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Estimating Persistent Betti Numbers For Discrete Shape Analysis

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Introduction

Persistent Topology is an innovative way of matching topology and geometry, and it proves to be an effective mathematical tool in Pattern Recognition [3, 24, 5]. This new research area is experiencing a period of intense theoretical progress, particularly in the form of the multidimensional *persistent Betti numbers* (PBNs; also called *rank invariant* in [6]). In order to express its full potential for applications, it has to interface with the typical environment of Computer Science: It must be possible to deal with a finite sampling of the object of interest, and with combinatorial representations of it.

A predecessor of the PBNs, the *size function* (i.e. PBNs at degree zero) already enjoys such a connection, in that it is possible to estimate it from a finite, sufficiently dense sampling [21], and it is possible to simplify the computation by processing a related graph [14]. Moreover, strict inequalities hold only in "blind strips", i.e. in the ω-neighbourhood of the discontinuity lines, where ω is the modulus of continuity of the measuring (filtering) function. Out of the blind strips, the values of the size function of the original object, of a ball covering of it, and of the related graph coincide.

The main purpose of this Thesis is to extend the previous result to the PBNs of any degree; to obtain that we have divided the work into three parts, corresponding to the three chapters. In the first one we shall introduce the preliminary results about persistent homology theory and the different tools that we shall need in the proof of the Blind Strips Theorem (Th. 2.2.1). Moreover we shall present, in Section 1.2, a further extension of a result concerning pseudo-critical points (see [9] for the original statements).

In the second chapter we shall provide the main results (Th. 2.2.1 and Th. 2.3.2), in which we merge the consequence of a proposition (Prop. 1.3.2) by P. Niyogi, S. Smale, and S. Weinberger in [27] and a special construction given in Lemma 2.1.3. Thanks to that, we shall be able to prove a double inequality that relates the PBNs of a compact Riemanniann submanifold X of $\mathbb{R}^m$ with a ball covering generated by the points of a suitable sampling of the object X itself. Such a result will allow us to re-introduce the idea of blind strips and, this time, extend it to any homology degree.

Following the previous concepts, in the last chapter we shall propose an alternative construction, with respect to chapter two; thanks to that we shall be able to decrease the error that we commit during all the process of approximation and association of the PBNs. To be more precise, we shall introduce a new way (see Th. 3.2.4) to relate the PBNs of our object of interest X (we consider only 1-submanifold in this case) and a simplicial complex S_ϵ generated by the points of the sampling [16]; the basic idea is that, now, we can use a more flexible approximation which allows us to decrease the number of points of the sampling or to reduce the error in the construction. As a result we obtain that the width of the blind strips is, in the first case, a half of the previous one and, in the second alternative, only a quarter.

To better clarify the meaning of the results presented in the last two chapters, in each of them an entire section, dedicated to examples, is present.

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Chapter 1

Preliminary Results

Before starting with the preliminary results, we think that it is appropriate to recall the most significant steps of the evolution of *persistent homology theory*. The first step, maybe the most important, was performed by P. Frosini in his PhD thesis [20]; in that work a method was introduced, to study an object not only as an entity itself, but as an entity endowed with a function that describes it. The main idea was to understand that the important item to study is not only the object itself, but the object seen through the eyes of an observer; in other words we have to focus our attention on what we want to analyze of the object, as e.g. colour or height. In order to do that, we need to define suitable functions, called *measuring functions*, that can encode the features of our analysis. At the beginning of the theory, the functions were built ad hoc, but soon the theory was generalized to any continuous functions with values in $\mathbb{R}$. The second passage was to build a method to numerically analyze the pair (object, function), the so called *size pair* (X, ϕ) (where, typically, X is a topological space and ϕ is a continuous function from X to $\mathbb{R}$). The method is based on the use of the lower level sets of ϕ ($\langle \phi \leq v \rangle = \{p \in X | \phi(p) \leq v\}$); in this way it is possible to filtrate the object X in portions of itself such that each portion is contained in the following one (according to the increasing values of the measuring function). Thanks to this construction it is possible to define a function $\rho : \mathbb{R}^2 \to \mathbb{N}$, called *size function*, that counts the number of path-connected components of the lower level set generated by v that contain at least one point in the lower level set generated by u, where $u, v \in \mathbb{R}^2$ and $u \leq v$. In this way all the topological information, related to the path-connected components, is encoded in a plane diagram. Moreover it was possible to prove that each diagram can be fully described by some particular points, called *corner points*. In other words all the information relative to a size pair (X, ϕ) is contained in a set of points of the plane.

Since path-connected components are also represented by the 0-degree homology, the second significant step in the theory was to generalize the result to all homology degrees. This passage has been introduced by E. Edelsbrunner in [13], [18] and this theory was named *persistent homology*. The main idea at the basis of this work is that, if you work with coefficients in a field $\mathbb{K}$, all the homology modules are vector spaces (in particular they have no torsion). This means that it is again possible to associate a number to each relation between lower level sets; in particular the theory studies the morphisms induced in homology by the inclusion of the associated lower level sets. Since all the homology modules are vector spaces, it is easy to compute the rank of these morphisms and, once again, we can fully describe our size pair with a set of points of the plane.

The first two steps take into account only functions with values in $\mathbb{R}$, but it is quite natural to analyze a feature of the object that is multidimensional, meaning that the feature is described by multiple values as, e.g. the RGB color (Red, Green, Blue). This idea has been developed by the team in Bologna in [3], but only in the case of size functions (0-degree homology).

The final step has been taken by the same team with [4]; this paper provides the final generalization that uses functions with values in $\mathbb{R}^n$ and analyzes all the homology degrees.

This chapter is divided into three sections. In the first one we shall state some definitions that are at the basis of persistent homology theory. In Section 1.2 we shall present a generalization of a result, introduced by P. Frosini in [9] that concerns the study of the so called pseudo-critical points. And, finally, in Section 1.3 we shall report some topological properties of the compact Riemannian submanifolds of $\mathbb{R}^m$.

Remark 1.0.1. To be more precise, it is important to underline the different types of functions used in the different steps. The theory of size functions (step 1) works with continuous functions. The theory of persistent homology (step 2) uses tame functions as in [13]. The theory of multidimensional size functions (step 3) operates again with continuous functions. The theory of multidimensional persistent homology (step 4) takes into account tame functions as described in [4], but a recent result [8] states that it is possible to extend to continuous functions in both one-dimensional and multidimensional persistent homology.

1.1 Persistent Betti Numbers

In this section, as the title suggests, we shall introduce and define the concept of persistent Betti numbers; after that, we shall define a function that counts these numbers (called persistent Betti numbers function) and we shall show some properties and methods to analyze it.
As already said in the introduction of the chapter, to avoid problems with torsion, we shall always work with coefficients in a field $\mathbb{K}$, so that all homology modules are vector spaces.
First we define the following partial relation $\prec$ (resp. $\preceq$) in $\mathbb{R}^n$: if $\vec{u} = (u_1, \ldots, u_n)$ and $\vec{v} = (v_1, \ldots, v_n)$, we write $\vec{u} \prec \vec{v}$ (resp. $\vec{u} \preceq \vec{v}$) if and only if $u_j < v_j$ (resp. $u_j \leq v_j$) for $j = 1, \ldots, n$. We also define Δ^+ as the open set $\{(\vec{u}, \vec{v}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \vec{u} \prec \vec{v}\}$.
As usual, a topological space X is *triangulable* if there is a finite simplicial complex, whose underlying space is homeomorphic to X; for a submanifold of a Euclidean space, it will mean that its triangulation can be extended to a domain containing it.
Studying the morphisms between homology modules induced by the inclusion

of the relative lower level sets, can also be seen as the study of the change in homology of the lower level sets relative to the increasing values of the measuring function. In other words we are interested in the changing of the values of the Betti numbers of the lower level sets through the inclusion of one in another. In fact it is exactly this change that tells us how the object X evolves with respect to the chosen function. In order to keep track of these changes, we need to introduce the following definition

Definition 1.1 (Persistent Betti Numbers)**.** *Let X be a triangulable space and $\vec{f} = (f_1, \ldots, f_n) : X \to \mathbb{R}^n$ be a continuous function. We denote by $X\langle \vec{f} \preceq \vec{u} \rangle$ the lower level subset $\{p \in X \mid f_j(p) \leq u_j, j = 1, \ldots, n\}$. Then, for each $i \in \mathbb{Z}$, the i-th persistent Betti number function of $(X, \vec{f})$ is $\beta_{(X,\vec{f},i)} : \Delta^+ \to \mathbb{N}$ defined as $\beta_{(X,\vec{f},i)}(\vec{u}, \vec{v}) = \dim(\mathrm{I}m f_{\vec{u}}^{\vec{v}})$, with*

$$f_{\vec{u}}^{\vec{v}} : \check{H}_i(X\langle \vec{f} \preceq \vec{u} \rangle) \to \check{H}_i(X\langle \vec{f} \preceq \vec{v} \rangle),$$

the homomorphism induced by the inclusion map of lower level sets $X\langle \vec{f} \preceq \vec{u} \rangle \subseteq X\langle \vec{f} \preceq \vec{v} \rangle$.

Here $\check{H}_i$ denotes the i-th Čech homology module. We use Čech homology because it guarantees some useful continuity properties at the limit [8].

For the sake of notation, from now on, we shall refer to persistent Betti numbers as PBNs, and to the persistent Betti numbers function as PBNF (PBNFs for the plural). Since we mostly work with a multidimensional setting (in the sense that the measuring functions are with values in $\mathbb{R}^n$) the PBNs and PBNFs are considered multidimensional. When we shall refer to the one-dimensional (or mono-dimensional) case we shall use 1d-PBNs and 1d-PBNFs respectively.

Remark 1.1.1. When the PBNFs were defined in terms of filtering functions in [13], these were so taken as to be tame (as said in Remark 1.0.1), but in our case, since we need continuous maps, we refer to ([Sect. 2.1][8]) for the relevant extension.

PBNs give us a way to analyze triangulable spaces through their homological properties. Then it is natural to introduce a distance for comparing them. As we shall show in Subsection 1.1.1 and 1.1.2, this has been done, and through this distance it has been possible to prove stability of the 1d-PBNFs and PBNFs under variations of the measuring function in the one-dimensional [13] and multidimensional case [8].

1.1.1 Stability Theorem

First we shall define the proper corner points and corner points at infinity, that are the root of the construction of the 1d-PBNFs representation and then we shall present the Representation Theorem (Th. 1.1.3) which allows to express the 1d-PBNFs in a compact way.
Next, we shall define the one-dimensional *matching distance* (or *bottleneck distance* in [13]); the choice of this distance has been done because, among all the other, this is the one that appears to be the best lower bound for the *natural pseudo-distance.* The natural pseudo-distance, introduced by the team in Bologna in [15], is a distance that arises when we try to compare two homeomorphic spaces in a direct way; in other words it comes out naturally when we take into account all possible homeomorphism between the two spaces and then we check which one minimizes the displacement of points (in relation to the measuring functions on each space). To be more precise, if $(X, \vec{f})$ and $(Y, \vec{f'})$ are two size pairs with X and Y homeomorphic, then the natural pseudo-distance is defined as $d((X, \vec{f}), (Y, \vec{f'})) = \inf_{\vec{f}} \max_{p \in X} ||\vec{f}(p) - \vec{f'}(\phi(p))||_\infty$, where ϕ varies among all the possible homeomorphisms between X and Y. The only problem in applications is that we need to look at all the possible homeomorphisms and that is computationally impossible; for this reason we use the matching distance that is easier to compute. For a complete dissertation we refer to [15].
After the definition, we shall continue showing how this distance can be used to state the one-dimensional *Stability Theorem* (Th. 1.1.5); the result of this theorem is extremely important for what concerns applications. In fact

the basic idea is that, even if we change a bit the values of the measuring functions (because e.g. there is some noise), also the distance between the 1d-PBNFs will change only a bit.

Definition 1.2 (Proper Corner Point). *For every point $d = (u, v) \in \Delta^+$ we define the number $\mu_i(d)$ as:*

$$\lim_{\varepsilon\to 0} \beta_{(X,f,i)}(u+\varepsilon, v-\varepsilon)-\beta_{(X,f,i)}(u-\varepsilon, v-\varepsilon)-\beta_{(X,f,i)}(u+\varepsilon, v+\varepsilon)+\beta_{(X,f,i)}(u-\varepsilon, v+\varepsilon).$$

The number $\mu_i(d)$ is called the multiplicity of d. Also if $\mu_i(d) > 0$ we call d a proper corner point of $\beta_{(X,f,i)}$.

Definition 1.3 (Corner Point At Infinity). *For every point of the form $(\overline{u}, \infty) \in \Delta^+$ we associate to them the line r of equation $u = \overline{u}, \overline{u} \in \mathbb{R}$, and we define $\mu_i(r)$ as:*

$$\lim_{\varepsilon\to 0} \beta_{(X,f,i)}\left(\bar{u} + \varepsilon, \frac{1}{\varepsilon}\right) - \beta_{(X,f,i)}\left(\bar{u} - \varepsilon, \frac{1}{\varepsilon}\right).$$

The number $\mu_i(r)$ will be called the multiplicity of r for $\beta_{(X,f,i)}$. When this finite number is strictly positive, we call r a corner point at infinity for $\beta_{(X,f,i)}$.

Remark 1.1.2. The multiset of all proper corner points and corner points at infinity for $\beta_{(X,f,i)}$, counted with their multiplicity, union the points of Δ (where Δ is the diagonal $\Delta = \{(u, v) \in \mathbb{R}^2 | u = v\}$), counted with infinite multiplicity, is the set of points of the persistent diagram $D_i(X, f)$. The complete description of persistence diagrams $D_i(X, f)$ can be found in [13]; in plain words they are the encoder of the information generated by the 1d-PBNFs (in the one-dimensional case they are exactly defined as a simple collection of points of $\mathbb{R}^2$ counted with multiplicity).

Using the idea of proper corner points and corner points at infinity it is possible to state a Representation Theorem, claiming that the value of the i-th 1d-PBNF can be fully determined by the set of these points and their multiplicity.

For the sake of simplicity, each line of equation $u = a$ will be identified to a point at infinity with coordinates (a, ∞).

Theorem 1.1.3 (Representation Theorem)**.** *For every $\bar{u} < \bar{v} < \infty$, it holds that*

$$\beta_{(X,f,i)}(\bar{u},\bar{v}) = \sum_{\substack{u \le \bar{u} \\ \bar{v} < v \le \infty}} \mu_i((u,v)).$$

The previous theorem claims that the value of $\beta_{(X,f,i)}(\bar{u},\bar{v})$ equals the number of corner points lying above and on the left of $(\bar{u},\bar{v})$. Thanks to that, we are able to compactly represent the i-th 1d-PBNF as formal series of corner points and corner lines.
For a detailed proof of the Theorem 1.1.3 we refer to [19][k-Triangle Lemma].
We are now ready to state the definition of the Matching Distance

Definition 1.4 (Matching Distance)**.** *Let X and Y be homotopically equivalent triangulable spaces, endowed with continuous functions $f : X \to \mathbb{R}$ and $f' : Y \to \mathbb{R}$. The* matching distance d_{match} *between $\beta_{(X,f,i)}$ and $\beta_{(Y,f',i)}$ is equal to the bottleneck distance between the persistence diagrams $D_i(X,f)$ and $D_i(Y,f')$ i.e.*

$$d_{match}\left(\beta_{(X,f,i)}, \beta_{(Y,f',i)}\right) = \inf_{\gamma} \max_{q \in D_i(X,f)} \|q - \gamma(q)\|_{\widetilde{\infty}},$$

where γ ranges over all multi-bijections between $D_i(X,f)$ and $D_i(Y,f')$, and for every $q = (u,v), q' = (u',v')$ in Δ^*,

$$\|q - q'\|_{\widetilde{\infty}} = \min\left\{\max\left\{|u-u'|, |v-v'|\right\}, \max\left\{\frac{v-u}{2}, \frac{v'-u'}{2}\right\}\right\},$$

with the convention about points at infinity that $\infty - y = y - \infty = \infty$ when $y \neq \infty$, $\infty - \infty = 0$, $\frac{\infty}{2} = \infty$, $|\infty| = \infty$, $\min\{c,\infty\} = c$ and $\max\{c,\infty\} = \infty$.

Remark 1.1.4. The classical matching distance d_{match} is defined only for the 1d-PBNFs associated with the same space. The use of $\beta_{(X,f,i)}$ and $\beta_{(Y,f',i)}$ is a quite new theoretical approach. The idea under this definition (also introduced in [11], but with a totally different setting) is that if the two spaces X and Y are homotopically equivalent, then the associated persistence diagrams can be totally compared. In other words, the homotopically equivalence guarantees that the number of points at infinity is the same in the two spaces, for all homology degrees. On the other hand, if the two spaces are not homotopically equivalent the matching distance becomes an extended distance with values in $\mathbb{R} \cup \infty$.

Using the previous definition, we can recall a result introduced in [8], that states the stability of the 1d-PBNFs with respect to perturbation of the measuring functions. The one-dimensional version is the following.

Theorem 1.1.5 (Stability Theorem)**.** *Let X be a triangulable space and $f, f' : X \to \mathbb{R}$ two continuous functions. Then, for every $i \in \mathbb{Z}$, we have that*

$$d_{match}\left(\beta_{(X,f,i)}, \beta_{(X,f',i)}\right) \leq \max_{p \in X} \operatorname{abs}(f(p) - f'(p)).$$

As said in the introduction of this subsection, this result implies an important stability property that is essential in the applications; in fact the theorem shows that the values encoded by the 1d-PBNFs are robust against small perturbations, thus the comparison with slightly noisy measuring functions can be done also in the worst case.
The previous dissertation is based on the one-dimensional setting, in the sense that all the measuring functions take values in $\mathbb{R}$; in the next subsection we shall extend to functions with values in $\mathbb{R}^n$.

1.1.2 Mono-dimensional Reduction

As has been briefly hinted in the introduction of the chapter, the multidimensional setting offers a wider opportunity in the choice of the measuring functions, this because we can now choose a function with values in $\mathbb{R}^n$ and that implies that we can describe more features at the same time.
On the other hand the multidimensional setting brings out new problems: e.g. there is not a direct description of the PBNFs into sets of points or a distance that gives a stability result. Nevertheless it is still possible to build a special construction that allows us to use the information generated

by the multidimensional setting, as shown by the team in Bologna in [10]. The main idea of this theory is to foliate the domain Δ^+ of the PBNFs with half-planes and then to compute the 1d-PBNFs on these half-planes. Since the half-planes are Δ^+ domains in $\mathbb{R}^2$, this method gives a way to reduce the computation of the multidimensional case to the computation of the one-dimensional one on each single leaf of the foliation (as stated in Theorem (Th. 1.1.6)). Moreover, with this construction, it is possible to define a multidimensional distance (Def. 1.6), based on the values of the one-dimensional distance on each leaf. Then, with this distance, we can state a stability result that guarantees the stability of the PBNFs respect to small changes of the multidimensional measuring functions (Multidimensional Stability Theorem (Th. 1.1.8)). It is also important to underline that it can be proved that the multidimensional matching distance is a better lower bound for the natural pseudo-distance, compared to the one-dimensional one. Thus the use of the multidimensional setting guarantees a better (or equal) discrimination power in our analysis.
To precisely define the multidimensional matching distance D_{match}, we need to technically define the concept of foliation and admissible pair. We also remember that all the following part has been introduced in [10], [3] and [8].

We start by recalling that the following parametrised family of half-planes in $\mathbb{R}^n \times \mathbb{R}^n$ is a foliation of Δ^+.

Definition 1.5 (Admissible Pairs)**.** *For every unit vector $\vec{l} = (l_1, \dots, l_n)$ of $\mathbb{R}^n$ such that $l_j > 0$ for $j = 1, \dots, n$, and for every vector $\vec{b} = (b_1, \dots, b_n)$ of $\mathbb{R}^n$ such that $\sum_{j=1}^n b_j = 0$, we shall say that the pair $(\vec{l}, \vec{b})$ is* admissible*. We shall denote the set of all admissible pairs in $\mathbb{R}^n \times \mathbb{R}^n$ by Adm_n. Given an admissible pair $(\vec{l}, \vec{b})$, we define the half-plane $\pi_{(\vec{l},\vec{b})}$ of $\mathbb{R}^n \times \mathbb{R}^n$ by the following parametric equations:*

$$\begin{cases} \vec{u} = s\vec{l} + \vec{b} \\ \vec{v} = t\vec{l} + \vec{b} \end{cases}$$

for $s, t \in \mathbb{R}$, with $s < t$.

The key property of this foliation is that the restriction of $\beta_{(X,\vec{f},i)}$ to each leaf can be seen as a particular 1d-PBNF, as the following theorem states.

Theorem 1.1.6 (Reduction Theorem). *Let $(\vec{l},\vec{b})$ be an admissible pair, and $F^{\vec{f}}_{(\vec{l},\vec{b})} : X \to \mathbb{R}$ be defined by setting*

$$F^{\vec{f}}_{(\vec{l},\vec{b})}(p) = \max_{j=1,\dots,n} \left\{ \frac{f_j(p) - b_j}{l_j} \right\} \ .$$

Then, for every $(\vec{u},\vec{v}) = (s\vec{l}+\vec{b}, t\vec{l}+\vec{b}) \in \pi_{(\vec{l},\vec{b})}$ the following equality holds:

$$\beta_{(X,\vec{f},i)}(\vec{u},\vec{v}) = \beta_{(X,F^{\vec{f}}_{(\vec{l},\vec{b})},i)}(s,t) \ .$$

As a consequence of the Reduction Theorem 1.1.6, we observe that the identity $\beta_{(X,\vec{f},i)} \equiv \beta_{(X,\vec{f}',i)}$ holds if and only if $d_{match}(\beta_{(X,F^{\vec{f}}_{(\vec{l},\vec{b})},i)}, \beta_{(X,F^{\vec{f}'}_{(\vec{l},\vec{b})},i)}) = 0$, for every admissible pair $(\vec{l},\vec{b})$.

Since we are working with a continuous foliation, it is normal to analyze what happens when we change leaf; luckily the next theorem gives a stability result respect to each leaf of the foliation, in the sense that a small change of the leaf implies a small change in the reduced 1d-PBNF.

Theorem 1.1.7 (Leaf Stability). *If $(X,\vec{f})$ is a size pair, $(\vec{l},\vec{b}) \in Adm_n$ and ϵ is a real number with $0 \leq \epsilon < \min_{i=1,\dots,n} l_i$, then for every admissible pair $(\vec{l}',\vec{b}')$ with $||(\vec{l},\vec{b}) - (\vec{l}',\vec{b}')||_\infty \leq \epsilon$, it hold that*

$$d_{match}(\beta_{(X,F^{\vec{f}}_{(\vec{l},\vec{b})},i)}, \beta_{(X,F^{\vec{f}}_{(\vec{l}',\vec{b}')},i)}) \leq \frac{\max_{p\in X} ||\vec{f}||_\infty + ||\vec{l}||_\infty + ||\vec{b}||_\infty}{\min_{i=1,\dots,n}(l_i(l_i-\epsilon))}$$

So, once again, we are guaranteed that our result is independent from the noise and that implies that the theory is compatible with the applications. Finally we are ready to define the multidimensional matching distance D_{match} and then state the Multidimensional Stability Theorem.

Definition 1.6 (Multidimensional Matching Distance). *Let $\vec{f},\vec{f}' : X \to \mathbb{R}^n$ be continuous functions and $\beta_{(X,\vec{f},i)}$, $\beta_{(X,\vec{f}',i)}$ be the corresponding PBNFs.*

Then:

$$D_{match}(\beta_{(X,\vec{f},i)},\beta_{(X,\vec{f}',i)}) = \sup_{(\vec{l},\vec{b})\in Adm_n} \min_j l_j \cdot d_{match}(\beta_{(X,F^{\vec{f}}_{(\vec{l},\vec{b})},i)},\beta_{(X,F^{\vec{f}'}_{(\vec{l},\vec{b})},i)}).$$

D_{match} *is a distance on* $\{\beta_{(X,\vec{f},i)} \,|\, \vec{f}: X \to \mathbb{R}^n$ *continuous*$\}$.

Theorem 1.1.8 (Multidimensional Stability Theorem)**.** *Let X be a triangulable space. For every $i \in \mathbb{Z}$, there exists a distance D_{match} on the set* $\{\beta_{(X,\vec{f},i)} \,|\, \vec{f}: X \to \mathbb{R}^n$ *continuous*$\}$ *such that*

$$D_{match}\left(\beta_{(X,\vec{f},i)},\beta_{(X,\vec{f}',i)}\right) \leq \max_{p\in X} \|\vec{f}(p) - \vec{f}'(p)\|_\infty.$$

Remark 1.1.9. Unfortunately in the multidimensional case it is not possible to obtain a full description, in terms of sets of points, for the PBNFs, but only an evaluation point by point. In our approach this is not a major problem; in fact we are interested in the shape comparison, not in the shape description. Thus, in this setting, we only need to compare the values of the PBNFs in a sufficiently large number of points, to obtain a suitable distance between the two objects.
For what concerns the shape description problem, the previous method gives only a theoretical way to get all the information, because we should compute the values of the PBNF in an infinite number of points relative to the leaves of the foliation.

1.2 Pseudo-critical Points

In this section we shall define what the pseudo-critical points of a C^1 function $\vec{f}$ are and we shall relate them to the discontinuities of the PBNF associated to $\vec{f}$. This construction will allow us to better understand the structure of the PBNFs and also will give us important information for the choice of the measuring functions in the applications.
The entire section is based on a theory, introduced by the team in Bologna in [9], that studies what happens in the one-dimensional setting. Thanks to

new results in the multidimensional theory, we are now able to present a generalization of that theory in the multidimensional setting. The generalization arises quite naturally from the original work; in fact the whole structure is basically the same and every proof is only adapted to the multidimensional setting; but the main ideas remain exactly the same.
The section is divided into two Subsections 1.2.1 and 1.2.2; in the first one we shall show some basic and well-known properties of the 1d-PBNFs; in the second one we shall state the main core of the theory. To better understand the construction of the theory we shall briefly resume the fundamental steps of Subsection 1.2.2: first of all we shall give the definition of pseudo-critical point (Def. 1.7) in relation to a particular leaf of the foliation as in Def. 1.5, thus we shall use the idea of the one-dimensional reduction (Subsection 1.1.2) and the Reduction Theorem 1.1.6. After that we shall relate the corner points of Def. 1.2 of the reduced PBNF (the PBNF generated by the function reduced according to the Reduction Theorem) to the pseudo-critical values of $\vec{f}$. The third step is to show an "if and only if" relation between the discontinuities of the PBNF of $\vec{f}$ and the discontinuities of the reduced PBNF, using the parametrisation values (s, t) of the foliation in Def. 1.5. Finally, using the two previous results we shall point out the relation between the discontinuities of the PBNF of $\vec{f}$ and the pseudo-critical values of $\vec{f}$ itself.
In the last part of the Subsection 1.2.2 we shall introduce a new definition of pseudo-critical point (Def. 1.8), that does not depend on the foliation. And thanks to that we shall state the main theorem (Th. 1.2.12) of the section, that encodes the general setting of the previous results.

From now on, let X be a triangulable space and let $f : X \to \mathbb{R}$ be, at least, a continuous function.

1.2.1 Monodimensional Case

In this subsection we shall report some definitions and propositions related to the 1d-PBNFs. The original statements have been introduced in [23] and then also in [8]. First of all we recall the definitions of proper corner

point as in Def. 1.2 and proper corner point at infinity as in Def. 1.3 and the Representation Theorem 1.1.3 of the previous section. Then, thanks to two corollaries (Cor. 1.2.1, Cor. 1.2.3) and a theorem (Th. 1.2.2), we shall be able to describe the connection between the discontinuities of the 1d-PBNFs and the proper corner points just recalled.
Before starting we underline that it is exactly the idea of the Representation Theorem 1.1.3 that leads to the following corollary, which gives us information about the structure of the discontinuities of the 1d-PBNFs. In particular it is easy to see that the following result implies that the representation of the 1d-PBNFs on the $\mathbb{R}^2$ domain consists of right triangles in which every discontinuity point is a point of a side and the hypotenuse is along the diagonal of $\mathbb{R}^2$.

Corollary 1.2.1. *Each discontinuity point $(\bar{u}, \bar{v})$ for $\beta_{(X,f,i)}$ is such that either $\bar{u}$ is a discontinuity point for $\beta_{(X,f,i)}(\cdot, \bar{v})$, or $\bar{v}$ is a discontinuity point for $\beta_{(X,f,i)}(\bar{u}, \cdot)$, or both these conditions hold.*

Furthermore, when we are in presence of a proper corner point we get additional information.

Theorem 1.2.2. *Let X be a closed C^1 Riemannian manifold, and let $f : X \to \mathbb{R}$ be a C^1 function. Then if $(\bar{u}, \bar{v})$ is a proper corner point for $\beta_{(X,f,i)}$, it follows that both $\bar{u}$ and $\bar{v}$ are critical values of f. If $(\bar{u}, \infty)$ is a corner point at infinity for $\beta_{(X,f,i)}$, it follows that $\bar{u}$ is a critical value of f.*

Proof. We confine ourselves to prove the former statement, since the proof of the latter is analogous.

First of all, let us remark that there exists a closed C^∞ Riemannian manifold $\widetilde{X}$ that is C^1-diffeomorphic to X through a C^1-diffeomorphism $h : \widetilde{X} \to X$ ([25, Th. 2.9]). Set $\tilde{f} = f \circ h$. Obviously, the i-th 1d-PBNF associated with the pairs $(\widetilde{X}, \tilde{f})$ and (X, f) coincide. Therefore, $(\bar{u}, \bar{v})$ is also a corner point for $\beta_{(\widetilde{X},\tilde{f},i)}$.

We observe that the claim of our theorem holds for a closed C^∞ Riemannian manifold endowed with a Morse function ([22, Th. 2.2]). Now, for every real value $\varepsilon > 0$ it is possible to find a Morse function $f_\varepsilon : \widetilde{X} \to \mathbb{R}$ such that $\max_{Q\in\widetilde{X}} |\tilde{f}(Q) - f_\varepsilon(Q)| \le \varepsilon$ and $\max_{Q\in\widetilde{X}} \left\|\nabla\tilde{f}(Q) - \nabla f_\varepsilon(Q)\right\| \le \varepsilon$: We can obtain f_ε by considering first the smooth function given by the convolution of $\tilde{f}$ with a suitable "regularizing" function, and then a Morse function f_ε approximating in $C^1(\widetilde{X}, \mathbb{R})$ the previous function ([26, Cor. 6.8]). Therefore, from the Stability Theorem 1.1.5 it follows that for every $\varepsilon > 0$ we can find a corner point $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ for the i-th 1d-PBNF $\beta_{(\widetilde{X}, f_\varepsilon, i)}$ with $\|(\bar{u}, \bar{v}) - (\bar{u}_\varepsilon, \bar{v}_\varepsilon)\|_\infty \le \varepsilon$ and $\bar{u}_\varepsilon, \bar{v}_\varepsilon$ as critical values for f_ε. Passing to the limit for $\varepsilon \to 0$ we obtain that both $\bar{u}$ and $\bar{v}$ are critical values for $\tilde{f}$. The claim follows by observing that, since $\tilde{f}$ and f have the same critical values, both $\bar{u}$ and $\bar{v}$ are also critical values for f. □

From the Representation Theorem 1.1.3 and Theorem 1.2.2 we can obtain the following corollary, refining Corollary 1.2.1 in the C^1 case (we skip the easy proof):

Corollary 1.2.3. *Let X be a closed C^1 Riemannian manifold, and let $f : X \to \mathbb{R}$ be a C^1 function. Let also $(\bar{u}, \bar{v})$ be a discontinuity point for $\beta_{(X,f,i)}$. Then at least one of the following statements holds:*

(i) *$\bar{u}$ is a discontinuity point for $\beta_{(X,f,i)}(\cdot, \bar{v})$ and $\bar{u}$ is a critical value for f;*

(ii) *$\bar{v}$ is a discontinuity point for $\beta_{(X,f,i)}(\bar{u}, \cdot)$ and $\bar{v}$ is a critical value for f.*

1.2.2 Discontinuities of PBNFs

In this subsection, as hinted in the introduction of the section, we are going to prove some new results about the discontinuities of $\beta_{(X,\vec{f},i)}$ and a way to connect them to a particular class of points, the so called pseudo-critical points. In order to do that, we shall follow three steps and then we shall state the general result; the description of each single passage will be given along the subsection.

Since we want to get information on how the PBNFs are represented in the multidimensional domain $\mathbb{R}^n \times \mathbb{R}^n$, we start by analyzing how they change

when the values of $\vec{u}$ and $\vec{v}$ change. The next lemma, that is a generalization of the one-dimensional case, shows a basic monotonicity property of the PBNFs.
We also recall that we need to work with a C^1 function $\vec{f}$ and a triangulable space X.

Lemma 1.2.4 (Monotonicity Lemma). *$\beta_{(X,\vec{f},i)}(\vec{u},\vec{v})$ is non-decreasing in the variable $\vec{u}$ and non-increasing in the variable $\vec{v}$.*

Proof. If $\vec{u}' \prec \vec{u} \prec \vec{v}$ we have that $H(X\langle \vec{f} \preceq \vec{u}'\rangle) \hookrightarrow H(X\langle \vec{f} \preceq \vec{u}\rangle) \hookrightarrow H(X\langle \vec{f} \preceq \vec{v}\rangle)$. Now if a cycle is born before $\vec{u}'$ and still alive till $\vec{v}$, it is also alive in $\vec{u}$. On the other hand, a cycle could born before $\vec{u}$ but after $\vec{u}'$, thus the associated PBNF cannot be decreasing.
If $\vec{u} \prec \vec{v} \prec \vec{v}'$ we have that $H(X\langle \vec{f} \preceq \vec{u}\rangle) \hookrightarrow H(X\langle \vec{f} \preceq \vec{v}\rangle) \hookrightarrow H(X\langle \vec{f} \preceq \vec{v}'\rangle)$. If a cycle is still alive at $\vec{v}'$ then it is also alive at $\vec{v}$, but a cycle could die after $\vec{v}$ and before $\vec{v}'$, thus the PBNF is non increasing. □

Before stating the next Definition we want to recall the Reduction Theorem 1.1.6. This is because we shall use the idea of the foliation with relative $(\vec{l},\vec{b})$-half-planes and also the reduced function g (for sake of notation we use g instead of $F^{\vec{f}}_{(\vec{l},\vec{b})}$). Now with all these tools we can finally define pseudo-critical points.

Definition 1.7 (Pseudocritical Points). *For every $Q \in X$, set $I_Q = \left\{ j \in \{1,\dots,n\} : \frac{f_j(Q)-b_j}{l_j} = g(Q) \right\}$. We shall say that Q is an $(\vec{l},\vec{b})$-*pseudo-critical point for $\vec{f}$ *if the convex hull of the gradients $\nabla f_j(Q)$, $j \in I_Q$, contains the null vector, i.e. for every $j \in I_Q$ there exists a real value λ_j such that $\sum_{j\in I_Q} \lambda_j \nabla f_j(Q) = \mathbf{0}$, with $0 \le \lambda_j \le 1$ and $\sum_{j\in I_Q} \lambda_j = 1$. If Q is an $(\vec{l},\vec{b})$-pseudo-critical point for $\vec{f}$, the value $g(Q)$ will be said an $(\vec{l},\vec{b})$-*pseudo-critical value for $\vec{f}$.

Remark 1.2.5. The concept of $(\vec{l},\vec{b})$-pseudo-critical point is strongly connected, via the function g introduced in Def. 1.7, with the notion of generalized gradient introduced by F. H. Clarke [12]. For a point $Q \in X$, the condition of being $(\vec{l},\vec{b})$-pseudo-critical for $\vec{f}$ corresponds to the one of being "critical" for the generalized gradient of g ([12, Prop. 2.3.12]). However, in this context we prefer to adopt a terminology highlighting the dependence on the considered half-plane.

With this definition we can state the following theorem, that is the first step to relate the discontinuities of the measuring function $\vec{f}$ to its $(\vec{l},\vec{b})$-pseudo-critical values. More precisely we start claiming the existence of a connection between the proper corner points of the reduced PBNF and the $(\vec{l},\vec{b})$-pseudo-critical value relative to $\vec{f}$.

Theorem 1.2.6. *Let (u,v) be a proper corner point of $\beta_{(X,g,i)}$, then both u and v are $(\vec{l},\vec{b})$-pseudo-critical value for $\vec{f}$. If (u,∞) is a corner point at infinity of $\beta_{(X,g,i)}$, then u is a $(\vec{l},\vec{b})$-pseudo-critical value for $\vec{f}$.*

Proof. The main idea of the proof is to show that the thesis holds for a C^1 function approximating the function $g : X \to \mathbb{R}$ in C^0, and prove that the property passes to the limit.

First of all we need to fix the degree i, then let's set $\Phi_j(Q) = \frac{f_j(Q)-b_j}{l_j}$ and choose $c \in \mathbb{R}$ such that $\min_{Q\in X} \Phi_j(Q) > -c$, for every $j = 1,\ldots,n$. Consider the function sequence (g_p), $p \in \mathbb{N}^+ = \mathbb{N}\setminus\{0\}$, where $g_p : X \to \mathbb{R}$ and $g_p(Q) = \left(\sum_{j=1}^n (\Phi_j(Q)+c)^p\right)^{\frac{1}{p}} - c$: Such a sequence converges uniformly to the function g. Indeed, for every $Q \in X$ and for every index p we have that

$$\begin{aligned}
|g(Q)-g_p(Q)| &= \left|\max_j \Phi_j(Q) - \left(\left(\sum_{j=1}^n (\Phi_j(Q)+c)^p\right)^{\frac{1}{p}} - c\right)\right| = \\
&= \left|\max_j\{\Phi_j(Q)+c\} - \left(\sum_{j=1}^n (\Phi_j(Q)+c)^p\right)^{\frac{1}{p}}\right| = \\
&= \left(\sum_{j=1}^n (\Phi_j(Q)+c)^p\right)^{\frac{1}{p}} - \max_j\{\Phi_j(Q)+c\} \le \\
&\le \max_j\{\Phi_j(Q)+c\}\cdot(n^{\frac{1}{p}}-1).
\end{aligned}$$

If we consider a proper corner point $\bar{C}$ of $\beta_{(X,g,i)}$, by the Stability Theorem 1.1.5 it follows that it is possible to find a large enough p and a proper corner point C_p of $\beta_{(X,g_p,i)}$, such that C_p is arbitrarily close to $\bar{C}$. Since C_p is a proper corner point of $\beta_{(X,g_p,i)}$, it follows from Corollary 1.2.3 that its coordinates are critical values of the C^1 function g_p. If we look at the abscissa of C_p (analogous considerations hold for the ordinate of C_p) it follows that there exists $Q_p \in X$ with $x(C_p) = g_p(Q_p)$ and (with respect to local coordinates $x_1, \ldots, x_m$ of the m-manifold X)

$$
\begin{aligned}
0 &= \frac{\partial g_p}{\partial x_1}(Q_p) = \left(\sum_{j=1}^{n}(\Phi_j(Q_p)+c)^p\right)^{\frac{1-p}{p}} \cdot \left(\sum_{j=1}^{n}(\Phi_j(Q_p)+c)^{p-1}\cdot\frac{\partial \Phi_j}{\partial x_1}(Q_p)\right)\\
&\vdots\\
0 &= \frac{\partial g_p}{\partial x_m}(Q_p) = \left(\sum_{j=1}^{n}(\Phi_j(Q_p)+c)^p\right)^{\frac{1-p}{p}} \cdot \left(\sum_{j=1}^{n}(\Phi_j(Q_p)+c)^{p-1}\cdot\frac{\partial \Phi_j}{\partial x_m}(Q_p)\right) .
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&\sum_{j=1}^{n} \ (\Phi_j(Q_p)+c)^{p-1}\cdot\frac{\partial \Phi_j}{\partial x_1}(Q_p) = 0\\
&\vdots\\
&\sum_{j=1}^{n} \ (\Phi_j(Q_p)+c)^{p-1}\cdot\frac{\partial \Phi_j}{\partial x_m}(Q_p) = 0 .
\end{aligned}
$$

Therefore, if we set

$$
\boldsymbol{v}_p = (v_p^1, \ldots, v_p^n) = \left((\Phi_1(Q_p)+c)^{p-1}, \ldots, (\Phi_n(Q_p)+c)^{p-1}\right),
$$

we can write ${}^tJ(Q_p) \cdot {}^t\boldsymbol{v}_p = \mathbf{0}$, where $J(Q_p)$ is the Jacobian matrix of $\vec{\Phi} = (\Phi_1, \ldots, \Phi_n)$ computed at the point Q_p. Thanks to the compactness of X, we can assume (possibly by extracting a subsequence) that (Q_p) converges to a point $\bar{Q}$. We can define $\boldsymbol{u}_p = \frac{\boldsymbol{v}_p}{\|\boldsymbol{v}_p\|_\infty}$. By compactness (recall that $\|\boldsymbol{u}_p\|_\infty = 1$) we can also assume (possibly by considering a subsequence) that the sequence

$(\boldsymbol{u}_p)$ converges to a vector $\bar{\boldsymbol{u}} = (\bar{u}^1, \dots, \bar{u}^n)$, where $\bar{u}^j = \lim_{p\to\infty} \frac{v_p^j}{\|\boldsymbol{v}_p\|_\infty}$ and $\|\bar{\boldsymbol{u}}\|_\infty = 1$. Obviously ${}^t\mathcal{J}(Q_p) \cdot {}^t\boldsymbol{u}_p = \mathbf{0}$ and hence we have

$$ {}^t\mathcal{J}(\bar{Q}) \cdot {}^t\bar{\boldsymbol{u}} = \mathbf{0}. \tag{1.1}$$

Since for every index p and for every $j = 1, \dots, n$ the relation $0 < u_p^j \le 1$ holds, for each $j = 1, \dots, n$ the condition $0 \le \bar{u}^j = \lim_{p\to\infty} u_p^j \le 1$ is satisfied. We can now recall that $g(\bar{Q}) = \max_j \Phi_j(\bar{Q})$, by definition, and consider the set $I_{\bar{Q}} = \{j \in \{1, \dots, n\} : \Phi_j(\bar{Q}) = g(\bar{Q})\} = \{j_1, \dots, j_h\}$. For every $r \notin I_{\bar{Q}}$ the component $\bar{u}^r$ is equal to 0, since $0 \le u_p^r = \left(\frac{\Phi_r(Q_p)+c}{\max_j\{\Phi_j(Q_p)+c\}}\right)^{p-1}$ and $\lim_{p\to\infty} \frac{\Phi_r(Q_p)+c}{\max_j\{\Phi_j(Q_p)+c\}} = \frac{\Phi_r(\bar{Q})+c}{g(\bar{Q})+c}$, which is strictly less than 1 for $\Phi_r(\bar{Q}) < g(\bar{Q})$. Hence we have $\bar{\boldsymbol{u}} = \bar{u}^{j_1} \cdot \boldsymbol{e}_{j_1} + \dots + \bar{u}^{j_h} \cdot \boldsymbol{e}_{j_h}$, where $\boldsymbol{e}_j$ is the j^{th} vector of the standard basis of $\mathbb{R}^n$. Thus, from equality (1.1) we have $\sum_{w=1}^h \bar{u}^{j_w} \cdot \frac{\partial \Phi_{j_w}}{\partial x_1}(\bar{Q}) = 0, \dots, \sum_{w=1}^h \bar{u}^{j_w} \cdot \frac{\partial \Phi_{j_w}}{\partial x_m}(\bar{Q}) = 0$, that is $\sum_{w=1}^h \frac{\bar{u}^{j_w}}{l_{j_w}} \cdot \frac{\partial f_{j_w}}{\partial x_1}(\bar{Q}) = 0, \dots, \sum_{w=1}^h \frac{\bar{u}^{j_w}}{l_{j_w}} \cdot \frac{\partial f_{j_w}}{\partial x_m}(\bar{Q}) = 0$, since $\Phi_j = \frac{f_j - b_j}{l_j}$. Hence, $\sum_{w=1}^h \frac{\bar{u}^{j_w}}{l_{j_w}} \nabla f_{j_w}(\bar{Q}) = \mathbf{0}$. By recalling that $\bar{u}^{j_w} \ge 0$, $l_{j_w} > 0$ and $\bar{\boldsymbol{u}}$ is a non–vanishing vector, it follows immediately that $\sum_{w=1}^h \frac{\bar{u}^{j_w}}{l_{j_w}} > 0$ and therefore the convex hull of the gradients $\nabla f_{j_1}(\bar{Q}), \dots, \nabla f_{j_h}(\bar{Q})$ contains the null vector. Thus, $\bar{Q}$ is an $(\vec{l}, \vec{b})$-pseudo-critical point for $\vec{f}$ and hence $g(\bar{Q})$ is an $(\vec{l}, \vec{b})$-pseudo-critical value for $\vec{f}$. Moreover, from the uniform convergence of the sequence (g_p) to g and from the continuity of the function g, we have (recall that $\bar{C} = \lim_{p\to\infty} C_p$)

$$x(\bar{C}) = \lim_{p\to\infty} x(C_p) = \lim_{p\to\infty} g_p(Q_p) = g(\bar{Q}).$$

In other words, the abscissa $x(\bar{C})$ of a proper corner point of the persistent diagram $D_i(X, g)$ is the image of an $(\vec{l}, \vec{b})$-pseudo-critical point $\bar{Q}$ through g, i.e. an $(\vec{l}, \vec{b})$-pseudo-critical value for $\vec{f}$. An analogous reasoning holds for the ordinate $y(\bar{C})$ of a persistent corner point.

□

The next lemma points out another property of the 1d-PBNFs, which will be used in the proof of Proposition (Prop.1.2.8).

Lemma 1.2.7. *Assume that* (X, f), (X, f') *are two size pairs, with* $f, f' : X \to \mathbb{R}$*. If* $d_{match}\big(\beta_{(X,f,i)}, \beta_{(X,f',i)}\big) \le 2\varepsilon$*, then it holds that*

$$\beta_{(X,f,i)}(u-\varepsilon, v+\varepsilon) \le \beta_{(X,f',i)}(u+\varepsilon, v-\varepsilon),$$

for every (u, v) *with* $u + \varepsilon < v - \varepsilon$*.*

Proof. Let Δ^* be the set given by $\Delta^+ \cup \{(a, \infty) : a \in \mathbb{R}\}$. For every (u, v) with $u < v$, let us define the set $L_{(u,v)} = \{(\sigma, \tau) \in \Delta^* : \sigma \le u, \tau > v\}$. By the Representation Theorem 1.1.3 we have that $\beta_{(X,f,i)}(u-\varepsilon, v+\varepsilon)$ equals the number of proper corner points and corner points at infinity for $\beta_{(X,f,i)}$ belonging to the set $L_{(u-\varepsilon,v+\varepsilon)}$. Since $d_{match}\left(\beta_{(X,f,i)}, \beta_{(X,f',i)}\right) \le 2\varepsilon$, the number of proper corner points and corner points at infinity for $\beta_{(X,f',i)}$ in the set $L_{(u+\varepsilon,v-\varepsilon)}$ is not less than $\beta_{(X,f,i)}(u-\varepsilon, v+\varepsilon)$. The reason is that the change from f to f' does not move the corner points more than 2ε, with respect to the max-norm, because of the Stability Theorem 1.1.5. By applying the Representation Theorem 1.1.3 once again to $\beta_{(X,f',i)}$, we get our thesis. □

Thanks to Lemma 1.2.7 we can state the following proposition, which represents another step in the direction of our final goal. In this particular case, we claim that there is a double implication between being a discontinuity point for the PBNF related to the original measuring function $\vec{f}$ and being a discontinuity point of the the reduced PBNF. This property holds when it is possible to express the chosen point $(\vec{u}, \vec{v})$ as a specific parametrisation of the leaf generated by (s, t).

Proposition 1.2.8. *A point* $(\vec{u}, \vec{v}) = (s\vec{l} + \vec{b}, t\vec{l} + \vec{b}) \in \pi_{(\vec{l},\vec{b})}$ *is a discontinuity point for* $\beta_{(X,\vec{f},i)}$ *if and only if* (s, t) *is a discontinuity point for* $\beta_{(X,g,i)}$*.*

Proof. Obviously, if (s, t) is a discontinuity point for $\beta_{(X,g,i)}$, then $(\vec{u}, \vec{v}) = (s\vec{l} + \vec{b}, t\vec{l} + \vec{b}) \in \pi_{(\vec{l},\vec{b})}$ is a discontinuity point for $\beta_{(X,\vec{f},i)}$, because of the Reduction Theorem 1.1.6. To prove the inverse implication, we shall verify the contrapositive statement, i.e. if (s, t) is not a discontinuity point for $\beta_{(X,g,i)}$, then $(s\vec{l} + \vec{b}, t\vec{l} + \vec{b})$ is not a discontinuity point for $\beta_{(X,\vec{f},i)}$. Indeed, if

(s,t) is not a discontinuity point for $\beta_{(X,g,i)}$, then $\beta_{(X,g,i)}$ is locally constant at (s,t). So we can choose a real number $\eta > 0$ such that

$$\beta_{(X,g,i)}(s-\eta, t+\eta) = \beta_{(X,g,i)}(s+\eta, t-\eta). \tag{1.2}$$

Thanks to the Leaf Stability Theorem 1.1.7, we can then consider a real value $\varepsilon = \varepsilon(\eta)$ with $0 < \varepsilon < \min_{j=1,\ldots,n} l_j$ such that for every admissible pair $(\vec{l}', \vec{b}')$ with $\left\|(\vec{l}, \vec{b}) - (\vec{l}', \vec{b}')\right\|_\infty \leq \varepsilon$, the relation $d(\beta_{(X,g,i)}, \beta_{(X,g',i)}) \leq \frac{\eta}{2}$ holds, where $\beta_{(X,g',i)}$ is the 1d-PBNF corresponding to the half-plane $\pi_{(\vec{l}', \vec{b}')}$. By applying Lemma 1.2.7 twice and the monotonicity Lemma 1.2.4 of $\beta_{(X,g',i)}$ in each variable, we have

$$\begin{aligned} \beta_{(X,g,i)}(s-\eta, t+\eta) &\leq \beta_{(X,g',i)}(s-\frac{\eta}{2}, t+\frac{\eta}{2}) \leq \\ &\leq \beta_{(X,g',i)}(s+\frac{\eta}{2}, t-\frac{\eta}{2}) \leq \beta_{(X,g,i)}(s+\eta, t-\eta). \end{aligned} \tag{1.3}$$

Because of equality (1.2) we have that the inequalities (1.3) imply

$$\begin{aligned} \beta_{(X,g,i)}(s-\eta, t+\eta) &= \beta_{(X,g',i)}(s-\frac{\eta}{2}, t+\frac{\eta}{2}) = \\ &= \beta_{(X,g',i)}(s+\frac{\eta}{2}, t-\frac{\eta}{2}) = \beta_{(X,g,i)}(s+\eta, t-\eta). \end{aligned} \tag{1.4}$$

Therefore, once again because of the monotonicity of $\beta_{(X,g',i)}$ in each variable, for every (s',t') with $\|(s,t)-(s',t')\|_\infty \leq \frac{\eta}{2}$ and for every $(\vec{l}', \vec{b}')$ with $\|(\vec{l}, \vec{b}) - (\vec{l}', \vec{b}')\|_\infty \leq \varepsilon$ the equality $\beta_{(X,g',i)}(s',t') = \beta_{(X,g,i)}(s,t)$ holds. By applying the Reduction Theorem 1.1.6 we get $\beta_{(X,\vec{f},i)}(s'\vec{l}'+\vec{b}', t'\vec{l}'+\vec{b}') = \beta_{(X,\vec{f},i)}(s\vec{l}+\vec{b}, t\vec{l}+\vec{b})$. In other words, $\beta_{(X,\vec{f},i)}$ is locally constant at the point $(\vec{u}, \vec{v})$, and hence $(\vec{u}, \vec{v})$ is not a discontinuity point for $\beta_{(X,\vec{f},i)}$. □

Remark 1.2.9. Let us observe that Proposition 1.2.8 holds under weaker hypotheses, i.e. in the case that X is a non-empty, compact and locally connected Hausdorff space. However, for the sake of simplicity, we prefer here to confine ourselves to the setting assumed at the beginning of the present section.

The following theorem is the final step relative to the study of the $(\vec{l}, \vec{b})$-pseudo-critical points, and it associates the discontinuities of $\beta_{(X,\vec{f},i)}$ to the $(\vec{l}, \vec{b})$-pseudo-critical values of $\vec{f}$.

Theorem 1.2.10. *Let $(\vec{u},\vec{v}) \in \Delta^+$ be a discontinuity point for $\beta_{(X,\vec{f},i)}$, with $(\vec{u},\vec{v}) = (s\vec{l}+\vec{b}, t\vec{l}+\vec{b}) \in \pi_{(\vec{l},\vec{b})}$. Then it follows that either $\vec{u}$ is a discontinuity point for $\beta_{(X,\vec{f},i)}(\cdot,\vec{v})$ and s is an $(\vec{l},\vec{b})$-pseudo-critical value for $\vec{f}$, or $\vec{v}$ is a discontinuity point for $\beta_{(X,\vec{f},i)}(\vec{u},\cdot)$ and t is an $(\vec{l},\vec{b})$-pseudo-critical value for $\vec{f}$, or both the previous conditions hold.*

Proof. By Proposition 1.2.8 we have that (s,t) is a discontinuity point for $\beta_{(X,g,i)}$, and from Proposition 1.2.1 it follows that either s is a discontinuity point for $\beta_{(X,g,i)}(\cdot,t)$ or t is a discontinuity point for $\beta_{(X,g,i)}(s,\cdot)$, or both these conditions hold. Let us now suppose that s is a discontinuity point for $\beta_{(X,g,i)}(\cdot,t)$. Since $\beta_{(X,g,i)}(\cdot,t)$ is monotonic, then there exists an arbitrarily small real value $\varepsilon > 0$ such that $\beta_{(X,g,i)}(s-\varepsilon,t) \neq \beta_{(X,g,i)}(s+\varepsilon,t)$. Moreover, the following equalities hold because of the Reduction Theorem 1.1.6:

$$\beta_{(X,g,i)}(s-\varepsilon,t) = \beta_{(X,\vec{f},i)}((s-\varepsilon)\vec{l}+\vec{b}, t\vec{l}+\vec{b}) = \beta_{(X,\vec{f},i)}(\vec{u}-\varepsilon\vec{l},\vec{v})$$
$$\beta_{(X,g,i)}(s+\varepsilon,t) = \beta_{(X,\vec{f},i)}((s+\varepsilon)\vec{l}+\vec{b}, t\vec{l}+\vec{b}) = \beta_{(X,\vec{f},i)}(\vec{u}+\varepsilon\vec{l},\vec{v}).$$

By setting $\vec{\varepsilon} = \varepsilon\vec{l}$, we get $\beta_{(X,\vec{f},i)}(\vec{u}-\vec{\varepsilon},\vec{v}) \neq \beta_{(X,\vec{f},i)}(\vec{u}+\vec{\varepsilon},\vec{v})$. Therefore $\vec{u}$ is a discontinuity point for $\beta_{(X,\vec{f},i)}(\cdot,\vec{v})$. Moreover, since s is a discontinuity point for $\beta_{(X,g,i)}(\cdot,t)$, from the Representation Theorem 1.1.6 it follows that s is the abscissa of a corner point (possibly at infinity), and hence by Theorem 1.2.6 we have that s is an $(\vec{l},\vec{b})$-pseudo-critical value for $\vec{f}$. Analogously we can examine the case that t is a discontinuity point for $\beta_{(X,g,i)}(s,\cdot)$, and get our statement. □

In order to generalize the previous result to the case of pseudo-critical points that do not depend on foliation, we need to give the following definition.

Definition 1.8. *Let $\vec{\chi} : X \to \mathbb{R}^n$ be a C^1 function. A point $P \in X$ is said to be a* pseudo-critical point for $\vec{\chi}$ *if the convex hull of the gradients $\nabla\chi_i(P)$, $j = 1,\ldots,n$, contains the null vector, i.e. there exist $\lambda_1,\ldots,\lambda_n \in \mathbb{R}$ such that $\sum_{j=i}^n \lambda_j \cdot \nabla\chi_j(P) = \mathbf{0}$, with $0 \leq \lambda_j \leq 1$ and $\sum_{j=1}^n \lambda_j = 1$. If P is a pseudo-critical point of $\vec{\chi}$, then $\vec{\chi}(P)$ will be called a* pseudo-critical value for $\vec{\chi}$.

Remark 1.2.11. Definition 1.8 corresponds to the Fritz John necessary condition for optimality in Nonlinear Programming [2]. We shall use the term "pseudo-critical" just for the sake of conciseness. For further references see [28]. The concept of pseudo-critical point is strongly related also to the one of Jacobi Set [17]. In literature, pseudo-critical points are also called Pareto critical points.

In the following, we shall say that $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ is a *projection* if there exist n indexes $j_1, \dots, j_n$ such that $\varphi((x_1, \dots, x_m)) = (x_{i_1}, \dots, x_{i_n})$, for every $\vec{x} = (x_1, \dots, x_m) \in \mathbb{R}^m$.

We are now ready to give the main result of this section.

Theorem 1.2.12. *Let $(\vec{u}, \vec{v}) \in \Delta^+$ be a discontinuity point for $\beta_{(X,\vec{f},i)}$. Then at least one of the following statements holds:*

(i) *$\vec{u}$ is a discontinuity point for $\beta_{(X,\vec{f},i)}(\cdot, \vec{v})$ and a projection φ exists such that $\varphi(\vec{u})$ is a pseudo-critical value for $\varphi \circ \vec{f}$;*

(ii) *$\vec{v}$ is a discontinuity point for $\beta_{(X,\vec{f},i)}(\vec{u}, \cdot)$ and a projection φ exists such that $\varphi(\vec{v})$ is a pseudo-critical value for $\varphi \circ \vec{f}$.*

Proof. By Theorem 1.2.10 we have that either $\vec{u}$ is a discontinuity point for $\beta_{(X,\vec{f},i)}(\cdot, \vec{v})$, or $\vec{v}$ is a discontinuity point for $\beta_{(X,\vec{f},i)}(\vec{u}, \cdot)$, or both these conditions hold. Let us now confine ourselves the assumption that $\vec{u}$ is a discontinuity point for $\beta_{(X,\vec{f},i)}(\cdot, \vec{v})$ and prove that a projection φ exists such that $\varphi(\vec{u})$ is a pseudo-critical value for $\varphi \circ \vec{f}$. The proof in the case that $\vec{v}$ is a discontinuity point for $\beta_{(X,\vec{f},i)}(\vec{u}, \cdot)$ proceeds in quite a similar way. Consider the half-plane $\pi_{(\vec{l},\vec{b})}$ of the foliation containing the point $(\vec{u}, \vec{v})$, and the pair (s, t) such that $(\vec{u}, \vec{v}) = (s \cdot \vec{l} + \vec{b}, t \cdot \vec{l} + \vec{b})$. Since $\vec{u}$ is a discontinuity point for $\beta_{(X,\vec{f},i)}(\cdot, \vec{v})$, by applying once more Theorem 1.2.10 we obtain that s is an $(\vec{l}, \vec{b})$-pseudo-critical value for $\vec{f}$. Therefore, by definition of g, there exist a point $P \in X$ and some indexes $j_1, \dots, j_n$ with $1 \le n \le m$, such that $s = g(P) = \frac{f_{j_1}(P) - b_{j_1}}{l_{j_1}} = \dots = \frac{f_{j_h}(P) - b_{j_h}}{l_{j_h}}$ and $\sum_{w=1}^{n} \lambda_w \cdot \nabla \vec{f}_{j_w}(P) = \mathbf{0}$, with $0 \le \lambda_w \le 1$ for $w = 1, \dots, n$, and $\sum_{w=1}^{n} \lambda_w = 1$. Let us now consider the projection $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ defined by setting $\varphi(\vec{u}) = (u_{j_1}, \dots, u_{j_n})$. Since $(\vec{u}, \vec{v}) = (u_1, \dots, u_m, v_1, \dots, v_m) = (s \cdot l_1 + b_1, \dots, s \cdot l_m + b_m, t \cdot l_1 + b_1, \dots, t \cdot l_m + b_m)$, we observe that $u_{j_w} = \left(\frac{f_{j_w}(P) - b_{j_w}}{l_{j_w}}\right) \cdot l_{j_w} + b_{j_w} = f_{j_w}(P)$, for every $w = 1, \dots, n$. Therefore it follows that $\varphi(\vec{u})$ is a pseudo-critical value for $\varphi \circ \vec{f}$. □

Remark 1.2.13. We stress that Theorem 1.2.12 restate the result obtained in Theorem 1.2.10, providing a necessary condition for discontinuities of multidimensional size functions that does not depend on the foliation of the domain Δ^+.

1.3 Topological Properties of Riemannian Submanifolds

As hinted in the introduction of the chapter, we are interested in getting information on a submanifold of $\mathbb{R}^m$. In particular we want to find relations between our submanifold and a related ball covering generated by the points of the sampling. For this purpose, we now state some properties of compact Riemannian submanifolds of $\mathbb{R}^m$, especially referred to such an approximating covering. Definition (Def. 1.9) and Proposition (Prop. 1.3.2) are due to P. Niyogi, S. Smale, and S. Weinberger [27]. The main idea is that, under suitable hypotheses, it is possible to get, from a sampling of a submanifold, a b covering whose union retracts on it.

Before moving to the proposition we need to introduce the concept of open normal bundle and condition number, this because they underlie the construction of the covering and they also give important information about the error of the approximation. Thus, by *normal bundle* N_s of radius s at a point $p \in X$, we mean the collection of all vectors of length less than s anchored at p and with direction normal to X (orthogonal to Tan_p, the tangent space of X in p).

The embedding of the open normal bundle N_s of radius s described above is a tubular neighbourhood of X in $\mathbb{R}^m$, $Tub_s = \{p + \eta \in \mathbb{R}^m \,|\, p \in X, \eta \in Tan_p^\perp, \|\eta\| < s\}$, where $Tan_p^\perp$ denotes the set of vectors normal to Tan_p. Tub_s can also be seen as $\bigcup_{p \in X}(Tan_p^\perp \cap B(p,s))$, where $B(p,s)$ is the ball of radius s centred at p. For a detailed definition and discussion on it we refer to [1].

A *condition number* $\frac{1}{\tau}$ is associated with a compact Riemannian submanifold X of $\mathbb{R}^m$.

Definition 1.9. *τ is the largest number such that every open normal bundle B about X of radius s is embedded in $\mathbb{R}^m$ for $s < \tau$.*

In plain words, the condition number tells when a submanifold is "better" than another to be approximated: the bigger τ is, the better we can approximate the submanifold, in the sense that we shall need less points to sample it in an accurate way. On the other hand, if τ is small, it means that the submanifold is badly conditioned and we shall need more points.

Remark 1.3.1. We notice that the number τ is directly correlated to both the local and the global curvature aspects of the submanifold, to be more precise τ controls the curvature of the submanifold at every point. Then, if we are interested in analyzing the curvature through the second fundamental form (as in [7]), we can take into account an important result shown in ([27, Sect. 6]) ; this result states that the norm of the second fundamental form is bounded by $\frac{1}{\tau}$.
Moreover, the number τ is related to another important tool for the computational geometry that is the concept of medial axis (and the relative local feature size). Now if we define $\mathcal{G}$ as the medial axis relative to X and for every $p \in X$ we define the local feature size $\sigma(p)$ as the distance of p from $\mathcal{G}$ (the distance is defined as $d(p,\mathcal{G}) = \inf_{g \in \mathcal{G}} ||p - g||$), then we obtain that

$$\tau = \inf_{p \in X} \sigma(p).$$

With τ as before, we can state the next proposition

Proposition 1.3.2. *[27, Prop. 3.1]*

Let X be a compact Riemannian submanifold of $\mathbb{R}^m$. Let $L = \{c_1, \ldots, c_k\}$ be a collection of points of X, and let $U = \bigcup_{j=1,\ldots,k} B(c_j, \delta)$ be the union of balls of $\mathbb{R}^m$ with centre at the points of L and radius δ. Now, if L is such that for every point $p \in X$ there exists an $c_j \in L$ such that $\|p - c_j\| < \frac{\delta}{2}$, then, for every $\delta < \sqrt{\frac{3}{5}}\tau$, X is a deformation retract of U. So they have the same homology.

The previous proposition states exactly what we need to relate the submanifold X to a ball covering U generated by the sampling L. Moreover, the deformation retract property and the fact that they share the same homology will be critical in the next chapter, in which we shall use those properties to correlate the PBNF of the submanifold with the one of its covering. Since the retraction will be an important tool too, in the next remark we shall try to give a brief explanation of its construction, as stated in the proof of the previous proposition.

Remark 1.3.3. The proof of Proposition 1.3.2 gives us a way to construct a retraction $\pi : U \to X$ and a homotopy $F : U \times I \to U$ such that $F(q, 0) = q$ and $F(q, 1) = \pi(q)$.
Let $\pi_0 : Tub_\tau \to X$ be the canonical projection from the tubular neighbourhood of radius τ of X onto X. Then π is the restriction of π_0 to U for which it holds:

$$\pi(q) = \arg\min_{p \in X} \|q - p\|.$$

Then the homotopy is given by

$$F(q, t) = (1 - t)q + t\pi(q).$$

It is also important to observe that the retraction π moves the points of U less than δ; this is because the trajectory of $\pi(q)$ always remains inside a ball of U that contains q (q can be contained in the intersection of different balls), for every $q \in U$. In fact $\pi^{-1}(q) = U \cap Tan_q^\perp \cap B(q, \tau)$ (for a complete argument we refer to ([27, Sect. 4]).

Besides the condition number τ, we need another tool to evaluate the error in the approximation. Since we work with continuous measuring functions, it could happen that the function that we chose changes suddenly; or, in other words, that the variation of the function could be high. In this case we need to take into account the fast variation and merge it with the error of the approximation, namely the radius of the balls of the covering. To do that we introduce the concept of *modulus of continuity* that encodes these information.

Definition 1.10 (Modulus of Continuity)**.** *Let $\vec{f} : \mathbb{R}^m \to \mathbb{R}^n$ be a continuous function. Then, for $\varepsilon \in \mathbb{R}^+$, the modulus of continuity $\Omega(\varepsilon)$ of $\vec{f}$ is:*

$$\Omega(\varepsilon) = \max_{j=1,\ldots,n} \sup \Big\{ \mathrm{abs}(f_j(\vec{p}) - f_j(\vec{p'})) \mid \vec{p}, \vec{p'} \in \mathbb{R}^m, \ \|\vec{p} - \vec{p'}\| \leq \varepsilon \Big\}.$$

In other words $\Omega(\varepsilon)$ is the maximum over all moduli of continuity of the single components of $\vec{f}$.

We shall conclude this section with an important remark, that underlines the differences between the two spaces that we use in our setting.

Remark 1.3.4. First, we notice that the spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ play two different rôles in our arguments: the ambient space of our submanifolds (which will always be $\mathbb{R}^m$) is endowed with the classical Euclidean norm and has no partial order relation on it. On the other hand, the range of the measuring functions ($\mathbb{R}^n$ throughout) is endowed with the max norm and with the partial order relation $\preceq$, as defined at the beginning of Section 1.1.

Chapter 2

Estimating Persistent Betti Numbers

In this chapter we shall present the main results of this Thesis; these results try to give a solution to a fundamental problem that arises in applications. That is, unluckily, that all the theories presented in Chapter 1 are based on the study of a topological space X (or worse of a submanifold of Euclidean space); obviously in the real application cases the object of interest does not meet those requirements. In fact, normally, we have access only to a finite approximation of the object, as for example the pixels of a photo or a 3D laser sampling. In order to remedy this situation we tried to find a way to relate the topological properties analyzed by the PBNFs of the original object (possibly unknown), with the ones of a particular covering generated by the points of the approximation of the object itself. More precisely, the covering U is generated by the union of balls (of the ambient space $\mathbb{R}^m$) centred in the points of the sampling. This peculiar covering has the property that the object X is a deformation retract of it and thus they share the same homology (the proof can be found in Proposition 1.3.2). Luckily, this kind of construction can be extended to the lower level sets of $\vec{f}$ (with $\vec{f}: X \to \mathbb{R}^n$ continuous) and consequently it can be used for the description of the PBNFs.

Thanks to these results we are able to approximate the PBNFs of X knowing only the ones of U; obviously, since we are working with a covering, our result will not be valid in the whole domain of the PBNF of U. In fact, when we are near to the discontinuity sets of the PBNF the theory does not give an accurate result, but when we are far from these sets we obtain the exact value of the PBNF of X, i.e. we are able to recover the topological information of the object X taking into account only the approximation U. It is also clear that the width (or dimension) of the areas of uncertainly depends on the quality of the approximation.
A similar argument can be made also in the case in which we consider, as an approximation, a particular finite simplicial complex built from the points of the sampling.

The chapter is divided into three sections. In the first one we shall present two lemmas that will be used in the proof of the main results. The second one is the core of chapter and of the entire Thesis, inside this section we shall state the main theorem (Th. 2.2.1) and we shall describe some examples. In the last section we shall extend the result of the main theorem when we consider the combinatorial nature of the problem.

2.1 Retracts

Aim of this section is to yield two rather general results, which will be specialized to Theorem (Th. 2.2.1) and Lemma (Lem. 2.3.1). Throughout this section, Y will be a compact triangulable submanifold of $\mathbb{R}^m$ and V will be a compact, triangulable subspace of $\mathbb{R}^m$ such that Y is a deformation retract of V, with retraction r and homotopy $G : V \times I \to V$ from the identity of V, 1_V, to r. Moreover $\forall y \in Y$, $\forall v \in r^{-1}(y)$, $\forall t \in I$ we assume that $(r \circ G)(v, t) = y$.
Let also $\vec{f} : \mathbb{R}^m \to \mathbb{R}^n$ be a continuous function, and $\vec{f}_Y$ and $\vec{f}_V$ be the restrictions of $\vec{f}$ to Y and V respectively.

Lemma 2.1.1. *$Y\langle \vec{f}_Y \preceq \vec{x}\rangle$ is a deformation retract of $V\langle \vec{f}_Y \circ r \preceq \vec{x}\rangle$.*

Proof. Let $r_{\vec{x}} : V\langle \vec{f}_Y \circ r \preceq \vec{x}\rangle \to Y\langle \vec{f}_Y \preceq \vec{x}\rangle$ be the restriction of r to $V\langle \vec{f}_Y \circ r \preceq \vec{x}\rangle$. It is well-defined since, by the definition of the two sets, $r_{\vec{x}}(V\langle \vec{f}_Y \circ r \preceq \vec{x}\rangle) \subseteq Y\langle \vec{f}_Y \preceq \vec{x}\rangle$. We now set $G_{\vec{x}} : V\langle \vec{f}_Y \circ r \preceq \vec{x}\rangle \times I \to V\langle \vec{f}_Y \circ r \preceq \vec{x}\rangle$ as the restriction of G to $V\langle \vec{f}_Y \circ r \preceq \vec{x}\rangle \times I$. This restriction is well-defined, because the path from v to $r_{\vec{x}}(v)$ is all contained in $V\langle \vec{f}_Y \circ r \preceq \vec{x}\rangle$, thanks to the assumptions on G and r. Moreover, it is continuous and for every $v \in V\langle \vec{f}_Y \circ r \preceq \vec{x}\rangle$, $G_{\vec{x}}(v, 0) = v$ and $G_{\vec{x}}(v, 1) = r_{\vec{x}}(v)$. So it is the searched for deformation retraction.

□

In simple terms, the previous lemma states that the lower level sets of Y, that in our specific case it can be identified with the original object, are a deformation retract of the lower level sets of V, that in our setting can be considered as the cover U; this kind of relation can be obtained thanks to the definition of the retraction and its properties.

Remark 2.1.2. Since the homotopy G is relative to Y (i.e. keeps the points of Y fixed throughout), this is what is called a *strong* deformation retract in [29].

Now let $\varepsilon = \max_{v\in V} \|r(v) - v\|$ and $\vec{\omega}(\varepsilon) = (\Omega(\varepsilon), \dots, \Omega(\varepsilon)) \in \mathbb{R}^n$, where Ω is the modulus of continuity of $\vec{f}$ as in 1.10.

Lemma 2.1.3. *If $(\vec{z}, \vec{w})$ is a point of Δ^+ and if $\vec{z} + \vec{\omega}(\varepsilon) \prec \vec{w} - \vec{\omega}(\varepsilon)$, then*

$$\beta_{(V,\vec{f}_V,i)}(\vec{z} - \vec{\omega}(\varepsilon), \vec{w} + \vec{\omega}(\varepsilon)) \le \beta_{(Y,\vec{f}_Y,i)}(\vec{z}, \vec{w}) \le \beta_{(V,\vec{f}_V,i)}(\vec{z} + \vec{\omega}(\varepsilon), \vec{w} - \vec{\omega}(\varepsilon))$$

Proof. First, observe that there are inclusions

$$\gamma : Y\langle \vec{f}_Y \preceq \vec{z}\rangle \to Y\langle \vec{f}_Y \preceq \vec{w}\rangle$$
$$\varphi : Y\langle \vec{f}_Y \preceq \vec{w}\rangle \to V\langle \vec{f}_V \preceq \vec{w} + \vec{\omega}(\varepsilon)\rangle$$
$$\psi : V\langle \vec{f}_V \preceq \vec{z} - \vec{\omega}(\varepsilon)\rangle \to V\langle \vec{f}_V \preceq \vec{w} + \vec{\omega}(\varepsilon)\rangle$$

The fact that r moves every point not more than ε (since $\varepsilon = \max_{v\in V} \|r(v) - v\|$) implies that also the inclusions $\eta : V\langle \vec{f}_V \preceq \vec{z} - \vec{\omega}(\varepsilon)\rangle \to$

$V\langle \vec{f}_Y \circ r \preceq \vec{z}\rangle$ and $\theta : V\langle \vec{f}_Y \circ r \preceq \vec{z}\rangle \to V\langle \vec{f}_V \preceq \vec{w} + \vec{\omega}(\varepsilon)\rangle$ make sense. Let further $\bar{r} = r_{\vec{z}} \circ \eta$, where $r_{\vec{z}}$ is as in the proof of Lemma 2.1.1 with $\vec{x} = \vec{z}$. Now we have the following (not necessarily commutative) diagram:

$$\begin{array}{ccc} V\langle \vec{f}_V \preceq \vec{z} - \vec{\omega}(\varepsilon)\rangle & \xrightarrow{\bar{r}} & Y\langle \vec{f}_Y \preceq \vec{z}\rangle \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\gamma} \\ V\langle \vec{f}_V \preceq \vec{w} + \vec{\omega}(\varepsilon)\rangle & \xleftarrow[\varphi]{} & Y\langle \vec{f}_Y \preceq \vec{w}\rangle \end{array}$$

The next step is to prove that ψ is homotopic to $\varphi \circ \gamma \circ \bar{r}$. Let $\bar{G} : V\langle \vec{f}_V \preceq \vec{z} - \vec{\omega}(\varepsilon)\rangle \times I \to V\langle \vec{f}_V \preceq \vec{w} + \vec{\omega}(\varepsilon)\rangle$ be the composition $\bar{G} = \theta \circ G_{\vec{z}} \circ (\eta \times 1_I)$, where $G_{\vec{z}}$ is as in the proof of Lemma 2.1.1. Now, for every $v \in V\langle \vec{f}_V \preceq \vec{z} - \vec{\omega}(\varepsilon)\rangle$, we have $\bar{G}(v, 0) = G(v, 0) = v = \psi(v)$ and $\bar{G}(v, 1) = G(v, 1) = r(v) = \bar{r}(v) = \varphi \circ \gamma \circ \bar{r}(v)$.

Since homotopic maps induce the same homomorphisms in homology, we have (setting $\psi^* = \psi^{w+\vec{\omega}(\varepsilon)}_{z-\vec{\omega}(\varepsilon)}$ and $\gamma^* = \gamma^{\vec{w}}_{\vec{z}}$, and φ^*, $\bar{r}^*$ are the homology homomorphisms induced by φ and $\bar{r}$ respectively)

$$\begin{aligned} \beta_{(V,\vec{f}_V,i)}(\vec{z} - \vec{\omega}(\varepsilon), \vec{w} + \vec{\omega}(\varepsilon)) &= \dim \mathrm{Im}(\psi^*) = \dim \mathrm{Im}(\varphi^* \circ \gamma^* \circ \bar{r}^*) \leq \\ &\leq \dim \mathrm{Im}(\gamma^*) = \beta_{(Y,\vec{f}_Y,i)}(\vec{z}, \vec{w}) \end{aligned}$$

concluding the first part of the proof.
For the second inequality we use the following commutative (as will be proved) diagram, with analogous definitions of maps γ', ψ', φ' and $\bar{r}'$:

$$\begin{array}{ccc} Y\langle \vec{f}_Y \preceq \vec{z}\rangle & \xrightarrow{\varphi'} & V\langle \vec{f}_V \preceq \vec{z} + \vec{\omega}(\varepsilon)\rangle \\ {\scriptstyle\gamma'}\downarrow & & \downarrow{\scriptstyle\psi'} \\ Y\langle \vec{f}_Y \preceq \vec{w}\rangle & \xleftarrow[\bar{r}']{} & V\langle \vec{f}_V \preceq \vec{w} - \vec{\omega}(\varepsilon)\rangle \end{array}$$

Here ψ' is well defined because we are assuming $\vec{z} + \vec{\omega}(\varepsilon) \prec \vec{w} - \vec{\omega}(\varepsilon)$.

Then, passing to homology, we have (with analogous settings for the starred symbols):

$$\begin{aligned}\beta_{(Y,\vec{f}_Y,i)}(\vec{z},\vec{w}) &= \dim \mathrm{I}m(\gamma'^*) = \dim \mathrm{I}m(\bar{r}'^* \circ \psi'^* \circ \varphi'^*) \leq \\ &\leq \dim \mathrm{I}m(\psi'^*) = \beta_{(V,\vec{f}_V,i)}(\vec{z}+\vec{\omega}(\varepsilon), \vec{w}-\vec{\omega}(\varepsilon))\end{aligned}$$

To prove the commutativity of the diagram, we observe that $\bar{r}'$ is the identity map on the points of Y. Since $Y\langle \vec{f}_Y \preceq \vec{z}\rangle \subseteq Y$, we have that $\bar{r}' \circ \psi' \circ \varphi'$ is the canonical inclusion of $Y\langle \vec{f}_Y \preceq \vec{z}\rangle$ in $Y\langle \vec{f}_Y \preceq \vec{w}\rangle$. □

The previous result shows how it can be possible to relate together, with a double inequality, the PBNFs of two different spaces, bound together by a particular retraction. The first part of the double inequality is proved through the use of the result of Lemma 2.1.1 and the fact that the retraction moves the points less than ϵ. The second part is proved in an analogous way, but this time it exploits the fact that the space Y is contained inside V (instead of the retraction we use the inclusion map).
This result will be included in the proof of the main theorem, where the setting of our interest will be introduced.

2.2 Ball Coverings

Throughout this section, X will be a compact Riemannian (triangulable) submanifold of $\mathbb{R}^m$. As hinted in the introduction of the chapter, we want to get information on X out of a finite set of points. First, the points will be sampled on X itself, then even in a (narrow) neighbourhood. In both cases, the idea is to consider a covering of X made of balls centred on the sampling points.

What we get, is a double inequality which yields an estimate of the PBNFs of X within a fixed distance from the discontinuity sets of the PBNFs (meant as integer functions on Δ^+) of the union U of the balls of the covering, but even offers the exact value of it at points sufficiently far from the discontinuity sets.

When we foliate the domain Δ^+ of the PBNFs as in [4, Sect. 2.1] or [8, Sect. 3] — or simply when $n = 1$ — the discontinuity sets are (possibly infinite) line segments, and the regions where only the inequality holds appear as strips around them (which we colloquially call "blind strips"). The width of such strips is a representation of the approximation error, in that it is directly related to $\Omega(\delta)$, where $1/\delta$ represents the density of the sampling.

2.2.1 Points on X

Let $\delta < \sqrt{\frac{3}{5}}\tau$ and let $L = \{c_1, \dots, c_k\}$ be a set of points of X such that for every $p \in X$ there exists an $c_j \in L$ for which $\|p - c_j\| < \dfrac{\delta}{2}$. Let U be the union of the balls $B(c_j, \delta)$ of radius δ centred at c_j, $j = 1, \dots, k$. So all conditions of Proposition 1.3.2 are satisfied.
For the remaining part of the chapter let $\vec{f} : \mathbb{R}^m \to \mathbb{R}^n$ be a continuous function and let $\vec{f_X}$ and $\vec{f_U}$ be the restriction of $\vec{f}$ to X and U respectively.

Theorem 2.2.1 (Blind Strips Theorem)**.** *If $(\vec{u}, \vec{v})$ is a point of Δ^+ and if $\vec{u} + \vec{\omega}(\delta) \prec \vec{v} - \vec{\omega}(\delta)$, where $\vec{\omega}(\delta) = (\Omega(\delta), \dots, \Omega(\delta)) \in \mathbb{R}^n$, then*

$$\beta_{(U,\vec{f_U},i)}(\vec{u} - \vec{\omega}(\delta), \vec{v} + \vec{\omega}(\delta)) \le \beta_{(X,\vec{f_X},i)}(\vec{u}, \vec{v}) \le \beta_{(U,\vec{f_U},i)}(\vec{u} + \vec{\omega}(\delta), \vec{v} - \vec{\omega}(\delta))$$

Proof. By Lemma 2.1.3, with $Y = X$ (recalling that a compact Riemanniann submanifold of $\mathbb{R}^m$ is triangulable), $V = U$. □

Blind Strips Theorem 2.2.1 represents the main result of the whole Thesis; therefore we think that it is appropriate to spend some words on it. The idea of comparing topological properties of different spaces, of which one is the approximation of the other, is not new. In fact a similar result can be found in [21], in which however only the 0-degree homology case is considered. The extension of such a result, that at the beginning, seemed to be natural and direct, has proved to be very complex and deep. The main problem is the one of working with an approximation that does not change the topological properties of the object, both from the global point of view and from the local one (but in a different way). This is because, if the local alteration is

controlled, then the double inequality presented in the theorem can be managed; otherwise if the global features are modified, every kind of connection disappears.
Once we passed the first hurdle, thanks to Proposition 1.3.2; we started to deal with the problem of transporting that kind of construction to the lower level sets. This step has been solved with the introduction of Lemma 2.1.1 and 2.1.3, which allowed us to finally state the result of the main theorem.
The most important consequence (as we shall show in the examples) is that, when both the extremes of the double inequality coincide, the value of all the three PBNFs will be the same, regardless of their origins. In other words, when we are far away from the discontinuity sets of the PBNF of the approximation we are able to express, with total exactness, the value of the PBNF relative to X. Remembering that the object X can be unknown or only virtually known (we know only some features), we underline that this result allows to study the object using only its approximation. Thus, since in the applications everything is an approximation, we can link this kind of field with the continuous one of the geometric abstraction.

2.2.2 Points near X

So far we have approximated X by points picked up on X itself, but it is also possible to choose the points near X, by respecting some constraints. Once more, this is possible thanks to a result of P. Niyogi, S. Smale, and S. Weinberger in [27].

Proposition 2.2.2. *[27, Prop. 7.1] Let $L = \{c_1, \dots, c_k\}$ be a set of points in the tubular neighbourhood of radius s around X and $U = \bigcup_{j=1,\dots,k} B(c_j, \delta)$ be the union of the balls of $\mathbb{R}^m$ centred at the points of L and with radius δ. If for every point $p \in X$, there exists a point $c_j \in L$ such that $\|p - c_j\| < s$, then U is a deformation retract of X, for all $s < (\sqrt{9} - \sqrt{8})\tau$ and $\delta \in \left(\frac{(s+\tau)-\sqrt{s^2+\tau^2-6s\tau}}{2}, \frac{(s+\tau)+\sqrt{s^2+\tau^2-6s\tau}}{2}\right)$.*

Then, as with Blind Strips Theorem 2.2.1, we have, with an analogous proof:

Theorem 2.2.3. *Under the hypotheses of Proposition 2.2.2, if* $(\vec{u}, \vec{v})$ *is a point of* Δ^+ *and if* $\vec{u} + \vec{\omega}(\delta + s) \prec \vec{v} - \vec{\omega}(\delta + s)$*, where* $\vec{\omega}(\delta + s) = (\Omega(\delta + s), \ldots, \Omega(\delta + s)) \in \mathbb{R}^n$ *, then*

$$\beta_{(U,\vec{f}_U,i)}(\vec{u} - \vec{\omega}(\delta + s), \vec{v} + \vec{\omega}(\delta + s)) \leq \beta_{(X,\vec{f}_X,i)}(\vec{u}, \vec{v}) \leq$$
$$\leq \beta_{(U,\vec{f}_U,i)}(\vec{u} + \vec{\omega}(\delta + s), \vec{v} - \vec{\omega}(\delta + s)).$$

Since this result is the extension of the Blind Strips Theorem 2.2.1, we do not think that it is necessary to repeat the qualitative discussion. Anyway, it is important to underline that this result permits to work and to manage errors due to the inaccuracy of the approximation. In other words, even if the points of the sampling are not on the object X, but we know their maximum distance from it, we can still build a double inequality that continues to give us some important topological information.

2.2.3 Examples

The following examples show how Blind Strips Theorem 2.2.1 can be used for applications. Let X be a circle of radius 4 in $\mathbb{R}^2$ (Fig. 2.1); we observe that τ is exactly the radius of X, so $\tau = 4$. In order to create a well defined approximation we need that $\delta < \sqrt{\frac{3}{5}}\tau$.

In the first example we have taken $\delta = 0.5$. Now, to satisfy the hypothesis of the Blind Strips Theorem 2.2.1 (that for every $p \in X$ there exist an $c_j \in L$ such that $\|p - c_j\| < \frac{\delta}{2}$), we have chosen 64 points c_j on X. Moreover we have sampled X uniformly, so that there is a point every $\frac{\pi}{32}$ radians Figure 2.2. We stick to the monodimensional case, choosing $f : \mathbb{R}^2 \to \mathbb{R}$, with $f(x, y) = \text{abs}(y)$. U is the resulting ball union.

Figures 2.3 and 2.4 represent the PBNF at degree zero of X and U respectively. Δ^+ is the half-plane above the diagonal line, and the numbers are the values of the PBNFs in the triangular regions they are written in. In Figure 2.3 there is only a big triangle where the value 2 signals the two different

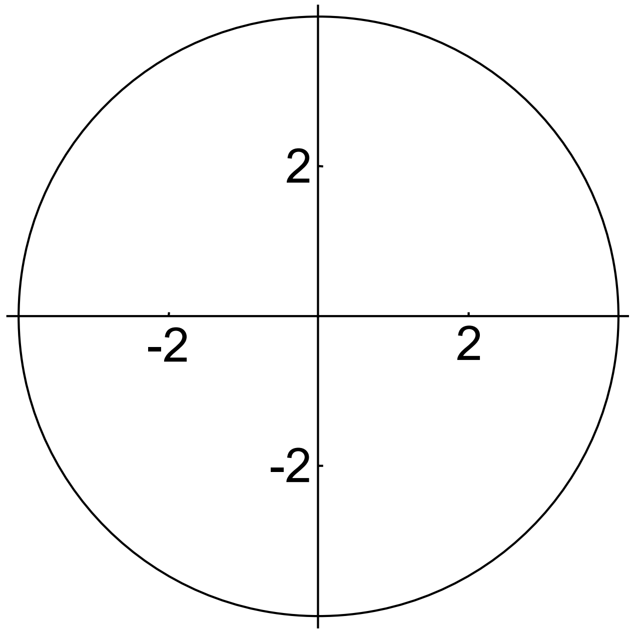

Figure 2.1: The circle of radius 4, X.

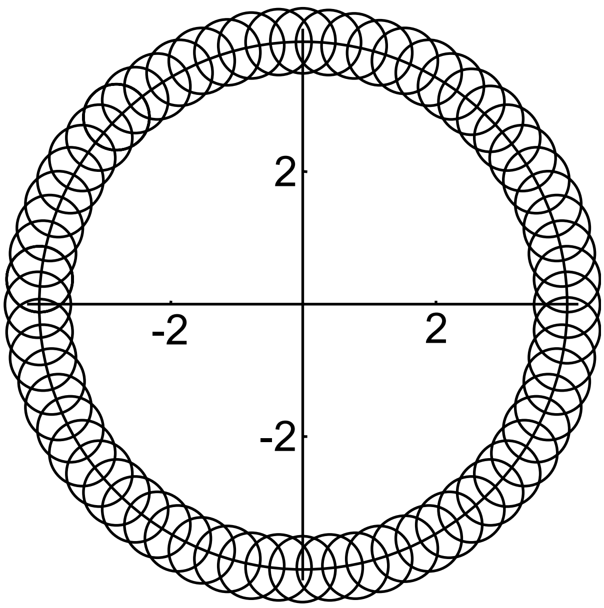

Figure 2.2: The ball union U.

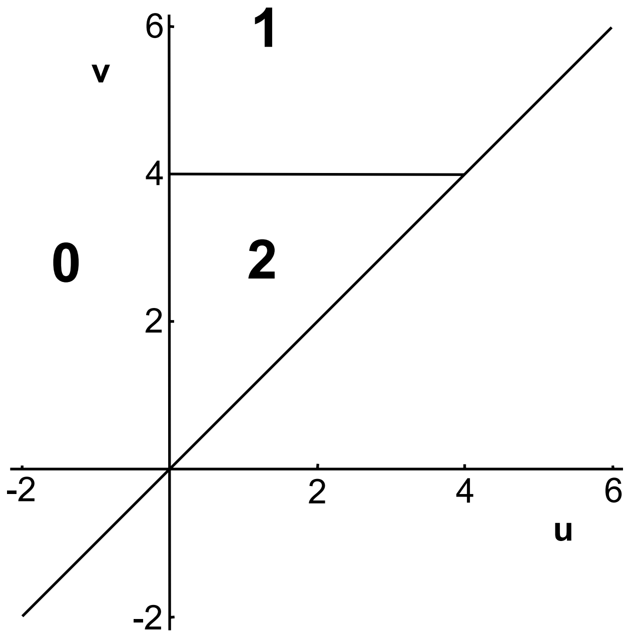

Figure 2.3: The representation of $\beta_{(X,f_X,0)}$, the 0-PBNs of X.

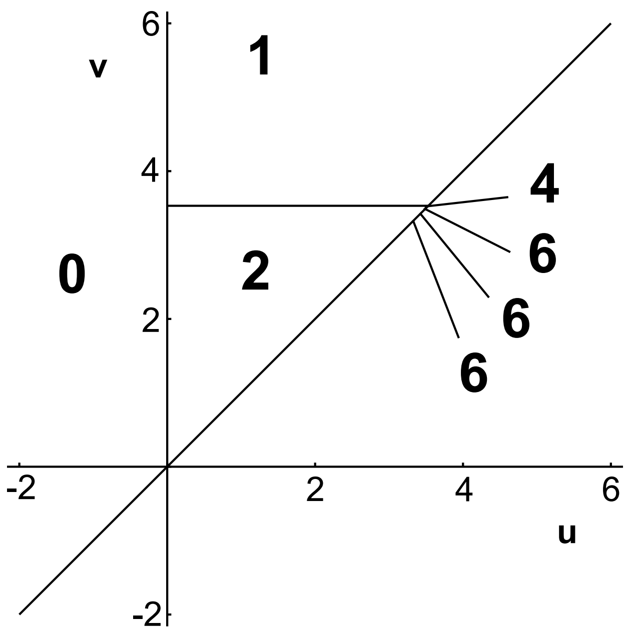

Figure 2.4: The representation of $\beta_{(U,f_U,0)}$, the 0-PBNs of the ball union U

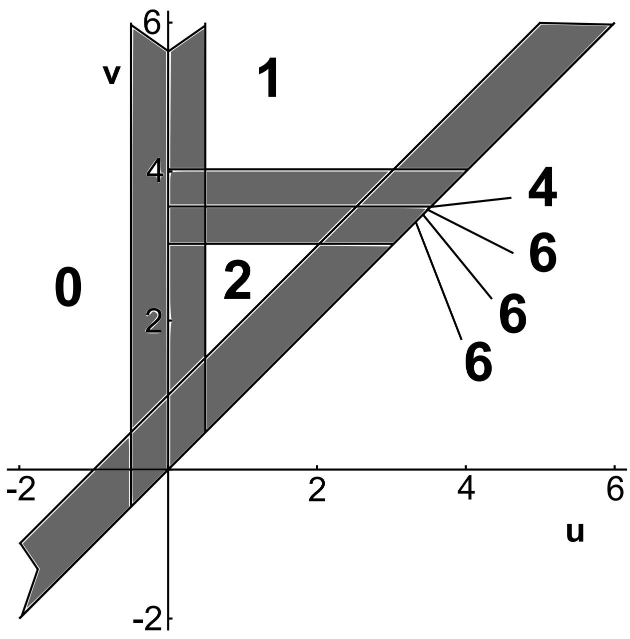

Figure 2.5: The blind strips of $\beta_{(U,f,0)}$.

connected components generated by f_X. The two connected components collapse to one at value 4. In Figure 2.4 there is also a big triangle representing the two connected components, but they collapse at value 3.53106. Moreover there are 4 other very small triangles near the diagonal, representing more connected components generated by the boundary of the small circle of the cover. In the last figure (Fig. 2.5) the blind strips around the discontinuity lines of $\beta_{(U,f_U,0)}$ are shown. The width of these strips, since $\Omega(\delta) = 0.5$, is equal to $2\Omega(\delta) = 1$. This figure illustrates the idea underlying the Blind Strips Theorem 2.2.1. Taken a point (u, v) outside the strips, the values of the PBNFs of U at $(u - \Omega(\delta), v + \Omega(\delta))$ and $(u + \Omega(\delta), v - \Omega(\delta))$ are the same. So also the value of the PBNFs of X at (u, v) is determined. Figures 2.6, 2.7, 2.8 depict, in analogous way, the (obviously much simpler) PBNFs of degree 1.

For a second example we have chosen the points c_j not necessarily on X. We have satisfied the hypothesis of Proposition 2.2.2, choosing $s = 0.25$ and $\delta = 0.55$. Then, in order to cover X well, we have chosen a point every $\frac{\pi}{48}$

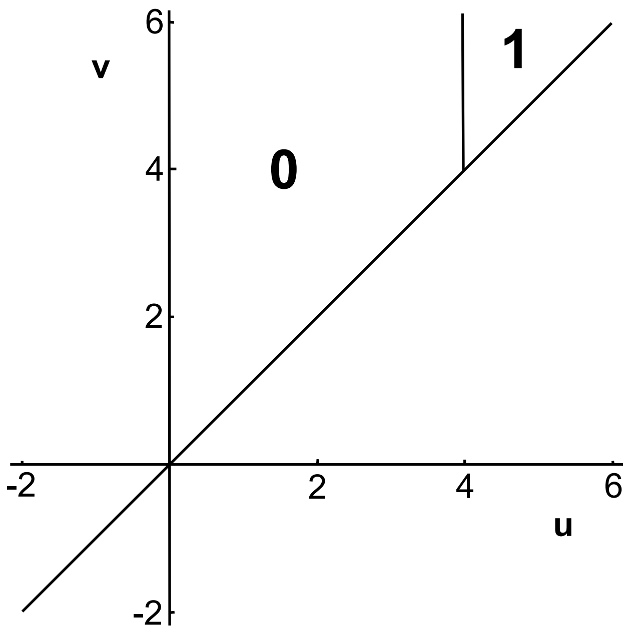

Figure 2.6: The representation of $\beta_{(X,f_X,1)}$, the 1-PBNs of X.

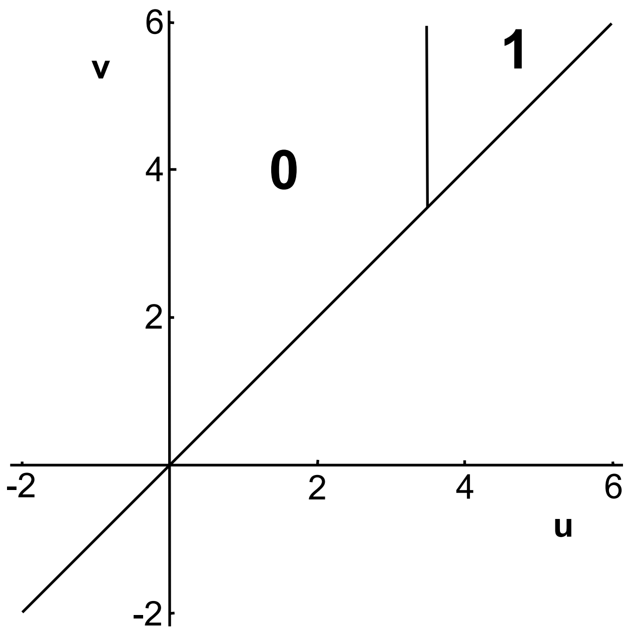

Figure 2.7: The representation of $\beta_{(U,f_U,1)}$, the 1-PBNs of the ball union U.

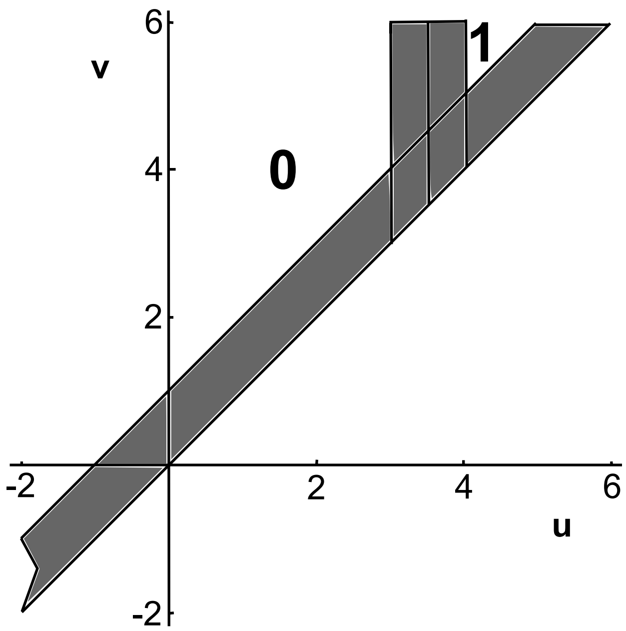

Figure 2.8: The blind strips of $\beta_{(U,f,1)}$.

radians, for a total of 96 points. But this time the points are either 0 or 0.1 or 0.2 away from X. Figure 2.9 shows the resulting ball union U'. As in the previous case, in the representation of $\beta_{(U',f_{U'},0)}$ (Fig. 2.10) there is a big triangle showing two connected components and this time they collapse at value 3.40955. Compared to Figure 2.4, there are many more small triangles generated by the asymmetry of the sampling. The width of the blind strips in Figure 2.11 is $2\Omega(\delta+s) = 1.6$, so there is still the central triangle. This means that, although the error in the approximation is much bigger, the blind strips do not cover the entire figure, leaving the topological information intact at least in some small areas of Δ^+.

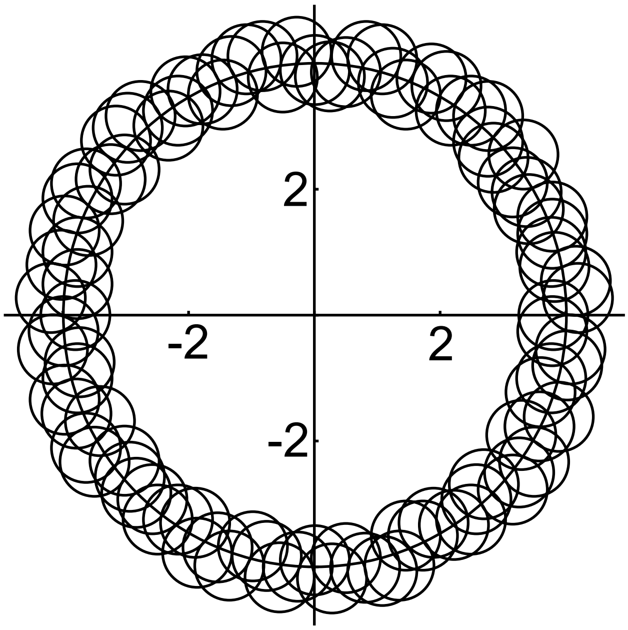

Figure 2.9: The ball union U'.

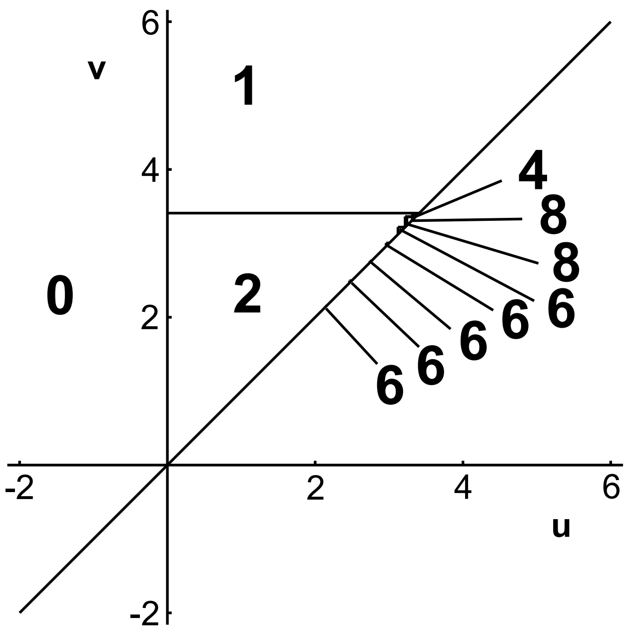

Figure 2.10: The representation of $\beta_{(U', f_{U'}, 0)}$, the 0-PBNs of the ball union U'.

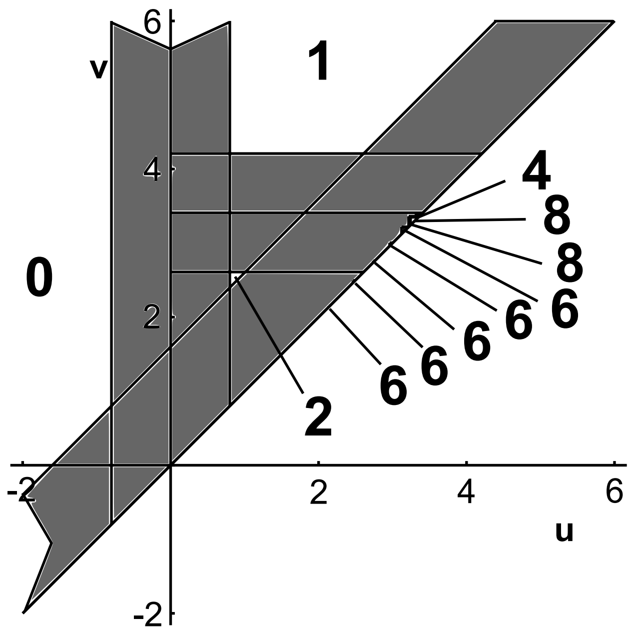

Figure 2.11: The blind strips of $\beta_{(U',f_{U'},0)}$.

2.3 A Combinatorial Representation

The ball unions of Section 2.2, although generated by finite sets, are still continuous objects. It is desirable that the topological information on X, up to a certain approximation, be condensed in a combinatorial object. For size functions (i.e. for PBNFs of degree 0) it was a graph; here, it has to be a simplicial complex. We shall build such a complex, by following [16], to which we refer for all definitions not reported here. Please note that [16] uses *weighted* Voronoi cells and diagrams, while we do not need to worry about that, since all of our balls have the same radius; so the customary Euclidean distance can be used instead of the power distance employed in that paper. Let X, $L = \{c_1, \ldots, c_k\}$ and δ be as in Section 2.2.1 (the case of Section 2.2.2 is an immediate extension). Moreover, let the points of L be in general position. For each $c_j \in L$, let $B(c_j, \delta)$ be the ball of radius δ, centred at c_j. The set $\mathcal{B} = \{B(c_1, \delta), \ldots, B(c_k, \delta)\}$ is a ball covering of X; denote by U the corresponding ball union. Let now V_j be the *Voronoi cell* of $B(c_j, \delta)$, i.e. the

set of points of $\mathbb{R}^m$ whose distance from c_j is not greater than the distance from any other $c_{j'}$.
The set $\mathcal{V} = \{V_1, \ldots, V_k\}$ is the *Voronoi diagram* of $\mathcal{B}$. From $\mathcal{V}$ we get the collection of cells $\mathcal{Q} = \{V_j \cap B(c_j, \delta) \mid j = 1, \ldots, k\}$, a decomposition of U.
The *nerve* $\mathcal{N}(\mathcal{V})$ of $\mathcal{V}$ is the abstract simplicial complex where vertices are the elements of $\mathcal{V}$ and, for a subset T of $\{1, \ldots, k\}$, the set of vertices $\{V_j \mid j \in T\}$ is a simplex if and only if $\bigcap_{j \in T} V_j \neq \emptyset$.
For any $T \subseteq \{1, \ldots, k\}, T \neq \emptyset$ we denote by σ_T the convex hull of $\{c_j \mid j \in T\}$.
The *dual complex* of $\mathcal{Q}$ is $\mathcal{K} = \{\sigma_T \mid \{V_j \cap B(c_j, \delta) \mid j \in T\} \in N(\mathcal{Q})\}$ and $\mathcal{S} = |\mathcal{K}|$, union of the simplices of $\mathcal{K}$, is the *dual shape* of U.

For a better understanding of the previous part we produce a toy example. Let X be a quarter of circle of radius 4 and U be the union of nine balls of radius 1, with centres near X (Fig. 2.12). The Voronoi Diagram $\mathcal{V}$ associated to this ball covering $\mathcal{B}$ is depicted in Figure 2.13.

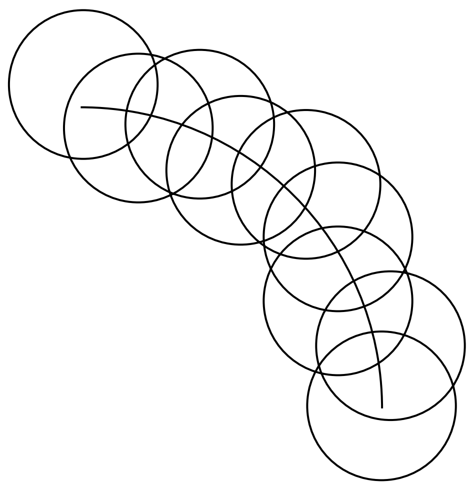

Figure 2.12: A quarter of circle of radius 4 covered by nine balls of radius 1.

Now the main idea is that we can associate the dual complex $\mathcal{K}$ with the submanifold X. In fact, by [16, Thm. 3.2], its space $\mathcal{S}$ is homotopically equivalent to U and, by transitivity, to X. Moreover, [16, Sect. 3] explicitly builds a retraction r from U to $\mathcal{S}$ and a homotopy H from the identity of U, to p, such that $\forall y \in \mathcal{S}$, $\forall v \in p^{-1}(y)$, $\forall t \in I$ we have $(p \circ H)(v, t) = y$. For

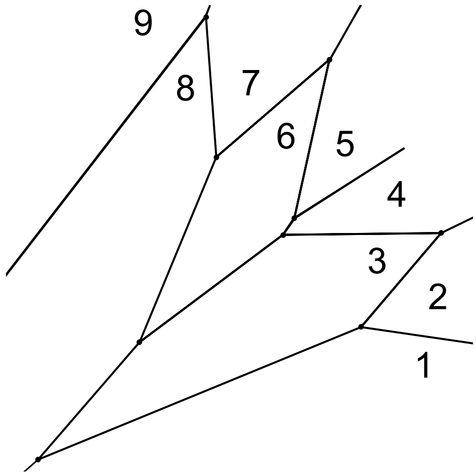

Figure 2.13: The Voronoi Diagram $\mathcal{V}$ of B.

a complete description of the homotopy H and the retraction p we refer to the original article.

$\mathcal{S}$ is shown in Figure 2.14 (we recall that in this example the dual complex $\mathcal{K}$ graphically coincides with $\mathcal{S}$).

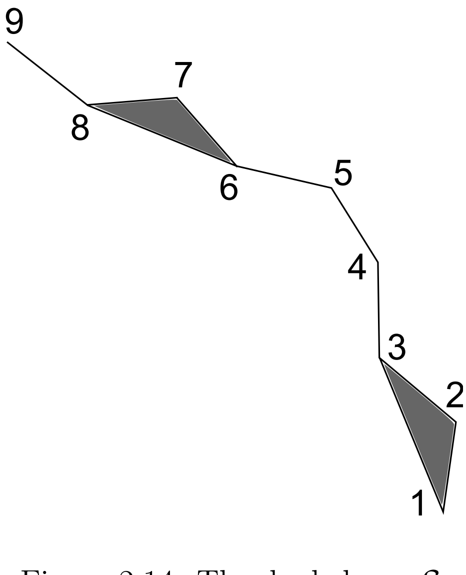

Figure 2.14: The dual shape $\mathcal{S}$.

2.3.1 Ball Union and Dual Shape

Since we are in an analogous situation of the Blind Strips Theorem 2.2.1 and since the hypothesis on the complex $\mathcal{S}$ and the retraction from U to $\mathcal{S}$ are coherent with the hypothesis of the Lemma 2.1.3; then we can directly apply it.

Let $\vec{f}: \mathbb{R}^m \to \mathbb{R}^n$ be the continuous function as in Section 2.2.1 and let $\vec{f}_{\mathcal{S}}$ be the restriction of $\vec{f}$ to $\mathcal{S}$.

Lemma 2.3.1. *If $(\vec{u}, \vec{v})$ is a point of Δ^+ and if $\vec{u} + \vec{\omega}(\delta) \prec \vec{v} - \vec{\omega}(\delta)$, where $\vec{\omega}(\delta) = (\Omega(\delta), \ldots, \Omega(\delta)) \in \mathbb{R}^n$, then*

$$\beta_{(U,\vec{f}_U,i)}(\vec{u} - \vec{\omega}(\delta), \vec{v} + \vec{\omega}(\delta)) \leq \beta_{(\mathcal{S},\vec{f}_{\mathcal{S}},i)}(\vec{u}, \vec{v}) \leq \beta_{(U,\vec{f}_U,i)}(\vec{u} + \vec{\omega}(\delta), \vec{v} - \vec{\omega}(\delta)).$$

Proof. By Lemma 2.1.3, with $Y = \mathcal{S}$, $V = U$.

□

Now we can get an estimate of the PBNFs of X from the ones of $\mathcal{S}$. The blind strips will be doubly wide, with respect to the ones previously considered. Still, this can leave some regions of Δ^+ where the computation is exact.

Theorem 2.3.2. *If $(\vec{u}, \vec{v})$ is a point of Δ^+ and if $\vec{u} + 2\vec{\omega}(\delta) \prec \vec{v} - 2\vec{\omega}(\delta)$, where $\vec{\omega}(\delta) = (\Omega(\delta), \ldots, \Omega(\delta)) \in \mathbb{R}^n$, then*

$$\beta_{(\mathcal{S},\vec{f}_{\mathcal{S}},i)}(\vec{u}-2\vec{\omega}(\delta), \vec{v}+2\vec{\omega}(\delta)) \leq \beta_{(X,\vec{f}_X,i)}(\vec{u}, \vec{v}) \leq \beta_{(\mathcal{S},\vec{f}_{\mathcal{S}},i)}(\vec{u}+2\vec{\omega}(\delta), \vec{v}-2\vec{\omega}(\delta)).$$

Proof. By the Blind Strips Theorem 2.2.1,

$$\beta_{(U,\vec{f}_U,i)}(\vec{u} - \vec{\omega}(\delta), \vec{v} + \vec{\omega}(\delta)) \leq \beta_{(X,\vec{f}_X,i)}(\vec{u}, \vec{v}) \leq \beta_{(U,\vec{f}_U,i)}(\vec{u} + \vec{\omega}(\delta), \vec{v} - \vec{\omega}(\delta))$$

Then we have

$$\beta_{(U,\vec{f}_U,i)}(\vec{u} + \vec{\omega}(\delta), \vec{v} - \vec{\omega}(\delta)) \leq \beta_{(\mathcal{S},\vec{f}_{\mathcal{S}},i)}(\vec{u} + 2\vec{\omega}(\delta), \vec{v} - 2\vec{\omega}(\delta))$$

by Lemma 2.3.1 by substituting $(\vec{u}, \vec{v})$ with $(\vec{u} + 2\omega(\delta), \vec{v} - 2\omega(\delta))$, and

$$\beta_{(\mathcal{S},\vec{f}_{\mathcal{S}},i)}(\vec{u} - 2\vec{\omega}(\delta), \vec{v} + 2\vec{\omega}(\delta)) \leq \beta_{(U,\vec{f}_U,i)}(\vec{u} - \vec{\omega}(\delta), \vec{v} + \vec{\omega}(\delta))$$

by Lemma 2.3.1 by substituting $(\vec{u}, \vec{v})$ with $(\vec{u} - 2\vec{\omega}(\delta), \vec{v} + 2\vec{\omega}(\delta))$. □

This result is probably the closest one to the application field, because it allows us to take as an input a structure that is purely combinatorial, thus very easy to compute directly. Obviously this possibility requires a cost of $2\omega(\delta)$, because the approximation is applied twice; nevertheless this does not affect the value of the result.

Chapter 3

Narrowing the Blind Strips in $\mathbb{R}^2$

The theorems of Chapter 2 opened a new possible way of research, in fact the proposed results can be analyzed and studied to make them more suitable for the application needs. A natural question is, when could it be possible to decrease the error and consequently to lessen the width of the blind strips. Therefore in this chapter we shall try to understand when there is a solution to that kind of problem.
The first step, as we shall see in Section 3.1, is to look for new relations between the well-known elements of the proof, as the ball covering or the tubular neighbourhood. Unluckily that kind of approach has not brought the desired result, but only an abstract one (Prop. 3.1.3). Nevertheless, since we are working with 1-submanifolds of $\mathbb{R}^2$, we were able, at least, to use some properties of the complex (Lem. 3.2.1); thanks to them we discovered a new relation between the original object X and the complex itself. With this new relation we stated the theorem (Th. 3.2.4) that represents the main result of this chapter.
Similarly to Theorem 2.3.2, this kind of result states a double inequality between the complex generated by the sampling points and the original object. The advantage is that, in this formulation, the error is a quarter of

the previous one, thanks to a particular construction that frees us from the use of Proposition 1.3.2 in which a more restrictive density in the sampling points is required. Moreover this freedom manifests itself in the possibility of managing the choice of the sampling points in a different manner, making this result more flexible for applications.

This chapter, in detail, is divided into three sections. In the first one some results are presented that contributed to the analysis of the problem. In the second one we shall present the main theorem and finally in the third one we shall illustrate some comparative examples related to the results of Chapter 2.

3.1 Deformation Retract

In this section we shall introduce a particular construction that will allow us to prove that, if X is a 1-submanifold of $\mathbb{R}^2$, the covering generated by the balls centred in the sampling points with radius ϵ is a deformation retract of the normal bundle of the same radius ϵ.

We start with some observations in the general case; let X be a compact Riemannian submanifold of $\mathbb{R}^m$ as before and let $L = \{c_1, \dots, c_n\}$ be a finite collection of points on X. We also request that for every point $p \in X$ there exist a $c_j \in L$ such that $||p - c_j||_{\mathbb{R}^m} < \epsilon$, where $\epsilon < \tau$ and it shall represent the radius of the balls of the covering. Then we take the condition number τ as in chapter 1, definition1.9 and moreover we assume that $\tau > 0$ to prevent infinitely many oscillations in the neighbourhood of any point. With this setting we can state that

Lemma 3.1.1. *For every $\epsilon < \tau$ X is a deformation retract of N_ϵ, where N_ϵ is the open normal bundle around X of radius ϵ.*

Proof. This holds by definition, using the retraction along the normal directions. □

It is important to underline that the retraction moves the points at most ϵ.

Remark 3.1.2. From now on we shall confuse throughout N_ϵ with its image Tub_ϵ, the tubular neighbourhood of radius ϵ.

First of all we know that, by definition of normal bundle, for every point c_j of L there exists a normal subspace at X (of dimension m) that intersects the boundary of N_ϵ in an $(m-1)$-dimensional sphere (in dimension one we have two points). Along this intersection the boundary of the ball centred in c_j is tangent (they have the same tangent space in each of these points) to the boundary of the normal bundle. Moreover, if we define U_ϵ as the union of the balls $B(c_j, \epsilon)$ centred in the points of L of radius ϵ, we can prove that the intersection of the boundary of N_ϵ with the boundary of U_ϵ is a finite union of submanifolds of $\mathbb{R}^{m+1}$ of dimension $(m-2)$ (in $\mathbb{R}^2$ we have just a finite collection of points). From now on we shall limit ourselves to work with X

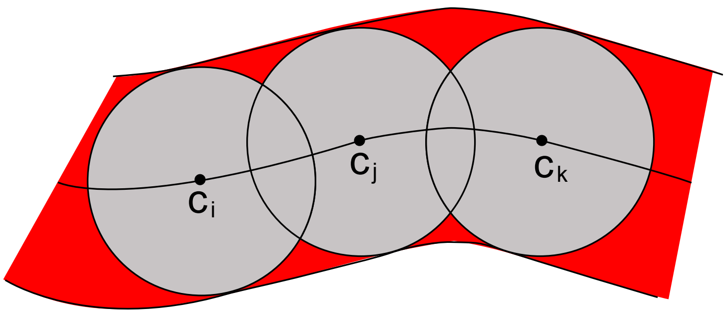

Figure 3.1: The 1-submanifold X, the ball covering U_ϵ and the tubular neighbourhood N_ϵ.

as a 1-dimensional submanifold of $\mathbb{R}^2$ (Fig. 3.1). Then

Proposition 3.1.3. *U_ϵ is a deformation retract of N_ϵ by a retraction that moves the points at most ϵ*

To prove the proposition we need to introduce a special construction relating U_ϵ and N_ϵ, in order to obtain the directions of the retraction. In our notation we shall call the two points of intersection of the boundary

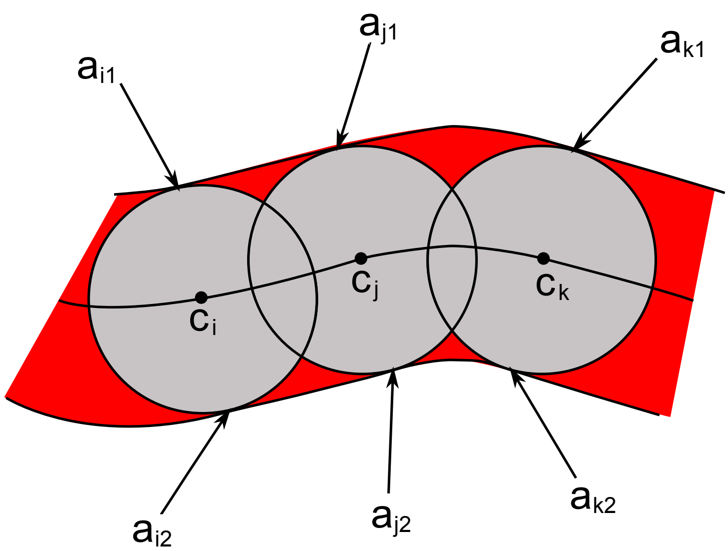

Figure 3.2: The intersection points of U_ϵ and N_ϵ.

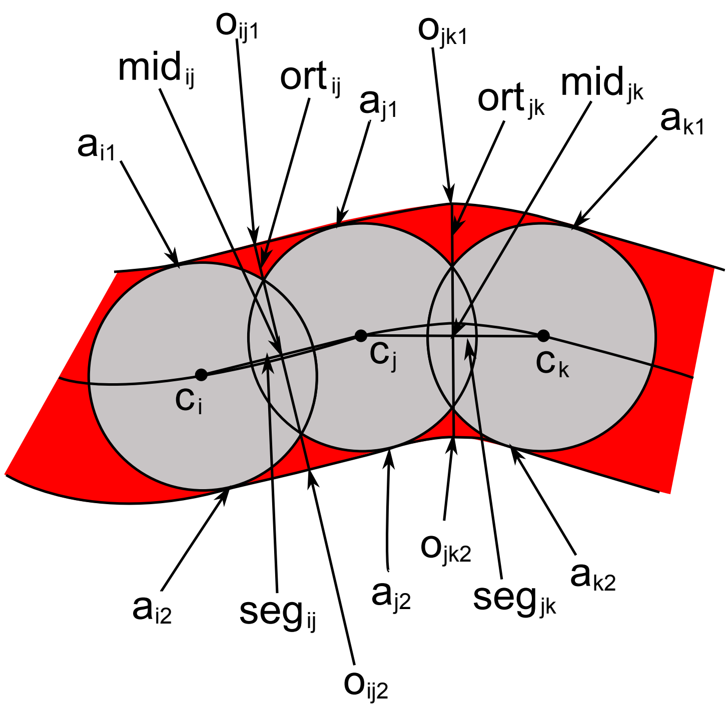

Figure 3.3: The description of all the elements involved.

of the ball $B(c_j, \epsilon)$ with the boundary of N_ϵ a_{j1} and a_{j2} (Fig. 3.2). Now for every intersection of $B(c_j, \epsilon)$ with other balls we construct the segment seg_{jk} between the two centres c_j and c_k, after that we build the orthogonal line ort_{jk} to the previous segment, through its middle point mid_{jk}. We call o_{jk1} and o_{jk2} the two intersection points of ort_{jk} with the boundary of N_ϵ as in Figure 3.3 (it could happen that the line ort_{jk} intersects the normal bundle in other points, but far from mid_{jk}. So we consider only the first two intersections of the line starting from mid_{jk} and following the two opposite directions).

In this way we obtain that all the normal bundle is divided into zones generated by three segments a_{j1} to c_j, c_j to mid_{jk}, mid_{jk} to o_{jk1} (also for a_{j2} and o_{jk2}) and the boundary of N_ϵ between a_{j1} and o_{jk1} (also for a_{j2} and a_{k2}), see Figure 3.4. The idea is to create a continuous foliation of the previous

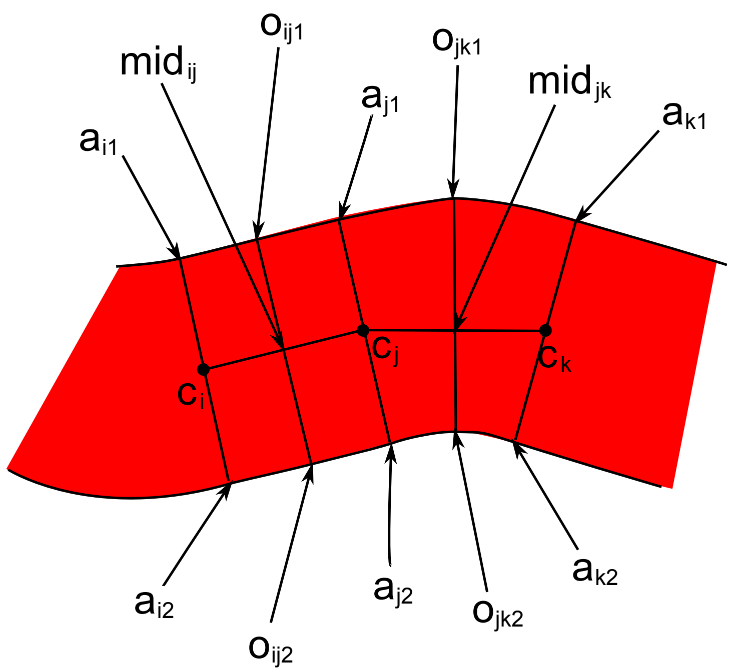

Figure 3.4: The zones of the tubular neighbourhood.

zones generated by the continuous transformation of the segment a_{j1}, a_{j2} into the segment o_{jk1}, o_{jk2}. In this way we can obtain a continuous retraction from the normal bundle to the union of balls, that moves the points at most ϵ as shown in Figure 3.5. To proceed in our construction, we shall show how to

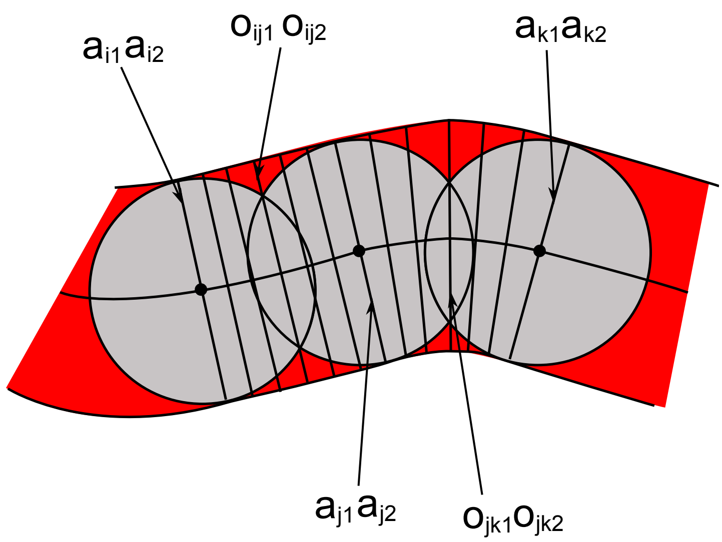

Figure 3.5: The segments of the continuous transformation.

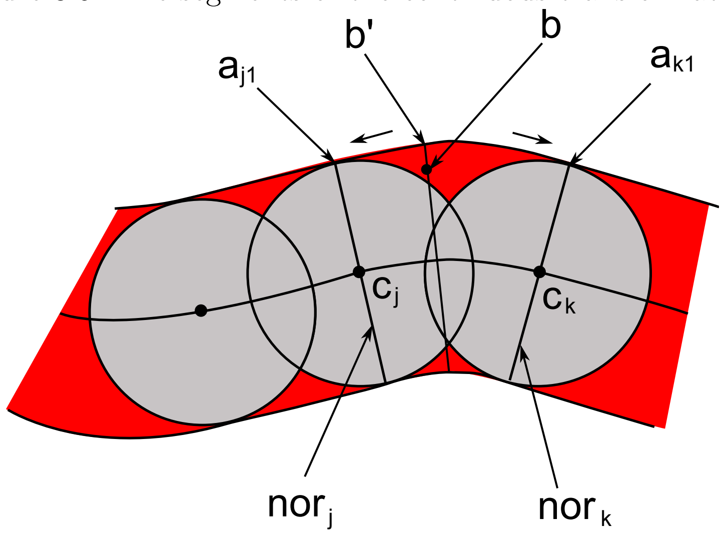

Figure 3.6: The construction for the choice of the balls.

choose the balls $B(c_j, \epsilon)$ and $B(c_k, \epsilon)$, that will be used in the next lemma, in relation with the point $b \in N_\epsilon$. If the point b lies on U_ϵ then we do not need to find any balls because the retraction does not move this point. On the other hand, if the point lies on $N_\epsilon \setminus U_\epsilon$ we need to define a little construction.

Let $b \in N_\epsilon \setminus U_\epsilon$ be the chosen point, we define b' as the closest intersection point of the normal line at X passing through b with the boundary of N_ϵ. Now we proceed along the two different directions of the boundary from b', until we find the intersection of a normal line nor_h of some $c_h \in L$ with the boundary in both directions. After that we call c_j and c_k these two points along the two different directions, so that nor_j and nor_k are the two normal lines that generate the searched for intersections, that we call a_{j1} and a_{k1} (Fig. 3.6). Before giving the proof of Proposition 3.1.3, we need to introduce the following lemma (an example of the construction can be found in Figure 3.7).

Lemma 3.1.4. *The intersection point int_j, if it exists, of the line nor_j and ort_{jk} is distant more than ϵ from c_j ($d(int_j, c_j) > \epsilon$, where $d(,)$ is the Euclidean distance in $\mathbb{R}^2$).*

Proof. Supposing that the intersection point int_j lies inside the normal bundle, we search for an absurd. We know that $int_j \in nor_j$ and $int_j \in N_\epsilon$, thus $d(int_j, c_j) = \rho < \epsilon$. Moreover we get that $int_j \in B(c_j, \epsilon)$ and since $int_j \in ort_{jk}$ then $int_j \in B(c_j, \epsilon) \cap B(c_k, \epsilon)$. Now we define $\tilde{n}$ the point of nor_j at distance exactly ϵ from c_j in the direction of int_j. By definition of normal bundle, this point has to belong to the boundary of N_ϵ, but this is absurd because this point lies inside $B(c_k, \epsilon)$ and that concludes the claim.
In the limit case in which the intersection point is exactly one of the intersection points k_1, k_2 of the two balls $B(c_j, \epsilon)$ and $B(c_k, \epsilon)$, we obtain a different absurd. In fact we know that the function that associates the submanifold X to the boundary of the normal bundle is a diffeomorphism (this is true by construction of the normal bundle). Thus, if $int_j = k_1$ (or k_2), there will be two points c_j and c_k with distance exactly ϵ from int_j and this is absurd (we also recall that, by definition of normal bundle, each ball $B(c_h, \epsilon)$ intersects the boundary of N_ϵ in two opposite points along the normal line passing through the centre c_h). □

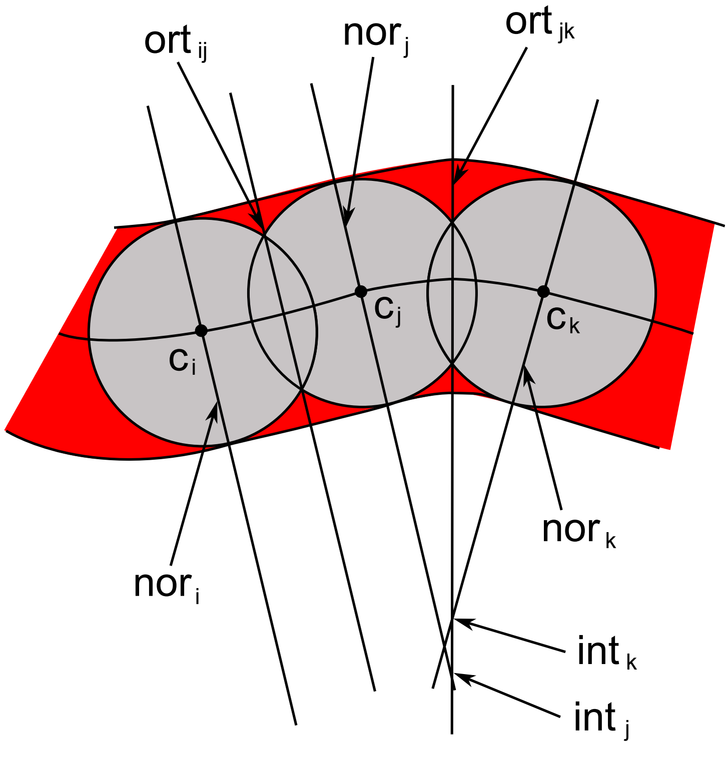

Figure 3.7: The construction related to Lemma 3.1.4.

Remark 3.1.5. It is important to underline that, thanks to Lemma 3.1.4 and the previous construction, the point o_{jk1} lies between the points a_{j1} and a_{k1} with respect to the boundary of the normal bundle (Fig. 3.4). This positioning of the points guarantees that the construction of the retraction in the next proof is well defined.

Finally we can state the proof of Proposition 3.1.3, using the notation of the previous lemma.

Proof of Proposition 3.1.3. Our goal is to find a homotopy between the identity of N_ϵ and a retraction of N_ϵ on U_ϵ; for this purpose we need a continuous retraction $r_\epsilon(b) : N_\epsilon \to U_\epsilon$. During the construction of this function we can encounter two different cases. In the first one the line nor_j is parallel to the line ort_{jk}, in the second one the two lines intersect in the point int_j.
In the first case the retraction will follow the direction parallel to nor_j, retracting in this way the zone bounded by $a_{j1}, c_j, mid_{jk}, o_{jk1}$ and the relative boundary of N_ϵ. In the second case the retraction will follow the radial direction from the point int_j, retracting the same zone as in the first case. We need to divide the normal bundle into zones so small, because it could happen that the retraction is defined in different ways in each zone depending on the choice of the balls $B(c_j, \epsilon), B(c_k, \epsilon)$.
To be more precise:
Case one: for each point b in the zone bounded by $a_{j1}, c_j, mid_{jk}, o_{jk1}$ and the relative boundary of N_ϵ, we can define the line n_b going through b and parallel to nor_j. After that we shall call int_{bn} the first intersection point of n_b with the boundary of U_ϵ starting from b and with direction that minimizes the distance to the segment seg_{jk}. Now r_ϵ is defined as $r_\epsilon(b) = int_{bn}$.
Case two: for every point b in the zone bounded by $a_{j1}, c_j, mid_{jk}, o_{jk1}$ and the relative boundary of N_ϵ we can define the line rad_b going through b and int_j. We shall call int_{br} the first intersection point of rad_b with the boundary of U_ϵ starting from b and with direction that minimizes the distance to the segment seg_{jk}, thus the retraction is defined as $r_\epsilon(b) = int_{br}$. For a graphical description we refer to Figure 3.8. Now we prove that the retraction built in

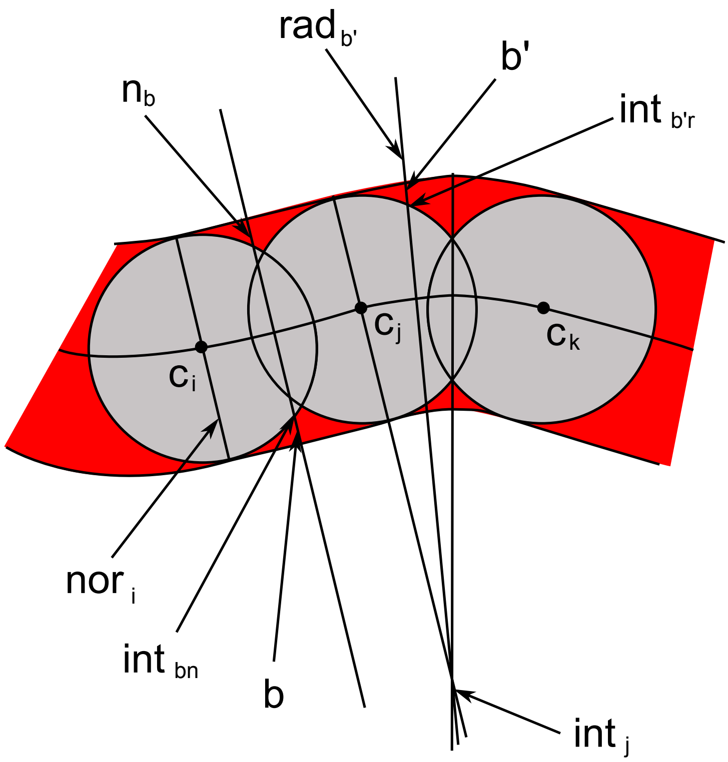

Figure 3.8: The graphical description of the retraction.

this way is continuous. Thanks to the way in which we define the retraction for each point $b \in N_\epsilon \setminus U_\epsilon$ and more precisely with respect to the choice of the balls, we can divide the domain of the retraction into small zones. After that we shall prove that the retraction is continuous in these zones and in the points of intersection of them. When we choose a point b in $N_\epsilon \setminus U_\epsilon$ the relative *zone* Z_{j1} *of type one* is defined as the portion of N_ϵ delimited by the segment mid_{jk}, o_{jk1}, the boundary of N_ϵ between o_{jk1} and a_{j1} and the boundary of U_ϵ from k_1 (or k_2) and a_{j1}. The adjacent zone Z_{k1} is generated by substituting j with k, so they share the segment mid_{jk}, o_{jk1} (Fig. 3.9). Thus we can divide the domain of the retraction into a union of pairs of the previous zones Z_{j1} and Z_{k1}; moreover this pair of zones intersects with two analogous ones at point a_{j1}, respectively at point a_{k1}. In each zone Z_{j1},

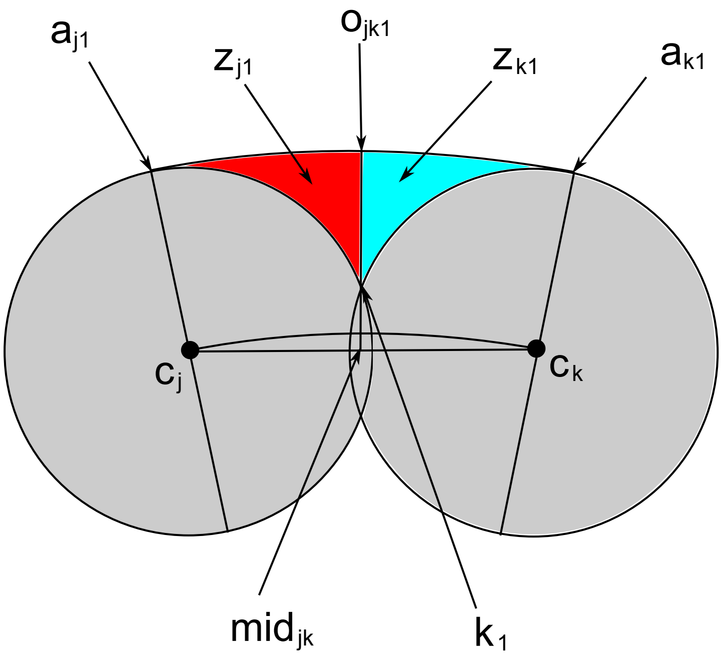

Figure 3.9: The zones of N_ϵ.

the retraction r_ϵ can be seen as a perspective transformation (from either a proper or an improper point), so it is continuous. These perspective transformations agree on the intersection of any two adjacent zones, so r_ϵ is globally continuous by the "gluing Lemma".

Finally we can define the homotopy $H_\epsilon(b,t) : N_\epsilon \times [0,1] \to N_\epsilon$ as $H_\epsilon(b,t) = b(1-t) + r_\epsilon(b)t$ and in this way conclude the proof. Moreover it is easy to check that the retraction moves the points of N_ϵ at most ϵ.

□

Remark 3.1.6. In the case that there is only one point in L and so only one ball in U_ϵ, the proof of the lemma is trivial.

As hinted in the introduction of the chapter, we are interested in decreasing the error generated by the approximation and in this way decreasing the width of the blind strips. Our first idea was to use the previous result to relate the object X to the covering U_ϵ and to the simplicial complex $\mathcal{S}_\epsilon$ generated by the points of the sampling as in the previous chapter, Section 3. And then we used, once again, the idea of Lemma 2.3.1, thus we got that $\mathcal{S}_\epsilon$ is a deformation retract of U_ϵ.
Next step was to glue together all the three retractions and to define a new double inequality similar to Theorem 2.3.2, but with this setting the error that we committed was of the order of $3\Omega(\epsilon)$ and the strips were $6\Omega(\epsilon)$-wide. This means that the the strips were wider than in the previous case, but at least in the construction we did not need to use the result of Proposition 1.3.2 with the consequence of having more freedom in the choice of the sampling points. Since this was not the searched for result, we continued in a different way and the core idea is developed in the next section.

3.2 Decreasing the Error

This is the main section of the chapter, in which we shall show the main results; the basic idea is to exploit a property of the simplicial complex, that rises in the particular setting of the one-dimension submanifolds of $\mathbb{R}^2$. In fact in this case, as the next lemma states, the simplicial complex is only made of simplices of dimension 0 (the vertices) and dimension 1 (the edges). Thanks to that we can create a special construction, in which we shall define a new subspace called W. More precisely, W is a subspace of $\mathbb{R}^2$ generated by

the union of some portions of the tubular neighbourhood Tub_ϵ; this subspace will play a fundamental role in the proof of the theorem (Th. 3.2.4) ensuring the continuity of the retractions.
The structure of the section contains three lemmas that will allow us to state the main theorem; the first one is the following.

Lemma 3.2.1. *The simplicial complex $\mathcal{K}$ generated from X as in section 2.3 is mono-dimensional.*

Proof. The main idea is to find an absurd in the construction of the simplicial complex. Thus let us suppose that there exists a 2-dimensional simplex (a triangle), generated by three balls $B(c_1, \epsilon)$, $B(c_2, \epsilon)$ and $B(c_3, \epsilon)$ (for simplicity we shall call them only B_1, B_2 and B_3). Without loss of generality we can assume that the curve X passes through the centre of the balls in increasing order (c_1, c_2 and c_3). Thanks to the properties of the Voronoi diagram, we know that the three cells generated by the balls meet in one point and also that this point lies inside B_2. This in turn implies that the normal segments at X in the centres c_j (i.e. the leaf of the tubular neighbourhood in that point) cannot pass through the boundary of the cells, because otherwise an end-point of the segment would be in the interior of a ball and this is not allowed.
As Figure 3.10 shows, let V_i ($i = 1, 2, 3$) be the intersection of the Voronoi cell relative to c_i with the ball B_i. Let then $A = V_1 \cap V_2$ and $C = V_3 \cap V_2$. Now the segment normal to X based at c_2 cannot cross A and C. We recall that the union of the balls B_1 and B_3 has to leave, at least, half of the ball B_2 not covered (at least one segment relative to a diameter must be preserved); if not the segment normal to X at c_2 would have, at least, an end point inside the union of the two balls and this is absurd. Then if we consider the line L generated by the segment at c_2 normal to X, L divides the plane into two half planes (H_1, H_2). It is easy to see that A and C lie in the same half-plane (H_1 say) and moreover, since the joining segments $\overline{c_2c_1}$, $\overline{c_2c_3}$ are the axes of A and C respectively, also c_1 and c_3 lie in H_1.
Since the curve lies locally (inside the three balls) in H_1, when it reaches the

point c_2 the line L is tangent to X, but this is absurd because L is instead normal to it. □

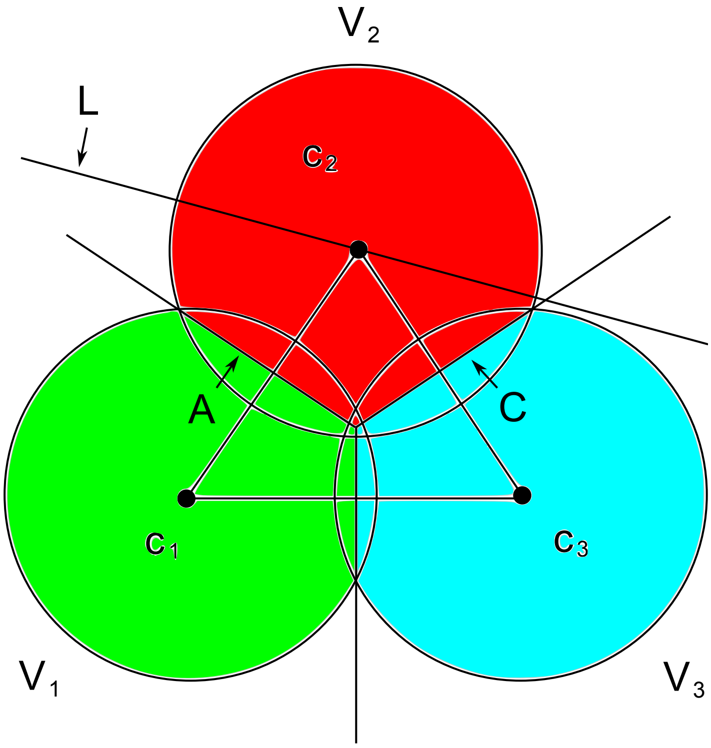

Figure 3.10: The three balls and the relative construction.

To proceed in our discussion we need to introduce a new construction based on the idea of normal bundle. First of all we generate the dual complex $\mathcal{K}$ from X as in section 2.3, then we consider the dual shape $\mathcal{S} = |\mathcal{K}|$.
Since $\mathcal{S}$ is mono-dimensional, thanks to Lemma 3.2.1, we obtain that the intersection of $\mathcal{S}$ with X can be of two types: either a finite set of points Q, or a union of a finite set of points and a finite set of segments $Q \cup Seg$. We shall analyze the second case, because it is more general. Now for every segment in Seg we consider only the two end points consecutive, in a fixed circular ordering induced by X and we add these points to the set Q (maintaining the ordering). In this way we obtain a new finite set of points QS, then for every two consecutive points (in the fixed circular ordering induced by

X) q, q' of QS it is possible to construct the relative open normal bundle of radius ϵ ($N_q^{q'}$). This relative open normal bundle is divided into two parts by the curve X; we now define $W_q^{q'}$ the part of $N_q^{q'}$ which contains the segment $\overline{qq'}$ (in the case in which the two consecutive points are the end-points of a segment of Seg we define $W_q^{q'} = N_q^{q'}$). Finally we define W as the union of all the $W_q^{q'}$ with $q, q' \in QS$ consecutive (Fig. 3.11).

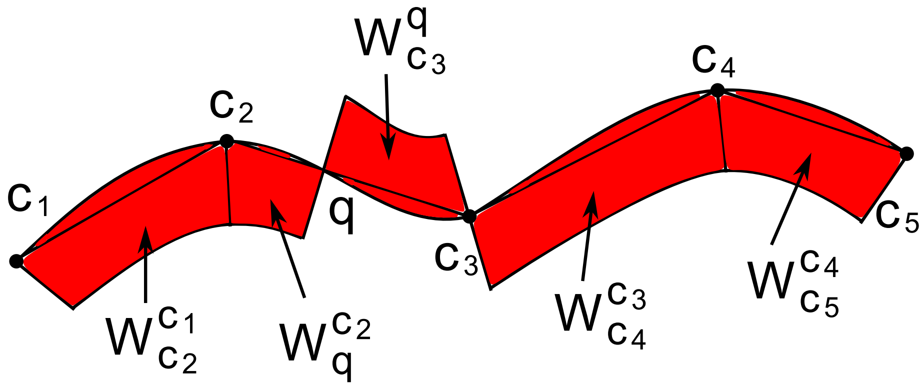

Figure 3.11: The representation of W.

With these new definitions we can state the two remaining lemmas:

Lemma 3.2.2. *X is a deformation retract of W and the retraction moves the points at most ϵ.*

Proof. By definition of normal bundle and using the retraction ($r_X : W \to X$) along the normal directions. (as in [27]) □

Lemma 3.2.3. *$\mathcal{S}$ is a deformation retract of W and the retraction moves the points at most ϵ.*

Proof. We need to prove that there exists a continuous retraction from W to $\mathcal{S}$ that moves the points less than ϵ. To do that we define $r_{\mathcal{S}} : W \to \mathcal{S}$ as the retraction along the normal direction (as $r : N_\epsilon \to X$). First of all we shall prove that the segments of the normal directions intersect each simplex only one time; one intersection is guaranteed by the construction of Tub_ϵ and by Jordan's Curve Theorem. Note that if the two segments met in more than

one point the intersection would be a whole segment. We shall show that this is not possible analyzing two cases for an intersection point x. In the first case x lies inside the simplex, in the second one x is a vertex.
In case one we observe that the maximum distance between two vertices is less than 2ϵ and the length of a normal segment is exactly 2ϵ; this means that the normal segment through x touches at least one vertex of the simplex and this is absurd by construction.
In case two if the other vertex is closer than ϵ we obtain the same absurd as in case one. If not we get that the simplex touches the boundary of N_ϵ, but this is absurd because $\mathcal{S} \subset intU_\epsilon \subseteq N_\epsilon$.
Thus the retraction is well-defined and also continuous. Moreover it is easy to check that the length of the segments of the retraction is at most ϵ so the retraction moves the points less than ϵ.
Finally we can define the homotopy $H_\mathcal{S} : W \times I \to W$ as $H_\mathcal{S}(w,t) = r_\mathcal{S}(w)t + w(1-t)$ which, for every $w \in W$ follows exactly the segment along the direction normal to X, thus is well-defined and is the searched for deformation retraction. □

Since this kind of construction passes to the lower level sets, we can state a new double inequality concerning the PBNFs of X and the ones of $\mathcal{S}$.

Theorem 3.2.4. *Let $\vec{f} : W \to \mathbb{R}^n$ be a continuous function and let f_X, $f_\mathcal{S}$ be the restriction to X, $\mathcal{S}$ respectively. If $(\vec{u}, \vec{w})$ is a point of Δ^+ and if $\vec{u} + \vec{\omega}(\varepsilon) \prec \vec{w} - \vec{\omega}(\varepsilon)$, then*

$$\beta_{(\mathcal{S},\vec{f}_\mathcal{S},i)}(\vec{u} - \vec{\omega}(\varepsilon), \vec{w} + \vec{\omega}(\varepsilon)) \leq \beta_{(X,\vec{f}_X,i)}(\vec{u}, \vec{w}) \leq \beta_{(\mathcal{S},\vec{f}_\mathcal{S},i)}(\vec{u} + \vec{\omega}(\varepsilon), \vec{w} - \vec{\omega}(\varepsilon)).$$

Proof. To prove this theorem we shall follow the idea of Lemma 2.1.3 constructing two functions, one from X to $\mathcal{S}$ and the other from $\mathcal{S}$ to X. These two functions will be the respective restriction of the retractions of the two previous lemmas. We define $r_{X\mathcal{S}} : X \to \mathcal{S}$ as the restriction of $r_\mathcal{S} : W \to \mathcal{S}$ and $r_{\mathcal{S}X} : \mathcal{S} \to X$ as the restriction of $r_X : W \to X$. It is easy to see that $r_{X\mathcal{S}} \circ r_{\mathcal{S}X} = id_\mathcal{S}$ and also $r_{\mathcal{S}X} \circ r_{X\mathcal{S}} = id_X$, because both the functions follow

the directions normal to X.
First, observe that there are inclusions

$$\gamma : X\langle \vec{f}_X \preceq \vec{u}\rangle \to X\langle \vec{f}_X \preceq \vec{w}\rangle$$
$$\psi : \mathcal{S}\langle \vec{f}_\mathcal{S} \preceq \vec{u} - \vec{\omega}(\varepsilon)\rangle \to \mathcal{S}\langle \vec{f}_\mathcal{S} \preceq \vec{w} + \vec{\omega}(\varepsilon)\rangle$$

The fact that $r_{X\mathcal{S}}$ and $r_{\mathcal{S}X}$ move every point not more than ε (by the previous lemmas) implies that also the maps $r_{X\mathcal{S}} : X\langle \vec{f}_X \preceq \vec{w}\rangle \to \mathcal{S}\langle \vec{f}_\mathcal{S} \preceq \vec{w} + \vec{\omega}(\varepsilon)\rangle$ and $r_{\mathcal{S}X} : \mathcal{S}\langle \vec{f}_\mathcal{S} \preceq \vec{u} - \vec{\omega}(\varepsilon)\rangle \to X\langle \vec{f}_X \preceq \vec{u}\rangle$ make sense. Now we have the following commutative diagram (as will be proved):

$$\begin{array}{ccc} \mathcal{S}\langle \vec{f}_\mathcal{S} \preceq \vec{u} - \vec{\omega}(\varepsilon)\rangle & \xrightarrow{r_{\mathcal{S}X}} & X\langle \vec{f}_X \preceq \vec{u}\rangle \\ \psi \downarrow & & \downarrow \gamma \\ \mathcal{S}\langle \vec{f}_\mathcal{S} \preceq \vec{w} + \vec{\omega}(\varepsilon)\rangle & \xleftarrow[r_{X\mathcal{S}}]{} & X\langle \vec{f}_X \preceq \vec{w}\rangle \end{array}$$

Then, passing to homology and setting $\psi^* = \psi_{u-\vec{\omega}(\varepsilon)}^{w+\vec{\omega}(\varepsilon)}$ and $\gamma^* = \gamma_{\vec{u}}^{\vec{w}}$, and $r^*_{X\mathcal{S}}$, $r^*_{\mathcal{S}X}$ as the homology homomorphisms induced by $r_{X\mathcal{S}}$ and $r_{\mathcal{S}X}$ respectively, we have

$$\begin{aligned} \beta_{(\mathcal{S},\vec{f}_\mathcal{S},i)}(\vec{u} - \vec{\omega}(\varepsilon), \vec{w} + \vec{\omega}(\varepsilon)) &= \dim \mathrm{I}m(\psi^*) = \dim \mathrm{I}m(r^*_{X\mathcal{S}} \circ \gamma^* \circ r^*_{\mathcal{S}X}) \leq \\ &\leq \dim \mathrm{I}m(\gamma^*) = \beta_{(X,\vec{f}_X,i)}(\vec{u}, \vec{w}) \end{aligned}$$

To prove the commutativity of the diagram, we observe that $r_{X\mathcal{S}} \circ r_{\mathcal{S}X}$ is the identity map on the points of $\mathcal{S}$. Thus we have that $r_{\mathcal{S}X} \circ \psi \circ r_{X\mathcal{S}}$ is the canonical inclusion of $\mathcal{S}\langle \vec{f}_\mathcal{S} \preceq \vec{u} - \omega(\epsilon)\rangle$ in $\mathcal{S}\langle \vec{f}_\mathcal{S} \preceq \vec{w} + \omega(\epsilon)\rangle$, concluding the first part of the proof.
For the second inequality we use the following commutative (as will be proved) diagram, with analogous definitions of maps $\gamma', \psi', r_{X\mathcal{S}}$ and $r_{\mathcal{S}X}$:

$$\begin{array}{ccc} X\langle \vec{f}_X \preceq \vec{u}\rangle & \xrightarrow{r_{X\mathcal{S}}} & \mathcal{S}\langle \vec{f}_\mathcal{S} \preceq \vec{u} + \vec{\omega}(\varepsilon)\rangle \\ \gamma' \downarrow & & \downarrow \psi' \\ X\langle \vec{f}_X \preceq \vec{w}\rangle & \xleftarrow[r_{\mathcal{S}X}]{} & \mathcal{S}\langle \vec{f}_\mathcal{S} \preceq \vec{w} - \vec{\omega}(\varepsilon)\rangle \end{array}$$

Here ψ' is well defined because we are assuming $\vec{u} + \vec{\omega}(\varepsilon) \prec \vec{w} - \vec{\omega}(\varepsilon)$.

Then, passing to homology, we have (with analogous settings for the starred symbols):

$$\begin{aligned}\beta_{(X,\vec{f}_X,i)}(\vec{u},\vec{w}) &= \dim \mathrm{I}m(\gamma'^*) = \dim \mathrm{I}m(r^*_{\mathcal{S}X} \circ \psi'^* \circ r^*_{X\mathcal{S}}) \leq \\ &\leq \dim \mathrm{I}m(\psi'^*) = \beta_{(\mathcal{S},\vec{f}_\mathcal{S},i)}(\vec{u}+\vec{\omega}(\varepsilon),\vec{w}-\vec{\omega}(\varepsilon)).\end{aligned}$$

To prove the commutativity of the diagram, we observe that $r_{\mathcal{S}X} \circ r_{X\mathcal{S}}$ is the identity map on the points of X. Thus we have that $r_{X\mathcal{S}} \circ \psi' \circ r_{\mathcal{S}X}$ is the canonical inclusion of $X\langle \vec{f}_X \preceq \vec{u}\rangle$ in $X\langle \vec{f}_X \preceq \vec{w}\rangle$. □

The previous proof follows the basic steps of the proof of Lemma 2.1.3, in fact the main idea is to obtain two functions one from X to $\mathcal{S}$ and the other from $\mathcal{S}$ to X such that they move the points less than ϵ. In this way it is possible to use the properties of the lower level sets and the monotonicity of the PBNFs to create the two commutative diagrams and then pass to the homology level to obtain the searched double inequality.
The two functions can be generated thanks to the special construction of W, that, in some way, works as a bigger space in which we can easily move and create our connection. As a last thing it is important to underline that all that is possible thanks to the fact that the simplicial complex $\mathcal{S}$ appears to be one-dimensional.

3.3 Examples

In this section we shall show how the previous results can decrease the error. In Figure 3.12 we have a circle of radius 4 covered by 64 balls of radius $\delta = 0.5$ as in the example of Section 2.2.3; we recall that the choice of the radius and the density of the sampling are conditioned by the hypothesis of Proposition 1.3.2. Since the construction of Theorem 3.2.4 gives us more freedom in the building of the sampling, we have decided to consider only 32 points for our sampling (Fig. 3.13).

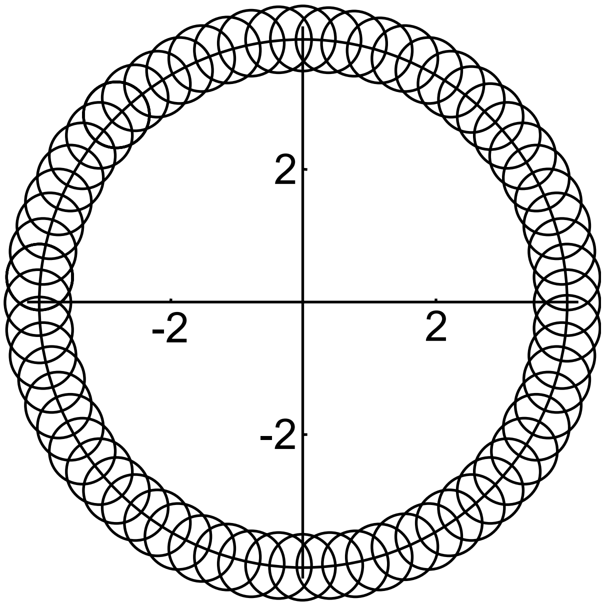

Figure 3.12: The ball union U_δ.

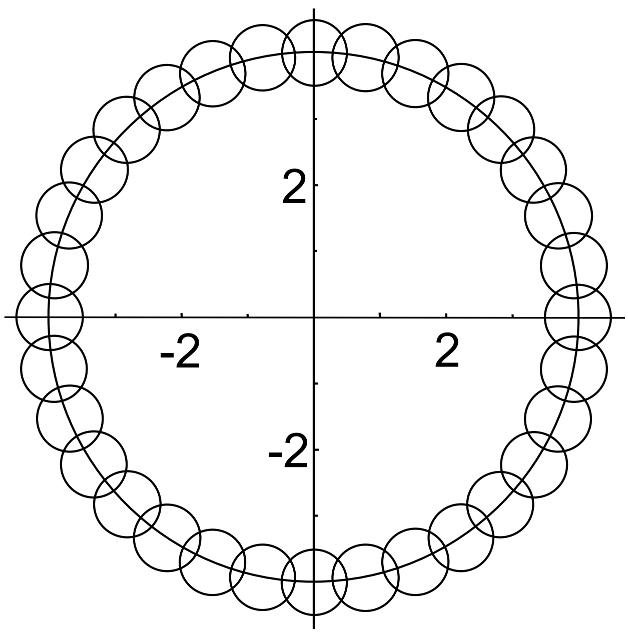

Figure 3.13: The ball union U_ϵ with 32 balls.

We notice that the width of the blind strips remains the same, because the radius of the balls is, in both cases, equal to 0.5; nevertheless we underline two important things: first of all the number of the sampling points is halved, in the sense that the second approximation is less accurate. Second, we are comparing, in Figure 3.14, the blind strips relative to the union of the balls U_δ and in Figure 3.15 the blind strips relative to $\mathcal{S}_\epsilon$; this, in plain words, means that we are working, in the second case, with a combinatorial object that is much easier to handle. We also recall that, if we want to directly compare $\mathcal{S}_\delta$ with the correspondent $\mathcal{S}_\epsilon$, the blind strips of the first one would be doubled with respect to the ones in Figure 3.14.

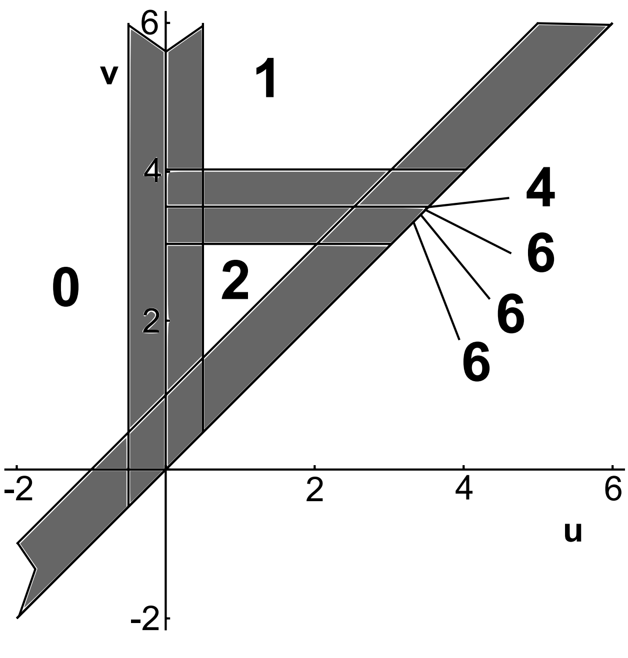

Figure 3.14: The blind strips of $\beta_{(U_\delta, f_{U_\delta}, 0)}$.

As a last example, we shall show the case in which we maintain the same number of points in the sampling, 64 points namely, but we reduce the radius of the balls from 0.5 to only 0.25 (the value is halved); now Figure 3.16 represents the new approximation.

Finally, it is easy to see the difference between the blind strips around the representation of $\beta_{(U_\delta, f, 0)}$ in Figure 3.14 and the blind strips around the

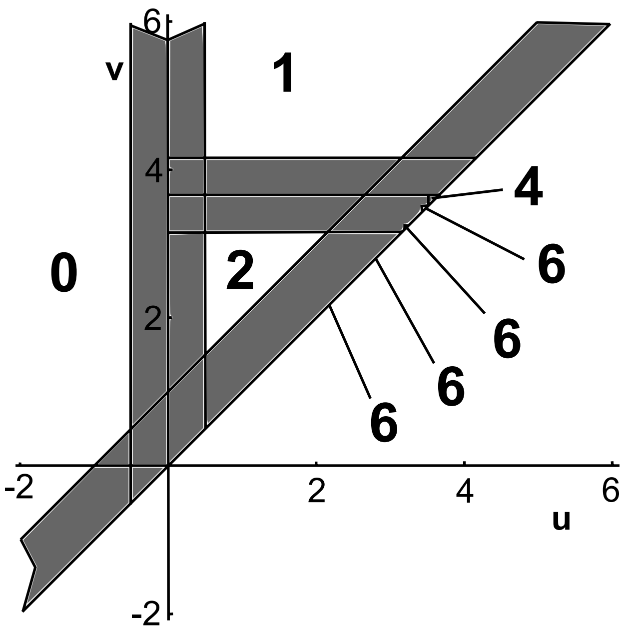

Figure 3.15: The blind strips of $\beta_{(\mathcal{S}_\epsilon, f_{\mathcal{S}_\epsilon}, 0)}$ with 32 balls.

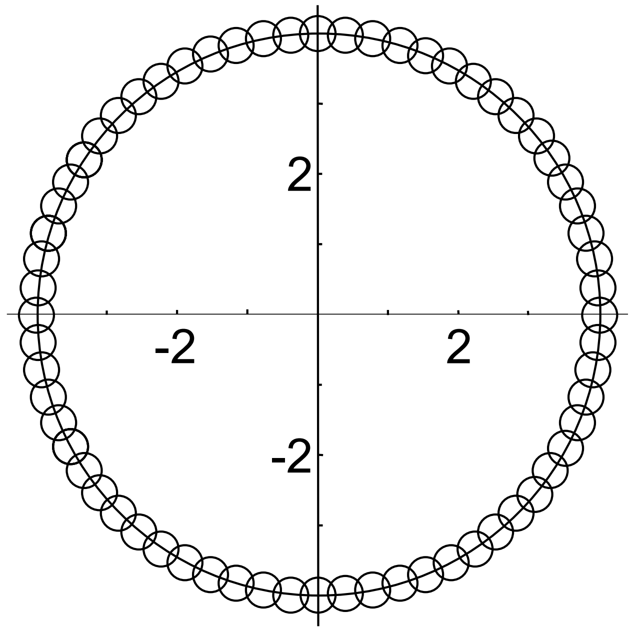

Figure 3.16: The ball union U_ϵ with 64 balls.

representation of $\beta_{(\mathcal{S}_\epsilon,f,0)}$ in Figure 3.17. We obtain that the width of the blind strips is exactly halved in the second case and this means that, with the same number of sampling points, we can halve the error. A similar remark on $\mathcal{S}_\delta$ is true also in this example; then, in this case, the error is reduced to a quarter.

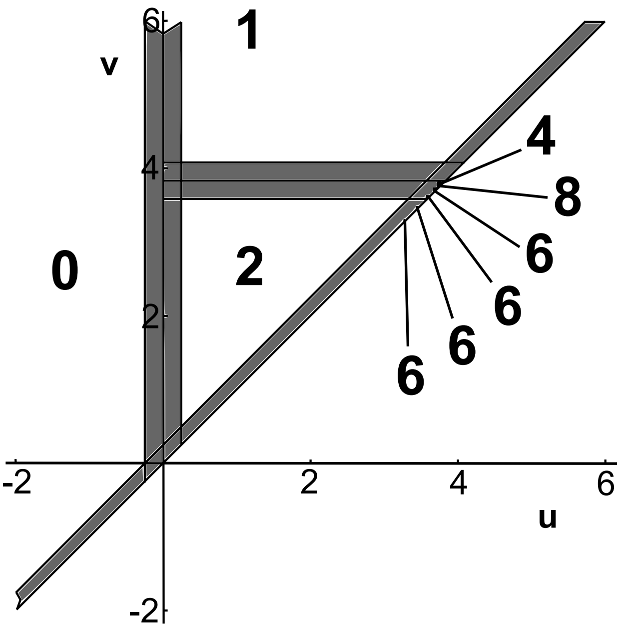

Figure 3.17: The blind strips of $\beta_{(\mathcal{S}_\epsilon,f_{\mathcal{S}_\epsilon},0)}$ with 64 balls.

Conclusion and Future Work

In this Thesis we have shown how it is possible to directly use a topological and geometrical tool, as the PBNs, to deal with a discrete problem concerning the shape analysis. More precisely, we have presented a way to link together the PBNs of a compact Riemanniann submanifold X of $\mathbb{R}^m$ with the PBNs of one of its approximations; in particular when the approximation is represented by a ball covering or even a combinatorial structure. In other words we have introduced a way to relate the continuous aspect of the original object X to the discrete one typical of applications, thanks to the use of a suitable point sampling.
Moreover we have shown that it is possible to find , in the limited setting of 1-submanifolds, a more accurate relation between the different PBNs; this new construction is more flexible for what concerns applications.

Other theoretical results deserve further investigation:

The first of them is the possibility to extend the result of chapter three to every m-submanifold with $m > 1$; creating, in this way, a complete study of the relation simplicial complex – submanifold.

Since we are working with combinatorial objects, a tool to reduce their complexity would be really helpful; maintaining, at the same time, all the topological properties.

Also a theoretical result that introduces a method to reduce the computation complexity of the PBNFs, when we are dealing with a simplicial complex, would be most welcome.

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