ALMA MATER STUDIORUM · UNIVERSITÀ DI BOLOGNA

DOTTORATO DI RICERCA

Ciclo XXII

Settore scientifico disciplinare: MAT/05

TITOLO TESI:

Prescribed mean curvature graphs on exterior domains of the hyperbolic plane

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Esame finale anno 2010

Introduction

This thesis deals with constant mean curvature surfaces in the Riemannian three dimensional manifold $\mathbb{H}^2 \times \mathbb{R}$, where $\mathbb{H}^2$ is the hyperbolic real plane. The modern study of classical constant mean curvature surfaces in three dimensional manifolds began in the XX century with the work of Radò [38] and Douglas [8] on Euclidean minimal surfaces. In the 60's Osserman [35] gave a relevant contribution to this theory: he proved that every complete properly embedded minimal surface of $\mathbb{R}^3$ with finite total curvature is a minimal embedding of a closed orientable surface pinched in a finite number of points. This result has the consequence that minimal surfaces of $\mathbb{R}^3$ are not bounded, hence in the following years the research in this field moved to the understanding of non bounded parts of minimal surfaces. Such a part is called *end* of the surface and it can be defined as a part that can be realized as a complete embedding of a pinched disc. In the 80's Schoen proved that finiteness of total curvature is enough to completely determine the asymptotic behavior of minimal ends. Indeed in [47] he proved that a finite total curvature minimal end is asymptotically a plane or a catenoid. This statement rises the converse question: what curve can be a boundary of an end? If by *exterior domain* we mean the closure of the complement of Ω', a compact set diffeomorphic to a disc in $\mathbb{R}^2$, a natural way to address the problem of existence of minimal ends is to look for solutions of the Dirichlet problem of constant zero mean curvature on an exterior domain. Hence one has to deal with minimal surfaces equation on non convex domains, and it is well known ([9] and [19]) that non-convexity can lead to non existence of the solution. In other words on non-convex domains it is possible to assign boundary data so that there is no minimal graph assuming these data. This Dirichlet problem on non convex domains has been considered by Krust [21], Kuwert [24] and Tomi and Ye in [50], Ripoll and Sauer in [39]. In the late 90's, Kutev and Tomi solved the problem in [23] and [22]. Indeed, in the second paper the authors give sufficient conditions on boundary data in order to obtain minimal exterior graphs with finite total curvature.
When in 2000 Perelman proved Thurston's Geometrization Conjecture for

three dimensional manifolds, the interest in constant mean curvature surfaces in non Euclidean spaces was renewed. One of the most relevant general problems that have been proposed is the existence of pairs (M, h), where M is a three dimensional manifold and h is a value for mean curvature, leading to a precise behavior for ends as in the $(\mathbb{R}^3, 0)$ case. In this thesis we consider the manifold $\mathbb{H}^2 \times \mathbb{R}$. The theory of constant mean curvature surfaces (that we will denote *cmc surfaces*) here is relatively not developed. What happens in $\mathbb{H}^2 \times \mathbb{R}$ is quite different from what happens in $\mathbb{R}^3$. Indeed, if we require the completeness of the surface, in the Euclidean case we have a very neat distinction between the minimal case and the $h > 0$ case. As it is well known, the theory of Euclidean minimal surfaces is wide, deep and it has connections with many branches of mathematics. An introduction to minimal surfaces can be found in [36]. Some recent results are due to Hildebrandt and Tromba [16], Hauswirth, Morabito and Rodriguez [15], Sa Earp and Toubiana [46], Meeks and Frohman [10], Meeks and Pérez [26].
For two different positive values of the mean curvature, the Euclidean cmc surfaces cannot be too different. It is a consequence of the fact that $\mathbb{R}^3$ is a vector space and hence we can rescale any cmc surface so that its (constant) mean curvature is equal to one. In this case we have both compact and non compact surfaces. One can find an introduction to this subject in [17], and [20].
In $\mathbb{H}^2 \times \mathbb{R}$ it is no more possible to reduce the study of positive cmc surfaces to a single value of mean curvature since rescaling is not defined. Roughly speaking we have at least three distinct cases, according to the value of the constant mean curvature. First of all, one has to consider the minimal case. The study of these surfaces has been started in the 00's by Nelli and Rosenberg [43] (and errata [44]). In this work the authors find some analogies with minimal Euclidean surfaces (existence of catenoids, Jenkins-Serrin theorems) and an important distinction: failure of Bernstein theorem. Other results are due to Hauswirth [14], Meeks and Rosenberg [42] and [27], Rosenberg [40], and, more recently, Daniel [6].
The theory for positive constant mean curvature started some years later with the introduction of the generalized Hopf differential proposed by Abresch and Rosenberg in [41]. This caused a fast development: Sa Earp and Toubiana in [45] found examples of rotational constant mean curvature surfaces, Fernandez and Mira in [29] proposed a construction of a Gauss Map for constant mean surfaces in $\mathbb{H}^2 \times \mathbb{R}$. Moreover Nelli and Rosenberg in [31] and [30] established theoretical results for these surfaces. Moreover for positive mean curvature h we have, in constrast with the Euclidean case, to distinguish between the two intervals $(0, \frac{1}{2}]$ and $[\frac{1}{2}, \infty)$. For $h = \frac{1}{2}$ two relevant phenomena occur. The first one was discovered by Daniel in [5]. In this work sufficient

conditions are given to immerge a surface in a 3−dimensional homogeneous manifold with isometry group of dimension 4. Moreover the author discovers a generalization of Lawson's correspondence [25]. Daniel's correspondence establishses a local isometry between cmc surfaces of pairs of 3−dimensional homogeneous manifold with isometry group of dimension 4. In particular it states that any $\frac{1}{2}$−surface of $\mathbb{H}^2 \times \mathbb{R}$ is locally isometric to a minimal surface of the (Riemannian) Heisenberg group. The other relevant phenomenon happening for $h = \frac{1}{2}$ is described in the remarks following theorem 1.4 of Spruck's work [49]. What happens for $h \in [\frac{1}{2}, +\infty)$ is that the mean curvature of graphs of a distance function from the boundary of a bounded set increases with distance. Thus these graphs can be useful to construct barriers.
But to study the problem of ends, the most important papers are from Nelli and Rosenberg [32], Sa Earp and Toubiana [45], Nelli and Sa Earp [33] and Spruck [49]. In the first work it is proved that, for mean curvature h between 0 and $\frac{1}{2}$, there are no compact h−surfaces, hence it makes sense to look for ends for these values of mean curvature. More precisely it is proved that any exterior graph is non bounded. In the second paper, for each $h \in \left(0, \frac{1}{2}\right)$, is given a one parameter family $\{H^h_\alpha\}_\alpha$ of rotational ends which have features similar to the ones of Euclidean catenoids.

Hence Nelli and Sa Earp in [33], guessing that the rotational surfaces have the standard asymptotic behavior, propose new examples of non rotational constant mean curvature graphs on exterior domains, obtained through suitable perturbations of elements of the $\{H^h_\alpha\}_\alpha$ family. Precisely they consider a curve lying on a rotational surface H^h_α, graph on the boundary of an exterior domain. Using the fact that for every α the functions H^h_α have a different asymptotic behavior, they considered an increasing sequence of boundary data on these functions, and they built a new surface growing faster than any element of the rotational family.

Our work starts here and has the objective of giving an existence theorem for exterior graphs with constant mean curvature $h \in (0, \frac{1}{2})$. The technique used by Nelli and Sa Earp cannot be extended to this case since for $h \in (0, \frac{1}{2})$ all surfaces grow with the same rate.

Hence if we started with a curve on a rotational surface, and we imposed conditions at infinity in terms of an increasing sequence of rotational surfaces, we would probably end up with a part of the first surface. Thus we decided to consider as a boundary of the end a Jordan curve not necessarly lying on a H^h_α surface. We end up with a Dirichlet problem on a non convex and unbounded domain in $\mathbb{H}^2$. Dirichlet problem for mean curvature on hyperbolic space is considered by Spruck in [49]. In this work the author gives

a priori estimates for constant mean curvature graphs on compact domains of $\mathbb{H}^2$. Moreover Spruck gives sufficient conditions on the domain to solve the Dirichlet problem of constant mean curvature. But these conditions exclude the case where the domain is an annulus, hence this theorem cannot be used in our case. Spruck establishes as well that when the mean curvature we are prescribing is greater than $\frac{1}{2}$, it is possible to build barriers for the solution using a distance function. This does not mean that for $h \in (0, \frac{1}{2})$ one cannot use distance function to do barriers, but it means that standard ways fail.
In this thesis two main results are proved. The first one is a non existence result for $h-$graphs on circular annuli of $\mathbb{H}^2$. This result follows from a-priori estimates that can be obtained in a way similar to the one proposed by Finn in [9] for the Euclidean case. The second result is an existence theorem for $h-$graphs on non bounded annuli of the hyperbolic plane. The hypotheses of this theorem are on the boundary γ of the exterior domain. The hypotheses are given in terms of the curvature of γ and by requiring that it satisfies a geometric condition associated to the $\{H^h_\alpha\}_\alpha$ surfaces. In order to complete the result, we need to establish fine properties of the asymptotic behavior of the family H^h_α, and a general result of evolution of curves under the flow of the gradient of a distance function. These results have an independent interest, since they describe properties of surfaces and curves, but they will also be used to build barriers from below and from above for our solution. This is done in terms of the H^h_α functions, at least if the boundary of the exterior domain satisfies a suitable geometrical assumption expressed in terms of growth of the elements of the family of rotational solutions(see the condition of r-admissibility below). In this way we will obtain a priori estimates in the spaces of Hölder continuous functions. With a suitable adaptation of Schauder theory and of the continuity method we conclude the proof of the existence theorem.

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Notation

We will use the following notation

- We write M^n when M is a $n-$dimensional manifold
- Ω' is the complement of the set Ω
- $\langle .,. \rangle$ denotes scalar product of vectors
- $B_p(r)$ is the closed ball of center p and radius r
- $S_p(r)$ is the boundary of $B_p(r)$
- $C^\infty(M)$ are the real valued C^∞ functions on the manifold M

Chapter 1

Some concepts of Riemannian geometry

In this chapter we define the geometrical objects that will be used all over the thesis. This part does not want to be comprehensive nor exhaustive. A complete treatment of the basic Riemannian geometry subjects can be found, for instance, in the books [7], [37], [2], [11] and [48]. We start with differential objects to move later to the Riemannian setting.

1.1 Differentiable facts

Definition 1. Topological manifold
A second countable Hausdorff topological space M^n is said a topological manifold of dimension n if each point in M has a neighborhood homeomorphic to an open set in $\mathbb{R}^n$.

Since our goal is to present calculus in manifolds, we have to deal with objects having more structure than the topological one.

Definition 2. Differentiable manifold of class C^k
Consider M^n a topological manifold and a family $\Big(U_\alpha, \phi_\alpha\Big)_{\alpha \in I}$ where $U_\alpha \subset M$ is an open set and $\phi_\alpha : U_\alpha \to V_\alpha := \phi_\alpha(U_\alpha) \subset \mathbb{R}^n$ is an homeomorphism. If for each α_1, α_2 such that $U_{\alpha_1} \cap U_{\alpha_2} \neq \emptyset$ the real valued map

$$\phi_{\alpha_1}^{-1} \circ \phi_{\alpha_2} : U_{\alpha_1} \cap U_{\alpha_2} \to \phi_{\alpha_1}^{-1} \circ \phi_{\alpha_2}\Big(U_{\alpha_1} \cap U_{\alpha_2}\Big)$$

is a diffeomorphism of class C^k, we say that $\Big(U_\alpha, \phi_\alpha\Big)_{\alpha \in I}$ is a differential structure of class C^k for M, provided the union of the U_α covers M. A differentiable manifold is a topological manifold together with a differentiable structure.

For the sake of clarity, we state what we mean by *parametrization* and *local coordinates*. Take $p \in M^n$ and an element (ϕ, U) of the differentiable structure of M such that $p \in U$. We then have an homeomorphism $\phi : U \to \phi(U) = V \subset \mathbb{R}^n$. We call the function ϕ^{-1} a *parametrization* for M near p and $\{(x_1, \ldots, x_n) \in V \subset \mathbb{R}^n\}$ *local coordinates* for M near p. As we shall see, some calculations are simpler in local coordinates, while others are simpler when a parametrization is chosen. Parametrizations help the visual intuition when are given for a submanifold of an ambient manifold, e. g. a surface in $\mathbb{R}^3$.

It is possible to give a natural definition of differential sub-manifold. A differentiable sub-manifold of dimension $k \leq n$ of M^n is a differentiable manifold of dimension k contained in M with a differential structure inherited, by intersection, from M.

A differentiable manifold is *smooth* if it is a C^∞ manifold.

On differentiable manifolds we can define the concept of tangent vector.

Definition 3. Tangent space and tangent bundle
Consider $p \in M^n$ and $\phi : U \to V \subset \mathbb{R}^n$ local coordinates for M near p. We consider two C^1 curves $\gamma_1(t)$, $\gamma_2(t) \subset M$ passing through p at $t = 0$ and we say that they are tangent at p if the Euclidean curves $\phi \circ \gamma_i$ have same tangent vector at time 0. The relation just defined is an equivalence relation on the set of all curves passing through p because it is defined by means of an equality. The tangent space of M at p, T_pM, is the set of all curves on M passing through p modulo this equivalence relation. This is a vector space provided we induce the vector space structure via the ϕ map.
Taking the disjoint union of all the tangent spaces we obtain a $2n-$ dimensional differentiable manifold, TM, called the tangent bundle of M

$$TM = \cup_{p \in M} T_pM = \{(p, v) : p \in M \text{ and } v \in T_pM\}$$

Now that we have tangent spaces, we can define the linearization of differentiable functions.

Definition 4. Differentiable function
Let be M^m, N^n two differentiable manifolds and $f : M \to N$ a function. f is differentiable if its composition with the inverse of a parametrization is differentiable.

Definition 5. Differential of a function
Consider $f : M^m \to N^n$ a differentiable function. Let be $p \in M$, $v \in T_pM$ and $q = f(p) \in N$. Consider a curve $\gamma(t) \subset M$ passing through p at time 0 and with tangent vector v at time 0. We define the differential of f at p applied to the vector v as the tangent vector of the curve which is the image of γ via f at time 0. Namely

$$d_pf(v) = \left(\frac{d}{dt} f \circ \gamma(t)\right)_{t=0}$$

We remark that the differential of function at a point p is a linear function mapping T_pM in T_qN.

It is useful to remark the following facts

Remark.

- It is a matter of computation to show that the definition just given does not depend on the coordinate system, i.e. the representations of a tangent vector in two different coordinate systems are related by the (real) differential of the coordinate change.
- Given a differentiable structure, a basis for each tangent space is given. Take $p \in M$ and $\phi : U \to V \subset \mathbb{R}^n$ near p. Then the canonical basis of $\mathbb{R}^n$ $\left\{e_i\right\}_{i=1,\dots,n}$ induces the basis for T_pM

$$\frac{\partial}{\partial x_i} = \left(d_{\phi(p)}\, \phi^{-1}\right)\left(e_i\right)$$

- The tangent space of a submanifold is a vector subspace of the ambient tangent space

We now define vectorf fields. The definition we give here can be found in [7, Chapter 0, Definition 5.1]. A more formal definition expressed in terms of sections can be found for instance in [11].

Definition 6. Vector field of class C^k
If M^n is a smooth manifold, a vector field on M is a function defined on M which associates to each p a vector of T_pM. We require this function to be of class C^k when thought as a function between M and its tangent bundle. We will use the symbol $\mathfrak{X}(M)$ for the vector space of all vector fields of M.

Given a submanifold $S \subset M$, a vector field along S is a vector field $X : S \to \mathfrak{X}(M)$, i. e. we do not require the vector field to be tangent to the submanifold.

Examples.

- The tangent vector to a curve is an example of a vector field along a submanifold.
- The unit normal field of an orientable surface of the Euclidean space is an example of a vector field along a submanifold.
- A local basis for T_pM given by a system of coordinates near p is an example of a frame, i.e. a local basis.

Vector fields generalize the concept of directional derivative of a function

Definition 7. Action of a field
Let X be a vector field on M and $f \in C^1(M)$ a smooth function. We define the action of X on f as the function

$$X(f)(p) = d_p f(X(p))$$

$\mathfrak{X}(M)$ as much more structure than a plain vector space. Indeed there is a natural operation defined on this space.

Definition 8. Consider X and Y two vector fields on M. For each $f \in C^2(M)$ we define

$$[X, Y](f) = X(Y(f)) - Y(X(f))$$

It is a matter of computation to verify that the object just defined is actually a vector field.

The concept of flow of a vector field will prove to be useful .

Definition 9. Flow of a vector field
Consider M^n is a smooth manifold, X a C^k vector field on M and $p \in M$ a point. An integral curve of X in p is a curve whose tangent vector is X. In other words it is a solution of the initial values problem:

$$\begin{cases} \varphi_p'(t) & = & X(\varphi_p(t)) \\ \varphi_p(0) & = & p \end{cases} \tag{1.1}$$

Choosing a local system of coordinates, this problem writes as an Euclidean $n-$dimensional Cauchy problem. So for each $p \in M$ it has solution defined on an open interval $I_p \subset \mathbb{R}$ containing $t = 0$. Moreover the solution has a C^k dependence on the point p. We can therefore define a map

$$\varphi : \bigcup_{p \in M} \Big(\{p\} \times I_p \Big) \to M$$
$$(q, t) \mapsto \varphi_q(t)$$

where $\varphi_q(t)$ is the solution of (1.1) with initial condition q at time t. We call this function the flow of X

The flow of a vector field has some important properties that we now summarize.

Proposition 1.
Let φ be the flow of the vector field X. Then $\forall\, p \in M$

- $\varphi_p(t+s) = \varphi_{\varphi_p(t)}(s)$ *for each t, s small enough*
- *if we fix the time t, the function $p \mapsto \varphi_p(t)$ is a diffeomorphism on any open set of its domain provided t is small enough.*

A proof of these facts can be found in [2, Chapter IV, Theorem 3.12]. What we have just recalled means that the flow is a one parameter (the time) group of local homeomorphisms. In other words we have given an action of $\mathbb{R}$ on M. It is time to give an example of action of a vector field and flow.

Example 1. Choose $M^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ and consider the field $X \in \mathfrak{X}(M)$

$$X(x,y) = \left(X_1(x,y)\, \frac{\partial}{\partial\, x} + X_2(x,y)\, \frac{\partial}{\partial\, y} \right) = \left(y - \frac{x^2+y^2}{2y} \right) \frac{\partial}{\partial\, x} - x\, \frac{\partial}{\partial\, y} =$$

then the flow of X is the function

$$\varphi_t(x,y) = r(x,y) \Big(-\sin\left(2\,\theta(x,y) - t\right),\, \cos\left(2\,\theta(x,y) - t\right) + 1 \Big)$$

where

$$r(x,y) = \frac{x^2+y^2}{2y} \qquad \text{and} \qquad \theta(x,y) = -\arctan\left(\frac{x}{y}\right)$$

This flow is defined $\forall\, (x,y) \in M$ and $\forall\, t \in \Big(-2\,\theta(x,y) - \pi, -2\,\theta(x,y) + \pi \Big)$. To show this fact we are going to use the defintion of flow and the global (cartesian) coordinates (x, y) for M. If we omit the dependence on (x, y), we write

$$\varphi_t = (\varphi_1(t), \varphi_2(t)) \quad \Longrightarrow \quad \varphi_t' = \varphi_1' \frac{\partial}{\partial x} + \varphi_2' \frac{\partial}{\partial y}$$

Hence to calculate the flow of X we have to solve the differential system

$$\begin{cases} \varphi_1' = X_1(\varphi_t) = \varphi_2 - \dfrac{\varphi_1^2 + \varphi_1^2}{2\,\varphi_2} \\ \varphi_2' = X_2(\varphi_t) = -\varphi_1 \\ \varphi_0(x, y) = (x, y) \end{cases}$$

Plugging the second equation into the first one we get

$$2\,\varphi_2\,\varphi_2'' - {\varphi_2'}^2 = -\varphi_2^2$$

whose solution is

$$\varphi_2(t) = \beta\,\left(\cos\left(\frac{1}{2}\,(2\,\alpha - t)\right)\right)^2$$

where α, β will be determined later to fulfill intial conditions. Using the bisection formula $\cos\left(\frac{\omega}{2}\right)^2 = \frac{1+\cos(\omega)}{2}$ and the duplication formula $\cos(2\,\omega) = \cos(\omega)^2 - \sin(\omega)^2$ we get

$$\begin{aligned} \varphi_2(t) &= \frac{\beta}{2}\,(1 + \cos\,(2\,\alpha - t)) \\ \varphi_2(0) &= \beta\,\cos(\alpha)^2 = y \end{aligned}$$

Using the second relation of the definig system of the flow and the duplication formula $\sin(2\,\omega) = 2\,\sin(\omega)\,\cos(\omega)$ we get

$$\begin{aligned} \varphi_1(t) &= -\varphi_2'(t) = -\frac{\beta}{2}\,\sin(2\,\alpha - t) \\ \varphi_1(0) &= -\beta\,\sin(\alpha)\,\cos(\alpha) = x \end{aligned}$$

Now we find explicit expressions for α and β. We have

$$\frac{x}{y} = \frac{\varphi_1(0)}{\varphi_2(0)} = -\tan(\alpha) \quad \Longrightarrow \quad \alpha = -\arctan\left(\frac{x}{y}\right)$$

Similarly we have

$$\beta = \frac{y}{\cos\left(-\arctan\left(\dfrac{x}{y}\right)\right)^2} \quad \Longrightarrow \quad \beta = \frac{x^2 + y^2}{y}$$

being $\cos(\arctan(\omega)) = \frac{1}{\sqrt{1+\omega^2}}$

1.2 Riemannian metrics and connections

So far we have dealt with differentiable objects, we are now introducing objects depending on a metric.

Definition 10. Riemannian metric
Given M^n a differentiable manifold, a Riemannian metric on M is the datum, for each $p \in M$, of a scalar product $g_p : T_pM \times T_pM \rightarrow \mathbb{R}$ smoothly dependending on p. Then, if $x = (x_1, \dots, x_n)$ are local coordinates near p, a metric is given in terms of the symmetric, smooth, non singular matrix of order n defined by:

$$g_{ij}(x) = g_{(x)}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) \quad i, j = 1, \dots, n$$

We will use the notations $g_p(\,.\,,\,.\,) = \langle\,.\,,\,.\,\rangle_g = \langle\,.\,,\,.\,\rangle_M$. For the elements of the inverse of the metric we write $g^{ij}(x)$. Moreover we set $|\,.\,|_g = \sqrt{g(\,.\,,\,.\,)}$.

We remark that, by means of the partition of unity, it is possible to show that each differentiable manifold admits a Riemannian metric. See, for example, [7, Proposition 2.10, Chapter 1].
Now that we can measure length of vectors, we can define distance on a manifold.

Definition 11.
Consider M a connected manifold equipped with a Riemannian metric g. Consider a curve $\gamma : [a, b] \rightarrow M$. We define the length of γ by setting

$$l(\gamma) = \int_a^b |\gamma'(t)|_g \, dt$$

A curve is said parametrized by arc length when the tangent vector has length constant and equal to one. As in the Euclidean case, it is possible to show that any curve has a unique parametrization by arc-length.
Moreover given two points $p, q \in M$ we can define their distance by setting

$$dist(p, q) = inf\{l(\gamma)\ s.t.\ \gamma : [0, 1] \rightarrow M \text{ with } \gamma(0) = p \text{ and } \gamma(1) = q\} \quad (1.2)$$

Then a Riemannian manifold is a metric space and the proof of this fact can be found, for example, in [7].

Together with a scalar product, we always have a definition of angle. Take v and w in T_pM and define the angle between v and w by setting

$$\cos \sphericalangle(v, w) = \frac{\langle v, w\rangle_g}{|v|_g \, |w|_g}$$

There is an important class of metrics characterized by the behavior on angles

Definition 12.
Consider M a differentiable manifold and g, h two Riemannian metrics on M. g is conformal with respect to h if there exists a positive function $\lambda : M \to \mathbb{R}^+$ such that, for each $p \in M$ it holds

$$g(p) = \lambda^2(p)h(p)$$

Angles measured with two conformal metrics are the same, in fact $\forall\, p \in M$ and $v \in T_pM$ we have $|v|_g = \lambda\, |v|_h$, hence

$$\frac{\langle v, w\rangle_g}{|v|_g\,|w|_g} = \frac{\lambda^2}{\lambda^2}\,\frac{\langle v, w\rangle_h}{|v|_h\,|w|_h}$$

Definition 13. Isometry
Consider $F : (M^n, g) \to (N^n, h)$ a orientation preserving diffeomorphism between orientable smooth Riemannian manifolds. F is an isometry if, for each $p \in M$ and $v,\, w \in T_pM$, it holds

$$g_p(v, w) = h_{F(p)}(d_pF(v), d_pF(w))$$

F is said a local isometry if the just given condition is satisfied locally.

We have an useful characterization of isometric manifolds via local coordinates.

Proposition 2.
Two Riemannian manifolds (M, g) and (N, h) are locally isometric if and only if the matrices of the metrics in some local coordinates coincide.

We are now ready to recall how to define calculus in Riemannian manifolds. A Riemannian metric is a good tool to give a coordinate independent definition of gradient of a function because we can use Riesz duality.

Definition 14. Gradient
Consider (M, g) a Riemannian manifold and $f \in C^1(M)$ a function. For each $p \in M$, we define the gradient of f in p as the vector ∇f satisfying

$$d_pf(v) = g_p(\nabla f, v) \qquad \forall\, v \in T_pM$$

From the variational point of view, the Riemannian gradient has the same useful properties of the gradient of Euclidean multi-variable calculus. We recall two of these properties:

f does not decrease along the integral curves of ∇f

If p is an extremal point for f, then $\nabla f(p) = 0$

We also write the formula for the gradient in local coordinates. One can easily check that in local coordinates it holds

$$\nabla f(x) = \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} g^{ij}(x) \frac{\partial f}{\partial x_i}(x) \Big) \frac{\partial}{\partial x_i} \tag{1.3}$$

where g^{ij} are the local expressions of the inverse of the matrix of the metric.

Now we have all what we need to introduce derivatives of order higher than one on manifolds.

Definition 15. Affine connection
An affine connection on the manifold M is a map $\nabla^M : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfying the following relations for each $X, Y, Z \in \mathfrak{X}(M)$

$C^\infty(M)$ linearity on the first argument. For all $f, g \in C^\infty(M)$

$$\nabla^M_{f X + g Y} Z = f \nabla^M_X Z + g \nabla^M_Y Z$$

$\mathbb{R}$ linearity on the second argument. For all $\alpha, \beta \in \mathbb{R}$

$$\nabla^M_X \Big(\alpha Y + \beta Z \Big) = \alpha \nabla^M_X Y + \beta \nabla^M_X Z$$

Liebniz rule on the second argument. For all $f \in C^\infty(M)$

$$\nabla^M_X f Y = df(X) Y + f \nabla^M_X Y$$

Moreover, a connection is said *torsion-free* if it is well behaved with Lie parenthesis:

$$[X, Y] = \nabla^M_X Y - \nabla^M_Y X$$

A connection is said *compatible* with the Riemannian metric g on M if it satisfies $X \langle Y, Z \rangle_g = \big\langle \nabla^M_X Y, Z \big\rangle_M + \big\langle Y, \nabla^M_X Z \big\rangle_M$ for all $X, Y, Z \in \mathfrak{X}(M)$. It is well known that given a metric on a manifold, there is precisely one torsion-free connection which is compatible with the metric.

Theorem 1. *Levi-Civita*
Given $(M, \langle \cdot, \cdot \rangle)$ a Riemannian manifold, there exists a unique torsion-free connection compatible with the Riemannian structure. This connection is called the Riemannian connection of M or the Levi-Civita connection of M. Moreover, this connection satisfies $\forall\, X, Y, Z, W \in \mathfrak{X}(M)$

$$2 \big\langle \nabla^M_X Y, Z \big\rangle = X \langle Y, Z \rangle + Y \langle X, Z \rangle - Z \langle X, Y \rangle + \\ + \langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle \tag{1.4}$$

For the proof we refer, for example, to [37, Chapter 2, Theorem 1.1]. The last theorem states that the metric of a manifold completely determines its connection. Then it is reasonable to expect the connection to well behave with isometries.

Proposition 3.
Let $F : \Big(M, g\Big) \to \Big(N, h\Big)$ an isometry. Then for each $X, Y \in \mathfrak{X}(M)$ we have

$$dF\left(\nabla^M_X Y\right) = \nabla^N_{dF(X)} dF(Y) \tag{1.5}$$

Proof. We can proceed by direct calculation. If we prove that $\forall W \in \mathfrak{X}(N)$ we have

$$h\Big(dF\Big(\nabla^N_X Y\Big), W\Big) = h\Big(\nabla^N_{dF(X)} dF(Y), W\Big)$$

we are done. We observe that, being F an isometry, we have $W = dF(Z)$ for a $Z \in \mathfrak{X}(M)$. Thus we can write

$$\begin{aligned} h\Big(dF\Big(\nabla^M_X Y\Big), W\Big) &= h\Big(dF\Big(\nabla^M_X Y\Big), dF(Z)\Big) \\ &= g\Big(\nabla^M_X Y, Z\Big) \\ &= \frac{1}{2}\Big(X\, g(Y, Z) + Y\, g(X, Z) - Z\, g(X, Y) + \\ &\qquad + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y)\Big) \end{aligned}$$

On the other hand, for each $f \in C^\infty(N)$ and $dF(X) \in \mathfrak{X}(N)$, we have

$$dF(X)\, f = X\Big(f \circ F\Big)$$

and

$$X\, g(Y, Z) = g(\nabla^M_X Y, Z) + g(Y, \nabla^M_X Z)$$

In the same way we can also prove that

$$dF\Big([X, Y]\Big) = [dF(X), dF(Y)]$$

Thus we have

$$\begin{aligned} 2\,h\Big(\nabla^N_{dF(X)}\,dF(Y), dF(Z)\Big) &= dF(X)\,h\Big(dF(Y), dF(Z)\Big)+ \\ &+ dF(Y)\,h\Big(dF(X), dF(Z)\Big) \\ &- dF(Z)\,h\Big(dF(X), dF(Y)\Big)+ \\ &+ h\Big([dF(X), dF(Y)], dF(Z)\Big) \\ &- h\Big([dF(Y), dF(Z)], dF(X)\Big)+ \\ &+ h\Big([dF(Z), dF(X)], dF(Y)\Big) \end{aligned}$$

which proves the claim. □

Since we will need to write the Riemannian connection in local coordinates, we introduce the Christoffel's Symbols

Definition 16.
Take $p \in M$ and $(x_1, \ldots, x_n)$ local coordinates near p. For each $1 \le i, j \le n$ we define

$$\nabla^M_{\frac{\partial}{\partial\,x_i}} \frac{\partial}{\partial\,x_j} = \sum_{k=1}^{n} \Gamma^k_{ij}\,\frac{\partial}{\partial x_k}$$

The functions Γ^k_{ij} are the Crhistoffel's symbols of the connection ∇^M in the basis given by $(x_1, \ldots, x_n)$. Being the connection torsion-free, these functions are symmetric in the lower indexes. Moreover, if g_{ij} is the expression of the metric in local coordinates, we have the following formula

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{h=1}^{n} g^{kh}\left(\frac{\partial g_{hi}}{\partial x_j} + \frac{\partial g_{hj}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_h}\right) \tag{1.6}$$

From now on when we will talk about a manifold, we will mean a Riemannian manifold with the associated Riemannian connection.
The definition of connection is what we will use to define calculus objects on Riemannian manifolds.

Definition 17. Divergence and laplacian
Consider $X \in \mathfrak{X}(M)$. We define the divergence of X as the trace of the linear function $Y \mapsto \nabla^M_Y\,X$. The trace of a linear function on a vector space does not depend on the base with respect to which it is calculated. Then we have

$$div(X) = \sum_{i=1}^{n} \left\langle \nabla^M_{E_i}\,X, E_i \right\rangle_M$$

provided E_i is a local orthonormal frame for the tangent space to M.
Consider $f \in C^{\infty}(M)$. We define the laplacian of f as the divergence of its gradient:

$$\Delta f = div(\nabla f)$$

One can esaily check that the expression of the laplacian in local coordinates is

$$\Delta f = \frac{1}{\sqrt{det\,(g)}} \left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{det\,(g)} \sum_{j=1}^{n} g^{ij} \frac{\partial f}{\partial x_j} \right) \right)$$

1.3 Geodesics and distance

The material defined until now is enough to make derivatives of any order on Riemannian manifold. We introduce some concepts descending from the Riemannian connection.

Definition 18. Covariant derivative and parallel field along a curve
Consider a curve $\gamma(t) \in M$ of class C^1 and X a vector field along γ. We define the covariant derivative of X along γ as

$$\frac{D}{dt} X = \nabla^M_{\gamma'} X$$

X is parallel along γ if it satisfies

$$\frac{D}{dt} X \equiv 0$$

The covariant derivative inherits all the properties of the Riemannian connection, so it acts as an usual temporal derivative.

We now introduce an extremely important concept: geodesics.

Definition 19. Geodesic
A C^2 curve γ in the Riemannian manifold M is a geodesics if it is not constant and its tangent vector is parallel along the curve, that is

$$\nabla^M_{\gamma'} \gamma' \equiv 0$$

We will write γ_v for the geodesic leaving p with tangent vector v. Geodesics are set of points together with a parametrization, in the sense that geodesics are curves with speed proportional to arc length. This is seen directly

$$\frac{D}{dt} |\gamma'|^2 = \nabla^M_{\gamma'} \langle \gamma', \gamma' \rangle_M = 2 \left\langle \nabla^M_{\gamma'} \gamma', \gamma' \right\rangle_M \equiv 0$$

We point out that geodesics have short time existence. More precisely, $\forall\, p \in M$ it exists a $V_p \subset T_pM$ a neighborhood of 0 such that for each $v \in V_p$ exists an $\epsilon(v) > 0$ and a unique geodesics $\gamma(t)$ defined for $t \in (\epsilon(v), \epsilon(v))$ satisfying $\gamma(0) = p$ and $\gamma'(0) = v$. A proof of this fact can be found, for example, in [37, Chapter 5, Theorem 2.1].

Geodesics allow to extend the Euclidean concepts of convexity of domains and curvature of a curve.

Definition 20. Geodesic curvature
Let $\gamma \subset M$ be a curve parametrized by arclength. We define the geodesic curvature of γ as the norm of the acceleration of γ

$$k_g(\gamma) = \left|\frac{D}{dt}\gamma'\right|$$

If M has dimension two and is orientable, we can assign to the geodesic curvature a sign. In order to do that we recall that, being M orientable, on its tangent plane is defined J the operator making a counter-clock rotation of $\frac{\pi}{2}$ radians. In other words $J(v)$ is the unique vector orthogonal to v, with same length as v that makes the ordered basis $(v, J(v))$ a positive basis of T_pM.

Definition 21. Normal curvature
Let M^2 be a surface and $\gamma \subset M$ be a curve not necessarily parametrized by arclength. If T is a unit tangent vector, we define

$$k_n(\gamma) = -\left\langle \nabla^M_T\, J\, T, T \right\rangle$$

The function we have just defined adds to the previous one regularity and sign.

Proposition 4. *Let $\gamma \subset M^2$ a curve arc-length parametrized. Then*

$$|k_n(\gamma)| \propto k_g(\gamma)^2$$

Proof. The proof is a simple verify. Being γ parametrized by arc-lenrgth, the acceleration is normal to the speed

$$\begin{aligned}\left\langle \frac{D}{dt}\gamma', \gamma' \right\rangle &= \frac{1}{2}\frac{D}{dt}\left\langle \gamma', \gamma' \right\rangle \\ &\equiv 0\end{aligned}$$

Thus it exists a non vanishing smooth function ψ satisfying

$$J\,\gamma' = \psi\,\frac{D}{dt}\,\gamma'$$

Then we have

$$\begin{aligned} k_n(\gamma) &= -\left\langle \nabla_{\gamma'}\, J\,\gamma', \gamma' \right\rangle \\ &= -\left\langle \psi'\,\frac{D}{dt}\gamma' + \psi\,\frac{D^2}{dt^2}\gamma', \gamma' \right\rangle \\ &= \psi\left(-\frac{D}{dt}\left\langle \frac{D}{dt}\,\gamma', \gamma' \right\rangle + \left\langle \frac{D}{dt}\,\gamma', \frac{D}{dt}\,\gamma' \right\rangle \right) \\ &= \psi\left\langle \frac{D}{dt}\,\gamma', \frac{D}{dt}\,\gamma' \right\rangle \end{aligned}$$

which yields

$$|k_n(\gamma)| = |\psi|\,k_g(\gamma)^2$$

□

But the most important feature of the normal curvature in dimension two is that it characterizes geodesics

Proposition 5.
Let $\gamma \subset M^2$ a curve parametrized by arc length. γ is a geodesic in M^2 if and only if its normal curvature vanishes identically

Proof. On one side being γ parametrized by arc length, its acceleration is orthogonal to its tangent space. On the other side we have

$$-\left\langle \nabla_{\gamma'}\, J\,\gamma', \gamma' \right\rangle = \left\langle J\,\gamma', \nabla_{\gamma'}\,\gamma' \right\rangle$$

which means that the geodesic curvature of γ is identically zero if and only if the normal component of the acceleration is identically zero. But this component is the only one that could be non zero. □

This property allows to think to the normal curvature as function quantifying the distance of a curve from being a geodesic. Thus, from now on, each time that we will deal with two dimensional manifolds, saying geodesic curvature, we will refer to the normal curvature.
Before goin on, we observe a technical fact that will be helpful in the calculations

Remark.
If $T = \frac{V}{|V|}$ we have

$$k_n(\gamma) = -\frac{1}{|V|^3} \langle \nabla_V JV, V \rangle$$

and the proof is straightforward:

$$\begin{aligned} k_n(\gamma) &= -\langle \nabla_T JT, T \rangle \\ &= -\frac{1}{|V|^2} \left\langle \nabla_V \frac{JV}{|V|}, V \right\rangle \\ &= -\frac{1}{|V|^2} \left\langle V \left(\frac{1}{|V|} \right) JV + \frac{1}{|V|} \nabla_V JV, V \right\rangle \\ &= -\frac{1}{|V|^3} \langle \nabla_V JV, V \rangle \end{aligned}$$

Definition 22. Geodesically convex domain
Let M^n be a manifold. Consider $\Omega \subset M^n$ a compact and orientable submanifold of dimension $n-1$. We call Ω a domain if it is connected and simply connected. Moreover, we say that Ω is geodesically convex, briefly convex, if for each $p, q \in \Omega$ exists a geodesic γ going from p to q contained in Ω

Geodesics are extremely useful because they locally minimize distance. In our setting geodesics will be distance minimizing for all times, but this is not the general situation.

Proposition 6.
Take $p \in M$. Then $\forall\, v \in T_pM$ exists $\epsilon(v) > 0$ such that $\gamma_v(t)$ is distance minimizing for each $|t| \leq \epsilon(v)$

The proof of this standard fact can be found, for instance, in [37, Theorem 5.1].

We now recall the definition of distance functions. These functions, that will be used in a substantial way in Chapter 3, are strongly related to geodesics.

Definition 23. Distance function
Consider $U \subset M$ an open set and $f \in C^\infty(U)$. f is a distance function if

$$|\nabla f|_M \equiv 1$$

The following definition should help justifying the name given to these functions.

Definition 24. Signed distance
Consider $S \subset M$ a closed and orientable hypersurface. We call *interior* of S the compact part of M bounded by S and *exterior* its complement. For each $p \in M$ we define $d_S(p)$ to be

$$d_S(p) = \begin{cases} -\min_{y \in S} d(p,y) & \text{if } p \in Interior(S) \\ \\ \min_{y \in S} d(p,y) & \text{if } p \in Exterior(S) \end{cases} \tag{1.7}$$

We call this function the signed distance from S

It is well-known that a signed distance function inherits the regularity from the associated closed hypersurface. This fact can be found, for example, in [28].

Remark.
We are going to prove that d_S is a distance function on some open set $U \subset M$.
Using the exponential coordinates based on S of definition 27 we obtain the following expression of d_S

$$d_S(p) = d_S(exp_y(t\,\eta(y))) = t$$

which means that the only derivative appearing in the gradient formula (1.3) is the one in the t direction. Thus, by orthogonality of $\frac{\partial}{\partial t}$ to all other elements of the frame induced by exponential coordinates, we get

$$\nabla f(x) = \frac{\partial}{\partial t}$$

since $\frac{\partial}{\partial t}$ is the vector tangent to the geodesic $exp_y(t\,\eta(y))$ and this geodesics leave y with speed $|\eta(y)| = 1$, we have

$$\left|\frac{\partial}{\partial t}\right| \equiv 1$$

We present another result which is the ultimate good reason to call distance function a map with gradient of length one.

Proposition 7.
The integral curves of a distance function are geodesics

Proof. We are following the proof given in [37, Chapter 5, Lemma 3.6]. Take $U \subset M$ an open subset and $f \in C^\infty(U)$ a distance function. Consider

$x, y \in U$ and consider $\gamma : [0, \epsilon] \to U$ going from x to y. We are going prove that

$$f(y) - f(x) \leq l(\gamma) \tag{1.8}$$

We remark that,when this fact will be proved, to conclude the proof it will be enough to show that integral curves of ∇f realizes equality. This is because $d(x, y)$ is defined as the infimum of the length of curves going from x to y. We have

$$\begin{aligned}
l(\gamma) &= \int_0^\epsilon |\gamma'(t)|\, dt \\
&= \int_0^\epsilon |\nabla f(\gamma(t))|\, |\gamma'(t)|\, dt \quad \text{since } |\nabla f| \equiv 1 \\
&\geq \int_0^\epsilon \langle \nabla f(\gamma(t)), \gamma'(t) \rangle\, dt \quad \text{by Cauchy-Schwarz inequality} \qquad (1.9) \\
&= \int_0^\epsilon \left(d_{\gamma(t)} f \right) (\gamma'(t))\, dt \quad \text{by definition of gradient} \\
&= \int_0^\epsilon \frac{d}{ds} \Big(f \circ \gamma(s) \Big)_{|s=t} dt \quad \text{by definition of differential} \\
&= f \circ \gamma(\epsilon) - f \circ \gamma(0) \\
&= f(y) - f(x) \qquad (1.10)
\end{aligned}$$

Moreover, by Cauchy-Schwarz inequality, equality is achieved if and only if $\nabla f(\gamma(t))$ and $\gamma'(t)$ are parallel. Then inequality (1.8) is an equality when γ is an integral curve of a distance function. □

It can happen that all geodesics emanating from each point of M are defined on all the tangent space for all times. A manifold with this feature is called geodesically complete. Completeness is a very important feature since a complete manifold has topological properties similar to the Euclidean ones.

Theorem 2. *Hopf-Rinow*
Let M be a Riemannian manifold. Then the following conditions are equivalent

1. *M is geodesically complete*
2. *The compact sets of M are the bounded and closed ones*
3. *For each $p, q \in M$ there exists a geodesic going from p to q whose length equals $d(p, q)$*

4. *M is a topological complete space with respect to the topology induced by d*

For the proof of this fact we refer to [37, Chapter 5, Theorem 7.1].

We are now going to make a technical assumption. Namely, we will assume that all geodesics emanating from any point are defined until time $t = 1$.

Remark.
The assumption just made is general. Assume $p \in M$ and consider the geodesic γ_v defined up to time $\epsilon(v)$. If we consider the geodesic $\gamma_{\alpha v}(t) = \gamma_v(\alpha t)$ we have that $\gamma_{\alpha v}$ is defined up to time $\frac{\epsilon(v)}{\alpha} > 0$. So by taking $\alpha = \frac{1}{\epsilon(v)}$ we have the claim. So the set $O_p \subset T_pM$ for which γ_v is defined until time $t = 1$ is non empty. It is also open because geodesics depend in a smooth way on the initial speed.

We now define the exponential map

Definition 25. Exponential Map
Take $p \in M$, and consider O_p as before. Then one defines

$$\begin{array}{cccc} exp_p(v): & O_p & \longrightarrow & M \\ & v & \mapsto & \gamma_v(1) \end{array}$$

where γ is the geodesic satisfying $\gamma_v(0) = p$ and $\gamma'(0) = v$.

This map induces a map on an open subset of TM, namely

$$O = \bigcup_{p \in M} O_p \tag{1.11}$$

The exponential map has many properties. We will only recall the ones that will be used later.

Theorem 3.
For each $p \in M$ the map exp_p is a diffeomorphism from a neighborhood of $0 \in T_pM$ onto its image, a neighborhood of $p \in M$.

Proof. This result follows from implicit function theorem. The proof consists in showing that the differential of the exponential map is non singular in $0 \in T_pM$. We are actually going to prove that this differential is the identity of

$T_0 T_pM$, provided we assume the identification $T_0 T_pM \simeq T_pM$. If $v \in T_pM$ we have

$$\begin{aligned} d_0\,(exp_p)\,(v) &= \frac{d}{dt}\Big(exp_p(t\,v)\Big)_{t=0} \\ &= \frac{d}{dt}\Big(\gamma_{tv}(1)\Big)_{t=0} \\ &= \frac{d}{dt}\Big(\gamma_v(t)\Big)_{t=0} \\ &= v \end{aligned}$$

□

Thus M can be locally described using geodesics.

Definition 26. Exponential Coordinates
Take $p \in M$ and suppose that $exp_p : V_p \to U_p$ is a diffeomorphism. Then U_p can be described using the coordinates given by

$$\begin{array}{rccc} (\exp_p)^{-1}: & U_p & \longrightarrow & V_p \\ & q & \mapsto & (x_1, \ldots, x_n) \end{array}$$

where $(x_1, \ldots, x_n)$ are real variables given by an identification of T_pM with $\mathbb{R}^n$.

Actually more can be done, in fact we can let move the base point of exp_p and still get local coordinates for a part of M. What we are going to see is that, if we have S an orientable hypersurface of M, we can describe a neighborhood of S with $n-1$ coordinates belonging to S and one coordinate moving in the direction normal to S.
Consider $S \subset M$ an orientable hypersurface and define the normal bundle of S

$$TS^{\perp} = \{(p, \eta) \in TM \,:\, p \in S \text{ and } \eta \in (T_pS)^{\perp} \subset T_pM\} \tag{1.12}$$

Then we can define the normal exponential map by setting

$$\begin{array}{rccc} \exp^{\perp}: & O \cap TN^{\perp} & \longrightarrow & M \\ & (p, \eta) & \mapsto & exp_p(\eta) \end{array}$$

where O is defined in (1.11). It is clear that $d_0\,(exp_p{}^{\perp})$ is non singular for each p. Then, being S embedded, $exp^{\perp}$ is a diffeomorphism of a neighborhood of $S \times 0 \subset TS^{\perp}$ onto its image $U \subset M$. We have just proven that the following definition is well posed.

Definition 27. Exponential coordinates based on an hypersurface
Let $S \subset M$ an orientable hypersurface. Then $\exists U$ a neighborhood of S in M such that, for each $p \in U$, we have

$$p = exp_y(t\,\eta(y)) \tag{1.13}$$

for a unique $y \in S$ and a unique $t \in \mathbb{R}$. Here $\eta(y)$ is one of the possible unit normal fields of S.

Remark.
If $(y_1, \ldots, y_{n-1})$ are local coordinates for S near y then we have local coordinates for U given by $(y_1, \ldots, y_{n-1}, t)$.

Exponential coordinates are extremely useful because allow to make calculus on manifolds in a way very similar to the Euclidean one. So it would be very useful if these coordinates were global, i. e. if chosen a point p we could describe the whole M by the coordinates of T_pM. Clearly for general manifolds this is not possible, but it is possible for a class of manifolds to which the hyperbolic plane belongs. Later we will state for what class of manifolds exponential coordinates are global. Now we see that for closed hypersurfaces we can use these coordinates to describe tubular neighbourhoods. Moreover we can associate a signed distance to any closed and orientable hypersurface.

Proposition 8. *Consider $S \subset M^n$ a closed hypersurface and $S \subset M$. Let d the signed distance function associated to S. If φ_t is the flow ot ∇d, then $\exists\, a > 0$ such that*

$$\begin{array}{rccc} \varphi(t): & S & \longrightarrow & C_t \\ & p & \mapsto & \varphi_p(t) \end{array}$$

is a diffeomorphism $\forall\, t \in [-a, a]$

Proof. The essence of the proof is to show that the flow of a distance function is gedesic so that one can use exponential coordinates. If a is small enough we can parametrize $\overline{V}_a$ via exponential coordinates in the following way ([13, pag 32])

$$\begin{array}{rccc} \phi: & S \times [-a, a] & \longrightarrow & \overline{V}_a \\ & (q, t) & \mapsto & exp_q(t\,\eta(q)) \end{array}$$

where η is a unit normal field on S. We now show that the flow of d is coincides with the exponential map on the normal direction to S. Take Sia $q \in S$. Since, by proposition 7, the flow of a distance function is geodesic and because two geodesics with same tangent field coincide, we verify that the two geodesics $exp_q(t\,\eta(q))$ and $\varphi_q(t)$ leave q with same tangent vector. We have

$$\begin{aligned}\varphi_q'(0) &= \nabla d(\varphi_q(0)) = \nabla d(q)\\ \frac{d}{dt}\left(exp_q(t\,\nabla d(q))\right)|_{t=0} &= (d_0 exp_q)\,(\nabla d(q))\\ &= \nabla d(q)\end{aligned}$$

since $d_0 exp_q = id_{T_qS}$ as in the proof of Theorem 3. □

Definition 28. Injectivity Radius
Consider a complete manifold M and $p \in M$. One defines the injectivity radius of M at p as

$$\rho_{inj}(p) = \sup\{r > 0\, :\, exp_p \text{ is a diffeomorphism from } B_0 r \subset T_pM\}$$

We can associate an injectivity radius to M by setting

$$\rho_{inj}(M) = \inf_{p \in M} \rho_{inj}(p)$$

It is clear from the definition that $\rho_{inj}(M) = \infty$ if and only if exp_p is a diffeomorphism defined on the whole T_pM which means, by completeness of M via Hopf-Rinow theorem, that $exp_p\Big(T_pM\Big) = M$.

1.4 Curvature

In this section we will omit the dependence of the connection on the manifold unless it could lead to a misunderstanding.

Definition 29. Curvature
Consider M^n a Riemannian manifold and $X, Y, Z \in \mathfrak{X}(M)$. One defines the curvature function associated to M by setting

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z \qquad (1.14)$$

If we consider one more field W we define the following notation

$$(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle$$

The curvature function is actually a tensor since it can be proved to be linear in each of its arguments. It can be considered as the core of Riemannian geometry. However, here we will only recall the facts that will be useful in the forecoming parts. One can find a handy treatment of the subject in [7].

Proposition 9.
For all $X, Y, Z, W \in \mathfrak{X}(M)$ *we have*

1. *Bianchi Identity*

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

2. $$(X,Y,Z,W) = -(X,Y,W,Z)$$

3. $$(X,Y,Z,W) = -(Y,X,Z,W)$$

4. $$(X,Y,Z,W) = (Z,W,X,Y)$$

Now we show that the curvature tensor can be understood by means of a simpler curvature. In order to do this we will need to a notation for planes spanned by vectors. Take $X, Y \in \mathfrak{X}(M)$ two linearly independent vector fields. For each $p \in M$ we are writing by $\pi_p(X,Y) \subset T_pM$ the plane spanned by $X(p)$ and $Y(p)$. From now on we are omitting the dependence on p.

Definition 30. Sectional curvature
Take $p \in M$ and two fields X, Y non zero in p. Then we define

$$K_{sect}(X,Y)(p) = \frac{(X,Y,X,Y)}{|X|^2 |Y|^2 - \langle X,Y \rangle^2}(p) \tag{1.15}$$

Actually sectional curvature only depends on the plane spanned by X, Y, and not on the specific choice of X and Y.

Proposition 10.
Take $X, Y \in \mathfrak{X}(M)$ *two linearly independent fields. If* $W, Z \in \mathfrak{X}(M)$ *span the same plane as* X, Y *we have*

$$K_{sect}(X,Y) = K_{sect}(W,Z)$$

The proof of this standard result can be found, for instance, in [7, Chapter 4, Proposition 3.1].

We now recall that sectional curvature completely determines the curvature tensor.

Theorem 4.
Consider V a vector space with a scalar product $\langle . , . \rangle$ and $R, R' : V \times V \times V \to V$ two trilinear functions satisfying the same algebraic properties of a curvature tensor, namely the properties of proposition 9. Taken $x, y \in V$ linearly independent, if we define

$$K(\pi(x,y)) = \frac{\langle R(x,y,x), y\rangle}{|x|^2 |y|^2 - \langle x,y\rangle^2} \qquad K'(\pi(x,y)) = \frac{\langle R'(x,y,x), y\rangle}{|x|^2 |y|^2 - \langle x,y\rangle^2}$$

we have the following implication

$$K(\pi) = K'(\pi) \quad \forall \;\; \textit{plane} \;\; \pi \subset V$$
$$\Downarrow$$
$$R = R'$$

The proof of this result is completely algebraic, so we refer to [7, Chapter 4, Lemma 3.3] for the proof.

1.5 Immersed manifolds

In this section we will consider submanifolds with the Riemannian structure induced from an ambient manifold. We will recall geometric functions determined by how the sub-manifold is placed in the ambient manifold.

1.5.1 Geometry on submanifolds

In this section M^n will be a complete Riemannian manifold and S^m, $m \leq n$ a differentiable manifold.

Definition 31.
A smooth map $\Phi : S \to M$ is an immersion of S provided its differential is non singular at each point of S. If in addition Φ is injective and is an homeorphism onto its image, Φ is called an embedding of S.

Definition 32. Induced Riemannian structure
Suppose $\Phi : S \to M$ is an immersion. Then if the Riemannian structure of M is given by the metric g, Φ induces a Riemannian structure on S defined as follows. Consider $p \in S$. For each $v, w \in T_pS$ we define

$$h_p(v,w) = g_{\Phi(p)}(d_p\Phi(v), d_p\Phi(w))$$

This metric on S is called the pull-back metric.
Roughly speaking, we define a metric on S such that Φ is an isometry.
It turns out that the Riemannian connection ∇^S is the projection of ∇^M on the tangent bundle of S. From now on we shall assume that all the hypersurfaces that we take into account are orientable.

Definition 33.
Let M be a Riemannian manifold and S an hypersurface with unit normal field η. For each $X, Y \in \mathfrak{X}(S)$ and $p \in S$ we define

$$\nabla^S_X Y(p) = \left(\nabla^M_{\widetilde{X}} \widetilde{Y}\right)^{tangent}(p) = \nabla^M_{\widetilde{X}} \widetilde{Y}(p) - \left\langle \nabla^M_{\widetilde{X}} \widetilde{Y}, \eta \right\rangle_M \eta(p)$$

where $\widetilde{X}$ and $\widetilde{Y}$ are extensions of X and Y to a neighborhood of p in M.

Proposition 11.

1. *∇^S is well defined, i. e. it does not depend on the extension of X and Y*

2. *∇^S is the unique connection compatible with the metric induced on S by M*

For the proof one can see, for example, [7, Chapter 6].

Definition 34. Shape operator
Let $S \subset M$ be an orientable hypersurface with unit normal field η. Then for each $p \in S$ one defines the *shape operator*

$$\begin{array}{rccc} A: & T_pS & \longrightarrow & T_pS \\ & v & \mapsto & \left(-\nabla^M_v \eta\right)^{tangent} \end{array}$$

The operator we have just defined is linear. Since we are going to prove it is self adjoint, we can say that this operator is the Riemannian generalization of the Euclidean differential of the Gauss map.

Proposition 12.
The Shape operator is self adjoint

Proof. We can proceed by direct computation. Clearly we have

$$-\left\langle \nabla^M_v \eta, w \right\rangle = \left\langle \nabla^M_v w, \eta \right\rangle$$

and

$$\langle A(v), w\rangle - \langle v, A(w)\rangle = \left\langle \nabla^M_v w - \nabla^M_w v, \eta \right\rangle$$
$$= 0$$

since the Lie bracket of two fields tangent to a submanifold is tangent to the submanifold.

□

It is useful to understand how the shape operator changes when moving away form the hypersurface. To state the result solving this problem we introduce some notation. Consider M a manifold and $S \subset M$ a sub-manifold. If η is a unit normal field along S and $X \in \mathfrak{X}(M)$ we define

$$R_\eta(X) = R(X, \eta)\eta \tag{1.16}$$

Then the change of the shape operator along normal directions is described by the following equation.

Theorem 5. *Radial Curvature Equation*
Consider $S \subset M$ a submanifold with a unit normal vetor field η. Thus we have

$$-\nabla_\eta A + A^2 = R_\eta \tag{1.17}$$

where we recall that $\nabla_\eta A$ is defined as

$$\Big(\nabla_\eta A\Big)(X) = \nabla_\eta \Big(A(X)\Big) - A\Big(\nabla_\eta X\Big)$$

We are not going to prove this fact. For a proof one can see, for example, [13, Chapter 3, Lemma 3].

1.5.2 Mean Curvature

We can finally introduce the main object of study of this thesis.

Definition 35. Mean Curvature
Consider $S \subset M^n$ an orientable hypersurface with unit normal field η. Then

for each $p \in S$ we define the mean curvature of S in x with respect to η as

$$\begin{aligned} H(p) &= -\frac{1}{n-1} trace(A)(p) \\ &= -\frac{1}{n-1} \sum_{i=1}^{n-1} \langle \nabla^M_{E_i} \eta, E_i \rangle \\ &= -\frac{1}{n-1} div(\eta)(p) \end{aligned}$$

where $E_1, \dots, E_{n-1}$ is an orthonormal frame of $T_y S$ for y near x

A few facts need to be observed.

Remark.

- The normal curvature for curves of definition 21 coincides with the mean curvature just defined.
- The sign of $H(x)$ changes if the unit normal field is changed.
- H is preserved by ambient isometries, i. e. if $F : M \to M$ is an orientation preserving isometry, we have, for each $x \in S$

  $$H(x) = \widetilde{H}(F(x))$$

  where $\widetilde{H}$ is the mean curvature of $F(S)$ calculated with respect to $\widetilde{\eta} = dF(\eta)$. This is true because mean curvature is a divergence. Indeed, choose $x \in S$ and $E_1, \dots, E_{n-1}$ an orthonormal frame for $T_y S$ for y near x. Then

  $$\begin{aligned} -n\,\widetilde{H} &= \sum_{i=1}^{n-1} \langle \nabla^M_{dF(E_i)} dF(\eta), dF(E_i) \rangle \\ &= \sum_{i=1}^{n-1} \left\langle dF\left(\nabla^M_{E_i} \eta\right), dF(E_i) \right\rangle \quad \text{by (1.5)} \\ &= \sum_{i=1}^{n-1} \langle \nabla^M_{E_i} \eta, E_i \rangle \\ &= -n\,H \end{aligned}$$

We end this general part on mean curvature proving a result describing the mean curvature for level surfaces of distance functions. This is because distance functions are a standard tool to build barriers for solutions of Dirichlet problems. We need some notation for level set manifolds. Let $U \subset M^n$ an open set and $f : U \to \mathbb{R}$ a smooth function on U. For each $0 < a$ we define

$$\overline{V}_a = \{p \in U : -a \leq f(p) \leq a\} \tag{1.18}$$
$$C_a = \{p \in U : f(p) = a\} \tag{1.19}$$

Proposition 13.
Consider $d : U \to \mathbb{R}$ a distance function on the open set $U \subset M^n$. Let $\psi \in C^2(\mathbb{R})$ a function such that $\psi(0) = 0$ and $\psi'(0) > 0$.

Assume $p \in M$ and $(\psi \circ d)(p) = t$. Thus $\forall\, q \in C_t$ we have

$$H(\psi \circ d)(q) = \frac{\psi'}{\sqrt{1+\psi'^2}} \left(H'(q) + \frac{\psi''}{\psi'\,(1+\psi'^2)} \right) \tag{1.20}$$

where H is calculated with respect to the normal field $-\frac{\psi'}{\sqrt{1+\psi'^2}}\nabla d$ and $H'(q)$ is the mean curvature of the hyper-surface $C_{d(q)}$ calculated with respect to the field $-\nabla d$.

Proof. We proceed by direct computation. If $q \in C_t$ we can write

$$\begin{aligned}
(n-1)\, H(\psi \circ d)(q) &= div \left(\frac{\psi'\,\nabla d}{\sqrt{1+\psi'^2}} \right) \\
&= \left\langle \nabla \left(\frac{\psi'}{\sqrt{1+\psi'^2}} \right), \nabla d \right\rangle + \frac{\psi'}{\sqrt{1+\psi'^2}}\, \Delta d(q) \\
&= \left(\frac{\psi''}{(1+\psi')^{\frac{3}{2}}} \right) |\nabla d|^2 + \frac{\psi'}{\sqrt{1+\psi'^2}}\, \Delta d(q) \\
&= \left(\frac{\psi''}{(1+\psi')^{\frac{3}{2}}} \right) + \frac{\psi'}{\sqrt{1+\psi'^2}}\, H'(q)
\end{aligned}$$

by definition of distance function. □

We end this section with a remark on how we can write mean curvature for two special parametrizations, namely the level-set case and the graph case.

We recall that if $U \subset M^n$ is an open set and $u : U \to \mathbb{R}$ is a smooth function, $y \in \mathbb{R}$ is a regular value for u, then the set $S_y = u^{-1}(y)$ is a smooth hypersurface of M^n. Moreover ∇u is a normal vector on S_y which vanishes nowhere, thus, if renormalized, it is a normal unit vector field along S_y.

$$\begin{aligned} H &= -\frac{1}{n-1}\, div\left(\frac{\nabla u}{|\nabla u|}\right) \\ &= -\frac{1}{n-1}\left(\frac{\Delta u}{|\nabla u|} - \frac{1}{|\nabla u|^2}\left\langle \nabla\,|\nabla u|, \nabla u\right\rangle\right) \end{aligned}$$

In local coordinates we have

$$\begin{aligned} \Delta u &= \frac{1}{|g|}\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\sqrt{|g|}\sum_{j=1}^{n} g^{ij}\,\frac{\partial f}{\partial x_j}\right) \\ |\nabla u|^2 &= \sum_{i,j=1}^{n} g^{ij}\,\frac{\partial f}{\partial x_i}\,\frac{\partial f}{\partial x_j} \\ \langle \nabla\,|\nabla u|, \nabla u\rangle &= \sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\,|\nabla u|\, g^{ij}\,\frac{\partial f}{\partial x_k} \end{aligned}$$

We are now going to deduce a formula for mean curvature of hypersurfaces expressed in form of a graph. Assume S is the graph of a function $u : N^{n-1} \to \mathbb{R}$ where $N \subset M$ is a smooth submanifold of dimension $n-1$. Precisely if $p \in S$ we consider $(x, x_n) = (x_1, \ldots, x_{n-1}, x_n)$ local coordinates for M, with $(x_1, \ldots, x_{n-1})$ local coordinates for N. We assume that $S = \{(x, f(x)) : x \in \Omega\}$ for some $f \in C^\infty(N)$. Thus if we define $u(x, x_n) = x_n - f(x)$ we have taht u has no singular points and thus $S = u^{-1}(0)$. We will calculate mean curvature of S with respect to the upward normal vector which we define as follows. Recall that $\nabla u \in TM$, and is orthogonal to TS. Thus, as before, we have $\nabla u = \dfrac{\partial}{\partial x_n} - \nabla f$. Clearly we have $\left|\dfrac{\partial}{\partial x_n} - \nabla f\right|^2 = 1 + |\nabla u|^2$. We define the upward normal vector to S as the vector

$$\eta = \frac{1}{\sqrt{g_{nn} + |\nabla f|^2}}\left(-\nabla f + \frac{\partial}{\partial x_n}\right)$$

We use the adjective *upward* to strengthen the fact that the component of η along $\frac{\partial}{\partial x_n}$ is positive.

Proposition 14. *$\forall\, q \in S$ close enough to p we have*

$$H(q) = -\frac{1}{n-1}\, div \left(\frac{\nabla f}{\sqrt{g_{nn} + |\nabla f|^2}} \right)$$
$$= -\frac{1}{n-1} \left(\frac{\Delta f}{\sqrt{g_{nn} + |\nabla f|^2}} - \frac{|\nabla u|}{(g_{nn} + |\nabla u|^2)^{3/2}} \langle \nabla |\nabla u|, \nabla u \rangle \right)$$

Chapter 2

The manifold $\mathbb{H}^2 \times \mathbb{R}$

In this chapter we present the Riemannian manifold in which we will study constant mean curvature surfaces. We will specialize the geometric quantities we introduced in the preceding chapter to $\mathbb{H}^2 \times \mathbb{R}$. Our manifold is a Riemannian product of a curved surface and the real line, then we begin presenting the geometry of the surface.

2.1 Hyperbolic plane

There are several models of the hyperbolic plane. Here we will introduce just two of them, the upper half-space and the Poincaré disc. We will introduce the two models at the same time and we will develop the geometry in parallel. We introduce two models because some aspects are more clear in the upper half space model and others are more clear in the disc model.
We will give global coordinates on $\mathbb{R}^2$, i. e. we will see the hyperbolic plane as a subset of $\mathbb{R}^2$ with a non flat metric. Thus, it is useful to use complex notation for points of $\mathbb{R}^2$. Throughout this section, we will write $z = x + i\,y = (x, y)$ for points of $\mathbb{R}^2$ and $\bar{z} = x - i\,y$ for the complex conjugate of z. Moreover we will see any function $f : \mathbb{R}^2 \to \mathbb{R}^2$ as depending on the two variables $z, \bar{z}$. In this notation, if $f = (f_1, f_2)$ with $f_i \in \mathbb{R}$, we have $f_1 = Re(f)$ and $f_2 = Im(f)$. One of the most relevant benefits coming from complex notation is that the differential of functions behaves like multiplication in $\mathbb{C}$. In order to prove this fact we associate two vector fields to the z and $\bar{z}$ coordinates

$$\frac{\partial}{\partial\, z} = \frac{1}{2}\left(\frac{\partial}{\partial\, x} - i\,\frac{\partial}{\partial\, y}\right)$$
$$\frac{\partial}{\partial\, \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial\, x} + i\,\frac{\partial}{\partial\, y}\right)$$

These definitions make sense because $\mathbb{R}^2$ is a vector space, hence any of its tangent spaces is intended as identified with $\mathbb{R}^2$.

Proposition 15.
Let $U \subset \mathbb{R}^2$ an open set and $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 function. Consider $\gamma : [0,1] \to U$ a C^1 curve. Then we have

$$\Big(d_\gamma f\Big)(\gamma') = \frac{\partial f}{\partial z}\Big(\gamma, \overline{\gamma}\Big)\, \gamma' + \frac{\partial f}{\partial \overline{z}}\Big(\gamma, \overline{\gamma}\Big)\, \overline{\gamma}'$$

Proof. The proof we suggest is a verify that one can switch from one notation to the other. Let's write the differential of f applied to γ' in the usual notation. Assume $\gamma(t) = (\gamma_1(t), \gamma_2(t)) = \gamma_1(t) + i\,\gamma_2(t)$ and $f(z) = (f_1(z), f_2(z)) = f_1(z) + i\,f_2(z)$. Then we have

$$\Big(d_\gamma f\Big)(\gamma') = \Big(\frac{\partial f_1}{\partial x}\,\gamma_1' + \frac{\partial f_1}{\partial y}\,\gamma_2',\ \frac{\partial f_2}{\partial x}\,\gamma_1' + \frac{\partial f_2}{\partial y}\,\gamma_2'\Big)$$

On the other hand we have

$$\begin{aligned}
\frac{\partial f}{\partial z} &= \frac{\partial f_1}{\partial z} + i\,\frac{\partial f_2}{\partial z} = \frac{1}{2}\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) \\
\frac{\partial f}{\partial \overline{z}} &= \frac{\partial f_1}{\partial \overline{z}} + i\,\frac{\partial f_2}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_2}{\partial x} + \frac{\partial f_1}{\partial y}\right)
\end{aligned}$$

Taking the product with the tangent vectors in $\mathbb{C}$ we get

$$\begin{aligned}
\frac{\partial f}{\partial z}\,\gamma' = \frac{1}{2}&\left(\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right)\gamma_1' - \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\gamma_2'\right) + \\
&+ \frac{i}{2}\left(\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\gamma_1' + \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right)\gamma_2'\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial f}{\partial \overline{z}}\,\overline{\gamma}' = \frac{1}{2}&\left(\left(\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial y}\right)\gamma_1' + \left(\frac{\partial f_2}{\partial x} + \frac{\partial f_1}{\partial y}\right)\gamma_2'\right) + \\
&+ \frac{i}{2}\left(\left(\frac{\partial f_2}{\partial x} + \frac{\partial f_1}{\partial y}\right)\gamma_1' - \left(\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial y}\right)\gamma_2'\right)
\end{aligned}$$

and hence

$$\frac{\partial f}{\partial z}\,\gamma' + \frac{\partial f}{\partial \overline{z}}\,\overline{\gamma}' = \frac{\partial f_1}{\partial x}\,\gamma_1' + \frac{\partial f_1}{\partial y}\,\gamma_2' + i\left(\frac{\partial f_2}{\partial x}\,\gamma_1' + \frac{\partial f_2}{\partial y}\,\gamma_2'\right)$$

□

Two models

This section is devoted to a presentation of the main results of hyperbolic planar geometry. For each model we will present isometries, geodesics and other constant curvature curves.

Definition 36. Poincaré disc model
The disc $\mathbb{D} = \{z \in \mathbb{R}^2 : |z| < 1\}$ with the metric

$$h(z) = \left(\frac{2}{1-|z|^2}\right)^2 \left(dx^2 + dy^2\right) = \lambda(z)^2 \left(dx^2 + dy^2\right)$$

is the Poincaré disc model of the hyperbolic real plane.

Definition 37. Upper half-space model
The upper real half plane $\mathbb{US} = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ with the metric $g(z) = \frac{1}{y^2}\left(dx^2 + dy^2\right)$ is the hyperbolic real plane.

The two metrics we have defined are conformal to the Euclidean one hence if $z = (x, y) \in \mathbb{US}$ and $v \in T_z\mathbb{US}$ we have $|v|_{\mathbb{US}} = \frac{1}{y}|v|_{\mathbb{R}^2}$. Moreover the angle between two vectors tangent to $\mathbb{US}$ in any point is the same when measured in the hyperbolic and in the Euclidean setting. Clearly all these remarks apply to the Poincaré model as well.

What we have defined are just two different parametrizations of the same Riemannian surface.

Proposition 16.
The function

$$\begin{array}{rccc} \Phi : & \mathbb{D} & \longrightarrow & \mathbb{US} \\ & z & \longmapsto & \dfrac{i\,z+1}{-z-i} \end{array} \tag{2.1}$$

is an isometry

Proof. One can check that the function

$$\Phi^{-1}(w) = \frac{-iz-1}{z+i}$$

is the inverse of Φ. Moreover $\Phi(\mathbb{D}) \subset \mathbb{US}$ since we have

$$Im(\Phi(z)) = \frac{1-|z|^2}{|z+i|^2}$$

We remark that this formula also shows that Φ sends $\partial\mathbb{D}$ in $\partial\mathbb{US}$.
To prove that Φ is an isometry, we show that it conserves lengths of curves. Consider $\gamma(t) : [0,1] \to \mathbb{D}$ a curve. Then

$$l_{\mathbb{D}}(\gamma) = \int_0^1 |\gamma'(t)|_{\mathbb{D}}\, dt$$

On the other side we have

$$\begin{aligned} l_{\mathbb{US}}(f \circ \gamma) &= \int_0^1 |\,(f \circ \gamma(t))'\,|_{\mathbb{US}}\, dt \\ &= \int_0^1 \left| \frac{\partial f}{\partial z}(\gamma(t))\, \gamma'(t) \right|_{\mathbb{US}} dt \\ &= \int_0^1 \frac{1}{Im(f \circ \gamma(t))} \left| \frac{\partial f}{\partial z}(\gamma(t))\, \gamma'(t) \right|_{\mathbb{R}^2} dt \end{aligned}$$

where

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{2}{(z+i)^2} \\ Im(f \circ \gamma(t)) &= \frac{1 - |\gamma(t)|^2}{|\gamma(t) + i|^2} \end{aligned}$$

So we can write

$$\begin{aligned} l_{\mathbb{US}}(f \circ \gamma) &= \int_0^1 \frac{2}{1 - |\gamma(t)|^2}\, |\gamma'(t)|_{\mathbb{R}^2} \\ &= \int_0^1 |\gamma'(t)|_{\mathbb{D}} \end{aligned}$$

□

We call the manifold parametrized by the two models the hyperbolic real plane and we write $\mathbb{H}^2$.

Hyperbolic Isometries

We begin the study of hyperbolic geometry by its isometries. To do that we need to do a digression in elementary Riemann surfaces theory.
With $\overline{\mathbb{C}}$ we mean the compactification of $\mathbb{C}$. In other words we define $\overline{\mathbb{C}} = S^2$, where the equality is realized via stereographic projection from the north pole of S^2. Thus it makes sense to indicate the north pole of S^2 with ∞ .

With stereographic projection in mind, one can see that a circle of $\overline{\mathbb{C}} = S^2$ is projected into a circle of $\mathbb{C}$ or into a straight line. One can be more precise: every circle in S^2 that does not contain the north pole is sent into a circle of $\mathbb{C}$ and every circle of S^2 containing the north pole is sent in a straight line of $\mathbb{C}$.
We can now define a class of functions that play a crucial role in our study of hyperbolic isometries.

Definition 38. Möbius function
Consider $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ a function. f is a Möbius function if there exist $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ satisfying

$$f(z) = \frac{az + b}{cz + d}$$

We write $M\ddot{o}b^+(\overline{\mathbb{C}})$ for the set of all such functions.

This definition makes sense because any holomorphic rational linear function has precisely one pole p and has limit at infinity. Thus we define

$$f(p) = \infty \in \overline{\mathbb{C}} \quad \text{and} \quad f(\infty) = \lim_{z \to \infty} f(z) \in \overline{\mathbb{C}}$$

Möbius functions have connections to many other theories, for example to Lie groups theory. However here we will only summarize the features that are relevant for hyperbolic geometry. An introduction to general theory of Möbius functions can be found in [1, Chapter 2].

Theorem. *Properties of Möbius functions*
$M\ddot{o}b^+(\overline{\mathbb{C}})$ is a group. Moreover if $f \in M\ddot{o}b^+(\overline{\mathbb{C}})$ then

1. *f is holomorphic and hence conformal*
2. *f acts transitively on triple of points of $\overline{\mathbb{C}}$*
3. *f maps circles of $\overline{\mathbb{C}}$ in circles of $\overline{\mathbb{C}}$*

For the proof of these facts we refer to [1, Chapter 2].
We specialize to Möbius functions which are diffeomorphsims of $\mathbb{US}$. In order to do that, we define

$$M\ddot{o}b^+(\mathbb{US})_{\mathbb{R}} = \left\{ f \in M\ddot{o}b^+(\mathbb{US}) : f(z) = \frac{az+b}{cz+d} \right. \\ \left. \text{for } a, b, c, d \in \mathbb{R} \text{ and } ad - bc \neq 0 \right\}$$

We have the following characterization.

Proposition 17.
$f \in M\ddot{o}b^+(\mathbb{US})_{\mathbb{R}} \iff \exists\, a, b, c, d \in \mathbb{R}$ satisfying $ad - bc = 1$ such that

$$f(z) = \frac{az + b}{cz + d}$$

The proof of this representation formula can be found in [1, Chapter 2, Theorem 2.26].
We are now ready to state why Möbius are important for us.

Theorem 6.
$Isom(\mathbb{H}^2) \simeq M\ddot{o}b^+(\mathbb{US})_{\mathbb{R}}$. Moreover the action of $Isom(\mathbb{H}^2)$ is transitive, hence $\mathbb{H}^2$ is an homogeneous manifold.

We are not going to prove this result because the proof would bring us too far from our goal. A proof can be found in [1, Chapter 3, Theorem 3.19]. The isomorphism between Möbius functions and hyperbolic isometries can be made precise. In the upper half space model the isomorphism is the identity. In the Poincaré disc model the isomorphism is made conjugating elements of $M\ddot{o}b^+(\mathbb{US})_{\mathbb{R}}$ with Φ.

What we are going to prove is a factorization of any hyperbolic isometry in terms of three special functions.

Definition 39.

1. $\forall\, s \in \mathbb{R}$ we define

$$\begin{array}{rccc} m_s : & \mathbb{US} & \longrightarrow & \mathbb{US} \\ & z & \mapsto & z + s \end{array} \tag{2.2}$$

2. $\forall\, t > 0$ we define

$$\begin{array}{rccc} \tau_t : & \mathbb{US} & \longrightarrow & \mathbb{US} \\ & z & \mapsto & t\, z \end{array} \tag{2.3}$$

3. We define

$$\begin{array}{rccc} R : & \mathbb{US} & \longrightarrow & \mathbb{US} \\ & z & \mapsto & -\dfrac{1}{z} \end{array} \tag{2.4}$$

To check that these definitions are well, we directly compute the imaginary part of all these functions to see it is strictly positive. We have

$$Im(m_s(z)) = Im(z)$$
$$Im(\tau_t(z)) = t\,Im(z)$$
$$Im(R(z)) = \frac{Im(z)}{|z|^2}$$

Before stating the factorization result, we show geometrical properties that allow to think about m as a translation, τ as a dilation and R as a rotation.

Proposition 18.

1. *m_s fixes horizontal Euclidean lines of* $\mathbb{US}$
2. *τ_t fixes the vertical Euclidean line of* $\mathbb{US}$ $\{Re(z) = 0\}$
3. *R fixes hyperbolic circles of center i*

Proof. Item 1 and 2 are a consequence of the linearity of Im. Being $s, t \in \mathbb{R}$ we have $Im(z + s) = Im(z)$ and $Im(tz) = tIm(z) > 0$.
Item 3 follows from the fact that $z = i$ is the only fixed point of R. □

It is useful to give a version of this result in the Poincaré disc model. Given $f \in Isom(\mathbb{US})$ we define $\widetilde{f} = \Phi^{-1} \circ f \circ \Phi$ the conjugated element of f in $Isom(\mathbb{D})$.

Proposition 19.

1. *$\widetilde{m_s}$ fixes Euclidean circles tangent to $-i \in \partial\mathbb{D}$*
2. *$\widetilde{\tau_t}$ fixes horizontal segments of $\mathbb{D}$ defined by $Im(z) = 0$*
3. *$\widetilde{R}$ fixes Euclidean circles of center* 0

Proof. Item 1 follows from the fact that Φ is a Möbius function and from the fact that an Euclidean horizontal line has one point at infinity.
Item 2 follows from the fact that Φ is a Möbius function and from the fact that the line $\{Re(z) = 0\}$ asymptotically crosses the asymptotic boundary of $\mathbb{US}$ orthogonally. Moreover it passes through i which is mapped to 0 by Φ^{-1}.
Item 3 is straightforward because Möbius functions preserve circles.

□

We can now show the factorization of isometries in terms of the translations, rotation and dilations.

Proposition 20.
Let $f \in Isom(\mathbb{US})$. Then f can be obtained composing rotations, translations and dilations.

Proof. Assume we have

$$f(z) = \frac{az+b}{cz+d} \quad \text{for } a,b,c,d \in \mathbb{R} \text{ and } ad-bc=1$$

we separate two cases according to the value of c.
If $c = 0$, we have

$$f(z) = a^2 z + ab = m_{ab} \circ \tau_{a^2}(z)$$

If $c \neq 0$, we have

$$f(z) = m_{\frac{a}{c}} \circ \tau_{\frac{1}{c^2}} \circ R \circ m_{\frac{d}{c}}(z)$$

Indeed it holds

$$\begin{aligned} m_{\frac{a}{c}} \circ \tau_{\frac{1}{c^2}} \circ R \circ m_{\frac{d}{c}}(z) &= m_{\frac{a}{c}} \circ \tau_{\frac{1}{c^2}} \circ R\left(z + \frac{d}{c}\right) \\ &= m_{\frac{a}{c}} \circ \tau_{\frac{1}{c^2}}\left(-\frac{1}{z+\frac{d}{c}}\right) \\ &= m_{\frac{a}{c}}\left(-\frac{1}{c^2 z + dc}\right) \\ &= \frac{1}{c}\left(-\frac{1}{cz+d} + a\right) \\ &= \frac{1}{c}\left(\frac{acz+bc}{cz+d}\right) \qquad \text{being } ad-bc=1 \end{aligned}$$

□

Sectional curvature

This section is devoted to showing that the sectional curvature of the hyperbolic plane is constant and equal to -1. In order to do that we need coordinate expressions of Christoffel symbols.

Proposition 21.
In the upper half-space model the Christoffel's symbols write

$$\Gamma^1_{11}(x,y) = 0 \qquad \Gamma^2_{11}(x,y) = \frac{1}{y}$$
$$\Gamma^1_{12}(x,y) = -\frac{1}{y} \qquad \Gamma^2_{12}(x,y) = 0$$
$$\Gamma^1_{22}(x,y) = 0 \qquad \Gamma^2_{22}(x,y) = -\frac{1}{y}$$

In the Poincaré disc model the Christoffel's symbols write

$$\begin{aligned} \Gamma^1_{11}(x,y) &= \frac{2x}{1-|z|^2} & \Gamma^2_{11}(x,y) &= -\frac{2y}{1-|z|^2} \\ \Gamma^1_{12}(x,y) &= -\Gamma^2_{11}(x,y) & \Gamma^2_{12}(x,y) &= \Gamma^1_{11}(x,y) \\ \Gamma^1_{22}(x,y) &= -\Gamma^1_{11}(x,y) & \Gamma^2_{22}(x,y) &= -\Gamma^2_{11}(x,y) \end{aligned} \tag{2.5}$$

This proposition can be proved by a direct computation based on formula 1.6.

Proposition 22.
The sectional curvature of $\mathbb{H}^2$ *is constant and equal to* -1*.*

Proof. We will proceed by direct computation in the half space model. We first recall that any two coordinate fields commute, i.e. their Lie bracket is zero. This is a completely differentiable fact. However, it can be proved using the fact that the Riemannian connection is torsion free and that the functions that we are considering are smooth.
Consider $z \in \mathbb{H}^2$ and the basis of T_zM induced by cartesian coordinates. Thus we have

$$R\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\frac{\partial}{\partial x} = -\frac{1}{y^2}\frac{\partial}{\partial y}$$

Indeed, by definition, we have

$$\begin{aligned}
R\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\frac{\partial}{\partial x} &= \nabla_{\frac{\partial}{\partial y}}\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} - \nabla_{\frac{\partial}{\partial x}}\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x} + \nabla_{\left[\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right]}\frac{\partial}{\partial x}\\
&= \nabla_{\frac{\partial}{\partial y}}\left(\Gamma^2_{11}\frac{\partial}{\partial y}\right) - \nabla_{\frac{\partial}{\partial x}}\left(\Gamma^1_{12}\frac{\partial}{\partial x}\right)\\
&= \nabla_{\frac{\partial}{\partial y}}\left(\frac{1}{y}\frac{\partial}{\partial y}\right) + \nabla_{\frac{\partial}{\partial x}}\left(\frac{1}{y}\frac{\partial}{\partial x}\right)\\
&= -\frac{1}{y^2}\frac{\partial}{\partial y} + \frac{1}{y}\left(\Gamma^2_{22}\frac{\partial}{\partial y}\right) + \frac{1}{y}\left(\Gamma^2_{11}\frac{\partial}{\partial y}\right)\\
&= -\frac{1}{y^2}\frac{\partial}{\partial y}
\end{aligned}$$

Thus

$$\begin{aligned}
K_{sect}\left(T_z\mathbb{H}^2\right) &= \frac{\left\langle R\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right\rangle}{\left|\frac{\partial}{\partial x}\right|^2\left|\frac{\partial}{\partial y}\right|^2}\\
&= -1
\end{aligned}$$

□

This fact has an extremely useful consequence, i. e. the globality of the exponential map. Since it is a general result concerning manifolds with non positive sectional curvature, we state it in the general form.

Theorem 7. *Cartan - Hadamard*
If M is a complete manifold with non positive sectional curvature, then $\rho_{inj}(M) = \infty$.

For a proof of this result we refer to [7, Chapter 7, Theorem 3.1].
Using this theorem we prove that, in a manifold with non positive curvature, the flow of the distance function from a convex domain is global, i.e. it is defined for all times.

Proposition 23.
Let M^n be a complete manifold with $K_{sect} \leq 0$. Let $\Omega \subset M$ a convex domain (see definition 22) and d_Ω the distance function associated to Ω. Then the flow of d is global, i.e.

1. *$\forall\, p \in \partial\Omega$ $\varphi_p(s)$, the flow of ∇d, is defined $\forall\, s \geq 0$*

2. *$\forall\, p \neq q \in \partial\Omega$ we have $\varphi_p(s_1) \neq \varphi_q(s_2) \quad \forall\, s_1, s_2 \in \mathbb{R}^+$*

Proof. Assertion 1 follows from the completeness of M together with the fact that the flow of a distance function is geodesic. Assertion 2 is proved by contradiction. Take $p, q \in \partial\Omega$ and $s_1, s_2 \in \mathbb{R}^+$ satisfying $\varphi_p(s_1) = \varphi_q(s_2) = m \in M$. Thus the flows φ_p and φ_q coincide. It is because they are two geodesics intersecting in a point where their tangent vectors are both equal to $\nabla d(m)$. Thus we have a geodesic from p to q not contained in Ω. This is a contradiction because if such a geodesic existed, it would have had a length not smaller than $l = d(p, q)$. Thus one of the two following cases would have been given

1. The geodesic we are considering has length greater than l. Then it would exist a time after which the geodesic would not minimize length anymore. But this would imply $\rho_{inj}(x) < \infty$ which contradicts Cartan-Hadamard Theorem.

2. The geodesic we are considering has length equal to l, thus we would have two distinct geodesics going from p to q minimizing distance. Indeed one geodesic would be the flow of the distance function, the other would exist by hypothesis of convex domain. But this would mean $\rho_{inj}(p) < \infty$

□

Constant curvature curves

We now study curves whose geodesic curvature (see remark following proposition 5) is constant. This is because in the hyperbolic plane there are three families of curves with constant geodesic curvature that will be used in the last chapter. We begin by finding all the geodesics. After that we will take into account curves with constant curvature equal or greater to one.

Proposition 24.
The geodesics of $\mathbb{US}$ are constant speed parametrizations of vertical Euclidean lines or Euclidean half circles intersecting the $\{y = 0\}$ line orthogonally.

Proof. We first show that that these curves are all the geodesics of the hyperbolic plane. It is an elementary geometric fact that, for each point z in $\mathbb{US}$ and for each vector $v \in T_z\mathbb{H}^2$, there exists precisely one curve passing trough z with tangent vector v belonging to the class of half circles with

center on the $\{y = 0\}$ line and vertical lines.
Then we prove that each one of these curves is a geodesic. We start from the vertical lines. Let $\gamma = \{(x, y) \in \mathbb{H}^2 : x = t\}$. Thus

$$V = \frac{\partial}{\partial y} \qquad \text{with} \qquad |V| = \frac{1}{y}$$

is a tangent field to γ. To parametrize the curve by arc length we use $\widetilde{\gamma}$, the reparametrization of γ whose tangent field is $T = \frac{V}{|V|}$. This can be done because any curve can be parametrized by arc length. Moreover two arc length parametrizations can differ only on the orientation they induce on the curve.
From $T = \widetilde{\gamma}'$ it follows

$$\begin{aligned}\nabla_T T &= y \nabla_{\frac{\partial}{\partial y}} y \frac{\partial}{\partial y} \\ &= y \left(\frac{\partial}{\partial y} + y \Gamma^2_{22} \frac{\partial}{\partial y} \right) \\ &= y \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y} \right) \\ &= 0\end{aligned}$$

If γ is an Euclidean circle with center on the $\{y = 0\}$ line, we can suppose that its center is $(0, 0)$. This is because using a suitable isometry m_s, defined in (2.2), we can reduce to this case. Thus we assume that $\gamma = \{(x, y) \in \mathbb{H}^2 : x^2 + y^2 = r^2\}$ for some $r > 0$. Then a tangent field is given by

$$V = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \qquad \text{with} \qquad |V| = \frac{\sqrt{x^2 + y^2}}{y} = \frac{r}{y}$$

As in the preceding case, if we define $T = \frac{V}{|V|}$, we can consider $\widetilde{\gamma}$ the arc length reparametrization of γ whose tangent field is T. Then we compute

$$\begin{aligned}
\nabla_T T &= \frac{y}{r^2}\Big(- y \nabla_{\frac{\partial}{\partial x}} \left(-y^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}\right) + \\
&\qquad + x \nabla_{\frac{\partial}{\partial y}} \left(-y^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}\right)\Big) \\
&= \frac{y}{r^2}\left(-y \left(-y^2 \Gamma^2_{11} \frac{\partial}{\partial y} + y \frac{\partial}{\partial y} + xy \Gamma^1_{12} \frac{\partial}{\partial x}\right) + \right. \\
&\qquad \left. +x \left(-2y \frac{\partial}{\partial x} - y^2 \Gamma^1_{12} \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + xy \Gamma^2_{22} \frac{\partial}{\partial y}\right)\right) \\
&= \frac{y}{r^2}\left(xy \frac{\partial}{\partial x} - 2xy \frac{\partial}{\partial x} + xy \frac{\partial}{\partial x}\right) \\
&= 0
\end{aligned}$$

□

It is useful to have the picture of what are geodesics in the Poincaré model.

Proposition 25.
The geodesics of $\mathbb{D}$ are constant speed parametrizations of segments emanating from 0 or arcs of circle approaching $\partial\mathbb{D}$ orthogonally.

Proof. We just transport geodesics of $\mathbb{US}$ using Φ^{-1}. Thus any geodesic $\widetilde{\gamma}$ in $\mathbb{D}$ is the image of a geodesic γ in $\mathbb{US}$. Hence γ is either a vertical line or an half circle intersecting the $\{y = 0\}$ line orthogonally. In both cases, being Φ^{-1} a Möbius function, $\widetilde{\gamma}$ is the intersection of a straight line or of a circle with $\mathbb{D}$. Moreover, since Φ is conformal as a function from $\overline{\mathbb{C}}$ to $\overline{\mathbb{C}}$, $\widetilde{\gamma}$ asymptotically touches $\partial\mathbb{D}$ making the same angle made by γ when it asymptotically touches $\{y = 0\}$. Thus we are done. □

We use geodesics of the Poincaré model to give a relation between hyperbolic and Euclideann distance from a fixed point.

Proposition 26.
Let be $z \in \mathbb{D}$ with $|z|_{\mathbb{R}^2} = r$ for some $0 < r$. Thus if $\rho = d_{\mathbb{H}^2}(z, 0)$ we have

$$\tanh\left(\frac{\rho}{2}\right) = r$$

Proof. We use the fact that suitable parametrizations of rays emanating from 0 are hyperbolic geodesics. One can esily check that the curve

$$\gamma(t) = \left(tanh\left(\frac{t}{2}\right), 0 \right)$$

is an hyperbolic geodesic starting in 0. It is a geodesic since it is a ray and

$$|\gamma'(t)|_{\mathbb{H}^2} = \frac{sech^2\left(\frac{t}{2}\right)}{1 - tanh^2\left(\frac{t}{2}\right)} = 1$$

Hence we have $\forall\, t \geq 0$

$$d_{\mathbb{H}^2}(\gamma(t), 0) = \int_0^t ds = t$$
$$d_{\mathbb{R}^2}(\gamma(t), 0) = tanh\left(\frac{t}{2}\right)$$

□

In figure 2.1 are shown geodesics and horocycles on $\mathbb{H}^2$ in the disc model.

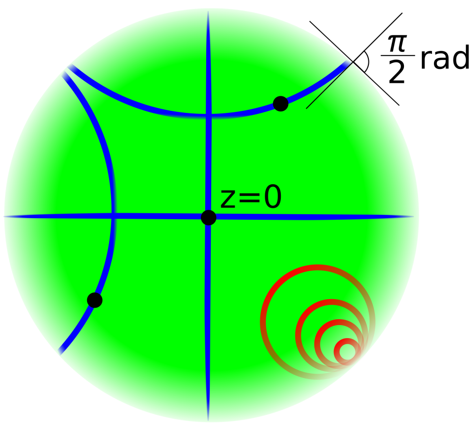

Figure 2.1: Geodesics in blue and horocycles in red)

2.1.1 Convexity and horocycle convexity

As we have seen in the definition 22, geodesics allow to define a concept of convexity. In the Euclidean case it is possible to characterize geodesic convexity in terms of the curvature of the boundary. Here an analogous result holds. We recall that we compute geodesic curvature of closed curves with respect to the normal field pointing toward the compact bounded by γ.

Proposition 27.
Let $\gamma : [0,1] \to \mathbb{H}^2$ a smooth closed curve. Call Ω the compact set bounded by γ. Ω is geodesically convex if and only if $k_g(\gamma) \geq 0$

Proof. Consider $\mathbb{H}^2$ in the Poincaré disc model. We show that each point of Ω has a convex neighborhood. Being $\mathbb{H}^2$ homogeneous, γ can be written locally as a graph on a interval I of the horizontal axis. Namely we write $\gamma = (x, f(x))$ for some function f. Recalling the definition of λ given in definition 36, one can directly verify that

$$k_g(\gamma) = \frac{f''}{\lambda\,(1+f'^2)^{\frac{3}{2}}} \qquad + \text{ terms in the first order derivatives } f \text{ and } \lambda$$

if Ω is in the super-graph of f

$$k_g(\gamma) = -\frac{f''}{\lambda\,(1+f'^2)^{\frac{3}{2}}} \quad + \text{ terms in the first order derivatives of } f \text{ and } \lambda$$

if Ω is in the subgraph of f

Thus the curvature operator written in these coordinates is respectively elliptic and anti-elliptic.
We prove the claim in the super-graph case. The other one can be ruled out in the same way. Consider $p \in \gamma$ near which Ω is in the super-graph of f and consider q_1, q_2 two points in γ close enough to p so that the graph representation still holds. Here the curvature operator is elliptic, hence if ϕ is the geodesic going from q_1 to q_2 minimizing length, the maximum principle implies

$$\begin{aligned} \phi \geq& f \quad \text{ in } q_1 \text{ and } q_2 \\ 0 = k_g(\phi) \leq& k_g(\gamma) \end{aligned}$$

then we have

$$\phi \geq f \quad \text{all over } \phi$$

which means ϕ is in the super-graph of f

We have to show that local convexity implies global convexity. We use the infinity of the injectivity radius in $\mathbb{H}^2$. In fact this allows us to reduce to the Euclidean case. We retract $\mathbb{H}^2$ to $T_0\mathbb{H}^2$ by means of the inverse of the exponential map. Now we observe that Ω is geodesically convex if and only if its retract on $T_0\mathbb{H}^2$ is convex by straight lines. But in $\mathbb{R}^2$ local convexity implies global convexity. □

We now turn our attention to curves with constant positive geodesic curvature. This is because they allow to give another definition of convexity that is crucial in the existence theorem in the third chapter.

Consider the upper half-space model of $\mathbb{H}^2$. We define an *horocycle* to be an horizontal Euclidean line or an Euclidean circle tangent to the $\{y = 0\}$ line. Moreover we can associate to each horocycle a relatively compact set, called horodisc. We proceed as follows: consider an horizontal line $\{y = t\}$. We call horodisc the set $\{y \geq t\}$. Similarly, if we consider an Euclidean disc γ tangent to $\{y = 0\}$, we call horodisc the Euclidean disc bounded by γ.
By means of Φ^{-1}, the isometry between the upper half space model and the Poincaré disc, we see that horocycles in the disc model are Euclidean circles tangent to ∂D. Indeed Φ^{-1} is a Möbius function, hence maps circles of $\overline{\mathbb{C}}$ in circles of $\overline{\mathbb{C}}$. Hence it maps intersection of these circles with $\mathbb{US}$ in intersection of circles with $\mathbb{D}$. To conclude we observe that horizontal lines of $\mathbb{US}$ and circles tangent to $\{y = 0\}$ are circles of $\overline{\mathbb{C}}$ with one point in the asymptotic boundary of $\mathbb{US}$. Being that Φ maps the asymptotic boundary of a model into the asymptotic boundary of the other, we are done.

We now compute geodesic curvature of horocycles.

Proposition 28.
Horocycles have constant geodesic curvature equal to 1

Proof. We will do computations in the upper half space model. We begin with the horizontal line case.
Suppose $t > 0$ and consider $\gamma = \{(x, y) \in \mathbb{H}^2 : y = t\}$. Thus we have

$$J\gamma' = \frac{\partial}{\partial y} \qquad \text{and then} \qquad \gamma' = \frac{\partial}{\partial x} \qquad\qquad |\gamma'| = \frac{1}{t}$$

moreover

$$\begin{aligned}\nabla_{\gamma'} J\gamma' &= \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y}\\ &= \Gamma^1_{12}\frac{\partial}{\partial x} + \Gamma^2_{12}\frac{\partial}{\partial y}\\ &= \frac{1}{t}\frac{\partial}{\partial x}\end{aligned}$$

thus

$$\begin{aligned}k_g(\gamma) &= -\frac{1}{|\gamma'|^3}\left\langle \nabla_{\gamma'} J\gamma', \gamma'\right\rangle\\ &= -t^3\,\Gamma^1_{12}\frac{1}{t^2}\\ &= 1\end{aligned}$$

Now we take into account the case of a circle tangent to the $\{y = 0\}$ line. As we have seen, horizontal translations are hyperbolic isometries. Thus we can assume that the horocycle we are considering is tangent to $\{y = 0\}$ in $(0, 0)$. Then, if $r > 0$, any such circle can be described by

$$\gamma = \{(x, y) \in \mathbb{H}^2 : x^2 + (y - r)^2 = r^2\}$$

Then a normal and tangent vector field are given by

$$JV = -\Big(x\frac{\partial}{\partial x} + (y-r)\frac{\partial}{\partial y}\Big) \qquad \text{and} \qquad V = -(y-r)\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

and

$$|V| = \frac{r}{y}$$

We claim that it holds

$$\nabla_V JV = -\Big(\Big(r + \frac{x^2 - r^2}{y}\Big)\frac{\partial}{\partial x} + \Big(x - \frac{2r\,x}{y}\Big)\frac{\partial}{\partial y}\Big) \tag{2.6}$$

To check the claim we make the explicit computations:

$$\begin{aligned}-\nabla_V JV &= \nabla_V\left(x\frac{\partial}{\partial x}\right) + \nabla_V\left((y-r)\frac{\partial}{\partial y}\right)\\ &= (y-r)\,\nabla_{\frac{\partial}{\partial x}}\left(x\frac{\partial}{\partial x}\right) - x\,\nabla_{\frac{\partial}{\partial y}}\left(x\frac{\partial}{\partial x}\right) +\\ &\quad + (y-r)\,\nabla_{\frac{\partial}{\partial x}}\left((y-r)\frac{\partial}{\partial y}\right) - x\,\nabla_{\frac{\partial}{\partial y}}\left((y-r)\frac{\partial}{\partial y}\right)\end{aligned}$$

using the expression of the Christoffel symbols in the half space model given in proposition 21 we write

$$\nabla_{\frac{\partial}{\partial x}} \left(x \frac{\partial}{\partial x} \right) = \frac{\partial}{\partial x} + \frac{x}{y} \frac{\partial}{\partial y}$$
$$\nabla_{\frac{\partial}{\partial y}} \left(x \frac{\partial}{\partial x} \right) = -\frac{x}{y} \frac{\partial}{\partial x}$$
$$\nabla_{\frac{\partial}{\partial x}} \left((y-r) \frac{\partial}{\partial y} \right) = -\left(\frac{y-r}{y} \right) \frac{\partial}{\partial x}$$
$$\nabla_{\frac{\partial}{\partial y}} \left((y-r) \frac{\partial}{\partial y} \right) = \left(1 - \frac{y-r}{y} \right) \frac{\partial}{\partial y}$$

Thus, we have obtained

$$\begin{aligned} -\nabla_V JV = &\left((y-r) + \frac{x^2}{y} - \frac{(y-r)^2}{y} \right) \frac{\partial}{\partial x} + \\ &+ \left(\frac{x\,(y-r)}{y} - x + \frac{x\,(y-r)}{y} \right) \frac{\partial}{\partial y} \end{aligned}$$

To prove the claim we use the relation $x^2 + (y-r)^2 = r^2$

According to (2.6) we compute

$$\begin{aligned} -\langle \nabla_V JV, V \rangle &= \frac{1}{y^2} \left((y-r) \left(r + \frac{x^2 - r^2}{y} \right) - x \left(x - \frac{2r\,x}{y} \right) \right) \\ &= \frac{1}{y^3} \left((y-r)\,(x^2 - r^2 + ry - x^2) + rx^2 \right) \\ &= \frac{r}{y^3} \left((y-r)^2 + x^2 \right) \\ &= \frac{r^3}{y^3} \end{aligned}$$

To conclude we use $|V| = \frac{y}{r}$ to get

$$\begin{aligned} k_g(\gamma) &= -\frac{1}{|V|^3} \langle \nabla_V JV, V \rangle \\ &= \frac{y^3}{r^3} \frac{r^3}{y^3} \\ &= 1 \end{aligned}$$

□

We can now give the definition of convexity descending from the existence of horocycles.

Definition 40. Horocycle convexity
Consider $\Omega \subset \mathbb{H}^2$ a compact and smooth domain. We say that Ω is horocycle $-$ convex, briefly $h-$convex, if $\forall p \in \partial\Omega$ there is an horocycle γ passing through p such that Ω is contained in the horodisc associated to γ.

Figure 2.2 shows a comparison between Euclidean convexity and hyperbolic $h-$convexity.

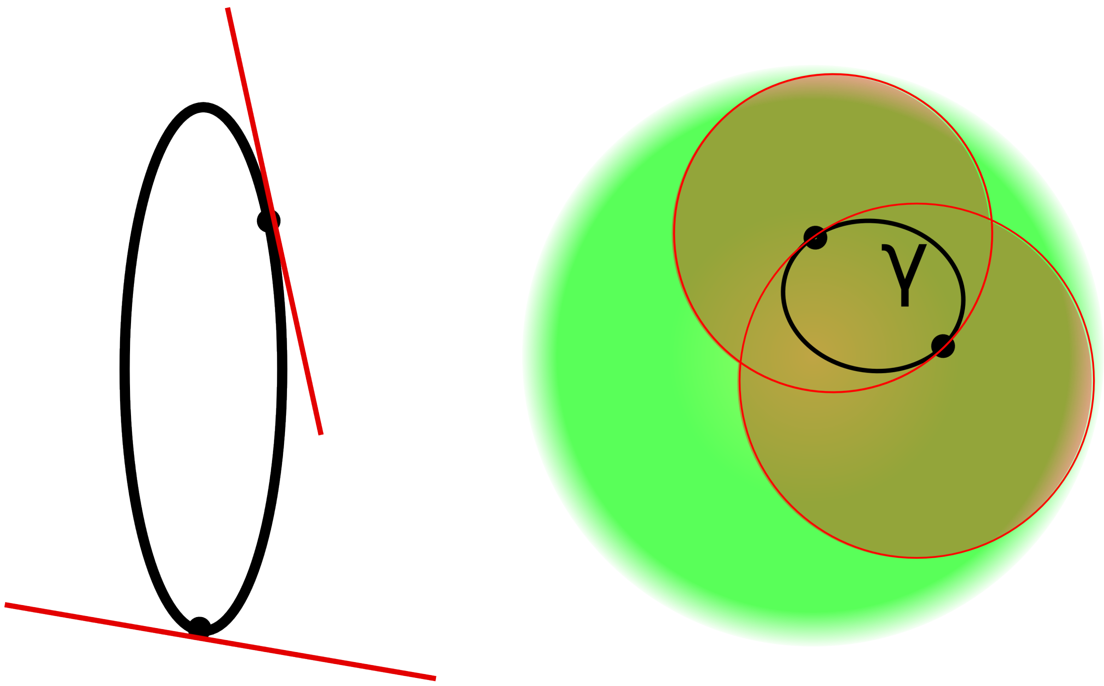

Figure 2.2: Euclidean convexity (on the left) versus hyperbolic $h-$convexity (on the right)

The $h-$convexity condition on Ω can be stated in terms of curvature of its boundary.

Proposition 29.
Let $\gamma : [0,1] \to \mathbb{H}^2$ a smooth closed curve. Call Ω the compact set bounded by γ. Ω is $h-$convex if and only if $k_g(\gamma) \geq 1$,

One implication can be showed as in the geodesically convex case. The other implication is less standard and can be found in [4].

We end this section showing that hyperbolic circles have constant geodesic curvature greater than one. Recall that we calculate geodesic curvature of closed curves with respect to the inner normal vector. The computation is made in the disc model because hyperbolic circles can be parametrized in a very efficient way.

Proposition 30.
Let $0 < \rho$, $z \in \mathbb{H}^2$ and $\gamma_\rho(z) = \{w \in \mathbb{H}^2 : d_{\mathbb{H}^2}(w, z) = \rho\}$. Then

$$k_g(\gamma_\rho(z)) = \frac{1 + \tanh\left(\frac{\rho}{2}\right)^2}{2\,\tanh\left(\frac{\rho}{2}\right)} \tag{2.7}$$

Proof. Let's consider the disc model of $\mathbb{H}^2$. Since $\mathbb{H}^2$ is homogeneous, we can reduce to the case $z = 0$. Moreover, as we have seen in proposition 26, if $r = \tanh\left(\frac{\rho}{2}\right)$ we have $\gamma_\rho(0) = \{w \in \mathbb{D} : |w|_{\mathbb{R}^2} = r\}$. We are going to prove the claim by direct computations. We will have to carry out the following steps

- Give an arc-length parametrization $\tilde{\gamma}(s)$ of the circle
- Write T e $N = JT$
- Compute
$$k_g(\gamma) = -\langle(\nabla_T N), T\rangle$$

We recall that

$$\lambda(z) = \frac{2}{1 - |z|^2_{\mathbb{R}^2}}$$

and then

$$\lambda(\widetilde{\gamma}) = \frac{2}{1 - r^2}$$

If we define
$$f_r(s) = \frac{1 - r^2}{2r}\, s$$

and we parametrize γ by

$$\widetilde{\gamma}(s) = r\left(\cos\left(f_r(s)\right), \sin\left(f_r(s)\right)\right)$$

s is an hyperbolic arc-length parameter. Indeed

$$T(s) = \widetilde{\gamma}'(s) = \frac{1-r^2}{2}\left(-\sin\left(f_r(s)\right)\frac{\partial}{\partial\, x} + \cos\left(f_r(s)\right)\frac{\partial}{\partial\, y}\right)$$

and, being $\forall\, v \in T_z\mathbb{H}^2$ $|v|_{\mathbb{H}^2} = \lambda(z)\,|v|_{\mathbb{R}^2}$, we have

$$|T(s)|_{\mathbb{H}^2} \equiv 1$$

Then if we define

$$N(s) = JT(s) = -\frac{1-r^2}{2}\left(\cos\left(f_r(s)\right)\frac{\partial}{\partial\, x} + \sin\left(f_r(s)\right)\frac{\partial}{\partial\, y}\right)$$

we obtain that the fields $\{T, N\}$ are an orthonormal basis $T\mathbb{H}^2$ along γ. We now compute $\nabla_T N = \dfrac{D}{ds}N$

$$\frac{D}{ds}N =: w_1 + w_2$$

where

$$\begin{aligned} w_1 &= -\frac{1}{r}\left(\frac{1-r^2}{2}\right)^2\left(-\sin\left(f_r(s)\right)\frac{\partial}{\partial\, x} + \cos\left(f_r(s)\right)\frac{\partial}{\partial\, y}\right) \\ w_2 &= -\frac{1-r^2}{2}\cos\left(f_r(s)\right)\frac{D}{ds}\frac{\partial}{\partial\, x} + \sin\left(f_r(s)\right)\frac{D}{ds}\frac{\partial}{\partial\, y} \end{aligned}$$

and clearly $\left\langle \dfrac{D}{ds}N, T\right\rangle = \langle w_1, T\rangle + \langle w_2, T\rangle$. But

$$\begin{aligned} \langle w_1, T\rangle &= -\frac{1}{r}\left(\frac{1-r^2}{2}\right)^3\lambda(\widetilde{\gamma})^2 \\ &= -\frac{1-r^2}{2r} \\ \langle w_2, T\rangle &= \left(\frac{1-r^2}{2}\right)^2\left(\sin^2(f_r(s))\left\langle\frac{D}{ds}\frac{\partial}{\partial\, y}, \frac{\partial}{\partial\, x}\right\rangle - \cos^2(f_r(s))\left\langle\frac{D}{ds}\frac{\partial}{\partial\, x}, \frac{\partial}{\partial\, y}\right\rangle + \right. \\ &\qquad \left. + \sin(f_r(s))\cos(f_r(s))\left(\left\langle\frac{D}{ds}\frac{\partial}{\partial\, x}, \frac{\partial}{\partial\, x}\right\rangle - \left\langle\frac{D}{ds}\frac{\partial}{\partial\, y}, \frac{\partial}{\partial\, y}\right\rangle\right)\right) \\ &= \frac{1-r^2}{2}\left(\sin^2(f_r(s))\Big(-\sin(f_r(s))\,\Gamma^2_{12} + \cos(f_r(s))\,\Gamma^1_{22}\Big) - \right. \\ &\qquad - \cos^2(f_r(s))\Big(-\sin(f_r(s))\,\Gamma^2_{11} + \cos(f_r(s))\,\Gamma^2_{12}\Big) + \\ &\qquad + \sin(f_r(s))\,\cos(f_r(s))\Big(\sin(f_r(s))\left(\Gamma^2_{12} - \Gamma^1_{11}\right) + \\ &\qquad\qquad \left. + \cos(f_r(s))\left(\Gamma^1_{12} - \Gamma^2_{22}\right)\Big)\right) \end{aligned}$$

The local coordinates expressions of Christoffel symbols given in (2.5) yields the following relations

$$\begin{array}{cc} \Gamma^2_{11} = -\Gamma^2_{12} & \Gamma^1_{22} = -\Gamma^2_{12} \\ \Gamma^2_{11} = \Gamma^2_{12} & \Gamma^1_{12} = \Gamma^2_{22} \end{array}$$

Thus we get

$$\langle w_2, T \rangle = -\frac{1-r^2}{2}\left(\sin(f_r(s))\,\Gamma^1_{12} + \cos(f_r(s))\,\Gamma^1_{11}\right)$$

Being the hyperbolic metric conformal to the Euclidean one, if we define $\mu = \lambda^2$ and $z = (x, y) \in \mathbb{H}^2$, we have

$$\Gamma^1_{11}(x,y) = \frac{1}{2\,\mu}\frac{\partial \mu}{\partial x} = x\,\lambda(x,y)$$
$$\Gamma^1_{12}(x,y) = \frac{1}{2\,\mu}\frac{\partial \mu}{\partial y} = y\,\lambda(x,y)$$

Thus

$$\langle w_2, T \rangle = -r$$

which yields

$$k_g\widetilde{\gamma} = -\langle \nabla_T N, T \rangle = \frac{1+r^2}{2r}$$

□

The computations we have just done imply the geodesic convexity of hyperbolic discs.

Proposition 31.
Consider the Poincaré model of the hyperbolic plane. Any hyperbolic disc is geodesically convex.

2.2 The product manifold

In this section we deal with the manifold $\mathbb{H}^2 \times \mathbb{R}$.
In the first part we will study its geometry. We will choose a model and then we will discuss geodesics, isometries and curvature.
In the second part we will finally start studying constant mean surfaces. From now on we will only use the Poincaré model for the hyperbolic plane.

Geometry of $\mathbb{H}^2 \times \mathbb{R}$

Definition 41.
$\mathbb{H}^2 \times \mathbb{R} = \{(z,t) : z \in \mathbb{D} \text{ and } t \in \mathbb{R}\}$. On this manifold one defines the product metric

$$ds^2(z,t) = \lambda^2(z)\Big(dx^2 + dy^2\Big) + dt^2$$

In a manifold which is a product with the flat real line, many objects split up in components. For example, if $p = (z,t) \in \mathbb{H}^2 \times \mathbb{R}$ we have $T_p\mathbb{H}^2 \times \mathbb{R} = T_z\mathbb{H}^2 \oplus T_t\mathbb{R}$ and the two tangent spaces are orthogonal. In other words, we have $\mathfrak{X}(\mathbb{H}^2 \times \mathbb{R}) = \mathfrak{X}(\mathbb{H}^2) \oplus \mathfrak{X}(\mathbb{R})$ and then if $X \in \mathfrak{X}(\mathbb{H}^2 \times \mathbb{R})$ we write $X = X_1 + X_2$ where $X_1 \in \mathfrak{X}(\mathbb{H}^2)$ and $X_2 \in \mathfrak{X}(\mathbb{R})$ are unique.

Clearly, the use of a product metric has consequences on the Riemannian connection we obtain on the manifold. The result is that product connection splits along components of the product.

Proposition 32.
Consider $X, Y \in \mathfrak{X}(\mathbb{H}^2 \times \mathbb{R})$. The affine connection defined by

$$\nabla_X^{\mathbb{H}^2 \times \mathbb{R}} Y = \nabla_{X_1}^{\mathbb{H}^2} Y_1 + \nabla_{X_2}^{\mathbb{R}} Y_2$$

is the Riemannian connection compatible with $\mathbb{H}^2 \times \mathbb{R}$.

This fact has as the immediate consequence that geodesics of $\mathbb{H}^2 \times \mathbb{R}$ are product of geodesics of $\mathbb{H}^2$ and $\mathbb{R}$.

Proposition 33.
$\gamma \in \mathbb{H}^2 \times \mathbb{R}$ is a geodesic if and only if $\gamma = (\gamma_1, \gamma_2)$ where γ_1 is a geodesic of $\mathbb{H}^2$ and γ_1 is a geodesic of $\mathbb{R}$.

Proof. If $\gamma = (\gamma_1, \gamma_2) \in \mathbb{H}^2 \times \mathbb{R}$ is a curve, we have

$$\nabla_{\gamma'}^{\mathbb{H}^2 \times \mathbb{R}} \gamma' = \nabla_{\gamma_1'}^{\mathbb{H}^2} \gamma_1' + \nabla_{\gamma_2'}^{\mathbb{R}} \gamma_2'$$

and

$$\nabla_{\gamma_1'}^{\mathbb{H}^2} \gamma_1' + \nabla_{\gamma_2'}^{\mathbb{R}} \gamma_2' = 0 \Longleftrightarrow \nabla_{\gamma_1'}^{\mathbb{H}^2} \gamma_1' = \nabla_{\gamma_2'}^{\mathbb{R}} \gamma_2' = 0$$

because vectors tangent to different elements of a Riemannian product are orthogonal. □

Surprisingly, also the isometries group splits in components.

Proposition 34. $Isom(\mathbb{H}^2 \times \mathbb{R}) = Isom(\mathbb{H}^2) \times Isom(\mathbb{R})$

A proof of this proposition can be found, for example, in [5].
We will call vertical translations the isometries $f(z,t) = (z, t+s)$, i.e. the isometries originating from the $\mathbb{R}$ component, horizontal translations the isometries $f(z,t) = (\widetilde{\tau}_t(z), t)$ where $\widetilde{\tau}_t$ is defined in proposition 19. The fact that isometries of a product factor in product of isometries of the components is not general. One can think to the product $\mathbb{R}^2 \times \mathbb{R}$ to see that, in general, multiplying Riemannian manifolds is a process setting up new isometries. We remark one more useful properties of this manifold, always coming from the product structure.

Proposition 35.
$\mathbb{H}^2 \times \mathbb{R}$ *is homogeneous*

Proof. Take $p \in \mathbb{H}^2 \times \mathbb{R}$. We can operate a vertical translation to bring p at height zero. Then we can use homogeneity of $\mathbb{H}^2 \times \{0\}$ to go in $(0,0) \in \mathbb{H}^2 \times \mathbb{R}$. □

We conclude this sketch of the geometry of $\mathbb{H}^2 \times \mathbb{R}$ recalling the values that sectional curvature can assume.

Proposition 36.
$-1 \leq K_{sect}(\mathbb{H}^2 \times \mathbb{R}) \leq 0$

2.3 CMC rotational surfaces: the H_α^h family

From now on, we will write *cmc* as a short for *constant mean curvature.* In this section we propose a quick review of the theory of *rotational* constant mean curvature surfaces in $\mathbb{H}^2 \times \mathbb{R}$, where by *rotational* we mean invariant with respect to R. Since the three-manifold we are considering is rotationally symmetric, it is natural to start the study of cmc surfaces considering the *rotational* examplesIndeed it is a standard fact that finding a rotational solution of a constant mean curvature problem reduces to an ODE. Requiring the rotational symmetry, we assign a value to one of the two principal curvatures in each point of the surface: the curvature of a circle. Hence finding rotational $h-$surfaces is a problem that usually has an explicit solution which is found integrating the associated ODE.
In this section we recall the rotational $h-$surfaces of $\mathbb{H}^2 \times \mathbb{R}$ introduced by Sa Earp and Toubiana in [45]. Consider $\mathbb{H}^2$ in the Poincaré disc model and recall that R, the rotation about the $\{(0,0)\} \times \mathbb{R}$ axis, is an isometry of $\mathbb{H}^2 \times \mathbb{R}$. Sa Earp and Toubina found, for each fixed value of the mean curvature $h \in (0, \frac{1}{2}]$, a one parameter (α) family of surfaces $\{H_\alpha^h\}_\alpha$ of radially symmetric solutions with constant mean curvature h. Here we will

consider only the $(0, \frac{1}{2})$ case because we are interested in using these surfaces as barriers for building $h-$graphs on exterior domains, and this problem for $h = \frac{1}{2}$ has been considered in [33]. We now recall the definition of the $\{H_\alpha^h\}_\alpha$ $h-$surfaces.

Almost each element of the family is a rotational graph defined on the complement of a disc of the hyperbolic plane. Hence we write $\rho^h(\alpha)$ for the radius of the disc and we call *base circle* of H_α^h the boundary of this disc.

Definition 42. H_α^h
Let be $h \in (0, \frac{1}{2})$ and $0 \leq \alpha$. Thus $\rho^h(\alpha)$ is defined as

$$\rho^h(\alpha) = arccosh\left(\frac{-2\,\alpha\,h + \sqrt{1 - 4h^2 + \alpha^2}}{1 - 4h^2}\right) \tag{2.8}$$

If ρ is the hyperbolic distance from $0 \in \mathbb{H}^2$, one defines $\forall\,\rho \geq \rho_h(\alpha)$

$$H_\alpha^h(\rho) = \int_{\rho^h(\alpha)}^{\rho} u_\alpha^h(r)\,dr \tag{2.9}$$

where

$$u_\alpha^h(\rho) = \frac{-\alpha + 2h\,\cosh(\rho)}{\sqrt{\sinh(\rho)^2 - (-\alpha + 2h\,\cosh(\rho))^2}} \tag{2.10}$$

Figure 2.3 shows the generating curves of these rotational surfaces.

For technical convenience we call ϕ^h the argument of the *cosh* giving ρ^h:

$$\phi^h(\alpha) =: arcosh(\phi^h(\alpha)) \tag{2.11}$$

We recall without proof some properties of these functions. A proof can be found in [45] and in the appendix of [34].

Proposition 37.
Let be $h \in (0, \frac{1}{2})$ and $0 \leq \alpha$. Then we have

1. *Each element of the familiy has constant mean curvature h*

$$H(H_\alpha^h) = \frac{1}{2}\,div\left(\frac{\nabla\,H_\alpha^h}{\sqrt{1 + |\nabla H_\alpha^h|^2}}\right) \equiv h$$

2. *All the surfaces are zero valued on their boundary, i.e. on the circle* $S_0(\rho^h(\alpha)$ *we have:*

$$H^h_\alpha(\rho^h_\alpha) = 0$$

3. *If* $\alpha < 2h$*, the* H^h_α *functions are positive and vertical on their base circle:*

$$\begin{aligned} \rho^h_\alpha &> 0 \\ u^h_\alpha(\rho) &> 0 \qquad \forall\, \rho > \rho^h_\alpha \\ u^h_\alpha(\rho^h_\alpha) &= +\infty \end{aligned}$$

4. *If* $\alpha = 2h$*, defining* $H^h_{2h} = S^h$ *and* $u_\alpha = u^h_\alpha$*, we have a simply connected entire graph on* $\mathbb{H}^2$*. Moreover* S^h *is positive and has horizontal tangent plane in* $0 \in \mathbb{H}^2$

$$\rho^h(2h) = 0 \tag{2.12}$$

$$u^h(\rho) > 0 \qquad \forall\, \rho > 0 \tag{2.13}$$

$$u_\alpha(0) = 0 \tag{2.14}$$

5. *If* $\alpha > 2h$*, the* H^h_α *functions are negative on a circular annulus and positive out of that annulus. Moreover all these surfaces are vertical on their base circle:*

$$\begin{aligned} \rho^h_\alpha &> 0 \\ u^h_\alpha(\rho) &> 0 \Leftrightarrow \rho > arccosh\left(\frac{\alpha}{2h}\right) \\ u^h_\alpha(\rho^h_\alpha) &= -\infty \end{aligned}$$

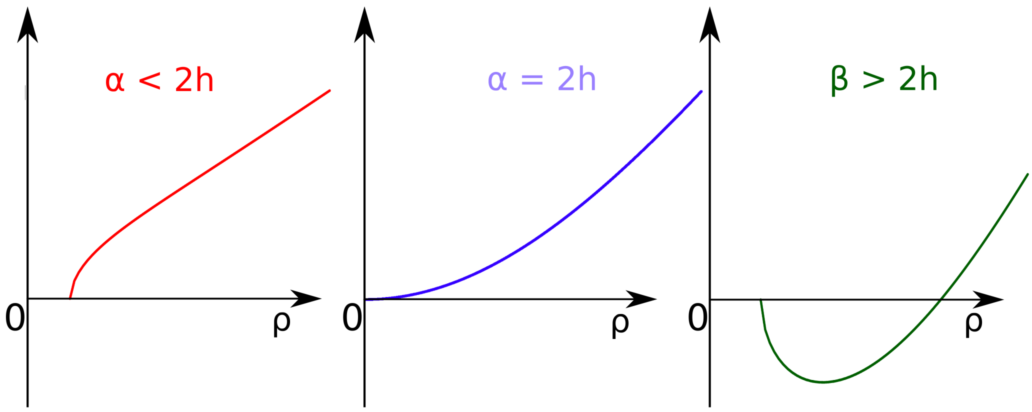

Figure 2.3: $\alpha < 2h$, $\alpha = 2h$ and $\alpha > 2h$

Chapter 3

Fine properties of curves and surfaces in $\mathbb{H}^2 \times \mathbb{R}$

3.1 Evolution of curves in $\mathbb{H}^2$

In this section we introduce two results in hyperbolic geometry less standard than what we have seen until now. What we are going to consider are two special evolutions of closed curves.

Evolution along the distance function

We are interested on the evolution of the geodesic curvature of a closed curve $\gamma \subset \mathbb{H}^2$ along the flow of the distance function associated to the compact bounded by γ. We now prove that in the hyperbolic setting we can give explicit formulas for the evolution of curvature along this flow.

Proposition 38.
Consider $\gamma \subset \mathbb{H}^2$ a smooth Jordan curve and d the distance function associated to Ω the compact bounded by γ, positive in Ω'. Let $a > 0$ small enough so that $\overline{V}_a$ can be parametrized by the flow of d. Let $q_0 \in \gamma$ and $k(q_0) = k_0$ the geodesic curvature of γ in q_0. If we write $k_{x_0}(t)$ for the geodesic curvature of the curve $C_t = \{p \in \Omega' : d(p) = t\}$ and $q(t) = \varphi_{q_0}(t)$ we have

$$\begin{aligned} |k_0| = 1 &\Longrightarrow |k_{q_0}(t)| \equiv 1 \\ |k_0| < 1 &\Longrightarrow k_{q_0}(t) = \tanh\left(t - \log\sqrt{\beta}\right) \\ |k_0| > 1 &\Longrightarrow k_{q_0}(t) = \coth\left(t - \log\sqrt{\beta}\right) \end{aligned} \tag{3.1}$$

where $\beta = \beta(k_0) = \left|\frac{1-k_0}{1+k_0}\right|$.

Proof. The proof is made of two parts. First of all we use the Radial Curvature Equation (1.17) to write the evolution equation of the geodesic curvature along the flow of d for a general manifold. After that, we will express the equation in the hyperbolic setting and find an explicit expression in the hyperbolic case.
Consider M^n an orientable manifold and S a closed hypersurface with unit normal field η and shape operator A. Call Ω the compact bounded by S and define on Ω' the distance function $d(p) = dist_M(p, S)$. Let $0 < a$ such that $\overline{V}_a$, as defined in (1.18), can be written in exponential coordinates on the tangent bundle of M along S. Thus we have

$$0 \leq t \leq a,\ q \in S \Longrightarrow \nabla d(q,t) \ \perp T_{(q,t)}S_t$$

We chose $\eta_t = \nabla d_t$ as a unit normal field along C_t, and we define A^t to be the shape operator associated to η_t. If we consider $v_1(t), \dots v_{n-1}(t)$ an orthonormal frame of $T_{(q,t)}S_t$ diagonalizing A^t, the Radial Curvature Equation (1.17) yields $\forall\, i = 1, \dots, n-1$

$$\left(-\nabla_{\nabla d} A^t\right) v_i(t) + \left(A^t\right)^2 v_i(t) = R\left(v_i(t), \nabla d\right) \nabla d$$

with

$$-\left(\nabla_{\nabla d} A^t\right) v_i(t) = \frac{D}{dt}\left(k_i(t) v_i(t)\right)$$

This is a general fact following form the fact that A is self-adjoint. $\forall\, X \in \mathfrak{X}(S_t)$ we have

$$\begin{aligned} (\nabla_{N_t} A^t)X &= \nabla_{N_t}(A^t X) - A^t(\nabla_{N_t} X) \\ &= \nabla_{N_t}(A^t X) \qquad\qquad \text{being } \nabla_{N_t} X \ \perp\ TC_t \end{aligned}$$

Thus taking the scalar product with v_i we get

$$-k_i'(t) + k_i(t)^2 = -sect(v_i(t), \nabla d) \tag{3.2}$$

The evolution equation we have just found is a Riccati equation. Being the sectional curvature constant we can calculate an explicit solution. To calculate this expression we plug hyperbolic planar geometry in this equation. Clearly $n - 1 = 1$ and $-sect(v_i(t), \nabla d) \equiv 1$, thus the evolution of the curvature is given by the solution of the following Cauchy problem:

$$\left\{ \begin{array}{rcl} -k'(t) + k(t)^2 & = & 1 \\ k(0) & = & k_0 \end{array} \right.$$

Separating variables and integrating we find the claim.

□

Remark.

- If $|k_g(p)| \overset{p}{\geq} 1$ the geodesic curvature is monotone decreasing along the flow of distance function. Thus if γ is a $h-$convex curve in $\mathbb{H}^2$, its geodesic curvature does not grow during the evolution along the distance flow.

Now we turn our attention to the case where we deform the curve by mean curvature flow. Actually the flow we will use is a mean curvature flow with one more constraint: conservation of volume. We briefly recall what is the mean curvature flow.

Definition 43. Mean curvature flow
Consider M^n an orientable manifold and $S \subset M^n$ an hypersurface oriented by the unit normal field η. Assume S is the immersion of a manifold $\widetilde{S}$ and $I \subset \mathbb{R}$ an open interval. Then, the mean curvature flow of S is a one parameter family $\{X(t)\}_{t\in I}$ of immersions of $\widetilde{S}$ satisfying the following differential system

$$\begin{cases} X(0) & = & S \\ \dfrac{d}{dt}X(t) & = & -H(t)\,\eta(t) \end{cases} \tag{3.3}$$

where $\eta(t)$ and $H(t)$ are the normal vector and the mean curvature of $X(t)\,S$.

We now turn our attention to the mean curvature flow with constant volume.

Definition 44. Constant volume mean curvature flow
Let be $S \subset M^n$ a closed and oriented hypersurface and assume M^n orientable. We define

$$\overline{H} = \frac{1}{Area(S)} \int_S H\,d\sigma$$

where $d\sigma$ is the area element induced on S by the metric of M. Assume $X : \widetilde{S} \to M$ is an immersion. Then the prescribed volume mean curvature flow of S is a family of immersions $\{X(t)\}_{t\in I}$ solving the following differential system

$$\begin{cases} X(0) & = & S \\ \dfrac{d}{dt}X(t) & = & \Big(\overline{H}(t) - H(t)\Big)\,\eta(t) \end{cases} \tag{3.4}$$

We will write cvmcf for this flow.

It is a matter of computation to show that the volume of the compact bounded by $X(t)\,\widetilde{S}$ is constant along this flow, whenever this compact exists. In the hyperbolic setting a very precise result has been established by Cabezas-Rivas and Miquel [3]. The statement of this result is more general than the one we recall here because it is true in the hyperbolic space of any dimension. But we state the two dimensional case because it is what we will use.

Theorem 8 (Number 1 in [3])**.**
Consider $C \subset \mathbb{H}^2$ a Jordan smooth h-convex curve (recall definition 40). Then the cvmcf of C has a unique solution C_t satisying:

1. *C_t is defined for all times*
2. *C_t is smooth and h-convex*
3. *C_t converges exponentially fast to an hyperbolic circle*

Figure 3.1 sketches this flow.

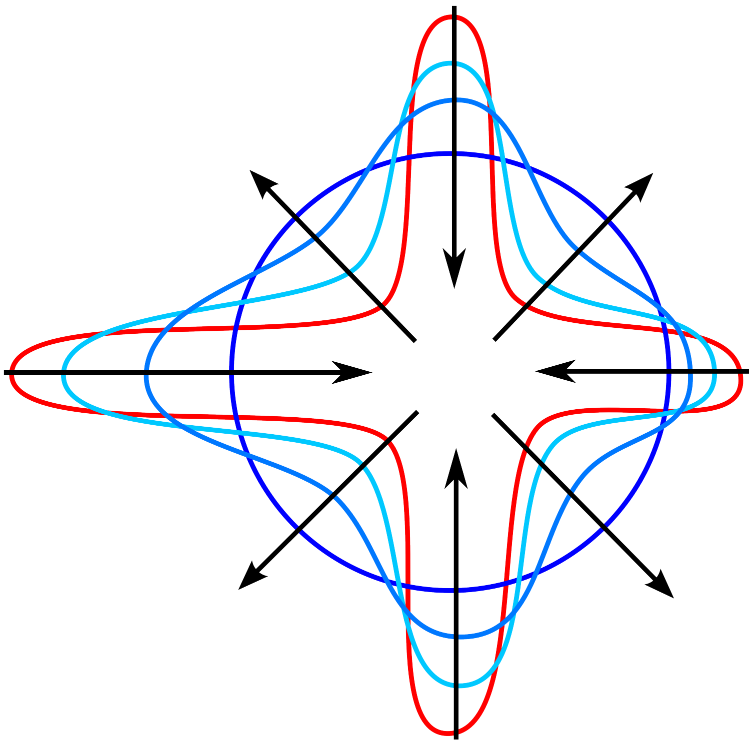

Figure 3.1: Constant volume mean curvature flow

Corollary.
Let $\gamma \subset \mathbb{H}^2$ a smooth Jordan curve and consider X its constant volume mean curvature flow and call C its limit circle. Then $\forall\, \varepsilon > 0$ exists $t(\varepsilon) \in \mathbb{R}^+$ such that $\forall t \geq t(\varepsilon)$ we have

$$d(X(t), C) \leq \varepsilon$$
$$|k_g(X(t)) - k_g(C)| \leq \varepsilon$$

The following corollary states that we can deform any $h-$convex curve of $\mathbb{H}^2$ to any hyperbolic circle

Corollary. Consider $C \subset \mathbb{H}^2$ a Jordan smooth h-convex curve. Then $\forall\, 0 < \rho\, \exists \{X(\sigma)\}_{\sigma \in [0,1]} : S^1 \to \mathbb{H}^2$ a family of Jordan curves such that

$$X(0) = \gamma$$
$$X(1) = S_\rho(0)$$

and $X(\sigma)$ is h-convex for each $\sigma \in [0, 1]$.

Proof. Up to a reparametrization we can assume that the cvmcf deforms γ in an hyperbolic circle in a finite time τ. Up to an hyperbolic translation along geodesics, we can also assume that this circle is centered in $0 \in \mathbb{D}$. Now we continue the deformation decreasing the radius of the circke to the desired one. The result follows from the fact that hyperbolic circles are $h-$convex. □

Remark. Here we make an assumption: we assume that the limit circle of the constant volume mean curvature flow is centered in 0. It means we can deform any h-convex curve into a circle centered in 0 by mean of the cvmcf and a dilation. This is not restrictive because if the limit circle $\widetilde{C}$ of a curve $\widetilde{\gamma}$ via cvmcf is not centered in 0, and τ is the translation sending $\widetilde{C}$ to C a circle centered in 0, the limit of the cvmcf of $\gamma = \tau(\widetilde{\gamma})$ is C. This follows from unicity.

3.2 The asymptotic behavior of the H^h_α family

In this section we prove an important asymptotical property of the $\{H^h_\alpha\}_\alpha$ surfaces. The explicit expression provided in (2.9), and standard Taylor approximation ensure that

$$H^h_\alpha(\rho) \approx \frac{2h}{\sqrt{1-4h^2}}\,\rho + c_{h,\alpha}\, e^{-\rho} + k^h + o(e^{-\rho}) \tag{3.5}$$

Since the leading term of the function H^h_α is independent of α, then all the functions H^h_α have the same asymptotic behavior at the first order. This is the main reason why the results of [33] cannot be applied for $h \in (0, \frac{1}{2})$. Hence we need to understand the dependence of the constant k^h on the variable α.

To study the asymptotic behavior we need to introduce some notation and to remark some facts.

Remark.

- We remark that if $\alpha \neq 2h$, H^h_α is zero valued and vertical on the circle $S_0(\rho^h_\alpha)$.
- The behavior of the elements of $\{H^h_\alpha\}_\alpha$ near the base circle changes if $\alpha \in [0, 2h)$ or if $(2h, +\infty)$. It is useful to use two different notations for these two intervals of the parameter. Thus we declare that, if the parameter of the surface we are considering is greater than $2h$, we use the letter β and we write H^h_β.
- For $\alpha \neq 2h$ these surfaces can be extended to complete $h-$surfaces of $\mathbb{H}^2 \times \mathbb{R}$. It is because they are vertical on their base circle. Hence if we consider a reflection how H^h_α in the zero height slice we obtain another $h-$surface that can be glued to the H^h_α along the base circle. The gluing is smooth because it is made where the surface is vertical. Moreover these complete surfaces are embedded for $\alpha < 2h$ and with self intersection for $\alpha > 2h$.
- Consider the completed surfaces associated to the $\{H^h_\alpha\}$ family. By standard low dimensional topology these surfaces disconnect $\mathbb{H}^2 \times \mathbb{R}$. Hence it makes sense to talk about *interior* and *exterior* of such a surface. We call *interior* of a completed H^h_α the region of $\mathbb{H}^2 \times \mathbb{R}$ bounded by H^h_α containing the $\{(0, 0)\} \times \mathbb{R}$ axis. We will also use the words *mean convex side* as a synonym of interior.
  By abuse of language we will talk about the mean convex side of a H^h_α meaning the part of $\mathbb{H}^2 \times \mathbb{R}^+$ where the upward normal vector of H^h_α is pointing.

We now study the dependence of the radius of the base circles on the parameter α.

Proposition 39.
Let $h \in (0, \frac{1}{2})$. Hence

- *For $\alpha \in [0, 2h]$ the radius $\rho^h(\alpha)$ is monotone decreasing with α*
- *For $\alpha \in [2h, +\infty)$ the radius $\rho^h(\alpha)$ is monotone increasing with α*

Proof. Given the definition of ϕ^h in (2.11) we show that

- ϕ^h is monotone decreasing when $\alpha \in [0, 2h]$
- ϕ^h is monotone increasing when $\alpha \in [2h, \infty)$

We have

$$\frac{\partial \phi^h}{\partial \alpha}(\alpha) = \frac{1}{1-4h^2}\left(-2h + \frac{\alpha}{\sqrt{1-4h^2+\alpha^2}}\right)$$

which is positive for $\alpha \geq 2h$. Being *arccosh* monotone increasing we are done. □

Corollary.
Let $h \in (0, \frac{1}{2})$. Hence we have

$$\rho^h\Big([0,2h]\Big) = \left[0, arccosh\left(\frac{1}{\sqrt{1-4h^2}}\right)\right]$$
$$\rho^h\Big([2h,+\infty)\Big) = \Big[0,+\infty\Big)$$

Proof. We have

$$\phi^h\Big([0,2h]\Big) = \Big[\phi^h(2h), \phi^h(0)\Big] = \left[1, \frac{1}{\sqrt{1-4h^2}}\right]$$

and

$$\phi^h\Big([0,2h]\Big) = \Big[\phi^h(2h), \lim_{\alpha\to+\infty} \phi^h(\alpha)\Big]$$

to conclude we apply *arccosh* to get

$$\rho^h\Big(\,[0,2h]\,\Big) = \left[\,0, arccosh\left(\frac{1}{\sqrt{1-4\,h^2}}\right)\right]$$

The other case can be done in the same way. □

We can now state and prove the theorem describing of H^h_α depends on α at infinity.

Theorem 9.
Let be $0 < \rho$ *large. Then* $\exists \overline{\beta} > 2h$ *such that* $\forall\, 0 < \widetilde{\alpha} \leq \overline{\beta}$ *we have*

$$\frac{\partial H^h_\alpha}{\partial\, \alpha}(\rho)_{|\, \alpha = \widetilde{\alpha}} < 0$$

Proof. This proof is completely technical and is made of two parts: first of all we prove that the derivative is negative for $\alpha < 2h$. After that, we will see it is equal to $-\infty$ for $\alpha = 2h$.
What we are going to do is a change of variable of the u^h_α so that the derivation with respect to α does not interact with the singularity in $\rho^h(\alpha)$. We define $s = Cosh(r)$ and get $dr = \frac{ds}{\sqrt{s^2-1}}$. Omitting the dependence on α we have

$$H^h_\alpha(\rho) = \int_\phi^{Cosh(\rho)} \frac{-\alpha + 2h\, s}{(s-b)^{\frac{1}{2}}\,(s-\phi)^{\frac{1}{2}}}\, \frac{ds}{\sqrt{s^2-1}}$$

where

$$b = -\frac{2h\,\alpha \,+\, \sqrt{1 - 4h^2 + \alpha^2}}{(1-4h^2)}$$

is the negative zero of the denominator of u^h_α defined in (2.10).
To remove the dependence of singularity on the parameter we define $z = s - \phi$, $z(\rho, \alpha) = Cosh(\rho) - \phi$ and we get

$$\begin{aligned} H^h_\alpha(\rho) &= \int_0^{z(\rho,\alpha)} \frac{-\alpha + 2h\,(z+\phi)}{\sqrt{z}\,\sqrt{z+\phi-b}\,\sqrt{(z+\phi)^2-1}}\, dz \\ &=: \int_0^{z(\rho,\alpha)} \widetilde{u}^h_\alpha(z)\, dz \end{aligned}$$

where $\phi > 1$ and $-b = |b| > 0$. Thus the only singularity of the integrand function in the interval $[0, Cosh(\rho) - \phi]$ is in 0. The only singularity that could appear when we move α is $z = 1$ for $\alpha = 2h$. But a simple calculation shows that this singularity is canceled by a term in the numerator.
Let's compute the derivative

$$\frac{\partial H^h_\alpha}{\partial \alpha}(\rho) = \int_0^{z(\rho,\alpha)} \frac{\partial \widetilde{u}_\alpha}{\partial \alpha}(z)\, dz \quad + \quad \widetilde{u}_\alpha(z(\rho,\alpha))\, \frac{\partial z}{\partial \alpha}(\rho, \alpha)$$

where, being ρ large, the second term can be neglected because $\dfrac{\partial z}{\partial \alpha}(\rho, \alpha)$ is a constant independent from ρ and

$$\widetilde{u}_\alpha(z(\rho,\alpha)) = \frac{1}{Cosh(\rho)} \frac{2h - \dfrac{\alpha}{Cosh(\rho)}}{\sqrt{1-\dfrac{\phi}{Cosh(\rho)}}\sqrt{1-\dfrac{b}{Cosh(\rho)}}\sqrt{1-\dfrac{1}{Cosh(\rho)}}}$$

The term that needs to be worked is the first one. We want to prove that it is negative when ρ is big enough. We notice that here we have a neat separation between the two cases $\alpha < 2h$ and $\beta > 2h$. In the first case the integrand function is negative and thus if the integral converges, it converges to a negative number. When $\beta > 2h$, the integrand function is positive near 0 and negative away form 0, so one cannot say anything about its sign.
To write the derivative of $\widetilde{u}_\alpha$, it is useful to use the following notation (we omit the dependence of b and ϕ on α). Here we will use the dot notation for derivatives with respect to α.

$$\begin{aligned}\psi_1(z) &= \frac{-1+2h\,\dot{\phi}}{(z+\phi-b)^{\frac{1}{2}}\left((z+\phi)^2-1\right)^{\frac{1}{2}}}\\ &= \frac{-1+2h\,\dot{\phi}}{(l-b)^{\frac{1}{2}}(l^2-1)^{\frac{1}{2}}}\end{aligned}$$

$$\begin{aligned}\psi_2(z) &= -\,(\dot{\phi}-\dot{b})\,\frac{-\alpha+2h\,(z+\phi)}{2\,(z+\phi-b)^{\frac{3}{2}}\,((z+\phi)^2-1)^{\frac{1}{2}}}\\ &= -\,(\dot{\phi}-\dot{b})\,\frac{-\alpha+2h\,l}{2\,(l-b)^{\frac{3}{2}}\,(l^{\,2}-1)^{\frac{1}{2}}}\end{aligned}$$

$$\begin{aligned}\psi_3(z) &= -\dot{\phi}\,\frac{(z+\phi)\,(-\alpha+2h\,(z+\phi))}{(z+\phi-b)^{\frac{1}{2}}\,((z+\phi)^2-1)^{\frac{3}{2}}}\\ &= -\dot{\phi}\,\frac{l\,(-\alpha+2h\,l)}{(l-b)^{\frac{1}{2}}\,(l^2-1)^{\frac{3}{2}}}\end{aligned}$$

where $l = z+\phi$. Thus we have

$$\frac{\partial\widetilde{u}_\alpha}{\partial\alpha}(z) = \frac{1}{\sqrt{z}}\Big(\psi_1(z)+\psi_2(z)+\psi_3(z)\Big) \tag{3.6}$$

Many of the terms involved have a sign which does not depend on h, α and z. We have $\forall z \geq 0$ e $\alpha \neq 2h$:

- $z + \phi - 1 \overset{h,\alpha}{>} 0$ being $\phi = Cosh(r^h(\alpha))$ e $r^h(\alpha) = 0 \Leftrightarrow \alpha = 2h$
- $z + \phi - b \overset{h,\alpha}{>} 0$ being $-b = |b|$ e $\phi - b = \dfrac{2\sqrt{1-4h^2+\alpha^2}}{1-4h^2}$
- $-1 + 2h\,\dot{\phi} \overset{h,\alpha}{<} 0$ being $\dot{\phi} = \dfrac{1}{1-4h^2}\Big(-2h + \dfrac{\alpha}{\sqrt{1-4h^2+\alpha^2}}\Big)$
- $\dot{b} < 0$ being $\dot{b} = \dfrac{1}{1-4h^2}\Big(-2h - \dfrac{\alpha}{\sqrt{1-4h^2+\alpha^2}}\Big)$

Now assume $\alpha < 2h$. We have the following equalities

- $\dot{\phi} \overset{h,\alpha}{<} 0$ being $\dot{\phi} \leq 0 \Leftrightarrow \alpha^2 \leq 4h^2$
- $-\alpha + 2h\,(z + \phi) \overset{h,\alpha}{>} 0$ being $z + \phi > 1$
- $\psi_2(z) \overset{h,\alpha}{>} 0$

Thus it is enough to prove that one of the following equivalent inequalities holds.

$$\psi_1(z) + \psi_3(z) \geq 0 \;\Leftrightarrow\; -\frac{\psi_1(z)}{\psi_3(z)} \geq 1$$

To simplify calculations we define

$$c_1 = -(-1 + 2h\,\dot{\phi}) > 0 \quad c_{31} = -2h\,\dot{\phi} > 0 \quad c_{32} - \alpha\dot{\phi} > 0$$

and thus we get

$$-\frac{\psi_1(z)}{\psi_3(z)} = c_1\,\frac{l^2 - 1}{c_{31}l^2 - c32l} \geq 1 \Longleftrightarrow (c_1 - c_{31})l^2 - c_{32}l - c_1 \geq 0$$

where

$$c_1 - c_{31} = 1 - 2h \quad \dot{\phi} + 2h\,\dot{\phi} = 1$$

The roots of the equation

$$l^2 + c_{32}l - c_1 = 0$$

are the two distinct real numbers

$$l_{\pm} = \frac{-c_{32} \pm \sqrt{{c_{32}}^2 + 4c_1}}{2}$$

If we prove that

$$max\{l_-, l_+\} = l_+ \overset{h,\alpha}{\leq} \phi$$

we have proven that the derivative is negative.

$$\begin{aligned} l_+ \leq \phi &\Leftrightarrow \sqrt{{c_{32}}^2 + 4c_1} \leq 2\phi + c_{32} \\ &\Leftrightarrow \phi^2 + c_{32}\phi - c_1 \geq 0 \\ &\Leftrightarrow \phi^2 - 1 - \dot{\phi}(\alpha\phi - 2h) \geq 0 \end{aligned}$$

which is always true because $\alpha\phi - 2h \overset{h,\alpha}{\leq} 0$. Indeed we have

$$\begin{aligned} \alpha\phi - 2h \leq 0 &\Leftrightarrow \alpha\sqrt{1 - 4h^2 + \alpha^2} \leq 2h\,(1 - 4h^2 + \alpha^2) \\ &\Leftrightarrow \alpha^2 \leq 4h^2\,(1 - 4h^2 + \alpha^2) \\ &\Leftrightarrow \alpha \leq 2h \end{aligned}$$

We have just shown that when ρ is big enough, and $\alpha \in (0, 2h)$ the height $H^h_\alpha(\rho)$ is strictly decreasing when α increases.
We now prove that for $\alpha = 2h$ the speed of descent is ∞. We remark that this fact implies the existence of $\overline{\beta} > 2h$ where the derivative is still negative. If we evaluate equation (3.6) in $\alpha = 2h$ we obtain a non integrable singularity. Being

$$\begin{aligned} &\dot{\phi}(2h) = 0 \\ &\phi(2h) = 1 \\ &b(2h) = -\frac{1 + 4h^2}{1 - 4h^2} \\ &\dot{b}(2h) = -\frac{4h}{1 - 4h^2} \\ &l(2h) - b(2h) = z + \frac{2}{1 - 4h^2} \\ &l(2h)^2 - 1 = z(z + 2) \\ &\dot{\phi}(2h) - \dot{b}(2h) = \frac{4h}{1 - 4h^2} \end{aligned}$$

We have $\psi_3 \equiv 0$ and

$$\begin{aligned}\psi_1(z)+\psi_2(z) &= -\frac{1}{\sqrt{z\,(z+2)}\sqrt{z+\dfrac{2}{1-4h^2}}}+ \\ &\quad + \frac{4h^2}{1-4h^2}\,\frac{z}{\sqrt{z\,(z+2)}\left(z+\dfrac{2}{1-4h^2}\right)^{\frac{3}{2}}} \\ &= -\frac{(1-4h^2)\left(z+\dfrac{2}{1-4h^2}\right)+4h^2\,z}{\sqrt{z}\,(1-4h^2)\,\sqrt{z+2}\,\left(z+\dfrac{2}{1-4h^2}\right)^{\frac{3}{2}}} \\ &= -\frac{1}{\sqrt{z}}\left(\frac{\sqrt{z+2}}{(1-4h^2)\,\left(z+\dfrac{2}{1-4h^2}\right)^{\frac{3}{2}}}\right)\end{aligned}$$

Hence $\dfrac{\partial\widetilde{u}_\alpha}{\partial\alpha}(z)_{|\alpha=2h}$ contains the non integrable singularity

$$\begin{aligned}\frac{\partial\widetilde{u}_\alpha}{\partial\alpha}(z)_{|\alpha=2h} &= \frac{\psi_1(z)+\psi_2(z)+\psi_3(z)}{\sqrt{z}} \\ &= -\frac{1}{z}\left(\frac{\sqrt{z+2}}{(1-4h^2)\,\left(z+\dfrac{2}{1-4h^2}\right)^{\frac{3}{2}}}\right)\end{aligned}$$

□

Remark.
This proof cannot be used in the case $\alpha > 2h$. Indeed here $\widetilde{u}_\alpha(z)$ is positive near $z=0$. We can prove that $\psi_1(0) = -\psi_3(0)$ and $\psi_2(0) > 0$:

$$\begin{aligned}\psi_1(0)+\psi_3(0) &= \frac{(-1+2h\,\dot{\phi})\,(\phi^2-1)-\dot{\phi}\,\phi\,(-\alpha+2h\,\phi)}{(\phi-b)^{\frac{1}{2}}\,(\phi^2-1)^{\frac{3}{2}}} \\ &= \frac{1-\phi^2-(2h-\alpha\,\phi\,)\dot{\phi}}{(\phi-b)^{\frac{1}{2}}\,(\phi^2-1)^{\frac{3}{2}}}\end{aligned}$$

where, if we define $g(h,\alpha) = 1 - 4h^2 + \alpha^2$, we have

$$1-\phi^2 = \frac{-4h^2\,\alpha^2 + 4h\,\alpha\,\sqrt{g(h,\alpha)} - g(h,\alpha) + \left(1-4h^2\right)^2}{\left(1-4h^2\right)^2}$$

$$\begin{aligned}(2h-\alpha\,\phi)\,\dot{\phi} &= \left(\frac{2h\,(1-4h^2) - \alpha\,(-2h\,\alpha + \sqrt{g(h,\alpha)}\,)}{1-4h^2}\right)\left(\frac{-2h + \dfrac{\alpha}{\sqrt{g(h,\alpha)}}}{1-4h^2}\right)\\ &= \frac{-(1+4h^2)\,\alpha^2 + 4h\,\alpha\sqrt{g(h,\alpha)} - 4h^2(1-4h^2)}{\left(1-4h^2\right)^2}\\ &= \frac{-4h^2\,\alpha^2 + 4h\,\alpha\,\sqrt{g(h,\alpha)} - g(h,\alpha) + \left(1-4h^2\right)^2}{\left(1-4h^2\right)^2}\\ &= 1-\phi^2\end{aligned}$$

This result allows us to associate two concepts of distance between these rotational surfaces.

Definition 45. Asymptotic vertical distance
Consider α such that $0 \leq \alpha < \overline{\beta}$. We define the asymptotic vertical distance between H^h_β S^h to be

$$\lim_{\rho\to+\infty} H^h_\beta(\rho)S^h(\rho)$$

Now we can state the consequences of the last theorem in terms of the relative positions of the element of the family according to the value of the parameter.

Corollary.
Let be $h \in (0, 2h]$.

- $\forall\,\alpha \in [0, 2h)$ $S^h \cap H^h_\alpha$ is a circle. Inside this circle we have $H^h_\alpha < S^h$, outside we have the opposite inequality.
- $\exists\,\overline{\beta} \in (2h, +\infty)$ such that $\forall\,\beta \in (2h, \overline{\beta})$ we have $S^h \cap H^h_\beta = \emptyset$ andS^h is in the mean convex side of H^h_β, and hence in the mean convex side of the H^h_α for $\alpha \in [0, 2h]$. In other words we have

$$H^h_\beta < H^h_\alpha \qquad \forall\,\beta \in (2h, \overline{\beta}) \text{ and } \alpha \in [0, 2h]$$

- The vertical asymptotic distance between S^h and H^h_α is a positive real number if $\alpha > 2h$, negative id $\alpha < 2h$. Moreover this number increases with α.

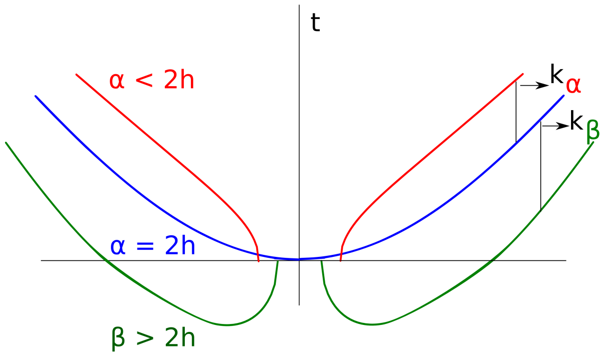

Figure 3.2: Asymptotic behavior

Figure 3.2 describes the situation.

We end this section defining the concept of asymptotic horizontal distance for element of the $\{H^h_\alpha\}_\alpha$ family. This is the limit for $t \to +\infty$ of the distance between the circles obtained intersecting two distinct elements of the family with the plane of height t.

Definition 46. Asymptotic horizontal distance
Consider $0 \le \alpha \le 2h$ and $2h < \beta \le \overline{\beta}$. H^h_α, and H^h_β are asymptotically invertible since their asymptotic behavior is linear. Thus their inverse functions $\left(H^h_\beta\right)^{-1}$ and $\left(H^h_\alpha\right)^{-1}$ are invertible and linear. We define the asymptotic distance between H^h_α and H^h_β to be

$$d_\infty(H^h_\alpha, H^h_\beta) = \lim_{t \to +\infty} \left| \left(H^h_\alpha\right)^{-1}(t) - \left(H^h_\beta\right)^{-1}(t) \right|$$

3.2.1 $r-$admissibility

What we have just established, together with the knowledge of the asymptotic behavior of the H^h_α, gives informations about the horizontal distance between two distinct elements of the family. Recall the asymptotic expansion (3.5):

$$H^h_\alpha(\rho) \approx \frac{2h}{\sqrt{1-4h^2}}\,\rho + c_{h,\alpha}\,e^{-\rho} + k^h_\alpha + o(e^{-\rho})$$

and write $\rho_{\alpha,t}$ for the solution of $H^h_\alpha(\rho) = t$. In other words ρ_t is the radius of the circle in $\mathbb{H}^2$ whose image via H^h_α is contained in the plane $\mathbb{H}^2 \times \{t\}$.

Thus if $\alpha \in [0, 2h]$ and $\beta \in (2h, \overline{\beta}\,]$ we have that

$$H_\alpha^h(\rho) = t = H_\beta^h(\rho)$$

is equivalent to

$$\frac{2h}{\sqrt{1-4h^2}}\,\rho_{\alpha,t} + c_{h,\alpha}\, e^{-\rho_{\alpha,t}} + k_\alpha^h = \frac{2h}{\sqrt{1-4h^2}}\,\rho_{\beta,t} + c_{h,\beta}\, e^{-\rho_{\beta,t}} + k_\beta^h + o(e^{-\rho})$$

If we define $c = \frac{2h}{\sqrt{1-4h^2}}$, we have

$$\rho_{\alpha,t} - \rho_{\beta,t} = \frac{k_\beta^h - k_\alpha^h}{c} + \frac{1}{c}\Big(c_{h,\beta}\, e^{-\rho_{\beta,t}} - c_{h,\alpha}\, e^{-\rho_{\alpha,t}}\Big) + o(e^{-\rho}) \tag{3.7}$$

Being $\rho_{\alpha,t} \longrightarrow +\infty$ when $t \to +\infty$ it is clear that the horizontal distance between two different element of the family is a non zero number.

Remark.
Take $h \in (0, \frac{1}{2})$. Consider $\alpha,\, \beta \in \mathbb{R}^+$, with β close to $2h$. We have proved that

$$d_\infty(H_\alpha^h, H_\beta^h) = d_\infty(\alpha, \beta) = \left|\frac{k_\beta^h - k_\alpha^h}{c}\right| \tag{3.8}$$

As we have mentioned, in the third chapter we will use an H_β^h surface to give a-priori C^1 estimates for a cmc graph on an annuls of $\mathbb{H}^2$. Assume γ is the inner boundary of the annulus. We will consider an horizontal translation of a precise H_β^h which makes the surface tangent to the γ. This $\tau(H_\beta^h)$ will be a barrier, provided it is below the graph. To guarantee this hierarchy, by the maximum principle, it will be enough to guarantee the hierarchy on the boundary. The aim of the $r-$admissibility condition is to guarantee that $\tau(H_\beta^h)$ is actually below the $h-$graph. Thus we will have to assure that, during the horizontal translation, the first contact between H_β^h and the $h-$graph is on γ and not on H_α^h. Moreover we have to guarantee that $\tau(H_\beta^h)_{|\gamma} \leq 0$. We now make these remarks precise.

Consider $\gamma \subset \mathbb{H}^2$ a smooth Jordan curve. Assume the curve satisfies an interior sphere condition of radius $0 < r$. We associate to this curve two elements of the $\{H_\alpha^h\}_\alpha$ family. First we consider the H_α^h whose base circle is the biggest between all the circles contained in the compact part of Ω'. More precisely we define

$$\alpha = \inf\Big\{\tilde{\alpha} \in [0, 2h) : H_{\tilde{\alpha}}^h \cap \{t = 0\} \subset \Omega\Big\}$$

The other surface is given by the $r-$sphere condition. Namely we take the $\beta \in (2h, +\infty)$ such that $\rho^h(\beta) = r$ if $\beta \leq \overline{\beta}$, where $\overline{\beta}$ is given in theorem 9. If the β such that $\rho^h(\beta) = r$ is bigger than $\overline{\beta}$, we reduce r to $\widetilde{r}$ until when $\beta(\widetilde{r}) \leq \overline{\beta}$. This is possible by proposition 39. Moreover, if γ satisfies an $r-$sphere conditions it also satisfies a $\widetilde{r}-$sphere conditions. To see that one can apply the maximum principle and recall that, by equation (2.7), the curvature of an hyperbolic circle increases as the radius decreases.
Now we define d to be the hyperbolic distance between the base circle of H^h_β and the circle where it has its minima (recall item 5 of proposition 37), i.e. the hyperbolic circle of radius $\frac{arccosh(\beta)}{2h}$. Thus we have

$$d = \frac{arccosh(\beta)}{2h} - \rho^h(\beta)$$

By other way we can define the quantity

$$x = \min\left\{\frac{d}{2},\ d_\infty(\alpha, \beta)\right\} \tag{3.9}$$

where d_∞ is given in (3.8). Remark that since α and β depend only on γ, the same holds for x. We can now say what we mean by $r-$admissibility.

Definition 47.
Let $\gamma \subset \mathbb{H}^2$ a smooth Jordan curve. Suppose γ satisfies an interior sphere condition of radius r. We say that γ is $r-$admissible if it is contained int he circular annulus A_γ with inner radius $r = \rho^h(\beta)$ and outer radius $\rho^h(\beta) + x$.

Figure 3.3 gives a description of the definition. We now check that a curve verifying this definition allows the H^h_β to be used as a barrier.

Proposition 40.
Let γ an $r-$admissible curve and take $z \in \gamma$. Consider $\tau_z(H^h_\beta)$ an horizontal translation of H^h_β tangent to γ in z.
Thus

1. *$\tau_z(H^h_\beta)$ is not in the mean convex side of H^h_α, i. e. $\tau_z(H^h_\beta) < H^h_\alpha$*
2. *$\tau_z(H^h_\beta)_{|\gamma} \leq 0$*

Proof. The first fact follows directly from the definition. In fact, if γ is in the circular annulus A_γ, the distance that has to be covered to make the circle $S_0(r)$ tangent to $S_0(r + \frac{d}{2})$ is smaller than $d_\infty(\alpha, \beta)$. Moreover the distance that needs to be covered to make $S_0(r)$ tangent to γ is strictly smaller than the distance between the two circles.

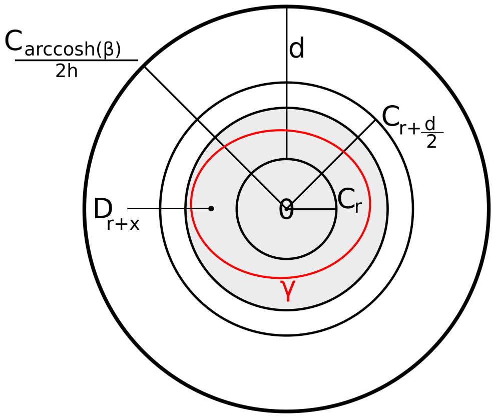

Figure 3.3: $r-$admissible curve

To prove the second statement we will proceed moving γ instead of H_β^h. If we prove that any translation making $S_0(r+\frac{d}{2})$ tangent to $S_0(r)$ is contained in the disc $B_0(\frac{arccosh(\beta)}{2h})$ we are done. It is because in this disc H_β^h is negative and, to make $S_0(\frac{d}{2}+r)$ tangent to $S_0(r)$, we need to cover a distance greater than the distance we have to cover to make γ tangent to $S_0(r)$. But $S_0(\frac{d}{2}+r) \subset B_0(arcosh\left(\frac{\beta}{2h}\right))$. □

An example of $r-$admissible curve is given by a small C^2 perturbation of a circle.

Chapter 4

CMC Graphs on non convex domains

In this chapter we prove a non existence and an existence result for surfaces with mean curvature constantly equal to $h \in (0, \frac{1}{2})$, on an exterior domain. Both results are obtained representing mean curvature as an elliptic operator (in the sense recalled in appendix A). Indeed if we think to $\mathbb{H}^2$ in the disc model, it is immediate to recognize that if we consider S a surface in $\mathbb{H}^2 \times \mathbb{R}$ which is a graph on $U \subset \mathbb{H}^2$:

$$S = \{(z, u(z)) \in \mathbb{H}^2 \times \mathbb{R} : z \in U\}$$

the mean curvature operator on S is elliptic and it is uniformly elliptic if $|\nabla^{H^2} u|$ is bounded

$$\begin{aligned} H(z) = Q(u)(z) &= \frac{1}{2}\, div^{\mathbb{H}^2} \left(\frac{\nabla^{\mathbb{H}^2} u}{\sqrt{1 + |\nabla^{\mathbb{H}^2} u|^2}} \right)(z) \\ &= \sum_{i,j=1}^{2} a_{ij}(z, \nabla u)\, \frac{\partial\, u}{\partial x_i\, \partial\, x_j} + b(z, \nabla u) \end{aligned}$$

4.1 A non existence result

The first non existence reult for mean curvature graphs is due to Finn [9] in the Euclidean setting. If f denotes a solution of he minimal surfaces problem on an annulus Ω, Finn established an estimate for $|f|$ depending only on the thickness of Ω and the value of f on the outer boundary of Ω. This estimate is obtained using two half catenoids as a bounding box for the

graph of f. Thus, for each fixed circular annulus, one can assign constant boundary data leading to non existence: it is enough to assign a constant boundary data very different on the two components of $\partial\Omega$.
In this section we prove an hyperbolic version of this result for $h-$graphs on circular annuli when $h \in (0, \frac{1}{2})$. In the Euclidean case the estimate follows from properties of the mean curvature operator and existence of minimal graphs with singular normal derivative defined in the complement of a disc. In our setting the mean curvature operator has the same properties of the Euclidean one and the H^h_α will play the role of catenoids.
To state the results leading to the estimate we need to define some notation. We recall that we are using the disc model of $\mathbb{H}^2$.
Consider $\Omega \subset \mathbb{D}$ a compact smooth annulus. Suppose $\partial\Omega = \gamma_1 \cup \gamma_2$ where γ_i is a Jordan curve for $i = 1, 2$. Assume γ_1 is contained in the compact set bounded by γ_2. We declare that we will compute normal derivatives along γ_1 with respect to the inward normal vector. Since we will not require differentiability on the boundary, we need to define this derivative precisely. We suppose that each $z \in \gamma_1$ is the end point of a geodesic contained in $int(\Omega)$. More precisely, we require $\forall\, z \in \gamma_1$ existence of a geodesic $\gamma : [0, a] \to \mathbb{H}^2$ such that $\gamma \subset int(\Omega)$ and $\gamma(a) = z$. Now consider $\phi \in C^1\,(int(\Omega))$ a function and $p \in \Omega$ a point such that $\gamma(t) = p$. Using Taylor formula, if σ is small enough, we can write

$$\begin{aligned}\Big(\phi \circ \gamma\Big)(t + \sigma) &= \phi(p) + (d_p\phi)\,(\gamma'(t))\ \sigma + o(\sigma)\\ &= \phi(p) + \langle \nabla\phi(p), \gamma'(t)\rangle\ \sigma + o(\sigma)\end{aligned}$$

and we define

$$= \phi(p) - \frac{\partial\phi}{\partial s}\,(\sigma) + o(\sigma)$$

In other words, $\frac{\partial}{\partial s}$ is the opposite of the covariant derivative induced by γ, provided that γ ends on γ_1.
We can now state our estimate of the solution of cmc problem on annuli.

Lemma.
Assume Ω is a domain satisfying the hypothesis just described. Consider $\phi_0, \phi : \Omega \to \mathbb{R}$ two functions such that:

1. $\phi_0 \in C^1(int(\Omega), \mathbb{R}) \cap C(\overline{\Omega})$

   $\lim_{z\to\gamma_1} \dfrac{\partial\phi_0}{\partial s}(z) = +\infty$ along all geodesics ending on γ_1

2. $\phi \in C^1(\overline{\Omega}, \mathbb{R})$

3. $\liminf_{x \to \gamma_2}(\phi_0 - \phi) \geq 0$

4. $Q(\phi_0) \leq Q(\phi)$ in $int(\Omega)$, where $Q(u, z)$ is a quasilinear and elliptic second order operator with coefficients depending only on space and on the gradient of u (see the Appendix for a definition of quasilinear operators).

Then we have

$$\liminf_{z \to \gamma_1}(\phi_0 - \phi) \geq 0$$

Proof. Since the proof proposed by Finn is based on topological properties of open sets of $\mathbb{R}^2$, quasi-linearity of the operator and Taylor formula, we can proceed in the same way. We are proving the result by contradiction.
Suppose $\liminf_{z \to \gamma_1}(\phi_0 - \phi) < 0$ and we will end up with points in Ω where the difference $\phi_0 - \phi$ is both positive and negative, which is a contradiction. First of all we remark that $\liminf_{z \to \gamma_1}(\phi_0 - \phi) = l$ is finite because if it was $-\infty$ we would have had a contradiction with the hypothesis on the derivative of ϕ_0 approaching γ_1. On the other hand, by definition of $\liminf$, it exists a sequence $\Omega \ni z_n \longrightarrow z_1$ with $z_1 \in \gamma_1$

$$\liminf_{z \to \gamma_1}(\phi_0(z) - \phi(z)) = \liminf_{n \to \infty}(\phi_0(z_n) - \phi(z_n))$$

Now define

$$\widetilde{\phi}_0(z) = \phi_0(z) - l \Longrightarrow \liminf_{z \to \gamma_1}(\widetilde{\phi}_0 - \phi) = 0$$

Thus, by hypothesis of $l < 0$ and by definition of $\liminf$, we have

$$\lim_{z \to \gamma_1}(\widetilde{\phi}_0 - \phi) \geq 0$$
$$\lim_{z \to \gamma_2}(\widetilde{\phi}_0 - \phi) \geq 0$$

Then the comparison principle yields

$$\widetilde{\phi}_0 - \phi \geq 0 \quad \text{on } \overline{\Omega}$$

On the other hand, if y is close to z_n, we can write

$$\widetilde{\phi}_0(y) - \phi(y) = \widetilde{\phi}_0(z_n) - \phi(z_n) + \left(\frac{\partial \phi}{\partial s}(z_n) - \frac{\partial \widetilde{\phi}_0}{\partial s}(z_n) \right) \sigma + o(\sigma)$$

where the first difference goes to zero in n while the second goes to $-\infty$. Thus there are point where $\widetilde{\phi}_0 - \phi$ is negative, which is a contradiction. □

We can now show an analogous to theorem 1 in [9]. For $0 < a < b$ we define $\Omega(a,b) = \{z \in \mathbb{D} : a \leq |z|_{\mathbb{H}^2} \leq b\}$.

Theorem 10.
Let be $0 < h < \frac{1}{2}$ *and consider* $u : \Omega(a,b) \to \mathbb{R}$ *a smooth function such that*

$$\frac{1}{2}\, div \left(\frac{\nabla^{\mathbb{H}^2} u}{\sqrt{1 + |\nabla^{\mathbb{H}^2} u|_{\mathbb{H}^2}}} \right) = h$$

Assume u *satisfies*

$$m \leq u_{|\,\{|z|=b\}} \leq M$$

Then $\exists\, \beta \in (2h, +\infty)$ *such that* $\forall z \in \Omega(a,b)$ *it holds*

$$-H^h_\beta(z) + H^h_\beta(b) - m \leq u(z) \leq H^h_\beta(z) - H^h_\beta(b) + M \tag{4.1}$$

Proof. The β which will do the work is determined by a: we consider the H^h_β with base circle equal to the inner boundary $\{|z| = a\}$. We define

$$\beta(a) = 2h\, \cosh(a) - \sqrt{\cosh(a)^2 - 1}$$

To prove inequalities (4.1) we show that they hold on every circular annulus $\Omega(a^*, b)$ for $a < a^*$. Let's start with the inequality on the right. We define

$$\phi_0(x,y) = H^h_{\beta(a^*)}(x,y) - H^h_{\beta(a^*)}(b) + M$$

so we obtain

$$\phi_{0|\{|z|=b\}} = M \geq \phi_{\{|z|=b\}}$$

To have the same inequality on $\{|z| = a^*\}$ we observe that $H^h_{\beta(a^*)}$ is vertical on $\{|z| = a^*\}$ and nearby this boundary negative, hence

$$lim_{z \to \{|z|=a^*\}} \frac{\partial H^h_\beta}{\partial s}(z) = +\infty$$

Moreover if Q is the mean curvature operator, we have

$$Q(\phi_0) = h = Q(\phi)$$

and thus the lemma implies

$$\phi_0(a^*) \geq \phi_{|\{|z|=a^*\}}$$

Iterating the maximum principle argument we get the inequality

$$u(z) \leq \phi_o(z) \quad \text{on } \Omega(a^*, b)$$

To prove the left inequality we proceed in the same way. If

$$\phi_1(z) = -H^h_{\beta(a^*)}(z) + H^h_{\beta(a^*)} + m$$

we are able to prove

$$\phi_1(z) \leq u(z) \quad \text{ on } \ \Omega(a^*, b)$$

□

Figure 4.1 shows a vertical section of the bounding box we have just built.

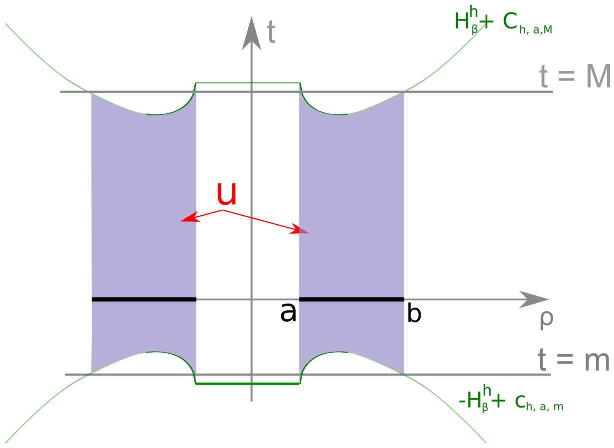

Figure 4.1: Part of the bounding box given in lemma 10. Vertical section.

It is easy to derive the following corollary stating non existence of a solution of the constant mean curvature problem on an hyperbolic circular annulus.

Corollary.
Let $\Omega(a, b)$ be a circular annulus in $\mathbb{D}$. Let be $\beta = \beta(a)$. Then $\forall \varepsilon > 0$ the following Dirichlet problem does not have a solution

$$\begin{cases} \frac{1}{2}\, div\left(\dfrac{\nabla^{\mathbb{H}^2}\phi}{\sqrt{1+|\nabla^{\mathbb{H}^2}\phi|_{\mathbb{H}^2}}}\right) & = h & \text{on } \Omega(a,b) \\ \phi & = c > 0 & \text{on } \{|z| = b\} \\ \phi & = -H^h_\beta(a) + H^h_\beta(b) - m - \varepsilon & \text{on } \{|z| = a\} \end{cases}$$

Proof. Define $\gamma_1 = \{|z| = a\}$. Since rays emanating from $0 \in \mathbb{H}^2$ are geodesics, each point $z \in \gamma_1$ is the end point of at least one geodesics in $\Omega(a, b)$. Thus $\Omega(a, b)$ is a domain where the theorem can be applied. Thus any solution of the differential equation associated to the Dirichlet problem is in the bounding box given by the inequalities (4.1) which does not depend on ε. □

Roughly speaking, the estimate we have just established means that h-graphs on annular domains cannot grow much faster than the rotational ones. Unfortunately this remark cannot be made more precise than that, and hence has to be taken as a (hopefully) suggestive interpretation of what we have just proven. We also remark that the Euclidean version of these estimates has a physical counterpart. It is known that minimal surfaces model soap bubbles (a pleasant reading on this subject is [18]) and it is a fact of everyday life that a laminar soap bubble bounded by two planar and concentric circles will blow if the two circles are too much departed along the axis through the centers.

4.2 A priori estimates for h-graphs on annuli

In this section we give the a priori estimates of $h-$graphs that will be a crucial step in the proof of existence of a such a graph, which will be presented in the next section. These estimates are given under geometrical hypothesis on the boundary of the domain and on boundary data. On one side, the $\{H^h_\alpha\}_\alpha$ family has shown to give constraints on the graphs one can build, on the other side they can be used as a building block for non rotational graphs. Here, again, the Poincaré model of $\mathbb{H}^2$ is used. In what follows, if $\Omega \subset \mathbb{D}$ is a compact annulus, we write $\partial\Omega = \gamma_1 \bigsqcup \gamma_2$ and we assume that γ_1 is contained in the compact domain bounded by γ_2.
Before stating the result, we recall that if γ is an $r-$admissible curve, see definition 47, we associate to it an $\alpha(\gamma) = \alpha(r) \in (0, 2h)$ and a $\beta(\gamma) = \beta(r) \in (2h, +\infty)$ such that:

- $\rho^h(\beta) = r$, where ρ^h_β is defined in (2.8)
- $H^h_\alpha < H^h_\beta$
- For each $z \in \gamma$ $\exists\, \tau_z(H^h_\beta)$ an horizontal translation of H^h_β such that $\tau_z\left(H^h_\beta\right) < H^h_\alpha$ and $\tau_z\left(H^h_\beta\right)_{|\gamma} < 0$

Theorem 11.
Let $\Omega \subset \mathbb{D}$ a compact smooth annulus and write . Assume γ_1 is $r-$ admissible, h-convex and γ_2 is a circle. If $h \in (0, \frac{1}{2})$ and $u \in C^{2,\delta}(\Omega)$ is a solution of the Dirichlet problem

$$\begin{cases} \frac{1}{2} div \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) &= h \quad in\ \Omega \\ u &= 0 \quad in\ \gamma_1 \\ u &= H^h_{\alpha(r)} \quad in\ \gamma_2 \end{cases}$$

then $\exists\, C > 0$ such that

$$||u||_{C^{2,\delta}(\Omega)} \leq C \tag{4.2}$$

and $C = C(h, \gamma_1)$.

Proof. By mean of item 1 and 2 of theorem 15 of the apendix, we reduce to C^1 estimates.

As a consequence we have to give estimates of u, boundary gradient estimates of u and interior gradient estimates of u.
Since $H^h_\alpha > 0$, we will call respectively γ_1 and $H^h_{\alpha|\gamma_2}$ the lower and the upper boundary of the graph of u. Before going through the proof we state that in this proof by H^h_α we mean a small negative vertical translation of H^h_α and byH^h_β we mean a small positive vertical translation of H^h_β. More precisely making an abuse of language we will call H^h_α a the surface $H^h_\alpha - \varepsilon$ with $0 < \varepsilon << 1$ and we will call H^h_β the surface $H^h_\beta + \varepsilon$ with $0 < \varepsilon << 1$. This is done because we are using these surfaces as barriers for the gradient of the solution along the boundary and hence we need the normal derivative to be finite.

- C^0 *estimates*
  We use the maximum principle.
  Let's prove that u is below H^h_α:

  $$\begin{aligned} u_{|\gamma_1} &= 0 < H^h_{\alpha|\gamma_1} && \text{being } H^h_\alpha \text{ positive on } \gamma_1 \\ u_{|\gamma_2} &= H^h_{\alpha|\gamma_2} && \text{by hypothesis} \\ Q(u) &= 2h = Q(H^h_\alpha) \end{aligned}$$

  and hence $u \leq H^h_\alpha$ all over Ω.

Let's prove that u is bigger than H^h_β:

$$\begin{aligned} u_{|\gamma_1} &= 0 > H^h_{\beta_{|\gamma_1}} && \text{by r-admissibility hypothesis} \\ u_{|\gamma_2} &= H^h_{\alpha|\gamma_2} > H^h_{\beta_{|\gamma_2}} && \text{by the hierarchy (theorem 9)} \\ Q(u) &= 2h = Q(H^h_\beta) \end{aligned}$$

and hence $u \geq H^h_\beta$ all over Ω.
We have proved that the graph of u is in the compact region contained between the graph of H^h_α and the graph of H^h_β.

- *C^1 estimates along $\partial\,\Omega$*
We will prove four estimates. Some will be done using H^h_α and H^h_β as barriers, others will be done using the same technique proposed in the chapter 14 of [12], i. e. we will make barriers using distance function.

C^1 estimates along γ_1
This is maybe on the of the most difficult estimate to establish. Actually the r-admissibility concept has been introduced to carry out this very estimate. We are building an upper barrier bending a distance function and we are using the H^h_β as a lower barrier. We observe that, even if the construction of the upper barrier is technically non trivial, the difficult point is the construction of the lower barrier. Consider $z \in \Omega$ and define $d(z) = d_{\mathbb{H}^2}(z, \gamma_1)$.

Consider $a \in \mathbb{R}^+$ and $\overline{V}_a$ the part of the tubular neighborhood of γ_1 of thickness a contained in Ω. Consider ψ a real C^2 function such that $\psi(0) = 0.$ and $\psi' > 0$. Let be $k \leq 0$ and $\nu \in \mathbb{R}$ and define

$$w^+(p) = (\psi \circ d)() = \frac{\arctan\left(e^{-k}\sqrt{e^{2\,c\,d} - e^{2\,k}}\right)}{c} + \nu \tag{4.3}$$

where

$$c = c(\gamma_1) = \max_{y_0\,\in\,\gamma_1} k_g(y_0) \tag{4.4}$$

We now show that we can choose ν and k so that $w^+$is a global upper barrier, i. e. it can be used as a barrier for u on γ_1. We are going to do that in two steps. First of all we show that the mean curvature of the graph of $\psi \circ d$ is non positive and then we will choose the parameters so that boundary conditions to have a barrier are fulfilled.

– In order for w^+ to be a barrier we need that

$$Q(w^+) \leq Q(u) = 2h \quad \text{in } \overline{V}_a$$

where Q is the mean curvature operator. Using (1.20) we get $\forall\, p \in \overline{V}_a$

$$\frac{1}{2}\, Q(w^+)(p) = \frac{1}{\sqrt{1+\psi'^2}} \Big(\psi'\, k_g(p) + \frac{\psi''}{1+\psi'^2} \Big)$$

where k_g is the geodesic curvature of the curve $\{q \in \Omega : d(q) = d(p)\}$ calculated with respect to the unit normal vector $\eta = -\nabla d$. To go on, we need to use the $h-$convexity of γ_1. According to proposition 38 we can write

$$Q(w^+)(q) \leq \frac{1}{\sqrt{1+\psi'^2}} \left(\psi'\, k_g(p) + \frac{\psi''}{1+\psi'^2} \right)$$

where $q = \varphi_p(t)$ for a unique $p \in \gamma_1$ and $0 \leq t \leq a$ ($\varphi_p(t)$ is the flow of d). Thus we get

$$Q(w^+)(q) \leq 0 \quad \Longleftrightarrow \quad c\,\psi' + \frac{\psi''}{1+\psi'^2} \leq 0 \tag{4.5}$$

To have an upper barrier, it is enough to find a solution of the differential equation associated to the differential inequality just written. We proceed separating variables and we do the following change of variables

$$\begin{cases} y = \psi' \\ c\,y + \dfrac{y'}{1+y^2} = 0 \end{cases}$$

The hypothesis $\psi' > 0$ yields

$$y(d) = \frac{e^k}{\sqrt{e^{2\,cd} - e^{2k}}}$$

where we impose $k \leq 0$ to have the solution defined $\forall\, d \geq 0$. We still have to solve

$$\widetilde{\psi}(d) = \int_0^d \frac{ds}{\sqrt{\frac{e^{2\,cs}}{e^{2k}} - 1}}$$

making the change of variable $x = \frac{e^{cs}}{e^k}$ and recalling that

$$\left(\arctan\left(\sqrt{x^2-1}\right)\right)' = \frac{1}{x\sqrt{x^2-1}}$$

we get

$$\widetilde{\psi}(d) = \frac{1}{c}\arctan\left(\frac{e^{-k}}{\sqrt{e^{2cd}-e^{2k}}}\right) + \nu$$

Thus we have found a two parameter family of surfaces with non positive mean curvature which are monotone increasing in the parameter k.

– Boundary conditions for w^+ to be an upper barrier along γ_1 write as

  ∗ ${w^+}_{|\gamma_1} \geq u_{|\gamma_1}$
  ∗ ${w^+}_{|\gamma_1} \geq u_{|\gamma_1}$

  We use ν to fulfill the first condition

$$\nu = -\widetilde{\psi}(0)$$

  and we define

$$w^+(d) = \psi(d) = \widetilde{\psi}(d) - \widetilde{\psi}(0)$$

  The first boundary condition for an upper barrier is satisfied because

$${w^+}_{|\gamma_1} = 0 = u_{|\gamma_1}$$

  Being $u \leq H^h_\alpha$,to fulfill the second condition we choose

$$w^+(a) \geq M \tag{4.6}$$

  where $M = \max_{\overline{V}_a} H^h_\alpha$.
  If we take k big in absolute value, we are done. It is because the argument of *arctanh* in (4.3) is a product of two terms one of which is positively divergent for $k \longrightarrow -\infty$ while the other converges to the positive constant e^{cd}.

We now prove the estimate from below. This will be done using H^h_β as a punctual lower barrier.
Take $p \in \gamma_1$. By $r-$admissibility hypothesis on γ_1 follows the existence of an horizontal translation $\tau_z\left(H^h_\beta\right)$ of H^h_β tangent to γ_1 in z with

$\tau_z\left(H^h_\beta\right) \leq H^h_\alpha$. This hypothesis implies as well that $H^h_{\beta_{\gamma_1}} < 0$. We have proven that

$$\tau_z\left(H^h_\beta\right)_{|\gamma_1} < 0 = u_{|\gamma_1}$$
$$\tau_z\left(H^h_\beta\right)_{|\gamma_2} < H^h_{\alpha|\gamma_2} = u_{|\gamma_2}$$
$$Q(\tau_z\left(H^h_\beta\right)) = 2h = Q(H^h_\alpha) \qquad \text{in } \Omega$$

hence, by the maximum principle, we have

$$\tau_z\left(H^h_\beta\right) \leq u \qquad \text{in } \Omega$$

but being $\tau_z\left(H^h_\beta\right)(p) = 0 = u(z)$ we have given a lower bound for the normal derivative of u in p. From the compactness of Ω together with the regularity of u we get a bound for $||\nabla u||_{C^0(\gamma_1)}$.

C^1 estimates on γ_2
These estimates follow directly from our construction.
To give a lower barrier we observe that, by the maximum principle, u is below the minimal slice $\Omega \times \{height(H^h_\alpha(\gamma_2))\}$, where *height* is the projection on the third component of $\mathbb{H}^2 \times \mathbb{R}$. Hence u assumes its global maximum on γ_2 and so here the normal derivative is non negative (calculated with respect to the exterior normal to γ_2) because if this derivative had been negative, the function would have had to grow when considering internal points close to γ_2.
On the other hand, H^h_α is clearly a global upper barrier for the normal derivative if u on γ_2.

- *C^1 estimates in $int(\Omega)$*
  These estimates are a consequence of Theorem 3.1 of Spruck's work [49]. Actually this theorem gives an interior estimate for ∇u in terms of the C^0 estimate of u and the C^1 norm on the boundary, provided $\partial\Omega$ is C^3.

All estimates depending only translations of H^h_α and H^h_β depends only on h and γ_1 because these two parameters are enough to determine H^h_α and H^h_β. The only estimate which is not made using these two surfaces is the one made via $\psi \circ d$ on γ_1. It only depends on γ because w^+ depends on the three parameters c, ν and k, where

- c is the maximum of the geodesic curvature of γ_1
- k depends only on H^h_α, as seen in (4.6).

- ν depends on c and k

□

In figure 4.2 we sketch how we carry out the height estimates

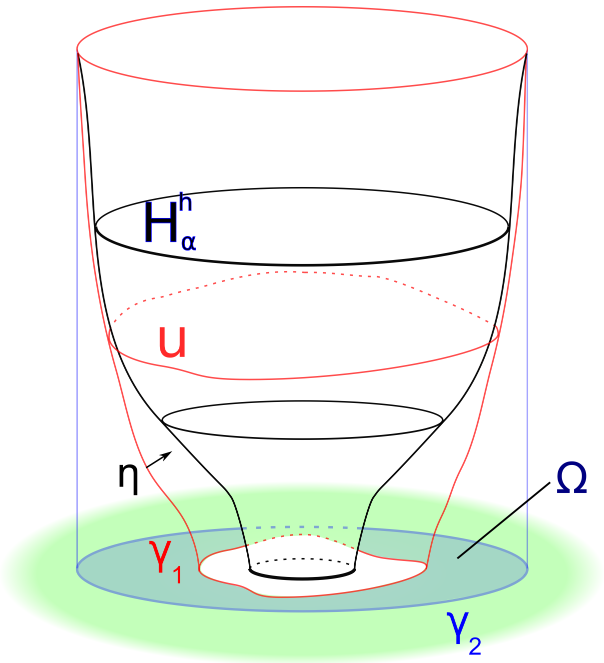

Figure 4.2: A priori estimates

Remark.
The C^0 estimate from below could have been done using S^h. Consider a negative vertical translation of S^h whose intersection with the plane $\{t = 0\}$ bounds a disc containing γ_2. Then the u is greater than this translation. Indeed:

$$u_{\gamma_1} \geq S^h \quad \text{because here } S^h \text{ is negative and } u \text{ is zero}$$
$$u_{\gamma_2} \geq S^h \quad \text{because we can assume the translation of } S^h \text{ is below } H^h_\alpha$$
$$Q(u) = 2h = (S^h)$$

4.3 A modified method of continuity

We give a proof of a deformation theorem for quasilinear operators. This result is standard but cannot be found in the literature, hence we propose a proof. This result tells that, if we have a solution of a Dirichlet problem, we can solve any problem obtained by a deformation of the boundary data, provided the a priori estimates along the deformation are uniformly bounded. Since the proof is after the ones given in [12, Chapter 17, Section 2], we will use the same language and notation. The theorem works for fully non linear operator, hence we allow the coefficients of the second order terms of the operator to depend on ∇^2.

The reason for which we need to prove such a result is that the deformations theorems proposed in [12, Chapter 17, Section 2] consider a family of operators along the deformation which depends linearly on the parameter of the deformation. We will need these operators to depend on the parameter in a more general way.

Theorem 12.
Consider $\Omega \subset \mathbb{R}^2$ a compact set and $\mathcal{U} \subset C^{2,\delta}(\Omega)$ an open set. Consider a second order Frechet differentiable operator

$$\begin{array}{rccc} F: & \mathcal{U} \times [0,1] & \longrightarrow & C^{0,\alpha}(\Omega) \\ & (u,\sigma) & \mapsto & F^\sigma(u) \end{array}$$

Assume the Dirichlet problem

$$\begin{cases} F(u,1) = 0 & in \quad \Omega \\ u = \phi & in \ \partial\Omega \quad for \quad \phi \in C^{2,\delta}(\partial\Omega) \end{cases}$$

has solution in $C^{2,\delta}(\Omega)$.
If

i) F^σ is strictly elliptic $\forall\, \sigma \in [0,1]$

ii) $F_z^\sigma \leq 0 \quad \forall\, \sigma \in [0,1]$, where F_z is the partial derivative of F

iii) $E = \{u \in \mathcal{U} \,/\, \exists\, \sigma \in [0,1] : F(u,\sigma) = 0 \quad with \quad u_{|\partial\Omega} = \varphi\}$ is bounded

iv) $\bar{E} \subset \mathcal{U}$

Then the Dirichlet problem

$$\begin{cases} F(u,0) = 0 & in \quad \Omega \\ u = \phi & in \quad \partial\Omega \end{cases}$$

has solution in $C^{2,\delta}(\Omega)$.

Proof. We are going to show that the set

$$S = \{\sigma \in [0,1] \,/\, \exists\, u \in \mathcal{U} : F(u,\sigma) = 0 \quad \text{with} \quad u_{|\partial\Omega} = \varphi\} \tag{4.7}$$

is non empty, open and closed, and hence it coincides with $[0,1]$.
Clearly hypothesis i implies that S is non empty.
We now prove that S is open. Let $\sigma_0 \in S$ and u^0 such that $F(u^0, \sigma_0) = 0$. If the Frechet derivative of F with respect to z is invertible, we can apply the local invertibility theorem [12, Chapter 17, Theorem 6] and obtain that S is open. Following the notation of [12] we write $F^1_{(u_0,\sigma_0)}$ for this derivative evaluated in (u_0, σ_0). Thus we have

$$F^1_{(u_0,\sigma_0)}(h) = F_{ij}(x;\sigma_0)\,\partial_{ij}h + b^i(x;\sigma_0)\,\partial_i h + c(x;\sigma_0)$$

where

$$\begin{aligned} F_{ij}(x;\sigma_0) &= F_{ij}(x, u_0(x), Du(x_0), D^2u(x_0); \sigma_0) \\ b^i(x;\sigma_0) &= F_{p_i}(x, u_0(x), Du(x_0), D^2u(x_0); \sigma_0) \\ c(x;\sigma_0) &= F_z(x, u_0(x), Du(x_0), D^2u(x_0); \sigma_0) \end{aligned}$$

Thus in order to have invertibility of F^1 in (u^0, σ_0) we have to solve a linear Dirichlet problem. This is done using the theorem of existence and uniqueness of the solution for the linear case [12, Chapter 6, Theorem 14].
If we take the regularity of the coefficients for granted, sufficient hypothesis for solving a linear problem are: strict ellipticity of the operator and negative zero order term. Since these hypotheses are assumed by ii) and iii), we have the invertibility of the derivative. Hence the local invertibility theorem [12, Chapter 17, Theorem 6] yields V a neighborhood of σ_0 in S where $F(u,\sigma) = 0$ is solvable, and so S is open.

We now show that S is closed. It is a consequence of the boundedness of E Ascoli-Arzelà theorem. This theorem establishes that E is compact. Indeed clearly $\overline{E}$ is compact. But E is closed as well. To see that, take $\{u_n\} \subset E$ and $u_0 \in \overline{E}$ such that $u_n \longrightarrow u_0$. We check that $u_0 \in E$. Being $u_n \in E$, it exists $\{\sigma_n\} \subset [0,1]$ such that $F(u_n, \sigma_n) = 0$. From the compactness of $[0,1]$ we deduce the existence of a converging subsequence $\sigma_n \longrightarrow \sigma_0 \in [0,1]$. Hence we have

$$\begin{aligned} \lim_{n\to+\infty} (u_n, \sigma_n) &= (u_0, \sigma_0) \\ 0 = \lim_{n\to+\infty} F(u_n, \sigma_n) &= F(u_0, \sigma_0) \quad \text{being } F \text{ continuous} \end{aligned}$$

which implies $u_0 \in E$. We still have to see that S is closed. We proceed in the same way: we take $\sigma_n \to \sigma_0 \in [0,1]$, and we prove that $\sigma_0 \in S$. By definition of S $\forall n \, \exists \, u_n \in E$ such that $F(u_n, \sigma_n) = 0$. Being E compact it exists a converging subsequence $u_n \to u \in E$. As in the preceding case this yields $\sigma_0 \in S$. □

4.4 Existence of h-graphs on annuli

In this section we prove two existence theorems. First of all we will prove an existence theorem for $h-$graph on compact annuli and after that we will prove an existence result on non compact annuli, more precisely on complement of compact domains of $\mathbb{H}^2$. The technique we use is completely new, and it is based on the idea of deforming the solutions H_α^h described in the previous chapters. Indeed the only analogous existing result is due to [33] and refer to the case $h = \frac{1}{2}$, but it is largely based on the fact that in that case all the functions H_α^h have a different behavior at infinity. Since we have seen that for $h \in (0, \frac{1}{2})$ all the solutions H_α^h have the same behavior at first order at infinity, a new approach to the problem is needed.

The first theorem is proved by means of a deformation argument shown in theorem 12, and the estimate established in the preceding section. The second theorem will easily follow from the existence on compact annuli.
As in the preceding section we take $h \in (0, \frac{1}{2})$ and we refer to the Poincaré model of the hyperbolic plane. As before, in the compact case we consider Ω a smooth compact annulus in $\mathbb{H}^2$ whose inner and outer boundaries are respectively γ_1 and γ_2.

Theorem 13.
Let $h \in (0, \frac{1}{2})$ and $r > 0$. Let $\Omega \subset \mathbb{H}^2$ be a compact annulus.
Assume γ_1 is r-admissible, $h-$convex and that γ_1 can be smoothly deformed into a circle, with $r-$admissibility and $h-$convexity preserved along the deformation.
Assume γ_2 is a circle.
Then the Dirichlet problem

$$\begin{cases} \dfrac{1}{2}\, div\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = h & in\ \Omega \\ u = 0 & in\ \gamma_1 \\ u = H^h_{\alpha(r)} & in\ \gamma_2 \end{cases} \tag{D}$$

has a solution $u \in C^{2,\delta}(\Omega)$.

Proof. This proof is done by the deformation method proved in theorem 12. Hence we need to define a family of Dirichlet problems linking our problem to one whose solution is known to exist. After that we will need to prove that this path in $C^{2,\delta}(\Omega)$ is bounded.
First of all we define the operator $F(\sigma, u)$. If by D_2 we mean the compact domain bounded by γ_2, we are going to show that we can deform $H^h_{\alpha|D_2\cap(B_0(\rho^h(\alpha))')}$ to a function on Ω preserving the mean curvature and without loosing any regularity. More precisely, what we are going to do is to deform γ_1 to the base circle of H^h_α, i. e. $S_0(\rho) = S_0(\rho^h_\alpha) = \{z \in \mathbb{D} : |z|_{\mathbb{H}^2} = \rho^h(\alpha)\}$. Without loss of generality we can suppose that the deformation assumed in the hypothesis is parametrized by $\sigma \in [0,1]$ and we write γ_1^σ for the evolution at time σ of γ_1. We also assume $\gamma_1^0 = \gamma_1$ and $\gamma_1^1 = S_0(\rho)$.
We define A^σ to be the annulus whose inner and outer boundaries are respectively γ_1^σ and γ_2. Here we could already define the family of Dirichlet problems linking problem (D) to the one whose solution is H^h_α. But since in theorem 12 all the problems have the same domain, we need a small technical effort to give a family of Dirichlet problems defined in the same domain. To do that, consider $A \subset \mathbb{R}^2$ a compact annulus with inner boundary a_1 and outer boundary a_2. Let $\{\phi^\sigma\}_{\sigma\in[0,1]}$ be a family of smooth orientation preserving embedding of A in $\mathbb{D}$ such that $\phi^\sigma(A) = A^\sigma$. The compactness of $[0,1]$ together with the smoothness of the family $\{\phi^\sigma\}$ yields the existence of $0 < C$ which does not depend on σ such that:

$$\|\varphi_\sigma\|_{C^{2,\delta}(A)} < C \tag{4.8}$$

We now define the operator $F(\sigma, u)$ to do the deformation argument.

$$F(u,\sigma) = Q(u \circ \phi^\sigma) - 2h$$

This definition and the remark on the $C^{2,\delta}$ norm of ϕ^σ imply that an estimate for $\|u\|_{C^{2,\delta}(A^\sigma)}$ is equivalent to an estimate of $\|u\circ\phi^\sigma\|_{C^{2,\delta}(A)}$. Hence we can define the family of Dirichlet problems. If we write $\forall\,\sigma \in [0,1] \quad u^\sigma = u\circ\phi^\sigma$, we introduce

$$\begin{cases} F(u^\sigma,\sigma) & = 0 & \text{in } A \\ u^\sigma & = 0 & \text{in } a_1 \\ u^\sigma & = H^h_\alpha & \text{in } a_2 \end{cases} \tag{D^σ}$$

We are going to show that, for each $\sigma \in [0,1]$, we can apply theorem 11 and get an a priori estimate not depending on σ. We observe that the preservation of $h-$convexity and $r-$admissibility along the deformation assumed in the hypothesis, allows to use the estimates of our preceding result. Clearly all the estimates given in terms of H^h_α and H^h_β do not depend on σ because neither H^h_α nor H^h_β do. Hence the only estimate that could depend on σ is the estimate of ∇u^σ on γ_1^σ because, as shown by (4.4) and (4.3), the barrier depends on the geodesic curvature of γ_1^σ. But to remove the dependence on σ we use the compactness of $[0,1]$ and the smoothness dependence of γ_1^σ on σ to redefine c in (4.3) to be

$$c = \max_{\substack{\sigma \in [0,1] \\ y_0 \in \gamma_1^\sigma}} k_g(\gamma_1^\sigma)(y_0)$$

which depends only on γ and h.

□

With this theorem proved we can state the existence result on exterior domains, which is the main geometrical result of this work. We are going to prove the existence of non rotational constant mean curvature h vertical ends. Figure 4.3 sketches the output of the result we are going to prove.

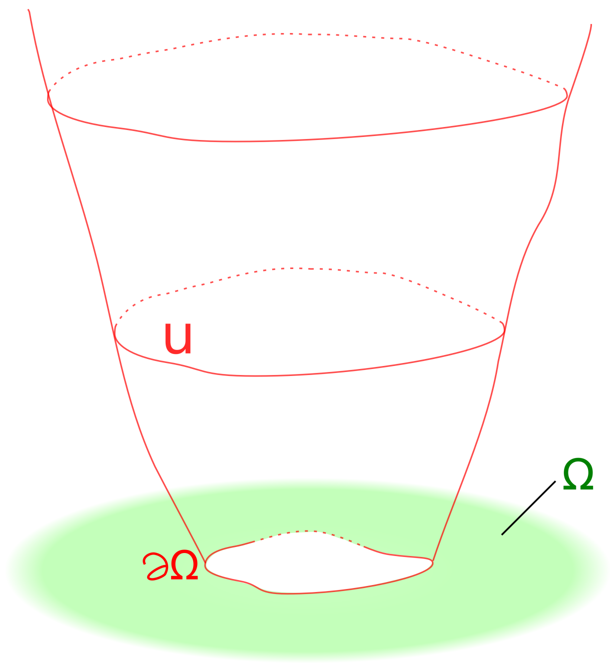

Figure 4.3: Constant mean curvature graph on an exterior domain

Remark.

- Recalling the non existence theorem 10, it should be clear why we introduced the $r-$admissibility definition 47. Indeed it is one of the two sufficient conditions for a curve to be the boundary of a vertical end of curvature $h \in (0, \frac{1}{2})$.

- An example of a curve satisfing all the hypotheses of the theorem is a small C^2 deformation of an hyperbolic circle. Indeed corollary 3.1 shows that we can use constant volume mean curvature flow to deform the curve into a circle without loosing any regularity nor $r-$admissibility along the flow.

Theorem 14.
Fix $h \in (0, \frac{1}{2})$. Let Ω be the complement of a compact domain of $\mathbb{H}^2$ with boundary a Jordan smooth curve γ_1. Assume γ_1 is r-admissible, $h-$convex and that γ can be smoothly deformed into a circle, with $r-$admissibility and $h-$convexity preserved along the deformation.
Then the following Dirichlet problem

$$\begin{cases} \frac{1}{2}\, div\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) & = & h & in\ \Omega \\ u & = & 0 & in\ \gamma_1 \\ \lim_{|z|_{\mathbb{H}^2}\to+\infty} u(z) & = & +\infty & \end{cases} \tag{E}$$

has a solution $u \in C^{2,\delta}(\Omega)$.

Proof. The proof consists in considering a sequence of compact annuli Ω_n converging to Ω, build a sequence of $C^{2,\delta}(\Omega)$ solutions by mean of theorem 13 and prove convergence in $C^{2,\delta}(\Omega)$.
To accomplish the first step we consider $\{\rho_n\}_{n\in\mathbb{N}}$ a sequence of positive reals monotonically diverging to $+\infty$. We define γ_n to be the circle $S_0(\rho_n)$. Then we define Ω_n to be the annulus whose inner and outer boundary are respectively γ_1 and γ_n. We introduce the Dirichlet problem

$$\begin{cases} div\left(\dfrac{\nabla u_n}{\sqrt{1+|\nabla u|^2}}\right) & = 2\,h & \text{in } \Omega_n \\ u_n & = 0 & \text{in } \gamma_1 \\ u_n & = H^h_\alpha & \text{in } \gamma_n \end{cases}$$

Because of theorem 13 this sequence exists, i.e. $\forall\, n \in \mathbb{N}$ we have $u_n \in C^{2,\delta}(\Omega_n)$ an $h-$graph on Ω_n. Moreover theorem 11 gives an estimate for

$\|u_n\|_{C^{2,\delta}(\Omega)}$ which does not depend on n. To see that assume we have

$$\|u_n\|_{C^{2,\delta}(\Omega_n)} < C$$

with C independent from n. Then we can consider an extension of u_n to Ω with $C^{2,\delta}$ norm proportional to the one of u_n. This is a well known result in a priori estimates that can be found in [12, Chapter 6, Lemma 37]. So if we prove the independence on n of the estimates given by theorem 11 we are done. In this case the independence from the parameter n is even simpler than in the compact case. Indeed we recall that the bound given in theorem 11 depend only on γ_1 and h.
Hence $\{u_n\}_n$ is bounded and equicontinuous sequence in $C^{2,\delta}(\Omega)$. By Ascoli - Arzelà theorem, up to passing to a subsequence, we have the convergence in $C^{2,\delta}(\Omega)$. □

Chapter 5

Appendix

In this appendix we recall some Dirichlet problem theory and we will specialize it to the $\mathbb{H}^2 \times \mathbb{R}$ space. In the the first section we recall some of the standard PDE theory for Dirichlet Problem, in the second we write the mean curvature operator in the hyperbolic setting. We will deal with classical Dirichlet problems, i.e. all derivatives are strong.

We begin with recalling the functional spaces which show to be natural when dealing with classical Dirichlet problem, Hölder differentiable functions. Before going through the material we recall some notation of multi-variables calculus. Assume $U \subset \mathbb{R}^2$ is an open domain described by coordinates $z = (x_1, x_2)$ and $f \in C^\infty(\Omega)$. If $h = (h_1, h_2) \in \mathbb{N}^2$ is a multi-index, we write

$$|h| = h_1 + h_2$$
$$\frac{\partial^h f}{\partial p^h} = \frac{\partial^{h_1} \partial^{h_2} f}{\partial {x_1}^{h_1} \partial {x_2}^{h_2}}$$

Moreover we will use $\nabla^2 u$ to mean the set of second derivatives of u. With this notation we can define Hölder continuous functions.

Definition 48. Hölder spaces
Consider $\Omega \subset \mathbb{R}^2$ a compact domain. Take $0 < \delta < 1$ and define

$$C^{k,\delta}(\Omega) = \Big\{ u \in C^k(\Omega) : \\ \forall\, z, w \in \Omega, \forall\, |j| \le k \quad \left| \frac{\partial^j u}{\partial x^j}(z) - \frac{\partial^j u}{\partial x^j}(w) \right| \le C\, |z - w|^\delta \Big\}$$

for some $0 < C$ independent from z, w.

Moreover on this space we define

$$\text{the semi-norm} \quad ||u||_{k,\delta} = \sup_{z\neq w} \frac{\left|\dfrac{\partial^j u}{\partial x^j}(z) - \dfrac{\partial^j u}{\partial x^j}(w)\right|}{|z-w|^\delta}$$

$$\text{the norm} \quad ||u||_{C^{k,\delta}(\Omega)} = \sum_{h=0}^{k} ||u||_{h,\delta}$$

We recall that all the $C^{k,\delta}(\Omega)$ spaces are Banach spaces when equipped with the norm just defined. Since the mean curvature operator involves derivtives of order at most two, we recall the definition of second order operator.

Definition 49. Second order elliptic quasi-linear operator
Consider $u \in C^{2,\delta}(\Omega)$ and a second order quasi-linear operator on $C^{2,\delta}(\Omega)$ is a function $Q : C^{2,\delta}(\Omega) \to C^{0,\delta}(\Omega)$ such that

$$Q(u) = \sum_{i,j}^{2} a_{ij}(x,u,\nabla u) \frac{\partial^2 u}{\partial x^i\, \partial x^j} + b(x,u,\nabla u) \tag{5.1}$$

where

$$a_{ij} : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$$
$$b : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$$

are $C^{0,\delta}(\Omega)$ functions.
If $\exists\, 0 < l \leq L \in \mathbb{R}$ such that for each $(x,z,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^2$ and $v \in \mathbb{R}^2$ we have

$$l\,|v|_{\mathbb{R}^2} \leq \langle A(x,z,p)\, v, v\rangle_{\mathbb{R}^2} \leq L|v|_{\mathbb{R}^2}$$

Moreover Q is *uniformly elliptic* if L/l is bounded in Ω.

It is well known that prescribing boundary conditions is necessary in order to have uniqueness of solutions of differential equations. Hence we recall the definition of Dirichlet problem which prescribes the value of the solution on the boundary.

Definition 50. Dirichlet problem
Let $0 < \delta < 1$ and consider $\Omega \subset \mathbb{R}^2$ a compact domain of class $C^{2,\delta}$. Consider Q a quasilinear elliptic operator and $\phi \in \mathbb{C}^0(\partial\Omega)$ a function. We call *Dirichlet problem* the problem of finding a $C^2(\Omega)$ function $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} Q(u) &= 0 \quad \text{in } \Omega \\ u &= \phi \quad \text{in } \partial\Omega \end{cases} \tag{P}$$

The theory of classical Dirichlet problem is enormous. A study of the elliptic case can be found in [12]. Here we will summarize, without proof, some facts.

Theorem 15.
Consider the Dirichlet problem (P). *Then*

- *Maximum principle:*

  If Q is locally uniformly elliptic, a_{ij} are independent from $u(x)$, b is non increasing in $u(x)$, a_{ij} are C^1

  if $u, v \in C^{2,\delta}(\Omega)$ satisfy

  $u \leq v$ on $\partial\Omega$

  $Q(u) \geq Q(v)$ on Ω

  Then we have

$$u \leq v \quad \text{in } \Omega$$

- *Schauder Estimates I:*
  Let $u \in C^{2,\delta}(\Omega)$ be a solution of (P). *Then, being Q quasi-linear and elliptic, we have*

$$||\nabla u||_{C^0(\Omega)} \leq \widetilde{C}_1 \Longrightarrow ||\nabla u||_{C^{0,\delta}(\Omega)} \leq \widetilde{C}$$

- *Schauder Estimates II:*
  Let $u \in C^{2,\delta}(\Omega)$ be a solution of (P). *Then, being Q quasi-linear and elliptic, we have*

$$||u||_{C^0(\Omega)} + ||\nabla u||_{C^0(\Omega)} + ||\nabla u||_{C^{0,\delta}(\Omega)} < C_1 \Longrightarrow ||u||_{C^{2,\delta}(\Omega)} \leq C$$

The proof of these facts can be found respectively in [12, Chapter 10, Theorem 1], [12, Chapter 6, Lemma 6] and [12, Chapter 11, Theorem 4], [12, Chapter 11].

Bibliography

[1] J. W. Anderson, *Hyperbolic geometry*, second ed., Springer, 2005.

[2] W. M. Boothby, *An introduction to differentiable manifolds and riemannian geometry*, second ed., Academic Press, 1986.

[3] E. Cabezas-Rivas and M. Vicente, *Volume preserving mean curvature flow in the hyperbolic space*, Indiana University Mathematical Journal **56** (2007), 2061–2086.

[4] R. J. Currier, *On hypersurfaces of hyperbolic space infinitesimally supported by horospheres*, Trans. American Math. Soc. (1989), no. 313, 419–431.

[5] B. Daniel, *Isometric immersions into 3-dimensional homogeneous manifolds*, Commentarii mathematici helvetici **82** (2007), no. 1.

[6] ———, *Isometric immersions into* $\mathbb{S}^n \times \mathbb{R}$ *and* $\mathbb{H}^n \times \mathbb{R}$ *and applications to minimal surfaces*, Trans. Amer. Math. Soc. (2009), no. 12, 6255–6282.

[7] M. P. Do Carmo, *Riemannian geometry*, Birkhauser, 1992.

[8] J. Douglas, *One-sided minimal surfaces with a given boundary*, Trans. Amer. Math. Soc. **34** (1932), no. 4, 731–756.

[9] R. Finn, *Remarks relevant to minimal surfaces, and to surfaces of prescribed mean curvature*, Journal d'Analyse Mathmatique **14** (1965), no. 1, 139–160.

[10] C. Forhman and W. H. III Meeks, *The topological classification of minimal surfaces in* $\mathbb{R}^3$, Ann. of Math. **167** (2008), no. 3, 681–700.

[11] S. Gallot, D. Hulin, and J. Lafontaine, *Riemannian geometry*, Springer, 2004.

[12] D. Gillbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, fourth ed., Springer, 2001.

[13] A. Grey, *Tubes*, Birkhauser, 2004.

[14] L. Hauswirth, *Minimal surfaces of riemann type in three-dimensional product manifolds*, Pacific J. Math. **224** (2006), no. 1, 91–117.

[15] L. Hauswirth, F. Morabito, and M. Rodriguez, *An end-to-end construction for singly periodic minimal surfaces*, Pacific J. Math. **241** (2009), no. 1, 1–61.

[16] S. Hildebrandt and A. J. Tromba, *On the branch point index of minimal surfaces*, Arch. Math. (Basel) **92** (2009), no. 5, 493–500.

[17] H. Hopf, *Lectures on differential geometry in the large*, Springer, 1989.

[18] C. Isenberg, *The science of soap films and soap bubbles*, Dover, 1978.

[19] H. Jenkins and J. Serrin, *The dirichlet problem for the minimal surface equation in higher dimensions*, J. Reine Angew. Math. **229** (1968), 170–187.

[20] N. J. Korevaar, R. Kusner, and B. Solomon, *The structure of complete embedded surfaces with constant mean curvature*, J. Differential Geom. **30** (1989), no. 2, 465–503.

[21] R. Krust, *Remarques sur le problème extérieur de plateau*, Duke Math. J. **59** (1989), no. 1, 161–173.

[22] N. Kutev and F. Tomi, *Nonexistence and instability in the exterior dirichlet problem for the minimal surface equation in the plane*, Pacific J. Math. **170** (1995), no. 2, 535–542.

[23] ______, *Existence and nonexistence for the exterior dirichlet problem for the minimal surface equation in the plane.*, Differential Integral Equations **11** (1998), no. 6, 917–928.

[24] E. C. Kwert, *Embedded solutions for exterior minimal surface problems*, Manuscripta Math. **70** (1990), no. 1, 51–65.

[25] H. B. Lawson, *Complete minimal surfaces in S^3*, Ann. of Math. **92** (1970), 335–374.

[26] W. H. III Meeks and J. Pérez, *Properly embedded minimal planar domains with infinite topology are riemann minimal examples*, Int. Press, Somerville, 2008.

[27] W. H. III Meeks and H. Rosenberg, *Stable minimal surfaces in* $M \times \mathbb{R}$, J. Differential Geom. **68** (2004), no. 3.

[28] J. Milnor, *Morse theory*, Princeton Univ. Press, 1963.

[29] P. Mira and I. Fernandez, *Harmonic maps and constant mean curvature surfaces in* $\mathbb{H}^2 \times \mathbb{R}$, Amer. J. Math. **129** (2007), no. 4, 1145–1181.

[30] S. Montaldo and F. Mercuri, *A weierstrass representation formula for minimal surfaces in* $\mathbb{H}_3$ *and* $\mathbb{H}^2 \times \mathbb{R}$, Acta Math. Sin. (Engl. Ser.) **22** (2006), no. 6, 1603–1612.

[31] B. Nelli and H. Rosenberg, *Global properties of constant mean curvature surfaces in* $\mathbb{H}^2 \times \mathbb{R}$., Pacific J. Math. **226** (2006), no. 1, 137–152.

[32] ———, *Simply connected constant mean curvature surfaces in* $\mathbb{H}^2 \times \mathbb{R}$, Michigan Math. J. **54** (2006), no. 3, 537–543.

[33] B. Nelli and R. Sa Earp, *Vertical ends of constant mean curvature* $H = \frac{1}{2}$ *in* $\mathbb{H}^2 \times \mathbb{R}$, Preprint (2007).

[34] B. Nelli, R. Sa Earp, W. Santos, and E. Toubiana, *Uniqueness of H-surfaces in* $\mathbb{H}^2 \times \mathbb{R}$, $|H| \leq 1/2$, *with boundary one or two parallel horizontal circles.*, Annals of Global Analysis and Geometry **33** (2008), no. 4, 307–321.

[35] R. Osserman, *Global properties of minimal surfaces in* E^3 *and* E^n, Annals of Mathematics **80** (1964), no. 2, 340–364.

[36] ———, *A survey of minimal surfaces*, second ed., A survey of minimal surfaces, 1986.

[37] P. Petersen, *Riemannian Geometry*, Springer, 1998.

[38] T. Radò, *On Plateau's problem*, Ann. of Math. **31** (1930), no. 2, 457–469.

[39] J. Ripoll and L. Sauer, *A note on the dirichlet problem for the minimal surface equation in nonconvex planar domains*, Mat. Contemp. **35** (2008), 177–183.

[40] H. Rosenberg, *Minimal surfaces in* $\mathbb{M}^2 \times \mathbb{R}$, Illinois J. Math. **46** (2002), 1177–1195.

[41] H. Rosenberg and U. Abresch, *A hopf differential for constant mean curvature surfaces in $S^2 \times R$ and $H^2 \times R$*, Acta Math. **193** (2004), no. 2, 141–174.

[42] H. Rosenberg and W. H. III Meeks, *The theory of minimal surfaces in $M \times \mathbb{R}$*, Comment. Math. Helv. **80** (2005), no. 4, 811–858.

[43] H. Rosenberg and B. Nelli, *Minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$*, Bull. Braz. Math. Soc. (N.S.) **33** (2002), no. 2, 263–292.

[44] ______, *Errata: "Minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$"*, Bull. Braz. Math. Soc. (N.S.) **38** (2007), no. 4.

[45] R. Sa Earp and E. Toubiana, *Screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$*, Illinois Journal of Mathematics **49** (2005), no. 4, 1323–1362.

[46] ______, *An asymptotic theorem for minimal surfaces and existence results for minimal graphs in $\mathbb{H}^2 \times \mathbb{R}$*, Math. Ann. **342** (2008), no. 2, 309–331.

[47] R. Schoen, *Uniqueness, symmetry, and embeddedness of minimal surfaces*, J. Differential Geom. **18** (1983), no. 4, 791–809.

[48] M. Spivak, *A Comprehensive Introduction di Differential Geometry*, third ed., Publish or Perish, 1999.

[49] J. Spruck, *Interior gradient estimates and existence theorems for constant mean curvature graphs in $M^n \times \mathbb{R}$*, Pure and Applied Mathematics Quarterly (2007), no. 3, 785–800.

[50] F. Tomi and R. Ye, *The exterior plateau problem*, Math. Z. **205** (1990), no. 2, 791–809.

Acknowledgements

The line of person to which I am thankful is growing day by day.
First of all I thank my advisor, Professor Giovanna Citti, for her great disponibility and generosity, mathematical and human. Without her hints this work wouldn't have been possible.
I would like to thank the director of my Ph. D. program, Professor Alberto Parmeggiani, for his constant support and encouragement and for his fruitfull effort in making this program better and better.
I thank Stefano Francaviglia for the time he spent trying to help me proving inequalities in mean curvature flow.
I also am thankful to Professor Luca Capogna, University of Arkansas, for all the questions he asked to me about this work. It helped my understanding of this material.
I thank the University of Bologna for the economic support of the last three years and the Marco Polo grant funded my visit to the University of Arkansas.

I thank my parents and my brothers because their support and friendship made possible my prosecution of the studies after the Graduation.
I thank Annachiara for all what we did together and for what we will do.