Alma Mater Studiorum · Università di Bologna

DOTTORATO DI RICERCA IN MATEMATICA

XXIII Ciclo

Settore scientifico disciplinare: MAT/05

LONG-TIME BEHAVIOR OF SOLUTIONS TO THE SYSTEM OF CRYSTAL ACOUSTICS FOR TETRAGONAL CRYSTALS

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Esame finale anno 2011

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Introduction

This thesis treats the global existence of solutions of the system of nonlinear elasticity, obtained as a perturbation of the linear system of crystal acoustics for tetragonal crystals. The system of crystal acoustics is a second-order hyperbolic system, with constant coefficients, which describes the propagation of waves in a crystal. Crystals are homogeneous, elastic and anisotropic solids, with a regular structure, characterized by a basic element: the unit cell. By filling repetition, the entire volume occupied by the crystal can be filled. It is well known from crystallography that there are seven crystal classes. Each class is characterized by the invariance of the unit cell under the action of some prescribed subgroup of $\mathbb{O}(3)$. In this thesis we will consider crystals belonging to the tetragonal crystal class (cf. [M] and the first chapter).
In the linear case we will take into account the Cauchy problem

$$\partial_t^2 u_i(t,x) = \sum_{j,k,l=1}^{3} c_{ijkl}\partial_{x_k}\partial_{x_l} u_j(t,x) \qquad i = 1,2,3, \tag{0.0.1}$$

$$u_i(0,x) = f_i(x) \qquad \partial_t u_i(0,x) = g_i(x) \qquad i = 1,2,3, \tag{0.0.2}$$

where $(t,x) \in \mathbb{R} \times \mathbb{R}^3$, $u_i(t,x)$ are the components of the displacement vector, f_i, g_i belong to $C_0^\infty(\mathbb{R}^3)$ and c_{ijkl} are the stiffness constants. We will assume some conditions on c_{ijkl} such that system (0.0.1) will be hyperbolic. Moreover, the symmetries of the tetragonal class are such there are at most six stiffness constants different from zero.
We can obtain the previous system from the equations

$$\partial_t^2 u_i(t,x) = \mathrm{div}\sigma_{ij}(\varepsilon(t,x)) \qquad i = 1,2,3, \tag{0.0.3}$$

if we apply Hooke's law and the relation between the strain tensor and the derivatives of the displacement vector (cf. [M], [Du] and chapter 1). Here we denote the strain and stress tensors by ε and σ respectively. Moreover, we recall that Hooke's law is the following linear relation between stress and strain

$$\sigma_{ij} = \sum_{k,l=1}^{3} c_{ijkl}\varepsilon_{kl}. \tag{0.0.4}$$

Therefore, if we assume the following nonlinear relation between stress and strain, instead of Hooke's law,

$$\sigma_{ij}(\varepsilon) = \sum_{k,l=1}^{3} c_{ijkl}\varepsilon_{kl} + H_{ij}(\varepsilon), \tag{0.0.5}$$

$$H : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}, \quad H_{ij}(\varepsilon(t,x)) = \mathcal{O}(||\varepsilon||_\infty^\kappa), \quad \mathbb{N}\ni\kappa\geq 4, \tag{0.0.6}$$

we can state the nonlinear problem of crystal acoustics. In particular, we will take into account the Cauchy problem formed by system (0.0.3) (where σ_{ij} is in the form (0.0.5)), and the initial data (0.0.2).
Thus, the first part of this thesis will be devoted to obtaining long time decay estimates of the solutions of the linear Cauchy problem. These estimates will have a key role in the second part of the thesis, where we will prove the global existence of solutions of the nonlinear problem.

A few words on the history of these problems are in order. The study of the propagation of elastic waves in a homogeneous solid began at the end of the XIXth century, with the works of Poisson and Stokes. Duff was the first who studied the same problem in the case of a homogeneous, but not isotropic, solid. He gave a detailed and qualitative description of the solution of the problem and he also studied the geometry of the wave surface associated with the problem (cf. [Du]).
Obviously, the first equation for which decay properties were studied in a systematic way was the wave equation (cf. e.g. [K2], [T1] and [Ho]). In the three-dimensional case, the results obtained give a decay of type ct^{-1} when t tends to infinity, whereas a decay of order $ct^{-1/2}$ is typical for the

wave equation in two dimensions. Similar results have been obtained for a number of related hyperbolic equations, such as the Klein-Gordon equation, or, sometimes, for more general classes of constant-coefficient hyperbolic operator or system (cf. e.g. [Su], [Sr], [Se], [G-L-Z]). All equations or systems mentioned so far have in common the characteristic of constant multiplicity. The case of crystal acoustics is different because the characteristic surface has singularities and thus the system has changing multiplicity. In this case, the study of the decays of solutions of Maxwell's system for optically biaxial crystals (cf. [L5] and [L-Z]), of the system of crystal acoustics for cubic crystals (cf. [L1]), of the system of crystal acoustics for exagonal crystals and of the system of thermo-elasticity in cubic crystals in two-dimension (cf. [R-W] and [W]) have been studied. Therefore, the study of the system of crystal acoustics for tetragonal crystal seems a natural continuation, in this direction.
As mentioned above, the study of the decay estimates of the solutions of the linear system is necessary to prove the existence of the global solution of the nonlinear system which is a perturbation of the linear one. In the cases of the wave equation and other evolution equations, there are many well known results (cf. e.g. [J1], [K1], [K-S], [K1], [Sh], [Si1] and [Si2]). In particular, in this thesis, the general line of argument will be close to the one from Klainerman and Ponce (cf. [K-P]). In this paper it is proved the global existence of solutions of nonlinear evolution equations, which are perturbations of linear ones, provided that the initial data are sufficiently small.
Finally, we recall that the nonlinear problem of elasticity has been studied, with different methods, from a variational point of view (cf. e.g. [E-N-S], [F-K-R]) and that in the engineering literature there are many results in the case of certain specific crystals.

Now, we give an overview of this thesis.
In the first chapter of this thesis, we will recall the standard notion of elasticity, that will be useful in the rest of the work, we will state the problem and we will write down explicitly the solution of the linear problem.

As mentioned above, the first part of the thesis will be devoted to obtaining the decay estimates of the solutions of the linear Cauchy problem (0.0.1), (0.0.2). The main difficulties in this study come from the presence of isolated singularities in the characteristic surface associated to the system. Thus, in the second chapter of the thesis, we will give a detailed geometrical study of the wave surface of tetragonal crystals. In particular, we will prove that there are three different types of singular points: the so called *uniplanar*, *biplanar* and *conical* points. The uniplanar and conical singular points are also present on the wave surface of cubic crystals, whereas the biplanar singular points appear only in the tetragonal case, precisely when the following relation for the stiffness constants holds: $c_{1111} = c_{1212}$. The name *biplanar* comes from the fact that the local second order approximation of the wave surface in those singular points is the union of two planes (cf. section 2.4). Moreover, we will prove that, in the case of tetragonal crystals, the singular points are located in a more general position with respect to the cubic case, where they are located only on the coordinate axes or on the space diagonals. Thus, the expressions which come up in the study of the nature of the singularities and in the study of the curvatures of the wave surface are more involved than in the cubic case. Moreover, in those calculations, in the tetragonal case we have to take into account six different stiffness constants, while in the cubic case there are only three.

The last part of the second chapter will be devoted to the study of the total and mean curvatures of the wave surface. This study is necessary because we want to use theorems about the decay of oscillatory integrals on surfaces in order to obtain the desired decay estimates. The expressions which appear in this study are very involved and thus a direct computation would seem very difficult. Therefore, we will consider only tetragonal crystals which are *near* (in some sense which will be clarified in the thesis) the cubic case. Indeed, we will obtain satisfying results only for small perturbations of the cubic case (cf. section 2.6).

In the third chapter of the thesis we will prove, with specific conditions on

the stiffness constants, theorem (3.1), about the decay of the solutions of the linear Cauchy problem (0.0.1), (0.0.2). In particular, we will obtain a decay of type $ct^{-1/2}$, as in the case of cubic crystals. In order to do this, we will first give a detailed study of the phase and amplitude functions of the oscillatory integrals which appear in the solution of the system (cf. section 3.1) and secondly we will reduce the estimates of this integrals to estimates of oscillatory integrals on the wave surface (cf. section 3.2). Then the prove follows in a standard way with the study of many sub-cases depending on the relative position of t, x, ξ (the time variable, the space variable and the dual variable respectively) and with the use of well known theorems about the decay estimates of oscillatory integrals which live on surfaces (cf. sections 3.2 and 3.4). Finally, we will prove, in analogy with the theorem used in section 3.4, theorem (3.15) about the decay estimate of oscillatory integrals on surfaces with singular biplanar points. In particular, we will obtain a decay of type $c|\xi|^{-1/2}\log(1+|\xi|)$.
In the fourth chapter we will treat the nonlinear problem. The main theorem is the following.

Theorem 0.1. *Assume that*

$$\sigma_{ij}(\varepsilon) = \sum_{k,s=1}^{3} c_{ijks}\varepsilon_{ks} + H_{ij}(\varepsilon), \quad H_{ij}(\varepsilon) = \mathcal{O}(||\varepsilon||_{\infty}^{5}).$$

Then we can find $s \in \mathbb{N}$ and $\delta > 0$ such that if

$$||\nabla_x f||_{2,s} + ||g||_{2,s} \leq \delta$$
$$||\nabla f||_{10/9,s} + ||g||_{10/9,s} \leq \delta$$

then there is a solution $u \in C([0,\infty[, H^{s+1}(\mathbb{R}^3)) \cap C^1([0,\infty[, H^s(\mathbb{R}^3))$ of the Chauchy's problem

$$\begin{cases} \dfrac{\partial^2 u_i}{\partial t^2}(t,x) = div\sigma_{ij}(\varepsilon(t,x)) & i = 1,2,3 \\ u_i(0,x) = f_i(x) & i = 1,2,3 \\ \partial_t u_i(0,x) = g_i(x) & i = 1,2,3 \end{cases}.$$

The prove is standard but quite technical and is close to the one from Klainerman and Ponce (cf. [K-P]). At first, we will obtain energy estimates (cf. sections 4.2 and 4.3), then we will transform the initial system into a first-order, symmetric, hyperbolic and quasi-linear system. Thus, for this kind of system we can apply well know theorems about the existence of local solutions. Then, combining the energy estimates and the decay estimates of the linear system, we will obtain an a priori estimate of the H^s-norm of the solution (cf. section 4.4). Finally, we will combine, using a *boot-strap* argument, the a priori estimate and the result of the existence of the local solution to conclude the prove.

Chapter 1

Tetragonal Crystal Acoustic

In this chapter we will consider the equations which describe the propagation of waves in tetragonal crystals. In order to do this we will recall the definition of tetragonal crystals, we will describe the physical properties of crystals and we will introduce some kinematic and dynamical entities which have a key role in the construction of the equations. Finally, we will write explicitly the Cauchy Problem we will take into account and we will write down its solution as an oscillatory integral.

1.1 Review of crystal structure

In this thesis we will treat a specific elastic material, the crystal, and we will now describe exactly the kind of material we have in mind. We say that some matter is in solid state if it is characterized by structural rigidity and resistance to changes of shape or volume. In many cases such solids are aggregates of small volumes which are identical in their structure. In these cases we call them *crystal solids*. We can take as a model a regular infinite and periodic lattice of points in $\mathbb{R}^3$. This model is a good approximation of a crystal where the ions (or molecules), which form the solid, may be considered to be located, at their mean positions, in the points which constitute the space lattice. As a result of the periodicity in the crystal structure, there

exists an elementary and irreducible volume which characterizes any given crystal: by space filling repetition, the entire volume occupied by the crystal can be filled. We call this basic element of the structure the *unit cell.* Using operations such as translation, rotation, and inversion, the unit cell of a crystal may be moved in such a way that it is brought into self-coincidence. Points representing molecular sites are exactly in the same position with respect to the molecular sites when the cell was in the original position. This feature of the unit cell gives the crystal its particular symmetry.
We require two other conditions: homogeneity and anisotropy. We say that a solid is homogeneous if its physical properties are invariant under translations. We say that a solid is anisotropic if its properties change with direction. It's easy to understand that the anisotropic quality of a crystal has its origin in the symmetry properties of the unit cell.
The last hypothesis on the solid is elasticity. We say that a solid is elastic if it returns to its original shape after the stress that causes some deformation has been removed.
Finally, in this work we will discuss waves whose wavelengths are very large compared with the intermolecular spacings or the dimension of the unit cell.

We now study with some details the structure and symmetry properties of some crystals.

Definition 1.1. *We call* three-dimensional lattice *a set of points* $p \in \mathbb{R}^3$ *such that* $p = n_1 a + n_2 b + n_3 c$ *where* $n_i \in \mathbb{Z}$, $i = 1, 2, 3$ *and* a, b, c *are linearly independent vectors of* $\mathbb{R}^3$, *called* primitive vectors.
We call unit cell *the set of points of the lattice such that* $n_i \in \{0, 1\}$ *for all* $i = 1, 2, 3$.

We want to classify the three-dimensional lattices with respect to the symmetries of the unit cell. We have the following definition.

Definition 1.2. *We denote by* $\mathbb{O}(3)$ *the group of all isometries that leave the origin fixed, or correspondingly, the group of orthogonal matrices. We call*

point group *a subgroup of* $\mathbb{O}(3)$.
We call crystal system *or* crystal lattice *the set of three-dimensional lattices which are invariant under the action of some prescribed point groups.*

It's well know in crystallography that there are seven classes of crystal systems. In table (1.1) we enumerate the seven classes of crystal systems with their prescribed symmetries and their corresponding unit cells. To describe the unit cell we use the length of the primitive vectors and the measures of the angles between them, with the following notation: $(a, b, c, \alpha, \beta, \gamma) \in \mathbb{R}^3 \times [0, \pi]^3$ where a, b, c are the lengths of the three primitive vectors and α, β, γ are the measures of angles between them.

crystal system	prescribed symmetries	unit cell
Triclinic	none	(a,b,c,α,β,γ)
Monoclinic	1 2-fold axis of rotation	(a,b,c,$\pi/2$,$\pi/2$,γ)
Orthorhombic	3 2-fold axis of rotation	(a,b,c,$\pi/2$,$\pi/2$,$\pi/2$)
Tetragonal	1 4-fold axis of rotation	(a,a,c,$\pi/2$,$\pi/2$,$\pi/2$)
Trigonal	1 3-fold axis of rotation	(a,a,a,α,α,α)
Hexagonal	1 6-fold axis of rotation	(a,a,c,$\pi/3$,$\pi/2$,$\pi/2$)
Cubic	4 3-fold axis of rotation	(a,a,a,$\pi/2$,$\pi/2$,$\pi/2$)

Table 1.1: Crystal systems with prescribed symmetries and unit cell

Remark 1.1. We can associate with each crystal class the subgroups of the isometries of $\mathbb{R}^3$ (we call them space groups) which bring the crystal structure to self-coincidence. However, we know that our primary interest is in the point group symmetry of the crystal structure, since the tensors representing macroscopic properties have to be invariant under the symmetry elements of the point group.

As described above, we can model a crystal solid with a lattice with molecules located in the points which form the space lattice, that is in the

vertices of the unit cells which fill the crystal. It is possible for the molecules of a crystal to be located not only in the vertices of the unit cells, but also in the center of one or all faces or in the barycentre of the cells.
We distinguish the following four possible distributions of lattice points in the unit cell:

(P) Primitive centering: lattice points on the cell vertices only.

(I) Body centered: one additional lattice point at the center of the cell.

(F) Face centered: one additional lattice point at the center of each face of the cell.

(C) Single face centered: one additional lattice point at the center of one of the cell faces and another one in the opposite face.

If we combine the seven crystal systems with the four possible distributions of points in the unit cell we find 14 different classes of lattices: the so called Bravais lattices (named after French physicist Auguste Bravais, 1811-1863).
We observe that there are in total 42 combinations of different classes of lattices, but it can be shown that several of these are in fact equivalent to each other. For example the monoclinic I lattice can be described by a monoclinic C lattice by a different choice of primitive vectors.
Our main interest is the study of point groups associated with the cubic and the tetragonal crystal classes.

Definition 1.3. *Using the Schönflies notation for the point groups (for details see [Co]), we say that a three-dimensional lattice belongs to the cubic class if it is invariant under the action of the following five subgroups of* $\mathbb{O}(3)$*:*

$$T, \quad T_h, \quad O, \quad T_d, \quad O_h. \tag{1.1.1}$$

We say that a three-dimensional lattice belongs to the tetragonal class if it is invariant under the action of the following seven point groups:

$$C_4, \quad S_4, \quad C_{4h}, \quad D_4, \quad C_{4v}, \quad D_{2d}, \quad D_{4h}. \tag{1.1.2}$$

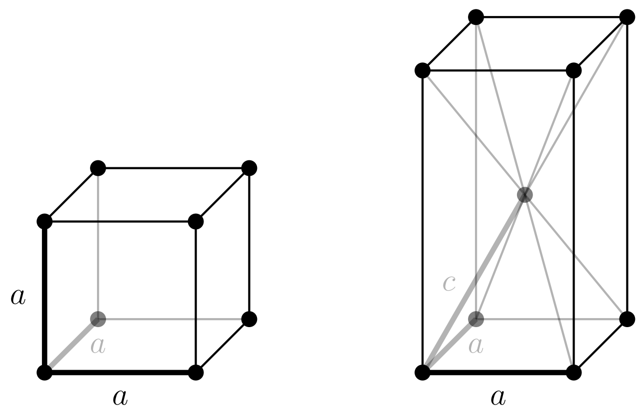


Figure 1.1: The cubic P lattice and the tetragonal I lattice unit cells, with relative primitive vectors.

Remark 1.2. In the Schönflies notation the letters O and T indicate that the group has the symmetry of an octahedron and of a tetrahedron respectively. The letters C_n, S_n and D_n indicate that the group has an n-fold rotation axis, an n-fold rotation-reflection axis and an n-fold rotation axis plus a twofold axis perpendicular to that axis, respectively. The subscript v and h indicate that the group has the symmetry, in addition to the other symmetries, of a mirror plane parallel or perpendicular to the axis of rotation. Moreover, the subscript d indicate that the group has the symmetries of a improper rotation.

Sometimes in the following, for simplicity, we may use the expression *cubic crystal* and *tetragonal crystal* to denote a crystal in the cubic or the tetragonal crystal class respectively.

1.2 The strain and stress tensors

To describe the wave propagation in crystals we have to study the mechanics of deformation in solids, that is the relationship between changes in the volume and the shape of solids and the forces which induce them.

Macroscopic deformation theory ignores the discrete-particle structure of matter and considers a solid as a continuum. Consequently, a load is most usefully expressed as force per unit area and it is also convenient to specify deformation in terms of fractional displacement which, for a continuum, can be written as partial derivatives.
Here we give the definition and a brief qualitative description of the strain and stress tensors. We don't intend to give a comprehensive treatment of this subject, for more details see [M] and [Ci].
To begin, we consider a body $\mathcal{C}$ in $\mathbb{R}^3$ and denote by $x = (x_1, x_2, x_3)$ the coordinates of a generic point. We suppose that the body undergoes the following linear transformation of $\mathbb{R}^3$

$$\varphi : \mathbb{R}^3 \to \mathbb{R}^3, \qquad \varphi(x) = x' = x + \tau + \chi x, \tag{1.2.1}$$

where $\tau \in \mathbb{R}^3$ is a vector which represents the translational displacement of the body and $\chi \in \mathbb{M}_3$ is a 3×3 matrix which represents a change in orientation and shape of the body.

Remark 1.3. We suppose, from the very beginning, that the general space deformation is the same for all the line elements within the body. This implies that the matrix χ and the vector τ do not depend on the space variables x_i.
These kind of space deformations are called homogeneous affine space transformations.
If τ and χ are not constant for all line elements within the body, the transformation is called inhomogeneous. It is not difficult to see that a variation in τ may be incorporated into the variation in χ. However, if we wish to represent a deformable body, restrictions must be placed on the variation of $\chi(x)$ with the space variable x. This means that we must have some conditions on the regularity of the transformation $\varphi(x)$.
We will examine these regularity conditions later on.

Now suppose that the linear transformation preserves the length, that is

$$|\varphi(y) - \varphi(x)| = |y - x|, \quad \text{for all } x, y \in \mathcal{C}. \tag{1.2.2}$$

The squared length may now be written in two ways

$$|y - x|^2 = |\varphi(y) - \varphi(x)|^2 = |(y - x) + \chi(y - x)|^2 =$$
$$= |y - x|^2 + 2\sum_{i,j=1}^{3} \chi_{ij}(y_i - x_i)(y_j - x_j) + R$$

where R contains quadratic powers of χ_{ij}. So, neglecting R, we obtain

$$\sum_{i,j=1}^{3} \chi_{ij}(y_i - x_i)(y_j - x_j) = 0,$$

which implies

$$\chi_{ii} = 0 \qquad \text{and} \qquad \chi_{ij} = -\chi_{ji},$$

that is χ is antisymmetric. It is clear that under hypothesis (1.2.2), $\mathcal{C}$ is subject to a common translation and rotation.
Now let restriction (1.2.2) be lifted and consider

$$|y' - x'|^2 - |y - x|^2 = 2\chi_{ij}(y_i - x_i)(y_j - x_j) = (\chi_{ij} + \chi_{ji})(y_i - x_i)(y_j - x_j). \tag{1.2.3}$$

This shows that the change in length of a line element is related to the symmetric part of matrix χ. Thus we can write $\chi = \varepsilon + \omega$, where $\varepsilon = \mathrm{Symm}(\chi)$ and $\omega = \mathrm{SkewSymm}(\chi)$, and where the symmetric matrix ε contains all the information about the deformation of any line element within a body subjected to the transformation χ, while the antisymmetric quantity ω specifies the rigid-body rotation. We give the following definition.

Definition 1.4. *Let $\chi \in \mathbb{M}_3$ be the matrix associated to a linear trasformation of $\mathbb{R}^3$. We call* strain matrix *the 3×3 symmetric matrix $\varepsilon = Symm(\chi)$.*

We observe that a matrix is a 2-tensor and so we often call ε the strain tensor.

We now investigate how the strain tensor changes if we change the basis of $\mathbb{R}^3$. Let e_i, $i = 1, 2, 3$ and e'_i, $i = 1, 2, 3$ be two bases of $\mathbb{R}^3$ and $A \in \mathbb{G}L(3)$ (here $\mathbb{G}L(3)$ is the group of 3×3 invertible matrices) the base changing matrix, then it is a standard result of linear algebra that

$$\varepsilon' = A\varepsilon A^{-1}$$

where ε' is the strain tensor related to the deformation in the space with base $\{e'_i\}_i$. Now, suppose that $A \in \mathbb{O}(3)$, that is e_i is mapped to e'_i by a rotation, then we have the following relation

$$\varepsilon' = A\varepsilon A^T.$$

We can rewrite the last equation using the following tensor notation

$$\varepsilon'_{lk} = \sum_{i,j=1}^{3} a_{li} a_{kj} \varepsilon_{ij}, \tag{1.2.4}$$

which will be useful in the following of this work (here $A = (a_{ij})_{i,j=1,2,3}$). Since ε is symmetric we can define its eigenvalues and eigenvectors.

Definition 1.5. *We call* principal axes of strain *and* principal strains *the eigenvectors and the eigenvalues of ε respectively.*

Now we want to give a brief geometrical interpretation of the strain. We start considering a unit cube of material, whose edges are parallel to the principal axes of the strain which is acting on the material. Suppose that $\varepsilon = \mathrm{diag}(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33})$, $\tau \equiv 0$ and $\omega \equiv 0$, then $\varphi(x) = x + \varepsilon x$. Thus

$$\varphi(e_i) = e_i + \varepsilon_{ii}, \quad \forall i \in \{1, 2, 3\}.$$

This means that the diagonal elements of the strain tensor indicate the displacement along the principal axes, and so if $\varepsilon = \mathrm{diag}(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33})$ then it represents exactly a dilatation and/or a contraction (according to the signs of ε_{ii}). Consequently if we call V the volume of the perturbed cube, we have

$$V = 1 + \mathrm{Tr}(\varepsilon) + \mathcal{O}(\varepsilon^2).$$

Now suppose as before that the linear space transformation is precisely a deformation such that $\varepsilon_{ij} = 0$ for all $ij \neq 23$. Thus the edges of the unit cube are transformed by

$$\varphi(e_1) = e_1, \quad \varphi(e_2) = e_2 + \varepsilon_{23} e_3, \quad \varphi(e_3) = e_3 + \varepsilon_{23} e_2.$$

This means that the (e_2, e_3) face is changed to a rhomb with angles $\pi/2 \pm 2\varepsilon_{23}$. Nonzero strains ε_{13} and ε_{12} give rise to corresponding deformations of the (e_1, e_3) and (e_1, e_2) faces. But this change is also specified by the change in the lengths of the diagonals from $\sqrt{2}$ to $\sqrt{2}(1 \pm \varepsilon_{23})$. Thus, it is possible to calculate the strain with respect to the diagonals as principal axes e'_2 and e'_3, and we obtain the strain tensor

$$\varepsilon' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \varepsilon_{23} & 0 \\ 0 & 0 & -\varepsilon_{23} \end{pmatrix},$$

so that the diagonals of the (e_2, e_3) face are seen to be the principal axes of the strain, and the e_1 axis is the remaining member of the orthogonal triad. Finally, consider a deformation such that $\varepsilon_{ii} \neq 0$ for all i, $\varepsilon_{23} \neq 0$ and $\varepsilon_{12} = \varepsilon_{13} = 0$. From the discussion above, the cube is deformed into a rectangular parallelepiped with the (e_2, e_3) face as a base. Thus, a strain tensor with all nonzero elements transforms a cube into a parallelepiped, and it is useful to note that the elements of the leading diagonal preserve the rectangular faces, while the other elements cause the change into parallelogram faces.

Remark 1.4. Obviously, the definitions, properties and geometrical meanings described above do not change if χ and ε depend on the space variable x_i.

We suppose, for the remainder of this thesis, that the space transformation φ is inhomogeneous, that is $\chi = \chi(x)$ and $\varepsilon = \varepsilon(x)$.

Definition 1.6. *Let* φ *be as in* (1.2.1), $\varphi \in C^3(\mathbb{R}^3)$. *We call* space displacement *the vector field*

$$u : \mathbb{R}^3 \to \mathbb{R}^3, \qquad u(x) = \varphi(x) - x.$$

In dealing with materials assumed to be continuous, we may expect to obtain a relationship between the strain tensor and space derivative of the displacement. To find it, we consider a line element ds of components dx_i which has length the square root of $\sum_{i=1}^3 dx_i^2$. After the space transformation the components of the transformed line element ds' are $dx_i' = dx_i + \sum_{j=1}^3 \partial_{x_j} u_i dx_j$ and so we have

$$(ds')^2 = \sum_{i=1}^3 (dx_i + \sum_{j=1}^3 \partial_{x_j} u_i dx_j)(dx_i + \sum_{k=1}^3 \partial_{x_k} u_i dx_k) =$$

$$= \sum_{i,j,k=1}^3 dx_i dx_j \left[\delta_{ij} + \partial_{x_j} u_{i,j} + \partial_{x_i} u_j + \partial_{x_i} u_k \partial_{x_j} u_k\right]$$

and

$$ds' - ds = \sum_{i,j,k=1}^3 dx_i dx_j (\partial_{x_j} u_i + \partial_{x_i} u_j + \partial_{x_i} u_k \partial_{x_j} u_k).$$

If we consider only the first-order term, according to (1.2.3), we can write the strain tensor in terms of the displacement derivative

$$\varepsilon_{ij} = \frac{1}{2}(\partial_{x_j} u_i + \partial_{x_i} u_j). \tag{1.2.5}$$

Remark 1.5. The elements of the rotational matrix ω_{ij} are of course given by the antisymmetric combination $1/2(\partial_{x_j} u_i - \partial_{x_i} u_j)$, but we will not use this fact in the following.

Remark 1.6. From the assumption $\varphi \in C^3$ in definition (1.6) it follows that $u \in C^3$. This condition ensures that the transformation does not create holes or voids in an originally continuous material, and that only one volume element of the deformed material is assigned to a given volume element of space. This regularity condition also implies that second derivatives of components of rotations and of strain matrices exist. Moreover, relation (1.2.5) and the regularity of u give compatibility equations:

$$\partial_{x_i}\omega_{jk} = (\partial_{x_k}\varepsilon_{ij} - \partial_{x_j}\varepsilon_{ik}),$$

$$\partial^2_{x_i x_l}\omega_{jk} = \partial^2_{x_l x_i}\omega_{jk},$$

$$\partial^2_{x_k x_l}\varepsilon_{ij} - \partial^2_{x_i x_k}\varepsilon_{jl} - \partial^2_{x_j x_l}\varepsilon_{ik} + \partial^2_{x_i x_l}\varepsilon_{kl} = 0.$$

In the last equation each suffix may take the values $1, 2, 3$, so we have 81 possible relations. However, symmetries of the suffixes render many of them trivial, so they can be reduced to six non trivial relations.

Now, suppose that the body $\mathcal{C}$ is subjected to an equilibrated system of external loads and, thereby, held in a state of deformation. We want to examine the surface forces acting on the body, that is the forces acting on an imaginary internal surface, that divides the body into two portions, as a result of the mechanical interaction between both parts of the body at each side of the surface, or similarly, the forces acting on the surface of a small volume element due to external loads. According to the Euler-Cauchy stress principle these forces can be represented by a vector field $T^n(x)$, called the stress vector, defined at each point x of the body and depending continuously on the normal unit vector n at imaginary surfaces passing through x. The state of stress at a point in the body is then defined by all the stress vectors T^n associated with all the planes that pass through that point. It's easy to see that the stress vector on any plane passing through that point can be obtained as the linear sum of three stress vectors on three mutually perpendicular planes. In particular it can be obtained as the sum of three stress vectors on the coordinate planes. Now, Cauchy's stress theorem states that there exists a second-order tensor field $\sigma(x)$, independent from n such that

$$T^n(x) = \sigma \cdot n$$

where

$$T^{e_i} = \sigma_{i1}e_1 + \sigma_{i2}e_2 + \sigma_{i3}e_3, \qquad i = 1, 2, 3$$

and where e_i are the vectors forming the base of $\mathbb{R}^3$.
This implies that the stress vector T^n, at any point x in a continuum associated with a plane with normal vector n, can be expressed as a function of the stress vectors on the planes perpendicular to the coordinate axes, i.e., in terms of the components σ_{ij} of σ.

Definition 1.7. *We call* stress tensor *the 2-tensor σ defined by Cauchy's stress theorem.*

Using the momentum conservation law and the divergence theorem it's possible to prove that the stress tensor is a symmetric matrix, i.e. $\sigma_{ij} = \sigma_{ji}$, for all $i, j \in \{1, 2, 3\}$. We will not show this standard result here (see e.g. [M] and [Ci]).
By analogy with the strain tensor, if $A = (a_{ij})_{i,j=1,2,3} \in \mathbb{O}(3)$ and a base $\{e_i\}_{i=1,2,3}$ of $\mathbb{R}^3$ is brought to $\{e'_j\}_{j=1,2,3}$ by the transformation associated with A, then

$$\sigma'_{lk} = \sum_{i,j=1}^{3} a_{li} a_{kl} \sigma_{ij} \tag{1.2.6}$$

where σ' is the stress tensor in the space with base e'_i.
Again, by analogy with the strain tensor we have the following definitions.

Definition 1.8. *We call* principal axes of stress *and* principal stresses *the eigenvectors and eigenvalues of σ respectively.*

Finally, we will study the relation between stress and strain. It's obvious that the application of stress to a deformable body will result in the material passing into a state of strain. Here we are concerned with the formation and the propagation of waves of small displacements in anisotropic media. We therefore adopt the most general linear relation, that is the generalized Hook's law. Such relation has the form

$$\sigma_{ij} = \sum_{k,l=1}^{3} c_{ijkl} \varepsilon_{kl}, \tag{1.2.7}$$

where c_{ijkl} is a real constant for all $i, j, k, l = 1, 2, 3$.

Definition 1.9. *We call* elastic stiffness *tensor the 4-tensor*

$$C = (c_{ijkl})_{i,j,k,l=1,2,3}.$$

Note that the stiffness tensors has 81 elements. The symmetries of the stress and strain tensors imply that there are 6 independent choices for the pairs of suffixes ij and kl. This implies the following relations

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{jilk}. \tag{1.2.8}$$

Another relation involving the strain energy $W = 1/2\sigma_{ij}\varepsilon_{kl}$ and the internal energy of a unit mass of material $\Phi(\varepsilon)$ (for details see [M]) gives us the following further symmetry:

$$c_{ijkl} = c_{klij}. \tag{1.2.9}$$

Hence, by (1.2.8) and (1.2.9), the stiffness tensor can have, at most, 21 independent elements.

Now it's useful to write the transformation law for the 4-tensor c_{ijkl}. Suppose as before that $A = (a_{ij})_{i,j=1,2,3} \in \mathbb{O}(3)$ and that a base e_i of $\mathbb{R}^3$ is brought to e_i' by the transformation associated to A. Then, using (1.2.4), (1.2.6) and (1.2.7) we obtain

$$\begin{aligned}\sigma'_{mn} &= \sum_{i,j=1}^{3} a_{mi}a_{nj}\sigma_{ij} = \sum_{i,j,k,l=1}^{3} a_{mi}a_{nj}c_{ijkl}\varepsilon_{kl} = \\ &= \sum_{i,j,k,l,r,s=1}^{3} a_{mi}a_{nj}c_{ijkl}a_{rk}a_{sl}\varepsilon'_{rs} = \sum_{r,s=1}^{3} c'_{mnrs}\varepsilon'_{rs}\end{aligned}$$

so that

$$c'_{mnrs} = \sum_{i,j,k,l=1}^{3} a_{mi}a_{nj}a_{rk}a_{sl}c_{ijkl}. \tag{1.2.10}$$

We conclude this section with some useful remarks about notations concerning the stiffness tensor. Indeed, the symmetries of the stiffness tensor allow us to write it as a 6×6 symmetric matrix in the following way: we substitute the first two indices and the second two indices of the 4-index notation with

4-index notation	2-index notation
11	1
22	2
33	3
23 or 32	4
31 or 13	5
21 or 12	6

Table 1.2: relation between 4-index and 2-index notations

one index each, using the relations defined in table (1.2). We can use the same relations to change a 2-index notation into a 1-index notation. Thus, we can write the stress and strain tensors as six-vectors:

$$\begin{aligned}\varepsilon^T &= (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{23}, 2\varepsilon_{13}, 2\varepsilon_{12}),\\ \sigma^T &= (\sigma_{11}, \sigma_{22}, \sigma_{33}, 2\sigma_{23}, 2\sigma_{13}, 2\sigma_{12})\end{aligned}$$

and so we can rewrite the generalized Hook's law as

$$\sigma = C\varepsilon$$

provided that σ and ε are six-vectors, while C is a 6×6 symmetric matrix, called the elastic stiffness matrix.

1.3 Equations of motion

In this section we will derive the equations of motion for a typical volume element of a crystal. Then we will write down explicitly the Cauchy problem in the case of cubic and tetragonal crystal classes.

In the following we will consider the stress and strain tensors and the displacement vector not only depending on the spatial coordinates x_i of $\mathbb{R}^3$, but also depending on the time $t \in \mathbb{R}$, such that it make sense to consider the time derivative, and so we will write the equation of motion.

Consider a typical volume element V of a homogeneous elastic crystal, bounded by a surface S. Let ρ be the density of the solid, F and σ the body force and the stress tensor acting on the volume element respectively. Let the acceleration produced be $\partial_t^2 u$, thus Euler's equation of motion is expresses as

$$\int_S \sum_{j=1}^{3} \sigma_{ij} n_j dS + \rho \int_V F_i dV = \rho \int_V \partial_t^2 u_i dV, \qquad i = 1, 2, 3 \tag{1.3.1}$$

where n is the outward normal to S. The regularity of σ described in section (1.2) allows us to use the Divergence Theorem and transform the surface integral into a volume integral. Thus the expression (1.3.1) may be written as

$$\int_V \left(\text{div } \sigma_i + \rho F_i - \rho \partial_t^2 u_i\right) dV = 0, \qquad i = 1, 2, 3. \tag{1.3.2}$$

Remark 1.7. Here and in the following, with the notation div σ_i we indicate the divergence of the vector $\sigma_i = (\sigma_{i1}, \sigma_{i2}, \sigma_{i3})$, i.e. $\sum_{j=1}^{3} \partial_{x_j} \sigma_{ij}$.

Since V contains continuous and homogeneous material, equation (1.3.2) requires

$$\text{div } \sigma_i + \rho F_i - \rho \partial_t^2 u_i = 0, \qquad i = 1, 2, 3, \tag{1.3.3}$$

for every volume element, however small, and thus represents the differential equation of motion.

Remark 1.8. It's also possible to derive equation (1.3.3) directly considering the forces acting on the unit volume of the crystal in each direction.

In the following we suppose that there are no body forces acting on the crystal and that the density is constant equal one. With these assumptions the equations (1.3.3) are reduced to

$$\text{div } \sigma_i = \partial_t^2 u_i \qquad i = 1, 2, 3. \tag{1.3.4}$$

Now we want to write equations (1.3.4) in terms of the displacement. To do this we recall the generalized Hook's law (1.2.7) and the relationship between the strain tensor and the displacement derivative (1.2.5): thus we can write

$$\sigma_{ij} = \sum_{k,l=1}^{3} c_{ijkl}\varepsilon_{kl} = \sum_{k,l=1}^{3} c_{ijkl}\frac{1}{2}(\partial_{x_l}u_k + \partial_{x_k}u_l) = \sum_{k,l=1}^{3} c_{ijkl}\partial_{x_l}u_k, \qquad (1.3.5)$$

where the last equality follows from the symmetries of the stiffness constants. Now, substituting (1.3.5) in (1.3.4), we obtain a system of linear partial differential equations of second order

$$\sum_{k,l,j=1}^{3} c_{ijkl}\frac{\partial^2 u_k}{\partial x_l \partial x_j} = \partial_t^2 u_i, \qquad i = 1, 2, 3. \qquad (1.3.6)$$

It's clear that the nature of the equations of motion depends on the symmetries of the stiffness tensors, which depend on the nature of the solid, i.e. on the crystal class. Thus, it's convenient to refer stress and strain to the symmetry axes of the crystal, and we will henceforth assume that the axes of reference for the dynamical equations coincide with these axes.
Consider the tetragonal crystal class: it's invariant under a four-fold rotation around the x_3-axis. Thus the stiffness constant in the equations of motion of tetragonal crystals must be invariant under the action of the matrix $R \in \mathbb{O}(3)$ associated with the four-fold rotation. We can write R as

$$R = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and we impose the invariance of c_{ijkl} under transformation law (1.2.10). Thus we obtain the following conditions for the stiffness tensor

$$c_{14} = c_{15} = c_{24} = c_{25} = c_{34} = c_{35} = c_{36} = c_{45} = c_{46} = c_{56} = 0, \qquad (1.3.7)$$

$$c_{11} = c_{22} \quad c_{44} = c_{55} \quad c_{13} = c_{23} \quad c_{16} = -c_{26}, \qquad (1.3.8)$$

where we used the two-index notation. If we apply all the symmetries of the tetragonal class we obtain in addition the condition $c_{16} = 0$, so the elastic stiffness matrix for the tetragonal crystal has 5 independent constants and it is

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix}. \tag{1.3.9}$$

Now we can explicitly write the system of linear elasticity for tetragonal crystals using the stiffness constants of (1.3.9) in (1.3.6). It has the following form

$$\begin{aligned}
\partial_t^2 u_1 &= (c_{11}\partial_{11}^2 + c_{66}\partial_{22}^2 + c_{44}\partial_{33}^2)u_1 + (c_{12}+c_{66})\partial_{12}^2 u_2 + (c_{13}+c_{44})\partial_{13}^2 u_3,\\
\partial_t^2 u_2 &= (c_{12}+c_{66})\partial_{12}^2 u_1 + (c_{66}\partial_{11}^2 + c_{11}\partial_{22}^2 + c_{44}\partial_{33}^2)u_2 + (c_{13}+c_{44})\partial_{23}^2 u_3,\\
\partial_t^2 u_3 &= (c_{13}+c_{44})\partial_{13}^2 u_1 + (c_{13}+c_{44})\partial_{23}^2 u_2 + (c_{44}\partial_{11}^2 + c_{44}\partial_{22}^2 + c_{33}\partial_{33}^2)u_3.
\end{aligned} \tag{1.3.10}$$

Here, for simplicity, we used the notation ∂_{ij} instead of $\partial_{x_i x_j}$.
With this system we now associate the initial conditions

$$u_i(0,x) \equiv 0, \qquad \frac{\partial u_i(0,x)}{\partial t} = g_i(x), \qquad i = 1,2,3, \tag{1.3.11}$$

where $x \in \mathbb{R}^3$ and we assume functions g_i, $i = 1,2,3$ are C^∞-functions on $\mathbb{R}^3$ and have compact support.

Remark 1.9. As is often done in similar situations, we can consider, without loss of generality, the initial condition $u_i(0,x) \equiv 0$ instead of $u_i(0,x) = f_i(x)$.

Now we want to explicitly write the solution of system (1.3.10) with the initial data (1.3.11). First of all we assume several restrictions on constants c_{ij}, which come from physical considerations, so that the system is hyperbolic. We will not write these conditions explicitly here, but we will assume the following implicit condition on the stiffness tensor: we suppose that the matrix

$$A(\xi) = \left(\sum_{j,l=1}^{3} c_{ijkl}\xi_j\xi_l \right)_{i,k=1,2,3} \tag{1.3.12}$$

is positive definite for all $\xi \in \mathbb{R}^3$.

Remark 1.10. Here and in the following, when we write expressions involving c_{ijkl}, we suppose that conditions (1.3.7) and (1.3.8) holds.

We recall that the characteristic polynomial of system (1.3.10) is given by the determinant of $P(\tau,\xi)$, where $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}^3$ and $P(\tau,\xi)$ is the following matrix:

$$\begin{pmatrix} \tau^2 - c_{11}\xi_1^2 - c_{66}\xi_2^2 - c_{44}\xi_3^2 & -(c_{12}+c_{66})\xi_1\xi_2 & -(c_{13}+c_{44})\xi_1\xi_3 \\ -(c_{12}+c_{66})\xi_1\xi_2 & \tau^2 - c_{66}\xi_1^2 - c_{11}\xi_2^2 - c_{44}\xi_3^2 & -(c_{13}+c_{44})\xi_2\xi_3 \\ -(c_{13}+c_{44})\xi_1\xi_3 & -(c_{13}+c_{44})\xi_1\xi_3 & \tau^2 - c_{44}\xi_1^2 - c_{44}\xi_2^2 - c_{33}\xi_3^2 \end{pmatrix}.$$

Moreover, the characteristic manifold associated with the system is

$$\left\{(\tau,\xi) \in \mathbb{R}^4;\ \det P(\tau,\xi) = 0\right\}.$$

We give some additional remarks.

Remark 1.11. Since the system (1.3.10) is hyperbolic with constant coefficients, it is known to admit global solutions in $\mathbb{R}^4$. Moreover, if we assume the initial conditions (1.3.11), the functions $x \mapsto u_i(t,x)$ are compactly supported in x for any fixed t.

Remark 1.12. The matrix $P(\tau,\xi)$ is equal to the matrix $\tau^2 I - A(\xi)$, where I is the identity matrix and $A(\xi)$is the matrix defined in (1.3.12).

Remark 1.13. An easy computation shows that the characteristic polynomial has the form

$$\begin{aligned} p(\tau,\xi) = n_1(\xi)d_2(\tau,\xi)d_3(\tau,\xi) + n_2(\xi)d_3(\tau,\xi)d_1(\tau,\xi) + \\ + n_3(\xi)d_1(\tau,\xi)d_2(\tau,\xi) - d_1(\tau,\xi)d_2(\tau,\xi)d_3(\xi), \end{aligned}$$

where

$$\begin{aligned} n_1(\xi) &= (c_{12}+c_{66})\xi_1^2, \\ n_2(\xi) &= (c_{12}+c_{66})\xi_2^2, \\ n_3(\xi) &= \frac{(c_{13}+c_{44})^2}{c_{12}+c_{66}}\xi_3^2, \end{aligned}$$

and

$$
\begin{aligned}
d_1(\tau,\xi) &= \tau^2 - d_1'(\xi), \quad d_1'(\xi) = c_{11}\xi_1^2 + c_{66}\xi_2^2 + c_{44}\xi_3^2 - (c_{12}+c_{66})\xi_1^2,\\
d_2(\tau,\xi) &= \tau^2 - d_2'(\xi), \quad d_2'(\xi) = c_{66}\xi_1^2 + c_{11}\xi_2^2 + c_{44}\xi_3^2 - (c_{12}+c_{66})\xi_2^2,\\
d_3(\tau,\xi) &= \tau^2 - d_3'(\xi), \quad d_3'(\xi) = c_{44}\xi_1^2 + c_{44}\xi_2^2 + c_{33}\xi_3^2 - \frac{(c_{13}+c_{44})^2}{c_{12}+c_{66}}\xi_3^2.
\end{aligned}
$$

With these notations the characteristic surface is given by $p(\tau,\xi)=0$.
This is often written in the so called "Kelvin's form":

$$
\frac{n_1(\xi)}{d_1(\tau,\xi)} + \frac{n_2(\xi)}{d_2(\tau,\xi)} + \frac{n_3(\xi)}{d_3(\tau,\xi)} = 1. \tag{1.3.13}
$$

Remark 1.14. From the previous two remarks it follows immediately that $p(\tau,\xi)$ is a homogeneous polynomial of degree six. Thus, the condition on hyperbolicity implies that for every fixed $\xi \in \mathbb{R}^3$ the equation $p(\tau,\xi)=0$ has 6 real roots, if multiplicities are counted, and it is obvious that for every fixed $\xi \neq 0$ three of them are positive and three negative. We denote these roots by $\tau_p(\xi)$, $p=1,2,3,4,5,6$, and label them in such a way that $0 < \tau_1(\xi) \le \tau_2(\xi) \le \tau_3(\xi)$, $\tau_4(\xi) = -\tau_1(\xi)$, $\tau_5(\xi) = -\tau_2(\xi)$, $\tau_6(\xi) = -\tau_3(\xi)$.
In particular, the quantities $\tau_p^2(\xi)$, with $p=1,2,3$ are, for each fixed ξ, the eigenvalues of the matrix $A(\xi) = (\sum_{j,l=1}^3 c_{ijkl}\xi_j\xi_l)_{i,k=1,2,3}$.
Finally, we note that the $\tau_p(\xi)$ are roots of a homogeneous polynomial of degree six in ξ and so they are homogeneous of degree one in ξ.

With these assumptions it is easy to find the solution of the Cauchy problem (1.3.10), (1.3.11) explicitly in terms of Fourier integrals involving the Fourier transforms of the initial data. Indeed, by remark (1.11) it makes sense to consider the partial Fourier transform in x of (1.3.6), with t considered as a parameter. Thus, after some calculations (for details see section (3.1) of this thesis, [M], [Du] and [L1]), we obtain the solutions of the Cauchy problem, in the form

$$
u_i(t,x) = \int_{\mathbb{R}^3} \sum_{p=1}^{6} \sum_{j=1}^{3} e^{it\tau_p(\xi)+i\langle x,\xi\rangle} T_{ipj}(\xi)\hat{g}_j(\xi)d\xi, \qquad i=1,2,3, \tag{1.3.14}
$$

where $\hat{g}_j(\xi)$ is the Fourier transform of the initial data g_j and $\tau_p(\xi)$ are the roots of $p(\tau,\xi)$ defined in remark (1.14). Furthermore, the map $\xi \mapsto T_{ipj}(\xi)$ is a measurable function and it is possible to write it explicitly as a function of the square root of the eigenvalues of $A(\xi)$ and of their associated eigenvectors.

Remark 1.15. We observe that the formula (1.3.14) gives a solution for the Cauchy problem of crystal acoustics (1.3.6), (1.3.11), for all crystal classes. So, different functions $\xi \mapsto T_{ipj}(\xi)$ are defined for each crystal class.

Remark 1.16. Since $u_i(0,x) = 0$ and $(\partial/\partial t)u_i(0,x) = g_i(x)$, using (1.3.14) we conclude that T_{ipj} has the following properties

$$\sum_{j=1}^{6} T_{ipj}(\xi) \equiv 0$$

$$\sum_{p=1}^{6} i\tau_p(\xi)T_{ipj}(\xi) \equiv (2\pi)^{-3/2}\delta_{ij}$$

for all i, j, where δ_{ij} is the Kroneker delta.

Moreover, following the method described in [Du] and [L1], it is possible to explicitly write the function $T_{ipj}(\xi)$ as a rational function of the $\tau_p(\xi)$ and of the quantities $n_i(\xi)$, $d_i'(\xi)$, with $i = 1,2,3$, defined in remark (1.13). In particular, in the tetragonal case, for $i,p,j = 1,2,3$ it has the following form

$$T_{ipj} = \frac{(n_i(\xi)n_j(\xi))^{1/2}}{2i\tau_p(\xi)} \cdot \frac{(\tau_p^2(\xi) - d_{j+1}'(\xi))(\tau_p^2(\xi) - d_{j+2}'(\xi))}{(\tau_p^2(\xi) - d_i'(\xi))(\tau_p^2(\xi) - \tau_{p+1}^2(\xi))(\tau_p^2(\xi) - \tau_{p+2}^2(\xi))}, \tag{1.3.15}$$

where $p+1$, $p+2$ and $j+1$, $j+2$ are calculated modulo 3. In addition, it follows from the properties of the $\tau_p(\xi)$, that $T_{i(p+3)j}(\xi) = -T_{ipj}(\xi)$ for $p = 1,2,3$. Now, if we denote

$$\Sigma(\xi) = \prod_{1\le p<k\le 3} (\tau_p(\xi) - \tau_k(\xi))^2,$$

we see that the T_{ipj} may become singular only when $\Sigma(\xi) = 0$. In particular, where $\Sigma(\xi)$ vanishes the system has characteristics with changing multiplicity and the functions $\xi \mapsto \tau_j(\xi)$ become singular.

Expression (1.3.15) is a little bit involved and we will not prove the properties of the function T_{ipj} starting from it, even if it is possible. In section (3.1), we will write T_{ipj} explicitly in another, simpler form and we will deduce from that expression its regularity properties.

The aim of this thesis is to prove the existence of a global solution for the non-linear system of crystal acoustic for tetragonal crystals. To do this, the decay estimate of the solution of the linear system play a key role. Thus, we are interested in the study of the decay for long time of the integral (1.3.14). So, we will have to study the regularity and give an estimate of the quantity $T_{ipj}(\xi)$, when $\Sigma(\xi)$ vanishes, which compare in the amplitude function of the oscillatory integral (1.3.14). Moreover, to obtain a decay estimate for the solution of the system we will take into account the oscillatory character of the exponential $\exp[i\tau_p(\xi) + i < x, \xi >]$, when (t, x) tends to infinity (we will show that, if t remains bounded, it's easy to find the desired estimate for the integral) and we will reduce the estimates of integrals as in (1.3.14) to integrals on the characteristic manifold intersected with the plane $\tau = 1$. Thus, we are interested in examining the geometrical structure of the characteristic manifold. The next chapter will be devoted to this.

Chapter 2

Geometrical properties of the slowness surface

In this chapter we will study the geometrical properties of the characteristic surface of the system of crystal acoustic for the tetragonal crystals. In particular, we will find where the singular points of the surface are located and we will describe their geometrical features. We will distinguish three different types of singular points: the so called *uniplanar*, *biplanar* and *conical* points. Moreover, we will give some necessary conditions on the stiffness constants c_{ij} for hyperbolicity. Finally, we will study the curvature properties of the characteristic surface near its singular points and also away from them.
Using the notation of the previous chapter, we give the following definition.

Definition 2.1. *The surface S, defined by the condition $p(\xi) = 0$, where $p(\xi) = p(1, \xi)$, is called the* slowness surface *of the crystal.*
Moreover, we define:

$$d_1(\xi) = d_1(1, \xi), \quad d_2(\xi) = d_2(1, \xi), \quad d_3(\xi) = d_3(1, \xi).$$

We observe that the slowness surface is essentially the intersection of the characteristic surface with the plane $\tau = 1$.
As in the case of the characteristic manifold we say that the equation which

define S the slowness surface is written in Kelvin's form if the equation $p(\xi) = 0$ is in the form

$$\frac{n_1(\xi)}{d_1(\xi)} + \frac{n_2(\xi)}{d_2(\xi)} + \frac{n_3(\xi)}{d_3(\xi)} = 1.$$

First of all we want to find the double roots of $p(\xi)$ and we want to give some conditions on the stiffness constant in order to avoid triple roots. The way in which we do this begin with the following proposition.

Proposition 2.1. *Consider constants $n_i > 0, d_i, i = 1, \ldots, k$ and denote by p the polynomial*

$$p(t) = \sum_{j=1}^{k} n_j (t - d_{j+1})(t - d_{j+2}) \cdots (t - d_{j+k}) - \prod_{j=1}^{k} (t - d_j), \qquad (2.0.1)$$

with indices calculated modulo k. If the d_j are all mutually distinct then the roots of the polynomial p are all simple and positive. Moreover, if we denote the roots by t_i and label them in such a way that $t_i < t_{i+1}$, and if σ is a permutation of the set $\{1, 2, \ldots, k\}$ such that $d_{\sigma(i)} \leq d_{\sigma(i+1)}$, then we have

$$d_{\sigma(1)} < t_1 < d_{\sigma(2)} < t_2 < d_{\sigma(3)} < \cdots < d_{\sigma(k)} < t_k.$$

It follows that if p is a polynomial of form (2.0.1) *for constants $n_i \geq 0, d_i \geq 0$, then it can have multiple roots only if some of the n_i vanish or if some of the d_j coincide. Further, if $d_j = d_\ell$ for two indices j and ℓ, then $t = d_j$ is a root of p, but if all other d_i are distinct and if all $n_i > 0$, then the roots are still simple. Double roots can therefore only appear if $d_j = d_\ell = d_s$ for three indices j, ℓ, s. In this case $t = d_j$ is a double root.*

Proof. We write $p(t) = 0$ in Kelvin's form $g(t) = 1$, where

$$g(t) = \sum_{j=1}^{k} \frac{n_j}{t - d_j}. \qquad (2.0.2)$$

In the interval $(-\infty, d_{\sigma(1)})$, g is negative and vanishes at $-\infty$, in the interval $(d_{\sigma(k)}, \infty)$, it is positive, vanishes at $+\infty$ and tends to $+\infty$ when t tends to

$d_{\sigma(k)}$. Finally, in any interval of type $(d_{\sigma(i)}, d_{\sigma(i+1)})$, g tends to $+\infty$ when $t \to d_{\sigma(i)}$ and tends to $-\infty$ when $t \to d_{\sigma(i+1)}$. The graph of g (see figure 2.1) in the (t,s)-plane must therefore intersect the line $s = 1$ k times, once in each of the intervals $(d_{\sigma(i)}, d_{\sigma(i+1)})$, $i = 1, \ldots, k-1$, $(d_{\sigma(k)}, \infty)$. This concludes the proof of the first part of the proposition.
The remaining statements are proved in a similar way. □

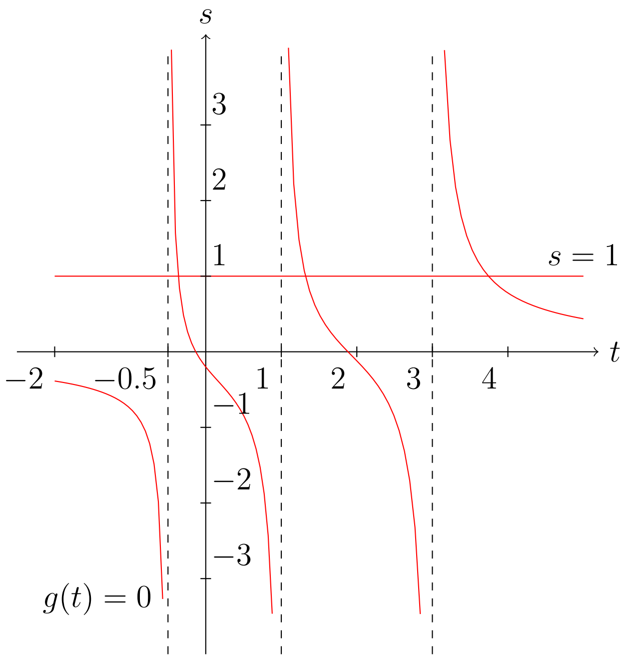


Figure 2.1: Graph of the function (2.0.2), with $k = 3$, $n_i = i/5$, $d_1 = -1/2$, $d_2 = 1$ and $d_3 = 3$.

Now we return to the polynomial in the Kelvin's form, which define the slowness surface S of a tetragonal crystal. It follows from the previous result that, if $n_i(\xi) \geq 0$ and $d'(\xi) \geq 0$ for all $\xi \in \mathbb{R}^3$, we can have a double root at $\tilde{\xi} \in \mathbb{R}^3$ only when

$$d'_1(\tilde{\xi}) = d'_2(\tilde{\xi}) = d'_3(\tilde{\xi}), \tag{2.0.3}$$

or when

$$n_1(\tilde{\xi}) n_2(\tilde{\xi}) n_3(\tilde{\xi}) = 0. \tag{2.0.4}$$

Note that, with the previous assumptions, the condition $n_1(\tilde{\xi})n_2(\tilde{\xi})n_3(\tilde{\xi}) = 0$ means that we are on a coordinate plane.
Now we observe that the points of intersection between the slowness surface and the axes are

$$(\pm\frac{1}{\sqrt{c_{66}}}, 0, 0), (\pm\frac{1}{\sqrt{c_{44}}}, 0, 0), (\pm\frac{1}{\sqrt{c_{11}}}, 0, 0), \tag{2.0.5}$$

$$(0, \pm\frac{1}{\sqrt{c_{66}}}, 0), (0, \pm\frac{1}{\sqrt{c_{44}}}, 0), (0, \pm\frac{1}{\sqrt{c_{11}}}, 0), \tag{2.0.6}$$

$$(0, 0, \pm\frac{1}{\sqrt{c_{44}}}), (0, 0, \pm\frac{1}{\sqrt{c_{44}}}), (0, 0, \pm\frac{1}{\sqrt{c_{33}}}). \tag{2.0.7}$$

This gives the geometrical interpretation of the quantities c_{ii}.
Also note that it follows from these expressions that we always have double roots on the ξ_3-axis. On the other hand, we have double roots on the other two axes only when we have $c_{66} = c_{44}$, $c_{66} = c_{11}$, or $c_{11} = c_{44}$.
So, considering this, the previous proposition and the condition on hyperbolicity of the system, we assume the following conditions on the stiffness constants

$$c_{ii} > 0, \text{ for } i = 1, 3, 4, 6, \quad c_{66} > c_{12}, \quad c_{44} \neq c_{13}, \tag{2.0.8}$$

$$c_{11} - c_{66} - c_{12} > 0, \quad c_{33} - \frac{(c_{13} + c_{44})^2}{c_{12} + c_{66}} > 0. \tag{2.0.9}$$

Moreover, in order to avoid triple roots on the axes, we assume

$$c_{33} \neq c_{44} \quad \text{and} \quad c_{11}, c_{66}, c_{44} \text{ not all equal.} \tag{2.0.10}$$

Remark 2.1. Here we want to write down explicitly the relation between the stiffness constant in the cubic case and in the tetragonal case. The cubic case is when we have

$$c_{11} = c_{33}, \quad c_{44} = c_{66}, \quad c_{12} = c_{13}.$$

With the conventions for the constants used in [L1] and [L2], the values of the constants a, b, c are: $a = c_{11} - c_{44}$, $b = c_{12} + c_{44}$ and $c = c_{44}$.

Remark 2.2. There is an easy interpretation of the constants c_{ii} in terms of plane waves. We call $u = (u_1, u_2, u_3)$ a plane wave solution of the system of crystal elasticity if we can find $v = (v_1, v_2, v_3) \in \mathbb{R}^3$, (τ, ξ) such that $u(t, x_1, x_2, x_3) = v \exp[it\tau + i\langle x, \xi\rangle]$ is a solution of the system. Here ξ gives the direction of propagation of the wave. When we consider propagation in the direction of the x_1-axis, then $\xi_2 = \xi_3 = 0$, and we can assume that $\xi_1 = 1$. The condition on τ is that it be a solution of $p(\tau, 1, 0, 0) = 0$, and v must be some kind of eigenvector. The solutions of $p(\tau, 1, 0, 0) = 0$ are given by (2.0.5). The c_{ii} are thus related to the speeds of the plane waves in the respective directions. This agrees with the fact that we must assume that the quantities c_{11}, c_{44}, c_{66} are positive. In a similar way we can justify the assumption that c_{33} is positive.

It follows from conditions (2.0.3) and (2.0.4) that S has double points only when we can write the sextic $p(\xi)$ as the product of two homogeneous polynomial of degree two and four respectively. Indeed, if $n_i = 0$ for some $i \in \{1, 2, 3\}$, then

$$p(\xi) = d_i(n_{i+1}d_{i+2} + n_{i+2}d_{i+1} - d_{i+1}d_{i+2}),$$

whereas if $d_1 = d_2 = d_3$, then

$$p(\xi) = d_1^2(n_i + n_{i+1} + n_{i+2} - d_1),$$

where in the previous two equations the indices are counted modulo three. In the next section we will study quartics on the plane of the type which appears in the factorization of $p(\xi)$.

2.1 Double roots of special quartics in the plane

In this section we will make some elementary remarks on plane quartics of form $\tilde{q}(x, y) = 0$, where $\tilde{q}$ is given by

$$\tilde{q}(x,y) = a_1x^4 + a_2y^4 + a_3x^2y^2 + b_1x^2 + b_2y^2 + c_1, \tag{2.1.1}$$

for some constants a_i, b_j, c_1 with $a_1 > 0$, $a_2 > 0$, $c_1 \neq 0$. Special emphasis will be put on the case $a_1 = a_2$, $b_1 = b_2$, when $\tilde{q} = 0$ can be reduced (after dividing by a_1 and with obvious new constants) to the form

$$q(x,y) = (x^2 + y^2)^2 + ax^2y^2 + b(x^2 + y^2) + c, \tag{2.1.2}$$

where $c \neq 0$.
Plane sections with coordinate planes of the slowness surfaces for tetragonal crystals which we consider in this thesis will have these form and in particular we need the results we mention here to study the singular points of our surfaces on the coordinate planes.

Definition 2.2. *A quartic on the plane is called of* slowness type *if each ray starting from the origin has exactly (when counted with multiplicities) two intersection points with the quartic. (Note that this means that each line which passes through the origin intersects the quartic in four points, which is the maximum possible number.)*

First, we want to see what restrictions we have to impose on the constants a_i , b_j and c_1 to make sure that $\tilde{q}$ is of the slowness type.

Proposition 2.2. *Let $\tilde{q}(x,y)$ be a quartic of the form* (2.1.1)*, if $\tilde{q}(x,y)$ is of the slowness type then the following conditions must be satisfied*

$$b_1 < 0, \quad b_2 < 0, \quad c_1 > 0, \tag{2.1.3}$$

$$b_1^2 - 4a_1c_1 \geq 0, \quad b_2^2 - 4a_2c_1 \geq 0. \tag{2.1.4}$$

Proof. Due to symmetry, it suffices to check the proposition in the first quadrant $(x > 0, y > 0)$. The problem is simplified if we make the change of variables $(x,y) \to (s,t)$, $s = x^2$, $t = y^2$. $\tilde{q}$ then transform to $\hat{q}(s,t) = a_1s^2 + a_2t^2 + a_3st + b_1s + b_2t + c_1$.

Suppose that $\tilde{q}$ is of the slowness type. Then $\hat{q}(s,t) = 0$ must intersect the positive half-axes in two points. If we restrict $\hat{q}$ on $x = 0$ and $y = 0$, we can easily find, assuming $a_1 > 0, a_2 > 0$, that the b_i, and c_1 must be positive and the conditions $b_1^2 - 4a_1c_1 \geq 0$, $b_2^2 - 4a_2c_1 \geq 0$ must be fulfilled.

□

Remark 2.3. Let $\hat{q}(s,t)$ be as in the proof of the previous proposition. If we assume that $\tilde{q}$ is of the slowness type, then we can have three cases. In the first case $\hat{q}(s,t)$ is an ellipse which intersects each positive half-axis in two points, i.e. $a_3^2 - 4a_1a_2 < 0$ and (2.1.3), (2.1.4) hold.
In the second case $\hat{q}(s,t)$ is a parabola which intersects each positive half-axis in two points, i.e. $a_3^2 - 4a_1a_2 = 0$ and (2.1.3), (2.1.4) hold.
In the third case $\hat{q}(s,t)$ is a hyperbola (or a degenerate hyperbola, i.e. the union of two lines) such that it intersects each positive half-axis in two points. In this case $a_3^2 - 4a_1a_2 > 0$ and (2.1.3), (2.1.4) hold.
Finally, we observe that in the first and second case, the necessary conditions for $\tilde{q}$ to be of the slowness type are also sufficient, whereas in the third case they are only necessary. Indeed, it is possible to find a hyperbola with the center in the first quadrant such that each branch intersects one of the positive semi-axis twice. Thus the equation $\hat{q} = 0$ associated with this hyperbola fulfills the conditions $a_3^2 - 4a_1a_2 > 0$, (2.1.3) and (2.1.4), but the associated quartic $\tilde{q}$ is not of the slowness type.

Our main concern with quartics as above is with double points. First of all we prove the following lemma.

Lemma 2.3. *Let $\tilde{q}(x,y)$ be a quartic of the form* (2.1.1)*, then $\tilde{q}(x,y)$ has double points if and only if it is the product of two factors of degree two.*

Proof. It is a simple and classical result (of G.Maclaurin, 1720), that the number of double points for a non degenerate quartic in two variables can be at most 3. (See [En], tome III, page 283 and [Sa].) Due to the symmetries our quartics have, double points, if at all present, must come in multiples of four. It follows that we can have double points only when they are degenerate. By degenerate we mean that $\tilde{q}$ is the product of two lower order factors. Since we are only interested in the case when the quartic is of the slowness type, and therefore in particular bounded, we have in the degenerate case exactly two factors of degree 2 and the quartic is the union of two ellipses defined by these factors. $\square$

Example 2.1. *To give an example when the situation in lemma* (2.3) *effectively occurs, assume that we are given two constants* $\alpha > 0, \beta > 0, \alpha \neq \beta$ *and consider the polynomial* $(\alpha x^2 + \beta y^2 - 1)(\beta x^2 + \alpha y^2 - 1)$. *The set where the polynomial vanishes is then the union of two ellipses which intersect at* $x^2 = y^2 = (\alpha + \beta)^{-1}$.

Starting from MacLaurin's result, it is not difficult to give explicit conditions on the coefficients of $\tilde{q}$ when it has real double points.

Proposition 2.4. *Let* $\tilde{q}(x, y)$ *be a quartic of the form* (2.1.1) *of the slowness type. If* $\tilde{q}(x, y)$ *has double points, these must lie on the axis, or else we have the following conditions*

$$a_3^2 - 4a_1a_2 > 0, \tag{2.1.5}$$

$$2b_2a_1 - b_1a_3 \geq 0 \tag{2.1.6}$$

$$2b_1a_2 - b_2a_3 \geq 0, \tag{2.1.7}$$

$$(a_3b_1 - 2b_2a_1)^2 = (b_1^2 - 4c_1a_1)(a_3^2 - 4a_2a_1) \tag{2.1.8}$$

with $(2b_2a_1 - b_1a_3)(2b_1a_2 - b_2a_3) \neq 0$. *In this case* $\tilde{q}(x, y)$ *has the following form:*

$$\begin{aligned}\tilde{q}(x, y) = &\left[a_1x^2 - \frac{1}{2}(-a_3y^2 - b_1 + \sqrt{a_3^2 - 4a_2a_1}(y^2 + \frac{a_3b_1 - 2a_1b_2}{a_3^2 - 4a_1a_2}))\right] \\ &\left[a_1x^2 - \frac{1}{2}(-a_3y^2 - b_1 - \sqrt{a_3^2 - 4a_2a_1}(y^2 + \frac{a_3b_1 - 2a_1b_2}{a_3^2 - 4a_1a_2}))\right].\end{aligned}$$

Proof. We assume that $\{(x, y) \in \mathbb{R}^2; \tilde{q}(x, y) = 0\}$ is the union of two ellipses which intersect. We denote x^2 by s and y^2 by t. Dividing by a_1 and after a re-notation for the constants we write $\tilde{q}(x, y) = 0$ in the variables (s, t) as

$$s^2 + (at + b)s + ct^2 + dt + e = 0. \tag{2.1.9}$$

This equation will have (one or two) positive solutions in s for any given $t > 0$ if the following two conditions are simultaneously satisfied (for t):

i) $at + b \leq 0$ or $\sqrt{D(t)} \geq (at + b) > 0$

ii) $D(t) = (at + b)^2 - 4(ct^2 + dt + e) \geq 0$.

In order to obtain a double real root we need of course i) and that $D(t)$ vanishes. Assume that this happens at some positive $t^0 > 0$ and denote $y^0 = (t^0)^2$. Since our quartic is the union of two ellipses, $D(t)$ must then be nonnegative in the interval $[0, t^0]$. Moreover, the graph of the function $t \to s = D(t)$ is a parabola which lies for $t \in [0, t^0]$ on the upper half plane $s \geq 0$. Two situations could now arise in principle:

a) $D(t)$ has negative values to the right of t^0. In this case, our quartic must lie completely in the region $\{(x, y) \in \mathbb{R}^2; y \in [-y^0, y^0]\}$. This is a case in which the double points occur precisely when $y = y^0$ and are easy to find since this situation corresponds to a double root on the y axis.

b) The parabola has a minimum at $t = t^0$ and the values of $D(t)$ are everywhere else strictly positive. In particular, we must have $D(t) = \delta^2(t - t^0)^2$ for some real constant $\delta \neq 0$ and the values of δ, t^0 are respectively

$$\delta = \sqrt{a^2 - 4c}, \quad t^0 = -(ab - 2d)/(a^2 - 4c).$$

In particular, $a^2 - 4c$ and t^0 must be positive. This implies that conditions (2.1.5) and (2.1.6) hold. The condition for $D(t)$ to be equal to $\delta^2(t - t^0)^2$ is

$$\delta^2(t^0)^2 = \frac{(ab - 2d)^2}{(a^2 - 4c)} = b^2 - 4e. \tag{2.1.10}$$

If this is satisfied we can write $s^2 + (at + b)s + ct^2 + dt + e$ as $[s - (1/2)(-at - b + \delta(t - t^0))][s - (1/2)(-at - b - \delta(t - t^0))]$.

Now, if $at^0 - b \leq 0$, then (2.1.9) has positive solutions. This condition implies

$2bc - da > 0$, i.e. (2.1.7). Back in the variables (x, y), the condition (2.1.10) is exactly (2.1.8) and the form of $\tilde{q}$ is then

$$\tilde{q}(x,y) = [x^2 - (1/2)(-ay^2 - b + \delta(y^2 - t^0))][x^2 - (1/2)(-ay^2 - b - \delta(y^2 - t^0))].$$

which is precisely the form we were looking for. $\square$

Remark 2.4. From the expression of $\tilde{q}(x, y)$ follows that it has four double points of the form $(\pm x, \pm y)$, where

$$y = \sqrt{t^0} = \sqrt{-\frac{ab - 2d}{a^2 - 4ac}}$$
$$x = \sqrt{\frac{2bc - ad}{a^2 - 4c}}.$$

Here we used the notations of the above proof.

Remark 2.5. We observe that, in case b) of the previous proof, when $D(t) > 0$ for all $t \geq 0$, $t \neq t_0$, there exist $t_1 > t_0$ such that for all $t > t_1$, the quantity $at + b$ is non negative. In fact, if $at + b \leq 0$ for all t, the quartic is not of the slowness type.

Remark 2.6. Since the expressions we have obtained are not simple, it might be useful to observe that they just say that the leading coefficient of $D(t)$, which is $a^2 - 4c$, is positive and that $D(t)$ has a positive double root.

We will now make some comments concerning quartics of type (2.1.2), since this case is simpler and the conditions are easier to understand. In addition to being biquadratic, the polynomial q is also symmetric in the variables (x, y).

Proposition 2.5. *Let $q(x, y)$ be a quartic of the form* (2.1.2) *of the slowness type. If $q(x, y)$ has double points, these must either lie on an axis or on a diagonal.*

Proof. If $P = (x^0, y^0)$ is a real double root of q, it follows that P must either lie on an axis or on a diagonal: otherwise we get by symmetries 7 additional real double points, which is too many, even in the case when the quartic is the union of two ellipses which intersect. $\square$

A first information on q can be obtained by restricting q to the axis $y = 0$, and to the diagonal $y = x$. We thus obtain the polynomials $q(x, 0) = x^4 + bx^2 + c$ and $q(x, x) = (4 + a)x^4 + 2bx^2 + c$ respectively. Thus, the conditions for these polynomials to have two positive roots are

$$c > 0, \quad b \leq 0, \quad b^2 - 4c \geq 0,$$

$$a + 4 > 0, \qquad b^2 - (a + 4)c \geq 0.$$

The conditions (2.1.13), (2.1.14) are thus necessary conditions for q to be of the slowness type. As will be clear from our discussion, they are also sufficient.

We observe that, when $b^2 - 4c = 0$, then we can write that $q(x, y) = (x^2 + y^2 + \frac{b}{2} - \sqrt{a}xy)(x^2 + y^2 + \frac{b}{2} + \sqrt{a}xy)$, which gives q as the product of two second order polynomials, which define two ellipses.

Further, when $b^2 - (a+4)c = 0$, then we have double roots on the diagonals. To decompose q into the product of two second order factors, we first write $(x^2 + y^2)^2 + ax^2y^2$ as

$$(x^2 + y^2)^2 + ax^2y^2 = (\alpha x^2 + \beta y^2)(\beta x^2 + \alpha y^2), \tag{2.1.11}$$

for suitable constants α, β. This is in fact easy to achieve: if τ is a root of the polynomial $t^2 + (a + 2)t + 1$, e.g.,

$$\tau = \frac{-a - 2 + \sqrt{a^2 + 4a}}{2},$$

then $1/\tau$ is the other root. Now we set $\alpha = \sqrt{-\tau}$ and $\beta = 1/\alpha$ and (2.1.11) is easily checked. This implies the relation

$$(\alpha + \beta)^2 = -\tau - 1/\tau + 2 = a + 4. \tag{2.1.12}$$

If we now set $\gamma = -b/(\alpha + \beta)$, then it follows that

$$(\alpha x^2 + \beta y^2 - \gamma)(\beta x^2 + \alpha y^2 - \gamma) = (x^2 + y^2)^2 + ax^2y^2 + b(x^2 + y^2) + \gamma^2.$$

This is precisely (2.1.2) if we have $\gamma^2 = b^2/(\alpha + \beta)^2 = c$. In view of (2.1.12), $(\alpha + \beta)^2 = a + 4$, so the condition on γ is exactly the one for which $b^2 - (a +$

$4)c = 0$.

We can conclude this section with the following proposition.

Proposition 2.6. *Let $q(x, y)$ be of the form (2.1.2). If $q(x, y)$ is of the slowness type, then the following conditions must hold:*

$$b \leq 0, \quad c > 0, \quad b^2 - 4c \geq 0, \tag{2.1.13}$$

$$a + 4 > 0, \quad b^2 - (a + 4)c \geq 0. \tag{2.1.14}$$

Moreover $q(x, y) = 0$ has double points if and only if either $b^2 - 4c = 0$ or $b^2 - (a + 4)c = 0$.

If $b^2 - 4c = 0$, then $q(x, y)$ has one double point on each axis and it is possible to write it in the following form as the product of two ellipses:

$$q(x, y) = (x^2 + y^2 + \frac{b}{2} - \sqrt{a}xy)(x^2 + y^2 + \frac{b}{2} + \sqrt{a}xy).$$

If $b^2 - (a + 4)c = 0$, than $q(x, y)$ has one double point on each semi-diagonal and it is possible to write it in the following form as the product of two ellipses:

$$q(x, y) = (\alpha x^2 + \beta y^2 - \gamma)(\beta x^2 + \alpha y^2 - \gamma),$$

where $(\alpha + \beta)^2 = a + 4$, $\gamma = -b/(\alpha + \beta)$ and $\gamma^2 = c$.

2.2 Double points of the slowness surface in the coordinate planes

In this section we will study the location of the double points on the sextics which appear when we restrict the slowness surface of a tetragonal crystal to the coordinate planes. If we now restrict to the coordinate plane $\{\xi \in \mathbb{R}^3; \xi_i = 0\}$ for some $i \in \{1, 2, 3\}$, then the terms in $p(\xi)$ which contain $n_i(\xi)$ as a factor vanish, and we obtain the curve

$$\{\xi \in \mathbb{R}^3; \xi_i = 0, d_i = 0\} \cup \{\xi \in \mathbb{R}^3; \xi_i = 0, n_{i+1}d_{i+2} + n_{i+2}d_{i+1} - d_{i+1}d_{i+2} = 0\},$$

with indices calculated modulo 3. Our restriction is thus the union of an ellipse with a bounded quartic. Real double points can appear then in principle in two ways: if we intersect the ellipse with the quartic, or if the quartic itself has double points.
We observe that $p(\xi)$ is completely symmetric in the variable ξ_1 and ξ_2. So, we will only study what happens on the planes $\xi_1 = 0$ and $\xi_3 = 0$.
Our first concern is to understand for which values of the constants c_{ij} we can have double points on the quartic. In fact, as we have already seen in the previous section, the quartic can have double points only if it is the union of two ellipses which intersect.

2.2.1 Double points of the quartic in the coordinate planes

Assume at first that $\xi_1 = 0$. Then the restriction of p to $\xi_1 = 0$ factors into the form $d_1(n_2d_3 + n_3d_2 - d_2d_3)$. This means that $\{(\xi_2, \xi_3); p(0, \xi_2, \xi_3) = 0\}$ is the union of the two curves $\mathcal{C}_1 = \{(\xi_2, \xi_3); d_1(0, \xi_2, \xi_3) = 0\}$ and $\mathcal{C}_2 = \{(\xi_2, \xi_3); (n_2d_3 + n_3d_2 - d_2d_3)(0, \xi_2, \xi_3) = 0\}$. $\mathcal{C}_1$ is the ellipse $s_1(\xi_2, \xi_3) = 0$, where

$$s_1(\xi_2, \xi_3) = 1 - c_{66}\xi_2^2 - c_{44}\xi_3^2, \tag{2.2.1}$$

whereas $\mathcal{C}_2$ is the quartic given by $q_1(\xi_2, \xi_3) = 0$, where

$$\begin{aligned} q_1(\xi_2, \xi_3) = c_{11}c_{44}\xi_2^4 + c_{33}c_{44}\xi_3^4 - (c_{13}^2 - c_{11}c_{33} + 2c_{13}c_{44})\xi_2^2\xi_3^2 \\ - c_{44}\xi_2^2 - c_{11}\xi_2^2 - c_{33}\xi_3^2 - c_{44}\xi_3^2 + 1. \end{aligned} \tag{2.2.2}$$

Following the proof of proposition (2.4) of the previous section we can write $q_1(\xi_2, \xi_3)$ as $X\xi_3^4 + Y(\xi_2)\xi_3^2 + Z(\xi_2)$, where

$$X = c_{44}c_{33},$$

$$Y(\xi_2) = (-2c_{13}c_{44} + c_{11}c_{33} - c_{13}^2)\xi_2^2 - c_{33} - c_{44},$$

$$Z(\xi_2) = c_{11}c_{44}\xi_2^4 - (c_{11} + c_{44})\xi_2^2 + 1.$$

We denote by $D(\xi_2)$ the quantity $Y(\xi_2)^2 - 4XZ(\xi_2)$. We have seen in remark (2.6) that a necessary condition for the quartic (2.2.2) to have double points is that $D(\xi_2)$ have positive double roots and its leading coefficient be positive. After some calculations, we can write $D(\xi_2)$ as

$$D(\xi_2) = A\xi_2^4 + B\xi_2^2 + C$$

with

$$A = (c_{11}c_{33} - 2c_{13}c_{44} - c_{13}^2)^2 - 4c_{44}^2c_{33}c_{11},$$
$$B = -2(c_{33} + c_{44})(c_{11}c_{33} - 2c_{13}c_{44} - c_{13}^2) + 4c_{44}c_{33}(c_{44} + c_{11}),$$
$$C = c_{44}^2 + c_{33}^2 - 2c_{44}c_{33}.$$

We can therefore have double roots only if $D_1 = B^2 - 4AC = 0$. The expression for D_1 is quite long, but it factors conveniently to

$$D_1 = 16c_{44}c_{33}(c_{13} + c_{44})^2(c_{13}^2 + 2c_{13}c_{44} + c_{44}c_{33} - c_{33}c_{11} + c_{44}c_{11}),$$

and so we have $D_1 = 0$ if and only if

$$D_1 = (c_{13}^2 + 2c_{13}c_{44} + c_{44}c_{33} - c_{33}c_{11} + c_{44}c_{11}) = 0. \tag{2.2.3}$$

Note that $c_{33}c_{44}$ is strictly positive and $c_{13} + c_{44} \neq 0$ by conditions (2.0.8). Moreover we have that the double root of $D(\xi_2)$ is positive if and only if

$$B = -2(c_{33} + c_{44})(c_{11}c_{33} - 2c_{13}c_{44} - c_{13}^2) + 4c_{44}c_{33}(c_{44} + c_{11}) \leq 0. \tag{2.2.4}$$

A further condition for the quartic (2.2.2) to have double points is that $A > 0$, i.e.

$$(c_{11}c_{33} - 2c_{13}c_{44} - c_{13}^2)^2 - 4c_{44}^2c_{33}c_{11} > 0. \tag{2.2.5}$$

If we denote by $\bar{\xi}_2$ the double root of D, the last condition for the quartic (2.2.2) to have double points is that $Y(\bar{\xi}_2) \leq 0$, i.e.

$$(c_{11} + c_{44})(c_{11}c_{33} - 2c_{13}c_{44} - c_{13}^2) - 2(c_{33} + c_{44})c_{11}c_{44} \geq 0. \tag{2.2.6}$$

Remark 2.7. We observe that the conditions (2.2.3), (2.2.4), (2.2.5) and (2.2.6) correspond to the conditions (2.1.8), (2.1.6), (2.1.5) and (2.1.7) respectively.

Remark 2.8. Here and in the following we assume another condition on the stiffness constants. This condition comes from physics and numerical examples of stiffness constants for tetragonal crystals agree with it. We assume that c_{12} and c_{13} are small when compared with c_{ii}, for $i = 1, 3, 4, 6$.

With the assumption of the previous remark the conditions (2.2.3), (2.2.4), (2.2.5) and (2.2.6) are reduced to the following

$$c_{44}c_{33} - c_{33}c_{11} + c_{44}c_{11} = 0, \tag{2.2.7}$$

$$c_{11}c_{33} > 4c_{44}^2, \tag{2.2.8}$$

$$c_{11}(c_{33} - c_{44}) > 2c_{44}^2. \tag{2.2.9}$$

$$c_{33}(c_{11} - c_{44}) > 2c_{44}^2. \tag{2.2.10}$$

From (2.2.7) follows that $c_{11} = c_{33}c_{44}/(c_{33} - c_{44})$ and, taking into account this condition, conditions (2.2.8), (2.2.9) and (2.2.10) yield $c_{33} < c_{44}$. But if $c_{33} < c_{44}$, then c_{11} must be negative and so we can conclude that $q_1(\xi_2, \xi_3)$ can have double points only on the axes.

We can now understand whether or not the quartic $q_1 = 0$ can have double points on the axes. We recall from (2.0.7) that the points on the positive ξ_3-axis are

$$(0, 0, \frac{1}{\sqrt{c_{44}}}) \text{ and } (0, 0, \frac{1}{\sqrt{c_{44}}}), (0, 0, \frac{1}{\sqrt{c_{33}}}).$$

Since the first one is a point on the ellipse (2.2.1), it follows that the double point on the positive ξ_3-axis is the result of the fact that the ellipse and the quartic touch. The points on the positive ξ_2-axis are

$$(0, \pm\frac{1}{\sqrt{c_{66}}}, 0), (0, \pm\frac{1}{\sqrt{c_{44}}}, 0), (0, \pm\frac{1}{\sqrt{c_{11}}}, 0).$$

Since the first one is a point on the ellipse (2.2.1), it follows that we have double points on the positive ξ_2-axis when $c_{11} = c_{44}$.

Thus, we have proved the following proposition.

Proposition 2.7. *Let $q_1(\xi_2, \xi_3) = 0$ be the quartic defined by (2.2.2). It has double points if and only if $c_{11} = c_{44}$. In this case $q_1(\xi_2, \xi_3) = 0$ has two double points of coordinates $(0, \pm 1/\sqrt{c_{44}}, 0)$, on the ξ_2-axis.*

We now deal with the restriction to $\xi_3 = 0$. Since the restriction of p to this plane factors into $d_3(n_1 d_2 + n_2 d_1 - d_1 d_2) = 0$, we then have to look at the ellipse

$$s_3(\xi_1, \xi_2) = d_3(\xi_1, \xi_2, 0) = 0 \tag{2.2.11}$$

and the quartic $q_3(\xi_1, \xi_2) = 0$, where

$$\begin{aligned} q_3(\xi_1, \xi_2) = &-c_{12}^2 \xi_1^2 \xi_2^2 + c_{11}^2 \xi_1^2 \xi_2^2 - c_{11}\xi_2^2 - c_{11}\xi_1^2 - c_{66}\xi_2^2 - c_{66}\xi_1^2 \\ &+ c_{11}c_{66}\xi_1^4 - 2c_{12}c_{66}\xi_1^2\xi_2^2 + c_{11}c_{66}\xi_2^4 + 1. \end{aligned} \tag{2.2.12}$$

Our first concern is to understand whether or not the quartic can have double points. We have seen in the previous section that such double points can only lie on the axes or on the diagonals. The double points on the axes are known from the relations (2.0.5)and will exist when $c_{11} = c_{66}$. The points on the positive principal diagonal $\xi_1 = \xi_2$, $\xi_1 \geq 0$ of the quartic are on the other hand

$$\left(\frac{1}{\sqrt{c_{11} - c_{12}}}, \frac{1}{\sqrt{c_{11} - c_{12}}}, 0\right), \quad \left(\frac{1}{\sqrt{c_{11} + c_{12} + 2c_{66}}}, \frac{1}{\sqrt{c_{11} + c_{12} + 2c_{66}}}, 0\right).$$

It follows from this that the quartic has double points only in the case when $c_{11} - c_{12} = c_{11} + c_{12} + 2c_{66}$, i.e., when $c_{12} + c_{66} = 0$. Since we assume that c_{12} is small compared with c_{66}, there will thus be no double points on the quartic and the double points of $p(\xi_1, \xi_2, 0) = 0$ must come from the intersection of the ellipse with the quartic which we will now compute. Thus, we have proved the following proposition.

Proposition 2.8. *Let $q_3(\xi_1, \xi_2) = 0$ be the quartic defined by* (2.2.12)*. It has double points if and only if $c_{11} = c_{66}$. In this case $q_3(\xi_1, \xi_2) = 0$ has four double points, two on the ξ_1-axis, and two on the ξ_2-axis, of coordinates $(\pm 1/\sqrt{c_{66}}, 0, 0)$ and $(0, \pm 1/\sqrt{c_{66}}, 0)$ respectively.*

2.2.2 On the intersection of the ellipses with the quartics in the coordinate planes

As before, we assume at first that $\xi_1 = 0$. We have the following proposition.

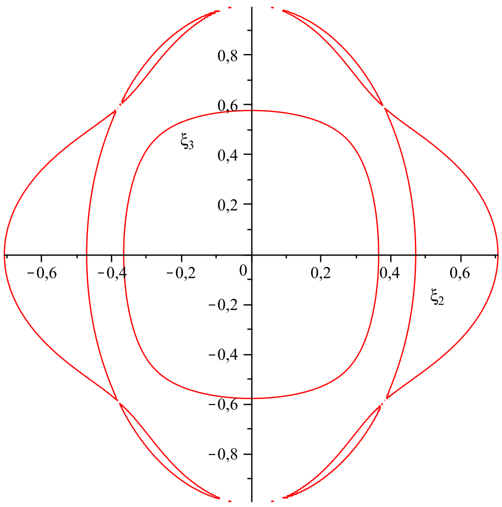
Proposition 2.9. *Let $s_1(\xi_2, \xi_3)$ and $q_1(\xi_2, \xi_3)$ be the polynomials defined in (2.2.1) and (2.2.2) respectively. We denote*

$$\begin{aligned} \tilde{\xi}_2^2 &= \frac{(c_{13}+c_{44})^2 + (c_{44}-c_{33})(c_{11}-c_{66})}{c_{11}c_{44}^2 + 2c_{13}c_{44}c_{66} + c_{13}^2 c_{66} - c_{11}c_{33}c_{66} + c_{33}c_{66}^2}, \\ \tilde{\xi}_3^2 &= \frac{(c_{44}-c_{66})(c_{11}-c_{66})}{c_{11}c_{44}^2 + 2c_{13}c_{44}c_{66} + c_{13}^2 c_{66} - c_{11}c_{33}c_{66} + c_{33}c_{66}^2}. \end{aligned}$$

If the stiffness constants c_{ij} are such that $\tilde{\xi}_2$ and $\tilde{\xi}_3$ are positive and $c_{11} \neq c_{66} \neq c_{44}$, then $s_1(\xi_2, \xi_3) = 0$ intersects $q_1(\xi_2, \xi_3) = 0$ in six points of coordinates:

$$(0, 0, \pm\sqrt{\frac{1}{c_{44}}}), \qquad (0, \pm\tilde{\xi}_2, \pm\tilde{\xi}_3).$$

If the stiffness constants c_{ij} are such that $\tilde{\xi}_2$ and $\tilde{\xi}_3$ are not positive, then $s_1(\xi_2, \xi_3) = 0$ intersects $q_1(\xi_2, \xi_3) = 0$ only in the points $(0, 0, \pm\sqrt{\frac{1}{c_{44}}})$.



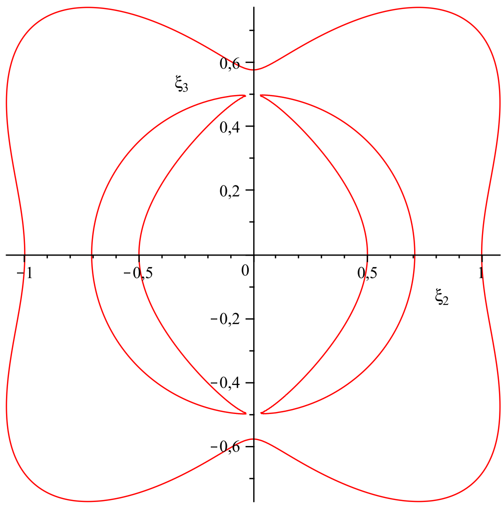


Figure 2.2: Restrictions of S on the plane $\xi_1 = 0$ with $(c_{11}, c_{33}, c_{44}, c_{66}, c_{12}, c_{13})$ equal to $(4, 3, 1, 2, -1/2, 1/5)$ and $(1, 3, 4, 2, -1/2, 1/5)$ respectively.

Proof. We denote $P = (0, \xi_2, \xi_3)$. $P' = (\xi_2, \xi_3)$ then corresponds to an intersection between $s_1 = 0$ and $q_1 = 0$ if we have simultaneously

$$d_1(P) = 0, \quad (n_2 d_3 + n_3 d_2 - d_2 d_3)(P) = 0,$$

with the usual notation. The condition $d_1(P) = 0$ means that $P' = (\xi_2, \xi_3)$ lies on the ellipse

$$1 - c_{66}\xi_2^2 - c_{44}\xi_3^2 = 0,$$

which gives

$$\xi_3^2 = g(\xi_2) = \frac{1 - c_{66}\xi_2^2}{c_{44}}. \tag{2.2.13}$$

We have to insert the value of ξ_3^2 given in (2.2.13) into the equation $(n_2d_3 + n_3d_2 - d_2d_3)(P) = 0$, and to solve the resulting equation for ξ_2. Calculations are simplified if we make the following preliminary remarks: the values of d_2, $n_2 - d_2$, and d_3, for $\xi_1 = 0$ and ξ_3 given by (2.2.13) are

$$d_2 = (-c_{11} + 2c_{66} + c_{12})\xi_2^2,$$

$$n_2 - d_2 = (c_{11} - c_{66})\xi_2^2,$$

$$d_3 = \frac{c_{12} + c_{66}}{c_{13} + c_{44}} - \frac{c_{12} + c_{66}}{c_{13} + c_{44}}(c_{44}\xi_2^2 + \frac{c_{33}}{c_{44}}(1 - c_{66}\xi_2^2)) + \frac{c_{13} + c_{44}}{c_{44}}(1 - c_{66}\xi_2^2).$$

After some calculations, it follows that $[n_3d_2 + d_3(n_2 - d_2)](0, \xi_2, g(\xi_2))$, is divisible by ξ_2^2 and that we have

$$\begin{aligned}
\frac{[n_3d_2 + d_3(n_2 - d_2)](0, \xi_2, g(\xi_2))}{\xi_2^2} &= \\
&= \frac{(c_{13} + c_{44})\,(1 - c_{66}\xi_2^2)\,(-c_{11} + 2c_{66} + c_{12})}{c_{44}} \\
&+ \frac{c_{12} + c_{66}}{c_{13} + c_{44}}\frac{1}{c_{44}}\left(c_{44} - \left(c_{44}^2\xi_2^2 + c_{33}\left(1 - c_{66}\xi_2^2\right)\right)\right. \\
&\qquad \left. + (c_{13} + c_{44})\left(1 - c_{66}\xi_2^2\right)\right)(c_{11} - c_{66})\,.
\end{aligned}$$

In particular, we see that $\xi_2 = 0$ is a solution of $[n_3d_2 + d_3(n_2 - d_2)](0, \xi_2, g(\xi_2)) = 0$ with multiplicity 2. When $\xi_1 = \xi_2 = 0$, the value of ξ_3^2 for which we have the intersection is $1/c_{44}$. Thus, the first part of the proposition is proved. The other solutions of $[n_3d_2 + d_3(n_2 - d_2)](0, \xi_2, g(\xi_2)) = 0$ are also easy to calculate, since $[n_3d_2 + d_3(n_2 - d_2)](0, \xi_2, g(\xi_2))/\xi_2^2$ is linear in the variable $s = \xi_2^2$. We obtain

$$\begin{aligned}
\xi_2^2 &= \frac{c_{33}c_{66} - c_{44}c_{66} + 2c_{13}c_{44} - c_{11}c_{33} + c_{13}^2 + c_{44}^2 + c_{11}c_{44}}{c_{11}c_{44}^2 + 2\,c_{13}c_{44}c_{66} + c_{13}^2c_{66} - c_{11}c_{33}c_{66} + c_{33}c_{66}^2} \\
&= \frac{(c_{13} + c_{44})^2 + (c_{44} - c_{33})(c_{11} - c_{66})}{c_{11}c_{44}^2 + 2c_{13}c_{44}c_{66} + c_{13}^2c_{66} - c_{11}c_{33}c_{66} + c_{33}c_{66}^2}.
\end{aligned} \tag{2.2.14}$$

Inserting this in $s_1(\xi_2, \xi_3)$, we obtain the value of ξ_3 corresponding to ξ_2 given by (2.2.14):

$$\xi_3^2 = \frac{(c_{44} - c_{66})(c_{11} - c_{66})}{c_{11}c_{44}^2 + 2c_{13}c_{44}c_{66} + c_{13}^2 c_{66} - c_{11}c_{33}c_{66} + c_{33}c_{66}^2}.$$

□

We now deal with the restriction to $\xi_3 = 0$.

Proposition 2.10. *Let $s_3(\xi_1, \xi_2)$ and $q_3(\xi_1, \xi_2)$ be the polynomials defined in* (2.2.11) *and* (2.2.12) *respectively. We denote*

$$R = \frac{(c_{12} + 2c_{44} - c_{11})(c_{12} + 2c_{66} - 2c_{44} + c_{11})}{(c_{12} + c_{11})(c_{12} - c_{11} + 2c_{66})}.$$

If the stiffness constants c_{ij} are such that $0 < R < 1$ and $c_{11} \neq c_{44} \neq c_{66}$, then $s_3(\xi_1, \xi_2)$ intersects $q_3(\xi_1, \xi_2)$ in eight points of coordinates

$$(\pm(\frac{1+\sqrt{R}}{2c_{44}}))^{1/2}, \pm(\frac{1-\sqrt{R}}{2c_{44}})^{1/2}, 0),$$
$$(\pm(\frac{1-\sqrt{R}}{2c_{44}}))^{1/2}, \pm(\frac{1+\sqrt{R}}{2c_{44}})^{1/2}, 0).$$

Proof. We proceed as in the proof of proposition (2.9). We denote $P = (\xi_1, \xi_2, 0)$. $P' = (\xi_1, \xi_2)$ then corresponds to an intersection between $s_1 = 0$ and $q_1 = 0$ if we have simultaneously

$$d_3(P) = 0, \quad (n_1 d_2 + n_2 d_1 - d_1 d_2)(P) = 0.$$

The condition $d_3(P) = 0$ means that $P' = (\xi_1, \xi_2)$ lies on the circle

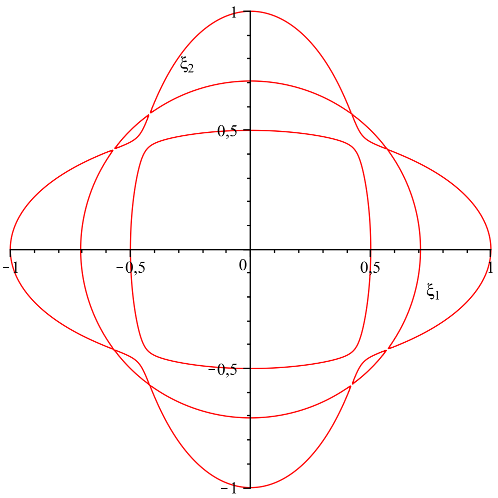
$$1 - c_{44}\xi_1^2 - c_{44}\xi_2^2 = 0,$$

which gives

$$\xi_2^2 = \frac{1}{c_{44}} - \xi_1^2. \tag{2.2.15}$$

Inserting the value of ξ_2^2 given by (2.2.15) into the equation $(n_1 d_2 + n_2 d_1 - d_1 d_2)(P) = 0$, we obtain the following expression for ξ_1^2:

$$\xi_1^2 = \frac{B \pm \sqrt{D}}{2A},$$



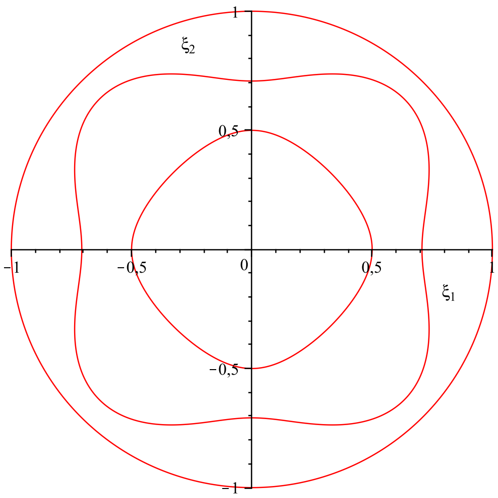


Figure 2.3: Restrictions of S on the plane $\xi_3 = 0$ with $(c_{11}, c_{33}, c_{44}, c_{66}, c_{12}, c_{13})$ equal to $(2, 3, 1, 4, -1/2, 1/5)$ and $(4, 3, 2, 1, -1/2, 1/5)$ respectively.

where

$$A = (c_{12} + c_{11})\,(c_{12} - c_{11} + 2\,c_{66})\,c_{44},$$

$$B = (c_{12} + c_{11})\,(c_{12} - c_{11} + 2\,c_{66})\,,$$

$$D = (c_{12} + c_{11})\,(c_{12} + 2\,c_{44} - c_{11})\,(c_{12} - c_{11} + 2\,c_{66})\,(c_{12} + 2\,c_{66} - 2\,c_{44} + c_{11})\,.$$

We conclude that

$$\xi_1 = \pm \left(\frac{1}{2c_{44}} \left(1 \pm \sqrt{\frac{(c_{12} + 2c_{44} - c_{11})\,(c_{12} + 2c_{66} - 2c_{44} + c_{11})}{(c_{12} + c_{11})\,(c_{12} - c_{11} + 2c_{66})}} \right) \right)^{1/2}.$$

Finally, the value of ξ_2 is obtained by inserting the value of ξ_1 into (2.2.15).

□

Remark 2.9. We have seen that, if $c_{11} = c_{44}$, then the quartic $q_1(\xi_2, \xi_3)$ has two double points on the coordinate ξ_2-axis and we note that the ellipse $e_3(\xi_1, \xi_2)$ intersects the quartic $q_3(\xi_1, \xi_2)$ in the coordinate axes. Similarly, if $c_{11} = c_{66}$, then the quartic $q_3(\xi_1, \xi_2)$ has four double points on the coordinate axes and the ellipse $e_1(\xi_2, \xi_3)$ intersects the quartic $q_1(\xi_2, \xi_3)$ in the coordinate ξ_2-axis.

Moreover, we note that, if $c_{44} = c_{66}$, the quartics $q_1(\xi_2, \xi_3)$ and $q_3(\xi_1, \xi_2)$ do not have double points, but the ellipses $e_1(\xi_2, \xi_3)$ and $e_3(\xi_1, \xi_2)$ intersect the quartics $q_1(\xi_2, \xi_3)$ and $q_3(\xi_1, \xi_2)$ respectively, on the ξ_1-axis and on the ξ_3-axis.

2.3 Double points of the slowness surface near the diagonal

In this section we will study the location of double points of the slowness surface of a tetragonal crystal, which do not lie in the coordinate planes. As shown above, singular points $\tilde{\xi}$ which do not lie in the coordinate planes can only occur when $d_1(\tilde{\xi}) = d_2(\tilde{\xi}) = d_3(\tilde{\xi})$. Now, suppose that this condition holds and $d_i(\tilde{\xi}) \neq 0$, then $\tilde{\xi}$ must be a double point of $(n_1+n_2+n_3-d_1)(\tilde{\xi}) = 0$, but this is absurd because, given our assumptions on c_{ij}, $(n_1 + n_2 + n_3 - d_1)(\tilde{\xi}) = 0$ is an ellipse in $\mathbb{R}^3$. Conversely, if we know that

$$d_1(\tilde{\xi}) = d_2(\tilde{\xi}) = d_3(\tilde{\xi}) = 0,$$

for some point $\tilde{\xi}$, then $\tilde{\xi}$ is a double point of S. Thus we have the following lemma.

Lemma 2.11. *Singular points $\tilde{\xi}$ of the slowness surface, which do not lie on the coordinate planes, can occur if and only if $d_1(\tilde{\xi}) = d_2(\tilde{\xi}) = d_3(\tilde{\xi}) = 0$.*

A first remark is that $d_1(\xi) = d_2(\xi)$ implies $\xi_1^2 = \xi_2^2$. Inserting this information into $d_1(\xi) = d_3(\xi)$ shows that ξ_1^2 and ξ_3^2 must be related by the condition

$$\xi_3^2 = \frac{(-c_{11} + c_{12} + 2c_{44})(c_{12} + c_{66})}{(c_{13} + c_{44})^2 + (c_{12} + c_{66})(c_{44} - c_{33})}\xi_1^2. \tag{2.3.1}$$

Using $\xi_1^2 = \xi_2^2$ and (2.3.1), $p(\xi) = 0$ reduces to a third-degree polynomial in $t = \xi_1^2$, which will have a double root.

Solving this equation we then obtain the following value for ξ_1^2

$$\xi_1^2 = -\frac{(c_{13} + c_{44})^2 + (c_{12} + c_{66})(c_{44} - c_{33})}{(c_{13} + c_{44})^2(c_{12} - c_{11}) + (c_{12} + c_{66})(c_{33}c_{11} - c_{12}c_{33} - 2c_{44}^2)}. \tag{2.3.2}$$

This gives the following value for ξ_3^2

$$\xi_3^2 = -\frac{(2c_{44}+c_{12}-c_{11})(c_{12}+c_{66})}{(c_{13}+c_{44})^2(c_{12}-c_{11})+(c_{12}+c_{66})(c_{33}c_{11}-c_{12}c_{33}-2c_{44}^2)}.$$

Thus we have the following proposition.

Proposition 2.12. *Let S be the slowness surface for the tetragonal crystal system. We denote*

$$\tilde{\xi}_1^2 = -\frac{(c_{13}+c_{44})^2+(c_{12}+c_{66})(c_{44}-c_{33})}{(c_{13}+c_{44})^2(c_{12}-c_{11})+(c_{12}+c_{66})(c_{33}c_{11}-c_{12}c_{33}-2c_{44}^2)},$$
$$\tilde{\xi}_3^2 = -\frac{(2c_{44}+c_{12}-c_{11})(c_{12}+c_{66})}{(c_{13}+c_{44})^2(c_{12}-c_{11})+(c_{12}+c_{66})(c_{33}c_{11}-c_{12}c_{33}-2c_{44}^2)}.$$

If the stiffness constants c_{ij} are such that $\tilde{\xi}_1^2$ and $\tilde{\xi}_3^2$ are positive, then S has eight double points, four on each plane $\{(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3 : \xi_1 = \pm\xi_2\}$, of coordinates:

$$(\tilde{\xi}_1,\tilde{\xi}_1,\pm\tilde{\xi}_3), \qquad (-\tilde{\xi}_1,-\tilde{\xi}_1,\pm\tilde{\xi}_3).$$
$$(\tilde{\xi}_1,-\tilde{\xi}_1,\pm\tilde{\xi}_3), \qquad (-\tilde{\xi}_1,\tilde{\xi}_1,\pm\tilde{\xi}_3).$$

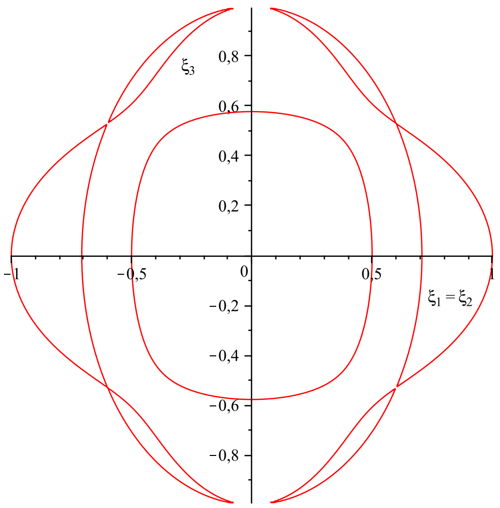


Figure 2.4: Restriction of S on the plane $\xi_1 = \xi_2$ with $(c_{11}, c_{33}, c_{44}, c_{66}, c_{12}, c_{13})$ equal to $(4, 3, 1, 2, -1/2, 1/5)$.

Remark 2.10. In the case of cubic crystals the condition $d_1 = d_2 = d_3$ implies that we must have $\xi_1^2 = \xi_2^2 = \xi_3^2$. So, if we call the eight lines defined by these conditions the space diagonals, in the cubic case we have eight double points, one on each space diagonal. Denote $F = (-c_{11} + c_{12} + 2c_{44})(c_{12} + c_{66}) - (c_{13} + c_{44})^2 - (c_{12} + c_{66})(c_{44} - c_{33})$. Then, in the tetragonal case, we have double points on the space diagonals if $F = 0$. Further, we can decompose F as

$$F = (-c_{11} + c_{12} + c_{44} + c_{33})(c_{12} + c_{66} - c_{13} - c_{44}) \\ + (c_{13} + c_{44})(-c_{13} - c_{11} + c_{12} + c_{33}).$$

It follows in particular that $F = 0$ if $c_{12} + c_{66} - c_{13} - c_{44} = 0$ and $-c_{13} - c_{11} + c_{12} + c_{33} = 0$ simultaneously . To put the conditions into a symmetric form we can also write them as

$$c_{66} - c_{44} = c_{13} - c_{12} = c_{33} - c_{11}. \tag{2.3.3}$$

Note however that these conditions are only sufficient to guarantee that the double points lie on the diagonals. The nice thing about the conditions in (2.3.3) is that the three quantities $c_{66} - c_{44}$, $c_{13} - c_{12}$, $c_{33} - c_{11}$ measure the "distance" to the cubic case. These conditions therefore say that the three quantities which determine this distance are equal, but do not necessarily vanish. Thus, if we are near the cubic case, we can expect the double points of tetragonal crystal on the planes $\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_1 = \pm\xi_2\}$ to be near the space diagonal.
We will explain later on what we mean with "distance to the cubic case" and "near the cubic case" (see e.g. remarks (2.1) and (2.16) and definition (2.4)).

2.4 Tetragonal crystals when $c_{11} = c_{66}$

Here and in the remainder of this section we will assume that the stiffness constants c_{11} and c_{66} are equal. In this case not only do we have some simplifications in the calculations, but also a particular type of double point

appears on the slowness surface. We will call it a "biplanar" double point (see definition (2.3)). It is a type of double point which does not appear in the cubic case, thus, it seems to be interesting to prove a theorem for the decay of oscillatory integrals on surfaces with double points such as this (see section (3.5)).
We begin the study of tetragonal crystal when we have $c_{11} = c_{66}$ with the description of where the double points of the slowness surface are located. The results of the previous two sections yield the following proposition.

Proposition 2.13. *Assume $c_{11} = c_{66}$ and let c_{ij} be such that the conditions (2.0.8), (2.0.9), (2.0.10) are satisfied and c_{ij}, with $i \neq j$, is small compared with c_{ii}. Moreover, let S be the slowness surface for the tetragonal crystal system. Then S has six double points, one on each semi-axis, of coordinates*

$$(\pm\frac{1}{\sqrt{c_{66}}}, 0, 0), \quad (0, \pm\frac{1}{\sqrt{c_{66}}}, 0), \quad (0, 0, \pm\frac{1}{\sqrt{c_{44}}}).$$

In addition, if $(c_{12}+2c_{44}-c_{66})(c_{12}+3c_{66}-2c_{44}) > 0$, then S has eight double points on the plane $\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_3 = 0\}$, of coordinates

$$(\pm(\frac{1+\sqrt{R}}{2c_{44}}))^{1/2}, \pm(\frac{1-\sqrt{R}}{2c_{44}})^{1/2}, 0),$$
$$(\pm(\frac{1-\sqrt{R}}{2c_{44}}))^{1/2}, \pm(\frac{1+\sqrt{R}}{2c_{44}})^{1/2}, 0),$$

where all combinations of signs are allowed and

$$R = \frac{(c_{12}+2c_{44}-c_{66})(c_{12}+3c_{66}-2c_{44})}{(c_{12}+c_{66})^2}.$$

Finally, if $\tilde{\xi}_1 > 0$ and $\tilde{\xi}_3 > 0$, then S has four double points on each of the planes $\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_1^2 = \xi_2^2\}$ of coordinates

$$(\tilde{\xi}_1, \tilde{\xi}_1, \pm\tilde{\xi}_3), \qquad (-\tilde{\xi}_1, -\tilde{\xi}_1, \pm\tilde{\xi}_3),$$
$$(\tilde{\xi}_1, -\tilde{\xi}_1, \pm\tilde{\xi}_3), \qquad (-\tilde{\xi}_1, \tilde{\xi}_1, \pm\tilde{\xi}_3),$$

where

$$\tilde{\xi}_1^2 = -\frac{(c_{13}+c_{44})^2 + (c_{12}+c_{66})(c_{44}-c_{33})}{(c_{13}+c_{44})^2(c_{12}-c_{66}) + (c_{12}+c_{66})(c_{33}c_{66}-c_{12}c_{33}-2c_{44}^2)},$$
$$\tilde{\xi}_3^2 = -\frac{(2c_{44}+c_{12}-c_{66})(c_{12}+c_{66})}{(c_{13}+c_{44})^2(c_{12}-c_{66}) + (c_{12}+c_{66})(c_{33}c_{66}-c_{12}c_{33}-2c_{44}^2)}.$$

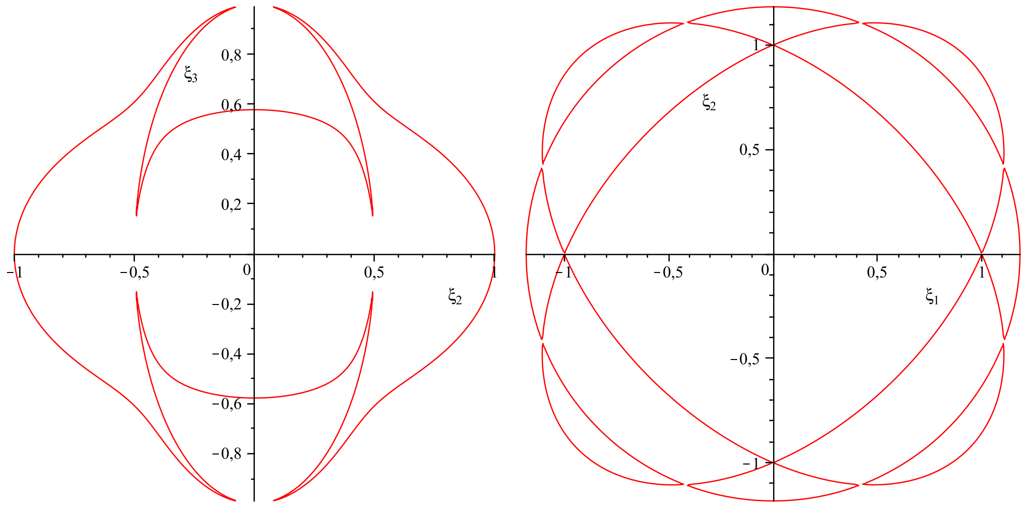


Figure 2.5: Restrictions of S on the plane $\xi_1 = 0$ and $\xi_3 = 0$, with $(c_{11}, c_{33}, c_{44}, c_{66}, c_{12}, c_{13})$ equal to $(4, 3, 1, 4, -1/2, 1/5)$ and $(1, 2, 5/7, 1, -1/7, 1/2)$ respectively.

Remark 2.11. We observe that there exist admissible values of the stiffness constants such that the conditions $\tilde{\xi}_i > 0$, with $i = 1, 3$, and $R > 0$ of the previous proposition can be either both satisfied or both not satisfied or one satisfied and the other not satisfied. Indeed if we choose $(c_{33}, c_{44}, c_{66}, c_{12}, c_{13}) = (2, 5/7, 1, -1/7, 1/2)$ we have $\tilde{\xi}_1 > 0$, $\tilde{\xi}_3 > 0$, $R > 0$. If we choose $(c_{33}, c_{44}, c_{66}, c_{12}, c_{13}) = (3, 11/7, 1, -1/7, 1/2)$ we have $\tilde{\xi}_1 > 0$, $\tilde{\xi}_3 > 0$, but $R < 0$. If we choose $(c_{33}, c_{44}, c_{66}, c_{12}, c_{13}) = (7, 9/7, 1, -1/7, 1/2)$ we have $R > 0$, but $\tilde{\xi}_3 < 0$. Finally, if we choose $(c_{33}, c_{44}, c_{66}, c_{12}, c_{13}) = (6/5, 3/7, 1, -1/7, 1/2)$ we have $R < 0$ and $\tilde{\xi}_3 < 0$.

Now we want to classify the double points of S into three different types, depending on their geometrical features. To do so, we need the following definitions.

Definition 2.3. *Let S be a surface in $\mathbb{R}^3$ on which linear coordinates are denoted by $\xi = (\xi_1, \xi_2, \xi_3)$. We assume that $P \in S$ and that in a neighborhood $\mathcal{U}$ of P, S is defined by an equation of form $\{\xi \in \mathcal{U} : f(\xi) = 0\}$ for some* *function* $f \in C^\infty(\mathcal{U})$. *We assume that* $\nabla f(\xi) = 0$ *precisely when* $\xi = P$ *and denote by* $J_k f(\xi) = \sum_{|\alpha|=k}(1/\alpha!)\partial_\xi^\alpha f(P)\xi^\alpha$ *the homogeneous part of degree* k *in the Taylor expansion of* f *at* P.

(i) We say that P *is a* conical singularity *if for some suitable choice of linear coordinates* $J_2 f$ *has the form* $J_2 f(\xi) = \xi_1^2 - \xi_2^2 - \xi_3^2$.

(ii) We say that P *is a* uniplanar singularity *if it is possible to find linear coordinates for which* $J_2 f(\xi) = \xi_3^2$ *and if* $f = 0$ *is locally equivalent to*

$$\xi_3^2 + A(\xi_1, \xi_2)\xi_3 + B(\xi_1, \xi_2) = 0$$

with $A(P_1, P_2) = 0$, $B(P_1, P_2) = 0$, $\nabla A(P_1, P_2) = 0$, *for some smooth function* A, B.
Moreover, we assume that if we denote by Δ *the quantity* $\Delta = A^2 - 4B$, *then we have* $\Delta(\xi_1, \xi_2) = \mathcal{O}(|\xi_1, \xi_2|^4)$ *for* $(\xi_1, \xi_2) \to (P_1, P_2)$.

(iii) We say that P *is a* biplanar singularity *if the following happens: for some suitable choice of linear coordinates* $J_2 f(\xi) = \xi_1^2 - \xi_2^2$.

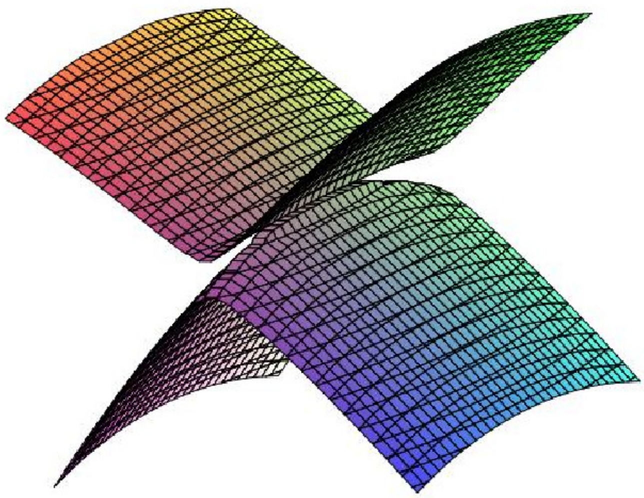


Figure 2.6: The biplanar double point at the origin of the surface defined by the equation $z^2 - (1/2)x^2 + 2yz^2 - 2zx^2 + x^4 + 2x^2y^2 + (1/2)y^4 = 0$.

In the next three subsections, we will prove the following proposition about the nature of the singular points of the slowness surface for the tetragonal crystal system, when we have $c_{11} = c_{66}$.

Proposition 2.14. *Let S be the slowness surface for the tetragonal crystal system.*
The double points of S on the ξ_3-axis are uniplanar singularities.
The double points of S on the ξ_1-axis and ξ_2-axis are biplanar singularities.
If S has double points on the plane $\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_3 = 0\}$, then they are conical singularities. If S has double points on the planes $\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_1^2 = \xi_2^2\}$, and $c_{66} - c_{44} = c_{13} - c_{12} = c_{33} - c_{11}$, then they are conical singularities.

2.4.1 Hessian at the singular points on the axes

In this sections we suppose that the assumptions on the stiffness constants made the proposition (2.13) hold.
We begin with a study of the singularity on the ξ_3-axis. We have the following result.

Proposition 2.15. *Let S be the slowness surface for the tetragonal crystal system defined by the condition $p(\xi) = 0$.*
Then the points $(0, 0, \pm 1/\sqrt{c_{44}}) \in S$ are uniplanar singularities.

Proof. We denote $P = (0, 0, 1/\sqrt{c_{44}})$. We will prove that $\nabla p(P) = 0$, $(\partial/\partial\xi_3)^2 p(P) \neq 0$, $(\partial^2/\partial\xi_i\partial\xi_j)^2 p(P) = 0$ if $i \in \{1, 2\}$, $j \in \{1, 2, 3\}$, $(\partial/\partial\xi)^\alpha p(P) = 0$ if $|\alpha| = 3$ and the order of derivations in (ξ_1, ξ_2) is odd.
The gradient of p at P vanishes since P is a double point of the slowness surface.
The following remarks help us simplify the calculations of second order derivatives:

(i) the factors n_1, n_2 vanish twice at P.

(ii) The expressions d_i vanish at P for $i = 1, 2$ (but not necessarily for $i = 3$).

(iii) When we derivate one of the d_i, $i = 1, 2, 3$, in one of the variables ξ_j, $j = 1$ or 2, then we obtain a factor ξ_j, and therefore this derivative will vanish at P.

(iv) $(\partial/\partial\xi_i)n_3 = 0$, $(\partial^2/\partial\xi_i\partial\xi_j)n_3 = 0$ for $i = 1, 2$, whatever j is.

We conclude from these remarks, that the terms $n_1d_2d_3$, $n_2d_3d_1$ vanish of order 3 at P. Therefore, they will not contribute to the Hessian of p at P. Moreover, when we calculate second order derivatives of type $(\partial^2/\partial\xi_i\partial\xi_j)$ of $(n_3 - d_3)d_1d_2$, then, in order to have a nontrivial contribution, we must derivate each one of the factors d_1 and d_2, since these factors vanish at P. However, first order derivatives of d_1, d_2 again vanish at P, so we do not have enough derivations to obtain a nontrivial contribution. In a similar way we conclude that derivatives of form $(\partial/\partial\xi_i)(\partial/\partial\xi_3)^k p(P)$ vanish when $i \in \{1, 2\}$, $k \geq 2$.
We next calculate $(\partial/\partial\xi_3)^2 p(P)$. Again, only $(n_3 - d_3)d_1d_2$ can give a nontrivial contribution. We must of course derivate each of the factors d_1 and d_2 once, to get a nontrivial contribution. Therefore $(\partial/\partial\xi_3)^2 p(P) = (n_3 - d_3)(P)(\partial/\partial\xi_1)d_1(P)(\partial/\partial\xi_1)d_2(P)$. After some calculations we obtain

$$\frac{\partial^2}{\partial\xi_3^2}p(P) = \frac{8(c_{33} - c_{44})(c_{12} + c_{66})}{c_{13} + c_{44}}.$$

By assumption on the stiffness constants this is non vanishing.
We still have to say something about third order derivatives. If we derivate once in ξ_3 and the remaining derivatives are in the variables ξ_1, ξ_2, then the result may be non vanishing. $\square$

We now turn to the case of the ξ_1-axis and of the ξ_2-axis. The two cases are of course symmetric.

Proposition 2.16. *Let S be the slowness surface for the tetragonal crystal system defined by the condition $p(\xi) = 0$.*

Then the points $(\pm 1/\sqrt{c_{66}}, 0, 0) \in S$ and $(0, \pm 1/\sqrt{c_{66}}, 0) \in S$ are biplanar singularities.

Proof. We denote $P = (1/\sqrt{c_{66}}, 0, 0)$. As in the case of proposition (2.15)we can simplify the calculations with some preliminary remarks. We now have:

(i) n_2, n_3 vanish of order 2 at P.

(ii) d_2 vanishes at P.

(iii) The first order derivatives of the d_i, $i = 1, 2, 3$, in the variables ξ_2, ξ_3 vanish at P.

(iv) $n_1 - d_1$ vanishes at P.

We first prove that

$$\begin{aligned} \frac{\partial^2 p(P)}{\partial \xi_1^2} &= \frac{8(c_{12} + c_{66})(c_{44} - c_{66})}{c_{13} + c_{44}}, \\ \frac{\partial^2 p(P)}{\partial \xi_2^2} &= -\frac{2(c_{12} + c_{66})^3 (c_{44} - c_{66})}{c_{66}^2 (c_{13} + c_{44})}, \\ \frac{\partial^2 p(P)}{\partial \xi_3^2} &= 0. \end{aligned}$$

It follows, by the assumptions on the stiffness constants, that $(\partial/\partial\xi_1)^2 p(P)$ and $(\partial/\partial\xi_2)^2 p(P)$ have opposite signs.

To calculate the second derivatives in ξ_1, we notice that the terms containing n_1, n_2 will not give any contribution: they contain factors of type ξ_2^2, ξ_3^2 and these factors are like constants if we derivate them in ξ_1. Since d_2 and $n_1 - d_1$ vanish at P, we have

$$\frac{\partial^2}{\partial \xi_1^2} p(P) = \frac{\partial}{\partial \xi_1} d_2(P) \frac{\partial}{\partial \xi_1} (n_1 d_3 - d_1 d_3)(P).$$

After some calculations, we obtain the result in the statement referring to $(\partial/\partial\xi_1)^2 p(P)$.

When we calculate $(\partial/\partial\xi_2)^2 p(P)$, the term $n_3 d_1 d_2$ gives no contribution due to the factor n_3, which behaves like a constant under derivations in ξ_2. We may thus write that

$$\frac{\partial^2}{\partial\xi_2^2}p(P) = \frac{\partial^2}{\partial\xi_2^2}[(n_1 - d_1)d_2d_3](P) + d_1(P)d_3(P)\frac{\partial^2}{\partial\xi_2^2}n_2(P).$$

Since $(n_1 - d_1)$ and d_2 both vanish at P, we have

$$\frac{\partial^2}{\partial\xi_2^2}[(n_1 - d_1)d_2d_3](P) = d_3(P)\frac{\partial}{\partial\xi_2}(n_1 - d_1)(P)\frac{\partial}{\partial\xi_2}d_2(P).$$

However, $(\partial/\partial\xi_2)d_2(P) = 0$. Therefore,

$$\frac{\partial^2}{\partial\xi_2^2}p(P) = d_1(P)d_3(P)\frac{\partial^2}{\partial\xi_2^2}n_2(P).$$

It follows after some calculations that $(\partial/\partial\xi_2)^2p(P)$ is as stated in the lemma. To calculate $(\partial/\partial\xi_3)^2p(P)$ we note that the term containing n_2 will give no contribution. The same is true for the term $n_3d_1d_2$: here we use the fact that n_3d_2 vanishes of order 3 at P. We are left with

$$\frac{\partial^2}{\partial\xi_3^2}[(n_1 - d_1)d_2d_3](P).$$

Since $(n_1 - d_1)$, d_2 both vanish at P, we must have that

$$[\frac{\partial^2}{\partial\xi_3^2}(n_1 - d_1)d_2d_3](P) = d_3(P)[\frac{\partial}{\partial\xi_3}(n_1 - d_1)(P)][\frac{\partial}{\partial\xi_3}d_2(P)].$$

We use again that $(\partial/\partial\xi_3)d_2(P) = 0$ and, in the end, we obtain 0.
Now we prove that

$$\begin{aligned}
\frac{\partial^2}{\partial\xi_1\partial\xi_2}p(P) &= 0, \\
\frac{\partial^2}{\partial\xi_1\partial\xi_3}p(P) &= 0, \\
\frac{\partial^2}{\partial\xi_2\partial\xi_3}p(P) &= 0.
\end{aligned}$$

To calculate $(\partial^2/\partial\xi_1\partial\xi_2)p(P)$, we note that the terms with n_2 and n_3 give no contribution. Thus,

$$\frac{\partial^2}{\partial\xi_1\partial\xi_2}p(P) = \frac{\partial^2}{\partial\xi_1\partial\xi_2}[(n_1 - d_1)d_2d_3](P).$$

We can now argue as above. The evaluation of $(\partial^2/\partial\xi_1\partial\xi_3)p(P)$, $(\partial^2/\partial\xi_3\partial\xi_3)p(P)$ is done in exactly the same way.
If we proceed as above, it is not difficult to prove that third order derivatives of p which contain odd order derivatives in one of the variables ξ_2 or ξ_3 vanish at P, i.e., we have

$$\frac{\partial^2}{\partial\xi_1^2}\frac{\partial}{\partial\xi_2}p(P) = \frac{\partial^2}{\partial\xi_1^2}\frac{\partial}{\partial\xi_3}p(P) = \frac{\partial^2}{\partial\xi_2^2}\frac{\partial}{\partial\xi_3}p(P) = \frac{\partial^2}{\partial\xi_3^2}\frac{\partial}{\partial\xi_2}p(P) = 0,$$

$$\frac{\partial}{\partial\xi_1}\frac{\partial}{\partial\xi_2}\frac{\partial}{\partial\xi_3}p(P) = \frac{\partial^3}{\partial\xi_2^3}p(P) = \frac{\partial^3}{\partial\xi_3^3}p(P) = 0.$$

Finally, after some calculations, we can prove in the same way that

$$\frac{\partial^3}{\partial\xi_1^3}p(P) = 24\sqrt{c_{66}}\frac{(c_{12}+c_{66})(-3c_{44}+c_{66})}{c_{13}+c_{44}},$$

$$\frac{\partial^2}{\partial\xi_2^2}\frac{\partial}{\partial\xi_1}p(P) = -\frac{4(c_{12}+c_{66})((c_{13}+c_{44})^2+2c_{44}(c_{66}-c_{44}))}{\sqrt{c_{66}}(c_{13}+c_{44})},$$

$$\frac{\partial^2}{\partial\xi_3^2}\frac{\partial}{\partial\xi_1}p(P) = \frac{4(c_{12}+c_{66})(2c_{12}(c_{44}c_{66}-c_{66}c_{12}+c_{44}c_{12})+c_{66}(c_{12}-c_{66})^2)}{\sqrt{c_{66}^3}(c_{13}+c_{44})},$$

$$\frac{\partial^4}{\partial\xi_3^4}p(P) = -\frac{24(c_{12}+c_{66})c_{44}((c_{13}+c_{44})^2+c_{44}(c_{66}-c_{44}))}{c_{66}(c_{13}+c_{44})}.$$

□

Remark 2.12. The localization polynomial of S at $P = (1/\sqrt{c_{66}}, 0, 0)$ is thus

$$\frac{8(c_{12}+c_{66})(c_{44}-c_{66})}{c_{13}+c_{44}}\xi_1^2 - \frac{2(c_{12}+c_{66})^3(c_{44}-c_{66})}{c_{66}^2(c_{13}+c_{44})}\xi_2^2.$$

The tangent set at S in P is then given by

$$\{v \in \mathbb{R}^3; \frac{8(c_{12}+c_{66})(c_{44}-c_{66})}{c_{13}+c_{44}}v_1^2 - \frac{2(c_{12}+c_{66})^3(c_{44}-c_{66})}{c_{66}^2(c_{13}+c_{44})}v_2^2 = 0, v_3 \in \mathbb{R}\}.$$

We have two planes and this is the reason to call them “biplanar”.

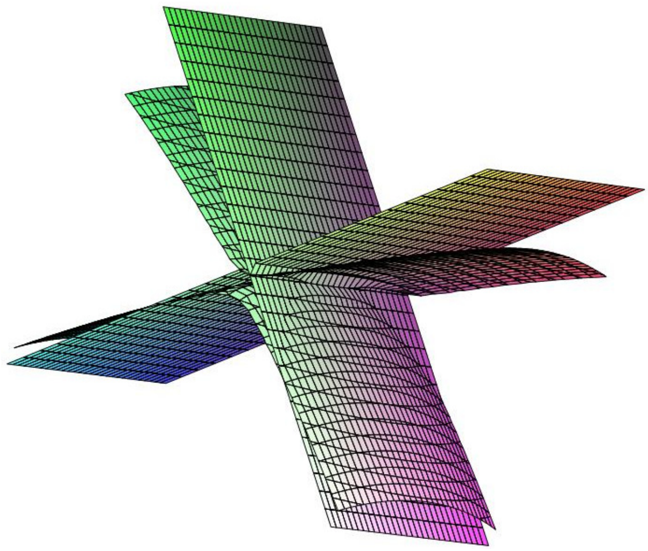


Figure 2.7: The biplanar double point of the surface defined by the equation $z^2 - (1/2)x^2 + 2yz^2 - 2zx^2 + x^4 + 2x^2y^2 + (1/2)y^4 = 0$, and its tangent set defined by $(z - x)(z + x) = 0$.

2.4.2 Hessian at the singular points in the coordinate planes, but not on the axes

We have to consider the singular points on the plane $\xi_3 = 0$. Let P be one of these points, e.g., $((\frac{1+\sqrt{R}}{2c_{44}}))^{1/2}, (\frac{1-\sqrt{R}}{2c_{44}})^{1/2}, 0)$. Here we want to calculate the determinant of the Hessian of $p(\xi)$ in P. Recall that P satisfies simultaneously

$$d_3(P) = 0, f_3(P) = 0, \tag{2.4.1}$$

where we denote $f_3 = n_1d_2 + n_2d_1 - d_1d_2$. Relation (2.4.1) can be used to simplify the calculation of the Hessians. Indeed, $(\partial^2/\partial\xi_3^2)p(P)$ is very easy to calculate. This is based on the following remarks:

(i) in view of (2.4.1)

$$\frac{\partial^2}{\partial\xi_3^2}[d_3(n_1d_2+n_2d_1-d_1d_2)](P) = 2\frac{\partial}{\partial\xi_3}d_3(P)\frac{\partial}{\partial\xi_3}(n_1d_2+n_2d_1-d_1d_2)(P).$$

(ii) If we derivate d_3, n_3 or $n_1d_2 + n_2d_1 - d_1d_2$ just once in ξ_3, then the expression which we obtain will be a multiple of ξ_3 and will therefore vanish on our coordinate plane.

It follows that

$$\frac{\partial^2}{\partial \xi_3^2}[d_3(P)(n_1d_2 + n_2d_1 - d_1d_2)(P)] = 0.$$

It is then clear that

$$\begin{aligned}\frac{\partial^2}{\partial \xi_3^2}p(P) = \frac{\partial^2 n_3}{\partial \xi_3^2}(P)d_1(P)d_2(P) &= \\ &= 2\frac{(c_{13}+c_{44})^2}{c_{12}+c_{66}}d_1(P)d_2(P) = \\ &= 2\frac{(c_{13}+c_{44})^2(c_{44}-c_{66})(c_{12}+2c_{44}-c_{66})}{c_{44}^2(c_{12}+c_{66})}\end{aligned} \tag{2.4.2}$$

Remark 2.13. Mixed derivatives of p which contain just one derivation in ξ_3 will vanish. This is proved with an argument similar to the one just used for the calculation of $(\partial/\partial\xi_3)^2 p(P)$.

We are now left with derivatives of form $(\partial^2/\partial\xi_i\partial\xi_j)$ where $i, j \in \{1, 2\}$. It is obvious that $(\partial^2/\partial\xi_i\partial\xi_j)(n_3d_1d_2)(P) = 0$. Now, we can again argue as above and conclude that

$$\frac{\partial^2}{\partial\xi_i\partial\xi_j}(d_3f_3)(P) = \frac{\partial}{\partial\xi_i}d_3(P)\frac{\partial}{\partial\xi_j}f_3(P) + \frac{\partial}{\partial\xi_j}d_3(P)\frac{\partial}{\partial\xi_i}f_3(P). \tag{2.4.3}$$

The situation is further simplified by noting that $(\partial/\partial\xi_i)d_3$ is divisible by ξ_i and $(\partial/\partial\xi_j)(n_1d_2+n_2d_1-d_1d_2)$ is divisible by ξ_j. In (2.4.3) we can therefore divide out a factor $\xi_i\xi_j$. If we also take into account that

$$\frac{1}{\xi_i}\frac{\partial d_3}{\partial\xi_i} = \frac{1}{\xi_j}\frac{\partial d_3}{\partial\xi_j} = -2c_{44}$$

then we obtain that

$$\frac{\partial^2}{\partial\xi_i\partial\xi_j}(d_3f_3)(P) = -2c_{44}\left[\xi_i\xi_j\left(\frac{1}{\xi_i}\frac{\partial f_3}{\partial\xi_i} + \frac{1}{\xi_j}\frac{\partial f_3}{\partial\xi_j}\right)\right](P).$$

An elementary calculation gives

$$\frac{1}{\xi_1}\frac{\partial}{\partial\xi_1}f_3(\xi) = -4c_{66}^2\xi_1^2 + 2(c_{12}^2 - c_{66}^2 + 2c_{12}c_{66})\xi_2^2 - 4c_{66}c_{44}\xi_3^2 + 4c_{66},$$
$$\frac{1}{\xi_2}\frac{\partial}{\partial\xi_2}f_3(\xi) = -4c_{66}^2\xi_2^2 + 2(c_{12}^2 - c_{66}^2 + 2c_{12}c_{66})\xi_1^2 - 4c_{66}c_{44}\xi_3^2 + 4c_{66}.$$

It also follows from this that the determinant of the Hessian in the variables ξ_1, ξ_2 is

$$4\xi_1^2\xi_2^2c_{44}^2\left(4\frac{1}{\xi_1}\frac{\partial f_3}{\partial\xi_1}\frac{1}{\xi_2}\frac{\partial f_3}{\partial\xi_2} - \left(\frac{1}{\xi_2}\frac{\partial f_3}{\partial\xi_2} + \frac{1}{\xi_1}\frac{\partial f_3}{\partial\xi_1}\right)^2\right)$$
$$= -4\xi_1^2\xi_2^2c_{44}^2\left(\frac{1}{\xi_2}\frac{\partial f_3}{\partial\xi_2} - \frac{1}{\xi_1}\frac{\partial f_3}{\partial\xi_1}\right)^2. \quad (2.4.4)$$

We are now ready to prove the following proposition

Proposition 2.17. *The double points of the slowness surface on the plane* $\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_3 = 0\}$ *are conical singularities.*

Proof. Let $P = ((\frac{1+\sqrt{R}}{2c_{44}}))^{1/2}, (\frac{1-\sqrt{R}}{2c_{44}})^{1/2}, 0)$ be one of the double points in the plane $\xi_3 = 0$, with the standard notation. From (2.4.4), (2.4.2) and remark (2.13), follows that the Hessian of $p = 0$ in P has the form

$$R = \begin{pmatrix} A & 0 \\ 0 & \frac{\partial^2 p}{\partial\xi_3^2}(P) \end{pmatrix},$$

where the determinant of A is negative. Thus, P is a conical singularity. For the other singular points on the plane $\xi_3 = 0$ we can argue as above. □

2.4.3 Hessian at the singular points in the planes $\xi_1^2 = \xi_2^2$

In principle it is not difficult to calculate the Hessian of p at these points. For the explicit numerical constants c_{ij} this is a simple arithmetic calculation, but for general constants the expressions which one obtains are not as easy to understand. We now begin our discussion recalling that, if P is a double point on the planes $\xi_1^2 = \xi_2^2$, then $d_i(P) = 0$ for $i = 1, 2, 3$. Then, the part $d_1 d_2 d_3$ of p will vanish of third order at P, and will not contribute to the Hessian at P. It is also clear that we have

$$\frac{\partial^2 (n_j d_{j+1} d_{j+2})(P)}{\partial\xi_i \partial\xi_l} = \\ = n_j(P)\left(\frac{\partial}{\partial\xi_i} d_{j+1}(P)\frac{\partial}{\partial\xi_l} d_{j+2}(P) + \frac{\partial}{\partial\xi_l} d_j(P)\frac{\partial}{\partial\xi_i} d_{j+2}(P)\right),$$

where the indices are calculated modulo 3, since $d_{j+1}d_{j+2}$ vanishes of order two at P. Calculations are quite complex and we will only discuss what happens under the additional assumption $c_{66} - c_{44} = c_{13} - c_{12} = c_{33} - c_{11}$ (see remark (2.10)). In this case, if we set $c_{13} + c_{44} = c_{12} + c_{66}$ and $c_{33} = 2c_{66} - c_{44}$, then the double points lie on the space diagonal. In particular they have the following coordinates

$$(\pm(c_{44} + c_{66} - c_{12})^{-1/2}, \pm(c_{44} + c_{66} - c_{12})^{-1/2}, \pm(c_{44} + c_{66} - c_{12})^{-1/2}).$$

Moreover the $n_i(P)$ have the same values and

$$\begin{aligned} d_1(\xi) &= 1 - c_{66}\xi_1^2 - c_{66}\xi_2^2 - c_{44}\xi_3^2 + (c_{12} + c_{66})\xi_1^2, \\ d_2(\xi) &= 1 - c_{66}\xi_1^2 - c_{66}\xi_2^2 - c_{44}\xi_3^2 + (c_{12} + c_{66})\xi_2^2, \\ d_3(\xi) &= 1 - c_{44}\xi_1^2 - c_{44}\xi_2^2 - (2c_{66} - c_{44})\xi_3^2 + (c_{12} + c_{66})\xi_3^2. \end{aligned}$$

Denoting $q = d_1 d_2 + d_2 d_3 + d_3 d_1$, it follows that

$$\frac{1}{n_1(P)\xi_i\xi_j}\frac{\partial^2 p(P)}{\partial\xi_i\partial\xi_j} = \frac{\partial^2 q(P)}{\partial\xi_i\partial\xi_j}, \qquad i = 1, 2, 3.$$

The Hessian of p at P is then proportional to the matrix

$$A = \begin{pmatrix} \dfrac{\partial^2 q}{\partial\xi_1^2} & \dfrac{\partial^2 q}{\partial\xi_1\partial\xi_2} & \dfrac{\partial^2 q}{\partial\xi_1\partial\xi_3} \\ \dfrac{\partial^2 q}{\partial\xi_2\partial\xi_1} & \dfrac{\partial^2 q}{\partial\xi_2^2} & \dfrac{\partial^2 q}{\partial\xi_2\partial\xi_3} \\ \dfrac{\partial^2 q}{\partial\xi_3\partial\xi_1} & \dfrac{\partial^2 q}{\partial\xi_3\partial\xi_2} & \dfrac{\partial^2 q}{\partial\xi_3^2} \end{pmatrix}.$$

We can easily obtain

$$\frac{1}{\xi_1}\frac{\partial d_1(P)}{\partial\xi_1} = 2c_{12}, \quad \frac{1}{\xi_2}\frac{\partial d_2(P)}{\partial\xi_2} = 2c_{12}, \quad \frac{1}{\xi_3}\frac{\partial d_3(P)}{\partial\xi_3} = 2c_{44} + 2c_{12} - 2c_{66},$$

and

$$\frac{1}{\xi_2}\frac{\partial d_1(P)}{\partial \xi_2} = -2c_{66}, \quad \frac{1}{\xi_3}\frac{\partial d_1(P)}{\partial \xi_3} = -2c_{44}, \quad \frac{1}{\xi_1}\frac{\partial d_2(P)}{\partial \xi_1} = -2c_{66},$$
$$\frac{1}{\xi_3}\frac{\partial d_2(P)}{\partial \xi_3} = -2c_{44}, \quad \frac{1}{\xi_1}\frac{\partial d_3(P)}{\partial \xi_1} = -2c_{44}, \quad \frac{1}{\xi_2}\frac{\partial d_3(P)}{\partial \xi_2} = -2c_{44},$$

Finally, explicit calculations give

$$(c_{44}+c_{66}-c_{12})\frac{\partial^2 q}{\partial \xi_i^2}(P) = 8(-c_{12}c_{66}+c_{44}c_{66}-c_{44}c_{12}), \quad i=1,2,$$

$$(c_{44}+c_{66}-c_{12})\frac{\partial^2 q}{\partial \xi_3^2}(P) = 8c_{44}(2c_{66}-2c_{12}-c_{44}),$$

$$(c_{44}+c_{66}-c_{12})\frac{\partial^2 q}{\partial \xi_1 \partial \xi_2}(P) = 4((c_{12}+c_{66})^2+2(-c_{12}c_{66}+c_{44}c_{66}-c_{44}c_{12})),$$

$$(c_{44}+c_{66}-c_{12})\frac{\partial^2 q}{\partial \xi_i \partial \xi_3}(P) = 4((c_{12}-c_{66}^2)+2c_{44}^2), \quad i=1,2.$$

Remark 2.14. Let A be a matrix of the following form:

$$A = \begin{pmatrix} a & b & c \\ b & a & c \\ c & c & d \end{pmatrix}. \tag{2.4.5}$$

where a, b, c and d are real numbers. We observe that the eigenvalues of A are $a-b$, $(1/2)(a+b+d \pm \sqrt{(a+b-d)^2+8c^2})$.

We note that the Hessian of p at P has the same form of (2.4.5) and so its eigenvalues are

$$4(c_{12}+c_{66})^2, \quad -8(c_{44}+c_{66}-c_{12})^2, \quad 4(2c_{44}-c_{66}+c_{12})^2. \tag{2.4.6}$$

Moreover, the determinant of the Hessian is equal to

$$128(c_{12}+c_{66})^2(c_{44}+c_{66}-c_{12})^2(2c_{44}-c_{66}+c_{12})^2. \tag{2.4.7}$$

Thus, if we assume $2c_{44}-c_{66}+c_{12} \neq 0$, using the symmetries between ξ_1 and ξ_2, it is easy to prove the following proposition.

Proposition 2.18. *Assume, in addition to all the conditions on the stiffness constants used in the previous sections, that $c_{13} + c_{44} = c_{12} + c_{66}$, $c_{33} = 2c_{66} - c_{44}$ and $2c_{44} - c_{66} + c_{12} \neq 0$. Then, the double points of the slowness surface on the planes $\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_1^2 = \xi_2^2\}$ lie on the space diagonals and are conical singularities.*

Remark 2.15. We recall that in [L2] the quantity $b - a$, denoted by d, is the measure of the anisotropy of the crystal. In particular, when $d = 0$ the slowness surface of the cubic crystals is reduced to the union of a double sphere with another different sphere. In the same work a property which holds when $|b - b_0| + |c - c_0| + |d| \leq \varepsilon$, for fixed b_0, c_0 and positive ε is said to hold in the nearly isotropic case. Moreover, the singular points on the ξ_3-axis and on the diagonal have coordinates $(0, 0, c^{-1/2})$ and $((3c-d)^{-1/2}, (3c-d)^{-1/2}, (3c-d)^{-1/2})$.
Using the notations of this thesis (see also remark (2.1)) and considering the assumption on the stiffness constants made in this section, we have $d = 2c_{44} - c_{66} + c_{12}$. So, it is possible to write the coordinates of the singular points on the ξ_3-axis and on the space diagonal precisely as in the cubic case, because $3c - d = c_{44} + c_{66} - c_{12}$. Moreover, if $d = 0$, the slowness surface of the tetragonal crystal does not become degenerate as in the cubic case but, in view of (2.4.6) and (2.4.7), it is easy to observe that the double points on the space diagonals become biplanar singular points. Indeed, if $d = 0$, then $2c_{44} - c_{66} + c_{12} = 0$ and so the determinant of the Hessian is zero and its two eigenvalues different from zero have opposite sign.
Thus, if the stiffness constants are such that d is small, i.e. we are in the nearly isotropic case, the conical singularities on the space diagonals, are near the biplanar case.

2.5 Hessian of singular points when $c_{11} \neq c_{66}$

In the previous section we studied the nature of the singular points which appear on the slowness surface, in the case when we have $c_{11} = c_{66}$. Using exactly the same arguments it is possible to verify the nature of the singular points in the case when $c_{11} \neq c_{66}$. We observe that the main difference between these two cases is that, if $c_{11} = c_{66}$, then the slowness surface has four biplanar singular points, one on each ξ_i-semi-axis, with $i \in \{1, 2\}$, whereas if $c_{11} \neq c_{66}$, then the slowness surface does not have biplanar singular points, but it may have eight more singular points on the planes $\xi_i = 0$, with $i \in \{1, 2\}$, as proposition (2.9) shows. Thus, with similar calculations, even if a little bit more involved, it is possible to prove the following proposition.

Proposition 2.19. *Let S be the slowness surface associated with the tetragonal crystal system. Assume that the stiffness constants c_{ij} satisfy the assumptions made in the previous sections and, in addition, suppose $c_{11} \neq c_{66}$. Then:*

(i) The double points of S on the ξ_3-axes are uniplanar singularities.

(ii) The double points of S on the plane $\xi_3 = 0$, which do not lie on the coordinate axes are conical singularities.

(iii) If, in addition, we assume that $c_{11} - c_{33} = c_{12} - c_{13} = c_{44} - c_{66}$, then the double points of S on the planes $\xi_1^2 = \xi_2^2$ are conical singularities.

Now we are left with the double points of S which lie on the planes $\xi_i = 0$, with $i \in \{1, 2\}$, but not on the axes. It is possible to prove the following statement.

Proposition 2.20. *Let S be the slowness surface associated with the tetragonal crystal system. Assume that the stiffness constants c_{ij} satisfy the assumptions made in the previous sections and, in addition, suppose $c_{11} \neq c_{66}$. Then the double points of S which lie on the planes $\xi_i = 0$, with $i \in \{1, 2\}$, but not on the axes, are conical singularities.*

Proof. We proceed exactly as in subsection (2.4.2). Moreover, we recall that we can prove the proposition only in the case of the double points on the plane $\xi_1 = 0$, because on the plane $\xi_2 = 0$ the situation is completely symmetric.

Let P be one of the double points of S on the plane $\xi_1 = 0$, with coordinates $(0, \tilde{\xi}_2, \tilde{\xi}_3)$ defined in proposition (2.9). We denote by f_1 the quantity $n_2 d_3 + n_3 d_2 - d_2 d_3$, and we recall that $d_1(P) = f_1(P) = 0$. In view of this

$$\frac{\partial^2}{\partial \xi_1^2}\left[d_1(n_2 d_3 + n_3 d_2 - d_2 d_3)\right](P) = 2\frac{\partial}{\partial \xi_1} d_1(P) \frac{\partial}{\partial \xi_1} f_1(P)$$

and if we derivate d_1, n_1 or f_1 just once in ξ_1, then the expression which we obtain will be a multiple of ξ_1 and will therefore vanish in our coordinate plane. It is then clear that

$$\begin{aligned}\frac{\partial^2}{\partial \xi_1^2} p(P) = \frac{\partial^2 n_1}{\partial \xi_1^2}(P) d_2(P) d_3(P) = \\ = 2(c_{13}+c_{44})^2(c_{44}-c_{66})(c_{11}-c_{12}-2c_{66})^2 \tilde{\xi}_2^2,\end{aligned}$$

and that the mixed derivatives of p which contain just one derivation in ξ_1 will vanish. We can argue as above (see also section (2.4.2)) and we can conclude that

$$\frac{\partial^2}{\partial \xi_i \partial \xi_j}(d_1 f_1)(P) = \frac{\partial}{\partial \xi_i} d_1(P) \frac{\partial}{\partial \xi_j} f_1(P) + \frac{\partial}{\partial \xi_j} d_1(P) \frac{\partial}{\partial \xi_i} f_1(P).$$

Thus, recalling that$(\partial/\partial \xi_i) d_1$ is divisible by ξ_i and $(\partial/\partial \xi_j) f_1$ is divisible by ξ_j, that $(\partial/\partial \xi_2) d_1 = -2c_{66}\xi_2$ and that $(\partial/\partial \xi_3) d_1 = -2c_{44}\xi_3$, we obtain

$$\frac{\partial^2 d_1 f_1}{\partial \xi_2 \partial \xi_3}(P) = -2\xi_2\xi_3\left(c_{66}\frac{\partial f_1}{\partial \xi_2}\frac{1}{\xi_2} + c_{44}\frac{\partial f_1}{\partial \xi_3}\frac{1}{\xi_3}\right)(P),$$

$$\frac{\partial^2 d_1 f_1}{\partial \xi_2^2}(P) = -4\xi_2^2 c_{66}\frac{\partial f_1}{\partial \xi_2}\frac{1}{\xi_2},$$

$$\frac{\partial^2 d_1 f_1}{\partial \xi_3^2}(P) = -4\xi_3^2 c_{44}\frac{\partial f_1}{\partial \xi_3}\frac{1}{\xi_3}.$$

So the determinant of the Hessian in the variable ξ_2, ξ_3 is

$$-4\xi_2^2\xi_3^2\left(c_{66}\frac{\partial f_1}{\partial \xi_2}\frac{1}{\xi_2} - c_{44}\frac{\partial f_1}{\partial \xi_3}\frac{1}{\xi_3}\right)^2.$$

This concludes the proof. □

Remark 2.16. We observe that the previous two propositions hold for generic tetragonal stiffness constants c_{ij} (Here we call tetragonal stiffness constants some constants c_{ij} such that the associated crystal system is hyperbolic and tetragonal, in particular c_{ij} must satisfy all the necessary conditions on the hyperbolicity of the system stated in the previous sections). It is not difficult to show that, if we consider the tetragonal system in the nearly cubic case, i.e. if we assume $c_{11} - c_{33} = c_{12} - c_{13} = c_{44} - c_{66} = e$, with $|e|$ sufficiently small, then the previous propositions still remain valid.

Now, we want to investigate the case when $c_{11} \neq c_{66}$, but $c_{44} = c_{66}$. We have seen that, in this case, the slowness surface has four double points, one on each semi-axis of the coordinate plane $\xi_3 = 0$. We have the following results about the nature of these double points.

Proposition 2.21. *Let S be the slowness surface associated with the tetragonal crystal system. Assume that the stiffness constants c_{ij} satisfy the assumption made in the previous sections and, in addition, suppose $c_{11} \neq c_{66}$, but $c_{44} = c_{66}$. Then the four double points of coordinate $(\pm 1/\sqrt{c_{44}}, 0, 0)$ and $(0, \pm 1/\sqrt{c_{44}}, 0)$ are uniplanar singularities.*

Proof. We denote $P = (1/\sqrt{c_{44}}, 0, 0)$. We will prove that $\nabla p(P) = 0$, $(\partial/\partial\xi_1)^2 p(P) \neq 0$, $(\partial^2/\partial\xi_i\partial\xi_j)^2 p(P) = 0$ if $i \in \{2, 3\}$, $j \in \{1, 2, 3\}$, $(\partial/\partial\xi)^\alpha p(P) = 0$ if $|\alpha| = 3$ and the order of derivations in (ξ_2, ξ_3) is odd.

The gradient of p at P vanishes since P is a double point of the slowness surface.

The following remarks help us simplify the calculations of second order derivatives:

(i) the factors n_2, n_3 vanish twice at P.

(ii) The expressions d_i vanish at P for $i = 2, 3$ (but not necessarily for $i = 1$).

(iii) When we derivate one of the d_i, $i = 1, 2, 3$, in one of the variables ξ_j, $j = 2$ or 3, then we obtain a factor ξ_j, and therefore this derivative will vanish at P.

(iv) $(\partial/\partial\xi_i)n_1 = 0$, $(\partial^2/\partial\xi_i\partial\xi_j)n_1 = 0$ for $i = 2, 3$, whatever j is.

We can argue precisely as in the proof of proposition (2.15) and conclude that derivatives of form $(\partial/\partial\xi_i\partial\xi_j)^2 p(P)$, and $(\partial/\partial\xi_i)(\partial/\partial\xi_1)^k p(P)$ vanish when $i, j \in \{2, 3\}$, $k \geq 2$. Moreover, after some calculations we obtain

$$\frac{\partial^2}{\partial\xi_1^2} p(P) = 8(c_{11} - c_{44}).$$

By assumption on the stiffness constants this is non vanishing.
We still have to say something about third order derivatives. If we derivate once in ξ_3 and the remaining derivatives are in the variables ξ_1, ξ_2, then the result may be non vanishing. □

We conclude this section with a proposition about the nature of the singular points on the plane $\xi_1^2 = \xi_2^2$ in the case when they are near the diagonal. Indeed, as we have observed in the previous section, it seems difficult to calculate the Hessian of p at these points in the general situation, whereas if $c_{11} - c_{33} = c_{12} - c_{13} = c_{44} - c_{66}$, the singular points lie on the space diagonals and it is easy to establish the conical nature of these singularities. Now, we will prove that, if these singular points remain near the space diagonals, then they still remain of conical type. In particular we will assume $c_{44} = c_{66}$, $c_{33} - c_{11} = e_1$ and $c_{13} - c_{12} = e_3$., with $|e_i|$, $i = 1, 3$ small. The choice of this assumption will become easy to understand in the following sections, when we will discuss the curvature properties of the slowness surface (cf. also the remarks (2.16) and (2.1) and definition (2.4)).

Proposition 2.22. *Let S be the slowness surface associated with the tetragonal crystal system. Assume that the stiffness constants c_{ij} satisfy the assumptions made in the previous sections and, in addition, suppose $c_{11} \neq c_{66}$, either $c_{12} - c_{13} = c_{44} - c_{66} = 0$ and $c_{11} - c_{33} = e$, or $c_{44} = c_{66}$, $c_{33} - c_{11} = e_1$ and $c_{13} - c_{12} = e_3$. Then, if $|e_i|$, $i = 1, 3$ and $|e|$ are sufficiently small, the double points of S on the planes $\xi_1^2 = \xi_2^2$ are conical singularities.*

Proof. We prove the proposition with the assumption $c_{12} - c_{13} = c_{44} - c_{66} = 0$ and $c_{11} - c_{33} = e$. The proof in the case when $c_{44} = c_{66}$, $c_{33} - c_{11} = e_1$ and

$c_{13} - c_{12} = e_3$ is exactly the same, with the expressions of f_2 and f_3 a little bit more involved.

We recall that, if $c_{12} - c_{13} = c_{44} - c_{66} = c_{11} - c_{33}$, then the double points have the following coordinates

$$(\pm(c_{44} + c_{11} - c_{12})^{-1/2}, \pm(c_{44} + c_{11} - c_{12})^{-1/2}, \pm(c_{44} + c_{11} - c_{12})^{-1/2}).$$

Now, if we assume $c_{12} - c_{13} = c_{44} - c_{66} = 0$ and $c_{11} - c_{33} = e$, it is not difficult to see that the double points have coordinates

$$(\pm\tilde{\xi}_2, \pm\tilde{\xi}_2, \pm\tilde{\xi}_3) \tag{2.5.1}$$

where

$$\tilde{\xi}_2^2 = \frac{1}{c_{44} + c_{11} - c_{12}} + ef_2(c_{ij}) \qquad \tilde{\xi}_3^2 = \frac{1}{c_{44} + c_{11} - c_{12}} + ef_3(c_{ij})$$

and

$$f_2(c_{ij}) = \frac{c_{44} + 2c_{11} - 2c_{12}}{(c_{44} + c_{11} - c_{12})((c_{11} - c_{12})^2 - (c_{11} - c_{12})(c_{44} - e) - 2c_{44}^2)}$$

$$f_3(c_{ij}) = \frac{c_{11} - c_{12}}{(c_{44} + c_{11} - c_{12})((c_{11} - c_{12})^2 - (c_{11} - c_{12})(c_{44} - e) - 2c_{44}^2)}.$$

Thus, as above, it is possible to write the quantities $(\partial^2 q/\partial\xi_i\partial\xi_j)(P)$, with $i = 1, 2, 3$, as the same quantities in the case when $c_{12} - c_{13} = c_{44} - c_{66} = c_{11} - c_{33}$ plus e times a rational function of the stiffness constants c_{ij} (here and in the following of the proof, P will be one of the singular points in (2.5.1)). So, we can argue in the same way and prove that, if e is small enough, the signs of the eigenvalues of the Hessian of p at P and of the determinant of the Hessians when we assume $c_{12} - c_{13} = c_{44} - c_{66} = c_{11} - c_{33}$, do not change if we assume $c_{12} - c_{13} = c_{44} - c_{66} = 0$ and $c_{11} - c_{33} = e$. This concludes the proof. □

2.6 Curvature properties of the slowness surface

In order to obtain a decay estimate for the solutions of the system of crystal acoustics, it is necessary to study the curvature of the slowness surface. In fact, we want to apply some theorems which assure a decay estimate for oscillatory integrals defined on a surface, assuming that the surface satisfies some prescribed curvature properties. Thus, in the following subsections, we will study the curvature properties of the slowness surface of tetragonal crystals near the uniplanar singularities and far from all different singularities.

2.6.1 Curvature properties near the uniplanar singularities

We denote by $P = (0, 0, c_{44}^{-1/2})$ the double point of the slowness surface S on the positive ξ_3-semi-axis. We recall that the slowness surface is defined by the equation $p(\xi) = 0$ and that from the definition of uniplanar double point we have $p(P) = \nabla p(P) = 0$, $\nabla p(\xi) \neq 0$ if $\xi \in U \setminus \{P\}$, where U is a small neighborhood of P, and $(\partial/\partial\xi_3)^2 p(P) \neq 0$. Thus it is possible to write p near P in the form

$$\varphi(\xi)(\xi_3^2 + a(\xi')\xi_3 + b(\xi'))$$

for some smooth functions φ, a and b, where $\varphi(P) \neq 0$ and $\xi' = (\xi_1, \xi_2)$. Now we consider the sets

$$\Gamma^{\pm} = \{\xi' \in \mathbb{R}^2 : -J_2 a(\xi') \pm \sqrt{J_4 \Delta(\xi')} = 1\}, \tag{2.6.1}$$

where we denote by Δ the quantity $a^2(\xi') - 4b(\xi')$ and where the expression $J_k f$ is defined (2.3). We want to prove that the curves $\Gamma^{\pm}$ are smooth and of nowhere vanishing Gaussian curvature. To do that, first of all we recall the following results obtained in [L2].

Proposition 2.23. *Assume that*

$$\beta > 0, \quad \gamma + 1 > 0, \quad \alpha - \beta > 0, \quad 2\alpha^2 > \beta^2(\gamma + 1).$$

Then the curve

$$\left\{(\xi_1, \xi_2) \in \mathbb{R}^2 : \alpha(\xi_1^2 + \xi_2^2) + \beta\sqrt{\xi_1^4 + 2\gamma\xi_1^2\xi_2^2 + \xi_2^4} = 1\right\}$$

has no inflection points. Furthermore, the curve

$$\left\{(\xi_1, \xi_2) \in \mathbb{R}^2 : \alpha(\xi_1^2 + \xi_2^2) - \beta\sqrt{\xi_1^4 + 2\gamma\xi_1^2\xi_2^2 + \xi_2^4} = 1\right\}$$

will have inflection points, only if

$$(\alpha - \beta\gamma)(-\alpha\sqrt{2 + 2\gamma} + \beta(3 - \gamma)) \geq 0.$$

Now we will calculate $J_2 a$ and $J_4\Delta$. To do this we parametrize S near P by ξ' and assume that the three sheets of S are given in a neighborhood of the ξ_3-axis by the graph of the functions $\rho_j(\xi')$, $j = 1, 2, 3$. We label j in such a way that

$$\rho_1(0) = \rho_2(0) = c_{44}^{-1/2} \quad \text{and} \quad \rho_3(0) = c_{33}^{-1/2}. \tag{2.6.2}$$

Now, we observe that if we write $p(\xi) = \varphi(\xi)(\xi_3^2 + a(\xi')\xi_3 + b(\xi'))$, then $\varphi(P) \neq 0$, so $\xi_3^2 + a(\xi')\xi_3 + b(\xi') = (\xi_3 - \rho_1(\xi'))(\xi_3 - \rho_2(\xi'))$ and this yields $-a(\xi') = \rho_1(\xi') + \rho_2(\xi')$.
Moreover, it is possible to write

$$p(\xi) = A_0(\xi')\xi_3^6 + A_1(\xi')\xi_3^4 + A_2(\xi')\xi_3^2 + A_3(\xi'),$$

where A_i are functions of ξ_1^2 and ξ_2^2 (here we recall that p is a homogeneous polynomial of order six). Thus, using Cardano's formula it is possible to write $\rho_3^2(\xi')$ as a function of ξ_1^2 and ξ_2^2. So, using the properties of polynomials of order three, we have $\rho_1^2 + \rho_2^2 + \rho_3^2 = -(A_1/A_0)$ and we have that $\rho_1^2 + \rho_2^2$ is again a function of ξ_1^2 and ξ_2^2. We can conclude that $J_2 a(\xi') = \alpha\xi_1^2 + \beta\xi_2^2$ and it is clear by symmetry that $\alpha = \beta$.
Thus, we need to calculate α. To do this, we consider $p(\xi_1, 0, \rho(\xi_1))$. We have

$$p(\xi_1, 0, \rho(\xi_1)) = d_2(\xi_1, 0, \rho(\xi_1))(n_1 d_3 + n_3 d_1 - d_1 d_3)(\xi_1, 0, \rho(\xi_1))$$

and from $d_2(\xi_1, 0, \rho(\xi_1)) = 0$ we obtain the smooth root

$$\nu_1(\xi_1) = c_{44}^{-1/2}\sqrt{1 - c_{66}\xi_1^2}.$$

Thus $\nu_1(0) = c_{44}^{-1/2}$, $\nu_1'(0) = 0$ and $\nu_1''(0) = -c_{66}/c_{44}^{1/2}$. In order to take into account the remaining sheet which passes through the singular point, we consider the smooth solution $\nu_2(\xi_1)$ of

$$(n_1 d_3 + n_3 d_1 - d_1 d_3)(\xi_1, 0, \rho(\xi_1)) = 0, \tag{2.6.3}$$

which satisfies $\nu_2(0) = c_{44}^{-1/2}$. It's clear that $\nu_2'(0) = 0$, so we want to compute $\nu_2''(0)$. To do this we can just derivate (2.6.3) twice in ξ_1 and then set $\xi_1 = 0$. It follows, after some calculations, that

$$\nu_2'(0) = \frac{(c_{13} + c_{44})^2 - c_{11}(c_{33} - c_{44})}{\sqrt{c_{44}}(c_{33} - c_{44})}.$$

Now, we observe that $\nu_1(\xi_1) + \nu_2(\xi_1) = \rho_1(\xi_1, 0) + \rho_2(\xi_1, 0)$ and so we will have $(\partial/\partial\xi_1)^2(\rho_1(0,0) + \rho_2(0,0)) = \nu_1''(0) + \nu_2''(0)$. Thus we conclude that

$$\alpha = \frac{(c_{13} + c_{44})^2 - c_{11}(c_{33} - c_{44})}{\sqrt{c_{44}}(c_{33} - c_{44})} - \frac{c_6 6}{\sqrt{c_{44}}} = \frac{(c_{13} + c_{44})^2 - (c_{11} + c_{66})(c_{33} - c_{44}))}{\sqrt{c_{44}}(c_{33} - c_{44})}$$

We now turn to the calculation of $J_4\Delta$. From the above discussion it follows that $\Delta = a^2 - 4b = (\rho_1 - \rho_2)^2$. Furthermore, we denote by D the discriminant of the polynomial $q(t) = A_0 t^3 + A_1 t^2 + A_2 t + A_3$. D can be written in terms of the coefficient of the polynomial q as

$$A_1^2 A_2^2 - 4A_0 A_2^3 - 4A_1^3 A_3 - 27A - 0^2 A_3^2 + 18 A_0 A_1 A_2 A_3, \tag{2.6.4}$$

and also in terms of the roots of q as

$$A_0^4 \left[(\rho_1^2 - \rho_2^2)(\rho_1^2 - \rho_3^2)(\rho_2^2 - \rho_3^2)\right]^2$$

(it is obvious that ρ_i is a root of $p(\xi)$ if and only if ρ_i^2 is a root of $q(t)$). Thus we can conclude that

$$J\Delta = A_0^4 \mu J_4 D$$

where

$$\mu = [(\rho_1 + \rho_2)^2(0)(\rho_1^2 - \rho_3^2)^2(0)(\rho_2^2 - \rho_3^2)^2(0)].$$

So it is easy to calculate μ if we take into account (2.6.2). We obtain

$$\mu = \frac{4}{c_{44}} \left(\frac{c_{33} - c_{44}}{c_{33} c_{44}}\right)^4 = \frac{4(c_{33} - c_{44})^4}{c_{33}^4 c_{44}^5}.$$

The expression of D can be calculated explicitly using (2.6.4). Here all the coefficients depend explicitly on ξ_1^2, ξ_2^2 and thus they do not directly depend on ξ_1, ξ_2. We conclude that J_4D is a polynomial in ξ_1^2, ξ_2^2 and we can calculate it using Maple. In particular we have

$$A_0 = c_{44}^2 c_{33},$$

$$\begin{aligned} A_1(\xi') = (c_{33}(c_{11}+c_{66}) - (c_{13}+c_{44})^2 + c_{44}^2))c_{44}(\xi_1^2+\xi_2^2) \\ - (c_{44}+2c_{33})c_{44}, \end{aligned}$$

$$\begin{aligned} A_2(\xi') = 2c_{44} + c_{33} + \left[(c_{13}+c_{44})^2 - (c_{44}+c_{33})(c_{11}+c_{66}) - 2c_{44}^2\right](\xi_1^2+\xi_2^2) \\ + (-2c_{11}c_{13}^2 + 4c_{66}c_{44}^2 - c_{12}^2c_{33} - 2c_{33}c_{66}c_{12} \\ + 2c_{12}c_{13}^2 + 2c_{12}c_{44}^2 + 2c_{66}c_{13}^2 - 4c_{11}c_{13}c_{44} + 4c_{12}c_{13}c_{44} + 4c_{66}c_{13}c_{44} + c_{11}^2c_{33})\xi_1^2\xi_2^2 \\ + (c_{11}c_{66}c_{33} - c_{66}c_{13}^2 + c_{44}^2c_{11} - 2c_{66}c_{13}c_{44})(\xi_1^4+\xi_2^4), \end{aligned}$$

$$\begin{aligned} A_3(\xi') = \left[c_{11}c_{66}(\xi_1^4+\xi_2^4) - (c_{66}+c_{11})(\xi_1^2+\xi_2^2)\right. \\ \left. + (c_{11}^2 - c_{12}^2 - 2c_{12}c_{66})\xi_1^2\xi_2^2\right]\left[c_{44}(\xi_1^2+\xi_2^2) - 1\right], \end{aligned}$$

and

$$J_4D = 2c_{44}^2\delta(\xi_1^2+\xi_2^2) - 2c_{44}^2\epsilon\xi_1^2\xi_2^2,$$

where ε and δ are polynomials in c_{ij}. In the general tetragonal case these expressions are quite involved, and thus it is quite difficult to study the curvature of $\Gamma^\pm$. Therefore we will only consider the nearly cubic case, i.e. we introduce the following relations on the stiffness constant

$$c_{11} - c_{33} = c_{44} - c_{66} = c_{12} - c_{13},$$

and we assume $e = c_{11} - c_{33} = c_{44} - c_{66} = c_{12} - c_{13}$ to be small. We observe that, if $e = 0$, we are precisely in the cubic case. We also return to the notations used for the cubic case. In particular, we set $c_{13} = c_{12} + e$,

$c_{66} = c_{44} + e$, $c_{33} = c_{11} + e$ and $c_{44} = c$, $c_{12} = b - c$ and $c_{11} = c + b - d$. With these notations we can write

$$\begin{aligned} J_4 D &= c^2(e+b-d)^2(2e^2+2eb-d^2+2bd)^2(\xi_1^4+\xi_2^4) - c^2(e+b-d)^2 \\ &(4e^2b^2+8eb^2d+2b^2d^2+8e^3b-8bd^2e+8e^2bd-4bd^3+d^4-6e^2d^2+4e^4)\xi_1^2\xi_2^2 \\ &= \left[c^2d^2(b-d)^2(2b-d)^2+ec^2D_1\right](\xi_1^4+\xi_2^4) \\ &\quad - \left[c^2d^2(2b^2+d^2-4bd)(b-d)^2+ec^2D_2\right]\xi_1^2\xi_2^2, \end{aligned} \tag{2.6.5}$$

where

$$\begin{aligned} D_1 = (2d^3 - ed^2 - 6bd^2 + 4b^2d - 2e^2d + 4e^2b + 2eb^2 + 2e^3) \\ (-d^2 + 2b^2 - 2ed + 4eb + 2e^2) \end{aligned}$$

and

$$\begin{aligned} D_2 = 4e^5 + 16e^4b - 8e^4d + 24e^3b^2 - 16e^3bd - 2e^3d^2 - 28e^2bd^2 + 16e^2b^3 + 12e^2d^3 \\ - 48eb^2d^2 - 5ed^4 + 32ebd^3 + 16eb^3d + 4eb^4 - 20b^3d^2 + 12b^2d^3 + 2bd^4 \\ - 2d^5 + 8b^4d. \end{aligned}$$

Thus, if we assume $d \neq 0$, it is possible to write $J_4 D = d^2 J_4' D$, where

$$\begin{aligned} J_4' D = \left[c^2(b-d)^2(2b-d)^2 + ed^{-2}c^2D_1\right](\xi_1^4+\xi_2^4) \\ - \left[c^2(2b^2+d^2-4bd)(b-d)^2 + ed^{-2}c^2D_2\right]\xi_1^2\xi_2^2. \end{aligned}$$

We observe that, if we set $e = 0$ in (2.6.5), we find exactly the expression of $J_4 D$ in the case of a cubic crystal (see [L2], page 185). Moreover, with this notation, we have

$$J_4\Delta = \frac{4(b+e-d)^4}{c^3(c+b+e-d)^3} J_4 D = \frac{4(b-d)^4}{c^3(c+b-d)^3} J_4 D + eD_3 J_4 D, \tag{2.6.6}$$

where D_3 is an opportune rational homogeneous polynomial of order -5 in the variable b, c, d, e. We can rewrite the constant α calculated above, in terms of b, c, d and e. We have

$$\begin{aligned} \alpha &= \frac{(e+b)^2}{\sqrt{c}(e+b-d)} - \frac{2c+b-d+3}{\sqrt{c}} = -2\sqrt{c} - d\frac{2b+2e-d}{\sqrt{c}(e+b-d)} \\ &= -2\sqrt{c} - d\frac{2b-d}{\sqrt{c}(b-d)} + e\frac{d^2\sqrt{c}}{c(b-d)^2+ce(b-d)}. \end{aligned} \tag{2.6.7}$$

As before, we observe that, if we set $e = 0$ in (2.6.7) and (2.6.6), again we find precisely the constants of the cubic case. Furthermore, we note that $c + b - d = c_{11} > 0$ and that, if $b - d = c_{11} - c_{44} = 0$, (here we are still assuming $e = 0$) then α tends to infinity and $J_4\Delta$ vanishes of order five and so proposition (2.23) is easily satisfied. Thus, we have shown that the quantities $-J_2 a \pm \sqrt{J_4\Delta}$ are of form

$$\begin{aligned} - J_2 a \pm \sqrt{J_4\Delta} = 2\sqrt{c}(\xi_1^2 + \xi_2^2) + dQ_1(\xi_1, \xi_2, d) + eQ_2(\xi_1, \xi_2, d, e) \\ \pm |d|\sqrt{Q_3(\xi_1, \xi_2, d) + eQ_4(\xi_1, \xi_2, d, e)}, \quad (2.6.8) \end{aligned}$$

where Q_1 and Q_2 are polynomials of order two, and Q_3 and Q_4 are polynomials of order four in (ξ_1^2, ξ_2^2), with coefficients which are analytic in d and e. We recall that the sets $\Gamma^{\pm}$ are defined in terms of $-J_2 a \pm \sqrt{J_4\Delta}$.
We recall that, if $e = 0$, then

$$-J_2 a \pm \sqrt{J_4\Delta} = 2\sqrt{c}(\xi_1^2 + \xi_2^2) + dQ_1(\xi_1, \xi_2, d) \pm |d|\sqrt{Q_3(\xi_1, \xi_2, d)}, \quad (2.6.9)$$

and it is well known that there exist constants b, c and d, such that the curves, defined by

$$2\sqrt{c}(\xi_1^2 + \xi_2^2) + dQ_1(\xi_1, \xi_2, d) \pm |d|\sqrt{Q_3(\xi_1, \xi_2, d)} = 1,$$

have Gaussian curvature different from zero everywhere (for detail see [L1], pages 17-20). In particular, this means that there exists a cubic crystal for which the required curvature properties near the uniplanar singularities hold. Now, we rewrite expression (2.6.9) in the form

$$\alpha(\xi_1^2 + \xi_2^2) + \beta\sqrt{\xi_1^4 + 2\gamma\xi_1^2\xi_2^2 + \xi_2^4},$$

whereas we rewrite expression (2.6.8) in the form

$$\alpha'(\xi_1^2 + \xi_2^2) + \beta'\sqrt{\xi_1^4 + 2\gamma'\xi_1^2\xi_2^2 + \xi_2^4},$$

where α, β, γ, α', β' and γ' are opportune functions of b, c, d and e. We are ready to state the following proposition.

Proposition 2.24. *Let b, c and d be fixed and such that α, β and γ satisfy the hypothesis of proposition* (2.23) *and the condition*

$$(\alpha-\beta\gamma)(-\alpha\sqrt{2+2\gamma}+\beta(3-\gamma))<0.$$

Then, there exists a small real number e such that α', β' and γ' still satisfy the hypothesis of proposition (2.23) *and the condition*

$$(\alpha'-\beta'\gamma')(-\alpha'\sqrt{2+2\gamma'}+\beta'(3-\gamma'))<0.$$

Proof. The proof is trivial because the hypothesis of the proposition (2.23) and the condition $(\alpha-\beta\gamma)(-\alpha\sqrt{2+2\gamma}+\beta(3-\gamma))<0$ define an open subset A of $\mathbb{R}^3$ and we can write, in view of (2.6.6), (2.6.7) and (2.6.8), $\alpha'=\alpha+eD_\alpha$, $\beta'=\beta+eD_\beta$ and $\gamma'=\gamma+eD_\gamma$, where D_α, D_β and D_γ are smooth functions of b, c, d and e. Thus, if $(\alpha,\beta,\gamma)\in A$, then there exists a small e such that $(\alpha',\beta',\gamma')\in A$. □

From this proposition follows the desired result.

Proposition 2.25. *Assume $c_{13}=c_{12}+e$, $c_{33}=c_{11}+e$ and $c_{66}=c_{44}+e$. Then there exist stiffness constants c_{12}, c_{44}, c_{11} and a small real number e such that the sets $\Gamma^\pm$ are smooth and of nowhere vanishing Gaussian curvature.*

Remark 2.17. It is easy to see (cf. proposition (2.22)) that, if we assume $c_{44}=c_{66}$, $c_{11}-c_{33}=e_1$, and $c_{12}-c_{13}=e_3$, the quantities $-J_2a\pm\sqrt{J_4\Delta}$ are of form

$$\begin{aligned}2\sqrt{c}(\xi_1^2+\xi_2^2)&+dQ_1(\xi_1,\xi_2,d)+e_1\tilde{Q}_2(\xi_1,\xi_2,d,e_1,e_3)+e_3\tilde{Q}_3(\xi_1,\xi_2,d,e_1,e_3)\\ &\pm|d|\sqrt{\tilde{Q}_4(\xi_1,\xi_2,d)+e_1\tilde{Q}_5(\xi_1,\xi_2,d,e_1,e_3)+e_3\tilde{Q}_6(\xi_1,\xi_2,d,e_1,e_3)},\end{aligned}$$

where Q_1, $\tilde{Q}_2$, $\tilde{Q}_3$ are polynomials of order two and $\tilde{Q}_4$, $\tilde{Q}_5$, $\tilde{Q}_6$ are polynomials of order four in (ξ_1,ξ_2), with coefficients which are analytic in d, e_1 and e_3.

Indeed, if we consider J_2a and $J_4\Delta$ as functions of (e_1,e_3), we have that, if (e_1,e_3) tends to $(0,0)$, then $J_2a(e_1,e_3)\pm\sqrt{J_4\Delta(e_1,e_3)}$ tends to (2.6.9). Thus,

if we use the Taylor expansion of $J_2 a(e_1, e_3)$ and $J_4\Delta(e_1, e_3)$ when (e_1, e_3) tends to $(0, 0)$, we obtain the desired result, with $\tilde{Q}_2 = \partial_{e_1} J_2 a(0, 0) + \mathcal{O}(e_1^2)$, $\tilde{Q}_3 = \partial_{e_3} J_2 a(0, 0) + \mathcal{O}(e_3^2)$, $\tilde{Q}_5 = \partial_{e_1} J_4\Delta(0, 0) + \mathcal{O}(e_1^2)$, $\tilde{Q}_6 = \partial_{e_3} J_4\Delta(0, 0) + \mathcal{O}(e_3^2)$ and $\tilde{Q}_4(\xi_1, \xi_2, d) = Q_3(\xi_1, \xi_2, d)$. Moreover, it is not difficult to write down the expressions of the $\tilde{Q}_i$, but they are quite involved and since we will not need to use them later, we will not write them here.
Thus, in this case, it is possible to prove a proposition similar to (2.24) and so the result of corollary (2.25) still remains valid if we assume $c_{44} = c_{66}$, $c_{11} - c_{33} = e_1$ and $c_{12} - c_{13} = e_3$.

We conclude this section proving that the total curvature near the uniplanar singularities of the external sheet of the slowness surface is positive.

Proposition 2.26. *Assume $c_{44} = c_{66}$, $c_{11} - c_{33} = e_1$ and $c_{12} - c_{13} = e_3$. If $|e_i|$, $i = 1, 3$ are sufficiently small, then the total curvature of the external sheet of the slowness surface is positive near the uniplanar singularities.*

Proof. We prove the proposition for the uniplanar singularity on the ξ_3-semi-axis. As before, we denote by $P = (0, 0, c_{44}^{-1/2})$ the double point of the slowness surface S on the positive ξ_3-semi-axis. It follows from our study of the local discriminant that the defining equation for the two sheets which pass through P can be written locally in the form:

$$\xi_3^{\pm} = Q_1 + e_1\tilde{Q}_2 + e_3\tilde{Q}_3 + f_1 \pm |d|\sqrt{\tilde{Q}_4 + e_1\tilde{Q}_5 + e_3\tilde{Q}_6 + f_2}$$

where the functions Q_1, $\tilde{Q}_i$, with $i = 2, 3, 4, 5, 6$ and f_j, with $j = 1, 2$, have the following properties:

(i) Q_1 is a positive definite quadratic form in the variable (ξ_1, ξ_2) with coefficients which depend in a C^∞ way on d.

(ii) $\tilde{Q}_i$, with $i = 2, 3$ are functions of $(\xi_1, \xi_2, d, e_1, e_2)$ and are homogeneous polynomials of order two in the variable (ξ_1, ξ_2) with coefficients which depend in a C^∞ way on d, e_1, e_3.

(iii) There exist constants c_i, with $i = 1, 2, 3$ such that $Q_1 \geq c_1(\xi_1^2 + \xi_2^2)$ and $\tilde{Q}_i \geq c_i(\xi_1^2 + \xi_2^2)$ for $i = 1, 2$.

(iv) $\tilde{Q}_4$ is a polynomial of order four in the variable (ξ_1, ξ_2) with coefficients which depend in a C^∞ way on d, whereas $\tilde{Q}_i$, with $i = 5, 6$ are homogeneous polynomials of order four in the variable (ξ_1, ξ_2) with coefficients which depend in a C^∞ way on d ,e_1, and e_3.

(v) There exist constants c_i such that $\tilde{Q}_i \geq c_i|(\xi_1, \xi_2)|^4$ for $i = 4, 5, 6$.

(vi) f_j with $j = 1, 2$ are C^∞-functions of $(\xi_1, \xi_2, d, e_1, e_2)$, defined in a neighborhood of zero, such that $\partial_{\xi_1}^k \partial_{\xi_2}^l f_1(0) = 0$ for $k + l \leq 2$ and $\partial_{\xi_1}^k \partial_{\xi_2}^l f_2(0) = 0$ for $k + l \leq 4$.

We also know that in the cubic case, i.e. if $e_1 = e_3 = 0$, there exists a small d such that the total curvature of ξ_3^+ calculated for (ξ_1, ξ_2) near $(0, 0)$ is positive. Now, we fix such d and we consider the total curvature of ξ_3^+ as a continuos function F of (e_1, e_3) which is positive in the origin. Indeed, we recall that the total curvature of ξ_3^+ is given by

$$\partial^2_{\xi_1\xi_1}\xi_3^+ \partial^2_{\xi_2\xi_2}\xi_3^+ - (\partial^2_{\xi_1\xi_2}\xi_3^+)^2$$

which is a continuous function of (e_1, e_3), in view of the previously enumerated properties. Thus, by continuity, there exists a neighborhood U of $(0, 0)$ such that, if $(e_1, e_3) \in U$, then $F(e_1, e_3) > 0$. This concludes the proof. □

2.6.2 Curvature properties near the conical singularities and in the regular regions

The aim of this section is to prove three main results. The first result is that, if we denote by S the slowness surface of some tetragonal crystal and if $P \in S$ is sufficiently away from the double points of S, then the Gaussian curvature of S at P is strictly positive. The second one is the following: if $P \in S$ is a regular point for which the Gaussian curvature vanishes, then the mean curvature of S at P does not vanish. The third one is that, generically,

there are no planes which are tangent to S along entire nontrivial curves. To prove these results we will use a perturbative argument. In fact, the slowness surface associated with a cubic crystal satisfies these curvature properties and so, by continuity, the slowness surface associated with a tetragonal crystal will also fulfill the same properties if we are in a nearly cubic case.
These three results on the curvature of S are crucial because we want to use, in the following of this work, some well known theorems which assure a decay estimate for oscillatory integrals on S, if the previous curvature properties are satisfied.
We start with the following definition which clarifies the notion of nearly cubic crystal, which we already used in the previous part of this work (cf. remark (2.15), proposition (2.22) and the previous section).

Definition 2.4. *We denote the first octant in* $\mathbb{R}^3$ *by* $\mathcal{O} = \{\xi; \xi_i > 0, i = 1, 2, 3\}$.
Let K *be a cubic crystal associated with the constants* $a = c_{11} - c_{44}, b = c_{12} + c_{44}$, $c = c_{44}$. *We denote by* $e_1 = c_{33} - c_{11}$, $e_2 = c_{66} - c_{44}$, $e_3 = c_{13} - c_{12}$. *We call the quantities* e_i *the tetragonal excess of the crystal and if all* e_i *are small, we say that the tetragonal crystal is nearly cubic.*
This is justified by the fact that when $e_i = 0$ *for* $i = 1, 2, 3$, *we have* $c_{11} = c_{44}$, $c_{44} = c_{66}$, $c_{12} = c_{13}$, *which means that the crystal is cubic (if we already know that it is tetragonal).*
Also we denote by $Q(e_1, e_2, e_3)$ *the tetragonal crystal associated with the constants* $c_{33} = c_{11} + e_1$, $c_{66} = c_{44} + e_2$, $c_{13} = c_{12} + e_3$.
Finally, we denote by $S(e_1, e_2, e_3)$ *the slowness surfaces of* $Q(e_1, e_2, e_3)$, *and by* $S_e(e_1, e_2, e_3)$, $S_m(e_1, e_2, e_3)$, $S_i(e_1, e_2, e_3)$ *the exterior, middle and inner sheets of* $S(e_1, e_2, e_3)$.

In this section we start from a cubic crystal K with slowness surface given as in (1.3.13). We recall that in (1.3.13), K is associated with the constants a, b, c, defined in the previous definition, and that we denote the quantity $b - a$ by d. The following result has been proved in [L2]:

Theorem 2.27. *Assume that $d = b - a$ is sufficiently small (compared to c), $b > 0$, and denote by S the slowness surface of the crystal defined in (1.3.13). Then the following conclusions hold:*

a) There is no plane which is tangent to S along an entire curve.

b) The total (Gaussian) and mean curvatures never vanish simultaneously in the smooth portion of S.

c) Let Γ be an open cone in $\mathbb{R}^3$ which does not contain the 14 singular points of the slowness surface S and such that if $(\xi_1, \xi_2, \xi_3) \in S$, then there exist $i \in \{1, 2, 3\}$ such that $|\xi_i| \geq c_i|(\xi_{i+1}, \xi_{i+2})|$ (here the indices are counted modulo 3) for some constant $c_i > 2^{-1/2}$. Then the total curvature of S does not vanish in $S \cap \Gamma$.

d) If Γ is fixed as in c), then the three sheets S_e, S_m, S_i of K stay at positive distance from one another in $S \cap \Gamma$.

e) If we denote by $P = ((3c-d)^{-1/2}, (3c-d)^{-1/2}, (3c-d)^{-1/2})$ the conically singular point of S in the first octant, then there is a small open cone Γ' which contains P, such that the total curvature of S_e is negative in Γ'.

f) If P is one of the points of K on an axis, then there is a conic neighborhood Γ of P such that the total curvature of P is strictly positive for smooth points in $S_e \cap \Gamma$.

The main result in this section is that statements similar to theorem (2.27) hold for a tetragonal crystal if this is sufficiently close to the crystal K. We may roughly say that K is *nearly isotropic* and that the tetragonal crystals which we study are *nearly cubic*. We should say that the argument in [L2] gives a direct proof of our result for tetragonal crystals which are just nearly isotropic (Of course the tetragonal crystals which we consider are close to the isotropic case, but closeness is defined in a two-step approach: firstly, d has to be small and then, once d is fixed, the d_i must be even smaller, with

smallness defined in terms of d).
Since we have no new contributions with respect to [L2], we think that it is not necessary to repeat the argument found in [L2] for tetragonal crystals (in order to extract all information which one can obtain by arguing directly for a given tetragonal crystal, rather than considering tetragonal crystals which are small perturbations of cubic crystals), and hope that better results can be obtained with a completely different approach.
Now we are ready to state the first result.

Proposition 2.28. *We fix open cones* Γ, Γ' *such that* $(1,1,1) \in \Gamma' \subset\subset \Gamma \subset \mathcal{O}$*, and such that the distance between the sheets of the slowness surface of the crystal* K *is positive in* $\Gamma \setminus \Gamma'$*. If* Γ *is small (it suffices to have that* Γ *does not intersect the* $\xi_3 = 0$ *plane), these sheets can be represented as graphs of functions* $(\xi_1, \xi_2) \to \xi_3(\xi_1, \xi_2)$ *in* Γ*. Assume* $|e_i| \leq \delta$*.*
Then if δ *is small enough, the singular point* $P(e_1, e_2, e_3)$ *of* $S(e_1, e_2, e_3)$ *in* $\mathcal{O}$ *will stay in* Γ'*. Moreover, the distance between the sheets* $S_e(e_1, e_2, e_3)$*,* $S_m(e_1, e_2, e_3)$*,* $S_i(e_1, e_2, e_3)$ *in* $\Gamma \setminus \Gamma'$ *will be bigger than* $\tilde{\delta} > 0$ *for some constant* $\tilde{\delta}$ *which does not depend on the* e_i*. In particular, the curvatures of* $S_*(e_1, e_2, e_3)$ *will depend in a continuous way on the constants* e_i *and therefore the total and mean curvatures cannot vanish simultaneously in* $\Gamma \setminus \Gamma'$ *if* δ *is small enough.*

Proof. The first statement follows from the expressions in proposition (2.12) (see also proposition (2.22)), which give the conically singular points near the diagonals of a tetragonal crystal. As for the second statement, we already know that the sheets of $S(0,0,0)$ stay at a positive distance from one another in $\Gamma \setminus \Gamma'$. If Γ is not too large, the sheets of the surfaces $S_*(e_1, e_2, e_3)$ can be represented in Γ as graphs of some functions in the variables (ξ_1, ξ_2), too. Again after assuming δ to be small, we may assume that these functions are smooth when their graphs stay in $\Gamma \setminus \Gamma'$ and depend in a continuous way on the constants e_i, since they are given by Cardano's formulas and we are staying away from the singularities (which correspond to the points where the discriminant in the Cardano's formulas vanishes). Thus, by continuity, there exists a neighborhood U of the origin in $\mathbb{R}^3$ such that, if $(e_1, e_2, e_3) \in U$, then the sheets of $S(e_1, e_2, e_3)$ stay at a positive distance from one another in $\Gamma \setminus \Gamma'$. Using the same argument it is possible to prove the last part of the proposition, because the total curvature of $S(0, 0, 0)$ is strictly positive and the expressions of the total curvatures of the sheets of $S(e_1, e_2, e_3)$ are continuous functions of (e_1, e_2, e_3). (See also the proposition (2.26)). □

Proposition 2.29. *Let Γ be an open cone in $\mathbb{R}^3$ with the properties of point c) in proposition* (2.27)*, i.e. Γ does not contain the singular directions of the slowness surface and it is not on the space diagonals, then the total curvature of the slowness surface $S(e_1, e_2, e_3)$ is strictly positive for all points $\xi \in S(e_1, e_2, e_3) \cap \Gamma$.*

Proof. We note that the total curvature of $S(0, 0, 0)$ is strictly positive for all points $\xi \in S(e_1, e_2, e_3) \cap \Gamma$. Thus the proof is precisely as in the second part of the previous proposition. □

Remark 2.18. Now we would state a proposition similar to (2.28), in the case of singular points near the ξ_i-axes, $i = 1, 2$. The main difference is that, in the cubic case, we have uniplanar singularities on the axes, whereas when we move to the tetragonal case each uniplanar singularity gives rise to two conical singularities which lie near the axes. First it doesn't seem difficult to prove a proposition like the previous one in this case, but there are some problems when we want to prove that there are no planes which are tangent to the slowness surface along an entire curve. Indeed, passing from a uniplanar to a conical singularities, the sign of the curvature near the singular point changes from positive to negative and so we cannot exclude the possibility that the tangency of a plane in one point became tangency along an entire curve.
Thus, we assume in the following that $c_{44} = c_{66}$, i.e. $e_2 = 0$. In fact, in this case, it is easy to see that the double points on the coordinate plane $\xi_3 = 0$ lie on the axes (see (2.0.5) and (2.0.6)) and that these singularities are of uniplanar type (see the proof of proposition (2.20)).

With this assumption, using arguments similar to those used in the proof of the previous proposition, it is possible to prove the desired curvature properties.

Proposition 2.30. *We assume here that $b > 0$ and pick an open cone Γ which contains an axis. If the e_i are sufficiently small, then the singular points of $S(e_1, e_2, e_3)$ close to the chosen axis stay in Γ. Moreover, the total curvature of $S_e(e_1, 0, e_3)$ in a neighborhood of the uniplanar singularities is strictly positive.*

Remark 2.19. It is possible to prove the same result with $b < 0$ if we replace S_e with S_i. For details see [L2].

Proof. If we now take a tetragonal crystal $Q(e_1, e_2, e_3)$ with e_1, e_2, e_3 sufficiently small, and if we fix an open cone Γ which contains the axes, then it follows, by an argument similar to the one used in the previous proof, that the singular points of the surfaces $S_*(e_1, e_2, e_3)$ near the axes have to stay in Γ if $|e_i| \leq \delta$ and δ is small. In the case when $e_2 = 0$, i.e. $c_{44} = c_{66}$, we have seen that the singular points in question are uniplanar and lie on the coordinate axes. Thus, the second part of the statement follows from proposition (2.26). $\square$

We now come to the main result in this section.

Theorem 2.31. *Let K be a cubic crystal as in theorem 2.27. If $\delta > 0$ is sufficiently small and $|e_i| < \delta$, then there is no plane which is tangent to $S(e_1, 0, e_3)$ along an entire curve.*

The proof of theorem (2.31) will be given later in this section. We start by recalling the following result from [L2].

Proposition 2.32. *Let f be a real-valued polynomial on $\mathbb{R}^3$ of degree six such that, except for a finite number of points $P^1, \ldots, P^s \in S = \{\xi \in \mathbb{R}^3; f(\xi) = 0\}$, we have that $f(\xi) = 0$ implies $\nabla_\xi f(\xi) \neq 0$. We also assume that S is bounded and that there is a plane Σ which is tangent to S along a smooth*

curve Γ.
Then Σ *is tangent to* S *along an ellipse which contains* Γ. *In addition to this ellipse of tangency, there can at most be finite additional points at which* Σ *is tangent to* S.

Now, before we start the proof of theorem (2.31), we will make some additional consideration.
We then assume that for every $j > 0$ there are $|e_1^j| \leq 1/j, |e_2^j| \leq 1/j, |e_3^j| \leq 1/j$, and ellipses $\gamma^j \subset S(e_1^j, e_2^j, e_3^j)$ such that for each j there is a common tangent plane $Z(e_1, e_2, e_3)$ to $S(e_1^j, e_2^j, e_3^j)$ for all points in γ^j. Our aim is to show that a subsequence of these ellipses tends to an ellipse $\gamma \subset S$ which has the property that there is a plane Z which is tangent to S at all points of γ. Here S is of course the slowness surface of the cubic crystal K from which we started. The meaning of the statement that $\gamma^j \to \gamma$ will be explained later. One of the problems with this approach is that we must exclude that the ellipses γ^j in the subsequence shrink until the limit set is a point. We do this by obtaining some preliminary information on the location of the ellipses γ^j when the e_i^j are small. We first observe that (by symmetry) we may assume that all γ^j have points in $\mathcal{O}$ and that they can not have points on parts of the $S(e_1^j, e_2^j, e_3^j)$ where the total curvature is non vanishing. (See e.g., proposition 8.2 in [L2].) In particular, they can have no points on the inner sheets. (This is also clear by elementary considerations on the number of points in which a line in the tangent plane to $S(e_1^j, e_2^j, e_3^j)$ along γ^j intersects $S(e_1^j, e_2^j, e_3^j)$.)
Further information can be obtained by studying the suitable plane curves in $S(e_1, e_2, e_3)$. We fix some singular point $P(e_1, e_2, e_3)$ in $S(e_1, e_2, e_3)$ in the closure of and denote the coordinates of the projection of $P(e_1, e_2, e_3)$ in the ξ_3-plane by ξ_1^0, ξ_2^0. Also denote by $\mathcal{L}(\alpha, \beta)$ the curves

$$\mathcal{L}(\alpha, \beta) = \{\xi \in S_e(e_1, e_2, e_3) \cup S_m(e_1, e_2, e_3); \\ \alpha(\xi_1 - \xi_1^0) = \beta(\xi_2 - \xi_2^0), \xi_i > 0, i = 1, 2, 3\}$$

with $\alpha, \beta \in \mathbb{R}, \alpha^2 + \beta^2 = 1$. Arguing as in [L2] it follows that these curves have no inflection points when the crystal K is sufficiently close to the isotropic case and $|e_i| < \delta$ with δ sufficiently small.

Lemma 2.33. *Assume that we are given some ellipse $\gamma \subset S(e_1, e_2, e_3)$ such that there is a plane which is tangent to $S(e_1, e_2, e_3)$ at all points of γ. Then γ must lie completely in a small conic neighborhood of some singular conical point. Moreover, it has to lie on the outer sheet $S_e(e_1, e_2, e_3)$.*

Proof. We know that γ is a curve of vanishing total curvature, so it must lie in a neigborhood of a conical singular point. Now consider the curves $\mathcal{L}(\alpha, \beta)$ associated with this point. Since smooth branches of $\mathcal{L}(\alpha, \beta)$ have non vanishing plane curvatures, γ cannot intersect $\mathcal{L}(\alpha, \beta)$ on one single smooth branch, and in fact one point of the intersection must be on one smooth branch, the other on another smooth branch. This is only possible if the two branches both lie on the outer sheet of $S(e_1, e_2, e_3)$ and both branches must lie in $\mathcal{O}$.
In the case of a conical singularity it implies at least that both points have to lie on the outer sheet of $S(e_1, e_2, e_3)$. (See [L2] for a similar situation.) $\square$

Lemma 2.34. *Let $\delta > 0$ and $\Gamma' \subset \Gamma$ be small open conic neighborhoods of the space diagonal, as in proposition (2.28), and assume that the conical singular point in $\mathcal{O}$ of $S_e(e_1, e_2, e_3)$ which is close to the space diagonal stays in Γ and that the total curvature of $S_e(e_1, e_2, e_3)$ is strictly positive when we stay outside Γ in $\mathcal{O}$. Also assume that it is strictly negative in Γ'.*
If $\gamma \subset S(e_1, e_2, e_3)$ is an ellipse in Γ such that the points of γ have a common tangent plane, and if $|e_i| \leq \delta$ with δ small enough, then $\gamma \subset \Gamma \setminus \Gamma'$ and γ is a curve which encircles the space diagonal.

Proof. γ cannot contain points where the total curvature $S(e_1, e_2, e_3)$ is vanishing. It must then lie completely in $\Gamma \setminus \Gamma'$. The fact that γ encircles the space diagonal follows then from the fact that it must contain for every α, β as above, points from different smooth branches of $\mathcal{L}(\alpha, \beta)$. This shows that γ encircles the singular direction in Γ' (see the picture in section 9 of [L2]) and therefore also the space diagonal, since it has no points in Γ'. $\square$

Before we continue, we must specify now what we mean by stating that a family of ellipses tends to another ellipse.

Definition 2.5. *Let $\Sigma \subset \mathbb{R}^3$ be a plane and consider an ellipse $\gamma \subset \Sigma$ with center $P \in \mathbb{R}^3$.*
We call "maximal diagonal" and "minimal diagonal" the line segment in Σ which passes through P, and has end points on the ellipse and its length is maximal and minimal respectively, among the lenghts of all line segments which pass through the center P and have endpoints on γ. (We tacitly assume that γ is not a circle. If it is, then there is no need for our terminology.)
Now, let $\gamma^j \in \Sigma^j$ be a family of plane ellipses. We say that γ^j tends to the plane ellipse $\gamma \in \Sigma$ if the planes Σ^j in which the γ^j lie tend in the Grassmanian topology to the plane Σ, the centers of the ellipses γ^j tend to the center of γ and the end points of the maximal and minimal diagonals in the ellipses γ^j tend respectively to the end points of the maximal and minimal diagonals on γ.

Lemma 2.35. *We consider two open cones $\Gamma' \subset\subset \Gamma$ such that $(1, 1, 1) \in \Gamma'$. We also fix $P \in \Gamma' \setminus \{0\}$ and consider the plane ellipses $\gamma^j \subset \Gamma \setminus \Gamma'$ which encircle Γ'. Moreover let $\delta > 0$ be small enough and such that the distances in the set $\Gamma \setminus \Gamma'$ between the plane orthogonal to the diagonal which passes through P and the planes Σ^j which contain γ^j are smaller then δ. Then there exists a subsequence j_k and a plane ellipse γ of center P such that the ellipses γ^{j_k} tend to γ in the sense of the previous definition.*

Proof. The condition that $P \in \Gamma' \setminus \{0\}$ together with the assumption on the distance between the planes and the fact that the γ^j encircle Γ' implies that the size of the minimal diagonal of the γ^j remains bounded from below. The assumption $\gamma^j \subset \Gamma \setminus \Gamma'$, on the other hand, shows that the maximal diagonal of the γ^j remains bounded from above.
The statement now follows from compactness. □

We are now ready to conclude the proof of the main theorem.

Proof. (of theorem (2.31)). We proceed by contradiction and we assume that the conclusion of the theorem is false. We will then show that there is a curve γ on the slowness surface S of K such that the points of γ admit a common tangent plane to S.
We pick two open cones $\Gamma' \subset\subset \Gamma$ in $\mathcal{O}$. Let $Q(e_1^j, 0, e_3^j)$ be a sequence of tetragonal crystals with $e_i^j \to 0$ when j tends to infinity and for which we can find plane curves γ^j in $S(e_1^j, 0, e_3^j)$ which have (for each fixed j) a common tangent plane. We observe that $S(e_1^j, 0, e_3^j)$ always have uniplanar singularities on the coordinate axes. We know that these curves are ellipses and lemma (2.33) states that the curves γ^j lie in a conical neighborhood of the conical singular points of the planes $\xi_1^2 = \xi_2^2$ and that they lie on the outer sheet $S_e(e_1^j, 0, e_3^j)$. Moreover, lemma (2.34) shows that $\gamma_j \subset \Gamma \setminus \Gamma'$ and that the γ_j encircle Γ'. We also know from lemma (2.35) that a subsequence of these ellipses tends to a nontrivial ellipse γ in $\Gamma \setminus \Gamma'$. By what we have seen above, the surfaces $S(e_1^j, 0, e_3^j)$ depend in $\Gamma \setminus \Gamma'$ in a smooth way on the parameters e_1^j and e_3^j if these are small enough. It follows that the limit ellipse γ lies on S (which is the $S(d_1, 0, d_3)$ with $d_i = 0$) and that there is a plane which is tangent to all points of γ, which is a contradiction. $\square$

Chapter 3

Decay estimate for the solutions of the linear system

The main result of this section will be the following decay estimate for the solution (1.3.14) of the system of crystal acoustic for tetragonal crystals.

Theorem 3.1. *Assume that the stiffness constants c_{ij} satisfy the conditions:*

$$c_{ii} > 0, \quad i = 1, 3, 4, 6, \qquad c_{44} \neq c_{33}, \qquad c_{44} \neq c_{11},$$
$$c_{11} - c_{66} - c_{12} > 0 \qquad c_{33}(c_{12} + c_{66}) - (c_{13} + c_{44})^2 > 0,$$
$$c_{12}, c_{13} \text{ small when compared with } c_{ii}, \text{ for } i = 1, 3, 4, 6,$$

together with

$$c_{44} = c_{66}, \qquad c_{33} = c_{11} + e_1, \qquad c_{13} = c_{12} + e_3,$$

with $|e_i|$ sufficiently small for $i = 1, 3$. Then, there is a constant C_1, such that

$$|u(t, x)| \leq C_1 (1 + |t|)^{-1/2} \sum_{j=1}^{3} \sum_{|\alpha| \leq k} \left(||\partial_x^\alpha f_j||_1 + ||\partial_x^\alpha g_j||_1 \right), \tag{3.0.1}$$

for all $(t, x) \in \mathbb{R}^4$, for any solution of the Cauchy problem of the system (1.3.10), *with the initial data $u_j(0, x) = f_j(x)$, $\partial_t u_j(0, x) = g_j(x)$, $j = 1, 2, 3$, where f_j and g_j are smooth functions on $\mathbb{R}^3$ and have compact support.*

The overall strategy for proving results like this one is well established and is in particular similar with the one used in the related case of crystal acoustics for cubic crystals in [L1] and in the case of crystal optics in [L5]. We recall (as we have already observed in the first chapter) that the solution of the Cauchy problem

$$\frac{\partial^2}{\partial t^2} u_i(t,x) = \sum_{j,k,l=1}^{3} c_{ijkl} \frac{\partial^2}{\partial_{x_l}\partial_{x_j}} u_k(t,x) \qquad i = 1,2,3. \tag{3.0.2}$$

$$u_i(0,x) \equiv 0, \qquad \frac{\partial}{\partial t} u_i(0,x) = g_i(x) \qquad i = 1,2,3 \tag{3.0.3}$$

where $x \in \mathbb{R}^3$ and g_i, $i = 1,2,3$ are $\mathcal{C}^\infty$-functions on $\mathbb{R}^3$ with compact support, admits an explicit representation in terms of oscillatory integrals involving the Fourier transform of the Cauchy data. In particular we have that the solution of the Cauchy problem can be written in the following form, already described in the first chapter:

$$u_i(x,t) = \int_{\mathbb{R}^3} \sum_{p=1}^{6} \sum_{j=1}^{3} e^{it\tau_p(\xi) + i\langle x,\xi\rangle} T_{ipj}(\xi)\hat{g}_j(\xi) d\xi \qquad i = 1,2,3. \tag{3.0.4}$$

Thus, first of all, in order to obtain the desired decay it is essential to have some information about the regularity of the amplitude function, i.e. the function $T_{ipj}(\xi)\hat{g}_j(\xi)$. In particular, we want to find an explicit representation of the function $\xi \to T_{ipj}(\xi)$ and we want to know as much as we can about its regularity. We will discuss the properties of the function $T_{ipj}(\xi)$ in the first section of this chapter. In the same section, we will also see that the regularity of the amplitude function is sufficient in order to obtain the estimate (3.0.1) when t remains bounded.

Remark 3.1. In fact the function $\hat{g}_j(\xi)$ is the Fourier transform of the initial data and so its properties are easily established. In particular, the functions $\hat{g}_j(\xi)$ are in $\mathcal{S}(\mathbb{R}^3)$. in order to understand the singularities of the function $T_{ipj}(\xi)\hat{g}_j(\xi)$ it will then suffice to study the factor $T_{ipj}(\xi)$. In a similar way the decay properties of $T_{ipj}(\xi)\hat{g}_j(\xi)$ are easily read derived those of $T_{ipj}(\xi)$.

It follows that, when we prove (3.0.1), we may assume that t is large. Indeed, to obtain the actual decay, we will take into account the oscillatory character of the exponential $e^{it\tau_j(\xi)+i\langle x,\xi\rangle}$ when (t,x) tends to infinity and the way in which we will do this will depend on the relative position of x, t, ξ. Actually, using the fact that the function $\xi \to \tau_j(\xi) + i\langle x,\xi\rangle$ is homogeneous in ξ, we can reduce estimates of the integrals in (3.0.4) to integrals on the slowness surface. We will perform this reduction in section (3.3).
As we have discussed in great detail in the second chapter, the slowness surface is a three-sheeted surface and the three sheets are smooth surfaces except for a finite number of double points, i.e. points where two of the three sheets touch each other. Thus, during the proof, it will be convenient to separate the contributions coming from the singular points of the slowness surface from those coming from the regular points. Thus, it will be possible to use well known theorems about the estimate for Fourier transforms of surface carried densities. We will use different theorems, depending on the geometrical properties of the slowness surface. We will discuss all these cases in section (3.4).
In order to achieve this, we consider some direction $\xi_0 \in \mathbb{R}^3 \setminus \{0\}$ and a smooth function χ of $\mathbb{R}^3 \setminus \{0\}$ with the following properties:

(i) $\chi(\lambda\xi) = \chi(\xi)$ for all positive λ.

(ii) There exists a conic neighborhood Γ_{ξ_0} of ξ_0 such that, $\chi(\xi) \equiv 0$ for all $\xi \in \mathbb{R}^3 \setminus \Gamma_{\xi_0}$.

(iii) There exists a conic neighborhood $\Gamma'_{\xi_0} \subset\subset \Gamma_{\xi_0}$ of ξ_0 such that, $\chi(\xi) \equiv 1$ for all $\xi \in \Gamma'_{\xi_0}$.

Therefore, rather than directly estimating the solution of the Cauchy problem, we will estimate the expressions

$$I_{ipj} = \int_{\mathbb{R}^3} e^{it\tau_p(\xi)+i\langle x,\xi\rangle} T_{ipj}(\xi)\chi(\xi)\hat{g}_j(\xi)d\xi, \tag{3.0.5}$$

for some fixed $i = 1,2,3$, $p = 1,2,3,4,5,6$ and $j = 1,2,3$. If we can do this for every ξ_0, thereby obtaining an estimate as in (3.0.1) for I_{ipj}, we will also have proved theorem (3.1). It is clear that the cases $p = 4, 5, 6$ will be similar to the cases $p = 1, 2, 3$, so we shall deal only with the latter.

To conclude the chapter, we will prove, in analogy with the theorem used in section (3.4), a theorem, about the decay estimate of oscillatory integrals on surfaces with singular biplanar points. Indeed, once we prove this theorem, we will be able to estimate the expressions (3.0.5), when ξ_0 is the direction of a biplanar singular point of the slowness surface for tetragonal crystals, provided that the amplitude function is sufficiently regular. In particular, we recall that the slowness surface for a tetragonal crystal has biplanar double points on two coordinate axes when the relation $c_{11} = c_{66}$ is fulfilled.

3.1 Estimate of the amplitude function

The aim of this section is to prove an estimate for the amplitude function T_{ipj}, which appears in the expression of the amplitude function of the solution of the Cauchy problem for the system of crystal acoustics (3.0.4) and to study what kind of regularity it has. Here we put ourselves in a more general contest with respect to the case of the tetragonal crystal acoustics in a 3-space. We compare the following system, which generalizes the system of tetragonal crystal acoustics in $\mathbb{R}^3$, with the case of some formally similar systems in $\mathbb{R}^n$. They have the form

$$\frac{\partial^2}{\partial t^2} u_i(t,x) = \sum_{j,k,\ell=1}^{n} c_{ijk\ell} \frac{\partial^2}{\partial x_j \partial x_\ell} u_k(t,x), \quad i = 1, \ldots, n. \tag{3.1.1}$$

for $x \in \mathbb{R}^n$ and for some constants $c_{ijk\ell}$ which satisfy the following property (compare with (1.3.12))

$$A(\xi) = (\sum_{j,\ell=1}^{n} c_{ijk\ell} \xi_j \xi_\ell)_{i,k=1,\ldots,n},$$

is symmetric and positive definite for all $\xi \in \mathbb{R}^n$. Moreover, in analogy with the case of $\mathbb{R}^3$, we associate with (3.1.1) and $\xi \in \mathbb{R}^n$ the polynomial matrix

$$Q(\lambda, \xi) = \lambda I - A(\xi), \tag{3.1.2}$$

and its characteristic polynomial $P(\lambda, \xi) = \det Q(\lambda, \xi)$. We also associate Cauchy conditions as in (3.0.3) with the system and write the solutions of the Cauchy problem in a form similar to (3.0.4)

$$u_i(x,t) = \int_{\mathbb{R}^n} \sum_{p=1}^{2n} \sum_{j=1}^{n} e^{it\tau_p(\xi) + i\langle x, \xi \rangle} T_{ipj}(\xi) \hat{g}_j(\xi) d\xi \qquad i = 1, 2, \ldots, n. \tag{3.1.3}$$

for suitable functions T_{ipj}. The main estimate of the expression T_{ipj} is given in the following proposition.

Proposition 3.2. *We assume that the eigenvalues of the matrix A are simple except for a finite number of directions ξ and that there is a constant $C_2 > 0$*

such that $|\lambda(\xi)| \geq C_2$ *if* $|\xi| = 1$. *Let* $T_{ipj}(\xi)$ *be the function in* (3.1.3). *Then, the function* $T_{ipj}(\xi)$ *is a smooth function of* ξ *where the discriminant of the polynomial* $P(\lambda, \xi)$ *in the variable* λ *is different from zero and there exists a constant* C_3 *such that*

$$|T_{ipj}(\xi)| \leq C_3 |\xi|^{-1} \tag{3.1.4}$$

for all $i, p, j \in \{1, \dots, n\}$.

The proof of proposition (3.2) will be given later in this section. We start by recalling some facts about the solution of the Cauchy problem in question.

By assumption, $Q(\lambda, \xi)$ is a symmetric positive definite matrix for every $\xi \in \dot{\mathbb{R}}^n$. It follows that it has n positive eigenvalues $\lambda_1(\xi), \dots, \lambda_n(\xi)$ with n associated orthonormal eigenvectors $v^1(\xi), \dots, v^n(\xi)$. The components of the $v^p(\xi)$ are denoted $v_i^p(\xi)$, $i = 1, \dots, n$. We now denote $\tau_p(\xi) = \sqrt{\lambda_p(\xi)}$, and $\tau_{n+p}(\xi) = -\sqrt{\lambda_p(\xi)}$, with $p = 1, \dots, n$, where the square roots are taken to be non negative. Moreover, we can suppose, at least after a permutation of the indices p, that $\tau_1(\xi) \leq \cdots \leq \tau_n(\xi)$. Thus, if we consider the function $\exp\left[i\langle x, \xi\rangle + it\tau_p(\xi)\right]$, using the fact that

$$\lambda_p(\xi) v_i^p(\xi) = \sum_{j,k,\ell=1}^{n} c_{ijk\ell} \xi_j \xi_\ell v_k^p(\xi)$$

for all $i = 1, \dots, n$ and $\xi \in \dot{\mathbb{R}}^n$, it follows that we have

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2} e^{i\langle x,\xi\rangle + it\tau_p(\xi)} v_i^p(\xi) = -e^{i\langle x,\xi\rangle + it\tau_p(\xi)} \lambda_p(\xi) v_i^p(\xi) = \\
&= -e^{i\langle x,\xi\rangle + it\tau_p(\xi)} \sum_{j,k,\ell=1}^{n} c_{ijk\ell} \xi_j \xi_\ell v_k^p(\xi) = \sum_{j,k,\ell=1}^{n} c_{ijk\ell} \frac{\partial^2}{\partial x_j \partial x_\ell} e^{i\langle x,\xi\rangle + it\tau_p(\xi)} v_k^p(\xi)
\end{aligned}$$

for all $i, p \in \{1, \dots, n\}$. This means that the function $\exp\left[i\langle x, \xi\rangle + it\tau_p(\xi)\right] v^p(\xi)$ satisfies the system (3.1.1). We now consider the Cauchy problem associated with (3.1.1) at $t = 0$,

$$\begin{gathered}
\frac{\partial^2}{\partial t^2} u_i(t, x) = \sum_{j,k,\ell=1}^{n} c_{ijk\ell} \frac{\partial^2}{\partial x_l \partial x_\ell} u_k(t, x), \\
u_i(0, x) = 0, \quad \frac{\partial}{\partial t} u_i(0, x) = g_i(x).
\end{gathered}$$

for $i = 1, \ldots, n$. If we assume here that the g_i are in $C_0^\infty(\mathbb{R}^n)$, then we can make a partial Fourier transform in x and then have to solve, with the notation $\psi_i(\xi) = \hat{g}_i(\xi)$, the problem

$$\frac{\partial^2}{\partial t^2}\hat{u}_i(t,\xi) = \sum_{j,k,\ell=1}^{n} c_{ijk\ell}\xi_j\xi_\ell\hat{u}_k(t,\xi), \tag{3.1.5}$$

$$\hat{u}_i(0,\xi) = 0, \quad \frac{\partial}{\partial t}\hat{u}_i(0,\xi) = \psi_i(\xi). \tag{3.1.6}$$

for $i = 1, \ldots, n$, where $\hat{u}_i$ denotes the partial Fourier transform of u_i. Thus, a solution to this problem can be chosen in the form

$$\hat{u}_i(t,\xi) = \sum_{p=1}^{2n} \varphi_p(\xi) e^{it\tau_p(\xi)} v_i^p(\xi), \quad i = 1, \ldots, n,$$

provided that the scalar functions φ_p satisfy the conditions:

$$\sum_{p=1}^{2n} \varphi_p(\xi) v_i^p(\xi) = 0, \tag{3.1.7}$$

$$\sum_{p=1}^{2n} i\tau_p(\xi)\varphi_p(\xi) v_i^p(\xi) = \psi_i(\xi), \tag{3.1.8}$$

for $i = 1, \ldots, n$ and where the functions $v_i^{p+n}(\xi)$ are precisely the $v_i^p(\xi)$ for all $p, i \in \{1, \ldots, n\}$. The φ_j are uniquely defined by (3.1.7), so it makes sense to guess part of them. Since $\tau_{p+n} = -\tau_p$ and $v_i^{p+n}(\xi) = v_i^p(\xi)$, the second set of equations can for example be written as

$$\sum_{p=1}^{n} i\tau_p(\xi)(\varphi_p - \varphi_{p+n})(\xi) v_i^p(\xi) = \psi_i(\xi).$$

It is clear that (3.1.7) will hold if we assume, at least where the eigenvalues are simple, and choosing eigenvectors of norm one for every ξ, that

$$\begin{aligned} \varphi_{p+n}(\xi) &= -\varphi_p(\xi), \\ \varphi_p(\xi) &= \frac{1}{2i\tau_p(\xi)}\langle \psi(\xi), v^p(\xi)\rangle, \end{aligned} \tag{3.1.9}$$

for all $p = 1, \dots, n$ and $\xi \in \dot{\mathbb{R}}^n$. Note that by assuming $|v^p(\xi)| = 1$, then there exists a positive constant c_1 such that

$$|v_i^p(\xi)| \leq c$$

for all $p, i \in \{1, \dots, n\}$. Moreover, it follows from (3.1.9) that

$$\varphi_p(\xi) = \sum_{j=1}^{n} \frac{1}{2i\tau_p(\xi)} v_j^p(\xi)\psi_j(\xi), \tag{3.1.10}$$

for all $p = 1, \dots, n$. Thus, it is not difficult to find an explicit expression for $T_{ipj}(\xi)$ in terms of the eigenvalues and eigenvectors of $A(\xi)$. We note that

$$u_i(t,x) = \int_{\mathbb{R}^n} \sum_{p=1}^{2n} \varphi_p(\xi) e^{i\langle x,\xi\rangle + it\tau_p(\xi)} v_i^p(\xi) d\xi, \quad i = 1, \dots, n.$$

Thus, using the relation (3.1.10), it is possible to write the solution in the form (3.0.4) and obtain the desired explicit expression for $T_{ipj}(\xi)$. Indeed, we can conclude that

$$\varphi_p v_i^p(\xi) = \frac{1}{2i\tau_p(\xi)} \left(\sum_{j=1}^{n} v_j^p(\xi)\psi_j(\xi) \right) v_i^p(\xi) = \sum_{j=1}^{n} T_{ipj}(\xi)\psi_j(\xi),$$

where

$$T_{ipj}(\xi) = \frac{1}{2i\tau_p(\xi)} v_i^p(\xi) v_j^p(\xi). \tag{3.1.11}$$

Thus, in view of (3.1.11), in order to obtain the desired result on the regularity of T_{ipj}, we have to study the regularity of the eigenvectors and eigenvalues of $A(\xi)$. The following two lemmas will allow us to find an explicit expression for the eigenvectors and give us some information about their regularity.

Lemma 3.3. *Consider a polynomial $n \times n$ matrix M in the variable ξ. Assume that we are given a locally defined smooth function $\xi \to \lambda(\xi)$ of simple eigenvalues of the matrix.*
Then the eigenvector $v(\xi)$ associated with the eigenvalue $\lambda(\xi)$ can be chosen of form $(q_1(\xi, \lambda(\xi)), \dots, q_n(\xi, \lambda(\xi)))$, where $(\xi, \lambda) \to q_j(\xi, \lambda)$ are polynomials in the variables (ξ, λ).

Proof. We denote by $M^j(\lambda,\xi)$, $j \in \{1,\dots,n\}$ the columns of $M(\xi)-\lambda I$. Then

$$\sum_{j=1}^{n-1} \mu_j(\xi) M^j(\lambda(\xi),\xi) = M^n(\lambda(\xi),\xi) \tag{3.1.12}$$

for functions $\mu(\xi)$ which are uniquely defined since the $\lambda(\xi)$ are simple eigenvalues: the eigenvector associated with $\lambda(\xi)$ is then $(\mu_1(\xi),\dots,\mu_{n-1}(\xi),1)$. (Here we suppose that it is possible to write the nth-column in terms of the first $n-1$. This is always possible up to a permutation of the columns of $M(\xi)-\lambda I$).

We recall that the solution of a linear system of form $Ax=b$ where x and b are n-vectors and A is an $n\times n$ matrix is given by the Cramer's rule in the following form:

$$x_i = \frac{detA_i}{detA} \quad i=1,\dots,n,$$

where we denote by A_i the matrix formed by replacing the i-th column of A by the column vector b. We can now calculate $\mu_j(\xi)$ with Cramer's rule from the $(n-1)\times(n-1)$ system obtained from (3.1.12) by eliminating a row such that the remaining system is determined. We assume for simplicity that the last row has this property. It follows that the $\mu_j(\xi)$ are of form $Q_j(\xi,\lambda(\xi))/R(\xi,\lambda(\xi))$ for some polynomials Q_j, R, in the variables (ξ,λ). The vector

$$v(\xi) = (Q_1(\xi,\lambda(\xi)),\dots,Q_{m-1}(\xi,\lambda(\xi)),-R(\xi,\lambda(\xi)))$$

is then an eigenvector associated with $\lambda(\xi)$ of the desired form. (Here $q_j(\xi,\lambda)=Q_j(\xi,\lambda)$ for $j=1,\dots,n-1$ and $q_n=-R(\xi,\lambda)$.) □

Lemma 3.4. *Let M be an $n\times n$ symmetric polynomial matrix in the variable ξ, homogenous of degree two in ξ and positive defined. Assume that its eigenvalues $\lambda_p(\xi)$ are all simple for ξ in an open cone $\Gamma \subset \dot{\mathbb{R}}^n$. Then they are smooth functions of ξ in Γ, homogenous of degree two.*

Proof. Smoothness follows from the implicit function theorem and homogeneity is obvious. □

Using the previous lemmas, we can now find an explicit expression for the eigenvectors of $A(\xi)$ and give additional information on their regularity. Since we work with normalized eigenvectors, we should first normalize the eigenvectors obtained in lemma (3.3), which leads, with obvious notations, to

$$v_i^p(\xi) = \frac{q_i^p(\xi, \lambda_p(\xi))}{||(q_1^p(\xi, \lambda_p(\xi)), \dots, q_n^p(\xi, \lambda_p(\xi)))||}, \tag{3.1.13}$$

for all ξ such that the discriminant of $P(\lambda, \xi)$ does not vanish. We observe that in the case of the system (3.1.1), the entries of the matrix $A(\xi)$ are homogeneous of degree two in ξ and so it follows from the proof of lemma (3.3) that the polynomials $q_i(\xi, \lambda^2)$, with $i = 1, \dots, n$ are homogeneous of degree $2(n-1)$ as a function of (ξ, λ). Moreover, it follows from lemma (3.4), that the eigenvalues $\lambda_p(\xi)$ are smooth functions and thus the $q_i(\xi, \lambda(\xi))$ are also smooth functions of ξ for all $i = 1, \dots, n$. This implies that the eigenvectors $v^p(\xi)$ are smooth functions of $\xi \neq 0$, as long as the discriminant of $P(\lambda, \xi)$ does not vanish.
We are now ready to prove proposition (3.2).

Proof. (of proposition (3.2)). We recall that there exists a positive constant c such that $|v_i^p(\xi)| \leq c$ for all $i, p \in \{1, \dots, n\}$, and that the $\tau_p(\xi)$ are homogeneous of degree one. Thus, from the expression (3.1.11) it follows that the T_{ipj} are homogeneous of degree -1 and satisfy the estimate

$$|T_{ipj}(\xi)| \leq C_3 |\xi|^{-1} \qquad \forall \quad i, p, j \in \{1, 2, 3\}.$$

Here we have also used the assumption that $|\tau_p(\xi)| \geq C > 0$ if $|\xi| = 1$.
Finally, we have that, when ξ is such that the discriminant of $P(\lambda, \xi)$ is different from zero, we can write the $v_i^p(\xi)$ in the form (3.1.13) which are a smooth functions of ξ. From the fact that the $\tau_p(\xi)$ are also smooth functions of ξ and are different from zero for all $\xi \in \dot{\mathbb{R}}^n$, we can prove the first part of the statement and conclude the proof. □

We continue this section with one more remark about the explicit form of the amplitude function T_{ipj}. In fact, it is possible to find another explicit expression (and so prove using another way the proposition (3.2)) for the amplitude function which is a function only of the eigenvalues and that it is such like the formula written in remark (1.16).

Remark 3.2. We start by associating with the system (3.1.1) a scalar equation of order $2n$. In fact, if we denote by Δ_x the $n \times n$ partial differential operator on the right hand side of (3.1.1) and if we calculate the formal determinant of $\partial_t^2 I - \Delta_x$, then we get the scalar operator $P(\partial_t, \partial_x)$ and the u_i satisfy $P(\partial_t, \partial_x)u_i = 0$ for every i. P corresponds of course to the determinant of $Q(\lambda, \xi)$ in (3.1.2).
Now, let v be a solution of $P(\partial_t, \partial_x)v = 0$ in $C^\infty(\mathbb{R}^{1+n})$ such that for every t the support in x of the function $x \to v(t, x)$ is compact. If we denote by $\hat{v}(t, \xi)$ the partial Fourier transform of v in the variable x, then it follows that we can find functions χ_j such that

$$\hat{v}(t, \xi) = \sum_{p=0}^{2n-1} \chi_p(\xi) \exp\left[t\tau_p(\xi)\right].$$

The χ_j are easily determined from the Cauchy data $\partial_t^i v(0, x) = g_i$, $i = 0, \ldots, 2n-1$. If we denote by γ_i the Fourier transfors of the g_i, then the Cauchy conditions give

$$\sum_{j=0}^{2n-1} \chi_p(\xi)\tau_p^i(\xi) = \gamma_i(\xi), \qquad \forall \quad i = 0, \ldots, 2n-1.$$

This is a linear system of the form $\chi\Theta_{i,p=0,\ldots,2n-1} = \Gamma$, where χ and Γ are the vectors of components $\chi_p(\xi)$ and $\gamma_i(\xi)$ respectively and $\Theta_{i,p=0,\ldots,2n-1}$ is the matrix of components $\tau_p^i(\xi)$. It follows that the χ_p have the form

$$\chi_p(\xi) = \sum_{i=0}^{2n-1} \frac{\rho_i(\tau_0(\xi), \ldots, \tau_{2n-1}(\xi))}{D(\xi)} \gamma_i(\xi),$$

where $D(\xi)$ is the determinant of $\Theta_{i,p=0,\ldots,2n-1}$ and $\rho_i(\tau_1(\xi), \ldots, \tau_{2n}(\xi))$ is the polynomial equal to $(-i)^{p+i}$ times the determinant of the matrix obtained by deleting the i-th column and the p-th row from $\Theta_{i,p=0,\ldots,2n-1}$. We note that

$D(\xi)$ is a homogeneous polynomial of order $n(2n-1)$ in ξ, whereas $\rho_i(\xi)$ is homogeneous of order $n(2n-1)-i$ in ξ , such that the order of homogeneity of ρ_i/D is precisely $-i$.

In order to apply these considerations for the system (3.1.1), we need to write down the Cauchy conditions of order ν, $\nu < 2n$, for the single components u_i of the solutions of (3.1.5). In order to obtain them, we derivate the system by $(\partial/\partial t)^\nu$ for $\nu = 1, \dots, 2n-1$. Next, we restrict the system and the relations obtained after these derivations, to $t = 0$. We then obtain by also using the Cauchy conditions (3.1.6)

$$\frac{\partial^{2k}}{\partial t^{2k}}\hat{u}_i(0,\xi) = 0, \qquad \frac{\partial^{2k+1}}{\partial t^{2k+1}}\hat{u}_i(0,\xi) = \Delta_\xi^k \psi,$$

for all $k = 0, \dots, n-1$ and $i = 1, \dots, n$. By putting these calculations together, we obtain

$$\varphi_p(\xi) v_i^p(\xi) = \sum_{k=0}^{n-1} \frac{\rho_k(\tau_1(\xi), \dots, \tau_n(\xi))}{D(\xi)} \Delta_\xi^k(\xi)\psi_i(\xi) \qquad \forall i, p \in \{1, \dots, n\}.$$

This implies

$$\varphi_p(\xi) v_i^p(\xi) = \sum_{j=1}^{n} \sum_{k=0}^{n-1} \frac{\rho_k(\tau_1(\xi), \dots, \tau_n(\xi))\Delta_\xi^k(\xi)}{D(\xi)} v_j^p(\xi)\psi_j(\xi),$$

for all $i, p \in \{1, \dots, n\}$. Thus, T_{ipj} must have the following structure:

$$T_{ipj}(\xi) = \frac{Q_{ipj}(\xi, \tau_1(\xi), \dots, \tau_n(\xi))}{D(\xi)}, \tag{3.1.14}$$

where $Q_{ipj}(\xi, \tau_1, \dots, \tau_n)$ is a homogeneous polynomial in the variables (ξ, τ) of total degree of homogeneity $n(2n-1)-1$. In fact, we observe that Δ_ξ^k is a homogeneous function of order $2k$ in ξ and ρ_k is homogeneous of order $2n(2n-1)-(2k+1)$ in ξ, so $\rho_k \Delta_\xi^k v_j^p$ has order of homogenity $n(2n-1)-1$. Moreover we note that the set of $\xi \in \mathbb{R}^n$ such that $D(\xi) = 0$ is precisely the same set as when the discriminant of $P(\lambda, \xi)$ vanishes. This means that the discriminant of $P(\lambda, \xi)$ and $D(\xi)$, are equal up to a multiplicative constant, since they are two polynomials of the same degree and with the same roots.

It is then clear that the $T_{ipj}(\xi)$ are smooth homogeneous functions of order -1 where the discriminant of $P(\lambda, \xi)$ is different from zero, and we also have the estimate $|T_{ipj}(\xi)| \leq C_3|\xi|^{-1}$, already obtained above.

We conclude this section with the following estimate of the solution of the three dimensional system of the crystal acoustic when t remains bounded.

Proposition 3.5. *Let $u(t,x)$ be the solution defined in* (3.0.4) *of the Cauchy problem* (3.0.2), (3.0.3). *If t remains bounded, then we have the following estimate:*

$$|u(t,x)| \leq C_4 \sum_{j=1}^{3} \left(\sup_{|\xi|\leq 1} |\hat{g}_j(\xi)| + \sum_{|\alpha|\leq 5} ||\partial_x^\alpha g_j||_1 \right). \tag{3.1.15}$$

Proof. We start by considering the expression (3.0.4). We note that estimate (3.1.4) implies the local integrability of the function $T_{ipj}(\xi)$. Thus, using the estimate (3.1.4) and the fact that t is bounded, we obtain

$$\begin{aligned}
|u_i(t,x)| &\leq \sum_{j=1}^{3} \left(c_2 \int_{|\xi|\leq 1} \frac{|\hat{g}_j(\xi)|}{|\xi|} d\xi + c_2 \int_{|\xi|\geq 1} \frac{|\hat{g}_j(\xi)|}{|\xi|} d\xi \right) \leq \\
&\leq \sum_{j=1}^{3} \left(c_3 \sup_{|\xi|\leq 1} |\hat{g}_j(\xi)| + c_2 \int_{|\xi|\geq 1} (1+|\xi|)|\hat{g}_j(\xi)| d\xi \right) \leq \\
&\leq \sum_{j=1}^{3} \left(c_3 \sup_{|\xi|\leq 1} |\hat{g}_j(\xi)| + c_2 \int_{\mathbb{R}^3} (1+|\xi|)|\hat{g}_j(\xi)| d\xi \right) \leq \\
&\leq \sum_{j=1}^{3} \left(c_3 \sup_{|\xi|\leq 1} |\hat{g}_j(\xi)| + c_4 \sup_{\xi\in\mathbb{R}^3} (1+|\xi|)^5|\hat{g}_j(\xi)| \right) \leq \\
&\leq \sum_{j=1}^{3} \left(c_3 \sup_{|\xi|\leq 1} |\hat{g}_j(\xi)| + c_4 \sum_{|\alpha|\leq 5} \int_{\mathbb{R}^3} |\partial_x^\alpha g_j(x)| dx \right) \leq \\
&\leq c_5 \sum_{j=1}^{3} \left(\sup_{|\xi|\leq 1} |\hat{g}_j(\xi)| + \sum_{|\alpha|\leq 5} ||\partial_x^\alpha g_j||_1 \right).
\end{aligned}$$

We observe that the supremum of $(1+|\xi|)^5|\hat{g}_j(\xi)|$ exists because the g_j are smooth functions with compact support and so $\hat{g}(\xi)$ belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^3)$. Summing in i, we now obtain the statement. $\square$

Obviously, the relation (3.1.15) is an estimate of type (3.0.1) when t remains bounded. It follows in particular that when we want to prove theorem (3.1), we may assume that t is large. This is of interest in that in some of the partial results which we shall use one needs to apply the method of stationary phase with the phases $t\tau_p(\xi) + \langle x, \xi \rangle$, which works best for large parameters.

3.2 The case when $|t|$ dominates $|x|$

As we mentioned at the beginning of this chapter, in order to obtain the desired decay for the solution of the system of crystal acoustic when t is large, we have to take into account the relative position of the variables t, x, ξ. In this section we will prove the estimate (3.0.1), when $|t|$ dominates $|x|$, i.e. when $|t| \geq c_1|x|$ for some large constant c_1. We can obtain the decay in a rather elementary way. In this section, we want to estimate the expression I_{ipj} (see (3.0.5)), of which is essential to write the solutions, as an oscillatory integral with parameter t large and phase function $\xi \mapsto t\tau_p(\xi) + \langle x, \xi \rangle$ and at the same time we can argue for any fixed ξ_0, i, p, j. In order to simplify notation we will change coordinates linearly in such a way as to have $\xi_0 = (0, 0, 1)$, but of course this does not necessarily mean that ξ_0 is a singular direction. We can essentially estimate I_{ipj} by using partial integration in the variable ξ_3. Before we state the main proposition, we will make some preliminary remarks.
We assume that

$$\operatorname{supp}(\chi) \subset \{\xi \in \mathbb{R}^3 : |\xi_i| \leq \xi_3, i = 1, 2\}.$$

Thus, in the support of χ, we have that $\xi_3 \neq 0$ and so we can consider the change of variable F which maps (ξ_1, ξ_2, ξ_3) into $(\omega_1, \omega_2, \xi_3)$, where $\omega_i = \xi_i/\xi_3$ with $i = 1, 2$. Thus, we have that $|\omega| \leq 1$ and *det* $\nabla F^{-1} = \xi_3^2$. We observe that in the new coordinates we can write ξ as $\xi_3(\omega, 1)$. Moreover, since $\tau_p(\xi)$ is homogeneous of order one, we have that in the new coordinates we write

$\tau_p(\xi)$ as $\xi_3\tau_p(\omega,1)$.

Now we are ready to state the main result, observing that the estimate we will prove is stronger than what is needed for theorem (3.1).

Proposition 3.6. *Let $u(t,x)$ be the solution of the Cauchy problem* (3.0.2), (3.0.3) *defined in* (3.0.4) *and let I_{ipj} be the oscillatory integral defined in* (3.0.5). *When there exist a large constant c_1 such that $|t|\geq c_1|x|$, then*

$$|I_{ipj}|\leq C_5|t|^{-1}\sup_{\xi\in\mathbb{R}^3}\left((1+|\xi|)^5\sum_{\alpha\leq 1}|\partial_\xi^\alpha\hat{g}_j(\xi)|\right), \tag{3.2.1}$$

for all $i,p,j\in\{1,2,3\}$.

Proof. If we use the change of variable F defined above, then the integral I_{ipj} becomes

$$I_{ipj}(t,x)=\int_\omega\int_0^\infty e^{it\xi_3\tau_p(\omega,1)+i\xi_3\langle x,(\omega,1)\rangle}T_{ipj}(\xi_3\omega,\xi_3)\chi(\xi_3\omega,\xi_3)\hat{g}_j(\xi_3\omega,\xi_3)\xi_3^2d\xi_3d\omega.$$

We note that for $|t|\geq c_1|x|$ and for c_1 sufficiently large, there exist a positive constant c_2 such that

$$|\partial_{\xi_3}(it\xi_3\tau_p(\omega,1)+i\xi_3\langle x,(\omega,1)\rangle)|=|it\tau_p(\omega,1)+i\langle x,(\omega,1)\rangle|\geq c_2|t|.$$

Therefore, with these assumptions, the phase function is non-stationary as a function of ξ_3. We can now integrate partially in ξ_3 the integral I_{ipj}, using the identity

$$\frac{\partial_{\xi_3}e^{it\xi_3\tau_p(\omega,1)+i\xi_3\langle x,(\omega,1)\rangle}}{i(t\tau_p(\omega,1)+\langle x,(\omega,1)\rangle)}=e^{it\xi_3\tau_p(\omega,1)+i\xi_3\langle x,(\omega,1)\rangle}.$$

Now we observe that, in the previous integral, the integration in the variable ξ_3 is only in the region $\xi_3\geq 0$. Moreover, as a function of ξ_3, $\hat{g}_j$ belongs to the space $\mathcal{S}(\mathbb{R}^3)$, $T_{ipj}(\xi_3\omega,\xi_3)=\xi_3^{-1}T_{ipj}(\omega,1)$ and $\chi(\xi_3\omega,\xi_3)=\chi(\omega,1)$, since they are homogeneous function of degree -1 and zero respectively. Moreover, no boundary terms at $\xi_3=0$ appear after the partial integration, due to the presence of the factor ξ_3^2 in the integral and the previous considerations. Thus, after integration by parts, we have to estimate the following integral

$$I_{ipj} = -\int_\omega \int_0^\infty \frac{e^{it\xi_3\tau_p(\omega,1)+i\xi_3\langle x,(\omega,1)\rangle}}{i(t\tau_p(\omega,1)+\langle x,(\omega,1)\rangle)} \\ T_{ipj}(\omega,1)\partial_{\xi_3}(\chi(\xi_3\omega,\xi_3)\hat{g}_j(\xi_3\omega,\xi_3)\xi_3)d\xi_3 d\omega.$$

Now we recall that:

(i) $|t\tau(\omega,1)+\langle x,(\omega,1)\rangle| \leq c_3|t|^{-1}$,

(ii) $T_{ipj}(\omega,1) \leq c_4|(\omega,1)|^{-1} \leq c_5$ because $\xi_i/\xi_3 \leq 1$, with $i=1,2$,

(iii) $\partial_{\xi_3}\chi(\xi_3\omega,\xi_3) \leq c_6|\xi|^{-1}$,

(iv) $|\chi(\xi)| \leq 1$.

Using these remarks, we can conclude the proof in the following way:

$$|I_{ipj}| \leq c_7|t|^{-1}\int_\omega\int_0^{+\infty} |\partial_{\xi_3}(\chi(\xi_3\omega,1)\hat{g}_j(\xi_3\omega,1)\xi_3)|d\xi_3 d\omega \leq \\ \leq c_8|t|^{-1}\int_{\mathbb{R}^3}(|\hat{g}_j(\xi)|+|\partial_{\xi_3}\hat{g}_j(\xi)||\xi|)d\xi \leq \\ \leq C_5|t|^{-1}\sup_{\xi\in\mathbb{R}^3}\left((1+|\xi|)^5\sum_{\alpha\leq 1}|\partial_\xi^\alpha \hat{g}_j(\xi)|\right).$$

□

Remark 3.3. It is well known how an estimate of type (3.2.1) implies an estimate of type (3.0.1). We will prove this fact in Appendix A.
In the following we will make no further comment about this fact and we will prove, in almost all cases, estimates of type $(A.0.1)$ in order to obtain the desired result in the main theorem.

3.3 Reduction of the estimate to the slowness surface

In this section we will show that it will suffice to study the decay of an integral defined on the slowness surface in order to obtain an estimate of the

integral I_{ipj}. To do this we will adapt the argument used in [L1] to the case of tetragonal crystals in $\mathbb{R}^3$.
We start by considering the integral

$$I_{ipj} = \int_\Gamma e^{it\tau_p(\xi)+i\langle x,\xi\rangle} T_{ipj}(\xi)\chi(\xi)\hat{g}_j(\xi)d\xi,$$

with the standard notations and i, p, j fixed. Here Γ is an open cone such that the function $\chi(\xi)$ vanishes outside Γ. Without loss of generality we can assume, at least after a linear change of variable, that

$$\Gamma = \{\xi \in \mathbb{R}^3 : |(\xi_1, \xi_2)| < \varepsilon\xi_3\}$$

for some positive constant ε but we shall feel free to shrink ε if needed (note that we are free to shrink the support of χ too). As seen in section (3.1), the integrand has a singularity at $\xi = 0$, so we will cut off to $\xi \neq 0$. More precisely, we consider a smooth function $\varphi : \mathbb{R}^n \to [0,1]$ which vanishes for $|\xi| \leq 1/|x|$ and is identically one for $|\xi| \geq 2/|x|$. Moreover, in order to control the derivative in $|\xi|$ of φ when x is large and ξ is small, we make the following additional assumption on φ. For any fixed x we assume

$$\varphi(\xi) = \vartheta(|x||\xi|)$$

for some smooth function $\vartheta(t)$, $\vartheta : \mathbb{R}^+ \to \mathbb{R}$, which is, of course, identically one for $t \geq 2$ and vanishes for $t \in [0,1]$. Now, we observe that, if $\partial_\xi^\alpha \varphi(\xi) \neq 0$, with $\alpha \neq 0$, then we have $1 \leq |x||\xi| \leq 2$, because elsewhere the function φ is constant. This implies that, if $\partial_\xi^\alpha \varphi(\xi) \neq 0$, then there exist some positive constants c_1 and c_2 such that $c_1|\xi| \leq |x|^{-1} \leq c_2|\xi|$. In addition, if we compute $\partial_\xi^\alpha \varphi(\xi)$ we have

$$|\partial_\xi^\alpha \varphi(\xi)| \leq |\partial_t^{|\alpha|}\vartheta(t)|\partial_\xi^\alpha |x||\xi| \leq c_3 \sum_{\beta\leq\alpha} |x|^{|\beta|}|\xi|^{-|\alpha|+|\beta|}.$$

Thus, if we fix $s \geq 0$ and x, then there is a constant c_4 such that

$$|\partial_\xi^\alpha \varphi(\xi)| \leq c_4|\xi|^{-|\alpha|} \qquad \text{for} \quad |\alpha| \leq s \tag{3.3.1}$$

Now we write $I_{ijp}(t,x) = I_1(t,x) + I_2(t,x)$ where $I_1(t,x)$ and $I_2(t,x)$ are the integrals

$$\begin{aligned} I_1(t,x) &= \int_{|\xi||x|\le 2} e^{it\tau_p(\xi)+i\langle x,\xi\rangle} T_{ipj}(\xi)\chi(\xi)\hat{g}_j(\xi)(1-\varphi(\xi))d\xi, \\ I_2(t,x) &= \int_{|\xi||x|\ge 1} e^{it\tau_p(\xi)+i\langle x,\xi\rangle} T_{ipj}(\xi)\chi(\xi)\hat{g}_j(\xi)\varphi(\xi)d\xi. \end{aligned}$$

Is it now easy to estimate $|I_1(t,x)|$. In fact we recall that $|\chi(\xi)| \le 1$, $|(1 - \varphi(\xi)| \le 1$, $|\hat{g}_j(\xi)|$ is bounded and $|T_{ipj}(\xi)| \le C_3|\xi|^{-1}$. It follows that

$$|I_1(t,x)| \le c_5 \int_{|\xi||x|\le 2} |\xi|^{-1} d\xi \le c_6|x|^{-2}.$$

Furthermore, we observe that, in view of what we have proved in the previous section, we may assume that $|t| \le c_7|x|$, for some constant $c_7 > 0$ and therefore the above estimate of I_1 is stronger then the estimate we want to obtain in our main theorem. Thus, we will essentially have to deal with I_2.
Now, we want to define an opportune C^1-diffeomorphism F from $\{(\nu,\theta) \in \mathbb{R}^3 : \nu > 0, |\theta| < \varepsilon\}$ to a conic neighborhood of $(0,0,1)$, such that we can pass in the integral $I_2(t,x)$ to the coordinates (ν,θ).
To do this, we recall that, for each fixed p, the equation $\tau_p(\xi) = 1$ defines a sheet S_p of the slowness surface and $\tau_p(\xi) \ne 0$ if $\xi \ne 0$. It follows that on every half-ray lying in Γ there is precisely one point in S_p. In addition, we denote by ξ' the 2-vector (ξ_1,ξ_2) and we introduce the function $\rho_p : \{\xi' \in \mathbb{R}^2 : |\xi'| < \varepsilon\} \to \mathbb{R}$ such that $\tau_p(\xi', \rho_p(\xi')) \equiv 1$. Now we consider a parametrization η of the surface S_p near the point $(0, \rho_p(0)) \in \Gamma$ (see the figure (3.1) below). In particular we define $\eta(\xi') = (\xi', \rho_p(\xi'))$. With these definitions we are ready to define the diffeomorphism

$$F : \left\{(\nu,\theta) \in \mathbb{R}\times\mathbb{R}^2 : |\theta| < \varepsilon, \nu > 0\right\} \to \mathbb{R}^3 \qquad F(\nu,\theta) = \nu\eta(\theta).$$

In [L1] it is proved, in a more general context, that the function F is really a C^1-diffeomorphism and that the determinant of the Jacobian of F is equal to $\nu^2 L(\theta)$ where $L(\theta) = (\partial_{\xi_3}\tau_p(\theta, \rho_p(\theta)))^{-1}$ and it is different from zero if $|\theta| < \varepsilon$. (Moreover we recall that the homogenity of τ_p implies that $\partial_{\xi_3}\tau_p(\xi) \ne 0$ in

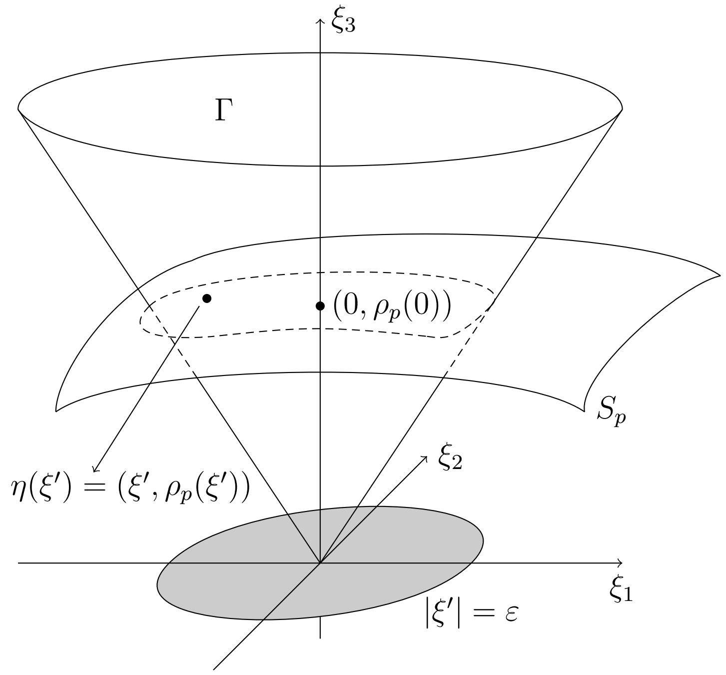


Figure 3.1: Local parametrization of the sheet S_p.

Γ, and then $L(\theta)$ is well defined). Now we are ready to pass in the integral I_2 to the coordinates (ν, θ). In particular, we can thus write $I_2(t, x)$ in the form

$$I_2(t,x) = \int_0^{+\infty} \int_{|\theta|\leq\varepsilon} e^{it\tau_p(\nu\eta(\theta))+i\langle x,\nu\eta(\theta)\rangle} T_{ipj}(\nu\eta(\theta))$$

$$(\chi\hat{g}_j\varphi)(\nu\eta(\theta))\nu^2 L(\theta)d\theta d\nu =$$

$$= \int_0^{+\infty} e^{it\nu}\nu d\nu \int_{|\theta|\leq\varepsilon} e^{ix_3\nu\rho_p(\theta)+i\langle\nu x',\theta\rangle} T_{ipj}(\eta(\theta))(\chi\hat{g}_j\varphi)(\nu\eta(\theta))L(\theta)d\theta, \quad (3.3.2)$$

where we use the homogenity of τ_p and T_{ipj}, i.e. the fact that $T_{ipj}(\nu\eta(\theta)) = \nu^{-1}T_{ipj}(\eta(\theta))$ and $\tau_p(\nu\eta(\theta)) = \nu\tau_p(\eta(\theta)) = \nu$. We recall that we are interested in the decay of I_2 when $|x|$ dominates t. In this case, we will prove that the decay of I_2 must come from the integral in θ, and in fact from the oscillatory character of $\exp(ix_3\nu\rho_p(\theta) + i\nu\langle x', \theta\rangle)$, taking $|\nu x|$ as a large parameter. Indeed in (3.3.2), $\varphi(\nu\eta(\theta)) \neq 0$ implies $|\nu\eta(\theta)| \geq |x|^{-1}$. In addition, since $\eta(0) = (0, \rho_p(0)) \neq 0$, if $|\theta| < \varepsilon$ there is a positive constant c_8 such that $|\eta(\theta)| \geq c_8$. Therefore, these facts imply that in the integral (3.3.2) we have $|\nu x| \geq c_8^{-1}$, because we are in the support of φ. Thus, $|\nu x|$ can not be very small.

These considerations allow us to write I_2 in the form

$$I_2(t,x) = \int_0^{+\infty} e^{it\nu} \nu I(\nu x) d\nu \tag{3.3.3}$$

where $I(\nu x)$ is the inner integral in (3.3.2) which is an oscillatory integral associated with the phase function $\theta \to (x_3 \nu \rho_p(\theta) + \langle x'\nu, \theta\rangle)$, with $\nu|x|$ as a large parameter.

Now we emphasize the fact that η is a parametrization of S_p near the point $(0, \rho(0)) \in \Gamma$, i.e. when $|\theta| < \varepsilon$. Thus, if we denote the variable on S_p by $\mu = \eta(\theta)$, νx by y and $(\chi \hat{g}_j \varphi)(\nu \eta(\theta))$ by $k(\mu, \nu)$, we obtain

$$I(y) = \int_{S_p} e^{i\langle y, \mu\rangle} \frac{T_{ipj}(\mu) k(\mu,\nu) L(\eta^{-1}(\mu))}{\sqrt{1 + \partial_{\theta_1} \rho_p^2(\eta^{-1}(\mu)) + \partial_{\theta_2} \rho_p^2(\eta^{-1}(\mu))}} d\sigma(\mu), \tag{3.3.4}$$

where the surface element $d\sigma$ on S_p is

$$d\sigma = \sqrt{1 + \partial_{\theta_1} \rho_p^2(\theta) + \partial_{\theta_2} \rho_p^2(\theta)} d\theta.$$

We conclude this section with the following proposition which shows in which way we can obtain the desired estimate of (3.3.3) from the estimate of (3.3.4).

Proposition 3.7. *Assume that for some C_6, s_1 and s_2 the following estimate is valid:*

$$|I(y)| \leq C_6 (1 + |y|)^{-1} (1 + \nu)^{-3} \sup_{\xi \in \mathbb{R}^3} \left(\sum_{\substack{|\alpha| \leq s_1 \\ |\beta| \leq s_2}} |\xi^\alpha| |\partial_\xi^\beta \hat{g}(\xi)| \right) \tag{3.3.5}$$

for all $g \in C_0^\infty(\mathbb{R}^3)$. Then we also have

$$|I_2(t,x)| \leq C_7 (1 + |x|)^{-1} \sup_{\xi \in \mathbb{R}^3} \left(\sum_{\substack{|\alpha| \leq s_1 \\ |\beta| \leq s_2}} |\xi^\alpha| |\partial_\xi^\beta \hat{g}(\xi)| \right),$$

for all $g \in C_0^\infty(\mathbb{R}^3)$.

Proof. The statement is also valid for more general exponents, but we state it in this form which we will use in the next section. In view of (3.3.3), the only estimate we have to prove is the following

$$\int_0^\infty \nu(1+\nu|x|)^{-1}(1+\nu)^{-3}d\nu \leq c_9(1+|x|)^{-1}$$

We distinguish a number different cases. First of all we suppose $|x| \leq 1$. In this case the proof is easy because we have:

$$\int_0^\infty \nu(1+\nu|x|)^{-1}(1+\nu)^{-3}d\nu \leq$$
$$\leq (1+|x|)^{-1}\int_0^\infty \frac{\nu(1+|x|)}{(1+\nu|x|)(1+\nu)^3}d\nu \leq c_{10}(1+|x|)^{-1}.$$

Now we suppose $|x| > 1$ and we consider three sub-cases. When $\nu \in [0, 1/|x|]$, $\nu \in [1/|x|, 1]$ and $\nu \geq 1$. We have respectively

$$\int_0^{1/|x|} \frac{\nu}{(1+\nu)^3(1+\nu|x|)}d\nu \leq \int_0^{1/|x|} \frac{\nu}{(1+\nu)^3}d\nu \leq$$
$$\leq c_{11}|x|^{-2} \leq c_{12}(1+|x|)^{-1},$$

$$\int_{1/|x|}^1 \frac{\nu}{(1+\nu)^3(1+\nu|x|)}d\nu \leq \int_{1/|x|}^1 \frac{\nu}{(1+\nu|x|)}d\nu =$$
$$= |x|^{-2}\int_1^{|x|} \frac{z}{1+z}dz \leq c_{13}(1+|x|)^{-1},$$

$$\int_1^\infty \frac{\nu}{(1+\nu)^3(1+\nu|x|)}d\nu \leq$$
$$\leq (1+\nu|x|)^{-1}\int_1^\infty \frac{\nu}{(1+\nu)^3}d\nu \leq c_{14}(1+\nu|x|)^{-1}.$$

This conclude the proof. $\square$

Thus, we will now study the integral (3.3.5) and we will separately discuss the cases when $(0, \rho_p(0))$ is a singular or a regular point.

3.4 End of proof of theorem 3.1

In this section we will conclude the proof of the main theorem (3.1) by discussing the cases in (3.0.5) not considered until now. In particular we will estimate the oscillatory integral (3.0.5) when ξ_0 is a singular or regular direction and $|x|$ dominates $|t|$.
In order to do this, we need to recall some results on estimate of Fourier transforms of densities which live on surfaces. We start with some notational remarks.
Let S be a surface in $\mathbb{R}^3$, as defined in definition (2.3). We denote by (ξ_1, ξ_2, ξ_3) the coordinates in $\mathbb{R}^3$. If $P \in S$ is a uniplanar singularity, we recall the definition of the curves $\Gamma^{\pm}$ given in (2.6.1):

$$\Gamma^{\pm} = \{\xi' \in \mathbb{R}^2 : -J_2 a(\xi') \pm \sqrt{J_4 \Delta(\xi')} = 1\},$$

with the notation used in definition (2.3). Now, suppose that we change variable and we pass from ξ to (ν, θ) as in the previous section and let $\rho(\theta)$ be such that $(\theta, \rho(\theta))$ is a local parametrization of S near P if $|\theta|$ is small enough. Now we make another change of variable and we pass from $\theta \in \mathbb{R}^2$ to (r, α), with $r \geq 0$, $\alpha \in U \subset [0, 2\pi]$ in such a way that $\rho(\theta) - \rho(0) = r^2$ and with U being an open subset of $[0, 2\pi]$ such that $((r, \alpha), \rho(r, \alpha))$ is a local parametrization of S near P if $0 \leq r \leq \varepsilon$ and $\alpha \in U$, for an appropriate small ε. With these notations we have the following theorem.

Theorem 3.8. *Assume that S has a uniplanar singularity at 0 and that the curves $\Gamma^{\pm}$ introduced above are smooth and of nowhere vanishing curvature. If $h : S \to \mathbb{R}$ is such that $h(r, \alpha)$ is smooth up to $r = 0$ and $h(\xi) = 0$ if $\xi \in S$, $|\xi| \geq \varepsilon$ for a sufficiently small ε, then there is a constant C_8 such that*

$$\left| \int_S e^{-i\langle x, \xi \rangle} h(\xi) d\sigma(\xi) \right| \leq C_8 (1 + |x|)^{-1}.$$

We start by using the previous theorem to estimate the integral (3.3.4) in the case when ξ_0 is the direction of a uniplanar singularity. In order to apply the theorem, we need some additional assumptions about the support of the amplitude function $T_{ipj}\chi\hat{g}_j\varphi L$ and some information concerning its regularity.

For this purpose we fix ξ_0 such that it is the direction of a uniplanar singularity P, and p such that S_p is one of the two sheets of S with $P \in S_p$. First of all we have to make sure that the amplitude function vanishes outside the set $|\xi| \leq \varepsilon$, where ε is given by the theorem. We just choose the conic neighborhood Γ_{ξ_0} of ξ_0 where χ is supported, such that if $\xi \in \Gamma_{\xi_0} \cap S_p$ then $|\xi| < \varepsilon$.

Secondly, we consider the absolute value of the integral (3.3.4) in the variable θ.

$$|I(y)| = |\int_{|\theta|<\varepsilon} e^{i\nu x_3(\rho_p(\theta)-\rho_p(0))+i\langle \nu x',\theta\rangle} T_{ipj}(\nu(\theta))(\chi\varphi\hat{g}_j)(\nu\eta(\theta))L(\theta)d\theta|, \tag{3.4.1}$$

where we use the fact that $|I(y)| = |e^{-i\nu x_3\rho_p(0)}I(y)|$. Thus, we can pass from θ to the new coordinates (r, α), introduced at the beginning of this section, such that $\rho_p(\theta) - \rho_p(0) = r^2$.

Now, if we want to apply theorem (3.8), we have to check that the amplitude function $T_{ipj}\chi\hat{g}_j\varphi L$ in the new coordinates (r, α) are smooth up to $r = 0$. This is immediate due to the following two lemmas.

Lemma 3.9. *(a) The change of variable $(r, \alpha) \to \theta(r, \alpha)$ is smooth up to $r = 0$ and the same is true for the surface element $d\sigma$ of S_p.*

(b) $\nabla_\theta \rho_p$ is smooth up to $r = 0$ as a function of (r, α).

(c) Let f be a smooth function in the variable ξ defined near P. Then $f(\theta(r, \alpha), \rho_p(\theta(r, \alpha)))$ is smooth in (r, α) up to $r = 0$.

Proof. The proofs are immediate and based on simple calculations. For details see [L1] and [L4]. □

Lemma 3.10. *(a) $L(\theta)$ is smooth up to $r = 0$ in the coordinates (r, α).*

(b) The functions $T_{ipj}(\eta(\theta))$ are smooth up to $r = 0$ in the coordinates (r, α).

(c) Fix $s, \varepsilon > 0$, ε small. Then there is a c_1 such that, for $\xi \neq 0$

$$|\partial_r^{\jmath}\partial_\alpha^{\ell}(\chi\varphi)(\nu\eta(\theta(r,\alpha)))| \leq c_1$$

if $\jmath + \ell \leq s$ and $r \leq \varepsilon$, $\alpha \in U$.

(d) For $\nu > 0$, the function $\hat{g}_j(\nu\eta(\theta(r,\alpha)))$ is smooth up to $r = 0$ and for any given s, σ, there is a constant c_2, such that for $r < \varepsilon$, $\alpha \in U$

$$(1+\nu)^{\sigma}|\partial_t^{\jmath}\partial_\alpha^{\ell}\hat{g}_j(\nu\eta(\theta(r,\alpha)))| \leq c_2 \sup_{\substack{|\gamma|\leq s \\ \xi\in\mathbb{R}^3}} (1+|\xi|)^{s+\sigma}|\partial_\xi^{\gamma}\hat{g}_j(\xi)|$$

if $\jmath + \ell \leq s$.

Proof. (a) It is possible to write L in the form $L(\theta) = \rho_p(\xi') - \langle \xi', \nabla_{\xi'}\rho_p(\xi')\rangle$ (see for details [L1]). We can then apply (a) and (b) of lemma (3.10).

(b) We use the expression (3.1.14) of T_{ipj} and we note that the denominator vanishes only when the discriminant of the characteristic polynomial vanishes, i.e. when $(\tau_i(\xi) - \tau_j(\xi))^2$ vanishes, with $i, j \in \{1,2,3\}$ such that S_i and S_j touch each other forming the uniplanar singularity under consideration. For a small r, we can write

$$\begin{aligned}(\tau_i(\xi) - \tau_j(\xi))^2 = |\xi_3|^2|\tau_i(\xi'/|\xi_3|, 1) - \tau_j(\xi'/|\xi_3|, 1)|^2 = \\ = \xi'^2(c_3 + \mathcal{O}(\xi')) = r^2(c_4 + \mathcal{O}(r)).\end{aligned}$$

Moreover, from proposition (3.2), we know that the T_{ipj} are uniformly bounded near $r = 0$, then we must have a factor of order r^2 at nominator. Thus, we can divide out the factor r^2 and the resulting function must be smooth up to $r = 0$.

(c) We know that there is a constant c_5, such that $|\partial_\xi^{\gamma}\chi(\xi)| \leq c_5|\xi|^{-|\gamma|}$, because χ is a smooth function which is homogeneous of degree zero function. Furthermore, in section 3.3 we proved the estimate (3.3.1) for the absolute value of the derivative of φ. Now, for instance, we have

$$|\partial_r(\chi\varphi)(\nu\eta(\theta(r,\alpha)))| = |\partial_\xi(\chi\varphi)\nu\partial_\theta\eta\partial_r\theta| \leq c_6|\xi|^{-1}\nu c_7 c_8 \leq c_9,$$

because $\nu \leq c_{10}|\xi|$ and because the change of variables η and θ is smooth. Thus, we can proceed in a similar way to estimate $|\partial_r^{\jmath}\partial_\alpha^\ell(\chi\varphi)(\nu\eta(\theta(r,\alpha)))|$ using the Faá di Bruno's formula.

(d) The proof is similar to case (c). We again use the Faá di Bruno's formula and we have, if we fix admissible r, α

$$\partial_t^{\jmath}\partial_\alpha^\ell \hat{g}_j(\nu\eta(\theta(r,\alpha))) = \sum_{1\leq q\leq s} \nu^q \sum_{|\gamma|=q} \partial_\xi^\gamma \hat{g}_j(\nu\eta(r,\alpha)) f_{q\gamma}(r,\alpha) \tag{3.4.2}$$

where $f_{q\gamma}(r,\alpha)$ are appropriate smooth functions in $r > 0$ up to $r = 0$. In addition, since $|\nu| \leq c_{11}|\nu\eta(r,\alpha)|$, we have

$$(1+\nu)^\sigma \nu^q \leq c_{12}(1+|\nu\eta(r,\alpha)|)^{\sigma+q}$$

and we note that $(1+|\nu\eta(r,\alpha)|)^{\sigma+q}|\partial_\xi^\gamma \hat{g}_j(\nu\eta(r,\alpha))| \leq \sup_{\xi\in\mathbb{R}^3}(1+|\xi|)^{\sigma+q}|\partial_\xi^\gamma \hat{g}_j(\xi)|$. Therefore, if we multiply the inequality (3.4.2) by $(1+\nu)^\sigma$ we easily obtain the desired estimate.

□

The last hypothesis to check before we use theorem (3.8) is that the curves $\Gamma^\pm$ must be smooth and of nowhere vanishing curvature. But this is precisely what we proved in proposition (2.25) of section 2.6.1 (note that proposition (2.25) holds if we assume that the hypotheses on the stiffness constant of theorem (3.1) are fulfilled, cf. remark (2.17)).
We have now checked that all the assumptions needed to apply theorem (3.8) for the integral (3.4.1) are satisfied. We apply the theorem, with ν as a parameter, in a such way that allow us to write the estimate of $|I(y)|$ in the form (3.3.5) (for the precise form of the estimate cf. section 5 in [L4]). In particular, we obtain, for suitable constants c_{13}, s,

$$|I(y)| \leq c_{13}(1+|y|)^{-1} \sup_{\substack{r\leq\varepsilon,\alpha\in U\\ \jmath+\ell\leq s}} |\partial_r^{\jmath}\partial_\alpha^\ell(T_{ipj}\chi\varphi\hat{g}_j)(\nu\eta(r,\alpha))|.$$

The following lemma shows how we can obtain an estimate of type (3.3.5) from the one above.

Lemma 3.11. *Let s be fixed. There is a constant c_{14} such that*

$$|\partial_r^{\jmath}\partial_\alpha^{\ell}(T_{ipj}\chi\varphi\hat{g}_j)(\nu\eta(r,\alpha))| \leq c_{14}(1+\nu)^{-m} \sup_{\substack{\xi\in\mathbb{R}^3 \\ |\gamma|\leq s}} (1+|\xi|)^{m+s}|\partial_\xi^{\gamma}\hat{g}_j(\xi)|,$$

for $|\jmath|+|\ell|\leq s$ and some positive m.

Proof. We use again lemma (3.10). In fact, from point (b) it follows that all the derivatives of $T_{ipj}(\nu\eta(r,\alpha))$ are bounded when r is small. In addition we have, from point (c) that the derivatives of $\chi\varphi$ are bounded by a constant. Now, we write

$$|\partial_r^{\jmath}\partial_\alpha^{\ell}\hat{g}_j(\nu\eta(r,\alpha))| = (1+\nu)^{-m}(1+\nu)^{m}|\partial_r^{\jmath}\partial_\alpha^{\ell}\hat{g}_j(\nu\eta(r,\alpha))|$$

and we estimate $(1+\nu)^{m}|\partial_r^{\jmath}\partial_\alpha^{\ell}\hat{g}_j(\nu\eta(r,\alpha))|$ using point (d) of lemma (3.10). $\square$

Thus, we have concluded the proof of theorem (3.1) as far as the contribution of the uniplanar singularities is concerned.
We continue our proof of the main theorem by looking at the contribution of the parts of the slowness surface near conical singularities. As above, we first recall the relevant theorem on estimates of Fourier transforms of densities which live on surfaces. Let S be a surface in $\mathbb{R}^3$ as described at the beginning of this section. We again pass from the coordinate $\theta\in\mathbb{R}^2$ to (r,α), but in this case (r,α) are precisely the polar coordinates in the ξ'-plane. With these assumptions we have the following theorem.

Theorem 3.12. *Let zero be a conical singularity of S. If $h^{\pm}: S\to\mathbb{R}$ are such that $h^{\pm}(r,\alpha)$ are smooth up to $r=0$ and $h^{\pm}(\xi)=0$ if $\xi\in S$, $|\xi|\geq\varepsilon'$ for a sufficiently small ε', then there are some constants C_9 and s such that*

$$\left|\int_S e^{-i\langle x,\xi\rangle}h^{\pm}(\xi)d\sigma(\xi)\right| \leq C_9(1+|x|)^{-1} \sup_{\substack{r\leq\varepsilon',\alpha\in U \\ \ell+\jmath\leq s}} |\partial_r^{\jmath}\partial_\alpha^{\ell}h^{\pm}(r,\alpha)|,$$

where we used the superscript $\pm$ to emphasize the fact that $h^{\pm}$ can be calculated in terms of h and the two functions which define the sheets of S near P.

We use the previous theorem to estimate the integral (3.3.4) in the case when ξ_0 is the direction of a conical singularity. Thus, we fix ξ_0 such that it is a direction of a conical double point P and p such that S_p is one of the two sheets of the slowness surface S which contain the conical double point. As before, we choose the conic neighborhood Γ_{ξ_0} of ξ_0 where χ is supported, such that if $\xi \in \Gamma_{\xi_0} \cap S_p$ then $|\xi| < \varepsilon'$. We introduce in the ξ'-plane the polar coordinates $\xi_1 = P_1 + r\cos\alpha$, $\xi_2 = P_2 + r\sin\alpha$ (here we denote $P = (P_1, P_2, P_3)$). Moreover, it is possible to prove that the amplitude function $(T_{ipj}\chi\varphi\hat{g}_j)(\nu\eta(\theta))L(\theta)$ is smooth in the new coordinates up to $r = 0$, precisely as in the uniplanar case. Thus, it is possible to use theorem (3.12) to obtain an estimate of the integral $I(y)$ of type (3.3.5), again with the aid of lemma (3.11).
We are left with the cases when either ξ_0 is a regular direction or it is the direction of a singular point, but we are going study the contribution of a sheet S_p of S which has no singular points on the ray with direction ξ^0. In these cases we need to use some further results on estimates of Fourier transforms of densities on surfaces.
First of all we, fix ξ_0 and we suppose that the total curvature of S_p at P does not vanish for all $P \in \Gamma_{\xi_0} \cap S_p$. In this case we can use the following theorem of Hlawka.

Theorem 3.13 (Hlawka, 1950)**.** *Let $S \subset \mathbb{R}^n$ be a smooth compact surface with nowhere vanishing total curvature. Also let $h : S \to \mathbb{C}$ be a smooth function on S. Then there is a constant $C_{10} > 0$ such that the Fourier transform $I(x)$ of $hd\sigma$, $d\sigma$ being the surface element on S, defined by*

$$I(y) = \int_S e^{i\langle y,\xi\rangle} h(\xi) d\sigma(\xi)$$

satisfies the estimate

$$|I(y)| \leq C_{10}(1 + |y|)^{(1-n)/2}.$$

Once more, we use this theorem to estimate the integral (3.3.4), with ξ_0 fixed as above. We observe that, in view of what we have proved in the second chapter of this thesis, the fact that the total curvature of S does not vanish means that we are far away from the double points of the slowness surface. Thus, it is easy to prove that the amplitude function is smooth in the support of χ and therefore all the hypotheses of the theorem of Hlawka are satisfied. Consequently, arguing as before, we obtain

$$|I(y)| \leq c_{15}(1+|y|)^{-1}(1+\nu)^{-3} \sup_{\substack{|\gamma|\leq s \\ \xi\in\mathbb{R}^3}} |\xi|^3 |\partial_\xi^\gamma \hat{g}_j(\xi)|.$$

Secondly, we fix ξ_0 and we suppose that there exists $P \in \Gamma_{\xi_0} \cap S_p$ such that the Gaussian curvature of S_p vanishes at P. We recall that, with our assumptions on the stiffness constants, the total curvature and the main curvature cannot vanish simultaneously (cf. proposition (2.28)). Moreover, we know from theorem (2.31) that there are no planes which are tangent to S_p along entire non trivial curves. Thus, in this situation, in order to estimate the integral (3.3.4) we can use the following result, proved in [L2].

Proposition 3.14. *Let $S \subset \mathbb{R}^3$ be a smooth algebraic surface given by a polynomial equation $S = \{\xi \in \mathbb{R}^3 : p(\xi) = 0\}$, let $U' \subset\subset U \subset \mathbb{R}^3$ be open and bounded and assume that the following assumptions are satisfied:*

(i) $\nabla_\xi p(\xi) \neq 0$ for $\xi \in S \cap U$ and the mean curvature of $S \cap U$ does not vanish.

(ii) There is no plane tangent to S along an entire curve.

Moreover, let $h : S \to \mathbb{C}$ be a smooth function such that $h(\xi) = 0$ if $\xi \ni U' \cap S$. If we denote by $I(x)$ the following integral

$$I(x) = \int_S e^{i\langle x,\mu\rangle} h(\mu) d\sigma(\mu),$$

where we denoted by $d\sigma$ the surface element on S, then there exists a natural number $k \geq 2$ and constants C_{11}, s such that

$$|I(x)| \leq C_{11}(1+|x|)^{-1/2-1/k} \sup_{|\gamma|\leq s, \mu\in S} |\partial_\mu^\gamma h(\mu)|.$$

(The derivatives of h are calculated locally in a suitably chosen coordinate set.)

Thus, again arguing as before, the previous result now gives

$$|I(y)| \leq c_{16}(1+|y|)^{-1/2-1/k}(1+\nu)^{-3} \sup_{\substack{|\gamma|\leq s \\ \xi\in\mathbb{R}^3}} |\xi|^3 |\partial_\xi^\gamma \hat{g}_j(\xi)|.$$

This concludes the proof of the theorem (3.1).

Remark 3.4. The natural number k, which shows up in the statement of proposition (3.14), depends on the nature of the curvature in the points of S in which the total curvature vanishes. Unfortunately, the proof of this is based on qualitative arguments and thus we have no estimate for the value of k.

3.5 Estimate for Fourier transforms of surface carried densities on surfaces with biplanar singular points

We conclude this chapter with the study of the decay estimate for Fourier transforms of surfaces carried densities on surface with biplanar singular points. We recall that the definition of biplanar singular points is given in definition (2.3) and that this kind of singularity appears in the case of the slowness surface of a tetragonal crystal, if we assume that the stiffness constant c_{11} is equal to c_{66}. Actually, in this case, we have four biplanar singularities, two on each semi-axis of the coordinate plane $\xi_3 = 0$ (cf. proposition 2.13).

Therefore, it seems interesting to try to obtain a result similar to theorems 3.8 and 3.12, in the case when the surface taken under consideration has a biplanar singularity.

We next describe the setting of our problem. We shall study decay estimates for Fourier transforms of densities which live on surfaces of form

$$S = \{(x, y, z); z = g(x, y), (x, y) \in U\}$$

where U is a neighborhood of the origin, $g : U \to \mathbb{R}$ is for some constants $a, b, c, a_1, b_1, c_1, \delta$, a function of form

$$ax^2 + 2bxy + cy^2 + g_1(x,y) + \delta\sqrt{a_1x^2 + 2b_1xy^2 + c_1y^4 + g_2(x,y)}.$$

Here $g_1, g_2 : U \to \mathbb{R}$ are two functions in $C^\infty(U)$. In addition, we suppose that:

(i) $g_1(x,y) = \mathcal{O}(|(x,y)|^3)$, for $(x,y) \to 0$,

(ii) $g_2(x,y) = o(|(x^2+y^4)|)$, for $(x,y) \to 0$,

(iii) the functions $(x,y) \to ax^2 + 2bxy + cy^2$, $(x,t) \to a_1x^2 + 2b_1xt + c_1t^2$ are strictly positive for $(x,y) \neq 0$, $(x,t) \neq 0$,

(iv) $\max(|a|,|b|,|c|) \leq 1$, $\max(|a_1|,|b_1|,|c_1|) \leq 1$ and δ is small and positive.

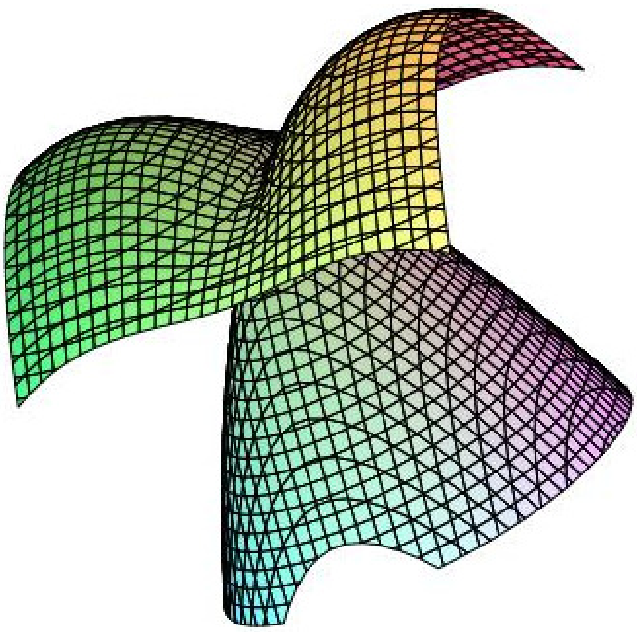


Figure 3.2: The two sheets of the surface defined by the equation $z^2 - (1/2)x^2 + 2yz^2 - 2zx^2 + x^4 + 2x^2y^2 + (1/2)y^4 = 0$ with a biplanar double point at the origin.

Remark 3.5. Condition (iv) means that, if we denote $a_1' = \delta^2 a_1, b_1' = \delta^2 b_1, c_1' = \delta^2 c_1$, then the constants a_1', b_1', c_1' are small compared with the constants a, b, c. As for the constant δ we shall assume it to be small. Indeed, what we need is that the second derivatives of the function $\delta\sqrt{a_1x^2 + 2b_1xy^2 + c_1y^4 + g_2(x,y)}$ be small when compared with the second derivatives of the terms $ax^2 + 2bxy + cy^2 + g_1(x,y)$.
Furthermore, the estimates which we will obtain later on will depend on δ, but must not depend on a_1, b_1, c_1.
Finally, the assumption (iii) implies in particular that $|ax^2 + 2bxy + cy^2| \geq c_1|(x,y)|^2$ and $|a_1x^2 + 2b_1xt + c_1t^2| \geq c_2|(x,t)|^2$, for some constants c_1, c_2.

Remark 3.6. The conditions that we assume on S are sufficient but not necessary to have a biplanar singularity in the origin.
However, we note that the singular biplanar double points of the slowness surface of the tetragonal crystals in the case when $c_{11} = c_{66}$ satisfy the hypothesis made on S (cf. the proof of proposition 2.16).

Remark 3.7. The condition on g_2 implies that we can find some C^∞- functions G_i, $i = 1, 2, 3, 4$ such that

$$g_2(x,y) = x^3G_1(x,y) + x^2yG_2(x,y) + xy^3G(x,y)_3 + y^5G_4(x,y). \qquad (3.5.1)$$

Thus, S has a singular point at $0 \in \mathbb{R}^3$ and the singularity is biplanar there. The integrals which we consider are of form

$$I(\xi, \eta, \tau) = \int_S \exp\left[i\tau g(x,y) + i\xi x + i\eta y\right] f(x,y,z) d\sigma.$$

The function f is defined on S, or, for convenience, in a neighborhood of S. We will assume that it vanishes outside some small neighborhood of 0, say, for $|(x,y)| \geq \rho$ for some small ρ. As for regularity, f will be assumed to be regular outside 0, but may have a singularity at 0. In view of future applications we shall study the following situation (cf. theorems 3.8 and 3.12, section 3.1, [Ba-Li] and [L4]).
We assume that f is of form $f(x,y) = f_1(x,y)h(x,y)$ where f_1 is C^∞ in

polar coordinates up to $r = 0$ and h is $C^\infty(\dot{\mathbb{R}}^2)$ and homogeneous of degree -1. It follows that in polar coordinates $rf(r\cos\alpha, r\sin\alpha)$ is bounded in a neighborhood of $r = 0$.

Before we state the main theorem of this section, we will give some additional remarks on the properties of the functions which we will consider in the following.

Remark 3.8. (i) In polar coordinates the surface element $d\sigma$ of S is of form $r\tilde{\sigma}(r,\alpha)drd\alpha$, with

$$\tilde{\sigma}(r,\alpha) = \sqrt{1 + g_x^2(r\cos\alpha, r\sin\alpha) + g_y^2(r\cos\alpha, r\sin\alpha)}.$$

In particular, $\tilde{\sigma}$ is easily seen to be C^∞ up to $r = 0$.

(ii) Let h be a function in $C^\infty(\dot{\mathbb{R}}^2)$ which is homogeneous of degree $-k$, where k is some natural number or 0. Then $r^k h(r\cos\alpha, r\sin\alpha)$ is C^∞ up to $r = 0$.

(iii) Let $f \in C_0^\infty(\dot{\mathbb{R}}^2)$ be a function which is C^1 up to $r = 0$ in polar coordinates. Then we have the estimate

$$|\nabla_{(x,y)} f(x,y)| \leq c_3 |(x,y)|^{-1}.$$

This follows by writing ∇f in polarcoordinates.

The main result of this section is the following proposition.

Proposition 3.15. *Assume that S and f are as above. If δ and ρ are small enough, then the integral*

$$I(\xi,\eta,\tau) = \int_S \exp\left[i\tau g(x,y) + i\xi x + i\eta y\right] f(x,y) d\sigma,$$

satisfies the estimate

$$|I(\xi,\eta,\tau)| \leq C_{12}(1 + |(\xi,\eta,\tau)|)^{-1/2} \ln(1 + |(\xi,\eta,\tau)|),$$

for some positive constant C_{12}.

As in many arguments used to prove estimates for Fourier transforms for densities we have to discuss a number of cases according to where the variables (ξ, η, τ) are placed and how they relate to $(x, y) \in U$. In particular we distinguish two cases: when $|\tau| \geq c_4(|\xi|+|\eta|)$ and when $|\tau| \leq c_5(|\xi|+|\eta|)$, for some, possibly large, constants $c_4 \leq c_5$. In the more elaborate parts of the argument we will employ the polar coordinates $(x = r\cos\theta, y = r\sin\theta)$. The integral defining I then becomes

$$I(\xi, \eta, \tau) = \int_{[0,2\pi]} \int_{[0,\rho]} \exp\left[i\tilde{g}(r\cos\theta, r\sin\theta)\right] (r\tilde{f}(r\cos\theta, r\sin\theta)) d\theta dr, \tag{3.5.2}$$

where

$$\tilde{g}(x, y) = \tau g(x, y) + x\xi + y\eta \quad \text{and} \quad \tilde{f}(x, y) = f(x, y)\sqrt{1 + g_x^2(x, y) + g_y^2(x, y)}$$

Thus, the factor $\sqrt{1 + g_x^2(x, y) + g_y^2(x, y)}$, which comes from the surface element is a factor in $\tilde{f}$. We now observe that

$$|\sup_\theta \frac{\partial}{\partial r}(r\tilde{f}(r\cos\theta, r\sin\theta))| \leq c_6 r^{-1}. \tag{3.5.3}$$

We now start to study the first case.

Case A We assume at first that $|\tau| \geq c_4(|\xi| + |\eta|)$ for some large constant c_4. We fix τ and we split the integral into the regions $|(x, y)| \leq |\tau|^{-1/2}$ and $|(x, y)| \geq |\tau|^{-1/2}$. For $|(x, y)| \leq |\tau|^{-1/2}$ we use a very rough estimate. The oscillatory character of the exponential is not taken into account in the argument and we estimate the integrand by its absolute value. The assumptions show that $|f(x, y)| \leq c_7 r^{-1}$, therefore passing to polar coordinates r, θ, we can estimate

$$\int_{|(x,y)| \leq \tau^{-1/2}} |f(x, y)| d(x, y)$$

by

$$c_7 \int_0^{2\pi} \int_0^{|\tau|^{-1/2}} dr d\theta = c_8 |\tau|^{-1/2}.$$

For the case $|(x,y)| \geq |\tau|^{-1/2}$ we use polar coordinates, as well. We shall argue on the curves $x = r\cos\theta, y = r\sin\theta,\ z = g(r\cos\theta, r\sin\theta)$, writing

$$I(\xi,\eta,\tau) = \int_{[0,2\pi]} II(\xi,\eta,\tau)d\theta$$

where $II(\xi,\eta,\tau)$ is defined by

$$\int_{0\leq r\leq\rho} \exp\left[i\tau g(r\cos\theta, r\sin\theta) + ir\xi\cos\theta + ir\eta\sin\theta\right]\tilde{f}(r\cos\theta, r\sin\theta)dr.$$

All the decay which we can obtain when $|\tau|$ dominates $|\xi| + |\nu|$ will come from the inner integral II, the one in the variable r. It will in fact suffice to show that the function II satisfies

$$|II(\xi,\eta,\tau)| \leq c_9(1 + |(\xi,\eta,\tau)|)^{-1/2}\ln(1 + |(\xi,\eta,\tau)|),$$

with constants uniform in θ.
We will prove this by using the following lemma.

Lemma 3.16 (Stein)**.** *Let φ be a real-valued function on the interval $[a,b]$ which is k times differentiable. Assume that $k \geq 2$ and that $|\varphi^{(k)}(x)| \geq 1$. Also consider $\psi \in C^1[a,b]$. Then it follows that*

$$|\int_a^b e^{it\varphi(x)}\psi(x)dx| \leq C_{13}t^{-1/k}\left[|\psi(b)| + \int_a^b |\psi'(x)|dx\right],\ \textit{for } t > 0,$$

for some constant C_{13}, which does not depend on φ, ψ, a and b.

The main step in the argument is to show that the hypotheses of the previous lemma are satisfied, in particular we will prove that

$$\left|\frac{d^2}{dr^2}[\tau g(r\cos\theta, r\sin\theta) + r\xi\cos\theta + r\eta\sin\theta]\right| \geq c_{10}(|\xi| + |\eta| + |\tau|), \tag{3.5.4}$$

uniformly in θ for $|\tau| \geq c(|\xi| + |\eta|)$ and c sufficiently large . Since the terms containing ξ, η are linear in r and since $|\tau|$ dominates $|\xi| + |\eta|$, 3.5.4 follows from $|(d/dr)^2 g(r\cos\theta, r\sin\theta)| \geq c_{11}$. We denote $\cos\theta$ by k and $\sin\theta$ by m. Then we write

$$\frac{d^2 g(rk, rm)}{dr^2} = I_1 + I_2 + I_3 + I_4,$$

where

$$
\begin{aligned}
I_1 &= 2ak^2 + 4bkm + cm^2 + \frac{d^2 g_1(rk, rm)}{dr^2}, \\
I_2 &= \frac{2b_1 km^2 + 3c_1 rm^4 + (d/dr)[g_2(rk, rm)/r^2]}{\sqrt{a_1 k^2 + 2b_1 rkm^2 + c_1 r^2 m^4 + g_2(rk, rm)/r^2}}, \\
I_3 &= -\frac{r}{4} \frac{\left(2b_1 km^2 + 2c_1 rm^4 + (d/dr)[g_2(rk, rm)/r^2]\right)^2}{[a_1 k^2 + 2b_1 rkm^2 + c_1 r^2 m^4 + g_2(rk, rm)/r^2]^{3/2}}, \\
I_4 &= \frac{r}{2} \frac{(d/dr)^2 [g_2(rk, rm)/r^2]}{\sqrt{a_1 k^2 + 2b_1 rkm^2 + c_1 r^2 m^4 + g_2(rk, rm)/r^2}}.
\end{aligned}
$$

We shall denote $a_1 k^2 + 2b_1 rkm^2 + c_1 r^2 m^4 + g_2(rk, rm)/r^2$ by $g_3(r, k, m)$. Thus, we have the following inequalities in which all estimates are uniform in k, m, provided $k^2 + m^2 = 1$:

Remark 3.9. (i) $|I_1| \geq c$ if $r < \rho$ and ρ is small enough. Indeed, $|I_1| = |2ak^2 + 4bkm + 2cm^2| \geq 2c_{12}(k^2 + m^2) = 2c_{12}$ by the assumptions on the polynomial $ax^2 + 4bxy + 2cy^2$ and the fact that $k^2 + m^2 = 1$ and $(d/dr)^2 g_1$ is as small as we want in the region $r < \rho$.

(ii) $|(d^2/dr^2) g_1(rk, rm)| \leq c_{13} r$ since g_1 vanishes of order 3 at 0.

(iii) $|a_1 k^2 + 2b_1 rkm^2 + c_1 r^2 m^4| \geq c_{14}(k^2 + r^2 m^4)$, for $k^2 + m^2 = 1$, with an argument as in (i).

(iv) $|g_3(r, k, m)| \geq c_{15}(k^2 + r^2 m^4)$, since $|g_2(rk, rm)/r^2| \leq O(r(k^2 + r^2 m^4))$, for $r \to 0$,

(v) $|a_1 k^2 + 2b_1 rkm^2 + c_1 r^2 m^4 + g_2(rk, rm)/r|^{3/2} \geq c_{16}(|k| + rm^2)^3 = c_{17}(|k|^3 + 3rk^2 m^2 + 3r^2 |k| m^4 + r^3 m^6)$

It is clear from this remark that the denominators in I_i, with $i = 2, 3, 4$, are small if k and rm^2 are simultaneously small. Conversely, if some term contains a factor of type k or rm^2, then we can estimate it by $\sqrt{g_3}$. We can now state the following lemma.

Lemma 3.17. *(vi)* $|2b_1 km^2| \leq c_{18} |g_3(r, k, m)|^{1/2}$, *(vii)* *in a trivial way,* $|2c_1 rm^4| \leq c_{19}|g_3(r,k,m)|^{1/2}$,

(viii) $r^2m^8 \leq c_{20}|g_3(r,k,m)|^{3/2}$,

(ix) $|(d/dr)g_2(rk,rm)/r^2| \leq c_{21}\sqrt{g_3(r,k,m)}$,

(x) $|(d/dr)(g_2(rk,rm)/r)|^2 \leq c_{22}(k^2+r^2m^4)r^2 \leq c_{23}|g_3(r,k,m)|^{3/2}$,

(xi) $r|(d/dr)^2(g_2(rk,rm)/r^2)| \leq c_{24}\sqrt{|g_3(r,k,m)|}$.

Proof. We begin with the proof of (ix) and (x). We use (3.5.1) and have then that $g_2(rk,rm)/r^2 = k^3rG_1(rk,rm) + k^2mrG_2(rk,rm) + km^3r^2G_3(rk,rm) + m^5r^3G_4(rk,rm)$. This gives

$$\begin{aligned}|(d/dr)g_2(rk,rm)/r^2| &\leq k^3|G_1|+k^2m|G_2|+2km^3r|G_3|+3m^5r^2|G_4|+\\&+r(k^4G_1'+k^3mG_1'')+rk^3mG_2'+rk^2m^2G_2''+r^2k^2m^3G_3'+\\&+r^2km^4G_3'+r^3km^5G_4'+r^3m^6G_4'',\end{aligned}$$

where $G_i' = (\partial/\partial x)G_i(rk,rm)$ and $G_i'' = (\partial/\partial y)G_i(rk,rm)$, $i=1,2,3,4$. All the terms in this sum are easily seen to be estimable by $c(|k|+rm^2)$, given that $|k|\leq 1$, $|m|\leq 1$. This gives (ix). The argument for (x) is similar. Now we prove (xi). The terms in $(d/dr)^2[g_2/r^2]$ all contain either factors of type k, or of type rm^2. These factors can be estimated by $\sqrt{g_3}$. Finally, (vi) easily follows from (iv) of the previous remark, wherease (vii) and $(viii)$ are trivial. □

We now return to the proof of proposition 3.15. Since I_1 is relatively large, while I_i, with $i=2,3,4$ are small quantities, we have now proved (3.5.4). If we also use (3.5.3) it follows from Stein's lemma that we have, if c is large enough,

$$\begin{aligned}&|\int_{|(x,y)|\geq|\tau|^{-1/2}} e^{i\tau g(x,y)+ix\xi+iy\eta}f(x,y)dxdy| =\\&= |\int_{|\tau|^{-1/2}\leq r\leq\rho,\alpha\in[0,2\pi]} e^{i\tau\varphi(r,\alpha,\xi,\tau)}r\tilde{f}(r\cos\alpha, r\sin\alpha)drd\alpha|\\&\leq c_{25}|\tau|^{-1/2}\int_{|\tau|^{-1/2}\leq r\leq\rho} r^{-1}dr,\end{aligned}$$

since we may assume that $\tilde{f}(\rho\cos\alpha, \rho\sin\alpha) = 0$ for every α. This concludes the argument in this case.

Case B We now may assume that $|\tau| \leq c_5(|\xi| + |\eta|)$ for some large constant c_5 and, as before, we split the integral into the regions $|(x, y)| \leq (|\xi| + |\eta|)^{-1/2}$ and $|(x, y)| \geq (|\xi| + |\eta|)^{-1/2}$. For $|(x, y)| \leq (|\xi| + |\eta|)^{-1/2}$ we argue precisely as in case A. This gives an estimate by $c_{26}(|\xi| + |\eta|)^{-1/2}$ for the contribution of this region. As for the remaining region we now denote by L, using a standard argument, the operator

$$L = \frac{\langle \nabla_{(x,y)}(\tau g(x, y) + \xi x + \eta y), \nabla_{(x,y)} \rangle}{|\nabla_{(x,y)}(\tau g(x, y) + \xi x + \eta y)|^2}$$

such that

$$L \exp\left[i(\tau g(x, y) + \xi x + \eta y)\right] = i \exp\left[i(\tau g(x, y) + \xi x + \eta y)\right].$$

Thus $L = A(x, y, \xi, \eta, \tau)(\partial/\partial x) + B(x, y, \xi, \eta, \tau)(\partial/\partial y)$ with

$$A(x, y, \xi, \eta, \tau) = \frac{(\partial/\partial x)(\tau g(x, y) + \xi x + \eta y)}{|\nabla_{(x,y)}(\tau g(x, y) + \xi x + \eta y)|^2}$$

and

$$B(x, y, \xi, \eta, \tau) = \frac{(\partial/\partial y)(\tau g(x, y) + \xi x + \eta y)}{|\nabla_{(x,y)}(\tau g(x, y) + \xi x + \eta y)|^2}.$$

Now, we have $|\nabla_{(x,y)}(\tau g(x, y) + \xi x + \eta y)| \geq |(\xi, \eta)| - |\tau \nabla_{(x,y)} g(x, y)|$. Since $|\nabla_{(x,y)} g(x, y)|$ is as small as we want if ρ and δ are sufficiently small, it follows that

$$|\nabla_{(x,y)}(\tau g(x, y) + \xi x + \eta y)| \geq (1/2)|(\xi, \eta)|. \tag{3.5.5}$$

From this and the explicit expressions of A and B we have $|A(x, y, \xi, \eta, \tau)| \leq c_{27}|(\xi, \eta, \tau)|^{-1}$ and $|B(x, y, \xi, \eta, \tau)| \leq c_{28}|(\xi, \eta, \tau)|^{-1}$. Moreover, since the Hessian in (x, y) of $\tau g(x, y) + \xi x + \eta y$ can be estimated by $c_{29}|\tau|$, it follows from (3.5.5) that we have

$$|(\partial/\partial x) A(x, y, \xi, \eta, \tau)| \leq c_{30}|(\xi, \eta, \tau)|^{-1},$$

$$|(\partial/\partial y)B(x,y,\xi,\eta,\tau)| \leq c_{31}|(\xi,\eta,\tau)|^{-1}.$$

Further, let $\rho > 0$ be such that $supp\tilde{f} \subset B(0,\rho)$, and write

$$\int_{|(x,y)|\geq(|\xi|+|\eta|)^{-1/2}} \exp\left[i(\tau g(x,y)+\xi x+\eta y)\right]$$

$$f(x,y,g(x,y))dxdy =$$

$$= \int_{(|\xi|+|\eta|)^{-1/2}\leq|(x,y)|\leq\rho} -iL\left(\exp\left[i(\tau g(x,y)+\xi x+\eta y)\right]\right)$$

$$f(x,y,g(x,y))dxdy =$$

$$= i\int_{(|\xi|+|\eta|)^{-1/2}\leq|(x,y)|\leq\rho} \exp\left[i(\tau g(x,y)+\xi x+\eta y)\right]$$

$$L^\star\left(f(x,y,g(x,y))\right)dxdy + \text{boundary term}$$

where $L^\star$ is the formal adjoint of L, i.e.,

$$L^\star = A'(x,y,\xi,\eta,\tau)(\partial/\partial x) + B'(x,y,\xi,\eta,\tau)(\partial/\partial y) + C'(x,y,\xi,\eta,\tau)$$

It is immediate that $A' = -A$, $B' = -B$ and that

$$C'(x,y,\xi,\eta,\tau) = \frac{\partial A}{\partial x}(x,y,\xi,\eta,\tau) + \frac{\partial B}{\partial y}(x,y,\xi,\eta,\tau).$$

Now, we pass to polar coordinates, and we observe that the boundary term is associated with the boundary set $r = (|\xi|+|\eta|)^{-1/2}$ and it is given by the product of the functions A, $\tilde{f}$, $\exp\left[i(\tau g+\xi x+\eta y)\right]$ plus the product of the functions B, $\tilde{f}$ and $\exp\left[i(\tau g+\xi x+\eta y)\right]$. We use the following observation: A and B can be estimated by $(|\xi|+|\eta|)^{-1}$, whereas $\tilde{f}$ can be estimated by $|(x,y)|^{-1}$. Therefore, since in the boundary region $r = (|\xi|+|\eta|)^{-1/2}$, $\tilde{f}$ can be estimated by $(|\xi|+|\eta|)^{1/2}$. We then integrate the products $A\tilde{f}$ and $B\tilde{f}$ over a circle of length $2\pi(|\xi|+|\eta|)^{-1/2}$. This shows that the boundary term is of order of magnitude $(|\xi|+|\eta|)^{-1}$.

Thus, we are left with the integral

$$\int_0^{2\pi}\int_{(|\xi|+|\eta|)^{-1/2}\leq r\leq\rho} e^{i(\tau g(r,\alpha)+\xi r\cos\alpha+\eta r\sin\alpha)} L^\star\left(\tilde{f}(r,\alpha)\right)d\alpha dr, \quad (3.5.6)$$

and we have to estimate $L^\star(\tilde{f}(r,\alpha))$. If we write it down explicitly, we find

$$L^\star(\tilde{f}) = -A\frac{\partial}{\partial x}\tilde{f} - B\frac{\partial}{\partial y}\tilde{f} + \tilde{f}\frac{\partial}{\partial x}A + \tilde{f}\frac{\partial}{\partial y}B.$$

In the following lemma we summarize the partial estimates which we have proved above:

Lemma 3.18. *If δ and ρ are small enough, we have, for suitable constants c_i and for $|(x,y)| \leq \rho$:*

(i) $|A(x,y,\xi,\eta,\tau)| \leq c_{27}|(\xi,\eta,\tau)|^{-1}$,

(ii) $|B(x,y,\xi,\eta,\tau)| \leq c_{28}|(\xi,\eta,\tau)|^{-1}$,

(iii) $|(\partial/\partial x)A(x,y,\xi,\eta,\tau)| \leq c_{30}|(\xi,\eta,\tau)|^{-1}$,

(iv) $|(\partial/\partial y)B(x,y,\xi,\eta,\tau)| \leq c_{31}|(\xi,\eta,\tau)|^{-1}$.

(v) $|\tilde{f}| \leq c_{32}|(x,y)|^{-1}$,

(vi) $|\nabla\tilde{f}| \leq c_{33}|(x,y)|^{-2}$,

The previous lemma shows that we have the following estimate

$$|L^\star(\tilde{f})| \leq c_{34}|(\xi,\eta,\tau)|^{-1}|(x,y)|^{-2},$$

thus, the absolute value of the integral (3.5.6) can be estimated by

$$\int_0^{2\pi}\int_{(|\xi|+|\eta|)^{-1/2}}^{\rho} c_{34}|(\xi,\eta,\tau)|^{-1}r^{-1}dr d\alpha \leq$$
$$\leq c_{35}|(\xi,\eta,\tau)|^{-1}\log\left(1+|(\xi,\eta,\tau)|\right).$$

This is an estimate of the desired type.

Remark 3.10. It seems difficult to apply this result to the study of the decay estimates of solutions of the tetragonal crystal system in the case when $c_{11} = c_{66}$, i.e. in the case when the slowness surface has biplanar double points.

Indeed, in the general case, we are not able to obtain results on the curvature properties near biplanar points using the same approach employed in the second chapter. In addition, a more specific study on the regularity properties of the amplitude function, which appears in the solution of the system in the case when $c_{11} = c_{66}$, is needed.

Chapter 4

Global existence of small solutions to nonlinear systems of crystal acoustics

In this chapter we will study the long-time existence of solutions to the nonlinear system of crystal acoustics which is a perturbation of classical linear system. Apart from the technical complications in the case at hand, we will follow the extensive literature on long-time existence for nonlinear wave equations initiated by F. John and S. Klainerman (cf. [J1], [J2], [J3], [K1], [Sh]). In particular the general line of argument here is close to the one in Klainerman and Ponce (cf. [K-P]). Our method of proof consists in combining the classical local existence theorem for quasilinear symmetric hyperbolic systems in L^2-norm with an a priori estimate for the solution of the nonlinear system in an appropriate L^p-norm in which the asymptotic properties of the solution are the same as those of the unperturbed equations. Since we have a worse decay estimates for the fundamental solution of the linear system of crystal acoustic with respect to the wave equation in $\mathbb{R}^3$, and since here we don't have any kind of *null condition* (cf. [Si2]), it seems natural that our assumptions on the nonlinearity of the system must be weaker then those taken in [K-P] and [Si2].

We will start our discussion by describing the main framework of the nonlinear acoustic, that is the assumptions on the nonlinearity and by writing down exactly the form of the nonlinear system of crystal acoustic. Then, we will define the energy function of the system, we will check its conservation in time and we will define and estimate the high order energy in terms of L^∞-norm of the strain tensor. Later, we will prove the a priori estimate for the solution of the nonlinear system and we will rewrite the system as a quasilinear symmetric system. Finally, we will combine the previous results and we will conclude the proof.

4.1 Nonlinear Elasticity

In the following we will consider a solid that behaves like a crystal with the same physical properties which we have described for the linear case (cf. section 1.1).
In this section we want to write down exactly the nonlinear equation of crystal acoustic and we want make our assumptions on the stress and strain tensors precise.
We start by recalling some notations and facts about the crystal acoustic.
We denote by $u = (u_1, u_2, u_3)$ the displacement vector and by $\varepsilon = (\varepsilon_{ij})_{i,j=1,2,3}$ the strain tensor of some elastic body (cf. section 1.2). We recall that, by definition

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

thus in particular, ε is symmetric.

Remark 4.1. In fact, the matrix built with the ε_{ij} is just the symmetric part of the Jacobian in x of the map $(t, x) \to u(t, x)$ (see section 1.2 and the relation (1.2.5)).

Moreover, we recall that in the linear theory the symmetric stress tensor $\sigma = (\sigma_{ij})_{i,j=1,2,3}$ is related to the strain tensor by the generalized Hooke's law (cf. (1.2.7))

$$\sigma_{ij}(\varepsilon) = \sum_{k,l=1}^{3} c_{ijkl}\varepsilon_{kl},$$

where the c_{ijks} are the elastic coefficients, called stiffness constants. We shall only consider homogeneous bodies such that the c_{ijks} are constant. In addition, we know that the c_{ijks} satisfy the following symmetry relations (cf. (1.2.8) and (1.2.9)):

$$c_{ijkl} = c_{klij} = c_{jikl}, \qquad \forall \quad p, q, r, s \in \{1, 2, 3\}.$$

Therefore, out of the 81 possible degrees of freedom for $p, q, r, s \in \{1, 2, 3\}$, we are left with 21 independent constants. In addition, we must also assume that the strain energy

$$W = \frac{\rho}{2}\sum_{ijkl} c_{ijkl}\varepsilon_{ij}\varepsilon_{kl},$$

where ρ is the density of the body, is a positive definite quadratic form in ε_{ij} for symmetric tensors ε. This is a physical condition, in that the strain energy should be given by

$$W = \frac{\rho}{2}\sum_{ij=1}^{3} \sigma_{ij}(\varepsilon)\varepsilon_{ij}, \tag{4.1.1}$$

and this should be positive definite for symmetric components $\varepsilon_{ij} = \varepsilon_{ji}$, if $\varepsilon_{ij} = \varepsilon_{ji} \neq 0$. It follows from the assumption on strain energy that for each y the matrix

$$A_{ik}(y) = \sum_{j,l} c_{ijkl} y_j y_l$$

is a positive definite symmetric matrix (cf. (1.3.12) and see [Du] for more details).

In the absence of body forces, the linear equations of motion are (cf. e.g., [M] and (1.3.6)),

$$\frac{\partial^2 u_i}{\partial t^2} = \sum_{j,k,l=1}^{3} \frac{\partial^2}{\partial x_l \partial x_j}(c_{ijkl}u_k),$$

where the u_i, $i = 1, 2, 3$, are the components of the elastic displacement vector. Accordingly, the energy contained in some measurable set $U \subset \mathbb{R}^3$ at the moment t is

$$E_U(t) = \frac{1}{2}\int_U \left(\sum_{i=1}^{3} \dot{u}_i(t,x)\dot{u}_i(t,x) + \sum_{i,j,k,l=1}^{3} c_{iikl}\partial_{x_j} u_i(t,x)\partial_{x_l} u_k(t,x) \right) dx.$$

We recall that the equation (1.3.6) come, using the generalized Hooke's law, from the relation (1.3.3), when there are no body forces. Thus, in the nonlinear case, it is possible to write the equation of motion in that form, showing the dependence of σ on ε. Here this dependence will not be linear as in Hooke's law. Thus, again assuming the absence of body forces, we obtain

$$\rho \frac{\partial^2 u_i(t,x)}{\partial t^2} = \mathrm{div}\sigma_{ij}(\varepsilon(t,x)) \qquad i = 1, 2, 3. \tag{4.1.2}$$

Here we will assume that the stress-strain relation is only *mildly* nonlinear, i.e. of form:

$$\sigma_{ij}(\varepsilon) = \sum_{k,l=1}^{3} c_{ijkl}\varepsilon_{kl} + H_{ij}(\varepsilon), \tag{4.1.3}$$

$$H : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \qquad H_{ij}(\varepsilon(t,x)) = \mathcal{O}(||\varepsilon||_\infty^\kappa). \tag{4.1.4}$$

Remark 4.2. In the previous equation we assume that ε is *small* This means that we take into account only small perturbations of the body. Actually, keeping ε small during time-evolution is one of our main concerns in the arguments below. We may also assume as a consequence that $|\sigma_{ij}(\varepsilon)| \leq C$ for some constant C, but this assumption depends on wether or not we can control the size of ε.

Remark 4.3. The number κ here is chosen $\kappa \geq 4$. This choice is due to technical reasons. In fact in the isotropic case, and in the presence of so-called *null conditions*, good results can be obtained already when $\kappa = 2$ (see [Si1]). In the present case, there is no known replacement for the null-conditions, and we will lose one order of vanishing of the nonlinear perturbation, when compared with the isotropic situation as a consequence of the fact that the decay estimates for the linear systems of elasticity is weaker than the decay estimate in the isotropic case.

Remark 4.4. We explicitly mention that (4.1.3) implies $\sigma_{ij}(0) = 0$ and $c_{ijks} = (\partial/\partial\varepsilon_{ks})\sigma_{ij}(0)$.

Of course we need some relations to replace the symmetry of the matrix of elastic stiffnesses and the positivity of the energy function. The method for formulating these conditions has been indicated by Green, who asked for the energy function to be a *function of the state*. Actually, what Green postulated is that the differential form

$$\sum_{i,j=1}^{3} \sigma_{ij}(\varepsilon) d\varepsilon_{ij} \tag{4.1.5}$$

be exact. Here we must also take into account that $\varepsilon_{ij} = \varepsilon_{ji}$ and that we also assume that $\sigma_{ij} = \sigma_{ji}$.
We may then consider σ_{ij} as a function of ε_{kl}, and the condition of exactness is then

$$\frac{\partial}{\partial\varepsilon_{kl}}\sigma_{ij} = \frac{\partial}{\partial\varepsilon_{ij}}\sigma_{kl}.$$

It follows that we can find a function $F(\varepsilon)$, such that

$$\sum_{i,j=1}^{3} \sigma_{ij}(\varepsilon) d\varepsilon_{ij} = dF(\varepsilon) = \sum_{i,j=1}^{3} \frac{\partial}{\partial\varepsilon_{ij}} F(\varepsilon) d\varepsilon_{ij},$$

if ε is small. Here $F(\varepsilon)$ should be the non linear strain energy (cf. (4.1.1)), and it is determined up to a constant. We can normalize it to $F(0) = 0$ and it is reasonable to assume that the strain energy is bigger than $F(0)$ when $\varepsilon \neq 0$, i.e., $F(\varepsilon) \geq 0$. Now, if we differentiate $F(\varepsilon)$ with respect to ε_{ij} we obtain

$$\frac{\partial F(\varepsilon)}{\partial\varepsilon_{ij}} = \sigma_{ij}(\varepsilon). \tag{4.1.6}$$

Again, if the strain is zero, we can reasonably expect the stress in the body to be zero, too. Therefore,

$$\frac{\partial F}{\partial\varepsilon_{ij}}(0) = \sigma_{ij}(0) = 0.$$

If we differentiate F twice with respect to ε, we obtain

$$\frac{\partial^2 F(\varepsilon)}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} = \frac{\partial \sigma_{ij}(\varepsilon)}{\partial \varepsilon_{kl}}$$

and thus, if $||\varepsilon||_\infty$ is small, we can write

$$F(\varepsilon) = \sum_{i,j,k,l=1}^{3} \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \varepsilon_{ij} \varepsilon_{kl} + \mathcal{O}(||\varepsilon||_\infty^3),$$

where

$$\sum_{i,j,k,l=1}^{3} \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \varepsilon_{ij} \varepsilon_{kl},$$

is a positive quantity due to the symmetry of ε, and the fact that F has a minimum at 0 if $\varepsilon \neq 0$. It follows that

$$c_1 ||\varepsilon||_\infty^2 \leq F(\varepsilon) \leq c_2 ||\varepsilon||_\infty^2 \tag{4.1.7}$$

for some constant c_1 and c_2, if $||\varepsilon||_\infty$ is small.

Thus, we have written the nonlinear equation for the crystal acoustic (4.1.2), we have replaced the generalized Hooke's law with the nonlinear relation (4.1.3), and finally we have assumed the exactness of the differential form (4.1.5) such that we could define the non linear strain energy $F(\varepsilon)$, which is positive by (4.1.7). In the next section we will prove the conservation of energy for the nonlinear system.

4.2 Conservation of energy

In this section we will define the functional of energy for the nonlinear system of crystal acoustics in $\mathbb{R}^3$ and we will check that it is constant in time.
For simplicity, and without loss of generality, we can assume $\rho \equiv 1$. We define the energy at the time t as

$$E(t) = \int_{\mathbb{R}^3} \left(\frac{1}{2} \sum_{i=1}^{3} \dot{u}_i^2(t,x) + F(\varepsilon(t,x)) \right) dx.$$

In the following proposition we will prove that the energy we have defined is constant in time.

Proposition 4.1. *Assume that u is a solution of*

$$\frac{\partial^2 u}{\partial t^2} = div\sigma$$

for $t \in [0,T]$. We assume that u "dies out" at infinity sufficiently rapidly, i.e. $|\sigma_{ij}|+|\partial_{x_j}\sigma_{ij}| \leq C$, for all t, such that $|\dot{u}_i(t,x)|+|\partial_{x_j}\dot{u}_i(t,x)| \leq c(1+|x|)^{-3-\delta}$, for some $\delta > 0$.
Then $E(t)$ is constant in time.

Proof. We derivate E in time and we have

$$\begin{aligned}
\frac{d}{dt}E(t) &= \int_{\mathbb{R}^3} \sum_{i=1}^{3} (\dot{u}_i \ddot{u}_i)(t,x) + \sum_{i,j=1}^{3} \frac{\partial F}{\partial \varepsilon_{ij}}(\varepsilon(t,x)) \frac{\partial}{\partial t}\varepsilon_{ij}(t,x)dx \\
&= \int_{\mathbb{R}^3} \sum_{i,j=1}^{3} \dot{u}_i \partial_{x_j}\sigma_{ij}(\varepsilon) + \sum_{i,j=1}^{3} \frac{\partial F}{\partial \varepsilon_{ij}}(\varepsilon(t,x)) \frac{\partial_{x_j}\dot{u}_i + \partial_{x_i}\dot{u}_j}{2} dx \\
&= \int_{\mathbb{R}^3} - \sum_{i,j=1}^{3} \partial_{x_j}\dot{u}_i \sigma_{ij}(\varepsilon) + \sum_{i,j=1}^{3} \frac{\partial F}{\partial \varepsilon_{ij}}(\varepsilon(t,x)) \partial_{x_j}\dot{u}_i dx \\
&= \int_{\mathbb{R}^3} \sum_{i,j=1}^{3} \partial_{x_j}\dot{u}_i \left(\frac{\partial F}{\partial \varepsilon_{ij}}(\varepsilon(t,x)) - \sigma_{ij}(\varepsilon) \right) dx = 0,
\end{aligned}$$

since

$$\frac{\partial F}{\partial \varepsilon_{ij}}(\varepsilon(t,x)) = \sigma_{ij}(\varepsilon(t,x)),$$

in view of (4.1.6) and so $E(t) = E(0)$. □

We note that, with the aid of (4.1.7), there exist some constants c_1 and c_2 such that

$$E(t) \geq c_1(||\dot{u}(t)||_2^2 + ||\varepsilon(t)||_2^2) \tag{4.2.1}$$

$$E(0) \leq c_2(||\dot{u}(0)||_2^2 + ||\varepsilon(0)||_2^2). \tag{4.2.2}$$

We conclude, based on the conservation of energy that

$$||\dot{u}(t)||_2 + ||\varepsilon(t)||_2 \leq c_3(||\dot{u}(0)||_2 + ||\varepsilon(0)||_2) \leq c_4(||\dot{u}(0)||_2 + ||\nabla_x u(0)||_2), \tag{4.2.3}$$

for some constants c_3, c_4, which do not depend on t.

Remark 4.5. We recall that

$$||\varepsilon||_2^2 = \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} |\varepsilon_{ij}|^2 dx = \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} |\frac{\partial_{x_i} u_j + \partial_{x_j} u_i}{2}|^2 dx$$
$$\leq c_5 \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} |\partial_{x_i} u_j|^2 dx = ||\nabla_x u||_2^2.$$

This prove the the right hand side of (4.2.3).

On the left hand side of (4.2.3) we want to replace $\varepsilon(t)$ by $\nabla_x u(t)$, changing, if needed, the constants c_3, c_4. The fact that this is possible is well known and goes under the name of *Korn's inequality*, i.e.: if $v = (v_1, v_2, v_3)$, then

$$||\nabla v||_2 \leq c_1 \left[\sum_{i=1}^{3} ||\frac{\partial v_i}{\partial x_i}||_2 + ||\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}||_2 + ||\frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3}||_2 + ||\frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3}||_2 \right], \tag{4.2.4}$$

if v satisfies the condition of the previous proposition (4.1). Now, in order to obtain what we want it suffices to estimate $||\frac{\partial v_j}{\partial x_i}||_2$, when $i \neq j$, by the right hand side of (4.2.4). We then observe that

$$\int_{\mathbb{R}^3} |\frac{\partial v_1}{\partial x_2}|^2 + |\frac{\partial v_2}{\partial x_1}|^2 dx =$$
$$= \int_{\mathbb{R}^3} |\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}|^2 - 2\frac{\partial v_1}{\partial x_2}\frac{\partial v_2}{\partial x_1} dx \leq$$
$$\leq \int_{\mathbb{R}^3} |\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}|^2 dx + 2||\frac{\partial v_1}{\partial x_1}||||\frac{\partial v_2}{\partial x_2}||,$$

since

$$\int_{\mathbb{R}^3} \frac{\partial v_1}{\partial x_2}\frac{\partial v_2}{\partial x_1}dx = \int_{\mathbb{R}^3} \frac{\partial v_1}{\partial x_1}\frac{\partial v_2}{\partial x_2}dx.$$

Thus, we can conclude from the conservation of energy that

$$||\dot{u}(t)||_2 + ||\nabla_x u(t)||_2 \leq C_1(||\dot{u}(0)||_2 + ||\nabla_x u(0)||_2). \tag{4.2.5}$$

4.3 High order energy estimate

We are now interested in seeing to what extent we can estimate $||\dot{u}(t)||_{2,s}$ and $||\nabla u(t)||_{2,s}$ for $s \geq 1$ by $||\dot{u}(0)||_{2,s}$ and $||\nabla u(0)||_{2,s}$. As before, it will suffice to estimate $||\dot{u}(t)||_{2,s}$ and $||\varepsilon(t)||_{2,s}$, but the estimate itself is more complicated. Let us first observe that we can write $\partial_{x_j}\sigma_{ij}$ in the form

$$\partial_{x_j}\sigma_{ij} = \sum_{k,l=1}^{3} \frac{\partial}{\partial \varepsilon_{kl}}\sigma_{ij}\frac{\partial}{\partial x_j}\varepsilon_{kl}$$

for small ε.Thus, the equation (4.1.2) becomes

$$\frac{\partial^2 u_i}{\partial t^2} = \sum_{k,l,j=1}^{3} \frac{\partial \sigma_{ij}(\varepsilon)}{\partial \varepsilon_{kl}}\frac{\partial}{\partial x_j}\varepsilon_{kl} \tag{4.3.1}$$

again with

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

This suggests considering, for a fixed multi-indiex α, with $|\alpha| \neq 0$, the following high order energy:

$$I_\alpha(t) = \int_{\mathbb{R}^3} \left[\sum_{i=1}^{3} \rho(\varepsilon)(\partial^\alpha \dot{u}_i)^2 + \sum_{k,l,i,j=1}^{3} \frac{\partial}{\partial \varepsilon_{kl}}\sigma_{ij}(\varepsilon)\partial^\alpha \varepsilon_{ij}\partial^\alpha \varepsilon_{kl}\right] dx.$$

Remark 4.6. Since we do not expect conservation of higher order energies, there is some freedom in defining them.

Remark 4.7. Here and in the following, if there is no subscript, we denote by ∂^α the derivative in the variables x of order α, where α is a fixed multi-indiex.

We will again assume $\rho(\varepsilon) \equiv 1$ and we obtain

$$I_\alpha(t) = \int_{\mathbb{R}^3} \left[\sum_{i=1}^{3} (\partial^\alpha \dot{u}_i)^2 + \sum_{k,l,i,j=1}^{3} \frac{\partial}{\partial \varepsilon_{kl}} \sigma_{ij}(\varepsilon) \partial^\alpha \varepsilon_{ij} \partial^\alpha \varepsilon_{kl} \right] dx.$$

It follows, by analogy with (4.2.1) and (4.2.2), that

$$\begin{aligned} I_\alpha(t) &\geq c_1(||\partial^\alpha \dot{u}(t)||_2^2 + ||\partial^\alpha \varepsilon(t)||_2^2) && (4.3.2) \\ I_\alpha(0) &\leq c_2(||\partial^\alpha \dot{u}(0)||_2^2 + ||\partial^\alpha \varepsilon(0)||_2^2) && (4.3.3) \end{aligned}$$

for some constants c_1 and c_2 which do not depend on t, as long as $||\varepsilon(t)||_\infty$ remain small.

Our next concern is to study the time evolution of I_α. The rest of this section will be devoted to prove the following result.

Proposition 4.2. *Let $s \in \mathbb{N}$ be fixed and assume that*

$$\sigma_{ij}(\varepsilon) = \sum_{k,s=1}^{3} c_{ijks} \varepsilon_{ks} + H_{ij}(\varepsilon) \quad \textit{with} \quad H_{ij}(\varepsilon) = O(||\varepsilon||_\infty^\kappa),$$

for some integer κ. Then there are some constants C, C_2, δ such that if u is a solution on $[0,T] \times \mathbb{R}^3$ of the system (4.1.2) *which has compact support in x for every t and for which*

$$\begin{aligned} &||\varepsilon(t)||_{\infty,s/2} \leq 1, \quad ||\dot{\varepsilon}(t)||_\infty \leq 1, \\ &||\varepsilon(t)||_\infty \leq \delta, \qquad \textit{for all } t \in [0,T] \end{aligned}$$

then

$$||\dot{u}(t)||_{2,s} + ||\varepsilon(t)||_{2,s} \leq$$
$$\leq C_2(||\dot{u}(0)||_{2,s} + ||\varepsilon(0)||_{2,s}) \exp[C \int_0^t ||\varepsilon(\tau)||_{\infty,s/2}^{\kappa-2} d\tau].$$

As a preparation, we calculate $\partial^\alpha \sigma_{ij}$ with Faa Di Bruno's formula. In fact we have that

$$\partial^\alpha \sigma_{ij} = \sum_{k,l=1}^{3} \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \partial^\alpha \varepsilon_{kl} + R_\alpha, \tag{4.3.4}$$

where R_α is a remainder term which shall be specified later on. It is a sum of terms which contain products of derivatives $\partial^\beta \varepsilon$ with $\beta \leq \alpha$ and $|\beta| < |\alpha|$: see the expression (4.3.7) later on.

Remark 4.8. Of course, the term R_α vanishes for $|\alpha| = 1$.

If u is a solution of (4.3.1) for $t \in [0, T]$, with compact support in x for any fixed t, then we have

$$\int_{\mathbb{R}^3} \sum_{i=1}^{3} \partial_t (\partial^\alpha \dot{u}_i)^2 dx =$$

$$\int_{\mathbb{R}^3} \sum_{i=1}^{3} 2\partial^\alpha \dot{u}_i \partial^\alpha \ddot{u}_i dx = \int_{\mathbb{R}^3} \sum_{i=1}^{3} 2\partial^\alpha \dot{u}_i \partial^\alpha \sum_{j=1}^{3} \partial_{x_j} \sigma_{ij} dx =$$

$$- \int_{\mathbb{R}^3} \sum_{i,j=1}^{3} 2\partial^\alpha \partial_{x_j} \dot{u}_i \partial^\alpha \sigma_{ij} dx = - \int_{\mathbb{R}^3} \sum_{i,j=1}^{3} 2\partial^\alpha \partial_{x_j} \dot{u}_i \sum_{k,l=1}^{3} \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \partial^\alpha \varepsilon_{kl} dx -$$

$$\int_{\mathbb{R}^3} \sum_{i,j=1}^{3} 2\partial^\alpha \partial_{x_j} \dot{u}_i R_\alpha dx.$$

On the other hand, from (4.1.3) it follows that

$$\frac{\partial^2}{\partial t \partial \varepsilon_{kl}} \sigma_{ij}(\varepsilon) = \frac{\partial^2}{\partial t \partial \varepsilon_{kl}} H_{ij}(\varepsilon).$$

We conclude that

$$\frac{d}{dt} I_\alpha(t) = \int_{\mathbb{R}^3} \sum_{i,j,k,l=1}^{3} -2\partial^\alpha \partial_{x_j} \dot{u}_i \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \partial^\alpha \varepsilon_{kl} dx - \int_{\mathbb{R}^3} \sum_{i,j=1}^{3} 2\partial^\alpha \partial_{x_j} \dot{u}_i R_\alpha dx +$$

$$\int_{\mathbb{R}^3} \sum_{k,l,i,j=1}^{3} \left(\frac{\partial^2}{\partial t \partial \varepsilon_{kl}} H_{ij}(\varepsilon) \right) \partial^\alpha \varepsilon_{ij} \partial^\alpha \varepsilon_{pq} dx +$$

$$\int_{\mathbb{R}^3} \sum_{k,l,i,j=1}^{3} \frac{\partial \sigma_{ij}(\varepsilon)}{\partial \varepsilon_{kl}} \left(\partial^\alpha \dot{\varepsilon}_{ij} \partial^\alpha \varepsilon_{kl} + \partial^\alpha \dot{\varepsilon}_{kl} \partial^\alpha \varepsilon_{ij} \right) dx =$$

$$\int_{\mathbb{R}^3} \sum_{i,j,k,l=1}^{3} \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \partial^\alpha \varepsilon_{kl} (2\partial^\alpha \dot{\varepsilon}_{ij} - 2\partial^\alpha \partial_{x_j} \dot{u}_i) dx +$$

$$\int_{\mathbb{R}^3} \sum_{i,j,k,l=1}^{3} \frac{\partial^2 H_{ij}}{\partial t \partial \varepsilon_{kl}} (\varepsilon) \partial^\alpha \varepsilon_{ij} \partial^\alpha \varepsilon_{kl} dx - \int_{\mathbb{R}^3} \sum_{i,j=1}^{3} 2\partial^\alpha \partial_{x_j} \dot{u}_i R_\alpha dx =$$

$$\int_{\mathbb{R}^3} \sum_{i,j,k,l=1}^{3} \partial^\alpha \varepsilon_{kl} \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} (\partial^\alpha \partial_{x_j} \dot{u}_i + \partial^\alpha \partial_{x_i} \dot{u}_j - 2\partial^\alpha \partial_{x_j} \dot{u}_i) dx +$$
$$\int_{\mathbb{R}^3} \sum_{i,j,k,l=1}^{3} \frac{\partial^2 H_{ij}}{\partial t \partial \varepsilon_{kl}} (\varepsilon) \partial^\alpha \varepsilon_{ij} \partial^\alpha \varepsilon_{kl} dx -$$
$$\int_{\mathbb{R}^3} \sum_{i,j=1}^{3} 2\partial^\alpha \partial_{x_j} \dot{u}_i R_\alpha dx = I - II, \quad (4.3.5)$$

with the following obvious notations

$$I = \int_{\mathbb{R}^3} \sum_{i,j,k,l=1}^{3} \frac{\partial^2 H_{ij}}{\partial t \partial \varepsilon_{kl}} (\varepsilon) \partial^\alpha \varepsilon_{ij} \partial^\alpha \varepsilon_{kl} dx,$$
$$II = \int_{\mathbb{R}^3} \sum_{i,j=1}^{3} 2\partial^\alpha \partial_{x_j} \dot{u}_i R_\alpha dx.$$

Now we have to estimate the two integrals I and II. We recall that we can write $H_{ij}(\varepsilon)$ in the form (4.1.4) and so, because our assumptions assure that $||\varepsilon||_\infty < \delta$ is small and $||\dot{\varepsilon}(t)||_\infty \le 1$, there exist some constants c_3 and c_4 such that

$$|I| \le c_3 ||\varepsilon||_\infty^{\kappa-2} ||\partial^\alpha \varepsilon||_2^2 \le c_4 ||\varepsilon||_{\infty,s/2}^{\kappa-2} \left(||\partial^\alpha \dot{u}||_2^2 + \sum_{|\beta| \le s} ||\partial^\beta \varepsilon||_2^2 \right). \quad (4.3.6)$$

Here and in the following, we denote $|\alpha|$ by s.

Remark 4.9. If $|\alpha| = 1$, then, as observed in remark (4.8), $II \equiv 0$. This gives the estimate
$$\left| \frac{dI_\alpha}{dt} \right| \le c_5 ||\varepsilon||_\infty^{k-2} ||\partial^\alpha \varepsilon||_2^2,$$
which, when combined with (4.3.2) leads to
$$\left| \frac{dI_\alpha}{dt} \right| \le c_6 ||\varepsilon||_\infty^{k-2} I_\alpha.$$
When we apply Gronwall's inequality to this we obtain
$$|I_\alpha(t)| \le c_7 I_\alpha(0) \exp\left[||\varepsilon||_\infty^{k-2} t \right].$$
This is the desired estimate for $I_\alpha(t)$, provided we already know that $||\varepsilon||_\infty$ is small.

We now consider the case $|\alpha| > 1$. Of course the estimate for I is still the same as above. In order to estimate II we write R_α explicitly as a sum in $i, j, k, l, \gamma_{kl}, \nu_{pq}$ of terms:

$$\prod_{p,q=1}^{3} \prod_{k,l=1}^{3} c_{ijkl\gamma_{kl}\nu_{pq}} \left(\frac{\partial}{\partial \varepsilon_{kl}}\right)^{\gamma_{kl}} \sigma_{ij} \left(\partial^{\nu_{pq}} \varepsilon_{pq}\right)^{\mu_{pq}}, \tag{4.3.7}$$

where $\gamma_{kl} \in \mathbb{N}$, $1 < \sum_{k,l} \gamma_{kl} = \gamma \leq |\alpha|$, $\mu_{pq} \in \mathbb{N}$ (or zero), $\sum_{p,q} \mu_{pq} = \gamma$, where the ν_{pq} are multi-indices such that $|\nu_{pq}| \leq |\alpha| - \gamma$, $\sum_{p,q} \mu_{pq}\nu_{pq} = \alpha$ and where the $c_{ijkl\gamma_{kl}\nu_{pq}}$ are natural numbers, not all different from zero, which in principle can be calculated explicitly. (The terms with $\sum \gamma_{kl} = 1$ are already taken care of in the first term on the right hand side of (4.3.4). We explicitly mention that when $\mu_{pq} = 0$, then we set $(\partial^{\nu_{pq}} \varepsilon_{pq})^{\mu_{pq}} = 0$. Moreover we do not specify the exact form of the constants $c_{ijkl\gamma_{kl}\nu_{pq}}$, since we shall only use the properties of the $\nu_{pq}, \mu_{pq}, \gamma_{kl}$ mentioned above.)
As before we observe that (we implicitly assume that we deal with terms for which $c_{ijkl\gamma_{kl}\nu_{pq}} \neq 0$)

$$\prod_{k,l} \left(\frac{\partial}{\partial \varepsilon_{kl}}\right)^{\gamma_{kl}} \sigma_{ij} = O(||\varepsilon||_\infty^{max\{0,\kappa-\gamma\}}). \tag{4.3.8}$$

Now, we write II as the sum of two integrals III and IV where III contains exactly the terms with derivatives of order two of σ. Therefore, we have

$$III = O(||\varepsilon||_\infty^{\kappa-2}) \int_{\mathbb{R}^3} (\sum_{i,j=1}^{3} \partial^\alpha \partial_{x_j} \dot{u}_i \sum_{\substack{\alpha_1+\alpha_2=\alpha, \\ \alpha_1\neq 0, \alpha_2\neq 0}} \sum_{\substack{\lambda_1,\lambda_2, \\ \mu_1,\mu_2=1}}^{3} \partial^{\alpha_1} \varepsilon_{\lambda_1\mu_1} \partial^{\alpha_2} \varepsilon_{\lambda_2\mu_2}) dx.$$

It follows in particular for the indices α_i that $|\alpha_i| \leq |\alpha| - 1$, with one of them smaller than or equal to $|\alpha|/2$. We integrate by parts and we obtain

$$III = O(||\varepsilon||_\infty^{\kappa-2}) \int_{\mathbb{R}^3} \sum_{i=1}^{3} \partial^\alpha \dot{u}_i \sum_{j=1}^{3} \partial_{x_j} \left(\sum_{\substack{\alpha_1+\alpha_2=\alpha, \\ \alpha_1\neq 0, \alpha_2\neq 0}} \sum_{\substack{\lambda_1,\lambda_2, \\ \mu_1,\mu_2}} \partial^{\alpha_1} \varepsilon_{\lambda_1\mu_1} \partial^{\alpha_2} \varepsilon_{\lambda_2\mu_2} \right) dx,$$

and therefore there exist some constants c_8, c_9 and c_{10} such that

$$|III| \leq c_8||\varepsilon||_\infty^{\kappa-2} \sum_{j=1}^{3} \sum_{\substack{\alpha_1+\alpha_2=\alpha+e_j,\\ \alpha_1\neq 0,\alpha_2\neq 0}} ||\partial^\alpha \dot{u}||_2 ||\partial^{\alpha_1}\varepsilon||_\infty ||\partial^{\alpha_2}\varepsilon||_2 \leq$$
$$c_9||\varepsilon||_{\infty,s/2}^{\kappa-1} \sum_{|\alpha_2|\leq|\alpha|} (||\partial^\alpha \dot{u}||_2 ||\partial^{\alpha_2}\varepsilon||_2) \leq$$
$$c_{10}||\varepsilon||_{\infty,s/2}^{\kappa-1}(||\partial^\alpha \dot{u}||_2^2 + \sum_{|\alpha_2|\leq|\alpha|} ||\partial^{\alpha_2}\varepsilon||_2^2), \quad (4.3.9)$$

where we have assumed that $|\alpha_1| \leq |\alpha_2|$ such that $|\alpha_1| \leq s/2$ and therefore $||\partial^{\alpha_1}\varepsilon||_\infty \leq 1$. We now consider the terms in IV. We again use the relation (4.3.8) and the fact that $||\varepsilon||_\infty$ is small, in order to estimate IV.

$$|IV| \leq \sum_{r=3}^{|\alpha|} O(||\varepsilon||_\infty)^{\kappa-r}$$
$$\int_{\mathbb{R}^3} |\sum_{i=1}^{3} \partial^\alpha \dot{u}_i \sum_{\substack{\lambda_1,\ldots,\lambda_r\\ \mu_1,\ldots,\mu_r=1}}^{3} \sum_{\substack{\alpha_1+\cdots+\alpha_r=\alpha'\\ \alpha_i\neq 0\forall i}} \partial^{\alpha_1}\varepsilon_{\lambda_1\mu_1}\cdots\partial^{\alpha_r}\varepsilon_{\lambda_r\mu_r} dx|$$

where $1 \leq \alpha_i \leq |\alpha| - 2$ and α' is such that $|\alpha'| = |\alpha| + 1$ (the length of α' is $|\alpha| + 1$ since we have also included in α' the derivative in $(\partial/\partial x_j)$), with at most one of $|\alpha_i|$ bigger than $|\alpha|/2$. Here we suppose that $\kappa > r$. Thus, there exists a constant c_{11} such that

$$|IV| \leq c_{11} \sum_{r=3}^{|\alpha|} ||\varepsilon||_\infty^{\kappa-r} ||\partial^{\alpha_{l_1}}\varepsilon||_\infty \cdots ||\partial^{\alpha_{l_{r-1}}}\varepsilon||_\infty$$
$$\int_{\mathbb{R}^3} \left| \sum_{i=1}^{3} \partial^\alpha \dot{u}_i \sum_{\lambda_{l_r},\mu_{l_r}=1}^{3} \partial^{\alpha_{l_r}}\varepsilon_{\lambda_{l_r}\mu_{l_r}} \right| dx$$

where $l_1, \ldots, l_r$ is a permutation of $1, \ldots, r$ such that $|\alpha_{l_r}| > |\alpha_{l_i}|$ for all $i \neq r$ and $|\alpha_{l_i}| \leq |\alpha_{l_j}|$ if $l_i < l_j$.
We note that $|\alpha_{l_i}| < |\alpha|/2$ for all $i \neq r$ and thus $||\partial^{\alpha_{l_i}}\varepsilon||_\infty \leq 1$ for all $i \neq r$.

Therefore we have, for some constants c_i,

$$
\begin{aligned}
|IV| \leq c_{12}||\varepsilon||^{\kappa-1}_{\infty,s/2}\sum_{r=3}^{|\alpha|}\int_{\mathbb{R}^3}\left|\sum_{i=1}^{3}\partial^\alpha \dot{u}_i \sum_{\lambda_{l_r},\mu_{l_r}=1}^{3}\partial^{\alpha_{l_r}}\varepsilon_{\lambda_{l_r}\mu_{l_r}}\right| dx \leq \\
c_{13}||\varepsilon||^{\kappa-1}_{\infty,s/2}\sum_{r=3}^{|\alpha|}\sum_{i=1}^{3}\sum_{\lambda_{l_r},\mu_{l_r}=1}^{3}\int_{\mathbb{R}^3}\left|\partial^\alpha \dot{u}_i\partial^{\alpha_{l_r}}\varepsilon_{\lambda_{l_r}\mu_{l_r}}\right| dx \leq \\
c_{14}||\varepsilon||^{\kappa-1}_{\infty,s/2}\sum_{r=3}^{|\alpha|}\sum_{i=1}^{3}\sum_{\lambda_{l_r},\mu_{l_r}=1}^{3}||\partial^\alpha \dot{u}_i||_2||\partial^{\alpha_{l_r}}\varepsilon_{\lambda_{l_r}\mu_{l_r}}||_2 \leq
\end{aligned}
$$

$$
\begin{aligned}
c_{15}||\varepsilon||^{\kappa-1}_{\infty,s/2}\sum_{r=3}^{|\alpha|}||\partial^\alpha \dot{u}||_2||\partial^{\alpha_{l_r}}\varepsilon||_2 \leq \\
c_{16}||\varepsilon||^{\kappa-1}_{\infty,s/2}\sum_{r=3}^{|\alpha|}\left(||\partial^\alpha \dot{u}||_2^2 + ||\partial^{\alpha_{l_r}}\varepsilon||_2^2\right) \leq \\
c_{17}||\varepsilon||^{\kappa-1}_{\infty,s/2}\left(||\partial^\alpha \dot{u}||_2^2 + \sum_{|\beta|\leq s}||\partial^\beta\varepsilon||_2^2\right),
\end{aligned} \quad (4.3.10)
$$

where we used the fact that

$$
\sum_{r=3}^{|\alpha|}||\varepsilon||^{\kappa-r}_{\infty}||\partial^{\alpha_{l_1}}\varepsilon||_\infty\cdots||\partial^{\alpha_{l_{r-1}}}\varepsilon||_\infty \leq c_{18}\sum_{r=3}^{|\alpha|}||\varepsilon||^{\kappa-1}_{\infty,l_{r-1}} \leq c_{19}||\varepsilon||^{\kappa-1}_{\infty,|\alpha|/2}.
$$

Now, if we combine (4.3.6), (4.3.9) and (4.3.10), in view of (4.3.5), we obtain the following estimate of I_α, for all fixed α:

$$
\left|\frac{dI_\alpha}{dt}(t)\right| \leq c_{20}||\varepsilon||^{\kappa-2}_{\infty,s/2}\left(||\partial^\alpha \dot{u}||_2^2 + \sum_{|\beta|\leq s}||\partial^\beta\varepsilon||_2^2\right) \quad (4.3.11)
$$

Now, if we fix an integer s and we define

$$
J_s(t) = \sum_{|\alpha|\leq s} I_\alpha(t),
$$

from estimate (4.3.11) and relation (4.3.2), it follows that

$$\left|\frac{d}{dt}J_s(t)\right| \leq \sum_{|\alpha|\leq s}\left|\frac{d}{dt}I_\alpha(t)\right| \leq$$

$$\sum_{|\alpha|\leq s} c_{20}||\varepsilon||^{\kappa-2}_{\infty,|\alpha|/2}\left(||\partial^\alpha \dot{u}||_2^2 + \sum_{|\beta|\leq|\alpha|}||\partial^\beta\varepsilon||_2^2\right) \leq$$

$$c_{21}||\varepsilon||^{\kappa-2}_{\infty,s/2}\sum_{|\alpha\leq s}\left(||\partial^\alpha\dot{u}||_2^2 + ||\partial^\alpha\varepsilon||_2^2\right) \leq$$

$$c_{22}||\varepsilon||^{\kappa-2}_{\infty,s/2}\sum_{|\alpha\leq s} I_\alpha(t) \leq$$

$$\leq C_3||\varepsilon||^{\kappa-2}_{\infty,s/2}|J_s(t)|,$$

and using Gronwall's inequality we obtain

$$|J_s(t)| \leq |J_s(0)|\exp[C_3\int_0^t ||\varepsilon(\tau)||^{\kappa-2}_{\infty,s/2}d\tau].$$

The inequality just obtained conclude the proof of proposition (4.2).

Remark 4.10. As before, we can replace $\varepsilon(t)$ by $\nabla_x u(t)$. Indeed we can use Korn's inequality to estimate the L^2-norm of ε from below and the fact that $||\varepsilon(t)||_\infty \leq c_{23}||\nabla_x u(t)||_\infty$ to estimate the norm under the sign of integral. Thus, there are constants C_4, C_5 such that

$$||\dot{u}(t)||_{2,s} + ||\nabla_x u(t)||_{2,s} \leq$$
$$\leq C_5(||\dot{u}(0)||_{2,s} + ||\nabla_x u(0)||_{2,s})\exp[C_4\int_0^t ||\nabla_x u(\tau)||^{\kappa-2}_{\infty,s/2}d\tau]. \quad (4.3.12)$$

Thus, using energy estimates, we control $||\nabla_x u(t)||_{2,s}$ in terms of $||\nabla_x u(\tau)||_{\infty,s/2}$. We are also able to control $||\nabla_x u||_{\infty,s/2}$ with $||\nabla_x u||_{q,s'}$, for a suitable choice of q and s', using Sobolev's inequality. Therefore we can use a bootstrap argument to prove the existence of a global solution for the Cauchy problem (4.1.2) with initial conditions $u_i(x,0) = f_i(x)$, $\partial_t u_i(x,0) = g_i(x)$ (here f_i and g_i are smooth functions with compact support) with $i = 1,2,3$, if we can estimate $||\nabla_x u(t)||_{q,s'}$ in terms of $||\nabla_x u(t)||_{2,s}$.

4.4 An a priori estimate in the H^s-norm for the solution

This step of the proof is the key of our argument. As a first step, we rewrite the inhomogeneous problem as a perturbation problem of the linearized equation. Indeed, if we recall the relation (4.1.3), we can write (4.1.2) as

$$\frac{\partial^2 u_i}{\partial t^2}(t,x) - \sum_{j,k,l=1}^{3} c_{ijkl}\frac{\partial u_k}{\partial x_l \partial x_j}(t,x) = \mathrm{div} H_{ij}(\varepsilon) \qquad i = 1,2,3. \tag{4.4.1}$$

Let us now denote by $U(t)$ the solution operator of the Cauchy problem

$$\frac{\partial^2 u_i}{\partial t^2}(t,x) - \sum_{j,k,l=1}^{3} c_{ijkl}\frac{\partial u_k}{\partial x_l \partial x_j}(t,x) = 0 \quad i = 1,2,3 \tag{4.4.2}$$

$$u_i(0,x) = f_i(x), \quad \partial_t u_i(0,x) = g_i(x), \quad i = 1,2,3, \tag{4.4.3}$$

i.e., let for any given f and g, $U(t)(f,g) = (u(t,x), \dot{u}(t,x))$, where $u(t,x)$ satisfies (4.4.2), (4.4.3) and $f = (f_1, f_2, f_3)$, $g = (g_1, g_2, g_3)$.
We also introduce the notation $V(t)(f,g) = (\nabla_x u(t,x), \dot{u}(t,x))$. In the case of cubic and tetragonal crystal classes studied (for details see e.g. [L1] and the chapter 3 of this thesis) we have the $L^1 - L^\infty$ decay estimate for the solution of the homogeneous system. In particular, there are a constant c_1 and a positive integer r such that

$$||U(t)(f,g)||_\infty \leq c_1(1+t)^{-1/2}(||f||_{1,r} + ||g||_{1,r})$$

From this we also obtain the estimate

$$||V(t)(f,g)||_\infty \leq c_2(1+t)^{-1/2}(||f||_{1,r+1} + ||g||_{1,r+1})$$

For the solution $U(t)(f,g)$ also holds the inequality (4.2.5), and in particular from that relation it follows

$$||V(t)(f,g)||_2 \leq c_3(||\nabla_x f||_2 + ||g||_2).$$

Here we are now interested in a combination of the previous two inequalities as given by interpolation theory. We apply the Riesz-Thorin inequality (for detail see [Be-Lo]) with the following choices

$$\begin{aligned} X &= \{(\partial_x^\alpha f, \partial_x^\alpha g)_{|\alpha|\le r+1} : ||(\partial_x^\alpha f, \partial_x^\alpha g)_{|\alpha|\le r+1}||_{L^s}\}, \\ Y &= \{(h_1, h_2) : ||(h_1, h_2)||_{L^\rho}\} \end{aligned}$$

and $T : X \to Y$, with

$$T((\partial_x^\alpha f, \partial_x^\alpha g)_{|\alpha|\le r+1}) = V(t)(f, g)$$

and the pairs $(s = \infty, \rho = 1), (s = 2, \rho = 2)$. The following remark then follows from the theorem of Riesz-Thorin.

Remark 4.11. There is a constant c_4 which does not depend on t such that

$$||V(t)(f, g)||_q \le c_4(1 + t)^{-1/2+1/q}(||f||_{p,r+1} + ||g||_{p,r+1}) \tag{4.4.4}$$

provided that $q \ge 2$ and $1/p + 1/q = 1$.

As a second step, we can write the solution $u(t, x)$ of the nonlinear Cauchy problem (4.4.1), (4.4.3), by using Duhamel's principle. In fact, with the previous notations, we have

$$u(t, x) = U(t)(f, g) + \int_0^t U(t - \tau)(0, \mathrm{div}H)d\tau$$

where $\mathrm{div}H$ is the vector $(\mathrm{div}H_{1j}, \mathrm{div}H_{2j}, \mathrm{div}H_{3j})$. Since the same relation holds for the partial derivative of u, we can easily obtain the following equation:

$$\nabla_x u(t, x) = V(t)(f, g) + \int_0^t V(t)(0, \mathrm{div}H)d\tau.$$

Thus, if we consider the L^q-norm of $\nabla_x u(t, x)$, we can apply the estimate (4.4.4), and we have

$$\begin{aligned} ||\nabla_x u(t, x)||_q \le c_4(1 + t)^{-1/2+1/q}(||\nabla_x f||_{p,r} + ||g||_{p,r})+ \\ + c_4 \int_0^t (1 + t - \tau)^{-1/2+1/q}||\mathrm{div}H||_{p,r}d\tau. \end{aligned}$$

As before, if we consider the $W^{p,s'}$-norm, we obtain

$$||\nabla_x u(t,x)||_{q,s'} \leq c_5(1+t)^{-1/2+1/q}(||\nabla_x f||_{p,s'+r} + ||g||_{p,s'+r}) + \\ + c_5 \int_0^t (1+t-\tau)^{-1/2+1/q} ||\mathrm{div} H||_{p,s'+r} d\tau. \quad (4.4.5)$$

Note that

$$\mathrm{div} H_{ij} = \sum_{j=1}^{3} \sum_{p,q=1}^{3} \frac{\partial H_{ij}}{\partial \varepsilon_{pq}} \frac{\partial \varepsilon_{pq}}{\partial x_j}.$$

Therefore, using the relation (4.1.4), and the same argument used in the estimate of IV in the previous section, we can prove that there exists a constant c_6 such that

$$\left| \partial^\alpha \left(\sum_{j=1}^{3} \sum_{p,q=1}^{3} \frac{\partial H_{ij}}{\partial \varepsilon_{pq}} \frac{\partial \varepsilon_{pq}}{\partial x_j} \right) \right| \leq c_6 \left(\sum_{|\beta| \leq |\alpha|+1} |\partial^\beta \varepsilon_{pq}| \right) \left(\sum_{|\beta| \leq |\alpha|/2} |\partial^\beta \varepsilon_{pq}| \right)^{\kappa-1},$$

if $||\varepsilon||_\infty \leq \delta$ and $||\varepsilon||_{\infty,|\alpha|/2} \leq 1$. Now, using Hölder's inequality, we obtain the following estimate of $(p, s'+r)$-norm of $\mathrm{div} H$:

$$||\mathrm{div} H||_{p,s'+r} \leq c_7 ||\varepsilon||_{2,s'+r+1} ||\varepsilon||^{\kappa-1}_{q,s'/2+r/2} \quad (4.4.6)$$

where c_7 is a suitable constant, provided that $1/p = 1/2 + (\kappa-1)/q$. From remark (4.11) we also have the condition $1/p + 1/q = 1$. This implies that p and q must be equal to $2\kappa/(2\kappa-1)$ and 2κ, respectively.
Now we use estimate (4.4.6) in the right hand side of inequality (4.4.5) and thus we have proved the following proposition.

Proposition 4.3. *Let $s' \in \mathbb{N}$, $\kappa \geq 3$ integer, $p = 2\kappa/(2\kappa-1)$ and $q = 2\kappa$ fixed. Then there are constants δ, r, c_8 and c_9 with the following property: assume $u(t,x)$ is a solution of (4.4.2), (4.4.3) on $[0,T]\times\mathbb{R}^3$ which has compact support on x for every t and*

$$||\nabla_x u(t,x)||_\infty \leq \delta, \qquad ||\nabla_x u(t,x)||_{\infty,s'/2+r/2} \leq 1$$

for all $t \in [0,T]$. Then it follows that

$$||\nabla_x u(t,x)||_{q,s'} \leq c_8(1+t)^{-1/2+1/q}(||\nabla_x f||_{p,s'+r} + ||g||_{p,s'+r}) + \\ + c_9 \int_0^t (1+t-\tau)^{-1/2+1/q} ||\varepsilon||_{2,s'+r+1} ||\varepsilon||^{\kappa-1}_{q,s'/2+r/2} d\tau \quad (4.4.7)$$

for all $t \in [0, T]$.

Now, in order to prove the existence of a global solution of the Cauchy problem, we give two elementary results as preparation.

Lemma 4.4. *We define, for some integers* q *and* s

$$M_s(t) := \sup_{0 \le \tau \le t} (1+\tau)^{1/2-1/q} ||\nabla_x u(x,\tau)||_{q,s}.$$

If $q > 6$, *there exist constants* c_{10} *and* c_{11} *such that*

$$\int_0^t ||\nabla_x u(\tau)||^3_{q,s} d\tau \le c_{10} M_s^4(t), \tag{4.4.8}$$

and

$$\int_0^t (1+t-\tau)^{-1/2+1/q} ||\nabla_x u(\tau)||^4_{q,s} d\tau \le c_{11}(1+t)^{-1/2+1/q} M_s^4(t). \tag{4.4.9}$$

Proof. We have

$$\begin{aligned}
\int_0^t ||\nabla_x u(\tau)||^3_{q,s} d\tau = \int_0^t (1+\tau)^{3/2-3/q}(1+\tau)^{-3/2+3/q} ||\nabla_x u(\tau)||^3_{q,s} d\tau \le \\
\le M_s^3(t) \int_0^t (1+\tau)^{-3/2+3/q} d\tau \le \\
\le M_s^3(t) \int_0^\infty (1+\tau)^{-3/2+3/q} d\tau \le c_{10} M_s^3(t),
\end{aligned}$$

because $q > 6$ implies $-3/2 + 3/q < -1$. Moreover

$$\begin{aligned}
\int_0^t (1+t-\tau)^{-1/2+1/q} ||\nabla_x u(\tau)||^4_{q,s} d\tau \le \\
\le M_s^4(t) \int_0^t (1+t-\tau)^{-1/2+1/q}(1+\tau)^{-2+4/q} d\tau \le \\
\le c_{11}(1+t)^{-1/2+1/q} M_s^4(t).
\end{aligned}$$

In order to prove the last line, we must show that

$$\int_0^t (1+t-\tau)^{-1/2+1/q}(1+\tau)^{-2+4/q} d\tau \le c_{11}(1+t)^{-1/2+1/q}.$$

In order to do this we discuss separately the two cases $|t-\tau| < (1/3)t$ and $|t-\tau| \geq (1/3)t$. In the first case $t \sim \tau$ and in the second $|t-\tau| \sim (t+\tau)$. We obtain for $|t-\tau| < (1/3)t$ that $(1+\tau)^{-2+4/q} \sim t^{-2+4/q}$ and therefore that $\int_0^t (1+t-\tau)^{-1/2+1/q}(1+\tau)^{-2+4/q} d\tau \leq c' \int_0^t (1+t-\tau)^{-1/2+1/q} t^{-2+4/q} d\tau \leq c'(1+t-\tau)^{1/2+1/q}|_0^t t^{-2+4/q} \leq c(1+t)^{-3/2+5/q}$. Thus, it remains to observe for this case that $-3/2+5/q < -1/2+1/q$ when $q > 6$. In the remaining case, i.e., when $|t-\tau| \geq (1/3)t$, we can estimate the integral by $c\int_0^t (1+t)^{-1/2+1/q}(1+\tau)^{-2+4/q} d\tau \leq c(1+t)^{-1/2+1/q}$ since $\int_0^t (1+\tau)^{-2+4/q} d\tau < \infty$. □

Remark 4.12. We recall that, by Sobolev embedding theorem, if $\sigma \geq 3/q$, there exists a constant c_{12} such that

$$||\nabla_x u(t,x)||_{\infty, s/2} \leq c_{12} ||\nabla_x u(t,x)||_{q, s/2+\sigma}.$$

Lemma 4.5. *Let q be fixed and r the same as in proposition* (4.3)*. It is then possible to choose s' and η such that the conditions $||\nabla_x u(t,x)||_{q,s'} \leq \eta$ and $||\nabla_x \dot{u}(t,x)||_{q,s'-1} \leq \eta$ imply implies*

$$||\nabla_x u(t,x)||_\infty \leq \delta,$$
$$||\nabla_x \dot{u}(t,x)||_\infty \leq 1$$
$$||\nabla_x u(t,x)||_{\infty, s'/2+r/2+1/2} \leq 1,$$
$$||\nabla_x u(t,x)||_{\infty, s'/2+r/2} \leq 1.$$

Proof. It is sufficient to choose $s' \geq s'/2 + r/2 + 3/q + 1/2$ and the argument easily follows from remark (4.12). □

Proposition 4.6. *Let $\kappa = 5$, r, $p = 10/9$, $q = 10$ as in* (4.3)*, s' and η as in the previous lemma. Suppose that*

$$||\nabla_x f||_{10/9, s'+r} + ||g||_{10/9, s'+r} \leq \delta', \tag{4.4.10}$$
$$||\nabla_x f||_{2, s'+r+1} + ||g||_{2, s'+r+1} \leq \delta', \tag{4.4.11}$$
$$||\dot{u}||_{10,s'} + ||\nabla_x u(t,x)||_{10,s'} \leq \eta, \quad \forall t \in [0,T] \tag{4.4.12}$$

where f and g are the initial data, δ' is a small positive quantity and $u(t,x)$ is a solution of (4.4.2), (4.4.3) *on $[0,T] \times \mathbb{R}^3$, with compact support in x for* *all fixed t. Then $M_{s'}(t)$ is bounded by some constant M_0 which is independent of T, for all $t \in [0, T]$.*

Proof. By lemma (4.5) it's easy to check that all hypoteses of proposition (4.2) are satisfied. Thus, if we consider the $H^{s'+r+1}$ norm of $\nabla_x u$, then we can, at first, use the estimate (4.3.12) and then the Sobolev embedding theorem (to estimate the $W^{\infty,2/s+r/2+1/2}$ norm of the $\nabla_x u$ under the sign of integral) to obtain

$$||\nabla_x u(t,x)||_{s'+r+1} \leq \\ c_{13}(||\nabla_x f||_{s'+r+1} + ||g||_{s'+r+1}) \exp[c_{14} \int_0^t c_{15} ||\nabla_x u(\tau)||^3_{10,s'} d\tau].$$

Now we use the inequality (4.4.8) of lemma (4.4) to estimate the integral in the exponential and we have

$$||\nabla_x u(t,x)||_{s'+r+1} \leq c_{13}(||\nabla_x f||_{s'+r+1} + ||g||_{s'+r+1}) \exp[c_{14} c_{10} c_{15} M^3_{s'}(t)]. \tag{4.4.13}$$

As before, lemma (4.5) ensures that all conditions in proposition (4.3) hold. Therefore, we can consider inequality (4.4.7) and use (4.4.13) to estimate the H^s-norm of the gradient of u. We obtain

$$||\nabla_x u(t,x)||_{10,s'} \leq \\ \leq c_8(1+t)^{-1/2+1/q}(||\nabla_x f||_{10/9,s'+r} + ||g||_{10/9,s'+r}) + \\ + c_9 c_{16}(||\nabla_x f||_{s'+r+1} + ||g||_{s'+r+1}) \exp[c_{14} c_{10} c_{15} M^3_{s'}(t)] \\ \int_0^t (1+t+\tau)^{-1/2+1/q} ||\nabla_x u(\tau)||^4_{10,s'}.$$

Now we use the hypoteses (4.4.10), (4.4.11) and the inequality (4.4.9) of lemma (4.4) to obtain

$$M_{s'}(t) \leq c_{17}\delta'(1 + M^4_{s'}(t) \exp[c_{18} M^3_{s'}(t)]). \tag{4.4.14}$$

for a suitable choice of the constant c. We can conclude from this that $M_{s'}(t)$ must be smaller than M_0 if δ' is sufficiently small. Indeed, let $f(x) = c\delta'(1+$

$x^4 \exp[cx^3]) - x$, then there exists $x = M_0 > 0$ such that $f(M_0) = 0$, provided that δ' is sufficient small. Now, if $\delta' < M_0$, then, by Sobolev's inequality, the conditions on the initial data imply that $M_{s'}(0) \leq M_0$. Moreover, let $\bar{t}$ be such that $M_0 = M_{s'}(\bar{t})$, then $\bar{t} > T$. Indeed, if we suppose, by absurd, that $\bar{t} < T$, then there exist $\bar{t} < \tilde{t} < T$ such that $f(x) < 0$ which is absurd because (4.4.14) is true for all $t \in [0, T]$. Thus, we can conclude that $M_{s'}(t) \leq M_0$ for all $t \in [0, T]$. The proof is therefore complete. $\square$

Remark 4.13. In the previous proposition we chose $\kappa = 5$. Note that it is the smallest possible choose for κ. Indeed if we take $\kappa = 4$ we can not obtain estimates as in lemma (4.4) and so the whole proof come down.
Moreover we observe that in the case of the nonlinear wave equation (cf. [Si2]) the existence of the global solution is proved in $\mathbb{R}^3$ for $\kappa = 2$. Thus, in the case of crystal acoustic we lose three *degrees of linearity*. This is due to the following three facts. The first is that in the linear system of crystal acoustic we have a weaker estimate for the decay of the solution, than in the case of the wave equation. The second is that we don't have any kind of null conditions for the nonlinear equation. The third is that we have a worse high order energy estimate, than in the case of the wave equation: in particular, the exponent of the supremum norm in the estimate (4.3.12) is $\kappa - 2$ and it can't be better, due to the estimate (4.3.6).

Now we can obtain an a priori estimate in the H^s-norm of the gradient of the solution of the nonlinear problem (4.4.1), (4.4.3).
Indeed, let $s' \geq s/2 + 3/10$, thus we can argue as in previous proof and we obtain:

$$\begin{aligned}
&||\dot{u}(t,x)||_s + ||\nabla_x u(t,x)||_s \leq \\
&\qquad \leq c_{19}(||\nabla_x f||_s + ||g||_s) \exp[c_{20} \int_0^t ||\nabla_x u(\tau)||^3_{\infty, s/2} d]\tau \leq \\
&\qquad \leq c_{19}(||\nabla_x f||_s + ||g||_s) \exp[c_{20} c_{17} c_{15} M^3_{s'}(T)] \leq \\
&\qquad \leq c_{19}(||\nabla_x f||_s + ||g||_s) \exp[c_{20} c_{10} c_{15} M^3_0]. \qquad (4.4.15)
\end{aligned}$$

We are now ready to prove the following theorem.

Theorem 4.7. *Given a solution u of the nonlinear problem* (4.4.1), (4.4.3) *in a time interval* $[0,T]$ *with compact support in x for all fixed t, there exist a $\delta > 0$ sufficiently small and $\eta > 0$ such that if*

$$||\nabla_x f||_{10/9,s} + ||g||_{10/9,s} \leq \delta, \tag{4.4.16}$$

$$||\nabla_x f||_s + ||g||_s \leq \delta, \tag{4.4.17}$$

$$||\dot{u}||_{10,s'} + ||\nabla_x u(t,x)||_{10,s'} \leq \eta, \quad \forall t \in [0,T] \tag{4.4.18}$$

then

$$||\dot{u}(t,x)||_s + ||\nabla_x u(t,x)||_s \leq K_s(||\nabla_x f||_s + ||g||_s) \quad \textit{for all} \quad t \in [0,T],$$

where K_s is a constant independent from T, provided that the conditions

$$s' \geq s/2 + 3/10 \tag{4.4.19}$$

$$s \geq s' + r + 1 \tag{4.4.20}$$

$$s' \geq s'/2 + r/2 + 4/5 \tag{4.4.21}$$

$$s, s', r > 0 \tag{4.4.22}$$

hold.

Proof. The condition $s \geq s' + r + 1$ assures that the hypothesis of smallness on the initial data imply the hypothesis of proposition (4.6). The condition $s' \geq s'/2 + r/2 + 4/5$ assures that we can use lemma (4.5) with condition (4.4.18). Moreover, the condition $s' \geq s/2 + 3/10$ assures that the estimate (4.4.15) holds. □

Remark 4.14. We observe that the conditions

$$s' \geq s/2 + 3/10, \qquad s \geq s' + r + 1,$$
$$s' \geq s'/2 + r/2 + 4/5, \qquad s, s', r > 0$$

can be satisfied simultaneously.

4.5 Reduction to a symmetric quasilinear hyperbolic system

The next step in order to prove the existence of a global solution of (4.4.1), (4.4.3) is to transform the second order system of three equations into a first order system of bigger dimension.

Remark 4.15. To avoid misunderstanding, in this section we will denote the vectors with a bold letter.

In order to do this we denote $\mathbf{u} = (u_1, u_2, u_3)$, thus we can write the system (4.4.1) in the following form:

$$\partial_t^2 \mathbf{u} - \sum_{lj=1}^{3} C_{lj} \mathbf{u} = \mathrm{div} H. \tag{4.5.1}$$

We recall that $\mathrm{div} H$ is a vector which depends only on ∇u, and we denote it by $\mathbf{g}(\nabla \mathbf{u})$. We also denote $\mathbf{W} = (\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, where

$$\mathbf{u}_0 = \partial_t \mathbf{u} \qquad \mathbf{u}_i = \partial_{x_i} \mathbf{u} \quad i = 1, 2, 3.$$

In this way, we obtain a first-order system of twelve equations for W, namely

$$\begin{aligned} \partial_t \mathbf{u}_0 &= \sum_{lj=1}^{3} C_{lj} \partial_{x_j} \mathbf{u}_l + \mathbf{g}(\nabla \mathbf{u}) \\ \partial_t \mathbf{u}_j &= \partial_{x_j} \mathbf{u}_0 \qquad j = 1, 2, 3, \end{aligned} \tag{4.5.2}$$

which is a system of the form

$$\partial_t \mathbf{W} = \sum_{j=1}^{3} C'_j \partial_{x_j} \mathbf{W} + F(\mathbf{W}). \tag{4.5.3}$$

It is well known that the previous system is a symmetrizable hyperbolic system provided there exists a matrix C_0, positive definite, such that $C_0 C'_j = A_j$ are all symmetric (for details see [T2]).

Note that, if C_{lj} have the form $c_{lj} I$ (here $c_{lj} \in \mathbb{R}$ and I is the identity matrix), then we can choose $C_0 = \mathrm{diag}(1, 1, C^{-1})$, where C^{-1} is the inverse of the matrix $C = c_{lj}$. In this case C_0 is positive definite as long as C is and, under these hypotheses , (4.5.3) is symmetrizable.

After some easy calculations, we can find that the C'_j are the following 12×12 matrices:

$$C'_1 = \left(\begin{array}{c|c|c|c} 0 & B_1 & D & E \\ \hline I & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right), \qquad C'_2 = \left(\begin{array}{c|c|c|c} 0 & D & B_2 & F \\ \hline 0 & 0 & 0 & 0 \\ \hline I & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

$$C'_1 = \left(\begin{array}{c|c|c|c} 0 & E & F & B_3 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline I & 0 & 0 & 0 \end{array}\right).$$

Here 0 is the 3×3 matrix where all the elements equal zero, I is the 3×3 identity matrix and A_i, B, D and E are the following 3×3 matrices:

$$B_1 = \begin{pmatrix} c_{11} & 0 & 0 \\ 0 & c_{66} & 0 \\ 0 & 0 & c_{44} \end{pmatrix}, \quad B_2 = \begin{pmatrix} c_{66} & 0 & 0 \\ 0 & c_{11} & 0 \\ 0 & 0 & c_{44} \end{pmatrix}, \quad B_3 = \begin{pmatrix} c_{44} & 0 & 0 \\ 0 & c_{44} & 0 \\ 0 & 0 & c_{33} \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & \dfrac{c_{12}+c_{66}}{2} & 0 \\ \dfrac{c_{12}+c_{66}}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 0 & \dfrac{c_{13}+c_{44}}{2} \\ 0 & 0 & 0 \\ \dfrac{c_{13}+c_{44}}{2} & 0 & 0 \end{pmatrix},$$

$$F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \dfrac{c_{13}+c_{44}}{2} \\ 0 & \dfrac{c_{13}+c_{44}}{2} & 0 \end{pmatrix}.$$

Now it is easy to find the matrix C_0 such that $C_0 C'_j = A_j$ are all symmetric. It has the form

$$C_0 = \left(\begin{array}{c|c|c|c} I & 0 & 0 & 0 \\ \hline 0 & B_1 & D & E \\ \hline 0 & D & B_2 & F \\ \hline 0 & E & F & B_3 \end{array}\right).$$

Moreover, C_0 is positive definite if and only if its right lower corner 9×9 sub-matrix is. Thus it is possible to prove that C_0 is positive definite if the following conditions hold:

$$\begin{aligned} c_{66} - c_{12} &> 0 \\ 3c_{66} + c_{12} &> 0 \\ 2c_{11} - c_{66} - c_{12} &> 0 \\ c_{44} - c_{13} &> 0 \\ 3c_{44} + c_{13} &> 0 \\ 2c_{11} + 2c_{33} + c_{66} + c_{12} &> \sqrt{(2c_{11} - 2c_{33} + c_{66} + c_{12})^2 + 8(c_{13} + c_{44})^2} \end{aligned}$$

Note that these conditions hold if the hyperbolic conditions on the initial system are fulfilled (cf. chapter 2). Thus, if the system (4.5.3) has a local solution W, it is possible to prove that the system (4.5.1) also has a local solution. We prove this result in the following proposition.

Proposition 4.8. *Suppose that the hyperbolicity conditions on* (4.4.1) *hold, $f_i \in H^s(\mathbb{R}^3)$, $g_i \in H^{s-1}(\mathbb{R}^3)$, for $i = 1, 2, 3$ and $s > 3/2 + 1$, then the system* (4.4.1) *with initial conditions* (4.4.3) *has a unique local solution*

$$\boldsymbol{u} \in C(I, H^s(\mathbb{R}^3)) \cap C^1(I, H^{s-1}(\mathbb{R}^3)).$$

Proof. We consider the solution $\mathbf{W} = (\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ of the system (4.5.3) with initial data

$$\mathbf{u}_0(0) = \mathbf{g}, \qquad \mathbf{u}_j(0) = \partial_{x_j}\mathbf{f}. \tag{4.5.4}$$

The system (4.5.3) is a symmetrizable hyperbolic system, thus, if we take the previous initial data, it is a standard result (see [T2] for details) that it has a unique local solution $\mathbf{W} \in C(I, H^{s-1}(\mathbb{R}^3))$.

Now, we define $\mathbf{u}(0, x) = \mathbf{f}(x)$ and $\partial_t\mathbf{u}(t, x) = \mathbf{u}_0(t, x)$. These definitions and initial data (4.5.4) yield $\partial_{x_j}\mathbf{u}(0, x) = \mathbf{u}_j(0, x)$ and $\partial_t\mathbf{u}(0, x) = \mathbf{g}(x)$. Moreover this implies that $\mathbf{u} \in C^1(I, H^{s-1})$, because $\mathbf{u}_0 \in C(I, H^{s-1})$.

It is remain to prove that $\mathbf{u}$ satisfies the system (4.4.1) and belongs to

$C(I, H^s)$. In order to do this we will prove that the existence of $\mathbf{W}$ implies that $\mathbf{u}_j = \partial_{x_j}\mathbf{u}$ on $I \times \mathbb{R}^3$. Note that this condition yields $\partial_{x_i}\mathbf{u}_j = \partial^2_{x_i x_j}\mathbf{u} = \partial^2_{x_j x_i}\mathbf{u} = \partial_{x_j}\mathbf{u}_i$ on $I \times \mathbb{R}^3$.
Let $\mathbf{v}_j = \mathbf{u}_j - \partial_j \mathbf{u}$, applying ∂_t to each side we obtain

$$\partial_t \mathbf{v}_j = \partial_t \mathbf{u}_j - \partial_{x_j}\mathbf{u}_0 = 0$$

by the second line of (4.5.2) and the definition of $\partial_t \mathbf{u}(t, x)$. Now, by the previous considerations, $\mathbf{u}_j(0) = \partial_{x_j}\mathbf{u}(0)$. It follows that $\mathbf{v}_j(t) = 0$ for all $t \in I$. This proves that $\mathbf{u}_j = \partial_{x_j}\mathbf{u}$ on $I \times \mathbb{R}^3$. Now, if we take $\partial_t \mathbf{u}$ for $\mathbf{u}_0$ and $\partial_{x_j}\mathbf{u}$ for $\mathbf{u}_j$ in the middle line of (4.5.2), we have that $\mathbf{u}$ solve the desired system (4.4.1).
Finally, since $\mathbf{u}_j \in C(I, H^{s-1})$, we have $\nabla_x \mathbf{u} \in C(I, H^{s-1})$ and consequently $\mathbf{u} \in C(I, H^s)$. This concludes the proof. □

The following corollary is a version of the theorem of existence of local solution for quasi-linear symmetric hyperbolic systems which will we useful in the sequel.

Corollary 4.9. *Given the initial data $\boldsymbol{W}_0 = (\boldsymbol{g}, \nabla_x \boldsymbol{f}) \in H^s(\mathbb{R}^3)$, with $s > 3/2 + 1$, and $||\,\boldsymbol{W}_0||_{2,s} \leq \delta$, with δ sufficiently small, there exist a finite time interval $I = [0, T]$ and a positive number η (which depends on T) such that the unique solution $\boldsymbol{W} \in C(I, H^s) \cap C^1(I, H^{s-1})$ of (4.4.1), (4.4.3) is such that $||\,\boldsymbol{W}(t)||_{2,s} \leq \eta$ for all $t \in I$.*

We note that $||\mathbf{W}(t)||_{p,s} = ||\dot{u}(t)||_{p,s} + ||\nabla_x u(t)||_{p,s}$, so the previous theorems allows us to state the following corollary.

Corollary 4.10. *There exists a $\delta > 0$ sufficiently small such that if*

$$||\,\boldsymbol{W}_0||_{2,s} + ||\,\boldsymbol{W}_0||_{10/9,s} \leq \delta,$$

$s \geq s' + 6/5$ and $s > 5/2$, then

$$||\,\boldsymbol{W}(t, x)||_{2,s} \leq K_s||\,\boldsymbol{W}_0||_{2,s}$$

for all $t \in [0, T]$, where K_s is a constant independent from T.

Proof. In order to prove the statement we want to apply theorem (4.7). Thus, we have to check that all hypotheses of that theorem are fulfilled.
Let $\mathbf{W}$ be the solution of (4.4.1), (4.4.3) which exists with the properties stated in the corollary (4.9) because $s > 5/2$ and $||\mathbf{W}_0||_s \leq \delta$. Moreover, since $s \geq s' + 6/5$ we can use the Sobolev imbedding theorem and the fact that $||\mathbf{W}(t)||_{2,s} \leq \eta$ for all $t \in [0, T]$, to prove that $||\mathbf{W}||_{10,s'} \leq \eta$ for all $t \in [0, T]$. This condition and the hypothesis of smallness on $\mathbf{W}_0$ assure that the hypotheses of the theorem (4.7) are satisfied. Finally, remark (4.16) assure that the conditions (4.4.19), (4.4.20), (4.4.21), (4.4.22) on s, s' and r are satisfied. Thus we can apply it and conclude the proof. □

Remark 4.16. Note that the conditions

$$s' \geq s/2 + 3/10, \qquad s' \geq s'/2 + r/2 + 4/5,$$
$$s' > 0, \quad r > 0, \quad s \geq s' + r + 1,$$
$$s \geq s' + 6/5 \qquad s > 5/2,$$

can be satisfied simultaneously. In particular, they are equivalent to the following:

$$s' > 9/5$$
$$s' + 6/5 \leq s \leq 2s' - 3/5$$
$$0 < r \leq s - s' - 1$$

Thus, we can re-apply the local existence corollary (4.9) with initial data with time close to T, in order to obtain the desired global solution. We can now prove the following main theorem.

Theorem 4.11. *Assume that*

$$\sigma_{ij}(\varepsilon) = \sum_{k,s=1}^{3} c_{ijks}\varepsilon_{ks} + H_{ij}(\varepsilon), \quad H_{ij}(\varepsilon) = \mathcal{O}(||\varepsilon||_\infty^5).$$

Then we can find s and δ such that if

$$||\nabla_x \boldsymbol{f}||_{2,s} + ||\boldsymbol{g}||_{2,s} \leq \delta$$
$$||\nabla \boldsymbol{f}||_{10/9,s} + ||\boldsymbol{g}||_{10/9,s} \leq \delta$$

then there is a solution $\boldsymbol{u} \in C([0,\infty[, H^{s+1}(\mathbb{R}^3)) \cap C^1([0,\infty[, H^s(\mathbb{R}^3))$ *of the Chauchy's problem*

$$\begin{cases} \dfrac{\partial^2 u_i}{\partial t^2}(t,x) = div\sigma_{ij}(\varepsilon(t,x)) & i = 1,2,3 \\ u_i(0,x) = f_i(x) & i = 1,2,3 \\ \partial_t u_i(0,x) = g_i(x) & i = 1,2,3 \end{cases}.$$

Proof. The condition $||\nabla_x \mathbf{f}||_s + ||\mathbf{g}||_s \leq \delta$, corollary (4.9) and theorem (4.8) imply that there exists a solution of the Cauchy's problem

$$\mathbf{u} \in C([0,T], H^{s+1}(\mathbb{R}^3)) \cap C^1([0,T], H^s(\mathbb{R}^3)).$$

The conditions $||\nabla_x \mathbf{f}||_s + ||\mathbf{g}||_s \leq \delta$, $||\nabla \mathbf{f}||_{10/9,s} + ||\mathbf{g}||_{10/9,s} \leq \delta$ and corollary (4.10) imply that $||\nabla_x \mathbf{u}(t,x)||_s + ||\dot{\mathbf{u}}(t,x)||_s \leq K_s\delta$, where K_s is a constant independent of T, for all $t \in [0,T]$. In δ is sufficiently small, we can reapply the corollary (4.9) with initial data $||\nabla_x \mathbf{u}(\bar{t},x)||_s + ||\dot{\mathbf{u}}(\bar{t},x)||_s \leq K_s\delta$, where $\bar{t}$ is close to T. Thus, there exist a solution of the Cauchy's problem $\mathbf{u} \in C([0,\bar{T}], H^{s+1}(\mathbb{R}^3)) \cap C^1([0,\bar{T}], H^s(\mathbb{R}^3))$, with $\bar{T} > T$ if $\bar{t}$ is sufficiently close to T. We can iterate this process because K_s does not depend on T and thus we obtain the desired global solution. □

Appendix A

Proposition A.1. *Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a scalar function solution of a Cauchy Problem with initial data $f(x,0)$, which belongs to $C^k(\mathbb{R}^n)$ and has compact support for all fixed t. If there exist j and k such that*

$$||f(x,t)||_\infty \leq c_1 t^{-j} \sup_{\xi \in \mathbb{R}^3} \sum_{|\alpha|,|\beta| \leq k} ||\xi^\alpha \partial_\xi^\beta \widehat{f}(\xi,0)||_\infty \tag{A.0.1}$$

where $\widehat{f}$ is the Fourier transform of f in the variable x, then the following inequality holds:

$$||f(x,t)||_\infty \leq c_2 t^{-j} \sum_{|\alpha| \leq k} ||\partial_x^\alpha f(x,0)||_1. \tag{A.0.2}$$

Proof. Using standard results on Fourier transform and integration by parts it is easy to see that

$$\sum_{|\alpha|,|\beta| \leq k} ||x^\alpha \partial_x^\beta f(x,0)||_1 \geq \sup_{\xi \in \mathbb{R}^3} \sum_{|\alpha|,|\beta| \leq k} ||\xi^\alpha \partial_\xi^\beta \widehat{f}(\xi,0)||_\infty$$

and thus $(A.0.1)$ implies

$$||f(x,t)||_\infty \leq c_3 t^{-j} \sum_{|\alpha|,|\beta| \leq k} ||x^\alpha \partial_x^\beta f(x,0)||_1 \tag{A.0.3}$$

Now, we observe that, if the support of f is a subset of the unit sphere, then $(A.0.3)$ trivially implies $(A.0.2)$. Moreover, using translation invariance in the space variable of the Cauchy problem, $(A.0.3)$ implies $(A.0.2)$ also in the cases when the diameter of the supports of the initial data are smaller then

two. We can now consider the case when no restriction is imposed on the supports of the initial data with the aid of a suitable partition of unity.
In fact, let h_i a partition of unity in $\mathbb{R}^3$ with the following properties:

(i) the diameter of the support of h_i is smaller then two for all i.

(ii) $|\partial^\alpha h_i| \leq c_\alpha$ for all i.

(iii) There is no point $x_0 \in \mathbb{R}^3$ such that it lies in the support of h_i for more then d different indices i.

Thus, we have

$$
\begin{aligned}
||f(x,t)||_\infty \leq c_4 t^{-j} \sum_{i\in\mathbb{N}} \sum_{|\alpha|\leq k} ||\partial_x^\alpha (h_i f(x,0))||_1 \leq \\
\leq c_5 t^{-j} \sum_{i\in A} \sum_{|\alpha|\leq k} \int_{supph_i} |\partial_x^\gamma f(x,0)| dx \leq \\
\leq c_6 t^{-j} \sum_{|\gamma|\leq k} \int_{\mathbb{R}^3} |\partial_x^\gamma f(x,0)| dx
\end{aligned}
$$

where A is a subset of $\mathbb{N}$ which contain the indices for which the support of h_i has a nontrivial intersection with the support of $f(x,0)$. $\square$

Appendix B

Notation

Geometric notation

- $\mathbb{R}^n$ = n-dimensional real Euclidean space, $\mathbb{R} = \mathbb{R}^1$.

- $e_i = (0, \ldots, 0, 1, \ldots, 0)$ = i-th standard coordinate vector.

- A typical point in $\mathbb{R}^n$ is $x = (x_1, \ldots, x_n)$.
  We will also, depending upon the context, regard x as a row or column vector.

- A point in $\mathbb{R}^{n+1}$ will be often be denoted as $(t, x) = (t, x_1, \ldots, x_n)$, and we usually interpret $t = \mathit{time}$.
  A point $x \in \mathbb{R}^n$ will sometimes be written $x = (x', x_n)$ for $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$.

- U and V usually denote open subset of $\mathbb{R}^n$. We write $U \subset\subset V$ if $U \subset \bar{U} \subset V$, and $\bar{U}$ is compact, and say U is compactly contained in V. Γ usually denote open cone in $\mathbb{R}^n$.

- ∂U = boundary of U, $\bar{U} = U \cup \partial U$ = closure of U.

- If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ belong to $\mathbb{R}^n$,

$$< x, y >= \sum_{i=1}^{n} x_i y_i, \qquad |x| = \left(\sum_{i=1}^{n} a_i^2 \right)^{1/2}.$$

- $\mathbb{N}$ = set of natural number.
  $\mathbb{Z}$ = group of integer number.
  $\mathbb{C}$ = complex plane.

Notation for matrices

- We write $A = (a_{ij})_{i,j=1,\ldots,n}$ to mean A is an $n \times n$ matrix with $(i, j)-$th entry a_{ij}.

- We write $A = (a_{ijkl})_{i,j,k,l=1,\ldots,n}$ to mean A is an four tensor with $(i, j, k, l)-$th entry a_{ijkl}.

- A diagonal matrix is denoted $\mathrm{diag}(d_1, \ldots, d_n)$.

- $\mathrm{Tr}(A)$ = trace of the matrix A.

- $\det A$ = determinant of the matrix A.

- A^T = transpose of the matrix A.

- A^{-1} = inverse of the matrix A.

- $\mathrm{Symm}(A) = (1/2)(A + A^T)$ denote the symmetric part of the matrix A and $\mathrm{SkweSymm}(A) = (1/2)(A - A^T)$ denote the skew-symmetric part of the matrix A.

- $\mathbb{M}_3$ = space of real $n \times n$ matrices.
  $\mathbb{O}(3)$ = space of real orthogonal 3×3 matrices.
  $\mathbb{G}L(3)$ = space of real 3×3 invertible matrices.

- If $A = (a_{ij})_{i,j=1,\ldots,n}$ is a $n \times n$ matrix, then

$$|A| = \left(\sum_{i,j=1}^{n} a_{ij}^2 \right)^{1/2}.$$

- If $A = (a_{ij})_{i,j=1,\ldots,n}$ is a $n \times n$ matrix and $x \in \mathbb{R}^n$, then $A \cdot x =$ standar matrix product.

Notation for functions

- If $u : U \to \mathbb{R}$, we write $u(x) = u(x_1, \ldots, x_n)$, with $x \in U$. We say u is *smooth* provided u is infinitely differentiable.

- The support of a function u is denoted $\text{supp}(u)$.

- $\hat{u}(\xi) =$ the Fourier transform of u.

- If $u : U \to \mathbb{R}^m$, we write $u(x) = (u_1(x), \ldots, u_m(x))$, with $x \in U$. The function u_k is the $k-$th *component* of u, $k = 1, \ldots, m$.
  In the last section of chapter four, we denote $\mathbf{u}$ the previous function.

- If S is a smooth surface in $\mathbb{R}^3$, we write

$$\int_S f dS$$

  for the integral of f over S, with respect to $2-$dimensional surface measure. If V is a open subset of $\mathbb{R}^3$ we sometimes write

$$\int_V f dV$$

  for the integral of f over V, with respect to the standard Lebesgue measure.

- We write

$$f = \mathcal{O}(g) \qquad \text{as } x \to x_0,$$

provided there exist a constant C such that $|f(x)| \leq C|g(x)|$ for all x sufficiently close to x_0.

- We write
$$f = o(g) \qquad \text{as } x \to x_0,$$
provided
$$\lim_{x \to x_0} \frac{|f(x)|}{|g(x)|} = 0.$$
- We write $f \sim g$ as $x \to x_0$, if and only if $(f - g) = o(g)$ as $x \to x_0$.

The expression $\mathcal{O}(g)$ (or $o(g)$) is not itself defined. There must always be an accompanying limit, for examples *as* $x \to x_0$ above, although this limit is often implicit.

Notation for derivatives

Assume $u : U \to \mathbb{R}$, $x \in U$.

- $\frac{\partial u}{\partial x_i}(x) = \lim_{h \to 0} \frac{u(x+he_i)-u(x)}{h}$, provided this limit exist.
- We usually write $\partial_{x_i} u$ or $\partial_i u$ for $\frac{\partial u}{\partial x_i}(x)$.
- Similarly, $\frac{\partial^2 u}{\partial x_i \partial x_j} = \partial_{x_i}\partial_{x_j} u = \partial_{ij} u$.
- A vector of the form $\alpha = (\alpha_1, \ldots, \alpha_n)$, where each component α_i is a nonnegative integer, is called a *multiindiex* of *orde* $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Given a multiindex α, define
$$\partial^\alpha u(x) := \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u(x).$$
- $\nabla u = (\partial_{x_1} u, \ldots, \partial_{x_n} u) =$ gradient vector.
- We sometimes employ a subscript attached to the symbols ∂, ∇, etc. to denote the variables being differentiated. For exemple if $u = u(t,x)$, $(t,x) \in \mathbb{R}^{n+1}$, then $\nabla_x u = (\partial_{x_1} u, \ldots, \partial_{x_n} u)$ and $\partial_t u = \lim_{h \to 0} \frac{u(t+h,x)-u(t,x)}{h}$.

- We sometimes denote $\partial_t u$ by $\dot{u}$.

- If now $m > 1$, $u : U \to \mathbb{R}^m$, $x \in U$ we define $\partial^\alpha u = (\partial^\alpha u_1, \ldots, \partial^\alpha u_m)$ for each muliindex α.

- We denote with u^{-1} the *inverse* function of u.

- We write

$$\nabla u = \begin{pmatrix} \partial_{x_1} u_1 & \cdots & \partial_{x_n} u_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} u_m & \cdots & \partial_{x_n} u_m \end{pmatrix} = gradient \quad matrix$$

- If $m = n$ we have $\mathrm{div} u := \mathrm{Tr}(\nabla u) = \sum_{i=1}^n \partial_{x_i} u_i = $ *divergence* of u.

- If $A = (a_{ij})_{i,j=1,\ldots,n}$ is a $n \times n$ matrix, we denote

$$\mathrm{div} A = \mathrm{div} A_i := (\sum_{j=1}^n \partial_{x_j} a_{1j}, \ldots, \sum_{j=1}^n \partial_{x_j} a_{nj}).$$

Function spaces

- $C(U) = \{u : U \to \mathbb{R} : u \text{ continuous}\}$.
  $C^k(U) = \{u : U \to \mathbb{R} : u \text{ is } k\text{-times continuously differentiable}\}$.
  $C^\infty(U) = \bigcap_{k=0}^\infty C^k(U)$.
  $C_0^\infty(U)$ denotes these functions in $C^\infty(U)$ with compact support.

- $\mathcal{S}(U)$ denotes the Schwartz space of rapidly decreasing functions.

- The spaces $C(U, \mathbb{R}^m), C^k(U, \mathbb{R}^m)$, etc. consist of those functions $u : U \to \mathbb{R}^m$, $u = (u_1, \ldots, u_m)$ with $u_i \in C(U), C^k(U)$, etc., $i = 1, \ldots, m$.

- $L^p(U) = \{u : U \to \mathbb{R} : u \text{ is a Lebesgue measurable}, ||u||_p < \infty\}$ where

$$||u||_p := \left(\int_U |u|^p dx \right)^{1/p}, \quad 1 \le p < \infty.$$

$L^\infty(U) = \{u : U \to \mathbb{R} : u$ is a Lebesgue measurable, $||u||_\infty < \infty\}$ where

$$||u||_\infty := \text{ess sup}_U |u|.$$

$L^p_{\text{loc}}(U) = \{u : U \to \mathbb{R} : u \in L^p(V) \text{ for each } V \subset\subset U\}$.

- The spaces $L^p(U, \mathbb{R}^m), L^\infty(U, \mathbb{R}^m)$, $1 \le p < \infty$ consist of those functions $u : U \to \mathbb{R}^m$, $u = (u_1, \dots, u_m)$ with $u_i \in L^p(U), L^\infty(U)$, $i = 1, \dots, m$, respectively and

$$||u||_p := \left(\sum_{i=1}^m ||u_i||_p^p \right)^{1/p} = \left(\sum_{i=1}^m \int_U |u_i|^p dx \right)^{1/p},$$

$$||u||_\infty := \max_{i=1,\dots,m} ||u_i||_\infty.$$

- $W^{p,s}(U) = \{u : U \to \mathbb{R} : u \in L^1_{\text{loc}}(U), \partial^\alpha u \in L^p(U),$ for each multi index α, with $|\alpha| \le s\}$, with $1 \le p \le \infty$ and s a nonnegative integer. Moreover, we define

$$||u||_{p,s} := \begin{cases} \left(\int_U \sum_{|\alpha| \le s} |\partial^\alpha u|^p dx \right)^{1/p} & 1 \le p < \infty \\ \sum_{|\alpha| \le s} \text{ess sup}_U |\partial^\alpha u| & p = \infty \end{cases}$$

- $H^s(U) := W^{2,s}(U)$, $s = 0, 1, \dots$.

- The spaces $W^{p,s}(U, \mathbb{R}^m), H^s(U, \mathbb{R}^m), W^{\infty,s}(U, \mathbb{R}^m)$, consist of those functions $u : U \to \mathbb{R}^m$, $u = (u_1, \dots, u_m)$ with $u_i \in W^{p,s}(U), H^s(U), W^{\infty,s}(U)$, $i = 1, \dots, m$, respectively and

$$||u||_{p,s} := \left(\sum_{i=1}^m ||u_i||_{p,s}^p \right)^{1/p},$$

$$||u||_{\infty,s} := \max_{i=1,\dots,m} ||u_i||_{\infty,s}.$$

- If $I = [0, T] \subset \mathbb{R}$, the spaces $C(I, H^s(\mathbb{R}^n))$, and $C^1(I, H^s(\mathbb{R}^n))$ consist of those continuous, respectively $C^1(I)$, functions with values in $H^s(\mathbb{R}^n)$.

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