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Refined Gelfand models for some finite complex reflection groups

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Contents

Introduction

A Gelfand model of a finite group G is a G-module affording each irreducible representation of G exactly once. The study of Gelfand models found its roots in [4] and awoke a wide interest in the case of reflection groups and other related groups (see, e.g.,[1, 2, 3, 12, 14, 15, 16]).

In the present work we will provide a refinement for a Gelfand model due to F. Caselli (see [6]). Such model applies to all *involutory* reflection groups $G(r,p,n)$ and to all their quotients modulo a scalar subgroup. Let us briefly introduce to this topic.

Given a vector space V of finite dimension over $\mathbb{C}$, we say that $r \in \mathrm{GL}(V)$ is a reflection if it has finite order and fixes a hyperplane of V pointwise. A finite complex group $G < \mathrm{GL}(V)$ is a reflection group if it is generated by reflections. Irreducible finite complex reflection groups were completely classified in the fifties [19]. They consist of:

- an infinite family of groups $G(r,p,n)$ depending on the three parameters r, p, n, where $p\,|r$;
- 34 more sporadic groups.

We may mention that the infinite families of irreducible Coxeter groups are of the form $G(r,p,n)$: $S_n = G(1,1,n)$, $B_n = G(2,1,n)$, $D_n = G(2,2,n)$, $I_2(r) = G(r,r,2)$. Whenever $p = 1$, we have the wreath product $G(r,1,n) = C_r \wr S_n$, which we will denote with $G(r,n)$.

We will deal with the groups $G(r,p,n)$, and, eventually, with the bigger family of groups $G(r,p,q,n)$ (see Section 2.1) which is a generalization of them. *Projective reflection groups*, first introduced by F. Caselli in [5], can be roughly described as quotients - modulo a scalar group - of finite complex reflection groups. If we quotient a group $G(r,p,n)$ modulo the cyclic scalar subgroup C_q, we find a new group $G(r,p,q,n)$, so that in this notation we have $G(r,p,n) = G(r,p,1,n)$. We define the *dual group* $G(r,p,q,n)^*$ as the group $G(r,q,p,n)$ obtained by simply exchanging the parameters p and q. It turns out that many objects related to the algebraic structure of a projective reflection group G can be naturally described by means of the combinatorics of its dual G^* (see [5, 6]). For example, its representations.

A finite subgroup of $\mathrm{GL}(n,\mathbb{C})$ is involutory if the number of its absolute involutions, i.e. elements g such that $g\bar{g} = 1$, coincides with the dimension of its Gelfand model. A group $G(r,p,n)$ turns out to be involutory if and only if $\mathrm{GCD}(p,n) = 1, 2$ (Theorem 2.4.5).

The model (M, ϱ) provided in [6] works for every group $G(r,p,q,n)$ with $\mathrm{GCD}(p,n) = 1, 2$ and looks like this:

- M is a formal vector space spanned by all absolute involutions $I(r,p,q,n)^*$ of the dual group $G(r,p,q,n)^*$:

$$M \stackrel{\text{def}}{=} \bigoplus_{v \in I(r,p,q,n)^*} \mathbb{C}C_v;$$

- $\varrho : G(r,p,q,n) \to \mathrm{GL}(M)$ works, basically, as an *absolute conjugation* of $G(r,p,q,n)$ on the elements indexing the basis of M:

$$\varrho(g)(C_v) \stackrel{\text{def}}{=} \psi(g,v) C_{|g|v|g|^{-1}}, \tag{0}$$

  $\psi(g,v)$ being a scalar and $|g|$ being the natural projection of g in the symmetric group S_n.

Let us now give an account of the new result appearing in this thesis. Our main goal is to refine the above model. If $g, h \in G(r,p,q,n)^*$ we say that g and h are S_n-*conjugate* if there exists $\sigma \in S_n$ such that $g = \sigma h \sigma^{-1}$, and we call S_n-*conjugacy classes* the corresponding equivalence classes. If c is a S_n-conjugacy class of absolute involutions in $I(r,p,q,n)^*$ we denote by $M(c)$ the subspace of M spanned by the basis elements C_v indexed by the absolute involutions v belonging to the class c. Then it is clear from () that we have a decomposition

$$M = \bigoplus_c M(c) \quad \text{as } G(r,p,q,n)\text{-modules},$$

where the sum runs through all S_n-conjugacy classes of absolute involutions in $I(r,p,q,n)^*$. It is natural to ask if we can describe the irreducible decomposition of the submodules $M(c)$, and our main goal is to answer to this question. The final description of the irreducible decomposition of the modules $M(c)$ has a rather elegant formulation due to its compatibility with the projective Robinson-Schensted correspondence. Namely, the irreducible subrepresentations of $M(c)$ are indexed by the shapes which are obtained when performing this correspondence to the elements in c. The special case of this result for the symmetric group $S_n = G(1,1,n)$ was established in [12]. Our task will be to furnish a much more general version of it, concerning all those groups for which the model [6] was constructed. We will do it step by step, from the easiest case up to the general one.

Chapter 0 contains some basic notation, as well as an introduction about finite complex reflection groups, their representations, and an account of the generalized Robinson-Schensted correspondence.

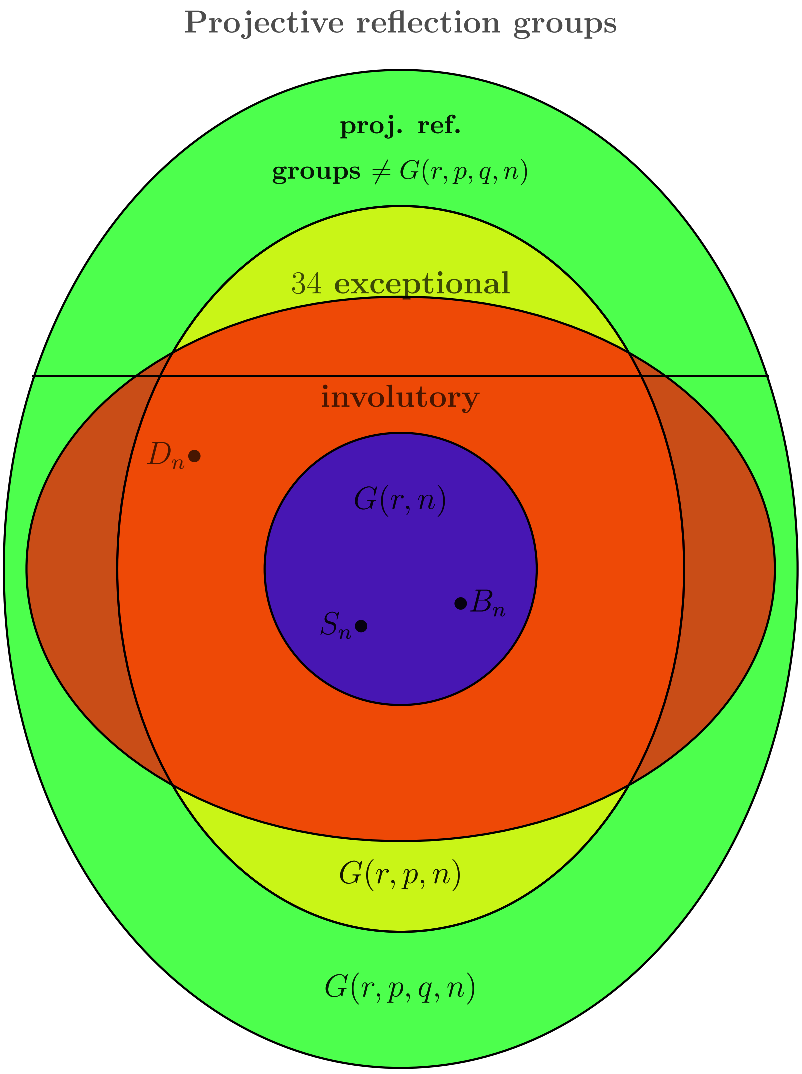
After the exposition of the necessary background in Chapter 0, we immediately turn to state and prove our main results about the decomposition of the model. Since the case of S_n was afforded in [12], the simplest new case to study is that of wreath products $G(r,n)$. This is done in Chapter 1, first for the special case of B_n, then for all groups $G(r,n)$. The description of Caselli's model for $G(r,n)$ is considerably more linear than in the more general setting of $G(r,p,n)$. This is due to the fact that $G(r,n)$ coincides with its dual.

Chapter 2 is devoted to the introduction of projective reflection groups $G(r,p,q,n)$ and their representations. After characterizing the involutory groups $G(r,p,q,n)$, we present the model built in [6] in its full generality.

In Chapter 3, the irreducible decomposition of the model for type D is afforded. Notice that, when n is even, $\mathrm{GCD}(p,n) = 2$, thus the group D_n furnishes a very good example of the main difficulties one meets when $\mathrm{GCD}(p,n)$ is not 1 anymore (as it was the case for $G(r,n)$). The decomposition of the submodules $M(c)$ in this wider setting is much more subtle. Indeed, when $\mathrm{GCD}(p,n) = 2$, the Gelfand model M splits first of all as the direct sum of two distinguished modules: the symmetric submodule M_{Sym}, which is spanned by the elements C_v indexed by symmetric absolute involutions, and the antisymmetric submodule M_{Asym}, which is defined similarly. This decomposition is compatible with the one described above: every submodule $M(c)$ is contained either in the symmetric or in the antisymmetric submodule. The existence of the antisymmetric submodule and of the submodules $M(c)$ contained therein will reflect in a very precise way the existence of split representations for these groups.

Chapter 4 treats the general case $G(r,p,n)$ with $\mathrm{GCD}(p,n) = 1, 2$. The study of the irreducible decomposition of $M(c)$, when c is made up of antisymmetric elements, requires a particular machinery developed in Sections 4.2-4.5. Such tools were not needed in the case of wreath products $G(r,n)$, where the antisymmetric submodule vanishes and so the Gelfand model coincides with its symmetric submodule. Once our main result is achieved for all groups $G(r,p,n)$ satisfying $\mathrm{GCD}(p,n) = 1, 2$, our arguments are generalized furtherly to the quotients $G(r,p,q,n)$.

Finally, we would like to highlight that the results appearing in this work were obtained in collaboration with F. Caselli.



Classical irreducible finite complex reflection groups are represented in the yellow set, involutory reflection groups in the red set, projective reflection groups in the green set. The lower part of the green set contains all projective reflection groups of the form $G(r,p,q,n)$. Thus, the upper part of the green set is made up of all the other projective reflection groups: namely, those obtained as quotients, modulo a scalar subgroup, of the 34 exceptional groups and of all classical non-irreducible finite complex reflection groups.

Chapter 0

Notation and prerequisites

In this chapter we collect some well-known results that will be essential to our exposition. First of all, in Section 0.1, we set some basic notation and we describe the sets Fer and $\mathcal{ST}$ of Ferrers diagrams and Young tableaux, which will be met continuously in what follows. In Section 0.2 we outline some general results concerning all finite complex reflection groups. Section 0.3 is entirely devoted to the description of the groups $G(r,p,n)$ and to the notation used for their elements. In Section 0.4 we focus on the groups $G(r,n)$, to parametrize their conjugacy classes and their irreducible representations. Finally, in Section 0.5, we introduce the generalized Robinson-Schensted correspondence for $G(r,p,n)$. This is a tool of crucial importance for our aims. In fact, the Gelfand model given in [6] will be decomposed in a way (see Theorem 1.1.3) which is well-behaved with respect to such correspondence.

0.1 Basic notation

We let $\mathbb{Z}$ be the set of integer numbers and $\mathbb{N}$ be the set of nonnegative integer numbers. For $a, b \in \mathbb{Z}$, with $a \leq b$ we let $[a,b] = \{a, a+1, \ldots, b\}$ and, for $n \in \mathbb{N}, n \neq 0$, we let $[n] \stackrel{\text{def}}{=} [1,n]$. For $r \in \mathbb{N}$, $r > 0$, we let $\mathbb{Z}_r \stackrel{\text{def}}{=} \mathbb{Z}/r\mathbb{Z}$. We denote by ζ_r the primitive r-th root of unity $\zeta_r \stackrel{\text{def}}{=} e^{\frac{2\pi i}{r}}$.

Definition. Let $n \in \mathbb{N}$, $l \in \mathbb{N}, l \neq 0$. A *partition* of n is a l-tuple λ of the form $\lambda = (\lambda_1, \ldots, \lambda_l)$, where $\lambda_1 \geq \ldots \geq \lambda_l > 0$ and each $\lambda_i \in \mathbb{N}$, such that $\sum_{i=1}^{l} \lambda_i = n$. l is called the *length* of λ and it is denoted by $\ell(\lambda)$; n is the *size* of λ and is denoted by $|\lambda|$.

A partition λ can be represented by the *Ferrers diagram of shape* λ: it is a collection of boxes, arranged in left-justified rows, with λ_i boxes in row i.

Since a Ferrers diagram represents a partition, it is clear that the number of boxes on its rows must be not increasing. Given a partition λ, we denote by λ' its conjugate partition, i.e. the partition obtained by λ exchanging rows and columns of its Ferrer diagram.

Example 0.1.1. Consider the partition $\lambda = (5, 3, 2)$ of size $|\lambda| = 10$. Its length is $\ell(\lambda) = 3$ and λ can be visualized as the Ferrers diagram $\ydiagram{5,3,2}$. Its conjugate λ' is represented by $\ydiagram{3,3,2,1,1}$.

Let now consider a r-tuple $\lambda = (\lambda^{(0)}, \dots, \lambda^{(r-1)})$ of partitions such that $\sum |\lambda^{(i)}| = n$. Notice that we use a subscript to denote the row of a single partition and a superscript in round brackets to denote the index of a partition inside a r-tuple. Thus, the h^{th} row of the j^{th} partition of a r-tuple of partitions λ is denoted $\lambda_h^{(j-1)}$.

Occasionally, each partition of a r-tuple may be denoted with a different greek letter. For example, when dealing with pairs of partitions, it will be convenient to refer to them as to objects of the form (λ, μ).

In its turn, an r-tuple $\lambda = (\lambda^{(0)}, \dots, \lambda^{(r-1)})$ can be represented by means of a r-tuple of Ferrers diagrams. We denote by $\mathrm{Fer}(r, n)$ the set of r-tuples of Ferrers diagrams λ such that $\sum |\lambda^{(i)}| = n$. In this case, n is called the *total size* of λ.

Example 0.1.2. If $\lambda = \big((3, 2), (4, 2, 2), (5, 1, 1)\big)$, the relevant notation in Ferrers diagrams is

$$\left(\ydiagram{3,2}\,,\ydiagram{4,2,2}\,,\ydiagram{5,1,1}\right) \in \mathrm{Fer}(3, 20)$$

Let us now introduce some notation concerning standard Young tableaux. If $\mu \in \mathrm{Fer}(r, n)$ we denote by $\mathcal{ST}_\mu$ the set of all possible fillings of the boxes in μ with all the integers from 1 to n appearing once, in such way that rows are increasing from left to right and columns are increasing from top to bottom in every single Ferrers diagram of μ. We also say that $\mathcal{ST}_\mu$ is the set of *standard multitableaux* P of *shape* $\mathrm{Sh}(P) = \mu$. Moreover we let $\mathcal{ST}(r, n) \stackrel{\text{def}}{=} \cup_{\mu \in \mathrm{Fer}(r,n)} \mathcal{ST}_\mu$.

Example 0.1.3. The multitableau

$$\left(\begin{ytableau} 9 & 3 & 4 \\ 4 & 2 \\ 8 \end{ytableau}\,,\begin{ytableau} 5 & 1 \\ 6 \end{ytableau}\right)$$

is not standard, whereas the following multitableau P is:

$$P = (P_0, P_1) = \left(\begin{ytableau} 1 & 2 & 5 \\ 3 \end{ytableau}\,,\begin{ytableau} 4 & 6 \\ 7 & 8 \end{ytableau}\right) \in \mathcal{ST}_{\left(\ydiagram{3,1},\ydiagram{2,2}\right)} \subset \mathcal{ST}(2, 8);$$

the pair $\left(\ydiagram{3,1}, \ydiagram{2,2}\right) \in \mathrm{Fer}(2,8)$ is the shape $\mathrm{Sh}(P)$ of P.

We conclude this section with a few more definitions.

Definition. Let $\lambda = (\lambda^{(0)}, \ldots, \lambda^{(r-1)}) \in \mathrm{Fer}(r,n)$. The color of λ is $z(\lambda) \stackrel{\text{def}}{=} \sum_i i|\lambda^{(i)}|$.

If $p|r$ we let $\mathrm{Fer}(r,p,n) \stackrel{\text{def}}{=} \{\lambda \in \mathrm{Fer}(r,n) : z(\lambda) \equiv 0 \mod p\}$. Finally, as above, we can associate to $\mathrm{Fer}(r,p,n)$ a set of multitableaux:

$$\mathcal{ST}(r,p,n) \stackrel{\text{def}}{=} \cup_{\mu \in \mathrm{Fer}(r,p,n)} \mathcal{ST}_\mu.$$

Example 0.1.4. The element of $\mathrm{Fer}(2,6)$ given by $\left(\ydiagram{2,1}, \ydiagram{3}\right)$ does not belong to $\mathrm{Fer}(2,2,6)$, whereas $\left(\ydiagram{2,1,1}, \ydiagram{2}\right) \in \mathrm{Fer}(2,2,6)$. Here is a possible filling of it, belonging to $\mathcal{ST}(2,2,6)$:

$$\left(\begin{ytableau} 2 & 5 \\ 3 \\ 6 \end{ytableau}, \begin{ytableau} 1 & 4 \end{ytableau}\right).$$

0.2 Finite complex reflection groups

All through this section, V stands for a finite-dimension vector space over $\mathbb{C}$, and $\mathrm{GL}(V)$ is the group of its endomorphisms.

Definition. Let s be an element of finite order of $\mathrm{GL}(V)$. s is a reflection if it fixes a hyperplane of V pointwise.

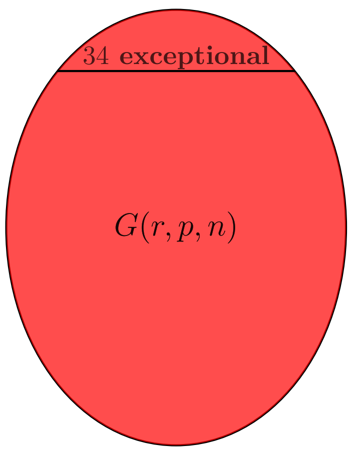
Definition. A group $G < \mathrm{GL}(V)$ is a complex reflection group if it is generated by reflections.

Definition. A finite reflection group $G < \mathrm{GL}(V)$ is called reducible if $G = G_1 \times G_2$ and V admits a non-trivial decomposition $V = V_1 \oplus V_2$, G_1 acting on V_1, G_2 acting on V_2. When such decomposition is not possible, G is irreducible.

Irreducible finite complex reflection groups were completely classified in the fifties by Shephard and Todd [19]:

Theorem 0.2.1. *Irreducible finite complex reflection groups consist of:*

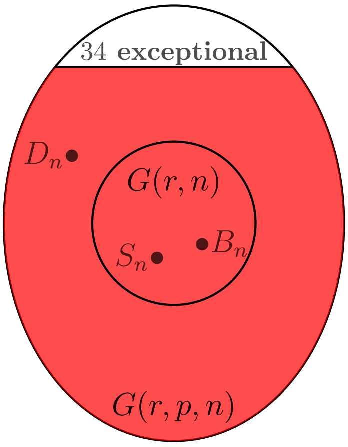
- *an infinite family of groups $G(r,p,n)$, where r, p, n are nonnegative integers and $p|r$;*
- *34 more exceptional groups.*



If G is a reflection group, its action on V induces an action on its dual V^* and hence on its polynomial algebra $S[V^*]$. We denote with $S[V^*]^G$ the subalgebra of $S[V^*]$ given by the polynomials on V that are invariant under the action of G. Finite complex reflection groups admit the following characterization, due to Chevalley [9] and Shephard-Todd [19]:

Theorem 0.2.2. *Let G be a finite subgroup of* $\mathrm{GL}(V)$. *Then G is a reflection group if and only if the invariant ring $S[V^*]^G$ is generated by $n = \dim(V)$ algebraically independent polynomials.*

0.3 The family $G(r, p, n)$



In the present work, we will not deal with the 34 sporadic groups mentioned in Theorem 0.2.1. We will focus on the infinite family $G(r, p, n)$. So let us turn to describe these groups.

When $r = p = 1$, the group $G(1, 1, n)$ is simply the symmetric group S_n of the $n \times n$ permutation matrices, i.e. matrices with exactly one 1 in every row and every column, and all the other entries equal to 0.

When $p = 1$, the group $G(r, n) \stackrel{\text{def}}{=} G(r, 1, n)$, also called the *generalized symmetric group*, is the wreath product $C_r \wr S_n$, where C_r is the cyclic group of order r. $G(r, n)$ consists of all the $n \times n$ complex matrices satisfying the following conditions:

- there is exactly one non-zero entry in every row and every column;
- the non-zero entries are r-th roots of unity.

Example 0.3.1. The group $B_n \stackrel{\text{def}}{=} G(2, n)$ of the signed permutations on n elements. For example, the matrix g given by

$$g = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

belongs to B_4.

Let now $p\,|r$. The group $G(r,p,n)$ is the subgroup of $G(r,n)$ of the elements verifying one extra condition:

- if we write every non-zero element as a power of ζ_r, the sum of all the exponents of ζ_r appearing in the matrix is a multiple of p.

Example 0.3.2. Consider the group $D_n = G(2,2,n) < G(2,n) = B_n$. Given $g \in B_n$, it belongs to D_n if -1 appears in the matrix of g an even number of times. For example, the following matrix $g \in B_4$

$$g = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

does not belong to D_4, while

$$h = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

does.

Definition. Given $g \in G(r,p,n)$, we denote by $z_i(g) \in \mathbb{Z}_r$ the exponent of ζ_r appearing in the i^{th} row of g. We say that $z_i(g)$ is the color of i in g and the sum $z(g) \stackrel{\text{def}}{=} z_1(g) + \cdots + z_n(g)$ will be called the color of g.

Thus, an element $g \in G(r,n)$ belongs to $G(r,p,n)$ if and only if $z(g) \equiv 0 \mod p$.

It is sometimes convenient to use alternative notation to denote an element in $G(r,n)$, other than the matrix representation.

Notation 0.3.3. We write $g = [(\sigma_1,\dots,\sigma_n); z_1,\dots,z_n]$ meaning that, for all $j \in [n]$, the unique nonzero entry in the j^{th} row appears in the σ_j^{th} column and equals $\zeta_r^{z_j}$ (i.e. $z_j(g) = z_j$). We call this the *window* notation of g.

Observe that $[(\sigma_1,\dots,\sigma_n); 0,\dots,0]$ is actually a permutation in S_n - namely, it is the element of S_n obtained from g forgetting its colors. We denote it by $|g|$.

Elements of $G(r,n)$ also have a cyclic decomposition which is analogous to the cyclic decomposition of permutations. A *cycle* c of $g \in G(r,n)$ is an object of the form $c = (a_1^{z_{a_1}},\dots,a_k^{z_{a_k}})$, where $(a_1,\dots,a_k)$ is a cycle of the permutation $|g|$, and $z_{a_i} = z_{a_i}(g)$ for all $i \in [k]$. Notice that we use square brackets (and round brackets on the permutation side) for the window notation and round brackets only for the cyclic notation.

We let k be the *length* of c, $z(c) \stackrel{\text{def}}{=} z_{a_1} + \cdots + z_{a_k}$ be the *color* of c, and $\text{Supp}(c) \stackrel{\text{def}}{=} \{a_1, \ldots, a_k\}$ be the *support* of c. We will sometimes write an element $g \in G(r,n)$ as the product of its cycles.

Example 0.3.4. Let $g \in G(3,6)$ be the matrix

$$g = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta_3 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta_3^2 & 0 \\ \zeta_3^2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then:

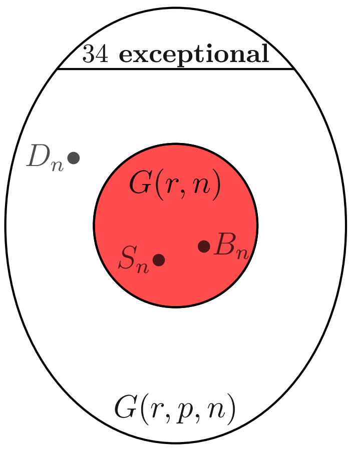
- $|g|$ is the matrix

$$|g| = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

- g has window notation $g = [(3,4,6,2,5,1); 0,1,1,0,2,2]$;
- the cyclic decomposition of g is given by $g = (1^0, 3^1, 6^2)(2^1, 4^0)(5^2)$.

0.4 The groups $G(r,n)$ and their irreducible representations

In this section we provide a parametrization for both the conjugacy classes and the irreducible representations of the group $G(r,n)$. These will turn up to be an essential tools in what follows.



The set of conjugacy classes of $G(r, n)$ is naturally parametrized by $\mathrm{Fer}(r, n)$ in the following way (see, for example, [13, §4]). If $(\alpha^{(0)}, \ldots, \alpha^{(r-1)}) \in \mathrm{Fer}(r, n)$ we let $m_{i,j}$ be the number of parts of $\alpha^{(i)}$ equal to j. Then the set

$$\mathrm{cl}_{\alpha^{(0)},\ldots,\alpha^{(r-1)}} = \{g \in G(r, n) : g \text{ has } m_{i,j} \text{ cycles of color } i \text{ and length } j\}$$

is a conjugacy class of $G(r, n)$, and all conjugacy classes are of this form.

Example 0.4.1. The element g given by $g = (2^0, 5^1, 7^2)(1^1, 4^0)(3^1)(6^2)$ belongs to the $G(3, 7)$-conjugacy class $\left(\yng(3), \yng(2,1), \yng(1)\right)$.

For what concerns the irreducible representations of $G(r, n)$, we have the following result:

Proposition 0.4.2. *Let* $\mathrm{Irr}(r, n)$ *be the set of the irreducible representations of* $G(r, n)$*. Then*

$$\mathrm{Irr}(r, n) = \{\rho_{\lambda^{(0)},\ldots,\lambda^{(r-1)}}, \text{ with } (\lambda^{(0)}, \ldots, \lambda^{(r-1)}) \in \mathrm{Fer}(r, n)\},$$

where the irreducible representation $\rho_{\lambda^{(0)},\ldots,\lambda^{(r-1)}}$ *of* $G(r, n)$ *is given by*

$$\rho_{\lambda^{(0)},\ldots,\lambda^{(r-1)}} = \mathrm{Ind}_{G(r,n_0)\times\cdots\times G(r,n_{r-1})}^{G(r,n)} \left(\bigodot_{i=0}^{r-1} (\gamma_{n_i}^{\otimes i} \otimes \tilde{\rho}_{\lambda^{(i)}}) \right),$$

with:

- $n_i = |\lambda^{(i)}|$;
- $\tilde{\rho}_{\lambda^{(i)}}$ *is the natural extension to* $G(r, n_i)$ *of the irreducible (Specht) representation* $\rho_{\lambda^{(i)}}$ *of* S_{n_i}*, i.e.* $\tilde{\rho}_{\lambda^{(i)}}(g) \stackrel{\mathrm{def}}{=} \rho_{\lambda^{(i)}}(|g|)$ *for all* $g \in G(r, n_i)$.
- γ_{n_i} *is the 1-dimensional 'color' representation of* $G(r, n_i)$ *given by*

$$\begin{aligned} \gamma_{n_i} : G(r, n_i) &\to \mathbb{C}^* \\ g &\mapsto \zeta_r^{z(g)}. \end{aligned}$$

Furthermore, the dimension of the representation $\rho_{\lambda^{(0)},\ldots,\lambda^{(r-1)}}$ *is given by* $|St_{(\lambda^{(0)},\ldots,\lambda^{(r-1)})}|$.

For the proof, see, for example, [13, §4], [20, §4].

Notation 0.4.3. Occasionally, a representation ρ_λ will be simply denoted by a λ. Also, sometimes we drop the round bracket on the subscript: $\rho_{\lambda^{(0)},\ldots,\lambda^{(r-1)}}$ stands for $\rho_{(\lambda^{(0)},\ldots,\lambda^{(r-1)})}$. This clarification is not redundant. It is important to remark that representations of $G(r, n)$ are parametrized by *ordered* r-tuples of diagrams. In the following chapters, we will also come across representations indexed by *unordered* pairs!

Example 0.4.4. The irreducible representations of B_n are parametrized by ordered pairs of Ferrers diagrams

$$\{(\lambda^{(0)}, \lambda^{(1)}) \text{ such that } |\lambda^{(0)}| + |\lambda^{(1)}| = n\} \subset \text{Fer}(2, n).$$

For example, here is a parametrization for the irreducible representations of B_3:

$$\left(\yng(3), \emptyset\right) \quad \left(\yng(2,1), \emptyset\right) \quad \left(\yng(1,1,1), \emptyset\right) \quad \left(\yng(2), \yng(1)\right) \quad \left(\yng(1,1), \yng(1)\right)$$

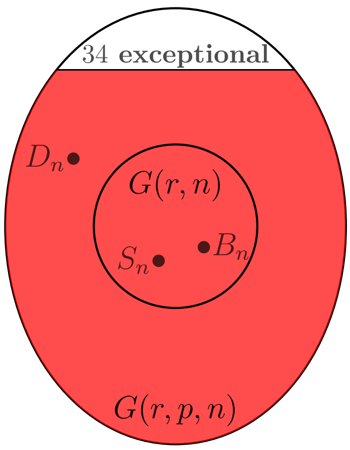
$$\left(\emptyset, \yng(3)\right) \quad \left(\emptyset, \yng(2,1)\right) \quad \left(\emptyset, \yng(1,1,1)\right) \quad \left(\yng(1), \yng(2)\right) \quad \left(\yng(1), \yng(1,1)\right).$$

The representation indexed by $\left(\yng(2,1), \emptyset\right)$ has, for example, dimension $|St_{\left(\yng(2,1), \emptyset\right)}| = 2$; the representations $\rho_{\left(\yng(3), \emptyset\right)}$ and $\rho_{\left(\emptyset, \yng(3)\right)}$ have dimension 1.

Notice that, in the case of $G(r, n)$, the conjugacy classes and the irreducible representations are parametrized by the same objects. This remark will be essential for the results exposed in the following chapter.

The parametization of the irreducible representations of the groups $G(r, p, n)$ finds a smart description by means of projective reflection groups and their dual. Thus, it is deferred to Chapters 2-4, as well as the treatment about $G(r, p, n)$-conjugacy classes for the involutory case, which we could not find anywhere else in literature.

0.5 The generalized Robinson-Schensted correspondence



The Robinson-Schensted correspondence will be met in more than one setting all along our treatment. In this section, we are going to see its classical version for the symmetric group and a first generalization of it to the case of $G(r, p, n)$. This will be immediately useful to provide a motivation for the form assumed by the Gelfand model due to F. Caselli [6]. Also, it will be of crucial importance to expose the nature of the refinement of such model (see section 1.1).

Recall the classical Robinson-Schensted correspondence. It is a bijection between S_n and the set of pairs of standard Young tableaux of size n of the same shape:

$$\begin{aligned} RS : S_n &\to \mathcal{ST}_n \times \mathcal{ST}_n \\ \sigma &\mapsto (P; Q), \end{aligned}$$

where $\mathrm{Sh}(P) = \mathrm{Sh}(Q)$ (see section 0.1 for the notation). An algorithm allows to construct P and Q from σ and vice versa (see [21, Section 7.11]).

It is possible to generalize the function RS to the case of wreath products: this was first done in [22]. The new bijection will be denoted with $\overline{RS}$:

$$\overline{RS} : G(r, n) \to \mathcal{ST}(r, n) \times \mathcal{ST}(r, n).$$

$\overline{RS}$ is defined as follows:

- split g into r double-rowed vectors $g_0, \dots, g_{r-1}$ according to the color;
- perform RS to the r double-rowed vectors;
- glue the images of $g_0, \dots, g_{r-1}$ together, thus obtaining one pair of elements of $\mathcal{ST}(r, n) \times \mathcal{ST}(r, n)$ with the same shape.

Example 0.5.1. $g = [(2, 4, 3, 1); 0, 1, 0, 0] \in G(2, 4) = B_4$.

$$g_0 = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 3 & 1 \end{pmatrix} \qquad g_1 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$\begin{aligned} g_0 &\xrightarrow{RS} (P_0; Q_0) = \left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} ; \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \right) \\ g_1 &\xrightarrow{RS} (P_1; Q_1) = \left(\begin{array}{|c|}\hline 4 \\ \hline \end{array} ; \begin{array}{|c|}\hline 2 \\ \hline \end{array} \right) \end{aligned}$$

$$g \xrightarrow{\overline{RS}} (P_0, P_1; Q_0, Q_1) = \left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} , \begin{array}{|c|}\hline 4 \\ \hline \end{array} ; \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} , \begin{array}{|c|}\hline 2 \\ \hline \end{array} \right)$$

Notation 0.5.2. Given $g \overset{\overline{RS}}{\mapsto} (P_0, \dots, P_{r-1}; Q_0, \dots, Q_{r-1})$, we denote by $\mathrm{Sh}(g)$ the element of $\mathrm{Fer}(r, n)$ which is the shape of $P_0, \dots, P_{r-1}$ and of $Q_0, \dots, Q_{r-1}$.

The generalized Robinson-Schensted correspondence satisfies the properties collected in the following lemma.

Lemma 0.5.3. *If* $g \mapsto (P_0, \dots, P_{r-1}; Q_0, \dots, Q_{r-1})$ *via* $\overline{RS}$*, then:*

- $\bar{g}^{-1} \mapsto (Q_0, \dots, Q_{r-1}; P_0, \dots, P_{r-1})$;

- $\zeta_r g \mapsto (P_1, \ldots, P_{r-1}, P_0; Q_1, \ldots, Q_{r-1}, Q_0)$.

Furthermore, given $g \in G(r,p,n)$*, it is easy to check that* $\overline{RS}(g) \in \mathcal{ST}(r,p,n) \times \mathcal{ST}(r,p,n)$*, so the function*

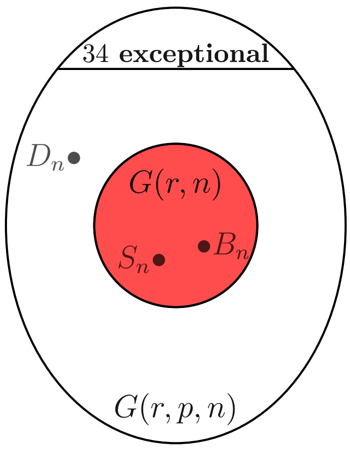
$$\overline{RS} : G(r,p,n) \to \mathcal{ST}(r,p,n) \times \mathcal{ST}(r,p,n)$$

is well defined.

Proof. The first point is an immediate consequence of the analogous result for the case of S_n. The second and the third can be easily derived by the way the correspondence is generalized to coloured permutations. $\square$

Chapter 1

The model and its decomposition for the groups $G(r, n)$



The aim of this chapter is to provide a refinement for the Gelfand model constructed in [6] for wreath products $G(r, n)$.

For our purpose, the groups $G(r, n)$ present an important advantage if compared to the groups of the more general form $G(r, p, n)$: they coincide with their *dual*. This circumstance simplifies a lot both the description of the model built in [6], and its refinement. The concepts of projective reflection group and of duality will not be required to read this chapter, which provides a complete, self-contained treatment for the special case of the groups $G(r, n)$, and gives the opportunity to shed some light on the subject without affording the background needed in the following chapters.

In Section 1.1, after describing the model constructed in [6] for the special case of $G(r,n)$, we state the main result of this chapter, Theorem 1.1.3. Such theorem provides a refinement of the model which is coherent with the generalized Robinson-Schensted correspondence. In the following section, we prove Theorem 1.1.3 for the particular case of B_n: this well-known example will give the reader a precise idea of our arguments. Finally, in the last section of this chapter, we will generalize the results already exposed for B_n to the case of all the groups $G(r,n)$.

1.1 A first statement

In the present section, we will illustrate the Gelfand model constructed in [6] for wreath products $G(r,n)$. With the model at hands, we will be able already to state our main result for what concerns the groups $G(r,n)$.

Definition. Let $G < \mathrm{GL}(n,\mathbb{C})$ and let $g \in G$. We say that g is an absolute involution if $g\bar{g} = \mathrm{Id}$. We set

$$I(r,n) \stackrel{\text{def}}{=} \{g \in G(r,n) \mid g \text{ is an absolute involution}\}.$$

Proposition 1.1.1. *Let $g \in G(r,n)$. Then $g \in I(r,n)$ if and only if $g \mapsto (P_0,\ldots,P_{r-1};P_0,\ldots,P_{r-1})$ via $\overline{RS}$.*

Proof. This follows immediately from the analogous result for S_n and the way $\overline{RS}$ is constructed. □

Our first remark is the following.

Remark 1. Let $G = G(r,n)$ and let M be a Gelfand model for G. Then

$$\dim M = \sharp\{\text{absolute involutions of } G\}. \tag{1.0}$$

Proof. We know from Proposition 0.4.2 that

$$\dim M = \sum_{\lambda\in \mathrm{Fer}(r,n)} \dim \rho_\lambda = \sum_{\lambda\in \mathrm{Fer}(r,n)} |\mathcal{ST}_\lambda|.$$

Thanks to Proposition 1.1.1, the absolute involutions of $G(r,n)$ are exactely as many as elements in $\mathcal{ST}(r,n)$. This proves the remark. □

The last observation gives a hint about a possible vector space structure on which one can build a Gelfand model for G: we can consider the formal vector space spanned by the absolute involutions of G,

$$M \stackrel{\text{def}}{=} \bigoplus_{v\in I(r,n)} \mathbb{C}C_v. \tag{1.0}$$

Let us now turn to describe the representation ϱ that will give M the structure of a G-module. To this aim, we need some more notation.

If $g, g' \in G(r,n)$ we define

$$< g, g' > \stackrel{\text{def}}{=} \sum_i z_i(g) z_i(g') \in \mathbb{Z}_r :$$

it is a sort of a scalar product between the color vectors of g and g'.

If $\sigma, \tau \in S_n$ with $\tau^2 = 1$ we let

$$\begin{aligned} \mathrm{Inv}(\sigma) &\stackrel{\text{def}}{=} \{\{i,j\} : (j-i)(\sigma(j)-\sigma(i)) < 0\}; \\ \mathrm{Pair}(\tau) &\stackrel{\text{def}}{=} \{\{i,j\} : \tau(i) = j \neq i\}; \\ \mathrm{inv}_\tau(\sigma) &\stackrel{\text{def}}{=} |\{\mathrm{Inv}(\sigma) \cap \mathrm{Pair}(\tau)\}| \end{aligned}$$

Finally, if $g, v \in G(r,n)$ with $v\bar{v} = 1$, we let $\mathrm{inv}_v(g) \stackrel{\text{def}}{=} \mathrm{inv}_{|v|}(|g|)$.

Theorem 1.1.2. *Consider the group $G = G(r,n)$. Let M be as in* (1.1)*, and let ϱ be the representation of G given by*

$$\begin{aligned} \varrho : G &\to GL(M) \\ g &\mapsto \varrho(g) : M \to M \\ &\qquad C_v \mapsto \varrho(g) C_v \stackrel{\text{def}}{=} \zeta_r^{<g,v>} (-1)^{\mathrm{inv}_v(g)} C_{|g|v|g|^{-1}}. \end{aligned}$$

Then (M, ϱ) is a G-model.

Proof. See [6, Theorem 3.2]. □

Let us have a closer look at the model. There is an immediate decomposition into smaller submodules.

Definition. If $g, h \in G(r,n)$, g and h are S_n-conjugate if there exists $\sigma \in S_n$ such that $g = \sigma h \sigma^{-1}$, and we call *S_n-conjugacy classes*, or *symmetric conjugacy classes*, the corresponding equivalence classes.

If c is a S_n-conjugacy class of absolute involutions in $I(r,n)$ we denote by $M(c)$ the subspace of M spanned by the basis elements C_v indexed by the absolute involutions v belonging to the class c, and it is clear that

$$M = \bigoplus_c M(c) \quad \text{as } G\text{-modules},$$

where the sum runs through all S_n-conjugacy classes of absolute involutions in $I(r,n)$. We are now ready to state the main result of this chapter.

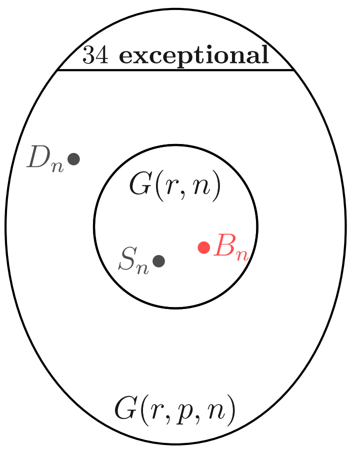
Theorem 1.1.3. *Consider the $G(r,n)$-model (M,ϱ) described in Theorem 1.1.2. Let c be a S_n-conjugacy class of absolute involutions in $G(r,n)$. Let $M(c)$ be the submodule of M spanned by the elements of c. Let $\mathrm{Sh}(c) \stackrel{\mathrm{def}}{=} \cup_{v\in c}\mathrm{Sh}(v)$ (see the notation in 0.5.2). Then the following decomposition holds:*

$$M(c) \cong \bigoplus_{(\lambda^{(0)},\ldots,\lambda^{(r-1)})\in \mathrm{Sh}(c)} \rho_{\lambda^{(0)},\ldots,\lambda^{(r-1)}}.$$

In words: if a submodule $M(c)$ of M is spanned by involutions whose images via $\overline{RS}$ have certain shapes, $M(c)$ affords the irreducible representations of $G(r,n)$ parametrized by those shapes.

We will first prove this result for the group of the signed permutations $B_n = G(2,n)$. We will devote to this the following section. The proof of the general case of $G(r,n)$ is in Section 1.3.

1.2 The case of B_n



We will now focus on the group $B_n = G(2,n)$. In particular, in Section 1.2.1 we prove a result (Proposition 1.2.5) which will be exploited in Section 1.2.2 to describe the irreducible decomposition of some particular submodules of the model. In Section 1.2.3 we complete the irreducible decomposition.

1.2.1 Some tools in B_n combinatorial representation theory

The main result of this section is Proposition 1.2.5 which is an extension of an idea appearing in [12] and will be of crucial importance to prove Theorem 1.1.3.

First of all we observe that, since B_n is given by real matrices, the absolute involutions in B_n are exactly the involutions in B_n. So, to understand our results, we need to describe and parametrize the S_n-conjugacy classes of involutions in B_n explicitly. To this aim, for all $v \in I(2,n)$ we let

$$\begin{aligned}
\mathrm{fix}_0(v) &\stackrel{\mathrm{def}}{=} |\{i : i > 0 \text{ and } v(i) = i\}|;\\
\mathrm{fix}_1(v) &\stackrel{\mathrm{def}}{=} |\{i : i > 0 \text{ and } v(i) = -i\}|;\\
\mathrm{pair}_0(v) &\stackrel{\mathrm{def}}{=} |\{(i,j) : 0 < i < j,\ v(i) = j \text{ and } v(j) = i\}|;\\
\mathrm{pair}_1(v) &\stackrel{\mathrm{def}}{=} |\{(i,j) : 0 < i < j,\ v(i) = -j \text{ and } v(j) = -i\}|.
\end{aligned}$$

Example 1.2.1. If $v = [(3,2,1,8,9,6,7,4,5); 1,0,1,0,1,1,0,0,1]$, we have $\mathrm{fix}_0(v) = 2$, $\mathrm{fix}_1(v) = 1$, $\mathrm{pair}_0(v) = 1$ and $\mathrm{pair}_1(v) = 2$.

Proposition 1.2.2. *Two involutions v, w of B_n are S_n-conjugate if and only if*

$$\begin{aligned}
\mathrm{fix}_0(v) = \mathrm{fix}_0(w), &\quad \mathrm{pair}_0(v) = \mathrm{pair}_0(w),\\
\mathrm{fix}_1(v) = \mathrm{fix}_1(w), &\quad \mathrm{pair}_1(v) = \mathrm{pair}_1(w).
\end{aligned}$$

Furthermore, given an involution v in B_n, let $\mathrm{Sh}(v) = (\lambda, \mu)$ (see Notation 0.5.2). Then λ has $\mathrm{fix}_0(v)$ columns of odd length and $\mathrm{fix}_0(v) + 2\,\mathrm{pair}_0(v)$ boxes, while μ has $\mathrm{fix}_1(v)$ columns of odd length and $\mathrm{fix}_1(v) + 2\,\mathrm{pair}_1(v)$ boxes.

Proof. The first part is clear, since conjugation of a cycle by an element in S_n does not alter the number of negative entries in the cycle. The second part follows easily from the corresponding result for the symmetric group due to Schützenberger (see [18] or [21, Exercise 7.28]) and the definition of the generalized Robinson-Schensted correspondence given in Section 0.5. □

We can thus name the S_n-conjugacy classes of the involutions of B_n in this way:

$$c_{f_0,f_1,p_0,p_1} \stackrel{\mathrm{def}}{=} \left\{ v \in I(2,n) \,\middle|\, \begin{array}{ll} \mathrm{fix}_0(v) = f_0 & \mathrm{fix}_1(v) = f_1 \\ \mathrm{pair}_0(v) = p_0 & \mathrm{pair}_1(v) = p_1 \end{array} \right\},$$

where $f_0, f_1, p_0, p_1 \in \mathbb{N}$ are such that $f_0 + f_1 + 2p_0 + 2p_1 = n$. The description given of the S_n-conjugacy classes ensures that the subspace of M generated by the involutions $v \in B_n$ with $\mathrm{fix}_0(v) = \mathrm{fix}_1(v) = 0$ - which is non trivial if n is even only - is a B_n-submodule. The crucial step in the proof of Theorem 1.1.3 is the partial result regarding this submodule.

Given $\lambda \in \mathrm{Fer}(1,n)$ we let

$$\begin{aligned}
R^-_\lambda &\stackrel{\mathrm{def}}{=} \{\sigma \in \mathrm{Fer}(n-1) : \sigma \text{ is obtained by deleting one box from } \lambda\}\\
R^+_\lambda &\stackrel{\mathrm{def}}{=} \{\sigma \in \mathrm{Fer}(n+1) : \sigma \text{ is obtained by adding one box to } \lambda\}.
\end{aligned}$$

Moreover, if $(\lambda, \mu) \in \mathrm{Fer}(2,n)$, we let

$$\begin{aligned}
R^-_{\lambda,\mu} &\stackrel{\mathrm{def}}{=} \{(\sigma,\mu) \in \mathrm{Fer}(2,n-1) : \sigma \in R^-_\lambda\} \cup \{(\lambda,\tau) \in \mathrm{Fer}(2,n-1) : \tau \in R^-_\mu\}\\
R^+_{\lambda,\mu} &\stackrel{\mathrm{def}}{=} \{(\sigma,\mu) \in \mathrm{Fer}(2,n+1) : \sigma \in R^+_\lambda\} \cup \{(\lambda,\tau) \in \mathrm{Fer}(2,n+1) : \tau \in R^+_\mu\}.
\end{aligned}$$

We always identify B_n as a subgroup of B_{n+1} as follows:

$$B_n = \{g \in B_{n+1} : g(n+1) = n+1\}.$$

Theorem 1.2.3 (Branching rule for B_n)**.** *Let* $(\lambda,\mu) \in \mathrm{Fer}(2,n)$. *Then the following holds:*

$$\rho_{\lambda,\mu} \downarrow_{B_{n-1}} = \bigoplus_{(\sigma,\tau)\in R^-_{\lambda,\mu}} \rho_{\sigma,\tau}$$

$$\rho_{\lambda,\mu} \uparrow^{B_{n+1}} = \bigoplus_{(\sigma,\tau)\in R^+_{\lambda,\mu}} \rho_{\sigma,\tau}.$$

Proof. See [11, §3]. □

Before stating the main result of this section we need some more notation:

Notation 1.2.4. A pair of diagrams $(\lambda,\mu) \in \mathrm{Fer}(2,n)$ will be called *even* if both λ and μ have all rows of even length. If ϕ and ψ are representations of a group G, we say that ϕ *contains* ψ if ψ is isomorphic to a subrepresentation of ϕ.

Proposition 1.2.5. *Let* Π_m *be representations of* B_{2m}, m *ranging in* $\mathbb{N}$. *Then the following are equivalent:*

a) for every m, Π_m *is isomorphic to the direct sum of all the irreducible representations of* B_{2m} *indexed by even diagrams of* $\mathrm{Fer}(2,2m)$, *each of such representations occurring once;*

b) for every m,

- *b_0) Π_0 is 1-dimensional (and B_0 is the group with one element);*
- *b_1) the module Π_m contains the irreducible representations $\rho_{\iota_{2m},\emptyset}$ and $\rho_{\emptyset,\iota_{2m}}$ of B_{2m}, where ι_k denotes the single-rowed Ferrers diagram with k boxes;*
- *b_2) the following isomorphism holds:*

$$\Pi_m \downarrow_{B_{2m-1}} \cong \Pi_{m-1} \uparrow^{B_{2m-1}} . \qquad (1.\text{-}4)$$

We explicitly observe that we are dealing here with even diagrams, i.e., with *rows* of even length. What we will need later are diagrams with *columns* of even length. This is a harmless difference which simplifies our computations and will be solved in §1.2.2.

Proof. a)$\Rightarrow$ b). Conditions b$_0$) and b$_1$) follow immediately.

Let us now compare $\Pi_m \downarrow_{B_{2m-1}}$ and $\Pi_{m-1} \uparrow^{B_{2m-1}}$. The branching rule ensures that $\Pi_m \downarrow_{B_{2m-1}}$ contains exactly the $\rho_{\lambda,\mu}$'s where the diagram (λ,μ) has exactly one row of odd length. Furthermore, the pair (α,β) such that $R^-_{\alpha,\beta} \ni (\lambda,\mu)$ is uniquely determined: to obtain it, it will only be allowed to add a box to the unique odd row of the diagram (λ,μ). This means that $\Pi_m \downarrow_{B_{2m-1}}$ is the multiplicity-free direct sum of all the representations of B_{2m-1} indexed by diagrams in $\mathrm{Fer}(2, 2m-1)$ with exactly one row of odd length.

Arguing analogously for $\Pi_{m-1} \uparrow^{B_{2m-1}}$, we can infer that it contains exactly the same irreducible representations with multiplicity 1 and it is thus isomorphic to $\Pi_m \downarrow_{B_{2m-1}}$.

b)$\Rightarrow$ a) Let us argue by induction.

The case $m = 0$ is given by b$_0$). Let us see also the case $m = 1$. We know that $\Pi_1 \downarrow_{B_1} \cong \Pi_0 \uparrow^{B_1} \cong \rho_{\iota_1,\emptyset} \oplus \rho_{\emptyset,\iota_1}$. But Π_1 contains $\rho_{\iota_2,\emptyset}$ and $\rho_{\emptyset,\iota_2}$ by b$_1$), and the isomorphism

$$\left(\rho_{\iota_2,\emptyset} \oplus \rho_{\emptyset,\iota_2}\right) \downarrow_{B_1} \cong \rho_{\iota_1,\emptyset} \oplus \rho_{\emptyset,\iota_1} \cong \Pi_0 \uparrow^{B_1}$$

ensures that

$$\Pi_1 \cong \rho_{\iota_2,\emptyset} \oplus \rho_{\emptyset,\iota_2}.$$

Let us show that, if Π_{m-1} is the direct sum of all the representations indexed by even diagrams, the same holds for Π_m. For notational convenience, we let

$$\Lambda_m \overset{\text{def}}{=} \{(\lambda,\mu) \in \mathrm{Fer}(2, 2m) : \ \rho_{\lambda,\mu} \text{ is a subrepresentation of } \Pi_m\}$$

First we shall see that, if $(\lambda,\mu) \in \mathrm{Fer}(2, 2m)$ is an even diagram, then $(\lambda,\mu) \in \Lambda_m$.

The set $\mathrm{Fer}(2, 2m)$ is totally ordered in this way: given two pairs $(\lambda,\mu), (\sigma,\tau) \in \mathrm{Fer}(2, 2m)$, we let $(\lambda,\mu) < (\sigma,\tau)$ if one of the following holds:

i) $\lambda < \sigma$ lexicographically;

ii) $\lambda = \sigma$ and $\mu < \tau$ lexicographically.

We observe that $(\iota_{2m}, \emptyset)$ is the maximum element of $\mathrm{Fer}(2, 2m)$ with respect to this order.

Claim. *If* $(\lambda,\mu) \in \mathrm{Fer}(2, 2m)$ *is such that:*

i) (λ,μ) *is even;*

ii) $(\lambda,\mu) \notin \{(\iota_{2m},\emptyset),(\emptyset,\iota_{2m})\}$;

iii) $(\sigma,\tau) \in \Lambda_m$ *for all* $(\sigma,\tau) \in \mathrm{Fer}(2,2m)$ *such that* (σ,τ) *is even and* $(\sigma,\tau) > (\lambda,\mu)$,

then $(\lambda,\mu) \in \Lambda_m$.

As we already know that $(\iota_{2m},\emptyset)$ and $(\emptyset,\iota_{2m})$ are contained in Λ_m, once we have proved the claim, all the even pairs will be too.
Proof of the claim. Let $(\lambda,\mu) \in \mathrm{Fer}(2,2m)$ be an even diagram satisfying *i), ii)* and *iii)*. Then the pair (λ,μ) has at least two rows. We let $(\sigma,\tau) \in \mathrm{Fer}(2,2m)$ be the pair obtained from (λ,μ) by deleting two boxes in the last non zero row and adding two boxes to the first non zero row.

As $(\sigma,\tau) > (\lambda,\mu)$, we have $(\sigma,\tau) \in \Lambda_m$, so the isomorphism (1.2.5), the induction hypothesis and the branching rule lead to the following:

$$\forall\ (\eta,\theta) \in R^-_{\sigma,\tau},\ R^+_{\eta,\theta} \cap \Lambda_m = \{(\sigma,\tau)\}. \tag{1.-4}$$

Now let $(\alpha,\beta) \in \mathrm{Fer}(2,2m-1)$ be obtained from (λ,μ) by deleting one box in the last nonzero row. Our induction hypothesis ensures that $\rho_{\alpha,\beta}$ is a subrepresentation of $\Pi_{m-1} \uparrow^{B_{2m-1}}$ with multiplicity 1. So the isomorphism (1.2.5) implies that

$$\text{there exists a unique } (\gamma,\delta) \in \mathrm{Fer}(2,2m) \text{ such that } \{(\gamma,\delta)\} = R^+_{\alpha,\beta} \cap \Lambda_m. \tag{1.-4}$$

The claim will be proved if we show that $(\gamma,\delta) = (\lambda,\mu)$.

The pair (γ,δ) is obtained from (α,β) by adding a single box, since $(\gamma,\delta) \in R^+_{\alpha,\beta}$. If such a box is not added in the first or in the last non zero rows of (α,β) then (γ,δ) has two rows of odd length and one can check that $R^-_{\gamma,\delta}$ contains at least a diagram with three rows of odd length. This contradicts (1.2.5).

Now assume that (γ,δ) is obtained by adding a box in the first nonzero row of (α,β). If we let (η,θ) be the pair obtained from (λ,μ) by deleting two boxes in the last nonzero row and adding one box in the first nonzero row, we have $(\eta,\theta) \in R^-_{\sigma,\tau}$, and $R^+_{\eta,\theta} \cap \Lambda_m \supseteq \{(\sigma,\tau),(\gamma,\delta)\}$ which contradicts (1.2.1).

Therefore (γ,δ) is obtained by adding a box in the last nonzero row of (α,β), i.e. $(\gamma,\delta) = (\lambda,\mu)$ and the claim is proved. □

We have just proved that if we let Π_m^{even} be the multiplicity-free sum of all irreducible representations of B_{2m} indexed by even diagrams we have that Π_m^{even} is a subrepresentation of Π_m. The result follows since we also have

$$\Pi_m^{\mathrm{even}} \downarrow_{B_{2m-1}} \cong \Pi_{m-1} \uparrow^{B_{2m-1}},$$

and so, in particular, $\dim(\Pi_m^{\mathrm{even}}) = \dim(\Pi_m)$.

□

1.2.2 A partial result for B_n

In the process of proving our main result we use the following auxiliary representation of B_n on M:

$$\begin{aligned}\varphi(g): M &\to M\\ C_v &\mapsto (-1)^{<g,v>} C_{|g|v|g|^{-1}}.\end{aligned}$$

Notice that the representation φ is just like the representation ϱ of the model (M, ϱ), apart from the factor $(-1)^{\mathrm{inv}_v(g)}$.

Let M_m be the subspace of M spanned by the elements C_v as v varies among all involutions in B_{2m} such that $\mathrm{fix}_0(v) = \mathrm{fix}_1(v) = 0$:

$$M_m \stackrel{\mathrm{def}}{=} \bigoplus_{p_0+p_1=m} M(c_{0,0,p_0,p_1}).$$

The main task of this section is to show that the representations (M_m, φ) satisfy the conditions of Proposition 1.2.5.

We first prove that the representation (M_m, φ) satisfies condition b_1) of Proposition 1.2.5. In fact, we will show explicitly that (M_m, φ) contains all irreducible representations indexed by an even pair of 1-rowed Ferrers diagrams. Recall from Proposition 0.4.2 that the irreducible representations of B_n are parametrized by pairs $(\lambda, \mu) \in \mathrm{Fer}(2, n)$, and that we have in this case

$$\rho_{\lambda,\mu} \simeq \mathrm{Ind}_{B_s \times B_{n-s}}^{B_n} (\tilde{\rho}_\lambda \odot (\gamma_{n-s} \otimes \tilde{\rho}_\mu)), \qquad (1.\text{-}4)$$

where $s = |\lambda|$.

For $S \subseteq [2m]$ let

$$\Delta_S \stackrel{\mathrm{def}}{=} \{g \in I(2, 2m) : \mathrm{fix}_0(g) = \mathrm{fix}_1(g) = 0 \text{ and } \{i \in [n] : z_i(g) = 0\} = S\},$$

and

$$C_S = \sum_{v \in \Delta_S} C_v \in M.$$

Lemma 1.2.6. *For all $p_0, p_1 \in \mathbb{N}$ such that $p_0 + p_1 = m$, the subspace of M_m spanned by all C_S with $|S| = 2p_0$, is an irreducible submodule of (M_m, φ) affording the representation $\rho_{\iota_{2p_0}, \iota_{2p_1}}$.*

Proof. Let us consider the 1-dimensional subspace $\mathbb{C}C_{[2p_0]}$ of M_m.

Let us identify the subgroup $B_{2p_0} \times B_{2p_1}$ of B_{2m} with the group of the elements permuting "separately" the first $2p_0$ integers and the remaining $2p_1$ integers:

$$B_{2p_0} \times B_{2p_1} \simeq \{g \in B_{2m} : |g|(i) \in [2p_0] \, \forall \, i \in [2p_0]\}.$$

Let $\psi=\varphi|_{B_{2p_0}\times B_{2p_1}}$. We have

$$\begin{aligned}\psi(g_1,g_2)(C_{[2p_0]}) &= \psi(g_1,g_2)\Big(\sum_{v\in\Delta_{[2p_0]}} C_v\Big)=\sum_{v\in\Delta_{[2p_0]}}\psi(g_1,g_2)(C_v)\\ &= \sum_{v\in\Delta_{[2p_0]}}(-1)^{<g_2,v>}|g_1g_2|v|g_1g_2|^{-1}=\sum_{v\in\Delta_{[2p_0]}}(-1)^{z(g_2)}|g_1g_2|v|g_1g_2|^{-1}\\ &= (-1)^{z(g_2)}\sum_{v\in\Delta_{[2p_0]}}|g_1g_2|v|g_1g_2|^{-1}=(-1)^{z(g_2)}C_{[2p_0]},\end{aligned}$$

since, clearly, the map $v\mapsto|g_1g_2|v|g_1g_2|^{-1}$ is a permutation of $\Delta_{[2p_0]}$. Therefore, we have that $(\mathbb{C}C_{[2p_0]},\psi)$ is a representation of $B_{2p_0}\times B_{2p_1}$ and that it is isomorphic to the representation $\tilde\rho_{\iota_{2p_0}}\odot(\gamma_{2p_1}\otimes\tilde\rho_{\iota_{2p_1}})$. By the description of the irreducible representations of B_n given in (1.2.2) we have that

$$\mathrm{Ind}_{B_{2p_0}\times B_{2p_1}}^{B_{2m}}(\mathbb{C}C_{[2p_0]},\psi)\cong\rho_{\iota_{2p_0},\iota_{2p_1}}.$$

Now we can observe that, by construction, $B_{2p_0}\times B_{2p_1}$ is the stabilizer in B_{2m} of v with respect to the absolute conjugation and that

$$\{C_S:|S|=2p_0\}=\{C\in M_m:C=\sum_{v\in\Delta_{[2p_0]}}C_{|g|v|g|^{-1}}\text{ for some }g\in B_{2m}\}.$$

From these facts we deduce that we also have

$$\mathrm{Ind}_{B_{2p_0}\times B_{2p_1}}^{B_{2m}}(\mathbb{C}\,C_{[2p_0]},\psi)=\bigoplus_{S\subseteq[2m],|S|=2p_0}\mathbb{C}\,C_S,$$

and the proof is complete. □

Proposition 1.2.7. *For all $m>0$, we have*

$$(M_m,\varphi)\downarrow_{B_{2m-1}}\cong(M_{m-1},\varphi)\uparrow^{B_{2m-1}}.$$

Proof. For brevity, for all $p_0,p_1\in\mathbb{N}$ such that $p_0+p_1=m$, we denote the B_{2m}-module $M(c_{0,0,p_0,p_1})$ with M_{p_0,p_1}. Via the representation φ, the vector space M_m naturally splits as a B_{2m}-module as it does via ϱ:

$$M_m=\bigoplus_{p_0+p_1=m}M_{p_0,p_1}.$$

We consider the action of B_{2m-1} on each class $c_{0,0,p_0,p_1}$ and it is clear that $z_{2m}(v)=z_{2m}(|g|v|g|^{-1})$ for all $v\in B_{2m}$ and $g\in B_{2m-1}$. In particular, each M_{p_0,p_1} splits, as a B_{2m-1}-module, into two submodules according to the color of $2m$. More precisely, if we denote by

$$\begin{aligned}M^0_{p_0,p_1} &\stackrel{\text{def}}{=} \mathrm{Span}\{C_v:v\in c_{0,0,p_0,p_1}\text{ and }z_{2m}(v)=0\};\\ M^1_{p_0,p_1} &\stackrel{\text{def}}{=} \mathrm{Span}\{C_v:v\in c_{0,0,p_0,p_1}\text{ and }z_{2m}(v)=1\},\end{aligned}$$

we have

$$M_{p_0,p_1} = M^0_{p_0,p_1} \oplus M^1_{p_0,p_1}$$

as B_{2m-1}-modules, and hence we also have the following decomposition of M_m as a B_{2m-1}-module

$$M_m \downarrow_{B_{2m-1}} = \bigoplus_{p_0+p_1=m} \left(M^0_{p_0,p_1} \bigoplus M^1_{p_0,p_1}\right).$$

Let us consider the involutions $v^0_{p_0,p_1}$, with $p_0 \neq 0$, and $v^1_{p_0,p_1}$, with $p_1 \neq 0$, given by

$$v^0_{p_0,p_1} \stackrel{\text{def}}{=} [(2,1,4,3,...,2m,2m-1); \underbrace{0,0,...0}_{2(p_0-1)}, \underbrace{1,...,1}_{2p_1}, 0,0];$$

$$v^1_{p_0,p_1} \stackrel{\text{def}}{=} [(2,1,4,3,...,2m,2m-1); \underbrace{0,0,...0}_{2p_0}, \underbrace{1,...,1}_{2p_1}].$$

We observe that $M^0_{p_0,p_1}$ and $M^1_{p_0,p_1}$ are spanned by all the elements C_v as v varies in the S_{2m-1}-conjugacy classes of $v^0_{p_0,p_1}$ and $v^1_{p_0,p_1}$ respectively, and so we can express them as induced representations of linear representations of the stabilizers of these elements with respect to the absolute conjugation in B_{2m-1}. Namely, if we let

$$H^0_{p_0,p_1} \stackrel{\text{def}}{=} \{g \in B_{2m-1} : |g| v^0_{p_0,p_1} |g|^{-1} = v^0_{p_0,p_1}\},$$

$$H^1_{p_0,p_1} \stackrel{\text{def}}{=} \{g \in B_{2m-1} : |g| v^1_{p_0,p_1} |g|^{-1} = v^1_{p_0,p_1}\},$$

we have

$$(M^0_{p_0,p_1}, \varphi) \cong \mathrm{Ind}^{B_{2m-1}}_{H^0_{p_0,p_1}}(\pi^0_{p_0,p_1}) \quad \text{and} \quad (M^1_{p_0,p_1}, \varphi) \cong \mathrm{Ind}^{B_{2m-1}}_{H^1_{p_0,p_1}}(\pi^1_{p_0,p_1}),$$

where

$$\begin{array}{rccc} \pi^0_{p_0,p_1}: & H^0_{p_0,p_1} & \rightarrow & \mathbb{C}^* \\ & g & \mapsto & (-1)^{<g, v^0_{p_0,p_1}>} \end{array} \quad \text{and} \quad \begin{array}{rccc} \pi^1_{p_0,p_1}: & H^1_{p_0,p_1} & \rightarrow & \mathbb{C}^* \\ & g & \mapsto & (-1)^{<g, v^1_{p_0,p_1}>}. \end{array}$$

Let us now turn to M_{m-1}: arguing as in M_m, we have

$$M_{m-1} = \bigoplus_{q_0+q_1=m-1} M_{q_0,q_1}.$$

As above, M_{q_0,q_1} can be written by means of an induction from the stabilizer of an involution in $c_{0,0,q_0,q_1}$ with respect to the absolute conjugation. For every q_0, q_1 such that $q_0 + q_1 = m-1$, let us consider the vector u_{q_0,q_1} given by

$$u_{q_0,q_1} \stackrel{\text{def}}{=} [(2,1,4,3,\ldots,2m-2,2m-3); \underbrace{0,0,\ldots,0}_{2q_0}, \underbrace{1,\ldots,1}_{2q_1}]$$

and let

$$K_{q_0,q_1} \stackrel{\text{def}}{=} \{g \in B_{2m-2} : |g|u_{q_0,q_1}|g|^{-1} = u_{q_0,q_1}\}.$$

Then

$$(M_{q_0,q_1}, \varphi) = \mathrm{Ind}_{K_{q_0,q_1}}^{B_{2m-2}}(\pi_{q_0,q_1}),$$

where

$$\begin{array}{rccl} \pi_{q_0,q_1} : & K_{q_0,q_1} & \to & \mathbb{C}^* \\ & g & \mapsto & (-1)^{<g,u_{q_0,q_1}>}. \end{array}$$

Summing up, observing that $M^0_{0,m} = M^1_{m,0} = \{0\}$, we have

$$\begin{aligned} M_m \downarrow_{B_{2m-1}} &= \bigoplus_{p_0+p_1=m} (M^0_{p_0,p_1} \oplus M^1_{p_0,p_1}) = \bigoplus_{q_0+q_1=m-1} (M^0_{q_0+1,q_1} \oplus M^1_{q_0,q_1+1}) \\ &\cong \bigoplus_{q_0+q_1=m-1} \left(\mathrm{Ind}_{H^0_{q_0+1,q_1}}^{B_{2m-1}}(\pi^0_{q_0+1,q_1}) \bigoplus \mathrm{Ind}_{H^1_{q_0,q_1+1}}^{B_{2m-1}}(\pi^1_{q_0,q_1+1}) \right) \end{aligned}$$

and

$$M_{m-1} \uparrow^{B_{2m-1}} \cong \mathrm{Ind}_{B_{2m-2}}^{B_{2m-1}} \Big(\bigoplus_{q_0+q_1=m-1} \mathrm{Ind}_{K_{q_0,q_1}}^{B_{2m-2}}(\pi_{q_0,q_1}) \Big).$$

So, to prove the statement it is enough to show that

$$\begin{aligned} \bigoplus_{q_0+q_1=m-1} &\left(\mathrm{Ind}_{H^0_{q_0+1,q_1}}^{B_{2m-1}}(\pi^0_{q_0+1,q_1}) \bigoplus \mathrm{Ind}_{H^1_{q_0,q_1+1}}^{B_{2m-1}}(\pi^1_{q_0,q_1+1}) \right) \\ &\cong \mathrm{Ind}_{B_{2m-2}}^{B_{2m-1}} \Big(\bigoplus_{q_0+q_1=m-1} \mathrm{Ind}_{K_{q_0,q_1}}^{B_{2m-2}}(\pi_{q_0,q_1}) \Big). \end{aligned}$$

As the induction commutes with the direct sum and has the transitivity property, the last equality is equivalent to

$$\bigoplus_{q_0+q_1=m-1} \big(\mathrm{Ind}_{H^0_{q_0+1,q_1}}^{B_{2m-1}}(\pi^0_{q_0+1,q_1}) \bigoplus \mathrm{Ind}_{H^1_{q_0,q_1+1}}^{B_{2m-1}}(\pi^1_{q_0,q_1+1}) \big) \cong \bigoplus_{q_0+q_1=m-1} \mathrm{Ind}_{K_{q_0,q_1}}^{B_{2m-1}}(\pi_{q_0,q_1}). \tag{1.-11}$$

The choice of the vectors $v^0_{p_0,p_1}$, $v^1_{p_0,p_1}$ and u_{q_0,q_1} leads to:

$$\begin{aligned} H^0_{p_0,p_1} &= \{g \in B_{2m-1} : |g| \in S_{2(p_0-1)} \times S_{2p_1}, |g|(i+1) = |g|(i) \pm 1 \,\forall\, i \text{ odd}, 0 < i < 2m\}; \\ H^1_{p_0,p_1} &= \{g \in B_{2m-1} : |g| \in S_{2p_0} \times S_{2(p_1-1)}, |g|(i+1) = |g|(i) \pm 1 \,\forall\, i \text{ odd}, 0 < i < 2m\}; \\ K_{q_0,q_1} &= \{g \in B_{2m-2} : |g| \in S_{2q_0} \times S_{2(q_1-1)}, |g|(i+1) = |g|(i) \pm 1 \,\forall\, i \text{ odd}, 0 < i < 2m-2\} \end{aligned}$$

where, as usual, $S_h \times S_k = \{\sigma \in S_{h+k} : \sigma(i) \leq h \text{ for all } i \leq h\}$. We therefore make the crucial observation that

$$H^0_{q_0+1,q_1} = H^1_{q_0,q_1+1},$$

so that to prove (1.2.2) it is enough to show that

$$\mathrm{Ind}_{H^1_{q_0,q_1+1}}^{B_{2m-1}} \big(\pi^0_{q_0+1,q_1} \bigoplus \pi^1_{q_0,q_1+1}\big) \cong \mathrm{Ind}_{K_{q_0,q_1}}^{B_{2m-1}}(\pi_{q_0,q_1}). \tag{1.-11}$$

Now we also observe that K_{q_0,q_1} is a subgroup of $H^1_{q_0,q_1+1}$ (of index 2), so that the right-hand side of (1.2.2) becomes $\mathrm{Ind}_{H^1_{q_0,q_1+1}}^{B_{2m-1}}\left(\mathrm{Ind}_{K_{q_0,q_1}}^{H^1_{q_0,q_1+1}}(\pi_{q_0,q_1})\right)$ and therefore we are left to prove that

$$\pi^0_{q_0+1,q_1}\bigoplus\pi^1_{q_0,q_1+1}=\mathrm{Ind}_{K_{q_0,q_1}}^{H^1_{q_0,q_1+1}}(\pi_{q_0.q_1}). \qquad (1.\text{-}11)$$

If we let χ_1 be the character of $\pi^0_{q_0+1,q_1}\bigoplus\pi^1_{q_0,q_1+1}$ and χ_2 be the character of $\mathrm{Ind}_{K_{q_0,q_1}}^{H^1_{q_0,q_1+1}}(\pi_{q_0,q_1})$ we only have to show that $\chi_1(g)=\chi_2(g)$ for all $g\in H^1_{q_0,q_1+1}$.

We have

$$\begin{aligned}\chi_1(g) &= (-1)^{<g,v^0_{q_0+1,q_1}>}+(-1)^{<g,v^1_{q_0,q_1+1}>}\\ &= (-1)^{\sum_{i=2q_0+1}^{2m-2}z_i(g)}+(-1)^{\sum_{i=2q_0+1}^{2m}z_i(g)}\\ &= (1+(-1)^{z_{2m-1}(g)})(-1)^{\sum_{i=2q_0+1}^{2m-2}z_i(g)},\end{aligned}$$

where we have used the fact that $z_{2m}(g)=0$, since $g\in B_{2m-1}$.

As for the character χ_2, we observe that K_{q_0,q_1} is the subgroup of $H^1_{q_0,q_1+1}$ of all the elements g with $z_{2m-1}(g)=0$. So we may take

$$C=\{\mathrm{Id}_{B_{2m-1}},\,\sigma\stackrel{\text{def}}{=}[(1,2,\ldots,2m-2,-(2m-1),2m);0,\ldots,0]\},$$

as a system of coset representatives of $H^1_{q_0,q_1+1}/K_{q_0,q_1}$. Therefore the induced character χ_2 is given by

$$\chi_2(g)=\sum_{\substack{h\in C\\ h^{-1}gh\in K_{p_0,p_1}}}\chi_{\pi_{q_0,q_1}}(h^{-1}gh).$$

Since $g(2m-1)=\pm(2m-1)$ we have that $g\notin K_{q_0,q_1}\Leftrightarrow\ \forall\, h\in C, h^{-1}gh\notin K_{q_0,q_1}$, and hence

$$\chi_2(g)=0\ \forall\, g\in H^1_{q_0,q_1+1}|z_{2m-1}(g)=1,$$

which agrees with $\chi_1(g)$.
So we are left to compute $\chi_2(g)$, where g satisfies $z_{2m-1}(g)=0$. In this case we have $g(2m-1)=2m-1$ which implies $\sigma^{-1}g\sigma=g$, and hence

$$\begin{aligned}\chi_2(g) &= (-1)^{<g,u_{q_0,q_1}>}+(-1)^{<\sigma^{-1}g\sigma,u_{q_0,q_1}>}\\ &= 2(-1)^{<g,u_{q_0,q_1}>}\\ &= 2(-1)^{\sum_{i=2q_0+1}^{2m-2}z_i(g)}.\end{aligned}$$

We conclude that $\chi_1(g)=\chi_2(g)$ for all $g\in H^1_{q_0,q_1+1}$, so (1.2.2) is satisfied and the proof is complete. □

Theorem 1.2.8. *For all $m \in \mathbb{N}$, (M_m, φ) is a B_{2m}-module isomorphic to the direct sum of all the irreducible representations of B_{2m} indexed by the even diagrams of* $\mathrm{Fer}(2,2m)$*, each of such representations occurring once.*

Proof. It is enough to check that the representations (M_m, φ) satisfy the conditions b_0), b_1), b_2) of Proposition 1.2.5.

Condition b_0) is trivial.

In order to check condition b_1), we have to find two submodules of M_m which are isomorphic to the representations indexed by $(\iota_{2m}, \emptyset)$ and $(\emptyset, \iota_{2m})$. By Lemma 1.2.6, they correspond respectively to

$$\rho_{\iota_{2m},\emptyset} = (\mathbb{C}\, C_{[2m]}, \varphi) \quad \text{and } \rho_{\emptyset,\iota_{2m}} = (\mathbb{C}\, C_{\emptyset}, \varphi).$$

Condition b_2) is the content of Proposition 1.2.7 and the proof is complete. □

We are now in a position to fully describe the irreducible decomposition of the submodules M_{p_0,p_1} of M_m via the representation φ.

Theorem 1.2.9. *We have*

$$(M_{p_0,p_1}, \varphi) \cong \bigoplus_{\substack{|\lambda|=2p_0, |\mu|=2p_1 \\ \lambda,\mu \text{ with no odd rows}}} \rho_{\lambda,\mu}.$$

Proof. We start by showing that there exist representations σ of S_{2p_0} and τ of S_{2p_1} such that

$$(M_{p_0,p_1}, \varphi) \cong \mathrm{Ind}_{B_{2p_0}\times B_{2p_1}}^{B_{2m}} (\tilde{\sigma} \odot (\gamma_{2p_1} \otimes \tilde{\tau})), \tag{1.-17}$$

where $\tilde{\sigma}$ and $\tilde{\tau}$ are the natural extensions of σ and τ to B_{2p_0} and to B_{2p_1}, respectively.

Recall the definition of Δ_S given before the statement of Lemma 1.2.6. If we let $M_S \stackrel{\text{def}}{=} \mathrm{Span}\{C_v : v \in \Delta_S\}$, it is clear that

$$M_{p_0,p_1} = M_{[2p_0]} \uparrow_{B_{2p_0}\times B_{2p_1}}^{B_{2m}}.$$

Now, since

$$\begin{aligned} \Delta_{[2p_0]} &= \{v \in B_{2m} : v \text{ is an involution in } S_{2p_0} \times -(S_{2p_1})\} \\ &= \{v : v = v'v'' \text{with } v' \text{ involution in } S_{2p_0} \text{ and } -v'' \text{ involution in } S_{2p_1}\}, \end{aligned}$$

we deduce the isomorphism of vector spaces $M_{[2p_0]} \cong M' \otimes M''$, where

$$M' = \mathrm{Span}\{C_{v'} : v' \text{ is an involution in } S_{2p_0}\}$$

and

$$M'' = \mathrm{Span}\{C_{v''} : v'' \text{ is an involution in } S_{2p_1}\},$$

the isomorphism being given by $C_{v'v''} \leftrightarrow C_{v'} \otimes C_{-v''}$. If $g = (g', g'') \in B_{2p_0} \times B_{2p_1}$ and $v = v'(-v'') \in \Delta_{[2p_0]}$ we have

$$\begin{aligned} \varphi(g)C_{v'} \otimes C_{v''} &\leftrightarrow \varphi(g)C_v \\ &= (-1)^{<g,v>} C_{|g|v|g|^{-1}} \\ &= (-1)^{<g'',-v''>} C_{|g'|v'|g'|^{-1}|g''|(-v'')|g''|^{-1}} \\ &\leftrightarrow C_{|g'|v'|g'|^{-1}} \otimes (-1)^{z(g_2)} C_{|g_2|v''|g_2|^{-1}}. \end{aligned}$$

and equation (1.2.2) follows. Now the full result is a direct consequence of the irreducible decomposition of the representations σ and τ, the description of the irreducible representations given in (1.2.2), and Theorem 1.2.8. □

The next goal is to describe the relationship between the irreducible decomposition of the representations φ and ϱ.

Recall that $\varrho(g)(C_v) = (-1)^{\mathrm{inv}_v(g)}\varphi(g)(C_v)$; we will show that the factor $(-1)^{\mathrm{inv}_v(g)}$ simply exchanges the roles of rows and columns of the Ferrers diagrams appearing in the irreducible decomposition of the B_{2m}-modules (M_m, φ) and (M_m, ϱ).

Lemma 1.2.10. *For $p_0, p_1 \in \mathbb{N}$ with $p_0 + p_1 = m$ let u_{p_0,p_1} and K_{p_0,p_1} be (as in Proposition 1.2.7):*

$$u_{p_0,p_1} = [(2,1,4,3,...,2m,2m-1); \underbrace{0,0,...0}_{2p_0}, \underbrace{1,...,1}_{2p_1}];$$

$$K_{p_0,p_1} = \{g \in B_{2m} : |g| \in S_{2p_0} \times S_{2p_1}, |g|(i+1) = |g|(i) \pm 1 \,\forall\, i \textit{ odd}, \ 0 < i < 2m\}.$$

Then, for every $g \in K_{p_0,p_1}$, we have

$$\mathrm{inv}_{u_{p_0,p_1}}(g) \equiv \mathrm{inv}(|g|) \mod 2.$$

Proof. We can clearly assume that $g = |g|$. Let $\{i,j\}$ be in $\mathrm{Inv}(g)$, but not in $\mathrm{Pair}(|u_{p_0,p_1}|)$. As u_{p_0,p_1} is an involution satisfying $\mathrm{fix}_0(u_{p_0,p_1}) = \mathrm{fix}_1(u_{p_0,p_1}) = 0$, there exist unique h and k such that $\{i,h\}$ and $\{j,k\}$ belong to $\mathrm{Pair}(|u_{p_0,p_1}|)$. We will show that $\{h,k\}$ - which does not belong to $\mathrm{Pair}(|u_{p_0,p_1}|)$ - is an element of $\mathrm{Inv}(g)$. In this way, every pair $\{i,j\} \in \mathrm{Inv}(g) \setminus \mathrm{Pair}(|u_{p_0,p_1}|)$ can be associated to exactly another, so $|\mathrm{Inv}(g) \setminus \mathrm{Pair}(|u_{p_0,p_1|})|$ is even and we get the result.

We can assume that $i < j$ (hence $g(i) > g(j)$) throughout. Observe that we know from the form of u_{p_0,p_1} that $i = h \pm 1$, and $j = k \pm 1$, depending on the parity of i and j. Nevertheless, in all cases, we always obtain $h < k$ (since the four integers i, j, h, k are distinct), so that the claim to prove is always $g(h) > g(k)$. But the definition of K_{p_0,p_1} ensures that $g(h) = g(i) \pm 1$ and $g(k) = g(j) \pm 1$. The result follows from $g(i) > g(j)$, and from the fact that the four integers $g(i), g(j), g(h), g(k)$ are distinct. □

We recall the following general result in representation theory. Let G be a finite group, $H < G$. Let ϑ, τ be representations respectively of G and of H. We have

$$(\vartheta \downarrow_H \otimes \tau) \uparrow^G \cong \vartheta \otimes (\tau \uparrow^G). \qquad (1.\text{-}23)$$

Let us denote by σ_n the linear representation of B_n given by $\sigma_n(g) = (-1)^{\mathrm{inv}(|g|)}$.

Lemma 1.2.11. *For all $(\lambda, \mu) \in \mathrm{Fer}(2, n)$ we have*

$$\sigma_n(g) \otimes \rho_{\lambda,\mu} = \rho_{\lambda',\mu'},$$

where λ' and μ' denote the conjugate partitions of λ and μ respectively (see the definition on page 2).

Proof. We recall the following well-known analogous fact for the symmetric group. We have

$$\epsilon \otimes \rho_\lambda = \rho_{\lambda'}, \qquad (1.\text{-}23)$$

where $\epsilon(g) \stackrel{\mathrm{def}}{=} (-1)^{\mathrm{inv}(g)}$ denotes the alternating representation. If we let $k = |\lambda|$ then, by Equations (1.2.2) and (1.2.2), we have

$$\begin{aligned}
\sigma_n \otimes \rho_{\lambda,\mu} &= \sigma_n \otimes \mathrm{Ind}_{B_k \times B_{n-k}}^{B_n} (\tilde{\rho}_\lambda \odot (\gamma_{n-k} \otimes \tilde{\rho}_\mu)) \\
&\cong \mathrm{Ind}_{B_k \times B_{n-k}}^{B_n} \big(\sigma_n \downarrow_{B_k \times B_{n-k}} \otimes (\tilde{\rho}_\lambda \odot (\gamma_{n-k} \otimes \tilde{\rho}_\mu))\big) \\
&= \mathrm{Ind}_{B_k \times B_{n-k}}^{B_n} \big((\sigma_n \downarrow_{B_k} \otimes \tilde{\rho}_\lambda) \odot (\sigma_n \downarrow_{B_{n-k}} \otimes \gamma_{n-k} \otimes \tilde{\rho}_\mu)\big) \\
&= \mathrm{Ind}_{B_k \times B_{n-k}}^{B_n} \big((\widetilde{\epsilon \otimes \rho_\lambda}) \odot (\gamma_{n-k} \otimes (\widetilde{\epsilon \otimes \rho_\mu}))\big) \\
&= \mathrm{Ind}_{B_k \times B_{n-k}}^{B_n} \big(\tilde{\rho}_{\lambda'} \odot (\gamma_{n-k} \otimes \tilde{\rho}_{\mu'})\big) \\
&= \rho_{\lambda',\mu'},
\end{aligned}$$

and the proof is complete. □

Theorem 1.2.12. *The submodule $M_{p_0,p_1} = M(c_{0,0,p_0,p_1})$ of (M, ϱ) satisfies*

$$(M_{p_0,p_1}, \varrho) \cong \bigoplus_{\substack{|\lambda|=2p_0, |\mu|=2p_1 \\ \lambda,\mu \text{ with no odd columns}}} \rho_{\lambda,\mu}.$$

Proof. Let us consider the linear representation of K_{p_0,p_1}

$$(-1)^{\mathrm{inv}_{u_{p_0,p_1}}(g)} \pi_{p_0,p_1}(g) = (-1)^{\mathrm{inv}_{u_{p_0,p_1}}(g)} (-1)^{<g, u_{p_0,p_1}>}.$$

We have

$$\begin{aligned}(M_{p_0,p_1},\varrho) &= ((-1)^{\mathrm{inv}_{u_{p_0,p_1}}(g)}\pi_{p_0,p_1})\big\uparrow_{K_{p_0,p_1}}^{B_{2m}} = ((-1)^{\mathrm{inv}(|g|)}\pi_{p_0,p_1})\big\uparrow_{K_{p_0,p_1}}^{B_{2m}} \\ &= \Big((-1)^{\mathrm{inv}(|g|)}\big\downarrow_{K_{p_0,p_1}} \otimes \ \pi_{p_0,p_1})\Big)\big\uparrow_{K_{p_0,p_1}}^{B_{2m}} \\ &\cong (-1)^{\mathrm{inv}(|g|)} \otimes \ (\pi_{p_0,p_1} \uparrow^{B_{2m}}) = (-1)^{\mathrm{inv}(|g|)} \otimes \ (M_{p_0,p_1},\varphi),\end{aligned}$$

where we have used Lemma 1.2.10 in the first line and equation (1.2.2) in the last line of the previous equalities. Now the result follows from Lemma 1.2.11 and Theorem 1.2.9. □

1.2.3 B_n: the proof of the full result

In this section we will give a complete proof in the case of B_n of Theorem 1.1.3 that, by Proposition 1.2.2, can be restated in the following slightly different but equivalent form.

Theorem 1.2.13. *For all $f_0, f_1, p_0, p_1 \in \mathbb{N}$ such that $f_0 + f_1 + 2p_0 + 2p_1 = n$ we have*

$$(M(c_{f_0,f_1,p_0,p_1}),\varrho) \cong \bigoplus_{\substack{|\lambda|=2p_0+f_0,|\mu|=2p_1+f_1 \\ \lambda \text{ with exactly } f_0 \text{ odd columns} \\ \mu \text{ with exactly } f_1 \text{ odd columns}}} \rho_{\lambda,\mu}.$$

Proof. Let $m = p_0+p_1$ and consider the space $M(c_{0,0,p_0,p_1})$: it is a B_{2m}-module via the representation

$$\Pi_{p_0,p_1} \stackrel{\text{def}}{=} (M(c_{0,0,p_0,p_1}),\varrho) = \mathrm{Ind}_{K_{p_0,p_1}}^{B_{2m}}(\tau_{p_0,p_1}),$$

where τ_{p_0,p_1} is the linear K_{p_0,p_1} representation given by $\tau_{p_0,p_1}(g) = (-1)^{\mathrm{inv}(|g|)}\pi_{p_0,p_1}(g)$. From Theorem 1.2.12, we know that it is the multiplicity-free direct sum of all representations indexed by pairs of diagrams (λ,μ) where λ and μ have even columns only, and $|\lambda| = 2p_0, |\mu| = 2p_1$.
We will first show that

$$(M(c_{f_0,f_1,p_0,p_1}),\varrho) = \mathrm{Ind}_{B_{2m}\times B_{n-2m}}^{B^n}(\Pi_{p_0,p_1} \odot \rho_{\iota_{f_0},\iota_{f_1}}). \qquad (1.\text{-}23)$$

Let us argue with the same strategy as in §1.2.2. We define the involution u representing the S_n-conjugacy class c_{f_0,f_1,p_0,p_1} as follows:

$$u = [(2,1,4,3,\ldots,2m,2m-1,2m+1,\ldots,n);\underbrace{0,\ldots,0}_{2p_0},\underbrace{1,\ldots,1}_{2p_1},\underbrace{0,\ldots,0}_{f_0},\underbrace{1,\ldots,1}_{f_1}].$$

We have that the stabilizer of u with respect to the absolute conjugation is $\{g \in B_n : |g|u|g|^{-1} = u\} = K_{p_0,p_1} \times B_{f_0} \times B_{f_1}$, and we can easily check that

$$(M(c_{f_0,f_1,p_0,p_1}),\varrho) = \mathrm{Ind}_{K_{p_0,p_1}\times B_{f_0}\times B_{f_1}}^{B_n}\big(\tau_{p_0,p_1} \odot \rho_{\iota_{f_0},\emptyset} \odot \rho_{\emptyset,\iota_{f_1}}\big).$$

We recall the following identity of induced representations: if $H < G$ and $H' < G'$ we have

$$\mathrm{Ind}_{H\times H'}^{G\times G'}(\rho\odot\rho')=\mathrm{Ind}_H^G(\rho)\odot\mathrm{Ind}_{H'}^{G'}(\rho'), \tag{1.-23}$$

where ρ is a representation of H and ρ' a representation of H'. So we have

$$\begin{aligned}(M(c_{f_0,f_1,p_0,p_1}),\varrho) &= \mathrm{Ind}_{K_{p_0,p_1}\times B_{f_0}\times B_{f_1}}^{B_n}\big(\tau_{p_0,p_1}\odot\rho_{\iota_{f_0},\emptyset}\odot\rho_{\emptyset,\iota_{f_1}}\big)\\ &= \mathrm{Ind}_{B_{2m}\times B_{n-2m}}^{B^n}\big(\mathrm{Ind}_{K_{p_0,p_1}\times B_{f_0}\times B_{f_1}}^{B_{2m}\times B_{n-2m}}(\tau_{p_0,p_1}\odot\rho_{\iota_{f_0},\emptyset}\odot\rho_{\emptyset,\iota_{f_1}})\big)\\ &= \mathrm{Ind}_{B_{2m}\times B_{n-2m}}^{B^n}\big(\mathrm{Ind}_{K_{p_0,p_1}}^{B_{2m}}(\tau_{p_0,p_1})\odot\mathrm{Ind}_{B_f\times B_{f_1}}^{B_{n-2m}}(\rho_{\iota_{f_0},\emptyset}\odot\rho_{\emptyset,\iota_{f_1}})\big)\\ &= \mathrm{Ind}_{B_{2m}\times B_{n-2m}}^{B^n}(\Pi_{p_0,p_1}\odot\rho_{\iota_{f_0},\iota_{f_1}})\end{aligned}$$

and equation (1.2.3) is achieved. Now the result follows from Theorem 1.2.12 and the following result which is the analogue in type B of the well-known Pieri rule:

Proposition 1.2.14. *Let $\rho_{\lambda,\mu}$ be any irreducible representation of B_h. Then*

$$\mathrm{Ind}_{B_h\times B_k}^{B_{h+k}}(\rho_{\lambda,\mu}\odot\rho_{\iota_f,\iota_{k-f}})=\bigoplus\rho_{\nu,\xi},$$

where the direct sum runs through all $(\nu,\xi)\in\mathrm{Fer}(2,h+k)$ such that ν is obtained from λ by adding f boxes to its Ferrers diagram, no two in the same column, and ξ is obtained from μ by adding $k-f$ boxes to its Ferrers diagram, no two in the same column.

For the proof, see [10, Lemma 6.1.3]. □

Example 1.2.15. For every $f_0,f_1\in[0,n]$, with $f_0+f_1=n$ let us consider the set $\mathrm{Sh}(c_{f_0,f_1,0,0})$. Since ι_k is the only k-boxed diagram with k odd columns, $\mathrm{Sh}(c_{f_0,f_1,0,0})$ contains the pair $(\iota_{f_0},\iota_{f_1})$ only. Thus we can explicitly find in (M,ϱ) the subspace $V_{\iota_{f_0},\iota_{f_1}}$ affording the representation $\rho_{\iota_{f_0},\iota_{f_1}}$: thanks to Theorem 1.1.3,

$$V_{\iota_{f_0},\iota_{f_1}}=M(c_{f_0,f_1,0,0})=\mathrm{Ind}_{B_{f_0}\times B_{f_1}}^{B_n}(\mathbb{C}\,C_{u_{f_0,f_1,0,0}}).$$

$u_{f_0,f_1,0,0}$ being the involution

$$u_{f_0,f_1,0,0}=[(1,2,\ldots,n);\underbrace{0,\ldots,0}_{f_0},\underbrace{1,\ldots,1}_{f_1}].$$

In other words

$$V_{\iota_{f_0},\iota_{f_1}}=\mathrm{Span}\{C_v : v\in B_n,\ |v|=\mathrm{Id},\ \#\{i:z(i)=0\}=f_0,\ \#\{i:z(i)=1\}=f_1\}.$$

Example 1.2.16. Let $v = [(6, 4, 3, 2, 5, 1); 1, 0, 0, 0, 1, 1] \in B_6$. In this case, $f_0 = f_1 = p_0 = p_1 = 1$ and the S_n-conjugacy class c of v has 180 elements. The B_n-module $M(c)$ is given by the sum of the irreducible representations indexed by $(\lambda, \mu) \in \mathrm{Fer}(2, n)$ such that both λ and μ have size 3, and exactly one column of odd length:

$$M(c) \cong \rho_{((2,1),(1,1,1))} \oplus \rho_{((1,1,1),(2,1))} \oplus \rho_{((2,1),(2,1))} \oplus \rho_{((1,1,1),(1,1,1))}.$$

1.3 The general case of wreath products $G(r, n)$

In this section we will treat the general case $G = G(r, n)$. To prove Theorem 1.1.3, we will be handling the same tools already used in the case of B_n. Nevertheless, as some of the results need to be slightly generalized, we will provide an outline of the whole argument in this wider setting.

Let M be the model for $G(r, n)$ described in Theorem 1.1.2. Let φ be the representation defined analogously to the case of B_n:

$$\begin{aligned} \varphi(g) : & M \to M \\ & C_v \mapsto (-1)^{<g,v>} C_{|g|v|g|^{-1}}. \end{aligned}$$

The S_n-conjugacy classes of absolute involutions of $G(r, n)$ are indexed by $2r$-tuples $(f_0, \dots, f_{r-1}, p_0, \dots, p_{r-1})$ satisfying $f_0 + \cdots + f_{r-1} + 2(p_0 + \cdots + p_{r-1}) = n$. These are given by

$$c_{f_0,\dots,f_{r-1},p_0,\dots,p_{r-1}} = \{v \in I(r, n) : \mathrm{fix}_i(v) = f_i, \mathrm{pair}_i(v) = p_i \, \forall i \in [0, r-1]\}.$$

where

$$\begin{aligned} \mathrm{fix}_i(v) &= |\{j \in [n] : v(j) = \zeta_r^i j\}| \\ \mathrm{pair}_i(v) &= |\{(h, k) : 1 \le h < k \le n, \, v(h) = \zeta_r^i k \text{ and } v(k) = \zeta_r^i h\}|. \end{aligned}$$

The main idea is, again, focusing on the submodule with no fixed points first. Our half-way result is

Theorem 1.3.1. *Let $M_{m,r}$ be the subspace of M spanned by the elements C_v as v varies among all involutions in $G(r, 2m)$ such that* $\mathrm{fix}_0(v) = \mathrm{fix}_1(v) = \ldots = \mathrm{fix}_{r-1}(v) = 0$*:*

$$M_{m,r} \stackrel{\mathrm{def}}{=} \bigoplus_{p_0 + \cdots + p_{r-1} = m} M(c_{0,\dots,0,p_0,\dots,p_{r-1}}).$$

Then $(M_{m,r}, \varphi)$ is a $G(r, 2m)$-module isomorphic to the direct sum of all the irreducible representations of $G(r, 2m)$ indexed by the diagrams of $\mathrm{Fer}(r, 2m)$

whose rows have an even number of boxes, each such representation occurring once.

We state here the $G(r,n)$-generalized version of Proposition 1.2.5, which will be applied to $M_{m,r}$.

Proposition 1.3.2. *Let Π_m^r be representations of $G(r,2m)$, m ranging in $\mathbb{N}$. Then the following are equivalent:*

a) for every m, Π_m^r is the direct sum of all the irreducible representations of $G(r,2m)$ indexed by r-tuples of even diagrams, each of such representations occurring once;

b) for every m,

b_0) Π_0^r is unidimensional;

b_1) the module Π_m^r contains the irreducible representations of $G(r,2m)$ indexed by the r r-tuples of diagrams $(\emptyset,\dots,\emptyset,\iota_{2m},\emptyset,\dots,\emptyset)$.

b_2) the following isomorphism holds:

$$\Pi_m^r\downarrow_{G(r,2m-1)}\cong\Pi_{m-1}^r\uparrow^{G(r,2m-1)}; \qquad (1.\text{-}29)$$

Here is the generalization of the branching rule for $G(r,n)$, which is an essential ingredient for the proof of Proposition 1.3.2.

Lemma 1.3.3 (Branching rule for $G(r,n)$)**.** *Let $(\lambda^{(0)},\dots,\lambda^{(r-1)})\in \mathrm{Fer}(r,n)$. Then the following holds:*

$$\rho_{\lambda^{(0)},\dots,\lambda^{(r-1)}}\downarrow_{G(r,n-1)}=\bigoplus_{(\mu^{(0)},\dots,\mu^{(r-1)})\in R^-_{\lambda^{(0)},\dots,\lambda^{(r-1)}}}\rho_{\mu^{(0)},\dots,\mu^{(r-1)}};$$

$$\rho_{\lambda^{(0)},\dots,\lambda^{(r-1)}}\uparrow^{G(r,n+1)}=\bigoplus_{(\mu^{(0)},\dots,\mu^{(r-1)})\in R^+_{\lambda^{(0)},\dots,\lambda^{(r-1)}}}\rho_{\mu^{(0)},\dots,\mu^{(r-1)}},$$

where we denote by $R^+_{\lambda^{(0)},\dots,\lambda^{(r-1)}}$ the set of diagrams in $\mathrm{Fer}(r,n+1)$ obtained by adding one box to the diagram $(\lambda^{(0)},\dots,\lambda^{(r-1)})$, and similarly for $R^-_{\lambda^{(0)},\dots,\lambda^{(r-1)}}$.

Proof of Theorem 1.3.2. a)$\Rightarrow$ b). b_0 and b_1 are trivial. So let us now turn to compare $\Pi_{m,r}\downarrow_{G(r,2m-1)}$ and $\Pi_{m-1,r}\uparrow^{G(r,2m-1)}$. The branching rule ensures that $\Pi_{m,r}\downarrow_{G(r,2m-1)}$ contains exactly the $\rho_{\lambda_0,\dots,\lambda_{r-1}}$'s where one of the Ferrers diagrams is 1-odd. Furthermore, the r-tuple whose restriction contains the pair $(\lambda_0,\dots,\lambda_{r-1})$ is uniquely determined: to obtain it, it will only be allowed to add a box on the odd row of the right λ_i. This means that $\Pi_{m,r}\downarrow_{G(r,2m-1)}$

is the multiplicity-free direct sum of all the representations of $G(r,2m-1)$ indexed by r-tuples of Ferrers diagrams with one 1-odd Ferrers diagram. Arguing analogously for $\Pi_{m-1,r}\uparrow^{G(r,2m-1)}$, we can infer that it contains exactly the same irreducible representations with multiplicity 1 and it is thus isomorphic to $\Pi_{m,r}\downarrow_{G(r,2m-1)}$.

b)$\Rightarrow$ a) Let us argue by induction.

The case $m=0$ is given by b3). Let us also see the case $m=1$. We know that

$$\Pi_{1,r}\downarrow_{G(r,1)}\cong \Pi_0\uparrow^{G(r,1)}\cong (\square,\emptyset,\dots,\emptyset)\oplus\dots\oplus(\emptyset,\dots,\emptyset,\square).$$

But $\Pi_{1,r}$ contains $(\square\square,\emptyset,\dots,\emptyset)\oplus\dots\oplus(\emptyset,\dots,\emptyset,\square\square)$ by b$_2$), and the isomorphism

$$\big((\square\square,\emptyset,\dots,\emptyset)\oplus\dots\oplus(\emptyset,\dots,\emptyset,\square\square)\big)\downarrow_{G(r,1)}\cong(\square,\emptyset,\dots,\emptyset)\oplus\dots\oplus(\emptyset,\dots,\emptyset,\square)\cong\Pi_{0,r}\uparrow^{G(r,1)}$$

ensures that

$$\Pi_{1,r}\cong(\square\square,\emptyset,\dots,\emptyset)\oplus\dots\oplus(\emptyset,\dots,\emptyset,\square\square).$$

Let us show that, if $\Pi_{m-1,r}$ is the direct sum of all the representations indexed by r-tuples of even Ferrers diagrams, the same holds for $\Pi_{m,r}$. For notational convenience, we let

$$\Lambda_{m,r}\stackrel{\text{def}}{=}\{(\lambda_0,\dots,\lambda_{r-1})\in \mathrm{Fer}(r,2m):\ \rho_{\lambda_0,\dots,\lambda_{r-1}}\text{ is a subrepresentation of }\Pi_{m,r}\}$$

First we shall see that, if $(\lambda_0,\dots,\lambda_{r-1})\in\mathrm{Fer}(r,2m)$ is an even diagram, then $(\lambda_0,\dots,\lambda_{r-1})\in\Lambda_{m,r}$.

The set $\mathrm{Fer}(r,2m)$ is totally ordered in this way: given two r-tuples $(\lambda_0,\dots,\lambda_{r-1})$, $(\mu_0,\dots,\mu_{r-1})\in\mathrm{Fer}(r,2m)$, we let $(\lambda_0,\dots,\lambda_{r-1})<(\mu_0,\dots,\mu_{r-1})$ if one of the following holds:

i) $\lambda_o<\mu_0$ lexicographically;

ii) there exists $k<r-1$ such that $\lambda_i=\mu_i$ for every $i<k$ and $\lambda_k<\mu_k$ lexicographically.

We observe that $(\iota_{2m},\emptyset,\dots,\emptyset)$ is the maximum element of $\mathrm{Fer}(r,2m)$ with respect to this order.

We claim that if $(\lambda_0,\dots,\lambda_{r-1})\in\mathrm{Fer}(r,2m)$ is such that:

i) $(\lambda_0,\dots,\lambda_{r-1})$ is even;

ii) $(\lambda_0,\dots,\lambda_{r-1})\neq(\emptyset,\dots,\emptyset,\iota_{2m},\emptyset,\dots,\emptyset)$;

iii) $(\mu_0,\dots,\mu_{r-1})\in\Lambda_{m,r}$ for all $(\mu_0,\dots,\mu_{r-1})\in\mathrm{Fer}(r,2m)$ such that

$$(\mu_0,\dots,\mu_{r-1})\text{ is even and }(\mu_0,\dots,\mu_{r-1})>(\lambda_0,\dots\lambda_{r-1}).$$

Then $(\lambda_0,\dots,\lambda_{r-1})\in\Lambda_{m,r}$.

As we already know that all the representations of the form $(\emptyset,\dots,\emptyset,\iota_{2m},\emptyset,\dots,\emptyset)$ are contained in $\Lambda_{m,r}$, once we prove the claim, all the even r-tuples will.

Proof of the claim. Let $(\lambda_0,\dots,\lambda_{r-1})\in \mathrm{Fer}(r,2m)$ be an even diagram satisfying i), ii) and iii). Then the r-tuple $(\lambda_0,\dots,\lambda_{r-1})$ has at least two rows. We let $(\mu_0,\dots,\mu_{r-1})\in\mathrm{Fer}(r,2m)$ be the pair obtained from $(\lambda_0,\dots,\lambda_{r-1})$ by deleting two boxes in the last non zero row and adding two boxes in the first non-zero row. As $(\mu_0,\dots,\mu_{r-1})>(\lambda_0,\dots,\lambda_{r-1})$, we have $(\mu_0,\dots,\mu_{r-1})\in\Lambda_{m,r}$, so the isomorphism (1.3.2) leads to the following:

$$\forall\ (\sigma_0,\dots,\sigma_{r-1})\in R^-_{\mu_0,\dots,\mu_{r-1}},\ R^+_{\sigma_0,\dots,\sigma_{r-1}}\cap\Lambda_{m,r}=\{(\mu_0,\dots,\mu_{r-1})\}. \quad (1.\text{-}29)$$

Now let $(\alpha_0,\dots,\alpha_{r-1})\in\mathrm{Fer}(r,2m-1)$ be obtained from $(\lambda_0,\dots,\lambda_{r-1})$ by deleting one box in the last nonzero row. Our induction hypothesis ensures that $\rho_{\alpha_0,\dots,\alpha_{r-1}}$ is a subrepresentation of $\Pi_{m-1,r}\uparrow^{G(2m-1,r)}$ with multiplicity one. So the isomorphism (1.3.2) implies that there exists a unique $(\gamma_0,\dots,\gamma_{r-1})\in \mathrm{Fer}(r,2m)$ such that

$$\{(\gamma_0,\dots,\gamma_{r-1})\}=R^+_{\alpha_0,\dots,\alpha_{r-1}}\cap\Lambda_{m,r}. \quad (1.\text{-}29)$$

The claim will be proved if we show that $(\gamma_0,\dots,\gamma_{r-1})=(\lambda_0,\dots,\lambda_{r-1})$.

The pair $(\gamma_0,\dots,\gamma_{r-1})$ is obtained from $(\alpha_0,\dots,\alpha_{r-1})$ by adding a single box, since $(\gamma_0,\dots,\gamma_{r-1})\in R^+_{\alpha_0,\dots,\alpha_{r-1}}$. If such box is not added in the first or in the last non zero rows of $(\alpha_0,\dots,\alpha_{r-1})$ then $(\gamma_0,\dots,\gamma_{r-1})$ has two rows of odd length and one can check that $R^-_{\gamma_0,\dots,\gamma_{r-1}}$ contains at least a diagram with three rows of odd length. This contradicts (1.3.2).

Now assume that $(\gamma_0,\dots,\gamma_{r-1})$ is obtained by adding a box in the first nonzero row of $(\alpha_0,\dots,\alpha_{r-1})$. If we let $(\sigma_0,\dots,\sigma_{r-1})$ be the pair obtained from $(\lambda_0,\dots,\lambda_{r-1})$ by deleting two boxes in the last nonzero row and adding one box in the first nonzero row, we have $(\sigma_0,\dots,\sigma_{r-1})\in R^-_{\mu_0,\dots,\mu_{r-1}}$, and

$$R^+_{\sigma_0,\dots,\sigma_{r-1}}\cap\Lambda_{m,r}\supseteq\{(\mu_0,\dots,\mu_{r-1}),(\gamma_0,\dots,\gamma_{r-1})\}$$

which contradicts the unicity of $(\gamma_0,\dots,\gamma_{r-1})$ in (1.3). Therefore $(\gamma_0,\dots,\gamma_{r-1})$ is obtained by adding a box in the last nonzero row of $(\alpha_0,\dots,\alpha_{r-1})$, i.e. $(\gamma_0,\dots,\gamma_{r-1})=(\lambda_0,\dots,\mu_{r-1})$ and the claim is proved.

We have just proved that if we let $\Pi^{even}_{m,r}$ the multiplicity free sum of all irreducible representations of $G(2m,r)$ indexed by even diagrams we have that $\Pi^{even}_{m,r}$ is a subrepresentation of $\Pi_{m,r}$. The result follows since we also have

$$\Pi^{even}_{m,r}\downarrow_{G(2m-1,r)}\cong\Pi_{m-1,r}\uparrow^{G(2m-1,r)},$$

and so, in particular, $\dim(\Pi_{m,r}^{even}) = \dim(\Pi_{m,r})$. □

Let us check that $M_{m,r}$ satisfies properties b) of Proposition 1.3.2, so that Theorem 1.3.1 follows.

Property b_0) is trivial and so we look at property b_1): for $S_0,\ldots,S_{r-1}$ disjoint subsets of $[2m]$ such that $\cup S_i = [2m]$ we let

$$\Delta_{S_0,\ldots,S_{r-1}} \stackrel{\text{def}}{=} \{v \mid v \text{ is an absolute involution of } G(r,2m) \text{ with:} \\ \text{fix}_0(v) = \ldots = \text{fix}_{r-1}(v) = 0; z_i(v) = j \text{ iff } i \in S_j\},$$

and

$$C_{S_0,\ldots,S_{r-1}} = \sum_{v\in\Delta_{S_0,\ldots,S_{r-1}}} C_v \in M.$$

Lemma 1.3.4. *The subspace of $M_{m,r}$ spanned by all $C_{S_0,\ldots,S_{r-1}}$, with $|S_i| = p_i$, is an irreducible submodule of $(M_{m,r},\varphi)$ affording the representation $\rho_{\iota_{2p_0},\ldots,\iota_{2p_{r-1}}}$.*

Proof. This proof can be carried on in the same way as in the case of B_n, relying on Proposition 0.4.2. □

Let us turn to property b_2). We have to check that

$$M_{m,r}\downarrow_{G(r,2m-1)} \cong M_{m-1,r}\uparrow^{G(r,2m-1)}.$$

We let $M_{p_0,\ldots,p_{r-1}} = M(c_{0,\ldots,0,p_0,\ldots,p_{r-1}})$. First of all, the following decomposition holds:

$$M_{m,r}\downarrow_{G(r,2m-1)} = \bigoplus_{p_0+\cdots+p_{r-1}=m} M_{p_0,\ldots,p_{r-1}}\downarrow_{G(r,2m-1)} \\ = \bigoplus_{p_0+\cdots+p_{r-1}=m}\bigoplus_{j=o}^{r-1} M^j_{p_0,\ldots,p_{r-1}},$$

$M^j_{p_0,\ldots,p_{r-1}}$ being the submodule of $M_{p_0,\ldots,p_{r-1}}$ spanned by the absolute involutions v such that $z_{2m}(v) = j$.

As the module $M^j_{p_0,\ldots,p_{r-1}}$ is trivial whenever $p_j = 0$, we can reduce ourselves to

$$\bigoplus_{q_0+\cdots+q_{r-1}=m-1}\bigoplus_{j=o}^{r-1} M^j_{q_0,\ldots,q_j+1,\ldots,q_{r-1}}$$

We introduce the absolute involution

$$v^j_{q_0,\ldots,q_{r-1}} \stackrel{\text{def}}{=} [(2,1,4,3,...,2m,2m-1);\underbrace{0,0,...0}_{2q_0},\underbrace{1,...,1}_{2q_1},\ldots,\underbrace{j,\ldots,j}_{2q_j},\ldots,\underbrace{r-1,\ldots,r-1}_{2q_{r-1}},j,j].$$

Its stabilizer with respect to the absolute conjugation does not depend on j: it is the subgroup of $G(r,2m-1)$ given by

$$H_{q_0,\ldots,q_{r-1}} = \{g\in G : |g| \in S_{2q_0}\times\ldots\times S_{2q_{r-1}}, |g|(i+1) = |g|(i)\pm 1\,\forall\, i \text{ odd}, 0<i<2m\}.$$

Thus, our module can be written as

$$\bigoplus_{q_0+\cdots+q_{r-1}=m-1}\bigoplus_{j=o}^{r-1} M^j_{q_0,\ldots,q_j+1,\ldots,q_{r-1}} = \bigoplus_{q_0+\cdots+q_{r-1}=m-1}\bigoplus_{j=o}^{r-1}(\mathbb{C}\, v^j_{q_0,\ldots,q_{r-1}})\big\uparrow^{G(r,2m-1)}_{H_{q_0,\ldots,q_{r-1}}}.$$

As for the right side of the isomorphism, we have

$$M_{m-1,r}\uparrow^{G(r,2m-1)} = \bigoplus_{q_0+\cdots+q_{r-1}=m-1} M_{q_0,\ldots,q_{r-1}}\big\uparrow^{G(r,2m-1)}.$$

We choose this time

$$u_{q_0,\ldots,q_{r-1}} \stackrel{\text{def}}{=} [(2,1,4,3,...,2m-2,2m-3);\underbrace{0,0,...0}_{2q_0},\underbrace{1,...,1}_{2q_1},\ldots,\underbrace{r-1,\ldots,r-1}_{2q_{r-1}}],$$

whose stabilizer with respect to the absolute conjugation in $G(r,2(m-1))$ is

$$K_{q_0,\ldots,q_{r-1}} = \{g\in G : |g|\in S_{2q_0}\times\ldots\times S_{2q_{r-1}}, |g|(i+1) = |g|(i)\pm 1\,\forall\, i \text{ odd}, 0<i<2m-2\}.$$

We observe that $K_{q_0,\ldots,q_{r-1}}$ is a subgroup of index r in $H_{q_0,\ldots,q_{r-1}}$, and a system of coset representatives is given by

$$C = \{\sigma_i \stackrel{\text{def}}{=} [(1,\ldots,2m);\underbrace{0,...,0}_{2(m-1)},i,0]\}_{i=0,\ldots,r-1}.$$

So we can split the induction into two steps, and we get

$$\begin{aligned} M_{m-1,r}\uparrow^{G(r,2m-1)} &= \bigoplus_{q_0+\cdots+q_{r-1}=m-1} M_{q_0,\ldots,q_{r-1}}\big\uparrow^{G(r,2m-1)} \\ &= \bigoplus_{q_0+\cdots+q_{r-1}=m-1} (\mathbb{C}\, u_{q_0,\ldots,q_{r-1}})\big\uparrow^{G(r,2m-1)}_{K_{q_0,\ldots,q_{r-1}}} \\ &= \bigoplus_{q_0+\cdots+q_{r-1}=m-1} \Big((\mathbb{C}\, u_{q_0,\ldots,q_{r-1}})\big\uparrow^{H_{q_0,\ldots,q_{r-1}}}_{K_{q_0,\ldots,q_{r-1}}}\Big)\Big\uparrow^{G(r,2m-1)}_{H_{q_0,\ldots,q_{r-1}}} \end{aligned}$$

We are enquiring if

$$\bigoplus_{q_0+\cdots+q_{r-1}=m-1}\bigoplus_{j=o}^{r-1}\Big(\mathbb{C}\, v^j_{q_0,\ldots,q_{r-1}}\Big)\Big\uparrow^{G(r,2m-1)}_{H_{q_0,\ldots,q_{r-1}}} \cong$$

$$\bigoplus_{q_0+\cdots+q_{r-1}=m-1}\Big((\mathbb{C}\, u_{q_0,\ldots,q_{r-1}})\big\uparrow^{H_{q_0,\ldots,q_{r-1}}}_{K_{q_0,\ldots,q_{r-1}}}\Big)\Big\uparrow^{G(r,2m-1)}_{H_{q_0,\ldots,q_{r-1}}},$$

and all we need to show is that

$$\bigoplus_{j=o}^{r-1}\mathbb{C}\, v^j_{q_0,\ldots,q_{r-1}} \cong (\mathbb{C}\, u_{q_0,\ldots,q_{r-1}})\big\uparrow^{H_{q_0,\ldots,q_{r-1}}}_{K_{q_0,\ldots,q_{r-1}}}$$

as $H_{q_0,\dots,q_{r-1}}$-modules. Let us compare their characters. The character χ_1 of the representation on the left is given by

$$\chi_1(g) = \sum_{j=0}^{r-1} \zeta_r^{<g,v^j_{q_0,\dots,q_{r-1}}>} = \zeta_r^{<g,u_{q_0,\dots,q_{r-1}}>} \sum_{j=0}^{r-1} \zeta_r^{jz_{2m-1}(g)}$$
$$= \begin{cases} 0 & \text{if } z_{2m-1}(g) \neq 0; \\ r\zeta_r^{<g,u_{q_0,\dots,q_{r-1}}>} & \text{if } z_{2m-1}(g) = 0. \end{cases}$$

As for the character χ_2 of the representation on the right, we have

$$\chi_2(g) = \sum_{\substack{h\in C \\ h^{-1}gh\in B_{q_0,\dots,q_{r-1}}}} \chi(h^{-1}gh)$$
$$= \begin{cases} 0 & \text{if } z_{2m-1}(g) \neq 0; \\ r\zeta_r^{<g,u_{q_0,\dots,q_{r-1}}>} & \text{if } z_{2m-1}(g) = 0, \end{cases}$$

so the two characters agree and the representations are isomorphic.

So we know that the modules $M_{m.r}$ satisfy the conditions of Proposition 1.3.2 and to complete the proof of Theorem 1.3.1, generalizing what was done for B_n, it suffices to show that there exist representations σ_0 of $S_{2p_0}, \dots, \sigma_{r-1}$ of $S_{2p_{r-1}}$ such that

$$(M_{p_0,\dots,p_{r-1}}, \varphi) \cong \mathrm{Ind}_{G(r,2p_0)\times\dots G(r,2p_{r-1})}^{G(r,2m)} (\tilde\sigma_0 \odot (\gamma_{2(p_1)} \otimes \tilde\sigma_1) \odot \cdots \odot (\gamma_{2(p_{r-1})}^{r-1} \otimes \tilde\sigma_{r-1})), \tag{1.-29}$$

where the $\tilde\sigma_i$'s are the natural extensions of σ_i to $G(r,2p_i)$.

If we set $S_i \stackrel{\text{def}}{=} [p_0 + \cdots + p_{i-1} + 1, p_0 + \cdots + p_{i-1} + p_i]$, we consider the vector space $M_{S_0,\dots,S_{r-1}} \stackrel{\text{def}}{=} \mathrm{Span}\{C_v : v \in \Delta_{S_0,\dots,S_{r-1}}\}$. We have

$$M_{p_0,\dots,p_{r-1}} = M_{S_0,\dots,S_{r-1}} \uparrow_{G(r,2p_0)\times\cdots\times G(r,2p_{r-1})}^{G(r,2m)} .$$

Let us define $M_i \stackrel{\text{def}}{=} \mathrm{Span}\{C_{v_i} : v_i \text{ is an involution in } S_{2p_i}\}$. Then

$$M_{S_0,\dots,S_{r-1}} \cong M_0 \times \cdots \times M_{r-1}$$
$$C_{v_0,\dots,v_{r-1}} \mapsto C_{v_0} \otimes \zeta_r C_{v_1} \otimes \cdots \otimes \zeta_r^{r-1} C_{v_{r-1}}$$

Arguing as for B_n, let $g = g_0, g_1, \dots, g_{r-1} \in G(r,2p_0) \times \cdots \times G(r,2p_{r-1})$. We get

$$\varphi(g) C_{v_0} \otimes \cdots \otimes C_{v_{r-1}} \leftrightarrow \varphi(g) C_v = (\zeta_r)^{<g,v>} C_{|g|v|g|^{-1}}$$
$$\leftrightarrow C_{|g_0|v_0|g_0|^{-1}} \otimes (\zeta_r)^{z(g_1)} C_{|g_1|v_1|g_1|^{-1}} \otimes \cdots \otimes (\zeta_r)^{(r-1)z(g_{r-1})} C_{|g_{r-1}|v_{r-1}|g_{r-1}|^{-1}}$$

and equation (1.3) is achieved. Our claim follows from the irreducible decomposition of the representations σ_i, the description of the irreducible representations of $G(r,n)$ in Proposition 0.4.2, and Theorem 1.3.1.

Before leaving the module $M_{m,r}$ with no fixed points and going on to study the decomposition of the whole model M, we only need to show that stepping from φ to ϱ is just like exchanging rows and columns. Up to obvious modifications, this result can be attained just as it was done in the case of B_n, so we will not treat it.

Summing up, at this point we can give for granted that:

$$(M_{p_0,\dots,p_{r-1}},\varrho) \cong \bigoplus_{\substack{|\lambda_i|=2p_i \\ \lambda_i \text{ with no odd columns}}} \rho_{\lambda_0,\dots,\lambda_{r-1}}. \tag{1.-31}$$

Let us take a step forward towards the proof of Theorem 1.1.3: we are now dealing with the modules $M(c_{f_0,\dots,f_{r-1},p_0,\dots,p_{r-1}})$, where $f_0+\ldots+f_{r-1}+2p_0,\ldots+2p_{r-1}=n$. Let $p_0+\ldots+p_{r-1}=m$ and let us consider the $G(r,2m)$-module

$$\Pi_{p_0,\dots,p_{r-1}} \stackrel{\text{def}}{=} \big(M_{p_0,\dots,p_{r-1}},\varrho\big).$$

We know its irreducible decomposition from (1.3). Arguing as above, we can infer that

$$(M(c_{f_0,\dots f_{r-1},p_0,\dots p'_{r-1}}),\varrho) \cong \mathrm{Ind}_{G(r,2m)\times G(r,n-2m)}^{G(r,n)}(\Pi_{m,r}^{p_0,\dots,p_{r-1}} \odot \rho_{\,\iota_{f_0},\dots,\iota_{f_{r-1}}}), \tag{1.-31}$$

and Theorem 1.1.3 follows from the $G(r,n)$-version of Pieri rule:

Proposition 1.3.5 (Pieri rule for $G(r,n)$)**.** *Let $\rho_{\lambda_0,\dots,\lambda_{r-1}}$ be any irreducible representation of $G(r,h)$. Then*

$$\mathrm{Ind}_{G(r,h)\times G(r,k)}^{G(r,h+k)}(\rho_{\lambda_0,\dots,\lambda_{r-1}} \odot \rho_{\,\iota_{f_0},\dots,\iota_{f_{r-1}}}) = \bigoplus \rho_{\nu_0,\dots,\nu_{r-1}},$$

where the direct sum runs through all $(\nu_0,\dots,\nu_{r-1}) \in \mathrm{Fer}(r,h+k)$ such that ν_i is obtained from λ_i by adding f_i boxes to its Ferrers diagram, no two in the same column.

Chapter 2

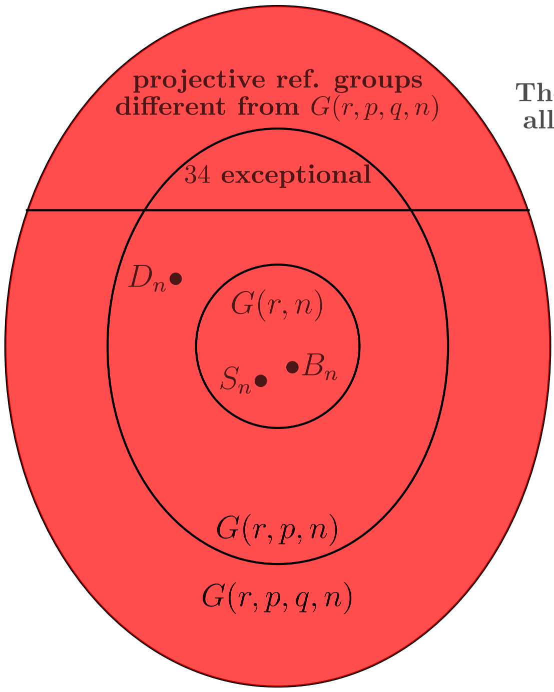
The model for involutory reflection groups

Our next goal is to state and prove a generalized version of Theorem 1.1.3, holding for a much bigger family of groups, including all those of the form $G(r,p,n)$ with $\mathrm{GCD}(p,n)=1,2$. Though, before doing this, we need to give a complete account of the Gelfand model constructed in [6] for these groups. This will be less immediate than it was for $G(r,n)$, because we are not dealing with autodual groups anymore. Therefore, the present chapter is entirely devoted to this explanation.

The description of the model will require, first of all, an introduction about *projective reflection groups* (Section 2.1). We will then immediately focus on the groups $G(r,p,q,n)$ (Section 2.2), and provide a parametrization for their irreducible representations. In Section 2.3 we explain how to generalize the Robinson-Schensted correspondence to all groups $G(r,p,q,n)$. In Section 2.4, we give the definition of involutory reflection group, and necessary and sufficient conditions for a group $G(r,p,q,n)$ to be involutory. Finally, in Section 2.5, we explicitly show the model constructed in [6] for all involutory reflection groups $G(r,p,n)$ and for all their quotients $G(r,p,q,n)$.

Projective reflection groups, jointly with the concept of *duality* (see Definition 2.2), are of crucial importance to describe the model of all involutory reflection groups. In this sense, projective reflection groups will be first of all used as a tool to prove results concerning classical reflection groups $G(r,p,n)$. Nevertheless, the results appearing here also hold for some projective reflection groups *that are not* classical reflection groups (see Theorems 2.5.1, 4.7.1). Furthermore, we may mention that the importance of projective reflection groups goes far beyond the aim of this work. For an outline of the many applications they have, we refer the reader to [5].

2.1 Projective reflection groups



The biggest circle represents all projective reflection groups

A projective reflection group G is obtained as a quotient of some finite complex reflection group W modulo a scalar subgroup. Notice that, so far, we are not requiring W to be of the form $G(r,p,n)$.

More precisely, let W be a finite complex reflection group acting on the finite-dimension complex vector space V. Let $S_q(V)$ be the q^{th} symmetric power of V. Consider the natural injection

$$\psi : \mathrm{GL}(V) \to \mathrm{GL}(S_q(V)),$$

whose kernel is $C_q = \langle \zeta_q \mathrm{Id} \rangle$. Once restricted to W, ψ induces the isomorphism

$$\frac{W}{C_q \cap W} \cong \psi(W).$$

When C_q is contained in W, this allows to see the quotient $G \stackrel{\text{def}}{=} \frac{W}{C_q}$ as a subgroup of $\mathrm{GL}(S_q(V))$, and G acts naturally on $S_q(V)$. Thus we can give the following

Definition. Let $G < \mathrm{GL}(S_q(V))$. The pair (G,q) is a projective reflection group if and only if there exists a finite complex reflection group W containing C_q and such that $G \cong \frac{W}{C_q}$.

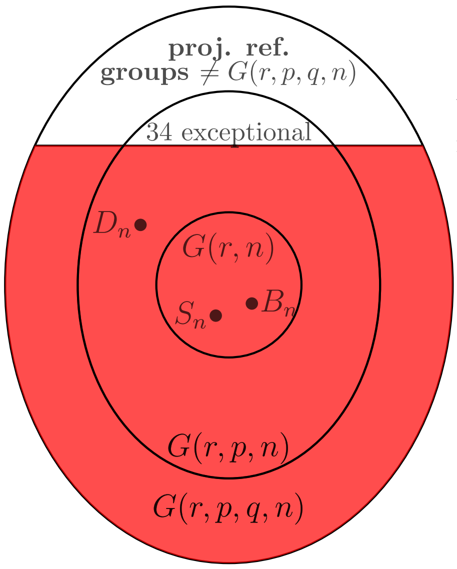
The dual action of G can be extended to $S_q[V^*]$, the algebra of polynomial functions on V generated by homogeneous polynomial functions of degree q. Theorem 0.2.2, characterizing classical reflection groups, generalizes to the case of projective reflection groups in the expected way:

Theorem 2.1.1. *Let V be a finite-dimension vector space over $\mathbb{C}$. Let G be a finite group of graded automorphisms $S_q[V^*]$. (G,q) is a projective reflection group if and only if its invariant algebra $S_q[V^*]^G$ is generated by $n = \dim V$ algebraically independent homogeneous polynomials.*

Proof. See [5, Theorem 2.1]. □

2.2 The groups $G(r,p,q,n)$ and their irreducible representations

From now on we will only consider projective reflection groups $\frac{W}{C_q}$ with $W = G(r,p,n)$. More precisely:



Definition. Let $r,p,q,n \in \mathbb{N}$ such that $p|r$, $q|r$ and $pq|rn$. Then we define the projective reflection group $G(r,p,q,n)$ as

$$G(r,p,q,n) \stackrel{\text{def}}{=} \frac{G(r,p,n)}{C_q}.$$

We observe that the first condition is required for the group $G(r,p,n)$ to exist, while the remaining two assure that the scalar group $C_q = \langle \zeta_q \mathrm{Id} \rangle$ is actually contained in $G(r,p,n)$.

In what follows we will always deal with projective reflection groups independently from their action on $S_q(V)$. We will therefore refer to a projective reflection group simply as the abstract group G itself, dropping the pair notation (G,q). Concerning this, we remark that two abstract projective reflection groups $G(r,p,q,n)$ and $G(\bar{r},\bar{p},\bar{q},\bar{n})$ may be isomorphic even if their parameters are not the same.

Notice that the conditions of existence of the group $G(r,p,q,n)$ given in Definition 2.2 are symmetric in p and q. This allows to give the following crucial

Definition. Let $G = G(r,p,q,n)$. We denote by G^* the group $G(r,q,p,n)$ and we call it the dual group of G.

We have seen in Chapter 1 that some objects related to the algebra of the groups $G(r,n)$ (namely, their irreducible representations and their conjugacy classes) can be described by means of $\mathrm{Fer}(r,n)$. Also, the generalized Robinson-Schensted correspondence associates to an element of $G(r,p,n)$ a pair of multi-tableaux in $\mathcal{ST}(r,p,n)$ (see lemma 0.5.3). With this motivation, let us introduce the new sets $\mathrm{Fer}(r,p,q,n)$ and $\mathcal{ST}(r,p,q,n)$.

Let the conditions of Definition 2.2 be satisfied. Consider the set $\mathrm{Fer}(r,p,n)$ and let the cyclic group C_q act on it by means of a shift of the diagrams of $\frac{r}{q}$ places:

$$C_q \circlearrowleft \mathrm{Fer}(r,p,n) \tag{2.0}$$

$$\zeta_q : \lambda^{(0)},\ldots,\lambda^{(r-1)} \mapsto \lambda^{(\frac{r}{q})},\ldots,\lambda^{(r-1)},\lambda^{(0)},\ldots,\lambda^{(\frac{r}{q}-1)}.$$

To check that the action is well posed, see [5, Lemma 6.1].

Example 2.2.1. Consider, for any $\lambda \in \mathrm{Fer}(r,p,n)$, its stabilizer $\mathrm{Stab}_{C_q}(\lambda)$ with respect to the action 2.2 of the group C_q. Given $\lambda = (\square\square\ \square\ \begin{smallmatrix}\square\\\square\end{smallmatrix}\ \square)$, if $q = 2$ we have $\mathrm{Stab}_{C_2}(\lambda) = C_2$; if $q = 4$, $\mathrm{Stab}_{C_4}(\lambda) = C_2$ again.

Definition. We call $\mathrm{Fer}(r,p,q,n)$ the quotient set $\mathrm{Fer}(r,p,n)$ modulo the action (2.2). Similarly, we define $\mathcal{ST}(r,p,q,n)$ as the quotient - modulo an analogous action - $\frac{\mathcal{ST}(r,p,n)}{C_q}$.

Example 2.2.2. The two elements of $\mathrm{Fer}(4,2,6)$ $(\begin{smallmatrix}\square\\\square\end{smallmatrix}\ \square\ \square\square\ \square)$ and $(\square\square\ \square\ \begin{smallmatrix}\square\\\square\end{smallmatrix}\ \square)$ coincide as elements of $\mathrm{Fer}(4,2,2,6)$.

The two elements of $\mathcal{ST}(4,2,4)$ $\left(\boxed{1}\,\boxed{2}\,\boxed{3}\,\boxed{4}\right)$ and $\left(\boxed{2}\,\boxed{3}\,\boxed{4}\,\boxed{1}\right)$ coincide as elements of $\mathcal{ST}(4,2,4,4)$, but not as elements of $\mathcal{ST}(4,2,2,4)$.

Notation 2.2.3. Since all elements in $\mathrm{Fer}(r,p,q,n)$ and $\mathcal{ST}(r,p,q,n)$ are equivalence classes of $\mathrm{Fer}(r,p,n)$ and $\mathcal{ST}(r,p,n)$ respectively, we will denote them by means of any of their lifts in square brackets. The two elements considered in Example 2.2.2 will be denoted with

$$\left[\begin{smallmatrix}\square\\\square\end{smallmatrix}\ \square\ \square\square\ \square\right] = \left[\square\square\ \square\ \begin{smallmatrix}\square\\\square\end{smallmatrix}\ \square\right] \in \mathrm{Fer}(4,2,2,6);$$

As for standard tableaux, an example may be

$$\left[\boxed{1}\boxed{2}\boxed{3}\boxed{4}\right] = \left[\boxed{3}\boxed{4}\boxed{1}\boxed{2}\right] \in \mathcal{ST}(4,2,2,4).$$

Notation 2.2.4. In analogy with what was done for the groups $G(r,p,q,n)$, we define the following dual sets:

- $\mathrm{Fer}(r,p,q,n)^* \stackrel{\text{def}}{=} \mathrm{Fer}(r,q,p,n)$
- $\mathcal{ST}(r,p,q,n)^* \stackrel{\text{def}}{=} \mathcal{ST}(r,q,p,n)$.

With this tools at hands, our aim is to provide a parametrization for the irreducible representations of the groups $G(r,p,q,n)$, as it was done for the groups $G(r,n)$ (see Proposition 0.4.2). The step from $G(r,n)$ to its quotient $\frac{G(r,n)}{C_q} = G(r,1,q,n)$ is actually quite easy. In fact, $\mathrm{Irr}(r,1,q,n)$ is given by those representations of $G(r,n)$ whose kernel contains the scalar cyclic subgroup C_q. It follows from this observation and Proposition 0.4.2 that

$$\mathrm{Irr}(r,1,q,n) = \{\rho_\lambda : \lambda \in \mathrm{Fer}(r,q,1,n) = \mathrm{Fer}(r,q,n)^*\}.$$

The following step, from $G(r,1,q,n)$ to $G(r,p,q,n)$, is a little more subtle. Once restricted to $G(r,p,q,n)$, the irreducible representations of $G(r,1,q,n)$ may not be irreducible anymore. We need to find out which of them split into more than one $G(r,p,q,n)$-module. This is the content of the following theorem, which fully descibes the parametrization of the irreducible representations of the groups $G(r,p,q,n)$. Here and in what follows, if $\lambda \in \mathrm{Fer}(r,p,n)$ we let $m_q(\lambda) = |\mathrm{Stab}_{C_q}(\lambda)|$. Observe that if $[\lambda] = [\mu] \in \mathrm{Fer}(r,p,q,n)$ then $m_q(\lambda) = m_q(\mu)$.

Theorem 2.2.5. *The set of irreducible representations* $\mathrm{Irr}(r,p,q,n)$ *of* $G(r,p,q,n)$ *can be parametrized in the following way*

$$\mathrm{Irr}(r,p,q,n) \quad = \quad \{\rho^j_{[\lambda]} : [\lambda] \in \mathrm{Fer}(r,q,p,n) = \mathrm{Fer}(r,p,q,n)^* \textit{ and } j \in [0, m_p(\lambda)-1]\},$$

so that the following conditions are satisfied:

- $\dim(\rho^j_{[\lambda]}) = |\mathcal{ST}_{[\lambda]}|$ *for all* $[\lambda] \in \mathrm{Fer}(r,q,p,n) = \mathrm{Fer}(r,p,q,n)^*$ *and* $j \in [0, m_p(\lambda)-1]$*;*
- $\mathrm{Res}^{G(r,1,q,n)}_{G(r,p,q,n)}(\rho_\lambda) \cong \bigoplus_j \rho^j_{[\lambda]}$ *for all* $\lambda \in \mathrm{Fer}(r,q,n)^* = \mathrm{Fer}(r,1,q,n)^*$.

Proof. See [6, Proposition 6.2]. □

If $m_p(\lambda) = 1$ we sometimes write $\rho_{[\lambda]}$ instead of $\rho^0_{[\lambda]}$ and we say that this is an *unsplit* representation, meaning that, once restricted from $G(r,1,q,n)$ to $G(r,p,q,n)$, λ remains irreducible and thus does not split into more than one irreducible representation. On the other hand, whenever $m_p(\lambda) > 1$, we say that all representations of the form $\rho^j_{[\lambda]}$ are *split* representations. We will come back to the description of the irreducible representations in Section 4.2 with more details.

Notice that the irreducible representations of both $G(r, 1, q, n)$ and of its subgroup $G(r, p, q, n)$ can be described by means of the *dual* set $\mathrm{Fer}(r, p, q, n)^*$. This is consistent with what happens in the case of $G(r, n)$, since this group and its related sets $\mathrm{Fer}(r, n)$ and $\mathcal{ST}(r, n)$ coincide with their dual.

2.3 The projective Robinson-Schensted correspondence

Let us now turn to give a brief account of the *projective Robinson-Schensted correspondence*, which is an extension of the generalized Robinson-Schensted correspondence [22] we presented in Section 0.5. While the generalized Robinson-Schensted correspondence works on the groups $G(r, p, n)$, the projective Robinson-Schensted correspondence applies to all projective reflection groups of the form $G(r, p, q, n)$.

There is a quite natural way to build this new function. Recall one of the properties of the generalized Robinson Schensted correspondence (see Proposition 0.5.3) :

$$\text{if } g \mapsto (P_0, \dots, P_{r-1}; Q_0, \dots, Q_{r-1}) \text{ via } \overline{RS},$$
$$\text{then } \zeta_r g \mapsto (P_1, \dots, P_{r-1}, P_0; Q_1, \dots, Q_{r-1}, Q_0) \text{ via } \overline{RS}.$$

For example, let $g \in B_n = G(2, 1, n)$:

$$g \in B_n \xrightarrow{\overline{RS}} (P_0, P_1; Q_0, Q_1). \text{ Then}$$
$$-g \in B_n \xrightarrow{\overline{RS}} (P_1, P_0; Q_1, Q_0).$$

Now consider the quotient group $\frac{B_n}{C_2} = G(2, 1, 2, n)$. If we choose the equivalence class $\bar{g}$ of g modulo the action of C_2, it is natural to associate to it one (ordered) pair of *unordered* pairs of tableaux:

$$\bar{g} \in \frac{B_n}{\pm Id} \xrightarrow{\overline{RS}} (\{P_0, P_1\}; \{Q_0, Q_1\}).$$

In the above example, we get unordered pairs because $q = r = 2$. In the general case, we will not obtain pairs of unordered sets of tableaux, but pairs of elements of $\mathcal{ST}(r, p, q, n)$.

The projective Robinson-Schensted correspondence is a surjective map RS_q

$$\begin{aligned} RS_q : G(r,p,q,n) &\to \mathcal{ST}(r,p,q,n) \times \mathcal{ST}(r,p,q,n) \\ g &\mapsto ([P];[Q]), \end{aligned}$$

where $[P]$ and $[Q]$ are determined as follows:

- choose any lift $\bar{g}$ of g in $G(r,p,n)$;
- perform the generalized Robinson-Schensted correspondence $\overline{RS}$ to $\bar{g}$, so obtaining a pair of tableaux $(P;Q) \in \mathcal{ST}(r,p,n) \times \mathcal{ST}(r,p,n)$;
- take the classes $[P],[Q]$ of P,Q modulo C_q w.r.t. its action on $\mathcal{ST}(r,p,n)$;
- set $RS_q(g) \stackrel{\text{def}}{=} ([P];[Q])$.

Proposition 2.3.1. *The projective Robinson-Schensted correspondence satisfies the following property: if* $[P],[Q] \in \mathcal{ST}_{[\lambda]}$ *then the cardinality of the inverse image of* $([P],[Q])$ *is equal to* $m_q(\lambda)$*. In particular we have that this correspondence is a bijection if and only if* $\mathrm{GCD}(q,n) = 1$.

Proof. See [5, Theorem 10.1]. □

Notation 2.3.2. Given $g \stackrel{RS_q}{\mapsto} ([P_0,\dots,P_{r-1}];[Q_0,\dots,Q_{r-1}])$, we denote by $\mathrm{Sh}(g)$ the element of $\mathrm{Fer}(r,p,q,n)$ which is the shape of $[P_0,\dots,P_{r-1}]$ and of $[Q_0,\dots,Q_{r-1}]$.

Notice that while in Theorem 2.2.5 we meet elements $[\lambda] \in \mathrm{Fer}(r,q,p,n) = \mathrm{Fer}(r,p,q,n)^*$, in the projective Robinson-Schensted correspondence the elements $[\lambda]$ involved belong to $\mathrm{Fer}(r,p,q,n)$. This is one of the reasons why it is natural to look at the dual groups when studying the combinatorial representation theory of any projective reflection group of the form $G(r,p,q,n)$.

2.4 Involutory projective reflection groups

Recall Definition 1.1: $g \in G(r,n)$ is an absolute involution if $g\bar{g} = \mathrm{Id}$, $\bar{g}$ being the complex conjugate of g. It is clear that the same definition holds for $G(r,p,n)$, since it is a subgroup of $G(r,n)$. Since complex conjugation stabilizes the cyclic scalar group C_q, we can give the same definition for projective reflection groups too:

Definition. Let G be a projective reflection group and let $g \in G$. g is an *absolute involution* if $g\bar{g} = 1$.

Notation 2.4.1. We denote by $I(r,p,q,n)$ the set of the absolute involutions of the group $G(r,p,q,n)$. The notation $I(r,p,n)$ stands for the set of the absolute involutions of $G(r,p,n)$. Moreover, we let

$$I(r,p,n)^* \stackrel{\text{def}}{=} \{\text{absolute involutions of } G(r,p,n)^*\}$$
$$I(r,p,q,n)^* \stackrel{\text{def}}{=} \{\text{absolute involutions of } G(r,p,q,n)^*\}.$$

The absolute involutions in $I(r,p,q,n)$ can be either symmetric or antisymmetric, according to the following definition:

Definition. Let $v \in G(r,p,q,n)$. We say that it is:

- symmetric, if every (any) lift of v in $G(r,n)$ is a symmetric matrix;
- antisymmetric, if every (any) lift of v in $G(r,n)$ is an antisymmetric matrix.

We observe that while a symmetric element is always an absolute involution, an antisymmetric element of $G(r,p,q,n)$ is an absolute involution if and only if q is even (see [6, Lemma 4.2]). Antisymmetric elements of $G(r,n)$ can also be characterized in terms of the Robinson-Schensted correspondence (see [6, Lemma 4.3]):

Lemma 2.4.2. *Let $v \in G(r,n)$. Then the following are equivalent*

1. *v is antisymmetric;*
2. *r is even and $v \mapsto (P_0,\ldots,P_{r-1};,P_{\frac{r}{2}},\ldots,P_{r-1},P_0,\ldots,P_{\frac{r}{2}-1})$ for some $(P_0,\ldots,P_{r-1}) \in \mathcal{ST}_\lambda$ and $\lambda \in \mathrm{Fer}(r,n)$ by the Robinson-Schensted correspondence.*

Now we can deduce the following combinatorial interpretation for the number of antisymmetric elements in a projective reflection group.

Notation 2.4.3. Since we often deal with even integers, here and in the rest of this work we let $k' \stackrel{\text{def}}{=} \frac{k}{2}$, whenever k is an even integer.

Proposition 2.4.4. *Let $\mathrm{asym}(r,q,p,n)$ be the number of antisymmetric elements in $G(r,q,p,n)$. Then*

$$\mathrm{asym}(r,q,p,n) = \sum_{[\mu,\mu]\in \mathrm{Fer}(r,q,p,n)} \mathcal{ST}_{[\mu,\mu]},$$

where $[\mu,\mu] \in \mathrm{Fer}(r,q,p,n)$ means that $[\mu,\mu]$ varies among all elements in $\mathrm{Fer}(r,q,p,n)$ of the form
$[\mu^{(0)},\ldots,\mu^{(r'-1)},\mu^{(0)},\ldots,\mu^{(r'-1)}]$, for some $\mu = (\mu^{(0)},\ldots,\mu^{(r'-1)}) \in \mathrm{Fer}(r',n')$.

Proof. Observe that if $v \in G(r,q,n)$ is antisymmetric and $(P_0,\dots,P_{r-1})$ and λ are as in Lemma 2.4.2, then necessarily $\lambda \in \mathrm{Fer}(r,q,1,n)$ is of the form $\lambda = (\mu,\mu)$, for some $\mu \in \mathrm{Fer}(r',n')$. So, if $v \mapsto (P;Q)$ is antisymmetric we have that P is an element in $\mathcal{ST}_{(\mu,\mu)}$ for some $\mu \in \mathrm{Fer}(r',n')$ whilst Q is uniquely determined by P. So we deduce that

$$\mathrm{asym}(r,q,1,n) = \sum_{(\mu,\mu)\in \mathrm{Fer}(r,q,1,n)} \mathcal{ST}_{(\mu,\mu)}.$$

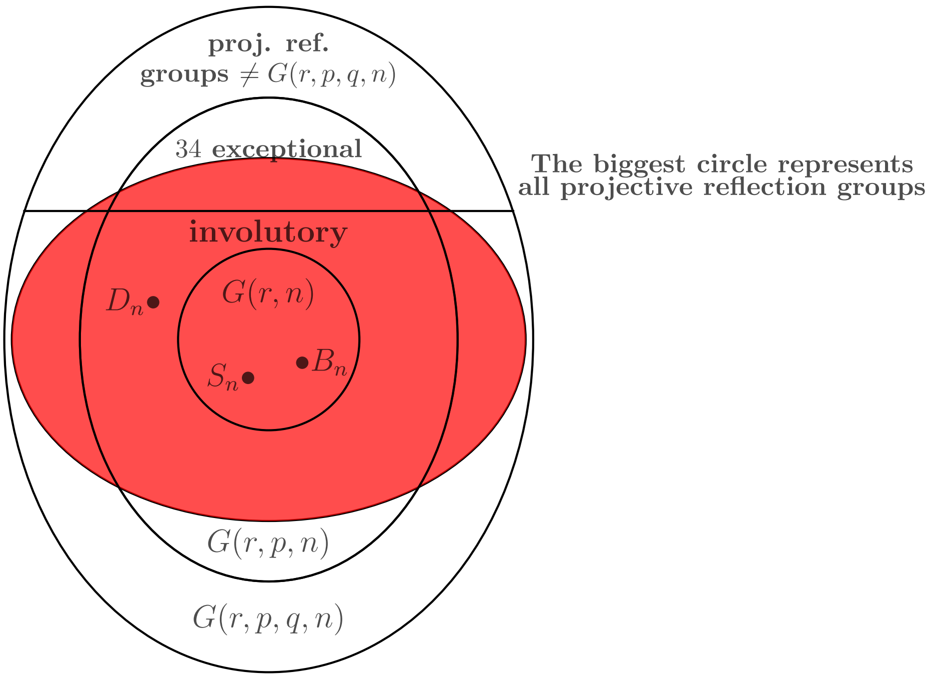
The result now follows since every antisymmetric element in $G(r,q,p,n)$ has p distinct lifts in $G(r,q,n)$ and any element in $\mathcal{ST}_{[\mu,\mu]}$ has p distinct lifts in $\cup_{(\nu,\nu)\in[\mu,\mu]}\mathcal{ST}_{(\nu,\nu)}$.

□

We are now ready to define and characterize involutory projective reflection groups $G(r,p,q,n)$.

Definition. A projective reflection group G is involutory if the dimension of its Gelfand model coincides with the number of its absolute involutions. When $G = G(r,p,q,n)$, G is involutory if

$$\sum_{\phi\in \mathrm{Irr}(G)} \dim\phi = |I(r,q,p,n)|.$$

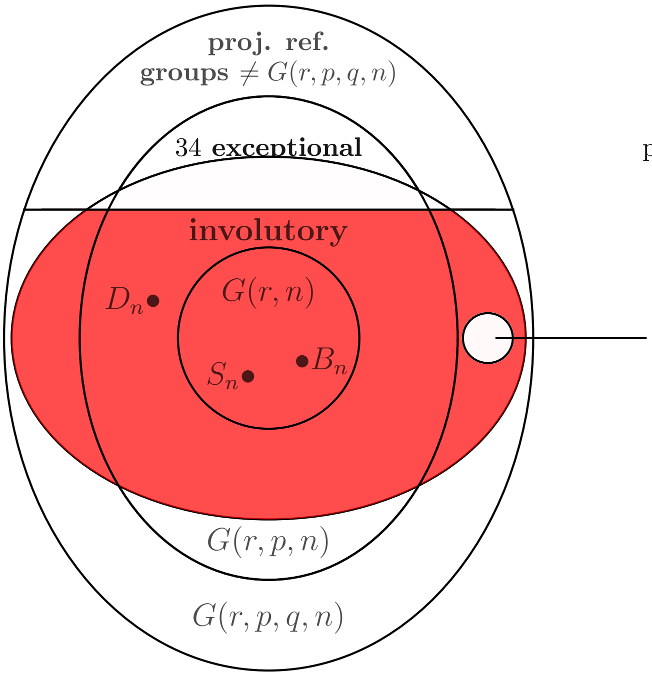


Theorem 2.4.5. *Let* $G = G(r,p,q,n)$*. Then* G *is involutory if and only if either* $\mathrm{GCD}(p,n) = 1, 2,$ *or* $\mathrm{GCD}(p,n) = 4$ *and* $r \equiv p \equiv q \equiv n \equiv 4 \mod 8$*. In particular, a classical reflection group* $G(r,p,n)$ *is involutory if and only if* $\mathrm{GCD}(p,n) = 1, 2$.

Proof. See [6, Theorem 4.5]. □

2.5 The model

In [6], a uniform Gelfand model is constructed for all involutory projective reflection groups $G(r,p,q,n)$ with $\mathrm{GCD}(p,n) = 1, 2$.



The model concerns all involutory projective reflection groups $G(r,p,q,n)$, except those satisfying $\mathrm{GCD}(p,n) = 4$

Before describing the model, we recall some notation from Chapter 1 and introduce some new objects. If $\sigma, \tau \in S_n$ with $\tau^2 = 1$ we let

- $\mathrm{Inv}(\sigma) \stackrel{\text{def}}{=} \{\{i,j\} : (j-i)(\sigma(j) - \sigma(i)) < 0\}$;
- $\mathrm{Pair}(\tau) \stackrel{\text{def}}{=} \{\{i,j\} : \tau(i) = j \neq i\}$;
- $\mathrm{inv}_\tau(\sigma) \stackrel{\text{def}}{=} |\{\mathrm{Inv}(\sigma) \cap \mathrm{Pair}(\tau)|$.

If $g \in G(r,p,q,n)$, $v \in I(r,q,p,n)$, $\tilde{g}$ any lift of g in $G(r,p,n)$ and $\tilde{v}$ any lift of v in $G(r,q,n)$, we let

- $\mathrm{inv}_v(g) \stackrel{\text{def}}{=} \mathrm{inv}_{|v|}(|g|)$;

- $\langle g, v\rangle \stackrel{\text{def}}{=} \sum_{i=1}^{n} z_i(\tilde{g}) z_i(\tilde{v}) \in \mathbb{Z}_r$;
- $a(g, v) \stackrel{\text{def}}{=} z_1(\tilde{v}) - z_{|g|^{-1}(1)}(\tilde{v}) \in \mathbb{Z}_r$.

The verification that $\langle g, v\rangle$ and $a(g, v)$ are well-defined is straightforward. We are now ready to present the Gelfand model constructed in [6].

Theorem 2.5.1. *Let* $\mathrm{GCD}(p, n) = 1, 2$ *and let*

$$M(r, q, p, n) \stackrel{\text{def}}{=} \bigoplus_{v \in I(r,q,p,n)} \mathbb{C}C_v$$

and $\varrho : G(r, p, q, n) \to \mathrm{GL}(M(r, q, p, n))$ *be defined by*

$$\varrho(g)(C_v) \stackrel{\text{def}}{=} \begin{cases} \zeta_r^{\langle g,v\rangle} (-1)^{\mathrm{inv}_v(g)} C_{|g|v|g|^{-1}} & \textit{if } v \textit{ is symmetric} \\ \zeta_r^{\langle g,v\rangle} \zeta_r^{a(g,v)} C_{|g|v|g|^{-1}} & \textit{if } v \textit{ is antisymmetric.} \end{cases} \tag{2.-1}$$

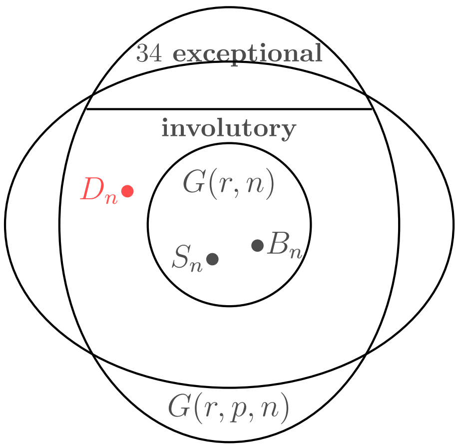
Then $(M(r, q, p, n), \varrho)$ *is a Gelfand model for* $G(r, p, q, n)$.

Proof. See [6, Theorem 5.4]. □

We explicitely remark that, as can be seen from the figure, this theorem also concerns some groups that are *not* classical reflection groups.

Chapter 3

Decomposition of the model for type D



We use this chapter to present our main result for the group D_n. Recall that D_n is the subgroup $G(2,2,n)$ of index 2 of the group $B_n = G(2,n)$ and observe that its dual group is given by $D_n^* = G(2,1,2,n) = B_n/\pm\mathrm{Id}$. Thus, the model space of theorem 2.5.1 is spanned by the absolute involutions in $B_n/\pm\mathrm{Id}$. Since we are now dealing with real matrices, absolute involutions are now simply involutions. Furthermore, if n is even, we observe that the antisymmetric involutions come into play. For this reason, the example of D_n is particularly enlightening.

The advantage of affording this particular case before the general one is twofold. On the one hand, we will shed some light on our arguments by means of a well-known example the reader is probably familiar with. On the other hand, we will make use here of some combinatorial results that, for D_n, are already known in literature. The general case will require to generalize such results - namely, the study of the split conjugacy classes (section 4.1) and of the characters of the split representations (Sections 4.2-4.3) of $G(r,p,n)$ with $\mathrm{GCD}(p,n) = 1, 2$.

As mentioned above, when n is even, the model constructed in Theorem 2.5.1 is spanned by both symmetric and antisymmetric involutions. Since a symmetric and an antisymmetric involution cannot be S_n-conjugate, we can immediately split the model into two natural submodules:

$$M = M_{\mathrm{Sym}} \oplus M_{\mathrm{Asym}},$$

where M_{Sym} is spanned by symmetric involutions of D_n^* and M_{Asym} is spanned by antisymmetric involutions of D_n^*. When n is odd, M_{Asym} vanishes and M coincide with M_{Sym}.

In Section 3.1 we will quickly revise the split conjugacy classes and the split characters of D_n.

In Section 3.2 below we will determine the irreducible representations afforded by the submodule M_{Asym}; thus, this section only concerns the case of n even. Observe that an antisymmetric element in B_{2m} is necessarily the product of cycles of length 2 and color 1, i.e. cycles of the form (a^0, b^1). It follows that the antisymmetric elements of B_{2m}, and hence also those of $B_{2m}/\pm I$, are all S_n-conjugate. For this reason, the submodule M_{Asym} will not furtherly decompose. This is a special feature of this case and is not true for generic involutory reflection groups (see Section 4.5).

The submodule M_{Sym}, on the other hand, admits a finer natural decomposition, which we will study in Section 3.3. This section, unlike the preceeding one, concerns both the case of n odd and the case of n even.

3.1 Split representations and split conjugacy classes

Recall from Proposition 2.2.5 that the representations of B_n, when restricted to D_n, always remain irreducible except for those of the form $\rho_{\lambda,\lambda}$, which exist if n is even only: in this case the representation splits into two irreducible representations that we denote, according to Theorem 2.2.5, $\rho^0_{[\lambda,\lambda]}$ and $\rho^1_{[\lambda,\lambda]}$. We will show that the simultaneous occurring of these two phenomena, the existence of antisymmetric elements and of the split representations when n is even, is not accidental.

Recall the parametrization of the conjugacy classes of the group $G(r,n)$ as seen in Section 0.4. It is easy to check that the conjugacy classes of B_n belonging to D_n are indexed by ordered pairs of partitions (α, β), with $|\beta| \equiv 0 \mod 2$.

Proposition 3.1.1. *Let $\mathrm{cl}_{\alpha,\beta}$ be a conjugacy class of B_n satisfying $|\beta| \equiv 0 \mod 2$, so that $\mathrm{cl}_{\alpha,\beta}$ is contained in D_n as well. Then $\mathrm{cl}_{\alpha,\beta}$ splits into two different D_n-conjugacy classes if and only if $\alpha = 2\gamma$ for some $\gamma \vdash \frac{n}{2}$, and $\beta = \emptyset$. In particular, if n is odd there are no B_n-conjugacy classes that split as D_n-conjugacy classes.*

For the proof, see [20, 17].

Notation 3.1.2. The pairs of the form $(2\alpha, \emptyset)$ label two D_n-conjugacy classes denoted $\mathrm{cl}^0_{2\alpha}$ and $\mathrm{cl}^1_{2\alpha}$. A representative of the conjugacy class $\mathrm{cl}^0_{2\alpha}$ is any element in S_n of cycle-type 2α. A representative of the conjugacy class $\mathrm{cl}^1_{2\alpha}$ is given by any element g such that $|g| \in S_n$ has cycle type 2α and

$$z_k(g) = \begin{cases} 1 & \text{if } k = i, j \\ 0 & \text{otherwise} \end{cases},$$

for some $i, j \in [n]$ such that $|g|(i) = j$.

Example 3.1.3. The element $[(4,7,5,3,1,8,2,6); 1,0,0,1,0,0,0,0] \in D_8$ belongs to the class $\mathrm{cl}^1_{\ydiagram{4,2,2}}$.

The characters of the unsplit representations are clearly the same as the corresponding representations of the groups B_n (being the corresponding restrictions). The characters of the split representations are given by the following result (see [20, 17]).

Lemma 3.1.4. *Let $g \in D_{2m}$, and $\mu \vdash m$. Then*

$$\chi^{\epsilon}_{[\mu,\mu]}(g) = \begin{cases} \frac{1}{2}\chi_{(\mu,\mu)}(2\alpha, \emptyset) + (-1)^{\epsilon+\eta} 2^{\ell(\alpha)-1} \chi_\mu(\alpha), & \textit{if } g \in \mathrm{cl}^{\eta}_{2\alpha}; \\ \frac{1}{2}\chi_{(\mu,\mu)}(g), & \textit{otherwise.} \end{cases}$$

where $\epsilon, \eta = 0, 1$, $\chi_{(\mu,\mu)}$ is the character of the B_{2m}-representation $\rho_{(\mu,\mu)}$, χ_μ is the character of S_m indexed by μ and $\ell(\alpha)$ is defined as in notation 0.3.3.

3.2 The case of n even: the submodule M_{Asym}

All through this section, we assume $n = 2m$. This section fully clarifies which of the irreducible representations of D_n are afforded by the submodule M_{Asym}. The crucial observation here is the following result, which is a straightforward consequence of Lemma 3.1.4.

Remark 2. Let $g \in D_{2m}$. Then

$$\sum_{\mu \vdash m} (\chi^0_{[\mu,\mu]} - \chi^1_{[\mu,\mu]})(g) = \begin{cases} (-1)^{\eta} 2^{\ell(\alpha)} \sum_{\mu \vdash m} \chi_\mu(\alpha), & \text{if } g \in \mathrm{cl}^{\eta}_{2\alpha}; \\ 0, & \text{otherwise.} \end{cases} \tag{3.0}$$

In order to study the representation $(M_{\mathrm{Asym}}, \varrho)$ we will need the following auxiliary representations of D_{2m} :

$$(M_{\mathrm{Asym}}, \phi^+) \quad \text{and} \quad (M_{\mathrm{Asym}}, \phi^-),$$

given by

$$\phi^+(g)(C_v) \overset{\mathrm{def}}{=} (-1)^{<g,v>} C_{|g|v|g|^{-1}}, \qquad \phi^-(g)(C_v) \overset{\mathrm{def}}{=} (-1)^{<g,v>}(-1)^{a(g,v)} C_{|g|v|g|^{-1}}$$

(notice that $\phi^-(g) = \varrho(g)|_{M_{\mathrm{Asym}}}$). The main result that we need to describe the irreducible decomposition of $(M_{\mathrm{Asym}}, \varrho)$ is an explicit combinatorial description of the difference character $\chi_{\phi^+} - \chi_{\phi^-}$ (Proposition 3.2.5). To this end we recall some ideas developed in [6] and some preliminary lemmas.

For every $g \in D_{2m}$, consider the set

$$\mathrm{A}(g) \overset{\mathrm{def}}{=} \{w \in B_{2m} : w^2 = -\mathrm{Id} \text{ and } |g|w|g|^{-1} = \pm w\}.$$

In other words, $\mathrm{A}(g)$ is the set of antisymmetric elements w in B_n (the condition $w^2 = -\mathrm{Id}$ is equivalent to w being antisymmetric) whose corresponding class in $B_n/\pm\mathrm{Id}$ is fixed by conjugation by $|g|$. Since any element in $B_n/\pm\mathrm{Id}$ has exactly two lifts in B_n, we have

$$\chi_{\phi^+}(g) - \chi_{\phi^-}(g) = \frac{1}{2} \sum_{w \in \mathrm{A}(g)} (-1)^{<g,w>}[1 - (-1)^{a(g,w)}]. \qquad (3.0)$$

The set $\mathrm{A}(g)$ was described in [6] and we recall here some of the needed notation.

Let $\Pi^{2,1}(g)$ be the set of partitions of the set of disjoint cycles of g into:

- singletons;
- pairs of cycles having the same length.

Recall the definition of Supp given on page 5. Any $w \in \mathrm{A}(g)$ determines a partition $\pi(w) \in \Pi^{2,1}(g)$: a cycle c is a singleton of $\pi(w)$ if the restriction of $|w|$ to $\mathrm{Supp}(c)$ is a permutation of $\mathrm{Supp}(c)$ and $\{c, c'\}$ is a pair of $\pi(w)$ if the restriction of $|w|$ to $\mathrm{Supp}(c)$ is a bijection between $\mathrm{Supp}(c)$ and $\mathrm{Supp}(c')$. Finally, let $\mathrm{A}_\pi \overset{\mathrm{def}}{=} \{w \in \mathrm{A}(g) : \pi(w) = \pi\}$. Then the set $\mathrm{A}(g)$ can be decomposed into the disjoint union

$$\mathrm{A}(g) = \bigcup_{\pi \in \Pi^{2,1}(g)} \mathrm{A}_\pi. \qquad (3.0)$$

Looking at the definition of the set $\mathrm{A}(g)$, for $\epsilon \in \mathbb{Z}_2$, we can also define

$$A_\pi^\epsilon \overset{\mathrm{def}}{=} \{w \in A_\pi : |g|w|g|^{-1} = (-1)^\epsilon w\},$$

so that partition (3.2) can be made still finer:

$$\mathrm{A}(g) = \bigcup_{\substack{\pi \in \Pi^{2,1}(g) \\ \epsilon = 0,1}} A_\pi^\epsilon. \tag{3.0}$$

end equation (3.2) can be rewritten as

$$\chi_{\phi^+}(g) - \chi_{\phi^-}(g) = \sum_{\pi \in \Pi^{2,1}(g)} F_\pi(g), \tag{3.0}$$

where

$$F_\pi(g) = \frac{1}{2} \sum_{\epsilon=0,1} \sum_{w \in A_\pi^\epsilon} (-1)^{<g,w>}[1 - (-1)^{a(g,w)}].$$

Notation 3.2.1. If $S \subseteq [n]$, we let

$$G(r, S) = \{[\sigma_1^{z_1}, \ldots, \sigma_n^{z_n}] \in G(r,n) : \sigma_i^{z_i} = i^0 \text{ for all } i \notin S\}.$$

Given $\pi = \{s_1, \ldots, s_h\} \in \Pi^{2,1}(g)$, we have $A_\pi^\epsilon = A_{s_1}^\epsilon \times \ldots \times A_{s_h}^\epsilon$, where the sets $A_s^\epsilon \subset G(2, \mathrm{Supp}(s))$ are described in Table 3.1. In the first column of the table we have all possible structures of the "absolute value" of a part s. It is clear from the definition that the sets A_π^0 and A_π^1 depend on $|g|$ only. Moreover the indices of the i's and of the j's should be considered $\mod d$, $k \in \mathbb{Z}_2$ and $l \in [d]$. For example, if $|s| = \{(i_1, \ldots, i_d), (j_1, \ldots, j_d)\}$ with d odd then

$$\mathrm{A}_s^0 = \bigcup_{k \in \mathbb{Z}_2} \bigcup_{l \in \mathbb{Z}_d} \{v \in G(2, \mathrm{Supp}(s)) : v(i_h) = (-1)^k j_{h+l} \text{ and } v(j_h) = -(-1)^k i_{h-l}\}.$$

Remark 3. It is a straightforward verification based on a case by case analisys of the table (see also [6, Proof of Lemma 5.7]) that

$$a(g,w) = \begin{cases} 0, & \text{if } w \in A_\pi^0(g); \\ 1, & \text{if } w \in A_\pi^1(g). \end{cases}$$

It is an immediate consequence of this remark that we can restrict the sum in (3.2) to all $w \in A_\pi^1$. In particular the definition of $F_\pi(g)$ can be pretty much simplified

$$F_\pi(g) = \frac{1}{2} \sum_{w \in A_\pi^1} (-1)^{<g,w>}[1 - (-1)^{a(g,w)}] = \sum_{w \in A_\pi^1} (-1)^{<g,w>}. \tag{3.0}$$

Now, if $\pi = \{s_1, \ldots, s_h\}$, every w in A_π^1 can be written as $w = w_1 \cdot \ldots \cdot w_h$, with $w_i \in A_{s_i}^1$ and viceversa, if $w_i \in A_{s_i}^1$ then $w = w_1 \cdots w_h \in A_\pi^1$. This will also

$\lvert s \rvert$	A_s^0	A_s^1
$\{(i_1,\ldots,i_d)\}$ with d odd	$\emptyset$	$\emptyset$
$\{(i_1,\ldots,i_d)\}$ with $d \equiv 2 \mod 4$	$\emptyset$	$i_h \mapsto (-1)^k(-1)^h i_{h+\frac{d}{2}}$
$\{(i_1,\ldots,i_d)\}$ with $d \equiv 0 \mod 4$	$\emptyset$	$\emptyset$
$\{(i_1,\ldots,i_d),(j_1,\ldots,j_d)\}$, with d odd	$i_h \mapsto (-1)^k j_{h+l}$ and $j_h \mapsto -(-1)^k i_{h-l}$	$\emptyset$
$\{(i_1,\ldots,i_d),(j_1,\ldots,j_d)\}$, with d even	$i_h \mapsto (-1)^k j_{h+l}$ and $j_h \mapsto -(-1)^k i_{h-l}$	$i_h \mapsto (-1)^k(-1)^h j_{h+l}$ and $j_h \mapsto -(-1)^k(-1)^{h-l} i_{h-l}$

Table 3.1: The sets A_s^ϵ as $\lvert s \rvert$ varies.

allow to focus on the single sets A^1_s, via the identity

$$\begin{aligned} F_\pi(g) &= \sum_{w\in A^1_\pi} (-1)^{<g,w>} \quad (3.1)\\ &= \sum_{w_1\in A^1_{s_1}} \cdots \sum_{w_h\in A^1_{s_h}} (-1)^{<g,w_1\cdots w_h>} \\ &= \sum_{w_1\in A^1_{s_1}} \cdots \sum_{w_h\in A^1_{s_h}} (-1)^{\sum_i <g_i,w_i>} \\ &= \prod_{i=1}^{h} \sum_{w_i\in A^1_{s_i}} (-1)^{<g_i,w_i>}, \end{aligned}$$

where $g_i \in G(r, \mathrm{Supp}(s_i))$ is the restriction of g to $\mathrm{Supp}(s_i)$ (if $s \in \Pi^{2,1}(g)$, we let $\mathrm{Supp}(s)$, the support of s, be the union of the supports of the cycles in s).

Lemma 3.2.2. *Let $\pi \in \Pi^{2,1}(g)$ and assume that one of the following conditions is satisfied:*

1. *π has a part which is a singleton of odd length;*
2. *π has a part which is a singleton of length $\equiv 0 \mod 4$;*
3. *π has a part which is a pair of cycles of odd length;*

Then $F_\pi(g) = 0$.

Proof. By Table 3.1 we have that π has a part s such that $A^1_s = \emptyset$. Therefore the result follows from Equation (3.1). □

So we are left to consider those $\pi \in \Pi^{2,1}(g)$ having only singletons of length $\equiv 2 \mod 4$ and pairs of cycles of even length.

Lemma 3.2.3. *Let $\pi \in \Pi^{2,1}(g)$ having only singletons of length $\equiv 2 \mod 4$ and pairs of cycles of even length. Assume further that g has at least one cycle c such that $z(c) = 1$. Then*

$$F_\pi(g) = 0$$

Proof. We split this result into two cases. Assume that the cycle of color 1 is a singleton $s_i = \{c\}$ of π. In this case we have $A^1_{s_i} = \{v, -v\}$ for some $v \in G(2, \mathrm{Supp}(s_i))$ (see Table 3.1). So, in this case the restriction of g_i to $\mathrm{Supp}(s_i)$ is the cycle c itself and therefore g_i has an odd number of negative signs. It follows that $< g_i, v >$ and $< g_i, -v >$ have opposite parity and we deduce that

$$\sum_{w_i\in A^1_{s_i}} (-1)^{<g_i,w_i>} = (-1)^{<g_i,v>} + (-1)^{<g_i,-v>} = 0$$

and the result follows from Equation (3.1). Assume now that the cycle c belongs to a pair $s_i = \{c_1, c\}$ of π. To fix the ideas we let $|c_1| = (i_1, \ldots, i_d)$ and $|c| = (j_1, \ldots, j_d)$. In this case we can consider the involution ψ with no fixed points on $A^1_{s_i}$ determined by

$$\psi : \left\{ \begin{array}{l} i_h \mapsto (-1)^k(-1)^h j_{h+l} \\ j_h \mapsto -(-1)^k(-1)^{h-l} i_{h-l} \end{array} \right\} \mapsto \left\{ \begin{array}{l} i_h \mapsto (-1)^k(-1)^h j_{h+l+1} \\ j_h \mapsto -(-1)^k(-1)^{h-l-1} i_{h-l-1} \end{array} \right\}$$

for all $l = [2d]$ even. In this case we have $z_{i_k}(w) = z_{i_k}(\psi(w))$ and $z_{j_k}(w) = z_{j_k}(\psi(w)) + 1$ for all $w \in A^1_{s_i}$ and so $(-1)^{<g_i,w>} + (-1)^{<g_i,\psi(w)>} = 0$ for all $w \in A^1_{s_i}$ and the result now follows as in the previous case. $\square$

If $\pi = \{s_1, \ldots, s_h\} \in \Pi^{2,1}(g)$ we let $\ell(\pi) = h$ and $\mathrm{pair}_j(\pi)$ be the number of parts of π which are pairs of cycles of length j.

Proposition 3.2.4. *If $g \in \mathrm{cl}^{\epsilon}_{2\alpha}$ then*

$$\chi_{\phi^+}(g) - \chi_{\phi^-}(g) = \epsilon \sum_{\pi \in \Pi^{2,1}(g)} 2^{\ell(\pi)} d^{\mathrm{pair}_d(\pi)},$$

where the sum is taken over all $\pi \in \Pi^{2,1}(g)$ having singletons of length $\equiv 2$ mod 4 *(and pairs of cycles of even length).*

Proof. If $g \in \mathrm{cl}^0_{2\alpha}$ then we may assume that $g = |g|$ (since the left hand side is a class function) and it is clear that in this case $(-1)^{<g,v>} = 1$ for all $v \in \mathrm{A}(g)$. Therefore, by Lemma 3.2.2, we have

$$\begin{aligned} \chi_{\phi^+}(g) - \chi_{\phi^-}(g) &= \sum_{\Pi^{2,1}(g)} F_\pi(g) \\ &= \sum_{\pi \in \Pi^{2,1}(g)} \sum_{w \in A^1_\pi} 1 \\ &= \sum_{(s_1,\ldots,s_h) \in \Pi^{2,1}(g)} \prod_{i=1}^{h} |A^1_{s_i}|, \end{aligned}$$

where the sum is taken over all $\pi \in \Pi^{2,1}(g)$ having singletons of length $\equiv 2$ mod 4 and pairs of cycles of even length. The result follows since, by Table 3.1, we have $|A^1_s| = 2$ if s is a singleton and $|A^1_s| = 2d$ if s is a pair of cycles of length d.

If $g \in \mathrm{cl}^1_{2\alpha}$ then we may assume that there exists a cycle $(i_1, \ldots, i_d)$ of $|g|$ such that $z_{i_1}(g) = z_{i_2}(g) = 1$ and $z_i(g) = 0$ for all $i \neq i_1, i_2$. From the description of A^1_π given in Table 3.1 it follows that $(-1)^{<g,v>} = -1$ for all $v \in A^1_\pi$ and the result follows as in the previous case. $\square$

We can now complete our description of the difference character $\chi_{\phi^+} - \chi_{\phi^-}$.

Proposition 3.2.5. *We have*

$$\chi_{\phi^+}(g) - \chi_{\phi^-}(g) = \sum_{\lambda \vdash m} (\chi^0_{[\lambda,\lambda]} - \chi^1_{[\lambda,\lambda]})(g) \qquad \forall\, g \in D_{2m}.$$

Proof. Let g belong to an unsplit class. Then g has at least a cycle of odd length or a cycle of color 1. It follows from Lemmas 3.2.2 and 3.2.3 that $F_\pi(g) = 0$ for all $\pi \in \Pi^{2,1}(g)$ and therefore $\chi_{\phi^+}(g) - \chi_{\phi^-}(g) = 0$ by Equation (3.2). Since we also have $\sum_{\lambda \vdash m}(\chi^0_{[\lambda,\lambda]} - \chi^1_{[\lambda,\lambda]})(g) = 0$ by Proposition 2 the proof is complete in the case of g in an unsplit class.

If g belongs to the class $\mathrm{cl}^\epsilon_{2\alpha}$ we make the simple observation that $\ell(\alpha) = \ell(\pi) + \sum_j \mathrm{pair}_j(\pi)$ for all $\pi \in \Pi^{2,1}(g)$ and we have, by Proposition 3.2.4,

$$\begin{aligned}
\chi_{\phi^+}(g) - \chi_{\phi^-}(g) &= \epsilon \sum_{\pi \in \Pi^{2,1}(g)} 2^{\ell(\pi)} (2j)^{\mathrm{pair}_{2j}(\pi)} \\
&= \epsilon \sum_{\pi \in \Pi^{2,1}(2\alpha)} 2^{\ell(\pi)} (2j)^{\mathrm{pair}_{2j}(\pi)} \\
&= \epsilon 2^{\ell(2\alpha)} \sum_{\pi \in \Pi^{2,1}(2\alpha)} j^{\mathrm{pair}_{2j}(\pi)} \\
&= \epsilon 2^{\ell(\alpha)} \sum_{\pi \in \Pi^{2,1}(\alpha)} j^{\mathrm{pair}_{j}(\pi)}
\end{aligned}$$

where the sum in the first three lines is taken over all $\pi \in \Pi^{2,1}(g)$ having singleton of length $\equiv 2 \mod 4$ and the sum in the last line is over all $\pi \in \Pi^{2,1}(\alpha)$ having singletons of odd length. The result now follows from Proposition 2 since it is known that

$$\sum_{\lambda \vdash n} \chi_\lambda(\alpha) = \sum_{\pi \in \Pi^{2,1}(\alpha)} j^{\mathrm{pair}_j(\pi)},$$

the sum being taken over all $\pi \in \Pi^{2,1}(\alpha)$ having singletons of odd length (see, e.g., [6, Proposition 3.6] specialized to the case $r = 1$). □

We are now ready to state and prove the main result of this section.

Theorem 3.2.6. *Let (M, ϱ) be the Gelfand model of D_{2m} constructed as in Theorem 2.5.1. Let $M_{\rm Asym}$ be the submodule of M spanned by antisymmetric involutions of D_n^*. We have*

$$M_{\rm Asym} \cong \bigoplus_{\mu \vdash m} \rho^1_{[\mu,\mu]}.$$

Proof. By Proposition 3.2.5 we have

$$\sum_{\lambda \vdash m} \chi_{[\lambda,\lambda]^0}(g) + \chi_{\phi^-}(g) = \sum_{\lambda \vdash m} \chi_{[\lambda,\lambda]^1}(g) + \chi_{\phi^+}(g) \qquad \forall\, g \in D_{2m}. \qquad (3.\text{-}10)$$

Now, recalling the linear independence of the characters of the irreducible representations of any finite group we deduce that ϕ^+ has a subrepresentation isomorphic to $\bigoplus_{\lambda\vdash m}\rho^0_{[\lambda,\lambda]}$, and ϕ^- has a subrepresentation isomorphic to $\bigoplus_{\lambda\vdash m}\rho^1_{[\lambda,\lambda]}$. By Proposition 2.2.5 we have

$$\sum_{\lambda\vdash m}\dim(\rho_{[\lambda,\lambda]^0})=\sum_{\lambda\vdash m}|\mathcal{ST}_{[\lambda,\lambda]}|=\frac{1}{2}\sum_{\lambda\vdash m}|\mathcal{ST}_{(\lambda,\lambda)}|.$$

On the other hand we have

$$\dim(\phi^+)=\dim(M_{\text{Asym}})=\frac{1}{2}\sum_{\lambda\vdash m}|\mathcal{ST}_{(\lambda,\lambda)}|$$

since, by Lemma 2.4.2, $v\in B_{2m}$ is antisymmetric $\iff v\overset{\overline{RS}}{\mapsto}(P_0,P_1;P_1,P_0)$ (see also Proposition 2.4.4). Therefore $\dim(\phi^+)=\sum_{\lambda\vdash m}\dim(\rho_{[\lambda,\lambda]^0})$ and the proof is now complete. $\square$

Example 3.2.7. If $G=D_4$, M_{Asym} is spanned by the antisymmetric elements in $B_4/\pm I$. These are

$$\begin{aligned}&[(2,1,4,3);0,1,0,1], \quad [(2,1,4,3);0,1,1,0], \quad [(3,4,1,2);0,0,1,1],\\ &[(3,4,1,2);0,1,1,0], \quad [(4,3,2,1);0,0,1,1], \quad [(4,3,2,1);0,1,0,1],\end{aligned}$$

so M_{Asym} has dimension 6 and the theorem says that as a D_4-module

$$M_{\text{Asym}}\cong\rho^1_{[\square\square,\,\square\square]}\oplus\rho^1_{\left[\begin{smallmatrix}\square\\\square\end{smallmatrix},\,\begin{smallmatrix}\square\\\square\end{smallmatrix}\right]}$$

3.3 The submodule M_{Sym}

An immediate consequence of Theorem 3.2.6 is the following

Theorem 3.3.1. *Let M_{Sym} be the submodule of M spanned by symmetric involutions of D_n^*. As a consequence of the natural decomposition $M=M_{\text{Sym}}\oplus M_{\text{Asym}}$, we obtain*

$$M_{\text{Sym}}\cong\bigoplus_{\mu\vdash m}\rho^0_{[\mu,\mu]}\oplus\bigoplus_{\substack{(\lambda,\mu)\vdash m\\ \lambda\neq\mu}}\rho_{[\lambda,\mu]}.$$

In this section we will provide the description of the irreducible decomposition of the D_n-modules $M(c)$, where c is any S_n-conjugacy class of symmetric involutions in $B_n/\pm\text{Id}$. Let v be a symmetric involution in $B_n/\pm\text{Id}$. Recall the notation 2.3.2: we let $\text{Sh}(v)$ be the element of $\text{Fer}(2,1,2,n)$ which is the shape of the tableaux of the image of v via the projective Robinson-Schensted correspondence. Namely, $\mathrm{Sh}(v) \stackrel{\mathrm{def}}{=} [\lambda, \mu]$ where

$$v \xrightarrow{RS_2} ([P];[P]), \quad P \in \mathcal{ST}(2,1,2,n), \quad P \text{ of shape } [\lambda,\mu].$$

If c is a S_n-conjugacy class of symmetric involutions in $B_n/\pm\mathrm{Id}$, we also let

$$\mathrm{Sh}(c) = \bigcup_{v \in c} \mathrm{Sh}(v).$$

Recall the analogous analysis made for B_n. In that setting, the S_n-conjugacy classes were parametrized as c_{f_0,f_1,p_0,p_1}. The corresponding S_n-conjugacy classes in the quotient $B_n/\pm\mathrm{Id}$ are indexed by *unordered* pairs $\{(f_0,p_0),(f_1,p_1)\}$ satisfying $f_0+f_1+2p_0+2p_1=n$. Paralleling the proof in the case of B_n one can show that if c is indexed by $\{(f_0,p_0),(f_1,p_1)\}$ then

$$\mathrm{Sh}(c) = \left\{ \begin{array}{c} [\lambda,\mu] \in \mathrm{Fer}(2,1,2,n) : \lambda \vdash (f_0+2p_0), \mu \vdash (f_1+2p_1), \\ \lambda \text{ has exactly } f_0 \text{ columns of odd length}, \\ \mu \text{ has exactly } f_1 \text{ columns of odd length}. \end{array} \right\}$$

We can state the main result of this section.

Theorem 3.3.2. *Let c be an S_n-conjugacy class of symmetric involutions in $B_n/\pm\mathrm{Id}$. Then*

$$M(c) = \bigoplus_{\substack{[\lambda,\mu]\in \mathrm{Sh}(c) \\ \lambda \neq \mu}} \rho_{[\lambda,\mu]} \oplus \bigoplus_{[\lambda,\lambda]\in \mathrm{Sh}(c)} \rho^0_{[\lambda,\lambda]}.$$

Proof. We first tackle the (easier) case $(f_0,p_0) \neq (f_1,p_1)$. In this case we let $\tilde{c} = c_{f_0,f_1,p_0,p_1}$ in the notation of Section 1.2.1, be one of the two S_n-conjugacy classes in B_n whose projection to $B_n/\pm\mathrm{Id}$ is c. Then the projection map $\tilde{c} \to c$ is a bijection, and it easily follows that $M(c) \cong M(\tilde{c}) \downarrow_{D_n}$. The result now is a consequence of Theorem 1.2.13 and the discussion on the irreducible representations of D_n given in Section 3.1.

Now assume that $f_0 = f_1$ and that $p_0 = p_1$. We denote by $\tilde{c} = c_{f_0,f_1,p_0,p_1}$ the unique S_n-conjugacy class in B_n whose projection to $B_n/\pm\mathrm{Id}$ is c. We remark in this case that the projection $\tilde{c} \to c$ is 2:1. Nevertheless we may notice that $M(c) \cong M' \downarrow_{D_n}$ where M' is the D_n-submodule of $M(\tilde{c})$ given by

$$M' = \mathrm{Span}\{C_v + C_{-v} : v \in \tilde{c}\}.$$

Therefore, by Theorem 1.2.13 and the same discussion on the irreducible representations of D_n cited above, we have that the irreducible components of $M(c)$

are some of the $\rho_{[\lambda,\mu]}$, with $[\lambda,\mu] \in \mathrm{Sh}(c)$ and $\lambda \neq \mu$, and some of the split representations $\rho^0_{[\lambda,\lambda]}$ and $\rho^1_{[\lambda,\lambda]}$, with $[\lambda,\lambda] \in \mathrm{Sh}(c)$.

But now we need to recall that, in the overall, $\oplus_c M(c)$ is a Gelfand model for D_n. Theorem 3.3.1 says that, as we already know that the split representations $\rho^1_{[\lambda,\lambda]}$ appear in M_{Asym}, they cannot appear in $M(c)$. And as the other irreducible representations listed above can appear in this submodule only (the sets $\mathrm{Sh}(c)$ are clearly all disjoint), they actually have to appear here (and with multiplicity 1) and the proof is complete. □

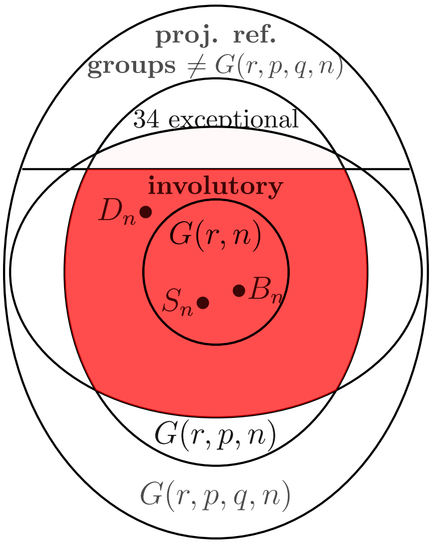
Example 3.3.3. Let $\tilde{v} \in B_6$ given by $|\tilde{v}| = [(6,4,3,2,5,1);1,0,0,0,1,1]$. Let v be the projection of $\tilde{v}$ in $B_n/\pm\mathrm{Id}$. Then the S_n-conjugacy class c containing v has 90 elements and the decomposition of the D_n-module $M(c)$ is given by all representations indexed by $[\lambda,\mu] \in \mathrm{Fer}(2,1,2,6)$ where both λ and μ are partitions of 3 and have exactly one column of odd length, with the additional condition that if $\lambda = \mu$ the split representation to be considered is $[\lambda,\lambda]^0$. Therefore

$$M(c) \cong \rho_{\left[\ydiagram{2,1},\,\ydiagram{1,1,1}\right]} \oplus \rho^0_{\left[\ydiagram{2,1},\,\ydiagram{2,1}\right]} \oplus \rho^0_{\left[\ydiagram{1,1,1},\,\ydiagram{1,1,1}\right]}.$$

Chapter 4

Decomposition of the model for the involutory groups $G(r, p, n)$

This chapter contains the main result of this work in its most general form. So far, we met it in various phrasings: for B_n (Theorem 1.2.13), for $G(r, n)$ (Theorem 1.1.3), for D_n (Theorems 3.2.6 and 3.3.2). In each of these occasions, we provided a proof which made somehow use of the special form of the group analyzed.

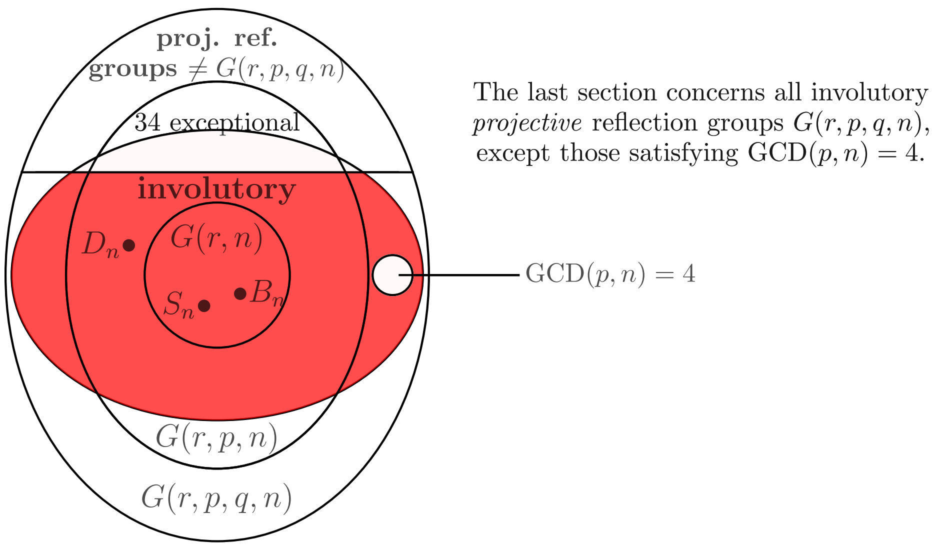


In the first part of this chapter, we will afford the general case of all involutory reflection groups $G(r, p, n)$, i.e. all $G(r, p, n)$ such that $\mathrm{GCD}(p, n) = 1, 2$ (see Theorem 2.4.5). Though the main result of this section is a generalization of Theorem 1.1.3, we should mention that the proof is not, in the sense that we will make use here of the main results of Chapter 1. This fact can also be observed in the proof of Theorem 3.3.2 for D_n, where the results already proved for B_n were actually exploited.

The strategy of our proof will be, in fact, similar to the one followed for D_n. Nevertheless, the general case presents some more difficulties if compared to the example of D_n. First of all, a characterization of those $G(r, n)$-conjugacy classes that split as $G(r, p, n)$-conjugacy classes is necessary. This is known for D_n but we could not find it in literature for the other involutory $G(r, p, n)$, and is the content of Section 4.1. Secondly, a study of characters of split representations is needed (see Sections 4.2-4.3), which was only known for D_n as well.

In Section 4.4, concerning the case $\mathrm{GCD}(p, n) = 2$, we are in a position to determine the irreducible representations appearing in the antisymmetric submodule. This result is furtherly refined in Section 4.5. Section 4.6, instead, is devoted to refining the symmetric submodule, and concerns both cases $\mathrm{GCD}(p, n) = 1, 2$. This completes our analysis for the groups $G(r, p, n)$.

In the very last section of this chapter we make a further generalization. We show how the results attained for the involutory $G(r, p, n)$ can also be extended to their quotients $G(r, p, q, n)$. This means that our main theorem in its final version (Theorem 4.7.1) concerns all involutory reflection groups $G(r, p, n)$ *and their quotients* modulo scalar sugroups, i.e. all $G(r, p, q, n)$ with $\mathrm{GCD}(p, n) = 1, 2$. In other words, we are dealing with all involutory projective reflection groups $G(r, p, q, n)$ except those satisfying $\mathrm{GCD}(p, n) = 4$ (see Theorem 2.4.5).



4.1 On the split conjugacy classes

Recall the parametrization of the $G(r, n)$-conjugacy classes given on page 7. In the case of any involutory reflection group $G(r, p, n)$, we have not been able to find the nature of the conjugacy classes that split from $G(r, n)$ to $G(r, p, n)$ in the literature. This is the content of the present section.

Definition. Let r be even. Let c be a cycle in $G(r,n)$ of even length and even color. Recall the cyclic notation on page 5. If $c = (i_1^{z_{i_1}}, i_2^{z_{i_2}}, \ldots, i_{2d}^{z_{i_{2d}}})$ we define the *signature* of c to be

$$\operatorname{sign}(c) \stackrel{\text{def}}{=} z_{i_1} + z_{i_3} + \cdots + z_{i_{2d-1}} = z_{i_2} + z_{i_4} + \cdots + z_{i_{2d}} \in \mathbb{Z}_2,$$

so that the signature of c can be either 0 or 1. If g is a product of disjoint cycles of even length and even color we define the *signature* $\operatorname{sign}(g)$ of g as the sum of the signatures of its cycles.

Proposition 4.1.1. *Let r be even and let c be a cycle in $G(r,n)$ of even length and even color. Let $h \in G(r,n)$. Then*

$$\operatorname{sign}(h^{-1}ch) = \operatorname{sign}(c) + \sum_{j \in |h|^{-1}(\operatorname{Supp}(c))} z_j(h) \in \mathbb{Z}_2$$

(see the definition of $\operatorname{Supp}(c)$ *on page 5). In particular, if $g \in G(r,n)$ is a product of cycles of even length and even color, then*

$$\operatorname{sign}(h^{-1}gh) = \operatorname{sign}(g) + z(h) \in \mathbb{Z}_2.$$

Proof. Let $|c| = (i_1, i_2, \ldots, i_{2d})$. We have that $h^{-1}ch$ is a cycle and $|h^{-1}ch| = (\tau^{-1}(i_1), \ldots, \tau^{-1}(i_{2d}))$, where $\tau = |h|$. Therefore

$$\begin{aligned}
\operatorname{sign}(h^{-1}ch) &= \sum_{j \text{ odd}} z_{\tau^{-1}(i_j)}(h^{-1}ch) \\
&= \sum_{j \text{ odd}} z_{\tau^{-1}(i_j)}(h) + z_{i_j}(c) - z_{\tau^{-1}(i_{j+1})}(h) \\
&= \operatorname{sign}(c) + \sum_{j \in |h|^{-1}(\operatorname{Supp}(c))} z_j(h),
\end{aligned}$$

where the sums in the first two lines are meant to be over all odd integers $j \in [2d]$. □

As a consequentce of Proposition 4.1.1, we have the following

Corollary 4.1.2. *The conjugacy classes of $G(r,n)$ contained in $G(r,p,n)$ of the special form*

$$\mathrm{cl}_{2\alpha} = \mathrm{cl}_{(2\alpha^{(0)}, \emptyset, 2\alpha^{(2)}, \emptyset, \ldots, 2\alpha^{(r-2)}, \emptyset)},$$

split in $G(r,p,n)$ into (at least) two conjugacy classes, according to the signature.

Let us now afford the $G(r,n)$-conjugacy classes of a different form, to determine if they split as $G(r,p,n)$-classes.

Notation 4.1.3. If G is a group and $g \in G$ we denote by $\mathrm{Cl}_G(g)$ the conjugacy class of g and by $C_G(g)$ the centralizer of g in G.

If $g \in G(r, p, n)$ then the $G(r, n)$-conjugacy class $\mathrm{Cl}_{G(r,n)}(g)$ of g splits into more than one $G(r, p, n)$-conjugacy class if and only if

$$\frac{|\mathrm{Cl}_{G(r,n)}(g)|}{|\mathrm{Cl}_{G(r,p,n)}(g)|} = \frac{[G(r, n) : G(r, p, n)]}{[C_{G(r,n)}(g) : C_{G(r,p,n)}(g)]} = \frac{p}{[C_{G(r,n)}(g) : C_{G(r,p,n)}(g)]} > 1.$$

Thus, $\mathrm{Cl}_{G(r,n)}(g)$ splits if and only if $[C_{G(r,n)}(g) : C_{G(r,p,n)}(g)] < p$. The following proposition clarifies which conjugacy classes of $G(r, n)$ split in $G(r, p, n)$, for every involutory $G(r, p, n)$.

Proposition 4.1.4. *Let $g \in G(r, p, n)$ and let $\mathrm{Cl}(g)$ be its conjugacy class in the group $G(r, n)$. Then the following holds:*

1. *if $\mathrm{GCD}(p, n) = 1$, $\mathrm{Cl}(g)$ does not split as a class of $G(r, p, n)$;*
2. *if $\mathrm{GCD}(p, n) = 2$, $\mathrm{Cl}(g)$ splits up into two different classes of $G(r, p, n)$ if and only if all the cycles of g have:*
   - *even length;*
   - *even color,*

i.e., if $g \in (2\alpha^{(0)}, \emptyset, 2\alpha^{(2)}, \emptyset, \ldots, 2\alpha^{(r-2)}, \emptyset)$.

Proof. Let $G = G(r, n)$ and $H = G(r, p, n)$. We first make a general observation. If $C_G(g)$ contains an element x such that $z(x) \equiv 1 \mod p$, we can split the group $C_G(g)$ into cosets modulo the subgroup $\langle x \rangle$: in each coset there is exactly one element having color $0 \mod p$ every p elements. Thus,

$$[C_G(g) : C_H(g)] = p$$

and $\mathrm{Cl}(g)$ does not split in H.

Now let $\mathrm{GCD}(p, n) = 1$. Thanks to Bézout identity, there exist a, b such that $an + bp = 1$, i.e. there exists a such that the scalar matrix $\zeta_r^a \mathrm{Id}$ has color $1 \mod p$, so that $\mathrm{Cl}(g)$ does not split thanks to the observation above.

Assume now that $\mathrm{GCD}(p, n) = 2$. Arguing as above, there exist a, b such that $ap + bn = 2$, so we know that $C_G(g)$ contains at least an element $\zeta_r^a \mathrm{Id}$ with color $2 \mod p$.

If there exists at least an element x of odd color in $C_G(g)$, the matrix $(\zeta_r^a \mathrm{Id})^i \cdot x$ has color 1 for some i, so again $\mathrm{Cl}(g)$ does not split in H.

On the other hand, if there are no elements of odd color in $C_G(g)$, every coset of $\langle \zeta_r^a \mathrm{Id} \rangle$ has exactly one element belonging to $G(r, p, n)$ out of p' elements (p' standing for $\frac{p}{2}$ as in Notation 2.4.3). Thus,

$$[C_G(g) : C_H(g)] = p'$$

and $\mathrm{Cl}(g)$ splits into $p/p' = 2$ classes.

Let us see when this happens according the the cyclic structure of g.

1. If g has at least a cycle of odd color, say c, c is in $C_G(g)$ and $\mathrm{Cl}(g)$ does not split.

2. If g has a cycle of odd length, say $(a_1^{z_1}, \ldots, a_{2d+1}^{z_{2d+1}})$, then $(a_1^1, \ldots, a_{2d+1}^1)$ has odd color and is in $C_G(g)$, so $\mathrm{Cl}(g)$ does not split.

3. We are left to study the case of g being a product of cycles all having even length and even color. Thanks to Lemma 4.1.1, every element in $C_G(g)$ has even color, so by the above argument $\mathrm{Cl}(g)$ splits into exactly two classes, and we are done.

□

Notation 4.1.5. If $2\alpha = (2\alpha^{(0)}, \emptyset, 2\alpha^{(2)}, \emptyset, \ldots, 2\alpha^{(r-2)}, \emptyset)$ is such that the $G(r,n)$-conjugacy class $\mathrm{cl}_{2\alpha}$ is contained in $G(r,p,n)$ (i.e. if $\sum 2i\ell(\alpha_{(i)}) \equiv 0 \mod p$), we denote by $\mathrm{cl}^0_{2\alpha}$ the $G(r,p,n)$-conjugacy class consisting of all elements in $\mathrm{cl}_{2\alpha}$ having signature 0, and we similarly define $\mathrm{cl}^1_{2\alpha}$. This choice is coherent with Notation 3.1.2 adopted for D_n.

4.2 The discrete Fourier transform

As we have seen in Section 2.2, there is an action of the cyclic group of order p on the set $\mathrm{Fer}(r,n)$, giving place to the quotient set $\mathrm{Fer}(r,p,n)^* = \mathrm{Fer}(r,1,p,n)$. We will now illustrate explicitely how this action can be constructed in terms of representations. Thus, the cyclic group will be acting not on the diagrams of $\mathrm{Fer}(r,n)$, but on the irreducible representations of $G(r,n)$. This action gives us the opportunity of introducing the concept of discrete Fourier transform, which will be essential in what follows about the case $\mathrm{GCD}(p,n) = 2$. We will parallel and generalize in this section an argument due to Stembridge [20, Sections 6 and 7B].

Notation 4.2.1. Given $n_0, \ldots, n_k \in \mathbb{N}$ such that $n_0 + \ldots + n_k = n$, consider the k-tuplet $\nu = (n_0, \ldots, n_k)$. We let $G(r,\nu) \stackrel{\text{def}}{=} G(r,n_0) \times \cdots \times G(r,n_k)$ be the (Young) subgroup of $G(r,n)$ given by

$$G(r,\nu) = \{[(\sigma_1, \ldots, \sigma_n); z_1, \ldots z_n] \in G(r,n) : \\ \sigma_i \le n_0 + \cdots + n_j \text{ if and only if } i \le n_0 + \cdots + n_j\}.$$

Consider the color representation γ_n mentioned in Proposition 0.4.2:

$$\begin{aligned}\gamma_n : G(r,n) &\to \mathbb{C}^* \\ g &\mapsto \zeta_r^{z(g)}.\end{aligned}$$

The representation γ_n has, of course, order r. Consider the $G(r,n)$ representation $\rho_{(\lambda^{(0)},\dots,\lambda^{(r-1)})}$, where $|\lambda_i| = n_i$. If $\nu = (n_0,\dots,n_{r-1})$, $\langle\gamma_n\rangle$ acts on $\rho_{(\lambda^{(0)},\dots,\lambda^{(r-1)})}$ in this way:

$$\begin{aligned}\gamma_n \otimes \rho_{(\lambda^{(0)},\dots,\lambda^{(r-1)})} =& \mathrm{Ind}_{G(r,\nu)}^{G(r,n)} \gamma_n|_{G(r,\nu)} \otimes \left(\bigodot_{i=0}^{r-1} (\gamma_{n_i}^{\otimes i} \otimes \tilde{\rho}_{\lambda^{(i)}})\right) \\ =& \mathrm{Ind}_{G(r,\nu)}^{G(r,n)} \Big((\gamma_{n_0} \otimes \tilde{\rho}_{\lambda^{(0)}}) \odot \cdots \odot (\gamma_{n_{r-1}}^{\otimes r} \otimes \tilde{\rho}_{\lambda^{(r-1)}})\Big) \\ =& \rho_{(\lambda^{(r-1)},\lambda^{(0)},\lambda^{(1)},\dots,\lambda^{(r-2)})}.\end{aligned}$$

Since $p|r$, the group $\langle\gamma_n\rangle$ contains a cyclic subgroup $\Gamma \stackrel{\text{def}}{=} \langle\gamma_n^{\frac{r}{p}}\rangle$ of order p, which, on its turn, acts on the set of the irreducible representations of $G(r,1,q,n)$:

$$\begin{aligned}\gamma_n^{r/p} \otimes \rho_{(\lambda^{(0)},\dots,\lambda^{(r-1)})} &= \mathrm{Ind}_{G(r,\nu)}^{G(r,n)} \Big(\big(\gamma_n^{r^p}|_{G(r,\nu)}\big) \otimes \bigodot_{i=0}^{r-1} (\gamma_{n_i}^{\otimes i} \otimes \tilde{\rho}_{\lambda^{(i)}})\Big) \quad (4.1) \\ &= \rho_{(\lambda^{(r-r/p)},\dots,\lambda^{(r-1)},\lambda^{(0)},\dots,\lambda^{(r-1-r/p)})},\end{aligned}$$

and so it corresponds to a shift of r/p of the indexing partitions.

We recall, according to [20], the following

Definition. Let $\lambda = (\lambda^{(0)},\dots,\lambda^{(r-1)}) \in \mathrm{Fer}(r,n)$, and let (V,ρ_λ) be a concrete realization of the irreducible $G(r,n)$-representation ρ_λ on the vector space V. Let γ be a generator for $\mathrm{Stab}_\Gamma(\rho_\lambda)$. An *associator* for the pair (V,γ) is an element $S \in GL(V)$ exhibiting an explicit isomorphism of $G(r,n)$-modules between

$$(V,\rho_\lambda) \quad \text{and} \quad (V,\gamma\otimes\rho_\lambda).$$

By Schur's lemma $S^{m_p(\lambda)}$ is a scalar, and therefore S can be normalized in such a way that $S^{m_p(\lambda)} = 1$.

Recall form Theorem 2.2.5 that a representation ρ_λ of $G(r,n)$ splits into exactly $m_p(\lambda)$ irreducible representations of $G(r,p,n)$.

Definition. Let $\lambda \in \mathrm{Fer}(r,n)$ and let S be an associator for the $G(r,n)$-module (V,ρ_λ). Then the *discrete Fourier transform* with respect to S is the family of $G(r,p,n)$-class functions $\Delta_\lambda^i : G(r,p,n) \to \mathbb{C}^*$ given by

$$\Delta_\lambda^i(h) \stackrel{\text{def}}{=} \mathrm{tr}(S^i \circ h), \qquad i \in [0, m_p(\lambda)-1].$$

A more accurate analysis of the associator shows that the irreducible representations $\rho^i_{[\lambda]}$ are exactly the eigenspaces of the associator S, and we make the convention that, once an associator S has been fixed, the representation $\rho^i_{[\lambda]}$ is the one afforded by the eigenspace of S of eigenvalue $\zeta^i_{m_p(\lambda)}$. Therefore

$$\Delta^i_\lambda(h) = \sum_{j=0}^{m_p(\lambda)-1} \zeta^{ij}_{m_p(\lambda)} \chi^j_{[\lambda]}(h), \tag{4.0}$$

for all $h \in G(r,p,n)$, $\chi^j_{[\lambda]}$ being the character of the split representation $\rho^j_{[\lambda]}$ of $G(r,p,n)$.

Now let us consider a representation ρ_λ. Looking at the action described in (4.1), we see that $m_p(\lambda) = |\mathrm{Stab}_\Gamma(\rho_\lambda)| = s$ only if $\lambda = (\lambda^{(0)}, \dots, \lambda^{(r-1)})$ consists of a smaller pattern repeated s times. It follows that $m_p(\lambda)$ is necessarily a divisor of both n and p.

In particular, if $\mathrm{GCD}(p,n) = 2$, $m_p(\lambda) = 1,2$ and so the stabilizer of a representation with respect to Γ can either be trivial or be $\{\mathrm{Id}, \gamma_n^{r'}\}$ (we recall once more that k' stands for $\frac{k}{2}$, according to Notation 2.4.3).

Notation 4.2.2. From now on, when r is even, we use for the representation $\gamma_n^{r'}(g) = (-1)^{z(g)}$ the notation $\delta(g)$.

When $\mathrm{Stab}_\Gamma(\rho_\lambda) = \{\mathrm{Id}, \delta\}$, we are dealing with representations of the form $\rho_{(\mu,\mu)}$, with $\mu \in \mathrm{Fer}(r', n')$. Notice that μ may be considered as belonging to $\mathrm{Fer}(r', 1, p', n')$: acting on μ with an element of $C_{p'}$ corresponds to acting on (μ,μ) with an element of C_p, and we know that elements of $\mathrm{Fer}(r,n)$ in the same class modulo C_p parametrize the same irreducible representation of $G(r,p,n)$. These $\rho_{(\mu,\mu)}$ are the representations of $G(r,n)$ that split as $G(r,p,n)$-modules. As in the case of D_n, they split into two different representations that we denote by $\rho^0_{[\mu,\mu]}$ and $\rho^1_{[\mu,\mu]}$. We also denote by $\chi^0_{[\mu,\mu]}$ and $\chi^1_{[\mu,\mu]}$ the corresponding characters. Then the discrete Fourier transform of $\rho_{(\mu,\mu)}$ is given by the two functions

$$\Delta^0_{(\mu,\mu)}(h) = \chi^0_{[\mu,\mu]}(h) + \chi^1_{[\mu,\mu]}(h); \qquad \Delta^1_{(\mu,\mu)}(h) = \chi^0_{[\mu,\mu]}(h) - \chi^1_{[\mu,\mu]}(h).$$

4.3 The difference character

In this section, we exploit the definition of $\Delta^1_{(\mu,\mu)}$ to provide an explicit computation of the difference character $\chi^0_{[\mu,\mu]} - \chi^1_{[\mu,\mu]}$ for every $G(r,p,n)$ with $(p,n) = 2$. This computation will turn up to be of crucial importance in the proof of Theorem 4.4.1.

Recall that when $\mathrm{GCD}(p,n) = 2$, the conjugacy classes of $G(r,n)$ of the form $\mathrm{cl}_{2\alpha}$ split into two distinct $G(r,p,n)$-classes $\mathrm{cl}^0_{2\alpha}$ and $\mathrm{cl}^1_{2\alpha}$, where $2\alpha = (2\alpha^{(0)}, \emptyset, 2\alpha^{(2)}, \emptyset, \ldots, 2\alpha^{(r-2)}, \emptyset)$ (see Section 4.1).

Notation 4.3.1. In what follows, we often need to compute class functions on $G(r,p,n)$. For this reason, it will be useful to fix one special element, that we call *normal*, for each $G(r,p,n)$-conjugacy class. If the conjugacy class is not of the form $\mathrm{cl}^1_{2\alpha}$, the normal element h is defined as follows:

- the elements of each cycle of h are chosen in increasing order, from the cycles of smallest color to the cycles of biggest color;
- in every cycle of color i, all the elements have color 0 but the biggest one whose color is i.

If the class is of the form $\mathrm{cl}^1_{2\alpha}$ the normal element h is defined similarly with the unique difference that if the cycle containing n has color $2j$ then the color of n is $2j-1$ and the color of $n-1$ is 1. For example, if

$$2\alpha = \left(\ydiagram{2}, \emptyset, \ydiagram{4}, \emptyset, \ydiagram{4,2}, \emptyset \right)$$

Then the normal element in $\mathrm{cl}^0_{2\alpha}$ is $(1,2)(3,4,5,6^2)(7,8,9,10^4)(11,12^4)$ and the normal element in $\mathrm{cl}^1_{2\alpha}$ is $(1,2)(3,4,5,6^2)(7,8,9,10^4)(11^1,12^3)$, where we have omitted all the exponents equal to 0.

Proposition 4.3.2. *Let $g \in G(r,p,n)$, and $\mu \in \mathrm{Fer}(r',n')$. Let χ_μ denote the character of the representation of $G(r',n')$ indexed by μ. Then*

$$\Delta^1_{(\mu,\mu)}(g) = \begin{cases} (-1)^\eta 2^{\ell(\alpha)} \chi_\mu(\alpha^{(0)}, \alpha^{(2)}, \ldots, \alpha^{(r-2)}), & \text{if } g \in \mathrm{cl}^\eta_{2\alpha}; \\ 0, & \text{otherwise} \end{cases}$$

where, if $\alpha = (\alpha^{(0)}, \emptyset, \alpha^{(2)}, \emptyset, \ldots, \alpha^{(r-2)}, \emptyset)$, $\ell(\alpha) = \sum \ell(\alpha^{(i)})$ (see the definition of ℓ in Notation 0.3.3).

Proof. When g does not belong to a split conjugacy class, $\Delta^1_{(\mu,\mu)}(g) = 0$. In fact, $\chi^0_{[\mu,\mu]}$ and $\chi^1_{[\mu,\mu]}$ are conjugate characters, so they must coincide on every element belonging to an unsplit class.

When g does belong to a split conjugacy class, this proof consists of three steps:

1. Provide an explicit description for the $G(r,n)$-module $\rho_{(\mu,\mu)}$.
2. Build an associator S for the $G(r,n)$-module $\rho_{(\mu,\mu)}$.
3. Compute the trace $\mathrm{tr}(S(g)) = \Delta^1_{(\mu,\mu)}(g)$.

Let us start with the first step.

Notation 4.3.3. For brevity, we set $\tau = (t_0, \ldots, t_{r'-1})$, where $t_i = |\mu^{(i)}|$ and $G(r, (\tau, \tau)) \overset{\text{def}}{=} G(r, t_0) \times \cdots \times G(r, t_{r'-1}) \times G(r, t_0) \times \cdots \times G(r, t_{r'-1}) < G(r, n)$.

Our representation $\rho_{(\mu,\mu)}$ looks like this (see Theorem 0.4.2):

$$\begin{aligned}\rho_{(\mu,\mu)} \quad &= \quad \mathrm{Ind}_{G(r,(\tau,\tau))}^{G(r,n)}\big(\tilde{\rho}_{\mu^{(0)}} \odot (\gamma_{n_1} \otimes \tilde{\rho}_{\mu^{(1)}}) \odot \cdots \odot (\gamma_{n_{r'-1}}^{\otimes r'-1} \otimes \tilde{\rho}_{\mu^{(r'-1)}}) \\ &\qquad \odot(\gamma_{n_0}^{\otimes r'} \odot \tilde{\rho}_{\mu^{(0)}}) \odot (\gamma_{n_1}^{\otimes r'+1} \otimes \tilde{\rho}_{\mu^{(1)}}) \odot \cdots \odot (\gamma_{n_{r'-1}}^{\otimes r-1} \otimes \tilde{\rho}_{\mu^{(r'-1)}})\big)\end{aligned}$$

Splitting the induction into two steps, using the intermediate subgroup $G(r, n') \times G(r, n')$, we obtain

$$\begin{aligned}&\mathrm{Ind}_{G(r,n')\times G(r,n')}^{G(r,n)}\Big(\mathrm{Ind}_{G(r,(\tau,\tau))}^{G(r,n')\times G(r,n')}\big(\tilde{\rho}_{\mu^{(0)}} \odot \cdots \odot (\gamma_{n_{r'-1}}^{\otimes r-1} \otimes \tilde{\rho}_{\mu^{(r'-1)}})\big)\Big) \\ =&\mathrm{Ind}_{G(r,n')\times G(r,n')}^{G(r,n)}\big(\rho_\mu \odot (\delta \otimes \rho_\mu)\big).\end{aligned}$$

We need to give an explicit description of this representation of $G(r, n)$. Consider the set Θ of two-rowed arrays $\begin{bmatrix} t_1 & \ldots & t_{n'} \\ t_{n'+1} & \ldots & t_n \end{bmatrix}$ such that $\{t_1, \ldots, t_n\} = \{1, \ldots, n\}$ and the t_i's increase on each of the two rows. Each element in Θ can be identified with the permutation whose window notation is $[(t_1, \ldots, t_n); 0, \ldots, 0]$.

Proposition 4.3.4. *Let $g \in G(r, n)$. Then*

$$\exists\ !\ t' \in \Theta : g = t'(x_1, x_2) \text{ with } (x_1, x_2) \in G(r, n') \times G(r, n').$$

Proof. Existence. Let $g = [(\sigma_1, \ldots, \sigma_n); z_1, \ldots, z_n]$, and let t be the tabloid whose first and second line are filled with the (reordered) integers $\sigma_1, \ldots, \sigma_{n'}$, and $\sigma_{n'+1}, \ldots, \sigma_n$ respectively. Since we need to obtain $g = t(x_1, x_2)$ with $(x_1, x_2) \in G(r, n') \times G(r, n')$, we have to check that $t^{-1}g \in G(r, n') \times G(r, n')$, i.e.,

$$\begin{aligned}1 \le t^{-1}|g|(i) = t^{-1}\sigma_i \le n', &\quad \text{if } i \in [n'] \\ n' < t^{-1}|g|(i)t^{-1}\sigma_i \le n, &\quad \text{if } i \in [n'+1, n],\end{aligned}$$

and this is an immediate consequence of the way t was constructed.

Uniqueness. This is due to cardinality reasons:

$$|\Theta|\,|G(r, n') \times G(r, n')| = \binom{n}{n'}(n'!r^{n'})^2 = |G(r, n)|\,.$$

□

Thanks to Proposition 4.3.4, a set of coset representatives for the quotient

$$\frac{G(r, n)}{G(r, n') \times G(r, n')}$$

is given by Θ. Let T be the vector space spanned by the elements of Θ. The vector space associated to the representation we are dealing with can be identified with $T \otimes V \otimes V$, and the action of $\rho_{(\mu,\mu)}$ on it is given by

$$\begin{aligned}\rho_{(\mu,\mu)} : G(r,n) &\longrightarrow GL(T \otimes V \otimes V)\\ x &\longmapsto \rho_{(\mu,\mu)}(x) : T \otimes V \otimes V \longrightarrow T \otimes V \otimes V\\ &\qquad t \otimes v_1 \otimes v_2 \longmapsto \delta(x_2)t' \otimes \rho_\mu(x_1)(v_1) \otimes \rho_\mu(x_2)(v_2),\end{aligned}$$

where t', x_1 and x_2 are uniquely determined by the relation $xt = t'(x_1, x_2)$ with $t \in \Theta$, $(x_1, x_2) \in G(r, n') \times G(r, n')$.

We are now ready for the second step.

Proposition 4.3.5. *The automorphism $S \in \mathrm{GL}(T \otimes V \otimes V)$ so defined:*

$$S(t \otimes v_1 \otimes v_2) = \hat{t} \otimes v_2 \otimes v_1,$$

where $\hat{t}$ is the element of Θ obtained from t by exchanging its rows, is an associator for $T \otimes V \otimes V$.

Proof. For brevity, all through this proof we set $\rho := \rho_{(\mu,\mu)}$. All we have to show is that S is an isomorphism of representations between ρ and $\delta \otimes \rho$, i.e.,

$$S \circ \rho(g) = \delta(g)\rho(g) \circ S.$$

The set of permutations together with the diagonal matrix $[(1, 2, \dots, n); 0, 0, \dots, 0, 1]$ generate $G(r, n)$, so this verification can be accomplished when g is one of these elements only.

Let g be a permutation first. In this case, g_1 and g_2 are permutations as well, so $\delta(g) = \delta(g_2) = 1$. Furthermore, notice that $\hat{t} = ts$ if s is the tabloid $\begin{bmatrix} \frac{n}{2}+1 & \dots & n \\ 1 & \dots & \frac{n}{2} \end{bmatrix}$. Thus

$$\begin{aligned}&S[\rho(g)(t \otimes v_1 \otimes v_2)] = S[\delta(g_2)t' \otimes g_1 v_1 \otimes g_2 v_2]\\ &= \widehat{t'} \otimes g_2 v_2 \otimes g_1 v_1 \overset{\bigstar}{=} [\rho(g)](\hat{t} \otimes v_2 \otimes v_1)\\ &= [\rho(g)](S(t \otimes v_1 \otimes v_2)) = [\delta(g)\rho(g)](S(t \otimes v_1 \otimes v_2)),\end{aligned}$$

where equality $\bigstar$ follows from

$$g\hat{t} = gts = t'(g_1, g_2)s = t's(g_2, g_1) = \widehat{t'}(g_2, g_1).$$

Let us now choose as g the diagonal matrix $[(1, 2, \dots, n); 0, 0, \dots, 0, 1]$. In this case, $|gt| = t$, while the colors of gt are all 0 but one: $z_{t^{-1}(n)}(gt) = 1$. We have $gt = t(x_1, x_2)$, where $|(x_1, x_2)|$ is the identity, and, again, all the colors are 0 but one: $z_{t^{-1}(n)}(x_1, x_2) = 1$. So $\delta(x_2) = \pm 1$ according to the value of $t^{-1}(n)$:

$$\delta(x_2) = \begin{cases} 0, & \text{if } t^{-1}(n) \leq \frac{n}{2} \\ 1 & \text{otherwise.} \end{cases}$$

Therefore, if $\delta(g_2) = -1$,

$$\begin{aligned} & S[\rho(g)(t \otimes v_1 \otimes v_2)] = S[\delta(g_2)t \otimes g_1 v_1 \otimes g_2 v_2] \\ & = S(-t \otimes g_1 v_1 \otimes g_2 v_2) = -\widehat{t} \otimes g_2 v_2 \otimes g_1 v_1 \\ & = -\rho(g)(\widehat{t} \otimes v_2 \otimes v_1) = [\delta(g)\rho(g)](S(t \otimes v_1 \otimes v_2)). \end{aligned}$$

If $\delta(g_2) = 1$,

$$\begin{aligned} & S[\rho(g)(t \otimes v_1 \otimes v_2)] = S[\delta(g_2)t \otimes g_1 v_1 \otimes g_2 v_2] \\ & = S(t \otimes g_1 v_1 \otimes g_2 v_2) = \widehat{t} \otimes g_2 v_2 \otimes g_1 v_1 \\ & = -(-\widehat{t} \otimes g_2 v_2 \otimes g_1 v_1) = -\rho(g)(\widehat{t} \otimes v_2 \otimes v_1) \\ & = [\delta(g)\rho(g)](S(t \otimes v_1 \otimes v_2)). \end{aligned}$$

In both cases, we used the equalities

$$g\widehat{t} = gts = t(g_1, g_2)s = ts(g_2, g_1) = \widehat{t}(g_2, g_1).$$

□

Finally, the last step: let us compute $\Delta^1_{(\mu,\mu)}(g)$, for every g belonging to a split conjugacy class. Since $\Delta^1_{(\mu,\mu)}$ is a class function, we can choose g to be the normal element of each $G(r,p,n)$-conjugacy class. In fact, even less is needed: it will be enough to choose the normal elements of the classes $\mathrm{cl}^0_{2\alpha}$ only, because of the useful relation (see [20], Proposition 6.2)

$$\Delta^1_{(\mu,\mu)}(ghg^{-1}) = \delta(g)\Delta^1_{(\mu,\mu)}(h) \;\; \forall g \in G(r,n), h \in G(r,p,n). \tag{4.-2}$$

So we compute $\Delta^1_{(\mu,\mu)}(h)$, where h is the normal element of the class $\mathrm{cl}^0_{2\alpha}$. By definition, $\Delta^1_{(\mu,\mu)}(h) = \mathrm{tr}(S \circ h)$. Now, given $t' \in \Theta$ and $(h_1, h_2) \in G(r, n') \times G(r, n')$ satisfying $ht = t'(h_1, h_2)$, if v_i, v_j are vectors of a basis of V,

$$\begin{aligned} S[h(t \otimes v_i \otimes v_j)] &= S[\delta(h_2)t' \otimes h_1 v_i \otimes h_2 v_j] \\ &= \delta(h_2)\hat{t'} \otimes h_2 v_j \otimes h_1 v_i \\ &= \hat{t'} \otimes h_2 v_j \otimes h_1 v_i, \end{aligned}$$

where the last equality depends on the special way we chose h. Namely, since $(h_1, h_2) = (t')^{-1}ht$ with $t, t' \in S_n$, the colors of (h_1, h_2) are the same as in h and they are simply permuted, so in (h_1,h_2) the colors are all even. So the trace we are computing is given by

$$\mathrm{tr}(S\circ h)=\sum_{i,j=1,\dots,n',t=\hat{t}'}(\rho_\mu(h_2))_{i,j}(\rho_\mu(h_1))_{j,i}=\sum_{t=\hat{t}'}\chi^\mu(h_1h_2).$$

Recall the way t' is constructed in the proof of Proposition 4.3.4: $t'=\hat{t}$ if and only if $|h|(t_i)$ belongs to $\{t_{n'+1},\dots,t_{2n}\}$ for every $i\in[n']$:

$$\{t_{n'+1},\dots,t_{2n}\}=\{|h|(t_i)\}_{1\le i\le n'},$$

and, viceversa,

$$\{t_1,\dots,t_{n'}\}=\{|h|(t_i)\}_{n'<i\le n}.$$

So the t's satisfying $t=\widehat{t'}$ are those t such that, for every cycle of h, the consecutive numbers are in opposites rows. We have two possibilities for each cycle, so they are $2^{\ell(\alpha)}$.

Furthermore, suppose h contains a cycle of length $2k$ and color $2j$. Then, according to which of the two possible choices is made for t, a cycle of length k and color j will be contained either in h_1 or in h_2. Thus, h_1h_2 belongs to the $G(r',n')$-class $\mathrm{cl}_{\alpha^{(0)},\alpha^{(2)},\dots,\alpha^{(r-2)}}$. So our final result is

$$\Delta^1_{(\mu,\mu)}(h)=2^{\ell(\alpha)}\chi^\mu(\alpha^{(0)},\alpha^{(2)},\dots,\alpha^{(r-2)}).$$

Let us now turn to the elements belonging to the other split conjugacy class $\mathrm{cl}^1_{2\alpha}$. If h belongs to $\mathrm{cl}^0_{2\alpha}$, thanks to Lemma 4.1.1,

$$ghg^{-1}\in\mathrm{cl}^1_{2\alpha}\Rightarrow z(g)=1 \mod 2\Rightarrow\delta(g)=-1,$$

therefore (see equality (4.3))

$$\Delta^1_{(\mu,\mu)}(ghg^{-1})=\delta(g)\Delta^1_{(\mu,\mu)}(h)=-2^{\ell(\alpha)}\chi^\mu(\alpha^{(0)},\alpha^{(2)},\dots,\alpha^{(r-2)}).$$

□

4.4 The antisymmetric submodule

In this section we study the irreducible decomposition of the antisymmetric submodule M_{Asym} (and hence also of the symmetric submodule M_{Sym}) of the Gelfand model $M(r,1,p,n)$ of the group $G(r,p,n)$ constructed in Theorem 2.5.1. More precisely we will show that M_{Asym} affords exactly one representation of each pair of split irreducible representations of $G(r,p,n)$; namely, the one labelled with 1. If $\mathrm{GCD}(p,n)=1$ the antisymmetric submodule vanishes (and there are no split representations) so in this section and in the following we can always assume $\mathrm{GCD}(p,n)=2$.

Theorem 4.4.1. *Let M_{Asym} be the antisymmetric submodule of the Gelfand model $M(r,1,p,n)$ of $G(r,p,n)$. Then*

$$(M_{\mathrm{Asym}},\varrho)\cong\bigoplus_{[\mu,\mu]\in\mathrm{Fer}(r,1,p,n)}\rho^1_{[\mu,\mu]}.$$

Proof. The strategy in this proof is the one outlined for the case of Weyl groups of type D. So we consider the two representations of $G(r,p,n)$

$$(M_{\mathrm{Asym}},\phi^+)\quad\text{and}\quad(M_{\mathrm{Asym}},\phi^-),$$

given by

$$\phi^+(g)(C_v)\stackrel{\mathrm{def}}{=}\zeta_r^{\langle g,v\rangle}C_{|g|v|g|^{-1}},\qquad\phi^-(g)(C_v)\stackrel{\mathrm{def}}{=}\zeta_r^{\langle g,v\rangle}\zeta_r^{a(g,v)}C_{|g|v|g|^{-1}}$$

(notice that $\phi^-(g)=\varrho(g)|_{M_{\mathrm{Asym}}}$). The main idea of this proof is to exploit Proposition 4.3.2 to show that

$$\chi_{\phi^+}(g)-\chi_{\phi^-}(g)=\sum_{[\mu]\in\mathrm{Fer}(r',1,p',n')}\chi^0_{[\mu,\mu]}(g)-\sum_{[\mu]\in\mathrm{Fer}(r',1,p',n')}\chi^1_{[\mu,\mu]}(g),\quad(4.\text{-}2)$$

where we observe that if $[\mu]$ ranges in $\mathrm{Fer}(r',1,p',n')$ then $[\mu,\mu]$ ranges in $\mathrm{Fer}(r,1,p,n)$. First of all, we will compute the right-hand side of (4.4). We already know that it vanishes on every g belonging to an unsplit conjugacy class. So let $g\in\mathrm{cl}^{\eta}_{2\alpha}$.

Let χ_M denote the character of a model for the group $G(r',n')$. Then

$$\begin{aligned}\sum_{[\mu]\in\mathrm{Fer}(r',1,p',n')}(\chi^0_{[\mu,\mu]}-\chi^1_{\mu,\mu})(g)&=\frac{1}{p'}\sum_{\mu\in\mathrm{Fer}(r',n')}(\chi^0_{[\mu,\mu]}-\chi^1_{[\mu,\mu]})(g)\\&=\frac{1}{p'}(-1)^{\eta}\sum_{\mu\in\mathrm{Fer}(r',n')}2^{\ell(\alpha)}\chi_{\mu}(\alpha)\\&=\frac{1}{p'}(-1)^{\eta}2^{\ell(\alpha)}\chi_M(\alpha),\end{aligned}$$

where the first equality holds because the contribution of every $\mu\in\mathrm{Fer}(r',n')$ provides p' copies of the same irreducible representation of $G(r,p,n)$, the second one follows from Proposition 4.3.2, and $\chi_M(\alpha)$ denotes the value of the character χ_M of a Gelfand model of $G(r',n')$ on any element of the conjugacy class cl_{α}.

Notation. We recall here some notation, which is used in [6] and which we already met in Chapter 3. For $g\in G(r,n)$ we denote by $\Pi^{2,1}(g)$ the set of partitions of the set of disjoint cycles of g into:

- singletons;
- pairs of cycles having the same length.

If $\pi \in \Pi^{2,1}(g)$ we let $\ell(\pi)$ be the number of parts of π and $\mathrm{pair}_j(\pi)$ be the number of parts of π which are pairs of cycles of length j. Moreover, if $s \in \pi$ is a part of π we let $z(s)$ be the sum of the colors of the (either 1 or 2) cycles in s. If g and g' belong to the same conjugacy class cl_α there is clearly a bijection between $\Pi^{2,1}(g)$ and $\Pi^{2,1}(g')$ preserving the statistics $\ell(\pi)$ and $\mathrm{pair}_j(\pi)$, and the colors $z(s)$ of the parts of π; therefore we sometimes write $\Pi^{2,1}(\alpha)$ meaning $\Pi^{2,1}(g)$ for some $g \in \mathrm{cl}_\alpha$.

If $S \subseteq [n]$, we let

$$G(r,S) = \{[\sigma_1^{z_1}, \dots, \sigma_n^{z_n}] \in G(r,n) : \sigma_i^{z_i} = i^0 \text{ for all } i \notin S\}.$$

If $s \in \Pi^{2,1}(g)$, we let $\mathrm{Supp}(s)$, the support of s, be the union of the supports of the cycles in s (when s is a cycle, see the definition of $\mathrm{Supp}(s)$ in Notation 0.3.3).

The set $\Pi^{2,1}(\alpha)$ can be used to describe the character of a Gelfand model of $G(r',n')$ (see [6, Proposition 3.6]):

$$\chi_M(\alpha) = \sum_\pi (r')^{\ell(\pi)} \prod_j j^{\mathrm{pair}_j(\pi)} \tag{4.-4}$$

for all $\alpha \in \mathrm{Fer}(r',n')$, where the sum is taken over all elements of $\Pi^{2,1}(\alpha)$ having no singletons of even length and such that $z(s) = 0 \in \mathbb{Z}_{r'}$ for all $s \in \pi$.

Let us now evaluate $\chi_{\phi^+}(g) - \chi_{\phi^-}(g)$. To this aim, we recall some further notation used in [6]. Consider, for every $g \in G(r,p,n)$ and $\epsilon \in \mathbb{Z}_2$, the set

$$A^\epsilon(g) \stackrel{\mathrm{def}}{=} \{w \in G(r,n) : w \text{ is antisymmetric and } |g|w|g|^{-1} = (-1)^\epsilon w\}.$$

Any $w \in A^\epsilon(g)$ determines a partition $\pi(w) \in \Pi^{2,1}(g)$: a cycle c is a singleton of $\pi(w)$ if the restriction of $|w|$ to $\mathrm{Supp}(c)$ is a permutation of $\mathrm{Supp}(c)$ and $\{c,c'\}$ is a pair of $\pi(w)$ if the restriction of $|w|$ to $\mathrm{Supp}(c)$ is a bijection between $\mathrm{Supp}(c)$ and $\mathrm{Supp}(c')$. Finally, if $\pi \in \Pi^{2,1}(g)$, we let $A^\epsilon_\pi \stackrel{\mathrm{def}}{=} \{w \in A^\epsilon(g) : \pi(w) = \pi\}$. Then the set $A^\epsilon(g)$ can be decomposed into the disjoint union

$$A^\epsilon(g) = \bigcup_{\pi \in \Pi^{2,1}(g)} A^\epsilon_\pi. \tag{4.-4}$$

Remark 4. With the above notation, we have

$$\chi_{\phi^+}(g) - \chi_{\phi^-}(g) = \frac{1}{p} \sum_{\pi \in \Pi^{2,1}(g)} \sum_{A^0_\pi \cup A^1_\pi} \zeta_r^{\langle g,w \rangle}(1 - \zeta_r^{a(g,w)}).$$

Since (see [6, Lemma 5.7])

$$a(g,w) = \begin{cases} 0, & \text{if } w \in A^0_\pi(g); \\ r', & \text{if } w \in A^1_\pi(g), \end{cases}$$

$\lvert s\rvert$	A^1_s
$\{(i_1,\ldots,i_d)\}$ with d odd	$\emptyset$
$\{(i_1,\ldots,i_d)\}$ with $d\equiv 2 \mod 4$	$i_h\mapsto \zeta_r^k(-1)^h i_{h+\frac{d}{2}}$
$\{(i_1,\ldots,i_d)\}$ with $d\not\equiv 2 \mod 4$	$\emptyset$
$\{(i_1,\ldots,i_d),(j_1,\ldots,j_d)\}$, with d odd	$\emptyset$
$\{(i_1,\ldots,i_d),(j_1,\ldots,j_d)\}$, with d even	$i_h\mapsto \zeta_r^k(-1)^h j_{h+l}$ and $j_h\mapsto -\zeta_r^k(-1)^{h-l} i_{h-l}$

Table 4.1: The sets A^1_s as $|s|$ varies.

we find

$$\chi_{\phi^+}(g)-\chi_{\phi^-}(g)=\frac{1}{p'}\sum_{\pi\in\Pi^{2,1}(g)}\sum_{w\in A^1_\pi}\zeta_r^{\langle g,w\rangle}.$$

If $\pi=\{s_1,\ldots,s_h\}$, it is shown in [6, Section 5] that the set A^1_π has a natural decomposition $A^1_\pi=A^1_{s_1}\times\ldots\times A^1_{s_h}$, i.e. every w in A^1_π can be written as a product $w=w_1\cdot\ldots\cdot w_h$, with $w_i\in A^1_{s_i}$. The sets $A^1_{s_i}$ depend on the structure of $|s_i|$ only, and are described in Table 4.1. The table is analogous to Table 3.1, but in the present case, thanks to Remark 4, we focus on A^1_s only. The indices of $i_1,\ldots,i_d,j_1,\ldots,j_d$ should be considered in $\mathbb{Z}_d$ and in any box of the table the parameters $k\in\mathbb{Z}_r$ and $l\in\mathbb{Z}_d$ are arbitrary but fixed. For example, if $s=\{(1,2),(3,4)\}$, and $r=4$, then A^1_s consists of the 8 elements having either the form $(1^{k+2},3^k)(2^k,4^{k+2})$ or the form $(1^{k+2},4^k)(2^k,3^{k+2})\}$, as k varies in

$\{0,1,2,3\}$. This allows to focus on the single sets A^1_s, via the identity

$$\begin{aligned}\sum_{w\in A^1_\pi}\zeta_r^{\langle g,w\rangle} &= \sum_{w_1\in A^1_{s_1},\dots,w_h\in A^1_{s_h}}\zeta_r^{\langle g,w_1\cdots w_h\rangle}\\ &= \sum_{w_1\in A^1_{s_1},\dots,w_h\in A^1_{s_h}}\zeta_r^{\sum_i\langle g_i,w_i\rangle}\\ &= \prod_{i=1}^{h}\sum_{w_i\in A^1_{s_i}}\zeta_r^{\langle g_i,w_i\rangle},\end{aligned}$$

where $g_i \in G(r,\mathrm{Supp}(s_i))$ is the restriction of g to $\mathrm{Supp}(s_i)$.

Lemma 4.4.2. *If $g \in G(r,n)$ has at least one cycle c of odd length, then*

$$\sum_{w\in A^1_\pi}\zeta_r^{\langle g,w\rangle} = 0$$

for all $\pi \in \Pi^{2,1}(g)$.

Proof. This is trivial since in these cases $A^1_\pi = \emptyset$ (see Table 4.1). □

By Lemma 4.4.2 we can restrict our attention to those elements g having all cycles of even length.

Lemma 4.4.3. *If $g \in G(r,p,n)$ has at least one cycle c of odd color, then*

$$\sum_{\pi\in\Pi_{2,1}(g)}\sum_{w\in A^1_\pi}\zeta_r^{\langle g,w\rangle} = 0.$$

for all $\pi \in \Pi_{2,1}(g)$.

Proof. Since the left-hand side is a class function (by equation (4)) we can assume that g is normal. We prove in this case the stronger statement that $\sum_{w\in A^1_\pi}\zeta_r^{\langle g,w\rangle} = 0$ for all $\pi \in \Pi_{2,1}(g)$. By Lemma 4.4.2, we can assume that the cycle c has even length. We split this result into two cases. Assume that the cycle c of odd color - say j - is a singleton $s_i = \{c\}$ of π. Then Table 4.1 furnishes the structure of $A^1_{s_i}$. In particular, if $\ell(c) \not\equiv 2 \mod 4$, $A^1_{s_i} = \emptyset$ and we are done; if $\ell(c) \equiv 2 \mod 4$, we find

$$\sum_{w\in A^1_{s_i}}\zeta_r^{\langle g_i,w_i\rangle} = \sum_{k=0}^{r-1}\zeta_r^{jk} = 0,$$

since j is odd and cannot be a multiple of r.

Now assume that the cycle c belongs to a pair s_i of π. Let us call a and b the two colors of the cycles in s_i, with b odd. Again, looking at Table 4.1,

$$\begin{aligned}\sum_{w_i \in A^1_{s_i}} \zeta_r^{\langle g_i, w_i\rangle} &= \sum_{k=0}^{r-1}\sum_{l=0}^{d-1} \zeta_r^{ak+b(k+(l+1)r')} \\ &= \sum_{k=0}^{r-1}\big(\frac{d}{2}\zeta_r^{(a+b)k} + \frac{d}{2}\zeta_r^{ak+b(k+r')}\big) \\ &= \frac{d}{2}\sum_{k=0}^{r-1}\zeta_r^{(a+b)k}(1+\zeta_r^{br'}) \\ &= \frac{d}{2}(1+\zeta_r^{br'})\sum_{k=0}^{r-1}\zeta_r^{(a+b)k}.\end{aligned}$$

Since b is odd, the factor $1+\zeta_r^{br'}$ vanishes and so does the whole sum. □

Lemma 4.4.4. *Let $g \in G(r,p,n)$ be normal and such that all cycles of g have even color and even length. Then, for all $\pi \in \Pi^{2,1}(g)$,*

$$\sum_{w \in A^1_\pi} \zeta_r^{\langle g, w\rangle} = \begin{cases} (-1)^{\mathrm{sign}(g)}|A^1_\pi|, & \textit{if } z(s)=0 \textit{ for all } s \in \pi; \\ 0, & \textit{otherwise.}\end{cases}$$

Proof. We first assume that $\mathrm{sign}(g)=0$. If s_i is a singleton of π of color $2j$ and length $\not\equiv 2 \mod 4$, then A_{s_i} is empty and the result clearly follows. So we can assume that $\ell(s_i) \equiv 2 \mod 4$ and we can derive the value of $\langle g_i, w_i\rangle$ from Table 4.1, and we obtain

$$\sum_{w_i \in A^1_{s_i}} \zeta_r^{\langle g_i, w_i\rangle} = \sum_{k=0}^{r-1}\zeta_r^{2jk} = \begin{cases} 0, & \text{if } 2j \not\equiv 0 \mod r; \\ \\ r = |A^1_{s_i}|, & \text{if } 2j \equiv 0 \mod r.\end{cases}$$

Let now $s_i = \{c_1, c_2\}$ be a pair of cycles of length d, and colors respectively $2a$ and $2b$.

$$\begin{aligned}\sum_{w_i \in A^1_{s_i}} \zeta_r^{\langle g_i, w_i\rangle} &= \sum_{k=0}^{r-1}\sum_{l=0}^{d-1}\zeta_r^{2ak+2b(k+(l+1)r')}) \\ &= d\sum_{k=0}^{r-1}\zeta_r^{2ak+2bk} = \begin{cases} 0, & \text{if } 2a+2b \not\equiv 0 \mod r; \\ \\ dr = |A^1_{s_i}|, & \text{if } 2a+2b \equiv 0 \mod r.\end{cases}\end{aligned}$$

The result follows from these computations together with equation (4.4). If $\mathrm{sign}(g)=1$ the proof is similar and is left to the reader. □

We are now ready to prove Theorem 4.4.1. Let first g belong to an unsplit class. Then g has a cycle of odd length or a cycle of odd color, and then equation (4), Lemmas 4.4.2 and 4.4.3 ensure that

$$\chi_{\phi^+}(g)-\chi_{\phi^-}(g)=0=\sum_{[\mu]\in\mathrm{Fer}(r',1,p',n')}\chi^0_{[\mu,\mu]}-\sum_{[\mu]\in\mathrm{Fer}(r',1,p',n')}\chi^1_{[\mu,\mu]}(g). \quad (4.\text{-}6)$$

So let g belong to the split conjugacy class of the form $c^{\eta}_{2\alpha}$. We are interested in the evaluation of the sum appearing in (4) and so we can assume that g is also a normal element. Thanks to Lemma 4.4.4, the only partitions $\pi\in\Pi^{2,1}(g)$ contributing to the sum (4) are those satisfying $z(s)=0 \mod r$ for all $s\in\pi$. Thus,

$$\begin{aligned}\chi_{\phi^+}(g)-\chi_{\phi^-}(g)&=\frac{1}{p'}(-1)^{\eta}\sum_{\pi\in\Pi^{2,1}(g)}|A^1_{\pi}|\\&=\frac{1}{p'}(-1)^{\eta}\sum_{\pi\in\Pi^{2,1}(g)}r^{\ell(\pi)}\prod_j(2j)^{\mathrm{pair}_{2j}(\pi)},\end{aligned}$$

where, by Table 4.1, the sum is taken over all partitions of $\Pi^{2,1}(g)$ such that:

- singletons have length $\equiv 2 \mod 4$;
- pairs have even length;
- $z(s)=0 \mod r$, for all $s\in\pi$.

Summing up, if $g\in \mathrm{cl}^{\eta}_{2\alpha}$, we have

$$\begin{aligned}\chi_{\phi^+}(g)-\chi_{\phi^-}(g)&=\frac{1}{p'}(-1)^{\eta}\sum_{\pi\in\Pi^{2,1}(2\alpha)}r^{\ell(\pi)}\prod_j(2j)^{\mathrm{pair}_{2j}(\pi)}\\&=\frac{1}{p'}(-1)^{\eta}\sum_{\pi\in\Pi^{2,1}(2\alpha)}(2r')^{\ell(\pi)}\prod_j(2j)^{\mathrm{pair}_{2j}(\pi)}\\&=\frac{1}{p'}(-1)^{\eta}2^{\ell(\pi)+\sum_j\mathrm{pair}_{2j}(\pi)}\sum_{\pi\in\Pi^{2,1}(2\alpha)}(r')^{\ell(\pi)}\prod_j j^{\mathrm{pair}_{2j}(\pi)}\\&=\frac{1}{p'}(-1)^{\eta}2^{\ell(\alpha)}\sum_{\pi\in\Pi^{2,1}(\alpha)}(r')^{\ell(\pi)}\prod_j j^{\mathrm{pair}_j(\pi)},\end{aligned}$$

where α has to be considered as an element in $\mathrm{Fer}(r',n')$ and the last sum is taken over all partitions of $\Pi^{2,1}(\alpha)$ whose singletons have odd length (and pairs have any length), and $z(s)=0\in\mathbb{Z}_{r'}$ for all $s\in\pi$.

The above computation, together with equations (4.4), (4.4) and (4.4), leads to

$$\sum_{[\mu]\in\mathrm{Fer}(r',1,p',n')}\chi^0_{[\mu,\mu]}(g)+\chi_{\phi^-}(g)=\sum_{[\mu]\in\mathrm{Fer}(r',1,p',n')}\chi^1_{[\mu,\mu]}(g)+\chi_{\phi^+}(g)\qquad\forall\, g\in G(r,p,n). \quad (4.\text{-}6)$$

Now, $\sum_{\mu\in\mathrm{Fer}(r',1,p',n')}\chi^0_{\mu,\mu}$ and $\sum_{\mu\in\mathrm{Fer}(r',1,p',n')}\chi^1_{\mu,\mu}$ are orthogonal characters. Therefore, by Theorem 2.2.5 and Proposition 2.4.4, we have that

$$\sum_{\mu\in\mathrm{Fer}(r',1,p',n')}\dim(\rho^0_{[\mu,\mu]}) = \sum_{\mu\in\mathrm{Fer}(r',1,p',n')}|\mathcal{ST}_{[\mu,\mu]}| = \dim(M_{\mathrm{Asym}}) = \dim(\phi^+),$$

and, analogously, $\sum_{\mu\in\mathrm{Fer}(r',1,p',n')}\dim(\rho^1_{[\mu,\mu]}) = \dim(\phi^-)$, we can conclude that

$$\sum_{\mu\in\mathrm{Fer}(r',1,p',n')}\chi^0_{[\mu,\mu]}(g) = \chi_{\phi^+}(g) \quad \text{and} \quad \sum_{\mu\in\mathrm{Fer}(r',1,p',n')}\chi^1_{[\mu,\mu]}(g) = \chi_{\phi^-}(g).$$

Recalling that $\phi^-(g) = \varrho(g)|_{M_{\mathrm{Asym}}}$, the above equality means that

$$(M_{\mathrm{Asym}}, \varrho) \cong \bigoplus_{\mu\vdash m}\rho^1_{[\mu,\mu]},$$

and Theorem 4.4.1 is proved. □

4.5 The antisymmetric classes

An antisymmetric element of $G(r,n)$ can be characterized by the structure of its cycles, namely, an element $v\in G(r,n)$ is antisymmetric if and only if every cycle c of v has length 2 and is of the form $c = (a_1^{z_{a_1}}, a_2^{z_{a_2}})$ with $z_{a_2} = z_{a_1} + r'$.

Definition. We say that the residue class of z_{a_1} and z_{a_2} modulo r' is the *type* of c. If the number of disjoint cycles of type i of an antisymmetric element v of $G(r,n)$ is t_i, then the integer vector $\tau(v) = (t_0,\dots,t_{r'-1})$ is called the *type* of v.

It is easy to check the following

Remark 5. Two antisymmetric elements in $G(r,n)$ are S_n-conjugate if and only if they have the same type.

Notation 4.5.1. We denote by $AC(r,n)$ the set of types of antisymmetric elements in $G(r,n)$, i.e. the set of vectors $(t_0,\dots,t_{r'})$ with nonnegative integer entries such that $t_0+\cdots+t_{r'-1} = n'$. If $\mathrm{GCD}(p,n)=2$ we let γ be the cyclic permutation of $AC(r,n)$ defined by $\gamma(t_0,\dots,t_{r'-1}) = (t_{r/p}, t_{1+r/p}\dots,t_{r'-1+r/p})$ where the indices must be intended as elements in $\mathbb{Z}_r$. We observe that γ has order p' and so we have an action of the cyclic group C'_p generated by γ on $AC(r,n)$. We denote the quotient set by $AC(r,p,n)^*$. The type of an antisymmetric element of $G(r,p,n)^*$ is then an element of $AC(r,p,n)^*$ and if $[\tau]\in AC(r,p,n)^*$ we let $c^1_{[\tau]}$ be the S_n-conjugacy class consisting of the antisymmetric absolute involutions in $G(r,p,n)^*$ of type $[\tau]$.

The main result of this section, Theorem 4.5.5, provides a compatibility between the coefficients of τ and the sizes of the indices of the irreducible components of the module $M(c^1_{[\tau]})$. For this, it will be helpful the following criterion.

Proposition 4.5.2. *Let $\nu = (n_0, \ldots, n_{r-1})$ be a composition of n into r parts, and ρ a representation of $G(r,n)$. Then the following are equivalent:*

1. *The irreducible subrepresentations of ρ are all of the form $\rho_{\lambda^{(0)},\ldots,\lambda^{(r-1)}}$ with $|\lambda^{(i)}| = n_i$ for all $i \in [0, r-1]$;*
2. *There exists a representation ϕ of $G(r,\nu)$ such that $\rho = \mathrm{Ind}_{G(r,\nu)}^{G(r,n)}(\phi)$ and $\phi(g) = \zeta_r^{\sum iz(g_i)}\phi(|g|)$, for all $g = (g_0, \ldots, g_{r-1}) \in G(r,\nu)$.*

Proof. In proving that (1) implies (2) we can clearly assume that ρ is irreducible and in this case the result is straightforward from the description in Proposition 0.4.2. In proving that (2) implies (1) we can assume that ϕ is irreducible. Then it is clear that $\phi\downarrow_{S_\nu}$ is also irreducible, where $S_\nu = S_{n_0}\times\cdots\times S_{n_{r-1}}$. In particular there exist $\lambda^{(0)}, \ldots, \lambda^{(r-1)}$, with $|\lambda^{(i)}| = n_i$ such that $\phi\downarrow_{S_\nu} \cong \rho_{\lambda^{(0)}}\odot\cdots\odot\rho_{\lambda^{(r-1)}}$. Now we can conclude that

$$\phi \cong (\gamma^0_{n_0}\otimes\tilde{\rho}_{\lambda^{(0)}})\odot\cdots\odot(\gamma^{r-1}_{n_{r-1}}\otimes\tilde{\rho}_{\lambda^{(0)}}),$$

and so the result follows again from Proposition 0.4.2. □

We now concentrate on the special case $p = 2$, so that $p' = 1$. The general case will then be a direct consequence. Since the index of $G(r,2,n)$ in $G(r,n)$ is 2 the induction $\psi = \mathrm{Ind}_{G(r,2,n)}^{G(r,n)}(M(c^1_\tau), \varrho)$ of the $G(r,2,n)$-representation $M(c^1_\tau)$ to $G(r,n)$ is a representation on the direct sum $V\oplus V'$ of two copies of $V \stackrel{\text{def}}{=} M(c^1_\tau)$. So a basis of $V\oplus V'$ consists of all the elements C_v, C'_v, as v varies in c^1_τ. If $x = [(1,2,\ldots,n); r-1, 0, \ldots, 0]$ is taken as a representative of the nontrivial coset of $G(r,2,n)$ and we impose that $x\cdot C_v = C'_v$, the representation ψ of $G(r,n)$ on $V\oplus V'$ will be as follows

$$g\cdot C_v = \begin{cases} \zeta_r^{\langle g,\tilde{v}\rangle}\zeta_r^{z_1(\tilde{v})-z_{|g|^{-1}(1)}(\tilde{v})}C_{|g|v|g|^{-1}} & \text{if } g\in G(r,2,n),\\ \zeta_r^{\langle g,\tilde{v}\rangle}\zeta_r^{z_1(\tilde{v})}C'_{|g|v|g|^{-1}} & \text{if } g\notin G(r,2,n),\end{cases}$$

and

$$g\cdot C'_v = \begin{cases} \zeta_r^{\langle g,\tilde{v}\rangle}C'_{|g|v|g|^{-1}} & \text{if } g\in G(r,2,n),\\ \zeta_r^{\langle g,\tilde{v}\rangle}\zeta_r^{-z_{|g|^{-1}(1)}(\tilde{v})}C_{|g|v|g|^{-1}} & \text{if } g\notin G(r,2,n);\end{cases}$$

where $\tilde{v}$ is any lift of v in $G(r,n)$. Now we want to show that this representation ψ of $G(r,n)$ is actually also induced from a particular representation of $G(r,(\tau,\tau)) = G(r,t_0)\times\cdots\times G(r,t_{r'-1})\times G(r,t_0)\times\cdots\times G(r,t_{r'-1})$. With this in mind we let $\mathcal{C}$ be the set of elements $v\in c^1_\tau$ having a lift $\tilde{v}$ in $G(r,n)$

satisfying the following condition: if $(a^i, b^{i+r'})$ is a cycle of $\tilde{v}$ of type i, then $a \in [t_0+\cdots+t_{i-1}+1, t_0+\cdots+t_i]$ and $b \in [n'+t_0+\cdots+t_{i-1}+1, n'+t_0+\cdots+t_i]$. Then, if $z \stackrel{\text{def}}{=} \min\{j : t_j \neq 0\}$ we let

$$W \stackrel{\text{def}}{=} \bigoplus_{v \in \mathcal{C}} \mathbb{C}(C_v + \zeta_r^z C'_v) \subseteq V \oplus V'.$$

Lemma 4.5.3. *The subspace W is invariant by the restriction of ψ to $G(r,(\tau,\tau))$.*

Proof. It is clear that if $v \in \mathcal{C}$ and $g \in G(r,(\tau,\tau))$, then $|g|v|g|^{-1} \in \mathcal{C}$. We observe that, by definition, $|g|$ permutes the elements in $\tilde{v}$ having the same color, and in particular $z_1(\tilde{v}) = z_{|g|^{-1}(1)}(\tilde{v})$. Moreover, by definition, we also have $z_1(\tilde{v}) = z$. In particular, if $g \in G(r,2,n) \cap G(r,(\tau,\tau))$ we have

$$\begin{aligned} g \cdot (C_v + \zeta_r^z C'_v) &= \zeta_r^{\langle g,\tilde{v}\rangle} \zeta_r^{z_1(\tilde{v}) - z_{|g|^{-1}(1)}(\tilde{v})} C_{|g|v|g|^{-1}} + \zeta_r^z \zeta_r^{\langle g,\tilde{v}\rangle} C'_{|g|v|g|^{-1}} \\ &= \zeta_r^{\langle g,\tilde{v}\rangle} (C_{|g|v|g|^{-1}} + \zeta_r^z C'_{|g|v|g|^{-1}}), \end{aligned}$$

and if $g \in G(r,(\tau,\tau)$ but $g \notin G(r,2,n)$ we have

$$\begin{aligned} g \cdot (C_v + \zeta_r^z C'_v) &= \zeta_r^{\langle g,\tilde{v}\rangle} \zeta_r^{z_1(\tilde{v})} C'_{|g|v|g|^{-1}} + \zeta_r^z \zeta_r^{\langle g,\tilde{v}\rangle} \zeta_r^{-z_{|g|^{-1}(1)}(\tilde{v})} C_{|g|v|g|^{-1}} \\ &= \zeta_r^{\langle g,\tilde{v}\rangle} (\zeta_r^z C'_{|g|v|g|^{-1}} + C_{|g|v|g|^{-1}}). \end{aligned}$$

The proof is now complete. □

Lemma 4.5.4. *We have $V \oplus V' = \mathrm{Ind}_{G(r,(\tau,\tau))}^{G(r,n)}(W)$.*

Proof. For this we need to prove that

$$V \oplus V' = \bigoplus_{g \in K} g \cdot W, \tag{4.-10}$$

where K is any set of coset representatives of $G(r,(\tau,\tau))$ in $G(r,n)$. But

$$[G(r,n) : G(r,(\tau,\tau))] = \frac{n! r^n}{(\tau!)^2 r^n} = \frac{n!}{(\tau!)^2},$$

where $\tau! = t_0! \cdots t_{r'-1}!$. Moreover

$$\dim(V \oplus V') = 2 \dim V = 2 \frac{\binom{n}{2}\binom{n-2}{2} \cdots \binom{n}{2} 2^{n'}}{\tau! 2} = \frac{n!}{\tau!} \quad \text{and} \quad \dim W = \tau!,$$

so that $[G(r,n) : G(r,(\tau,\tau))] = \frac{\dim(V \oplus V')}{\dim W}$, and hence to prove (4.5) it is enough to show that $V \oplus V' \subset G(r,n)W$. To show this we take $\sigma = [n'+1, n'+2, \ldots, n, 1, 2, \ldots, n'] \in S_n$. Then it follows that conjugation by σ stabilizes $\mathcal{C}$, although $\sigma \notin G(r,(\tau,\tau))$. Then we have

$$\begin{aligned} \sigma \cdot (\zeta_r^z C_v + C'_v) &= \zeta_r^z C_{\sigma v \sigma^{-1}} + \zeta_r^{z - z_{n'+1}(\tilde{v})} C'_{\sigma v \sigma^{-1}} \\ &= \zeta_r^z C_{\sigma v \sigma^{-1}} + \zeta_r^{r'(\tilde{v})} C'_{\sigma v \sigma^{-1}} \\ &= \zeta_r^z C_{\sigma v \sigma^{-1}} - C'_{\sigma v \sigma^{-1}}. \end{aligned}$$

Since also $\zeta_r^z C_{\sigma v\sigma^{-1}} + C'_{\sigma v\sigma^{-1}} \in W$ we conclude that both $C_{\sigma v\sigma^{-1}}$ and $C'_{\sigma v\sigma^{-1}}$ belong to $G(r,n)W$ for all $v \in \mathcal{C}$, and the proof is complete. □

We are now ready to prove the main result of this section.

Theorem 4.5.5. *Let* $\mathrm{GCD}(p,n) = 2$ *and* $[\tau] = [t_0, \ldots, t_{r'}] \in AC(r,p,n)^*$ *(see Notation 4.5.1). Then*

$$M(c^1_{[\tau]}) = \bigoplus_{\substack{[\lambda^{(0)},\ldots,\lambda^{(r'-1)}]\in \mathrm{Fer}(r',1,p',n'):\\ |\lambda^{(i)}|=t_i\ \forall i\in[0,r'-1]}} \rho^1_{[\lambda^{(0)},\ldots,\lambda^{(r'-1)},\lambda^{(0)},\ldots,\lambda^{(r'-1)}]}.$$

Proof. If $p = 2$ we need to give a closer look at the $G(r,(\tau,\tau))$-representation W. From the proof of Lemma 4.5.3 we have that $gD_v = \zeta_r^{\langle g,v\rangle} D_{|g|v|g|^{-1}} = \prod_i \zeta_r^{iz(g_i)} D_{|g|v|g|^{-1}}$, where $D_v \stackrel{\mathrm{def}}{=} C_v + \zeta_r^z C'_v$ for $v \in \mathcal{C}$ are the basis elements of W. In particular, condition (2) of Proposition 4.5.2 are satisfied and the result is a straightforward consequence of Theorem 4.4.1.

If $p > 2$ we simply have to observe that the $G(r,p,n)$-module $M(c^1_{[\tau]})$ is a quotient of the restriction to $G(r,p,n)$ of the $G(r,2,n)$-module $M(c^1_\tau)$. Since $\mathrm{GCD}(p,n) = 2$, the irreducible representations of $G(r,2,n)$ restricted to $G(r,p,n)$ remain irreducible (and are indexed in the "same" way). The result is then a consequence of the case $p = 2$ and Theorem 4.4.1. □

4.6 The symmetric classes

In this section we complete our discussion with the description of the $G(r,p,n)$-module $M(c)$, where c is any S_n-conjugacy class of symmetric absolute involutions in $G(r,p,n)^*$. Despite the case $p = 1$ considered in Chapter 1 and the case of antisymmetric classes treated in Section 4.5, where a self-contained proof of the irreducible decomposition of the module $M(c)$ was given, we will need here to make use of all the main results that we have obtained so far, namely the construction of the complete Gelfand model in Section 2.5, the study of the submodules $M(c)$ for wreath products in Section 1.3, as well as the discussion of the antisymmetric submodule in Section 4.4.

We first observe that, by Theorems 2.5.1 and 4.4.1, the symmetric submodule has the following decomposition into irreducible representations

$$M_{\mathrm{Sym}} \cong \bigoplus_{[\lambda]\in \mathrm{Fer}(r,1,p,n)} \rho^0_{[\lambda]}.$$

Recall notation 2.3.2: if v is a symmetric absolute involution in $G(r,p,n)^*$ we denote by $\mathrm{Sh}(v)$ the element of $\mathrm{Fer}(r,1,p,n)$ which is the shape of the multitableaux of the image of v, via the projective Robinson-Schensted correspondence. Namely, we let

$$\mathrm{Sh}(v) \stackrel{\mathrm{def}}{=} [\lambda] \in \mathrm{Fer}(r,1,p,n),$$

where

$$v \stackrel{RS_p}{\longrightarrow} (P,P),$$

with $P \in \mathcal{ST}_{[\lambda]}$. For notational convenience, if c is a S_n-conjugacy class of symmetric absolute involutions in $G(r,p,n)^*$ we also let $\mathrm{Sh}(c) = \cup_{v\in c}\mathrm{Sh}(v) \subset \mathrm{Fer}(r,1,p,n)$.

We are now ready to state the main result of this section.

Theorem 4.6.1. *Let c be a S_n-conjugacy class of symmetric absolute involutions in $G(r,p,n)^*$ and* $\mathrm{GCD}(p,n) = 1,2$. *Then the following decomposition holds:*

$$M(c) \cong \bigoplus_{[\lambda]\in \mathrm{Sh}(c)} \rho^0_{[\lambda]}.$$

Before proving this theorem we need some further preliminary observations. Fix an arbitrary S_n-conjugacy class c of symmetric absolute involutions in $G(r,p,n)^*$, and let $c_1,\ldots,c_s$ be the S_n-conjugacy classes of $G(r,n)$ which are lifts of c in $G(r,n)$ (one may observe that s can be either p or $p/2$, though this is not needed). We will need to consider the following restriction to $G(r,p,n)$ of the submodule of the Gelfand model for $G(r,n)$ associated to the classes $c_1,\ldots,c_s$,

$$\tilde{M}(c) \stackrel{\mathrm{def}}{=} (M(c_1)\oplus\cdots\oplus M(c_s))\downarrow_{G(r,p,n)}$$

Now the crucial observation is the following.

Lemma 4.6.2. *The $G(r,p,n)$-module $M(c)$ is a quotient (and hence is isomorphic to a subrepresentation) of $\tilde{M}(c)$.*

Proof. Let $K(c)$ be the vector subspace of $\tilde{M}(c)$ spanned by the elements $C_v - C_{\zeta^{r/p}v}$ as v varies among all elements in $c_1,\ldots,c_s$. Then it is clear that, as a vector space, $M(c)$ is the quotient of $\tilde{M}(c)$ by the vector subspace $K(c)$. Moreover, if $g \in G(r,p,n)$ then

$$\begin{aligned}\varrho(g)(C_v - C_{\zeta^{r/p}v}) &= \zeta_r^{\langle g,v\rangle}(-1)^{\mathrm{inv}_v(g)}C_{|g|v|g|^{-1}} - \zeta_r^{\langle g,\zeta^{r/p}v\rangle}(-1)^{\mathrm{inv}_{\zeta^{r/p}v}(g)}C_{|g|\zeta^{r/p}v|g|^{-1}}\\ &= \zeta_r^{\langle g,v\rangle}(-1)^{\mathrm{inv}_v(g)}(C_{|g|v|g|^{-1}} - C_{\zeta^{r/p}|g|v|g|^{-1}}),\end{aligned}$$

since $g \in G(r,p,n)$ implies $\langle g,\zeta^{r/p}v\rangle = \langle g,v\rangle$. In particular we deduce that $K(c)$ is also a submodule of $\tilde{M}(c)$ (as $G(r,p,n)$-modules). The fact that $M(c) \cong \tilde{M}(c)/K(c)$ is now a direct consequence of the definition of the structures of $G(r,p,n)$-modules. □

We are now ready to complete the proof of the main result of this section.

Proof of Theorem 4.6.1. By Theorem 1.1.3 the $G(r,n)$-module $M(c_1)\oplus\cdots\oplus M(c_s)$ is the sum of all representations ρ_λ with $\lambda\in \mathrm{Sh}(c_i)$, for some $i\in[s]$ or, equivalently, with $[\lambda]\in\mathrm{Sh}(c)$. It follows that the restriction $\tilde{M}(c)$ of this representation to $G(r,p,n)$ has the following decomposition

$$\tilde{M}(c)\cong \bigoplus_{\substack{[\lambda]\in\mathrm{Sh}(c):\\ m_p(\lambda)=1}}(\rho^0_{[\lambda]})^{\oplus p}\oplus\bigoplus_{\substack{[\lambda]\in\mathrm{Sh}(c):\\ m_p(\lambda)=2}}(\rho^0_{[\lambda]}\oplus\rho^1_{[\lambda]})^{\oplus p/2}. \qquad (4.\text{-}15)$$

We recall that $M(c)$ is a submodule of a Gelfand model for $G(r,p,n)$ and also a submodule of $\tilde{M}(c)$ by Lemma 4.6.2, and hence, by equation (4.6), we have that $M(c)$ is isomorphic to a subrepresentation of

$$\bigoplus_{\substack{[\lambda]\in\mathrm{Sh}(c):\\ m_p(\lambda)=1}}\rho^0_{[\lambda]}\oplus\bigoplus_{\substack{[\lambda]\in\mathrm{Sh}(c):\\ m_p(\lambda)=2}}(\rho^0_{[\lambda]}\oplus\rho^1_{[\lambda]}).$$

Furthermore, we already know that the split representations $\rho^1_{[\lambda]}$ appear in the antisymmetric submodule by Theorem 4.4.1 and so they can not appear in $M(c)$. For completing the proof it is now sufficient to observe that, if c and c' are two distinct S_n-conjugacy classes of symmetric absolute involutions in $G(r,p,n)^*$, then the two sets $\mathrm{Sh}(c)$ and $\mathrm{Sh}(c')$ are disjoint. □

We can also give an explicit combinatorial description of the set $\mathrm{Sh}(c)$ for a given S_n-conjugacy class of symmetric absolute involutions in $G(r,p,n)^*$.

Notation 4.6.3. Let $SC(r,n)=\{(f_0,\dots,f_{r-1},q_0,\dots,q_{r-1})\in\mathbb{N}^{2r}:\ f_0+\cdots+f_{r-1}+2(q_0+\cdots+q_{r-1})=n\}$. In fact, the set $SC(r,n)$ has already been used in Section 1.3 to parametrize the S_n-conjugacy classes of absolute involutions in $G(r,n)$. Let γ be the permutation of $SC(r,n)$ defined by

$$\gamma(f_0,\dots,f_{r-1},q_0,\dots,q_{r-1})=(f_{r/p},f_{1+r/p}\dots,f_{r-1+r/p},q_{r/p},q_{1+r/p}\dots,q_{r-1+r/p}),$$

where the indices must be intended as elements in $\mathbb{Z}_r$. We observe that γ has order p and so we have an action of the cyclic group C_p generated by γ on $SC(r,n)$. We denote the quotient set by $SC(r,p,n)^*$.

The set $SC(r,p,n)^*$ parametrizes the S_n-conjugacy classes of symmetric absolute involutions in $G(r,p,n)^*$ in the following way. Let $v\in I(r,p,n)^*$ be symmetric and $\tilde{v}$ be any lift of v in $I(r,n)$. Then the *type* of v is given by

$$[f_0(\tilde{v}),\dots,f_{r-1}(\tilde{v}),q_0(\tilde{v}),\dots,q_{r-1}(\tilde{v})]\in SC(r,p,n)^*,$$

where

$$\begin{aligned} f_i(\tilde{v}) &= |\{j\in[n]:\tilde{v}_j=j^i\}|\\ q_i(\tilde{v}) &= |\{(h,k):1\le h<k\le n,\ \tilde{v}_h=k^i \text{ and } \tilde{v}_k=h^i\}|. \end{aligned}$$

It is clear that this is well-defined and we have that two symmetric elements in $I(r,p,n)^*$ are S_n-conjugate if and only if they have the same type (see also [7, §6] for the special case $p=1$).

By Proposition 1.2.2 we can now conclude that, if

$$[\nu] = [f_0, \dots, f_{r-1}, q_0, \dots, q_{r-1}] \in SC(r,p,n)^*$$

and

$$c = \{v \in I(r,p,n)^* : \ v \text{ is symmetric of type } [\nu]\},$$

then

$$\mathrm{Sh}(c) = \left\{ \begin{array}{l} [\lambda^{(0)}, \dots, \lambda^{(r-1)}] \in \mathrm{Fer}(r,p,n)^* : \ \text{ for all } i \in [0, r-1], \\ |\lambda_i| = f_i + 2q_i \text{ and } \lambda^{(i)} \text{ has exactly } f_i \text{ columns of odd length} \end{array} \right\}.$$

Example 4.6.4. Consider $v \in G(6,6,14)^*$ given by

$$v = [(1,3,2,4,5,7,6,8,10,9,11,12,14,13); 0,1,1,1,1,2,2,3,4,4,4,4,5,5].$$

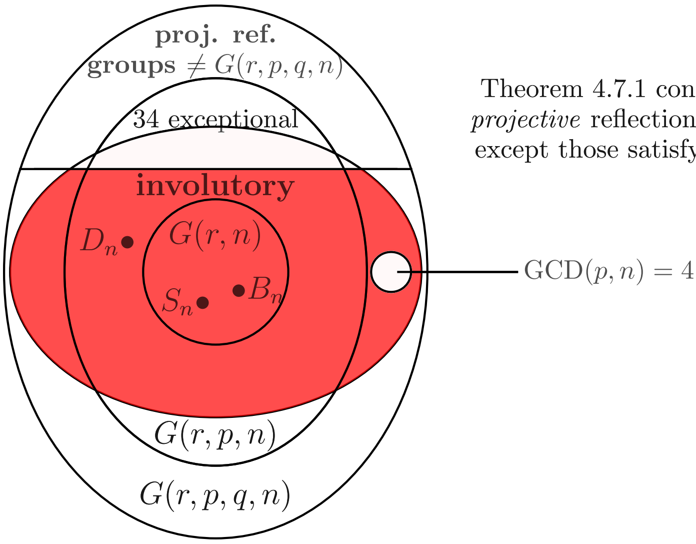
Then the type of v is $[\nu] = [1,2,0,1,2,0; 0,1,1,0,1,1]$. Therefore if c is the S_n-conjugacy class of v in $G(6,6,14)^*$ we have that $\mathrm{Sh}(c)$ is given by all elements $[\lambda^{(0)}, \dots, \lambda^{(5)}] \in \mathrm{Fer}(6,1,6,14)$ such that $\lambda^{(0)}$ and $\lambda^{(3)}$ have 1 box (and 1 column of odd length), $\lambda^{(1)}$ and $\lambda^{(4)}$ have 4 boxes and 2 columns of odd length, $\lambda^{(2)}$ and $\lambda^{(5)}$ have 2 boxes and no columns of odd length, i.e.

$$\mathrm{Sh}(c) = \left\{ \left[\yng(1), \yng(3,1), \yng(1,1), \yng(1), \yng(2,1,1), \yng(1,1)\right], \left[\yng(1), \yng(2,1,1), \yng(1,1), \yng(1), \yng(2,1,1), \yng(1,1)\right], \left[\yng(1), \yng(3,1), \yng(1,1), \yng(1), \yng(3,1), \yng(1,1)\right] \right\}$$

Therefore we have the following decomposition of $M(c)$ into irreducible representations

$$M(c) \cong \rho_{\left[\yng(1), \yng(3,1), \yng(1,1), \yng(1), \yng(2,1,1), \yng(1,1)\right]} \oplus \rho^0_{\left[\yng(1), \yng(2,1,1), \yng(1,1), \yng(1), \yng(2,1,1), \yng(1,1)\right]} \oplus \rho^0_{\left[\yng(1), \yng(3,1), \yng(1,1), \yng(1), \yng(3,1), \yng(1,1)\right]}.$$

4.7 A final survey and a further generalization



Theorem 4.7.1 concerns all involutory *projective* reflection groups $G(r,p,q,n)$, except those satisfying $\mathrm{GCD}(p,n)=4$.

The aim of this section is to provide a statement containing all the results that we have collected in this thesis and holding for all the groups $G(r,p,n)$ with $\mathrm{GCD}(p,n)=1,2$. Furthermore, we use this occasion to observe that the above results apply, in fact, to all the projective groups $G(r,p,q,n)$ with $\mathrm{GCD}(p,n)=1,2$: see the above diagram.

If v is an absolute involution in $G(r,q,p,n)$ and c is any (symmetric or antisymmetric) S_n-conjugacy class of absolute involutions in $G(r,q,p,n)$, we define $\mathrm{Sh}(v)\in\mathrm{Fer}(r,q,p,n)$ and $\mathrm{Sh}(c)\subset\mathrm{Fer}(r,q,p,n)$ as in Notation 2.3.2. Moreover, we let $\iota(c)=0$ if the elements of c are symmetric and $\iota(c)=1$ if the elements of c are antisymmetric.

Theorem 4.7.1. *Let* $G=G(r,p,q,n)$ *with* $\mathrm{GCD}(p,n)=1,2$, *and consider its Gelfand model* $(M(r,q,p,n),\varrho)$ *defined in Theorem 2.5.1. Given an* S_n-*conjugacy class* c *of absolute involutions in* G^*, *let* $M(c)=\mathrm{Span}\{C_v : v\in c\}$ *so that* $M(r,q,p,n)$ *naturally splits as a* G-*module into the direct sum*

$$M(r,q,p,n)=\bigoplus_c M(c).$$

Then the submodule $M(c)$ *has the following decomposition into irreducibles*

$$M(c)\cong\bigoplus_{[\lambda]\in\mathrm{Sh}(c)}\rho_{[\lambda]}^{\iota(c)}.$$

Proof. We have already established this result if $q=1$. In fact, if $\iota(c)=0$ this is the content of Theorem 4.6.1, and, if $\iota(c)=1$, the result follows directly from Theorem 4.5.5 with the further observation that if v is an antisymmetric element of type $[t_0,\dots,t_{r'-1}]$, then $\mathrm{Sh}(v)=[\lambda^{(0)},\dots,\lambda^{(r-1)}]$, with $|\lambda^{(i)}|=|\lambda^{(i+r')}|=t_i$.

If $q \neq 1$ the result is straightforward since an S_n-conjugacy class of absolute involutions in $G(r,q,p,n)$ is also an S_n-conjugacy class of absolute involutions in $G(r,1,p,n)$ and the definition of the Gelfand models for $G(r,p,q,n)$ and $G(r,p,1,n)$ are compatible with the projection $G(r,p,1,n) \to G(r,p,q,n)$. $\square$

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