O;\hat{\mathcal{L}}_T W)$ we have to analyze are the same as those of

$$\int_{V(u,\underline{u})} \mathbf{Div} Q(\hat{\mathcal{L}}_O W)_{\beta\gamma\delta}(\bar{K}^\beta \bar{K}^\gamma T^\delta)$$

with $\hat{\mathcal{L}}_O W$ replaced by $\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W$ and $J^1(O,W)$ replaced by $J^1(O;\hat{\mathcal{L}}_T W)$. So we have to consider the following integrals

$$\begin{aligned}
&\int_{V(u,\underline{u})} \tau_+^6 \alpha(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)\cdot\Theta^1(O;\hat{\mathcal{L}}_T W) \\
&\int_{V(u,\underline{u})} \tau_+^6 \beta(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)\cdot\Xi^1(O;\hat{\mathcal{L}}_T W)
\end{aligned} \tag{3.5.47}$$

The main difference with respect to the proposition 3.3.3 is that the Weyl null components present in $\Theta^1(O;\hat{\mathcal{L}}_T W)$ and in $\Xi^1(O;\hat{\mathcal{L}}_T W)$ involve terms of the form $\not{D}_3\not{D}_3 W, \not{D}_3\not{D}_4 W, \not{D}_4\not{D}_3 W, \not{D}_4\not{D}_4 W$.[1] So we have to control the following integrals:

$$\int_{u_0}^{u} du' \int_{C(u';[\underline{u}_0,\underline{u}])} r^6 |\not{D}_T \alpha(\hat{\mathcal{L}}_T W)|^2$$
$$\int_{\underline{u}_0}^{\underline{u}} d\underline{u}' \int_{\underline{C}(\underline{u}';[u_0,u])} r^6 |\not{D}_T \underline{\alpha}(\hat{\mathcal{L}}_T W)|^2$$

The terms associated with $J^2(O;\hat{\mathcal{L}}_T W)$ and $J^3(O;\hat{\mathcal{L}}_T W)$ don't present any second derivatives of W, so they are treated in the same way as $\int_{V(u,\underline{u})} \mathbf{Div} Q(\hat{\mathcal{L}}_O W)_{\beta\gamma\delta}(\bar{K}^\beta \bar{K}^\gamma T^\delta)$ with the obvious modifications.
Finally, we have to prove the boundedness and the smallness of the terms associated with $\hat{\mathcal{L}}_O J(T;W)$, that are:

$$\int_{V(u,\underline{u})} \tau_+^6 \alpha(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)(\hat{\mathcal{L}}_O {}^{(T)}p_3)\alpha(W)$$
$$\int_{V(u,\underline{u})} \tau_+^6 \alpha(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)(\hat{\mathcal{L}}_O {}^{(T)}\not{p})\beta(W)$$
$$\int_{V(u,\underline{u})} \tau_+^6 \alpha(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)(\hat{\mathcal{L}}_O {}^{(T)}p_4)(\rho,\sigma)(W).$$

Their estimates are proved in the following proposition.

Proposition 3.5.2. *In Kerr spacetime the following inequalities hold for a Weyl field W that satisfies Bianchi equations:*

$$\begin{aligned}
&\int_{V(u,\underline{u})} \tau_+^6 \alpha(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)(\hat{\mathcal{L}}_O {}^{(T)}p_3)\alpha(W)\\
&\int_{V(u,\underline{u})} \tau_+^6 \alpha(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)(\hat{\mathcal{L}}_O {}^{(T)}\not{p})\beta(W) \qquad (3.5.48)\\
&\int_{V(u,\underline{u})} \tau_+^6 \alpha(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)(\hat{\mathcal{L}}_O {}^{(T)}p_4)(\rho,\sigma)(W).
\end{aligned}$$

[1] In particular the terms $\not{D}_4\not{D}_4\alpha$ and $\not{D}_3\not{D}_3\underline{\alpha}$ are present, but they don't appear in Bianchi equations. We recall for these terms the equation (**??**).

Proof. The first term is the more delicate one. Recalling equation (1.3.15), and applying Schwartz inequality it follows that

$$\left|\int_{V(u,\underline{u})} \tau_+^6 \alpha(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)(\hat{\mathcal{L}}_O {}^{(T)}p_3)\alpha(W)\right| \leq c\int_{u_0}^{u} du'\left(\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^6|\alpha(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)|^2\right)^{\frac{1}{2}}$$

$$\left(\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^6\big(|{}^{(T)}p_3|^2 + r^2|\nabla\!\!\!/{}^{(T)}p_3|^2\big)|\alpha(W)|^2\right)^{\frac{1}{2}}$$

$$\leq c\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\cdot\int_{u_0}^{u} du'\frac{1}{u'^2}\left(\int_{C(u';[\underline{u}_0,\underline{u}])} |\tau_-^{2}\,{}^{(T)}p_3|^2\tau_+^6|\alpha(W)|^2\right)^{\frac{1}{2}} \leq \left(\sup_{V(u,\underline{u})} \tau_+^{\frac{7}{2}}|\alpha(W)|\right)$$

$$\cdot\int_{u_0}^{u} du'\frac{1}{u'^2}\left(\int_{C(u';[\underline{u}_0,\underline{u}])} \frac{1}{\tau_+}|\tau_-^2\big(|{}^{(T)}p_3|^2 + r^2|_{p=2,S}\nabla\!\!\!/{}^{(T)}p_3|^2_{p=2,S})\right)^{\frac{1}{2}}$$

$$\leq cr_0^{\frac{3}{2}}\mathcal{Q}_{\mathcal{K}},$$

since proposition 3.2.2 holds. This completes the proof. □

3.5.3 Estimate of $\int_{V(u,\bar{u})}\mathbf{Div}Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)_{\beta\gamma\delta}(\bar{K}^\beta\bar{K}^\gamma\bar{K}^\delta)$

This time we have to study the terms involving

$$J(T,S;W) = J^0(T,S;W) + \frac{1}{2}\big(J^1(T,S;W) + J^2(T,S;W) + J^3(T,S;W)\big).$$

Proceeding as in subsection 3.5.1, we have to control

$$\int_{V(u,\underline{u})} \tau_+^6 D(T,S;W)_{444}, \quad \int_{V(u,\underline{u})} \tau_+^4\tau_-^2 D(T,S;W)_{344}$$
$$\int_{V(u,\underline{u})} \tau_+^2\tau_-^4 D(T,S;W)_{334}, \quad \int_{V(u,\underline{u})} \tau_-^6 D(T,S;W)_{333}.$$

Let us examine only the first. Since

3.5.4 Estimate of the remaining terms

As far as the other terms of $\mathcal{E}_2$ are concerned, it suffices to observe that they are treated as the terms present in $\mathcal{E}_1$, with the obvious substitutions.
In fact:

$$\int_{V(u,\underline{u})} Q(\hat{\mathcal{L}}_O^2 W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma T^\delta)$$

is treated as

$$\int_{V(u,\underline{u})} Q(\hat{\mathcal{L}}_O W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma T^\delta)$$

and it is estimated by $\mathcal{Q}_2$ instead of $\mathcal{Q}_1$.
The term

$$\int_{V(u,\underline{u})} Q(\hat{\mathcal{L}}_O^2 W)_{\alpha\beta\gamma\delta}({}^{(T)}\pi^{\alpha\beta}\bar{K}^\gamma \bar{K}^\delta)$$

is of the same form of

$$\int_{V(u,\underline{u})} Q(\hat{\mathcal{L}}_O W)_{\alpha\beta\gamma\delta}({}^{(T)}\pi^{\alpha\beta}\bar{K}^\gamma \bar{K}^\delta)$$

by substituting $\mathcal{Q}_1$ with $\mathcal{Q}_2$.
The term

$$\int_{V(u,\underline{u})} Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma \bar{K}^\delta)$$

is estimated in the same way as

$$\int_{V(u,\underline{u})} Q(\hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma \bar{K}^\delta).$$

The final result is the same with the obvious substitutions of the factors $\mathcal{Q}_1$ with $\mathcal{Q}_2$.
The estimate of the integral

$$\int_{V(u,\underline{u})} Q(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma \bar{K}^\delta)$$

is made exactly in the same way as the estimate of

$$\int_{V(u,\underline{u})} Q(\hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma \bar{K}^\delta)$$

with the substitutions of $\mathcal{Q}_1$ with $\mathcal{Q}_2$.

3.6 Appendix

3.6.1 Null components of $J(X,W)$

The explicit expressions of the components of $J(X,W)$ are

$$
\begin{aligned}
\Xi(J^1)(X,W) &= Qr[{}^{(X)}i;\not\nabla\alpha]+Qr[{}^{(X)}\underline{m};\alpha_4]=Qr[{}^{(X)}m;\alpha_3]\\
&+ Qr[{}^{(X)}m,\not\nabla\beta]=Qr[{}^{(X)}j;\beta_4]+Qr[{}^{(X)}n;\beta_3]\\
&+ \mathrm{tr}\chi Qr[{}^{(X)}\underline{m};\alpha]+\mathrm{tr}\underline{\chi}\big(Qr[{}^{(X)}m;\alpha]\\
&+ Qr[{}^{(X)}n;\beta]\big)+l.o.t.\\
\Theta(J^1)(X,W) &= Qr[{}^{(X)}\underline{m};\not\nabla\alpha]+Qr[{}^{(X)}\underline{n};\alpha_4]+Qr[{}^{(X)}j;\alpha_3]\\
&+ Qr[{}^{(X)}i;\not\nabla\alpha]+Qr[{}^{(X)}\underline{m};\beta_4]+Qr[{}^{(X)}m;\beta_3]\\
&+ Qr[{}^{(X)}m;\not\nabla(\rho,\sigma)]+Qr[{}^{(X)}j;(\rho_4,\sigma_4)]+Qr[{}^{(X)}n;(\rho_3,\sigma_3)]\\
&+ \mathrm{tr}\chi\big(Qr[{}^{(X)}\underline{n};\alpha]+Qr[{}^{(X)}\underline{m};\beta]\big)+\mathrm{tr}\underline{\chi}\big(Qr[{}^{(X)}j;\alpha]\\
&+ Qr[{}^{(X)}m;\beta]+Qr[{}^{(X)}n;(\rho,\sigma)]\big)+l.o.t.\\
\Lambda(J^1)(X,W) &= Qr[{}^{(X)}i;\not\nabla\beta]+Qr[{}^{(X)}\underline{m};\beta_4]+Qr[{}^{(X)}m;\beta_3]\\
&+ Qr[{}^{(X)}m;\not\nabla(\rho,\sigma)]+Qr[{}^{(X)}j;(\rho_4,\sigma_4)]+Qr[{}^{(X)}n;(\rho_3,\sigma_3)]\\
&+ \mathrm{tr}\chi Qr[{}^{(X)}\underline{m};\beta]=\mathrm{tr}\underline{\chi}\big(Qr[{}^{(X)}m;\beta]+Qr[{}^{(X)}n;(\rho,\sigma)]\big)+l.o.t.\\
K(J^1)(X,W) &+ Qr[{}^{(X)}i,\not\nabla\beta]+Qr[{}^{(X)}\underline{m};\beta_4]+Qr[{}^{(X)}m;\beta_3]\\
&+ Qr[{}^{(X)}m;\not\nabla(\rho,\sigma)]=Qr[{}^{(X)}j;(\rho_4,\sigma_4)]+Qr[{}^{(X)}n;(\rho_3,\sigma_3)]\\
&+ \mathrm{tr}\chi Qr[{}^{(X)}\underline{m};\beta]+\mathrm{tr}\underline{\chi}\big(Qr[{}^{(X)}m;\beta]+Qr[{}^{(X)}n;(\rho,\sigma)]\big)+l.o.t.\\
I(J^1)(X,W) &= Qr[{}^{(X)}\underline{m};\not\nabla\beta]+Qr[{}^{(X)}\underline{n};\beta_4]+Qr[{}^{(X)}j;\beta_3]\\
&+ Qr[{}^{(X)}i;\not\nabla(\rho,\sigma)]+Qr[{}^{(X)}\underline{m};(\rho_4,\sigma_4)]+Qr[{}^{(X)}m;(\rho_3,\sigma_3)]\\
&+ \mathrm{tr}\chi\big(Qr[{}^{(X)}\underline{n};\beta]+Qr[{}^{(X)}\underline{m};(\rho,\sigma)]\big)\\
&+ \mathrm{tr}\underline{\chi}Qr[{}^{(X)}m;(\rho,\sigma)]+l.o.t.
\end{aligned}
$$

$$
\begin{aligned}
\underline{\Xi}(J^1)(X,W) &= Qr[{}^{(X)}i;\not\nabla\underline{\alpha}] + Qr[{}^{(X)}\underline{m};\underline{\alpha}_4] = Qr[{}^{(X)}m;\underline{\alpha}_3]\\
&+ Qr[{}^{(X)}\underline{m},\not\nabla\underline{\beta}] = Qr[{}^{(X)}\underline{n};\underline{\beta}_4] + Qr[{}^{(X)}j;\underline{\beta}_3]\\
&+ \mathrm{tr}\underline{\chi}Qr[{}^{(X)}m;\underline{\alpha}] + \mathrm{tr}\chi\big(Qr[{}^{(X)}\underline{m};\underline{\alpha}]\\
&+ Qr[{}^{(X)}\underline{n};\underline{\beta}]\big) + l.o.t.\\
\underline{\Theta}(J^1)(X,W) &= Qr[{}^{(X)}m;\not\nabla\underline{\alpha}] + Qr[{}^{(X)}n;\underline{\alpha}_3] + Qr[{}^{(X)}j;\underline{\alpha}_4]\\
&+ Qr[{}^{(X)}i;\not\nabla\underline{\beta}] + Qr[{}^{(X)}m;\underline{\beta}_3] + Qr[{}^{(X)}\underline{m};\beta_4]\\
&+ Qr[{}^{(X)}\underline{m};\not\nabla(\rho,\sigma)] + Qr[{}^{(X)}j;(\rho_3,\sigma_3)] + Qr[{}^{(X)}\underline{n};(\rho_4,\sigma_4)]\\
&+ \mathrm{tr}\underline{\chi}\big(Qr[{}^{(X)}n;\underline{\alpha}] + Qr[{}^{(X)}m;\underline{\beta}]\big)] + \mathrm{tr}\chi\big(Qr[{}^{(X)}j;\underline{\alpha}]\\
&+ Qr[{}^{(X)}\underline{m};\underline{\beta}] + Qr[{}^{(X)}\underline{n};(\rho,\sigma)]\big) + l.o.t.\\
\underline{\Lambda}(J^1)(X,W) &= Qr[{}^{(X)}i;\not\nabla\underline{\beta}] + Qr[{}^{(X)}\underline{m};\underline{\beta}_4] + Qr[{}^{(X)}m;\underline{\beta}_3]\\
&+ Qr[{}^{(X)}\underline{m};\not\nabla(\rho,\sigma)] + Qr[{}^{(X)}j;(\rho_3,\sigma_3)] + Qr[{}^{(X)}\underline{n};(\rho_4,\sigma_4)]\\
&+ \mathrm{tr}\underline{\chi}Qr[{}^{(X)}m;\underline{\beta}] + \mathrm{tr}\chi\big(Qr[{}^{(X)}\underline{m};\underline{\beta}] + Qr[{}^{(X)}\underline{n};(\rho,\sigma)]\big) + l.o.t.\\
\underline{K}(J^1)(X,W) &= Qr[{}^{(X)}i,\not\nabla\underline{\beta}] + Qr[{}^{(X)}\underline{m};\underline{m}_4] + Qr[{}^{(X)}m;\underline{\beta}_3]\\
&+ Qr[{}^{(X)}\underline{m};\not\nabla(\rho,\sigma)] = Qr[{}^{(X)}j;(\rho_3,\sigma_3)] + Qr[{}^{(X)}\underline{[}n];(\rho_4,\sigma_4)]\\
&+ \mathrm{tr}\underline{\chi}Qr[{}^{(X)}m;\underline{\beta}] + \mathrm{tr}\chi\big(Qr[{}^{(X)}\underline{m};\underline{\beta}] + Qr[{}^{(X)}\underline{n};(\rho,\sigma)]\big) + l.o.t.\\
\underline{I}(J^1)(X,W) &= Qr[{}^{(X)}m;\not\nabla\underline{\beta}] + Qr[{}^{(X)}n;\underline{\beta}_3] + Qr[{}^{(X)}j;\underline{\beta}_4]\\
&+ Qr[{}^{(X)}i;\not\nabla(\rho,\sigma)] + Qr[{}^{(X)}\underline{m};(\rho_4,\sigma_4)] + Qr[{}^{(X)}m;(\rho_3,\sigma_3)]\\
&+ \mathrm{tr}\underline{\chi}\big(Qr[{}^{(X)}n;\underline{\beta}] + Qr[{}^{(X)}m;(\rho,\sigma)]\big)\\
&+ \mathrm{tr}\chi Qr[{}^{(X)}\underline{m};(\rho,\sigma)] + l.o.t.
\end{aligned}
$$

Remark 3.6.1. *Qr[;] is a generic notation for any quadratic form with coefficients depending only on the induced metric and area form of* $S(u,\underline{u})$*, the terms which we denote with l.o.t. are cubic with respect to the connection coefficients and are linear with regard to each of them separately. They have an order of decay along the null outgoing hypersurfaces that is lower than the one of all other terms. Hereafter we will disregard them.*

The null decomposition of J^2 is given by

$$\begin{aligned}
\Xi(J^2)(X,W) &= Qr[{}^{(X)}\not{p};\alpha] + Qr[{}^{(X)}p_4;\beta] \\
\Theta(J^2)(X,W) &= Qr[{}^{(X)}p_3;\alpha] + Qr[{}^{(X)}\not{p};\beta] + Qr[{}^{(X)}p_4;\beta] \\
\Lambda(J^2)(X,W) &= Qr[{}^{(X)}\not{p};\beta] + Qr[{}^{(X)}p_4;(\rho,\sigma)] \\
K(J^2)(X,W) &= Qr[{}^{(X)}\not{p};\beta] + Qr[{}^{(X)}p_4;(\rho,\sigma)] \\
I(J^2)(X,W) &= Qr[{}^{(X)}p_3;\beta] + Qr[{}^{(X)}\not{p};(\rho,\sigma)] \\
\underline{\Xi}(J^2)(X,W) &= Qr[{}^{(X)}\not{p};\underline{\alpha}] + Qr[{}^{(X)}p_3;\underline{\beta}] \\
\underline{\Theta}(J^2)(X,W) &= Qr[{}^{(X)}p_4;\underline{\alpha}] + Qr[{}^{(X)}\not{p};\underline{\beta}] + Qr[{}^{(X)}p_3;(\rho,\sigma)] \\
\underline{\Lambda}(J^2)(X,W) &= Qr[{}^{(X)}\not{p};\underline{\beta}] + Qr[{}^{(X)}p_3;(\rho,\sigma)] \\
\underline{K}(J^2)(X,W) &= Qr[{}^{(X)}\not{p};\underline{\beta}] + Qr[{}^{(X)}p_3;(\rho,\sigma)] \\
\underline{I}(J^2)(X,W) &= Qr[{}^{(X)}p_4;\underline{\beta}] + Qr[{}^{(X)}\not{p};(\rho,\sigma)].
\end{aligned}$$

The null decomposition of J^3 is given by

$$\begin{aligned}
\Xi(J^3)(X,W) &= Qr[\alpha;(I,\underline{I})({}^{(X)}q)] + Qr[\beta;(K,\Lambda,\Theta)({}^{(X)}q)] + Qr[(\rho,\sigma);\Xi({}^{(X)}q)] \\
\Theta(J^3)(X,W) &= Qr[\alpha;(\underline{\Lambda},\underline{K})({}^{(X)}q)] \\
&\quad + Qr[\beta;(I,\underline{I})({}^{(X)}q)] + Qr[(\rho,\sigma);\Theta({}^{(X)}q)] \\
\Lambda(J^3)(X,W) &= Qr[\alpha;\underline{\Theta}({}^{(X)}q)] + Qr[\underline{\beta};\Xi({}^{(X)}q)] \\
&\quad + Qr[(\rho,\sigma);(K,\Lambda)({}^{(X)}q)] \qquad (3.6.49) \\
K(J^3)(X,W) &= Qr[\alpha;\underline{\Theta}({}^{(X)}q)] + Qr[\underline{\beta};\Xi({}^{(X)}q)] \\
&\quad + Qr[(\rho,\sigma);(K,\Lambda)({}^{(X)}q)] \\
I(J^3)(X,W) &= Qr[\underline{\beta};(K,\Lambda,\Theta)({}^{(X)}q)] + Qr[(\rho,\sigma);(I,\underline{I})({}^{(X)}q)] \\
&\quad + Qr[\beta;(\underline{K},\underline{\Lambda},\underline{\Theta})({}^{(X)}q)] + Qr[\alpha;\underline{\Xi}({}^{(X)}q)]
\end{aligned}$$

$$
\begin{aligned}
\underline{\Xi}(J^3)(X,W) &= Qr[\underline{\alpha};(I,\underline{I})(^{(X)}q)] + Qr[\underline{\beta};(\underline{K},\underline{\Lambda},\underline{\Theta})(^{(X)}q)] + Qr[(\rho,\sigma);\underline{\Xi}(^{(X)}q)] \\
\underline{\Theta}(J^3)(X,W) &= Qr[\underline{\alpha};(\Lambda,K)(^{(X)}q)] \\
&\quad + Qr[\underline{\beta};(I,\underline{I})(^{(X)}q)] + Qr[(\rho,\sigma);\underline{\Theta}(^{(X)}q)] \\
\underline{\Lambda}(J^3)(X,W) &= Qr[\underline{\alpha};\Theta(^{(X)}q)] + Qr[\beta;\underline{\Xi}(^{(X)}q)] \\
&\quad + Qr[(\rho,\sigma);(\underline{K},\underline{\Lambda})(^{(X)}q)] \qquad (3.6.50) \\
\underline{K}(J^3)(X,W) &= Qr[\underline{\alpha};\Theta(^{(X)}q)] + Qr[\beta;\underline{\Xi}(^{(X)}q)] \\
&\quad + Qr[(\rho,\sigma);(\underline{K},\underline{\Lambda})(^{(X)}q)] \\
\underline{I}(J^3)(X,W) &= Qr[\underline{\beta};(K,\Lambda,\Theta)(^{(X)}q)] + Qr[(\rho,\sigma);(I,\underline{I})(^{(X)}q)] \\
&\quad + Qr[\beta;(\underline{K},\underline{\Lambda},\underline{\Theta})(^{(X)}q)] + Qr[\underline{\alpha};\Xi(^{(X)}q)],
\end{aligned}
$$

where the null components of $^{(X)}q$ are expressed in the following way:

$$
\begin{aligned}
\Lambda(^{(X)}q) &= \frac{1}{4}(\mathbf{D}_3{}^{(X)}n + 4\underline{\omega}^{(X)}n - 4\eta\cdot{}^{(X)}m) \\
&- \frac{1}{4}(\mathbf{D}_4{}^{(X)}j - 2\underline{\eta}\cdot{}^{(X)}m) + \frac{2}{3}{}^{(X)}p_4 \\
K(^{(X)}q)_{ab} &= \frac{1}{2}(\nabla\!\!\!/_a{}^{(X)}m_b - \nabla\!\!\!/_b{}^{(X)}m_a) + \frac{1}{2}(\zeta_a{}^{(X)}m_b - \zeta_b{}^{(X)}m_a) \\
&- \frac{1}{2}(\hat{\chi}_{ac}i_{cb} - \hat{\chi}_{bc}i_{ca}) \\
\Xi(^{(X)}q)_a &= \frac{1}{2}\mathbf{D}\!\!\!/_4{}^{(X)}m_a - \frac{1}{2}\nabla\!\!\!/_a{}^{(X)}n - \frac{1}{2}\underline{\eta}_a{}^{(X)}n + \omega^{(X)}m_a \\
&+ \frac{1}{2}\mathrm{tr}\chi^{(X)}m_a + \hat{\chi}_{ac}{}^{(X)}m_c \qquad (3.6.51) \\
I(^{(X)}q)_a &= \frac{1}{2}\mathbf{D}\!\!\!/_4{}^{(X)}\underline{m}_a - \frac{1}{2}\nabla\!\!\!/_a{}^{(X)}j + \omega^{(X)}\underline{m}_a + \frac{1}{4}\mathrm{tr}\chi^{(X)}\underline{m}_a \\
&+ \frac{1}{2}\hat{\chi}_{ac}{}^{(X)}\underline{m}_c + \frac{1}{4}tr\underline{\chi}^{(X)}m_a + \frac{1}{2}\underline{\hat{\chi}}_{ac}{}^{(X)}m_c - \frac{1}{2}\underline{\eta}_c{}^{(X)}i_{ca} + \frac{3}{2}{}^{(X)}p\!\!\!/_a \\
\Theta(^{(X)}q)_{ab} &= 2\bigg(\mathbf{D}\!\!\!/_4{}^{(X)}i_{ab} - \frac{1}{2}\delta_{ab}\mathrm{tr}(\mathbf{D}\!\!\!/_4{}^{(X)}i)\bigg) - \bigg(\nabla\!\!\!/_a{}^{(X)}m_b + \nabla\!\!\!/_b{}^{(X)}m_a - \delta_{ab}\nabla\!\!\!/_c{}^{(X)}m_c\bigg) \\
&- 2\bigg(\underline{\eta}_a{}^{(X)}m_b + \underline{\eta}_b{}^{(X)}m_a - \delta_{ab}\underline{\eta}_c{}^{(X)}m_c\bigg) - \bigg(\zeta_a{}^{(X)}m_b + \zeta_b{}^{(X)}m_a \\
&- \delta_{ab}\zeta_c{}^{(X)}m_c\bigg) + \mathrm{tr}\chi^{(X)}i_{ab} + \hat{\chi}_{ab}(\mathrm{tr}^{(X)}i + {}^{(X)}j) + \underline{\hat{\chi}}_{ab}{}^{(X)}n,
\end{aligned}
$$

$$\begin{aligned}
\underline{\Lambda}(^{(X)}q) &= \frac{1}{4}(\mathbf{D}_n{}^{(X)}\underline{n} + 4\omega^{(X)}\underline{n} - 4\underline{\eta}\cdot{}^{(X)}\underline{m}) \\
&- \frac{1}{4}(\mathbf{D}_3{}^{(X)}j - 2\eta\cdot{}^{(X)}\underline{m}) + \frac{2}{3}{}^{(X)}p_3 \\
\underline{K}(^{(X)}q)_{ab} &= \frac{1}{2}(\nabla\!\!\!/_a{}^{(X)}\underline{m}_b - \nabla\!\!\!/_b{}^{(X)}\underline{m}_a) - \frac{1}{2}(\zeta_a{}^{(X)}\underline{m}_b - \zeta_b{}^{(X)}\underline{m}_a) \\
&- \frac{1}{2}(\underline{\hat{\chi}}_{ac}i_{cb} - \underline{\hat{\chi}}_{bc}i_{ca}) \\
\underline{\Xi}(^{(X)}q)_a &= \frac{1}{2}\mathbf{D}\!\!\!/_3{}^{(X)}\underline{m}_a - \frac{1}{2}\nabla\!\!\!/_a{}^{(X)}\underline{n} - \frac{1}{2}\eta_a{}^{(X)}\underline{n} + \underline{\omega}^{(X)}\underline{m}_a \\
&+ \frac{1}{2}\mathrm{tr}\underline{\chi}^{(X)}\underline{m}_a + \underline{\hat{\chi}}_{ac}{}^{(X)}\underline{m}_c \qquad (3.6.52) \\
\underline{I}(^{(X)}q)_a &= \frac{1}{2}\mathbf{D}\!\!\!/_3{}^{(X)}m_a - \frac{1}{2}\nabla\!\!\!/_a{}^{(X)}j + \underline{\omega}^{(X)}m_a + \frac{1}{4}\mathrm{tr}\underline{\chi}^{(X)}m_a \\
&+ \frac{1}{2}\underline{\hat{\chi}}_{ac}{}^{(X)}m_c + \frac{1}{4}\mathrm{tr}\chi^{(X)}\underline{m}_a + \frac{1}{2}\hat{\chi}_{ac}{}^{(X)}\underline{m}_c - \frac{1}{2}\eta_c{}^{(X)}i_{ca} + \frac{3}{2}{}^{(X)}p\!\!\!/_a \\
\underline{\Theta}(^{(X)}q)_{ab} &= 2\left(\mathbf{D}\!\!\!/_3{}^{(X)}i_{ab} - \frac{1}{2}\delta_{ab}\mathrm{tr}(\mathbf{D}\!\!\!/_3{}^{(X)}i)\right) - \left(\nabla\!\!\!/_a{}^{(X)}\underline{m}_b + \nabla\!\!\!/_b{}^{(X)}\underline{m}_a - \delta_{ab}\nabla\!\!\!/_c{}^{(X)}\underline{m}_c\right) \\
&- 2\left(\eta_a{}^{(X)}\underline{m}_b + \eta_b{}^{(X)}\underline{m}_a - \delta_{ab}\eta_c{}^{(X)}\underline{m}_c\right) + \left(\zeta_a{}^{(X)}\underline{m}_b + \zeta_b{}^{(X)}\underline{m}_a\right. \\
&- \left.\delta_{ab}\zeta_c{}^{(X)}\underline{m}_c\right) + \mathrm{tr}\underline{\chi}^{(X)}i_{ab} + \underline{\hat{\chi}}_{ab}(\mathrm{tr}^{(X)}i + {}^{(X)}j) + \hat{\chi}_{ab}{}^{(X)}\underline{n},.
\end{aligned}$$

3.6.2 Explicit expression of Div$Q(\hat{\mathcal{L}}_X\hat{\mathcal{L}}_Y W)$

Let X, Y two vector fields in $T\mathcal{M}$, then substituting X in place of T in (3.1.10), we find the following identities hold:

$$\begin{aligned}
D(X,Y;W)_{444} &= 4\alpha(\hat{\mathcal{L}}_O^2 W)\cdot\Theta(X,Y;W) - 8\beta(\hat{\mathcal{L}}_X\hat{\mathcal{L}}_Y W)\cdot\Xi(X,Y;W) \\
D(X,Y,W)_{443} &= 8\rho(\hat{\mathcal{L}}_O^2 W)\Lambda(X,Y;W) + 8\sigma(\hat{\mathcal{L}}_X\hat{\mathcal{L}}_Y W)K(X,Y;W) \\
&+ 8\beta(\hat{\mathcal{L}}_X\hat{\mathcal{L}}_Y W)\cdot I(X,Y;W) \\
D(O,O;W)_{334} &= 8\rho(\hat{\mathcal{L}}_X\hat{\mathcal{L}}_Y W)\underline{\Lambda}(X,Y;W) - 8\sigma(\hat{\mathcal{L}}_X\hat{\mathcal{L}}_Y W)\underline{K}(X,Y;W) \qquad (3.6.53) \\
&- 8\underline{\beta}(\hat{\mathcal{L}}_O^2 W)\cdot\underline{I}(O,O;W) \\
D(X,Y;W)_{333} &= 4\underline{\alpha}(\hat{\mathcal{L}}_X\hat{\mathcal{L}}_Y W)\cdot\underline{\Theta}(X,Y;W) + 8\underline{\beta}(\hat{\mathcal{L}}_O^2 W)\cdot\underline{\Xi}(X,Y;W).
\end{aligned}$$

Chapter 4

The Peeling theorem

4.1 Proof of the theorem (1.2.1)

In the previous chapter, we proved the norms defined from the Bel-Robinson tensor of a Weyl field W (see (1.4.19)), are bounded in terms of the initial data norms. Now we are showing how this fact is related to the asymptotic behavior of the null components of W, in particular we are going to demonstrate the theorem 1.2.1, enunciated in the second chapter.
Let us recall it:

Theorem 4.1.1. *Let W be a Weyl field in a spacetime with assigned metric of Kerr, that satisfies the 2-spin and zero-rest mass field equations*

$$D^{\mu}W_{\mu\nu\rho\sigma} = 0.$$

Let us suppose W null components for t fixed decay like $r^{-(\frac{7}{2})}$, all except $\rho(W)$, which behaves as r^{-3}.
Then the null components of W have the following asymptotic behavior along the future null infinity

$$\begin{aligned}
&\sup_{\mathcal{K}} r^{\frac{7}{2}}|\alpha| \leq C_0, \quad \sup_{\mathcal{K}} r^{\frac{7}{2}}|\beta| \leq C_0 \\
&\sup_{\mathcal{K}} r^{3}|\rho| \leq C_0, \quad \sup r^{3}|u|^{\frac{1}{2}}|(\rho-\bar{\rho},\sigma)| \leq C_0 \\
&\sup_{\mathcal{K}} r^{2}|u|^{\frac{3}{2}}\underline{\beta} \leq C_0, \quad \sup_{\mathcal{K}} r|u|^{\frac{5}{2}}|\underline{\alpha}| \leq C_0,
\end{aligned} \tag{4.1.1}$$

where C_0 is a constant that depends on the initial data.

Proof. In order to find the right estimates for null components of the tensor field W, we need proposition 1.3.2. In particular, let us start with the asymptotic behavior of $\alpha(W)$. Posing in (1.3.13) $F = r^2\alpha(W)$, we obtain the following Sobolev estimate:

$$\begin{aligned}
\sup_{S(u,\underline{u})}(r^{\frac{7}{2}}\alpha) &\leq c\Big[\Big(\int_{S(u_0,\underline{u})} r^{12}|\alpha|^4\Big)^{\frac{1}{4}}\Big) + \Big(\int_{S(u_0,\underline{u})} r^4|r\nabla\!\!\!/(r^2\alpha)|^4\Big)^{\frac{1}{4}} \\
&+ \Big(\int_{\underline{C}(\underline{u}';[u_0,u])} r^4|\alpha|^2 + r^2|\nabla\!\!\!/(r^2\alpha)|^2 + r^2|\mathbf{D}\!\!\!\!/_3(r^2\alpha)|^2 \\
&\quad + r^4|\nabla\!\!\!/^2(r^2\alpha)|^2 + r^4|\nabla\!\!\!/\mathbf{D}\!\!\!\!/_3(r^2\alpha)|^2\Big)^{\frac{1}{2}}\Big] \\
&\leq c\Big[\Big(\int_{S(u_0,\underline{u})} r^{12}|\alpha|^4\Big)^{\frac{1}{4}}\Big) + \Big(\int_{S(u_0,\underline{u})} r^4|r^3\nabla\!\!\!/\alpha|^4\Big)^{\frac{1}{4}} \\
&+ \Big(\int_{\underline{C}(\underline{u}';[u_0,u])} r^4|\alpha|^2 + r^6|\nabla\!\!\!/\alpha|^2 + r^6|\mathbf{D}\!\!\!\!/_3\alpha|^2 \\
&\quad + r^8|\nabla\!\!\!/^2\alpha|^2 + r^4|\nabla\!\!\!/\alpha| + r^8|\nabla\!\!\!/\mathbf{D}\!\!\!\!/_3\alpha|^2\Big)^{\frac{1}{2}}\Big]. \qquad (4.1.2)
\end{aligned}$$

The integrals in r.h.s. are shown to be bounded by the $\underline{\mathcal{Q}}_{\mathcal{K}}$ norms, using the same techniques as in [8], (cap.5), provided that $|r^3\alpha(W)|_{4,S(u_0,\underline{u})}$ is bounded, together with the norm $|r^4\nabla\!\!\!/\alpha(W)|_{4,S(u_0,\underline{u})}$. In fact the integral terms are in substance L_2 norms on the ingoing null hypersurfaces and, as we have remarked in chapter 1, section 1.3, the norms we are building was chosen in a suitable way just to control them. In particular the tangential derivatives of α at the first order are estimated by the $\underline{\mathcal{Q}}_1$ norms, recalling (1.3.15), (1.3.16) hold, while $\mathbf{D}\!\!\!\!/_3\alpha$ is controlled with the use of the Bianchi equation α_3 along the null ingoing cones, i.e. recalling that

$$\alpha_3 = \mathbf{D}\!\!\!\!/_3\alpha + \mathrm{tr}\underline{\chi}\alpha = \nabla\!\!\!/\hat{\otimes}\beta + \big[4\underline{\omega}\alpha - 3(\hat{\chi}\rho + {}^*\hat{\chi}\sigma) + (\zeta + 4\eta)\hat{\otimes}\beta\big]$$

it follows to estimate $\int_{\underline{C}(\underline{u}');[u_0,u]} r^6|\mathbf{D}\!\!\!\!/_3\underline{\alpha}|^2$, we have to control the following:

$$\int_{\underline{C}(\underline{u}';[u_0,u])} r^6|\nabla\!\!\!/\beta|^2 + r^6|\underline{\omega}\alpha|^2 + r^6|\hat{\chi}(\rho,\sigma)|^2 + r^6|(\zeta+\eta)\beta|^2.$$

Then to control $\alpha(W)$ we have to control the norms $L_2(\underline{C}(\underline{u}';[u_0,u]))$ of $\nabla\!\!\!/\beta(W)$ and of $\rho(W), \sigma(W)$ that are estimated by the $\underline{\mathcal{Q}}_1$ norms. As far as

the terms $\nabla\!\!\!\!/^2\alpha$ and $\nabla\!\!\!\!/ \mathbf{D}\!\!\!\!/_4\alpha$ are concerned, they are estimated in an analogous way with the help of the $\underline{\mathcal{Q}}_2$ norms.
Let us make a last remark: we have used the Sobolev estimate on the incoming null cones because an evolution equation for α along outgoing null cones doesn't exist. For an analogous reason, in order to estimate $\sup_{S(u,\underline{u})}|r\tau_-^{\frac{5}{2}}\underline{\alpha}(W)|$ we have to substitute F with τ_-^2 in the inequality (1.3.12), so one have to study the integral of $\underline{\alpha}_4$ along the null outgoing cones. More precisely, provided that the L_4 norms on S of the initial data of $\underline{\alpha}(W)$ decay as $r^{-\frac{1}{2}}\tau_-^{-\frac{5}{2}}$, we have to control the following integral

$$\left(\int_{C(u;[\underline{u}_0,\underline{u})}\tau_-^4|\underline{\alpha}|^2+r^2\tau_-^4|\nabla\!\!\!\!/\underline{\alpha}|^2+\tau_-^6|\mathbf{D}\!\!\!\!/_4\underline{\alpha}|^2\right.$$
$$\left.+r^4\tau_-^4|\nabla\!\!\!\!/\underline{\alpha}|^2+r^2\tau_-^6|\nabla\!\!\!\!/\mathbf{D}\!\!\!\!/_4\underline{\alpha}|^2\right)^{\frac{1}{2}}$$

and actually every term in the sum is controlled by the $\mathcal{Q}_{\mathcal{K}}$ norms.
The null component $\beta(W)$ is shown to satisfy the prescribed decay by posing in (1.3.11) (or equivalently in (1.3.13)) $F = r^2\beta$, while the decays of ρ,σ are estimated studying the equation (1.3.12) (or equivalently (1.3.14)) with $F = r^2\rho, r^2\sigma$. □

4.2 Definition of the modified energy norms

As we have already seen, with the previous techniques, the asymptotic behavior we are able to find for the null components of W is not in agreement with the one prescribed by the Peeling theorem. A way to improve the decay of the null components could be insert in the energy norms a weight factor with a power of τ_-^{α} that allows a better asymptotic behavior of W null components of the type $\tau_-^{-\frac{\alpha}{2}}$, and later on such an improved decay can be transformed in a better decay in r, by making use of the Bianchi equations along the incoming null cones. But, provided that null components of W on Σ_0 are in agreement with the initial null components of the Kerr curvature tensor, it follows the norms builded in such a way don't be bounded on the initial hypersurface.
The main difficult is given by the null component $\rho(W)$, that in the case of Kerr spacetime, it contains the angular momentum J term, which is not

zero. Moreover the crucial fact is that J is not spherically symmetric, so even if we study $\rho - \bar{\rho}$, we can't exclude it.

Remark 4.2.1. *In the case of Schwarzschild spacetime, this problem doesn't appear, because even if a linear momentum of dipole is present, it is spherically symmetric, so, considering $\rho - \bar{\rho}$, its "bad" contribute is eliminated (see [8]).*

However, it is conjectured that the Kerr spacetime is asymptotically simple (see definition 0.1.3), then the null components of an external Weyl field that propagates itself in the background Kerr spacetime, even if they have initial data according with the initial data of the Weyl tensor of the Kerr spacetime, has to satisfy the peeling decays. One way to solve this problem would be to eliminate the part containing the angular momentum. This can be done, making some geometrical considerations about the nature of the Kerr spacetime. Let us recall it is static, that means it does'nt depend on the time, and so every its peculiar quantities is time-independent. In particular the Riemann tensor of the Kerr spacetime is time-independent, then its conformal part W is it, then we are saying that null components of the Weyl field of the Kerr spacetime are static. With this in mind, recalling we are studying the linearized problem of the global solutions of Einstein equations, with initial data near to the Kerr spacetime, when W is an external Weyl field on Kerr spacetime, and we want it simulates the non linear problem, we can suppose W is shaped by an independent-time part and one time-depending:

$$W = W^{(Kerr)} + \delta W, \tag{4.2.3}$$

where $W^{(Kerr)}$ is the static part, and it will just coincide with the Weyl tensor of the Kerr spacetime, while we indicate with δW the non static part.
Then, let us consider the following tensor field:

$$\tilde{W} := \hat{\mathcal{L}}_{T_0} W,$$

where T_0 is the Killing vector field generating time symmetries and let us observe the following fact holds true:
If W is a solution of Bianchi equations, then $\hat{\mathcal{L}}_{T_0} W$ is a solution too.
This is due to the fact T_0 commutes with the null frame's vector fields, then

$$\hat{\mathcal{L}}_{T_0} W_{\mu\nu\rho\sigma} = \frac{\partial}{\partial_t} W_{\mu\nu\rho\sigma},$$

and so it follows that

$$D^\mu(\hat{\mathcal{L}}_{T_0}W)_{\mu\nu\rho\sigma} = D^\mu(\frac{\partial}{\partial_t}W)_{\mu\nu\rho\sigma} = \frac{\partial}{\partial t}(D^\mu W_{\mu\nu\rho\sigma}) = 0.$$

Then, let us define a new family of integral norms, builded from $\tilde{(}W)$:

$$\begin{aligned}
\tilde{\mathcal{Q}}_1(u,\underline{u}) &\equiv \int_{C(u)\cap V(u,\underline{u})} \tau_-^{5+\epsilon} Q(\hat{\mathcal{L}}_T\tilde{W})(\bar{K},\bar{K},\bar{K},e_4) \\
&\quad + \int_{C(u)\cap V(u,\underline{u})} \tau_-^{5+\epsilon} Q(\hat{\mathcal{L}}_O\tilde{W})(\bar{K},\bar{K},T,e_4) \\
\tilde{\mathcal{Q}}_2(u,\underline{u}) &\equiv \int_{C(u)\cap V(u,\underline{u})} \tau_-^{5+\epsilon} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T\tilde{W})(\bar{K},\bar{K},\bar{K},e_4) \\
&\quad + \int_{C(u)\cap V(u,\underline{u})} \tau_-^{5+\epsilon} Q(\hat{\mathcal{L}}_O^2\tilde{W})(\bar{K},\bar{K},T,e_4) \\
&\quad + \int_{C(u)\cap V(u,\underline{u})} \tau_-^{5+\epsilon} Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T\tilde{W})(\bar{K},\bar{K},\bar{K},e_4)
\end{aligned} \tag{4.2.4}$$

$$\begin{aligned}
\underline{\tilde{\mathcal{Q}}}_1(u,\underline{u}) &\equiv \int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} \tau_-^{5+\epsilon} Q(\hat{\mathcal{L}}_T\tilde{W})(\bar{K},\bar{K},\bar{K},e_3) \\
&\quad + \int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} \tau_-^{5+\epsilon} Q(\hat{\mathcal{L}}_O\tilde{W})(\bar{K},\bar{K},T,e_3)
\end{aligned}$$

$$\begin{aligned}
\underline{\tilde{\mathcal{Q}}}_2(u,\underline{u}) &\equiv \int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} \tau_-^{5+\epsilon} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T\tilde{W})(\bar{K},\bar{K},\bar{K},e_3) \\
&\quad + \int_{\underline{C}(u)\cap V(u,\underline{u})} \tau_-^{5+\epsilon} Q(\hat{\mathcal{L}}_O^2\tilde{W})(\bar{K},\bar{K},T,e_3) \\
&\quad + \int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} \tau_-^{5+\epsilon} Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T\tilde{W})(\bar{K},\bar{K},\bar{K},e_3)
\end{aligned} \tag{4.2.5}$$

and

$$\begin{aligned}
\tilde{\mathcal{Q}}_{1_{\Sigma_0\cap V(u,\underline{u})}} &\equiv \int_{\Sigma_0\cap V(u,\underline{u})} \tau_-^{5+\epsilon} Q(\hat{\mathcal{L}}_T\tilde{W})(\bar{K},\bar{K},\bar{K},T) \\
&\quad + \int_{\Sigma_0\cap V(u,\underline{u})} \tau_-^{5+\epsilon} Q(\hat{\mathcal{L}}_O\tilde{W})(\bar{K},\bar{K},T,T) \qquad (4.2.6)\\
\tilde{\mathcal{Q}}_{2_{\Sigma_0\cap V(u,\underline{u})}} &\equiv \int_{\Sigma_0\cap V(u,\underline{u})} \tau_-^{5+\epsilon} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T\tilde{W})(\bar{K},\bar{K},\bar{K},T) \\
&\quad + \int_{\Sigma_0\cap V(u,\underline{u})} \tau_-^{5+\epsilon} Q(\hat{\mathcal{L}}_O^2\tilde{W})(\bar{K},\bar{K},T,T) \\
&\quad + \int_{\Sigma_0\cap V(u,\underline{u})} \tau_-^{5+\epsilon} Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T\tilde{W})(\bar{K},\bar{K},\bar{K},T). \qquad (4.2.7)
\end{aligned}$$

Remark 4.2.2. *In order to show these new norms on the null hypersurfaces $C(u), \underline{C}(\underline{u})$ are estimated in terms of the same norms on the initial hypersurface Σ_0, we have to prove the error term $\epsilon(u,\underline{u})$ is bounded (with the same estimate found in chapter 3, (3.1.6)).*
Substantially we have to reproduce the same estimates of the previous chapter with these new family of norms. Let's us note we don't need to make all steps again, but we only emphasize the differences.
First we define the 1-form

$$\tilde{P}_\mu = \tau_-^{5+\epsilon} P_\mu,$$

where P_μ is as in (1.1.6). Then

$$\begin{aligned}
\mathbf{Div}\tilde{P} &= \mathbf{Div}Q_{\beta\gamma\delta}X^\beta Y^\gamma Z^\delta + div(\tau_-^{5+\epsilon})Q_{\beta\gamma\delta}X^\beta Y^\gamma Z^\delta \qquad (4.2.8)\\
&+ \frac{1}{2}Q_{\alpha\beta\gamma\delta}\big({}^{(X)}\pi^{\alpha\beta}Y^\gamma Z^\delta + {}^{(Y)}\pi^{\alpha\gamma}X^\beta Z^\delta + {}^{(Z)}\pi^{\alpha\delta}X^\beta Y^\gamma\big).
\end{aligned}$$

The term $div(\tau_-^{5+\epsilon}Q)_{\beta\gamma\delta}X^\beta Y^\gamma Z^\delta$ is not a problem in the estimate of the error *because it is shown it is negative, then by Stokes' theorem, it follows that:*

$$\begin{aligned}
&\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} \tau_-^{5+\epsilon} Q(W)(X,Y,Z,e_3) \; + \; \int_{C(u)\cap V(u,\underline{u})} \tau_-^{5+\epsilon} Q(W)(X,Y,Z,e_4) \\
&- \int_{\Sigma_0\cap V(u,\underline{u})} \tau_-^{5+\epsilon} Q(W)(X,Y,Z,T) \\
&= \int_{V(u,\underline{u})} \tau_-^{5+\epsilon}[\mathbf{Div} Q(W)_{\beta\gamma\delta} X^\beta Y^\gamma Z^\delta \; + \; \frac{1}{2} Q^{\alpha\beta\gamma\delta}(W)({}^{(X)}\pi_{\alpha\beta} Y_\gamma Z_\delta \\
&\qquad + {}^{(Y)}\pi_{\alpha\beta} Y_\gamma Z_\delta + {}^{(Z)}\pi_{\alpha\beta} X_\gamma Y_\delta)] \\
&-(5+\epsilon) \int_{V(u,\underline{u})} (2\Omega^{-1})|\tau_-|^{4+\epsilon} Q(W)(X,Y,Z,e_4) \quad . \qquad (4.2.9)
\end{aligned}$$

Then the last term can be ignored, so we have only to consider $\tau_-^{5+\epsilon}\mathbf{Div}P$. *Let's note that the error term with this extra factor is harmless because of it is automatically absorbed in the definition of the* $\tilde{\mathcal{Q}}$ *norms. Let us make an example to show this last fact. Consider the corresponding of* $\int_{V(u,\underline{u})} Q(\hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta)$, *then we have to prove the following proposition holds:*

Proposition 4.2.1. *In Kerr spacetime the following inequalities holds:*

$$\int_{V(u,\underline{u})} |\tau_-^{5+\epsilon} Q(\hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta)| \leq \frac{c}{r^0}\tilde{\mathcal{Q}}_{\mathcal{K}} \qquad (4.2.10)$$

Proof. For the complete explicit expression of the integrand, see [11, **?**](6.2.27)-(6.2.29).
All factors are cubic terms, quadratic in the null components of $\hat{\mathcal{L}}_T W$ and linear in ${}^{(\bar{K})}\pi$. Let us discuss the integral of terms that behave worst. They are those one involving ${}^{(\bar{K})}\underline{n}$ and $\rho(\hat{\mathcal{L}}_T W)$ and with the highest weight factor τ_+, exactly they are the integral relative to $\tau_-^{5+\epsilon}\tau_+^{4\,(\bar{K})}\underline{n}|\alpha(\hat{\mathcal{L}}_T W)|^2$ and the one relative to $\tau_-^{5+\epsilon}\tau_+^{4\,(\bar{K})}n|\rho(\hat{\mathcal{L}}_T W)|^2$. As far as the first integral is concerned, we obtain the following inequality:

$$\begin{aligned}
&\int_{V(u,\underline{u})} \tau_-^{5+\epsilon}\tau_+^4|\alpha(\hat{\mathcal{L}}_T W)|^2|\underline{n}| \leq c\int_{u_0}^u du' \int_{C(u';[\underline{u}_0,\underline{u}])} \tau_-^{5+\epsilon}\tau_+^6|\alpha(\hat{\mathcal{L}}_T W)|^2 \frac{1}{r^2}|{}^{(\bar{K})}\underline{n}| \\
&\leq \; c\left(\sup_{\mathcal{K}} {}^{(\bar{K})}\underline{n}\right)\left(\sup_{\mathcal{K}} \int_{C(u';[\underline{u}_0,\underline{u}])} \tau_-^{5+\epsilon}\tau_+^6|\alpha(\hat{\mathcal{L}}_T W)|^2\right)\int_{u_0}^u du'\frac{1}{u'^2} \leq \frac{c}{r_0}\tilde{\mathcal{Q}}_{\mathcal{K}}.
\end{aligned}$$

For the estimate of the other integral, we proceed in the same way,

$$\int_{V(u,\underline{u})} \tau_-^{5+\epsilon}\tau_+^4|\rho(\hat{\mathcal{L}}_T W)|^2|^{(\bar{K})}n| \leq c\int_{u_0}^{u} du'\frac{1}{u'^2}\int_{C(u';[\underline{u}_0,\underline{u}])}|\tau_-^{5+\epsilon}\frac{r^2}{\tau_-^2}{}^{(\bar{K})}n|\tau_+^2\tau_-^4|\rho(\hat{\mathcal{L}}_T W)|^2$$
$$\leq \; c\Big(\sup_{\mathcal{K}}|\frac{r^2}{\tau_-^2}{}^{(\bar{K})}n|\Big)\Big(\sup_{\mathcal{K}}\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_-^{5+\epsilon}\tau_+^2\tau_-^4|\rho(\hat{\mathcal{L}}_T W)|^2\Big)\int_{u_0}^{u} du'\frac{1}{u'} \leq \frac{c}{r_0}\tilde{\mathcal{Q}}_{\mathcal{K}}.$$

□

It remains to show that the $\tilde{\mathcal{Q}}_{\Sigma_0}$ are bounded. As far as we have previously seen, if we simply consider $\tilde{\mathcal{Q}}_{\Sigma_0}[W]$, with W mimics the asymptotic behavior of $W^{(kerr)}$, we don't find this boundedness. But, in considering $\hat{\mathcal{L}}_{T_0}W$ we forgot the static part (including the Kerr part of W that we have indicated $W^{(kerr)}$), and so it suffices that initial data of $\hat{\mathcal{L}}_{T_0}$ null components decay at spacelike null infinity as $r^{-(6+\epsilon)}$ to have the boundedness of the $\tilde{\mathcal{Q}}$ norms on Σ_0.

4.3 Proof of the Peeling theorem for $\hat{\mathcal{L}}_{T_0W}$

With the help of the new family of energy norms, we are able to prove the Peeling theorem for the null components of a zero-rest mass field of spin 2 propagating on the Kerr spacetime holds:

Theorem 4.3.1 (Peeling Theorem). *Let W be a tensor field with spin 2 that satisfies the null mass equations in Kerr spacetime. Let us suppose any null component of $\hat{\mathcal{L}}_T W$ calculated on Σ_0 decay like $r^{-(6+\epsilon)}$.*
Then W null components satisfy the following peeling decays

$$\begin{aligned}
&\sup_{\mathcal{K}} r^5|u|^{\epsilon'}|\alpha| \leq C_0, \quad \sup_{\mathcal{K}} r^4|u|^{1+\epsilon'}|\beta| \leq C_0\\
&\sup_{\mathcal{K}} r^3|\rho| \leq C_0, \quad \sup r^3|u|^{2+\epsilon'}|\sigma| \leq C_0\\
&\sup_{\mathcal{K}} r^2|u|^{3+\epsilon'}\underline{\beta} \leq C_0, \quad \sup_{\mathcal{K}} r|u|^{4+\epsilon'}|\underline{\alpha}| \leq C_0,
\end{aligned} \tag{4.3.11}$$

with $\epsilon' = \frac{1}{2}\epsilon$ and C_0 a constant depending on the initial data.

Chapter 5

The Schwarzschild spacetime

In this chapter we study the asymptotic behavior of the solutions of Maxwell equations in the empty external Schwarzschild spacetime. The aim of this last part of the work is to show that, with the help of a family of integral norms associated to the stress-energy tensor of the electromagnetic field (the analogous of the $\tilde{\mathcal{Q}}$ and $\underline{\tilde{\mathcal{Q}}}$ norms that in Kerr spacetime allowed us to prove the Peeling theorem is satisfied for an external Weyl field), we can demonstrate under what hypothesis on the $\hat{\mathcal{L}}_{T_0}O$ initial data, the Maxwell tensor propagating itself in Schwarzschild spacetime (supposed as a background space)satisfies the Peeling theorem.

5.1 Most important quantities of Schwarzschild spacetime

Schwarzschild spacetime describes the metric corresponding to the exterior gravitational field of a static, spherically symmetric body.
In spherical coordinates (t, r, θ, ϕ) the Schwarzschild metric has the following form:

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where M is the gravitational mass and the S.U. is chosen requiring that $c = G = 1$. At a first sight, this metric seems to have two singularities: when $r = 0$ and when $r = 2M$. Really the true curvature singularity is only for $r = 0$, while, for $r = 2M$, there is only a failure of coordinates to properly

cover a region of the spacetime. We do not analyze the problem in detail because in the following we will be interested in the region $r \geq 2M$, called Schwarzschild external spacetime.
As well as Kerr spacetime, also the external Schwarzschild spacetime can be foliated with a double null foliation

$$C(u) = \{p \in \mathcal{M} | u(p) = u\} \underline{C}(\underline{u}) \{p \in \mathcal{M} | \underline{u}(p) = \underline{u}\}$$

and the optical functions $u, \underline{u}$ are:

$$\underline{u} = t + r_*, \qquad u = t - r_*$$

where:

$$r_* = r + 2M \log(\frac{r}{2M} - 1)$$

is the radius parameter of the metric.
Let us define the following scalar function: $\Phi^2 = (1 - \frac{2M}{r})$, then the vector fields tangent to the null geodesics are

$$\begin{aligned} L &= \Phi^{-2}\frac{\partial}{\partial t} + \frac{\partial}{\partial r} = 2\Phi^{-2}\frac{\partial}{\partial \underline{u}} \\ \underline{L} &= \Phi^{-2}\frac{\partial}{\partial t} - \frac{\partial}{\partial r} = 2\Phi^{-2}\frac{\partial}{\partial u}. \end{aligned}$$

Moreover $g(L, \underline{L}) = -2\Phi^{-2}$. Starting from the null vector fields L and $\underline{L}$, we can find a null couple e_3, e_4, defined in the following way

$$\begin{aligned} e_4 &= \Phi L = 2\Phi^{-1}\frac{\partial}{\partial \underline{u}}, \\ e_3 &= \Phi \underline{L} = 2\Phi^{-1}\frac{\partial}{\partial u}. \end{aligned}$$

One can easily prove that these vector fields satisfy the relations:

$$g(e_3, e_3) = g(e_4, e_4) = 0, \quad g(e_3, e_4) = -2.$$

Adding to $(e_3, e_4)$the orthonormal vector fields

$$e_\theta = \frac{1}{r}\frac{\partial}{\partial \theta} \tag{5.1.1}$$

$$e_\phi = \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}, \tag{5.1.2}$$

we obtain a null frame relative to the foliation as in (1.1.5). Starting from the double null foliation we can define the 2-dimensional spacelike surfaces:

$$S(u,\underline{u}) = C(u) \cap \underline{C}(\underline{u})$$

which generate a 2-dimensional foliation of the spacetime.
Besides another foliation is given by the spacelike hypersurfaces:

$$\Sigma_t = \{p \in \mathcal{M} | t(p) = t\}$$

and $S(u,\underline{u}) = S(u,t) = \Sigma_t \cap C(u)$.

5.2 Killing and pseudo-Killing vector fields

As far as Schwarzschild spacetime isometries are concerned, because of it is static, they are the diffeomorphisms associated to timelike translations, and, because of it is spherically symmetric, they are the elements of $SO(3)$. Killing vector fields of Schwarzschid spacetime are: first, the generator of time translations

$$T_0 = \frac{\partial}{\partial t},$$

second, the generators of the Lorenz spatial rotations:

$$^{(i)}O = x^j \frac{\partial}{\partial x^k} - x^k \frac{\partial}{\partial x^j}.$$

Together with them let us introduce the pseudo-Killing vector fields, i.e. the analogous of the vector fields (2.4.26) of the Kerr spacetime. They are defined in the following way:

$$\begin{aligned} S &= t\frac{\partial}{\partial t} + r_* \frac{\partial}{\partial r_*} \\ K_0 &= 2tS + (r_*^2 - t^2)T_0. \end{aligned}$$

Now we can express T_0, S, K_0 and $\bar{K} = T_0 + K_0$ in terms of e_3, e_4 in the following way:

$$\begin{aligned} T_0 &= \frac{\Phi}{2}(e_4 + e_3) && (5.2.3)\\ S &= \frac{1}{2}(\underline{u}e_4 + ue_3) && (5.2.4)\\ K_0 &= \frac{1}{2}(\underline{u}^2 e_4 + u^2 e_3) && (5.2.5)\\ \bar{K} &= \frac{1}{2}(\tau_+^2 e_4 + \tau_-^2 e_3), && (5.2.6)\end{aligned}$$

where $\tau_\pm = 1 + (r_* \pm t)^2$.
At last, we consider the connection coefficients of Schwarzschild spacetime. It is easy to show that thanks to its symmetry properties the only nonzero are:

$$\begin{aligned} \chi_{ab} &= \frac{1}{2}\delta_{ab} tr\chi = \delta_{ab}\frac{\Phi}{r}, \quad \underline{\chi}_{ab} = \frac{1}{2}\delta_{ab} tr\underline{\chi} = -\delta_{ab}\frac{\Phi}{r}\\ \mathrm{tr}\chi &= -\mathrm{tr}\underline{\chi} = \frac{2\phi}{r}\\ \omega &= -\frac{1}{2}D_4 \ln\Phi = -\frac{M}{4r^2} && (5.2.7)\\ \underline{\omega} &= -\frac{1}{2}D_3 \ln\Phi = \frac{M}{4r^2}.\end{aligned}$$

These coefficients satisfy manifold' s structure equations.

Let us give in the following the asymptotic behavior of deformation tensors null components that we will need to estimate the error terms.
Recalling the general expressions relative to ${}^{(\bar{K})}\hat{\pi}$ given by (2.5.39) and (2.5.40), and the form of the Schwarzschild connection coefficients (see (5.2.7)), the

following expressions hold true. If we set $X = S$, we find the following:

$$
\begin{aligned}
&{}^{(S)}\quad i_{ab} = \frac{1}{2}\delta_{ab}\big(\frac{1}{2}(\underline{u}-u)2\frac{\phi}{r} - (\underline{u}-u)4\frac{M}{r^2}\big) - \frac{2}{\Phi} \\
&{}^{(S)}\quad j = \frac{1}{2}(\underline{u}-u)2\frac{\phi}{r} - (\underline{u}-u)4\frac{M}{r^2} - \frac{2}{\Phi} \\
&{}^{(S)}\quad m_a = 0 \\
&{}^{(S)}\quad \underline{m}_a = 0 \\
&{}^{(S)}\quad n = \frac{M}{r^2}u \\
&{}^{(S)}\quad \underline{n} = -\frac{M}{r^2}\underline{u}.
\end{aligned}
$$

Then, observing that:

$$
\begin{aligned}
&\underline{u} - u = 2r_* \\
&2r_*\frac{\phi}{r} - \frac{2}{\phi} = O\big(4M\frac{\log r}{r}\big),
\end{aligned}
$$

it results they have the following explicit form at the highest decay order:

$$
\begin{aligned}
&{}^{(S)}\quad i_{\theta\theta} = O\big(4M\frac{\log r}{r}\big) \\
&{}^{(S)}\quad i_{\phi\phi} ={}^{(S)}\ i_{\theta\theta} \\
&{}^{(S)}\quad i_{\theta\phi} = 0 \\
&{}^{(S)}\quad j = O\big(4M\frac{\log r}{r}\big) \\
&{}^{(S)}\quad m_a = 0 \\
&{}^{(S)}\quad \underline{m}_a = 0 \\
&{}^{(S)}\quad n = O\big(\frac{M}{r^2}u\big) \\
&{}^{(S)}\quad \underline{n} = O\big(-2\frac{M}{r}\big).
\end{aligned}
\tag{5.2.8}
$$

If $X = \bar{K}$, then:

$$
\begin{aligned}
&{}^{(\bar{K})}\ i_{ab} = \frac{1}{2}\delta_{ab}\big(\frac{1}{2}4tr_*2\frac{\phi}{r} - 4tr_*\frac{M}{4r^2} - \frac{4t}{\phi}\big)\\
&{}^{(\bar{K})}\ j = \frac{1}{2}4tr_*2\frac{\phi}{r} - 4tr_*\frac{M}{4r^2} - \frac{2t}{\phi}\\
&{}^{(\bar{K})}\ m_a = 0\\
&{}^{(\bar{K})}\ \underline{m}_a = 0\\
&{}^{(\bar{K})}\ n = \tau_-^2\frac{M}{r^2}\\
&{}^{(\bar{K})}\ \underline{n} = -\tau_+^2\frac{M}{r^2},
\end{aligned}
$$

that imply the following asymptotic behaviors hold:

$$
\begin{aligned}
&{}^{(\bar{K})}\ i_{\theta\theta} = O\big(4Mt\frac{\log r}{r}\big)\\
&{}^{(\bar{K})}\ i_{\phi\phi} = {}^{(\bar{K})}\ i_{\theta\theta}\\
&{}^{(\bar{K})}\ i_{\theta\phi} = 0\\
&{}^{(\bar{K})}\ j = O\big(4Mt\frac{\log r}{r}\big)\\
&{}^{(\bar{K})}\ m_a = 0 \qquad (5.2.9)\\
&{}^{(\bar{K})}\ \underline{m}_a = 0\\
&{}^{(\bar{K})}\ n = O\big(\frac{M\tau_-^2}{r^2}\big) \qquad (5.2.10)\\
&{}^{(\bar{K})}\ \underline{n} = O\big(4M\big).
\end{aligned}
$$

Remark 5.2.1. *Obviously* ${}^{(T_0)}\pi = {}^{(O)}\pi = 0.$

We give in the following some propositions about L_p estimates holding for any $S \subset \mathcal{K}$, satisfied from null components of deformation tensors relative to $S, \bar{K}$ and from their first derivatives.

Proposition 5.2.1. *The components of* ${}^{(S)}\hat{\pi}_{\alpha\beta}$ *satisfy the following esti-*

mates, for any $S \subset \mathcal{K}$, with $p \in [2,4]$:

$$
\begin{aligned}
&|\frac{r^{1-\frac{2}{p}}}{\log r}{}^{(S)}i|_{p,S} \leq c\\
&|\frac{r^{1-\frac{2}{p}}}{\log r}{}^{(S)}j|_{p,S} \leq c\\
&|\frac{r^{2-\frac{2}{p}}}{u}{}^{(S)}n|_{p,S} \leq c\\
&|\frac{r^{2-\frac{2}{p}}}{\underline{u}}{}^{(S)}\underline{n}|_{p,S} \leq c.
\end{aligned}
\tag{5.2.11}
$$

Besides, their first derivatives satisfy the following estimates:

$$
\begin{aligned}
&|r^{2-\frac{2}{p}}\not{D}_4{}^{(S)}i|_{p,S} \leq c\\
&|r^{2-\frac{2}{p}}\not{D}_4{}^{(S)}j|_{p,S} \leq c\\
&|\frac{r^{3-\frac{2}{p}}}{u}\not{D}_4{}^{(S)}n|_{p,S} \leq c\\
&|r^{2-\frac{2}{p}}\not{D}_4{}^{(S)}\underline{n}|_{p,S} \leq c.
\end{aligned}
\tag{5.2.12}
$$

and

$$
\begin{aligned}
&|r^{2-\frac{2}{p}}\not{D}_3{}^{(S)}i|_{p,S} \leq c\\
&|r^{2-\frac{2}{p}}\not{D}_3{}^{(S)}j|_{p,S} \leq c\\
&|r^{2-\frac{2}{p}}\not{D}_3{}^{(S)}n|_{p,S} \leq c\\
&|\frac{r^{3-\frac{2}{p}}}{\underline{u}}\not{D}_3{}^{(S)}\underline{n}|_{p,S} \leq c.
\end{aligned}
\tag{5.2.13}
$$

As far as tangential derivatives are concerned, they are equal to 0, since null components of ${}^{(S)}\pi$ don't depend on the angles.

Proposition 5.2.2. *On every 2-dimensional surface $S \subset \mathcal{K}$, the components* *of* ${}^{(\bar{K})}\pi$ *satisfy the following inequalities, with* $p \in [2,4]$*:*

$$
\begin{aligned}
&|\frac{r^{1-\frac{2}{p}}}{\log r} t^{(\bar{K})} i|_{p,S} \leq c \\
&|\frac{r^{1-\frac{2}{p}}}{\log r} t^{(\bar{K})} j|_{p,S} \leq c \\
&|\frac{r^{2-\frac{2}{p}}}{\tau_-^2} {}^{(\bar{K})} n|_{p,S} \leq c \\
&|\frac{r^{2-\frac{2}{p}}}{\tau_+^2} {}^{(\bar{K})} \underline{n}|_{p,S} \leq c,
\end{aligned} \tag{5.2.14}
$$

$$
\begin{aligned}
&|\frac{r^{1-\frac{2}{p}}}{\log r} \not{D}_4^{(\bar{K})} i|_{p,S} \leq c \\
&|\frac{r^{1-\frac{2}{p}}}{\log r} \not{D}_4^{(\bar{K})} j|_{p,S} \leq c \\
&|\frac{r^{3-\frac{2}{p}}}{\tau_-^2} \not{D}_4^{(\bar{K})} n|_{p,S} \leq c \\
&|\frac{r^{2-\frac{2}{p}}}{\tau_+} \not{D}_4^{(\bar{K})} \underline{n}|_{p,S} \leq c,
\end{aligned} \tag{5.2.15}
$$

and

$$
\begin{aligned}
&|\frac{r^{1-\frac{2}{p}}}{\log r} \not{D}_3^{(\bar{K})} i|_{p,S} \leq c \\
&|\frac{r^{1-\frac{2}{p}}}{\log r} \not{D}_3^{(\bar{K})} j|_{p,S} \leq c \\
&|\frac{r^{2-\frac{2}{p}}}{\tau_-} \not{D}_3^{(\bar{K})} n|_{p,S} \leq c \\
&|\frac{r^{2-\frac{2}{p}}}{\tau_+} \not{D}_3^{(\bar{K})} \underline{n}|_{p,S} \leq c,
\end{aligned} \tag{5.2.16}
$$

at las their tangential derivatives are null.

5.3 Maxwell equations and stress-energy tensor

The electromagnetic tensor $F_{\mu\nu}$ is a 2-form and its left Hodge dual is defined by:

$$^{*}F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}, \tag{5.3.17}$$

where $\epsilon_{\mu\nu\rho\sigma} = \sqrt{|det\{g_{\mu\nu}\}|}\delta^1_{[\mu}\delta^2_{\nu}\delta^3_{\rho}\delta^4_{\sigma]}$ F is a solution of vacuum Maxwell equations, if it satisfies the following tensorial equations:

$$D^{\mu}F_{\mu\nu} = 0, \qquad D^{\mu *}F_{\mu\nu} = 0. \tag{5.3.18}$$

It is easy to show, they are equivalent to the set of equations:

$$\begin{aligned} D_{[\lambda}F_{\mu\nu]} &= 0 \\ D^{\star}_{[\lambda}F_{\mu\nu]} &= 0. \end{aligned} \tag{5.3.19}$$

Given a vector field X we can define the following 1-form $i_X F$ in a way such that $i_X F \equiv F(\cdot, X)$ and in a similar way we introduce $i_X{}^*F$; then $i_X F$ and $i_X{}^*F$ completely determine F at every point where $g(X,X)$ is different from 0.

Remark 5.3.1. *If $X = \Phi^{-1}T_0 \equiv T$, the 1-forms $E = i_{\tilde{T}_0}F$ and $B = i_T{}^*F$ respectively represent the electric and the magnetic part ofF and they are tangent to the hypersurfaces: Σ_t.*

Definition 5.3.1. *We can decompose electromagnetic tensor in terms of the null frame, and so, in terms of the hypersurfaces that foliate the spacetime. Such a null decomposition results to be*

$$\alpha_a \equiv \alpha(F)(e_a) = F(e_a, e_4) \tag{5.3.20}$$
$$\underline{\alpha}_a \equiv \underline{\alpha}(F)(e_a) = F(e_a, e_3) \tag{5.3.21}$$
$$\rho \equiv \rho(F) = \frac{1}{2}F(e_3, e_4) \tag{5.3.22}$$
$$\sigma \equiv \sigma(F) = F(e_\theta, e_\phi). \tag{5.3.23}$$

where $\alpha, \underline{\alpha}$ are 1-form tangent to the spheres intersection of $C(u)$ and $\underline{C}(\underline{u})$ and ρ,   are scalar functions.

They can be expressed in terms of electric and the magnetic part of F, by holding the following identities:

$$\begin{aligned} \alpha_a &= (E_a + \epsilon_{ab} H_{ab}) \\ \underline{\alpha}_a &= (E_a - \epsilon_{ab} H_b) \\ \rho &= -E_\perp \\ \sigma &= -H_\perp . \end{aligned}$$

Maxwell equations (5.3.18), projected on the incoming and outgoing null cones assume the form:

$$\begin{aligned} \underline{\alpha}_4 = \not{D}_4 \underline{\alpha} \quad &+ \quad (\partial_r \Phi + \frac{\Phi}{r}) \underline{\alpha} = -\not{\nabla} \rho + {}^*\not{\nabla} \sigma \\ \alpha_3 = \not{D}_3 \alpha \quad &- \quad (\partial_r + \frac{\Phi}{r}) \alpha = \not{\nabla} \rho + {}^*\not{\nabla} \sigma \\ \sigma_4 = D_4 \sigma \quad &+ \quad 2\frac{\Phi}{r} \sigma = -\not{\mathrm{curl}}\, \alpha \\ \rho_4 = D_4 \rho \quad &+ \quad 2\frac{\Phi}{r} \rho = \not{\mathrm{div}}\, \alpha \\ \sigma_3 = D_3 \sigma \quad &- \quad 2\frac{\Phi}{r} \sigma = -\not{\mathrm{curl}}\, \underline{\alpha} \\ \rho_3 = D_3 \rho \quad &- \quad 2\frac{\Phi}{r} \rho = -\not{\mathrm{div}}\, \underline{\alpha}. \end{aligned} \tag{5.3.24}$$

Let us give the following definition:

Definition 5.3.2. *Let F be a solution of Maxwell vacuum equations and let X, Y be two vector fields. We define Q the stress-energy tensor relative to F, in the following way*

$$Q(X,Y) = \langle i_X F, i_Y F \rangle + \langle i_X {}^*F, i_Y {}^*F \rangle,$$

that, written respect to a coordinate basis, takes the form

$$Q_{\mu\nu} = F_{\mu\rho} F_\nu{}^\rho + {}^*F_{\mu\rho} {}^*F_\nu{}^\rho = 2F_{\mu\rho} F_\nu{}^\rho - \frac{1}{2} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}.$$

Lemma 5.3.1. *The stress- energy tensor associated to the Maxwell tensor has the following properties:*
i) Q is symmetric and traceless.

ii)For every couple of timelike or null future-directed vector fields, Q satisfies the inequality:

$$Q(X,Y) \geq 0.$$

iii)Because of F is a solution of the Maxwell equations, then Q has divergence equal to zero, i.e.,

$$D^\alpha Q_{\alpha\beta} = 0.$$

Proof. *i*) The symmetry of Q trivially derives from its definition.
Moreover, its traceless character is a consequence of the identity

$$F_{\alpha\beta}F^{\alpha\beta} +^\star F_{\alpha\beta}{}^\star F^{\alpha\beta} = 0. \tag{5.3.25}$$

For the proof of property *ii*), we first observe that, given two non-spacelike future directed vectors X, Y, in the plane spanned by X, Y, there are a couple of future directed null vectors $L, \underline{L}$ such that

$$g(L, \underline{L}) = -2,$$

then we can express X and Y as linear combinations of $\{L, \underline{L}\}$ with coefficients non-negatives. So, let $\{\alpha', \underline{\alpha}', \rho', \sigma'\}$ be the null decomposition of F respect to such vectors. Then

$$\begin{aligned} Q(L,L) &= 2|\alpha'|^2 \\ Q(\underline{L},\underline{L}) &= 2|\underline{\alpha}'|^2 \\ Q(L,\underline{L}) &= 2(\rho'^2 + \sigma'^2). \end{aligned}$$

So $Q(X,Y)$ is a linear combination, with non-negative coefficients, of these three positive quantities.
iii)

$$\begin{aligned} &D^\alpha Q_{\alpha\beta} = D^\alpha F_{\alpha\rho}F_\beta{}^\rho + F_{\alpha\rho}D^\alpha F_\beta{}^\rho + D^{\alpha\star}F^\star_{\alpha\rho}F_\beta{}^\rho +^\star F_{\alpha\rho}D^{\alpha\star}F_\beta{}^\rho \\ &= F^{\alpha\rho}D_\alpha F_{\beta\rho} +^\star F^{\alpha\rho}D_\alpha F_{\beta\rho} \\ &= \frac{1}{2}F^{\alpha\rho}(D_\alpha F_{\beta\rho} - D_\rho F_{\beta\alpha}) + \frac{1}{2}^\star F^{\alpha\rho}(D^\star_\alpha F_{\beta\rho} - D^\star_\rho F_{\beta\alpha}) \\ &= \frac{1}{2}F^{\alpha\rho}D_\beta F_{\alpha\rho} + \frac{1}{2}^\star F^{\alpha\rho}D^\star_\beta F_{\alpha\rho} \\ &= \frac{1}{4}D_\beta(F^{\alpha\rho}F_{\alpha\rho} + {}^\star F^{\alpha\rho\star}F_{\alpha\rho}) = 0, \end{aligned}$$

where we have used (5.3.18) and (5.3.25). □

When Q acts on the null frame $\{e_3, e_4, e_a\}$, we obtain the following :

$$\begin{aligned} Q(e_3, e_3) &= Q_{33} = 2|\underline{\alpha}|^2 \\ Q(e_4, e_4) &= Q_{44} = 2|\alpha|^2 \\ Q(e_3, e_4) &= Q_{34} = 2(\rho^2 + \sigma^2) \\ Q(e_\theta, e_\theta) &= Q_{11} = \sigma^2 - \alpha_\theta \underline{\alpha}_\theta + \alpha_\phi \underline{\alpha}_\phi \\ Q(e_\phi, e_\phi) &= Q_{22} = \sigma^2 + \alpha_\theta \underline{\alpha}_\theta - \alpha_\phi \underline{\alpha}_\phi \\ Q(e_\theta, e_\phi) &= Q_{12} = -(\alpha_\theta \underline{\alpha}_\phi + \alpha_\phi \underline{\alpha}_\theta). \end{aligned} \tag{5.3.26}$$

With these relations in mind we can obtain for multilinearity:

$$\begin{aligned} Q(T_0, e_4) &= \Phi\{|\alpha|^2 + (\rho^2 + \sigma^2)\} \\ Q(\bar{K}, e_4) &= \Phi\{\tau_+^2 |\alpha|^2 + \tau_-^2(\rho^2 + \sigma^2)\} \\ Q(T_0, e_3) &= \Phi\{|\underline{\alpha}|^2 + (\rho^2 + \sigma^2)\} \\ Q(\bar{K}, e_3) &= \Phi\{\tau_-^2 |\underline{\alpha}|^2 + \tau_+^2(\rho^2 + \sigma^2)\} \\ Q(T_0, T) &= (\Phi/2)\{|\underline{\alpha}|^2 + 2(\rho^2 + \sigma^2)\} \\ Q(\bar{K}, T) &= (\Phi/2)\{\tau_-^2 |\underline{\alpha}|^2 + \tau_+^2 |\alpha|^2 + (\tau_+^2 \tau_-^2)(\rho^2 + \sigma^2)\} \end{aligned}$$

with $T = \Phi^{-1} T_0$.

Remark 5.3.2. *In view of the fact that e_3, e_4 are null and $\bar{K}, T_0$ are timelike, both future directed, the above quantities are nonnegative.*

5.4 Energy norms relative to $\mathcal{L}_{T_0} F$

In this section we introduce a new family of energy norms, quite similar to the $\tilde{\underline{Q}}$ norms in Kerr spacetime. They will be related to the tensor field $L_{T_0} F$, instead of F, because in this way, as we will see in the following, we can improve the decays required for the initial date of the F null components, by holding peeling theorem anyway. Second in their definition, it shall appear a $\tau_-^{3+\epsilon}$ factor which will be counterbalanced by the better decay of the null components of $\mathcal{L}_{T_0} F$.
In fact, before to define the energy norms, we observe that the electromagnetic tensor can be divided in a static part and in a time depending part:

$$F_{\mu\nu(tot)} = F_{\mu\nu(stat)} + F_{\mu\nu(t)} \tag{5.4.27}$$

Obviously in $F_{\mu\nu(stat)}$ are included the fields generated by dipoles and by fixed charges. From now we will work on the components of $\tilde{F} = \mathcal{L}_{T_0}F$, so that in the results will not occur properties of the static part of the field. It immediately follows, from the symmetry properties of the Schwarzschild metric, and from the equations (5.4.27) that:

$$\mathcal{L}_{T_0}F_{\mu\nu(tot)} = \partial_t F_{\mu\nu(tot)} = \partial_t F_{\mu\nu(t)} = \mathcal{L}_{T_0}F_{\mu\nu(t)} \tag{5.4.28}$$

(being $\mathcal{L}_{T_0}F_{\mu\nu(stat)} = \partial_t F_{\mu\nu(stat)} = 0$.)

Definition 5.4.1. *We call generalized energy norms associated to the stress-energy tensor $Q_{\mu\nu}$ the following integral quantities:*

$$\begin{aligned}
\tilde{\mathcal{Q}}^O_k(u;[\underline{u}_0,\underline{u}]) &= \sum_{1\le a\le k+2}\int_{C(u;[\underline{u}_0,\underline{u}])} \tau_-^{3+\epsilon} Q(\mathcal{L}^a_O\tilde{F})(\bar{K},e_4)\\
\tilde{\mathcal{Q}}^S_k(u;[\underline{u}_0,\underline{u}]) &= \sum_{1\le a\le k+1}\int_{C(u;[\underline{u}_0,\underline{u}])} \tau_-^{3+\epsilon} Q(\mathcal{L}_S\mathcal{L}^a_O\tilde{F})(\bar{K},e_4)\\
\underline{\tilde{\mathcal{Q}}}^O_k(\underline{u};[u_0,u]) &= \sum_{1\le a\le k+2}\int_{\underline{C}(\underline{u};[u_0,u])} \tau_-^{3+\epsilon} Q(\mathcal{L}^a_O\tilde{F})(\bar{K},e_3) \qquad (5.4.29)\\
\underline{\tilde{\mathcal{Q}}}^S_k(\underline{u};[u_0,u]) &= \sum_{1\le a\le k+1}\int_{\underline{C}(\underline{u};[u_0,u])} \tau_-^{3+\epsilon} Q(\mathcal{L}_S\mathcal{L}^a_O\tilde{F})(\bar{K},e_3)\\
\tilde{\mathcal{Q}}^O_k(\Sigma_t) &= \sum_{1\le a\le k+2}\int_{\Sigma_t} \tau_-^{3+\epsilon} Q(\mathcal{L}^a_O\tilde{F})(\bar{K},T)\\
\tilde{\mathcal{Q}}^S_k(\Sigma_t) &= \sum_{1\le a\le k+1}\int_{\Sigma_t} \tau_-^{3+\epsilon} Q(\mathcal{L}_{T_0}\mathcal{L}^a_O\tilde{F})(T_0,T).
\end{aligned}$$

We can divide in a part which depends on the Lie derivative of the first order of F, that we will call $\tilde{\mathcal{Q}}_1, \tilde{\mathcal{Q}}b_1$ and a part depending on the second order Lie derivatives of F, that we'll indicate with $\tilde{\mathcal{Q}}_2, \tilde{\mathcal{Q}}_2$, in the following way:

$$\begin{aligned}
\tilde{\mathcal{Q}}_1(u,\underline{u}) &= \int_{C(u;[\underline{u}_0,\underline{u}])} \tau_-^{3+\epsilon} Q(\mathcal{L}_O\tilde{F})(\bar{K},e_4) \qquad (5.4.30)\\
\underline{\tilde{\mathcal{Q}}}_1(u,\underline{u}) &= \int_{\underline{C}(\underline{u};[u_0,u])} \tau_-^{3+\epsilon} Q(\mathcal{L}_O\tilde{F})(\bar{K},e_3)
\end{aligned}$$

and

$$
\begin{aligned}
\tilde{\mathcal{Q}}_2(u,\underline{u}) &= \int_{C(u;[\underline{u}_0,\underline{u}])} \tau_-^{3+\epsilon} Q(\mathcal{L}_O^2 \tilde{F})(\bar{K},e_4)\\
&+ \int_{C(u;[\underline{u}_0,\underline{u}])} \tau_-^{3+\epsilon} Q(\mathcal{L}_S\mathcal{L}_O \tilde{F})(\bar{K},e_4)\\
\underline{\tilde{\mathcal{Q}}}_2 &= \int_{\underline{C}(\underline{u};[u_0,u])} \tau_-^{3+\epsilon} Q(\mathcal{L}_O^2 \tilde{F})(\bar{K},e_3) \qquad (5.4.31)\\
&+ \int_{\underline{C}(\underline{u};[u_0,u])} \tau_-^{3+\epsilon} Q(\mathcal{L}_S\mathcal{L}_O \tilde{F})(\bar{K},e_3).
\end{aligned}
$$

Remark 5.4.1. *Since F is a solution of the Maxwell equations in the vacuum, also $\tilde{F}$ is it, in fact*

$$\mathcal{L}_{T_0} F_{\mu\nu} = \partial_t F_{\mu\nu},$$

(remember that $\mathcal{L}_{T_0} e_3 = \mathcal{L}_{T_0} e_4 = 0$) then, if

$$D^\mu F_{\mu\nu} = 0,$$

it follows that

$$D^\mu \tilde{F}_{\mu\nu} = D^\mu(\partial_t F)_{\mu\nu} = \partial_t(D^\mu F_{\mu\nu}) = 0.$$

Moreover, in view of the fact that Schwarzschild metric is invariant under spatial rotations, it follows that $\mathcal{L}_O^a \tilde{F}$ also satisfy the Maxwell equation in the vacuum, while $\mathcal{L}_S \tilde{F}$ will has a nonzero divergence.

First of all, in view of the fact that the asymptotic behavior of the components of F will be estimated by the integral energy norms, it is essential prove that these quantities are bounded.
To do it, we considerate the1-form P_μ, defined by:

$$P_\mu = \tau_-^{3+\epsilon} Q(G)_{\mu\nu} X^\nu, \qquad (5.4.32)$$

where Q is the stress-energy tensor relative to the antisymmetric 2-form G, that is one of $\mathcal{L}_O^a \tilde{F}, \mathcal{L}_S \mathcal{L}_O^a \tilde{F}, \mathcal{L}_{T_0} \mathcal{L}_O^a \tilde{F}$ and X is a vector field. Since Q is symmetric and traceless, the following equality holds:

$$
\begin{aligned}
\mathbf{Div} P &= div(\tau_-^{3+\epsilon} Q)_\nu X^\nu + \frac{1}{2}\tau_-^{3+\epsilon} Q_{\mu\nu}{}^{(X)}\pi^{\mu\nu}\\
&= \tau_-^{3+\epsilon}(\mathbf{Div} Q)_\nu \overline{K}^\nu + (3+\epsilon)\tau_-^{2+\epsilon} g^{\sigma\mu} D_\sigma \tau_- Q_{\mu\nu} X^\nu\\
&+ \frac{1}{2}\tau_-^{3+\epsilon} Q_{\mu\nu}{}^{(X)}\hat{\pi}^{\mu\nu}
\end{aligned}
$$

with ${}^{(X)}\hat{\pi}$ that represents the traceless part of the deformation tensor relative to X. Because of

$$D_\sigma\tau_- = \frac{1}{\tau_-}uD_\sigma u = -\frac{|u|}{\tau_-}\partial_\sigma u$$

it follows that, if G is a solution of the Maxwell equation in the vacuum and if X is a Killing vector field, we obtain:

$$\mathbf{Div}P = (3+\epsilon)\tau_-^{2+\epsilon}\frac{|u|}{\tau_-}(-g^{\sigma\mu}\partial_\sigma u)Q_{\mu\nu}X^\nu + (3+\epsilon)\tau_-^{2+\epsilon}\frac{|u|}{\tau_-}L^\mu Q_{\mu\nu}X^\nu.$$

Moreover, if X is a null or a timelike vector,the precedent lemma guarantees us positivity of these quantities. Integrating $\mathbf{Div}P$ in the bounded region $V(u,\underline{u}) = \{p \in \mathcal{M}|u(p) \in [u_0,u], \underline{u}(p) \in [\underline{u}_0,\underline{u}]\}$ and then applying Stokes' theorem, we obtain the following:

Lemma 5.4.1. *Let P_μ be defined as in (5.4.32), and let us take X as the vector field $\bar{K}$. Then Stokes' theorem implies:*

$$\begin{aligned}\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} \tau_-^{3+\epsilon}\Phi^{-1}Q(G)(\bar{K},e_3) &+ \int_{C(u)\cap V(u,\underline{u})} \tau_-^{3+\epsilon}\Phi^{-1}Q(G)(\bar{K},e_4)\\ -\int_{\Sigma_t\cap V(u,\underline{u})} \tau_-^{3+\epsilon}Q(G)(\bar{K},T) &= -\int_{V(u,\underline{u})} [\tau_-^{3+\epsilon}(\mathbf{Div}Q(G))_\beta X^\beta +\\ +(3+\epsilon)\tau_-^{2+\epsilon}g^{\sigma\alpha}D_\sigma\tau_- Q_{\alpha\beta}\bar{K}^\beta &+ \tau_-^{3+\epsilon}\frac{1}{2}Q(G)^{\alpha\beta(\bar{K})}\hat{\pi}_{\alpha\beta}]. \end{aligned} \tag{5.4.33}$$

Choosing G as the tensor fields $\mathcal{L}_O^k\tilde{F}$, for $k=1,2$, (which are solutions of the Maxwell equations, and for this reason have divergence equal to zero), and since $\bar{K}$ is a null vector, the following inequality holds:

$$\begin{aligned}&\tilde{\mathcal{Q}}_k^O(u;[\underline{u}_0,\underline{u}]) + \underline{\tilde{\mathcal{Q}}}_k^O(\underline{u};[u_0,u])\\ &-\tilde{\mathcal{Q}}_k^O(\Sigma_t\cap V(u,\underline{u})) \leq Err^{(O)}(V(u,\underline{u})),\end{aligned} \tag{5.4.34}$$

where

$$|Err^{(O)}(V(u,\underline{u}))| \leq \frac{1}{2}\sum_{1\leq a\leq 2}\int_{V(u,\underline{u})} \tau_-^{3+\epsilon}|^{(\bar{K})}\hat{\pi}^{\alpha\beta}Q(\mathcal{L}_O^a\tilde{F})_{\alpha\beta}|, \tag{5.4.35}$$

while, if $G = \mathcal{L}_S\mathcal{L}_O^k\tilde{F}$, (because of $\mathbf{Div}G \neq 0$) we obtain that

$$\tilde{\mathcal{Q}}_k^S(u;[\underline{u}_0,\underline{u}]) + \underline{\tilde{\mathcal{Q}}}_k^S(\underline{u};[u_0,u]) - \mathcal{Q}_k^S(\Sigma_t\cap V(u,\underline{u})) \leq Err^{(S)}(V(u,\underline{u})) \tag{5.4.36}$$

(5.4.37)

where

$$\begin{aligned} |Err^{(S)}(V(u,\underline{u}))| &\leq \frac{1}{2}\int_{V(u,\underline{u})} \tau_-^{3+\epsilon}|{}^{(\bar{K})}\hat{\pi}^{\alpha\beta}Q(\mathcal{L}_S\mathcal{L}_O\tilde{F})_{\alpha\beta}| \\ &+ \int_{V(u,\underline{u})} \tau_-^{3+\epsilon}|(\mathbf{Div}Q(\mathcal{L}_S\mathcal{L}_O\tilde{F}))_\alpha\bar{K}^\alpha|. \quad (5.4.38) \end{aligned}$$

In analogy with the error estimate for the energy norms in Kerr spacetime, we divide the error into two terms, depending on the number of derivatives which appear. We define the following:

$$\mathcal{E}_1(u,\underline{u}) = \int_{V(u,\underline{u})} \tau_-^{3+\epsilon}|{}^{(\bar{K})}\hat{\pi}^{\alpha\beta}Q(\mathcal{L}_O\tilde{F})_{\alpha\beta}|, \quad (5.4.39)$$

and

$$\begin{aligned} \mathcal{E}_2(u,\underline{u}) &= \int_{V(u,\underline{u})} \tau_-^{3+\epsilon}(\mathbf{Div}Q(\mathcal{L}_S\mathcal{L}_O\tilde{F}))_\alpha\bar{K}^\alpha \\ &+ \int_{V(u,\underline{u})} \tau_-^{3+\epsilon(\bar{K})}\hat{\pi}^{\alpha\beta}Q(\mathcal{L}_O^2\tilde{F})_{\alpha\beta} \quad (5.4.40) \\ &+ \int_{V(u,\underline{u})} \tau_-^{3+\epsilon(\bar{K})}\hat{\pi}^{\alpha\beta}Q(\mathcal{L}_S\mathcal{L}_O\tilde{F})_{\alpha\beta}. \end{aligned}$$

5.5 The error estimate

In order to estimate $\mathcal{E}_1$ and $\mathcal{E}_2$, we have to control the decay of the deformation tensor relative to $\bar{K}$ and of that one relative to S. For the estimate of $\mathcal{E}_1$, we have to study $\int_{V(u,\underline{u})} \tau_-^{3+\epsilon}Q(L_O\tilde{F})^{\mu\nu(\bar{K})}\pi_{\mu\nu}$.

Remark 5.5.1. *Since*

$$Q^{\mu\nu} = g^{\mu\tau}g^{\nu\sigma}Q_{\tau\sigma},$$

from the structure of the null frame in which we are writing any quantities, we can easily verify the following relations hold:

$$\begin{aligned} Q^{ab} &= Q_{ab} \\ Q^{33} &= \frac{1}{4}Q_{44} \\ Q^{44} &= \frac{1}{4}Q_{44} \\ Q^{34} &= \frac{1}{4}Q_{34}. \end{aligned}$$

With these in mind, recalling the asymptotic behavior of null components on ${}^{(\bar K)}\hat\pi$, (see proposition 5.2.2) and with the help of (5.3.26), we can write the error term $\mathcal{E}_1$ like an integral of the following type:

$$
\begin{aligned}
\mathcal{E}_1 \;&\leq\; c\int_{V(u,\underline{u})} \tau_-^{3+\epsilon}\Big[\frac{\log r}{r} t(\sigma^2(\hat{\mathcal{L}}_O F)+\alpha(\hat{\mathcal{L}}_O F)\underline{\alpha}(\hat{\mathcal{L}}_O F)+\rho^2(\hat{\mathcal{L}}_O F))\\
&+\; \frac{\tau_-^2}{r^2}|\underline{\alpha}(\hat{\mathcal{L}}_O F)|^2+\frac{\tau_+^2}{r^2}|\alpha(\hat{\mathcal{L}}_O F)|^2\Big]
\end{aligned}
$$

Now we want to show that all the quantities in the integral of r.h.s. are integrable. Observing that

$$\alpha\underline{\alpha}\leq|\alpha|^2, \qquad \alpha\underline{\alpha}\leq|\underline{\alpha}|^2,$$

it follows we have to control the following integral terms:

$$
\begin{aligned}
&\int_{V(u,\underline{u})} \tau_-^{3+\epsilon}\frac{\log r}{r} t(\sigma(\hat{\mathcal{L}}_O F)^2+\rho(\hat{\mathcal{L}}_O F)^2)\\
&\int_{V(u,\underline{u})} \tau_-^{3+\epsilon}\frac{\log r}{r} t|\alpha(\hat{\mathcal{L}}_O F)|^2\\
&\int_{V(u,\underline{u})} \tau_-^{3+\epsilon}\frac{\tau_-^2}{r^2}|\underline{\alpha}(\hat{\mathcal{L}}_O F)|^2\\
&\int_{V(u,\underline{u})} \tau_-^{3+\epsilon}\frac{\tau_+^2}{r^2}|\alpha(\hat{\mathcal{L}}_O F)|^2.
\end{aligned}
$$

The first of it is controlled in the following way:

$$
\begin{aligned}
&\int_{V(u,\underline{u})} \tau_-^{3+\epsilon}\frac{\log r}{r} t(\sigma^2+\rho^2)(\hat{\mathcal{L}}_O F)\\
&\leq c\int_{\underline{u}_0}^{\underline{u}} d\underline{u}'\int_{\underline{(C)}(\underline{u}';[u_0,u])} \tau_-^{3+\epsilon}\frac{\log r}{r^3} t\tau_+^2(\rho^2(\hat{\mathcal{L}}_O F)+\sigma^2(\hat{\mathcal{L}}_O F))\\
&c\int_{\underline{u}_0}^{\underline{u}} \frac{d\underline{u}'}{ur}\sup_{V(u,\underline{u})}\left(\frac{\tau_-}{r}\log r\right)\sup_{(} V(u,\underline{u}))\int_{\underline{C}(\underline{u}';[u_0,u])} \tau_-^{3+\epsilon}\tau_+^2(\rho(\hat{\mathcal{L}}_O^2)+\sigma(\hat{\mathcal{L}}_O)^2)\\
&\leq \frac{c}{r_0}\tilde{\mathcal{Q}}_1.
\end{aligned}
$$

The other terms are bounded by QQ_1 too. We see into details only the last integral, which has the highest weight factor τ_+ (then the worst behavior):

$$\begin{aligned}&\int_{V(u,\underline{u})} \tau_-^{3+\epsilon}\frac{\tau_+^2}{r^2}|\alpha(\hat{\mathcal{L}}_O F)|^2 \\ &\leq cint_{\underline{u}_0}^{\underline{u}}\frac{d\underline{u}'}{r^2}\int_{\underline{C}(\underline{u}';[u_0,u])}\tau_-^{3+\epsilon}\tau_+^2|\alpha(\hat{\mathcal{L}}_O F)|^2 \\ &\leq \frac{c}{r_0}\tilde{\mathcal{Q}}_1\end{aligned}$$

Then we have shown $\mathcal{E}_1$ is bounded in terms of a constant c and the inverse of r_0. As far as $\mathcal{E}_2$ is concerned we have to estimate the following quantities:

$$\begin{aligned}&\int_{V(u,\underline{u})} \tau_-^{3+\epsilon}(\mathbf{Div}Q(\mathcal{L}_S\mathcal{L}_O\tilde{F}))_\alpha\bar{K}^\alpha \\ &\int_{V(u,\underline{u})} \tau_-^{3+\epsilon(\bar{K})}\hat{\pi}^{\alpha\beta}Q(\mathcal{L}_O^2\tilde{F})_{\alpha\beta} \\ &\int_{V(u,\underline{u})} \tau_-^{3+\epsilon(\bar{K})}\hat{\pi}^{\alpha\beta}Q(\mathcal{L}_S\mathcal{L}_O\tilde{F})_{\alpha\beta}.\end{aligned} \tag{5.5.41}$$

The second term in (5.5.41) has the following form:

$$\begin{aligned}\mathcal{E}_1 \;\leq\; & c\int_{V(u,\underline{u})}\tau_-^{3+\epsilon}\Big[\frac{\log r}{r}t(\sigma^2(\hat{\mathcal{L}}_O^2F)+\alpha(\hat{\mathcal{L}}_O^2F)\underline{\alpha}(\hat{\mathcal{L}}_O^2F)+\rho^2(\hat{\mathcal{L}}_O^2F)) \\ + \;& \frac{\tau_-^2}{r^2}|\underline{\alpha}(\hat{\mathcal{L}}_O^2F)|^2+\frac{\tau_+^2}{r^2}|\alpha(\hat{\mathcal{L}}_O^2F)|^2\Big]\end{aligned}$$

that is exact the same as the error term $\mathcal{E}_1$ with the difference that there is $\hat{\mathcal{L}}_O^2$ instead of $\hat{\mathcal{L}}_O$. Then it can be estimated in the same way, changing $\tilde{\mathcal{Q}}_1$ with $\tilde{\mathcal{Q}}_2$. As far as the third integral is concerned, it is controlled exactly in the same way as the precedent one, and it results to be bounded by $\tilde{\mathcal{Q}}_2$ too. In order to estimate the first integral in (5.5.41), see [8].

5.6 The Peeling Theorem for $\mathcal{L}_{T_0}F$ and for F

In this section we are finding the asymptotic behavior along infinity null directions of the null components of electromagnetic tensor field F, which

propagates itself in the external Schwarzschild spacetime (seen as a background spacetime). We will see that they are in accord with the peeling results, provided to do some required on the initial conditions which the tensor field $\mathcal{L}_{T_0}F$ has to satisfy. The techniques that we are employing essentially are of an analytic nature and they are based on type Sobolev estimates.
Now we are able to prove null components of the tensor field $mathcalL_{T_0}F$ satisfy the Peeling theorem:

Theorem 5.6.1. *Let F a regular solution of the vacuum Maxwell equations, that propagate itself in the Schwarzschild spacetime (thought as background space). Then for every time t the components of the tensor field $\tilde{F} = L_{T_0}F$ satisfy the following asymptotic estimates:*

$$\begin{aligned} r^{\frac{5}{2}}\tau_-^{\frac{3}{2}+\frac{\epsilon}{2}}|\alpha(\tilde{F})| &\leq C\{[\mathcal{Q}_0^O(u;[\underline{u}_0,\underline{u}]) + \mathcal{Q}_0^S(u;[\underline{u}_0,\underline{u}])]^{\frac{1}{2}} \\ &+ [\mathcal{Q}_0^O(\Sigma_t) + \mathcal{Q}_0^S(\Sigma_t)]^{\frac{1}{2}}\} \qquad (5.6.42) \\ r^2(|\rho(\tilde{F})|,|\sigma(\tilde{F})|) &\leq C\sup_{\Sigma_t} r^2|\bar{\rho}|,|\bar{\sigma}| + C\tau_-^{-(2+\frac{\epsilon}{2})} \\ &\cdot \{[\underline{\mathcal{Q}}_0^O(\underline{u};[u_0,u])]^{\frac{1}{2}} + [\mathcal{Q}_0^O(\Sigma_t) + \mathcal{Q}_0^S(\Sigma_t)]^{\frac{1}{2}}\} \qquad (5.6.43) \\ r\tau_-^{3+\frac{\epsilon}{2}}|(\underline{\alpha}(\tilde{F})| &\leq C\{[\underline{\mathcal{Q}}_0^O(\underline{u};[u_0,u]) + \underline{\mathcal{Q}}_0^S(\underline{u};[u_0,u])]^{\frac{1}{2}} \\ &+ [\mathcal{Q}_0^O(\Sigma_t) + \mathcal{Q}_0^S(\Sigma_t)]^{\frac{1}{2}}\}. \qquad (5.6.44) \end{aligned}$$

Sketch of the proof: We are going to give just an idea of the theorem' s proof, and we invite the reader to see [4] for any details. The substantial difference with that article is due to the fact that, because of the reasons specified in section 2, we will work with the null components of $\tilde{F} = \mathcal{L}_{T_0}F$. We start with the component α. First we note the following relations hold:

$$\begin{aligned} \alpha(\mathcal{L}_{\Omega_{ij}}\tilde{F}) &= \mathcal{L}_{\Omega_{ij}}\alpha(\tilde{F}) \qquad (5.6.45) \\ |\mathcal{L}_O\alpha|^2 &= r^2|\nabla\!\!\!/\alpha|^2 + |\alpha|^2, \qquad (5.6.46) \end{aligned}$$

where $|\mathcal{L}_O\alpha|^2 = \Sigma_{i<j}|\mathcal{L}_{\Omega_{ij}}\alpha|^2$. Posing in 1.3.11 $F = r\tau_-^{\frac{1}{2}(3+\epsilon)}\alpha(\tilde{F})$, it follows

we have to control the following integrals

$$\int_{C(u;[\underline{u}_0,\underline{u}])} r^2\tau_-^{3+\epsilon}|\alpha(\tilde{F})|^2 \quad , \quad \int C(u;[\underline{u}_0,\underline{u}]) r^4\tau_-^{3+\epsilon}|\nabla\!\!\!/\alpha(\tilde{F})|^2$$
$$\int_{C(u;[\underline{u}_0,\underline{u}])} r^6\tau_-^{3+\epsilon}|\nabla\!\!\!/^2\alpha(\tilde{F})|^2 \quad , \quad \int_{C(u;[\underline{u}_0,\underline{u}])} r^4\tau_-^{3+\epsilon}|\mathbf{D}\!\!\!/_4\alpha(\tilde{F})|^2$$
$$\int_{C(u;[\underline{u}_0,\underline{u}])} r^6\tau_-^{3+\epsilon}|\nabla\!\!\!/\mathbf{D}\!\!\!/_4\alpha(\tilde{F})|^2 \quad .$$

Thanks to equations 5.6.46, the following estimates hold:

$$\begin{aligned}
\int_{C(u;[\underline{u}_0,\underline{u}])} r^2\tau_-^{3+\epsilon}|\alpha|^2 &\leq \Phi^{-1}(r(u,\underline{u}_0))\int_{C(u;[\underline{u}_0,\underline{u}])} Q(\mathcal{L}_O\tilde{F})(\bar{K},e_4)\\
\int_{C(u;[\underline{u}_0,\underline{u}])} r^4\tau_-^{3+\epsilon}|\nabla\!\!\!/\alpha|^2 &\leq \Phi^{-1}(r(u,\underline{u}_0))\int_{C(u;[\underline{u}_0,\underline{u}])} Q(\mathcal{L}_O\tilde{F})(\bar{K},e_4)\\
\int_{C(u;[\underline{u}_0,\underline{u}])} r^6\tau_-3+\epsilon|\nabla\!\!\!/^2\alpha|^2 &\leq \Phi^{-1}(r(u,\underline{u}_0))\int_{C(u:[\underline{u}_0,\underline{u}])} Q(\mathcal{L}_O^2\tilde{F})(\bar{K},e_4).
\end{aligned}$$

In order to estimate the forth integral we use the vector field S, as:

$$r^2|\mathbf{D}\!\!\!/_4\alpha|^2 \leq c(\Phi^{-2}|\mathbf{D}\!\!\!/_S\alpha|^2 + \tau_-^2|\mathbf{D}\!\!\!/_3\alpha|^2),$$

so for the first quantity of r.h.s. we obtain

$$\begin{aligned}
&\int_{C(u;[\underline{u}_0,\underline{u}])} \Phi^{-2}r^2\tau_-^{3+\epsilon}|\mathbf{D}\!\!\!/_S\alpha|^2 \leq c(\Phi^-(3+\delta)(r(u,\underline{u}_0))\\
&\int_{C(u;[\underline{u}_0,\underline{u}])} (Q(\mathcal{L}_S\mathcal{L}_O\tilde{F})(\bar{K},e_4) + Q(\mathcal{L}_O\tilde{F})(\bar{K},e_4)))
\end{aligned}$$

for any $\delta > 0$. To estimate the other involved quantity, we use Maxwell equation

$$\nabla\!\!\!/_3\alpha - (\partial_r\Phi + \frac{\Phi}{r})\alpha - \nabla\!\!\!/\rho - {}^*\nabla\!\!\!/\sigma = 0$$

(we remember that $\tilde{F}$ is a solution of the vacuum Maxwell equations too), to find

$$\begin{aligned}
\int_{C(u;[\underline{u}_0,\underline{u}])} r^2\tau_-^{5+\epsilon}|Dh_3\alpha|^2 &\leq c(\Phi^{-1}(r(u,\underline{u}_0))\int_{C(u;[\underline{u}_0,\underline{u}])} Q(\mathcal{L}_O\tilde{F})(\bar{k},e_4)\\
&+ \int_{C(u;[\underline{u}_0,\underline{u}])} [r^2\tau_-^{5+\epsilon}(|\nabla\!\!\!/\rho|^2 + |\nabla\!\!\!/\sigma|^2)].
\end{aligned}$$

It means that to control the behavior of α, we need to control $|\nabla\!\!\!/\rho|, |\nabla\!\!\!/\sigma|$, but they are proved to be bounded from

$$c\Phi^{-1}(r(u,\underline{u}_0))r^{-2}\tau_-^{-(5+\epsilon)}\int_{C(u;[\underline{u}_0,\underline{u}])} Q(\mathcal{L}_O\tilde{F})(\bar{K},e_4).$$

Collecting all these estimates we find 5.6.42. As far as the component $\underline{\alpha}$ is concerned, we substitute in 1.3.14 $U=\tau_-^{\frac{1}{2}(5+\epsilon)}\underline{\alpha}$ and we find that we have to control

$$\Phi^{-2}(r(u,\underline{u}))\int C(\underline{u};[u_0,u])\tau_-^{5+\epsilon}|\underline{\alpha}|^2 \quad , \quad \int C(\underline{u};[u_0,u])\tau_-^{5+\epsilon}r^2|\nabla\!\!\!/\underline{\alpha}|^2$$
$$\int C(\underline{u};[u_0,u])\tau_-^{7+\epsilon}|Dh_3\underline{\alpha}|^2 \quad , \quad \int C(\underline{u};[u_0,u])\tau_-^{5+\epsilon}r^4|\nabla\!\!\!/^2\underline{\alpha}|^2$$
$$\int C(\underline{u};[u_0,u])\tau_-^{7+\epsilon}r^2|\nabla\!\!\!/ D\!\!\!/_3\underline{\alpha}|^2.$$

Using the same techniques employed for α (with the obvious modifies), we finally obtain 5.6.44.
The components ρ and σ behave at the infinity in the same way, so in the following we are going to find only the decay of ρ. First we note

$$|\rho(\mathcal{L}_O\tilde{F})|^2 = r^2|\nabla\!\!\!/\rho|^2,$$

therefore the $\mathcal{Q}$-norms do not suffice to control ρ. But from Poincaré inequality, it follows

$$\int_{S(u,\underline{u})} r^2|\rho-\bar{\rho}|^2 \leq s\int_{S(u,\underline{u})} r^2|\rho(\mathcal{L}_O\tilde{F})|^2$$

so we expect we are able to estimate $\rho-\bar{\rho}$ and then we will use

$$|\rho| \leq |\rho-\bar{\rho}| + |\bar{\rho}|.$$

Substituing in 1.3.12 F with $r\tau_-^{\frac{1}{2}(3+\epsilon)}(\rho-\bar{\rho})$, and then using the equation

$$D_3\rho - 2\frac{\Phi}{r}\rho + d\!\!\!/\mathrm{iv}\,\underline{\alpha} = 0,$$

it is shown that

$$r^2\tau_-^{2+\epsilon}|\rho-\bar{\rho}(\tilde{F})| \leq c\Phi^{-1}(r(u,\underline{u})(\underline{\mathcal{Q}}_O^O(u,\underline{u}))^{\frac{1}{2}}.$$

Finally, as far $\bar{\rho}$ is concerned, it is easy to show it is bounded from

$$r^{-2}(u,\underline{u})\sup_{\Sigma_{t=0}}|r^2\bar{\rho}|^2$$

and this concludes the proof.

The decay we have found for $\alpha(\tilde{F})$ is not the one expected by the peeling theorem,but we are able to modify it by considering the Maxwell equation relative to the evolution of α on the null incoming cones and multiplying it times r:

$$\underbrace{r\not{D}_3\alpha-\alpha}_{2\partial_u(r\alpha)}=r(\not\nabla\rho+{}^*\not\nabla\sigma).$$

Integrating it on a finite portion of a null incoming cone, and considering the norms of the quantities, we obtain:

$$|r\alpha|(u)\leq|r_0\alpha(u_0)|+\int_{\underline{C}(\underline{u};[u_0,u])}r(|\not\nabla\rho|+|\not\nabla\sigma|),$$

This equation, multiplied for r^2, has the following interpretation: if the initial data of $\alpha(\tilde{F})$ decay along the spatial infinity like $r^{-(4+\epsilon)}$, then $\alpha(\tilde{F})$ has the asymptotic behavior prescribed from the Peeling theorem, that is the one expected cause of the asymptotic simplicity of the Schwarzschild specetime. Now,to prove that the components of F satisfy the peeling decay, decay, first of all let us observe the following relations hold:

Remark 5.6.1. *It easily follows from the symmetry properties of the Schwarzschild metric that the components of $\tilde{F}$ satisfy the following equalities:*

$$\begin{aligned}
\alpha(\mathcal{L}_{T_0}F) &= \alpha(\mathcal{L}_{T_0}F_{(t)})=\mathcal{L}_{T_0}\alpha(F_{(t)})=\partial_t\alpha(F_{(t)})\\
\rho(\mathcal{L}_{T_0}F) &= \rho(\mathcal{L}_{T_0}F_{(t)})=\mathcal{L}_{T_0}\rho(F_{(t)})=\partial_t\rho(F_{(t)})\\
\sigma(\mathcal{L}_{T_0}F) &= \sigma(\mathcal{L}_{T_0}F_{(t)})=\mathcal{L}_{T_0}\sigma(F_{(t)})=\partial_t\sigma(F_{(t)})\\
\underline{\alpha}(\mathcal{L}_{T_0}F) &= \underline{\alpha}(\mathcal{L}_{T_0}F_{(t)})=\mathcal{L}_{T_0}\underline{\alpha}(F_{(t)})=\partial_t\underline{\alpha}(F_{(t)}).
\end{aligned}$$

Therefore integrating in the time the components of $\tilde{F}$, in view of the fact that it is like integrate on u,it follows that the components of the non stationary part of F have in r the decay expected from the peeling theorem,While in u they decay more slowly with respect to the components of $\tilde{F}$ of a factor τ_-.

Hence on the initial hypersurface Σ_t (dove $\tau_-\sim r$),the components of $F_{(t)}$ behave as $r^{-(3+\frac{\epsilon}{2})}$, while there is nothing that describes the order of decay of the static part of F, and so we have no limitations about the presence of a dipole in the field F.

5.6.1 Maxwell tensor in Minkowski spacetime

The same work was done using as a background metric the flat one, in coordinates $(u, \underline{u}, \theta, \phi)$, such that:

$$u = t - r$$

retarded parameter,

$$\underline{u} = t + r$$

advanced parameter. In these coordinates the Minkowski metric assumes the form:

$$ds^2 = -dud\underline{u} + \frac{1}{4}(\underline{u} - u)^2(d\theta^2 + \sin^2\theta d\phi^2).$$

The connection coefficients of the metric are all null but χ e $\underline{\chi}$, which have only the trace part, in particular:

$$tr\chi = -tr\underline{\chi} = \frac{2}{r}.$$

The Maxwell equations in Minkowski spacetime, projected on the null outgoing cones, assume the form:

$$\not{D}_4\underline{\alpha} + \frac{1}{r}\underline{\alpha} + \not{\nabla}\rho - {}^*\not{\nabla}\sigma = 0 \quad (5.6.47)$$

$$\not{D}_3\alpha - \frac{1}{r}\alpha - \not{\nabla}\rho - {}^*\not{\nabla}\sigma = 0 \quad (5.6.48)$$

$$D_4\sigma + \frac{2}{r}\sigma + \mathrm{c\not{u}rl}\,\alpha = 0 \quad (5.6.49)$$

$$D_4\rho + \frac{2}{r} - \mathrm{d\not{i}v}\,\alpha = 0 \quad (5.6.50)$$

$$D_3\sigma - \frac{2}{r}\sigma + \mathrm{c\not{u}rl}\,\underline{\alpha} = 0 \quad (5.6.51)$$

$$D_3\rho - \frac{2}{r}\rho + \mathrm{d\not{i}v}\,\underline{\alpha} = 0. \quad (5.6.52)$$

The order of decay we have found for the components of F are the same we have found in this work. The substantial difference, which has considerably simplified the problem, is that in the Minkowski spacetime the energy norms that we use are notably simplified. In fact $\bar{K}$ is a Killing vector field,therefore:

$$\begin{aligned} Err^{(O)} &= \frac{1}{2}\sum_{1\le a\le k+2}\int_{V(u,\underline{u})}\tau_-^{3+\epsilon}|^{(\bar{K})}\hat{\pi}^{\alpha\beta}Q(\mathcal{L}_O^a\tilde{F})_{\alpha\beta}| \\ &= 0. \end{aligned}$$

Moreover Lie derivative and covariant derivative are ordinary derivatives then, if F satisfies the Maxwell equations, also $\mathcal{L}_S\mathcal{L}_O^aF$ satisfies them:

$$\begin{aligned} D^\mu(\mathcal{L}_S\mathcal{L}_O^aF)_{\mu\nu} &= \partial^\mu\partial_s\partial_O^aF \\ = \partial_S\partial_O^aD^\mu F &= 0. \end{aligned}$$

Therefore $Err^{(S)}$ is zero too.

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