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Cosmological Perturbations in Generalized Theories of Gravity

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Abstract

The first part of this thesis concerns the study of inflation in the context of a theory of gravity called "Induced Gravity" in which the gravitational coupling varies in time according to the dynamics of the very same scalar field (the "inflaton") driving inflation, while taking on the value measured today since the end of inflation. Through the analytical and numerical analysis of scalar and tensor cosmological perturbations we show that the model leads to consistent predictions for a broad variety of symmetry-breaking inflaton's potentials, once that a dimensionless parameter γ entering into the action is properly constrained. We also discuss the average expansion of the Universe after inflation (when the inflaton undergoes coherent oscillations about the minimum of its potential) and determine the effective equation of state. Finally, we analyze the resonant and perturbative decay of the inflaton during (p)reheating.

The second part is devoted to the study of a proposal for a quantum theory of gravity dubbed "Hořava-Lifshitz (HL) Gravity" which relies on power-counting renormalizability while explicitly breaking Lorentz invariance. We test a pair of variants of the theory ("projectable" and "non-projectable") on a cosmological background and with the inclusion of scalar field matter. By inspecting the quadratic action for the linear scalar cosmological perturbations we determine the actual number of propagating degrees of freedom and realize that the theory, being endowed with less symmetries than General Relativity, does admit an extra gravitational degree of freedom which is potentially unstable. More specifically, we conclude that in the case of projectable HL Gravity the extra mode is either a ghost or a tachyon, whereas in the case of non-projectable HL Gravity the extra mode can be made well-behaved for suitable choices of a pair of free dimensionless parameters and, moreover, turns out to decouple from the low-energy Physics.

Contents

Preface

This thesis addresses some aspects of a couple of theories of gravity, “Induced Gravity” and “Hořava-Lifshitz Gravity”, which have in common the property of being generalizations of Einstein’s General Relativity (GR). In the case of Induced Gravity, we discuss the occurrence of Cosmological Inflation and work out accurate predictions. In the case of Hořava-Lifshitz Gravity, a recent proposal for a quantum theory of gravity, we characterize its propagating degrees of freedom and comment on its viability. In both cases the investigation deeply relies on use of the Theory of Cosmological Perturbations.

At first, one might question whether GR needed to be “generalized” or not. A simple answer can be stated as follows: although, on one side, GR has proven to be very successful in explaining several gravitational phenomena with unprecedented accuracy, and also in predicting new ones (cf. for instance the very famous book Ref. [1]), on the other side there is a series of crucial points (*e.g.* the explanation of the present-day accelerated cosmic expansion, the quantizability of the gravitational interaction, the unification of gravity with the other three interactions under a common, quantum, framework, etc.) which GR fails to unravel in a convincing way. Such a pair of opposite considerations led several physicists think of GR as the low-energy limit of some other more general and more complete theory of gravitation. In other words, people tried to look for theories of gravity beyond GR which could address (at least some of) those questions left unanswered by GR, while preserving the agreement with GR on possibly all the standard low-energy tests.

For what concerns the Generalized Theories of Gravity considered in this thesis, the open issues we make reference to can be briefly summarized as follows:

(1) Rather recently (with respect to Edwin Hubble’s first discovery of cosmic expansion, back in 1925) observations of Type Ia Supernovae showed that not only our Universe is currently expanding but also that the rate of expansion is increasing in time [2]. Such an accelerated expansion constitutes one of the most puzzling questions in Cosmology since it cannot be accounted by GR unless one introduces either a cosmological constant or some sort of exotic energy (commonly dubbed “Dark Energy”) whose ratio between the pressure and the energy density is negative and lower than $-1/3$,

something which is not attainable with ordinary matter. An unknown "ingredient" is also required as a potential explanation of some other evidences coming from several independent observations (Cosmic Microwave Background Radiation, Baryon Acoustic Oscillations signal in the Sloan Digital Sky Survey, etc.), so that - according to the most widely accepted cosmological model (the so-called ΛCDM model) the Universe is supposed to be dominated by a cosmological constant (Λ, accounting for roughly the 73% of the entire energy density) and largely populated by Cold Dark Matter (CDM, $\sim 23\%$). Let us add that the ΛCDM model is endowed with a finite number of cosmological parameters which fit observations very well and thus, overall, it appears like a fairly good theoretical framework to explain the observed Universe. Nevertheless, as one begins to think of the cosmological constant as the vacuum energy of spacetime and tries to make an estimate based on Quantum Field Theory finds a value which is 10^{120} times bigger than the observed one! Of course such a point is critical and deserve thorough investigations.

(2) For the time being, GR is the best theory of gravitation we have at disposal. Yet it does not provide a quantum description of the gravitational interaction, thereby escaping the unification with the other three fundamental interactions of Nature which, on the contrary, have already been unified under a common framework which is intrinsically quantum. One reason for GR not to be quantizable is that it is not renormalizable, meaning that, at the high (ultra-violet) energy scales where quantum effects are supposed to be relevant, the coupling constants of the (linearized) theory diverge. Therefore, the stunning predictive power of GR at low-energy is somewhat lost at those higher energy scales ($\sim$ Planck mass, 10^{19} GeV) where quantum effects do play a role (see Ref. [3] for a pedagogical review).

Along with the previous points, let us add that of Cosmological Inflation, which is an epoch of accelerated expansion of spacetime required to anticipate the Standard Cosmology's Hot Big Bang so as to solve a list of important shortcomings (e.g. the horizon, flatness and entropy problems) which afflict the latter theory (cf. Ref. [4] for a thorough and updated presentation of the subject). The inflationary paradigm also provides us with a causal explanation of the Large Scale Structure of the Universe, in that the present-day macroscopic large-scale inhomogeneities (seen in galaxies and clusters of galaxies) can be traced back to tiny fluctuations during Inflation whose ultimate origin is supposed to be quantum [5]. Nevertheless, also the inflationary paradigms comes with its own shortcomings (initial singularity problem, trans-Planckian problem for cosmological fluctuations, . . .) and, at the same time, currently available observational data leave some fine "details" concerning Early Universe Cosmology so poorly constrained that it is not so easy to distinguish between inflation and alternatives to it [6] nor between different realizations of inflation itself.

Concerning point (1), we considered a theory of gravity commonly referred to as "Induced Gravity" (or "Spontaneously Generated Gravity") mostly because of some previous studies (in particular those in which the Ph.D. candidate's collaborators were directly involved, Refs. [7, 8]) which showed how to account for the present-day cosmic acceleration within the framework of Induced Gravity plus a scalar field with quartic potential. Given such an interesting result concerning late-times Cosmology, we wanted to study Inflation (thus early-times Cosmology) in the context of Induced Gravity in order to check under which conditions it occurs and whether predictions are in agreement or not with the available observational data.

Induced Gravity consists of a theory which differs from GR because the gravitational coupling is in principle time-dependent and, more specifically, dynamically generated by the time-evolution of a scalar field. It is then expected that the dynamics of the scalar field ceases at some fixed value at which the Newton's constant G_N takes on the value we measure nowadays. Taking inspiration from Ref. [9], we wanted to use the very same scalar field as the scalar field driving Inflation, namely the "inflaton".

The first part of the thesis mostly overlaps with the content of a couple of papers, Refs. [10, 11], which were worked out in collaboration with Prof. Giovanni Venturi, Dr. Fabio Finelli and Dr. Alessandro Tronconi.

Concerning point (2), we tackled a recent proposal for a quantum theory of gravity which was first developed by Prof. Petr Hořava and soon after dubbed as "Hořava-Lifshitz (HL) Gravity" because of a class of condensed-matter models exhibiting an anisotropy between space and time as in Hořava's proposal and whose prototype is just the theory of a Lifshitz scalar [12].

The construction of the theory explicitly abandons Lorentz invariance while demanding power-counting renormalizability. The need to break Lorentz invariance, which at first may sound quite disappointing for Lorentz invariance is actually the cornerstone of Einstein's Special and General Relativity, becomes less arguable if one recalls studies like the one in Ref. [13]: long time ago it was indeed realized that, on adding higher derivatives terms to the GR action, on one side its ultra-violet behavior turned out to be improved but, on the other, one had to face the emergence of ghost-like (= with negative kinetic energy) degrees of freedom which spoiled the unitarity of the theory. After Hořava's encouraging proposal it was hoped that giving up with Lorentz invariance as a fundamental principle might have helped skipping such a problem, even though to date there are more indications against than pro (as discussed also in this thesis).

Given the significance of its purpose, since it first appeared in January 2009 the theory attracted the attention of many enthusiasts who studied in detail different aspects, also including those regarding cosmological implications [14]. At the same time several concerns began to come out, like those concerning the existence of an unwanted bad-behaved degree of freedom, the strong coupling issue, *et cetera*. The study presented in this thesis mostly aimed at determining the actual number of propagating degrees of freedom and checking potential ghost-like or tachionic instabilities through the analysis of linear perturbations about a cosmological background with scalar matter, as an upgrade over previous investigations which considered the (less physically motivated) Minkowski background with no matter. Eventually our study partly confirmed the concerns above. More specifically, two different variants of the theory were considered and, while one was definitely proved "wrong" (in the sense that an extra ghost-like degree of freedom due to gravity survives down to low-energies where it is not expected to exist), the other was declared "safe" as long as the range of some of the free parameters is properly restricted.

The second part of the thesis mostly covers the content of a couple of papers, Ref. [15, 16], which were worked out in collaboration with Prof. Robert Brandenberger while visiting McGill University in Montréal, Canada.

Notation

- $c = 1 = \hbar$
- G_N is the Newton's constant
- $g = \det g_{\mu\nu}$
- $\dot{(\cdots)} \equiv \frac{\mathrm{d}}{\mathrm{d}t}(\cdots)$, $\ddot{(\cdots)} \equiv \frac{\mathrm{d}^2}{\mathrm{d}t^2}(\cdots)$
- $H = \dot{a}/a$ is the Hubble parameter

Unless otherwise stated, we will adopt the following spatially flat Friedmann-Robertson-Walker metric as the background spacetime metric of cosmological interest:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a(t)^2 d\vec{x}^2 \,, \tag{1}$$

where $a(t)$ is the scale factor.

Part I

Induced Gravity

Chapter 1

Introduction

A model with a time-varying gravitational coupling was first introduced long time ago by Brans and Dicke [17], and consisted of a massless scalar field whose inverse was associated with the gravitational coupling. Such a field was proven to have a non-trivial dynamics in presence of matter and the model led to cosmological predictions differing from GR in that one generally obtained a power-law time dependence for the gravitational coupling.

In later papers by Zee and, independently, by Smolin [18, 19, 20], a different model than Brans and Dicke's was presented, where basically the usual Einstein-Hilbert (EH) action,

$$S_{\rm EH} = \int d^4x\sqrt{-g}\,\frac{M_{\rm Pl}^2}{2}R, \tag{1.1}$$

R being the Ricci curvature scalar and $M_{\rm Pl}$ the reduced Planck mass, equal to $(8\pi G_N)^{-1/2}$ in natural units, was replaced by the following Induced Gravity (IG) action,

$$S_{\rm IG} = \int d^4x\sqrt{-g}\,\frac{\gamma}{2}\sigma^2 R, \tag{1.2}$$

where γ is a dimensionless parameter required to be positive, and $\sigma = \sigma(x)$ is a scalar field, eventually enriched with a kinetic and potential term so as to enable a non-trivial dynamics:

$$S = \int d^4x\sqrt{-g}\left[-\frac{g^{\mu\nu}}{2}\partial_\mu\sigma\partial_\nu\sigma + \frac{\gamma}{2}\sigma^2 R - V(\sigma)\right]. \tag{1.3}$$

This is actually the model we will stick to in the following.

It can be noted that in case, for some reason, the field σ takes on a constant value $\sigma_0 \neq 0$, we are left with

$$S = \int d^4x\sqrt{-g}\left[\frac{\gamma}{2}\sigma_0^2 R - V(\sigma_0)\right], \tag{1.4}$$

which exactly matches with GR plus a cosmological constant Λ once that we set

$$\begin{cases} \gamma\sigma_0^2 \equiv M_{\rm Pl}^2 \\ V(\sigma_0) \equiv 2\Lambda \end{cases} . \tag{1.5}$$

One natural way to make the field settle at σ_0 invokes the mechanism of spontaneous symmetry-breaking, which in fact would endow the field with a double-well potential,

$$V(\sigma) = \frac{\mu}{4}\left(\sigma^2 - \sigma_0^2\right)^2 \tag{1.6}$$

with minima at $\sigma = \pm\sigma_0$. It is then expected that the field dynamically evolves from some arbitrary initial value towards one of the minima.

The dynamical generation of the gravitational coupling through symmetry breaking in the framework of IG actually counts a vast literature (see Ref. [21] for a review).

Remarkably enough, the special choice of a quartic potential $V(\sigma) = \lambda\sigma^4/4$ would render the model globally scale invariant since no dimensional coupling would appear in the action. Such a setting was considered in Ref. [7] and it was shown that, in absence of matter and in a cosmological setting where the background metric is as in Eq. (1), the model admits a de Sitter solution in which σ has a constant value. On adding matter,

$$S \to S + \int d^4x\,\sqrt{-g}\,\mathcal{L}_M \tag{1.7}$$

the equation of motion of σ, again in the cosmological setting, turns out to be as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(a^3\sigma\dot{\sigma}\right) = \frac{a^3}{1+6\gamma}\sum_j \left(1 - 3\omega_j\right)\rho_j, \tag{1.8}$$

where the index j runs over different types of matter, ρ_j stands for the energy density of j-th component, and ω_j represents the so-called "equation of state parameter" which measures the ratio between its pressure and energy density. Supposing, for a while, that we begin with a vacuum configuration (de Sitter-like, as stated above), we see that relativistic matter (radiation), whose equation of state parameter is $\omega_R = 1/3$, in no way can affect the field's dynamics. Non-relativistic matter, instead, having $\omega_M = 0$, may actually kick the field out of the vacuum solution and lead to an interesting phenomenology. Such a case was considered in Ref. [7], where matter was introduced as a small perturbation around the de Sitter solution and consistent results were found.

In Ref. [8] a very detailed analysis of such a simple model with the inclusion of both radiation and matter showed that it yields GR plus a cosmological constant as a stable attractor among homogeneous cosmologies and, therefore, is a viable Dark Energy model for a range of scalar field's initial conditions and positive γ's. However, in that study the values of the scalar field which were considered were sufficiently close to the spontaneously broken symmetry equilibrium values and the magnitude of λ was such to explain the present cosmological constraints. In the later works [10, 11] here reported, instead, the basic idea was that of using σ as the field driving inflation (the "inflaton"), thus we started with field values sufficiently far from the equilibrium value so as to allow for a sufficiently long inflationary epoch.

Unfortunately, as will be shortly argued, the point $\sigma = 0$ is a singular one in the inflationary dynamics. It would in fact correspond to $M_{\rm Pl}^2 \to 0$ (following the analogy with GR) or, equivalently, $G_N \to \infty$, which clearly sounds unacceptable. Therefore, on setting up a viable inflationary dynamics, one has to consider a potential whose minima are at $\sigma \neq 0$ from the very beginning of the study. The possibility of crossing the origin is automatically excluded given the structure of the equations of motion.

Without any loss of generality, we will restrict our analysis to inflationary dynamics occurring along positive values of σ. Nevertheless inflation can occur in both the large field ($\sigma > \sigma_0$) and small field ($\sigma < \sigma_0$) regimes.

For the sake of completeness, let us remark that the theory of gravity here considered is clearly different from GR plus a non-minimally coupled field,

$$S_{\rm NMC} = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2 + \gamma\sigma^2}{2}R - \frac{g^{\mu\nu}}{2}\partial_\mu\sigma\partial_\nu\sigma - V(\sigma)\right], \tag{1.9}$$

because of the absence of the term in $M_{\rm Pl}^2$.

Moreover, it is known [22, 23] that through a conformal transformation,

$$\begin{aligned} \tilde{g}_{\mu\nu} &= \Omega^2 g_{\mu\nu} \\ d\tilde{\sigma}^2 &= \frac{(1+6\gamma)}{\Omega^2}d\sigma^2 \\ \tilde{V} &= \Omega^{-4}V\,. \end{aligned} \tag{1.10}$$

where $\Omega^2 = \gamma\sigma^2/{M_{\rm Pl}}^2$, we can rewrite (up to a boundary term) the action in Eq. (1.3) as

$$S_{\rm E} = \int d^4x\sqrt{-\tilde{g}}\left[-\frac{\tilde{g}^{\mu\nu}}{2}\partial_\mu\tilde{\sigma}\partial_\nu\tilde{\sigma} + \frac{\tilde{R}\,{M_{\rm Pl}}^2}{2} - \tilde{V}(\tilde{\sigma})\right], \tag{1.11}$$

where E stands for "Einstein frame", in contrast with "Jordan frame" which conventionally corresponds to the action in Eq. (1.3). We point out that inflationary computations are often performed in the Einstein frame where they are simpler. In principle that is correct because the spectrum of curvature perturbations and the amplitude of gravitational waves are both invariant under conformal transformations [24]. Nevertheless, other important quantities in cosmology are not left invariant under conformal transformation, which is the case, for instance, of the Hubble parameter H. If we are interested in late-times cosmology and wish to use observational data to constrain H in a scalar-tensor theory [8], this should be done for the Hubble parameter in the Jordan frame, which is different from the Hubble parameter in the Einstein frame. In this sense the Jordan frame may be judged "more physical" than the Einstein one. For this reason we will stick to the former.

This first part of the thesis is organized as follows:

- In Chapter 2 we will set up a suitable inflationary background, discussing both the "slow-roll" and power-law solutions. Then we will move on to consider inflationary predictions and the comparison with observational data, proving that inflationary predictions are generically in agreement with observations for a broad range of options for the inflaton's potential, whereas some other potentials turn out to be unacceptable.
- In Chapter 3 we will focus on the post-inflationary phenomenology, studying the average expansion of the Universe when the inflaton coherently oscillates about the minimum of its potential and tackling both its perturbative and resonant decay. Such an analysis will let us learn that, on average, the Universe expands as if it were dominated by pressureless matter, although its effective equation of state parameter is not exactly zero and the Hubble parameters exhibits oscillations even at the leading order in contrast with what would happen in General Relativity.
- Finally, Chapter 4 will be devoted to a summary of conclusions.

Chapter 2

Inflation in Induced Gravity

2.1 Homogeneous inflationary dynamics

The action in Eq. (1.3) yields the following constraint equation:

$$H^2 = \frac{1}{3\gamma\sigma^2}\left[\frac{\dot{\sigma}^2}{2} + V(\sigma)\right] - 2H\frac{\dot{\sigma}}{\sigma}\,, \tag{2.1}$$

generalizing the well-known Friedmann equation, and the following pair of independent dynamical equations:

$$\dot{H} = -\frac{1}{2\gamma}\frac{\dot{\sigma}^2}{\sigma^2} + 4H\frac{\dot{\sigma}}{\sigma} + \frac{1}{(1+6\gamma)}\frac{V_{\mathrm{eff},\sigma}}{\sigma} \quad (\sim \text{2nd Friedmann eq.}) \tag{2.2}$$

$$\ddot{\sigma} + 3H\dot{\sigma} + \frac{\dot{\sigma}^2}{\sigma} = -\frac{V_{\mathrm{eff},\sigma}}{1+6\gamma} \quad (\sim \text{Klein-Gordon eq.}) \tag{2.3}$$

where

$$V_{\mathrm{eff},\sigma} \equiv \frac{\mathrm{d}V(\sigma)}{\mathrm{d}\sigma} - 4\frac{V(\sigma)}{\sigma}. \tag{2.4}$$

We refer to Ref. [25] (and references therein) for the derivation of the previous equations from first principles. Since we are now concerned with the background homogeneous dynamics, the field σ appearing in the previous formulas is a function of time only, $\sigma = \sigma(t)$, as well as all the other quantities.

Since the l.h.s. of Eq. (2.1) is positive definite or null, whereas the r.h.s. of the same equation can in principle also be negative, the allowed phase-space must be restricted as follows, otherwise Eq. (2.1) is not well-defined:

$$6\gamma - \sqrt{6\gamma\,(1+6\gamma)} < \frac{\dot{\sigma}}{H\sigma} < 6\gamma + \sqrt{6\gamma\,(1+6\gamma)}\,. \tag{2.5}$$

A couple of observations are in order: (1) the r.h.s. of Eq. (2.3) vanishes in case of quartic potential $V(\sigma) \propto \sigma^4$, thereby allowing for σ = constant as a stable solution; (2) the point $\sigma = 0$ renders singular all the Eqs. (2.1), (2.3), (2.2). We recall that realizations of inflation in the more conventional context of GR typically end with the inflaton undergoing coherent oscillations around the minimum/a of the potential until all the initial energy is dissipated (both because of the friction offered by the expansion of the spacetime and because of the inflaton decay into other fields' excitations). Once more we stress on the fact that in order to enable such a mechanism in the present framework one has to select a potential whose minima are not in $\sigma = 0$. Later on we will provide a few examples of such suitable potentials.

We will now briefly discuss the dynamical conditions under which inflation occurs, so as to set proper initial conditions (see [9] for further details). By definition, inflation is taking place as long as the following inequality is satisfied:

$$\frac{\ddot{a}}{a} = H^2 + \dot{H} > 0, \tag{2.6}$$

meaning, precisely, that the Universe is expanding at an accelerated rate. On introducing the following adimensional parameter,

$$\epsilon_1 \equiv -\frac{\dot{H}}{H^2} \tag{2.7}$$

the condition for inflation can be re-stated as follows:

$$\epsilon_1 < 1 \Rightarrow \text{inflation}. \tag{2.8}$$

In order to relate ϵ_1 to the field dynamics in a convenient way, it is worth introducing the following hierarchies of so-called "flow functions", one referring to the scalar field's dynamics and the other one to the Hubble parameter's. The former is defined as

$$\delta_{n+1} \equiv \frac{\dot{\delta}_n}{H\delta_n}, \quad n = 0, 1, 2, \ldots; \quad \delta_0 \equiv \frac{\sigma(t)}{\sigma(t_i)}, \tag{2.9}$$

whereas the latter is, analogously, defined as follows:

$$\epsilon_{n+1} \equiv \frac{\dot{\epsilon}_n}{H\epsilon_n}, \quad n = 0, 1, 2, \ldots; \quad \epsilon_0 \equiv \frac{H(t_i)}{H(t)}. \tag{2.10}$$

In both the definitions t_i stands for some arbitrary initial time. With these definitions, higher-order time-derivatives of both σ and H can be recast in quite a simple form. For example,

$$\frac{\ddot{\sigma}}{H\dot{\sigma}} = \delta_1 + \delta_2 - \epsilon_1, \qquad \frac{\ddot{H}}{H\dot{H}} = \epsilon_2 - 2\epsilon_1. \tag{2.11}$$

On making use of Eqs. (2.3) and (2.2) one eventually obtains the following relationship between the two hierarchies:

$$\epsilon_1 = \frac{\delta_1}{1+\delta_1}\left(\frac{\delta_1}{2\gamma} + 2\delta_1 + \delta_2 - 1\right). \tag{2.12}$$

Therefore, the condition (2.8) can now be linked to the inflaton dynamics (more "fundamental", in the sense discussed below) as follows:

$$|\delta_1|, |\delta_2|, \frac{\delta_1^2}{2\gamma} \ll 1 \Rightarrow \epsilon_1 < 1 \Rightarrow \text{inflation}. \tag{2.13}$$

Note that we have also required $\delta_1^2/(2\gamma) \ll 1$ in order to account for those cases in which γ is that small that $\delta_1^2/(2\gamma) = \mathcal{O}(\delta_1)$ in spite of $|\delta_1| \ll 1$. That said, the two hierarchies of δ_n's and ϵ_n's naturally provide a collection of adimensional parameters which are all arbitrarily small during inflation and which thus allow to approximate inflationary predictions by means of "simple enough" series expansions. This point also regards inflationary models in the more conventional framework of GR. Since on tracing the condition (2.13) back to the inflaton dynamics one obtains

$$|\dot{\sigma}| \ll H|\sigma|, \quad |\ddot{\sigma}| \ll H|\dot{\sigma}|, \tag{2.14}$$

we can, as usual, say that *inflation occurs as long as the inflaton field slowly rolls down its potential*. We can thus refer to the approximation in (2.13) as a "slow-roll approximation" and to the set $\{\delta_n, \epsilon_n \,|\, n = 0, 1, 2, \dots\}$ as "slow-roll parameters".

Let us point out that the above slow-roll parameters are related to the (first two) Hubble flow functions in the Einstein frame ($\tilde{\epsilon}_n$) as follows:

$$\tilde{\epsilon}_1 = \frac{(1+6\gamma)\,\delta_1^2}{2\gamma\,(1+\delta_1)^2}, \tag{2.15}$$

$$\tilde{\epsilon}_2 = \frac{2\delta_2}{(1+\delta_1)^2}. \tag{2.16}$$

We thus see that the condition (2.13) is consistent with the more familiar smallness of the ϵ_n's in the Einstein frame. Nevertheless, in the Einstein frame it is not necessary to introduce the hierarchy of δ_n's, since the ϵ_n's are already enough to describe the background dynamics (as explicitly shown in the next subsection) and - which is even more physically relevant - the shape of the inflaton potential can be completely characterized in terms of ϵ_n's [26]:

$$\tilde{V} = 3M_{\rm Pl}^2 \tilde{H}^2 \left(1 - \frac{\tilde{\epsilon}_1}{3}\right) \tag{2.17a}$$

$$\frac{M_{\rm Pl}^2}{\tilde{V}^2}\left(\frac{\mathrm{d}\tilde{V}}{\mathrm{d}\tilde{\sigma}}\right)^2 = 2\tilde{\epsilon}_1 \frac{\left(1 - \frac{\tilde{\epsilon}_1}{3} + \frac{\tilde{\epsilon}_2}{6}\right)^2}{\left(1 - \frac{\tilde{\epsilon}_1}{3}\right)^2} \tag{2.17b}$$

$$\frac{M_{\rm Pl}^2}{\tilde{V}}\frac{\mathrm{d}^2\tilde{V}}{\mathrm{d}\tilde{\sigma}^2} = \frac{2\tilde{\epsilon}_1 - \frac{\tilde{\epsilon}_2}{2} - \frac{2\tilde{\epsilon}_1^2}{3} + \frac{5\tilde{\epsilon}_1\tilde{\epsilon}_2}{6} - \frac{\tilde{\epsilon}_2^2}{12} - \frac{\tilde{\epsilon}_2\tilde{\epsilon}_3}{6}}{1 - \frac{\tilde{\epsilon}_1}{3}} \tag{2.17c}$$

$$\vdots$$

In the case of IG, instead, we obtain the following:

$$V = 3\gamma\sigma^2 H^2 \left(1 + 2\delta_1 - \frac{\delta_1^2}{6\gamma}\right) \tag{2.18a}$$

$$\sigma\frac{V_{\mathrm{eff},\sigma}}{V} = -\frac{1+6\gamma}{3\gamma}\cdot\frac{\delta_1\left(2\delta_1+\delta_2-\epsilon_1+3\right)}{1+2\delta_1-\frac{\delta_1^2}{6\gamma}} \tag{2.18b}$$

$$\sigma^2\frac{V_{\mathrm{eff},\sigma\sigma}}{V} = -\frac{1+6\gamma}{3\gamma}\cdot\frac{\left(2\delta_1+\delta_2-\epsilon_1+3\right)\left(\delta_1+\delta_2-2\epsilon_1\right)+2\delta_1\delta_2+\delta_2\delta_3-\epsilon_1\epsilon_2}{1+2\delta_1-\frac{\delta_1^2}{6\gamma}} \tag{2.18c}$$

$$\vdots$$

Using Eq. (2.12) and its first time-derivative, which would yield a highly non-linear relationship between $\{\epsilon_1, \epsilon_2\}$ and $\{\delta_1, \delta_2, \delta_3\}$, one may eliminate $\{\epsilon_1, \epsilon_2\}$ from the above formulas and eventually express the derivatives of the inflaton (effective) potential only in terms of δ_n's. On the contrary, it is not possible to write them only in terms of ϵ_n's since the relationships linking the two hierarchies are not invertible. In this sense the hierarchy of δ_n's is somewhat more fundamental than that of ϵ_n's, as previously claimed.

2.1.1 Power-law solutions

In a while we will see that the homogeneous dynamics of the field-gravity system and the slow-roll conditions for inflation have a peculiar role in the theory of cosmological perturbations, in that they provide an approximate method to determine the dynamics of those perturbations and to compare theoretical predictions with observations. Nevertheless, exact solutions for such a dynamics can also be found for particular choices of the inflaton potential, both in the GR and IG frameworks.

Concerning GR with a minimally coupled scalar field, one finds the following background equations of motion:

$$\begin{cases} \tilde{H}^2 = \frac{1}{3M_{\mathrm{Pl}}^2}\left[\frac{\dot{\tilde{\sigma}}^2}{2} + \tilde{V}(\tilde{\sigma})\right] \\ \dot{\tilde{H}} = -\frac{\dot{\tilde{\sigma}}^2}{2M_{\mathrm{Pl}}^2} \\ \ddot{\tilde{\sigma}} + 3\tilde{H}\dot{\tilde{\sigma}} = -\frac{\mathrm{d}\tilde{V}}{\mathrm{d}\tilde{\sigma}} \end{cases} \tag{2.19}$$

The first two equations can be merged into the following one,

$$\tilde{\delta}_1^2\,\frac{\tilde{\sigma}^2}{2M_{\mathrm{Pl}}^2} = \tilde{\epsilon}_1\,, \tag{2.20}$$

and, on deriving both sides with respect to time, one discovers that

$$\tilde{\delta}_1 + \tilde{\delta}_2 = \frac{\tilde{\epsilon}_2}{2}\,. \tag{2.21}$$

Finally, the third equation in (2.19) can be rewritten as

$$\tilde{\delta}_2 + \tilde{\delta}_1 - \tilde{\epsilon}_1 + 3 + \frac{\tilde{\delta}_1}{\tilde{\epsilon}_1}\frac{\mathrm{d}\ln\tilde{V}}{\mathrm{d}\ln\tilde{\sigma}}\,(3-\tilde{\epsilon}_1) = 0. \tag{2.22}$$

From Eqs. (2.20), (2.21), (2.22) one easily observes that no solution with $\tilde{\delta}_1$ and $\tilde{\epsilon}_1$ simultaneously constant and different from zero exists. A non-trivial solution can instead be found for the case $\tilde{\epsilon}_1$ = constant $\Rightarrow \tilde{\epsilon}_2 = 0$, $\tilde{\delta}_1 = \pm\sqrt{2\tilde{\epsilon}_1}\,M_{\rm Pl}/\tilde{\sigma}$ and $\tilde{V} \propto \exp\left(\mp\sqrt{2\tilde{\epsilon}_1}\tilde{\sigma}/M_{\rm Pl}\right)$. Such a solution corresponds to the well-know power-law solution where $\tilde{a}(t) \propto t^p$, $\tilde{\epsilon}_1 = 1/p$ and inflation occurs for $p > 1$.

In the IG context, Eqs. (2.20), (2.21), (2.22) are replaced by Eq. (2.12) and by the following one:

$$\epsilon_1 \;=\; \delta_2 + \frac{-12\gamma + 3\,\delta_1 - 6\gamma\,\delta_1 + 4\,\delta_1^2 + 12\gamma\,\delta_1^2}{(1+6\gamma)\delta_1} + \frac{6\gamma + 12\gamma\,\delta_1 - \delta_1^2}{2(1+6\gamma)\delta_1}\frac{\mathrm{d}\ln V}{\mathrm{d}\ln\sigma}\,. \tag{2.23}$$

Despite their quite involved form one can still see that an exact non-trivial solution does exist in case of $V = \lambda_n\sigma^n/n$, with $\epsilon_2 = 0 = \delta_2$. Namely:

$$\delta_1 = -\frac{\gamma\,(n-4)}{1+\gamma\,(n+2)}, \qquad \epsilon_1 = \frac{\gamma\,(n-2)\,(n-4)}{2+2\gamma\,(n+2)} \tag{2.24}$$

and

$$\delta_1 = 6\gamma \pm \sqrt{6\gamma\,(1+6\gamma)}, \qquad \epsilon_1 = 3 + 2\delta_1. \tag{2.25}$$

However, the latter are not compatible with the phase-space constraint in Eq. (2.5). In the case of the former solution, instead, it is straightforward to prove that for $n \neq \{2,4\}$ it is

$$\sigma(t) = S\,t^{-\frac{2}{n-2}} \tag{2.26}$$

with

$$S^{n-2} = \frac{n}{\lambda_n}\frac{2(1+6\gamma)}{(n-2)^2(n-4)}\left[6\frac{1+(n+2)\gamma}{(n-4)\gamma} - (n+2)\right]. \tag{2.27}$$

In order for the amplitude S to be well-defined also for even values of n we must require the positivity of the r.h.s. of the previous formula and thus restrict n as follows:

$$4 - \sqrt{6(6+1/\gamma)} < n < 4 + \sqrt{6(6+1/\gamma)} \tag{2.28}$$

On requiring that the exact solutions be actually inflationary, one can either think of $a(t) \sim t^p$ with $t > 0$, $p > 1$, or $a(t) \sim (-t)^p$ with $t < 0$, $p < 0$. In the first case, n must be within the following intervals:

$$n^{(-)} < n < 2 \quad \text{or} \quad 4 < n < n^{(+)}\,, \qquad n^{(\pm)} \equiv 4 \pm \sqrt{2(6+1/\gamma)}\,. \tag{2.29}$$

In the second case, instead, inflation occurs within $2 < n < 4$. Such a solution characterizes a super-inflationary stage with $\dot{H} > 0$ and ends up in a future singularity (the Ricci scalar grows in time instead of decreasing). Let us specify that we have implicitly excluded those possibilities which are not consistent with the condition (2.28).

The poles at $n = \{2, 4\}$ in the above equations correspond to de Sitter solutions with $a(t) \propto e^{Ht}$. More specifically, the case $n = 4$ leads to the solution with both σ and H constant in time, whereas the solution for $n = 2$ is still de Sitter-like ($H =$ const.) but the field grows exponentially in time, $\sigma \propto \exp(H\delta_1 t)$ with $\delta_1 = 2\gamma/(1+4\gamma)$. The Hubble parameter during inflation is set by the mass of the inflaton, $m \equiv \sqrt{\lambda_2}$, and a γ-dependent pre-factor:

$$H = \frac{1+4\gamma}{\sqrt{2\gamma(1+6\gamma)(3+16\gamma)}} \cdot m\,. \tag{2.30}$$

We observe that H diverges as $m/\sqrt{2\gamma}$ for $\gamma \to 0$ and decreases as $m/\sqrt{12\gamma}$ for $\gamma \to \infty$. Remarkably enough, as already claimed above such a de Sitter-like solution does not exist in the case of GR with a massive scalar field since setting $\dot{H} = 0$ would imply $\dot{\sigma} = 0$ which is not compatible with the scalar field's equation of motion.

In Fig. 2.1 we show the behaviour of ϵ_1 in Eq. (2.24) as γ and n vary. It is possible to realize that decelerated, accelerated and super-accelerated solutions can all be found inside the boundaries of phase-space. We finally note that the exact solutions with $n \geq 4$ have $\delta_1 \leq 0$ (large field dynamics) while those for $n < 4$ have $\delta_1 > 0$ (small field dynamics).

2.2 Cosmological perturbations

In order to evaluate the inflationary predictions and to proceed with the comparison against observational data, we need to introduce scalar and tensor perturbations in both the field σ and in the metric tensor:

$$\sigma = \sigma(t) + \delta\sigma(t, \vec{x}) \tag{2.31}$$
$$g_{\mu\nu} = g_{\mu\nu}(t) + \delta g_{\mu\nu}(t, \vec{x})\,. \tag{2.32}$$

These perturbations, whose origin is ultimately supposed to be quantum, are mutually coupled via a set of equations of motion which generalizes the perturbed Einstein's field equations. After having eliminated the metric fluctuations through the use of all the available equations, one is left with the following differential equation which governs the dynamics of the Fourier component $\delta\sigma_k$ of the inflaton (quantum) fluctuations [27]:

$$\delta\ddot{\sigma}_k + \left(3H + \frac{\dot{Z}}{Z}\right)\delta\dot{\sigma}_k + \left[\frac{k^2}{a^2} - \frac{1}{a^3 Z\sigma\delta_1}\left(a^3 Z\,(\sigma\delta_1)^{\cdot}\right)^{\cdot}\right]\delta\sigma_k = 0, \tag{2.33}$$

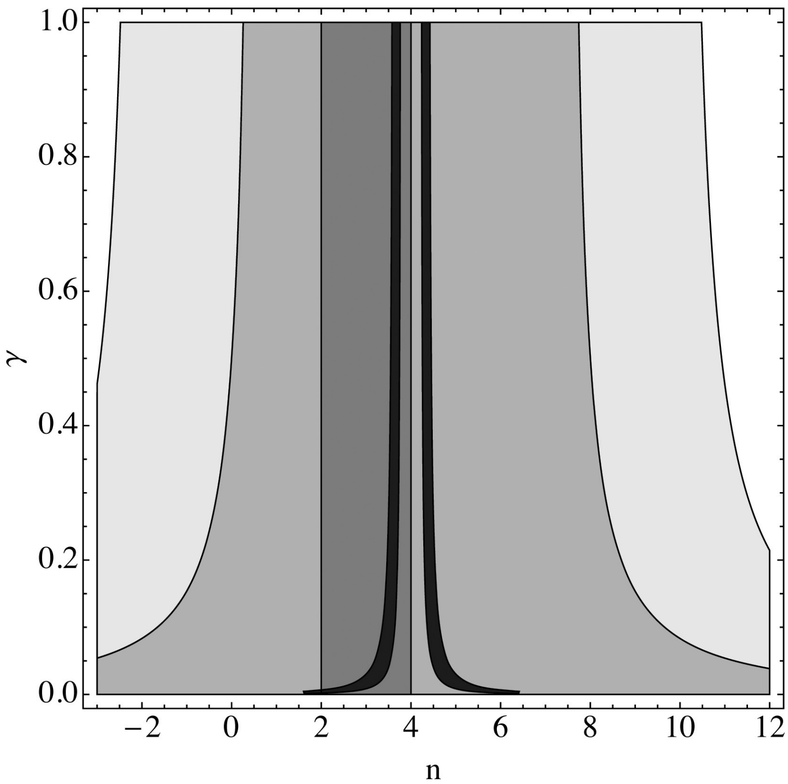


Figure 2.1: The figure represents ϵ_1 in Eq. (2.24) as a function of n and γ. Within the lighter grey region is $\epsilon_1 > 1$ (decelerated solution) and its boundaries delimit the allowed phase-space - cf. Eq. (2.28). The darker grey region is for $\epsilon_1 < 0$ (super-accelerated solution) and the intermediate grey region is for $0 < \epsilon_1 < 1$ (accelerated solution). The lines for $n = 2, 4$ represents the de Sitter solutions. The darkest black areas are the domains allowed by the present experimental constraints - cf. Eq. (2.75).

where

$$Z \equiv \frac{H^2\sigma^2(1+6\gamma)}{(\dot{\sigma}+H\sigma)^2} = \frac{1+6\gamma}{(1+\delta_1)^2}\,. \tag{2.34}$$

The friction term in the above equation can be eliminated by rescaling $\delta\sigma_k$ as $S_k \equiv a\sqrt{Z}\,\delta\sigma_k$ and by replacing derivatives with respect to the cosmic time t with derivatives with respect to the conformal time η (recall the definition $a(\eta)\mathrm{d}\eta \equiv \mathrm{d}t$):

$$\frac{\mathrm{d}^2 S_k}{\mathrm{d}\eta^2} + \left[k^2 + M_S^2(\eta)\right] S_k = 0\,, \tag{2.35}$$

where the effective mass squared is exactly

$$\begin{aligned} M_S^2(\eta) \equiv -\,\mathcal{H}^2\Bigg[& -1 + (3-\epsilon_1)(\delta_1+\delta_2+1) + \delta_1^2 + \delta_2^2 + \delta_2\delta_3 + \\ & + \frac{\delta_1\delta_2}{1+\delta_1}\left(\epsilon_1 + \delta_1 - 3\delta_2 - \delta_3 + \frac{2\delta_1\delta_2}{1+\delta_1} - 2\right)\Bigg]\,, \end{aligned} \tag{2.36}$$

being

$$\mathcal{H} \equiv \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}\eta}\,. \tag{2.37}$$

Gravitational waves are also produced during inflation. The Fourier modes of tensor perturbations satisfy the following equation:

$$\ddot{h}_{s,k} + (3H + 2H\delta_1)\dot{h}_{s,k} + \frac{k^2}{a^2}h_{s,k} = 0\,, \tag{2.38}$$

where $s = +\,,\times$ denotes the two polarization states. On setting $T_{s,k} \equiv a\sigma\sqrt{\gamma}\,h_{s,k}/\sqrt{2}$ the above equation can be rewritten as

$$\frac{\mathrm{d}^2 T_{s,k}}{\mathrm{d}\eta^2} + \left[k^2 + M_T^2(\eta)\right] T_{s,k} = 0\,, \tag{2.39}$$

with

$$M_T^2(\eta) \equiv -\mathcal{H}^2\left[2 - \epsilon_1 + \delta_1(3 + \delta_1 + \delta_2 - \epsilon_1)\right]. \tag{2.40}$$

The power-spectra of both scalar and tensor perturbations can be conveniently characterizable by a couple of parameters (amplitude at a fixed scale and spectral index) which are then constrained by observations. Concerning tensor perturbations, the power-spectrum is defined as

$$\mathcal{P}_h(k) \equiv 2 \times \frac{k^3}{2\pi^2}\left(|h_{+,k}|^2 + |h_{\times,k}|^2\right) \simeq \mathcal{P}_h(k_*)\left(\frac{k}{k_*}\right)^{n_t} \tag{2.41}$$

where k_* is a suitable pivot scale and n_t is the “tensor spectral index”.

Concerning scalar perturbations, it is not correct to simply consider the field's fluctuations because these are not gauge-invariant, meaning that one could use the available gauge freedom to switch to a reference frame where $\delta\sigma_k$ is null even though metric scalar fluctuations are not. In that case the actual physics of the problem would be erroneously missed. In order to properly account for scalar fluctuations (no matter the gauge freedom), one has to consider a mixture of metric and scalar field perturbations which is gauge-invariant. One example of such a well-defined quantity is the so-called "comoving curvature perturbation" $\mathcal{R}$ defined as

$$\mathcal{R} = \psi + \frac{H}{\dot{\sigma}}\delta\sigma, \tag{2.42}$$

where ψ is a metric scalar degree of freedom which enters into $\delta g_{\mu\nu}$ as follows:

$$\delta g_{\mu\nu} = \begin{pmatrix} \cdots & \cdots \\ \cdots & a^2\left[-2\psi\,\delta_{ij} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)E\right] \end{pmatrix}. \tag{2.43}$$

(We have left completely unspecified all those components which are not relevant for the argument here addressed.)

The power-spectrum of scalar perturbations is finally defined as

$$\mathcal{P}_{\mathcal{R}}(k) \equiv \frac{k^3}{2\pi^2}|\mathcal{R}_k|^2 \simeq \mathcal{P}_{\mathcal{R}}(k_*)\left(\frac{k}{k_*}\right)^{n_s-1}, \tag{2.44}$$

where n_s is the "scalar spectral index".

Amplitude and spectral index of both scalar and tensor perturbations can be analytically evaluated after having cast Eqs. (2.35) and (2.39) in the form of a Bessel equation:

$$\frac{\mathrm{d}^2 f(\eta)}{\mathrm{d}\eta^2} + \left[k^2 - \frac{1}{\eta^2}\left(\nu^2 - \frac{1}{4}\right)\right] f(\eta) = 0. \tag{2.45}$$

Such an equation has a general solution of the following type:

$$f(\eta) = \sqrt{-\eta}\left[c_1(k)\,\mathrm{H}_\nu^{(1)}(-k\eta) + c_2(k)\,\mathrm{H}_\nu^{(2)}(-k\eta)\right], \tag{2.46}$$

where $\mathrm{H}_\nu^{(1),(2)}$ are Hankel functions of the 1st and 2nd kind and order ν, and the arbitrary integration constant $c_{1,2}(k)$ must be set to

$$c_1(k) = \frac{\sqrt{\pi}}{2}e^{i\frac{\pi}{2}(\nu+\frac{1}{2})} \tag{2.47}$$

$$c_2(k) = 0 \tag{2.48}$$

so as to select positive frequency modes and to match with the Bunch-Davies vacuum solution at very short scales ($|k\eta| \gg 1$). The comparison with observations involves large-scale modes ($|k\eta| \ll 1$) for which the Hankel function of the 1st kind has the following asymptotic behaviour:

$$\mathrm{H}_\nu^{(1)}(-k\eta) \overset{|k\eta|\ll 1}{\sim} \frac{e^{-i\frac{\pi}{2}}}{\pi}\Gamma(\nu)\left(-\frac{1}{2}k\eta\right)^{-\nu}. \tag{2.49}$$

Therefore, according to the definition (2.44), we are now able to state that the spectral indices are related to the order of the Hankel function as follows:

$$n_s - 1 = 3 - 2\,\nu_s, \qquad n_t = 3 - 2\,\nu_t. \tag{2.50}$$

2.2.1 Slow-roll predictions

For generic potentials Eqs. (2.12), (2.23) are not exactly solvable. Still, on employing the slow-roll approximation (which, in the end, guarantees inflation) one can obtain analytical estimates of the spectra of perturbations. In such an approximation Eqs. (2.1), (2.3) become

$$H^2 \simeq \frac{V(\sigma)}{3\gamma\sigma^2}, \tag{2.51}$$

$$3H\dot{\sigma} \simeq -\frac{V_{\mathrm{eff},\sigma}}{1+6\gamma}. \tag{2.52}$$

Moreover, it is straightforward to prove that

$$\frac{1}{\mathcal{H}^2}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}\eta} = 1 - \epsilon_1 \tag{2.53}$$

and, on assuming the constancy in time of ϵ_1 during inflation, one can easily integrate the previous formula and discover that

$$\mathcal{H} = \frac{-1}{(1-\epsilon_1)\eta} \tag{2.54}$$

On plugging this result into Eqs. (2.35), (2.39) and on series expanding around $\epsilon_1 = 0, \delta_n = 0$ up to 1st-order one obtains:

$$M_S^2(\eta) \simeq -\frac{1}{\eta^2}\left(2 + 3\,\epsilon_1 + 3\,\delta_1 + 3\,\delta_2\right); \tag{2.55}$$

$$M_T^2(\eta) \simeq -\frac{1}{\eta^2}\left(2 + 3\,\epsilon_1 + 3\,\delta_1\right). \tag{2.56}$$

Consequently,

$$\nu_s \simeq \frac{3}{2}\left[1 + \frac{2}{3}\left(\epsilon_1 + \delta_1 + \delta_2\right)\right]; \tag{2.57}$$

$$\nu_t \simeq \frac{3}{2}\left[1 + \frac{2}{3}\left(\epsilon_1 + \delta_1\right)\right], \tag{2.58}$$

and finally

$$n_s - 1 = n_t - 2\delta_2 = -2\left(\delta_1 + \delta_2 + \epsilon_1\right) . \tag{2.59}$$

The amplitudes at the fixed scale k_*, which is chosen as the scale realizing the condition of "Hubble horizon crossing" ($-k_*\eta = 1$) are, to first order,

$$\mathcal{P}_{\mathcal{R}}(k_*) \simeq \left.\frac{A\,H^2}{4\pi^2 Z_s\,\delta_1^2\sigma^2}\right|_* , \tag{2.60}$$

and

$$\mathcal{P}_h(k_*) \simeq \left.\frac{2(A - C\delta_2)H}{\pi^2\gamma\sigma^2}\right|_* , \tag{2.61}$$

where

$$A \equiv \left[1 - 2\epsilon_1 + C\left(\delta_1 + \delta_2 + \epsilon_1\right)\right] , \tag{2.62}$$

$$C \equiv 2\left(2 - \ln 2 - \gamma_E\right) , \tag{2.63}$$

and γ_E is the Euler-Mascheroni constant.

The tensor-to-scalar ratio, defined as

$$r \equiv \frac{\mathcal{P}_h(k_*)}{\mathcal{P}_{\mathcal{R}}(k_*)}, \tag{2.64}$$

evaluates to

$$r = 8\frac{(1+6\gamma)\delta_1^2}{\gamma(1+\delta_1)^2}, \tag{2.65}$$

and can be shown to fulfill the following consistency relation:

$$r = -8\,n_t. \tag{2.66}$$

In the slow-roll regime the dynamics of the scalar field can be approximated as follows:

$$\delta_1 \simeq -\gamma\sigma\frac{V_{\mathrm{eff},\sigma}}{(1+6\gamma)\,V} , \tag{2.67}$$

$$\delta_2 \simeq -\gamma\sigma^2\frac{V_{\mathrm{eff},\sigma\sigma}}{(1+6\gamma)\,V} + \delta_1\left(\frac{1+6\gamma}{\gamma}\delta_1 - 3\right) , \tag{2.68}$$

and

$$\epsilon_1 \simeq -\delta_1 + \frac{1+6\gamma}{2\gamma}\delta_1^2. \tag{2.69}$$

Second-order terms in the above expressions are retained in order to better interpolate the regime from large to small γ's. Remarkably enough, we see that the slow-roll parameter

δ_1 is roughly the square root of what one would expect in the Einstein frame where, in fact,

$$\tilde{\epsilon}_1 \simeq \tilde{\epsilon}_V \equiv \frac{M_{\rm Pl}^2}{2}\left(\frac{V_{,\sigma}}{V}\right)^2 \quad . \tag{2.70}$$

Looking back at Eq. (2.15) one immediately finds a confirmation of what has just been stated.

Using the expressions above and keeping the first order contributions one is finally led to

$$n_s - 1 \simeq \frac{2\gamma\sigma_*^2}{1+6\gamma}\left[\frac{V_{{\rm eff},\sigma\sigma*}}{V_*} - \frac{3V_{{\rm eff},\sigma*}}{\sigma_* V_*} - \frac{3V_{{\rm eff},\sigma*}^2}{2V_*^2}\right] \tag{2.71}$$

and

$$n_t \simeq -\frac{\gamma\sigma_*^2}{1+6\gamma}\frac{V_{{\rm eff},\sigma*}^2}{V_*^2}\,, \tag{2.72}$$

where terms proportional to $V_{{\rm eff},\sigma*}^2/V_*^2$ are of 1st-order for $\gamma \ll 1$ and 2nd-order for $\gamma \gtrsim 1$.

2.2.2 Exact solutions for $V(\sigma) \propto \sigma^n$

Following the very same steps above in the case of power-law exact solutions one finds the following results:

$$n_s - 1 = n_t = \frac{2\gamma\,(n-4)^2}{\gamma\,(n-4)^2 - 2\,(6\gamma+1)}. \tag{2.73}$$

The consistency relation reads as

$$r = -\frac{8\,n_t}{1-\frac{n_t}{2}} \tag{2.74}$$

which is in full agreement with the one found in GR.

From the most recent compilation of data [26] we obtain the following constraint (at the 95% confidence interval), which is independent of γ for γ large:

$$39 < \frac{6\gamma+1}{\gamma(n-4)^2} < 123\,. \tag{2.75}$$

2.3 Constraints on different potentials

Using the expressions above for the quantities associated to the scalar and tensor power-spectra, one can investigate which constraints are put by observations on different potentials for the scalar field. In particular we have tested diverse symmetry-breaking potentials which allow to fix the Planck mass after inflation:

- Landau-Ginzburg (LG) potential:

$$V_{LG}(\sigma) = \frac{\mu}{4}\left(\sigma^2 - \sigma_0^2\right)^2 ; \tag{2.76}$$

- Coleman-Weinberg (CW) type potential:

$$V_{CW}(\sigma) = \frac{\mu}{8}\sigma^4\left(\log\frac{\sigma^4}{\sigma_0^4} - 1\right) + \frac{\mu}{8}\sigma_0^4 ; \tag{2.77}$$

- cosine potential (CO):

$$V_{CO}(\sigma) = \Lambda\left[1 + \cos\left(\pi\frac{\sigma}{\sigma_0}\right)\right] . \tag{2.78}$$

We have also considered some generalization of the potentials above, such as

$$V_1(\sigma) = \frac{\Lambda}{4n}\left(\sigma^2 - \sigma_0^2\right)^{2n} \tag{2.79}$$

and

$$V_2(\sigma) = \Lambda\left[\left(\frac{\sigma}{\sigma_0}\right)^n - 1\right]^2 , \tag{2.80}$$

with $n > 1$. In particular we have tested the potential (2.79) with $n = 2$ and the potential (2.80) with $n = 3/2$ and $n = 5/2$.

These potentials share a few interesting features: they have a minimum at $\sigma = \sigma_0$ and a relative maximum at $\sigma = 0$; $V(\sigma) \geq 0$; they all allow both small and large field inflation. We further note that we have included the CO potential although once expanded to lowest order around σ_0 exactly matches with the LG potential. Nonetheless it leads to slightly different results because of higher-order effects.

We have already anticipated that the comparison with observational data involves a fixed scale k_*, but we have not yet clarified which scale should be selected. It is usually demanded that inflation last enough for the Universe to expand of a factor $\sim e^{70}$. The amount of inflation is measured in terms of the number N of e-folds:

$$\frac{a(t_{\rm end})}{a(t_{\rm i})} \equiv e^N , \tag{2.81}$$

where $t_{\rm i}$, $t_{\rm end}$ represent the initial and final time of inflation, respectively. Differentiating the previous formula (after having replaced $t_{\rm end} \to t$, $N \to N(t)$), one easily finds the following differential relation between t and N:

$$\mathrm{d}N = H\mathrm{d}t. \tag{2.82}$$

Another relevant quantity is the number N_* of e-folds between the time at which the mode k_* crosses the Hubble horizon $(a(t_*)H(t_*) = k_*)$ and the end of inflation (set by the condition $\epsilon_1(t_{\rm end}) = 1$). Scales nowadays observed are such that $50 < N_* < 60$ (we refer to Ref. [28] for an extensive discussion on this point). In our analysis we have considered a wider interval, namely $50 < N_* < 70$, and concerning observations we made reference to the WMAP5+BAO+SN combined observational constraints [29]:

$$n_s = 0.963 \pm 0.014 \quad \text{(68\% confidence interval)} \tag{2.83}$$

$$n_s = 0.963 \pm 0.028 \quad \text{(95\% confidence interval)} \tag{2.84}$$

$$r < 0.22 \quad \text{(95\% confidence interval).} \tag{2.85}$$

In Figs. 2.2, 2.3 we plotted the trajectories of (n_s, r) for $N_* = 50$ and $N_* = 70$ as γ varies. The dashed line in each figure represents the exact consistency condition (2.74) which is $r\, n_s = 3r + 16\, n_s - 16$. A list of interesting conclusions can be drawn from Figs. 2.2 and 2.3:

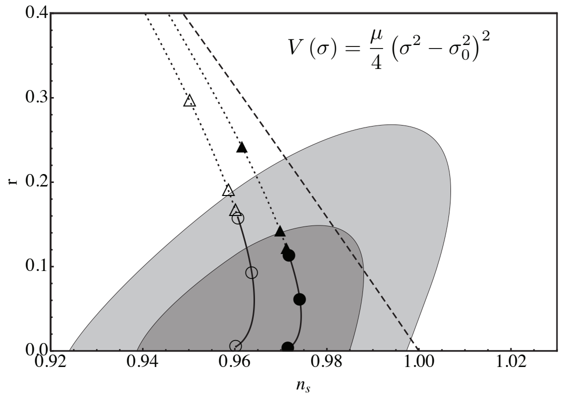
1. The LG and CW potentials do not constrain γ in the large field regime, while it is needed $\operatorname{Log}\gamma < -4$ for acceptable predictions in the small field case. In fact, from Fig. 2.2 one can see that the markers for $\operatorname{Log}\gamma = 3$ lie outside the 68% confidence interval.

2. Except for the potential (2.79) with $n = 2$, which is not compatible with observations for any γ, it can be observed that other choices of the potential are acceptable as long as the value of γ is properly costrained. In particular for $50 \leq N_* \leq 70$, $\operatorname{Log}_\gamma \lesssim -4$ is needed in the SF case and $\operatorname{Log}\gamma \lesssim -2$ in the LF case.

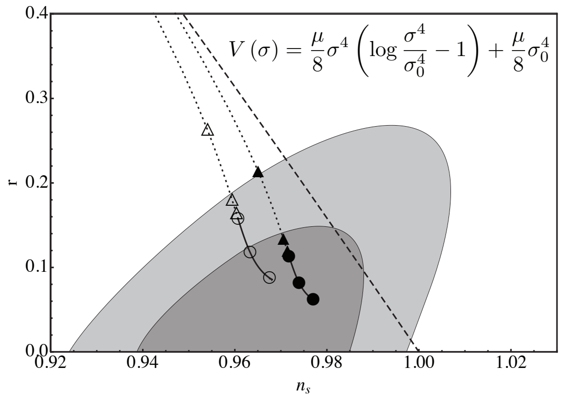
The coordinate of relevant points on the trajectories in Figs. 2.2, 2.3 can, in principle, be estimated as a function of N_*, although the task is not always simple nor doable analytically. On requiring a fixed amount of inflation ($N \simeq 70$) the required initial displacement of σ from the (positive) minimum of the potential strongly depends on the value of γ and, more specifically, the smaller is γ the smaller is $|\sigma_{\rm i} - \sigma_0|$. In the limit $\gamma \ll 1$ it can be safely assumed that $|\sigma_* - \sigma_0|/\sigma_0 \ll 1$. In order to relate σ_* and N_* one has to integrate $\delta_1 \equiv {\rm d}\log(\sigma)/{\rm d}N$ which, during slow-roll inflation, is given by Eq. (2.67). On further approximating $\sigma_{\rm end}$ (the value of σ at the end of inflation) with σ_0 - since $\sigma_{\rm end} = \sigma_0(1 \pm \mathcal{O}(\gamma))$ - we are left with the following integral:

$$\int_{\sigma_*}^{\sigma_0} \frac{1 + \gamma\,[n(\sigma) + 2]}{\gamma\sigma\,[4 - n(\sigma)]}\, {\rm d}\sigma = \int_0^{N_*} {\rm d}N = N_* \tag{2.86}$$

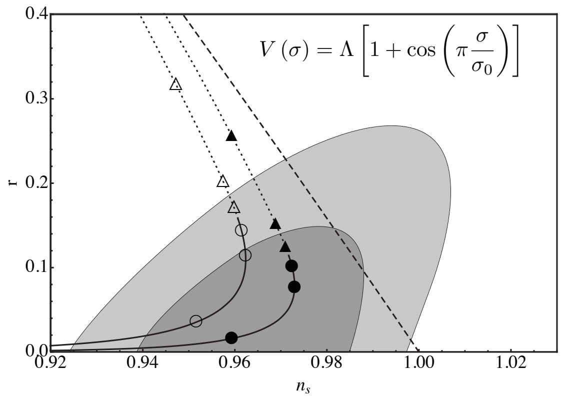
where $n(\sigma) \equiv {\rm d}\log V(\sigma)/{\rm d}\log\sigma$. The integral on the l.h.s. of Eq. (2.86) cannot be solved exactly for any potential $V(\sigma)$ and, even when possible, one further needs to invert the result in order to obtain $\sigma_* = \sigma(N_*)$ which is often a difficult task. However, for $\gamma \ll 1$ some acceptable simplifications can be made and a double series expansion around $\gamma = 0$ and $(\sigma_* - \sigma_0)/\sigma_0 = 0$ proved to yield good analytical predictions.



(a) The markers are for $\operatorname{Log}\gamma = -5, -4, -3$ and $\operatorname{Log}\gamma = -7, -3, +1$ respectively in the small field (dotted line) and large field (solid line) regimes.

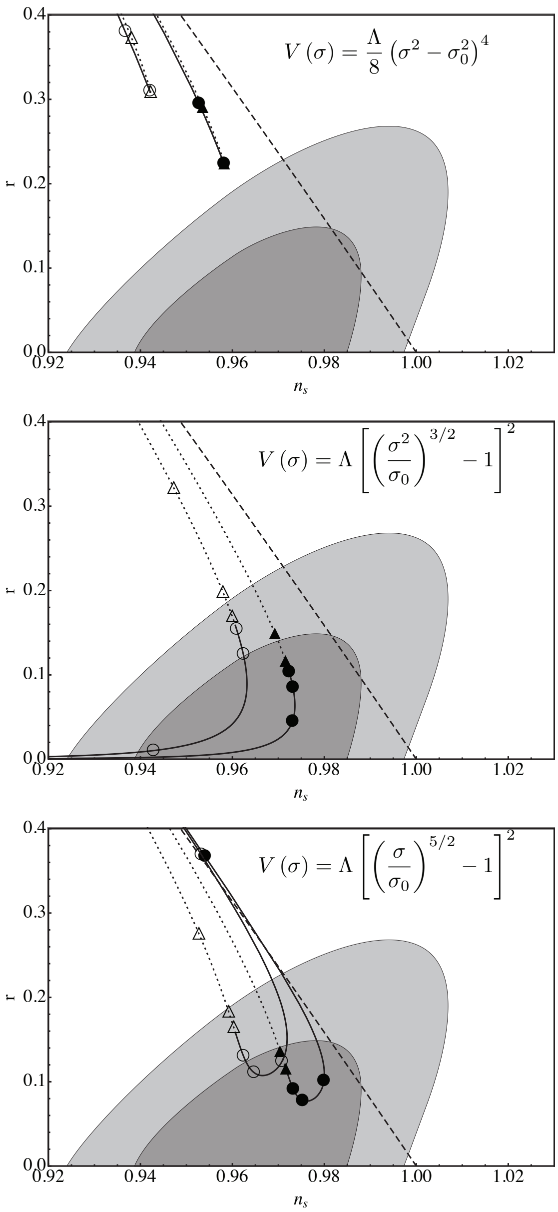


(b) The markers are for $\operatorname{Log}\gamma = -5, -4, -3$ and $\operatorname{Log}\gamma = -7, -3, +1$ respectively in the small field (dotted line) and large field (solid line) regimes.



(c) The markers are for $\operatorname{Log}\gamma = -5, -4, -3$ both in the small field (dotted line) and large field (solid line) regimes.

Figure 2.2: Trajectories of the vector (n_s, r) for different choices of the potential and different values of γ. The brighter grey region represents the 95% confidence interval while the darker one represent the 68% confidence interval. The ordering of the markers is such that dotted and continuous lines join for $\operatorname{Log}\gamma \to -\infty$. Empty and filled markers correspond to $N_* = 50$ and $N_* = 70$, respectively.



(a) The markers are for $\text{Log}_\gamma = -5, -3$ both in the small field (dotted line) and large field (solid line) regimes.

(b) The markers are for $\text{Log}_\gamma = -5, -4, -3$ in the small field regime (dotted line) and $\text{Log}_\gamma = -6, -4, -2$ in the large field regime (solid line).

(c) The markers are for $\text{Log}_\gamma = -5, -4, -3$ in the small field regime (dotted line) and $\text{Log}_\gamma = -3.6, -2.8, -2.0, -1.2$ in the large field regime (solid line).

Figure 2.3: Trajectories of the vector (n_s, r) as γ varies for potential (2.79) with $n = 2$ and for potential (2.80) with $n = 3/2$ and $n = 5/2$ respectively. The brighter grey region represents the 95% confidence interval while the darker one represent the 68% confidence interval. The ordering of the markers is such that dotted and continuous lines join for $L_\gamma \to -\infty$. Empty and filled markers correspond to $N_* = 50$ and $N_* = 70$, respectively.

In particular, one finds what follows (the symbol $\pm$ distinguishes between LF and SF dynamics):

$$\left(\begin{array}{c} V_{LG}, \quad V_{CW}, \\ V_2 \text{ with } n = 3/2 \text{ and } n = 5/2 \end{array} \right) \quad \gamma \ll 1 \rightarrow \begin{cases} \sigma\,(N_*) \simeq \sigma_0\,(1 \pm 2\sqrt{\gamma N_*}) \\ n_s \simeq 1 - 2/N_* \\ r \simeq 8/N_* \end{cases} \tag{2.87}$$

$$\left(\; V_1 \text{ with } n = 2 \; \right) \qquad \gamma \ll 1 \rightarrow \begin{cases} \sigma\,(N_*) \simeq \sigma_0\,(1 \pm 2\sqrt{2\gamma N_*}) \\ n_s \simeq 1 - 3/N_* \\ r \simeq 16/N_* \end{cases} \tag{2.88}$$

We conclude that, for small γ's, predictions on the tensor-to-scalar ratio can be incompatible with observations depending on the shape of the potential. Predictions on the scalar spectral index are fine, instead.

For large γ's inflation takes place only in the LF regime and it is possible to invert Eq. (2.86) on assuming $\sigma_* \gg \sigma_0$ and $\gamma^{-1} \simeq 0$. One obtains:

$$(V_{LG}) \qquad \gamma \gg 1 \rightarrow \begin{cases} n_s \simeq 1 - 2/N_* \\ r \simeq 12/N_*^2 \end{cases} \tag{2.89}$$

$$(V_{CW}) \qquad \gamma \gg 1 \rightarrow \begin{cases} n_s \simeq 1 - 1.5/N_* \\ r \simeq 4/N_* \end{cases} \tag{2.90}$$

In both cases predictions are compatible with observations but results are quite different.

We see that so far no constraint is put on the adimensional parameter μ (or Λ) entering into the potentials and somehow related to the value of $V(\sigma = 0)$. Such a parameter is in fact constrained by the data on the amplitude of scalar perturbations, namely [29]:

$$P_{\mathcal{R}}(k_*) = (2.445 \pm 0.096) \times 10^{-9}. \tag{2.91}$$

The pairs of values of (γ, μ) which allow for an agreement with observations are easily deducible from the plots in Figs. 2.4 and 2.5. Looking at those plots we can conclude that, except for some more involved example, it is generically sufficient to require that γ and μ be related by a piecewise function such as

$$\mu = \begin{cases} A\,\gamma^{\alpha}, & \text{if } \gamma \ll 1 \\ \\ B\,\gamma^{\beta}, & \text{if } \gamma \gtrsim 1 \end{cases} \tag{2.92}$$

We looked for an analytical proof of such a behavior and, restricting the analysis to the Landau-Ginzburg potential (2.76) for the sake of simplicity, we indeed found out that:

$$P_{\mathcal{R}}(k_*) \simeq \begin{cases} \frac{\mu}{3\pi^2\gamma} N_*^2, & \gamma \ll 1 \quad \text{(large and small field)} \\ \\ \frac{\mu}{72\pi^2\gamma^2} N_*^2, & \gamma \gg 1 \quad \text{(large field)} \end{cases} \tag{2.93}$$

meaning that, at fixed $\mathcal{P}_\mathcal{R}$ and N_*, we should require $\mu \propto \gamma$ in the case of $\gamma \ll 1$ and $\mu \propto \gamma^2$ in the opposite case.

2.3.1 Comparison with General Relativity

It is interesting to compare predictions computed in the IG framework to those of General Relativity with identical potentials. Since there is no γ parameter in GR to compare with, we introduce a fictitious parameter ξ measuring the ratio between σ_0^2 and $M_{\rm Pl}^2$:

$$\xi \equiv \frac{\sigma_0^2}{M_{\rm Pl}^2}. \tag{2.94}$$

In Figure 2.3.1 we compared the WMAP5+BAO+SN constraints at the 68% (dotted contour) and the 95% (continuous contour) confidence intervals [29] with the predictions for the CW potential (2.77) and the LG potential (2.76) both in the IG (continuous lines) and GR (dashed lines) frameworks.

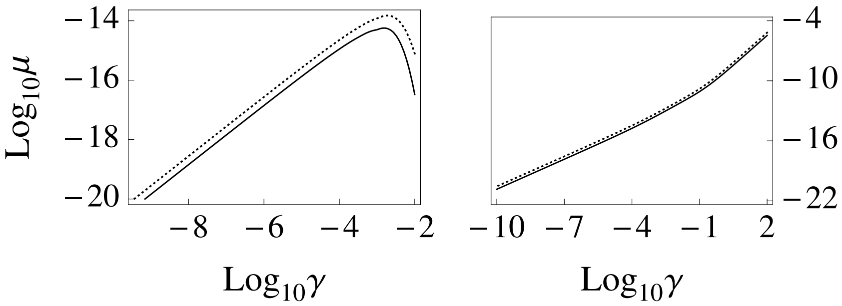
We can conclude that, in the case of large field inflation, GR's predictions differ from IG's in that the agreement with observations requires $\xi \geq 10^3$ as opposed to large field inflation in IG which fits observations independently of γ. It is however worth noting that both the potentials (2.77), (2.76), although very similar to a simple quartic potential in this regime, fit observations very well independently of μ and σ_0. This sounds like an improvement over GR where the quartic potential leads to results lying far away from the 95% region in the same interval of e-folds.

Another interesting comparison between IG and GR regards the tensor-to-scalar ratio:

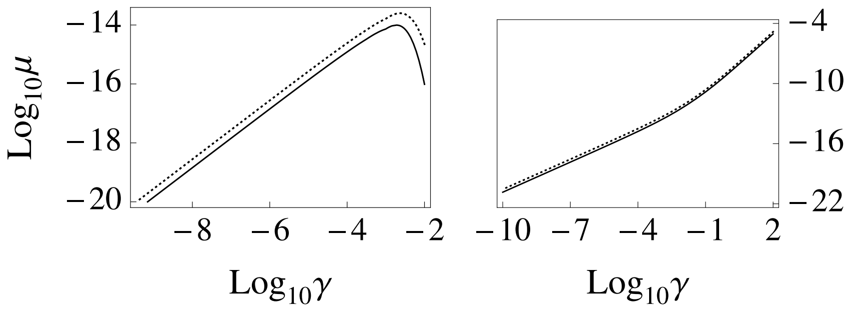
$$r^{(IG)} \simeq \frac{\gamma\sigma_*^2}{1+6\gamma}\frac{V_{{\rm eff},\sigma*}^2}{V_*^2} \overset{\gamma\sigma_0^2 \equiv M_{\rm Pl}^2}{=} \frac{8M_{\rm Pl}^2}{1+6\gamma}\frac{V_{{\rm eff},\sigma*}^2}{V_*^2}\frac{\sigma_*^2}{\sigma_0^2}\,, \tag{2.95}$$

$$r^{(GR)} \simeq 8M_{\rm Pl}^2\frac{V_{,\sigma_*}^2}{V_*^2}. \tag{2.96}$$

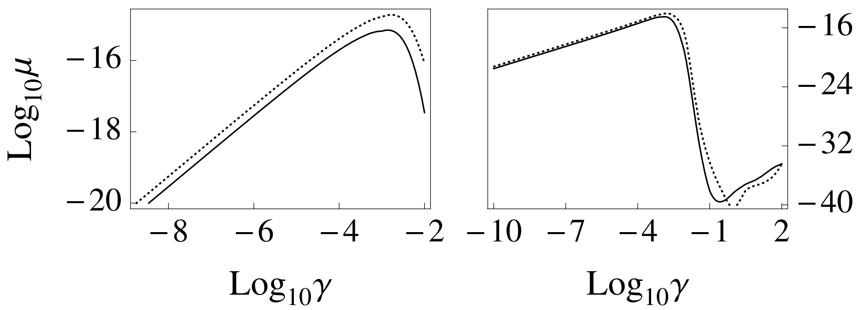
As a first point we see that the first derivative of the potential appearing in the numerator of $r^{(GR)}$ is replaced by the first derivative of the effective potential in IG. Such a difference can actually have a non-trivial impact on r. For instance, r is reduced for $V \propto \sigma^4$ in LF configurations while being potentially increased for SF configurations where $4V_*/\sigma_*$ can be actually much larger than $V_{,\sigma_*}$. Secondly, it is interesting to note that factor $(1+6\gamma)^{-1}$ appearing in the result for IG can drastically decrease r with respect to GR, in particular for $\gamma \gg 1$. Thirdly, the factor σ_*^2/σ_0^2 acts as a (de)amplifying factor for (small) large field configurations.



(a) Potential (2.76).

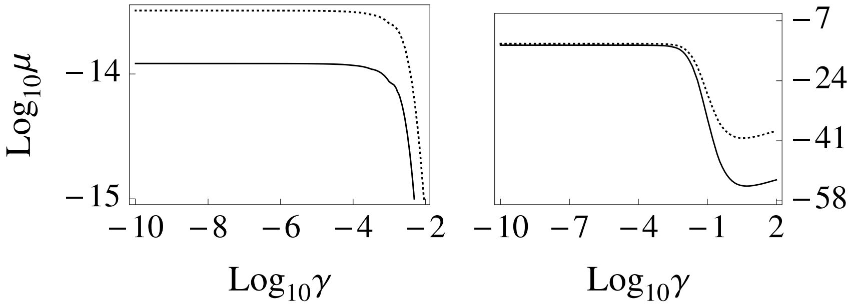


(b) Potential (2.77).

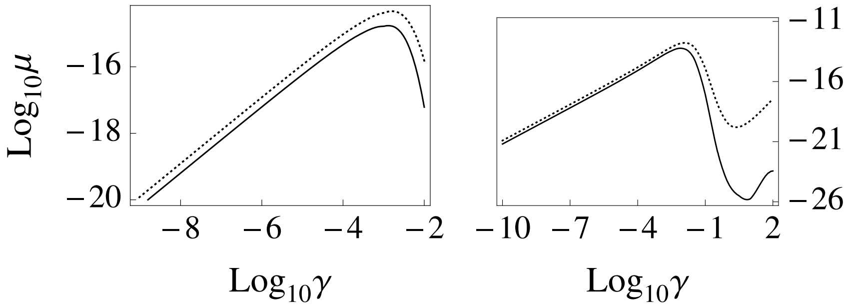


(c) Potential (2.78) with $\mu \equiv \Lambda/\sigma_0^4$.

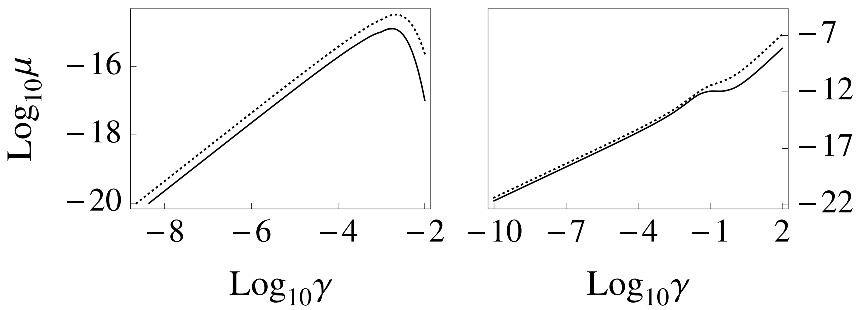
Figure 2.4: Constraints on μ from the amplitude of the scalar perturbations - cf. Eq. (2.91) - for different potentials. The dotted line is for $N_* = 50$ while the solid line is for $N_* = 70$. The plots on the left refer to the SF case whereas the ones on the right to the LF case.



(a) Potential (2.79) with $n = 2$ and $\mu \equiv \Lambda\,\sigma_0^4$.



(b) Potential (2.80) with $n = 3/2$ and $\mu \equiv \Lambda/\sigma_0^4$



(c) Potential (2.80) with $n = 3/2$ and $\mu \equiv \Lambda/\sigma_0^4$.

Figure 2.5: Constraints on μ from the amplitude of the scalar perturbations - cf. Eq. (2.91) - for different potentials. The dotted line is for $N_* = 50$ and the solid line is for $N_* = 70$. The plots on the left refer to the SF case whereas the ones on the right to the LF case.

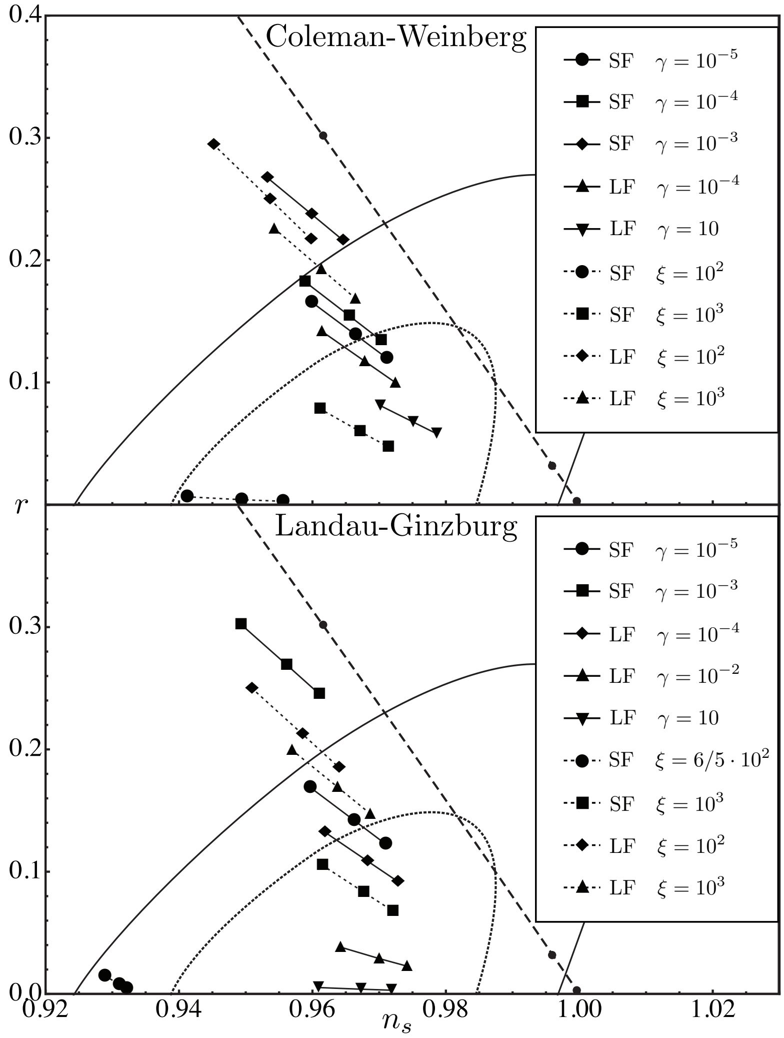


Figure 2.6: We compare predictions of IG (as γ varies) and of General Relativity (as $\xi \equiv \sigma_0^2/M_{\rm Pl}^2$ varies) with the WMAP5+BAO+SN constraints in the (n_s, r) plane. Identical markers identify (from left to right) $N_* = 50, 60, 70$. As usual SF stands for small field dynamics and LF for large field dynamics. The dashed straight line plots the consistency relation and the points on such a line correspond to exact power-law solutions with $n = 6$ and $\gamma = 10^{-2}, 10^{-3}, 10^{-4}$ from left to right.

Chapter 3

(P)reheating

At the end of the scalar field-driven inflation we would be left with an empty and cold Universe unless the residual energy stored in the inflaton were converted into the excitations (particles) of some other field because of some intervening process. In other words, we expect inflation to eventually set up the initial conditions for the subsequent stage of "standard" Big Bang Cosmology, which are that of a very dense and hot Universe.

The link between the highly non-thermal state of the Universe during inflation and the subsequent thermal state is provided by the so-called "reheating" mechanism. Since there is actually a couple of different physical phenomena that come into play during such a process, commonly one makes a precise distinction between preheating and reheating:

Preheating: At the end of inflation the inflaton has still enough energy to undergo rapid oscillations around the minimum/a of its potential. Most of the energy is stored in its zero-mode, meaning that the field oscillates in a coherent way. Moreover, the amplitude of oscillations decreases in time only slightly, because at this stage the only damping term is provided by the Hubble parameter H which, after all, is not that large from the end of inflation on.

As we saw in Section 2.2, the form of the equation governing the evolution of the inflation's fluctuations is similar to that of a (damped) harmonic oscillator whose effective mass (or angular frequency) is time-dependent and ultimately related to the inflation's coherent oscillations. Stated differently, the coherently oscillating inflaton field act as a (roughly periodic) external force which drives its own fluctuations and those of the other fields coupled to the inflaton. In such a configuration it is in principle possible that resonant phenomena occur and, if that is the case, the inflaton fluctuation can be amplified in a really significant way. Furthermore, the inflaton field might even decay into particles that are heavier than it is, which sounds like a very interesting possibility to consider.

Reheating: During the stage which is properly called "reheating", the inflaton field is supposed to be coupled to other fields in an effective way, namely through a dissipation term Γ which enters into its equation of motion. Such an effective coupling mimics all those decay channels of the inflaton into lighter fields. Since, as we will see, the dissipation term Γ appears next to the Hubble parameter H, it only becomes relevant when the Hubble parameter has reached a value such that $H \lesssim \Gamma$. At that point it is also assumed that the energy density of the Universe is mostly stored in a perfect fluid of radiation which accounts for all the relativistic species of matter. It is then possible to evaluate the so-called "reheating temperature" through the Stefan-Boltzmann law.

Let us add that reheating may be thought of as the perturbative final stage of preheating (see Ref. [30]). While referring to Ref. [31] for further details on the theory and phenomenology of (p)reheating, in what follows we address such an important process in the context of Induced Gravity Inflation.

3.1 Coherent oscillations

The dynamics of the inflaton field σ during the regime of coherent oscillations around σ_0 can be approximated analytically using a time-averaging procedure which relies on the fact that the frequency of the scalar field oscillations is much larger than the Hubble parameter which, instead, determines the falloff in time of the amplitude of oscillations. In other words, the dynamics of the field can be approximated with that of an undamped harmonic oscillator, since the modulation in time of the amplitude is a next-to-leading order effect. Therefore, it is acceptable to plug the following *ansatz* into the equation of motion of σ, Eq. (2.3):

$$\sigma(t) = \sigma_0 + A \sin(\omega_0 t) \,. \tag{3.1}$$

On also considering the Friedmann-like equation Eq. (2.1) and on approximating a generic potential $V(\sigma)$ around σ_0 as a massive potential centered at σ_0,

$$V(\sigma) \simeq \frac{m^2}{2}(\sigma - \sigma_0)^2, \tag{3.2}$$

it is indeed possible to show that the Eq. (2.3) reduces to

$$\ddot{\delta\sigma} + \omega_0^2 \delta\sigma = 0 \,, \tag{3.3}$$

where $\delta\sigma \equiv \sigma - \sigma_0$ and

$$\omega_0 = \sqrt{\frac{m^2}{1 + 6\gamma}} \,. \tag{3.4}$$

Let us note that $m^2 = 2\mu\,\sigma_0^2$ for both the Landau-Ginzburg - Eq. (2.76) - and the Coleman-Weinberg - Eq. (2.77) - potentials. In order to unveil the next-to-leading order effects one just has to promote the constant A to a simple function of time:

$$A \to A(t) \propto \frac{1}{t^n}. \tag{3.5}$$

It turns out that the next-to-leading order terms decay as t^{-1}, meaning that in the previous formula it is $n = 1$. In the end it is straightforward to prove that:

$$\sigma(t) = \sigma_0 + \frac{2}{t}\sqrt{\frac{\gamma}{3\mu}}\sin(\omega_0 t) + \mathcal{O}\left(\frac{1}{t^2}\right); \tag{3.6}$$

$$H(t) \simeq \frac{2}{3t}\left[1 - \sqrt{\frac{6\gamma}{1+6\gamma}}\cos(\omega_0 t)\right] + \mathcal{O}\left(\frac{1}{t^2}\right). \tag{3.7}$$

It is interesting to note that oscillations in the Hubble parameter appear already at order t^{-1}, as opposed to what happens in GR with a massive scalar field where such oscillations decay as t^{-2}. Still the overall factor $2/(3t)$ at the leading order in the $1/t$ expansion let us think of a matter-dominated Universe according to an argument that we now briefly discuss.

First of all, a remark is in order. In the GR framework, the 1st and 2nd Friedmann equations read as

$$\begin{aligned} H^2 &= \frac{1}{3M_{\rm Pl}^2}\rho\,, \\ \dot{H} &= -\frac{1}{2M_{\rm Pl}^2}(\rho + p)\,, \end{aligned} \tag{3.8}$$

where ρ and p respectively denote the energy density and the pressure of the matter content of the Universe in the perfect fluid approximation. The ratio between p and ρ,

$$w \equiv \frac{p}{\rho} \tag{3.9}$$

is commonly called "state parameter" and distinguishes between different kinds of matter. For instance, $w = 0$ in the case of non-relativistic matter, while $w = 1/3$ for relativistic matter or radiation. It is easy to check that the following relation connects the Universe's effective equation of state parameter of state to its dynamics:

$$w = -\frac{2}{3}\frac{\dot{H}}{H^2} - 1. \tag{3.10}$$

Quite a long time ago it was realized that, on averaging over oscillations, an oscillating scalar field mimics different kinds of matter depending on the functional form of its potential [32]. More specifically, an oscillating massive scalar field, $V(\varphi) \propto m^2\varphi^2$, makes the Universe effectively expand as if it were matter-dominated ($w_{\rm eff} = 0$), whereas an oscillating massless scalar field, $V(\varphi) \propto \lambda\varphi^4$, behaves like radiation ($w_{\rm eff} = 1/3$).

The presence of oscillations in H at the leading order in the $1/t$ expansion makes the problem at hand more complex than that in GR. For instance (we denote with angular brackets the operation of averaging over oscillations),

$$\langle \dot{H} + \frac{3}{2}H^2 \rangle \neq 0 \tag{3.11}$$

unlike what we would expect in GR with an oscillating massive scalar field.

Nevertheless, it is possible to obtain an analytical estimate of the effective state parameter. The derivation is laid out in what follows. We also managed to obtain a more refined (and more mathematically rigorous) approximation than the one in Eqs. (3.6) and (3.7) through the Multiple Scale Analysis method.

3.1.1 Multiple Scale Analysis (MSA)

The MSA method can be applied when the dynamics of the system under investigation is determined by oscillating behaviors which occur on different timescales. In our case, we can in fact identify two relevant timescales: (1) the frequency of the oscillations of σ around the minimum and (2) the damping rate of their amplitude, the former being much bigger than the latter.

The Friedmann-like equation, Eq. (2.1), can actually be solved for H. On selecting the positive root and then plugging the result into the Klein-Gordon equation, Eq. (2.3), one ends up with:

$$\ddot{\sigma} + \frac{1}{1+6\gamma}\left(\frac{\mathrm{d}V(\sigma)}{\mathrm{d}\sigma} - 4\frac{V(\sigma)}{\sigma}\right) - 2\frac{\dot{\sigma}^2}{\sigma} + \frac{\dot{\sigma}}{\sigma}\sqrt{\frac{3}{2\gamma}\left[2\,V(\sigma) + (1+6\gamma)\,\dot{\sigma}^2\right]} = 0. \tag{3.12}$$

If the potential $V(\sigma)$ has a minimum for $\sigma = \sigma_0$, and we are interested in obtaining a solution valid during the regime of small oscillations around such a minimum, it is convenient to introduce the variable $\delta\sigma = \sigma - \sigma_0$ and to series expand around $\delta\sigma = 0$. Concerning the potential and its first derivative we have:

$$V(\sigma) \equiv \frac{m^2}{2}\delta\sigma^2 + \mathcal{O}\left(\delta\sigma^3\right), \tag{3.13}$$

$$\frac{\mathrm{d}V(\sigma)}{\mathrm{d}\sigma} \equiv m^2\delta\sigma + \frac{\bar{n}}{2}\delta\sigma^2 + \mathcal{O}\left(\delta\sigma^3\right), \tag{3.14}$$

where $\bar{n}$ and m have the same units. The equation of motion for $\delta\sigma$ then reads, up to 2nd-order, as follows:

$$\ddot{\delta\sigma} + \omega_0^2\delta\sigma + \frac{\bar{n} - 4m^2/\sigma_0}{2(1+6\gamma)}\delta\sigma^2 - 2\frac{\dot{\delta\sigma}^2}{\sigma_0} + \frac{\dot{\delta\sigma}}{\sigma_0}\sqrt{\frac{3(1+6\gamma)}{2\gamma}\left[\omega_0^2\,\delta\sigma^2 + \dot{\delta\sigma}^2\right]} = 0. \tag{3.15}$$

Note that due to the presence of friction terms in Eq. (2.3) we can safely argue that such a 2nd-order equation will correctly reproduce the actual dynamics at least after a certain number of oscillations.

Examining Eq. (3.15) we observe that second-order contributions are responsible for the "slow" timescale evolution while the "fast" dynamics is that of a harmonic oscillator with frequency ω_0.

The MSA method tells us how to deal with such two timescales. The presence of a couple of timescales (one "fast", another "slow") can be made explicit as follows:

$$\delta\sigma = \delta\sigma(t,\tau) = \delta\sigma_0(t,\tau) + \epsilon\,\delta\sigma_1(t,\tau) + \mathcal{O}\left(\epsilon^2\right)\,, \tag{3.16}$$

where a fictitious time-coordinate $\tau \equiv \epsilon\, t$ has been introduced (ϵ will be then set to unity at the end of the calculation). On replacing $\delta\sigma/\sigma_0 \to \epsilon\,\delta\sigma/\sigma_0$ in Eq. (3.15) - because $\delta\sigma$ is itself small and we can mark its smallness with the same parameter ϵ introduced above - one is finally led to the following set of partial differential equations for $\delta\sigma_0$ and $\delta\sigma_1$, each one corresponding to different orders in the $\epsilon \ll 1$ expansion:

$$\epsilon^1\,)\quad \delta\ddot{\sigma}_0 + \omega_0^2\delta\sigma_0 = 0 \tag{3.17}$$

$$\begin{aligned}\epsilon^2\,)\quad \delta\ddot{\sigma}_1 + \omega_0^2\delta\sigma_1 &= -2\frac{\partial^2\delta\sigma_0}{\partial t\,\partial\tau} - \frac{\bar{n} - 4m^2/\sigma_0}{2(1+6\gamma)}\delta\sigma_0^2 + 2\frac{\dot{\delta\sigma_0}^2}{\sigma_0} \\ &\quad -\frac{\dot{\delta\sigma_0}}{\sigma_0}\sqrt{\frac{3(1+6\gamma)}{2\gamma}}\left[\omega_0^2\,\delta\sigma_0^2 + \dot{\delta\sigma_0}^2\right] \\ &\;\;\vdots\end{aligned} \tag{3.18}$$

Note that we have implicitly used the following substitution rules:

$$\frac{\mathrm{d}}{\mathrm{d}t} \to \frac{\partial}{\partial t} + \frac{\partial\tau}{\partial t}\frac{\partial}{\partial\tau} = \frac{\partial}{\partial t} + \epsilon\frac{\partial}{\partial\tau}; \tag{3.19}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \to \frac{\partial^2}{\partial t^2} + 2\epsilon\frac{\partial^2}{\partial t\partial\tau} + \epsilon^2\frac{\partial^2}{\partial\tau^2}. \tag{3.20}$$

The standard MSA method consists in writing the general solution for Eq. (3.17) as

$$\delta\sigma_0 = A^*(\tau)\,\mathrm{e}^{i\,\omega_0\,t} + c.c. \tag{3.21}$$

and in determining $A(\tau)$ by requiring the cancellation of secular terms[1] in the next-to-leading order equation, Eq. (3.18) (see Ref. [33] for further details). Applying this

[1]Secular term would be responsible for the unbounded growth of the solution, which we wish to forbid given that the solutions we are willing to approximate are known (from the numerical analysis) to be bounded. Secular terms are easily identifiable: they show up in the r.h.s. of Eq. (3.18) as terms multiplying the leading-order solution ($e^{\pm i\,\omega_0\,t}$ in the case at hand).

procedure one finds out that $A(\tau)$, $A^*(\tau)$ must fulfill the following pair of differential equations:

$$\frac{\mathrm{d}A}{\mathrm{d}\tau} + \sqrt{\frac{3\,(1+6\gamma)}{2\gamma}}\frac{\omega_0}{\sigma_0}\,|A|\,A = 0; \tag{3.22}$$

$$\frac{\mathrm{d}A^*}{\mathrm{d}\tau} + \sqrt{\frac{3\,(1+6\gamma)}{2\gamma}}\frac{\omega_0}{\sigma_0}\,|A|\,A^* = 0. \tag{3.23}$$

Setting $A(\tau) = R(\tau)\,\mathrm{e}^{i\,\theta(\tau)}$ and then summing/subtracting the two equations above and eventually integrating one obtains

$$\theta(\tau) = \theta_0\,; \tag{3.24}$$

$$R(\tau) = \frac{R_0}{1+\frac{R_0}{\sigma_0}\sqrt{\frac{3(1+6\gamma)}{2\gamma}}\,\omega_0\,\tau} = \sigma_0\frac{f\,r}{1+r\,\omega_0\,t}\,, \tag{3.25}$$

where

$$f \equiv \sqrt{\frac{2\gamma}{3\,(1+6\gamma)}}\,, \qquad r \equiv \frac{\Omega_0}{\omega_0} = \frac{R_0}{f\,\sigma_0}\,, \qquad \Omega_0 \equiv R_0\sqrt{3\mu/\gamma}. \tag{3.26}$$

Note that Ω_0 can be thought of as the inverse of the "slow" timescale. Moreover, the ratio r between the two timescales depends on the ratio R_0/σ_0 and on f, where f is a function of γ which takes on values between 0 and $\sqrt{1/3}$. On setting $\sigma_0 = M_{\rm Pl}/\sqrt{\gamma}$ one finds

$$r = \sqrt{\frac{1+6\gamma}{2}}\frac{R_0}{M_{\rm Pl}} \ll 1 \Longrightarrow \frac{R_0}{M_{\rm Pl}} \ll \sqrt{\frac{2}{1+6\gamma}} \tag{3.27}$$

which is precisely the condition required by the MSA method so to provide a good approximation of the dynamics. In the end the general solution for Eq. (3.15) turns out to be as follows:

$$\delta\sigma = \sigma_0\frac{2\,f\,r}{1+r\,\omega_0\,t}\cos\left(\omega_0\,t+\theta_0\right). \tag{3.28}$$

The constants R_0 and θ_0 are related to the initial conditions.

Although the MSA method is usually applied to first-order, thanks to the numerical analysis we realized that our results were significantly improved once that the second-order contributions on the r.h.s. of Eq. (3.18) were also taken into account. These terms, to be added to the r.h.s. of Eq. (3.28), slowly vary in time and amount to a shift of the centre of the oscillations given by

$$\Delta_{\bar{n}} = |A|^2\left(\frac{8}{\sigma_0} - \frac{\bar{n}}{m^2}\right). \tag{3.29}$$

It is straightforward to show that:

$$\bar{n}_{\rm LG} = 6\,\mu\,\sigma_0\,; \tag{3.30}$$

$$\bar{n}_{\rm CW} = 10\,\mu\,\sigma_0\,; \tag{3.31}$$

$$m^2_{\rm LG,CW} = 2\,\mu\,\sigma_0^2\,, \tag{3.32}$$

where, as usual, LG denotes the Landau-Ginzburg potential and CW the one *à la* Coleman-Weinberg.

3.1.2 Average equation of state

A first application of the above results involves the calculation of the average equation of state of the scalar field during the regime of coherent oscillations.

We have already said that the averaged massive oscillations of a scalar field in GR are effectively equivalent to a fluid with null parameter of state. Yet the very same setting in IG leads to somewhat different results.

Let us provide a suitable definition of the energy density and of the pressure of a scalar field in the framework of IG [8]:

$$\rho_\sigma \equiv 3\gamma\sigma_0^2 H^2, \qquad P_\sigma \equiv -2\gamma\sigma_0^2\left(\dot{H} + \frac{3}{2}H^2\right). \tag{3.33}$$

Such a definition matches with the well-known one in GR (once that $\gamma\sigma_0^2 \to M_{\rm Pl}^2$) and also satisfies the following continuity equation, as expected:

$$\dot{\rho}_\sigma + 3H(\rho_\sigma + P_\sigma) = 0. \tag{3.34}$$

Since after the end of inflation all the functions appearing on the r.h.s. of Eq. (3.33) are oscillating , it is convenient to deal with the averaged quantities $\langle\rho_\sigma\rangle$ and $\langle P_\sigma\rangle$. We define

$$\langle A(t)\rangle \equiv \frac{1}{T}\int_{t-T/2}^{t+T/2} A(t')\,\mathrm{d}t', \qquad T = 2\pi/\omega_0. \tag{3.35}$$

Furthermore we define the average equation of state parameter as

$$w \equiv \frac{\langle P_\sigma\rangle}{\langle\rho_\sigma\rangle}. \tag{3.36}$$

Using the solution for $\delta\sigma + \Delta_{\bar{n}}$ and keeping contributions up to the 2nd-order in $R(t)/\sigma_0 \ll 1$, after a lengthy calculations one finds:

$$\langle\rho_\sigma\rangle = (1+9\gamma)\langle\dot{\delta\sigma}^2\rangle \tag{3.37}$$

$$\langle P_\sigma\rangle = -\left[3\gamma - \sqrt{6\gamma\,(1+6\gamma)}\sin\omega_0\,t\right]\langle\dot{\delta\sigma}^2\rangle\,, \tag{3.38}$$

where

$$\langle\dot{\delta\sigma}^2\rangle = 4\,\omega_0^2\,\sigma_0^2\frac{f^2 r^2}{(1+r\,\omega_0\,t)^2}\,. \tag{3.39}$$

Discarding the sinusoidal term in Eq. (3.38) (since it averages out to zero), we finally end up with the following average effective equation of state parameter:

$$\langle w\rangle = -\frac{3\gamma}{1+9\gamma}. \tag{3.40}$$

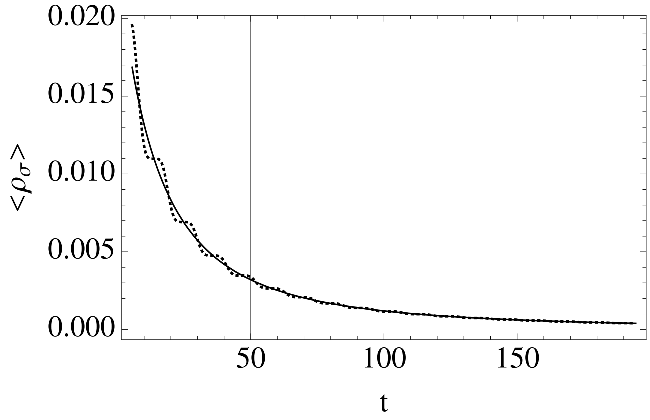


Figure 3.1: An example of the average value of the energy density of the scalar field σ during its massive oscillations around σ_0 with potential (2.76) and parameters set to $\gamma = 10$, $\mu = 10$. The dotted line represents the exact (numerical) ρ_σ compared with its average (solid line) evaluated analytically via the MSA method - cf. Eq. (3.33).

Remarkably enough, such a result does not depend on the fine details of the field's potential. It is just sufficient that the oscillations be "massive".

The analytical results so far presented are largely confirmed by the numerical analysis, as can be clearly seen in Figs. 3.1 and 3.2.

Let us finally observe that, in contrast with what happens in GR, here the average equation of state parameter (3.40) seems to be at odds with the average expansion (3.7) which is that of a matter dominated Universe, since $\langle H(t)\rangle \simeq 2/(3t)$. This apparent discrepancy stems from the fact that the average expansion is given by $\langle H\rangle$, while the computation of the average equation of state involves $\langle H^2\rangle$. Here $\langle H_{(IG)}\rangle \neq \sqrt{\langle H^2_{(IG)}\rangle}$, whereas $\langle H_{(GR)}\rangle \simeq \sqrt{\langle H^2_{(GR)}\rangle}$.

3.2 Preheating

The fluctuations of the inflaton field during the regime of coherent oscillations, as well as those of gravitational waves and of any other field coupled to the inflaton, are described by differential equations which resemble that of a harmonic oscillator with (quasi-periodic) time-dependent frequency. More specifically, through a suitable change of variables the equation of motion of the diverse fields can be recast in a Mathieu-like form:

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}\,(\Omega t)^2} + [A(t) - 2\,q(t)\cos 2\Omega t]\,y(t) = 0 \tag{3.41}$$

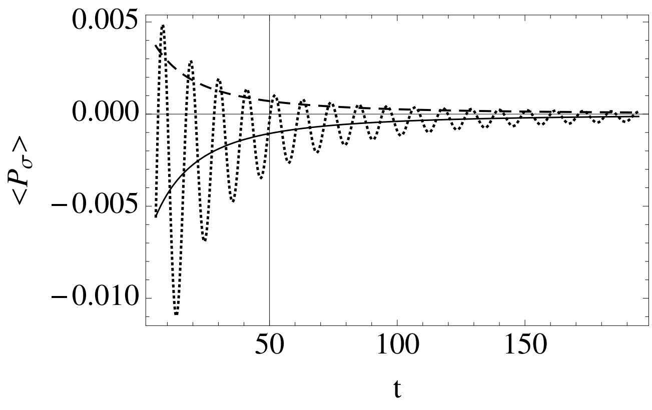


Figure 3.2: We plot the time evolution of the average pressure exhibited by the scalar field σ with potential (2.76) and parameters set to $\gamma = 10$, $\mu = 10$ during its massive oscillations around σ_0. The dotted line represents both the exact pressure of the scalar field as in Eq. (3.33) and its approximation as in Eq. (3.38), since they coincide within the resolution of the figure. The solid line represents the average of Eq. (3.38), whereas the dashed line represents the same quantity calculated without taking the correction term in Eq. (3.29) into account. It is therefore confirmed that the inclusion of such a correction leads to a clear improvement.

where Ω is an angular frequency which may depend on the field under consideration[2]. We refer to such a differential equation as a "Mathieu-like equation" because in the true Mathieu equation both A (referred to as "characteristic value") and q ("parameter") are fixed parameters not depending on time. In our case, instead, both A and q will depend on time because of the cosmic expansion. In particular, we anticipate that $A(t)$ is proportional to the physical wavenumber k/a and that $q(t)$ decays in time. Consequently the fluctuations characterized by a comoving wavenumber k move along some trajectory in the (q, A) plane and in case such a trajectory crosses any of the instability bands of the Mathieu equation plotted in Fig. 3.3 the solution $y(t)$ grows exponentially in time (see also [33]).

The occurrence of parametric resonance may thus lead to a severe amplification of the fluctuations, which is the point we now wish to address in detail. Usually one makes a distinction between the broad resonance regime (dubbed *stochastic resonance* when it takes place in an expanding Universe [30]) which is that occurring for $q \gg 1$ and the intermediate ($q \sim 1$) or narrow ($q \ll 1$) resonance regimes. Let us recall that the

[2]Note that also the solutions of the following equation can be mapped to that of a Mathieu-like equation with $A'(t) = A(t)/4$ and $q'(t) = q(t)/4$:

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}\,(\Omega t)^2} + \left[A'(t) - 2\,q'(t)\cos\Omega t\right] y(t) = 0\,. \tag{3.42}$$

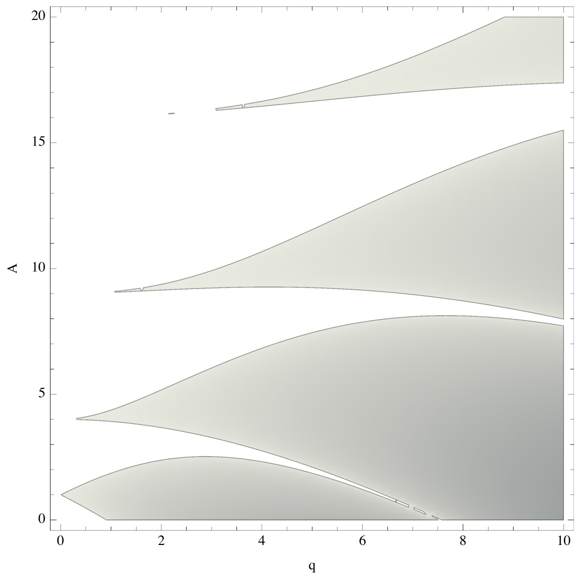


Figure 3.3: Instability chart of the Mathieu equation. The shaded area represents the region of instability and becomes darker as the imaginary part of Mathieu equation's characteristic exponent increases (in modulus).

small-q region of the first instability band (around $A \simeq 1$) is described by the following boundaries:

$$1 - q - \frac{q^2}{8} - \mathcal{O}(q^3) < A < 1 + q + \frac{q^2}{8} + \mathcal{O}(q^3)\,. \tag{3.43}$$

3.2.1 Gravitational waves

We begin with the analysis of gravitational waves. The equation governing the dynamics of gravitational waves, Eq. (2.38), reads as a Mathieu-like equation once that we define $\tilde{h}_k \equiv a^{3/2}\sigma\, h_k$:

$$\frac{\mathrm{d}^2\tilde{h}_k}{\mathrm{d}t^2} + m^2_{\mathrm{eff,t}}\,(t)\,\tilde{h}_k = 0\,, \tag{3.44}$$

where

$$m^2_{\mathrm{eff,t}}\,(t) = \omega_0^2\left(\frac{k^2}{a^2\omega_0^2} - \frac{f\,r}{1 + r\,\omega_0\,t}\cos\omega_0 t\right) + \mathcal{O}\left(\frac{1}{t^2}\right)\,. \tag{3.45}$$

For $r\omega_0 t \gg 1$ this equation can be rewritten as

$$\frac{d^2\tilde{h_k}}{d(\omega_0 t/2)^2} + \left[A_h - 2q_h\cos\left(\omega_0 t\right)\right]\tilde{h_k} = 0 \tag{3.46}$$

where A_h and q_h are time-dependent functions, namely (in what follows we omit terms of order $1/t^2$):

$$A_h(t) = \frac{4k^2}{a^2\omega_0^2}\,, \qquad q_h = \frac{f}{2\omega_0 t}\,, \tag{3.47}$$

ω_0 and f being the same as in Eq. (3.4) and (3.26). The trajectory in the (q, A) plane is

$$A_h(t) \propto k^2 q_h^{4/3}(t)\,, \tag{3.48}$$

meaning that the modes pass through the first resonance band but eventually end in the stability region, leading to no interesting effects.

3.2.2 Inflaton fluctuations

The dynamics of the inflaton fluctuations - cf. Eq. (2.33) - during the regime of coherent oscillations turns out to be governed by the following Mathieu-like equation:

$$\frac{\mathrm{d}^2\tilde{\delta\sigma}_k}{\mathrm{d}t^2} + m^2_{\mathrm{eff}\,,\sigma}\,(t)\;\tilde{\delta\sigma}_k = 0\,, \tag{3.49}$$

where $\tilde{\delta\sigma}_k \equiv \sqrt{a^3 Z}\delta\sigma_k$ and

$$m^2_{\mathrm{eff,s}}\,(t) \equiv \omega_0^2\left[1 - \frac{9\,f\,r}{1 + r\,\omega_0\,t}\cos\omega_0 t\left(1 + \frac{4\sqrt{6\gamma\,(1+6\gamma)}}{9\gamma}\sin\omega_0 t\right)\right] + \mathcal{O}\left(\frac{1}{t^2}\right) \tag{3.50}$$

with f and r as in Eq. (3.26). Thus, in the $r\,\omega_0\,t \gg 1$ regime, Eq. (3.49) becomes

$$\frac{d^2\tilde{\delta\sigma}_k}{d(\omega_0 t)^2} + \left[A_\sigma - 2q_{\sigma,1}\sin\left(2\omega_0 t\right) - 2q_{\sigma,2}\cos\left(\omega_0 t\right)\right]\tilde{\delta\sigma}_k = 0\,, \tag{3.51}$$

where A_σ, $q_{\sigma\,1}$ and $q_{\sigma\,2}$ are time-dependent functions given by (terms of order $1/t^2$ are omitted):

$$A_\sigma = \frac{k^2}{a^2\omega_0^2} + 1\,, \qquad q_{\sigma,1} = \frac{2}{\omega_0 t}\,, \qquad q_{\sigma,2} = \frac{9}{2}\frac{f}{\omega_0 t} = \frac{\sqrt{\frac{27\gamma}{2(1+6\gamma)}}}{\omega_0 t}. \tag{3.52}$$

We thus see that the effective mass squared of the rescaled inflaton fluctuations include two oscillating terms. A numerical study let us discover that the oscillating term multiplied by $q_{\sigma,2}$ can be fairly dropped since it does not affect the behavior of the solution. One is then left with the following trajectory in the (q, A) plane:

$$A_\sigma(t) = 1 + \frac{k^2}{a^2\omega_0^2} = 1 + \frac{k^2}{a_0^2\omega_0^2}\left(\frac{\omega_0 t_0}{2}\right)^{4/3} q_{\sigma,1}^{4/3}(t)\,, \tag{3.53}$$

which shows how fluctuations end in the first resonance band asymptotically, leading to a stably growing solution for $\tilde{\delta\sigma}_k$. The rate of growth can be found analytically by applying, once more, the MSA method.

As far as the first resonance band is concerned, the Mathieu-like equation governing the inflation fluctuations can be cast in the following form:

$$\ddot{y} + [1 + \epsilon\, q\,(t) \sin 2\,t]\, y = 0 \tag{3.54}$$

where ϵ is a small dimensionless parameter. Since $\epsilon\, q\,(t) \ll 1$ one can look for an approximate solution such as

$$y = y_0\,(t, \tau) + \epsilon\, y_1\,(t, \tau) \tag{3.55}$$

where $\tau \equiv \epsilon\, t$. Order by order in the $\epsilon \ll 1$ series expansion one finds the following equations:

$$\epsilon^0\,) \qquad \frac{\partial^2 y_0}{\partial t^2} + y_0 = 0 \Longrightarrow y_0 = A\,(\tau)\; \mathrm{e}^{i\,t} + \text{c.c.} \tag{3.56}$$

$$\epsilon^1\,) \qquad \frac{\partial^2 y_1}{\partial t^2} + y_1 + 2\frac{\partial^2 y_0}{\partial t \partial \tau} + q\,(t)\, \frac{\mathrm{e}^{2\,i\,t} - \mathrm{e}^{-2\,i\,t}}{2\,i} y_0 = 0 \tag{3.57}$$

$$\vdots$$

The requirement that secular contributions cancel out in Eq. (3.57) leads to the following pair of differential equations:

$$4\frac{\mathrm{d}A}{\mathrm{d}t} - q\,(t)\, A^* = 0\,, \qquad 4\frac{\mathrm{d}A^*}{\mathrm{d}t} - q\,(t)\, A = 0\,. \tag{3.58}$$

Splitting A into its real and imaginary parts, $A = B + i\,C$, the growing and decaying modes also separate:

$$4\frac{\mathrm{d}B}{\mathrm{d}t} - q\,(t)\, B \;=\; 0\,, \tag{3.59}$$

$$4\frac{\mathrm{d}C}{\mathrm{d}t} + q\,(t)\, C \;=\; 0\,. \tag{3.60}$$

The imaginary part decays for any $q\,(t) > 0$. Usually the expansion of the Universe during reheating leads to $q\,(t) \sim p/t^n$, and in that case the growing mode $B(t)$ has the following general solution:

$$B\,(t) = B_0 \exp \int_{t_0}^{t} \frac{p}{4\,\tilde{t}^n} \mathrm{d}\tilde{t}\,. \tag{3.61}$$

Three different scenarios are possible depending on the value of n:

1. If $n > 1$ the amplitude of oscillations increases in time but asymptotically tends to $B \sim B_0 \exp\left[\frac{p}{4(n-1)} t_0^{1-n}\right]$.

2. If $n < 1$ the amplitude increases exponentially as $B \sim B_0 \exp\left[\frac{p}{4(1-n)} t^{1-n}\right]$. This case also covers that of a true Mathieu equation (q = constant), since the latter would simply correspond to $n = 0$.

3. If $n = 1$ one obtains the following power-law solution:

$$B = B_0 \left(\frac{t}{t_0}\right)^{p/4} . \tag{3.62}$$

The case of the rescaled inflaton fluctuations $\delta\tilde{\sigma}_k$ exactly corresponds to the third one in the list above with $p = 4$, meaning that $\delta\tilde{\sigma}_k \sim t$. As a consequence the inflaton fluctuations $\delta\sigma_k$ oscillate with constant amplitude, rather than decaying as $a(t)^{-3/2}$ as it would do in absence of resonant effects.

Let us finally observe that, since $q_{\sigma,1}$ does not depend on γ, our analysis also applies to the coherent oscillations regime of chaotic inflation in General Relativity, in agreement with Ref. [34]. Therefore, during such a regime the gauge-invariant inflaton fluctuations oscillate with constant amplitude not only on large scales but also on small scales [35].

3.2.3 Scalar test-field

We finally consider the evolution of a scalar test-field χ non-minimally coupled to gravity and interacting with the inflaton. Let

$$S_\chi = \int d^4x\sqrt{-g}\left[-\frac{g^{\mu\nu}}{2}\partial_\mu\chi\partial_\nu\chi - \frac{m_\chi^2}{2}\chi^2 + \mathcal{L}_{\rm int}\right] \tag{3.63}$$

be the action for such a field and

$$\mathcal{L}_{\rm int} = -\frac{\xi}{2}R\chi^2 + \frac{g^2}{2}\sigma^2\chi^2 \tag{3.64}$$

the interaction Lagrangian with $\xi > 0$. On considering only the interaction of the modes of the field χ with the homogeneous part of the inflaton one eventually obtains

$$\frac{\mathrm{d}^2\tilde{\chi}_k}{\mathrm{d}t^2} + m^2_{\mathrm{eff},\chi}(t)\,\tilde{\chi}_k = 0\,, \tag{3.65}$$

where

$$\begin{aligned} m^2_{\mathrm{eff},\chi}(t) = \omega_0^2 \Bigg[& \frac{k^2}{a^2\omega_0^2} + \frac{3\,(4\xi-1)\ f\,r}{1+r\,\omega_0\,t}\cos\omega_0 t + 2g^2\frac{1+6\gamma}{\mu}\frac{f\,r}{1+r\,\omega_0\,t}\cos\omega_0 t \\ & + \frac{m_\chi^2}{\omega_0^2} + g^2\frac{1+6\gamma}{2\mu}\Bigg] + \mathcal{O}\left(\frac{1}{t^2}\right) \end{aligned} \tag{3.66}$$

and $\tilde{\chi}_k = a^{3/2}\chi_k$. From Eq. (3.66) we see that the scalar field does not end up in the first instability band ($A \simeq 1 \Leftrightarrow \tilde{A} \simeq 1/4$) unless

$$\frac{m_\chi^2}{\omega_0^2} + g^2\frac{1+6\gamma}{2\mu} = \frac{1+6\gamma}{2\mu}\left(\frac{m_\chi^2}{\sigma_0^2} + g^2\right) \simeq \frac{1}{4}, \tag{3.67}$$

meaning that the occurrence of resonant effects strongly depends on the values of several parameters, thus is not generic at all. Note also that a broad resonance regime is possible for $m_\chi = 0 = g$ and $\xi \gg 1$.

3.3 Perturbative decay

The perturbative decay of the inflation field into the excitations of lighter fields is usually modeled with a phenomenological decay-width Γ which enters into the equation of motion for the scalar field as an additional friction term:

$$\ddot{\sigma} + (3H + \Gamma)\,\dot{\sigma} + \frac{\mathrm{d}V}{\mathrm{d}\sigma} - 6\gamma\left(H^2 + \frac{\ddot{a}}{a}\right)\sigma = 0. \tag{3.68}$$

The energy loss due to the newly added friction term is supposed to be transferred to a perfect fluid of relativistic matter ($\sim$ radiation), whose continuity equation should read as

$$\dot{\rho}_R = -3H\,(\rho_R + P_R) + \Gamma\dot{\sigma}^2 = -4H\rho_R + \Gamma\dot{\sigma}^2 \tag{3.69}$$

in order for Eq. (3.68) to be consistent with the 1st and 2nd Friedmann equations which now also include relativistic matter [27]:

$$\begin{aligned} H^2 &= \frac{1}{3\gamma\sigma^2}\left(\frac{\dot{\sigma}^2}{2} + V + \rho_R\right) - 2H\frac{\dot{\sigma}}{\sigma}; \\ \dot{H} &= -\frac{1}{2\gamma\sigma^2}\,(\rho_R + P_R) - \frac{1+2\gamma}{2\gamma}\frac{\dot{\sigma}^2}{\sigma^2} + H\frac{\dot{\sigma}}{\sigma} - \frac{\ddot{\sigma}}{\sigma}. \end{aligned} \tag{3.70}$$

Assuming that for $\sigma = \sigma_0$ the Friedmann equations assume the same form as in GR with $M_{\mathrm{Pl}}^2 \to \gamma\sigma_0^2$, the energy density and pressure associated to the scalar field are easily identifiable as follows (cf. Eq. (3.8)):

$$\rho_\sigma = 3\gamma\sigma_0^2 H^2 - \rho_R; \tag{3.71}$$

$$P_\sigma = -2\gamma\sigma_0^2\left(\dot{H} + \frac{3}{2}H^2\right) - \rho_R - P_R. \tag{3.72}$$

With these definitions the continuity equation for ρ_σ, P_σ is still automatically preserved but the introduction of the decay-width Γ yields an extra term:

$$\dot{\rho}_\sigma = -3H\,(\rho_\sigma + P_\sigma) - \Gamma\dot{\sigma}^2. \tag{3.73}$$

Let us note that Eqs. (3.69) and (3.73) are formally the same as in GR, yet we will shortly see that they lead to different predictions.

We now proceed by looking for the average evolution of ρ_σ and ρ_M so to be able to provide an estimate of the reheating temperature. We anticipate that usually such a temperature is found by assuming that the energy stored in the coherently oscillating inflaton is converted into radiation *instantaneously* at the moment in which $3H \sim \Gamma$. Then one equals the energy density of the scalar field to that of the perfect fluid of radiation and finally the reheating temperature is found as $T_{\rm RH} \sim \rho_R^{1/4}$, according to the Stefan-Boltzmann law. In the framework of GR one finds the well-known estimate $T_{RH}^{(GR)} \sim \sqrt{\Gamma M_{\rm Pl}}$ [4].

Basically we now have to extend the computation already laid out in Section 3.1.1 so to also include radiation and the extra dissipative term Γ. We thus start from the equation of motion for σ which is obtained from Eq. (3.68) together with Eq. (3.70),

$$\ddot{\sigma}+\frac{1}{1+6\gamma}\left(\frac{{\rm d}V(\sigma)}{{\rm d}\sigma}-4\frac{V(\sigma)}{\sigma}\right)-2\frac{\dot{\sigma}^2}{\sigma}+\frac{\dot{\sigma}}{\sigma}\sqrt{\frac{3}{2\gamma}\left[2\,V(\sigma)+(1+6\gamma)\,\dot{\sigma}^2+2\,\rho_R\right]}+\frac{\Gamma}{1+6\gamma}\,\dot{\sigma}=0,$$

then expand up to 2nd-order in $\delta\sigma \equiv \sigma-\sigma_0$ and apply the MSA method to solve it:

$$\begin{aligned}\ddot{\delta\sigma} \quad &+ \quad \omega_0^2\delta\sigma+\frac{\bar{n}-4m^2/\sigma_0}{2(1+6\gamma)}\delta\sigma^2-2\frac{\dot{\delta\sigma}^2}{\sigma_0}+\frac{\Gamma}{1+6\gamma}\dot{\delta\sigma}+\frac{\dot{\delta\sigma}}{\sigma_0}\\ &\times \quad \sqrt{\frac{3(1+6\gamma)}{2\gamma}\left[\omega_0^2\,\delta\sigma^2+\dot{\delta\sigma}^2+\frac{2\,\rho_R}{1+6\gamma}\right]}=0. \end{aligned} \tag{3.74}$$

We can assume that $\rho_R \sim V(\sigma) \sim \dot{\delta\sigma}^2$ and that $\Gamma \sim H \sim \dot{\delta\sigma}/\sigma_0$, as confirmed by the numerical analysis.

The "fast" time evolution is still determined by the leading order terms $\ddot{\delta\sigma}_0+\omega_0^2\,\delta\sigma_0=0$. Setting $\delta\sigma_0 = A\,(\tau)\;y(t)$ and requiring the cancellation of secular terms one obtains the following differential equation and its complex conjugate:

$$2\dot{A}+\frac{\Gamma}{1+6\gamma}\,A+3\sqrt{\frac{\rho_R+(1+6\gamma)\,A^2\,E_F}{3\gamma\sigma_0^2}}A=0\,, \tag{3.75}$$

where

$$E_F \equiv \frac{1}{2}\dot{y}^2+\frac{\omega_0^2}{2}y^2, \tag{3.76}$$

and, consequently,

$$\langle\dot{\delta\sigma}^2\rangle = A^2 E_F\,. \tag{3.77}$$

Keeping the 2nd-order contributions in the continuity equation for relativistic matter, Eq. (3.69), and averaging over the "fast" oscillations of y one finds

$$\dot{\rho}_R-\Gamma\,A^2\,E_F+4\sqrt{\frac{\rho_R+(1+6\gamma)\,A^2\,E_F}{3\gamma\sigma_0^2}}\rho_R=0\,. \tag{3.78}$$

After a quick comparison with Eq. (3.69) we see that the average Hubble parameter is given by

$$\langle H\rangle = \sqrt{\frac{\rho_R+(1+6\gamma)\,A^2\,E_F}{3\gamma\sigma_0^2}}, \tag{3.79}$$

whereas the averaged 1st Friedmann equation let us estimate the inflaton average energy density as follows:

$$\langle\rho_\sigma\rangle = A^2 E_F\,(1+9\gamma)\,. \tag{3.80}$$

Finally we are left with the following system of coupled average equations:

$$\dot\rho_R = -4\langle H\rangle\rho_R + \frac{\Gamma}{1+9\gamma}\langle\rho_\sigma\rangle\,; \tag{3.81}$$

$$\langle\dot\rho_\sigma\rangle = -3\langle H\rangle\langle\rho_\sigma\rangle - \frac{\Gamma}{1+6\gamma}\langle\rho_\sigma\rangle\,; \tag{3.82}$$

$$\langle H\rangle = \sqrt{\frac{\rho_R + \frac{1+6\gamma}{1+9\gamma}\langle\rho_\sigma\rangle}{3\gamma\sigma_0^2}}\,, \tag{3.83}$$

where

$$\frac{\mathrm{d}\langle\rho_\sigma\rangle}{\mathrm{d}t} = \langle\dot\rho_\sigma\rangle. \tag{3.84}$$

The equations above differ from those arising in GR for an oscillating massive inflaton because of the extra factors $(1+9\gamma)^{-1}$ and $(1+6\gamma)^{-1}$ in both Eqs. (3.81) and (3.82) which make the energy transfer less efficient.

In Fig. 3.4 we compare the energy transfer obtained by the numerical exact solution of Eqs. (3.68) and (3.69) with that calculated by solving numerically the average equations (3.81-3.83). The continuous line represents

$$\Omega_\sigma \equiv \frac{\langle\rho_\sigma\rangle}{\rho_R + \langle\rho_\sigma\rangle}\,, \tag{3.85}$$

while the dashed one stands for

$$\Omega_R \equiv \frac{\rho_R}{\rho_R + \langle\rho_\sigma\rangle}\,. \tag{3.86}$$

Those plots prove that the assumptions we made are definitely reasonable. A second numerical check is provided in Fig. 3.5.

3.3.1 The reheating temperature

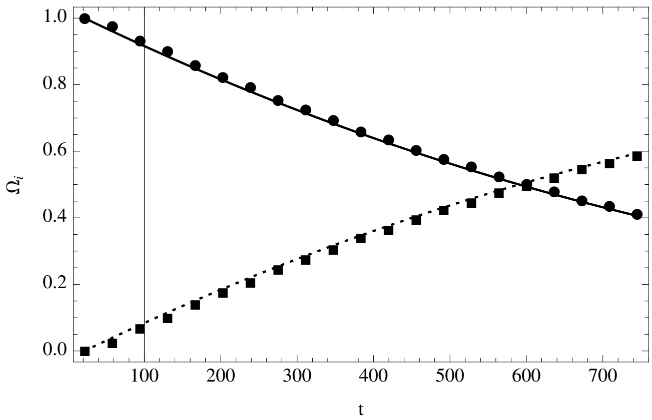
At the onset of reheating the averaged Friedmann equation (3.83) can be approximated as

$$\langle H\rangle \simeq \sqrt{\frac{\frac{1+6\gamma}{1+9\gamma}\langle\rho_\sigma\rangle}{3\gamma\sigma_0^2}} \tag{3.87}$$

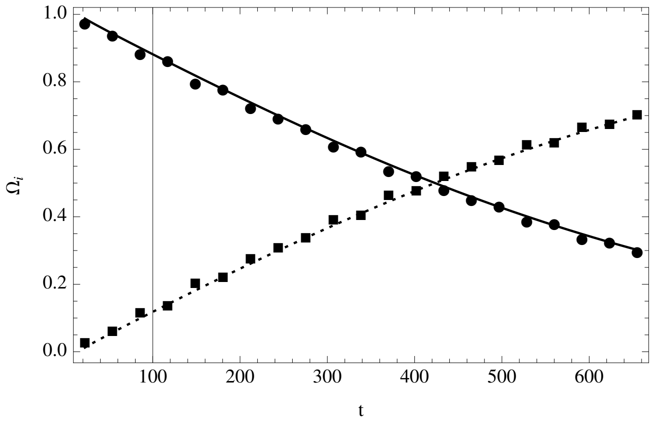
since $\rho_R \ll \langle\rho_\sigma\rangle$. The system of equations (3.81), (3.82), (3.87) can then be solved also analytically. Having defined

$$\tilde\rho_\sigma \equiv \frac{1+6\gamma}{1+9\gamma}\langle\rho_\sigma\rangle\,, \tag{3.88}$$

$$\tilde\Gamma \equiv (1+6\gamma)^{-1}\,\Gamma\,, \tag{3.89}$$

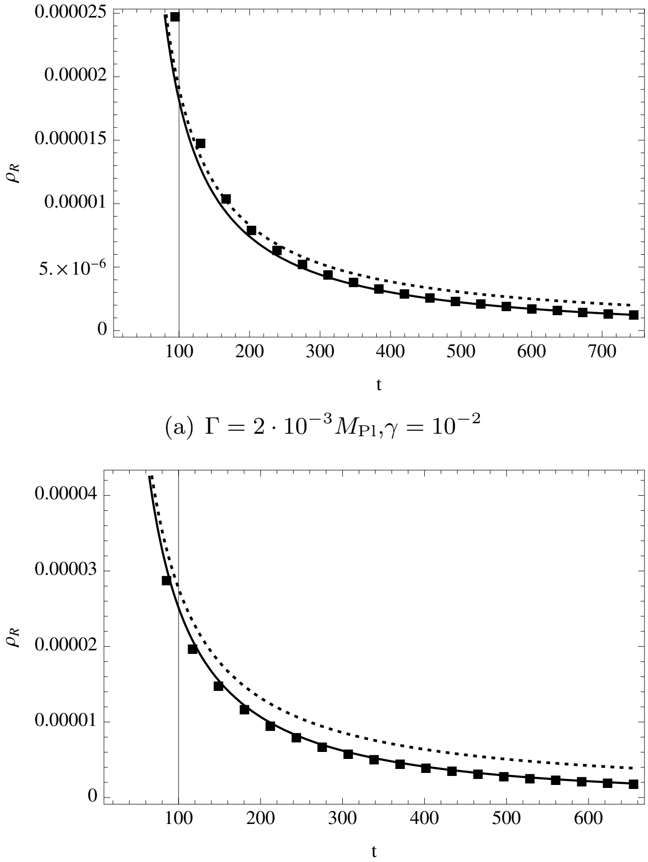


(a) $\Gamma = 2 \cdot 10^{-3} M_{\rm Pl}$, $\gamma = 10^{-2}$



(b) $\Gamma = 2 \cdot 10^{-1} M_{\rm Pl}$, $\gamma = 10$

Figure 3.4: In these figures the solid line is for Ω_σ - defined in Eq. (3.85) - while the dotted line is for Ω_R - defined in Eq. (3.86) - when both are calculated as numerical solutions of the average Eqs. (3.81-3.83). Circles and squares, instead, represent the same quantities obtained by solving the exact Eqs. (3.74), (3.69) and then by averaging ρ_σ as given by Eq. (3.71). The time evolution is expressed in the units of ${M_{\rm Pl}}^{-1}$.



(a) $\Gamma = 2 \cdot 10^{-3} M_{\rm Pl}, \gamma = 10^{-2}$

(b) $\Gamma = 2 \cdot 10^{-1} M_{\rm Pl}$, $\gamma = 10$

Figure 3.5: In the above figures the continuous line is ρ_R (in the units of ${M_{\rm Pl}}^4$) as a function of t (in the units of ${M_{\rm Pl}}^{-1}$) obtained by solving numerically the average Eqs. (3.81-3.83). The squares represent the same quantity evaluated by solving the exact Eqs. (3.74), (3.69) and the dotted line is ρ_R calculated as a numerical solution of Eqs. (3.90), (3.91).The plots show that the assumptions leading to the expression (3.94) yield good approximate results regardless of the value of γ.

Eqs. (3.81), (3.82) read as

$$\frac{\mathrm{d}\rho_R}{\mathrm{d}t} = -4\sqrt{\frac{\tilde{\rho}_\sigma}{3\gamma\sigma_0^2}}\,\rho_R + \tilde{\Gamma}\,\tilde{\rho}_\sigma\,, \tag{3.90}$$

$$\frac{\mathrm{d}\tilde{\rho}_\sigma}{\mathrm{d}t} = -3\sqrt{\frac{\tilde{\rho}_\sigma}{3\gamma\sigma_0^2}}\,\tilde{\rho}_\sigma - \tilde{\Gamma}\,\tilde{\rho}_\sigma\,, \tag{3.91}$$

exactly as in GR with a minimally coupled massive scalar field. As a consequence of this formal analogy one expects $\tilde{\rho}_\sigma \sim \rho_R$ when $\sqrt{3\tilde{\rho}_\sigma} \simeq \Gamma M_{\mathrm{Pl}}/\left(1+6\gamma\right)$ or, equivalently, when $t \sim \tilde{\Gamma}^{-1}$. Eq. (3.91) can easily be solved and leads to:

$$\langle\rho_\sigma\rangle = \frac{\gamma\sigma_0^2}{3}\,\frac{1+9\gamma}{\left(1+6\gamma\right)^3}\,\frac{\Gamma^2}{\left[C_0\exp\left(\frac{\Gamma}{2(1+6\gamma)}t\right)-1\right]^2} \tag{3.92}$$

where C_0 is an integration constant which depends on the initial energy density $\langle\rho_\sigma\rangle_0$ stored in the scalar field at $t=0$ when, in our notation, the energy transfer is supposed to begin:

$$C_0 = 1 + \frac{\Gamma}{3}\sqrt{\frac{3\gamma\sigma_0^2}{\langle\rho_\sigma\rangle_0}\cdot\frac{1+9\gamma}{\left(1+6\gamma\right)^3}} \tag{3.93}$$

Also Eq. (3.90) can be solved and, setting $\rho_R\left(t=0\right)=0$, one obtains:

$$\rho_R = \frac{\Gamma^2\,\gamma\,\sigma_0^2}{20\left(1+6\gamma\right)^2}\,\frac{\left[\left(\mathrm{e}^{\frac{\Gamma t}{2(1+6\gamma)}}B_0-1\right)^{5/3}\left(3\,\mathrm{e}^{\frac{\Gamma t}{2(1+6\gamma)}}B_0+5\right)-b_0^{5/3}\mathrm{e}^{\frac{4\Gamma t}{3(1+6\gamma)}}\left(3\,b_0+8\right)\right]}{\left(\mathrm{e}^{\frac{\Gamma t}{2(1+6\gamma)}}B_0-1\right)^{8/3}}\,, \tag{3.94}$$

where

$$B_0 = 1 + \Gamma\sqrt{\frac{\gamma\,\sigma_0^2}{3\langle\rho_\sigma\rangle_0}\cdot\frac{1+9\gamma}{\left(1+6\gamma\right)^2}} \tag{3.95}$$

and $b_0 = B_0 - 1$. We now set $\gamma\,\sigma_0^2 = {M_{\mathrm{Pl}}}^2$ and note that

$$b_0 = \frac{\Gamma\,M_{\mathrm{Pl}}}{\langle\rho_\sigma\rangle_0^{1/2}}\sqrt{\frac{1+9\gamma}{3\left(1+6\gamma\right)^3}} \simeq \mathcal{O}\left(\frac{\Gamma}{\langle H\rangle_0}\right) \ll 1 \tag{3.96}$$

since it is generally assumed that $\Gamma \ll \langle H\rangle_0$, where $\langle H\rangle_0$ is the average value of H at the beginning of reheating. The formula for the energy density of relativistic matter (3.94) then simplifies and becomes

$$\rho_R \simeq \frac{\Gamma^2\,\gamma\,\sigma_0^2}{20\left(1+6\gamma\right)^2}\cdot\frac{3\,\mathrm{e}^{\frac{\Gamma\,t}{2(1+6\gamma)}}+5}{\mathrm{e}^{\frac{\Gamma\,t}{2(1+6\gamma)}}-1}. \tag{3.97}$$

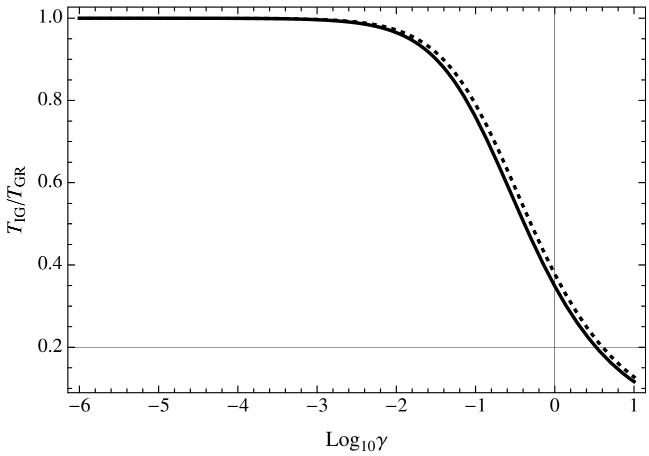


Figure 3.6: The continuous line represents the ratio $T_{\rm reh}^{(IG)}/T_{\rm reh}^{(GR)}$ as is evaluated numerically on using the exact equations for the time of equality t_*. The dotted line represents the analytical estimate of the same ratio obtained from Eqs. (3.98), (3.99) as $T_{\rm reh}^{(IG)}/T_{\rm reh}^{(GR)} = (1+6\gamma)^{-1/2}$.

Evaluating ρ_R at $t \sim (1+6\gamma)\,\Gamma^{-1}$ one finally obtains:

$$\rho_R\left((1+6\gamma)\,\Gamma^{-1}\right) \simeq \frac{3\,\Gamma^2\,{M_{\rm Pl}}^2}{(1+6\gamma)^2}\,, \tag{3.98}$$

while the same calculations in GR leads to

$$\rho_R^{(GR)}\left(\Gamma^{-1}\right) \simeq 3\,\Gamma^2\,{M_{\rm Pl}}^2\,. \tag{3.99}$$

In Fig. (3.6) we plot the ratio between the reheating temperatures in IG and in GR as γ varies. Such a ratio is computed as follows:

$$\frac{T_{\rm reh}^{(IG)}}{T_{\rm reh}^{(GR)}} = \left[\frac{\rho_R^{(IG)}\,(t_{*,IG})}{\rho_R^{(EG)}\,(t_{*,GR})}\right]^{1/4}\,, \tag{3.100}$$

where $\rho_R^{(IG)}$ and $\rho_R^{(EG)}$ are the numerical solutions of the exact equations and t_* is such that $\rho_R\,(t_*) = \rho_\sigma\,(t_*)$. For comparison we also plot the analytical prediction obtained from Eq. (3.98) and (3.99), observing once more that our analytical estimates well reproduce the exact behavior.

Chapter 4

Conclusions (I)

We have investigated in great detail the topic of inflation in the IG framework, where inflation is driven by the same scalar field supposed to generate, dynamically, the measured value of Newton's constant. Inflation takes place in the slow-roll regime as in General Relativity and leads to a nearly flat spectrum of scalar perturbations and a small tensor-to-scalar ratio, in full agreement with observations. However, the slow-roll conditions are not simply associated with the shape of the inflaton potential since the dynamics of the scalar field strongly depends on its coupling γ to gravity. We have made accurate comparisons with observations for different "symmetry-breaking" potentials and discovered that some of them lead to acceptable predictions whereas others do not. In particular, we have seen that the Landau-Ginzburg and Coleman-Weinberg potentials are compatible with observational data in the case of large field inflation and, for suitable values of the parameters, also in the case of small field inflation. The agreement with observations for potentials involving higher powers of the scalar field (higher than in the already mentioned cases) is more problematic and, if attainable, requires constraints on γ both in the case of large and small field dynamics.

We have also studied the post-inflationary regime of coherent oscillations, obtaining via the Multiple Scale Analysis method an accurate analytical solution for the dynamics of the inflaton and of the Hubble parameter. We employed the solution to first investigate the evolution of the Universe once that oscillations are averaged out, discovering that its expansion is characterized by an average Hubble parameter $\langle H(t)\rangle = 2/(3t)$, exactly as in the case of a matter-dominated Universe although the average equation of state parameter is not exactly zero - cf. Eq. (3.40). Then, we have studied both the resonant and perturbative reheating, finding out that parametric resonance makes the inflaton small-scale scalar fluctuations oscillate with constant amplitude instead of decaying as the Universe expands. Tensor perturbations (gravitational waves), instead, are not affected by resonant effects. Concerning the evolution of the fluctuations of a generic test-field coupled to the inflaton and to gravity (in a non-minimal way), we have seen that the occurrence of parametric resonance strongly depends on the parameters of the test-field's Lagrangian.

Finally, we have found that the value of γ may actually tweak the efficiency of the energy transfer from the inflaton into ordinary matter during perturbative reheating, making it less efficient the larger γ is. Moreover, the reheating temperature found in the IG framework is comparable to that in GR only for small γ's, otherwise the former is smaller.

Part II

Hořava-Lifshitz Gravity

Chapter 5

Introduction

Hořava-Lifshitz Gravity is a proposal for a candidate theory for quantum gravity which was first presented in a paper by Prof. Petr Hořava back in January 2009 [36]. Being it based on the anisotropic scaling between space and time as is the case of a class of condensed-matter models whose prototype is just the theory of a Lifshitz scalar [12], the theory was soon dubbed "Hořava-Lifshitz Gravity". More specifically, the theory is required to be invariant under the following transformation,

$$\begin{cases} \vec{x} & \rightarrow & b\,\vec{x} \\ t & \rightarrow & b^z t \end{cases} , \tag{5.1}$$

where z is a "critical exponent", as well as under parity transformation ($\vec{x} \rightarrow -\vec{x}$) and time reversal ($t \rightarrow -t$). The other fundamental building block of the theory is the power-counting renormalizability with respect to such an anisotropic scaling. Later on we will point out that the critical exponent z should be set equal to the number D of spatial dimensions in order to let the free theory be power-counting renormalizable. In what follows we will stick to the case $D = 3$. It is then clear that Lorentz invariance is explicitly broken, being $z \neq 1$. Nevertheless it is desired that z flow to 1 in the infrared, thereby restoring Lorentz symmetry at low energies. Unfortunately, it has not yet been shown explicitly whether and how this might occur (see Refs. [37, 38, 39] for a study of the renormalization group flow in scalar field theories of Lifshitz type).

The basic fields in Hořava-Lifshitz (HL) gravity are the same as in GR. However, there are less symmetries. More specifically, one loses the gauge symmetry consisting on space-dependent time rescalings. The theory is indeed invariant only under "foliation preserving diffeomorphisms":

$$x^k \rightarrow \tilde{x}^k = \tilde{x}^k(t, x^i)\,, \qquad t \rightarrow \tilde{t} = \tilde{t}(t)\,. \tag{5.2}$$

As a consequence there might be more dynamical degrees of freedom of gravitational origin than in GR. Such extra degrees of freedom are likely to cause serious problems for the theory in case they are unstable. In what follows we will address such a point in a pair of possible variants of the theory, the so-called "projectable" and "non-projectable" versions, defined below.

Given that in HL Gravity space and time are not on an equal footing it is convenient to adopt the Arnowitt-Deser-Misner (ADM) decomposition of the metric [40]:

$$ds^2 = -N^2 c^2 dt^2 + g_{ij} \left(dx^i + N^i dt\right) \left(dx^j + N^j dt\right) , \tag{5.3}$$

where t is the physical time, the x^i coordinates $(i = 1, ..., 3)$ are comoving spatial coordinates, and g_{ij} is the metric on the constant time hypersurfaces. The gravitational dynamical degrees of freedom are the lapse function N, the shift vector N^i and the spatial metric g_{ij}. In principle N, N^i and g_{ij} can be functions of both space and time, unless one introduces additional conditions. In particular, the "projectability condition" precisely requires N to be a function of time only, *i.e.* $N \equiv N(t)$.

Whether or not the projectability condition should be retained is a point which has been extensively debated in the literature. For instance, we mention that the projectability condition is supported in Ref. [41] because it prevents a non-relativistic theory of gravity from developing inconsistencies, whereas, on the other side, it is opposed in Ref. [42] because it gives rise to a non-local Hamiltonian constraint, potentially making it harder to recover GR (in which the constraint is local) in the infrared limit. Another reason in favor of imposing the projectability condition is that the algebra of constraints appears consistent only if this condition is imposed [43]. We refer to Ref. [44] for a more detailed discussion on these points.

HL Gravity has attracted the attention of lots of cosmologists for several reasons. It may lead to new solutions of some old cosmological problems (see *e.g.* [45, 46, 47, 48]) by providing alternatives to the inflationary paradigm of the very early Universe. In particular, in the case of a spatially curved background it is natural to obtain a bouncing cosmology [49, 50, 51] and, with the addition of scalar field matter, HL Gravity can yield a scale-invariant spectrum of curvature perturbations in the ultraviolet limit even for a non-inflationary expansion of the background spacetime [46].

At the same time, the emergence of an unwanted extra degree of freedom in HL Gravity has been studied in a number of works, but most of them only considered linear perturbations about flat spacetime, *i.e.* in the absence of matter (see *e.g.* [52, 53, 42, 44, 54, 55, 56, 57, 58, 59, 60]), instead of taking into full account the presence of matter and a proper cosmological background metric. Interestingly, in Ref. [61] it was found that in the non-projectable version of the theory the extra scalar gravitational mode is not dynamical when expanding about a Friedmann-Robertson-Walker (FRW) background metric in the presence of matter[1]. This conclusion holds both for flat and for curved spatial

[1]However, the later work of Ref. [62] indicates that the dangerous extra mode will appear at higher order in perturbation theory, and also at linear order if the background is not maximally symmetric.

sections [63]. Nevertheless, in Refs. [61, 63] not all the possible terms compatible with the power-counting renormalizability criterion were actually included in the Lagrangian, simply because of "historical" reasons (in fact only later it was understood how the full Lagrangian of the non-projectable version might have looked like).

Our approach can be briefly summarized as follows: after having laid out the full action of the gravity sector and of scalar matter, both in the projectable and non-projectable cases, we will expand the variables up to first-order and look for the action which is quadratic in those linear perturbations. We will make use of all the available constraint equations and take all the symmetries into account so as to isolate the actual physical degrees of freedom. A degree of freedom will be judged "dynamical" according to whether it enters the action with a kinetic term or not. Finally, the study of the coefficient matrices of both the kinetic part of the Lagrangian and of the mass terms will let us identify potential ghost-like or tachyonic states. The study will only concern scalar modes (vector and tensor perturbations are studied in Ref. [64]).

We point out that our conclusions can also be drawn through a different method which involves the identification of the constraint algebra and a suitable classification of the constraints as in the Dirac's method (or Hamiltonian formalism, cf. Refs. [65]) or through the Faddeev-Jackiw's Hamiltonian reduction approach [66]. We refer to Refs. [64, 67] for studies of Hořava-Lifshitz Gravity in such a spirit.

Let us add that there are several other concerns about fluctuations in HL Gravity which go beyond the scope of the study here reported. One is the issue of strong coupling, which would spoil the recovery of GR in the infrared limit [42]. In the infrared limit indeed the couplings of the extra degree of freedom diverge, indicating that the cosmological perturbation theory breaks down. For further details we refer to Refs. [68, 69, 59, 70, 71, 72].

This second part of the thesis is organized as follows:

- In the next section we will show how to build from basic principles the most general action for Hořava-Lifshitz Gravity plus scalar matter. We will then derive the two available constraints (Hamiltonian and super-momentum) and introduce the cosmological setup. Such considerations are common to both the versions of the theory, projectable and non-projectable.

- Cosmological perturbations will be considered separately for the projectable version, Chapter 6, and for the non-projectable one, Chapter 7. In both cases we will look for the action which is quadratic in the linear cosmological fluctuation variables and, after having made use of the available gauge freedom and of the constraint equations, we will count the actual number of propagating degrees of freedom by inspecting the kinetic terms in the action. We will eventually prove that the theory admits an unwanted scalar mode of gravitational origin which is potentially unstable, hence the eligibility of the theory as the ultimate theory of quantum gravity may be severely compromised.

- Chapter 8 is devoted to conclusions.

5.1 Hořava-Lifshitz Gravity plus scalar matter

The construction of the most general action of Hořava-Lifshitz Gravity relies on the requirement that the theory be power-counting renormalizable with respect to the scaling symmetry (5.1). As reviewed *e.g.* in Ref. [73], to render the free HL theory of gravity power-counting renormalizable in D spatial dimensions we need to set $z = D$ (see also Ref. [74] for a more general analysis). Given the choice of z, one then builds up the action by adding all those terms which are renormalizable (or relevant) or super-renormalizable (or marginal) and which are consistent with the residual symmetries. The whole procedure is very clearly described in Refs. [44, 75] and summarized in what follows.

Denoting the scaling dimensions with square brackets and a "s" subscript, we have:

$$[t]_s = -z, \qquad [x^k]_s = -1\,, \tag{5.4}$$

as can be read off from Eq. (5.1). As usual we require the action to be dimensionless. Note that

$$[S]_s = \left[\int dtd^Dx\,\mathcal{L}\right]_s = 0 \iff [\mathcal{L}]_s = z + D\,. \tag{5.5}$$

In $3+1$ dimensions the scaling dimension of the Lagrangian must be equal to 6. It is interesting to note that in GR the dimensionality of each term of the action is usually weighed according to the *mass/energy* (instead of *scaling*) dimensions, and that the Lagrangian needs to have mass dimensions equal to 4 in order for the action to be dimensionless. In the case of HL Gravity the mass dimensions of both space and time coordinates are still equal to -1, hence they clearly do not coincide with the scaling dimensions. The fact that the Lagrangian should have scaling dimension equal to 6 and that time- and space-derivatives weigh differently open up the possibility of including higher-order space-derivatives even though no time-derivatives of order higher than 2 are allowed, exactly as in GR.

The scaling dimensions of the metric coefficients appearing in Eq. (5.3) are as follows:

$$[g_{ij}]_s = 0, \qquad [N^i]_s = z-1, \qquad [N]_s = 0\,. \tag{5.6}$$

As in Refs. [60, 76], we consider the following action in $3+1$ dimensions:

$$S = \chi^2 \int dtd^3xN\sqrt{g}\left(\mathcal{L}_K - \mathcal{L}_V - \mathcal{L}_E + \chi^{-2}\mathcal{L}_M\right) \tag{5.7}$$

where $g \equiv \det(g_{ij})$, $\chi^2 \equiv (16\pi G_N)^{-1}$ and the content of the kinetic ($\mathcal{L}_K$), potential ($\mathcal{L}_V$) and matter ($\mathcal{L}_M$) Lagrangians will be shortly unveiled. We have denoted with $\mathcal{L}_E$ a collection of terms which are present only in the case of non-projectable HL Gravity ("E" stands for "extension").

We begin with the derivation of the kinetic matrix, starting from the Einstein-Hilbert action for GR,

$$S_{EH} = \chi^2 \int d^4x \sqrt{-g}\, {}^{(4)}R, \tag{5.8}$$

where ${}^{(4)}R$ is the Ricci scalar built from the four-dimensional metric $g_{\mu\nu}$. Making it explicit in terms of the ADM variables one obtains (see for instance [77]):

$$S_{EH} = \chi^2 \int dt d^3x N \sqrt{g} \left(K_{ij}K^{ij} - K^2 + {}^{(3)}R \right) , \tag{5.9}$$

where ${}^{(3)}R$ is Ricci scalar built from the three-dimensional g_{ij} and

$$K_{ij} \equiv \frac{1}{2N} \left(-\dot{g}_{ij} + \nabla_i N_j + \nabla_j N_i \right) \tag{5.10}$$

is the "extrinsic curvature". We see that the dynamics only enters into the extrinsic curvature (because of the presence of $\dot{g}_{ij}$), whereas the term ${}^{(3)}R$ is purely potential.

In the case of HL Gravity there is no fundamental principle to prevent $K_{ij}K^{ij}$ and K^2 from entering into the kinetic part of the action with different pre-factors, while in GR the pre-factors are exactly the same as a consequence of Lorentz invariance. It is thus reasonable to allow for a deviation from GR generalizing the kinetic part of the action in $D+1$ dimensions as follows:

$$S_K \propto \int dt d^Dx N \sqrt{g}\, g_K \left(K_{ij}K^{ij} - \lambda\, K^2 \right) , \tag{5.11}$$

where λ is a dimensionless parameter which is expected to flow to 1 under some (not yet specified in the literature) renormalization group. Being

$$[K_{ij}]_s = z, \qquad [g_K]_s = D - z \tag{5.12}$$

we realize that the coupling g_K is dimensionless for

$$z = D \tag{5.13}$$

which is the case we will stick to, since the fact that the kinetic term enters into the action with no dimensional coupling guarantees the power-counting renormalizability of the free theory with respect to the anisotropic scaling in Eq. (5.1). Conversely, in GR the kinetic term enters into the action with a dimensional coupling,

$$\left[\chi^2\right]_{s,GR} = 2\,, \tag{5.14}$$

which renders the theory non-renormalizable (see Ref. [3] for a pedagogical review). In the case at hand, instead,

$$[\chi]_{s,HL} = 0\,, \tag{5.15}$$

while

$$[\chi]_{\text{mass/energy},HL} = 1\,, \tag{5.16}$$

hence suitable powers of χ will be used to adjust the mass/energy dimensions of the several terms appearing in the Lagrangian so as to make them dimensionally homogeneous.

Finally, the kinetic Lagrangian reads as follows:

$$\mathcal{L}_K = K_{ij}K^{ij} - \lambda K^2\,. \tag{5.17}$$

Concerning the potential term of the gravity sector, the power-counting renormalizability criterion allows the inclusion of all those terms (built from metric variables) whose coupling constants g_i are such that

$$[g_i]_s \geq 0\,. \tag{5.18}$$

Consequently, the most general potential Lagrangian can include all the following terms:

$$\begin{aligned}\mathcal{L}_V =\; & \underbrace{g_0\chi^2 + g_1 R + \frac{1}{\chi^2}(g_2R^2 + g_3R_{ij}R^{ij})}_{\text{Super-renormalizable terms}} + \underbrace{\frac{1}{\chi^4}\left(g_4R^3 + g_5RR_{ij}R^{ij} + g_6R^i_jR^j_kR^k_i\right)}_{\text{Renormalizable terms}} + \\ & + \underbrace{\frac{1}{\chi^4}\left[g_7R\nabla^2R + g_8(\nabla_iR_{jk})(\nabla^iR^{jk})\right]}_{\text{Renormalizable terms}}\,;\end{aligned} \tag{5.19}$$

$$\begin{aligned}\mathcal{L}_E =\; & \underbrace{-\eta a_ia^i + \frac{1}{\chi^2}\left(\eta_2a_i\Delta a^i + \eta_3R\nabla_ia^i\right)}_{\text{Super-renormalizable terms}} + \\ & + \underbrace{\frac{1}{\chi^4}\left(\eta_4a_i\Delta^2a^i + \eta_5\Delta R\nabla_ia^i + \eta_6R^2\nabla_ia^i\right)}_{\text{Renormalizable terms}} + \dots\,,\end{aligned} \tag{5.20}$$

where a_i is defined as

$$a_i \equiv \frac{\partial_iN}{N}. \tag{5.21}$$

We have distinguished between $\mathcal{L}_V$ and $\mathcal{L}_E$ because the terms in $\mathcal{L}_E$ are present only in the non-projectable version of the theory, where $N = N(t,\vec{x})$. The dots in Eq. (5.20) stand for extra terms which become relevant to quadratic order only in the case of spatially curved background, a case which we will not consider. Let us point out that the dimensionalities of the couplings are:

$$\begin{aligned}&[g_0]_s = 6,\ [g_1]_s = 4,\ [g_2]_s = [g_3]_s = 2,\ [g_4]_s = [g_5]_s = [g_6]_s = [g_7]_s = [g_8]_s = 0\,,\\ &[\eta]_s = 4,\ [\eta_2]_s = [\eta_3]_s = 2,\ [\eta_4]_s = [\eta_5]_s = [\eta_6]_s = 0\,.\end{aligned} \tag{5.22}$$

Any other term with more than six spatial derivatives would require a coupling constant with negative scaling dimension, hence is forbidden.

We note that the first two terms in $\mathcal{L}_V$ also appear in GR (the very first one playing the role of the cosmological constant term), whereas all the others are extra. Usually one sets

$$g_0\chi^2 \equiv 2\Lambda, \qquad g_1 \equiv -1 \tag{5.23}$$

in order to let GR be (hopefully) recovered in the infrared limit. Given that g_1 is not (scaling) dimensionless, it can be set equal to -1 only if an overall dimension-four coupling constant is factored out and absorbed into an implicit rescaling of space and time coordinates.

The Lagrangian for scalar field matter can be found along the same lines and turns out to be as follows [60, 76, 75]:

$$\mathcal{L}_M = \frac{1}{2N^2}\left(\dot{\varphi} - N^i\nabla_i\varphi\right)^2 - V(g_{ij}, \mathcal{P}_n, \varphi)\,, \tag{5.24}$$

where

$$\begin{aligned} V =&V_0(\varphi) + V_1(\varphi)\mathcal{P}_0 + V_2(\varphi)\mathcal{P}_1^2 + \\ &+V_3(\varphi)\mathcal{P}_1^3 + V_4(\varphi)\mathcal{P}_2 + V_5(\varphi)\mathcal{P}_0\mathcal{P}_2 + V_6(\varphi)\mathcal{P}_1\mathcal{P}_2\,, \end{aligned} \tag{5.25}$$

with

$$\mathcal{P}_0 \equiv (\nabla\varphi)^2, \quad \mathcal{P}_i \equiv \Delta^i\varphi, \quad \Delta \equiv g^{ij}\nabla_i\nabla_j\,. \tag{5.26}$$

Note that

$$[\varphi]_s = \frac{D-z}{2}, \tag{5.27}$$

meaning that the scalar field φ is dimensionless in the case at hand (cf. Eq. (5.13) and the motivations above).

5.2 Constraints

The requirement that the variation of the action with respect to N be zero leads to the Hamiltonian constraint, whose form strongly depends on which version of the theory is considered. More specifically, in the case of projectable HL Gravity it is $N = N(t)$, hence

$$\begin{aligned} 0 &\equiv \frac{\delta S(t,\vec{x})}{\delta N(t')} \\ &= \int dt\, d^3x\sqrt{g}\left(\mathcal{L}_K + \mathcal{L}_V - \frac{1}{2\chi^2}J^t\right)\delta(t-t') \\ &= \int d^3x\sqrt{g}\left(\mathcal{L}_K + \mathcal{L}_V - \frac{1}{2\chi^2}J^t\right)\,, \end{aligned} \tag{5.28}$$

whereas in the case of non-projectable HL Gravity it is $N = N(t, \vec{x})$ and

$$\begin{aligned}
0 &\equiv \frac{\delta S(t,\vec{x})}{\delta N(t',\vec{x}')} \\
&= \int dt\, d^3x \sqrt{g} \left(\mathcal{L}_K + \mathcal{L}_V + \mathcal{L}_E + N \frac{\delta \mathcal{L}_E}{\delta N} - \frac{1}{2\chi^2} J^t \right) \delta(t-t')\delta^{(3)}(\vec{x}-\vec{x}') \\
&= \mathcal{L}_K + \mathcal{L}_V + \mathcal{L}_E + N \frac{\delta \mathcal{L}_E}{\delta N} - \frac{1}{2\chi^2} J^t \,.
\end{aligned} \tag{5.29}$$

In both cases,

$$J^t \equiv 2 \left(N \frac{\delta \mathcal{L}_M}{\delta N} + \mathcal{L}_M \right) . \tag{5.30}$$

Remarkably enough, we observe that the Hamiltonian constraint is *non-local* in the projectable version and *local* in the non-projectable version. We anticipate that, as consequence, the Hamiltonian constraint is trivial at linear order in cosmological perturbations in the projectable version because, by definition, we expect perturbations to average out to zero when integrated over the whole volume.

The cancellation of the variation of the action with respect to $N^i(t, \vec{x})$ yields the following super-momentum constraint:

$$\nabla_i \pi^{ij} = \frac{1}{2\chi^2} J^j \,, \tag{5.31}$$

where the super-momentum π^{ij} and the matter current are respectively given by

$$\pi^{ij} \equiv N \frac{\delta \mathcal{L}_K}{\delta \dot{g}_{ij}} = -K^{ij} + \lambda K g^{ij} \tag{5.32}$$

and

$$J_i \equiv -N \frac{\delta \mathcal{L}_M}{\delta N^i} = \frac{1}{N} \left(\dot{\varphi} - N^k \nabla_k \varphi \right) \nabla_i \varphi \,. \tag{5.33}$$

We will make explicit both constraints later on.

5.3 Cosmological background

The metric in the ADM form as in Eq. (5.3) exactly matches with a spatially flat FRW metric if one sets the values of N, N^i and g_{ij} as follows:

$$\begin{aligned}
N &= 1 + \delta N \,, \\
N^i &= 0 + \delta N^i \,, \\
g_{ij} &= a^2 \delta_{ij} + \delta g_{ij} \,.
\end{aligned} \tag{5.34}$$

The perturbations δN, δN^i, δg_{ij} will be specified later on.

On evaluating the Hamiltonian constraint in Eq. (5.28) to zeroth-order one obtains

$$(3\lambda - 1)\, H^2 \;=\; \frac{1}{3}\left(\frac{\rho_M}{\chi^2} + 2\Lambda\right) \tag{5.35}$$

which generalizes the first Friedmann equation. Here ρ_M is the energy density associated to the matter sector. Note that, in absence of Λ, H^2 is strictly positive only for $\lambda > 1/3$. Otherwise, if $\lambda < 1/3$, H^2 is positive only in the presence of a sufficiently negative cosmological constant, $\Lambda < -\rho_M/M_{\rm Pl}$. However, the range $\lambda < 1/3$ is not phenomenologically interesting because it is disconnected by the singular point $\lambda = 1/3$ from the value $\lambda = 1$ for which one wishes to recover GR.

The dynamical equation for the scale factor $a(t)$, namely the generalization of the second Friedmann equation, can be obtained by varying the action with respect to g_{ij} and evaluating the result in the homogeneous limit. The result is:

$$(3\lambda - 1)\frac{\ddot{a}}{a} = -\frac{1}{6\chi^2}(\rho_M + 3p_M) + \frac{2}{3}\Lambda\,, \tag{5.36}$$

where p_M is the (background) pressure associated with matter. For scalar field matter we have

$$\begin{aligned}\rho_M \;&=\; \frac{\dot{\varphi}_0^2}{2} + V_0(\varphi_0),\\ p_M \;&=\; \frac{\dot{\varphi}_0^2}{2} - V_0(\varphi_0)\,,\end{aligned} \tag{5.37}$$

and the background equation of motion becomes:

$$\ddot{\varphi}_0 + 3H\dot{\varphi}_0 \;=\; -\frac{dV_0(\varphi_0)}{d\varphi_0} \tag{5.38}$$

exactly as in GR.

We omit the derivation of the results above, while referring the reader to Ref. [76].

Chapter 6

Projectable version

We recall that the "projectable version" of Hořava-Lifshitz Gravity is the one in which the lapse function N appearing in Eq. (5.3) only depends on the time coordinate, $N = N(t)$.

6.1 Cosmological Perturbations

The basic scalar fluctuation variables are the same as in the case of GR:

$$\delta N(t) = \nu(t)\,; \tag{6.1a}$$

$$\delta N_i(t,\vec{x}) = \partial_i B(t,\vec{x})\,; \tag{6.1b}$$

$$\delta g_{ij}(t,\vec{x}) = a^2(t)\left[-2\,\psi(t,\vec{x})\,\delta_{ij}\ \ + 2\,E(t,\vec{x})_{|ij}\right], \tag{6.1c}$$

where the subscript $_{|i}$ denotes the covariant derivative.

Correspondingly, also matter fluctuations must be taken into account:

$$\varphi(t,\vec{x}) = \varphi_0(t) + \delta\varphi(t,\vec{x})\,. \tag{6.2}$$

We point out that the are two major differences with respect to GR:

1. the variable ν depends only on time because of the projectability condition;
2. as already pointed out several times, the symmetry group of HL Gravity is reduced in comparison with GR, since one loses the space-dependent time reparameterizations while maintaining (only) one space-dependent gauge mode - cf. Eq. (5.2). As a consequence, we can realize the gauge choice $E = 0$ but we cannot set to zero also the variable B, unlike what happens in GR. Nevertheless, we can in addition make use of space-independent time reparameterizations to set $\nu = 0$.

In conclusion, we can use the gauge freedom to set

$$\nu = 0, \qquad E = 0, \tag{6.3}$$

which corresponds to the choice of the quasi-longitudinal gauge (see also Refs. [60, 76]).

Expanding the Hamiltonian constraint (5.28) to first-order one finds

$$\int d^3x\, a^3 \left[2\Delta\psi - (3\lambda - 1)H(\Delta B + 3\dot{\psi}) - \frac{\delta\rho_M}{2\chi^2}\right] = 0\,, \tag{6.4}$$

which, as already anticipated, is actually trivially satisfied when dealing with linear cosmological perturbations since the spatial average of linear fluctuations must vanish (any non-vanishing term would in fact contribute to the background solution).

In contrast, the super-momentum constraint (5.31) is trivial at the homogeneous level but non-trivial at first-order, where it becomes

$$\partial_j \left[(\lambda - 1)\Delta B + (3\lambda - 1)\dot{\psi} - \frac{1}{2\chi^2} q_M\right] = 0\,, \tag{6.5}$$

with

$$q_M = \dot{\varphi}_0 \delta\varphi\,. \tag{6.6}$$

In linear perturbation theory we can work in Fourier space where the spatial derivative ∂_j is replaced by $(-i\,k_j)$. Hence, the quantities inside the square brackets of Eq. (7.10) must sum to zero.

Note that we started with five scalar degrees of freedom as in Eqs. (7.3), (6.2), and then we have decreased their number by two by making use of the gauge freedom. The number of degrees of freedom can be further reduced by one using the first-order super-momentum constraint (7.10) to remove B. Eventually only two physical degrees of freedom - ψ and $\delta\varphi$ - survive.

6.2 Second-order action

In the following we will insert the *ansatz* for cosmological fluctuations discussed in the previous section into the action for HL Gravity and determine the 2nd-order action, namely the action including terms quadratic in the 1st-order perturbative variables. This will let us find the canonically normalized fluctuation variables and determine whether they are stable or not.

The 2nd-order action receives three contributions, namely:

$$\begin{aligned} \delta_2 S^{(s)} &= \chi^2 \int dt d^3x \left[\delta_0(\sqrt{g})\,\delta_2\mathcal{L}^{(s)} + \delta_1(\sqrt{g})\delta_1\mathcal{L}^{(s)} + \delta_2(\sqrt{g})\delta_0\mathcal{L}^{(s)}\right] \\ &\equiv \chi^2 \int dt d^3x\, a^3\, \mathcal{L}_2^{(s)}\,, \end{aligned} \tag{6.7}$$

where we have implicitly introduced the following notation to denote the orders in the perturbative expansion:

$$f \equiv \sum_{i=0}^{\infty} \delta_i f \,. \tag{6.8}$$

Concerning the expansion of $\sqrt{g}$ one readily has

$$\delta_0(\sqrt{g}) = a^3, \qquad \delta_1(\sqrt{g}) = -3a^3\psi, \qquad \delta_2(\sqrt{g}) = \frac{3}{2}a^3\psi^2 \,. \tag{6.9}$$

After making use of the gauge freedom to eliminate ν and E, and of the constraint equation to express B in terms of the two remaining scalar degrees of freedom, the 2nd-order scalar action acquires the following form in terms of ψ and $\delta\varphi$:

$$\begin{aligned}\mathcal{L}_2^{(s)}[\psi, \delta\varphi] =& \frac{4(3\lambda-1)}{(\lambda-1)}\frac{\dot{\psi}^2}{2} + \frac{\dot{\delta\varphi}^2}{2\chi^2} + f_\psi \psi\dot{\psi} + f_{\varphi\psi}\psi\dot{\delta\varphi} + \tilde{f}_{\varphi\psi}\dot{\psi}\delta\varphi + \\ & - m_\psi^2\psi^2 - m_\varphi^2\delta\varphi^2 - m_{\varphi\psi}^2\psi\delta\varphi \\ & + \omega_\varphi\delta\varphi\Delta\delta\varphi + \omega_\psi\psi\Delta\psi \\ & + d_\psi(\Delta\psi)^2 + d_\varphi(\Delta\delta\varphi)^2 + \tilde{d}_\psi\Delta\psi\Delta^2\psi + \tilde{d}_\varphi\Delta\delta\varphi\Delta^2\delta\varphi\end{aligned} \tag{6.10}$$

where the various coefficients are listed in Appendix 6.A. In order to obtain this result we did some integrations by parts in intermediate steps and used the background dynamical equations for $a(t)$ and $\varphi_0(t)$, Eqs. (5.36) and (5.38).

We observe that the coefficient multiplying $\dot{\psi}^2$ has a "wrong" negative sign for $1/3 < \lambda < 1$, which will give rise to ghost instability as reported in almost all the literature about HL Gravity (see for instance [55, 60, 59, 44, 48]).

In order to compare our result with previous analyses worked out in the absence of matter, we can set the matter terms to zero and consider the remaining pieces in the 2nd-order action:

$$\begin{aligned}\delta_2 S^{(s)}[\psi] =& \chi^2 \int dt d^3x\, a^3 \left\{ \frac{4(3\lambda-1)}{(\lambda-1)}\frac{\dot{\psi}^2}{2} + 6H(1-3\lambda)\psi\dot{\psi} - 15(1-3\lambda)H^2\psi^2 + \right. \\ & \left. - 2\psi\Delta\psi - (16g_2 + 6g_3)\frac{(\Delta\psi)^2}{\chi^2} + (6g_8 - 16g_7)\frac{\Delta\psi\Delta^2\psi}{\chi^4} \right\}.\end{aligned} \tag{6.11}$$

This result is very similar to Eq. (33) of Ref. [48] and to Eq. (39) of Ref. [52] once that the spatial derivatives are set to zero, except for a discrepancy in the coefficient multiplying $(1-3\lambda)H^2\psi^2$ which is -15 in our result instead of 27 appearing in the cited references.

In order to draw definite conclusions about the ghost nature of the fluctuation modes, we must identify the canonically normalized variables. For values of λ which lie in the regions $\lambda < 1/3$ or $\lambda > 1$ we can rescale the fields as

$$\tilde{\psi} \equiv \sqrt{\frac{4(3\lambda-1)}{\lambda-1}}\psi, \qquad \widetilde{\delta\varphi} \equiv \frac{\delta\varphi}{\chi} \tag{6.12}$$

and obtain the following 2nd-order Lagrangian now in terms of canonically normalized variables:

$$\begin{aligned}\mathcal{L}_2^{(s)}[\psi,\delta\varphi] =& \frac{1}{2}\dot{\tilde{\psi}}^2 + \frac{1}{2}\dot{\widetilde{\delta\varphi}}^2 + f_{\tilde{\psi}}\tilde{\psi}\dot{\tilde{\psi}} + f_{\widetilde{\varphi}\tilde{\psi}}\tilde{\psi}\dot{\widetilde{\delta\varphi}} + \tilde{f}_{\widetilde{\varphi}\tilde{\psi}}\dot{\tilde{\psi}}\widetilde{\delta\varphi} + \\ & - m_{\tilde{\psi}}^2\tilde{\psi}^2 - m_{\widetilde{\varphi}}^2\widetilde{\delta\varphi}^2 - m_{\widetilde{\varphi}\tilde{\psi}}^2\tilde{\psi}\widetilde{\delta\varphi} \\ & + \omega_{\widetilde{\varphi}}\widetilde{\delta\varphi}\Delta\widetilde{\delta\varphi} + \omega_{\tilde{\psi}}\tilde{\psi}\Delta\tilde{\psi} \\ & + d_{\tilde{\psi}}(\Delta\tilde{\psi})^2 + d_{\widetilde{\varphi}}(\Delta\widetilde{\delta\varphi})^2 + \tilde{d}_{\tilde{\psi}}\Delta\tilde{\psi}\Delta^2\tilde{\psi} + \tilde{d}_{\widetilde{\varphi}}\Delta\widetilde{\delta\varphi}\Delta^2\widetilde{\delta\varphi}\,.\end{aligned} \tag{6.13}$$

The coefficients can be found in Appendix 6.B. The two degrees of freedom now have positive kinetic terms, thus there are no ghosts. Note that in the range $1/3 < \lambda < 1$ we should use the rescalings

$$\tilde{\psi} \equiv \sqrt{\frac{-4(3\lambda-1)}{\lambda-1}}\psi, \qquad \widetilde{\delta\varphi} \equiv \frac{\delta\varphi}{\chi}, \tag{6.14}$$

which make the sign of the kinetic term of ψ flip. Hence, in this range of values of λ the extra gravitational degree of freedom is ghost-like.

6.3 Number of physical degrees of freedom

In Ref. [61] perturbations in the non-projectable version of HL Gravity were analyzed, and it was shown that not all the degrees of freedom which naïvely appear in an expansion similar to that in Eq. (6.10) are really dynamical. Indeed, introducing the Sasaki-Mukhanov [78, 79] variable ζ defined as

$$\zeta \equiv -\psi - \frac{H}{\dot{\varphi}_0}\delta\varphi \tag{6.15}$$

and substituting for $\delta\varphi$ in terms of ζ, there remained only one variable which entered the Lagrangian with a proper kinetic term. Thus, the potentially dangerous degree of freedom was in fact not dynamical. The same “trick” turns out not to be successful in the present case. Written in terms of ζ, the Lagrangian takes the following form:

$$\begin{aligned}
\mathcal{L}_2^{(s)}[\psi,\zeta] =& \frac{\dot{\varphi}_0^2}{H^2\chi^2}\frac{\dot{\zeta}^2}{2} + \left[\frac{4(3\lambda-1)}{(\lambda-1)} + \frac{\dot{\varphi}_0^2}{H^2\chi^2}\right]\frac{\dot{\psi}^2}{2} + \\
&+ f_\zeta\zeta\dot{\zeta} + f_\psi\psi\dot{\psi} + f_{\zeta\psi}\psi\dot{\zeta} + \tilde{f}_{\zeta\psi}\zeta\dot{\psi} + g_{\zeta\psi}\dot{\zeta}\dot{\psi} + \\
&- m_\psi^2\psi^2 - m_\zeta^2\zeta^2 - m_{\zeta\psi}^2\zeta\psi \\
&+ \omega_\zeta\zeta\Delta\zeta + \omega_\psi\psi\Delta\psi + \omega_{\zeta\psi}\psi\Delta\zeta + \tilde{\omega}_{\zeta\psi}\zeta\Delta\psi + \\
&+ d_\psi(\Delta\psi)^2 + d_\zeta(\Delta\zeta)^2 + d_{\zeta\psi}\Delta\zeta\Delta\psi + \\
&+ \tilde{d}_{\zeta\psi}\Delta\zeta\Delta^2\psi + \tilde{d}_\psi\Delta\psi\Delta^2\psi + \tilde{d}_\zeta\Delta\zeta\Delta^2\zeta\,.
\end{aligned} \tag{6.16}$$

Once again, the various coefficients are listed in Appendix 6.C. We observe that, even in absence of any matter field, ψ is still a dynamical (gravitational) degree of freedom.

We wish to emphasize the fact that - as opposed to the situation in the non-projectable version - in the projectable version of HL Gravity it is not possible to reduce to one the number of physical degrees of freedom, in agreement with the results of the general analysis of Ref. [64] and with the conclusions reached in many other studies in which perturbations around Minkowski background were considered.

6.4 Discussion on tachyonic instabilities

We now want to investigate the issue of tachyonic (classical) instabilities. We do this by looking at the signs of the eigenvalues of the mass matrix. In general it is difficult to diagonalize the mass matrix. Hence, we will specialize to a couple of simple cases, both of them with $\Lambda = 0$ and a massive potential, $V_0(\varphi_0) = m^2\varphi_0^2/2$. The first example will be that of a static field, the second that of a scalar field oscillating around $\varphi_0 = 0$.

6.4.1 Static field and $\Lambda = 0$

Setting $\varphi_0 = x\chi$, where x is a dimensionless constant, the mass terms in Appendix 6.B read as follows:

$$m^2_{\tilde{\psi}} = -\frac{5}{8}\frac{\lambda-1}{3\lambda-1}m^2x^2\,; \tag{6.17a}$$

$$m^2_{\tilde{\varphi}} = \frac{m^2}{2}\,; \tag{6.17b}$$

$$m^2_{\tilde{\varphi}\tilde{\psi}} = -\frac{3}{2}\sqrt{\frac{\lambda-1}{3\lambda-1}}m^2x\,. \tag{6.17c}$$

The mass matrix, defined as

$$\widetilde{M}^2_{\widetilde{\varphi}\widetilde{\psi}} \equiv \begin{pmatrix} m^2_{\widetilde{\varphi}} & m^2_{\widetilde{\varphi}\widetilde{\psi}}/2 \\ m^2_{\widetilde{\varphi}\widetilde{\psi}}/2 & m^2_{\widetilde{\psi}} \end{pmatrix}, \tag{6.18}$$

can be easily diagonalized and its eigenvalues are

$$\begin{aligned} \widetilde{M}^2_{\pm} \ = \ & \frac{m^2}{16(3\lambda-1)} \Big[4(3\lambda-1) - 5x^2(\lambda-1) \pm \\ & \pm \sqrt{16(3\lambda-1)^2 + 25x^4(\lambda-1)^2 + 184x^2(\lambda-1)(3\lambda-1)} \Big] \end{aligned} \tag{6.19}$$

In both the ranges $\lambda > 1$ and $\lambda < 1/3$ and for any value of the scalar field one eigenvalue is positive ($\widetilde{M}^2_+$) while the other is negative. Thus, the extra scalar metric degree of freedom exhibits a tachyonic instability in these regions of λ (the ones which do not suffer from the ghost problem), as is also known from previous works which considered fluctuations in a theory without matter.

In terms of the variables ζ, ψ we obtain the following eigenvalues:

$$M^2_{\pm} \ = \ -\frac{m^2}{4} \left[12(3\lambda-1) + 5x^2 \pm \sqrt{25x^4 + 144(3\lambda-1)^2} \right] \tag{6.20}$$

which are both negative for any $\lambda > 1/3$ and for any x.

6.4.2 Oscillating field and $\Lambda = 0$

We set $\varphi_0 = A\cos(mt)$ and then average over field oscillations as follows:

$$\langle f(t) \rangle \equiv \frac{m}{2\pi} \int_{-\pi/m}^{\pi/m} dt\, f(t)\,. \tag{6.21}$$

We obtain the following result for the Hubble parameter,

$$\langle H^2 \rangle = \frac{1}{3\chi^2(3\lambda-1)} \frac{m^2A^2}{2}\,, \tag{6.22}$$

while the average mass terms amount to

$$\langle m^2_{\widetilde{\psi}} \rangle = -\frac{13}{6} \frac{\lambda-1}{3\lambda-1} \frac{m^2A^2}{\chi^2}\,; \tag{6.23a}$$

$$\langle m^2_{\widetilde{\varphi}} \rangle = \frac{m^2}{8} \left(4 - \frac{1}{\lambda-1} \frac{A^2}{\chi^2} \right); \tag{6.23b}$$

$$\langle m^2_{\widetilde{\psi}\widetilde{\varphi}} \rangle = 0\,. \tag{6.23c}$$

We see that $\langle m^2_{\widetilde{\psi}} \rangle$ is negative in both the regions $\lambda > 1$ and $\lambda < 1/3$ which are ghost-free, thus $\widetilde{\psi}$ displays tachyonic instability.

In terms of ζ and ψ we obtain

$$\langle m^2_\zeta \rangle = -\frac{9m^2 A^2(7\lambda - 3)}{32(\lambda - 1)\chi^2}\,; \tag{6.24a}$$

$$\langle m^2_\psi \rangle = -\frac{m^2 A^2(131\lambda - 95)}{32(\lambda - 1)\chi^2}\,; \tag{6.24b}$$

$$\langle m^2_{\zeta\psi} \rangle = -\frac{9m^2 A^2(5\lambda - 1)}{16(\lambda - 1)\chi^2}\,, \tag{6.24c}$$

and the following eigenvalues,

$$M^2_\pm = -\frac{m^2 A^2}{32(\lambda - 1)\chi^2}\left(97\lambda - 61 \pm \sqrt{1237 - 3122\lambda + 3181\lambda^2}\right)\,, \tag{6.25}$$

which are both negative for any $\lambda > 1$.

6.A Coefficients in Eq. (6.10)

$$f_\psi = -6H(3\lambda - 1), \quad f_{\varphi\psi} = -3\frac{\dot{\varphi}_0}{\chi^2}, \quad \tilde{f}_{\varphi\psi} = -\frac{3\lambda - 1}{\lambda - 1}\frac{\dot{\varphi}_0}{\chi^2}$$
$$m^2_\psi = -\frac{39}{2}(3\lambda - 1)H^2 + 3\Lambda + \frac{3}{2}\frac{V_0(\varphi_0)}{\chi^2} - \frac{3}{4}\frac{\dot{\varphi}_0^2}{\chi^2}$$
$$m^2_\varphi = -\frac{1}{4(\lambda - 1)}\frac{\dot{\varphi}_0^2}{\chi^4} + \frac{1}{2}\frac{V_{0,\varphi\varphi}(\varphi_0)}{\chi^2}$$
$$m^2_{\psi\varphi} = -3\frac{V_{0,\varphi}(\varphi_0)}{\chi^2}$$
$$w_\psi = -2, \quad w_\varphi = \frac{V_1(\varphi_0)}{\chi^2}$$
$$d_\varphi = -\frac{V_{4,\varphi}(\varphi_0)}{\chi^2} - \frac{V_2(\varphi_0)}{\chi^2}, \quad d_\psi = -16\frac{g_2}{\chi^2} - 6\frac{g_3}{\chi^2}$$
$$\tilde{d}_\psi = 6\frac{g_8}{\chi^4} - 16\frac{g_7}{\chi^4}, \quad \tilde{d}_{\widetilde{\varphi}} = -\frac{V_6(\varphi_0)}{\chi^2}$$

6.B Coefficients in Eq. (6.13)

$$
\begin{aligned}
&f_{\widetilde{\psi}} = -\frac{3}{2}H(\lambda-1), \quad f_{\widetilde{\varphi}\widetilde{\psi}} = -\frac{3}{2}\sqrt{\frac{\lambda-1}{3\lambda-1}}\frac{\dot{\varphi}_0}{\chi}, \quad \tilde{f}_{\widetilde{\varphi}\widetilde{\psi}} = -\frac{1}{2}\sqrt{\frac{3\lambda-1}{\lambda-1}}\frac{\dot{\varphi}_0}{\chi} \\
&m^2_{\widetilde{\psi}} = -\frac{3}{16}\frac{\lambda-1}{3\lambda-1}\frac{\dot{\varphi}_0^2}{\chi^2} + \frac{3}{8}\frac{\lambda-1}{3\lambda-1}\frac{V_0(\varphi_0)}{\chi^2} - \frac{39}{8}(\lambda-1)H^2 + \frac{3}{4}\frac{\lambda-1}{3\lambda-1}\Lambda \\
&m^2_{\delta\widetilde{\varphi}} = -\frac{1}{4(\lambda-1)}\frac{\dot{\varphi}_0^2}{\chi^2} + \frac{1}{2}V_{0,\varphi\varphi}(\varphi_0) \\
&m^2_{\tilde{\psi}\delta\tilde{\varphi}} = -\frac{3}{2}\sqrt{\frac{\lambda-1}{3\lambda-1}}\frac{V_{0,\varphi}(\varphi_0)}{\chi} \\
&w_{\widetilde{\psi}} = -\frac{1}{2}\frac{\lambda-1}{3\lambda-1}, \quad \omega_{\widetilde{\varphi}} = V_1(\varphi_0) \\
&d_{\widetilde{\varphi}} = -V_{4,\varphi}(\varphi_0) - V_2(\varphi_0), \quad d_{\widetilde{\psi}} = -4\frac{\lambda-1}{3\lambda-1}\frac{g_2}{\chi^2} - \frac{3}{2}\frac{\lambda-1}{3\lambda-1}\frac{g_3}{\chi^2} \\
&\tilde{d}_{\widetilde{\psi}} = -4\frac{\lambda-1}{3\lambda-1}\frac{g_7}{\chi^4} + \frac{3}{2}\frac{\lambda-1}{3\lambda-1}\frac{g_8}{\chi^4}, \quad \tilde{d}_{\widetilde{\varphi}} = -V_6(\varphi_0)
\end{aligned}
$$

6.C Coefficients in Eq. (6.16)

In what follows the function $F(\varphi_0)$ is defined as:

$$
F(\varphi_0) \equiv 2\Lambda\chi^2 - \dot{\varphi}_0^2 + V_0(\varphi_0)\,. \tag{6.26}
$$

$$
\begin{aligned}
f_\zeta =& -\frac{1}{3(3\lambda-1)}\frac{\dot{\varphi}_0^2}{\chi^4 H^3}F(\varphi_0) + \frac{\dot{\varphi}_0^2}{\chi^2 H} - \frac{\dot{\varphi}_0}{\chi^2 H^2}(3H\dot{\varphi}_0 + V_{0,\varphi}) \\
f_\psi =& -\frac{1}{3(3\lambda-1)}\frac{\dot{\varphi}_0^2}{\chi^4 H^3}F(\varphi_0) + \left(4 + \frac{3\lambda-1}{\lambda-1}\right)\frac{\dot{\varphi}_0^2}{\chi^2 H} - \frac{\dot{\varphi}_0}{\chi^2 H^2}(3H\dot{\varphi}_0 + V_{0,\varphi}) + \\
& - 6H(3\lambda-1) \\
f_{\zeta\psi} =& f_\zeta \\
\tilde{f}_{\zeta\psi} =& -\frac{1}{3(3\lambda-1)}\frac{\dot{\varphi}_0^2}{\chi^4 H^3}F(\varphi_0) + \left(1 + \frac{3\lambda-1}{\lambda-1}\right)\frac{\dot{\varphi}_0^2}{\chi^2 H} - \frac{\dot{\varphi}_0}{\chi^2 H^2}(3H\dot{\varphi}_0 + V_{0,\varphi}) \\
g_{\zeta\psi} =& \frac{\dot{\varphi}_0^2}{\chi^2 H^2} \\
\omega_\zeta =& \frac{V_1(\varphi_0)\dot{\varphi}_0^2}{\chi^2 H^2}, \quad \omega_\psi = \omega_\zeta - 2, \quad \tilde{\omega}_{\zeta\psi} = \omega_\zeta, \quad \omega_{\zeta\psi} = \omega_\zeta
\end{aligned}
$$

$$
\begin{aligned}
m_\zeta^2 &= -\frac{1}{4(\lambda-1)}\frac{\dot\varphi_0^4}{\chi^4 H^2} - \frac{1}{2}\frac{\dot\varphi_0^2}{\chi^2} + \frac{1}{2}\frac{V_{0,\varphi\varphi}\dot\varphi_0^2}{\chi^2 H^2} + \\
&\quad -\frac{1}{3(3\lambda-1)}\frac{\dot\varphi_0}{\chi^4 H^3}(3H\dot\varphi_0 + V_{0,\varphi})F(\varphi_0) - \frac{1}{2}\frac{(3H\dot\varphi_0 + V_{0,\varphi})^2}{\chi^2 H^2} + \\
&\quad +\frac{\dot\varphi_0}{\chi^2 H}(3H\dot\varphi_0 + V_{0,\varphi}) - \frac{1}{18(3\lambda-1)^2}\frac{\dot\varphi_0^2}{\chi^6 H^4}F(\varphi_0)^2 + \frac{1}{3(3\lambda-1)}\frac{\dot\varphi_0^2}{\chi^4 H^2}F(\varphi_0) \\
m_\psi^2 &= -\frac{39}{2}H^2(3\lambda-1) + 3\Lambda + 3\frac{\dot\varphi_0}{H\chi^2}V_{0,\varphi} - \frac{1}{4(\lambda-1)}\frac{\dot\varphi_0^4}{\chi^4 H^2} + \frac{1}{2}\frac{V_{0,\varphi\varphi}\dot\varphi_0^2}{\chi^2 H^2} - \frac{17}{4}\frac{\dot\varphi_0^2}{\chi^2} + \\
&\quad +\frac{3}{2}\frac{V_0(\varphi_0)}{\chi^2} - \frac{1}{3(3\lambda-1)}\frac{\dot\varphi_0}{\chi^4 H^3}(3H\dot\varphi_0 + V_{0,\varphi})F(\varphi_0) - \frac{1}{2}\frac{(3H\dot\varphi_0 + V_{0,\varphi})^2}{\chi^2 H^2} + \\
&\quad +4\frac{\dot\varphi_0}{\chi^2 H}(3H\dot\varphi_0 + V_{0,\varphi}) - \frac{1}{18(3\lambda-1)^2}\frac{\dot\varphi_0^2}{\chi^6 H^4}F(\varphi_0)^2 + \frac{4}{3(3\lambda-1)}\frac{\dot\varphi_0^2}{\chi^4 H^2}F(\varphi) \\
m_{\zeta\psi}^2 &= +\frac{5}{3(3\lambda-1)}\frac{\dot\varphi_0^2}{\chi^4 H^2}F(\varphi_0) - \frac{2}{3(3\lambda-1)}\frac{\dot\varphi_0}{\chi^4 H^3}(3H\dot\varphi_0 + V_{0,\varphi})F(\varphi_0) + \\
&\quad -\frac{1}{2(\lambda-1)}\frac{\dot\varphi_0^4}{\chi^4 H^2} - 4\frac{\dot\varphi_0^2}{\chi^2} + \frac{V_{0,\varphi\varphi}\dot\varphi_0^2}{\chi^2 H^2} - \frac{1}{\chi^2 H^2}(3H\dot\varphi_0 + V_{0,\varphi})^2 + \\
&\quad +\frac{5\dot\varphi_0}{\chi^2 H}(3H\dot\varphi_0 + V_{0,\varphi}) - \frac{1}{9(3\lambda-1)^2}\frac{\dot\varphi_0^2}{\chi^6 H^4}F(\varphi_0)^2 + 3\frac{\dot\varphi_0 V_{0,\varphi}}{H\chi^2}
\end{aligned}
$$

$$
\begin{aligned}
d_\psi &= -4\frac{\dot\varphi_0 V_4(\varphi_0)}{\chi^2 H} + 4\frac{V_4(\varphi_0)\dot\varphi_0}{\chi^2 H} - \frac{[V_{4,\varphi}(\varphi_0) + V_2(\varphi_0)]\dot\varphi_0^2}{\chi^2 H^2} - \frac{2}{\chi^2}(8g_2 + 3g_3) \\
d_{\zeta\psi} &= -4\frac{\dot\varphi_0 V_4(\varphi_0)}{\chi^2 H} + 4\frac{V_4(\varphi_0)\dot\varphi_0}{\chi^2 H} - 2\frac{[V_{4,\varphi}(\varphi_0) + V_2(\varphi_0)]\dot\varphi_0^2}{\chi^2 H^2} \\
d_\zeta &= -\frac{[V_{4,\varphi}(\varphi_0) + V_2(\varphi_0)]\dot\varphi_0^2}{\chi^2 H^2} \\
\tilde d_{\zeta\psi} &= -2\frac{V_6(\varphi_0)\dot\varphi_0^2}{\chi^2 H^2} \\
\tilde d_\psi &= -\frac{V_6(\varphi_0)\dot\varphi_0^2}{\chi^2 H^2} - \frac{2}{\chi^4}(8g_7 - 3g_8) \\
\tilde d_\zeta &= -\frac{V_6(\varphi_0)\dot\varphi_0^2}{\chi^2 H^2}
\end{aligned}
$$

Chapter 7

Non-projectable version

We recall that with "non-projectable version" we refer to the version of Hořava-Lifshitz Gravity in which the lapse function N is both time- and space-dependent, $N = N(t, \vec{x})$.

Let us now spend a few words to clarify the nomenclature. Historically, the very first non-projectable version which was considered did not actually include in the Lagrangian all the possible terms compatible with the symmetries of the theory. Only afterwards it was realized that the non-projectable version could have hosted also an additional set of potential terms, precisely the ones we have written in Eq. (5.20). Such terms were first introduced in a paper by Blas, Pujolas and Sibiryakov [80] who made reference to such an extended version as the "healthy extended" version of Hořava-Lifshitz Gravity. "Healthy" because, at least as far as perturbations about Minkowski space-time are considered, the additional terms improve the infrared behavior of the theory, as we now briefly motivate. We will in fact prove that the equation of motion of the gravitational extra degree of freedom reads, in the infrared limit, as follows:

$$\frac{3\lambda - 1}{\lambda - 1}\ddot{\psi}_k + \frac{2-\eta}{\eta}k^2\psi_k = 0. \tag{7.1}$$

The presence of a pair of coupling constants, λ and η, makes it possible to end up with a ghost- *and* tachyon-free theory by requiring

$$\lambda \notin (1/3, 1) \quad \text{and} \quad 0 < \eta < 2. \tag{7.2}$$

In the "unhealthy" version the coupling η was absent and, consequently, the theory was plagued by the presence of either a ghost-like or a tachyonic gravitational scalar degree of freedom.

The study here reported will let us conclude that similar conclusions also hold for linear fluctuations about the cosmological background: there is a dynamical extra scalar mode, but it can be made both ghost-free and non-tachyonic. In addition, it decouples in the infrared limit (a limit which cannot be seen when expanding about Minkowski spacetime, yet it is crucial for actual cosmological perturbations). Let us point out that we were not the first to consider cosmological fluctuations in the full non-projectable version of HL Gravity. In [81] the evolution of super- and sub-Hubble curvature fluctuations was studied both analytically and numerically. The emphasis in our work is on the general properties of the perturbation modes rather than on the specific form of the solutions of the equations of motion.

7.1 Cosmological Perturbations

The scalar metric perturbations can be written as follows, in full analogy with the projectable version - cf. Eq. (7.3) - except for the fact that the function ν is now both space- and time-dependent:

$$\delta N(t,\vec{x}) = \nu(t,\vec{x}) \tag{7.3a}$$

$$\delta N_i(t,\vec{x}) = \partial_i B(t,\vec{x}) \tag{7.3b}$$

$$\delta g_{ij}(t,\vec{x}) = a^2(t)\left[-2\,\psi(t,\vec{x})\,\delta_{ij} \;\; + 2\,E(t,\vec{x})_{|ij}\right] \tag{7.3c}$$

where the subscript $_{|i}$ denotes the covariant derivative[1]. Matter fluctuations are, once more, parameterized as in Eq. (6.2).

Given that under a scalar gauge transformation, such as

$$t \to t + f(t)\,, \quad x^k \to x^k + \partial^k \xi(t,\vec{x})\,, \tag{7.4}$$

the metric variables transform as [81]

$$\nu \to \nu - \dot{f}\,, \quad B \to B - a^2\dot{\xi}\,, \quad \psi \to \psi + H\,f\,, \quad E \to E - \xi\,, \tag{7.5}$$

we can use the spatial gauge transformation to set

$$E = 0\,. \tag{7.6}$$

However, the temporal gauge transformation cannot help removing neither ν nor ψ because f only depends on t while both ν and ψ are space- and time-dependent.

7.2 Constraints

We have already presented the Hamiltonian and super-momentum constraints in their most general form in Eqs. (5.29) and (5.31) respectively. We now move on to unveil their 1st-order expansion in turn.

[1] Note that in Ref. [81] the authors use the variable $\beta \equiv B/a^2$ instead of B.

7.2.1 Hamiltonian constraint

Expanding the Hamiltonian constraint (5.29) up to 1st-order we obtain the following result:

$$\begin{aligned}&3H(3\lambda-1)(\dot{\psi}+H\nu)+(3\lambda-1)H\Delta B-2\Delta\psi+\frac{\delta\rho_M}{2\chi^2}+\\&+\eta\Delta\nu-\frac{\eta_2}{\chi^2}\Delta^2\nu+2\frac{\eta_3}{\chi^2}\Delta^2\psi-\frac{\eta_4}{\chi^4}\Delta^3\nu+2\frac{\eta_5}{\chi^4}\Delta^3\psi=0\,,\end{aligned} \tag{7.7}$$

where

$$\beta_1\equiv\frac{3g_2+g_3}{\chi^2},\qquad \beta_2\equiv\frac{9g_4+3g_5+g_6}{\chi^4}, \tag{7.8}$$

and

$$\delta\rho_M \;=\; \dot{\varphi}_0\dot{\delta\varphi}-\nu\dot{\varphi}_0^2+V_{0,\varphi}(\varphi_0)\delta\varphi+V_4(\varphi_0)\Delta^2\delta\varphi\,. \tag{7.9}$$

We observe that the coupling constants η_i's, present in the healthy extension but absent in the original non-projectable version of HL Gravity, turn out to multiply only higher order derivatives of the gravitational degrees of freedom, even in the case of a spatially flat background.

7.2.2 Momentum constraint

To 1st-order the momentum constraint (5.31) reads as follows:

$$\partial_j\left[(\lambda-1)\Delta B+(3\lambda-1)\left(\dot{\psi}+H\nu\right)-\frac{1}{2\chi^2}q_M\right]\;=\;0\,, \tag{7.10}$$

where

$$q_M\;=\;\dot{\varphi}_0\delta\varphi\,. \tag{7.11}$$

7.2.3 Solving the constraints in a spatially flat background

We now make use of the constraints to solve for two of the metric degrees of freedom, namely ν and B. Since both variables enters the constraints as argument of some differential operator, it is convenient to switch to the Fourier space so to deal with algebraic equations. The results can be written in a more compact form if we introduce the notation

$$f_1(\bar{k})\;\equiv\;\eta+\eta_2\frac{\bar{k}^2}{\chi^2}-\eta_4\frac{\bar{k}^4}{\chi^4}\,, \tag{7.12}$$

$$f_2(\bar{k})\;\equiv\;1+\eta_3\frac{\bar{k}^2}{\chi^2}-\eta_5\frac{\bar{k}^4}{\chi^4}\,, \tag{7.13}$$

and

$$d(\bar{k}) \equiv 4(3\lambda-1)H^2+(\lambda-1)\left[\frac{\dot{\varphi}_0^2}{\chi^2}+2f_1(\bar{k})\bar{k}^2\right]. \tag{7.14}$$

where the $\bar{k}$ is the physical momentum, $\bar{k}\equiv k/a$. We then obtain:

$$\begin{aligned}
d(\bar{k})\,B_k(t) =&(3\lambda-1)\left[\frac{\dot{\varphi}_0^2}{\chi^2\bar{k}^2}+2f_1(\bar{k})\right]\dot{\psi}_k(t)+(3\lambda-1)\frac{H\dot{\varphi}_0}{\chi^2\bar{k}^2}\delta\dot{\varphi}_k(t)\\
&+4(3\lambda-1)f_2(\bar{k})H\psi_k(t)+\Big\{(3\lambda-1)[V_{0,\varphi}(\varphi_0)+V_4(\varphi_0)\bar{k}^4]H+\\
&+3(3\lambda-1)\dot{\varphi}_0H^2-\frac{\dot{\varphi}_0^3}{2\chi^2}-\dot{\varphi}_0f_1(\bar{k})\bar{k}^2\Big\}\frac{\delta\varphi_k(t)}{\chi^2\bar{k}^2}\,; \qquad (7.15)\\
d(\bar{k})\,\nu_k(t) =&(\lambda-1)\frac{\dot{\varphi}_0}{\chi^2}\delta\dot{\varphi}_k(t)-4(3\lambda-1)H\dot{\psi}_k(t)+\\
&+\Big\{(3\lambda-1)\dot{\varphi}_0H+(\lambda-1)\left[V_{0,\varphi}(\varphi_0)+V_4(\varphi_0)\bar{k}^4\right]\Big\}\frac{\delta\varphi_k(t)}{\chi^2}+\\
&+4(\lambda-1)f_2(\bar{k})\bar{k}^2\psi_k(t)\,.
\end{aligned}$$

Given the form of the common denominator (7.14), the solutions for $B_k(t)$ and $\nu_k(t)$ are both regular in the limit $\lambda\to 1$ whenever $H\neq 0$, meaning that the neglect of the cosmic expansion in presence of matter could lead to erroneous conclusions on potential singularities of the theory.

In order to better understand how to interpret the low- and the high-momentum limits, it is useful to rewrite the coefficient function $d(\bar{k})$ of Eq. (7.14) as follows (valid for $H\neq 0$ and $\lambda\neq 1/3$):

$$d(\bar{k}) = 4(3\lambda-1)H^2\left[1+\frac{\lambda-1}{2(3\lambda-1)}\frac{\dot{\varphi}_0^2}{\chi^2H^2}+\frac{\lambda-1}{2(3\lambda-1)}\left(\eta+\eta_2\frac{\bar{k}^2}{\chi^2}-\eta_4\frac{\bar{k}^4}{\chi^4}\right)\frac{\bar{k}^2}{H^2}\right]. \tag{7.16}$$

From this form of the expression we can see that the value of $\bar{k}=k/a$ which separates the low-momentum from the high-momentum region is the Hubble momentum H. This is not surprising since we expect fluctuations to behave differently for wavelengths larger and smaller than the Hubble radius. Note that in the short wavelength region $k>aH$, the next-to-leading order terms in the expression for $d(\bar{k})$ are controlled by the ratio k/χ (as long as $H<\chi$, which will hold in the region of validity of the effective field theory).

In the long wavelength (IR) limit, the expression for $d(\bar{k})$ becomes

$$\begin{aligned}
d(\bar{k}) &\sim 4(3\lambda-1)H^2+(\lambda-1)\dot{\varphi}_0^2/\chi^2 \qquad (7.17)\\
&= \frac{3\lambda-1}{3}\frac{\dot{\varphi}_0^2}{\chi^2}+\frac{4}{3}\frac{V_0(\varphi_0)}{\chi^2}+\frac{8}{3}\Lambda\,.
\end{aligned}$$

We see that the sign of the first term changes when λ crosses the critical value $\lambda = 1/3$. In the IR limit the first term dominates and hence $d(\bar{k})$ is positive. More generally, sufficient conditions for the positivity of $d(\bar{k})$ are $\lambda > 1/3$, $V_0(\varphi_0) > 0$ and $\Lambda > 0$.

In the short wavelength (UV) limit, the expression for $d(\bar{k})$ becomes

$$d(\bar{k}) \overset{\bar{k}\to\infty}{\longrightarrow} 2(\lambda - 1) f_1(\bar{k})\bar{k}^2 \,, \tag{7.18}$$

which changes sign as λ crosses the value $\lambda = 1$.

7.3 Second-order action

7.3.1 The action

We are now ready to discuss the 2nd-order action for cosmological fluctuations. We insert the metric *ansatz* including fluctuations into the full action, make use of the constraint equations to eliminate the variables ν and B, and series expand. Working in Fourier space, after a lot of algebra the terms in the total action which are quadratic in the perturbation variables are:

$$\begin{aligned}\delta_2 S^{(s)} = \chi^2 \int dt \frac{d^3k}{(2\pi)^3} a^3 \Big\{ & c_\varphi \dot{\delta\varphi}_k^2 + c_\psi \dot{\psi}_k^2 + c_{\varphi\psi}\dot{\psi}_k \dot{\delta\varphi}_k + f_\varphi \delta\varphi_k \dot{\delta\varphi}_k + f_\psi \psi_k \dot{\psi}_k + \\ & + f_{\varphi\psi}\psi_k \dot{\delta\varphi}_k + \tilde{f}_{\varphi\psi}\dot{\psi}_k \delta\varphi_k - m_\varphi^2 \delta\varphi_k^2 - m_\psi^2 \psi_k^2 - m_{\varphi\psi}^2 \psi_k \delta\varphi_k \Big\} \,. \end{aligned} \tag{7.19}$$

The coefficients of the kinetic terms are given by

$$d(\bar{k})\, c_\varphi = 2(3\lambda - 1)\frac{H^2}{\chi^2} + (\lambda - 1) f_1(\bar{k})\frac{\bar{k}^2}{\chi^2} \,; \tag{7.20}$$

$$d(\bar{k})\, c_\psi = 2(3\lambda - 1)\left[\frac{\dot{\varphi}_0^2}{\chi^2} + 2 f_1(\bar{k})\bar{k}^2\right] \,; \tag{7.21}$$

$$d(\bar{k})\, c_{\varphi\psi} = 4(3\lambda - 1)\frac{H\dot{\varphi}_0}{\chi^2} \,. \tag{7.22}$$

while the coefficients of the terms involving one time-derivative of a dynamical variable are:

$$d(\bar{k})\, f_\varphi = -(3\lambda-1)\frac{\dot{\varphi}_0^2 H}{\chi^4} - (\lambda-1)\left[V_{0,\varphi}(\varphi_0) + V_4(\varphi_0)\bar{k}^4\right]\frac{\dot{\varphi}_0}{\chi^4}\,; \tag{7.23}$$

$$\begin{aligned} d(\bar{k})\, f_\psi &= -24(3\lambda-1)H\Lambda + 12(3\lambda-1)^2 H^3 \\ &\quad -6\lambda(3\lambda-1)\frac{\dot{\varphi}_0^2 H}{\chi^2} - 12(3\lambda-1)\frac{V_0(\varphi_0)H}{\chi^2}\,; \end{aligned} \tag{7.24}$$

$$\begin{aligned} d(\bar{k})\, f_{\varphi\psi} &= 6(\lambda-1)\frac{\dot{\varphi}_0\Lambda}{\chi^2} - 3(3\lambda-1)(3\lambda+1)\frac{\dot{\varphi}_0 H^2}{\chi^2} - \frac{3}{2}(\lambda-1)\frac{\dot{\varphi}_0^3}{\chi^4} \\ &\quad +3(\lambda-1)\frac{V_0(\varphi_0)\dot{\varphi}_0}{\chi^4} - 2(\lambda-1)\left[3f_1(\bar{k}) + 2f_2(\bar{k})\right]\frac{\dot{\varphi}_0}{\chi^2}\bar{k}^2\,; \end{aligned} \tag{7.25}$$

$$\begin{aligned} d(\bar{k})\, \tilde{f}_{\varphi\psi} &= 4(3\lambda-1)\frac{V_{0,\varphi}(\varphi_0)H}{\chi^2} - (3\lambda-1)\frac{\dot{\varphi}_0^3}{\chi^4} + \\ &\quad -2(3\lambda-1)f_1(\bar{k})\frac{\dot{\varphi}_0}{\chi^2}\bar{k}^2 + 4(3\lambda-1)\frac{V_4(\varphi_0)H}{\chi^2}\bar{k}^4\,. \end{aligned} \tag{7.26}$$

Finally, the full expressions of the mass matrix coefficients are:

$$\begin{aligned} d(\bar{k})\, m_\varphi^2 = {}& 2(3\lambda-1)\frac{V_{0,\varphi\varphi}(\varphi_0)H^2}{\chi^2} + \frac{3}{2}(3\lambda-1)\frac{\dot{\varphi}_0^2 H^2}{\chi^4} + (3\lambda-1)\frac{V_{0,\varphi}(\varphi_0)\dot{\varphi}_0 H}{\chi^4} + \\ & + \frac{1}{2}(\lambda-1)\frac{V_{0,\varphi\varphi}(\varphi_0)\dot{\varphi}_0^2}{\chi^4} + \frac{1}{2}(\lambda-1)\frac{V_{0,\varphi}(\varphi_0)^2}{\chi^4} - \frac{1}{4}\frac{\dot{\varphi}_0^4}{\chi^6} + \\ & + \Bigg\{4(3\lambda-1)\frac{V_1(\varphi_0)H^2}{\chi^2} + (\lambda-1)f_1(\bar{k})\frac{V_{0,\varphi\varphi}(\varphi_0)}{\chi^2} + \\ & - \frac{1}{2}\left[f_1(\bar{k}) - 2(\lambda-1)V_1(\varphi_0)\right]\frac{\dot{\varphi}_0^2}{\chi^4}\Bigg\}\bar{k}^2 + \\ & + \Bigg\{4(3\lambda-1)\left[V_{4,\varphi}(\varphi_0) + V_2(\varphi_0)\right]H^2 + \\ & + 2(\lambda-1)f_1(\bar{k})V_1(\varphi_0) + (3\lambda-1)\frac{V_4(\varphi_0)\dot{\varphi}_0 H}{\chi^2} + \\ & + (\lambda-1)[V_{4,\varphi}(\varphi_0) + V_2(\varphi_0)]\frac{\dot{\varphi}_0^2}{\chi^2}\Bigg\}\frac{\bar{k}^4}{\chi^2} + \Bigg\{ - 4(3\lambda-1)V_6(\varphi_0)H^2\chi^2 + \\ & + 2(\lambda-1)f_1(\bar{k})[V_2(\varphi_0) + V_{4,\varphi}(\varphi_0)]\chi^2 - (\lambda-1)V_6(\varphi_0)\dot{\varphi}_0^2\Bigg\}\frac{\bar{k}^6}{\chi^4} + \\ & + \Bigg\{ - 2(\lambda-1)f_1(\bar{k})V_6(\varphi_0)\chi^4 + \frac{1}{2}(\lambda-1)V_4(\varphi_0)^2\chi^2\Bigg\}\frac{\bar{k}^8}{\chi^6}\,; \end{aligned} \tag{7.27}$$

$$
\begin{aligned}
d(\bar{k})\, m^2_{\varphi\psi} = {} & \frac{9}{2}(3\lambda-1)^2\frac{\dot{\varphi}_0 H^3}{\chi^2} + \frac{3}{2}(3\lambda-1)(3\lambda-7)\frac{V_{0,\varphi}(\varphi_0)H^2}{\chi^2} + \\
& - \frac{3}{4}(3\lambda-1)\left[\dot{\varphi}_0^2 + 2V_0(\varphi_0)\right]\frac{\dot{\varphi}_0 H}{\chi^4} + \\
& - \frac{3}{2}(\lambda-1)\left[\frac{3}{2}\dot{\varphi}_0^2 + V_0(\varphi_0)\right]\frac{V_{0,\varphi}(\varphi_0)}{\chi^4} + \\
& - 3(3\lambda-1)\frac{\dot{\varphi}_0 H\Lambda}{\chi^2} - 3(\lambda-1)\frac{V_{0,\varphi}(\varphi_0)\Lambda}{\chi^2} + \\
& + \left\{2(3\lambda-1)f_2(\bar{k})\frac{\dot{\varphi}_0 H}{\chi^2} - (\lambda-1)[3f_1(\bar{k}) - 2f_2(\bar{k})]\frac{V_{0,\varphi}(\varphi_0)}{\chi^2}\right\}\bar{k}^2 + \\
& - \left\{3\frac{V_4(\varphi_0)\Lambda}{\chi^2} - \frac{9}{2}(3\lambda-1)\frac{V_4(\varphi_0)H^2}{\chi^2} + \right. \\
& \left. + \frac{3}{4}\left[\dot{\varphi}_0^2 + 2V_0(\varphi_0)\ \right]\frac{V_4(\varphi_0}{\chi^4}\right\}(\lambda-1)\bar{k}^4 + 2(\lambda-1)f_2(\bar{k})V_4(\varphi_0)\frac{\bar{k}^6}{\chi^2}\,;
\end{aligned}
\tag{7.28}
$$

$$
\begin{aligned}
d(\bar{k})\, m^2_{\psi} = {} & 3\Lambda\left[4(3\lambda-1)H^2 + (\lambda-1)\frac{\dot{\varphi}_0^2}{\chi^2}\ \right] + \\
& - \left[\frac{3}{2}(13\lambda-11)\dot{\varphi}_0^2 - 6V_0(\varphi_0)\right](3\lambda-1)\frac{H^2}{\chi^2} + \\
& - 78(3\lambda-1)^2H^4 - \frac{3}{4}(\lambda-1)\frac{\dot{\varphi}_0^4}{\chi^4} + \frac{3}{2}(\lambda-1)V_0(\varphi_0)\frac{\dot{\varphi}_0^2}{\chi^4} + \\
& + \left\{6[f_1(\bar{k}) - 4f_2(\bar{k})]\Lambda - 3(3\lambda-1)[13f_1(\bar{k}) - 12f_2(\bar{k})]H^2 + \right. \\
& - \frac{1}{2}[3f_1(\bar{k}) + 12f_2(\bar{k}) + 4]\frac{\dot{\varphi}_0^2}{\chi^2} - 8\frac{3\lambda-1}{\lambda-1}H^2 + \\
& \left. + 3[f_1(\bar{k}) - 4f_2(\bar{k})]\frac{V_0(\varphi_0)}{\chi^2}\right\}(\lambda-1)\bar{k}^2 + \\
& + \left\{4(\lambda-1)[-f_1(\bar{k}) + 2f_2(\bar{k})^2] + 8(3\lambda-1)(8g_2 + 3g_3)\frac{H^2}{\chi^2} + \right. \\
& \left. + 2(\lambda-1)(8g_2 + 3g_3)\frac{\dot{\varphi}_0^2}{\chi^4}\right\}\bar{k}^4 + \left\{4(\lambda-1)f_1(\bar{k})(8g_2 + 3g_3) + \right. \\
& \left. - 8(3\lambda-1)(8g_7 - 3g_8)\frac{H^2}{\chi^2} - 2(\lambda-1)(8g_7 - 3g_8)\frac{\dot{\varphi}_0^2}{\chi^4}\right\}\frac{\bar{k}^6}{\chi^2} + \\
& - 4(\lambda-1)f_1(\bar{k})(8g_7 - 3g_8)\frac{\bar{k}^8}{\chi^4}\,.
\end{aligned}
\tag{7.29}
$$

Note that all the coefficients of the mass matrix remain finite in the IR limit $\bar{k} \to 0$.

7.3.2 Observations

Remarkably enough, on setting $\bar{k} = 0$ the kinetic part of the Lagrangian becomes

$$c_\varphi \dot{\delta\varphi}_k^2 + c_\psi \dot{\psi}_k^2 + c_{\varphi\psi} \dot{\psi}_k \dot{\delta\varphi}_k \Big|_{\bar{k}=0} \propto \left(\frac{H}{\dot{\varphi}_0} \dot{\delta\varphi}_k + \dot{\psi}_k \right)^2 . \tag{7.30}$$

This suggests that the introduction of the Sasaki-Mukhanov variable ζ, defined as in Eq. (6.15), may reduce to just one the actual number of dynamical degrees of freedom, as also happened in the original formulation of the non-projectable version of HL Gravity [61], prior to the so-called "healthy extension".

In terms of ζ and ψ, the kinetic part of the Lagrangian is a quadratic form with coefficients given by:

$$d(\bar{k})\, c_\zeta = \left[2(3\lambda - 1) + (\lambda - 1) f_1(\bar{k}) \frac{\bar{k}^2}{H^2} \right] \frac{\dot{\varphi}_0^2}{\chi^2} \tag{7.31}$$

$$d(\bar{k})\, c'_\psi = \left[4(3\lambda - 1) + (\lambda - 1) \frac{\dot{\varphi}_0^2}{\chi^2 H^2} \right] f_1(\bar{k}) \bar{k}^2 \tag{7.32}$$

$$d(\bar{k})\, c_{\zeta\psi} = 2(\lambda - 1) \frac{\dot{\varphi}_0^2}{\chi^2 H^2} f_1(\bar{k}) \bar{k}^2 . \tag{7.33}$$

Observe that c'_ψ and $c_{\zeta\psi}$ both tend to zero as $\bar{k} \to 0$, whereas c_ζ is non-trivial as long as the matter field is present. However, we see how the presence of the term $f_1(\bar{k}) \neq 0$ - which is present only in the full non-projectable version of HL Gravity - alters the findings of Ref. [61], in that here both the metric degrees of freedom survive as dynamical variables.

To exclude the possibility that there may be a single dynamical metric fluctuation variable (at this point different from ζ), we need to evaluate the eigenvalues of the coefficient matrix of the kinetic part of the Lagrangian. Returning to the original variables φ and ψ, we consider the kinetic matrix[2]

$$\begin{pmatrix} \chi^2 c_\varphi & \frac{\chi c_{\varphi\psi}}{2} \\ \frac{\chi c_{\varphi\psi}}{2} & c_\psi \end{pmatrix} \tag{7.34}$$

[2]We have multiplied the c's by proper powers of χ in order to make the matrix dimensionally homogeneous. Such a rescaling is equivalent to considering $\delta\varphi_k/\chi$ and ψ as the two dynamical variables.

which has the following eigenvalues:

$$
\begin{aligned}
d(\bar{k})c_{1,2} = (3\lambda-1)\left(\frac{\dot{\varphi}_0^2}{\chi^2}+H^2\right)+\frac{13\lambda-5}{2}f_1(\bar{k})\bar{k}^2 \pm \\
\pm \left[(3\lambda-1)^2\left(\frac{\dot{\varphi}_0^2}{\chi^2}+H^2\right)^2 + \right. \\
\left. + (11\lambda-3)(3\lambda-1)\left(\frac{\dot{\varphi}_0^2}{\chi^2}-H^2\right)f_1(\bar{k})\bar{k}^2+\left(\frac{11\lambda-3}{2}\right)^2 f_1(\bar{k})^2\bar{k}^4\right]^{1/2}
\end{aligned}
\tag{7.35}
$$

It is easy to see that only for $f_1(\bar{k})=0$ one eigenvalue is exactly zero. However, in general both eigenvalues are non-vanishing and hence both degrees of freedom are dynamical. This sounds like bad news for the model. However, we shall now show that in the infrared limit one of the modes decouples (its mass tends to infinity).

First, however, let us consider under which conditions the linear cosmological perturbations are ghost-free. We realize that if

$$
\frac{1}{d(\bar{k})}\left[(3\lambda-1)\left(\frac{\dot{\varphi}_0^2}{\chi^2}+H^2\right)+\frac{13\lambda-5}{2}f_1(\bar{k})\bar{k}^2\right] < 0\,, \tag{7.36}
$$

then for sure we will have one negative eigenvalue, meaning that the extra dynamical degree of freedom will be ghost-like. In the opposite case, in which the expression on the l.h.s. of Eq. (7.36) is positive, one needs to look more carefully at the expressions. For convenience, we rewrite $c_{1,2}$ as

$$
c_{1,2} \;=\; A \pm \sqrt{B} \tag{7.37}
$$

and then check whether A^2-B is positive or negative in case $A>0$. The difference is

$$
A^2 - B \;=\; 2(3\lambda-1)\frac{f_1(\bar{k})\bar{k}^2}{d(\bar{k})}\,. \tag{7.38}
$$

In the IR limit, $f_1(\bar{k})$ tends to η, and thus the condition for ghost freeness is simply

$$
\eta \,>\, 0\,. \tag{7.39}
$$

For larger values of $\bar{k}$ it is not so easy to estimate the sign of the difference in (7.38), since we are dealing with several parameters - $\lambda, \eta, \eta_2, \eta_4$ - and in the most general case each one can have an arbitrary sign. The same difficulty arises when wishing to determine the region in parameter space where Eq. (7.36) is satisfied.

In the following we will assume that the inequality in Eq. (7.36) is reversed and that $\eta > 0$, so that the theory is ghost-free in the infrared. On introducing the approximation $H^2 \gg \dot{\varphi}_0^2/\chi^2$, reasonable because the right hand side of the inequality consists of just one of the positive terms appearing in the expression for H^2, the IR limit of the eigenvalues $c_{1,2}$ reads as follows:

$$c_1 \simeq \frac{1}{2}; \tag{7.40}$$

$$c_2 \simeq \eta\left(\frac{\bar{k}}{H}\right)^2 . \tag{7.41}$$

Note that the eigenvalue c_2 of the extra degree of freedom goes to zero in the IR limit, while from the expressions for the mass matrix coefficients - Eqs. (7.27), (7.28), (7.29) - we see that they do not tend to zero in the very same limit. Hence, if we suppose to rescale the new scalar gravitational degree of freedom such that it has canonical kinetic normalization in the IR, we see that its effective mass would diverge as $H/\bar{k}$. In this sense *the extra scalar degree of freedom decouples in the IR limit.* Hence, at late times it will not contribute to cosmological perturbations on the scales relevant to current cosmological observations. This also explains the results of Ref. [81], where it is found that the cosmological fluctuations in the non-projectable HL Gravity agree quite well with those in GR. On short-wavelength scales, however, the extra scalar gravitational mode has a chance to play a role. More specifically, it may effect the early evolution of fluctuations in inflationary cosmology on sub-Hubble scales and hence be relevant to the "trans-Planckian" problem [82, 83, 84] of the inflationary Universe scenario.

In the UV limit we have

$$d(\bar{k}) \simeq 2(\lambda - 1)f_1(\bar{k})\bar{k}^2 , \tag{7.42}$$

and from this we can easily show that the eigenvalues are

$$c_1 \simeq \frac{1}{2}; \tag{7.43}$$

$$c_2 \simeq 2\frac{3\lambda - 1}{\lambda - 1} . \tag{7.44}$$

The eigenvalue labeled as c_2 is negative for all values of λ between $1/3$ and 1. In this range of values of λ the extra scalar degree of freedom will be a ghost.

The transition between the IR and UV scales occurs, as expected and as already discussed, at $\bar{k} = H$. Thus, for applications to Cosmology the theory should be ghost-free both in the IR and UV. We can then conclude that it should be

$$\eta > 0 \quad \text{and} \quad \lambda > 1 \tag{7.45}$$

in order for the extra degree of freedom to be well-behaved.

Once more we stress on the fact that the limit $\lambda \to 1$ is smooth as long as $H \neq 0$.

7.4 Minkowski limit

We finally discuss the Minkowski limit of our analysis, which can be obtained simply by dropping the matter contribution from the action ($\varphi_0 = 0 = \delta\varphi$). We are then left with just one dynamical degree of freedom for scalar metric fluctuations, namely ψ. It can be observed that $m_\psi^2 = 0$ in the absence of matter and of the cosmological constant, *i.e.* the scalar mode is massless.

The constraint equations can easily be solved and yield:

$$\bar{k}^2 B_k(t) \;=\; \frac{3\lambda - 1}{\lambda - 1}\dot{\psi}_k(t) \tag{7.46}$$

and

$$\nu_k(t) \;=\; \frac{2f_2(\bar{k})}{f_1(\bar{k})}\psi_k(t)\,. \tag{7.47}$$

Inserting these expressions into the action for fluctuations and dropping the matter terms we are finally left with

$$\begin{aligned}\delta_2 S^{(s)} = 2\chi^2 \int dt \frac{d^3k}{(2\pi)^3}\Bigg\{ & \frac{3\lambda-1}{\lambda-1}\dot{\psi}_k^2 + \\ & + \left[\left(1 - 2\frac{f_2(\bar{k})^2}{f_1(\bar{k})}\right)\bar{k}^2 - (8g_2 + 3g_3)\frac{\bar{k}^4}{\chi^2} \; + (8g_7 - 3g_8)\frac{\bar{k}^6}{\chi^4}\right]\psi_k(t)^2\Bigg\}\,.\end{aligned} \tag{7.48}$$

In the IR limit the equation of motion becomes

$$\frac{3\lambda - 1}{\lambda - 1}\ddot{\psi}_k + \frac{2-\eta}{\eta}\bar{k}^2\psi_k \;=\; 0\,, \tag{7.49}$$

as we have already anticipated at the very beginning of this chapter. Let us restate here a pair of important remarks concerning the extra scalar mode:

1. it is ghost-like for $1/3 < \lambda < 1$;
2. it is classically unstable (tachyonic) unless $0 < \eta < 2$.

These conclusions are in perfect agreement with those in Ref. [81].

Chapter 8

Conclusions (II)

The study of linear cosmological perturbations in Hořava-Lifshitz Gravity (plus scalar field matter) let us acknowledge that the theory, as opposed to General Relavity, generically admits a scalar propagating degree of freedom of gravitational origin which may be either ghost-like or tachyonic (thus pathological) according to the way how some free parameters are chosen. The emergence of such an unwanted scalar degree of freedom is due to the fact that HL Gravity, which explicitly abandons Lorentz-invariance, is endowed with less symmetries than GR.

More specifically, concerning the projectable version of the theory, we found that the extra scalar mode is either ghost-like for $1/3 < \lambda < 1$, where λ is a dimensionless parameter which appears *e.g.* in Eq. (5.17), or tachyonic for $\lambda < 1/3$ and $\lambda > 1$. Hence, the linear cosmological perturbation theory is sick for all values of λ except for the value $\lambda = 1$ which corresponds to GR. Nevertheless, turning to the "strong coupling problem" first discussed in Ref. [42], we notice that the strong coupling problem in the limit $\lambda \to 1$ manifests itself in the divergence of the coefficients in the 2nd-order action for the extra degree of freedom. Tracing back the origin of this divergence, we see that it comes from the factor $(\lambda - 1)$ which multiplies the variable B in the super-momentum constraint, Eq. (7.10). The super-momentum constraint is used to solve for B, and hence a divergence arises in the limit $\lambda \to 1$. In General Relativity, instead, B is a pure gauge mode and can be set to zero from the outset.

Concerning the non-projectable version, we proved that there is a chance for the extra mode to be well-behaved. In particular, the theory is healthy both in the IR and in the UV limit if we require $\lambda > 1$ (or $\lambda < 1/3$ which, however, becomes the undesired option since GR is expected to be recovered for $\lambda = 1$) and $\eta > 0$, where η is another coupling which enters into the action, cf. Eq. (5.20).

The issue of strong-coupling does not emerge (at least) at the linear level, since in the case of non-projectable HL Gravity we use a combination of the super-momentum and Hamiltonian constraints to solve for B and in such a combination the limit $\lambda \to 1$ does not yield any singularity as long as $H \neq 0$.

Finally, we proved that in the IR limit the mass of the canonically normalized extra scalar gravitational degree of freedom tends to infinity, meaning that the extra mode decouples from the low-energy Physics or, if Cosmology is concerned, that the extra gravitational degree of freedom is harmless for late-times cosmological perturbations. Its presence at high-momenta may, however, lead to interesting consequences for Early Universe Cosmology which deserve further investigations.

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