Alma Mater Studiorum · Università di Bologna
Universitat Politècnica de Catalunya

FACOLTÀ DI SCIENZE MATEMATICHE, FISICHE E NATURALI
Dottorato di ricerca in Matematica XXII ciclo
MAT 05: Analisi Matematica

FACULTAD DE MATEMÁTICAS Y ESTADÍSTICAS
Doctorado en Matemáticas

BISTABLE ELLIPTIC EQUATIONS WITH FRACTIONAL DIFFUSION

ELEONORA CINTI

Coordinatore del dottorato:
ALBERTO PARMEGGIANI

Advisers:
XAVIER CABRÉ VILAGUT

BRUNO FRANCHI

BARCELONA –BOLOGNA
April 2010

To my parents Bianca and Claudio.

Un uomo senza sogni, senza utopie,
senza ideali, sarebbe un mostruoso animale,
un cinghiale laureato in matematica pura.

Fabrizio De André

ABSTRACT

This work concerns the study of bounded solutions to elliptic nonlinear equations with fractional diffusion of the form $(-\Delta)^s u = f(u)$ in $\mathbb{R}^n$. More precisely, the aim of this thesis is to investigate some open questions related to the analog of a conjecture of De Giorgi for these equations. The conjecture concerns the one-dimensional (or 1-D) symmetry of bounded monotone solutions in all space, at least up to dimension 8. Of special interest is the bistable elliptic or Allen-Cahn equation, involving fractional Laplacians, which models phase transitions.

This property on 1-D symmetry of monotone solutions for the fractional equation was known when $n = 2$ for every fractional power $0 < s < 1$. The question remained open for $n > 2$.

Recently the fractional Laplacians attract much interest in nonlinear analysis. Caffarelli and Silvestre have given a new formulation of the fractional Laplacians through Dirichlet-Neumann maps. To study the nonlocal problem $(-\Delta)^s u = f(u)$ in $\mathbb{R}^n$, we use this formulation which let us to realize it as a local problem in $\mathbb{R}^{n+1}_+$ with a nonlinear Neumann condition.

In this work we focus our attention in two directions.

First, in chapter 2, we study a particular type of solutions of $(-\Delta)^s u = f(u)$ for $s = 1/2$, which are called *saddle-shaped solutions*. A crucial property of saddle-shaped solutions is that their 0-level set is the Simons cone. This cone appears in the theory of minimal surfaces and its variational properties motivated the conjecture of De Giorgi, namely the fact that the Simons cone is a minimal cone in dimensions $2m \geq 8$. We are interested in the study of saddle-shaped solutions, because they are the candidates to be global minimizers not 1-D in dimensions $n \geq 8$ (open problem). In this first part the main results are: existence of saddle-shaped solutions in every even dimension $2m$, as well as their asymptotic behaviour, monotonicity properties, and instability in dimensions $2m = 4$ and $2m = 6$.

In the second part of this thesis, we give a positive answer to the analog of the conjecture of De Giorgi for the fractional equation in dimension $n = 3$. To prove this 1-D symmetry result, we use a Liouville-type argument. In this approach the two principal ingredients in the proof of our 1-D symmetry result are the stability of solutions and an energy estimate. In chapters 3 and 4 we establish sharp energy estimates for global minimizers and bounded monotone solutions of our fractional equation for every $0 < s < 1$. As a consequence we deduce the analog of the conjecture of De Giorgi for the fractional equation $(-\Delta)^s u = f(u)$, in dimension $n = 3$ for every $1/2 \leq s < 1$. To prove our energy estimates we use a comparison argument combined with some extension results for functions belonging to fractional Sobolev spaces.

ACKNOWLEDGEMENTS

I would like to thank my advisers Professors Xavier Cabré and Bruno Franchi for their guidance all these years, for their belief in my work and encouragement.

I am grateful to the Departament de Matematica Aplicada I of the Universitat Politècnica de Catalunya for making me feel welcome and to the Dipartimento di Matematica of the Università di Bologna, where I began my studies.

I wish to thank Professor Joan Solà-Morales and Maria del Mar Gonzalez from UPC, and Professors Ermanno Lanconelli and Alberto Parmeggiani from Bologna for always being available.

Finally, a special thanks to my family, to Simone and to all my friends from Porretta, Bologna and Barcelona.

Table of Contents

Chapter 1

Introduction and summary of results

This work concerns the study of bounded solutions to elliptic nonlinear equations with fractional diffusion of the form

$$(-\Delta)^s u = f(u) \quad \text{in } \mathbb{R}^n. \tag{1.0.1}$$

We are interested in some questions related to the analog for fractional equations of a conjecture of De Giorgi on one-dimensional (or 1-D for short) symmetry of bounded monotone solutions.

In 1978 De Giorgi conjectured that the level sets of every bounded monotone solution of the Allen-Cahn equation $-\Delta u = u - u^3$ are hyperplanes, at least if the space dimension satisfies $n \leq 8$. This equation is a semilinear model for phase transitions and, as we will see, it is related to minimal surfaces theory.

This 1-D symmetry result for the fractional case, that is for equation (1.0.1), has been proven to be true when $n = 2$ and $s = 1/2$ by Cabré and Solà-Morales [10], and when $n = 2$ and for every $0 < s < 1$ by Cabré and Sire [8], and by Sire and Valdinoci [37]. The question remained open for dimensions $n > 2$.

In this work we focus our attention in two directions. First, in chapter 2, we study a particular type of solutions of problem (1.0.1) for $s = 1/2$, which are called *saddle-shaped solutions*. These solutions are the candidates to be global minimizers not 1-D in dimensions $n \geq 8$. This is an open problem and it is expected to be true from the classical theory of minimal surfaces.

Second, we give a positive answer to the analog of the conjecture of De Giorgi for the fractional equation (1.0.1), in dimension $n = 3$. The principal ingredient in the proof of our 1-D symmetry result is an optimal energy estimate for global minimizers and for monotone solutions, that we present in chapters 3 and 4.

In the following two sections we explain the connection between the semilinear model for phase transitions and the theory of minimal surfaces, and we recall some known results about the original conjecture of De Giorgi, as well as some properties of saddle-shaped solutions for the Allen-Cahn equation.

Later, we introduce the fractional Laplacians and we recall the known results concerning 1-D symmetry of bounded monotone solutions for fractional equations. Finally, we give a summary of the results contained in this thesis.

1.1 A conjecture of De Giorgi for the Allen-Cahn equation

The following is the conjectured raised by De Giorgi in 1978.

Conjecture([21]) *Let* $u \in C^2(\mathbb{R}^n)$ *be a solution of*

$$-\Delta u = u - u^3 \qquad in\ \mathbb{R}^n \tag{1.1.1}$$

such that

$$|u| \leq 1 \quad and \quad \partial_{x_n} u > 0$$

in the whole $\mathbb{R}^n$. *Then, all level sets* $\{u = \lambda\}$ *of* u *are hyperplanes, at least if* $n \leq 8$. *Equivalently,* u *is a function depending only on one Euclidian variable.*

If u satisfies this property, we will say that u is one-dimensional (1-D for short). Equation (1.1.1) is the Allen-Cahn equation, which models phase transitions.

The conjecture has been proven to be true in dimension $n = 2$ by Ghoussoub and Gui [24] and in dimension $n = 3$ by Ambrosio and Cabré [3]. For $4 \leq n \leq 8$, if $\partial_{x_n} u > 0$, and assuming the additional condition

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1 \quad \text{for all } x' \in \mathbb{R}^{n-1}, \tag{1.1.2}$$

it has been established by Savin [34]. Recently a counterexample to the conjecture for $n \geq 9$ has been announced by del Pino, Kowalczyk and Wei [22].

An heuristic motivation of the conjecture of De Giorgi is given by a $\Gamma-$convergence result of Modica and Mortola [29]. Given a solution u of (1.1.1), consider the blow-down family of functions $\{u_\epsilon\}$ defined by $u_\epsilon(x) = u\left(\frac{x}{\epsilon}\right)$, which are bounded solutions of the rescaled equation

$$-\Delta u_\epsilon = \frac{1}{\epsilon^2}(u_\epsilon - u_\epsilon^3) \text{ in } \mathbb{R}^n. \tag{1.1.3}$$

Equation (1.1.3) above can be viewed as the Euler-Lagrange equation associated to the functional

$$\mathcal{E}_\epsilon(v, \Omega) := \int_\Omega \left\{ \frac{\epsilon}{2}|\nabla v|^2 + \frac{1}{\epsilon} G(v) \right\} dx, \tag{1.1.4}$$

with $G(v) = (1/4)(1-v^2)^2$. These functionals have been studied by Modica and Mortola, who proved that they Γ-converge to a multiple of the perimeter functional. Thus, if u is a minimizer, the sequence $\{u_\epsilon\}$ should converge in some sense to a characteristic function (with values ± 1) of a set E such that $\partial E \cap \Omega$ is a minimal surface. If we further assume that u is monotone, say $\partial_{x_n} u > 0$ in $\mathbb{R}^n$, then the level sets of u, and hence those of u_ϵ, are graphs converging to a minimal graph of a function $g : \mathbb{R}^{n-1} \to \mathbb{R}$.

In [34] Simons classified all entire minimal graphs and proved that entire minimal graphs of functions $\mathbb{R}^k \to \mathbb{R}$ are hyperplanes for $k \leq 7$. This implies that, in our problem, the limiting minimal surface is a hyperplane if $k = n-1 \leq 7$, i.e., $n \leq 8$.

Concerning the problem of finding a counterexample to the conjecture of De Giorgi, we recall that Jerison and Monneau [25] proved that if there exists a global minimizers in $\mathbb{R}^{n-1}$, even in each variable x_i, then one can construct a bounded monotone solution of the Allen-Cahn equation in $\mathbb{R}^n$ which is not one-dimensional.

On the other hand, by a deep result of Savin [34], up to dimension $n \leq 7$ every global minimizer is an odd function of only one Euclidean variable. In particular, a global minimizer even with respect to each coordinate can not exist in $\mathbb{R}^n$ for $n \leq 7$.

Thus, the crucial remaining question (still open) is whether a global minimizer of (1.1.1), even with respect to each coordinate, exists in higher dimensions. For this, a natural candidate is expected to be found in the class of saddle-shaped solutions, that is, solutions that depend only on two radial coordinates $s = |x^1|$ and $t = |x^2|$, change sign in $\mathbb{R}^{2m} = \{x = (x^1, x^2) \in \mathbb{R}^m \times \mathbb{R}^m\}$, and vanish only on the Simons cone $\mathcal{C} = \{s = t\}$.

1.2 Saddle-shaped solutions for the Allen-Cahn equation

In this section we give the definition of saddle-shaped solution for the Allen-Cahn equation, and we recall some known results about their qualitative properties.

The saddle-shaped solutions that we consider are even with respect to the coordinate axes and odd with respect to the Simons cone, which is defined as follows. For $n = 2m$ the Simons cone $\mathcal{C}$ is given by:

$$\mathcal{C} = \{x \in \mathbb{R}^{2m} : x_1^2 + ... + x_m^2 = x_{m+1}^2 + ... + x_{2m}^2\}.$$

We recall that the Simons cone has zero mean curvature at every point $x \in \mathcal{C} \setminus \{0\}$, in every dimension $2m \geq 2$. Moreover, in dimensions $2m \geq 8$ (and only in these dimensions) it is a minimizer of the area functional, that is, it is a minimal cone (in the variational sense).

We define two new variables

$$s = \sqrt{x_1^2 + \cdots + x_m^2} \quad \text{and} \quad t = \sqrt{x_{m+1}^2 + \cdots + x_{2m}^2},$$

for which the Simons cone becomes $\mathcal{C} = \{s = t\}$.

We now introduce our notion of saddle solution. These solutions depend only on s and t, and are odd with respect to the Simons cone.

Definition 1.2.1. Let u be a bounded solution of $-\Delta u = f(u)$ in $\mathbb{R}^{2m}$, where $f \in C^1$ is odd. We say that $u : \mathbb{R}^{2m} \to \mathbb{R}$ is a *saddle solution* if

(a) u depends only on the variables s and t. We write $u = u(s,t)$;

(b) $u > 0$ for $s > t$;

(c) $u(s,t) = -u(t,s)$.

Remark 1.2.2. If u is a saddle solution then, in particular, $u = 0$ on the Simons cone $\mathcal{C} = \{s = t\}$. In other words, $\mathcal{C}$ is the zero level set of u.

Saddle solutions for the classical equation $-\Delta u = f(u)$ were first studied by Dang, Fife, and Peletier in [20] in dimension $n = 2$ for f odd, bistable, and $f(u)/u$ decreasing for $u \in (0,1)$. They proved the existence and uniqueness of saddle-shaped solutions and established monotonicity properties and the asymptotic behaviour. The instability property of saddle solutions in dimension $n = 2$ was studied by Schatzman [35]. In two recent works [12, 13], Cabré and Terra proved the existence of saddle-shaped solutions for the equation $-\Delta u = f(u)$ in $\mathbb{R}^n$, where f is of bistable type, in every even dimension $n = 2m$. Moreover they established some qualitative properties of these solutions, such as monotonicity properties, asymptotic behaviour and instability in dimensions $2m = 4, 6$. Finally, they proved the asymptotic stability of saddle-shaped solutions in dimensions $2m \geq 8$. This is an indication that these solutions might be stable in dimension 8 and higher (which is still an open question).

In chapter 2 we establish some analog results for saddle-shaped solutions of the equation $(-\Delta)^{1/2} u = f(u)$ in $\mathbb{R}^{2m}$.

1.3 The fractional Laplacian

The fractional powers of the Laplacian, which are called fractional Laplacians and are the infinitesimal generators of the Lévy stable diffusion processes, appear in anomalous diffusion phenomena in physics, biology, as well as other areas. They occur in flame propagation, chemical reaction in liquids, and population dynamics. Lévy diffusion processes have discontinuous sample paths and heavy tails, while

Brownian motion has continuous sample paths and exponential decaying tails. These processes have been applied to American options in mathematical finance for modelling the jump processes of financial derivatives, such as futures, forwards, options, and swaps; see [4] and references therein. Moreover, fractional Laplacians play an important role in the study of the quasi-geostrophic equations in geophysical fluid dynamics.

Recently the fractional Laplacians attract much interest in nonlinear analysis. Caffarelli and Silvestre [15] have given a new formulation of the fractional Laplacians through Dirichlet-Neumann maps. In [36] Silvestre has established regularity results for solutions of an obstacle problem for the operator $(-\Delta)^s$ with $0 < s < 1$. In [18] Caffarelli and Vasseur study the regularity for solutions of the quasi-geostrophic equation with critical diffusion $(-\Delta)^{1/2}$, using the De Giorgi-Nash-Moser method. In [14] Caffarelli, Roquejoffre, and Savin have developed a regularity theory for nonlocal minimal surfaces. This surfaces can be interpreted as a non-infinitesimal version of classical minimal surfaces and can be attained by minimizing the H^s-norm of the indicator function when $0 < s < 1/2$. As a last reference we remind that in [11] Cabré and Tan have established existence and regularity results for problems involving the square root of the Laplacian in bounded domain with zero Dirichlet boundary conditions.

The fractional Laplacian of a function $u : \mathbb{R}^n \to \mathbb{R}$ is expressed by the formula

$$(-\Delta)^s u(x) = c_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(\overline{x})}{|x - \overline{x}|^{n+2s}} d\overline{x},$$

where the parameter s is a real number between 0 and 1, and $c_{n,s}$ is some normalization constant depending on the dimension n and on the power s.

It can also be defined as a pseudo-differential operator via Fourier transform:

$$\widehat{(-\Delta)^s} u(\xi) = |\xi|^{2s} \widehat{u}(\xi).$$

In this work, we will study the equation $(-\Delta)^s u = f(u)$ in $\mathbb{R}^n$ by realizing it as a local problem in $\mathbb{R}^{n+1}_+$.

For the case of the half-Laplacian, or square root of the Laplacian, that is for $s = 1/2$, the local formulation was well known. Let $\mathbb{R}^{n+1}_+ := \{(x, \lambda) | x \in \mathbb{R}^n, \lambda > 0\}$. A function u is a solution of the equation $(-\Delta)^{1/2} u = f(u)$ in $\mathbb{R}^n$ if and only if its harmonic extension in the half-space $\mathbb{R}^{n+1}_+$ is a solution of the problem

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ -\dfrac{\partial v}{\partial \lambda} = f(v) & \text{on } \mathbb{R}^n = \partial \mathbb{R}^{n+1}_+. \end{cases} \tag{1.3.1}$$

Indeed, let u be a bounded continuous function in $\mathbb{R}^n$. There is a unique harmonic extension v of u in the half space. It is the solution of the following Laplacian problem:

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ v = u & \text{on } \mathbb{R}^n = \partial \mathbb{R}^{n+1}_+. \end{cases}$$

Consider the operator $T : u \mapsto -\partial_\lambda v(\cdot, 0)$. Since $\partial_\lambda v$ is still a harmonic function, if we apply the operator twice, we obtain

$$T \circ Tu = \partial_{\lambda\lambda} v \mid_{\lambda=0} = -\Delta_x v \mid_{\lambda=0} = -\Delta_x u \quad \text{in } \mathbb{R}^n.$$

Thus the operator T that maps the Dirichlet-type data u to the Neumann-type data $-\partial_\lambda v(x, 0)$ is actually the half Laplacian.

For the other fractional powers of the Laplacian $0 < s < 1$, the local formulation associated to equation (1.0.1), was established by Caffarelli and Silvestre in [15].

They proved that u is a solution of problem (1.0.1) in $\mathbb{R}^n$ if and only if v defined in $\mathbb{R}^{n+1}_+$ is a solution of the problem

$$\begin{cases} \operatorname{div}(\lambda^{1-2s} \nabla v) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ -\lim\limits_{\lambda \to 0} \lambda^{1-2s} \dfrac{\partial v}{\partial \lambda} = d_{n,s} f(v), \end{cases} \tag{1.3.2}$$

where $d_{n,s} > 0$ is a positive constant depending only on n and s, and $v(x, 0) = u(x)$ on $\mathbb{R}^n = \partial \mathbb{R}^{n+1}_+$. Later, we will study problem (1.3.2) with $d_{n,s} = 1$. In the sequel we will call the extension v of u in $\mathbb{R}^{n+1}_+$ satisfying $\operatorname{div}(\lambda^{1-2s} \nabla v) = 0$, the *s-extension* of u.

Observe that for every $0 < s < 1$, we have that $-1 < 1 - 2s < 1$ and thus the weight λ^{1-2s} which appears in (1.3.2), belongs to the Muckenoupt class A_2. Thus the theory developed by Fabes, Kenig and Serapioni [23] applies to problem (1.3.2) and hence a Poincaré inequality, a Harnack inequality, and Hölder regularity hold for solutions of our problem.

Moreover, the weight λ^{1-2s} does not depend on the horizontal variables $x_1, ..., x_n$. Hence problem (1.3.2) is invariant under translations in the directions $x_1, ..., x_n$, and then, for instance, the sliding method can be applied in these directions.

1.4 Stability and global minimality

Problem (1.3.2), associated to the nonlocal equation (1.0.1), allows us to introduce the notions of *energy*, *stability*, and *global minimality* for a solution u of problem (1.0.1).

In this work we will use the following notations. We denote by

$$\widetilde{B}_r^+ = \{(x, \lambda) \in \mathbb{R}^{2m+1} : \lambda > 0, \ |(x, \lambda)| < r\}$$

and by $\widetilde{B}_r^+(x,\lambda) = (x,\lambda) + \widetilde{B}_r^+$. Let $\Omega \subset \mathbb{R}^{n+1}_+$ be a bounded domain. We define the following subsets of $\partial\Omega$:

$$\partial^0\Omega := \{(x,0) \in \partial\mathbb{R}^{n+1}_+ : \widetilde{B}_\varepsilon^+(x,0) \subset \Omega \text{ for some } \varepsilon > 0\} \tag{1.4.1}$$

and

$$\partial^+\Omega := \overline{\partial\Omega \cap \mathbb{R}^{n+1}_+}. \tag{1.4.2}$$

Given a $C^{1,\alpha}$ nonlinearity $f : \mathbb{R} \to \mathbb{R}$, for some $0 < \alpha < 1$, define

$$G(u) = \int_u^1 f.$$

We have that $G \in C^2(\mathbb{R})$ and $G' = -f$.

Let v be a $C^1(\overline{\Omega})$ function. We consider the energy functional

$$\mathcal{E}_{s,\Omega}(v) = \int_\Omega \frac{1}{2}\lambda^{1-2s}|\nabla v|^2 + \int_{\partial^0\Omega} G(v). \tag{1.4.3}$$

Observe that the potential energy is computed only on the boundary $\partial^0\Omega$. This is a quite different situation from the one of interior reactions.

When $s = 1/2$, the energy becomes

$$\mathcal{E}_\Omega(v) = \int_\Omega \frac{1}{2}|\nabla v|^2 + \int_{\partial^0\Omega} G(v).$$

We start by recalling that problem (1.3.2) can be viewed as the Euler-Lagrange equation associated to the energy functional $\mathcal{E}_s$ (we consider $d_{n,s} = 1$).

Definition 1.4.1. a) We say that a bounded solution v of (1.3.2) is *stable* if the second variation of energy $\delta^2\mathcal{E}_s/\delta^2\xi$, with respect to perturbations ξ compactly supported in $\overline{\mathbb{R}^{n+1}_+}$, is nonnegative. That is, if

$$Q_{s,v}(\xi) := \int_{\mathbb{R}^{n+1}_+} \lambda^{1-2s}|\nabla\xi|^2 - \int_{\partial\mathbb{R}^{n+1}_+} f'(v)\xi^2 \geq 0 \tag{1.4.4}$$

for every $\xi \in C_c^\infty(\overline{\mathbb{R}^{n+1}_+})$.

We say that v is *unstable* if and only if v is not stable.

b) We say that a bounded solution u of (1.0.1) in $\mathbb{R}^{2m}$ is *stable* (respectively, *unstable*) if its s-extension v is a stable (respectively, unstable) solution for the problem (1.3.2).

Another important notion related to the energy functional $\mathcal{E}_s$ is the one of global minimality.

Definition 1.4.2. a) We say that a bounded $C^1(\overline{\mathbb{R}^{n+1}_+})$ function v in $\mathbb{R}^{n+1}_+$ is a *global minimizer* of (1.3.2) if

$$\mathcal{E}_{s,\Omega}(v) \le \mathcal{E}_{s,\Omega}(v+\xi),$$

for every bounded domain $\Omega \subset \overline{\mathbb{R}^{n+1}_+}$ and every C^∞ function ξ with compact support in $\Omega \cup \partial^0\Omega$.

b) We say that a bounded C^1 function u in $\mathbb{R}^n$ is a *global minimizer* of (1.0.1) if its s-extension v is a global minimizer of (1.3.2).

Observe that the perturbations ξ need not vanish on $\partial^0\Omega$, in contrast from interior reactions.

In some references, global minimizers are called "local minimizers", where local stands for the fact that the energy is computed in bounded domains. Clearly, every global minimizer is a stable solution.

In some parts of this work we will assume some, or all, of the following conditions on f:

$$f \text{ is odd;} \tag{1.4.5}$$

$$G \ge 0 = G(\pm 1) \;\text{ in }\; \mathbb{R}, \text{and }\; G > 0 \;\text{ in }\; (-1,1); \tag{1.4.6}$$

$$f' \;\text{ is decreasing in }\; (0,1). \tag{1.4.7}$$

Note that, if (1.4.5) and (1.4.6) hold, then $f(0) = f(\pm 1) = 0$. Conversely, if f is odd in $\mathbb{R}$, positive with f' decreasing in $(0,1)$ and negative in $(1,\infty)$ then f satisfies (1.4.5), (1.4.6) and (1.4.7). Hence, most of the nonlinearities f that we consider are of "balanced bistable type", while the potentials G are of "double well type". Our three assumptions (1.4.5), (1.4.6), (1.4.7) are satisfied by $f(u) = u - u^3$. In this case we have that $G(u) = (1/4)(1-u^2)^2$. The three hypothesis also hold for the Peierls-Nabarro problem

$$(-\Delta)^{1/2}u = \sin(\pi u), \tag{1.4.8}$$

for which $G(u) = (1/\pi)(1+\cos(\pi u))$.

In [10], Cabré and Solà-Morales studied bounded monotone solutions in one direction, $\partial_{x_n} u \ge 0$, and satisfying

$$\lim_{x_n \to \pm\infty} u(x', x_n) \pm 1 \quad \text{for all } \; x' \in \mathbb{R}^{n-1},$$

of the fractional equation

$$(-\Delta)^{1/2}u = f(u) \quad \text{in } \mathbb{R}^n. \tag{1.4.9}$$

They called these solutions *layer solutions.* They established their existence, uniqueness, symmetry, and variational properties, as well as their asymptotic behaviour. They proved that assumption (1.4.6) on G guarantees the existence of a *layer* solution. In addition, such solution is unique up to translations if $f'(\pm 1) < 0$. The following is the precise result established in [10].

Theorem 1.4.3. *([10]) Let f be any $C^{1,\alpha}$ function with $0 < \alpha < 1$. Let $G' = -f$. Then:*

- *There exists a layer solution u_0 of $(-\Delta)^{1/2}u_0 = f(u_0)$ in $\mathbb{R}$ if and only if*
$$G'(-1) = G'(1) = 0, \quad \text{and } G > G(-1) = G(1) \text{ in } (-1, 1).$$
- *If $f'(\pm 1) < 0$, then a layer solution of (1.3.1) is unique up to translations.*
- *If f is odd and $f'(\pm 1) < 0$, then every layer solution of (1.3.1) is odd with respect to some half-axis. That is, $u(x + b) = -u(-x + b)$ for some $b \in \mathbb{R}$.*

Normalizing the layer solution to vanishing at the origin, we call it u_0 and its harmonic extension in the half-plane v_0. Thus we have

$$\begin{cases} u_0 : \mathbb{R} \to (-1, 1) \\ u_0(0) = 0, \quad u_0' > 0 \\ (-\Delta)^{1/2}u_0 = f(u_0) \quad \text{in } \mathbb{R}. \end{cases} \tag{1.4.10}$$

The monotone bounded solution of the Peierls-Nabarro problem (1.4.8) is explicit; if we call v_0 the harmonic extension of u_0 in $\mathbb{R}^2_+$, we have

$$v_0(x, \lambda) = \frac{2}{\pi} \arctan \frac{x}{\lambda + 1/\pi}.$$

For the other fractional powers $0 < s < 1$, the study of layer solutions and their properties is contained in two works of Cabré and Sire [8, 9].

1.5 1-D symmetry for fractional equations

In this section we recall some recent results concerning the 1-D symmetry property of bounded monotone solutions of the fractional equation

$$(-\Delta)^s u = f(u) \quad \text{in } \mathbb{R}^n.$$

In [10], Cabré and Solà-Morales proved that, when $n = 2$ and $s = 1/2$, stable solutions of (1.0.1) are one-dimensional.

The same result has been proven to be true for $n = 2$ and for every fractional power $0 < s < 1$, by Cabré and Sire [8], and by Sire and Valdinoci [37].

In dimension $n = 3$ the analog of the conjecture of De Giorgi for fractional equations was completely open. In this work we give an affirmative answer to this question for the powers $1/2 \leq s < 1$ (see chapters 3 and 4).

In [10] and [8], the proof of 1-D symmetry property uses a method introduced in [5] and [3], which is based on a Liouville type result. In [37], the authors used a different method, which makes use of a geometric inequality of Poincaré type.

In both approaches, the two fundamental ingredients in the proof are:

- the stability of the solution,
- an estimate for its Dirichlet energy.

More precisely, the following energy estimate is enough to prove 1-D symmetry:

$$\int_{C_R} \frac{1}{2}\lambda^{1-2s}|\nabla v|^2 \leq CR^2 \log R, \tag{1.5.1}$$

where $C_R = B_R \times (0, R)$ is the cylinder of radius R and height R in $\mathbb{R}^{n+1}_+$. Here, B_R denotes the ball centered at the origin and of radius R in $\mathbb{R}^n$. It is easy to see that, when $n = 2$ and for every $0 < s < 1$, the following estimate holds

$$\int_{C_R} \frac{1}{2}\lambda^{1-2s}|\nabla v|^2 \leq CR^2,$$

and thus (1.5.1) also does hold.

The main difficulty in proving 1-D symmetry in dimension $n = 3$ is the proof of the energy estimate (1.5.1).

In chapters 3 and 4, we establish new sharp energy estimates for bounded monotone solutions and for global minimizers of fractional equations. As a consequence, we deduce the analog of the conjecture of De Giorgi for equation (1.0.1) in dimension $n = 3$ for every $1/2 \leq s < 1$.

In [14] Caffarelli, Roquejoffre, and Savin develop a regularity theory for nonlocal minimal surfaces. This surfaces can be interpreted as a non-infinitesimal version of classical minimal surfaces and can be attained by minimizing the H^s-norm of the indicator function when $0 < s < 1/2$. They also prove sharp energy estimates related to ours in some sense: our equation is the Allen-Cahn approximation of these nonlocal minimal surfaces. Instead, when $1/2 \leq s < 1$, our equation is an approximation of classical minimal surfaces. In [28] Maria del Mar Gonzalez prove that an energy functional related to fractional powers s of the Laplacian for $1/2 < s < 1$ Γ-converges to the classical perimeter functional. The same result for $s = 1/2$ was proven by Alberti, Bouchitté, and Seppecher in [2]. Finally, in [16] Caffarelli and Souganidis prove that scaled threshold dynamics-type algorithms corresponding to fractional Laplacians converge to moving fronts. More precisely, when $1/2 \leq s < 1$ the resulting interface moves by weighted mean curvature, while for $0 < s < 1/2$ the normal velocity is nonlocal of fractional-type.

1.6 Summary of results

In this section we present the main results of this thesis. For this, we provide a brief explanation of their meaning and importance.

This thesis can be divided into two parts.

In chapter 2 we establish existence and qualitative properties of saddle-shaped solutions for the problem

$$(-\Delta)^{1/2}u = f(u) \quad \text{in } \mathbb{R}^{2m}. \tag{1.6.1}$$

The interest in these solutions is related to the possibility of finding a global minimizer not 1-D in dimensions $2m \geq 8$.

In chapters 4 and 5, we establish new sharp energy estimates for bounded monotone solutions and for global minimizers of fractional equations. As a consequence, we establish the 1-D symmetry result for the equation

$$(-\Delta)^s u = f(u) \quad \text{in } \mathbb{R}^3,$$

for every $1/2 \leq s < 1$.

1.6.1 Saddle-shaped solutions for the half-Laplacian

In chapter 2 (which corresponds to [19]) we study saddle-shaped solution for the problem

$$(-\Delta)^{1/2}u = f(u) \quad \text{in } \mathbb{R}^n,$$

where $n = 2m$ is an even integer, and f is of bistable type.

We define two new variables

$$s = \sqrt{x_1^2 + \cdots + x_m^2} \quad \text{and} \quad t = \sqrt{x_{m+1}^2 + \cdots + x_{2m}^2},$$

for which the Simons cone becomes $\mathcal{C} = \{s = t\}$.

By saddle solution we mean a bounded solution of (1.6.1) such that

- u depends only on s and t,
- $u > 0$ for $s > t$,
- $u(s,t) = -u(t,s)$.

In particular $u = 0$ on $\mathcal{C}$.

Set $G(u) := \int_u^1 f$. In our results we assume some, or all, of the following conditions on f:

$$f \text{ is odd ;} \tag{1.6.2}$$

$$G \geq 0 = G(\pm 1) \text{ in } \mathbb{R}, \text{ and } G > 0 \text{ in } (-1,1); \tag{1.6.3}$$

$$f' \text{ is decreasing in } (0,1). \tag{1.6.4}$$

These conditions are related to the bistable and balanced character of f and are all satisfied by the Allen-Cahn model $f(u) = u - u^3$.

We recall that the local problem associated to equation (1.6.1) is the following:

$$\begin{cases} -\Delta v = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ -\dfrac{\partial v}{\partial \lambda} = f(v) & \text{on } \partial\mathbb{R}^{n+1}_+. \end{cases} \tag{1.6.5}$$

Our first result establishes the existence of a saddle solution in all even dimensions $n = 2m$. We use the following notations:

$$\mathcal{O} := \{x \in \mathbb{R}^{2m} : s > t\} \subset \mathbb{R}^{2m}$$
$$\widetilde{\mathcal{O}} := \{(x,\lambda) \in \mathbb{R}^{2m+1}_+ : x \in \mathcal{O}\} \subset \mathbb{R}^{2m+1}_+.$$

Note that

$$\partial\mathcal{O} = \mathcal{C} \quad \text{and} \quad \partial\widetilde{\mathcal{O}} = \mathcal{C} \times [0,\infty).$$

Finally, we define the cylinder $C_{R,L} = B_R \times (0, L)$, where B_R is the open ball in $\mathbb{R}^{2m}$ centered at the origin and of radius R.

Theorem 1.6.1. ***[see Theorem 2.1.6]*** *For every dimension $2m \geq 2$ and every nonlinearity f satisfying (1.6.2) and (1.6.3), there exists a saddle solution u of $(-\Delta)^{1/2}u = f(u)$ in $\mathbb{R}^{2m}$, such that $|u| < 1$ in $\mathbb{R}^{2m}$.*

Moreover, let v be the harmonic extension in $\mathbb{R}^{2m+1}_+$ of the saddle solution u. If in addition f satisfies (1.6.4), then the second variation of the energy $Q_v(\xi)$ at v, as defined in (1.4.4) with $s = 1/2$, is nonnegative for all function $\xi \in C^1(\overline{\mathbb{R}^{2m+1}_+})$ with compact support in $\overline{\mathbb{R}^{2m+1}_+}$ and vanishing on $\mathcal{C} \times [0, +\infty)$.

We prove the existence of a saddle solution u for problem (1.6.1), by proving the existence of a solution v for problem (1.6.5) with the following properties:

1. v depends only on the variables s, t and λ. We write $v = v(s, t, \lambda)$;
2. $v > 0$ for $s > t$;
3. $v(s, t, \lambda) = -v(t, s, \lambda)$.

Using a variational technique we construct a solution v in $\widetilde{\mathcal{O}}$ for the problem (1.6.5), satisfying $v > 0$ in $\widetilde{\mathcal{O}}$ and $v = 0$ on $\partial\widetilde{\mathcal{O}} = \mathcal{C} \times [0, +\infty)$. Then, since f is odd, by odd reflection with respect to $\partial\widetilde{\mathcal{O}}$ we obtain a solution v in the whole space which satisfies properties (1), (2), (3) above. Clearly the function $u(x) = v(x, 0)$ is a saddle solution for the problem (1.6.1).

To prove this existence result, we will use a energy estimate for v, which is not sharp but it is enough to prove the existence of a saddle solution. Instead in

Theorem 3.0.8 of chapter 3, we will establish the following sharp energy estimates for saddle-shaped solutions,

$$\mathcal{E}_{C_{S,S}}(v) \leq CS^{2m-1} \log S.$$

On the other hand, given the way in which it is constructed, the solution u is stable with respect to perturbations that vanish on the Simons cone $\mathcal{C}$. This however means nothing regarding the stability of u for general perturbations. In fact, we will see that u is actually unstable in dimensions 4 and 6.

In section 5 of chapter 2, we prove the existence and monotonicity properties of a maximal saddle solution of problem (1.6.1).

To establish these results, we need to introduce a new nonlocal operator $D_{H,\varphi}$, which is the square root of the Laplacian for functions defined in a domain $H \subset \mathbb{R}^n$ which agree with a given function φ on ∂H. We introduce this operator and we establish maximum principles for it, in section 4.

We introduce the new variables

$$\begin{cases} y = \dfrac{s+t}{\sqrt{2}} \\ z = \dfrac{s-t}{\sqrt{2}}. \end{cases} \tag{1.6.6}$$

Note that $|z| \leq y$ and that we may write the Simons cone as $\mathcal{C} = \{z = 0\}$.

The following theorem concerns the existence and monotonicity properties of the maximal saddle solution.

Theorem 1.6.2. ***[see Theorem 2.1.7]*** *Let f satisfy conditions* (1.6.2), (1.6.3), *and* (1.6.4).

Then, there exists a saddle solution $\overline{u}$ of $(-\Delta)^{1/2}\overline{u} = f(\overline{u})$ in $\mathbb{R}^{2m}$, with $|u| < 1$, which is maximal in the following sense. For every solution u of $(-\Delta)^{1/2}u = f(u)$ in $\mathbb{R}^{2m}$, vanishing on the Simons cone and such that u has the same sign as $s-t$, we have

$$0 < u < \overline{u} \quad in \ \mathcal{O}.$$

As a consequence, we also have

$$0 < |u| < |\overline{u}| \quad in \ \mathbb{R}^{2m}.$$

In addition, if $\overline{v}$ is the harmonic extension of $\overline{u}$ in $\mathbb{R}^{2m+1}_+$, then $\overline{v}$ satisfies:

(a) $\partial_s \overline{v} \geq 0$ in $\overline{\mathbb{R}^{2m+1}_+}$. Furthermore $\partial_s \overline{v} > 0$ in $\overline{\mathbb{R}^{2m+1}_+} \setminus \{s = 0\}$ and $\partial_s \overline{v} = 0$ in $\{s = 0\}$;

(b) $\partial_t \overline{v} \leq 0$ in $\overline{\mathbb{R}^{2m+1}_+}$. Furthermore $\partial_t \overline{v} < 0$ in $\overline{\mathbb{R}^{2m+1}_+} \setminus \{t = 0\}$ and $\partial_t \overline{v} = 0$ in $\{t = 0\}$;

(c) $\partial_z \overline{v} > 0$ *in* $\overline{\mathbb{R}^{2m+1}_+} \setminus \{0\}$*;*

(d) $\partial_y \overline{v} > 0$ *in* $\{s > t > 0\} \times [0, +\infty)$*.*

As a consequence, for every direction $\partial_\eta = \alpha\partial_y - \beta\partial_t$*, with* α *and* β *nonnegative constants,* $\partial_\eta \overline{v} > 0$ *in* $\{s > t > 0\} \times [0, +\infty)$*.*

In sections 6 and 7 of chapter 2, we focus our attention on the asymptotic behaviour and the instability properties of saddle-shaped solutions. We remind that u_0 satisfies (1.4.10). In [12] it is proved that $|z| = |s-t|/\sqrt{2} = d(x, \mathcal{C})$, where $d(\cdot, \mathcal{C})$ denotes the distance to the Simons cone. The following result concerns the asymptotic behaviour at infinity, of a class of solutions which contains saddle-shaped solutions.

Theorem 1.6.3. ***[see Theorem 2.1.9]*** *Let* f *satisfy conditions* (1.6.2)*,* (1.6.3)*, and* (1.6.4)*, and let* u *be a bounded solution of* $(-\Delta)^{1/2}u = f(u)$ *in* $\mathbb{R}^{2m}$ *such that* $u \equiv 0$ *on* $\mathcal{C}$*,* $u > 0$ *in* $\mathcal{O} = \{s > t\}$ *and* u *is odd with respect to* $\mathcal{C}$*.*

Then, denoting

$$U(x) := u_0((s-t)/\sqrt{2}) = u_0(z)$$

we have,

$$u(x) - U(x) \to 0 \quad \text{and} \quad \nabla u(x) - \nabla U(x) \to 0, \tag{1.6.7}$$

uniformly as $|x| \to \infty$*. That is,*

$$||u - U||_{L^\infty(\mathbb{R}^{2m}\setminus B_R)} + ||\nabla u - \nabla U||_{L^\infty(\mathbb{R}^{2m}\setminus B_R)} \to 0 \ \text{ as } \ R \to \infty. \tag{1.6.8}$$

Our proof of Theorem 1.6.3 follows the one given by Cabré and Terra in [13], and uses a compactness argument based on translations of the solutions, combined with two crucial Liouville-type results for nonlinear equations in the half-space and in a quarter of space.

Finally, in section 7, we prove that saddle-shaped solutions are unstable in dimensions $2m = 4,\ 6$.

In dimension $2m = 2$, instability of saddle-shaped solutions follows by the 1-D symmetry result established by Cabré and Solà-Morales in [10]. The following theorem concerns the instability of saddle solutions in dimensions 4 and 6. The principal ingredients in the proof are the monotonicity properties for the maximal saddle solution established in Theorem 1.6.2 and the asymptotic behaviour of Theorem 1.6.3.

Theorem 1.6.4. ***[see Theorem 2.1.10]*** *Let* f *satisfy conditions* (1.6.2)*,* (1.6.3)*,* (1.6.4)*. Then, every bounded solution of* $(-\Delta)^{1/2}u = f(u)$ *in* $\mathbb{R}^{2m}$ *such that* $u = 0$ *on the Simons cone* $\mathcal{C} = \{s = t\}$ *and* u *has the same sign as* $s - t$*, is unstable in dimensions* $2m = 4$ *and* $2m = 6$*.*

The stability of saddle solutions in dimensions $2m \geq 8$ is an open problem. However, at the end of section 7, we will establish some kind of asymptotic stability for the maximal saddle solution in dimensions 8 and higher. This is an indication that saddle solutions might be stable in dimensions higher or equal than 8.

We call $\overline{v}$ the harmonic extension of the maximal solution $\overline{u}$ in $\mathbb{R}^{2m+1}_+$. In the proof of Theorem 1.6.4, we use the maximal solution $\overline{v}$ of problem (1.6.5) and more importantly the equation satisfied by $\overline{v}_z = \partial_z \overline{v}$. We prove that $\overline{v}$ is unstable by constructing a test function $\xi(y,z,\lambda) = \eta(y,\lambda)\overline{v}_z(y,z,\lambda)$ such that $Q_{\overline{v}}(\xi) < 0$. We use the asymptotic convergence and monotonicity results for $\overline{v}$ (Theorems 1.6.3 and 1.6.2). Since $\overline{v}$ is maximal, we deduce that all bounded solutions v vanishing on $\mathcal{C} \times \mathbb{R}^+$ and having the same sign as $s-t$ are also unstable.

1.6.2 Energy estimates for equations involving the half-Laplacian

In chapter 3 (which corresponds to [6]), we establish sharp energy estimates for some solutions, such as global minimizers, monotone solutions and saddle-shaped solutions, of the fractional nonlinear equation $(-\Delta)^{1/2}u = f(u)$ in $\mathbb{R}^n$. Our energy estimates hold for every nonlinearity f and are sharp since they are optimal for one-dimensional solutions, that is for solutions depending only on one Euclidian variable.

As a consequence, in dimension $n = 3$, we deduce the one-dimensional symmetry of every global minimizer and of every monotone solution.

In the following theorem we establish a sharp energy estimate for global minimizers in every dimension n, and for monotone solutions in dimension $n = 3$. We will see that monotone solutions, without limits ± 1 at $\pm\infty$, are minimizers in some sense to be explained later, but, in case that they exist, they are not known to be global minimizers.

Given a bounded function u defined on $\mathbb{R}^n$, set $G(u) = \int_u^1 f$ and

$$c_u = \min\{G(s) : \inf_{\mathbb{R}^n} u \leq s \leq \sup_{\mathbb{R}^n} u\}. \tag{1.6.9}$$

In all this subsection by $||f||_{C^1}$ we mean $||f||_{C^1([\inf u, \sup u])}$.

Theorem 1.6.5. ***[see Theorems 3.0.3 and 3.0.4]*** *Let f be any $C^{1,\beta}$ nonlinearity, with $\beta \in (0,1)$, and $u \in L^\infty(\mathbb{R}^n)$ be a solution of (1.6.1). Let v be the harmonic extension of u in $\mathbb{R}^{n+1}_+$.*
a) If u is a global minimizer, then for all $R > 2$,

$$\int_{C_R} \frac{1}{2}|\nabla v|^2 dx d\lambda + \int_{B_R} \{G(u) - c_u\} dx \leq C R^{n-1} \log R, \tag{1.6.10}$$

where c_u is defined by (1.6.9) and C is a constant depending only on n, $||f||_{C^1}$, and $||u||_{L^\infty(\mathbb{R}^n)}$.

b) Let $n = 3$ and suppose that u is a bounded solution of (1.6.1) such that $\partial_e u > 0$ in $\mathbb{R}^3$ for some direction $e \in \mathbb{R}^3$, $|e| = 1$. Then, for all $R > 2$, v satisfies the energy estimate (1.6.10) *with $n = 3$.*

This energy estimate is sharp since it can not be improved for one-dimensional layer solutions by a result of [10].

As a consequence of Theorem 1.6.5, we deduce the analog of the conjecture of De Giorgi for the equation involving the half-Laplacian in $\mathbb{R}^3$.

Theorem 1.6.6. ***[see Theorem 3.0.5]*** *Let $n = 3$ and f be any $C^{1,\beta}$ nonlinearity with $\beta \in (0, 1)$. Let u be either a bounded global minimizer of* (1.6.1)*, or a bounded solution of* (1.6.1) *monotone in some direction $e \in \mathbb{R}^3$, $|e| = 1$.*

Then, u depends only on one variable, i.e., there exists $a \in \mathbb{R}^3$ and $g : \mathbb{R} \to \mathbb{R}$, such that $u(x) = g(a \cdot x)$ for all $x \in \mathbb{R}^3$. Equivalently, the level sets of u are planes.

The proof of 1-D symmetry result follows a method introduced in [5] and in [3], which is based on a Liouville type theorem. Using this approach the two fundamental ingredient in the proof are stability of solutions and our energy estimate.

The method that we use to prove the energy estimate also applies in the case of saddle-shaped solutions in $\mathbb{R}^{2m}$. These solutions, in principle, are not global minimizers (this is indeed the case in dimensions $2m \leq 6$ by Theorem 1.6.4), but at least one of them is a minimizer under perturbations vanishing on the Simons cone and this will be enough to prove a sharp energy estimate for such saddle solution.

Theorem 1.6.7. ***[see Theorem 3.0.8]*** *Let f be a $C^{1,\beta}$ function for some $0 < \beta < 1$ satisfying* (1.4.5), (1.4.6)*. Then there exists a saddle solution u of $(-\Delta)^{1/2}u = f(u)$ in $\mathbb{R}^{2m}$, with $|u| < 1$, satisfying*

$$\mathcal{E}_{C_R}(v) \leq CR^{2m-1} \log R,$$

for every $R > 2$, where v is the harmonic extension of u in $\mathbb{R}^{2m+1}_+$.

Now, we explain the main idea in the proof of Theorems 1.6.5 and 1.6.7.

To prove the energy estimate for global minimizers, we use a comparison argument, as follows. We construct a comparison function $\overline{w}$, which takes the same value of v on $\partial^+ C_R$ and thus, by minimality of v,

$$\mathcal{E}_{C_R}(v) \leq \mathcal{E}_{C_R}(\overline{w}).$$

Then, we give an estimate for the energy of $\overline{w}$. The main difficulties arise when proving an estimate for the Dirichlet energy.

For this aim we use extension theorems for functions belonging to the fractional Sobolev space $H^{1/2}$.

Let us recall the definition of the $H^{1/2}(A)$-norm, where A is either a Lipschitz open set of $\mathbb{R}^n$, or $A = \partial\Omega$ and Ω is a Lipschitz open set of $\mathbb{R}^{n+1}_+$. It is given by

$$||w||^2_{H^{1/2}(A)} = ||w||^2_{L^2(A)} + \int_A \int_A \frac{|w(z) - w(\overline{z})|^2}{|z - \overline{z}|^{n+1}} d\sigma_z d\sigma_{\overline{z}}.$$

In the sequel we will use it for $\Omega = C_R$ and $A = \partial C_R$.

In the proof of Theorem 1.6.5 a crucial point will be the following well known result.

If w is a function belonging to $H^{1/2}(\partial\Omega)$, where Ω is a bounded subset of $\mathbb{R}^{n+1}$ with Lipschitz boundary, then the harmonic extension $\overline{w}$ of w in Ω, satisfies the following inequality

$$\int_\Omega |\nabla \overline{w}|^2 \leq C||w||^2_{H^{1/2}(\partial\Omega)}. \tag{1.6.11}$$

After rescaling, we will apply this result for $\Omega = C_1$.

Finally, to give an estimate of the quantity $||w||^2_{H^{1/2}(\partial C_1)}$, we use the following result.

Theorem 1.6.8. ***[see Theorem 3.0.7]*** *Let A be either a bounded Lipschitz domain in $\mathbb{R}^n$ or $A = \partial\Omega$, where Ω is a bounded open set of $\mathbb{R}^{n+1}$ with Lipschitz boundary. Let $M \subset A$ be an open set (relative to A) with Lipschitz boundary (relative to A) $\Gamma \subset A$. Let $\varepsilon \in (0, 1/2)$.*

Let $w : A \to \mathbb{R}$ be a Lipschitz function such that, for almost every $x \in A$,

$$|w(x)| \leq c_0 \tag{1.6.12}$$

and

$$|Dw(x)| \leq c_0 \min\left\{\frac{1}{\varepsilon}, \frac{1}{\mathrm{dist}(x, \Gamma)}\right\}, \tag{1.6.13}$$

where D are all tangential derivatives to A, $\mathrm{dist}(x, \Gamma)$ is the distance from the point x to the set Γ (either in $\mathbb{R}^n$ or in $\mathbb{R}^{n+1}$), and c_0 is a positive constant.

Then,

$$||w||^2_{H^{1/2}(A)} = ||w||^2_{L^2(A)} + \int_A \int_A \frac{|w(z) - w(\overline{z})|^2}{|z - \overline{z}|^{n+1}} d\sigma_z d\sigma_{\overline{z}} \leq c_0^2 C|\log\varepsilon|, \tag{1.6.14}$$

where C is a positive constant depending only on A and Γ.

Later we will use this result for $A = \partial C_1$ and $\Gamma = \partial B_1 \times \{\lambda = 0\}$. Thus in this case the constant C that appear in (1.6.14), only depends on the dimension n.

The proof of energy estimate for monotone solutions in dimension $n = 3$ is, essentially, the same used for global minimizers. The only difficult is that, here, our solutions are not global minimizers and thus, in principle, we cannot apply a comparison argument. But in section 5 of chapter 3 (see Theorem 3.4.3), we will prove that a monotone solution is a minimizer in a suitable set. Then, we prove that our comparison function $\overline{w}$ belongs to this set (this result holds only when $n = 3$), and thus we can apply the argument as for global minimizers.

Finally, to prove the energy estimate for some saddle-shaped solutions, we use a comparison argument as before, but now in the set $\widetilde{\mathcal{O}}_R = \mathcal{O}_R \times (0,R) = (\mathcal{O} \cap B_R) \times (0,R)$, instead of C_R. Indeed, by construction, the saddle-shaped solution that we construct is not a global minimizer in $\mathbb{R}^{2m}$, but it is a minimizer in $\mathcal{O} = \{s > t\}$.

1.6.3 Energy estimates for equations with fractional diffusion

In chapter 4 (which corresponds to [7]) we continue the study of nonlinear fractional equations, extending some results contained in chapter 3 to the more general equation

$$(-\Delta)^s u = f(u) \quad \text{in } \mathbb{R}^n,$$

for every f and for $0 < s < 1$. We obtain sharp energy estimates for every fractional power $s \in (0,1)$. As a consequence, we deduce the one-dimensional symmetry property for global minimizers and monotone solutions in dimension $n = 3$, for every $1/2 < s < 1$.

The following is the main result of chapter 5. In all this subsection by $||f||_{C^{1,\beta}}$ we mean $||f||_{C^{1,\beta}([\inf u, \sup u])}$.

Theorem 1.6.9. *[see Theorem 4.1.2] Let f be any $C^{1,\beta}$ nonlinearity, with $\beta > \max\{0, 1-2s\}$, and $u : \mathbb{R}^n \to \mathbb{R}$ be a solution of (1.0.1). Let v be the s-extension of u in $\mathbb{R}^{n+1}_+$.*
a) If u is a global minimizer, then for all $R > 2$

$$\begin{aligned} &\int_{C_R} \frac{1}{2}\lambda^{1-2s}|\nabla v|^2 dx d\lambda + \int_{B_R} \{G(u) - c_u\} dx \le CR^{n-2s} \quad \text{if } 0 < s < 1/2 \\ &\int_{C_R} \frac{1}{2}\lambda^{1-2s}|\nabla v|^2 dx d\lambda + \int_{B_R} \{G(u) - c_u\} dx \le CR^{n-1} \quad \text{if } 1/2 < s < 1, \end{aligned} \tag{1.6.15}$$

where c_u is defined by (1.6.9) and C denotes different positive constants depending only on n, $||f||_{C^{1,\beta}}$, $||u||_{L^\infty(\mathbb{R}^n)}$ and s.
b) Let $n = 3$ and suppose that u is a bounded solution of (1.0.1) such that $\partial_e u > 0$ in $\mathbb{R}^3$ for some direction $e \in \mathbb{R}^3$, $|e| = 1$. Then, for all $R > 2$, v satisfies the energy estimates (1.6.15) with $n = 3$.

In dimension $n = 3$ and for every $1/2 < s < 1$, Theorems 1.6.5 lead to 1-D symmetry of global minimizers and of bounded monotone solutions of problem (1.0.1).

Theorem 1.6.10. ***[see Theorem 4.1.4]*** *Suppose $n = 3$ and $1/2 \leq s < 1$. Let f be any $C^{1,\beta}$ nonlinearity with $\beta > \max\{0, 1-2s\}$ and u be either a bounded global minimizer of* (1.0.1)*, or a bounded solution monotone in some direction $e \in \mathbb{R}^3$, $|e| = 1$.*

Then, u depends only on one variable, i.e., there exists $a \in \mathbb{R}^3$ and $g : \mathbb{R} \to \mathbb{R}$, such that $u(x) = g(a \cdot x)$ for all $x \in \mathbb{R}^3$, or equivalently the level sets of u are planes.

The previous result for $s = 1/2$ is Theorem 1.6.6.

Remark 1.6.11. In [14] Caffarelli, Roquejoffre, and Savin develop a regularity theory for nonlocal minimal surfaces. This surfaces can be interpreted as a non-infinitesimal version of classical minimal surfaces and can be attained by minimizing the H^s-norm of the indicator function when $0 < s < 1/2$. A crucial fact here is that when $0 < s < 1/2$ the indicator functions belong to the space H^s and the extension problem (1.3.2) is a well posed problem for indicator functions. The authors also prove a sharp energy estimate CR^{n-2s} related to ours in some sense: our equation is the Allen-Cahn approximation of these nonlocal minimal surfaces.

As for the case of the half-Laplacian, to prove the 1-D symmetry result above, we use a method, based on a Liouville-type result, which requires the following estimate for the Dirichlet energy

$$\int_{C_R} \frac{1}{2}\lambda^{1-2s}|\nabla v|^2 dx d\lambda \leq CR^2 \log R.$$

This is the reason for which our 1-D symmetry result in dimension $n = 3$ holds only for $1/2 < s < 1$.

The proof of Theorem 1.6.9 follows the one given for the case of the half-Laplacian.

We have seen that, in the proof of the estimate for the Dirichlet energy for the case $s = 1/2$, we use a well known extension result for functions belonging to $H^{1/2}(\partial C_1)$. Here, the Dirichlet energy contains a weight and we have to use a new extension theorem.

First we recall the definition of the $H^s(A)$-norm, for $0 < s < 1$, where A is either a Lipschitz open set of $\mathbb{R}^n$, or $A = \partial\Omega$ and Ω is a Lipschitz open set of $\mathbb{R}^{n+1}_+$. It is given by

$$||w||^2_{H^s(A)} = ||w||^2_{L^2(A)} + \int_A \int_A \frac{|w(z) - w(\overline{z})|^2}{|z - \overline{z}|^{n+2s}} d\sigma_z d\sigma_{\overline{z}}.$$

In the sequel we will use it for $\Omega = C_1$ and $A = \partial C_1$.

We fix some notations. Let A be either a Lipschitz domain D in $\mathbb{R}^n$ or $A = \partial\Omega$ where Ω is a bounded subset of $\mathbb{R}^{n+1}$ with Lipschitz boundary. Let $M \subset A$ be an open set (relative to A) with Lipschitz boundary (relative to A) $\Gamma = \partial M$.

We define the following two sets:

$$B_s = \begin{cases} A \times A & \text{if } 0 < s < 1/2 \\ M \times M & \text{if } 1/2 < s < 1; \end{cases} \tag{1.6.16}$$

$$B_w = \begin{cases} (A \setminus M) \times (A \setminus M) & \text{if } 0 < s < 1/2 \\ (A \setminus M) \times A & \text{if } 1/2 < s < 1. \end{cases} \tag{1.6.17}$$

The following is the crucial extension Theorem, that we will apply in the proof of our energy estimates.

Theorem 1.6.12. ***[see Theorem 4.1.6]*** *Let Ω be a bounded subset of $\mathbb{R}^{n+1}$ with Lipschitz boundary $\partial\Omega$ and M a Lipschitz subset of $\partial\Omega$. For $z \in \mathbb{R}^{n+1}$, let $d_M(z)$ denote the Euclidean distance from the point z to the set M. Let w belong to $C(\partial\Omega)$.*

Then, there exists an extension $\widetilde{w}$ of w in Ω belonging to $C^1(\Omega) \cap C(\overline{\Omega})$, such that

$$\begin{aligned} \int_\Omega d_M(z)^{1-2s} |\nabla \widetilde{w}|^2 dz \leq C||w||^2_{L^2(\partial\Omega)} + C \int\int_{B_s} \frac{|w(z) - w(\overline{z})|^2}{|z - \overline{z}|^{n+2s}} d\sigma_z d\sigma_{\overline{z}} \\ + C \int\int_{B_w} d_M(z)^{1-2s} \frac{|w(z) - w(\overline{z})|^2}{|z - \overline{z}|^{n+1}} d\sigma_z d\sigma_{\overline{z}}, \end{aligned} \tag{1.6.18}$$

where B_s and B_w are defined, respectively, in (1.6.16) *and* (1.6.17) *with $A = \partial\Omega$, and C denotes different positive constants depending on Ω, M and s.*

We have used the notations B_s and B_w to indicate, respectively, the set in which we compute the H^s-norm of w and the set in which we compute the weighted $H^{1/2}_{d^a_M}$-norm of w.

Finally, the following lemma is the analog of Lemma 1.6.8, for the other fractional powers of the Laplacian.

Theorem 1.6.13. ***[see Theorem 4.1.8]*** *Let A be either a Lipschitz domain in $\mathbb{R}^n$ or $A = \partial\Omega$ where Ω is a bounded subset of $\mathbb{R}^{n+1}$ with Lipschitz boundary. Let $M \subset A$ be an open set (relative to A) with Lipschitz boundary (relative o A)*

$\Gamma = \partial M$. Let $\varepsilon \in (0,1/2)$. Let $w : A \to \mathbb{R}$ be a Lipschitz function such that for almost every $z \in A$,

$$|w(z)| \le c_0 \tag{1.6.19}$$

and

$$|Dw(z)| \le \frac{c_0}{d_\Gamma(z)} \min\left\{1, \left(\frac{d_\Gamma(z)}{\varepsilon}\right)^{\min\{1,2s\}}\right\} \tag{1.6.20}$$

where D are all tangential derivatives to A, $d_\Gamma(z)$ is the Euclidean distance from the point z to the set Γ (either in $\mathbb{R}^n$ or in $\mathbb{R}^{n+1}$), and c_0 is a positive constant.

Then,

$$||w||_{L^2(A)} + \int\int_{B_s} \frac{|w(z) - w(\overline{z})|^2}{|z - \overline{z}|^{n+2s}} d\sigma_z d\sigma_{\overline{z}} + \int\int_{B_w} d_M(z)^{1-2s} \frac{|w(z) - w(\overline{z})|^2}{|z - \overline{z}|^{n+1}} d\sigma_z d\sigma_{\overline{z}}$$
$$\le \begin{cases} Cc_0^2 & \text{if } 0 < s < 1/2, \\ Cc_0^2 \varepsilon^{1-2s} & \text{if } 1/2 < s < 1. \end{cases} \tag{1.6.21}$$

where C denotes a positive constant depending only on A, M, and s and the sets B_s and B_w are defined in (1.6.16) *and* (1.6.17)*.*

We will use these two results for $A = \partial C_1$, $M = B_1 \times \{\lambda = 0\}$ and $\Gamma = \partial B_1 \times \{\lambda = 0\}$, thus in this case the constants C that appears in (1.6.18) and (1.6.21), only depend on the dimension n and the power s.

Chapter 2

Saddle-shaped solutions for the half-Laplacian

2.1 Introduction and results

This chapter (which corresponds to [19]) concerns the study of saddle-shaped solutions of elliptic equations with fractional diffusion

$$(-\Delta)^{1/2}u = f(u) \quad \text{in } \mathbb{R}^n, \tag{2.1.1}$$

where $n = 2m$ is an even integer and f is of bistable type.

Our interest in these solutions originates from the following conjecture of De Giorgi. Consider the nonlinear elliptic equation

$$-\Delta u = u - u^3 \quad \text{in } \mathbb{R}^n, \tag{2.1.2}$$

which is called the Allen-Cahn equation modelling phase transitions. In 1978 De Giorgi conjectured that the level sets of every bounded solution of (2.1.2), which is monotone in one direction, must be hyperplanes, at least if $n \leq 8$. That is, such solutions depend only on one Euclidian variable.

The conjecture has been proven to be true in dimension $n = 2$ by Ghoussoub and Gui [24] and in dimension $n = 3$ by Ambrosio and Cabré [3]. For $4 \leq n \leq 8$, if $\partial_{x_n} u > 0$, and assuming the additional condition

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1 \quad \text{for all } x' \in \mathbb{R}^{n-1},$$

it has been established by Savin [34]. Recently a counterexample to the conjecture for $n \geq 9$ has been announced by del Pino, Kowalczyk and Wei [22].

For the fractional equation $(-\Delta)^s u = f(u)$ in $\mathbb{R}^n$ with $0 < s < 1$, the conjecture has been proven to be true when $n = 2$ and $s = 1/2$ by Cabré and Solà-Morales

[10], and when $n = 2$ and for every $0 < s < 1$ by Cabré and Sire [8], and by Sire and Valdinoci [37]. In chapters 3 and 4 we will prove the conjecture in dimension $n = 3$ for every power $1/2 \leq s < 1$.

The study of saddle-shaped solutions is related to the possibility of finding a counterexample to the conjecture of De Giorgi in large dimensions.

More precisely, by a deep result by Savin [34], if $n \leq 7$ then every global minimizer of the equation $-\Delta u = u - u^3$ in $\mathbb{R}^n$ is one-dimensional. A natural question arises: is there a global minimizer in $\mathbb{R}^8$ (or higher dimensions) which is not one-dimensional? Saddle-shaped solutions are the candidates to give a positive answer to this question.

Moreover, by a result of Jerison and Monneau [25], if one could prove that saddle-shaped solutions are global minimizers in $\mathbb{R}^8$, one would have a counterexample to the conjecture of De Giorgi in $\mathbb{R}^9$, in an alternative way to that of [22].

Saddle-shaped solutions are expected to have relevant variational properties due to a well known connection between nonlinear equations modelling phase transitions and the theory of minimal surfaces. This connection also motivated the conjecture of De Giorgi.

More precisely the saddle-shaped solutions that we consider are even with respect to the coordinate axes and odd with respect to the Simons cone, which is defined as follows. For $n = 2m$ the Simons cone $\mathcal{C}$ is given by:

$$\mathcal{C} = \{x \in \mathbb{R}^{2m} : x_1^2 + ... + x_m^2 = x_{m+1}^2 + ... + x_{2m}^2\}.$$

We recall that the Simons cone has zero mean curvature at every point $x \in \mathcal{C} \setminus \{0\}$, in every dimension $2m \geq 2$. Moreover in dimensions $2m \geq 8$ it is a minimizer of the area functional, that is, it is a minimal cone (in the variational sense).

We define two new variables

$$s = \sqrt{x_1^2 + \cdots + x_m^2} \quad \text{and} \quad t = \sqrt{x_{m+1}^2 + \cdots + x_{2m}^2},$$

for which the Simons cone becomes $\mathcal{C} = \{s = t\}$.

We now introduce our notion of saddle-shaped solution. These solutions depend only on s and t, and are odd with respect to the Simons cone.

Definition 2.1.1. Let u be a bounded solution of $(-\Delta)^{1/2} u = f(u)$ in $\mathbb{R}^{2m}$, where $f \in C^1$ is odd. We say that $u : \mathbb{R}^{2m} \to \mathbb{R}$ is a *saddle-shaped* (or simply *saddle*) solution if

(a) u depends only on the variables s and t. We write $u = u(s,t)$;

(b) $u > 0$ for $s > t$;

(c) $u(s,t) = -u(t,s)$.

Remark 2.1.2. If u is a saddle solution then, in particular, $u = 0$ on the Simons cone $\mathcal{C} = \{s = t\}$. In other words, $\mathcal{C}$ is the zero level set of u.

Saddle solutions for the classical equation $-\Delta u = f(u)$ were first studied by Dang, Fife, and Peletier in [20] in dimension $n = 2$ for f odd, bistable and $f(u)/u$ decreasing for $u \in (0, 1)$. They proved the existence and uniqueness of saddle-shaped solutions and established monotonicity properties and the asymptotic behaviour. The instability property of saddle solutions in dimension $n = 2$ was studied by Schatzman [35]. In two recent works [12, 13], Cabré and Terra proved the existence of saddle-shaped solutions for the equation $-\Delta u = f(u)$ in $\mathbb{R}^n$, where f is of bistable type, in every even dimension $n = 2m$. Moreover they established some qualitative properties of these solutions as monotonicity properties, asymptotic behaviour and instability in dimensions $2m = 4$ and $2m = 6$.

In this work, we establish existence and qualitative properties of saddle-shaped solutions for the bistable fractional equation (2.1.1).

To study the nonlocal problem (2.1.1) we will realize it as a local problem in $\mathbb{R}^{n+1}_+$ with a nonlinear Neumann condition on $\partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n$. More precisely, if $u = u(x)$ is a function defined on $\mathbb{R}^n$, we consider its harmonic extension $v = v(x, \lambda)$ in $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, +\infty)$. It is well known (see [10, 15]) that u is a solution of (2.1.1) if and only if v satisfies

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ -\partial_\lambda v = f(v) & \text{on } \mathbb{R}^n = \partial\mathbb{R}^{n+1}_+. \end{cases} \tag{2.1.3}$$

Problem (2.1.3), associated to the nonlocal equation (2.1.1), allows to introduce the notions of *energy* and *global minimality* for a solution u of problem (2.1.1).

Let $\Omega \subset \mathbb{R}^{n+1}_+$ be a bounded domain. We denote by

$$\widetilde{B}^+_r = \{(x, \lambda) \in \mathbb{R}^{2m+1} : \lambda > 0,\ |(x, \lambda)| < r\}$$

and by $\widetilde{B}^+_r(x, \lambda) = (x, \lambda) + \widetilde{B}^+_r$.

We define the following subset of $\partial\Omega$:

$$\partial^0\Omega := \{(x, 0) \in \mathbb{R}^{n+1}_+ : \widetilde{B}^+_\varepsilon(x, 0) \subset \Omega \text{ for some } \varepsilon > 0\} \tag{2.1.4}$$

and

$$\partial^+\Omega := \overline{\partial\Omega \cap \mathbb{R}^{n+1}_+}. \tag{2.1.5}$$

Given a $C^{1,\alpha}$ nonlinearity $f : \mathbb{R} \to \mathbb{R}$, for some $0 < \alpha < 1$, define

$$G(u) = \int_u^1 f.$$

We have that $G \in C^2(\mathbb{R})$ and $G' = -f$.

Let v be a $C^1(\overline{\Omega})$ function with $|v| \leq 1$. We consider the energy functional

$$\mathcal{E}_\Omega(v) = \int_\Omega \frac{1}{2}|\nabla v|^2 + \int_{\partial^0\Omega} G(v). \tag{2.1.6}$$

Observe that the potential energy is computed only on the boundary $\partial^0\Omega$. This is a quite different situation from the one of interior reactions.

We start by recalling that problem (2.1.3) can be viewed as the Euler-Lagrange equation associated to the energy functional $\mathcal{E}$.

Definition 2.1.3. *a)* We say that a bounded solution v of (2.1.3) is *stable* if the second variation of energy $\delta^2\mathcal{E}/\delta^2\xi$, with respect to perturbations ξ compactly supported in $\overline{\mathbb{R}^{n+1}_+}$, is nonnegative. That is, if

$$Q_v(\xi) := \int_{\mathbb{R}^{n+1}_+} |\nabla\xi|^2 - \int_{\partial\mathbb{R}^{n+1}_+} f'(v)\xi^2 \geq 0 \tag{2.1.7}$$

for every $\xi \in C^\infty_0(\overline{\mathbb{R}^{n+1}_+})$.

We say that v is *unstable* if and only if v is not stable.

b) We say that a bounded solution u of (2.1.1) in $\mathbb{R}^{2m}$ is *stable* (*unstable*) if its harmonic extension v is a stable (unstable) solution for the problem (2.1.3).

Another important notion related to the energy functional $\mathcal{E}$ is the one of global minimality.

Definition 2.1.4. *a)* We say that a bounded $C^1(\overline{\mathbb{R}^{n+1}_+})$ function v in $\mathbb{R}^{n+1}_+$ is a *global minimizer* of (2.1.3) if

$$\mathcal{E}_\Omega(v) \leq \mathcal{E}_\Omega(v+\xi),$$

for every bounded domain $\Omega \subset \overline{\mathbb{R}^{n+1}_+}$ and every C^∞ function ξ with compact support in $\Omega \cup \partial^0\Omega$.

b) We say that a bounded C^1 function u in $\mathbb{R}^n$ is a *global minimizer* of (2.1.1) if its harmonic extension v is a global minimizer of (2.1.3).

Observe that the perturbations ξ need not vanish on $\partial^0\Omega$, in contrast from interior reactions.

In some references, global minimizers are called “local minimizers”, where local stands for the fact that the energy is computed in bounded domains. Clearly, every global minimizer is a stable solution.

Assume that

$$f \text{ is odd;} \tag{2.1.8}$$

$$G \geq 0 = G(\pm 1) \text{ in } \mathbb{R}, \text{ and } G > 0 \text{ in } (-1,1); \tag{2.1.9}$$

$$f' \text{ is decreasing in } (0,1). \tag{2.1.10}$$

Note that, if (2.1.8) and (2.1.9) hold, then $f(0) = f(\pm 1) = 0$. Conversely, if f is odd in $\mathbb{R}$, positive with f' decreasing in $(0,1)$ and negative in $(1,\infty)$ then f satisfies (2.1.8), (2.1.9) and (2.1.10). Hence, the nonlinearities f that we consider are of "balanced bistable type", while the potentials G are of "double well type". Our three assumptions (2.1.8), (2.1.9), (2.1.10) are satisfied for the scalar Allen-Cahn type equation

$$(-\Delta)^{1/2} u = u - u^3. \tag{2.1.11}$$

In this case we have that $G(u) = (1/4)(1-u^2)^2$ and (2.1.8), (2.1.9), (2.1.10) hold. The three hypothesis also hold for the Peierls-Nabarro problem

$$(-\Delta)^{1/2} u = \sin(\pi u), \tag{2.1.12}$$

for which $G(u) = (1/\pi)(1 + \cos(\pi u))$.

By a result of Cabré and Solà-Morales [10], assumption (2.1.9) on G guarantees the existence of an increasing solution, from -1 to 1, of (2.1.1) in $\mathbb{R}$. We call these solutions *layer* solutions. In addition, such an increasing solution is unique up to translations.

The following is the precise result established in [10].

Theorem 2.1.5. *([10]) Let f be any $C^{1,\alpha}$ function with $0 < \alpha < 1$ and $G' = -f$. Then:*

- *There exists a layer solution u_0 of $(-\Delta)^{1/2} u_0 = f(u_0)$, if and only if*

$$G'(-1) = G'(1) = 0, \quad \textit{and } G > G(-1) = G(1) \textit{ in } (-1,1).$$

- *If $f'(\pm 1) < 0$, then a layer solution of (2.1.3) is unique up to translations.*
- *If f is odd and $f'(\pm 1) < 0$, then every layer solution of (2.1.3) is odd in x with respect to some half-axis. That is, $u(x+b) = -u(-x+b)$ for some $b \in \mathbb{R}$.*

Normalizing the layer solution to vanishing at the origin, we call it u_0 and its harmonic extension in the half-plane v_0. Thus we have

$$\begin{cases} u_0 : \mathbb{R} \to (-1,1) \\ u_0(0) = 0, \quad u_0' > 0 \\ (-\Delta)^{1/2} u_0 = f(u_0) \quad \text{in } \mathbb{R}. \end{cases} \tag{2.1.13}$$

The monotone bounded solution u_0 of the Peierls-Nabarro problem (2.1.12) in $\mathbb{R}$ is explicit; calling v_0 its harmonic extension in $\mathbb{R}^2_+$ we have that

$$v_0(x,\lambda) = \frac{2}{\pi} \arctan \frac{x}{\lambda + 1/\pi}.$$

In the following theorem, we establish the existence of a saddle-shaped solution for problem (2.1.1) in every even dimension $n = 2m$. We use the following notations:

$$\mathcal{O} := \{x \in \mathbb{R}^{2m} : s > t\} \subset \mathbb{R}^{2m}$$

$$\widetilde{\mathcal{O}} := \{(x, \lambda) \in \mathbb{R}^{2m+1}_+ : x \in \mathcal{O}\} \subset \mathbb{R}^{2m+1}_+$$

Note that

$$\partial \mathcal{O} = \mathcal{C}.$$

We define the cylinder $C_{R,L} = B_R \times (0, L)$, where B_R is the open ball in $\mathbb{R}^{2m}$ centered at the origin and of radius R.

Theorem 2.1.6. *For every dimension $2m \geq 2$ and every nonlinearity f satisfying* (2.1.8) *and* (2.1.9)*, there exists a saddle solution u of $(-\Delta)^{1/2} u = f(u)$ in $\mathbb{R}^{2m}$, such that $|u| < 1$ in $\mathbb{R}^{2m}$.*

Let v be the harmonic extension of the saddle solution u in $\mathbb{R}^{2m+1}_+$. If in addition f satisfies (2.1.10)*, then the second variation of the energy $Q_v(\xi)$ at v, as defined in* (2.1.7)*, is nonnegative for all function $\xi \in C^1(\overline{\mathbb{R}^{2m+1}_+})$ with compact support in $\overline{\mathbb{R}^{2m+1}_+}$ and vanishing on $\mathcal{C} \times [0, +\infty)$.*

We prove the existence of a saddle solution u for problem (2.1.1), by proving the existence of a solution v for problem (2.1.3), with the following properties:

1. v depends only on the variables s, t and λ. We write $v = v(s, t, \lambda)$;
2. $v > 0$ for $s > t$;
3. $v(s, t, \lambda) = -v(t, s, \lambda)$.

Using a variational technique we construct a solution v in $\widetilde{\mathcal{O}}$ for the problem (2.1.3), satisfying $v > 0$ in $\widetilde{\mathcal{O}}$ and $v = 0$ on $\partial\widetilde{\mathcal{O}} = \mathcal{C} \times \mathbb{R}^+$. Then, since f is odd, by odd reflection with respect to $\partial\widetilde{\mathcal{O}}$ we obtain a solution v in the whole space which satisfies properties (1), (2), (3). Clearly the function $u(x) = v(x, 0)$ is a saddle solution for the problem (2.1.1). To prove this existence result, we will use the following not-sharp energy estimate for v. Given $1/2 \leq \gamma < 1$, there exists $\varepsilon = \varepsilon(\gamma) > 0$ such that

$$\mathcal{E}_{C_{S,S^\gamma}}(v) \leq C S^{2m-\varepsilon}. \tag{2.1.14}$$

In Theorem 3.0.8 of chapter 3, we establish the following sharp energy estimates for saddle-shaped solutions,

$$\mathcal{E}_{C_{S,S}}(v) \leq C S^{2m-1} \log S.$$

Here, (2.1.14) is not sharp, but it is enough to prove the existence of a saddle solution.

For solutions of problem (2.1.3) depending only on the coordinates s, t and λ, problem (2.1.3) becomes

$$\begin{cases} -(v_{ss}+v_{tt}+v_{\lambda\lambda})-(m-1)\left(\dfrac{v_s}{s}+\dfrac{v_t}{t}\right)=0, & \text{in } \mathbb{R}^{2m+1}_+ \\ -\partial_\lambda v = f(v) & \text{on } \partial\mathbb{R}^{2m+1}_+. \end{cases} \tag{2.1.15}$$

while the energy functional becomes

$$\mathcal{E}(v,\Omega)=c_m\left\{\int_\Omega s^{m-1}t^{m-1}\frac{1}{2}(v_s^2+v_t^2+v_\lambda^2)dsdtd\lambda+\int_{\partial^0\Omega}s^{m-1}t^{m-1}G(v)dsdt\right\}, \tag{2.1.16}$$

where c_m is a positive constant depending only on m—here we have assumed that $\Omega\subset\mathbb{R}^{2m+1}$ is radially symmetric in the first m variables and also in the last m variables, and we have abused notation by identifying Ω with its projection in the (s,t,λ) variables.

In section 5, we prove the existence and monotonicity properties of the maximal saddle solution.

To establish these results, we need to introduce a new nonlocal operator $D_{H,\varphi}$, which is the square root of the Laplacian, for functions defined in domains $H\in\mathbb{R}^n$ which do not vanish on ∂H. We introduce this operator and we establish maximum principles for it, in section 4.

We define the new variables

$$\begin{cases} y=\dfrac{s+t}{\sqrt{2}} \\ z=\dfrac{s-t}{\sqrt{2}}. \end{cases} \tag{2.1.17}$$

Note that $|z|\le y$ and that we may write the Simons cone as $\mathcal{C}=\{z=0\}$.

The following theorem concerns the existence and monotonicity properties of a maximal saddle solution.

Theorem 2.1.7. *Let f satisfy conditions* (2.1.8), (2.1.9), *and* (2.1.10).

Then, there exists a saddle solution $\overline{u}$ of $(-\Delta)^{1/2}\overline{u}=f(\overline{u})$ in $\mathbb{R}^{2m}$, with $|u|<1$, which is maximal in the following sense. For every solution u of $(-\Delta)^{1/2}u=f(u)$ in $\mathbb{R}^{2m}$, vanishing on the Simons cone and such that u has the same sign as $s-t$, we have

$$0<u<\overline{u}\quad in\ \mathcal{O}.$$

As a consequence, we also have

$$0<|u|<|\overline{u}|\quad in\ \mathbb{R}^{2m}.$$

In addition, if $\overline{v}$ is the harmonic extension of $\overline{u}$ in $\mathbb{R}^{2m+1}_+$, then $\overline{v}$ satisfies:

(a) $\partial_s \overline{v} \geq 0$ *in* $\overline{\mathbb{R}^{2m+1}_+}$. *Furthermore* $\partial_s \overline{v} > 0$ *in* $\overline{\mathbb{R}^{2m+1}_+} \setminus \{s = 0\}$ *and* $\partial_s \overline{v} = 0$ *in* $\{s = 0\}$*;*

(b) $\partial_t \overline{v} \leq 0$ *in* $\overline{\mathbb{R}^{2m+1}_+}$. *Furthermore* $\partial_t \overline{v} < 0$ *in* $\overline{\mathbb{R}^{2m+1}_+} \setminus \{t = 0\}$ *and* $\partial_t \overline{v} = 0$ *in* $\{t = 0\}$*;*

(c) $\partial_z \overline{v} > 0$ *in* $\overline{\mathbb{R}^{2m+1}_+} \setminus \{0\}$*;*

(d) $\partial_y \overline{v} > 0$ *in* $\{s > t > 0\} \times [0, +\infty)$.

As a consequence, for every direction $\partial_\eta = \alpha \partial_y - \beta \partial_t$*, with* α *and* β *nonnegative constants,* $\partial_\eta \overline{v} > 0$ *in* $\{s > t > 0\} \times [0, +\infty)$.

In the proof of Theorem 2.1.7 we will use the following proposition, which gives a supersolution for problem (2.1.3) in the set $\widetilde{\mathcal{O}}$.

Proposition 2.1.8. *Let* f *satisfy hypothesis* (2.1.8), (2.1.9), (2.1.10). *Let* u_0 *be the layer solution, vanishing at the origin, of problem* (2.1.1) *in* $\mathbb{R}$ *and let* v_0 *be its harmonic extension in* $\mathbb{R}^{2m+1}_+$.

Then, the function $v_0(z, \lambda) = v_0\left(\frac{s-t}{\sqrt{2}}, \lambda\right)$ *satisfies*

$$\begin{cases} -\Delta v_0 \geq 0 & \text{in } \widetilde{\mathcal{O}} \\ -\partial_\lambda v_0 \geq f(v_0) & \text{on } \mathcal{O} \times \{0\}. \end{cases}$$

In [13] an important ingredient in the proof of the existence of a maximal solution for interior reactions is the following pointwise estimate. Let u_1 be a saddle solution of $-\Delta u_1 = f(u_1)$ in $\mathbb{R}^{2m}$, with f bistable, and let $u_{1,0}$ be the layer solution in dimension $n = 1$ of $-u''_{1,0} = f(u_{1,0})$ (whose existence is guaranteed by hypothesis (2.1.9) on f). Then

$$|u_1(x)| \leq |u_{1,0}\left(d(x, \mathcal{C})\right)| = \left| u_{1,0}\left(\frac{|s-t|}{\sqrt{2}}\right)\right| \quad \text{for every } x \in \mathbb{R}^{2m}, \tag{2.1.18}$$

where $d(\cdot, \mathcal{C})$ denotes the distance to the Simons cone. This estimate follows by an important gradient bound of Modica [30] for the classical equation $-\Delta u = f(u)$ in $\mathbb{R}^n$. Moreover, another important ingredient in the proof of the existence of a maximal solution for interior reactions is that $u_{1,0}\left((s-t)/\sqrt{2}\right)$ is a supersolution in $\mathcal{O}$.

In the fractional case the Modica gradient estimate is not available. In [10] a non-local Modica-type estimate is established, but only in dimension $n = 1$. Thus we cannot prove the analog of (2.1.18) for solutions of the equation $(-\Delta)^{1/2} u = f(u)$. For this reason, to give an upper bound for saddle solutions, we need to consider the function $\min\{K v_0((s-t)/\sqrt{2}, \lambda), 1\}$ where $K \geq 1$ is a large constant depending only on n and f. Proposition 2.1.8 implies that this function is a supersolution in $\widetilde{\mathcal{O}}$. Moreover, we will show that there exists $K \geq 1$, depending only on n and f, such that if v is a bounded solution of problem (2.1.3), then

$$|v(x,\lambda)| \leq \min\{Kv_0(|s-t|/\sqrt{2},\lambda),1\}, \quad \text{for every } (x,\lambda) \in \mathbb{R}^{2m+1}_+. \qquad (2.1.19)$$

Estimate (2.1.19) follows by regularity results established in [10]. More precisely, in [10] Cabré and Solà-Morales proved that if v is a bounded solution of (2.1.3), then there exists a constant C depending on n and $||f||_{C^1}$ such that

$$|\nabla v(x,\lambda)| \leq C \quad \text{for every } x \in \mathbb{R}^n \text{ and } \lambda \geq 0.$$

If v is a saddle solution, then $v(x,\lambda) = v(y,z,\lambda) = 0$ on $\mathcal{C} = \{z=0\}$ and then

$$|v(x,\lambda)| = |v(y,z,\lambda)| \leq C|z| \quad \text{for every } x \in \mathbb{R}^n \text{ and } \lambda \geq 0.$$

Thus, we can choose K big enough such that

$$\min\{C|z|,1\} \leq \min\{Kv_0(z,\lambda),1\}.$$

This is possible since $v_0(z,\lambda) > 0$ for every $z > 0$, $\lambda \geq 0$ and $\partial_z v_0(0,\lambda) > 0$ for every $\lambda \geq 0$.

In section 6, we prove the following theorem concerning the asymptotic behaviour at infinity for a class of solutions which contains saddle-shaped solutions.

Theorem 2.1.9. *Let f satisfy conditions* (2.1.8), (2.1.9), *and* (2.1.10), *and let u be a bounded solution of $(-\Delta)^{1/2}u = f(u)$ in $\mathbb{R}^{2m}$ such that $u \equiv 0$ on $\mathcal{C}$, $u > 0$ in $\mathcal{O} = \{s > t\}$ and u is odd with respect to $\mathcal{C}$.*

Then, denoting $U(x) := u_0((s-t)/\sqrt{2}) = u_0(z)$ we have,

$$u(x) - U(x) \to 0 \quad \text{and} \quad \nabla u(x) - \nabla U(x) \to 0, \qquad (2.1.20)$$

uniformly as $|x| \to \infty$. That is,

$$||u-U||_{L^\infty(\mathbb{R}^{2m}\setminus B_R)} + ||\nabla u - \nabla U||_{L^\infty(\mathbb{R}^{2m}\setminus B_R)} \to 0 \ \text{ as } R \to \infty. \qquad (2.1.21)$$

Our proof of Theorem 2.1.9 follows the one given by Cabré and Terra in [13], and uses a compactness argument based on translations of the solutions, combined with two crucial Liouville-type results for nonlinear equations in the half-space and in a quarter of space.

Finally, in section 7 we establish that saddle-shaped solutions are instable in dimension $2m = 4$ and $2m = 6$.

Theorem 2.1.10. *Let f satisfy conditions* (2.1.8), (2.1.9), (2.1.10). *Then, every bounded solution u of $(-\Delta)^{1/2}u = f(u)$ in $\mathbb{R}^{2m}$ such that $u = 0$ on the Simons cone $\mathcal{C} = \{s = t\}$ and u has the same sign as $s - t$, is unstable in dimension $2m = 4$ and $2m = 6$.*

Instability in dimension $2m = 2$ follows by a result of Cabré and Solà Morales [10] which asserts that every stable solution of (2.1.1) in dimension $n = 2$ is one-dimensional. This is the analog of the conjecture of De Giorgi in dimension $n = 2$ for the half-Laplacian.

In [12], Cabré and Terra proved instability in dimension $2m = 4$ for saddle-shaped solutions of the classical equation $-\Delta u = f(u)$ in $\mathbb{R}^4$. A crucial ingredient in the proof of this result is the pointwise estimate (3.2.8).

However, in dimension $2m = 6$, this estimate is not enough to prove instability and thus Cabré and Terra used a more precise argument, based on some monotonicity properties and asymptotic behaviour of a maximal saddle solution.

As said before, we cannot prove the analog of (3.2.8) for solutions of the equation $(-\Delta)^{1/2}u = f(u)$.

Thus, here, we follow the argument introduced by Cabré and Terra in dimension $2m = 6$, both for the case $2m = 4$ and $2m = 6$.

Using this approach, the crucial ingredients in the proof of Theorem 2.1.10 are:

i) the equation satisfied by $\overline{v}$, which is the harmonic extension of the maximal saddle solution $\overline{u}$ in $\mathbb{R}^{2m+1}_+$;

ii) a monotonicity property of $\overline{v}$;

iii) the asymptotic behaviour at infinity of $\overline{v}$.

The chapter is organized as follows:

- In section 2 we prove Theorem 2.1.6 concerning the existence of a saddle solution for the equation (2.1.1) in every dimension $2m$.
- In section 3, we give a supersolution and a subsolution for the square root of the Laplacian in a domain $H \subset \mathbb{R}^n$. In particular we prove Proposition 2.1.8.
- In section 4, we introduce the operator $D_{H,\varphi}$ and we establish maximum principles.
- In section 5, we prove the existence of a maximal saddle solution $\overline{u}$ and its monotonicity properties (Theorem 2.1.7).
- In section 6, we prove Theorem 2.1.9, concerning the asymptotic behaviour.
- In section 7, we prove Theorem 2.1.10 about the instability of saddle solutions in dimensions $2m = 4$ and $2m = 6$.

2.2 Existence of a saddle solution in $\mathbb{R}^{2m}$

In this section we prove the existence of a saddle solution u for problem (2.1.1), by proving the existence of a solution v for problem (2.1.3) with the following properties:

1. v depends only on the variables s, t and λ. We write $v = v(s,t,\lambda)$;
2. $v > 0$ for $s > t$;
3. $v(s,t,\lambda) = -v(t,s,\lambda)$.

We recall that we have defined the sets:

$$\mathcal{O} = \{x \in \mathbb{R}^{2m} : s > t\} \subset \mathbb{R}^{2m}, \quad \widetilde{\mathcal{O}} = \{(x,\lambda) \in \mathbb{R}^{2m+1}_+ : x \in \mathcal{O}\} \subset \mathbb{R}^{2m+1}_+.$$

Let B_R be the open ball in $\mathbb{R}^{2m}$ centered at the origin and of radius R. We will consider the open bounded sets

$$\mathcal{O}_R := \mathcal{O} \cap B_R = \{s > t, |x|^2 = s^2 + t^2 < R^2\} \subset \mathbb{R}^{2m}.$$

$$\widetilde{\mathcal{O}}_{R,L} := (\mathcal{O} \cap B_R) \times (0,L) = \{(x,\lambda) \in \mathbb{R}^{2m+1}_+ : s > t,\ |x|^2 = s^2 + t^2 < R^2, \lambda < L\}.$$

Note that

$$\partial \mathcal{O}_R = (\mathcal{C} \cap \overline{B}_R) \cup (\partial B_R \cap \mathcal{O}).$$

Before giving the proof of Theorem 2.1.6, we recall some results established in [10] concerning the regularity of weak solutions of problem (2.1.3). Cabré and Solà-Morales [10] proved that every bounded weak solution v of problem (2.1.3) with $f \in C^{1,\alpha}$, satisfies $v \in C^{1,\alpha}$, for all $0 < \alpha < 1$. This result was deduced using the auxiliary function

$$w(x,\lambda) = \int_0^\lambda v(x,t)dt,$$

which is a solution of the Dirichlet problem

$$\begin{cases} -\Delta w = f(v(x,0)) & \text{in } \mathbb{R}^{2m+1}_+ \\ w(x,0) = 0 & \text{on } \partial\mathbb{R}^{2m+1}_+ \end{cases}.$$

Applying the standard regularity results for the Dirichlet problem above, they deduce regularity for the solution v of problem (2.1.3). Moreover, using standard elliptic estimates for bounded harmonic functions, we have that the following gradient bound for v holds:

$$|\nabla v(x,\lambda)| \le \frac{C}{1+\lambda} \quad \text{for every } (x,\lambda) \in \mathbb{R}^{2m+1}_+. \tag{2.2.1}$$

We define now the sets

$$\widetilde{L}^2(\widetilde{\mathcal{O}}_{R,L}) = \{v \in L^2(\widetilde{\mathcal{O}}_{R,L}) : v = v(s,t,\lambda) \text{ a.e.}\}$$

and

$$\widetilde{H}^1_0(\widetilde{\mathcal{O}}_{R,L}) = \{v \in H^1(\widetilde{\mathcal{O}}_{R,L}) : v \equiv 0 \text{ on } \partial^+\widetilde{\mathcal{O}}_{R,L},\ v = v(s,t,\lambda) \text{ a.e.}\}.$$

They are, respectively, the set of L^2 functions in the bounded open set $\widetilde{\mathcal{O}}_{R,L}$ which depend only on s, t, and λ, and the set of H^1 functions in the bounded open set $\widetilde{\mathcal{O}}_{R,L}$ which depend only on s, t and λ and which vanish on the positive boundary $\partial^+\widetilde{\mathcal{O}}_{R,L}$ in the weak sense.

We recall that the inclusion $\widetilde{H}^1_0(\widetilde{\mathcal{O}}_{R,L}) \subset\subset L^2(\widetilde{\mathcal{O}}_{R,L})$ is compact (see [10]). Indeed, let $v \in \widetilde{H}^1_0(\widetilde{\mathcal{O}}_{R,L})$. Since $v \equiv 0$ on $\partial^+\widetilde{\mathcal{O}}_{R,L}$, we can extend v to be identically 0 in $\mathbb{R}^{2m+1}_+ \setminus \widetilde{\mathcal{O}}_{R,L}$, and we have $v \in \widetilde{H}^1(\mathbb{R}^{2m+1}_+) = \{v \in H^1(\mathbb{R}^{2m+1}_+) : v = v(s,t,\lambda) \text{ a.e.}\}$. We have

$$\int_{\partial^0\widetilde{\mathcal{O}}_{R,L}} |v(x,0)|^2 dx = -\int_{\mathbb{R}^{n+1}_+} \partial_\lambda(|v|^2) = -2\int_{\mathbb{R}^{n+1}_+} v\partial_\lambda v \leq C||v||_{\widetilde{L}^2(\widetilde{\mathcal{O}}_{R,L})}||v||_{\widetilde{H}^1(\widetilde{\mathcal{O}}_{R,L})}.$$

Now, the compactness of the inclusion, follows from the fact that since $v \equiv 0$ on $\partial^+\widetilde{\mathcal{O}}_{R,L}$ a.e., then $\widetilde{H}^1_0(\widetilde{\mathcal{O}}_{R,L}) \subset\subset \widetilde{L}^2(\widetilde{\mathcal{O}}_{R,L})$ is compact (to see this it is enough to extend v to be identically zero in a $A \setminus \widetilde{\mathcal{O}}_{R,L}$, where $A \subset \mathbb{R}^{n+1}_+$ is a Lipschitz set containing $\widetilde{\mathcal{O}}_{R,L}$).

We can now give the proof of Theorem 2.1.6.

Proof of Theorem 2.1.6. As already mentioned, we prove the existence of a solution v for the problem (2.1.3) such that $v = v(s,t,\lambda)$ and $v(s,t,\lambda) = -v(-t,s,\lambda)$. The space $\widetilde{H}^1_0(\widetilde{\mathcal{O}}_{R,L})$, defined above, is a weakly closed subspace of $H^1(\widetilde{\mathcal{O}}_{R,L})$.

Consider the energy functional in $\widetilde{\mathcal{O}}_{R,L}$,

$$\mathcal{E}_{\widetilde{\mathcal{O}}_{R,L}}(v) = \int_{\widetilde{\mathcal{O}}_{R,L}} \frac{1}{2}|\nabla v|^2 + \int_{\partial^0\widetilde{\mathcal{O}}_{R,L}} G(v) \qquad \text{for every } v \in \widetilde{H}^1_0(\widetilde{\mathcal{O}}_{R,L}).$$

Next, we prove the existence of a minimizer of the functional among functions in this space. Recall that we assume condition (2.1.9) on G, that is,

$$G(\pm 1) = 0 \quad \text{and } G > 0 \text{ in } (-1,1).$$

We define a continuous extension $\widetilde{G}$ of G in $\mathbb{R}$ such that

- $\widetilde{G} = G$ in $[-1,1]$,
- $\widetilde{G} > 0$ in $\mathbb{R} \setminus [-1,1]$,

- $\widetilde{G}$ is even,
- $\widetilde{G}$ has linear growth at infinity.

We consider the new energy functional

$$\widetilde{\mathcal{E}}_{\widetilde{\mathcal{O}}_{R,L}}(v) = \int_{\widetilde{\mathcal{O}}_{R,L}} \frac{1}{2}|\nabla v|^2 + \int_{\partial^0\widetilde{\mathcal{O}}_{R,L}} \widetilde{G}(v) \qquad \text{for every} \quad v \in \widetilde{H}^1_0(\widetilde{\mathcal{O}}_{R,L}).$$

Note that every minimizer w of $\widetilde{\mathcal{E}}_{\widetilde{\mathcal{O}}_{R,L}}(\cdot)$ in $\widetilde{H}^1_0(\widetilde{\mathcal{O}}_{R,L})$ such that $-1 \le w \le 1$ is also a minimizer of $\mathcal{E}_{\widetilde{\mathcal{O}}_{R,L}}(\cdot)$ in the set

$$\{v \in \widetilde{H}^1_0(\widetilde{\mathcal{O}}_{R,L}) : v = v(s,t,\lambda), -1 \le v \le 1\}$$

We show that $\widetilde{\mathcal{E}}_{\widetilde{\mathcal{O}}_{R,L}}(\cdot)$ admits a minimizer in $\widetilde{H}^1_0(\widetilde{\mathcal{O}}_{R,L})$. Indeed, by the properties of $\widetilde{G}$, it follows that $\widetilde{\mathcal{E}}_{\widetilde{\mathcal{O}}_{R,L}}(\cdot)$ is well-defined, bounded below and coercive in $\widetilde{H}^1_0(\widetilde{\mathcal{O}}_{R,L})$. Hence, using the compactness of the inclusion $\widetilde{H}^1_0(\widetilde{\mathcal{O}}_{R,L}) \subset\subset \widetilde{L}^2(\partial^0\widetilde{\mathcal{O}}_{R,L})$, taking a minimizing sequence $\{v^k_{R,L}\} \in \widetilde{H}^1_0(\widetilde{\mathcal{O}}_{R,L})$ and a subsequence convergent in $\widetilde{L}^2(\partial^0\widetilde{\mathcal{O}}_{R,L})$, we conclude that $\widetilde{\mathcal{E}}_{\widetilde{\mathcal{O}}_{R,L}}(\cdot)$ admits an absolute minimizer $v_{R,L}$ in $\widetilde{H}^1_0(\widetilde{\mathcal{O}}_{R,L})$.

Note moreover that, without loss of generality, we may assume that $0 \le v^k_{R,L} \le 1$ in $\widetilde{\mathcal{O}}_{R,L}$ because, if not, we can replace the minimizing sequence $v^k_{R,L}$ with the sequence $\min\{|v^k_{R,L}|, 1\} \in \widetilde{H}^1_0(\widetilde{\mathcal{O}}_{R,L})$. Indeed, it is also minimizing because $\widetilde{G}$ is even and $\widetilde{G} \ge \widetilde{G}(1)$. Then the absolute minimizer $v_{R,L}$ is such that $0 \le v_{R,L} \le 1$ in $\widetilde{\mathcal{O}}_{R,L}$.

Next, we can consider perturbations $v_{R,L} + \xi$ of $v_{R,L}$, with ξ depending only on s, t and λ, and having compact support in $\widetilde{\mathcal{O}}_{R,L} \cap \{t > 0\}$. In particular ξ vanishes in a neighborhood of $\{t = 0\}$. Since the problem (2.1.3) in (s,t,λ) coordinates is the first variation of $\mathcal{E}_{\widetilde{\mathcal{O}}_{R,L}}(v)$ —recall that $\mathcal{E}$ has the form (2.1.16) on $\widetilde{H}^1_0$ functions— and the equation is not singular away from $\{s = 0\}$ and $\{t = 0\}$, we deduce that $v_{R,L}$ is a solution of (2.1.15) in $\widetilde{\mathcal{O}}_{R,L} \cap \{t > 0\}$.

We now prove that $v_{R,L}$ is also a solution in all of $\widetilde{\mathcal{O}}_{R,L}$, that is, also across $\{t = 0\}$. To see this for dimensions $2m + 1 \ge 5$, let ξ_ε be a smooth function of t alone being identically 0 in $\{t < \varepsilon/2\}$ and identically 1 in $\{t > \varepsilon\}$. Let $\varphi \in C^\infty_0(\widetilde{\mathcal{O}}_{R,L} \cup \partial^0\widetilde{\mathcal{O}}_{R,L})$, multiply the equation $-\Delta v_{R,L} = 0$ by $\varphi\xi_\varepsilon$ and integrate by parts to obtain

$$\int_{\widetilde{\mathcal{O}}_{R,L}} \nabla v_{R,L} \nabla\varphi\, \xi_\varepsilon + \int_{\widetilde{\mathcal{O}}_{R,L}\cap\{t<\varepsilon\}} \nabla v_{R,L}\, \varphi\, \nabla\xi_\varepsilon + \int_{\partial^0\widetilde{\mathcal{O}}_{R,L}} \partial_\lambda v_{R,L}\, \varphi\, \xi_\varepsilon = 0.$$

Reminding that $v_{R,L}$ satisfies the Neumann condition $-\partial_\lambda v_{R,L} = f(v_{R,L})$ on $\partial^0\widetilde{\mathcal{O}}_{R,L}$, we get

$$\int_{\widetilde{\mathcal{O}}_{R,L}} \nabla v_{R,L}\nabla\varphi\,\xi_\varepsilon + \int_{\widetilde{\mathcal{O}}_{R,L}\cap\{t<\varepsilon\}} \nabla v_{R,L}\,\varphi\,\nabla\xi_\varepsilon = \int_{\partial^0\widetilde{\mathcal{O}}_{R,L}} f(v_{R,L})\,\varphi\,\xi_\varepsilon. \qquad (2.2.2)$$

We conclude by seeing that the second integral on the left hand side goes to zero as $\varepsilon \to 0$. Indeed, by Cauchy-Schwartz inequality,

$$\left|\int_{\widetilde{\mathcal{O}}_{R,L}\cap\{t<\varepsilon\}} \nabla v_{R,L}\varphi\nabla\xi_\varepsilon\;dxd\lambda\right|^2 \\ \le C\int_{\widetilde{\mathcal{O}}_{R,L}\cap\{t<\varepsilon\}} |\nabla v_{R,L}|^2\;dxd\lambda \int_{\widetilde{\mathcal{O}}_{R,L}\cap\{t<\varepsilon\}} |\nabla\xi_\varepsilon|^2\;dxd\lambda. \qquad (2.2.3)$$

Since $|\nabla\xi_\varepsilon|^2 \le C/\varepsilon^2$, $|\widetilde{\mathcal{O}}_{R,L}\cap\{t<\varepsilon\}| \le C_R\varepsilon^m\,L$, and $m \ge 2$, the second factor in the previous bound, is bounded independently of ε. At the same time, the first factor tends to zero as $\varepsilon \to 0$, since $|\nabla v_{R,L}|^2$ is integrable in $\widetilde{\mathcal{O}}_{R,L}$.

In dimension $2m+1=3$, the previous proof does not apply and we argue as follows. We consider perturbations $\xi \in \widetilde{H}^1_{0,l}(\widetilde{\mathcal{O}}_{R,L})$ which do not vanish on $\{t=0\}$. Considering the first variation of energy and integrating by parts, we find that the boundary flux $s^{m-1}t^{m-1}\partial_t v_{R,L} = \partial_t v_{R,L}$ (here $m-1=0$) must be identically 0 on $\{t=0\}$. This implies that $v_{R,L}$ is a solution also across $\{t=0\}$.

We have established the existence of a solution $v_{R,L}$ in $\widetilde{\mathcal{O}}_{R,L}$ with $0 \le v_{R,L} \le 1$. Considering the odd reflection of $v_{R,L}$ with respect to $\mathcal{C}\times\mathbb{R}^+$,

$$v_{R,L}(s,t,\lambda) = -v_{R,L}(t,s,\lambda),$$

we obtain a solution in $B_R\setminus\{0\}\times(0,L)$. Using the same cut-off argument as above, but choosing now $1-\xi_\varepsilon$ to have support in the ball of radius ε around 0, we conclude that $v_{R,L}$ is also solution around 0, and hence in all of $B_R\times(0,L)$. Here, the cut-off argument also applies in dimension 3.

We now wish to pass to the limit in R and L, and obtain a solution in all of $\mathbb{R}^{2m+1}_+$. Let $S>0$, $L'>0$ and consider the family $\{v_{R,L}\}$ of solutions in $B_{S+2}\times[0,L'+2]$, with $R>S+2$ and $L>L'+2$. Since $|v_{R,L}|\le 1$, regularity results proved in [10], applied in $B_2\times[0,2]$ where B_2 is centered at points in $\overline{B}_S\times[0,L']$, give a uniform $C^{2,\alpha}(\overline{B}_S\times[0,L'])$ bound for $v_{R,L}$ (uniform with respect to R). We have

$$|\nabla v_{R,L}| \le C \quad \text{in } B_S\times[0,L'], \qquad \text{for all } R>S+2,\;\; L>L'+2 \qquad (2.2.4)$$

for some constant C independent of S, R, L and L'. Moreover since $v_{R,L}$ is harmonic and bounded we have that

$$|\nabla v_{R,L}(x,\lambda)| \le \frac{C}{\lambda} \quad \text{in } B_R\times(1,L). \qquad (2.2.5)$$

Choose now $L = R^\gamma$, with $1/2 < \gamma < 1$. By the Arzelà-Ascoli Theorem, a subsequence of $\{v_{R,L}\}$ converges in $C^2(\overline{B}_S \times [0, S^\gamma])$ to a solution in $B_S \times (0, S^\gamma)$. Taking $S = 1, 2, 3, \dots$ and making a Cantor diagonal argument, we obtain a sequence v_{R_j, R_j^γ} converging in $C^2_{loc}(\mathbb{R}^{2m+1}_+)$ to a solution $v \in C^2(\mathbb{R}^{2m+1}_+)$. By construction we have found a solution v in $\mathbb{R}^{2m+1}_+$ depending only on s, t and λ, such that $v(s,t,\lambda) = -v(t,s,\lambda)$, $|v| \leq 1$ and $v \geq 0$ in $\{s > t\}$. We want to prove now that $|v| < 1$. Indeed remind that v satisfies

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^{2m+1}_+ \\ -\partial_\lambda v = f(v) & \text{on } \partial\mathbb{R}^{2m+1} \end{cases}$$

Since $f(1) = 0$ and v is not identically 1 (because $v \equiv 0$ on $\mathcal{C} \times \mathbb{R}^+$), using that $v \leq 1$ and applying the maximum principle and Hopf's Lemma, we conclude that $v < 1$. In the same way we prove that $v > -1$.

It only remains to prove that $v \not\equiv 0$ in $\mathbb{R}^{2m+1}_+$. Then, the strong maximum principle and Hopf's Lemma lead to $v > 0$ in $\{s > t\} \times \mathbb{R}^+$ since $f(0) = 0$ and $v \geq 0$ in $\{s > t\} \times \mathbb{R}^+$.

To prove that $v \not\equiv 0$ in $\mathbb{R}^{2m+1}_+$, we establish an energy estimate for the saddle solution constructed above, which is not sharp, but it is enough to prove $v \not\equiv 0$ in $\mathcal{O} = \{s > t\} \times \mathbb{R}^+$.

We use a comparison argument, based on the minimality property of $v_{R,L}$ in the set $\widetilde{\mathcal{O}}_{R,L}$.

Let $1/2 \leq \gamma \leq 1$ and β be a positive real number depending only on γ and such that $1/2 \leq \beta < \gamma < 1$. Let $S < R - 2$, since we have chosen before $L = R^\gamma$, then $S^\gamma < L$. We consider a C^1 function $g : \widetilde{\mathcal{O}}_{S,S^\gamma} \to \mathbb{R}$ defined as follows:

$$g(x,\lambda) = g(s,t,\lambda) = \eta(s,t) \min\left\{1, \frac{s-t}{\sqrt{2}}\right\} + (1 - \eta(s,t)) v_{R,L}(s,t,\lambda),$$

where η is a smooth function depending only on $r^2 = s^2 + t^2$ such that $\eta \equiv 1$ in B_{S-1} and $\eta \equiv 0$ outside B_S. Observe that g agrees with $v_{R,L}$ on the lateral boundary of $\widetilde{\mathcal{O}}_{S,S^\gamma}$ and g is identically 1 inside $[\mathcal{O}_{S-1} \cap \{(s-t)/\sqrt{2} > 1\}] \times (0, S^\gamma)$. By (2.2.4) and (2.2.5), we have that

$$|\nabla g(x,\lambda)| \leq \frac{C}{\lambda + 1} \quad \text{for every } (x,\lambda) \in \widetilde{\mathcal{O}}_{S,S^\gamma}. \tag{2.2.6}$$

Next we consider a C^1 function $\xi : (0, S^\gamma) \to (0, +\infty)$, such that

$$\xi(\lambda) = \begin{cases} 1 & \text{if } 0 < \lambda \leq S^\gamma - S^\beta \\ \dfrac{\log S^\gamma - \log \lambda}{\log S^\gamma - \log(S^\gamma - S^\beta)} & \text{if } S^\gamma - S^\beta < \lambda \leq S^\gamma \end{cases}$$

Then, we define $w : \widetilde{\mathcal{O}}_{S,S^\gamma} \to (-1,1)$ as follows

$$w(x,\lambda) = \xi(\lambda)g(x,\lambda) + [1-\xi(\lambda)]v_{R,L}(x,\lambda). \tag{2.2.7}$$

Observe that w agree with $v_{R,L}$ on $\partial^+\widetilde{\mathcal{O}}_{S,S^\gamma}$ and $w \equiv 1$ in $\widetilde{\mathcal{O}}_{S-1,S^\gamma-S^\beta}$. We extend w to be identically equal to $v_{R,L}$ in $\widetilde{\mathcal{O}}_{R,L} \setminus \widetilde{\mathcal{O}}_{S,S^\gamma}$. By minimality of $v_{R,L}$ in $\widetilde{\mathcal{O}}_{R,L}$, we have

$$\mathcal{E}_{\widetilde{\mathcal{O}}_{R,L}}(v_{R,L}) \leq \mathcal{E}_{\widetilde{\mathcal{O}}_{R,L}}(w).$$

Thus, since $w = v_{R,L}$ in $\widetilde{\mathcal{O}}_{R,L} \setminus \widetilde{\mathcal{O}}_{S,S^\gamma}$, we get

$$\mathcal{E}_{\widetilde{\mathcal{O}}_{S,S^\gamma}}(v_{R,L}) \leq \mathcal{E}_{\widetilde{\mathcal{O}}_{S,S^\gamma}}(w).$$

We give now an estimate for $\mathcal{E}_{\widetilde{\mathcal{O}}_{S,S^\gamma}}(w)$. First, observe that, since $w \equiv 1$ on $\mathcal{O}_{S-1}$, then

$$\int_{\mathcal{O}_S} G(w) = \int_{\mathcal{O}_S \setminus \mathcal{O}_{S-1}} G(w) \leq C|\mathcal{O}_S \setminus \mathcal{O}_{S-1}| \leq CS^{2m-1}. \tag{2.2.8}$$

Next, we give a bound for the Dirichlet energy of w. We have

$$\begin{aligned}\int_{\widetilde{\mathcal{O}}_{S,S^\gamma}} |\nabla w(x,\lambda)|^2 dx d\lambda &= \int_{\widetilde{\mathcal{O}}_{S,S^\gamma-S^\beta}} |\nabla w(x,\lambda)|^2 dx d\lambda \\ &\quad + \int_{\widetilde{\mathcal{O}}_{S,S^\gamma} \setminus \widetilde{\mathcal{O}}_{S,S^\gamma-S^\beta}} |\nabla w(x,\lambda)|^2 dx d\lambda. \end{aligned} \tag{2.2.9}$$

Since $w \equiv 1$ in $\widetilde{\mathcal{O}}_{S-1,S^\gamma-S^\beta}$, we get

$$\int_{\widetilde{\mathcal{O}}_{S,S^\gamma}} |\nabla w(x,\lambda)|^2 dx d\lambda \leq CS^{2m-1+\gamma} + \int_{\widetilde{\mathcal{O}}_{S,S^\gamma} \setminus \widetilde{\mathcal{O}}_{S,S^\gamma-S^\beta}} |\nabla w(x,\lambda)|^2 dx d\lambda. \tag{2.2.10}$$

Consider now the integral on the right-hand side of (2.2.10). By the definition of w (2.2.7), we have that

$$|\nabla w(x,\lambda)|^2 \leq |\xi'(\lambda)|^2 [g(x,\lambda) + v_{R,L}(x,\lambda)]^2 + \{|\nabla g|^2 + |\nabla v_{R,L}(x,\lambda)|^2\}[1+\xi(\lambda)]^2.$$

Integrating in $\widetilde{\mathcal{O}}_{S,S^\gamma} \setminus \widetilde{\mathcal{O}}_{S,S^\gamma-S^\beta}$, using that g, $|\nabla g|$, v, and ξ are bounded, the definition of ξ, and the gradient bounds (2.2.5) and (2.2.6) for $v_{R,L}$ and for g, we get

$$\begin{aligned}
\int_{\widetilde{\mathcal{O}}_{S,S^\gamma}\setminus\widetilde{\mathcal{O}}_{S,S^\gamma-S^\beta}} |\nabla w(x,\lambda)|^2 &\leq C\int_{\mathcal{O}_S}\int_{S^\gamma-S^\beta}^{S^\gamma}|\xi'(\lambda)|^2 d\lambda dx + C\int_{\mathcal{O}_S}\int_{S^\gamma-S^\beta}^{S^\gamma}\frac{1}{\lambda^2}d\lambda dx\\
&\leq C\left[\frac{1}{\left(\log\frac{S^\gamma}{S^\gamma-S^\beta}\right)^2}+1\right]\int_{\mathcal{O}_S}\int_{S^\gamma-S^\beta}^{S^\gamma}\frac{1}{\lambda^2}d\lambda dx\\
&\leq CS^{2m}\left[\frac{1}{(-\log(1-S^{\beta-\gamma}))^2}+1\right]\left[\frac{1}{S^\gamma-S^\beta}-\frac{1}{S^\gamma}\right]\\
&\leq CS^{2m}\cdot S^{2(\gamma-\beta)}\cdot S^{-\gamma}\leq CS^{2m+\gamma-2\beta}. \qquad (2.2.11)
\end{aligned}$$

Combining (2.2.8), (2.2.10) and (2.2.11), we get

$$\mathcal{E}_{\widetilde{\mathcal{O}}_{S,S^\gamma}}(w)\leq C(S^{2m-1}+S^{2m-1+\gamma}+S^{2m+\gamma-2\beta}). \qquad (2.2.12)$$

Since, by hypothesis, γ and $\beta=\beta(\gamma)$ satisfy $1/2\leq\beta<\gamma<1$, then there exists $\varepsilon=\varepsilon(\gamma)>0$ such that

$$\mathcal{E}_{\widetilde{\mathcal{O}}_{S,S^\gamma}}(w)\leq CS^{2m-\varepsilon}.$$

Thus by minimality of $v_{R,L}$, we get

$$\mathcal{E}_{\widetilde{\mathcal{O}}_{S,S^\gamma}}(v_{R,L})\leq CS^{2m-\varepsilon}.$$

We now let R and $L=R^\gamma$ tend to infinity to obtain

$$\mathcal{E}_{\widetilde{\mathcal{O}}_{S,S^\gamma}}(v)\leq CS^{2m-\varepsilon}.$$

Note that this bound, after odd reflection with respect to $\mathcal{C}$, leads to the energy bound (2.1.14)

$$\mathcal{E}_{C_{S,S^\gamma}}(v)\leq CS^{2m-\varepsilon}.$$

Using this estimate we prove the claim. Suppose that $v\equiv 0$. Then we would have

$$c_m G(0)S^{2m}=\mathcal{E}_{C_{S,S^\gamma}}(v)\leq CS^{2m-\varepsilon}.$$

This is a contradiction for S large, and thus $v\not\equiv 0$.

We give now the prove of the last part of the statement, that is, we prove stability of saddle-shaped solutions under perturbations vanishing on $\mathcal{C}\times(0,+\infty)$.

Since $f(0)=0$, concavity leads to $f'(w)\leq f(w)/w$ for all real numbers $w\in(0,1)$. Hence we have

$$\begin{cases}-\Delta v=0 & \text{in } \widetilde{\mathcal{O}}\\ -\dfrac{\partial v}{\partial\lambda}\geq f'(v)v & \text{on } \mathcal{O}\times\{0\}.\end{cases}$$

By a simple argument (see the proof of Proposition 4.2 of [1]), it follows that the value of the quadratic form $Q_v(\xi)$ is nonnegative for all $\xi \in C^1$ with compact support in $\widetilde{\mathcal{O}} \cup \partial^0 \widetilde{\mathcal{O}}$ (and not necessarily depending only on s, t and λ). Indeed, multiply the equation $-\Delta v = 0$ by ξ^2/v, where $\xi \in C^1(\mathbb{R}^{2m+1}_+)$ with compact support in $\widetilde{\mathcal{O}} \cup \partial^0 \widetilde{\mathcal{O}}$, and integrate by parts in $\widetilde{\mathcal{O}}$, we get:

$$\begin{aligned} 0 &= \int_0^{+\infty} \int_{\mathcal{O}} (-\Delta v) \frac{\xi^2}{v} = \int_0^{+\infty} \int_{\mathcal{O}} \nabla v \cdot \nabla \xi \frac{2\xi}{v} \\ &\quad - \int_0^{+\infty} \int_{\mathcal{O}} |\nabla v|^2 \frac{\xi^2}{v^2} + \int_{\mathcal{O}} \frac{\xi^2}{v} \frac{\partial v}{\partial \lambda} \\ &\leq \int_0^{+\infty} \int_{\mathcal{O}} |\nabla \xi|^2 - \int_{\mathcal{O}} f'(v) \xi^2 = Q_v(\xi). \end{aligned}$$

By an approximation argument, the same holds for all $\xi \in C^1$ with compact support in $\overline{\widetilde{\mathcal{O}}}$ and vanishing on $\mathcal{C} \times \mathbb{R}^+$. Finally, by odd symmetry with respect to $\mathcal{C} \times \mathbb{R}^+$, the same is true for all C^1 functions ξ with compact support in $\overline{\mathbb{R}^{2m+1}}_+$ and vanishing on $\mathcal{C} \times \mathbb{R}^+$. □

Remark 2.2.1. Observe that, if $\gamma \to 1$, estimate (2.2.12) tends to

$$\mathcal{E}_{C_{S,S}}(v) \leq C S^{2m}.$$

This is a not sharp energy estimate, indeed in Theorem 3.0.8 of Chapter 3, we prove that the saddle solution v satisfies

$$\mathcal{E}_{C_{S,S}}(v) \leq C S^{2m-1} \log S.$$

2.3 Supersolution and subsolution for $A_{1/2}$

In [11], Cabré and Tan introduced the operator $A_{1/2}$, which is the square root of the Laplacian for functions defined on a bounded set and that vanish on the boundary. Let u be defined in a bounded set $H \subset \mathbb{R}^n$ and $u \equiv 0$ on ∂H. Consider the harmonic extension v of u in the half-cylinder $H \times (0, \infty)$ vanishing on the lateral boundary $\partial H \times [0, \infty)$. Define the operator $A_{1/2}$ as follows

$$A_{1/2} u := - \frac{\partial v}{\partial \lambda}_{|H \times \{0\}} \tag{2.3.1}$$

Then, since $\partial_\lambda v$ is harmonic and also vanishes on the lateral boundary, as for the case of the all space, the Dirichlet-Neumann map of the harmonic extension v on the bottom of the half cylinder is the square root of the Laplacian. That is, we have the property:

$$A_{1/2} \circ A_{1/2} = -\Delta_H$$

where $-\Delta_H$ is the Laplacian in H with zero Dirichlet boundary value on ∂H.

Hence, we can study the problem

$$\begin{cases} A_{1/2}u = f(u) & \text{in } H \\ u = 0 & \text{on } \partial H \\ u > 0 & \text{in } H, \end{cases} \tag{2.3.2}$$

by studying the local problem:

$$\begin{cases} -\Delta v = 0 & \text{in } \quad \Omega = H \times (0, \infty) \\ v = 0 & \text{on } \quad \partial_L \Omega = \partial H \times [0, \infty) \\ -\dfrac{\partial v}{\partial \lambda} = f(v) & \text{on } \quad H \times \{0\} \\ v > 0 & \text{in } \quad \Omega. \end{cases} \tag{2.3.3}$$

In [11] some results (Lemma 3.2.3 and Lemma 3.2.4) need H bounded. But for our aim, definition (2.3.1) is enough and it can be given also in the case H not bounded. Thus, we can consider problem (2.3.2) and (2.3.3) for a general open set H.

In this section we give a subsolution and supersolution for the problem

$$\begin{cases} A_{1/2}u = f(u) & \text{in } \mathcal{O} \\ u = 0 & \text{on } \partial\mathcal{O} \\ u > 0 & \text{in } \mathcal{O}, \end{cases} \tag{2.3.4}$$

In what follows it will be useful to use the following variables:

$$\begin{cases} y = \dfrac{s+t}{\sqrt{2}} \\ z = \dfrac{s-t}{\sqrt{2}} \end{cases} . \tag{2.3.5}$$

Note that $|z| \le y$ and that we may write the Simons cone as $\mathcal{C} = \{z = 0\}$.

If we take into account these new variables, the problem (2.1.15) becomes

$$\begin{cases} v_{yy} + v_{zz} + v_{\lambda\lambda} + \dfrac{2(m-1)}{y^2 - z^2}(yv_y - zv_z) = 0 & \text{in } \mathbb{R}^{2m+1}_+ \\ -\partial_\lambda v = f(v) & \text{on } \partial\mathbb{R}^{2m+1}_+ \end{cases} \tag{2.3.6}$$

We give the definition of supersolution and subsolution for the problem (2.3.2) by using the associated local formulation (2.3.3).

Definition 2.3.1. a) We say that a function w, defined on $H \times [0, +\infty)$, $w \equiv 0$ on $\partial H \times [0, +\infty)$ is a *supersolution* (*subsolution*) for the problem (2.3.3) if

$$\begin{cases} -\Delta w \geq \ (\leq)\, 0 & \text{in } \ H \times (0, +\infty) \\ w > 0 & \text{in } \ H \times (0, +\infty) \\ -\dfrac{\partial w}{\partial \lambda} \geq \ (\leq)\, f(w) & \text{on } \ H \times \{0\}. \end{cases}$$

b) We say that a function u, defined on H, $u \equiv 0$ on ∂H, $u > 0$ in H, is a *supersolution* (*subsolution*) for the problem (2.3.2) if its harmonic extension v such that $v \equiv 0$ on $\partial H \times [0, +\infty)$, is a supersolution (subsolution) for the problem (2.3.3).

Lemma 2.3.2. *The following assertions are equivalent:*

i) u is a subsolution (supersolution) for problem (2.3.2);

ii) there exists an extension w of u on $H \times (0, +\infty)$ vanishing on $\partial H \times (0, +\infty)$, such that w is a subsolution (supersolution) for problem (2.3.3).

Proof. The first implication $i) \Rightarrow ii)$ is trivial (just take the harmonic extension v of u with zero Dirichlet data on the lateral boundary).

It remains to show that $ii) \Rightarrow i)$. We consider the case of supersolution. Suppose that there exists a function w defined on $\mathbb{R}^{n+1}_+$ such that:

$$\begin{cases} -\Delta w \geq 0 & \text{in } \ H \times (0, +\infty) \\ w \equiv 0 & \text{on } \ \partial H \times (0, +\infty) \\ w > 0 & \text{in } \ H \times (0, +\infty) \\ w(x, 0) = u(x) & \text{on } \ H \times \{0\} \\ -\dfrac{\partial w}{\partial \lambda} \geq f(w) & \text{on } \ H \times \{0\}. \end{cases}$$

Now consider the harmonic extension v of u in $H \times (0, +\infty)$, with $v \equiv 0$ on $\partial H \times (0, +\infty)$. Then by the maximum principle we have that $v \leq w$ in $H \times (0, +\infty)$. This implies that

$$-\frac{\partial v}{\partial \lambda} \geq -\frac{\partial w}{\partial \lambda} \ \text{ on } \ \partial H \times (0, +\infty)$$

and hence that

$$-\frac{\partial v}{\partial \lambda} \geq f(v) \ \text{ on } \ \partial H \times (0, +\infty).$$

For the case of subsolution the proof is the same. □

We recall that in [10] it is proved that, under hypothesis (2.1.9), there exists a layer solution (i.e., a monotone increasing solution, from -1 to 1), for the problem (2.1.3) in dimension $n = 1$. Normalizing it to vanish at $\{x = 0\}$, we call it u_0 (see (2.1.13)).

Moreover we remind that $|s - t|/\sqrt{2}$ is the distance to the Simons cone (see [12]).

We can give now the following proposition. The first part of the statement, which gives a supersolution for problem (2.3.2) in $H = \mathcal{O}$, is equivalent to Proposition 2.1.8 in the Introduction.

Proposition 2.3.3. *Let f satisfy hypothesis* (2.1.8), (2.1.9), (2.1.10). *Let u_0 be the layer solution, vanishing at the origin, of problem* (2.1.1) *in* $\mathbb{R}$.

Then, the function $u_0(z) = u_0\left(\frac{s-t}{\sqrt{2}}\right)$ is a supersolution of problem (2.3.2) in the set $H = \mathcal{O} = \{s > t\}$.

Moreover when $2m = 2$ the function $\omega(x_1, x_2) = u_0\left(\frac{x_1 + x_2}{\sqrt{2}}\right) u_0\left(\frac{x_2 - x_1}{\sqrt{2}}\right)$ is a subsolution of problem (2.3.2) in the set $\mathcal{O}$.

Remark 2.3.4. We observe that, if f satisfies hypothesis (2.1.8), (2.1.9), (2.1.10), then $f(\rho)/\rho$ is non-increasing in $(0, 1)$. Indeed, given $0 < \rho < 1$, there exists ρ_1, with $0 < \rho_1 < \rho$, such that

$$\frac{f(\rho)}{\rho} = \frac{f(\rho) - f(0)}{\rho - 0} = f'(\rho_1) > f'(\rho).$$

Therefore

$$\left(\frac{f(\rho)}{\rho}\right)' = \frac{f'(\rho)\rho - f(\rho)}{\rho^2} = \frac{f'(\rho) - f'(\rho_1)}{\rho} < 0.$$

Proof of Proposition 2.3.3. We begin by considering the function $v_0\left((s-t)/\sqrt{2}, \lambda\right)$ and we show that it is a supersolution of the problem (2.3.3) in the set $\widetilde{\mathcal{O}}$.

First, we remind that the problem (2.3.3) in the (s, t, λ) variables reads

$$\begin{cases} -(v_{ss} + v_{tt} + v_{\lambda\lambda}) - (m-1)\left(\frac{v_s}{s} + \frac{v_t}{t}\right) = 0 & \text{in } \widetilde{\mathcal{O}} \\ -\frac{\partial v}{\partial \lambda} = f(v) & \text{on } \partial^0 \widetilde{\mathcal{O}} \end{cases} \tag{2.3.7}$$

By a direct computation, we have that $v_0\left((s-t)/\sqrt{2}, \lambda\right)$ is superharmonic in the set $\{(s, t, \lambda) : s > t > 0\}$ and satisfies the Neumann condition $\partial_\lambda v = f(v)$. In dimension $2m + 1 \geq 5$ there is nothing else to be checked, by a cut-off argument used as in (2.2.2).

In dimension $2m+1=3$, $v_0\left((s-t)/\sqrt{2},\lambda\right)$ is a supersolution in $\widetilde{\mathcal{O}}$ because the outer flux $-\partial_t v_0\left((s-t)/\sqrt{2},\lambda\right)=\partial_x v_0\left((s-t)/\sqrt{2},\lambda\right)>0$ is positive.

We prove now the second part of the statement. We introduce the coordinates

$$\begin{cases} \widetilde{x}_1 = \dfrac{x_1+x_2}{\sqrt{2}} \\ \widetilde{x}_2 = \dfrac{x_2-x_1}{\sqrt{2}} \end{cases}$$

We consider the function $v_0(x,\lambda)$ which is the solution of the problem (2.1.3) in dimension $n+1=2$ and we prove that the function

$$\widetilde{\omega}(x_1,x_2,\lambda) := v_0\left(\frac{\widetilde{x}_1}{2},\frac{\lambda}{2}\right) v_0\left(\frac{\widetilde{x}_2}{2},\frac{\lambda}{2}\right)$$

is a subsolution of (2.3.3) in the set $\widetilde{O}=\{(\widetilde{x}_1,\widetilde{x}_2,\lambda),\widetilde{x}_1>0,\widetilde{x}_2>0,\lambda>0\}$.

Since the Laplace operator is invariant under rotations and by a direct calculation we have

$$\Delta\widetilde{\omega}(x_1,x_2,\lambda) = \frac{1}{2}\frac{\partial v_0}{\partial\lambda}\left(\frac{\widetilde{x}_1}{2},\frac{\lambda}{2}\right)\frac{\partial v_0}{\partial\lambda}\left(\frac{\widetilde{x}_2}{2},\frac{\lambda}{2}\right).$$

Then to prove that $\Delta\widetilde{\omega}\geq 0$ it is enough to prove that $\dfrac{\partial v_0}{\partial\lambda}(x,\lambda)$ does not change sign in the set $\{x>0\}$. This can be easily shown using the maximum principle. Consider the function $-\dfrac{\partial v_0}{\partial\lambda}(x,\lambda)$ which satisfies the Dirichlet problem

$$\begin{cases} \Delta\left(-\dfrac{\partial v_0}{\partial\lambda}\right)=0 & \text{in } \mathbb{R}^2_{++}=\{(x,\lambda):x>0,\ \lambda>0\} \\ -\partial_\lambda v_0 = f(v_0) & \text{on } \{\lambda=0\} \\ -\partial_\lambda v_0 = 0 & \text{on } \{x=0\} \end{cases}.$$

Indeed, we recall that $v_0(0,\lambda)=0$. Moreover, by the fact that $v_0(x,\lambda)>0$ for $x>0$, we deduce that $f(v_0(x,0))>0$ for $x>0$ and then that $-\dfrac{\partial v_0}{\partial\lambda}\geq 0$ on $\partial\mathbb{R}^2_{++}$. By the maximum principle we get $\dfrac{\partial v_0}{\partial\lambda}\leq 0$ in $\mathbb{R}^2_{++}$.

To conclude the prove it remains to show that $\widetilde{\omega}$ satisfies the Neumann condition $-\dfrac{\partial\widetilde{\omega}}{\partial\lambda}(\widetilde{x}_1,\widetilde{x}_2,\lambda)\leq f(\widetilde{\omega})$ for $\lambda=0$. Here we follow an argument used by Schatzman for the equation involving the Laplacian, instead of the half Laplacian. Observe that this argument can be applied only in dimension 2.

We have

$$-\frac{\partial\widetilde{\omega}}{\partial\lambda}(x_1,x_2,\lambda)=-\frac{1}{2}\frac{\partial v_0}{\partial\lambda}\left(\frac{\widetilde{x}_1}{2},\frac{\lambda}{2}\right)v_0\left(\frac{\widetilde{x}_2}{2},\frac{\lambda}{2}\right)-\frac{1}{2}\frac{\partial v_0}{\partial\lambda}\left(\frac{\widetilde{x}_2}{2},\frac{\lambda}{2}\right)v_0\left(\frac{\widetilde{x}_1}{2},\frac{\lambda}{2}\right).$$

Putting $\lambda=0$ in the previous equality we get

$$-\frac{\partial\widetilde{\omega}}{\partial\lambda}(x_1,x_2,0)=\frac{1}{2}f\left(v_0\left(\frac{\widetilde{x}_1}{2},0\right)\right)v_0\left(\frac{\widetilde{x}_2}{2},0\right)+\frac{1}{2}f\left(v_0\left(\frac{\widetilde{x}_2}{2},0\right)\right)v_0\left(\frac{\widetilde{x}_1}{2},0\right).$$

We recall that $0<v_0(x,\lambda)<1$ for $x>0$. Set for simplicity $v_0\left(\frac{\widetilde{x}_1}{2},0\right)=a$ and $v_0\left(\frac{\widetilde{x}_2}{2},0\right)=b$. By Remark 2.3.4 $f(u)/u$ is non increasing, then we have that for $a,\ b\in(0,1)$

$$\frac{f(ab)}{ab}\geq\max\left\{\frac{f(a)}{a},\frac{f(b)}{b}\right\}\geq\frac{1}{2}\left(\frac{f(a)}{a}+\frac{f(b)}{b}\right)$$

Coming back to our notation we conclude

$$-\frac{\partial\widetilde{\omega}}{\partial\lambda}(x_1,x_2,0)\leq f(\widetilde{\omega}).$$

□

Remark 2.3.5. Observe that in dimension $2m=2$, $v_0\left((s-t)/\sqrt{2},\lambda\right)$ is a solution of problem (2.1.3) away from the sets $\{s=0\}$, $\{t=0\}$, while in higher dimensions it is a strict supersolution.

Corollary 2.3.6. *Let f satisfy hypothesis* (2.1.8), (2.1.9), (2.1.10). *Let u_0 be the layer solution, vanishing at the origin, of problem* (2.1.1) *in $\mathbb{R}$ and suppose $K\geq1$.*

Then, the function $\min\{Ku_0(z),1\}=\min\{u_0(s-t/\sqrt{2}),1\}$ *is a supersolution of problem (2.3.2) in the set $H=\mathcal{O}=\{s>t\}$.*

Proof. Proceeding as in the proof of Proposition 2.3.3, we consider the function $\min\{Kv_0(z,\lambda),1\}$. To prove that it is a supersolution of problem (2.3.3) in $\widetilde{\mathcal{O}}$, it is enough to prove that it is a supersolution of problem (2.3.3) in the set $\{(x,\lambda)\in\widetilde{\mathcal{O}}:Kv_0(z,\lambda)<1\}$.

First of all, in the proof of Proposition 2.3.3, we have seen that $v_0(z,\lambda)$ is superharmonic in $\widetilde{\mathcal{O}}$, and thus $\min\{Kv_0(z,\lambda),1\}=Kv_0(z,\lambda)$ is superharmonic in the set $\{(x,\lambda)\in\widetilde{\mathcal{O}}:Kv_0(z,\lambda)<1\}$.

Moreover

$$-\partial_\lambda(Kv_0(z,0))=Kf(v_0(z,0))\quad\text{on }\{(x,0)\in\widetilde{\mathcal{O}}:Kv_0(z,0)<1\}.$$

By Remark 2.3.4, we have that $f(u)/u$ is decreasing and then for every $K \geq 1$ we get

$$\frac{Kf(u_0)}{Ku_0} = \frac{f(u_0)}{u_0} \geq \frac{f(Ku_0)}{Ku_0} \quad \text{if } Ku_0 < 1.$$

This let us to conclude the proof, indeed

$$-\partial_\lambda(Kv_0(z,0)) = Kf(v_0(z,0)) \geq f(Kv_0(z,0)) \quad \text{on } \{(x,0) \in \widetilde{\mathcal{O}} : Kv_0(z,0) < 1\}.$$

□

2.4 The operator $D_{H,\varphi}$ and maximum principles

In what follows we need to introduce a new nonlocal operator $D_{H,\varphi}$, which is the analogue of $A_{1/2}$ but it can be applied to functions which do not vanish on the boundary of H.

Let φ be a function defined in $\overline{H} \subset \mathbb{R}^n$. Consider a function u defined in H such that $u = \varphi$ on ∂H. As in the case of $A_{1/2}$ we want to consider the harmonic extension v of u in the cylinder $\Omega = H \times (0, +\infty)$ and we have to give Dirichlet data on the lateral boundary of the cylinder $\partial_L\Omega = \partial H \times (0, +\infty)$. We do it in the following way: we put $v(x,\lambda) = \varphi(x)$ for every $(x,\lambda) \in \partial_L\Omega$.

As before we define $D_{H,\varphi}$ as follows:

$$D_{H,\varphi}u := -\partial_\lambda v_{|\Omega\times\{0\}}.$$

We observe that v is independent on λ on $\partial_L\Omega$, then $v_\lambda = 0$ on the lateral boundary. Thus, we can apply the operator $A_{1/2}$ to $v_\lambda(x,0)$ and we get, as before

$$A_{1/2} \circ D_{H,\varphi} = -\Delta_{H,\varphi}$$

where $-\Delta_{H,\varphi}$ is the Laplacian in H with Dirichlet boundary value φ.

If we have a nonlinear problem of the type

$$\begin{cases} D_{H,\varphi}u = f(u) & \text{in } H \\ u = \varphi & \text{on } \partial H, \end{cases}$$

then it can be restated in the local problem,

$$\begin{cases} -\Delta v = 0 & \text{in } \Omega \\ v(x,\lambda) = \varphi(x) & \text{on } \partial_L\Omega \\ -\dfrac{\partial v}{\partial \lambda} = f(v) & \text{on } H \times \{0\}. \end{cases} \tag{2.4.1}$$

Consider now the harmonic extension ψ of φ in the cylinder Ω, with boundary data $\psi(x,\lambda) = \varphi(x)$ for every $(x,\lambda) \in \partial\Omega$.
If we set $w(x,\lambda) := v(x,\lambda) - \psi(x,\lambda)$, then w satisfies

$$\begin{cases} -\Delta w = 0 & \text{in } \Omega \\ w(x,\lambda) = 0 & \text{on } \partial_L\Omega \\ -\dfrac{\partial w}{\partial \lambda} = -\dfrac{\partial v}{\partial \lambda} + \dfrac{\partial \psi}{\partial \lambda} = f(w+\psi) + \dfrac{\partial \psi}{\partial \lambda} & \text{on } H \times \{0\}. \end{cases} \tag{2.4.2}$$

Now w vanishes on $\partial_L\Omega$ and problem (2.4.2) can be seen as the local formulation of the non local problem

$$A_{1/2}\omega = f(\omega + \varphi) + \frac{\partial \psi}{\partial \lambda},$$

where $\omega := u - \varphi$. Then, if we set $g(x,u) := f(\omega+\varphi) + \dfrac{\partial \psi}{\partial \lambda}(x,0)$, our problem

$$\begin{cases} D_{H,\varphi}u = f(u) & \text{in } H \\ u = \varphi & \text{on } \partial H \end{cases}$$

can be reformulated as

$$\begin{cases} A_{1/2}\omega = g(x,\omega) & \text{in } H \\ \omega = 0 & \text{on } \partial H, \end{cases}$$

which is a problem with zero Dirichlet boundary condition and in which the non-linearity depends also on x. Observe that the operator $D_{H,\varphi}$ coincides with $A_{1/2}$ if the boundary data φ is identically zero.

Next, we give some maximum principles for the operator $D_{H,\varphi}$.

Lemma 2.4.1. *Let $\Omega = H \times \mathbb{R}^+$ be a cylinder in $\mathbb{R}^{n+1}_+$, where $H \subset \mathbb{R}^n$ is a bounded domain. Let $v \in C^2(\Omega) \cap C^(\overline{\Omega})$ be a bounded harmonic function in Ω. Then,*

$$\inf_{\Omega} v = \inf_{\partial\Omega} v.$$

Proof. Substracting a constant from v, we may assume that v is nonnegative on $\partial\Omega$ and we need to show $v \geq 0$ in Ω.

To prove this fact, we follow a classical argument, constructing a strictly positive harmonic function ψ in Ω tending to infinity as $|(x,\lambda)| \to \infty$. We proceed in the following way.

First, since $H \subset \mathbb{R}^n$ is bounded, then there exists a ball B_R of radius R in $\mathbb{R}^n$ such that $\overline{H} \subset B_R$. Let μ_R and ϕ_R be, respectively, the first eigenvalue and the corresponding eigenfunction of the Laplacian $-\Delta$ in B_R with $0-$Dirichlet value on ∂B_R.

We define the function $\psi : B_R \times \mathbb{R}^+ \to \mathbb{R}$ as follows

$$\psi(x,\lambda) = \phi_R(x)e^{\sqrt{\mu_R}\lambda}.$$

Then the restriction of ψ in Ω is a strictly positive harmonic function.

Moreover, since ϕ_R is bounded, we have that

$$\lim_{|(x,\lambda)|\to+\infty} \psi(x,\lambda) = \lim_{\lambda\to+\infty} \psi(x,\lambda) = +\infty. \tag{2.4.3}$$

We consider now the function $w = v/\psi$. Then w satisfies

$$\begin{cases} -\Delta w - 2\dfrac{\nabla\psi}{\psi}\cdot\nabla w = 0 & \text{in } \Omega \\ w \geq 0 & \text{on } \partial\Omega. \end{cases}$$

Note that w has the same sign as v. In addition, by (2.4.3), $w(x,\lambda) \to 0$ as $|(x,\lambda)| \to +\infty$ and thus, by the strong maximum principle (applied, by a contradiction argument, to a possible negative minimum) $w \geq 0$ in Ω, which implies $v \geq 0$ in Ω. □

From the previous result we deduce the following lemma.

Lemma 2.4.2. *Assume that* $u \in C^2(H) \cap C(\overline{H})$ *satisfies*

$$\begin{cases} D_{H,\varphi}u + c(x)u \geq 0 & \textit{in } H, \\ u = \varphi & \textit{on } \partial H, \end{cases}$$

where H is a bounded domain in $\mathbb{R}^n$ and $c(x) \geq 0$ in H. Suppose that $\varphi \geq 0$ on ∂H. Then $u \geq 0$ in H.

Proof. Consider the harmonic extension v of u in $\Omega = H \times (0,+\infty)$ with Dirichlet data $v(x,\lambda) = \varphi(x)$ on the lateral boundary $\partial_L\Omega = \partial H \times (0,+\infty)$ (as in the definition of the operator $D_{H,\varphi}$). We prove that $v \geq 0$ in Ω, then in particular $u \geq 0$ in H.

Suppose by contradiction that v is negative somewhere in $\Omega \times \mathbb{R}^+$. Since v is harmonic, by Lemma 2.4.1 the $\inf_\Omega v < 0$ will be achieved at some point $(x_0, 0) \in H \times \{0\}$. Thus, we have

$$\inf_\Omega v = v(x_0,0) < 0.$$

By Hopf's lemma,

$$v_\lambda(x_0,0) > 0.$$

It follows

$$-v_\lambda(x_0, 0) = D_{H,\varphi} v(x_0, 0) < 0.$$

Therefore, since $c \geq 0$,

$$D_{H,\varphi} v(x_0, 0) + c(x_0) v(x_0, 0) < 0.$$

This is a contradiction with the hypothesis $D_{H,\varphi} u + c(x) u \geq 0$. □

The following corollary follows directly by the previous lemma.

Corollary 2.4.3. *Let H be a bounded domain in $\mathbb{R}^n$. Suppose that u_1 and u_2 are two bounded functions, $u_1,\ u_2 \in C^2(H) \cap C(\overline{H})$, which satisfy*

$$\begin{cases} D_{H,\varphi} u_1 \leq D_{H,\varphi} u_2 & \text{in } H \\ u_1 = u_2 = \varphi & \text{on } \partial H. \end{cases}$$

Then, $u_1 \leq u_2$ in H.

We conclude this section with the following strong maximum principle.

Lemma 2.4.4. *Assume that $u \in C^2(H) \cap C(\overline{H})$ satisfies*

$$\begin{cases} D_{H,\varphi} u + c(x) u \geq 0 & \text{in } \Omega, \\ u \geq 0 & \text{in } H, \\ u = \varphi & \text{on } \partial H, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$ and $c \in L^\infty(H)$. Suppose $\varphi \geq 0$ on ∂H.

Then, either $u > 0$ in H, or $u \equiv 0$ in H.

Proof. The proof is similar to the one of Lemma 2.4.2.

Consider the harmonic extension v of u with lateral boundary data $v = \varphi$ on $\partial_L \Omega$. We observe that $v \geq 0$ in Ω. Suppose that $v \not\equiv 0$ but $v = 0$ somewhere in Ω. Then there exists a minimum point $x_0 \in H$ such that $v(x_0, 0) = 0$. Hence by Hopf's lemma we see that $\frac{\partial v}{\partial \lambda}(x_0, 0) > 0$. This implies that $D_{H,\varphi} u(x_0) + c(x_0) u(x_0) < 0$, since $v(x_0, 0) = u(x_0) = 0$, which is a contradiction. □

2.5 Maximal saddle solution and monotonicity properties

Let $R > 0$ and consider the open region

$$T_R = \{x \in \mathbb{R}^{2m} : 0 < t < s < R\}. \tag{2.5.1}$$

Note that $T_R \supset \mathcal{O}_R = \mathcal{O} \cap B_R$.

Let, as before, v be the harmonic extension of a saddle solution u in the half-space $\mathbb{R}^{2m+1}_+$. The regularity results given in [10] give a uniform upper bound for $|\nabla v|$ (see (2.2.1)). Then, since $v = 0$ on $\mathcal{C} \times \mathbb{R}^+ = \{z = 0\} \times \mathbb{R}^+$, there exists a constant C, depending only on n and $||f||_{C^1}$, such that

$$|v(x, \lambda)| = |v(y, z, \lambda)| \leq C|z|.$$

In particular, we have that $|u(x)| = |v(x, 0)| \leq C|z|$.

Observe that there exists a real number $K \geq 1$ such that $\min\{1, C|z|\} \leq \min\{1, K|u_0(z)|\}$ for every z. Indeed it is enough to choose

$$K \geq \max\{C/u_0'(0), 1/u_0(C^{-1})\}. \tag{2.5.2}$$

Observe that the quantities $u_0'(0)$ and $u_0(C^{-1})$ are strictly positive.

If we choose K as in (2.5.2), then the harmonic extension v in $\mathbb{R}^{2m+1}_+$ of every saddle solution u of (2.1.1) satisfies

$$|v(x, \lambda)| \leq \min\{1, K|u_0(z)|\} \quad \text{in } \mathbb{R}^n. \tag{2.5.3}$$

We define

$$u_b(z) = \min\{1, K|u_0(z)|\}, \tag{2.5.4}$$

where K satisfies (2.5.2). Note that $u_b = 0$ on $\mathcal{C} \cap \overline{T_R}$.

Lemma 2.5.1. *Let f satisfies conditions* (2.1.8), (2.1.9), (2.1.10).

Then, there exists a positive solution $\overline{u}_R$ of

$$\begin{cases} D_{T_R, u_b} u = f(u) & \text{in } T_R \\ u = u_b & \text{on } \partial T_R. \end{cases}$$

which is maximal in T_R in the following sense. We have that $\overline{u}_R \geq u$ in T_R (and hence in $\mathcal{O}_R$) for every bounded solution u of $(-\Delta)^{1/2} u = f(u)$ in $\mathbb{R}^{2m}$ that vanishes on the Simons cone and has the same sign as $s - t$. In addition $\overline{u}_R$ depends only on s and t.

Proof. We construct a sequence of solutions of linear problems involving the operator D_{T_R, u_b} and, by the iterative use of the maximum principle, we prove that this sequence is non increasing and it converges to the maximal solution $\overline{u}_R$.

We put

$$Lw := (D_{T_R, u_b} + a)\, w, \quad \text{and} \quad g(w) := f(w) + aw,$$

where a is a positive constant chosen such that $g'(w) = f'(w) + a$ is positive for every w.

Next we define a sequence of functions $\overline{u}_{R,j}$ as follows: $\overline{u}_{R,0}(x) = u_b = \min\{1, Ku_0(z)\}$ and $\overline{u}_{R,j+1}$ solves the linear problem

$$\begin{cases} L\overline{u}_{R,j+1} &= g(\overline{u}_{R,j}) \quad \text{in } T_R \\ \overline{u}_{R,j+1} &= u_b \quad\quad\ \ \text{on } \partial T_R. \end{cases} \tag{2.5.5}$$

Since L is obtained by adding a positive constant to D_{T_R,u_b}, it satisfies the maximum principles (Lemma 2.4.2 and Corollary 2.4.3) and hence the above problem admits a unique solution $\overline{u}_{R,j+1} = \overline{u}_{R,j+1}(x)$. Furthermore (and here we argue by induction), since the problem and its data are invariant by orthogonal transformations in the first (respectively, in the last) m variables x_i, the solution $\overline{u}_{R,j+1}$ depends only on s and t.

First, observe that by Corollary 2.3.6, the function $\overline{u}_{R,0} = \min\{1, Ku_0(z)\}$ is a supersolution of problem $Lw = g(w)$, i.e., $L\overline{u}_{R_0} \geq g(\overline{u}_{R,0})$. This implies that $L\overline{u}_{R_1} = g(\overline{u}_{R,0}) \leq L\overline{u}_{R,0}$ and then $\overline{u}_{R,1} \leq \overline{u}_{R,0} \leq 1$ in T_R. Moreover $u_b \geq 0$ on ∂T_R and therefore, by Lemma 2.4.2, $\overline{u}_{R,1} \geq 0$ in T_R.

Assume now that $0 \leq \overline{u}_{R,j} \leq \overline{u}_{R,j-1} \leq 1$ for some $j \geq 1$. Therefore, by the choice of a, $g(\overline{u}_{R,j}) \leq g(\overline{u}_{R,j-1})$. We have

$$L\overline{u}_{R,j+1} = g(\overline{u}_{R,j}) \leq g(\overline{u}_{R,j-1}) = L\overline{u}_{R,j}.$$

Again by the maximum principle (Corollary (2.4.3)) $\overline{u}_{R,j+1} \leq \overline{u}_{R,j}$. Besides, $\overline{u}_{R,j+1} \geq 0$ since $g(\overline{u}_{R,j}) \geq 0$. Therefore, by induction we have proven that the sequence $\overline{u}_{R,j}$ is nonincreasing, that is

$$1 = \overline{u}_{R,0}(x) \geq \overline{u}_{R,1}(x) \geq \cdots \geq \overline{u}_{R,j}(x) \geq \overline{u}_{R,j+1}(x) \geq \cdots \geq 0.$$

By monotone convergence, this sequence converges to a nonnegative solution in T_R, $\overline{u}_R$, which depends only on s and t, and such that $\overline{u}_R = u_b(z)$ on ∂T_R. Thus, the strong maximum principle (Lemma 2.4.4) leads to $\overline{u}_R > 0$ in T_R.

Moreover, $\overline{u}_R$ is maximal with respect to any bounded solution u, $|u| < 1$ in $\mathbb{R}^{2m}$, that vanishes on the Simons cone and has the same sign as $s - t$. Indeed, let $\overline{v}_{R,1}$ be the harmonic extension of $\overline{u}_{R,1}$ in $T_R \times \mathbb{R}^+$ which is equal to u_b on the lateral boundary $\partial T_R \times \mathbb{R}^+$. It is the solution of the following problem

$$\begin{cases} \Delta \overline{v}_{R,1} = 0 & \text{in } T_R \times \mathbb{R}^+ \\ \overline{v}_{R,1} = u_b & \text{on } \partial T_R \times \mathbb{R}^+ \\ -\dfrac{\partial \overline{v}_{R,1}}{\partial \lambda} + a\overline{v}_{R,1} = g(\overline{u}_{R,0}) = g(u_b) & \text{on } T_R \times \{0\}. \end{cases} \tag{2.5.6}$$

Consider now v the harmonic extension of u in $\mathbb{R}^{2m+1}_+$. Then the restriction of v to T_R, which we still call v, is the solution of the problem

$$\begin{cases} \Delta v = 0 & \text{in } T_R \times \mathbb{R}^+ \\ -\dfrac{\partial v}{\partial \lambda} + av = g(u) & \text{on } T_R \times \{0\}. \end{cases} \tag{2.5.7}$$

Recall that by (2.5.3), we have that $v \leq u_b$ in $T_R \times \mathbb{R}^+$ and in particular $u \leq u_b$ on $T_R \times \{0\}$. Since g is increasing, then the difference $v - \overline{v}_{R,1}$ is a solution of

$$\begin{cases} \Delta(v - \overline{v}_{R,1}) = 0 & \text{in } T_R \times \mathbb{R}^+ \\ v - \overline{v}_{R,1} = v - u_b \leq 0 & \text{on } \partial T_R \times \mathbb{R}^+ \\ -\dfrac{\partial(v - \overline{v}_{R,1})}{\partial\lambda} + a(v - \overline{v}_{R,1}) = g(u) - g(u_b) \leq 0 & \text{on } T_R \times \{0\}. \end{cases} \tag{2.5.8}$$

We claim that $v \leq \overline{v}_{R,1}$ in $T_R \times \mathbb{R}^+$. Indeed, suppose by contradiction that $v - \overline{v}_{R,1}$ is positive somewhere in $T_R \times \mathbb{R}^+$. Then, by the maximum principle (Lemma 2.4.2), the $\sup(v - \overline{v}_{R,1}) > 0$ will be achieved at some point $(x_0, 0) \in T_R \times \{0\}$. By Hopf's Lemma, we would have

$$-\frac{\partial(v - \overline{v}_{R,1})}{\partial\lambda}(x_0, 0) + a(v - \overline{v}_{R,1})(x_0, 0) > 0.$$

Since a is positive, this is a contradiction with the last inequality of (2.5.8). Thus we have proved that $v \leq \overline{v}_{R,1}$ in $T_R \times \mathbb{R}^+$.

Suppose now that $v \leq \overline{v}_{R,j}$. Arguing as before, we consider the problem satisfied by $(v - \overline{v}_{R,j+1})$. Using the maximum principle and Hopf's Lemma we deduce that $v \leq \overline{v}_{R,j+1}$ in $T_R \times \mathbb{R}^+$.

Then, by induction, $v \leq \overline{v}_{R,j}$ for every j and, in particular, $u \leq \overline{u}_{R,j}$ for every j. Thus,

$$v \leq \overline{v}_R := \lim_{j\to\infty} \overline{v}_{R,j} \quad \text{in } T_R \times (0, +\infty).$$

We set $\overline{u}(x) = \overline{v}(x, 0)$. □

The following are monotonicity results for the maximal solution constructed above.

Lemma 2.5.2. *Let $\overline{u}_R$ be the function constructed in Lemma 2.5.1. Let $\overline{v}_R$ be the harmonic function in $T_R \times (0, +\infty)$ such that $\overline{v}_R(x, 0) = \overline{u}_R(x)$ for every $x \in T_R$ and $v(x, \lambda) = u_b(x)$ for every $(x, \lambda) \in \partial T_R \times (0, +\infty)$.*

Then $\partial_t \overline{v}_R \leq 0$.

Proof. We consider the nonincreasing sequence of function $\overline{u}_{R,j}$ constructed in the proof of Lemma 2.5.1. We call as before $\overline{v}_{R,j}$ the harmonic extension of $\overline{u}_{R,j}$ in $T_R \times (0, +\infty)$ such that $\overline{v}_{R,j}(x, \lambda) = u_b(x)$ for every $(x, \lambda) \in \partial T_R \times (0, +\infty)$.

The function $\overline{v}_{R,j}$ is a solution in coordinates s and t of the problem

$$\begin{cases} \partial_{ss}\overline{v}_{R,j} + \partial_{tt}\overline{v}_{R,j} + \partial_{\lambda\lambda}\overline{v}_{R,j} + \dfrac{(m-1)}{s}\partial_s\overline{v}_{R,j} + \dfrac{(m-1)}{t}\partial_t\overline{v}_{R,j} = 0 & \text{in } T_R \times (0, \infty) \\ \overline{v}_{R,j} = u_b & \text{on } \partial T_R \times (0, +\infty), \\ -\partial_\lambda \overline{v}_{R,j} + a\overline{v}_{R,j} = g(\overline{v}_{R,j-1}) & \text{on } T_R \times \{0\} \end{cases}$$

Differentiating with respect to t we get:

$$\begin{cases} -\Delta(\partial_t \overline{v}_{R,j}) + \dfrac{(m-1)}{t^2}\partial_t \overline{v}_{R,j} = 0 & \text{in } T_R \times (0,\infty) \\ -\partial_\lambda(\partial_t \overline{v}_{R,j}) + a\partial_t \overline{v}_{R,j} = g'(\overline{v}_{R,j-1})\partial_t \overline{v}_{R,j-1} & \text{on } T_R \times \{0\}. \end{cases} \tag{2.5.9}$$

We observe that $\partial_t \overline{v}_{R,j} \le 0$ on $\partial T_R \times (0,+\infty)$. Indeed $\overline{v}_{R,j} \equiv 0$ on $(\mathcal{C} \cap \partial T_R) \times (0,+\infty)$ and $\overline{v}_{R,j} > 0$ inside $T_R \times (0,+\infty)$. Then, $\partial_t \overline{v}_{R,j} \le 0$ on $\{t = s < R\} \times (0,+\infty)$.

Moreover $\overline{v}_{R,j} = \min\{Ku_0(z), 1\} = \min\{Ku_0((R-t)/\sqrt{2}), 1\}$ on $\{t < s = R\}$ and thus $\partial_t \overline{v}_{R,j} \le 0$ on $\{t < s = R\} \times (0,+\infty)$.

Now, we argue by induction. First, recall that

$$\overline{v}_{R,0} = \min\{Ku_0(z), 1\} = \min\{Ku_0((s-t)/\sqrt{2}), 1\},$$

then $\partial_t \overline{v}_{R,0} \le 0$.

Suppose that $\partial_t \overline{v}_{R,j-1} \le 0$, we prove that $\partial_t \overline{v}_{R,j} \le 0$. Indeed, first observe that $(m-1)/t^2 \ge 0$. Then, remind that $\partial_t \overline{v}_{R,j} \le 0$ on the lateral boundary of the set $T_R \times (0,+\infty)$ and it satisfies the Neumann condition

$$-\partial_\lambda(\partial_t \overline{v}_{R,j}) + a\partial_t \overline{v}_{R,j} = g'(\overline{v}_{R,j-1})\partial_t \overline{v}_{R,j-1}. \tag{2.5.10}$$

Assume by contradiction that $\partial_t \overline{v}_{R,j}$ is positive somewhere in $T_R \times \mathbb{R}^+$, then, by the maximum principle the $\sup \overline{v}_{R,j} > 0$ will be achieved at some point $(x_0, 0)$ in $T_R \times \{0\}$. Since $g' > 0$ and $a > 0$, applying Hopf's Lemma we get a contradiction with (2.5.10). This implies that $\partial_t \overline{v}_{R,j} \le 0$ for every j and then, passing to the limit, that $\partial_t \overline{v}_R \le 0$. □

Lemma 2.5.3. *Let $\overline{u}_R$ be the function constructed in Lemma 2.5.1. Let $\overline{v}_R$ be the harmonic function in $T_R \times (0,+\infty)$ such that $\overline{v}_R(x,0) = \overline{u}_R(x)$ for every $x \in T_R$ and $v(x,\lambda) = u_b(x)$ for every $(x,\lambda) \in \partial T_R \times (0,+\infty)$.*

Then, $\partial_y \overline{v}_R \ge 0$.

Proof. Consider as before the sequences of functions $\overline{v}_{R,j}$ and $\overline{u}_{R,j}$. We first observe that $\partial_y \overline{v}_{R,j} \ge 0$ on $\partial T_R \times (0,+\infty)$. Indeed $\overline{v}_{R,j} \equiv 0$ on the part of the boundary $\{t = s < R\} \times (0,+\infty)$. Thus, since ∂_y is a tangential derivative here, we have $\partial_y \overline{v}_{R,j} \equiv 0$ on $\{t = s < R\} \times (0,+\infty)$.

Take now a point $(s = R, t, \lambda)$, with $0 < t < R$, on the remaining part of the boundary.

Recall that $\overline{u}_{R,j} \le \overline{u}_{R,0} = u_b = \min\{Ku_0(z), 1\}$ in all of T_R. Then, applying the maximum principle (Lemma 2.4.1), we deduce that $\overline{v}_{R,j} \le \overline{u}_{R,0} = \min\{Ku_0(z), 1\} = \min\{Ku_0((s-t)/\sqrt{2}), 1\}$ in all of $T_R \times (0,+\infty)$.

Then, for every $0 < \delta < t$ we have

$$\begin{aligned} \overline{v}_{R,j}(R-\delta, t-\delta, \lambda) &\leq \min\left\{ K u_0 \left(\frac{R-\delta-(t-\delta)}{\sqrt{2}} \right), 1 \right\} \\ &= \min\left\{ K u_0 \left(\frac{R-t}{\sqrt{2}} \right), 1 \right\} = u_b(R,t). \end{aligned}$$

Then $\partial_y \overline{v}_{R,j} \geq 0$ on $\{t < s = R\} \times (0, +\infty)$.

Next, we consider the problem satisfied by $\partial_t \overline{v}_{R,j}$ and $\partial_s \overline{v}_{R,j}$. We recall that $\partial_t \overline{v}_{R,j}$ is a solution of (2.5.9) and $\partial_s \overline{v}_{R,j}$ satisfies

$$\begin{cases} -\Delta(\partial_s \overline{v}_{R,j}) + \dfrac{(m-1)}{s^2} \partial_s \overline{v}_{R,j} = 0 & \text{in } T_R \times (0, \infty) \\ -\partial_\lambda(\partial_s \overline{v}_{R,j}) + a \partial_s \overline{v}_{R,j} = g'(\overline{v}_{R,j-1}) \partial_s \overline{v}_{R,j-1} & \text{on } T_R \times \{0\}. \end{cases} \tag{2.5.11}$$

Thus, since $\partial_y = (\partial_s + \partial_t)/\sqrt{2}$, we have that $\partial_y \overline{v}_{R,j}$ satisfies the equation

$$\begin{aligned} -\Delta(\partial_y \overline{v}_{R,j}) &= -\frac{m-1}{\sqrt{2}} \left(\frac{\partial_s \overline{v}_{R,j}}{s^2} + \frac{\partial_t \overline{v}_{R,j}}{t^2} \right) \\ &= -\frac{m-1}{s^2} \partial_y \overline{v}_{R,j} - \frac{(m-1)(s^2-t^2)}{\sqrt{2} s^2 t^2} \partial_t \overline{v}_{R,j}. \end{aligned}$$

Then $\partial_y \overline{v}_{R,j}$ is a solution of the problem

$$\begin{cases} -\Delta(\partial_y \overline{v}_{R,j}) + \dfrac{(m-1)}{s^2} \partial_y \overline{v}_{R,j} + \dfrac{(m-1)(s^2-t^2)}{\sqrt{2} s^2 t^2} \partial_t \overline{v}_{R,j} = 0 & \text{in } T_R \times (0, \infty) \\ \partial_y \overline{v}_{R,j} \geq 0 & \text{on } \partial T_R \\ -\partial_\lambda(\partial_y \overline{v}_{R,j}) + a \partial_y \overline{v}_{R,j} = g'(\overline{v}_{R,j-1}) \partial_y \overline{v}_{R,j-1} & \text{on } T_R \times \{0\}. \end{cases}$$

By Lemma 2.5.2 we have that $\partial_t \overline{v} \leq 0$ in $T_R \times (0, +\infty)$ and thus

$$\frac{(m-1)(s^2-t^2)}{\sqrt{2} s^2 t^2} \partial_t \overline{v}_{R,j} \leq 0, \quad \text{in } T_R \times (0, +\infty).$$

Then, we can apply, as in the proof of Lemma 2.5.2, the maximum principle and Hopf's Lemma, to obtain $\partial_y \overline{v}_{R,j} \geq 0$ for every j. Finally, passing to the limit for $j \to \infty$, we get $\partial_y \overline{v}_{R,j} \geq 0$ in $T_R \times (0, +\infty)$. □

We can give now the proof of Proposition 2.1.7.

Proof of Proposition 2.1.7. In Lemma 2.5.1 we established the existence of a maximal solution $\overline{u}_R$ in T_R, that is, $\overline{u}_R$ is a solution of $D_{T_R,u_b} \overline{u}_R = f(\overline{u}_R)$ in T_R and

$$\overline{u}_R \geq u$$

for every bounded solution $u \le 1$ in $\mathbb{R}^{2m}$ that vanishes on $\mathcal{C}$ and has the same sign as $s-t$.

By standard elliptic estimates and the compactness arguments as in the proof of Theorem 2.1.6, up to a subsequence we can take the limit as $R \to +\infty$ and obtain a solution $\overline{u}$ in $\mathcal{O} = \{s > t\}$, with $\overline{u} = 0$ on $\mathcal{C}$. By construction,

$$u \le \overline{u} := \lim_{R_j \to \infty} \overline{u}_{R_j},$$

for all solutions u as above. In addition, $\overline{u}$ depends only on s and t.

By maximality of $\overline{u}$ and the existence of saddle solution of Theorem 2.1.6, we deduce that $\overline{u} > 0$ in $\mathcal{O}$.

Since f is odd, by odd reflection with respect to the Simons cone, we obtain a maximal solution $\overline{u}$ in $\mathbb{R}^{2m}$ such that $|u| \le |\overline{u}|$ in $\mathbb{R}^{2m}$.

Let $\overline{v}$ be the harmonic extension of $\overline{u}$ in $\mathbb{R}^{2m+1}_+$. We prove now the monotonicity properties of $\overline{v}$.

By Lemmas 2.5.2 and 2.5.3, we have that $\partial_t \overline{v}_R \ge 0$ and $\partial_y \overline{v}_R \le 0$ in $T_R \times (0, +\infty)$. Letting $R \to +\infty$, we get $\partial_t \overline{v} \ge 0$ and $\partial_y \overline{v} \le 0$ in $\widetilde{\mathcal{O}}$. As a consequence $\partial_s \overline{v} \ge 0$ in $\widetilde{\mathcal{O}}$.

Since $v(s,t,\lambda) = -v(t,s,\lambda)$, it follows that $\partial_s \overline{v} \ge 0$ and $\partial_t \overline{v} \le 0$ in $\mathbb{R}^{2m+1}_+$.

Now, $\partial_t \overline{v} \le 0$ in $\mathbb{R}^{2m+1}_+$ and satisfies

$$-\Delta \partial_t \overline{v} + \frac{m-1}{t^2} \partial_t \overline{v} = 0 \quad \text{in } \mathbb{R}^{2m+1}_+.$$

Then, the strong maximum principle implies that $\partial_t \overline{v} < 0$ in $\mathbb{R}^{2m+1}_+ \setminus \{t = 0\}$. Moreover we multiply by t the following equation satisfied by $\overline{v}$ in $\mathbb{R}^{2m+1}_+$

$$\partial_{ss} \overline{v} + \partial_{tt} \overline{v} + \partial_{\lambda\lambda} \overline{v} + \frac{m-1}{s} \overline{v}_s + \frac{m-1}{t} \overline{v}_t = 0.$$

Using that $\overline{v} \in C^2$ and letting $t \to 0$, we get $\partial_t \overline{v} = 0$ on $\{t = 0\}$.

In the same way we deduce that $\partial_s \overline{v} > 0$ in $\mathbb{R}^{2m+1}_+ \setminus \{s = 0\}$ and $\partial_s \overline{v} = 0$ on $\{s = 0\}$.

Recalling that $\partial_z = (\partial_s - \partial_t)/\sqrt{2}$, statement c) follows directly by a) and b).

Finally, we remind that $\partial_y \overline{v}$ satisfies

$$-\Delta \partial_y \overline{v} = -\frac{m-1}{s^2} \partial_y \overline{v} - \frac{(m-1)(s^2 - t^2)}{\sqrt{2} s^2 t^2} \partial_t \overline{v} \ge -\frac{m-1}{s^2} \partial_y \overline{v}, \tag{2.5.12}$$

in $\{s > t > 0\} \times \mathbb{R}^+$, since $\partial_t \overline{v} \le 0$ in this set. Since we have already proved that $\partial_y \overline{v} \ge 0$ in $\{s > t > 0\} \times \mathbb{R}^+$, the strong maximum principle implies $\partial_y \overline{v} > 0$ in $\{s > t > 0\} \times \mathbb{R}^+$. □

2.6 Asymptotic behaviour of saddle solutions in $\mathbb{R}^{2m}$

In this section we study the asymptotic behaviour at infinity of solutions which are odd with respect to the Simons cone and positive in the set $\mathcal{O} = \{s > t\}$. In particular our result holds for saddle solutions.

We will consider the (y, z) system of coordinates. Recall that we have defined in (2.1.17) y and z by

$$\begin{cases} y = \dfrac{s+t}{\sqrt{2}} \\ z = \dfrac{s-t}{\sqrt{2}}, \end{cases} \tag{2.6.1}$$

which satisfy $y \geq 0$ and $-y \leq z \leq y$.

We give the proof of Theorem 2.1.9, which states that any solution u as above tends to infinity to the function

$$U(x) := u_0(z) = u_0(d(x, \mathcal{C})),$$

uniformly outside compact sets. We recall that u_0 is the layer solution of $(-\Delta)^{1/2}u_0 = f(u_0)$ in $\mathbb{R}$ which vanishes at the origin, and $d(\cdot, \mathcal{C})$ denotes the distance to the Simons cone. Similarly ∇u converges to ∇U. We will use this fact in the proof of instability of saddle solutions in dimension $2m = 4$ and $2m = 6$.

Our proof of the asymptotic behaviour follows a method used by Cabré and Terra for the classical equation $-\Delta u = f(u)$. They use a compactness argument based on translations of the solution, combined with two crucial Liouville-type results for nonlinear equations. Here, we use analog Liouville results for the nonlinear Neumann problem satisfied by the harmonic extension v of our saddle solutions u of equation (2.1.1). Both results were proved using the moving planes method.

The first result establishes a symmetry property for solutions of a nonlinear Neumann problem in the half-space, and it was proven in [26].

Theorem 2.6.1. *([26])*

Let $\mathbb{R}^{n+1}_+ = \{\xi = (x_1, x_2, \cdots, x_n, \lambda) \mid \lambda > 0\}$ and let f be such that $f(u)/u^{\frac{n}{n-2}}$ is non-increasing. Assume that v is a solution of problem

$$\begin{cases} -\Delta v = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ -\frac{\partial v}{\partial \lambda} = f(v) & \text{on } \{\lambda = 0\}, \\ v > 0 & \text{in } \mathbb{R}^{n+1}_+. \end{cases} \tag{2.6.2}$$

Then v depends only on λ.

More precisely, there exist $a \geq 0$ and $b > 0$ such that

$$v(x, \lambda) = v(\lambda) = a\lambda + b \quad \text{and} \quad f(b) = a.$$

Corollary 2.6.2. *Let f satisfy* (2.1.8)*,* (2.1.9)*,* (2.1.10)*. Let v be a bounded solution of problem* (2.6.2)*.*

Then, $v \equiv 0$ or $v \equiv 1$.

Proof of Corollary 2.6.2. By Remark 2.6.4, f satisfies the hypothesis of Theorem 2.6.1. Moreover since f is bistable, we have that f is odd, $f(0) = f(\pm 1) = 0$, $f > 0$ in $(0, 1)$ and $f < 0$ in $(1, +\infty)$. Then, since v is bounded, it has to be $v(x, \lambda) = b$ with $f(b) = 0$, that is $v \equiv 0$ or $v \equiv 1$. □

The following theorem establishes an analog symmetry properties but for solutions in a quarter of space, and was proven in [11].

Theorem 2.6.3. *([11]) Let $\mathbb{R}^{n+1}_{++} = \{\xi = (x_1, x_2, \cdots, x_n, \lambda) \mid x_n > 0, \lambda > 0\}$ and let f be such that $f(u)/u^{\frac{n}{n-2}}$ is non-increasing. Assume that v is a solution of problem*

$$\begin{cases} -\Delta v = 0 & \text{in } \mathbb{R}^{n+1}_{++}, \\ -\frac{\partial v}{\partial \lambda} = f(v) & \text{on } \{x_n > 0, \lambda = 0\}, \\ v = 0 & \text{on } \{x_n = 0, \lambda \geq 0\}, \\ v > 0 & \text{in } \mathbb{R}^{n+1}_{++}, \end{cases}$$

Then v depends only on x_n and λ.

Remark 2.6.4. We claim that if f satisfies hypothesis (2.1.8), (2.1.9), (2.1.10), then $f(u)/u^{\frac{n}{n-2}}$ is non-increasing.

First, we recall that, by Remark 2.3.4, $f(u)/u$ is non-increasing in $(0, 1)$. Moreover, we can write

$$\frac{f(u)}{u^{\frac{n}{n-2}}} = \frac{f(u)}{u} \cdot u^{1-\frac{n}{n-2}}.$$

Since $\dfrac{n}{n-2} > 1$, then $u^{1-\frac{n}{n-2}}$ is non-increasing, and thus f satisfies the hypothesis of Theorems 2.6.1 and 2.6.3.

Now, we can give the proof of our asymptotic behaviour result.

Proof of Theorem 2.1.9. Consider the harmonic extension $v(x, \lambda)$ of $u(x)$ in $\mathbb{R}^{2m+1}_+$, that satisfies

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^{2m+1}_+ \\ -\partial_\lambda v = f(v) & \text{on } \{\lambda = 0\}. \end{cases} \tag{2.6.3}$$

Set $V(x, \lambda) := v_0(z, \lambda)$. We want to prove that

$$v(x, \lambda) - V(x, \lambda) \to 0 \quad \text{and} \quad \nabla v(x, \lambda) - \nabla V(x, \lambda) \to 0,$$

uniformly as $|x| \to \infty,\ \lambda \in \mathbb{R}^+$.
Suppose that the theorem does not hold. Thus, there exists $\epsilon > 0$ and a sequence $\{x_k\}$ with

$$|x_k| \to \infty \quad \text{and} \quad |v(x_k,\lambda) - V(x_k,\lambda)| + |\nabla v(x_k,\lambda) - \nabla V(x_k,\lambda)| \geq \epsilon. \quad (2.6.4)$$

By continuity we may move slightly x_k and assume $x_k \notin \mathcal{C}$ for all k. Moreover, up to a subsequence (which we still denote by $\{x_k\}$), either $\{x_k\} \subset \{s > t\}$ or $\{x_k\} \subset \{s < t\}$. By the symmetries of the problem we may assume $\{x_k\} \subset \{s > t\} = \mathcal{O}$.

We distinguish two cases:

Case 1 $\{\operatorname{dist}(x_k, \mathcal{C}) = d_k\}$ is unbounded.

In this case, since $0 < z_k = \operatorname{dist}(x_k, \mathcal{C}) = d_k \to +\infty$ (for a subsequence), we have that $V(x_k,\lambda) = v_0(z_k,\lambda) = v_0(d_k,\lambda)$ tends to 1 and $|\nabla V(x_k,\lambda)|$ tends to 0, that is,

$$V(x_k,\lambda) \to 1 \quad \text{and} \quad |\nabla V(x_k,\lambda)| \to 0.$$

From this and (2.6.4) we have

$$|v(x_k,\lambda) - 1| + |\nabla v(x_k,\lambda)| \geq \frac{\epsilon}{2}, \quad (2.6.5)$$

for k large enough. Taking subsequence (and relabeling the subindex) we may assume $\operatorname{dist}(x_k, \mathcal{C}) = d_k \geq 2k$.

Consider the ball $B_k(0) \subset \mathbb{R}^{2m}$ of radius k centered at $x = 0$, and define

$$w_k(\tilde{x},\lambda) = v(\tilde{x} + x_k, \lambda), \text{ for every } (\widetilde{x},\lambda) \in B_k(0) \times (0,+\infty).$$

Since $B_k(0) + x_k \subset \{s > t\}$, we have that $0 < w_k < 1$ in $B_k(0) \times (0,+\infty)$ and

$$\begin{cases} \Delta w_k = 0 & \text{in } B_k(0) \times (0,+\infty) \\ -\partial_\lambda w_k = f(v) & \text{on } \{\lambda = 0\}. \end{cases} \quad (2.6.6)$$

Letting k tend to infinity we obtain, through a subsequence, a nonnegative solution w the problem in all of $\mathbb{R}^{2m+1}_+$. That is, w satisfies

$$\begin{cases} -\Delta w = 0 & \text{in } \mathbb{R}^{2m+1}_+ \\ -\partial_\lambda w = f(v) & \text{on } \{\lambda = 0\} \\ w > 0 & \text{in } \mathbb{R}^{2m+1}_+ \end{cases}. \quad (2.6.7)$$

Since f satisfies (2.1.8), (2.1.9), (2.1.10), we have that, by Corollary 2.6.2, $w \equiv 0$ or $w \equiv 1$. In either case, $\nabla w(0) = 0$, that is, $|\nabla v(x_k,\lambda)|$ tends to 0.

Next we show that $w \not\equiv 0$. By Theorem 2.1.6 we have that v is stable in $\mathcal{O} \times (0, +\infty)$. Hence, w_k is semi-stable in $B_k(0) \times (0, +\infty)$ (since $B_k(0) + x_k \subset \mathcal{O}$). This implies that w is stable in all of $\mathbb{R}^{2m+1}_+$ and therefore $w \not\equiv 0$ (otherwise, since $f'(0) > 0$ we could construct a test function ξ such that $Q_w(\xi) < 0$ which would be a contradiction with the fact that w is stable).

Hence, it must be $w \equiv 1$. But this implies that $w(0, \lambda) = 1$ and so $v(x_k, \lambda)$ tends to 1. Hence, we have that $v(x_k, \lambda)$ tends to 1 and $|\nabla v(x_k, \lambda)|$ tends to 0, which is a contradiction with (2.6.5). Therefore, we have proved the theorem in this case 1.

Case 2 $\{\operatorname{dist}(x_k, \mathcal{C}) = d_k\}$ is bounded.

The points x_k remain at a finite distance to the cone. Then, at least for a subsequence,

$$d_k \to d \geq 0 \quad \text{as } k \to \infty.$$

Let $x_k^0 \in \mathcal{C}$ be a point that realizes the distance to the cone, that is,

$$\operatorname{dist}(x_k, \mathcal{C}) = |x_k - x_k^0| = d_k, \tag{2.6.8}$$

and let ν_k^0 be the inner unit normal to $\mathcal{C} = \partial\mathcal{O}$ at x_k^0. Note that $B_{d_k}(x_k) \subset \mathcal{O} \subset \mathbb{R}^{2m} \setminus \mathcal{C}$ and $x_k^0 \in \partial B_{d_k}(x_k) \cap \mathcal{C}$, i.e., x_k^0 is the point where the sphere $\partial B_{d_k}(x_k)$ is tangent to the cone $\mathcal{C}$. It follows that $x_k^0 \neq 0$ and that $(x_k - x_k^0)/d_k$ is the unit normal ν_k^0 to $\mathcal{C}$ at x_k^0. That is, $x_k = x_k^0 + d_k \nu_k^0$.

Now, since the sequence $\{\nu_k^0\}$ is bounded, there exists a subsequence such that

$$\nu_k^0 \to \nu \in \mathbb{R}^{2m}, \quad |\nu| = 1.$$

Write $w_k(\tilde{x}, \lambda) = v(\tilde{x} + x_k^0, \lambda)$, for $\tilde{x} \in \mathbb{R}^{2m}$. The functions w_k are all solutions of

$$\begin{cases} \Delta w_k = 0 & \text{in } \mathbb{R}^{2m+1}_+ \\ -\partial_\lambda w_k = f(w_k) & \text{on } \{\lambda = 0\} \end{cases}. \tag{2.6.9}$$

and are uniformly bounded. Hence, by elliptic estimates, the sequence $\{w_k\}$ converges locally in space in C^2, up to a subsequence, to a solution w in $\mathbb{R}^{2m+1}_+$. Therefore we have that, as k tends to infinity and up to a subsequence,

$$w_k \to w \quad \text{and} \quad \nabla w_k \to \nabla w \text{ uniformly on compact sets of } \mathbb{R}^{2m+1}_+,$$

where w is a solution

$$\begin{cases} \Delta w = 0 & \text{in } \mathbb{R}^{2m+1}_+ \\ -\partial_\lambda w = f(w) & \text{on } \{\lambda = 0\} \end{cases} \tag{2.6.10}$$

Note that the curvature of $\mathcal{C}$ at x_k^0 goes to zero as k tends to infinity, since $\mathcal{C}$ is a cone and $|x_k| \to \infty$ (note that $|x_k^0| \to \infty$ due to $|x_k| \to \infty$ and $|x_k - x_k^0| = d_k \to d < \infty$).

Thus, $\mathcal{C}$ at x_k^0 is flatter and flatter as $k \to \infty$ and since we translate x_k^0 to 0, the limiting solution w satisfies

$$\begin{cases} \Delta w = 0 & \text{in } H := \{(x,\lambda) \in \mathbb{R}^{2m+1}_+ : \widetilde{x}\cdot\nu = 0, \lambda > 0\} \\ w \geq 0 & \text{in } H \\ w = 0 & \text{in } \{\widetilde{x}\cdot\nu = 0\} \\ -\partial_\lambda w = f(w) & \text{in } \{\lambda = 0\}. \end{cases} \tag{2.6.11}$$

For the details of the proof of this fact see [13].

Now, since v is stable for perturbations vanishing on $\partial\mathcal{O}\times\mathbb{R}^+$, it follows that w is stable for perturbations with compact support in H, and therefore w can not be identically zero. By Theorem 2.6.3, since f satisfies (2.1.8), (2.1.9), (2.1.10), we deduce that w is symmetric, that is, it is a function of only two variable (the orthogonal direction to H and λ). It follows that

$$w(\tilde{x},\lambda) = v_0(\tilde{x}\cdot\nu,\lambda) \quad \text{for all } (\tilde{x},\lambda) \in H.$$

From the definition of w_k, and using that $z_k = d_k = |x_k - x_k^0|$ is a bounded sequence and that $x_k - x_k^0 = d_k\nu_k^0$, we have that

$$\begin{aligned} v(x_k,\lambda) &= w_k(x_k - x_k^0,\lambda) = w(x_k - x_k^0,\lambda) + \text{ o}(1) = v_0((x_k - x_k^0)\cdot\nu,\lambda) + \text{ o}(1) \\ &= v_0((x_k - x_k^0)\cdot\nu_k^0,\lambda) + \text{ o}(1) = v_0(d_k,\lambda) + \text{ o}(1) \\ &= v_0(z_k,\lambda) + \text{ o}(1) = V(x_k,\lambda) + \text{ o}(1). \end{aligned}$$

The same argument can be done for $\nabla v(x_k,\lambda)$ and $\nabla V(x_k,\lambda)$. We arrive to a contradiction with (2.6.4). □

2.7 Instability in dimensions 4 and 6

Before proving the theorem on the instability of saddle solutions in dimensions 4 and 6, we establish a lemma that will be useful later.

Lemma 2.7.1. *Assume f satisfy conditions* (2.1.8), (2.1.9), (2.1.10). *Let v be a bounded solution of (2.1.3) in $\mathbb{R}^{n+1}_+$ and w a function such that $|v| \leq |w| \leq 1$ in $\mathbb{R}^{n+1}_+$. Then,*

$$Q_v(\xi) \leq Q_w(\xi) \quad \textit{for all } \xi \in C_0^\infty(\overline{\mathbb{R}^{n+1}_+}),$$

where Q_w is defined by

$$Q_w(\xi) = \int_{\mathbb{R}^{n+1}_+} |\nabla\xi|^2 dx d\lambda - \int_{\partial\mathbb{R}^{n+1}_+} f'(w)\xi^2 dx.$$

In particular, if there exists a function $\xi \in C_0^\infty(\overline{\mathbb{R}^{n+1}_+})$ such that $Q_w(\xi) < 0$, then v is an unstable solution.

Proof. Let v be such a bounded solution and w a function with $|v| \leq |w| \leq 1$.

Since f' is decreasing in $(0,1)$ we have that

$$f'(|v|) \geq f'(|w|) \quad \text{in } \mathbb{R}^{n+1}_{+}.$$

Moreover, f' being even yields,

$$f'(v) \geq f'(w) \quad \text{in } \mathbb{R}^{n+1}_{+},$$

so that

$$Q_v(\xi) \leq Q_w(\xi),$$

for every test function $\xi \in C_0^\infty(\overline{\mathbb{R}^{n+1}_{+}})$.

Hence, if there exists ξ_0 such that $Q_w(\xi_0) < 0$, then also $Q_v(\xi_0) < 0$. That is, v is unstable. □

In the proof of the instability results for dimension 4 and 6 we use the maximal solution $\overline{v}$ of problem (2.1.3) and, more importantly, the equation satisfied by $\overline{v}_z = \partial_z \overline{v}$. We prove that this solution $\overline{v}$ is unstable by constructing a test function $\xi(y,z,\lambda) = \eta(y,\lambda)\overline{v}_z(y,z,\lambda)$ such that $Q_{\overline{v}}(\xi) < 0$. Two crucial ingredients will be the asymptotic behaviour and monotonicity results for $\overline{v}$ (Theorems 2.1.9 and 2.1.7). Since $\overline{v}$ is maximal, Lemma 2.7.1 implies that all bounded solutions $v \leq 1$ vanishing on $\mathcal{C} \times \mathbb{R}^+$ and having the same sign as $s-t$ are also unstable.

We recall that, if v is a function depending only on s, t and λ, then the second variation of the energy is given by

$$\begin{aligned} c_m Q_v(\xi) &= \int_0^{+\infty} \int_{\{s>0,\, t>0\}} s^{m-1}t^{m-1}(\xi_s^2 + \xi_t^2 + \xi_\lambda^2)\,ds\,dt\,d\lambda \\ &\quad - \int_{\{s>0,\, t>0\}} s^{m-1}t^{m-1} f'(v)\xi^2\,ds\,dt, \end{aligned}$$

where c_m is a positive constant depending on m. Here, the perturbations are of the form $\xi = \xi(s,t,\lambda)$ and vanishes for λ large enough.

Moreover, if we change to variables (y,z,λ), for a different constant c_m we get,

$$\begin{aligned} c_m Q_v(\xi) &= \int_0^{+\infty} \int_{\{-y<z<y\}} (y^2-z^2)^{m-1}(\xi_y^2 + \xi_z^2 + \xi_\lambda^2)\,dy\,dtz\,d\lambda \\ &\quad - \int_{\{-y<z<y\}} (y^2-z^2)^{m-1} f'(v)\xi^2\,dy\,dz, \end{aligned}$$

where $\xi = \xi(y,z,\lambda)$ vanishes for y and λ large enough.

Proof of Theorem 2.1.10. We begin by establishing that the maximal solution $\overline{v}$ is unstable in dimension $2m = 4$ and $2m = 6$. Moreover, the maximality of $\overline{v}$ leads to $|v| \leq |\overline{v}|$ in $\mathbb{R}^{2m+1}_+$ for every solution v vanishing on $\mathcal{C} \times \mathbb{R}^+$ and with the same sign as $s - t$. Then, Lemma 2.7.1 leads to $Q_v \leq Q_{\overline{v}}$ and thus v is also unstable in dimension $2m = 4$ and $2m = 6$.

We have, for every test function ξ,

$$Q_{\overline{v}}(\xi) = \int_{\mathbb{R}^{2m+1}_+} |\nabla \xi|^2 dx d\lambda - \int_{\partial \mathbb{R}^{2m+1}_+} f'(\overline{v})\xi^2 dx.$$

Suppose now that $\xi = \xi(y, z, \lambda) = \eta(y, z, \lambda)\psi(y, z, \lambda)$. For ξ to be Lipschitz and of compact support in $\overline{\mathbb{R}^{2m+1}_+}$, we need η and ψ to be Lipschitz functions of compact support in $y \in [0, +\infty)$ and $\lambda \in [0, +\infty)$. The expression for $Q_{\overline{v}}$ becomes,

$$\begin{aligned} Q_{\overline{v}}(\xi) = \int_0^{+\infty} \int_{\mathbb{R}^{2m}} & \left(|\nabla \eta|^2 \psi^2 + \eta^2 |\nabla \psi|^2 + 2\eta\psi \nabla \eta \cdot \nabla \psi \right) dx d\lambda \\ & - \int_{\mathbb{R}^{2m}} f'(\overline{v}) \eta^2 \psi^2 dx. \end{aligned}$$

Using that $2\eta\psi\nabla\eta \cdot \nabla\psi = \psi\nabla(\eta^2) \cdot \nabla\psi$, and integrating by parts this term we have

$$\begin{aligned} Q_{\overline{v}}(\xi) = \int_0^{+\infty} \int_{\mathbb{R}^{2m}} & \left(|\nabla \eta|^2 \psi^2 - \eta^2 \psi \Delta \psi \right) dx d\lambda \\ & - \int_{\mathbb{R}^{2m}} \left(\psi(y, z, 0) \eta^2 \partial_\lambda \psi(y, z, 0) + f'(\overline{v}) \eta^2 \psi^2 \right) dx, \end{aligned}$$

that is,

$$Q_{\overline{v}}(\xi) = \int_0^{+\infty} \int_{\mathbb{R}^{2m}} \left(|\nabla \eta|^2 \psi^2 - \eta^2 \psi \Delta \psi \right) dx d\lambda - \int_{\mathbb{R}^{2m}} \eta^2 \psi (\partial_\lambda \psi + f'(\overline{v}) \psi) dx.$$

Choose $\psi(y, z, \lambda) = \overline{v}_z(y, z, \lambda)$. We consider now problem (2.1.3), which is satisfied by $\overline{v}$, written in the (y, z, λ) variables

$$\begin{cases} \overline{v}_{yy} + \overline{v}_{zz} + \overline{v}_{\lambda\lambda} + \dfrac{2(m-1)}{y^2 - z^2}(y\overline{v}_y - z\overline{v}_z) = 0 & \text{in } \mathbb{R}^{2m+1}_+ \\ -\partial_\lambda \overline{v} = f(\overline{v}) & \text{on } \partial \mathbb{R}^{2m+1}_+. \end{cases} \tag{2.7.1}$$

If we differentiate these equations written in (y, z, λ) variables with respect to z, we find

$$\begin{cases} \Delta \overline{v}_z - \dfrac{2(m-1)}{y^2 - z^2}\overline{v}_z + \dfrac{4(m-1)z}{(y^2 - z^2)^2}(y\overline{v}_y - z\overline{v}_z) = 0 & \text{in } \mathbb{R}^{2m+1}_+ \\ -\partial_\lambda \overline{v}_z = f'(\overline{v})\overline{v}_z & \text{on } \partial \mathbb{R}^{2m+1}_+. \end{cases} \tag{2.7.2}$$

Replacing in the expression for $Q_{\overline{v}}$ we obtain,

$$Q_{\overline{v}}(\xi)=\int_0^{+\infty}\int_{\mathbb{R}^{2m}}\Big(|\nabla\eta|^2\overline{v}_z^2-$$
$$-\eta^2\Big\{\frac{2(m-1)(y^2+z^2)}{(y^2-z^2)^2}\overline{v}_z^2-\frac{4(m-1)zy}{(y^2-z^2)^2}\overline{v}_y\overline{v}_z\Big\}\Big)dxd\lambda.$$

Next we change coordinates to (y,z,λ) and we have, for some positive constant c_m,

$$c_mQ_{\overline{v}}(\xi)=\int_0^{+\infty}\int_{\{-y<z<y\}}(y^2-z^2)^{m-1}\Big(|\nabla\eta|^2\overline{v}_z^2-$$
$$-\eta^2\Big\{\frac{2(m-1)(y^2+z^2)}{(y^2-z^2)^2}\overline{v}_z^2-\frac{4(m-1)zy}{(y^2-z^2)^2}\overline{v}_y\overline{v}_z\Big\}\Big)dydzd\lambda.$$

Now choose $\eta(y,z,\lambda)=\eta_1(y)\eta_2(\lambda)$, where η_1 and η_2 are smooth functions with compact support in $[0,+\infty)$. Moreover η_2 is such that $\eta_2(\lambda)\equiv 1$ for $\lambda<N$ and $\eta_2(\lambda)\equiv 0$ for $\lambda>N+1$, where N is a large positive number that we will choose later. For $a>1$, a constant that we will make tend to infinity, let $\phi=\phi(\rho)$ be a Lipschitz function of $\rho:=y/a$ with compact support $[\rho_1,\rho_2]\subset[0,+\infty)$. Let us denote by

$$\eta_1^a(y):=\phi(y/a)\quad\text{ and }$$
$$\xi_a(y,z,\lambda)=\eta_1^a(y)\eta_2(\lambda)\overline{v}_z(y,z,\lambda)=\phi(y/a)\eta_2(\lambda)\overline{v}_z(y,z,\lambda),$$

The change $y=a\rho, dy=ad\rho$ yields,

$$c_mQ_{\overline{v}}(\xi_a)=a^{2m-3}\int_0^{N+1}\int_{\{-a\rho<z<a\rho\}}\rho^{2(m-1)}\left(1-\frac{z^2}{a^2\rho^2}\right)^{m-1}\Big(\phi_\rho^2\eta_2^2(\lambda)\overline{v}_z^2$$
$$+a^2\phi^2(\rho)(\eta_2')^2\overline{v}_z^2-\phi^2\eta_2^2\Big\{\frac{2(m-1)(1+\frac{z^2}{a^2\rho^2})}{\rho^2(1-\frac{z^2}{a^2\rho^2})^2}\overline{v}_z^2-\frac{4(m-1)z}{a\rho^3(1-\frac{z^2}{a^2\rho^2})^2}\overline{v}_y\overline{v}_z\Big\}\Big)d\rho dz. \tag{2.7.3}$$

Dividing by $a^{2m-3}N$ and using that $\left(1-\frac{z^2}{a^2\rho^2}\right)^2\leq 1$ and $1+\frac{z^2}{a^2\rho^2}\geq 1$, we obtain

$$\frac{c_mQ_{\overline{u}}(\xi_a)}{a^{2m-3}N}\leq$$
$$\leq\frac{1}{N}\int_0^{N+1}\int_{\{-a\rho<z<a\rho\}}\rho^{2(m-1)}\eta_2^2\overline{v}_z^2(a\rho,z,\lambda)\left(\phi_\rho^2-\frac{2(m-1)}{\rho^2}\phi^2\right)d\rho dzd\lambda$$
$$+\frac{a^2}{N}\int_N^{N+1}\int_{\{-a\rho<z<a\rho\}}\rho^{2(m-1)}\phi^2(\eta_2')^2\overline{v}_z^2d\rho dzd\lambda$$
$$+\frac{1}{N}\int_0^{N+1}\int_{\{-a\rho<z<a\rho\}}\frac{4(m-1)z\rho\eta_2^2\phi^2(\rho)}{a}\overline{v}_y(a\rho,z,\lambda)\overline{v}_z(a\rho,z,\lambda)d\rho dzd\lambda.$$
$$=I_1+I_2+I_3.$$

We study these three integrals separately.

Consider first I_3. From Theorem 2.1.9 we have that $\overline{v}_y(a\rho, z, \lambda) \to 0$ uniformly, for all $\rho \in [\rho_1, \rho_2] = \mathrm{supp}\phi$, as a tends to infinity. Hence, given $\epsilon > 0$, for a sufficiently large, $|\overline{v}_y(a\rho, z)| \leq \epsilon$. Moreover, we have seen in Theorem 2.1.7 that $\overline{v}_z \geq 0$. Hence, since ϕ is bounded, for a large we have

$$\begin{aligned} I_3 \leq & \left| \frac{1}{N} \int_0^{N+1} \eta_2^2 \int \frac{4(m-1)z\rho\phi^2(\rho)}{a} \overline{v}_y \overline{v}_z d\rho dz d\lambda \right| \leq \\ & \leq \frac{1}{N} \int_0^{N+1} \eta_2^2 \int \left| \frac{4(m-1)z\rho\phi^2(\rho)}{a} \right| |\overline{v}_y| \overline{v}_z d\rho dz d\lambda \\ & \leq \frac{1}{N} \int_0^{N+1} \eta_2^2 \int 4(m-1)\rho^2\phi^2(\rho) |\overline{v}_y| \overline{v}_z d\rho dz d\lambda \\ & \leq \frac{C\epsilon}{N} \int_{\rho_1}^{\rho_2} \rho^2 d\rho \int_0^{N+1} \eta_2^2 d\lambda \int_{-a\rho}^{a\rho} \overline{v}_z dz \\ & = \frac{C\epsilon}{N} \int_0^{N+1} \eta_2^2 \int_{\rho_1}^{\rho_2} \left(\overline{v}(a\rho, a\rho, \lambda) - \overline{v}(a\rho, -a\rho, \lambda) \right) d\rho d\lambda \\ & \leq C\epsilon, \end{aligned}$$

where C are different constants depending on ρ_1 and ρ_2. Hence, as a tends to infinity, this integral converges to zero.

Now, consider the I_2 and choose $N = N(a)$ such that $a^2/N(a) \leq 1/a^2$. With this choice of N, we have

$$\begin{aligned} I_2 = & \frac{a^2}{N} \int_N^{N+1} \int_{\rho_1}^{\rho_2} \int_{\{-a\rho < z < a\rho\}} \rho^{2(m-1)} \phi^2 (\eta_2')^2 \overline{v}_z^2 \leq \\ & \leq \frac{1}{a^2} \int_N^{N+1} \int_{\rho_1}^{\rho_2} \int_{\{-a\rho < z < a\rho\}} \rho^{2(m-1)} \phi^2 (\eta_2')^2 \overline{v}_z^2 \\ & \leq \frac{C}{a} \sup \overline{v}_z^2. \end{aligned}$$

Thus, I_2 tends to 0 as $a \to \infty$.

Next, consider I_1. We have that, again by Theorem 2.1.9, $\overline{v}_z(a\rho, z, \lambda)$ converges to $\partial_z v_0(z,\lambda)$ which is a bounded positive integrable function. We write

$$\begin{aligned} I_1 = \frac{1}{N}\int_0^{N+1} \eta_2^2 \int_{\{-a\rho<z<a\rho\}} \rho^{2(m-1)}\overline{v}_z^2(a\rho,z,\lambda)\left(\phi_\rho^2 - \frac{2(m-1)}{\rho^2}\phi^2\right) d\rho dz d\lambda \ = \\ = \frac{1}{N}\int_0^{N+1} \eta_2^2 \int_{\{-a\rho<z<a\rho\}} (\partial_z v_0)^2 \rho^{2(m-1)}\left(\phi_\rho^2 - \frac{2(m-1)}{\rho^2}\phi^2\right) d\rho dz d\lambda \\ +\frac{1}{N}\int_0^{N+1} \eta_2^2 \int_{\{-a\rho<z<a\rho\}} \rho^{2(m-1)}(\overline{v}_z(a\rho,z,\lambda) - \partial_z v_0(z,\lambda))(\overline{v}_z(a\rho,z,\lambda) \\ +\partial_z v_0(z,\lambda))\left(\phi_\rho^2 - \frac{2(m-1)}{\rho^2}\phi^2\right) d\rho dz d\lambda. \end{aligned}$$

For a large, $|\overline{v}_z(a\rho,z,\lambda) - \partial_z v_0(z,\lambda)| \le \epsilon$ in $[\rho_1,\rho_2]$. In addition $\overline{v}_z(a\rho,z,\lambda) + \partial_z v_0(z,\lambda)$ is positive and is a derivative with respect to z of a bounded function, thus it is integrable in z. Hence, since $\phi = \phi(\rho)$ is smooth with compact support, the second integral converges to zero as a tends to infinity.

Therefore, letting a tend to infinity, we obtain

$$\begin{aligned} &\limsup_{a\to\infty} \frac{c_m Q_{\overline{v}}(\xi_a)}{a^{2m-3}N} \le \qquad (2.7.4) \\ &\le \limsup_{a\to\infty} \frac{1}{N}\left(\int_0^{N+1} \eta_2^2 \int_0^{+\infty} (\partial_z v_0)^2(z) dz d\lambda\right) \int \rho^{2(m-1)}\left(\phi_\rho^2 - \frac{2(m-1)}{\rho^2}\phi^2\right) d\rho \\ &\le C\int_0^{+\infty} (\partial_z v_0)^2(z) dz d\lambda \int \rho^{2(m-1)}\left(\phi_\rho^2 - \frac{2(m-1)}{\rho^2}\phi^2\right) d\rho \end{aligned}$$

Finally, we prove that when $2m = 4$ and $2m = 6$, there exists a test function ϕ for which

$$\int \rho^{2(m-1)}\left(\phi_\rho^2 - \frac{2(m-1)}{\rho^2}\phi^2\right) d\rho < 0. \qquad (2.7.5)$$

The integral in ρ can be seen as an integral in $\mathbb{R}^{2m-1}$ of radial functions $\phi = \phi(|x|) = \phi(\rho)$.

Using Hardy's inequality we have that the integral in (2.7.5) is positive for all Lipschitz ϕ with compact support if and only if

$$2(m-1) \le \frac{(2m-1-2)^2}{4}.$$

Writing $n = 2m$, the above inequality holds if and only if

$$n^2 - 10n + 17 \ge 0,$$

that is, $n \ge 8$. Thus, when $2m = 4$ and $2m = 6$, we have that the integral (2.7.5) is negative for some compactly supported Lipschitz function $\phi = \phi(\rho)$ and then we conclude that the limsup in (2.7.4) is negative for such ϕ and hence that $\overline{u}$ is unstable.

On the other hand for $n \geq 8$ the limsup in (2.7.4) is nonnegative for every ϕ and we conclude some kind of asymptotic stability of $\overline{v}$. □

Chapter 3

Energy estimates for equations involving the half-Laplacian

In this chapter (which corresponds to [6]) we establish sharp energy estimates for some solutions of the fractional nonlinear equation

$$(-\Delta)^{1/2}u = f(u) \quad \text{in } \mathbb{R}^n, \tag{3.0.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a $C^{1,\beta}$ function with $0 < \beta < 1$. In the particular case in which $f(u) = u - u^3$, we call equation (3.0.1) of Allen-Cahn type by the analogy with the corresponding equation involving the Laplacian instead of the half-Laplacian,

$$-\Delta u = u - u^3 \quad \text{in } \mathbb{R}^n. \tag{3.0.2}$$

In 1978 De Giorgi conjectured that the level sets of every bounded solution of (3.0.2), which is monotone in one direction, must be hyperplanes, at least if $n \leq 8$. That is, such solutions depend only on one Euclidian variable. The conjecture has been proven to be true in dimension $n = 2$ by Ghoussoub and Gui [24] and in dimension $n = 3$ by Ambrosio and Cabré [3]. For $4 \leq n \leq 8$, if $\partial_{x_n} u > 0$, and assuming the additional condition

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1 \quad \text{for all } x' \in \mathbb{R}^{n-1},$$

it has been established by Savin [34]. Recently a counterexample to the conjecture for $n \geq 9$ has been announced by del Pino, Kowalczyk and Wei [22].

In this chapter (see Theorem 3.0.5 below), we establish the one-dimensional symmetry of bounded monotone solutions of (3.0.1) in dimension $n = 3$, that is, the analog of the conjecture of De Giorgi for the half-Laplacian in dimension 3. We recall that one-dimensional (or 1-D) symmetry for bounded stable solutions of (3.0.1) in dimension $n = 2$ has been proven by Cabré and Solà-Morales [10]. The same result in dimension $n = 2$ for the other fractional powers of the Laplacian, i.e., for the equation

$$(-\Delta)^s u = f(u) \quad \text{in } \mathbb{R}^2, \quad \text{with } 0 < s < 1,$$

has been established by Cabré and Sire [8, 9] and by Sire and Valdinoci [37].

A crucial ingredient in our proof of 1-D symmetry in $\mathbb{R}^3$, is a sharp energy estimate for global minimizers and for monotone solutions, that we state in Theorems 3.0.3 and 3.0.4 below. It is interesting to note that our method to prove the energy estimate also applies to the case of saddle-shaped solutions in $\mathbb{R}^{2m}$. These solutions are not global minimizers in general (this is indeed the case in dimensions $2m \leq 6$ by Theorem 2.1.10 in chapter 2), but they are minimizers under perturbations vanishing on a suitable subset of $\mathbb{R}^{2m}$. We treat these solutions and their corresponding energy estimate at the end of this introduction.

To study the nonlocal problem (3.0.1) we realize it as a local problem in $\mathbb{R}^{n+1}_+$ with a nonlinear Neumann condition on $\partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n$. More precisely, if $u = u(x)$ is a function defined on $\mathbb{R}^n$, we consider its harmonic extension $v = v(x, \lambda)$ in $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, +\infty)$. It is well known (see [10, 15]) that u is a solution of (3.0.1) if and only if v satisfies

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ -\partial_\lambda v = f(v) & \text{on } \mathbb{R}^n = \partial\mathbb{R}^{n+1}_+. \end{cases} \tag{3.0.3}$$

Problem (3.0.3) allows to introduce the notions of *energy* and *global minimality* for a solution u of problem (3.0.1). Consider the cylinder

$$C_R = B_R \times (0, R) \subset \mathbb{R}^{n+1}_+,$$

where B_R is the ball of radius R centered at 0 in $\mathbb{R}^n$. We consider the energy functional

$$\mathcal{E}_{C_R}(v) = \int_{C_R} \frac{1}{2}|\nabla v|^2 dx d\lambda + \int_{B_R} G(v) dx, \tag{3.0.4}$$

whose Euler-Lagrange equation is problem (3.0.3). The potential G, defined up to an additive constant, is given by

$$G(v) = \int_v^1 f(t) dt.$$

Using the energy functional (3.0.4), we introduce the notions of *global minimizer* and of *layer solution* of (3.0.1). We call layer solutions of (3.0.1) bounded solutions that are monotone increasing, say from -1 to 1, in one of the $x-$variables. After rotation, we can suppose that the direction of monotonicity is the x_n-direction, as in point c) of the following definition.

Definition 3.0.2. a) We say that a bounded $C^1(\overline{\mathbb{R}^{n+1}_+})$ function v is a *global minimizer* of (3.0.3) if, for all $R > 0$,

$$\mathcal{E}_{C_R}(v) \leq \mathcal{E}_{C_R}(w)$$

for every $C^1(\overline{\mathbb{R}^{n+1}_+})$ function w such that $v \equiv w$ in $\mathbb{R}^{n+1}_+ \setminus \overline{C_R}$.
b) We say that a bounded C^1 function u in $\mathbb{R}^n$ is a *global minimizer* of (3.0.1) if its harmonic extension v is a global minimizer of (3.0.3).
c) We say that a bounded function u is a *layer solution* of (3.0.1) if $\partial_{x_n} u > 0$ in $\mathbb{R}^n$ and

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1 \quad \text{for every } x' \in \mathbb{R}^{n-1}. \tag{3.0.5}$$

Note that the functions w in point a) of Definition 3.0.2 need to agree with the solution v on the lateral boundary and on the top of the cylinder C_R, but not on the base. Since it will be useful in the sequel, we set

$$\partial^+ C_R = \partial C_R \cap \{\lambda > 0\}.$$

In some references, global minimizers are called "local minimizers", where local stands for the fact that the energy is computed in bounded domains.

We remind that every layer solution is a global minimizer (see Theorem 1.4 in [10]).

Our main result is the following energy estimate for global minimizers of problem (3.0.1). Given a bounded function u defined on $\mathbb{R}^n$, set

$$c_u = \min\{G(s) : \inf_{\mathbb{R}^n} u \leq s \leq \sup_{\mathbb{R}^n} u\}. \tag{3.0.6}$$

Theorem 3.0.3. *Let f be any $C^{1,\beta}$ nonlinearity, with $\beta \in (0,1)$, and $u \in L^\infty(\mathbb{R}^n)$ be a global minimizer of (3.0.1). Let v be the harmonic extension of u in $\mathbb{R}^{n+1}_+$.*

Then, for all $R > 2$,

$$\int_{C_R} \frac{1}{2}|\nabla v|^2 dx d\lambda + \int_{B_R} \{G(u) - c_u\} dx \leq C R^{n-1} \log R, \tag{3.0.7}$$

where c_u is defined by (3.0.6) and C is a constant depending only on n, $||f||_{C^1}$, and $||u||_{L^\infty(\mathbb{R}^n)}$. In particular, we have that

$$\int_{C_R} \frac{1}{2}|\nabla v|^2 dx d\lambda \leq C R^{n-1} \log R. \tag{3.0.8}$$

As a consequence, (3.0.7) and (3.0.8) also hold for layer solutions. We stress that this energy estimate is sharp because it is optimal for 1-D solutions, in the sense that for some explicit 1-D solutions the energy is also bounded below by

$cR^{n-1}\log R$, for some constant $c > 0$, when they are seen as solutions in $\mathbb{R}^n$ (see section 2.1 of [10]).

In dimensions $n = 1$ and $n = 2$ estimate (3.0.7) was established by Cabré and Solà-Morales in [10].

In dimension $n = 3$, the energy estimate (3.0.7) holds also for monotone solutions which do not satisfy the limit assumption (3.0.5). These solutions are minimizers in some sense to be explained later, but, in case that they exist, they are not known to be global minimizers as defined before.

Theorem 3.0.4. *Let $n = 3$, f be any $C^{1,\beta}$ nonlinearity with $\beta \in (0, 1)$, and u be a bounded solution of (3.0.1) such that $\partial_e u > 0$ in $\mathbb{R}^3$ for some direction $e \in \mathbb{R}^3$, $|e| = 1$. Let v be its harmonic extension in $\mathbb{R}^4_+$.*

Then, for all $R > 2$,

$$\int_{C_R} \frac{1}{2}|\nabla v|^2 dx d\lambda + \int_{B_R} \{G(u) - c_u\} dx \leq CR^2 \log R, \tag{3.0.9}$$

where c_u is defined by (3.0.6) and C is a constant depending only on $||f||_{C^1}$ and $||u||_{L^\infty(\mathbb{R}^3)}$.

In dimension $n = 3$, Theorems 3.0.3 and 3.0.4 lead to the 1-D symmetry of global minimizers and of monotone solutions to problem (3.0.1).

Theorem 3.0.5. *Let $n = 3$ and f be any $C^{1,\beta}$ nonlinearity with $\beta \in (0, 1)$. Let u be either a bounded global minimizer of* (3.0.1)*, or a bounded solution of* (3.0.1) *monotone in some direction $e \in \mathbb{R}^3$, $|e| = 1$.*

Then, u depends only on one variable, i.e., there exists $a \in \mathbb{R}^3$ and $g : \mathbb{R} \to \mathbb{R}$, such that $u(x) = g(a \cdot x)$ for all $x \in \mathbb{R}^3$. Equivalently, the level sets of u are planes.

To prove 1-D symmetry, we use a standard Liouville type argument which requires an appropriate estimate for the kinetic energy. By a result of Moschini [31] (see Proposition 3.5.1 in section 6 below), our energy estimate in $\mathbb{R}^3$,

$$\int_{C_R} |\nabla v|^2 dx d\lambda \leq CR^2 \log R,$$

allows to use such Liouville type result and deduce 1-D symmetry in $\mathbb{R}^3$ for global minimizers and for solutions monotone in one direction.

Remark 3.0.6. As a consequence of Theorem 3.0.5, we obtain that for all $R > 2$,

$$\int_{B_R} G(v(x, 0)) dx \leq CR^{n-1} \quad \text{if } 1 \leq n \leq 3, \tag{3.0.10}$$

if v is a bounded global minimizer or a bounded monotone solution of (3.0.3). This was proven in [10] for $n = 1$ and $n = 2$. For $n = 3$, (3.0.10) follows from the $n = 1$

case after using Theorems 3.0.3 and 3.0.4. In dimension $n \geq 4$ we do not know if the potential energy can be bounded by CR^{n-1} (instead of $CR^{n-1}\log R$) as in (3.0.10).

In chapter 4, using similar techniques, we establish sharp energy estimates for the other fractional powers of the Laplacian. More precisely, we prove that if u is a bounded global minimizer of

$$(-\Delta)^s u = f(u) \text{ in } \mathbb{R}^n, \text{ with } 0 < s < 1, \tag{3.0.11}$$

then the following energy estimate holds:

$$\mathcal{E}_{s,C_R}(u) \leq CR^{n-2s} \quad \text{for } 0 < s < \frac{1}{2},$$

$$\mathcal{E}_{s,C_R}(u) \leq CR^{n-1} \quad \text{for } \frac{1}{2} < s < 1.$$

Here the energy functional is defined using a local formulation in $\mathbb{R}^{n+1}_+$ of problem (3.0.11), found by Caffarelli and Silvestre in [15]. If $1/2 < s < 1$ then $\mathcal{E}_{s,C_R}(u) \leq CR^{n-1}$; in this case we can deduce 1-D symmetry for global minimizers and monotone solutions in dimension $n = 3$.

Back to the case $s = 1/2$, we have two different proofs of the energy estimate $CR^{n-1}\log R$.

The first one is very simple but applies only to Allen-Cahn type nonlinearities (such as $f(u) = u - u^3$) and to monotone solutions satisfying the limit assumption (3.0.5) or the more general (3.1.2) below. We present this very simple proof in section 2. It was found by Ambrosio and Cabré [3] to prove the optimal energy estimate for $-\Delta u = u - u^3$ in $\mathbb{R}^n$.

Our second proof applies in more general situations and will lead to Theorems 3.0.3 and 3.0.4. It is based on controlling the $H^1(\Omega)$-norm of a function by its fractional Sobolev norm $H^{1/2}(\partial\Omega)$ on the boundary.

Let us recall the definition of the $H^{1/2}(A)$ norm, where A is either a Lipschitz open set of $\mathbb{R}^n$, or $A = \partial\Omega$ and Ω is a Lipschitz open set of $\mathbb{R}^{n+1}$. It is given by

$$||w||^2_{H^{1/2}(A)} = ||w||^2_{L^2(A)} + \int_A \int_A \frac{|w(z) - w(\overline{z})|^2}{|z - \overline{z}|^{n+1}} d\sigma_z d\sigma_{\overline{z}}.$$

In our proof we will have $A = \partial C_R$, the boundary of the cylinder $\Omega = C_R$.

In the proof of Theorem 3.0.3 a crucial point will be the following well known result. If w is a function in $H^{1/2}(\partial\Omega)$, where Ω is a bounded subset of $\mathbb{R}^{n+1}$ with Lipschitz boundary, then the harmonic extension $\overline{w}$ of w in Ω satisfies:

$$\int_\Omega |\nabla \overline{w}|^2 \leq C(\Omega)||w||^2_{H^{1/2}(\partial\Omega)}. \tag{3.0.12}$$

For the sake of completeness (and since the proof will be important in the next chapter), we will recall a proof of this result in section 3 (see Proposition 3.2.1).

To prove the sharp energy estimate for a global minimizer v in $\mathbb{R}^{n+1}_+$, we will bound its energy in the cylinder $C_R = B_R \times (0, R)$, using (3.0.12), by that of the harmonic extension $\overline{w}$ in C_R of a well chosen function w defined on ∂C_R. This function w must agree with v on the lateral and top boundaries of C_R, while it will be identically 1 on the portion $B_{R-1} \times \{0\}$ of the bottom boundary. In this way, it will not pay potential energy in this portion of the bottom boundary.

By (3.0.12), we will need to control $||w||_{H^{1/2}(\partial C_R)}$. After rescaling ∂C_R to ∂C_1, we will control the $H^{1/2}$-norm of w using the following key result. We will apply it in the sets

$$A = \partial C_1 \quad \text{and} \quad \Gamma = \partial B_1 \times \{\lambda = 0\},$$

with a small parameter $\varepsilon = 1/R$. Other examples in which the following theorem applies are, among many others, $A = B_1 \subset \mathbb{R}^n$ the unit ball and $\Gamma = B_1 \cap \{x_n = 0\}$, and also $A = B_1 \subset \mathbb{R}^n$ and $\Gamma = \partial B_r$ for some $r \in (0, 1)$.

Theorem 3.0.7. *Let A be either a bounded Lipschitz domain in $\mathbb{R}^n$ or $A = \partial\Omega$, where Ω is a bounded open set of $\mathbb{R}^{n+1}$ with Lipschitz boundary. Let $M \subset A$ be an open set (relative to A) with Lipschitz boundary (relative to A) $\Gamma \subset A$. Let $\varepsilon \in (0, 1/2)$.*

Let $w : A \to \mathbb{R}$ be a Lipschitz function such that, for almost every $x \in A$,

$$|w(x)| \leq c_0 \tag{3.0.13}$$

and

$$|Dw(x)| \leq c_0 \min\left\{\frac{1}{\varepsilon}, \frac{1}{\operatorname{dist}(x,\Gamma)}\right\}, \tag{3.0.14}$$

where D are all tangential derivatives to A, $\operatorname{dist}(x, \Gamma)$ is the distance from the point x to the set Γ (either in $\mathbb{R}^n$ or in $\mathbb{R}^{n+1}$), and c_0 is a positive constant.

Then,

$$||w||^2_{H^{1/2}(A)} = ||w||^2_{L^2(A)} + \int_A \int_A \frac{|w(z) - w(\overline{z})|^2}{|z - \overline{z}|^{n+1}} d\sigma_z d\sigma_{\overline{z}} \leq c_0^2 C |\log \varepsilon|, \tag{3.0.15}$$

where C is a positive constant depending only on A and Γ.

As we said, we will use this result with $A = \partial C_1$ and $\Gamma = \partial B_1 \times \{\lambda = 0\}$. Thus, in this case the constant C in (3.0.15) only depends on the dimension n. The gradient estimate (3.0.14), after rescaling ∂C_R to ∂C_1 and taking $\varepsilon = 1/R$, will follow from the bound

$$|\nabla v(x, \lambda)| \leq \frac{C}{1 + \lambda} \quad \text{for all } x \in \mathbb{R}^n \text{ and } \lambda \geq 0, \tag{3.0.16}$$

satisfied by every bounded solution v of (3.0.3). Here the constant C depends only on n, $||f||_{C^1}$, and $||v||_{L^\infty(\mathbb{R}^{n+1}_+)}$. For $\lambda \geq 1$, (3.0.16) follows immediately from the fact that v is bounded and harmonic in $B_\lambda(x, \lambda) \subset \mathbb{R}^{n+1}_+$. For $\lambda \leq 1$, estimate (3.0.16) is proven in Lemma 2.3 of [10].

Our method to prove sharp energy estimates also applies to solutions which are minimizers under perturbations vanishing on a suitable subset of $\mathbb{R}^n$, even if they are not in general global minimizers as defined before. An important example of this are some *saddle-shaped solutions* (or *saddle solutions* for short) of

$$(-\Delta)^{1/2} u = f(u) \quad \text{in } \mathbb{R}^{2m}.$$

These solutions have been studied in chapter 2.

Saddle solutions are even with respect to the coordinate axes and odd with respect to the Simons cone, which is defined as follows. For $n = 2m$ the Simons cone $\mathcal{C}$ is given by

$$\mathcal{C} = \{x \in \mathbb{R}^{2m} : x_1^2 + ... + x_m^2 = x_{m+1}^2 + ... + x_{2m}^2\}.$$

We define two new variables

$$s = \sqrt{x_1^2 + \cdots + x_m^2} \quad \text{and} \quad t = \sqrt{x_{m+1}^2 + \cdots + x_{2m}^2},$$

for which the Simons cone becomes $\mathcal{C} = \{s = t\}$.

The existence of saddle solutions of (3.0.1) has been proven chapter 2 under the following hypotheses on f:

$$f \ \text{ is odd}; \tag{3.0.17}$$

$$G \geq 0 = G(\pm 1) \text{ in } \mathbb{R}, \text{ and } G > 0 \text{ in } (-1, 1); \tag{3.0.18}$$

$$f' \ \text{ is decreasing in } (0, 1). \tag{3.0.19}$$

Note that, if (3.0.17) and (3.0.18) hold, then $f(0) = f(\pm 1) = 0$. Conversely, if f is odd in $\mathbb{R}$, positive with f' decreasing in $(0, 1)$ and negative in $(1, \infty)$ then f satisfies (3.0.17), (3.0.18) and (3.0.19). Hence, the nonlinearities f that we consider are of "balanced bistable type", while the potentials G are of "double well type". Our three assumptions (3.0.17), (3.0.18), (3.0.19) are satisfied by the scalar Allen-Cahn type equation

$$(-\Delta)^{1/2} u = u - u^3.$$

In this case we have that $G(u) = (1/4)(1 - u^2)^2$. The three hypothesis also hold for the equation $(-\Delta)^{1/2} u = \sin(\pi u)$, for which $G(u) = (1/\pi)(1 + \cos(\pi u))$.

The following result states the existence of at least one saddle solution for which our energy estimates holds.

Theorem 3.0.8. *Let f be a $C^{1,\beta}$ function for some $0 < \beta < 1$, satisfying* (3.0.17), (3.0.18), *and* (3.0.19). *Then there exists a saddle solution u of $(-\Delta)^{1/2}u = f(u)$ in $\mathbb{R}^{2m}$, i.e., a bounded solution u such that*

(*a*) *u depends only on the variables s and t. We write $u = u(s,t)$;*

(*b*) *$u > 0$ for $s > t$;*

(*c*) *$u(s,t) = -u(t,s)$.*

Moreover, $|u| \leq 1$ in $\mathbb{R}^n$ and for every $R > 2$,

$$\mathcal{E}_{C_R}(v) \leq CR^{2m-1}\log R,$$

where v is the harmonic extension of u in $\mathbb{R}^{2m+1}_+$ and C is a constant depending only on m and $||f||_{C^1([-1,1])}$.

Remark 3.0.9. Observe that the saddle solution of the theorem satisfies the same optimal energy estimate as global minimizers do, that is, $CR^{n-1}\log R = CR^{2m-1}\log R$, even that in low dimensions it is known that saddle solutions are not global minimizers. Indeed saddle solutions are not stable in dimension 2 (by a result of Cabré and Solà-Morales [10]) and in dimensions 4 and 6 (by Theorem 2.1.10 in chapter 2). As we will explain in the last section, saddle solutions are minimizers under perturbations vanishing on the Simons cone, and this will be enough to prove that they satisfy the sharp energy estimate.

The chapter is organized as follows:

- In section 2 we prove the energy estimate for layer solutions of Allen-Cahn type equations, using a simple argument found by Ambrosio and Cabré [3].

- In section 3 we give the proof of the extension theorem and of the key Theorem 3.0.7.

- In section 4 we prove energy estimate (3.0.7) for global minimizers and for every nonlinearity f, that is, Theorem 3.0.3.

- In section 5 we establish energy estimates for monotone solutions in $\mathbb{R}^3$, Theorem 3.0.4.

- In section 6 we prove the 1-D symmetry result, that is, Theorem 3.0.5.

- In section 7 we prove the energy estimate for saddle solutions, Theorem 3.0.8.

3.1 Energy estimate for monotone solutions of Allen-Cahn type equations

In this section we consider potentials G which satisfy hypothesis (3.0.18), i.e., $G \geq 0 = G(\pm 1)$ in $\mathbb{R}$ and $G > 0$ in $(-1, 1)$. In the sequel we consider the energy $\mathcal{E}_{C_R}$ defined by

$$\mathcal{E}_{C_R}(v) = \int_{C_R} \frac{1}{2}|\nabla v|^2 dx d\lambda + \int_{B_R} G(v) dx.$$

In general, it can be defined up to an additive constant c in the potential $G(v) - c$, but in this case, by the assumption (3.0.18) on G, we take $c = 0$.

Theorem 3.1.1. *Let f be a $C^{1,\beta}$ function, for some $0 < \beta < 1$, satisfying* (3.0.18), *where $G' = -f$. Let u be a bounded solution of problem* (3.0.1) *in $\mathbb{R}^n$, with $|u| < 1$ in $\mathbb{R}^n$, and let v be the harmonic extension of u in $\mathbb{R}^{n+1}_+$. Assume that*

$$u_{x_n} > 0 \quad in \quad \mathbb{R}^n \tag{3.1.1}$$

and

$$\lim_{x_n \to +\infty} u(x', x_n) = 1 \quad for\ all \quad x' \in \mathbb{R}^{n-1}. \tag{3.1.2}$$

Then, for every $R > 2$,

$$\int_{C_R} \frac{1}{2}|\nabla v|^2 dx d\lambda \leq \mathcal{E}_{C_R}(v) \leq C R^{n-1} \log R,$$

for some constant C depending only on n and $||f||_{C^1([-1,1])}$.

Remark 3.1.2. This energy estimate in dimension $n = 1$ has been proven by Cabré and Solà-Morales [10], using the gradient bound

$$|\nabla v(x, \lambda)| \leq \frac{C}{1 + |(x, \lambda)|} \quad \text{for all} \quad x \in \mathbb{R} \text{ and } \lambda \geq 0, \tag{3.1.3}$$

(see estimate (1.14) of [10]). Indeed, we next see that (3.1.3) leads to

$$\int_{C_R} |\nabla v|^2 dx d\lambda \leq C \log R$$

and also

$$\int_0^{+\infty} d\lambda \int_{B_R} dx |\nabla v|^2 \leq C \log R. \tag{3.1.4}$$

That is, for $n = 1$, the energy estimate holds not only in the cylinder C_R, but also in the infinite cylinder $B_R \times (0, +\infty)$. Let us mention that for the explicit layer solutions in section 2.1 of [10], the upper bound $(1+|(x,\lambda)|)^{-1}$ for $|\nabla v|$ is also a lower bound for $|\nabla v|$, modulo a smaller multiplicative constant. As a consequence, the two upper bounds $\log R$ are also lower bounds for the Dirichlet energy, after multiplying $\log R$ by a smaller constant.

Estimate (3.1.4) holds, indeed:

$$\begin{aligned}\int_0^{+\infty} d\lambda \int_{-R}^{R} dx |\nabla v|^2 &\leq C \int_0^{+\infty} d\lambda \int_{-R}^{R} dx \frac{1}{1+x^2+\lambda^2} \\ &\leq C \int_{-R}^{R} dx \int_0^{+\infty} d\lambda \frac{1}{(1+x)^2} \cdot \frac{1}{1+\left(\frac{\lambda}{1+x}\right)^2} \\ &\leq C \int_{-R}^{R} \left[\frac{1}{1+x} \arctan \frac{\lambda}{1+x}\right]_{\lambda=0}^{\lambda=+\infty} dx \\ &\leq C \int_{-R}^{R} \frac{\pi}{2} \frac{1}{1+x} dx \leq C \log R.\end{aligned}$$

In higher dimensions, an analog of (3.1.3) is not available and therefore we need another method to prove Theorem 3.1.1.

Proof of Theorem 3.1.1. We follow an argument found by Ambrosio and Cabré [3] to prove the energy estimate for layer solutions of the analog problem $-\Delta u = f(u)$ in $\mathbb{R}^n$. It is based on sliding the function v, which is the harmonic extension of the solution u, in the direction x_n.

Consider the function

$$v^t(x,\lambda) := v(x', x_n + t, \lambda)$$

defined for $(x,\lambda) = (x', x_n, \lambda) \in \mathbb{R}^n \times [0,+\infty)$, where $t \in \mathbb{R}$. For each t we have

$$\begin{cases} \Delta v^t = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ -\partial_\lambda v^t = f(v^t) & \text{on } \mathbb{R}^n = \partial \mathbb{R}^{n+1}_+. \end{cases} \tag{3.1.5}$$

Moreover, as stated in (3.0.16), the following bounds hold:

$$|v^t| \leq 1 \quad \text{and} \quad |\nabla v^t| \leq \frac{C}{1+\lambda}. \tag{3.1.6}$$

Throughout the proof, C will denote different positive constants depending only on n and $||f||_{C^1([-1,1])}$.

A simple compactness argument implies that

$$\lim_{t\to+\infty} \{|v^t - 1| + |\nabla v^t|\} = 0 \tag{3.1.7}$$

uniformly in compact sets of $\overline{\mathbb{R}^{n+1}_+}$. Indeed, arguing by contradiction, assume that there exist $R > 0$, $\varepsilon > 0$, and a sequence $t_m \to \infty$ such that

$$||v^{t_m} - 1||_{L^\infty(C_R)} + ||\nabla v^{t_m}||_{L^\infty(C_R)} \geq \varepsilon \tag{3.1.8}$$

for every m, where $C_R = B_R \times (0, R)$. Since v^{t_m} are all solutions of (3.0.3) in all the halfspace, the regularity results in [10] give $C^2_{loc}(\overline{\mathbb{R}^{n+1}_+})$ estimates for v^{t_m} uniform in m. Thus, there exists a subsequence that converges in $C^2_{\text{loc}}(\overline{\mathbb{R}^{n+1}_+})$ to a bounded harmonic function v^∞. By hypothesis (3.1.2) $v^\infty \equiv 1$ on $\partial\mathbb{R}^{n+1}_+$, and thus by the maximum principle, $v^\infty \equiv 1$ in all of $\mathbb{R}^{n+1}_+$. This contradicts (3.1.8), by C^1 convergence in compact sets of v^{t_m} towards $v^\infty \equiv 1$.

Denoting the derivative of $v^t(x, \lambda)$ with respect to t by $\partial_t v^t(x, \lambda)$, we have

$$\partial_t v^t(x, \lambda) = v_{x_n}(x', x_n + t, \lambda) > 0 \quad \text{for all } \; x \in \mathbb{R}^n, \; \lambda \geq 0.$$

Note that $v_{x_n} > 0$, since it is the harmonic extension of the bounded function $u_{x_n} > 0$. We consider the energy of v^t in the cylinder $C_R = B_R \times (0, R)$,

$$\mathcal{E}_{C_R}(v^t) = \int_{C_R} \frac{1}{2}|\nabla v^t|^2 dx d\lambda + \int_{B_R} G(v^t) dx.$$

Note that, by (3.1.7), we have

$$\lim_{t \to +\infty} \mathcal{E}_{C_R}(v^t) = 0. \tag{3.1.9}$$

Next, we bound the derivative of $\mathcal{E}_{C_R}(v^t)$ with respect to t. We use that v^t is a solution of problem (3.0.3), the bound (3.1.6) for $|v^t|$ and $|\nabla v^t|$, and the crucial fact that $\partial_t v^t > 0$. Let ν denote the exterior normal to the lateral boundary $\partial B_R \times (0, R)$ of the cylinder C_R. We have

$$\begin{aligned}
\partial_t \mathcal{E}_{C_R}(v^t) &= \int_0^R d\lambda \int_{B_R} dx \nabla v^t \cdot \nabla(\partial_t v^t) + \int_{B_R} G'(v^t)\partial_t v^t dx \\
&= \int_0^R d\lambda \int_{\partial B_R} d\sigma \frac{\partial v^t}{\partial \nu} \partial_t v^t(x, \lambda) + \int_{B_R \times \{\lambda = R\}} \frac{\partial v^t}{\partial \lambda} \partial_t v^t(x, R) dx \\
&\geq -C \int_0^R \frac{d\lambda}{1 + \lambda} \int_{\partial B_R} d\sigma \partial_t v^t - \frac{C}{R} \int_{B_R \times \{\lambda = R\}} \partial_t v^t(x, R) dx.
\end{aligned}$$

Hence, for every $T > 0$, we have

$$\begin{aligned}
\mathcal{E}_{C_R}(v) &= \mathcal{E}_{C_R}(v^T) - \int_0^T \partial_t \mathcal{E}_{C_R}(v^t)dt \\
&\leq \mathcal{E}_{C_R}(v^T) + C\int_0^T dt \int_0^R \frac{d\lambda}{1+\lambda}\int_{\partial B_R} d\sigma \partial_t v^t \\
&\quad + \frac{C}{R}\int_0^T dt \int_{B_R\times\{\lambda=R\}} dx \partial_t v^t(x,R) \\
&= \mathcal{E}_{C_R}(v^T) + C\int_0^R \frac{d\lambda}{1+\lambda}\int_{\partial B_R} d\sigma \int_0^T dt \partial_t v^t \\
&\quad + \frac{C}{R}\int_{B_R\times\{\lambda=R\}} dx \int_0^T dt \partial_t v^t(x,R) \\
&= \mathcal{E}_{C_R}(v^T) + C\int_0^R \frac{d\lambda}{1+\lambda}\int_{\partial B_R} d\sigma (v^T - v^0) \\
&\quad + \frac{C}{R}\int_{B_R\times\{\lambda=R\}} dx (v^T - v^0) \\
&\leq \mathcal{E}_{C_R}(v^T) + CR^{n-1}\log R + CR^{n-1}.
\end{aligned}$$

Letting $T \to +\infty$ and using (3.1.9), we obtain the desired estimate. □

3.2 $H^{1/2}$ estimate

In this section we recall some definitions and properties about the spaces $H^{1/2}(\mathbb{R}^n)$ and $H^{1/2}(\partial\Omega)$, where Ω is a bounded subset of $\mathbb{R}^{n+1}$ with Lipschitz boundary $\partial\Omega$ (see [27]).

$H^{1/2}(\mathbb{R}^n)$ is the space of functions $u \in L^2(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x) - u(\overline{x})|^2}{|x-\overline{x}|^{n+1}} dx d\overline{x} < +\infty,$$

equipped with the norm

$$||u||_{H^{1/2}(\mathbb{R}^n)} = \left(||u||^2_{L^2(\mathbb{R}^n)} + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x) - u(\overline{x})|^2}{|x-\overline{x}|^{n+1}} dx d\overline{x} \right)^{\frac{1}{2}}.$$

Let now Ω be a bounded subset of $\mathbb{R}^{n+1}$ with Lipschitz boundary $\partial\Omega$. To define $H^{1/2}(\partial\Omega)$, consider an atlas $\{(O_j, \varphi_j); j = 1, ..., m\}$ where $\{O_j\}$ is a family of open bounded sets in $\mathbb{R}^{n+1}$ such that $\{O_j \cap \partial\Omega; j = 1, ..., m\}$ cover $\partial\Omega$. The functions φ_j are Lipschitz diffeomorphisms such that

- $\varphi_j : O_j \to U := \{(y,\mu) \in \mathbb{R}^{n+1} : |y| < 1,\ -1 < \mu < 1\}$,
- $\varphi_j : O_j \cap \Omega \to U^+ := \{(y,\mu) \in \mathbb{R}^{n+1} : |y| < 1,\ 0 < \mu < 1\}$,
- $\varphi_j : O_j \cap \partial\Omega \to \{(y,\mu) \in \mathbb{R}^{n+1} : |y| < 1, \mu = 0\}$,
- in $O_i \cap O_j \neq \emptyset$ the compatibility conditions hold.

Let $\{\alpha_j\}$ be a partition of unity on $\partial\Omega$ such that $\alpha_j \in C_c^\infty(O_j)$, $\sum_{j=1}^m \alpha_j = 1$ in $O_j \cap \partial\Omega$. If u is a function on $\partial\Omega$ decompose $u = \sum_{j=1}^m u\alpha_j$ and define the function

$$(u\alpha_j) \circ \varphi_j^{-1}(y,0) := (u\alpha_j)(\varphi_j^{-1}(y,0)), \quad \text{for every } (y,0) \in U \cap \{\mu = 0\}.$$

Since α_j has compact support in O_j, the function $(u\alpha_j) \circ \varphi_j^{-1}(\cdot,0)$ has compact support in $U \cap \{\mu = 0\}$ and therefore we may consider $((u\alpha_j)\varphi_j^{-1})(\cdot,0)$ to be defined in $\mathbb{R}^n$ extending it by zero out of $U \cap \{\mu = 0\}$. Now we define

$$H^{1/2}(\partial\Omega) := \{u : (u\alpha_j) \circ \varphi_j^{-1}(\cdot,0) \in H^{1/2}(\mathbb{R}^n),\ j = 1,...,m\}$$

equipped with the norm

$$\left(\sum_{j=1}^m ||(u\alpha_j) \circ \varphi_j^{-1}(\cdot,0)||^2_{H^{1/2}(\mathbb{R}^n)} \right)^{\frac{1}{2}}.$$

All these norms are independent of the choice of the system of local maps $\{O_j, \varphi_j\}$ and of the partition of unity $\{\alpha_j\}$, and are all equivalent to

$$||u||_{H^{1/2}(\partial\Omega)} := \left(||u||^2_{L^2(\partial\Omega)} + \int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(z) - u(\overline{z})|^2}{|z - \overline{z}|^{n+1}} d\sigma_z d\sigma_{\overline{z}} \right)^{\frac{1}{2}}.$$

We recall now the classical extension result that we will use in the proof of Theorem 3.0.3.

Proposition 3.2.1. *Let $\Omega = \mathbb{R}^{n+1}_+$ or Ω be a bounded subset of $\mathbb{R}^{n+1}$ with Lipschitz boundary $\partial\Omega$, and let w belong to $H^{1/2}(\partial\Omega)$.*

Then, there exists a Lipschitz extension $\widetilde{w}$ of w in $\overline{\Omega}$ such that

$$\int_\Omega |\nabla \widetilde{w}|^2 \leq C||w||^2_{H^{1/2}(\partial\Omega)}, \tag{3.2.1}$$

where C is a constant depending only on Ω.

For the sake of completeness (and since the proof will be important in Chapter 4) we give the proof of this proposition.

Proof of Proposition 3.2.1. **Case 1:** $\Omega = \mathbb{R}^{n+1}_+$. Let ζ be a function belonging to $H^{1/2}(\mathbb{R}^n)$. We prove that there exists a Lipschitz extension $\widetilde{\zeta}$ of ζ in $\overline{\mathbb{R}^{n+1}_+}$ such that

$$\int_{\mathbb{R}^{n+1}_+} |\nabla\widetilde{\zeta}|^2 dx d\lambda \le C \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\zeta(x)-\zeta(\overline{x})|^2}{|x-\overline{x}|^{n+1}} dx d\overline{x}. \tag{3.2.2}$$

Let $K(x)$ be a nonnegative C^∞ function defined on $\mathbb{R}^n$ with compact support in B_1 and such that $\int_{\mathbb{R}^n} K(x)dx = 1$. Define $\widetilde{K}(x,\lambda)$ on $\mathbb{R}^{n+1}_+$ by

$$\widetilde{K}(x,\lambda) := \frac{1}{\lambda^n} K\left(\frac{x}{\lambda}\right).$$

Then, since

$$\int_{\mathbb{R}^n} \widetilde{K}(x,\lambda)dx = 1 \ \text{ for all } \lambda > 0, \tag{3.2.3}$$

we obtain, differentiating with respect to x_i and λ,

$$\int_{\mathbb{R}^n} \partial_{x_i}\widetilde{K}(x,\lambda)dx = 0 \ \text{ and } \int_{\mathbb{R}^n} \partial_{\lambda}\widetilde{K}(x,\lambda)dx = 0 \ \text{ for all } \lambda > 0. \tag{3.2.4}$$

In addition, for a constant C depending only on n, we have

$$|\nabla\widetilde{K}(x,\lambda)| \le \frac{C}{\lambda^{n+1}} \quad \text{for all } (x,\lambda) \in \mathbb{R}^{n+1}_+.$$

This holds, since the support of $\widetilde{K}$ is contained in $\{|x| < \lambda\}$ and, in this set,

$$|\nabla_x\widetilde{K}(x,\lambda)| \le \frac{C}{\lambda^{n+1}} \ \text{ and}$$

$$|\partial_\lambda K(x,\lambda)| = \left| -\frac{n}{\lambda^{n+1}} K\left(\frac{x}{\lambda}\right) - \frac{1}{\lambda^n}\nabla K\left(\frac{x}{\lambda}\right)\cdot\frac{x}{\lambda^2}\right| \le \frac{C}{\lambda^{n+1}}.$$

Now we define the extension $\widetilde{\zeta}$ as

$$\widetilde{\zeta}(x,\lambda) = \int_{\mathbb{R}^n} \widetilde{K}(x-\overline{x},\lambda)\zeta(\overline{x})d\overline{x},$$

and we show that this function satisfies (3.2.2). Note also that, by (3.2.3), for every $\lambda \ge 0$

$$||\widetilde{\zeta}(\cdot,\lambda)||_{L^2(\mathbb{R}^n)} \le ||\zeta||_{L^2(\mathbb{R}^n)}. \tag{3.2.5}$$

To show (3.2.2), observe that, by (3.2.4),

$$\begin{aligned} \partial_{x_i}\widetilde{\zeta}(x,\lambda) &= \int_{\mathbb{R}^n} \partial_{x_i}\widetilde{K}(x-\overline{x},\lambda)\zeta(\overline{x})d\overline{x} \\ &= \int_{\mathbb{R}^n} \partial_{x_i}\widetilde{K}(x-\overline{x},\lambda)\{\zeta(\overline{x})-\zeta(x)\}d\overline{x}, \end{aligned}$$

and thus

$$|\partial_{x_i}\widetilde{\zeta}(x,\lambda)| \le C\int_{\{|x-\overline{x}|<\lambda\}} \frac{|\zeta(\overline{x})-\zeta(x)|}{\lambda^{n+1}}d\overline{x}.$$

In the same way

$$|\partial_{\lambda}\widetilde{\zeta}(x,\lambda)| \le C\int_{\{|x-\overline{x}|<\lambda\}} \frac{|\zeta(\overline{x})-\zeta(x)|}{\lambda^{n+1}}d\overline{x}.$$

Hence, by Cauchy-Schwarz,

$$|\nabla\widetilde{\zeta}(x,\lambda)|^2 \le C\int_{\{|x-\overline{x}|<\lambda\}} \frac{|\zeta(\overline{x})-\zeta(x)|^2}{\lambda^{n+2}}d\overline{x},$$

and then

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+}|\nabla\widetilde{\zeta}|^2 dx d\lambda &\le C\int_0^{+\infty} d\lambda\int_{\mathbb{R}^n} dx\int_{\{|x-\overline{x}|<\lambda\}} d\overline{x}\frac{|\zeta(\overline{x})-\zeta(x)|^2}{\lambda^{n+2}}\\ &\le C\int_{\mathbb{R}^n} dx\int_{\mathbb{R}^n} d\overline{x}\int_{\{\lambda>|x-\overline{x}|\}} d\lambda\frac{|\zeta(\overline{x})-\zeta(x)|^2}{\lambda^{n+2}}\\ &\le C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\zeta(x)-\zeta(\overline{x})|^2}{|x-\overline{x}|^{n+1}}dx d\overline{x}.\end{aligned}$$

Case 2. Consider now the general case of a function w belonging to $H^{1/2}(\partial\Omega)$, where Ω is a bounded subset of $\mathbb{R}^{n+1}$ with Lipschitz boundary.

Using the partition of unity $\{\alpha_j\}$ introduced in the beginning of this section, we write $w=\sum_{j=1}^m w\alpha_j$. Observe that, for every $j=1,...,m$,

$$\int_{\partial\Omega}\int_{\partial\Omega}\frac{|(w\alpha_j)(z)-(w\alpha_j)(\overline{z})|^2}{|z-\overline{z}|^{n+1}}d\sigma_z d\sigma_{\overline{z}} \le C||w||^2_{H^{1/2}(\partial\Omega)}, \tag{3.2.6}$$

where all constants C in the proof depend only on Ω.

Indeed,

$$\begin{aligned}&\int_{\partial\Omega}\int_{\partial\Omega}\frac{|(w\alpha_j)(z)-(w\alpha_j)(\overline{z})|^2}{|z-\overline{z}|^{n+1}}d\sigma_z d\sigma_{\overline{z}}\\ &=\int_{\partial\Omega}\int_{\partial\Omega}\frac{|(w\alpha_j)(z)-w(z)\alpha_j(\overline{z})+w(z)\alpha_j(\overline{z})-(w\alpha_j)(\overline{z})|^2}{|z-\overline{z}|^{n+1}}d\sigma_z d\sigma_{\overline{z}}\\ &\le 2\int_{\partial\Omega}\int_{\partial\Omega}\frac{|\alpha_j(z)-\alpha_j(\overline{z})|^2|w(z)|^2}{|z-\overline{z}|^{n+1}}d\sigma_z d\sigma_{\overline{z}}\\ &\quad+2\int_{\partial\Omega}\int_{\partial\Omega}\frac{|w(z)-w(\overline{z})|^2|\alpha_j(\overline{z})|^2}{|z-\overline{z}|^{n+1}}d\sigma_z d\sigma_{\overline{z}}.\end{aligned} \tag{3.2.7}$$

The first integral is bounded by $C||w||^2_{L^2(\partial\Omega)}$. Indeed using that α_j is Lipschitz, we get that the first integral is controlled by

$$C||w||_{L^2(\partial\Omega)}\int_{O_j\cap\partial\Omega}d\sigma_z\frac{1}{|z-\overline{z}|^{n-1}}\le C||w||_{L^2(\partial\Omega)}.$$

From this, (3.2.6) follows

We flatten the boundary $\partial\Omega$ using the local maps φ_j introduced in the beginning of this section, and consider the functions

$$\zeta_j(y,0):=(w\alpha_j)(\varphi_j^{-1}(y,0)),$$

which are defined for $(y,0)\in U\cap\{\mu=0\}$. Now $\zeta_j(\cdot,0)$, extended by 0 outside of $U\cap\{\mu=0\}$, is defined in all of $\mathbb{R}^n$, and we are in the situation of case 1. We make the extension $\widetilde{\zeta_j}$ of ζ_j as in case 1, and we define

$$\widetilde{w}=\sum_{j=1}^m\alpha_j\widetilde{\zeta_j}\circ\varphi_j\quad\text{in }\mathcal{A}=\overline{\Omega}\cap\bigcup_{j=1}^m O_j,$$

an open set of $\Omega\cup\partial\Omega$ containing $\partial\Omega$. Thus, $\widetilde{w}$ has compact support in $\mathcal{A}$ and, extending it by 0, $\widetilde{w}$ is defined in all $\Omega\cup\partial\Omega$.

Observe that, since φ_j is a bilipschitz map and $\alpha_j\in C_c^\infty(O_j)$ for every $j=1,...,m$, we have

$$|\nabla\widetilde{w}|\le C\sum_{j=1}^m\left\{|\nabla\alpha_j||\widetilde{\zeta_j}\circ\varphi_j|+|\alpha_j||(\nabla\widetilde{\zeta_j})\circ\varphi_j|\right\},$$

and thus

$$\int_\Omega|\nabla\widetilde{w}|^2\le C\sum_{j=1}^m\left\{||\widetilde{\zeta_j}||^2_{L^2(B_1\times(0,1))}+\int_{\mathbb{R}^{n+1}_+}|\nabla\widetilde{\zeta_j}|^2\right\}.$$

By (3.2.5) and (3.2.2) of case 1, we have for every $j=1,...m$,

$$\begin{aligned}||\widetilde{\zeta_j}||^2_{L^2(B_1\times(0,1))}+\int_{\mathbb{R}^{n+1}_+}|\nabla\widetilde{\zeta_j}|^2&\le C\left\{||\zeta_j||^2_{L^2(B_1)}+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\zeta_j(y)-\zeta_j(\overline{y})|^2}{|y-\overline{y}|^{n+1}}dyd\overline{y}\right\}\\ &\le C\left\{||w||^2_{L^2(\Omega)}+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\zeta_j(y)-\zeta_j(\overline{y})|^2}{|y-\overline{y}|^{n+1}}dyd\overline{y}\right\}.\end{aligned}$$

Finally, using that φ_j is a bilipschitz map for every $j=1,...,m$, the definition of

ζ_j and (3.2.6), we get

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\zeta_j(y)-\zeta_j(\overline{y})|^2}{|y-\overline{y}|^{n+1}} dy d\overline{y} \\
&\quad = \int_{B_1}\int_{B_1} \frac{|(w\alpha_j)(\varphi_j^{-1}(y,0)) - (w\alpha_j)(\varphi_j^{-1}(\overline{y},0))|}{|y-\overline{y}|^{n+1}} dy d\overline{y} \\
&\quad \le \int_{O_j\cap\partial\Omega}\int_{O_j\cap\partial\Omega} \frac{|(w\alpha_j)(z)-(w\alpha_j)(\overline{z})|}{|\varphi_j(z)-\varphi_j(\overline{z})|^{n+1}} d\sigma_z d\sigma_{\overline{z}} \\
&\quad \le C\int_{O_j\cap\partial\Omega}\int_{O_j\cap\partial\Omega} \frac{|(w\alpha_j)(z)-(w\alpha_j)(\overline{z})|}{|z-\overline{z}|^{n+1}} d\sigma_z d\sigma_{\overline{z}} \le C||w||^2_{H^{1/2}(\partial\Omega)}.
\end{aligned}$$

□

Remark 3.2.2. Let $\overline{w}$ be the harmonic extension of w in Ω. Since $\overline{w}$ is the extension with minimal $L^2(\Omega)$-norm of $\nabla\overline{w}$, then we have that

$$\int_\Omega |\nabla\overline{w}|^2 dx d\lambda \le \int_\Omega |\nabla\widetilde{w}|^2 dx d\lambda \le C||w||_{H^{1/2}(\partial\Omega)}.$$

We give now the proof of the crucial Theorem 3.0.7.

Proof of Theorem 3.0.7. The proof consists of two steps.

Step 1. Suppose that

$$A = Q_1 = \{x\in\mathbb{R}^n : |x_i| < 1 \text{ for all } i = 1,...,n\}$$

is a cube in $\mathbb{R}^n$, and that

$$\Gamma = \{x_n = 0\}\cap Q_1,$$

where $x = (x', x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}$. We may assume $c_0 = 1$ by replacing w by w/c_0. By hypothesis we have that $|w|\le 1$ in A and that

$$\begin{cases} |Dw(x)| \le 1/\varepsilon & \text{for a.e. } x\in Q_1 \text{ with } |x_n| < \varepsilon \\ |Dw(x)| \le 1/|x_n| & \text{for a.e. } x\in Q_1 \text{ with } |x_n| > \varepsilon. \end{cases} \tag{3.2.8}$$

We need to estimate the $H^{1/2}$-norm of w in Q_1, given by

$$||w||^2_{H^{1/2}(Q_1)} = ||w||^2_{L^2(Q_1)} + \int_{Q_1}\int_{Q_1} \frac{|w(x)-w(\overline{x})|^2}{|x-\overline{x}|^{n+1}} dx d\overline{x}.$$

All constants C in step 1 depend only on n and differ from line to line. In this step, we take $0 < \varepsilon \le 1/2$.

First observe that $||w||^2_{L^2(Q_1)} \le 2^n$. Let $x \in Q_1^+ = \{x \in Q_1 : x_n > 0\}$ and let R_x be a radius depending on the point x, defined by

$$R_x = \begin{cases} \varepsilon & \text{if } 0 < x_n < \varepsilon \\ x_n/2 & \text{if } \varepsilon < x_n < 1. \end{cases}$$

To bound $||w||_{H^{1/2}(Q_1)}$, we consider the two cases $\overline{x} \in B_{R_x}(x)$ and $\overline{x} \notin B_{R_x}(x)$, as follows:

$$\begin{aligned}
&\int_{Q_1^+} dx \int_{Q_1} d\overline{x} \frac{|w(x) - w(\overline{x})|^2}{|x - \overline{x}|^{n+1}} = \\
&\quad = \int_{Q_1^+} dx \int_{Q_1 \cap B_{R_x}(x)} d\overline{x} \frac{|w(x) - w(\overline{x})|^2}{|x - \overline{x}|^{n+1}} + \int_{Q_1^+} dx \int_{Q_1 \setminus B_{R_x}(x)} d\overline{x} \frac{|w(x) - w(\overline{x})|^2}{|x - \overline{x}|^{n+1}} \\
&\quad := I_1 + I_2.
\end{aligned}$$

We use $|w| \le 1$ to bound I_2, and the gradient estimate (3.2.8) for w to bound I_1. In both cases we use spherical coordinates, centered at x, calling $r = |x - \overline{x}|$ the radial coordinate. We have

$$\begin{aligned}
I_2 &\le \int_{Q_1^+} dx \int_{Q_1 \setminus B_{R_x}(x)} d\overline{x} \frac{4}{|x - \overline{x}|^{n+1}} \le C \int_{Q_1^+} dx \int_{R_x}^{2\sqrt{n}} dr \frac{1}{r^2} \\
&\le C \int_{Q_1^+} \frac{1}{R_x} dx = C \left(\int_0^{\varepsilon} \frac{1}{\varepsilon} dx_n + \int_{\varepsilon}^1 \frac{2}{x_n} dx_n \right) \le C |\log \varepsilon|.
\end{aligned}$$

Next, we bound I_1. We have

$$I_1 = \int_{Q_1^+} dx \int_{Q_1 \cap B_{R_x}(x)} d\overline{x} \frac{|w(x) - w(\overline{x})|^2}{|x - \overline{x}|^{n+1}} = \int_{Q_1^+} dx \int_{Q_1 \cap B_{R_x}(x)} d\overline{x} \frac{|Dw(y(x,\overline{x}))|^2}{|x - \overline{x}|^{n-1}},$$

where $y(x, \overline{x}) \in Q_1 \cap B_{R_x}(x)$ is a point of the segment joining x and $\overline{x}$.

Now, (3.2.8) reads $|Dw(y)| \le \min\{1/\varepsilon, 1/|y_n|\}$ for a.e. $y \in Q_1$. We use the bound $|Dw(y)| \le 1/\varepsilon$ when $0 < x_n < \varepsilon$. For $\varepsilon < x_n < 1$, since $y(x, \overline{x}) \in B_{R_x}(x) = B_{x_n/2}(x)$, we have $y_n(x, \overline{x}) \ge x_n - R_x = x_n/2$, and thus $|Dw(y(x, \overline{x}))| \le 1/y_n(x, \overline{x}) \le 2/x_n$. Thus, using spherical coordinates centered at x,

$$\begin{aligned}
I_1 &\le C \int_0^{\varepsilon} dx_n \int_0^{\varepsilon} dr \frac{1}{\varepsilon^2} + C \int_{\varepsilon}^1 dx_n \int_0^{x_n/2} dr \frac{4}{x_n^2} \\
&\le C + C \int_{\varepsilon}^1 \frac{1}{x_n} dx_n \le C |\log \varepsilon|.
\end{aligned}$$

Finally, for $x \in Q_1^- = \{x \in Q_1 : x_n < 0\}$ we proceed in the same way, and thus we conclude the proof of step 1.

Step 2. Suppose now the general situation of the theorem: $A \subset \mathbb{R}^n$ is a bounded Lipschitz domain, or $A = \partial\Omega$, where Ω is an open bounded subset of $\mathbb{R}^{n+1}$ with Lipschitz boundary. Recall that $\Gamma \subset A$ is the boundary (relative to A) of a Lipschitz open (relative to A) subset M of A. From now on, we denote by $B_r(p)$, the ball in $\mathbb{R}^n$ or in $\mathbb{R}^{n+1}$ indifferently, since we are considering together the cases $A \subset \mathbb{R}^n$ and $A = \partial\Omega$ with $\Omega \subset \mathbb{R}^{n+1}$. We define a finite open covering of A in the following way.

First, for every $p \in \Gamma$, we choose a radius r_p, for which there exists a bilipschitz diffeomorphism $\varphi_p : B_{r_p}(p) \cap A \to Q_1$, (where Q_1 is the unit cube of $\mathbb{R}^n$), such that $\varphi(B_{r_p}(p) \cap \Gamma) = \{x \in Q_1 : x_n = 0\}$.

Let $\overline{\Gamma}$ be the closure of Γ in $\mathbb{R}^n$ or $\mathbb{R}^{n+1}$. Only in the case $A \subset \mathbb{R}^n$, it may happen that $\Gamma \subset \overline{\Gamma}$. In such case, for $p \in \overline{\Gamma} \setminus \Gamma$, there exists a radius r_p and a bilipschitz diffeomorphism $\varphi_p : B_{r_p}(p) \to (-3, 1) \times (-1, 1)^{n-1}$ such that $\varphi_p(p) = (-1, 0, ..., 0)$, $\varphi_p(B_{r_p}(p) \cap A) = Q_1 = (-1, 1)^n$ and $\varphi_p(B_{r_p}(p) \cap \Gamma) = Q_1 \cap \{x_n = 0\}$. Thus, these last two properties hold for $p \in \overline{\Gamma} \setminus \Gamma$, as for the points $p \in \Gamma$ treated before.

Since $\overline{\Gamma}$ is compact, we can cover it by a finite number m of open balls $B_{r_{p_i}/2}(p_i)$, $i = 1, ..., m$, with half the radius r_{p_i}. We set $A^{(1)}_{r_i/2} := B_{r_{p_i}/2}(p_i) \cap A$ and $A^{(1)}_{r_i} := B_{r_{p_i}}(p_i) \cap A$. Observe that the number m of balls and the Lipschitz constant of φ_{p_i} depend only on A and Γ, as all constants from now on.

Next, consider the compact set $\mathcal{K} := \overline{A} \setminus \bigcup_{i=1}^m A^{(1)}_{r_i/2}$. For every $q \in \mathcal{K}$, take a radius $0 < s_q \le \frac{2}{3}\text{dist}(q, \Gamma)$, for which there exists a bilipschitz diffeomorphism $\varphi_q : B_{s_q} \cap A \to Q_1$. This is possible both if $q \in A$ or if $q \in \partial A$. Cover $\mathcal{K}$ by l balls $B_{s_{q_j}/2}(q_j)$, $j = 1, ..., l$, with center $q_j \in \mathcal{K}$ and half of the radius s_{q_j}. Set $A^{(2)}_{s_j/2} := B_{s_{q_j}/2}(q_j) \cap A$ and $A^{(2)}_{s_j} := B_{s_{q_j}}(q_j) \cap A$.

Thus, $\{A^{(1)}_{r_i/2}, A^{(2)}_{s_j/2}\}$ is a finite open covering of A. Set $\varepsilon_0 := \min_{i,j}\{r_i/2, s_j/2, 1/2\}$. If z and $\overline{z}$ are two points belonging to A, such that $|z - \overline{z}| < \varepsilon_0$, then there exists a set $A^{(1)}_{r_i}$, or $A^{(2)}_{s_j}$, such that both z and $\overline{z}$ belong to $A^{(1)}_{r_i}$, or to $A^{(2)}_{s_j}$. Hence we have

$$\{(z, \overline{z}) \in A \times A : |z - \overline{z}| < \varepsilon_0\} \subset \left(\bigcup_{i=1}^m A^{(1)}_{r_i} \times A^{(1)}_{r_i}\right) \cup \left(\bigcup_{j=1}^l A^{(2)}_{s_j} \times A^{(2)}_{s_j}\right) \quad (3.2.9)$$

. Observe that

$$\text{dist}(y, \Gamma) \ge \text{dist}(q_j, \Gamma) - |y - q_j| \ge \frac{3}{2}s_{q_j} - s_{q_j} = s_j/2 \ge \varepsilon_0 \text{ for every } y \in A^{(2)}_{s_j}. \quad (3.2.10)$$

Let $L > 1$ be a bound for the Lipschitz constants of all functions $\varphi_{p_1}, ..., \varphi_{p_m}, \varphi^{-1}_{p_1}, ..., \varphi^{-1}_{p_m}$.

Now, let w as in the statement of the theorem. Let us first treat the case $0 < \varepsilon \le 1/(2L)$. Since

$$\int_A d\sigma_z \int_{\{\overline{z} \in A : |z - \overline{z}| > \varepsilon_0\}} d\sigma_{\overline{z}} \frac{|w(z) - w(\overline{z})|^2}{|z - \overline{z}|^{n+1}} \le \frac{4c_0^2}{\varepsilon_0^{n+1}}|A|^2 = Cc_0^2,$$

we only need to bound the double integral in $\{z \in A\} \times \{\overline{z} \in A : |z-\overline{z}| < \varepsilon_0\}$. By (3.2.9), it suffices to bound the integrals in each $A^{(1)}_{r_i} \times A^{(1)}_{r_i}$ and in each $A^{(2)}_{s_j} \times A^{(2)}_{s_j}$.

Thus, for every i, consider

$$\int_{A^{(1)}_{r_i}} \int_{A^{(1)}_{r_i}} \frac{|w(z)-w(\overline{z})|^2}{|z-\overline{z}|^{n+1}} d\sigma_z d\sigma_{\overline{z}}.$$

Recall that, by construction, there exists a bilipschitz map $\varphi_{p_i} : A^{(1)}_{r_i} \to Q_1$ such that $\varphi_{p_i}(\Gamma \cap A^{(1)}_{r_i}) = \{x \in Q_1 : x_n = 0\}$. Thus, flattening the set $A^{(1)}_{r_i}$ using φ_{p_i}, we are in the situation of step 1. More precisely, since φ_{p_i} is bilipschitz, we have that

$$\int_{A^{(1)}_{r_i}} \int_{A^{(1)}_{r_i}} \frac{|w(z)-w(\overline{z})|^2}{|z-\overline{z}|^{n+1}} d\sigma_z d\sigma_{\overline{z}} \le C \int_{Q_1} \int_{Q_1} \frac{|w_i(z)-w_i(\overline{z})|^2}{|z-\overline{z}|^{n+1}} dz d\overline{z},$$

where we have set $w_i = w \circ \varphi_{p_i}^{-1}$.

Given $x \in Q_1$, let $y = \varphi_{p_i}^{-1}(x) \in A^{(1)}_{r_i}$. Recalling the definition of the Lipschitz constant L above, we have $|x_n| \le L\mathrm{dist}(y,\Gamma)$ and hence

$$\begin{aligned} |Dw_i(x)| &\le L|Dw(y)| \le Lc_0 \min\left\{\frac{1}{\varepsilon}, \frac{1}{\mathrm{dist}(y,\Gamma)}\right\} \\ &\le Lc_0 \min\left\{\frac{1}{\varepsilon}, \frac{L}{|x_n|}\right\} = L^2 c_0 \min\left\{\frac{1}{\varepsilon L}, \frac{1}{|x_n|}\right\}. \end{aligned}$$

Thus we can apply the result proved in Step 1, with ε replaced by εL (note that we have $\varepsilon L \le 1/2$, as in Step 1), to the function $w_i/[(1+L^2)c_0]$. We obtain the desired bound $Cc_0^2|\log(\varepsilon L)| \le Cc_0^2|\log(\varepsilon)|$.

Finally, we consider the double integral in $A^{(2)}_{s_j} \times A^{(2)}_{s_j}$, for any $j \in \{1,...,l\}$. Recall that there exists a bilipschitz diffeomorphism $\varphi_{q_j} : A^{(2)}_{s_j} \to Q_1$. Thus

$$\int_{A^{(2)}_{s_j}} \int_{A^{(2)}_{s_j}} \frac{|w(z)-w(\overline{z})|^2}{|z-\overline{z}|^{n+1}} d\sigma_z d\sigma_{\overline{z}} \le C \int_{Q_1} \int_{Q_1} \frac{|v_j(x)-v_j(\overline{x})|^2}{|x-\overline{x}|^{n+1}} d\sigma_x d\sigma_{\overline{x}},$$

where now $v_j := w \circ \varphi_{q_j}^{-1}$. By (3.2.10) and (3.0.14), $|Dw(y)| \le c_0/\varepsilon_0$ a.e. in $A^{(2)}_{s_j}$, and $|Dv_j| \le C$ a.e. in Q_1. From this, the last double integral is bounded by

$$C \int_{Q_1} dx \int_{Q_1} \frac{d\overline{x}}{|x-\overline{x}|^{n-1}} \le C \int_{Q_1} dx \int_0^{2\sqrt{n}} dr \le C.$$

This conclude the proof in case $\varepsilon \le 1/(2L)$.

Finally, given $\varepsilon \in (0,1/2)$ with $\varepsilon > 1/(2L)$, since (3.0.14) holds with such ε, it also holds with ε replaced by $1/(2L)$. By the previous proof with ε taken to be $1/(2L)$, the energy is bounded by $C|\log 1/(2L)| \le C \le C|\log(\varepsilon)|$ since $\varepsilon < 1/2$. □

3.3 Energy estimate for global minimizers

In this section we give the proof of Theorem 3.0.3. It is based on a comparison argument. The proof can be resumed in 3 steps. Let v be a global minimizer of (3.0.3).

i) construct a comparison function $\overline{w}$, harmonic in C_R, which takes the same values as v on $\partial^+ C_R = \partial C_R \cap \{\lambda > 0\}$ and thus, by minimality of v,

$$\mathcal{E}_{C_R}(v) \leq \mathcal{E}_{C_R}(\overline{w});$$

ii) use estimate (3.0.12):

$$\int_{C_R} |\nabla \overline{w}|^2 \leq C||\overline{w}||^2_{H^{1/2}(\partial C_R)},$$

iii) establish, using Theorem 3.0.7 the key estimate

$$||\overline{w}||^2_{H^{1/2}(\partial C_R)} \leq CR^{n-1} \log R.$$

Proof of Theorem 3.0.3. Let v be a bounded global minimizer of (3.0.3). Let u be its trace on $\partial \mathbb{R}^{n+1}_+$. Recall the definition (3.0.6) of the constant c_u. Let $s \in [\inf u, \sup u]$ be such that $G(s) = c_u$.

Throughout the proof, C denotes positive constants depending only on n, $||f||_{C^1([\inf u, \sup u])}$ and $||u||_{L^\infty(\mathbb{R}^n)}$. As explained in (3.0.16), v satisfies the following bounds:

$$|v| \leq C \quad \text{and} \quad |\nabla v(x,\lambda)| \leq \frac{C}{1+\lambda} \quad \text{for every } x \in \mathbb{R}^n, \;\; \lambda \geq 0. \tag{3.3.1}$$

We estimate the energy $\mathcal{E}_{C_R}(v)$ of v using a comparison argument. We define a function $\overline{w} = \overline{w}(x,\lambda)$ on C_R in the following way. First we define $\overline{w}(x,0)$ on the base of the cylinder to be equal to a smooth function $g(x)$ which is identically equal to s in B_{R-1} and $g(x) = v(x,0)$ for $|x| = R$. The function g is defined as follows:

$$g = s\eta_R + (1-\eta_R)v, \tag{3.3.2}$$

where η_R is a smooth function depending only on $r = |x|$ such that $\eta \equiv 1$ in B_{R-1} and $\eta \equiv 0$ outside B_R. Thus, g satisfies

$$g \in [inf u, \sup u] \quad \text{and} \quad |\nabla g| \leq C \;\text{ in }\; B_R. \tag{3.3.3}$$

Then we define $\overline{w}(x,\lambda)$ as the unique solution of the Dirichlet problem

$$\begin{cases} \Delta \overline{w} = 0 & \text{in } C_R \\ \overline{w}(x,0) = g(x) & \text{on } B_R \times \{\lambda = 0\} \\ \overline{w}(x,\lambda) = v(x,\lambda) & \text{on } \partial C_R \cap \{\lambda > 0\}. \end{cases} \tag{3.3.4}$$

Since v is a global minimizer of $\mathcal{E}_{C_R}$ and $\overline{w} = v$ on $\partial C_R \cap \{\lambda > 0\}$, then

$$\int_{C_R} \frac{1}{2}|\nabla v|^2 dx d\lambda + \int_{B_R} \{G(u) - c_u\} dx \\ \leq \int_{C_R} \frac{1}{2}|\nabla \overline{w}|^2 dx d\lambda + \int_{B_R} \{G(\overline{w}(x,0)) - c_u\} dx.$$

We prove next that

$$\int_{C_R} \frac{1}{2}|\nabla \overline{w}|^2 dx d\lambda + \int_{B_R} \{G(\overline{w}(x,0)) - c_u\} dx \leq C R^{n-1} \log R.$$

Observe that the potential energy is bounded by CR^{n-1}. Indeed, by definition $\overline{w}(x,0) = s$ in B_{R-1}, and hence

$$\int_{B_R} \{G(\overline{w}(x,0)) - c_u\} dx = \int_{B_R \setminus B_{R-1}} \{G(g(x)) - c_u\} dx \\ \leq C|B_R \setminus B_{R-1}| \leq C R^{n-1}.$$

Thus, we only need to bound the Dirichlet energy. First of all, rescaling, we set

$$w_1(x,\lambda) = \overline{w}(Rx, R\lambda),$$

for $(x,\lambda) \in C_1 = B_1 \times (0,1)$. Set

$$\varepsilon = 1/R.$$

Observe that

$$\int_{C_R} |\nabla \overline{w}|^2 = C R^{n-1} \int_{C_1} |\nabla w_1|^2.$$

Thus, we need to prove that

$$\int_{C_1} |\nabla w_1|^2 \leq C \log R = C|\log \varepsilon|. \tag{3.3.5}$$

Since w_1 is harmonic in C_1, Proposition 3.2.1 gives that

$$\int_{C_1} |\nabla w_1|^2 dx d\lambda \leq C||w_1||_{H^{1/2}(\partial C_1)}.$$

To control $||w_1||_{H^{1/2}(\partial C_1)}$, we apply Theorem 3.0.7 to $w_{1|_{\partial C_1}}$ in $A = \partial C_1$, taking $\Gamma = \partial B_1 \times \{\lambda = 0\}$.

Since $|w_1| \leq C$, we only need to check (3.0.14) in ∂C_1. In the bottom boundary, $B_1 \times \{0\}$, this is simple. Indeed $w_1 \equiv s$ in $B_{1-\varepsilon}$, and thus we need only to control

$|\nabla w_1(x,0)| = \varepsilon^{-1}|\nabla g(Rx)| \le C\varepsilon^{-1}$ for $|x| > 1-\varepsilon$, by (3.3.2). Here $\operatorname{dist}(x, \partial B_1) < \varepsilon$, and thus (3.0.14) holds here.

Next, to verify (3.0.14) in $\partial C_1 \cap \{\lambda > 0\}$ we use that $\overline{w} = v$ here and that we know

$$|\nabla \overline{v}(x,\lambda)| \le \frac{C}{1+\lambda} \quad \text{for every } (x,\lambda) \in C_R,$$

as stated in (3.0.16). Thus, the tangential derivatives of w_1 in $\partial C_1 \cap \{\lambda > 0\}$ satisfy

$$|\nabla w_1(x,\lambda)| \le \frac{CR}{1+R\lambda} = \frac{C}{\varepsilon+\lambda} \le C \min\left\{\frac{1}{\varepsilon}, \frac{1}{\lambda}\right\}.$$

Since $\operatorname{dist}((x,\lambda),\Gamma) \ge \lambda$ on $\partial C_1 \cap \{\lambda > 0\}$, $w_{1|_{\partial C_1}}$ satisfies the hypothesis of Theorem 3.0.7. We conclude that (3.3.5) holds. □

3.4 Energy estimate for monotone solutions in $\mathbb{R}^3$

The following lemma will play a key role in this section to establish the energy estimate for monotone solutions in dimension $n = 3$.

Lemma 3.4.1. *Let f be a $C^{1,\beta}$ function, for some $0 < \beta < 1$, and u a bounded solution of equation (3.0.1) in $\mathbb{R}^3$, such that $u_{x_3} > 0$. Let v be the harmonic extension of u in $\mathbb{R}^4_+$. Set*

$$\underline{v}(x_1,x_2,\lambda) := \lim_{x_3\to-\infty} v(x,\lambda) \ \text{ and } \ \overline{v}(x_1,x_2,\lambda) := \lim_{x_3\to+\infty} v(x,\lambda).$$

Then, $\underline{v}$ and $\overline{v}$ are solutions of (3.0.3) in $\mathbb{R}^3_+$, and each of them is either constant or it depends only on λ and one Euclidian variable in the $(x_1,x_2)-$plane. As a consequence, each $\underline{u} = \underline{v}(\cdot,0)$ and $\overline{u} = \overline{v}(\cdot,0)$ is either constant or 1-D.

Moreover, set $m = \inf \underline{u} \le \widetilde{m} = \sup \underline{u}$ and $\widetilde{M} = \inf \overline{u} \le M = \sup \overline{u}$. Then, $G > G(\widetilde{m}) = G(m)$ in $(m,\widetilde{m})$, $G'(\widetilde{m}) = G'(m) = 0$ and $G > G(\widetilde{M}) = G(M)$ in $(\widetilde{M}, M)$, $G'(\widetilde{M}) = G'(M) = 0$.

Proof. The function $\overline{v}(x',\lambda) = \lim_{x_3\to+\infty} v(x',x_3,\lambda)$ is the harmonic extension of $\overline{u}$. The key point of the proof is to verify that $\overline{v}$ is a stable solution of problem (3.0.3) in $\mathbb{R}^3_+$ and then apply Theorem 1.5 of [10] on 1-D symmetry in $\mathbb{R}^3_+$.

The fact that $\overline{v}$ is a solution of problem (3.0.3) in $\mathbb{R}^3_+$ is easily verified viewing $\overline{v}$ as a function of 4 variables, limit as $t \to +\infty$ of the solutions $v^t(x',x_3,\lambda) = v(x',x_3+t,\lambda)$. By standard elliptic theory, $v^t \to \overline{v}$ uniformly in the C^2 sense on compact sets of $\overline{\mathbb{R}^4_+}$.

Now we prove that $\overline{v}(x',\lambda)$ is a stable solution of problem (3.0.3) in $\mathbb{R}^3_+$. By Lemma 4.1 of [10], the stability of $\overline{v}$ is equivalent to the existence of a positive function φ which satisfies

$$\begin{cases} \Delta\varphi = 0 & \text{in } \mathbb{R}^3_+ \\ -\frac{\partial\varphi}{\partial\lambda} = f'(\overline{v})\varphi & \text{on } \partial\mathbb{R}^3_+. \end{cases} \tag{3.4.1}$$

To check the existence of $\varphi > 0$ satisfying (3.4.1), we use that $v_{x_3} > 0$ and that satisfies the problem

$$\begin{cases} \Delta v_{x_3} = 0 & \text{in } \mathbb{R}^4_+ \\ -\frac{\partial v_{x_3}}{\partial\lambda} = f'(v)v_{x_3} & \text{on } \partial\mathbb{R}^4_+. \end{cases}$$

This gives v is stable in $\mathbb{R}^4_+$, i.e.

$$\int_{\mathbb{R}^4_+} |\nabla\xi|^2 dx d\lambda - \int_{\mathbb{R}^3} f'(v)\xi^2 dx \geq 0, \quad \text{for every } \xi \in C_c^\infty(\overline{\mathbb{R}^4_+}).+ \tag{3.4.2}$$

Next, we claim that

$$\int_{\mathbb{R}^3_+} |\nabla\eta|^2 dx' d\lambda - \int_{\mathbb{R}^2} f'(\overline{v})\eta^2 dx' \geq 0, \quad \text{for every } \eta \in C_c^\infty(\overline{\mathbb{R}^3_+}). \tag{3.4.3}$$

To show this, we take $\rho > 0$ and $\psi_\rho \in C^\infty(\mathbb{R})$ with $0 \leq \psi_\rho \leq 1$, $0 \leq \psi'_\rho \leq 2$, $\psi_\rho = 0$ in $(-\infty,\rho) \cup (2\rho+2,+\infty)$, and $\psi_\rho = 1$ in $(\rho+1, 2\rho+1)$, and we apply (3.4.2) with $\xi(x,\lambda) = \eta(x',\lambda)\psi_\rho(x)$. We obtain after dividing the expression by $\alpha_\rho = \int \psi_\rho^2$, that

$$\int_{\mathbb{R}^3_+} |\nabla\eta(x',\lambda)|^2 dx' d\lambda + \int_{\mathbb{R}^3_+} dx' d\lambda \eta^2(x',\lambda) \int_{\mathbb{R}} \frac{(\psi'_\rho)^2(x_3)}{\alpha_\rho} dx_3$$
$$- \int_{\mathbb{R}^3_+} \eta^2(x',\lambda) dx' d\lambda \int_{\mathbb{R}} f'(v(x',x_3,0))\frac{\psi_\rho^2}{\alpha_\rho} dx_3 \geq 0$$

Passing to the limit as $\rho \to +\infty$, and using $f \in C^1$ and that $v(x',x_3,\lambda) \to \overline{v}(x',\lambda)$ as $x_3 \to +\infty$ uniformly in compact sets of $\overline{\mathbb{R}^4}_+$, we obtain (3.4.3).

Since $\overline{v}(x',\lambda)$ is a stable solution of problem (3.0.3) in $\mathbb{R}^3_+$, by Theorem 1.5 (point b)) in [10], we deduce that $\overline{v}$ is constant or $\overline{v}$ depends only on λ and one Euclidian variable in the x'-plane. Now note that the function $2(\overline{v}-\widetilde{M})/(M-\widetilde{M})-1$ is a layer solution for a new nonlinearity. Using Theorem 1.2 a) of [10], which characterizes the nonlinearities f for which there exists a layer solutions for problem (3.0.3) in dimension $n = 1$, and restating the conclusion for $\overline{v}$, we get $G'(\widetilde{M}) = G'(M) = 0$ and $G > G(\widetilde{M}) = G(M)$ in $(\widetilde{M}, M)$.

In the same way, we prove that the conclusion holds for $\underline{v}$ and that $G'(\widetilde{m}) = G'(m) = 0$ and $G > G(\widetilde{m}) = G(m)$ in $(m,\widetilde{m})$. $\square$

Remark 3.4.2. We claim that in the case of Allen-Cahn type equations, we could prove the energy estimate (3.0.9) for monotone solutions in dimension $n = 3$ using the same argument as in the proof of Theorem 3.1.1. The only difficulty is that in this section we do not assume $\lim_{x_3\to+\infty} v = 1$, and then we do not know if $\lim_{T\to\infty} \mathcal{E}_{C_R}(v^T) = 0$ (see (3.1.9) in the proof of Theorem 3.1.1). Using Lemma 3.4.1, we have that $\overline{u}(x_1, x_2) = \lim_{x_3\to+\infty} u(x_1, x_2, x_3)$ is either a constant or it depends only on one variable. Then, applying Theorem 1.6 of [10], which gives the energy bounds for 1-D solutions, we deduce that $\lim_{T\to+\infty} \mathcal{E}_{C_R}(v^T) \le CR \log R$, and this is enough to carry out the proof of Theorem 3.1.1 in the present setting.

Before giving the proof of Theorem 3.0.4, we need the following proposition. It is the analog of Theorem 4.4 of [1] and asserts that the monotonicity of a solution implies its minimality among a suitable family of functions.

Proposition 3.4.3. *Let f be any $C^{1,\beta}$ nonlinearity, with $\beta \in (0, 1)$. Let u be a bounded solution of (3.0.1) in $\mathbb{R}^n$ such that $u_{x_n} > 0$, and let v be its harmonic extension in $\mathbb{R}^{n+1}_+$.*
Then,

$$\begin{aligned}\int_{C_R} \frac{1}{2}|\nabla v(x,\lambda)|^2 dx d\lambda \ &+ \int_{B_R} G(v(x,0))dx \\ &\le \int_{C_R} \frac{1}{2}|\nabla w(x,\lambda)|^2 dx d\lambda + \int_{B_R} G(w(x,0))dx,\end{aligned}$$

for every $w \in C^1(\overline{\mathbb{R}^{n+1}_+})$ such that $w = v$ on $\partial^+ C_R = \partial C_R \cap \{\lambda > 0\}$ and $\underline{v} \le w \le \overline{v}$ in C_R, where $\underline{v}$ and $\overline{v}$ are defined by

$$\underline{v}(x',\lambda) := \lim_{x_n\to-\infty} v(x', x_n, \lambda) \ \text{ and } \ \overline{v}(x',\lambda) := \lim_{x_n\to+\infty} v(x', x_n, \lambda).$$

Proof. This property of minimality of monotone solutions among functions w such that $\underline{v} \le w \le \overline{v}$ follows from the following two results:

i) Uniqueness of solution to the problem

$$\begin{cases} \Delta w = 0 & \text{in } C_R, \\ w = v & \text{on } \partial^+ C_R, \\ -\partial_\lambda w = f(w) & \text{on } \partial^0 C_R, \\ \underline{v} \le w \le \overline{v} & \text{in } C_R. \end{cases} \tag{3.4.4}$$

Thus, the solution must be $w \equiv v$. This is the analog of Lemma 3.1 of [10], and below we comment on its proof.

ii) Existence of an absolute minimizer for $\mathcal{E}_{C_R}$ in the set

$$C_v = \{w \in H^1(C_R) | w \equiv v \text{ on } \partial^+ C_R,\ \underline{v} \le w \le \overline{v} \text{ in } C_R\}.$$

This is the analog of Lemma 2.10 of [10].

The statement of the proposition follows from the fact that by i) and ii), the monotone solution v, by uniqueness, must agree with the absolute minimizer in C_R.

To prove points i) and ii), we proceed exactly as in [10], with the difference that here we do not assume $\lim_{x_n \to \pm\infty} = \pm 1$. We have only to substitute -1 and $+1$, by $\underline{v}$ and $\overline{v}$ respectively in the proofs of Lemma 3.1 and Lemma 2.10 in [10]. For this, it is important that $\underline{v}$ and $\overline{v}$ are respectively, a strict subsolution and a strict supersolution of the Dirichlet- Neumann mixed problem (3.4.4). We make a short comment about these proofs.

i) The proof of uniqueness is based, as in Lemma 3.1 of [10], on sliding the function $v(x, \lambda)$ in the direction x_n. We set

$$v^t(x_1, ..., x_n, \lambda) = v(x_1, ..., x_n + t, \lambda) \quad \text{for every } (x, \lambda) \in \overline{C}_R.$$

Since $v^t \to \overline{v}$ as $t \to +\infty$ uniformly in $\overline{C}_R$ and $\underline{v} < w < \overline{v}$, then $w < v^t$ in $\overline{C}_R$, for t large enough. We want to prove that $w < v^t$ in $\overline{C}_R$ for every $t > 0$. Suppose that $s > 0$ is the infimum of those $t > 0$ such that $w < v^t$ in $\overline{C}_R$. Then by applying maximum principle and Hopf's lemma we get a contradiction, since one would have $w \le v^s$ in $\overline{C}_R$ and $w = v^s$ at some point in $\overline{C}_R \setminus \partial^+ C_R$.

ii) To prove the existence of an absolute minimizer for $\mathcal{E}_{C_R}$ in the convex set C_v, we proceed exactly as in the proof of Lemma 2.10 of [10], substituting -1 and $+1$ by the subsolutions and supersolution $\underline{v}$ and $\overline{v}$, respectively.

□

We give now the proof of the energy estimate in dimension 3 for monotone solutions without the limit assumptions.

Proof of Theorem 3.0.4. We follow the proof of Theorem 5.2 of [1]. We need to prove that the comparison function $\overline{w}$, used in the proof of Theorem 3.0.3, satisfies $\underline{v} \le \overline{w} \le \overline{v}$. Then we can apply Proposition 3.4.3 to make the comparison argument with the function $\overline{w}$ (as for global minimizers). We recall that $\overline{w}$ is the solution of problem (3.3.4);

$$\begin{cases} \Delta \overline{w} = 0 & \text{in } C_R \\ \overline{w}(x, 0) = g(x) & \text{on } B_R \times \{\lambda = 0\} \\ \overline{w}(x, \lambda) = v(x, \lambda) & \text{on } \partial C_R \cap \{\lambda > 0\}, \end{cases} \tag{3.4.5}$$

where $g = s\eta_R + (1-\eta_R)v$. Thus, if we prove that $\sup \underline{v} \le s \le \inf \overline{v}$, then $\underline{v} \le g \le \overline{v}$ and hence $\underline{v}$ and $\overline{v}$ are respectively, subsolution and supersolutions of (3.4.5). It follows that $\underline{v} \le \overline{w} \le \overline{v}$, as desired.

To show that $\sup \underline{v} \le s \le \inf \overline{v}$, let $m = \inf u = \inf \underline{u}$ and $M = \sup u = \sup \overline{u}$, where $\underline{u}$ and $\overline{u}$ are defined in Lemma 3.4.1. Set $\widetilde{m} = \sup \underline{u}$ and $\widetilde{M} = \inf \overline{u}$, obviously $\widetilde{m}$ and $\widetilde{M}$ belong to $[m, M]$. By Lemma 3.4.1, $\underline{u}$ and $\overline{u}$ are either constant or monotone 1-D solutions, moreover

$$G > G(m) = G(\widetilde{m}) \quad \text{in } (m, \widetilde{m}) \tag{3.4.6}$$

in case $m < \widetilde{m}$ (i.e. $\underline{u}$ not constant), and

$$G > G(M) = G(\widetilde{M}) \quad \text{in } (\widetilde{M}, M) \tag{3.4.7}$$

in case $\widetilde{M} < M$ (i.e. $\overline{u}$ not constant).

In all four possible cases (that is, each $\underline{u}$ and $\overline{u}$ is constant or one-dimensional), we deduce from (3.4.6) and (3.4.7) that $\widetilde{m} \le \widetilde{M}$ and that there exists $s \in [\widetilde{m}, \widetilde{M}]$ such that $G(s) = c_u$ (recall that c_u is the infimum of G in the range of u). We conclude that

$$\sup \underline{u} = \sup \underline{v} \le \widetilde{m} \le s \le \widetilde{M} \le \inf \overline{v} = \inf \overline{u}.$$

Hence we can apply Proposition 3.4.3 to make comparison argument with the function $\overline{w}$ and obtain the desired energy estimate. □

3.5 1-D symmetry in $\mathbb{R}^3$

In this section we present the Liouville result due to Moschini [31], that we will use in the proof of 1-D symmetry in dimension $n = 3$. Set

$$\mathcal{F} = \left\{ F : \mathbb{R}^+ \to \mathbb{R}^+, F \text{ is nondecreasing and } \int_2^{+\infty} \frac{1}{rF(r)} = +\infty \right\}.$$

Note that $\mathcal{F}$ includes the function $F(r) = \log(r)$.

Proposition 3.5.1. *([31]) Let $\varphi \in L^\infty_{loc}(\mathbb{R}^{n+1}_+)$ be a positive function. Suppose that $\sigma \in H^1_{loc}(\mathbb{R}^{n+1}_+)$ satisfies*

$$\begin{cases} -\sigma \mathrm{div}(\varphi^2 \nabla \sigma) \le 0 & \text{in } \mathbb{R}^{n+1}_+ \\ -\sigma \partial_\lambda \sigma \le 0 & \text{on } \partial\mathbb{R}^{n+1}_+ \end{cases} \tag{3.5.1}$$

in the weak sense. Let the following condition hold:

$$\limsup_{R \to +\infty} \frac{1}{R^2 F(R)} \int_{C_R} (\varphi\sigma)^2 dx < \infty \tag{3.5.2}$$

for some $F \in \mathcal{F}$.

Then, σ *is constant.*

In particular, this statement holds with $F(R) = \log(R)$.

Remark 3.5.2. In [31], the author proves the previous result under the assumption

$$\sum_{j=0}^{+\infty} \frac{1}{F(2^{j+1})} = +\infty \tag{3.5.3}$$

on F. This is equivalent to $\int_2^{+\infty} (rF(r))^{-1} dr = +\infty$. Indeed, since the function $F(2^{j+1})$ is nondecreasing, we have that

$$\sum_{j=3}^{+\infty} \frac{1}{F(2^{j+1})} \leq \int_2^{+\infty} \frac{ds}{F(2^{s+1})} = \frac{1}{\log 2} \int_8^{\infty} \frac{dr}{rF(r)} \leq \sum_{j=2}^{+\infty} \frac{1}{F(2^{j+1})}.$$

Thus, (3.5.1) holds if and only if $F \in \mathcal{F}$.

Proof of Proposition 3.5.1. We present the proof following that of Theorem 5.1 of [31], here in C_R instead of B_R. Set $\partial^+ C_R := \partial C_R \cap \{\lambda > 0\}$. Since σ satisfies (3.5.1), we have

$$\operatorname{div}(\sigma \varphi^2 \nabla \sigma) \geq \varphi^2 |\nabla \sigma|^2. \tag{3.5.4}$$

On the other hand

$$\int_{\partial^+ C_R} \sigma \varphi^2 \frac{\partial \sigma}{\partial \nu} ds \leq \left(\int_{\partial^+ C_R} \varphi^2 |\nabla \sigma|^2 ds \right)^{\frac{1}{2}} \left(\int_{\partial^+ C_R} (\varphi \sigma)^2 ds \right)^{\frac{1}{2}}, \tag{3.5.5}$$

where ν denotes the outer normal vector on $\partial^+ C_R$. Now, set, as in [31],

$$D(R) = \int_{C_R} \varphi^2 |\nabla \sigma|^2 dx.$$

Integrating (3.5.4) over C_R, using that $-\sigma \partial_\lambda \sigma \leq 0$ on the bottom boundary $\partial C_R \cap \{\lambda = 0\}$, and using (3.5.5), we get

$$D(R) \leq D'(R)^{\frac{1}{2}} \left(\int_{\partial^+ C_R} (\varphi \sigma)^2 ds \right)^{\frac{1}{2}}, \tag{3.5.6}$$

which is the analog of (5.5) in [31] on $\partial^+ C_R$ instead of ∂B_R.

Assume that σ is not constant. Then, there exists $R_0 > 0$ such that $D(R) > 0$ for every $R > R_0$. Integrating (3.5.6) and using Schwarz inequality, we get that, for every $r_2 > r_1 > R_0$,

$$\begin{aligned} (r_2 - r_1)^2 \left(\int_{C_{r_2} \setminus C_{r_1}} (\varphi \sigma)^2 dx \right)^{-1} &= (r_2 - r_1)^2 \left(\int_{r_1}^{r_2} dr \int_{\partial^+ C_R} ds (\varphi \sigma)^2 \right)^{-1} \\ \leq \int_{r_1}^{r_2} dr \left(\int_{\partial^+ C_R} ds (\varphi \sigma)^2 \right)^{-1} &\leq \frac{1}{D(r_1)} - \frac{1}{D(r_2)}. \end{aligned} \tag{3.5.7}$$

Next, choose $r_2 = 2^{j+1}r_*$ and $r_1 = 2^j r_*$, for some $r_* > R_0$, for every $j = 0, ..., N-1$. Using (3.5.2), (3.5.7) and summing over j, we find that

$$\frac{1}{D(r_*)} \geq C \sum_{j=0}^{N-1} \frac{1}{F(2^{j+1}r_*)}. \tag{3.5.8}$$

If j_0 is such that $r_* \leq 2^{j_0}$, then, by hypothesis on F, $F(2^{j+1}r_*) \leq F(2^{j+j_0+1})$. Thus, by (3.5.3), the sum in (3.5.8) diverges as $N \to \infty$ and hence $D(r_*) = 0$ for every $r_* > R_0$, which is a contradiction. □

We can give now the proof of the 1-D symmetry result.

Proof of Theorem 3.0.5. Without loss of generality we can suppose $e = (0, 0, 1)$. We follow the proof of Lemma 4.2 in [10].

First of all observe that both global minimizers and monotone solutions are stable. Then, in both cases, by Lemma 4.1 in [10], there exists a function $\varphi \in C^1_{loc}(\overline{\mathbb{R}^4_+}) \cap C^2(\mathbb{R}^4_+)$ such that $\varphi > 0$ in $\overline{\mathbb{R}^4_+}$ and

$$\begin{cases} \Delta\varphi = 0 & \text{in } \mathbb{R}^4_+ \\ -\frac{\partial\varphi}{\partial\lambda} = f'(v)\varphi & \text{on } \partial\mathbb{R}^4_+. \end{cases}$$

Note that, if u is a monotone solution in the direction x_3, then we can choose $\varphi = v_{x_3}$, where v is the harmonic extension of u in the half space. For $i = 1, 2, 3$ fixed, consider the function

$$\sigma_i = \frac{v_{x_i}}{\varphi}.$$

We prove that σ_i is constant in $\mathbb{R}^4_+$, using the Liouville result of Proposition 3.5.1 and our energy estimate.

Since

$$\varphi^2 \nabla\sigma_i = \varphi\nabla v_{x_i} - v_{x_i}\nabla\varphi,$$

we have that

$$\operatorname{div}(\varphi^2\nabla\sigma_i) = 0 \quad \text{in } \mathbb{R}^4_+.$$

Moreover, the normal derivative $-\partial_\lambda\sigma_i$ is zero on $\partial\mathbb{R}^4_+$. Indeed,

$$\varphi^2\partial_\lambda\sigma_i = \varphi v_{\lambda x_i} - v_{x_i}\varphi_\lambda = 0$$

since both v_{x_i} and φ satisfy the same boundary condition

$$-\partial_\lambda v_{x_i} - f'(v)v_{x_i} = 0, \quad -\partial_\lambda\varphi - f'(v)\varphi = 0.$$

Now, using our energy estimates (3.0.7) or (3.0.9), we have for $n = 3$,

$$\int_{C_R} (\varphi\sigma_i)^2 \leq \int_{C_R} |\nabla v|^2 \leq CR^2 \log R, \quad \text{for every } R > 2.$$

Thus, using Proposition 3.5.1, we deduce that σ_i is constant for every $i = 1, 2, 3$, i.e.,

$$v_{x_i} = c_i\varphi \quad \text{for some constant } c_i, \quad \text{with } i = 1, 2, 3.$$

We conclude the proof observing that if $c_1 = c_2 = c_3 = 0$ then v is constant. Otherwise we have

$$c_i v_{x_j} - c_j v_{x_i} = 0 \quad \text{for every } i \neq j,$$

and we deduce that v depends only on λ and on the variable parallel to the vector (c_1, c_2, c_3). Thus, $u(x) = v(x, 0)$ is 1-D. □

3.6 Energy estimate for saddle-shaped solutions

In this section we prove that energy estimate (3.0.7) holds also for saddle solutions (which are known, by Theorem 2.1.10 in chapter 2, not to be global minimizers in dimensions $2m \leq 6$) of the problem

$$(-\Delta)^{1/2} u = f(u) \quad \text{in } \mathbb{R}^{2m}.$$

Here, we suppose that f is balanced and bistable, that is f satisfies hypothesis (3.0.17), (3.0.18), and (3.0.19).

We recall that saddle solutions are even with respect to the coordinate axes and odd with respect to the Simons cone, which is defined as follows:

$$\mathcal{C} = \{x \in \mathbb{R}^{2m} : x_1^2 + ... + x_m^2 = x_{m+1}^2 + ... + x_{2m}^2\}.$$

If we set

$$s = \sqrt{x_1^2 + \cdots + x_m^2} \quad \text{and} \quad t = \sqrt{x_{m+1}^2 + \cdots + x_{2m}^2},$$

then the Simons cone becomes $\mathcal{C} = \{s = t\}$. We say that a solution u of problem (3.0.1), is a saddle solution if it satisfies the following properties:

$$u \text{ depends only on the variables } s \text{ and } t. \text{ We write } u = u(s, t); \tag{3.6.1}$$

$$u > 0 \text{ for } s > t; \tag{3.6.2}$$

$$u(s, t) = -u(t, s). \tag{3.6.3}$$

In chapter 2, we have proven the existence of a saddle solution $u = u(x)$ to problem (3.0.1), by proving the existence of a solution $v = v(x, \lambda)$ to problem (3.0.3) with the following properties:

a) v depends only on the variables s, t and λ. We write $v = v(s, t, \lambda)$;

b) $v > 0$ for $s > t$;

c) $v(s,t,\lambda) = -v(t,s,\lambda)$.

The proof of the existence of such function v is simple and it uses a non-sharp energy estimates. Next, we sketch the proof.

We use the following notations:

$$\mathcal{O} := \{x \in \mathbb{R}^{2m} : s > t\} \subset \mathbb{R}^{2m}$$

$$\widetilde{\mathcal{O}} := \{(x,\lambda) \in \mathbb{R}^{2m+1}_+ : x \in \mathcal{O}\} \subset \mathbb{R}^{2m+1}_+.$$

Note that

$$\partial\mathcal{O} = \mathcal{C}.$$

Let B_R be the open ball in $\mathbb{R}^{2m}$ centered at the origin and of radius R. We will consider the open bounded sets

$$\mathcal{O}_R := \mathcal{O} \cap B_R = \{s > t, |x|^2 = s^2 + t^2 < R^2\} \subset \mathbb{R}^{2m},$$

$$\widetilde{\mathcal{O}}_R := \mathcal{O}_R \times (0,R), \quad \text{and} \quad \widetilde{\mathcal{O}}_{R,L} := \mathcal{O}_R \times (0,L).$$

Note that

$$\partial\mathcal{O}_R = (\mathcal{C} \cap \overline{B}_R) \cup (\partial B_R \cap \mathcal{O}).$$

Moreover we define the set

$$\widetilde{H}^1_0(\widetilde{\mathcal{O}}_{R,L}) = \{v \in H^1(\widetilde{\mathcal{O}}_{R,L}) : v \equiv 0 \text{ on } \partial^+\widetilde{\mathcal{O}}_{R,L},\ v = v(s,t,\lambda) \text{ a.e.}\}.$$

Proof of Theorem 3.0.8. The proof of existence of the saddle solution v in $\mathbb{R}^{2m+1}_+$ can be resumed in three steps:
Step a) For every $R > 0$, $L > 0$ consider the minimizer $v_{R,L}$ of the energy functional

$$\mathcal{E}_{\widetilde{\mathcal{O}}_{R,L}}(v) = \int_{\widetilde{\mathcal{O}}_{R,L}} \frac{1}{2}|\nabla v|^2 + \int_{\mathcal{O}_R} G(v)$$

among all functions belonging to the space $\widetilde{H}^1_0(\widetilde{\mathcal{O}}_{R,L})$. The existence of such minimizer, that may be taken to satisfy $|v_{R,L}| \le 1$ by hypothesis (3.0.18), follows by lower semicontinuity of the energy functional. The minimizer $v_{R,L}$ is a solution of the equation (3.0.3) written in the (s,t,λ) variables and we can assume that $v_{R,L} \ge 0$ in $\widetilde{\mathcal{O}}_{R,L}$.
Step b) Extend $v_{R,L}$ to $B_R \times (0,L)$ by odd reflection with respect to $\mathcal{C} \times (0,L)$, that is, $v_{R,L}(s,t,\lambda) = -v_{R,L}(t,s,\lambda)$. Then, $v_{R,L}$ is a solution in $B_R \times (0,L)$.
Step c) Define v as the limit of the sequence $v_{R,L}$ as $R \to +\infty$, taking $L = R^\gamma \to +\infty$ with $1/2 \le \gamma < 1$. With the aid of a non-sharp energy estimate, verify that

$v \not\equiv 0$ and, as a consequence, that v is a saddle solution. This step could be carried out using the sharp energy estimate that we prove next.

Here, it is important to observe that the solution v constructed in this way is not a global minimizer in $\mathbb{R}^{2m+1}_+$ (indeed it is not stable in dimensions $2m = 4, 6$ by Theorem 2.1.10), but it is a minimizer in $\widetilde{\mathcal{O}}$, or in other words, it is a minimizer under perturbations vanishing on the Simons cone. Next, we use this fact to prove the energy estimate $\mathcal{E}_{\widetilde{\mathcal{O}}_R}(v) \le CR^{2m-1}\log R$ in the set $\widetilde{\mathcal{O}}_R = \mathcal{O}_R \times (0, R)$, using a comparison argument as for global minimizers.

As before, we want to construct a comparison function $\overline{w}$ in $\widetilde{\mathcal{O}}_R$ which agrees with v on $\partial^+\widetilde{\mathcal{O}}_R$ and such that

$$\mathcal{E}_{\widetilde{\mathcal{O}}_R}(\overline{w}) = \int_{\widetilde{\mathcal{O}}_R} \frac{1}{2}|\nabla \overline{w}|^2 + \int_{\mathcal{O}_R} G(\overline{w}) \le CR^{2m-1}\log R. \tag{3.6.4}$$

We define the function $\overline{w} = \overline{w}(x, \lambda) = \overline{w}(s, t, \lambda)$ in $\widetilde{\mathcal{O}}_R$ in the following way.

First we define $\overline{w}(x, 0)$ on the base $\mathcal{O}_R$ of $\widetilde{\mathcal{O}}_R$ to be equal to a smooth function $g(x)$ which is identically equal to 1 in $\mathcal{O}_{R-1} \cap \{(s-t)/\sqrt{2} > 1\}$ and $g(x) = v(x, 0)$ on $\partial\mathcal{O}_R$. The function g is defined as follows:

$$g = \eta_R \min\left\{1, \frac{s-t}{\sqrt{2}}\right\} + (1 - \eta_R)v, \tag{3.6.5}$$

where η_R is a smooth function depending only on $r = |x|$ such that $\eta_R \equiv 1$ in $\mathcal{O}_{R-1}$ and $\eta_R \equiv 0$ outside $\mathcal{O}_R$. Then we define $\overline{w}(x, \lambda)$ as the unique solution of the Dirichlet problem

$$\begin{cases} \Delta \overline{w} = 0 & \text{in } \widetilde{\mathcal{O}}_R \\ \overline{w}(x, 0) = g(x) & \text{on } \mathcal{O}_R \times \{\lambda = 0\} \\ \overline{w}(x, \lambda) = v(x, \lambda) & \text{on } \partial\widetilde{\mathcal{O}}_R \cap \{\lambda > 0\}. \end{cases} \tag{3.6.6}$$

Since v is a global minimizer of $\mathcal{E}_{\widetilde{\mathcal{O}}_R}$ and $\overline{w} = v$ on $\partial\widetilde{\mathcal{O}}_R \cap \{\lambda > 0\}$, then

$$\begin{aligned} &\int_{\widetilde{\mathcal{O}}_R} \frac{1}{2}|\nabla v|^2 dx d\lambda + \int_{\mathcal{O}_R} G(u) dx \\ &\qquad \le \int_{\widetilde{\mathcal{O}}_R} \frac{1}{2}|\nabla \overline{w}|^2 dx d\lambda + \int_{\mathcal{O}_R} G(\overline{w}(x, 0)) dx. \end{aligned}$$

We establish now the bound (3.6.4) for the energy $\mathcal{E}_{\widetilde{\mathcal{O}}_R}(\overline{w})$ of $\overline{w}$.

Observe that the potential energy of $\overline{w}$ is bounded by CR^{2m-1}, indeed

$$\begin{aligned} \int_{\mathcal{O}_R} G(\overline{w}(x, 0)) dx &\le C\left|\mathcal{O}_{R-1} \cap \left\{\frac{s-t}{\sqrt{2} < 1}\right\}\right| + C\left|\mathcal{O}_R \setminus \mathcal{O}_{R-1}\right| \\ &\le C\int_0^{R-1} \{(t + \sqrt{2})^m - t^m\} t^{m-1} dt + CR^{2m-1} \le CR^{2m-1}. \end{aligned}$$

Next, we bound the Dirichlet energy of $\overline{w}$. First of all, as in the proof of the energy estimate for global minimizers, we rescale and set

$$w_1(x,\lambda) = \overline{w}(Rx, R\lambda) \quad \text{for every} \ (x,\lambda) \in \widetilde{\mathcal{O}}_1.$$

Thus, the Dirichlet energy of $\overline{w}$ in $\widetilde{\mathcal{O}}_R$, satisfies

$$\int_{\widetilde{\mathcal{O}}_R} \frac{1}{2}|\nabla \overline{w}|^2 = CR^{n-1} \int_{\widetilde{\mathcal{O}}_1} \frac{1}{2}|\nabla w_1|^2.$$

Setting $\varepsilon = 1/R$, we need to prove that

$$\int_{\widetilde{\mathcal{O}}_1} \frac{1}{2}|\nabla w_1|^2 \leq C|\log \varepsilon|. \tag{3.6.7}$$

Set $s = |(x_1, ..., x_m)|$ and $t = |(x_{m+1}, ..., x_{2m})|$, for every $x = (x_1, ..., x_{2m}) \in \mathcal{O}_1$. We observe that

$$\begin{aligned}
&\int_{\widetilde{\mathcal{O}}_1} \frac{1}{2}|\nabla w_1|^2 dx d\lambda = \\
&\quad = C \int_0^1 d\lambda \int_{\{s^2+t^2<1, s>t\geq 0\}} \left\{(\partial_s w_1)^2 + (\partial_t w_1)^2 + (\partial_\lambda w_1)^2\right\} s^{m-1} t^{m-1} ds dt \\
&\quad \leq C \int_0^1 d\lambda \int_{\{s^2+t^2<1, s>t\geq 0\}} \left\{(\partial_s w_1)^2 + (\partial_t w_1)^2 + (\partial_\lambda w_1)^2\right\} ds dt.
\end{aligned}$$

We can see the last integral as an integral in the set

$$\{(s,t,\lambda) \in \mathbb{R}^3 : s^2 + t^2 < 1, s > t \geq 0, 0 < \lambda < 1\} \subset \mathbb{R}^3.$$

We consider now w_2 the even reflection of w_1 with respect to $\{t = 0\}$. We set

$$\begin{cases} s = z_1 \\ t = |z_2|, \end{cases}$$

and we define $w_2(z,\lambda) = w_2(z_1, z_2, \lambda) =: w_1(s,t,\lambda)$ in the set

$$\Omega = \{(z_1, z_2, \lambda) : z_1^2 + z_2^2 < 1, z_1 > |z_2|, 0 < \lambda < 1\} \subset \mathbb{R}^3.$$

We have that

$$\begin{aligned}
&\int_0^1 d\lambda \int_{\{s^2+t^2<1, s>t>0\}} \left\{(\partial_s w_1)^2 + (\partial_t w_1)^2 + (\partial_\lambda w_1)^2\right\} ds dt \\
&\qquad \leq \int_0^1 d\lambda \int_{\{z_1^2+z_2^2<1, z_1>|z_2|\}} |\nabla w_2|^2 dz_1 dz_2.
\end{aligned}$$

Next we apply Proposition 3.2.1 and Theorem 3.0.7 to the function w_2 in Ω. Observe that Ω is Lipschitz as a subset of $\mathbb{R}^3$, but it is not Lipschitz if seen as a subset of $\mathbb{R}^{2m+1}$. Since w_2 is harmonic in Ω, Proposition 3.2.1 gives that

$$\int_{\Omega} |\nabla w_2|^2 dz_1 dz_2 d\lambda \leq C||w_2||^2_{H^{1/2}(\partial\Omega)}.$$

To bound the quantity $||w_2||_{H^{1/2}(\partial\Omega)}$, we apply Theorem 3.0.7 with $A = \partial\Omega$ and

$$\Gamma = \left(\left\{z_1^2 + z_2^2 < 1, z_1 = |z_2|\right\} \times \{\lambda = 0\}\right) \cup \left(\left\{z_1^2 + z_2^2 = 1, z_1 > |z_2|\right\} \times \{\lambda = 0\}\right).$$

Since $|w_2| \leq 1$, we need only to check (3.0.14) in $\partial\Omega$. By the definition of w_2, we have that $w_2(z, 0) \equiv 1$ if $\operatorname{dist}(z, \Gamma) < \varepsilon$ and

$$|\nabla w_2(z_1, z_2, 0)| = |\nabla w_1(s, t, 0)| = \varepsilon^{-1}|\nabla g(Rx, 0)| \leq C\varepsilon^{-1} = C \min\{\varepsilon^{-1}, (\operatorname{dist}(z, \Gamma))^{-1}\}.$$

Moreover, as in the proof of Theorem 3.0.3, to verify (3.0.14) in $\partial\Omega \cap \{\lambda > 0\}$ we use that $\overline{w} \equiv v$ here and the gradient bound (3.0.16) for v. Thus,

$$|\nabla w_2(z_1, z_2, \lambda)| \leq \frac{CR}{1 + R\lambda} = \frac{C}{\varepsilon + \lambda} \leq C \min\left\{\frac{1}{\varepsilon}, \frac{1}{\lambda}\right\}.$$

Hence, w_2 satisfies the hypothesis of Theorem 3.0.7 and we conclude that (3.6.7) holds. □

Chapter 4

Energy estimates for equations with fractional diffusion

4.1 Introduction and results

In this chapter (which corresponds to [7]) we establish energy estimates for some solutions of the fractional nonlinear equation

$$(-\Delta)^s u = f(u) \quad \text{in } \mathbb{R}^n, \tag{4.1.1}$$

for every $0 < s < 1$, where $f : \mathbb{R} \to \mathbb{R}$ is a $C^{1,\beta}$ function for some $\beta > \max(0, 1-2s)$.

In Chapter 3, we considered the case $s = 1/2$ and established sharp energy estimates for global minimizers in every dimension n, and for monotone solutions in dimension $n = 3$. As a consequence, we deduced one-dimensional (or 1-D) symmetry for these types of solutions in dimension $n = 3$.

This result about 1-D symmetry is the analog of a conjecture of De Giorgi for the Allen-Cahn equation $-\Delta u = u - u^3$ in $\mathbb{R}^n$. More precisely, in 1978 De Giorgi conjectured that the level sets of every bounded, monotone in one direction solution of the Allen-Cahn equation must be hyperplanes, at least if $n \leq 8$. That is, such solutions depend only on one Euclidian variable. The conjecture has been proven to be true in dimension $n = 2$ by Ghoussoub and Gui [24] and in dimension $n = 3$ by Ambrosio and Cabré [3]. For $4 \leq n \leq 8$, if $\partial_{x_n} u > 0$, and assuming the additional condition

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1 \quad \text{for all } x' \in \mathbb{R}^{n-1},$$

it has been established by Savin [34]. Recently a counterexample to the conjecture for $n \geq 9$ has been announced by Del Pino, Kowalczyk and Wei [22].

When $f(u) = u - u^3$ in our equation (4.1.1), we will call it of Allen-Cahn type.

In this chapter (see Theorem 4.1.4 below), we establish one-dimensional symmetry for global minimizers and for bounded monotone solutions of (4.1.1) in dimension $n = 3$, for every $1/2 < s < 1$. This is the analog of the conjecture of De Giorgi for the operator $(-\Delta)^s$, for $1/2 < s < 1$. We recall that in [28] Maria del Mar Gonzalez prove that an energy functional related to fractional powers s of the Laplacian for $1/2 < s < 1$ Γ-converges to the classical perimeter functional. The same result for $s = 1/2$ was proven by Alberti, Bouchitté, and Seppecher in [2]. Moreover, in [16] Caffarelli and Souganidis prove that scaled threshold dynamics-type algorithms corresponding to fractional Laplacians converge to moving fronts. More precisely, when $1/2 \leq s < 1$ the resulting interface moves by weighted mean curvature, while for $0 < s < 1/2$ the normal velocity is nonlocal of fractional-type.

As said before, in dimension $n = 3$, the same result for $s = 1/2$ has been proven in Chapter 3. We recall that for $s = 1/2$, one-dimensional symmetry for stable solutions of (4.1.1) in dimension $n = 2$ has been proved by Cabré and Solà-Morales [10]. The same result in dimension $n = 2$ for every fractional power $0 < s < 1$ has been established by Cabré and Sire [9] and by Sire and Valdinoci [37].

As in Chapter 3, a crucial ingredient in the proof of 1-D symmetry is a sharp energy estimate for global minimizers. This is Theorem 4.1.2.

To study the nonlocal problem (4.1.1), we will realize it as a local problem in $\mathbb{R}^{n+1}_+$ with a nonlinear Neumann condition. More precisely, Caffarelli and Silvestre [15] proved that u is a solution of problem (4.1.1) in $\mathbb{R}^n$ if and only if v, defined on $\mathbb{R}^{n+1}_+ = \{(x,\lambda) : x \in \mathbb{R}^n,\ \lambda > 0\}$, is a solution of the problem

$$\begin{cases} \operatorname{div}(\lambda^{1-2s}\nabla v) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ -\lim_{\lambda\to 0} \lambda^{1-2s}\partial_\lambda v = d_{n,s} f(v) \end{cases} \tag{4.1.2}$$

where $d_{n,s} > 0$ is a positive constant depending only on n and s and $v(x,0) = u(x)$ on $\mathbb{R}^n = \partial\mathbb{R}^{n+1}_+$. Later we will study problem (4.1.2) for $d_{n,s} = 1$. In the sequel we will call the extension v of u in $\mathbb{R}^{n+1}_+$ satisfying $\operatorname{div}(\lambda^{1-2s}\nabla v) = 0$ the *s-extension* of u.

Observe that for every $0 < s < 1$, we have that $-1 < 1 - 2s < 1$ and thus the weight λ^{1-2s} which appears in (4.1.2) belongs to the Muckenoupt class A_2. As a consequence the theory developed by Fabes, Kenig and Serapioni [23] applies to problem (4.1.2) and thus a Poincaré inequality, a Harnack inequality, and Hölder regularity hold for solutions of our problem. There are some gradient estimates for solutions of problem (4.1.2) (see Remark 4.1.9) that will be important in the proof of our energy estimates.

Problem (4.1.2) associated to the nonlocal equation (4.1.1) allows to introduce the notions of *energy* and *global minimality* for a solution u of problem (4.1.1).

Consider the cylinder

$$C_R = B_R \times (0, R) \subset \mathbb{R}^{n+1}_+,$$

where B_R is the ball of radius R centerd at 0 in $\mathbb{R}^n$, and the energy functional

$$\mathcal{E}_{s,C_R}(v) = \int_{C_R} \frac{1}{2}\lambda^{1-2s}|\nabla v|^2 dx d\lambda + \int_{B_R} G(v(x,0))dx, \tag{4.1.3}$$

where $G' = -f$.

Definition 4.1.1. a) We say that a bounded $C^1(\overline{\mathbb{R}^{n+1}_+})$ function v is a *global minimizer* of (4.1.2) if, for all $R > 0$,

$$\mathcal{E}_{s,C_R}(v) \le \mathcal{E}_{s,C_R}(w),$$

for every $C^1(\overline{\mathbb{R}^{n+1}_+})$ function w such that $v \equiv w$ in $\mathbb{R}^{n+1}_+ \setminus \overline{C_R}$.

b) We say that a bounded C^1 function u in $\mathbb{R}^n$ is a *global minimizer* of (4.1.1) if its s-extension v is a global minimizer of (4.1.2).

c) We say that a bounded function u is a *layer solution* for the problem (4.1.1) if u is monotone increasing in one of the x-variables, say $\partial_{x_n} u > 0$ in $\mathbb{R}^n$, and

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1 \quad \text{for every } x' \in \mathbb{R}^{n-1}. \tag{4.1.4}$$

We remind that every layer solution is a global minimizer (see [9]). In this respect, one uses that the weight λ^{1-2s} does not depend on the horizontal variables $x_1, ..., x_n$, hence problem (4.1.2) is invariant under translations in the directions $x_1, ..., x_n$.

Our main result is the following energy estimate for global minimizers of problem (4.1.1). Given a bounded function u defined on $\mathbb{R}^n$, set

$$G(u) = \int_u^1 f \quad \text{and}$$

$$c_u = \min\{G(s) : \inf_{\mathbb{R}^n} u \le s \le \sup_{\mathbb{R}^n} u\}. \tag{4.1.5}$$

Theorem 4.1.2. *Let f be any $C^{1,\beta}$ nonlinearity, with $\beta > \max\{0, 1-2s\}$, and $u : \mathbb{R}^n \to \mathbb{R}$ be a global minimizer of (4.1.1). Let v be the s-extension of u in $\mathbb{R}^{n+1}_+$. Then, for all $R > 2$,*

$$\begin{aligned} \int_{C_R} \frac{1}{2}\lambda^{1-2s}|\nabla v|^2 dx d\lambda + \int_{B_R} \{G(u) - c_u\}dx &\le CR^{n-2s} \quad \text{if } 0 < s < 1/2 \\ \int_{C_R} \frac{1}{2}\lambda^{1-2s}|\nabla v|^2 dx d\lambda + \int_{B_R} \{G(u) - c_u\}dx &\le CR^{n-1} \quad \text{if } 1/2 < s < 1, \end{aligned} \tag{4.1.6}$$

where c_u is defined by (4.1.5), and C denotes different positive constants depending only on n, $||f||_{C_1}$, $||u||_{L^\infty(\mathbb{R}^n)}$ and s.

Here by $||f||_{C^{1,\beta}}$ we mean $||f||_{C^{1,\beta}([\inf_{\mathbb{R}^n} u, \sup_{\mathbb{R}^n} u])}$. As a consequence (4.1.6) also holds for layer solutions.

Moreover, we will prove that, in dimension $n = 3$, the energy estimate (4.1.6) holds also for monotone solutions without the limit assumption (4.1.4). These solutions are minimizers among a certain class of functions, but they are not, in general, global minimizers as defined before.

Theorem 4.1.3. *Let $n = 3$, f be any $C^{1,\beta}$ nonlinearity with $\beta \in (0,1)$ and u be a bounded solution of (4.1.1) such that $\partial_e u > 0$ in $\mathbb{R}^3$ for some direction $e \in \mathbb{R}^3$, $|e| = 1$. Let v be the s-extension of u in $\mathbb{R}^4_+$.*

Then, for all $R > 2$,

$$\begin{array}{ll} \displaystyle\int_{C_R} \frac{1}{2}\lambda^{1-2s}|\nabla v|^2 dx d\lambda + \int_{B_R} \{G(u) - c_u\} dx \le C R^{3-2s} & \text{if } 0 < s < 1/2 \\ \displaystyle\int_{C_R} \frac{1}{2}\lambda^{1-2s}|\nabla v|^2 dx d\lambda + \int_{B_R} \{G(u) - c_u\} dx \le C R^{2} & \text{if } 1/2 < s < 1, \end{array} \tag{4.1.7}$$

where c_u is defined by (4.1.5), and C denotes different positive constants depending only on $||u||_{L^\infty(\mathbb{R}^3)}$, $||f||_{C^{1,\beta}}$, and s.

In dimension $n = 3$, for every $1/2 < s < 1$, Theorems 4.1.2 and 4.1.3 lead to the 1-D symmetry of global minimizers and of bounded monotone solutions of problem (4.1.1). For $s = 1/2$ this was proved in Chapter 3.

Theorem 4.1.4. *Suppose $n = 3$ and $1/2 \le s < 1$. Let f be any $C^{1,\beta}$ nonlinearity with $\beta > \max\{0, 1-2s\}$ and u be either a bounded global minimizer of (4.1.1), or a bounded solution monotone in some direction $e \in \mathbb{R}^3$, $|e| = 1$.*

Then, u depends only on one variable, i.e., there exists $a \in \mathbb{R}^3$ and $g : \mathbb{R} \to \mathbb{R}$, such that $u(x) = g(a \cdot x)$ for all $x \in \mathbb{R}^3$. Equivalently, the level sets of u are planes.

Remark 4.1.5. In [14] Caffarelli, Roquejoffre, and Savin develop a regularity theory for nonlocal minimal surfaces. This surfaces can be interpreted as a non-infinitesimal version of classical minimal surfaces and can be attained by minimizing the H^s-norm of the indicator function when $0 < s < 1/2$. A crucial fact here is that when $0 < s < 1/2$ the indicator functions belong to the space H^s and the extension problem (1.3.2) is a well posed problem for indicator functions. The authors also prove a sharp energy estimate CR^{n-2s} related to ours in some sense: our equation is the Allen-Cahn approximation of these nonlocal minimal surfaces. The flatness of these nonlocal minimal surfaces is an open problem even in dimension $n = 2$ (while the 1-D symmetry property of monotone solution of the semilinear problem is known in dimension $n = 2$ for every s). Recently, Caffarelli and Valdinoci [17] have proven that nonlocal minimal cones converge to standard minimal surfaces when s tends to $1/2$. From this, they obtain that all the nonlocal minimal cones are flat and all the nonlocal minimal surfaces are smooth when the dimension n is lower or equal than 7 and s is closed to $1/2$.

To prove 1-D symmetry, we use a Liouville type argument, which requires an appropriate estimate for the Dirichlet energy. By a result of Moschini [31], the energy estimate

$$\int_{C_R} |\nabla v|^2 dx d\lambda \leq CR^2 \log R,$$

allows to use such Liouville type result and deduce 1-D symmetry in $\mathbb{R}^3$ for global minimizers and for solutions monotone in one direction. By Theorems 4.1.2 and 4.1.3, we have that for every $1/2 < s < 1$ and for $n = 3$,

$$\int_{C_R} \frac{1}{2} \lambda^{1-2s} |\nabla v|^2 dx d\lambda \leq CR^2.$$

If $0 < s < 1/2$, our estimates lead to

$$\int_{C_R} \frac{1}{2} \lambda^{1-2s} |\nabla v|^2 dx d\lambda \leq CR^{3-2s}.$$

Since $3 - 2s > 2$ when $0 < s < 1/2$, then we cannot use this Liouville argument. This is the reason why we can prove 1-D symmetry only for $1/2 \leq s < 1$.

We have two different proofs of our energy estimates (4.1.6).

The first one is very simple but applies only to Allen-Cahn type nonlinearities (such as $f(u) = u - u^3$) and to monotone solutions satisfying the limit assumption (4.1.4) or the more general (4.2.3) below. We present this very simple proof in section 2. It was found by Ambrosio and Cabré [3] to prove the optimal energy estimate for $-\Delta u = u - u^3$ in $\mathbb{R}^n$.

Our second proof applies in more general situations and will lead to Theorems 4.1.2 and 4.1.3. It is based on controlling a weighted $H^1(\Omega)$-norm of a function by some fractional Sobolev norms on the boundary.

Let us recall now the definition of the $H^s(A)$-norm of a function, for $0 < s < 1$, where A is either a Lipschitz open set of $\mathbb{R}^n$, or $A = \partial\Omega$ and Ω is a bounded Lipschitz open set of $\mathbb{R}^{n+1}$. It is given by

$$||w||^2_{H^s(A)} = ||w||^2_{L^2(A)} + \int_A \int_A \frac{|w(z) - w(\overline{z})|^2}{|z - \overline{z}|^{n+2s}} d\sigma_z d\sigma_{\overline{z}}.$$

In the sequel we will use it for $\Omega = C_1 = B_1 \times (0,1) \subset \mathbb{R}^{n+1}$ and $A = \partial C_1$.

To prove Theorem 4.1.2, we use the following comparison argument. We construct a comparison function $\overline{w}$ which takes the same values of v on $\partial C_R \cap \{\lambda > 0\}$ and thus, by minimality of v,

$$\mathcal{E}_{C_R}(v) \leq \mathcal{E}_{C_R}(\overline{w}).$$

Then, it is enough to estimate the energy of $\overline{w}$.

For simplicity consider the case of the Allen-Cahn type equation. We define the function $\overline{w}(x,\lambda)$ in C_R in the following way. First we define $\overline{w}(x,0)$ on the base of the cylinder as a smooth function $g(x)$ which is identically equal to 1 in B_{R-1} and $g(x)=v(x,0)$ for $|x|=R$; then we define $\overline{w}(x,\lambda)$ as the unique solution of the Dirichlet problem

$$\begin{cases} \operatorname{div}(\lambda^{1-2s}\nabla\overline{w})=0 & \text{in } C_R \\ \overline{w}(x,0)=g(x) & \text{on } B_R\times\{\lambda=0\} \\ \overline{w}(x,\lambda)=v(x,\lambda) & \text{on } \partial C_R\cap\{\lambda>0\}. \end{cases} \tag{4.1.8}$$

Since by definition $\overline{w}\equiv 1$ on $B_{R-1}\times\{0\}$, then the potential energy is bounded by CR^{n-1}. Thus it remains to estimate the Dirichlet energy.

To do this we proceed in two steps. First, after rescaling, we apply Theorem 4.1.6 below, to control the Dirichlet norm of $\overline{w}$ in C_1 by some fractional Sobolev norms of its trace on ∂C_1. Then, we use Theorem 4.1.8 below to give an estimate of these fractional norms.

More precisely, we recall that in the proof of the estimate for the Dirichlet energy for $s=1/2$ a crucial point was an extension theorem which let us to control the $H^1(\Omega)$-norm of a function with the $H^{1/2}(\partial\Omega)$-norm of its trace. Here we are in a more complicated situation, since we need to control a weighted $H^1(\Omega)$-norm, with a weight which degenerates on a subset of $\partial\Omega$.

We consider a bounded subset Ω of $\mathbb{R}^{n+1}$ with Lipschitz boundary $\partial\Omega$, and M a Lipschitz subset of $\partial\Omega$. For every $z\in\Omega$, we denote $d_M(z)$ the distance from the point z to the set M. Set

$$a := 1-2s \in (-1,1).$$

Here we want to control the $H^1_{d_M^a}$-norm of a function $\widetilde{w}$ defined in Ω, with some weighted fractional Sobolev norm of its trace. Observe that the weight d_M^a vanishes only on a subset M of the boundary $\partial\Omega$. Later we will consider $\Omega=C_1$, $A=\partial C_1$ and $M=B_1\times\{0\}$. In this case we have $d_M(x,\lambda)=\lambda$.

In Theorem 4.1.6 we establish that, given a function w defined on all $\partial\Omega$, then there exists an extension $\widetilde{w}$ of w in Ω, which $H^1_{d_M^a}$-norm is controlled by a combination of a H^s-norm and a $H^{1/2}_{d_M^a}$-norm of its trace w. If ω is a weight, we indicate with $H^s_\omega(\partial\Omega)$ the weighted Sobolev space of functions f such that

$$\int_{\partial\Omega}\omega\, f^2<\infty \quad \text{and}$$

$$\int_{\partial\Omega}\int_{\partial\Omega}\omega(z)\,\frac{|f(z)-f(\overline{z})|^2}{|z-\overline{z}|^{n+2s}}d\sigma_z d\sigma_{\overline{z}}<+\infty.$$

We fix some notations. Let A be either a Lipschitz domain in $\mathbb{R}^n$ or $A = \partial\Omega$ where Ω is a bounded subset of $\mathbb{R}^{n+1}$ with Lipschitz boundary. Let $M \subset A$ be an open set (relative to A) with Lipschitz boundary (relative to A) $\Gamma = \partial M$.

We define the following two sets:

$$B_s = \begin{cases} A \times A & \text{if } 0 < s < 1/2 \\ M \times M & \text{if } 1/2 < s < 1, \end{cases} \tag{4.1.9}$$

and

$$B_w = \begin{cases} (A \setminus M) \times (A \setminus M) & \text{if } 0 < s < 1/2 \\ (A \setminus M) \times A & \text{if } 1/2 < s < 1. \end{cases} \tag{4.1.10}$$

The following is the extension theorem that we will use to prove our energy estimates.

Theorem 4.1.6. *Let Ω be a bounded subset of $\mathbb{R}^{n+1}$ with Lipschitz boundary $\partial\Omega$ and M a Lipschitz subset of $\partial\Omega$. For $z \in \mathbb{R}^{n+1}$, let $d_M(z)$ denote the Euclidean distance from the point z to the set M. Let w belong to $C(\partial\Omega)$.*

Then, there exists an extension $\widetilde{w}$ of w in Ω belonging to $C^1(\Omega) \cap C(\overline{\Omega})$, such that

$$\begin{aligned} \int_\Omega d_M(z)^{1-2s} |\nabla \widetilde{w}|^2 dz \le C||w||^2_{L^2(\partial\Omega)} + C \int\int_{B_s} \frac{|w(z) - w(\overline{z})|^2}{|z - \overline{z}|^{n+2s}} d\sigma_z d\sigma_{\overline{z}} \\ + C \int\int_{B_w} d_M(z)^{1-2s} \frac{|w(z) - w(\overline{z})|^2}{|z - \overline{z}|^{n+1}} d\sigma_z d\sigma_{\overline{z}}, \end{aligned} \tag{4.1.11}$$

where B_s and B_w are defined, respectively, in (4.1.9) *and* (4.1.10) *with $A = \partial\Omega$, and C denotes a positive constant depending on Ω, M and s.*

We have used the notations B_s and B_w to indicate, respectively, the set in which we compute the H^s-norm of w and the set in which we compute the weighted $H^{1/2}_{d_M^a}$-norm of w.

Remark 4.1.7. We denote by $\overline{w}$ the s-extension of w in Ω. Since $\overline{w}$ is the extension of w in Ω which minimizes the quantity

$$\int_\Omega d_M(z)^{1-2s} |\nabla \widetilde{w}|^2,$$

then inequality (4.1.11) holds, in particular, with $\widetilde{w}$ replaced by $\overline{w}$.

In two articles [32, 33], Nekvinda treated some extension and trace problems for functions belonging to fractional Sobolev spaces, but his results are not applicable to our situation.

In [33], the author proved an extension theorem for functions belonging to $H^s(M)$, where M is, as before, a subset of $\partial\Omega$. More precisely he proved that if $w \in H^s(M)$ then there exists an extension $\widetilde{w}$ of w in Ω such that

$$\int_\Omega d_M(z)^{1-2s}|\nabla \widetilde{w}|^2 \leq C||w||_{H^s(M)}.$$

In [32], he considered the case of a function w defined on $\partial\Omega \setminus M$ and established that there exists an extension $\widetilde{w}$ of w in Ω which $H^1_{d^a_M}$-norm is controlled by some weighted fractional norm of w in $\partial\Omega \setminus M$.

Here we need an extension result to all of Ω for functions w defined in $\partial\Omega$, and thus we cannot apply the results of Nekvinda.

We conclude giving the key result in the proof of Theorem 4.1.2.

Theorem 4.1.8. *Let A be either a Lipschitz domain in $\mathbb{R}^n$ or $A = \partial\Omega$ where Ω is a bounded subset of $\mathbb{R}^{n+1}$ with Lipschitz boundary. Let $M \subset A$ be an open set (relative to A) with Lipschitz boundary (relative o A) $\Gamma = \partial M$. Let $\varepsilon \in (0, 1/2)$. Let $w : A \to \mathbb{R}$ be a Lipschitz function such that for almost every $z \in A$,*

$$|w(z)| \leq c_0 \tag{4.1.12}$$

and

$$|Dw(z)| \leq \frac{c_0}{d_\Gamma(z)} \min\left\{1, \left(\frac{d_\Gamma(z)}{\varepsilon}\right)^{\min\{1,2s\}}\right\} \tag{4.1.13}$$

where D are all tangential derivatives to A, $d_\Gamma(z)$ is the Euclidean distance from the point z to the set Γ (either in $\mathbb{R}^n$ or in $\mathbb{R}^{n+1}$), and c_0 is a positive constant.

Then,

$$\begin{aligned} ||w||_{L^2(A)} + \int\int_{B_s} \frac{|w(z) - w(\overline{z})|^2}{|z - \overline{z}|^{n+2s}} d\sigma_z d\sigma_{\overline{z}} + \int\int_{B_w} d_M(z)^{1-2s} \frac{|w(z) - w(\overline{z})|^2}{|z - \overline{z}|^{n+1}} d\sigma_z d\sigma_{\overline{z}} \\ \leq \begin{cases} Cc_0^2 & \text{if } 0 < s < 1/2, \\ Cc_0^2 \varepsilon^{1-2s} & \text{if } 1/2 < s < 1. \end{cases} \end{aligned} \tag{4.1.14}$$

where C denotes a positive constant depending only on A, M, and s and the sets B_s and B_w are defined in (4.1.9) *and* (4.1.10).

Later we will use this result for $A = \partial C_1$, $M = B_1 \times \{\lambda = 0\}$ and $\Gamma = \partial B_1 \times \{\lambda = 0\}$. Thus in this case the constant C that appears in (4.1.14) only depends on the dimension n and the power s.

Remark 4.1.9. In the proof of Theorem 4.1.8 the following gradient estimates for every bounded solution v of problem (4.1.2) will be of utmost importance. Let $f \in C^{1,\beta}$ for some $\beta > \max\{0, 1-2s\}$, then every bounded solution v of (4.1.2) satisfies

$$\begin{cases} |\nabla_x v(x,\lambda)| \leq \dfrac{C}{1+\lambda} & \text{for every } \ x \in \mathbb{R}^n \ \text{ and } \ \lambda \geq 0 \\ |\partial_\lambda v(x,\lambda)| \leq \dfrac{C}{\lambda} & \text{for every } \ x \in \mathbb{R}^n \ \text{ and } \ \lambda > 1 \\ |\lambda^{1-2s}\partial_\lambda v| \leq C & \text{for every } \ x \in \mathbb{R}^n \ \text{ and } \ 0 \leq \lambda < 1. \end{cases} \tag{4.1.15}$$

The bound $|\nabla_x v(x,0)| \leq C$ for every $x \in \mathbb{R}^n$ has been proven by Silvestre (see Lemmas 2.8 and 2.9 in [36]). Using the maximum principle we can extend this bound for every $\lambda > 0$ and deduce $|\nabla_x v(x,\lambda)| \leq C$ for every $x \in \mathbb{R}^n$ and $\lambda \geq 0$. The bound $|\nabla v(x,\lambda)| \leq C/\lambda$ for every $x \in \mathbb{R}^n$ and $\lambda > 1$ follows, after rescaling, by interior elliptic estimates, since equation (4.1.2) is uniformly elliptic for $\lambda > 1$. Finally, the last bound $|\lambda^{1-2s}\partial_\lambda v(x,\lambda)| \leq C$ for every $x \in \mathbb{R}^n$ and $0 \leq \lambda < 1$ is established by Cabré and Sire in [8], using that the function $\omega = \lambda^{1-2s}\partial_\lambda v$ satisfies the dual problem (with Dirichlet boundary condition)

$$\begin{cases} \operatorname{div}(\lambda^{1-2s}\nabla\omega) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ \omega = f & \text{on } \partial\mathbb{R}^{n+1}_+ \end{cases}.$$

The chapter is organized as follows:

- In section 2 we prove the energy estimate for layer solutions of the Allen-Cahn type equation, using a simple argument introduced by Ambrosio and Cabré [3].

- In section 3 we give the proof of the extension Theorem 4.1.6 and of the key Theorem 4.1.8.

- In section 4 we prove energy estimate (4.1.6) for global minimizers and for every nonlinearity f.

- In section 5 we establish energy estimates for bounded monotone solutions in $\mathbb{R}^3$.

- In section 6 we prove the 1-D symmetry result, that is Theorem 4.1.4.

4.2 Energy estimate for monotone solutions of Allen-Cahn type equations

In this section we consider potentials $G(u) = \int_u^1 f$ satisfying the following hypothesis:

$$G \geq 0 = G(\pm 1) \quad \text{in } \mathbb{R} \quad \text{and} \;\; G > 0 \quad \text{in } (-1,1). \tag{4.2.1}$$

An example is $G(u) = \frac{1}{4}(1-u^2)^2$. In this case the nonlinearity is given by $f(u) = u - u^3$.

In the sequel we consider the energy

$$\mathcal{E}_{s,C_R}(v) = \int_{C_R} \frac{1}{2}\lambda^{1-2s}|\nabla v|^2 dx d\lambda + \int_{B_R} G(v)dx.$$

In the following theorem we establish energy estimates for monotone solutions of (4.1.1) such that $\lim_{x_n \to +\infty} u(x', x_n) = 1$, in the case in which the potential G satisfies (4.2.1). Recall that we have defined the cylinder $C_R = B_R \times (0,R)$, where B_R is the ball of radius R in $\mathbb{R}^n$.

Theorem 4.2.1. *Let f be a $C^{1,\beta}$ function, with $\beta > \max\{0, 1-2s\}$. Suppose that $G(u) = \int_u^1 f$ satisfies (4.2.1). Let u be a solution of problem (4.1.1) in $\mathbb{R}^n$, with $|u| < 1$, and let v be the s-extension of u in $\mathbb{R}^{n+1}_+$. Assume that*

$$u_{x_n} > 0 \;\; \text{in } \mathbb{R}^n \tag{4.2.2}$$

and

$$\lim_{x_n \to +\infty} u(x', x_n) = 1 \;\; \text{for all } x' \in \mathbb{R}^{n-1}. \tag{4.2.3}$$

Then, for every $R > 2$,

$$\mathcal{E}_{s,C_R}(v) \leq CR^{n-2s}, \quad \text{if } 0 < s < 1/2,$$

and

$$\mathcal{E}_{s,C_R}(v) \leq CR^{n-1}, \quad \text{if } 1/2 < s < 1,$$

for some constants C depending only on n, $||u||_{L^\infty(\mathbb{R}^n)}$, $||f||_{C^1}$ and s.

Proof. As in Chapter 3, the proof uses an argument found by Ambrosio and Cabré [3] to prove an energy estimate for layer solutions of the analog problem $-\Delta u = f(u)$. This method is based on sliding the function v in the direction x_n. Throughout the proof, C will denote different positive constants depending only on n and $||f||_{C^1}$. Consider the function

$$v^t(x,\lambda) := v(x', x_n + t, \lambda)$$

defined for $(x', x_n, \lambda) \in \mathbb{R}^n$ and $t \in \mathbb{R}$. For each t we have

$$\begin{cases} \operatorname{div}(\lambda^{1-2s}\nabla v^t) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ -\lambda^{1-2s}\partial_\lambda v^t = f(v^t) & \text{on } \mathbb{R}^n = \partial\mathbb{R}^{n+1}_+. \end{cases} \tag{4.2.4}$$

Moreover, here we use the gradient estimates (4.1.15) for the solution v of problem (4.1.2) (see [8, 36]). We have that for every t $|v^t| < C$ and

$$|\nabla_x v^t(x,\lambda)| \leq \frac{C}{1+\lambda} \text{ for every } (x,\lambda) \in \mathbb{R}^{n+1}_+ \tag{4.2.5}$$

$$|\partial_\lambda v^t(x,\lambda)| \leq \frac{C}{\lambda} \text{ for every } x \in \mathbb{R}^n \text{ and } \lambda > 1 \tag{4.2.6}$$

$$|\lambda^{1-2s}\partial_\lambda v^t| \leq C \text{ for every } x \in \mathbb{R}^n \text{ and } 0 < \lambda < 1. \tag{4.2.7}$$

In addition (see [8])

$$\lim_{t\to+\infty} \left\{ |v^t(x,\lambda) - 1| + |\nabla v^t(x,\lambda)| \right\} = 0 \tag{4.2.8}$$

for all $x \in \mathbb{R}^n$ and all $\lambda \geq 0$.

Note that, hypothesis (4.2.2) and the maximum principle imply that $v_{x_n} > 0$ in $\mathbb{R}^{n+1}_+$. Thus, denoting the derivative of $v^t(x,\lambda)$ with respect to t by $\partial_t v^t(x,\lambda)$, we have

$$\partial_t v^t(x,\lambda) = v_{x_n}(x', x_n + t, \lambda) > 0 \quad \text{for all } x \in \mathbb{R}^n,\ \lambda \geq 0.$$

By (4.2.8), we have that

$$\lim_{t\to+\infty} \mathcal{E}_{s,C_R}(v^t) = 0.$$

Next, we bound the derivative of $\mathcal{E}_{s,C_R}(v^t)$ with respect to t. Recall that we have set $a = 1-2s$. We use that v^t is a solution of problem (4.1.2), the bounds (4.2.5), (4.2.6), (4.2.7) for v^t and the derivatives of v^t, and the crucial fact that $\partial_t v^t > 0$. Let ν denote the exterior normal to the lateral boundary $\partial B_R \times (0,R)$ of the cylinder C_R.

We have

$$\begin{aligned} \partial_t \mathcal{E}_{s,C_R}(v^t) &= \int_0^R d\lambda \int_{B_R} dx \lambda^a \nabla v^t \cdot \nabla(\partial_t v^t) + \int_{B_R} G'(v^t)\partial_t v^t dx \\ &= \int_0^R d\lambda \int_{\partial B_R} d\sigma \lambda^a \frac{\partial v^t}{\partial \nu}\partial_t v^t + \int_{B_R\times\{\lambda=R\}} \lambda^a \frac{\partial v^t}{\partial\lambda}\partial_t v^t(x,R)dx \\ &\geq -C\int_0^R \frac{\lambda^a}{1+\lambda}\int_{\partial B_R} d\sigma \partial_t v^t - CR^{-2s}\int_{B_R\times\{\lambda=R\}} dx \partial_t v^t(x,R). \end{aligned}$$

Hence, for every $T > 0$, we have

$$\begin{aligned}
\mathcal{E}_{s,C_R}(v) &= \mathcal{E}_{s,C_R}(v^T) - \int_0^T \partial_t \mathcal{E}_{s,C_R}(v^t)dt \\
&\leq \mathcal{E}_{s,C_R}(v^T) + C\int_0^T dt \int_0^R d\lambda \frac{\lambda^a}{1+\lambda}\int_{\partial B_R} d\sigma \partial_t v^t \\
&\quad +CR^{-2s}\int_0^T dt \int_{B_R\times\{\lambda=R\}} dx\partial_t v^t(x,R) \\
&= \mathcal{E}_{s,C_R}(v^T) + C\int_{\partial B_R} d\sigma \int_0^R d\lambda \frac{\lambda^a}{1+\lambda}\int_0^T dt\partial_t v^t(x,\lambda) \\
&\quad +CR^{-2s}\int_{B_R\times\{\lambda=R\}} dx \int_0^T dt\partial_t v^t(x,R) \\
&= \mathcal{E}_{s,C_R}(v^T) + C\int_{\partial B_R} d\sigma \int_0^R d\lambda \frac{\lambda^a}{1+\lambda}(v^T - v^0)(x,\lambda) \\
&\quad +CR^{-2s}\int_{B_R\times\{\lambda=R\}} dx(v^T - v^0)(x,\lambda) \\
&\leq \mathcal{E}_{s,C_R}(v^T) + CR^{n-1} + CR^{n-2s}.
\end{aligned}$$

Letting $T \to +\infty$, we obtain the desired estimates. Indeed, if $0 < s < 1/2$ then $\mathcal{E}_{s,C_R}(v) \leq CR^{n-2s}$, and if $1/2 < s < 1$ then $\mathcal{E}_{s,C_R}(v) \leq CR^{n-1}$. □

4.3 H^s estimate

In this section we recall some definitions and properties of the spaces $H^s(\mathbb{R}^n)$ and $H^s(\partial\Omega)$, where Ω is a bounded subset of $\mathbb{R}^{n+1}$ with Lipschitz boundary $\partial\Omega$ (see [27]).

$H^s(\mathbb{R}^n)$ is the space of functions $u \in L^2(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(\overline{x})|^2}{|x-\overline{x}|^{n+2s}}dxd\overline{x} < +\infty,$$

equipped with the norm

$$||u||_{H^s(\mathbb{R}^n)} = \left(||u||^2_{L^2(\mathbb{R}^n)} + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(\overline{x})|^2}{|x-\overline{x}|^{n+2s}}dxd\overline{x}\right)^{\frac{1}{2}}.$$

As in section 3 of Chapter 3, using a family of charts and a partition of unity, we can define the space $H^s(\partial\Omega)$, where Ω is a bounded subset of $\mathbb{R}^{n+1}$ with Lipschitz boundary.

We use the same notations of Chapter 3.

Consider an atlas $\{(O_j, \varphi_j), j = 1, ..., m\}$ where $\{O_j\}$ is a family of open bounded sets in $\mathbb{R}^{n+1}$ such that $\{O_j \cap \partial\Omega; j = 1, ..., m\}$ cover $\partial\Omega$. The functions φ_j are the corresponding Lipschitz diffeomorphism such that

- $\varphi_j : O_j \to U := \{(y, \mu) \in \mathbb{R}^{n+1} : |y| < 1,\ -1 < \mu < 1\}$,
- $\varphi_j : O_j \cap \Omega \to U^+ := \{(y, \mu) \in \mathbb{R}^{n+1} : |y| < 1,\ 0 < \mu < 1\}$,
- $\varphi_j : O_j \cap \partial\Omega \to \{(y, \mu) \in \mathbb{R}^{n+1} : |y| < 1, \mu = 0\}$,
- in $O_i \cap O_j \neq \emptyset$ the compatibility conditions hold.

Let $\{\alpha_j\}$ be a partition of unity on $\partial\Omega$ such that $\alpha_j \in C_0^\infty(O_j)$, $\sum_{j=1}^m \alpha_j = 1$ in $O_j \cap \partial\Omega$. If u is a function on $\partial\Omega$ decompose $u = \sum_{j=1}^m u\alpha_j$ and define the function

$$(u\alpha_j) \circ \varphi_j^{-1}(y, 0) := (u\alpha_j)(\varphi_j^{-1}(y, 0)), \quad \text{for every } (y, 0) \in U \cap \{\mu = 0\}.$$

Since α_j has compact support in O_j, the function $(u\alpha_j) \circ \varphi_j^{-1}(\cdot, 0)$ has compact support in $U \cap \{\mu = 0\}$ and therefore we may consider $((u\alpha_j) \circ \varphi_j^{-1})(\cdot, 0)$ to be defined in $\mathbb{R}^n$ extending it by zero out of $U \cap \{\mu = 0\}$. Now we define

$$H^s(\partial\Omega) := \{u | (u\alpha_j) \circ \varphi_j^{-1}(\cdot, 0) \in H^s(\mathbb{R}^n),\ j = 1, ..., m\}$$

equipped with the norm

$$\left(\sum_{j=1}^m ||(u\alpha_j) \circ \varphi_j^{-1}(\cdot, 0)||^2_{H^s(\mathbb{R}^n)} \right)^{\frac{1}{2}}.$$

All these norms are independent of the choice of the system of local maps $\{O_j, \varphi_j\}$ and of the partition of unity $\{\alpha_j\}$, and are all equivalent to

$$||u||_{H^s(\partial\Omega)} := \left(||u||^2_{L^2(\partial\Omega)} + \int_{\partial\Omega} \int_{\partial\Omega} \frac{|u(z) - u(\overline{z})|^2}{|z - \overline{z}|^{n+2s}} d\sigma_z d\sigma_{\overline{z}} \right)^{\frac{1}{2}}.$$

We can give now the proof of Theorem 4.1.6.

Proof of Theorem 4.1.6. **Case 1**: $\Omega = \mathbb{R}^{n+1}_+$.

We first consider the case of a half space $\Omega = \mathbb{R}^{n+1}_+$ and $M = \{(x', x_n) \in \mathbb{R}^n : x_n < 0\}$. Let ζ be a bounded function belonging to $C(\mathbb{R}^n)$. Following the first part of the proof of Proposition 3.2.1 in Chapter 3, we consider a C^∞ function $K(x)$, defined on $\mathbb{R}^n$ with compact support in B_1 and such that $\int_{\mathbb{R}^n} K(x)dx = 1$. Define $\widetilde{K}(x, \lambda)$ on $\mathbb{R}^{n+1}_+$ in the following way:

$$\widetilde{K}(x, \lambda) := \frac{1}{\lambda^n} K\left(\frac{x}{\lambda}\right)$$

and finally define the extension $\widetilde{\zeta}$ as

$$\widetilde{\zeta}(x,\lambda) = \int_{\mathbb{R}^n} \widetilde{K}(x-\overline{x},\lambda)\zeta(\overline{x})d\overline{x}. \tag{4.3.1}$$

Note that, since $\int_{\mathbb{R}^n} \widetilde{K}(x,\lambda)dx = 1$, we have

$$||\widetilde{\zeta}(\cdot,\lambda)||_{L^2(\mathbb{R}^n)} \leq ||\zeta||_{L^2(\mathbb{R}^n)} \quad \text{for every } \lambda \geq 0, \tag{4.3.2}$$

and

$$\int_0^1 d\lambda \lambda^{1-2s} \int_{\mathbb{R}^n} dx |\widetilde{\zeta}(x,\lambda)|^2 \leq C||\zeta||_{L^2(\mathbb{R}^n)}. \tag{4.3.3}$$

In chapter 3 a simple calculation leads to the following estimate for the gradient of $\widetilde{\zeta}$

$$|\nabla\widetilde{\zeta}(x,\lambda)|^2 \leq C \int_{\{|x-\overline{x}|<\lambda\}} \frac{|\zeta(x)-\zeta(\overline{x})|^2}{\lambda^{n+2}} d\overline{x}. \tag{4.3.4}$$

If $M = \{(x',x_n) \in \mathbb{R}^n : x_n < 0\}$, then $d_M(x,\lambda) = [(x_n)_+^2 + \lambda^2]^{1/2}$, where as usually, $(x_n)_+ = \max\{x_n, 0\}$.

Consider now, separately, the two cases $0 < s < 1/2$ and $1/2 < s < 1$.

If $0 < s < 1/2$ then $a = 1-2s > 0$ and we have that $d_M^a(x,\lambda) \leq (x_n)_+^a + \lambda^a$. Using (4.3.4), we get

$$\begin{aligned}
\int_{\mathbb{R}^{n+1}_+} d_M^a(x,\lambda)|\nabla\widetilde{\zeta}(x,\lambda)|^2 dx d\lambda &\leq C \int_{\mathbb{R}^{n+1}_+} ((x_n)_+^a + \lambda^a)|\nabla\widetilde{\zeta}(x,\lambda)|^2 dx d\lambda \\
&\leq C \int_0^{+\infty} d\lambda \int\int_{\{|x-\overline{x}|<\lambda\}} dx d\overline{x}\, \frac{(x_n)_+^a + \lambda^a}{\lambda^{n+2}} |\zeta(x)-\zeta(\overline{x})|^2 \\
&\leq C \int_0^{+\infty} d\lambda \int\int_{\{|x-\overline{x}|<\lambda\}} dx d\overline{x}\, \frac{1}{\lambda^{n+2-a}} |\zeta(x)-\zeta(\overline{x})|^2 \\
&\quad + C \int_0^{+\infty} d\lambda \int\int_{\{|x-\overline{x}|<\lambda\}} dx d\overline{x}\, \frac{(x_n)_+^a}{\lambda^{n+2}} |\zeta(x)-\zeta(\overline{x})|^2 \\
&\leq C \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} dx d\overline{x} |\zeta(x)-\zeta(\overline{x})|^2 [\lambda^{-n-1+a}]_{+\infty}^{|x-\overline{x}|} \\
&\quad + C \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} dx d\overline{x}\, (x_n)_+^a |\zeta(x)-\zeta(\overline{x})|^2 [\lambda^{-n-1}]_{+\infty}^{|x-\overline{x}|} \\
&\leq C \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\zeta(x)-\zeta(\overline{x})|^2}{|x-\overline{x}|^{n+2s}} dx d\overline{x} \\
&\quad + C \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (x_n)_+^{1-2s} \frac{|\zeta(x)-\zeta(\overline{x})|^2}{|x-\overline{x}|^{n+1}} dx d\overline{x}.
\end{aligned}$$

Next, we observe that the last integral can be computed only on the set $\{(x,\overline{x}) \in \mathbb{R}^n \times \mathbb{R}^n : |x-\overline{x}| < (x_n)_+/2\}$, which is contained in $(\mathbb{R}^n \setminus M) \times (\mathbb{R}^n \setminus M)$. Indeed

$$\int\int_{\{|x-\overline{x}|\geq \frac{(x_n)_+}{2}\}} (x_n)_+^{1-2s} \frac{|\zeta(x)-\zeta(\overline{x})|^2}{|x-\overline{x}|^{n+1}} dx d\overline{x} \leq \int\int_{\{|x-\overline{x}|\geq \frac{(x_n)_+}{2}\}} \frac{|\zeta(x)-\zeta(\overline{x})|^2}{|x-\overline{x}|^{n+2s}} dx d\overline{x}.$$

Thus, if $0 < s < 1/2$, we have

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}_+} d_M(x,\lambda)^{1-2s} |\nabla\widetilde{\zeta}(x,\lambda)|^2 dx d\lambda \\ &\quad\leq C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\zeta(x)-\zeta(\overline{x})|^2}{|x-\overline{x}|^{n+2s}} dx d\overline{x} + C\int_{\mathbb{R}^n\setminus M}\int_{\mathbb{R}^n\setminus M} d_M(x)^{1-2s}\frac{|\zeta(x)-\zeta(\overline{x})|^2}{|x-\overline{x}|^{n+1}} dx d\overline{x}.\end{aligned}$$

If $1/2 < s < 1$, set

$$b = -a = 2s-1 > 0.$$

In this case we use that $d_M(x,\lambda) \geq \max\{(x_n)_+, \lambda\}$, which implies $d_M^a(x,\lambda) = \frac{1}{d_M^b(x,\lambda)} \leq \frac{1}{(\max\{(x_n)_+,\lambda\})^b}$. In what follows we will use $d_M^a(x,\lambda) \leq 1/\lambda^b$ if $(x_n)_+ = 0$ and $d_M^a(x,\lambda) \leq 1/(x_n)_+^b$ if $(x_n)_+ > 0$. We have

$$\int_{\mathbb{R}^{n+1}_+} d_M^a(x,\lambda)|\nabla\widetilde{\zeta}(x,\lambda)|^2 dx d\lambda \leq \tag{4.3.5}$$

$$\begin{aligned}&\leq C\int_0^{+\infty} d\lambda \int_{\{(x_n)_+=0\}}\int_{\{|x-\overline{x}|<\lambda\}} dx d\overline{x} \frac{|\zeta(x)-\zeta(\overline{x})|^2}{\lambda^{n+2+b}} \\ &\quad + C\int_0^{+\infty} d\lambda \int_{\{(x_n)_+>0\}}\int_{\{|x-\overline{x}|<\lambda\}} dx d\overline{x} \frac{|\zeta(x)-\zeta(\overline{x})|^2}{(x_n)_+^b \lambda^{n+2}} \\ &\leq +C\int_{\{(x_n)_+=0\}} dx \int_{\mathbb{R}^n} d\overline{x} \frac{|\zeta(x)-\zeta(\overline{x})|^2}{|x-\overline{x}|^{n+2s}} \\ &\quad + C\int_{\{(x_n)_+>0\}} dx \int_{\mathbb{R}^n} d\overline{x} \frac{1}{(x_n)_+^b}\frac{|\zeta(x)-\zeta(\overline{x})|^2}{|x-\overline{x}|^{n+1}}\end{aligned}$$

$$\begin{aligned}&\leq C\int_{\{(x_n)_+=0\}} dx \int_{\{(\overline{x}_n)_+=0\}} d\overline{x} \frac{|\zeta(x)-\zeta(\overline{x})|^2}{|x-\overline{x}|^{n+2s}} \\ &\quad + C\int_{\{(x_n)_+>0\}} dx \int_{\mathbb{R}^n} d\overline{x} \frac{1}{(x_n)_+^b}\frac{|\zeta(x)-\zeta(\overline{x})|^2}{|x-\overline{x}|^{n+1}}.\end{aligned} \tag{4.3.6}$$

Observe that the integral in (4.3.6) is computed only on the set $\{(x,\overline{x}) \in \mathbb{R}^n \times \mathbb{R}^n | (x_n)_+ = 0, (\overline{x}_n)_+ = 0\}$. Indeed the set $L := M \times (\mathbb{R}^n \setminus M) = \{(x,\overline{x}) \in \mathbb{R}^n \times \mathbb{R}^n | (x_n)_+ = 0, (\overline{x}_n)_+ > 0\} \subseteq \{(x,\overline{x}) \in \mathbb{R}^n \times \mathbb{R}^n | (\overline{x}_n)_+ \leq |x-\overline{x}|\}$. Then if $(x,\overline{x}) \in L$

$$\frac{1}{|x-\overline{x}|^{n+1+b}} \leq \frac{1}{(\overline{x}_n)_+^b}\cdot\frac{1}{|x-\overline{x}|^{n+1}}$$

and hence we have that

$$\int_{\{(x_n)_+=0\}} dx \int_{\{(x_n)_+>0\}} d\overline{x} \frac{|\zeta(x)-\zeta(\overline{x})|^2}{|x-\overline{x}|^{n+2s}} \leq C \int_{\{(x_n)_+>0\}} dx \int_{\mathbb{R}^n} d\overline{x} \frac{1}{(x_n)_+^b} \frac{|\zeta(x)-\zeta(\overline{x})|^2}{|x-\overline{x}|^{n+1}}.$$

This conclude the proof in the case of the half space.

Case 2: Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded open set with Lipschitz boundary $A = \partial\Omega$, and let $w \in C(\partial\Omega)$.

Let Γ be the boundary (relative to A) of M and let $\widetilde{B}_{r_i} = \widetilde{B}_{r_i}(p_i) \in \mathbb{R}^{n+1}$ be the ball centered at $p_i \in \partial\Omega$ and of radius r_i. We set $A_{r_i} := \widetilde{B}_{r_i} \cap \partial\Omega$. Let Q_1 denote the unite cube in $\mathbb{R}^n$.

Since $\partial\Omega$ is compact, we can consider a finite open covering of $\partial\Omega$

$$\bigcup_{i=1}^{m} A_{r_i} := \bigcup_{i=1}^{m} (\widetilde{B}_{r_i} \cap \partial\Omega)$$

such that for every i there exists a bilipschitz function $\varphi_i : \widetilde{B}_{2r_i} \cap \Omega \to Q_1 \times (0,1)$ which satisfies

$$\varphi_i(A_{2r_i}) = Q_1 \times \{0\}. \tag{4.3.7}$$

Moreover we require that

- if $\Gamma_i = A_{2r_i} \cap \Gamma \neq \emptyset$, then

$$\varphi_i(\Gamma_i) = \{x \in Q_1 : x_n = 0\}; \tag{4.3.8}$$

$$\varphi_i(M \cap A_{2r_i}) = \{x \in Q_1 : x_n < 0\} = Q_1^-; \tag{4.3.9}$$

- if $\overline{A}_{r_i} \cap \Gamma = \emptyset$, then

$$r_i = \frac{1}{3} d_\Gamma(p_i),$$

  where p_i and r_i are respectively the center and the radius of the ball $\widetilde{B}_{r_i}$.

To construct this finite covering of $\partial\Omega$, we first cover Γ with a finite number l of balls $\widetilde{B}_{r_i}$ centered at $p_i \in \Gamma$ and of radius r_i such that there exists a bilipschitz function $\varphi_i : \widetilde{B}_{2r_i} \cap \Omega \to Q_1 \times (-1,1)$, which satisfies (4.3.7), (4.3.8), and (4.3.9).

Then, we consider the compact set

$$\mathcal{K} := \partial\Omega \setminus \bigcup_{i=1}^{l} A_{r_i}$$

and we cover it with a finite number of sets $A_{r_i} = \widetilde{B}_{r_i} \cap \partial\Omega$, where $\widetilde{B}_{r_i}$ are balls centered at $p_i \in \mathcal{K}$ and of radius $r_i = \frac{1}{3} d_\Gamma(p_i)$.

Observe that the number m of sets A_{r_i} which cover $\partial\Omega$, and the Lipschitz constant of φ_i, depend only on $\partial\Omega$ and Γ.

We consider now a partition of unity $\{\alpha_i\}_{i=1,...,m}$ relative to the covering $\{\widetilde{B_{r_i}}\}_{i=1,...,m}$, where $\alpha_i \in C_0^\infty(\widetilde{B_{r_i}})$ and $\sum_{i=1}^m \alpha_i = 1$ on A_{r_i}.

If w is a function defined on $\partial\Omega$, we write

$$w = \sum_{i=1}^m w\alpha_i = \sum_{i=1}^m w_i.$$

Using the bilipschitz map φ_i, we define

$$\zeta_i(y) := w_i(\varphi_i^{-1}(y,0)) \quad \text{for every} \;\; y \in Q_1.$$

Then ζ_i has compact support in Q_1 and can be extend by 0 outside Q_1 in all $\mathbb{R}^n$.

Next, we consider $\widetilde{\zeta_i}$, the extension of ζ in $\mathbb{R}_+^{n+1}$ defined by the convolution in (4.3.1), and we define $\widetilde{w_i}$ the extension of w_i in $\widetilde{B}_{r_i} \cap \Omega$ as follows:

$$\widetilde{w_i}(z) = \alpha_i(z)\widetilde{\zeta_i}(\varphi_i(z)) \quad \text{for every} \;\; z \in \widetilde{B}_{r_i} \cap \Omega.$$

Finally, we set

$$\widetilde{w} = \begin{cases} \sum_{i=1}^m \widetilde{w_i} & \text{in} \;\; \bigcup_{i=1}^m (\widetilde{B}_{r_i} \cap \Omega) \\ 0 & \text{in} \;\; \Omega \setminus \bigcup_{i=1}^m (\widetilde{B}_{r_i} \cap \Omega) \end{cases}$$

Observe that, since φ_j is a bilipschitz map and $\alpha_j \in C_c^\infty(O_j)$ for every $j = 1, ..., m$, we have

$$|\nabla \widetilde{w_i}| \le C \left\{ |\nabla \alpha_j||\widetilde{\zeta_j} \circ \varphi_j| + |\alpha_j||(\nabla \widetilde{\zeta_j}) \circ \varphi_j| \right\},$$

and thus

$$\int_{\widetilde{B}_{r_i}\cap\Omega} d_M^{1-2s}(z)|\nabla \widetilde{w_i}|^2 dz \le C \int_{\widetilde{B}_{r_i}\cap\Omega} d_M^{1-2s}(z)|\widetilde{\zeta_i} \circ \varphi_i|^2 dz + C \int_{\mathbb{R}_+^{n+1}} d_M^{1-2s}(z)|(\nabla \widetilde{\zeta_i}) \circ \varphi_i|^2 dz.$$

Observe that when $0 < s < 1/2$, we have $d_M^{1-2s} \le C$ in Ω, and thus using (4.3.2) we get

$$\int_{\widetilde{B}_{r_i}\cap\Omega} d_M^{1-2s}(z)|\widetilde{\zeta_i} \circ \varphi_i|^2 dz \le \int_{\widetilde{B}_{r_i}\cap\Omega} |\widetilde{\zeta_i} \circ \varphi_i|^2 dz \le ||\zeta||_{L^2(\mathbb{R}^n)} \le C||w||_{L^2(\partial\Omega)}.$$

On the other hand, when $1/2 < s < 1$, we use (4.3.3), to obtain

$$\int_{\widetilde{B}_{r_i}\cap\Omega} d_M^{1-2s}(z)|\widetilde{\zeta_i} \circ \varphi_i|^2 dz \le \int_0^1 \int_{Q_1} \lambda^{1-2s}|\widetilde{\zeta_i}(x,\lambda)|^2 dx d\lambda \le ||\zeta||_{L^2(\mathbb{R}^n)} \le C||w||_{L^2(\partial\Omega)}.$$

Thus

$$\int_{\widetilde{B}_{r_i}\cap\Omega} d_M^{1-2s}(z)|\nabla\widetilde{w}_i|^2dz \le C||w||_{L^2(\partial\Omega)} + C\int_{\mathbb{R}^{n+1}_+} d_M^{1-2s}(z)|(\nabla\widetilde{\zeta}_i)\circ\varphi_i|^2dz. \quad (4.3.10)$$

Claim: (4.1.11) holds with $\widetilde{w}$ and w replaced by $\widetilde{w}_i$ and w_i, which have compact support in $\widetilde{B}_{r_i}\cap\Omega$ and A_{r_i} respectively.

It is enough to prove the claim. Indeed, note first that

$$\int_\Omega d_M(z)^{1-2s}|\nabla\widetilde{w}|^2dz \le C\sum_{i=1}^m\int_{\widetilde{B}_{r_i}\cap\Omega} d_M(z)^{1-2s}|\nabla\widetilde{w}_i|^2dz.$$

Moreover, for every $i=1,...,m$,

$$\int\int_{B_s}\frac{|(w_i)(z)-(w_i)(\overline{z})|^2}{|z-\overline{z}|^{n+2s}}d\sigma_z d\sigma_{\overline{z}} \le C||w||^2_{H^s(\partial\Omega)}.$$

since,

$$\begin{aligned}
&\int\int_{B_s}\frac{|(w\alpha_i)(z)-(w\alpha_i)(\overline{z})|^2}{|z-\overline{z}|^{n+2s}}d\sigma_z d\sigma_{\overline{z}}\\
&=\int\int_{B_s}\frac{|(w\alpha_i)(z)-w(z)\alpha_i(\overline{z})+w(z)\alpha_i(\overline{z})-(w\alpha_i)(\overline{z})|^2}{|z-\overline{z}|^{n+2s}}d\sigma_z d\sigma_{\overline{z}}\\
&\le 2\int\int_{B_s}\frac{|\alpha_i(z)-\alpha_i(\overline{z})|^2|w(z)|^2}{|z-\overline{z}|^{n+2s}}d\sigma_z d\sigma_{\overline{z}}\\
&+2\int\int_{B_s}\frac{|w(z)-w(\overline{z})|^2|\alpha_i(\overline{z})|^2}{|z-\overline{z}|^{n+2s}}d\sigma_z d\sigma_{\overline{z}}\\
&\le C||w||^2_{L^2(\partial\Omega)} + C\int\int_{B_s}\frac{|w(z)-w(\overline{z})|^2}{|z-\overline{z}|^{n+2s}}d\sigma_z d\sigma_{\overline{z}},
\end{aligned}$$

where C denotes different positive constants depending on Ω. To get the bound $C||w||^2_{L^2(\partial\Omega)}$ for the first term, we have used that α_i is Lipschitz, and spherical coordinates centered at z.

In the same way, using that $\int_{B_w} d_M(z)^{1-2s}dz$ is bounded, we get

$$\begin{aligned}
&\int\int_{B_w} d_M(z)^{1-2s}\frac{|(w_i)(z)-(w_i)(\overline{z})|^2}{|z-\overline{z}|^{n+1}}d\sigma_z d\sigma_{\overline{z}}\\
&\qquad\le C||w||^2_{L^2(\partial\Omega)} + \int\int_{B_w} d_M(z)^{1-2s}\frac{|w(z)-w(\overline{z})|^2}{|z-\overline{z}|^{n+1}}d\sigma_z d\sigma_{\overline{z}}.
\end{aligned}$$

Next, we prove the claim.

Observe that we have three different cases, depending on the relative positions between the sets A_{r_i} and M.

Case a). First, consider the case $\Gamma_i = A_{2r_i} \cap \Gamma \neq \emptyset$. By (4.3.10), we have that

$$\int_{\widetilde{B}_{r_i}\cap\Omega} d_M(z)^{1-2s}|\nabla\widetilde{w}_i|^2dz \leq C||w||_{L^2(\partial\Omega)} + C\int_{\mathbb{R}^{n+1}_+}[(x_n)_+ + \lambda]^{1-2s}|\nabla\widetilde{\zeta}_i|^2dxd\lambda. \tag{4.3.11}$$

Then, using the result in case 1, applied to $\widetilde{\zeta}_i$, we get

$$\begin{aligned}\int_{\widetilde{B}_{r_i}\cap\Omega} d_M(z)^{1-2s}|\nabla\widetilde{w}_i|^2dz &\leq C||w||_{L^2(\partial\Omega)} + C\int_{\mathbb{R}^{n+1}_+}[(x_n)_+ + \lambda]^{1-2s}|\nabla\zeta_i|^2dxd\lambda \\ &\leq C||w||_{L^2(\partial\Omega)} + C\int\int_{B_s}\frac{|\zeta_i(x)-\zeta_i(\overline{x})|}{|x-\overline{x}|^{n+2s}}dxd\overline{x} \\ &\quad + \int\int_{B_w}(x_n)_+^{1-2s}\frac{|\zeta_i(x)-\zeta_i(\overline{x})|}{|x-\overline{x}|^{n+1}}dxd\overline{x},\end{aligned}$$

where B_s and B_w are defined as in (4.1.9) and (4.1.10), with $A = \mathbb{R}^n$ and $M = \{(x', x_n) \in \mathbb{R}^n : x_n < 0\}$.

Using the bilipschitz map φ_i^{-1}, we have

$$\begin{aligned}&\int_{\widetilde{B}_{r_i}\cap\Omega} d_M(z)^{1-2s}|\nabla\widetilde{w}_i|^2dz \\ &\quad\leq C||w||_{L^2(\partial\Omega)} + C\int\int_{B_s}\frac{|\widetilde{w}_i(z)-\widetilde{w}_i(\overline{z})|}{|z-\overline{z}|^{n+2s}}d\sigma_z d\sigma_{\overline{z}} \\ &\quad\quad + \int\int_{B_w} d_M(z)^{1-2s}\frac{|\widetilde{w}_i(z)-\widetilde{w}_i(\overline{z})|}{|z-\overline{z}|^{n+1}}d\sigma_z d\sigma_{\overline{z}},\end{aligned}$$

where now, B_s and B_w are defined as in (4.1.9) and (4.1.10), with $A = \partial\Omega$.

Case b). Second, consider the case $A_{r_i} \subset M$. In this case, the claim follows exactly as in case a), with $(x_n)_+ = 0$ in (4.3.11).

Case c). Finally, consider the case $A_{r_i} \subset \partial\Omega \setminus M$.

We recall that, by construction $A_{r_i} = \widetilde{B}_{r_i} \cap \partial\Omega$ where $\widetilde{B}_{r_i}$ is the ball centered at $p_i \in \partial\Omega \setminus M$ and of radius $r_i = \frac{1}{3}d_\Gamma(p_i)$.

Thus, for every $z \in \widetilde{B}_i \cap \Omega$, we have that

$$\frac{2}{3}d_\Gamma(p_i) \leq d_\Gamma(z) \leq \frac{4}{3}d_\Gamma(p_i).$$

Then, for every $i = 1, ..., m$

$$\int_{\widetilde{B}_{r_i}\cap\Omega} d_\Gamma(z)^{1-2s}|\nabla\widetilde{w}_i|^2dz \leq Cd_\Gamma(p_i)^{1-2s}\int_{\widetilde{B}_{r_i}\cap\Omega}|\nabla\widetilde{w}_i|^2dz.$$

Observe that the integral on the right-hand side, does not contain weights. Moreover, we recall that the extension $\widetilde{w}_i$ is defined as for the case $s = 1/2$. Thus, applying the extension result given in chapter 3 for $s = 1/2$, we get

$$\int_{\widetilde{B}_{r_i}\cap\Omega} d_\Gamma(z)^{1-2s}|\nabla\widetilde{w}_i|^2 dz$$
$$\leq C||w||_{L^2(\partial\Omega)} + Cd_M(p_i)^{1-2s}\int\int_{B_w}\frac{|\widetilde{w}_i(z)-\widetilde{w}_i(\overline{z})|}{|z-\overline{z}|^{n+1}}d\sigma_z d\sigma_{\overline{z}}$$
$$\leq C||w||_{L^2(\partial\Omega)} + C\int\int_{B_w} d_M(z)^{1-2s}\frac{|\widetilde{w}_i(z)-\widetilde{w}_i(\overline{z})|}{|z-\overline{z}|^{n+1}}d\sigma_z d\sigma_{\overline{z}}.$$

This conclude the proof of the claim. □

We give now the proof of the crucial Theorem 4.1.8.

Proof of Theorem 4.1.8. **Step 1**: suppose that $A = Q_1 = \{x \in \mathbb{R}^n : |x| < 1\}$ is the unit cube in $\mathbb{R}^n$. We may assume $c_0 = 1$ by replacing w by w/c_0. Let, as before, $(x', x_n) \in \mathbb{R}^n$. Recall that $M = Q_1^- = \{x \in Q_1 | x_n < 0\}$ and $\Gamma = \{x_n = 0\} \cap Q_1$.

Case $0 < s < 1/2$. By hypothesis we have that $|w(x)| \leq 1$ and

$$|\nabla w(x)| \leq \frac{C}{d_\Gamma(x)} = \frac{C}{|x_n|} \quad \text{in all } A = \mathbb{R}^n, \tag{4.3.12}$$

$$|\nabla w(x)| \leq \frac{C}{\varepsilon^{2s}}d_\Gamma^{2s-1}(x) = \frac{C}{\varepsilon^{2s}}|x_n|^{2s-1} \quad \text{in all } A = \mathbb{R}^n, \tag{4.3.13}$$

Let $Q_1^+ = \{x \in Q_1 | x_n > 0\}$. We prove that I is bounded, where I is given by

$$I := \int_{Q_1}\int_{Q_1}\frac{|w(x)-w(\overline{x})|^2}{|x-\overline{x}|^{n+2s}}dxd\overline{x} + \int_{Q_1^+}\int_{Q_1^+}(x_n)_+^{1-2s}\frac{|w(x)-w(\overline{x})|^2}{|x-\overline{x}|^{n+1}}dxd\overline{x}.$$

Since hypothesis (4.3.12) and (4.3.13) are symmetric in x_n and $-x_n$, we have that

$$I \leq \int_{Q_1}\int_{Q_1}\frac{|w(x)-w(\overline{x})|^2}{|x-\overline{x}|^{n+2s}}dxd\overline{x} + \int_{Q_1}\int_{Q_1}|x_n|^{1-2s}\frac{|w(x)-w(\overline{x})|^2}{|x-\overline{x}|^{n+1}}dxd\overline{x}. \tag{4.3.14}$$

Observe that in the set $\{|x-\overline{x}| < |x_n|/2\}$

$$\frac{|w(x)-w(\overline{x})|^2}{|x-\overline{x}|^{n+2s}} \leq C|x_n|^{1-2s}\frac{|w(x)-w(\overline{x})|^2}{|x-\overline{x}|^{n+1}},$$

while the reverse inequality holds in $\{|x-\overline{x}| \geq (x_n)_+/2\}$. We deduce that

$$I \leq 2\int_{Q_1}\int_{Q_1\cap\{\overline{x}:|x-\overline{x}|>|x_n|/2\}}\frac{|w(x)-w(\overline{x})|^2}{|x-\overline{x}|^{n+2s}}dxd\overline{x}$$
$$+2\int_{Q_1}\int_{Q_1\cap\{\overline{x}:|x-\overline{x}|<|x_n|/2\}}|x_n|^{1-2s}\frac{|w(x)-w(\overline{x})|^2}{|x-\overline{x}|^{n+1}}dxd\overline{x} =: I_1 + I_2. \tag{4.3.15}$$

We bound I_1 using the L^∞ estimate for w and spherical coordinates centered at x,

$$\begin{aligned}\int_{Q_1}\int_{\{\overline{x}\in Q_1:|x-\overline{x}|>|x_n|/2\}}\frac{|w(x)-w(\overline{x})|^2}{|x-\overline{x}|^{n+2s}}dxd\overline{x}&\le C\int_{Q_1}dx\int_{|x_n|/2}^{2\sqrt{n}}dr\frac{1}{r^{2s+1}}\\ &\le C\int_{-1}^{1}\frac{1}{|x_n|^{2s}}dx_n<C,\end{aligned}\tag{4.3.16}$$

where the finiteness of the last integral comes from the fact that, here, $0<s<1/2$. Next, we consider I_2. By (4.3.12) and (4.3.13) we have that $|\nabla w(x)|\le \frac{C}{\varepsilon}\left(\frac{|x_n|}{\varepsilon}\right)^{2s-1}$ if $0<|x_n|<\varepsilon$ and $|\nabla w(x)|\le\frac{C}{|x_n|}$ if $|x_n|>\varepsilon$. By symmetry between x_n and $-x_n$ we can suppose $x\in Q_1^+$. Using the gradient bounds above for w and spherical coordinates, we get

$$\begin{aligned}&\int_{Q_1}\int_{\{\overline{x}\in Q_1:|x-\overline{x}|<|x_n|/2\}}|x_n|^{1-2s}\frac{|w(x)-w(\overline{x})|^2}{|x-\overline{x}|^{n+1}}dxd\overline{x}\\ &\le C\int_0^{\varepsilon}dx_nx_n^{1-2s}\int_0^{x_n/2}dr\frac{1}{\varepsilon^2}\left(\frac{|y_n|}{\varepsilon}\right)^{2(2s-1)}+C\int_{\varepsilon}^{1}dx_nx_n^{1-2s}\int_0^{x_n/2}dr\frac{1}{y_n^2},\end{aligned}$$

where $y\in B_{|x_n|/2}(x)$ is a point of the segment joining x and $\overline{x}$. Since $y\in B_{|x_n|/2}(x)$, we have that $|y_n|\ge|x_n|/2$ and thus we deduce that

$$\begin{aligned}&\int_{Q_1}\int_{\{\overline{x}\in Q_1:|x-\overline{x}|<|x_n|/2\}}|x_n|^{1-2s}\frac{|w(x)-w(\overline{x})|^2}{|x-\overline{x}|^{n+1}}dxd\overline{x}\\ &\qquad\le C\int_0^{\varepsilon}dx_nx_n^{1-2s}\int_0^{x_n/2}dr\frac{1}{\varepsilon^2}\left(\frac{x_n}{\varepsilon}\right)^{2(2s-1)}+C\int_{\varepsilon}^{1}dx_nx_n^{1-2s}\int_0^{x_n/2}dr\frac{1}{x_n^2}\\ &\qquad C\frac{1}{\varepsilon^{4s}}\int_0^{\varepsilon}x_n^{2s}dx_n+C\int_{\varepsilon}^{1}x_n^{-2s}dx_n\le C\varepsilon^{1-2s}\le C.\end{aligned}\tag{4.3.17}$$

Using (4.3.15), (4.3.16), and (4.3.17), we conclude that $I_1\le C$.

Consider, now, the case $1/2<s<1$. We want to prove that

$$I=\int_{Q_1^-}\int_{Q_1^-}\frac{|w(x)-w(\overline{x})|^2}{|x-\overline{x}|^{n+2s}}dxd\overline{x}+\int_{Q_1^+}dx\int_{Q_1}d\overline{x}\,x_n^{1-2s}\frac{|w(x)-w(\overline{x})|^2}{|x-\overline{x}|^{n+1}}\le C\varepsilon^{1-2s}.$$

We recall that in this case, (4.3.12) and (4.3.13) imply that

$$\begin{cases}|Dw(x)|\le C/\varepsilon & \text{for every } x\in Q_1 \text{ s.t. } |x_n|<\varepsilon\\ |Dw(x)|\le C/|x_n| & \text{for every } x\in Q_1 \text{ s.t. } |x_n|>\varepsilon.\end{cases}\tag{4.3.18}$$

We have that

$$\begin{aligned}
I &\leq \int_{Q_1}\int_{Q_1} \frac{|w(x)-w(\overline{x})|^2}{|x-\overline{x}|^{n+2s}} dx d\overline{x} + \int_{Q_1} dx \int_{Q_1} d\overline{x}\, |x_n|^{1-2s} \frac{|w(x)-w(\overline{x})|^2}{|x-\overline{x}|^{n+1}} \\
&\leq 2\int\int_{\{|x-\overline{x}|\leq |x_n|/2\}} \frac{|w(x)-w(\overline{x})|^2}{|x-\overline{x}|^{n+2s}} dx d\overline{x} \\
&\quad +2\int\int_{\{|x-\overline{x}|\geq |x_n|/2\}} |x_n|^{1-2s} \frac{|w(x)-w(\overline{x})|^2}{|x-\overline{x}|^{n+1}} dx d\overline{x} \\
&\leq 2\int\int_{\{|x-\overline{x}|\leq |x_n|/2\}} \frac{|w(x)-w(\overline{x})|^2}{|x-\overline{x}|^{n+2s}} dx d\overline{x} \\
&\quad +2\int\int_{\{|x_n|/2\leq|x-\overline{x}|\leq \max\{\varepsilon/2,|x_n|/2\}\}} |x_n|^{1-2s} \frac{|w(x)-w(\overline{x})|^2}{|x-\overline{x}|^{n+1}} dx d\overline{x} \\
&\quad +2\int\int_{\{|x-\overline{x}|\geq \max\{\varepsilon/2,|x_n|/2\}\}} |x_n|^{1-2s} \frac{|w(x)-w(\overline{x})|^2}{|x-\overline{x}|^{n+1}} dx d\overline{x} = I_1 + I_2 + I_3.
\end{aligned}$$

We first bound I_1. Using the gradient bound (4.3.18) for w we have

$$I_1 \leq \int\int_{\{|x-\overline{x}|\leq |x_n|/2\}} \frac{|Dw(y)|^2}{|x-\overline{x}|^{n-2-2s}} dx d\overline{x},$$

where $y \in B_{|x_n|/2}(x)$ is a point of the segment joining x and $\overline{x}$. Now, the gradient bound (4.3.18) reads $|Dw(y)| \leq \min\{\varepsilon^{-1}, |y_n|^{-1}\}$ for a.e. $y \in Q_1$. Since $y \in B_{|x_n|/2}(x)$, we have $|y_n| \geq |x_n|/2$ and thus $|Dw(y)| \leq \min\{\varepsilon^{-1}, |y_n|^{-1}\} \leq \min\{\varepsilon^{-1}, 2|x_n|^{-1}\}$. Using spherical coordinates centered at x, we get

$$\begin{aligned}
I_1 &\leq C\int_{Q_1} dx \int_0^{|x_n|/2} dr r^{1-2s} \min\left\{\frac{1}{\varepsilon^2}, \frac{1}{|x_n|^2}\right\} \\
&\leq \int_{-1}^{1} dx_n \min\left\{\frac{1}{\varepsilon^2}, \frac{1}{|x_n|^2}\right\} |x_n|^{2-2s} \\
&\leq C\int_0^{\varepsilon} \frac{1}{\varepsilon^2}|x_n|^{2-2s} dx_n + C\int_{\varepsilon}^{1} |x_n|^{2-2s} dx_n \leq C\varepsilon^{1-2s}.
\end{aligned}$$

Consider now I_2. Here $|x_n| < \varepsilon$ (if not $\{|x_n|/2 < \max\{|x_n|/2, \varepsilon/2\}\} = \emptyset$). Using the gradient bound (4.1.13) and spherical coordinates centered at x, we get

$$I_2 \leq C\int_0^{\varepsilon} dx_n |x_n|^{1-2s} \int_{|x_n|/2}^{\varepsilon/2} dr \frac{C}{\varepsilon^2} \leq \frac{C}{\varepsilon}\varepsilon^{2-2s} \leq C\varepsilon^{1-2s}.$$

Finally, using that $|w| \leq C$ in Q_1 and spherical coordinates centered at x, we get the following bound for I_3:

$$
\begin{aligned}
I_3 &\le \int_{Q_1} dx_n |x_n|^{1-2s} \int_{\max\{|x_n|/2,\varepsilon/2\}}^{2\sqrt{n}} dr \frac{C}{r^2} \le C \int_{-1}^{1} dx_n |x_n|^{1-2s} \min\left\{\frac{1}{|x_n|}, \frac{1}{\varepsilon}\right\} \\
&\le \int_0^{\varepsilon} \frac{1}{\varepsilon} |x_n|^{1-2s} + C \int_{\varepsilon} |x_n|^{-2s} \le C\varepsilon^{1-2s}.
\end{aligned}
$$

Step 2. Suppose now that A is a Lipschitz subset of $\mathbb{R}^n$ or $A = \partial\Omega$, where Ω is an open bounded subset of $\mathbb{R}^{n+1}$ with Lipschitz boundary.

We consider the finite open covering $\{A_{r_i}\}_{i=1,\dots,m} = \{B_{r_i} \cap A\}_{i=1,\dots,m}$, constructed in the proof of Theorem 4.1.6 case 2. Here, for sake of simplicity, B_{r_i} denotes both the ball in $\mathbb{R}^n$ or $\mathbb{R}^{n+1}$. We set $\varepsilon_0 = \min\{r_i, 1/2\}$.

If z and $\overline{z}$ are two points belonging to A such that $|z - \overline{z}| < \varepsilon_0$, then there exists a set $A_{2r_i} = B_{2r_i} \cap A$ such that both z and $\overline{z}$ belong to A_{2r_i}. Hence

$$
\{(z, \overline{z}) \in A \times A : |z - \overline{z}| < \varepsilon_0\} \subset \bigcup_{i=1}^{m} A_{2r_i} \times A_{2r_i}.
$$

Let $L > 1$ be a bound for the Lipschitz constants of all functions $\varphi_1, ..., \varphi_m, \varphi_1^{-1}, ..., \varphi_m^{-1}$. Let us first treat the case $0 < \varepsilon \le 1/(2L)$.

We write

$$
\begin{aligned}
&\int\int_{B_s} \frac{|w(z) - w(\overline{z})|^2}{|z - \overline{z}|^{n+2s}} d\sigma_z d\sigma_{\overline{z}} + \int\int_{B_w} d_M(z)^{1-2s} \frac{|w(z) - w(\overline{z})|^2}{|z - \overline{z}|^{n+1}} d\sigma_z d\sigma_{\overline{z}} \\
&= \int\int_{B_s \cap \{\overline{z}: |z - \overline{z}| < \varepsilon_0\}} \frac{|w(z) - w(\overline{z})|^2}{|z - \overline{z}|^{n+2s}} d\sigma_z d\sigma_{\overline{z}} \\
&\quad + \int\int_{B_s \cap \{\overline{z}: |z - \overline{z}| > \varepsilon_0\}} \frac{|w(z) - w(\overline{z})|^2}{|z - \overline{z}|^{n+2s}} d\sigma_z d\sigma_{\overline{z}} \\
&\quad + \int\int_{B_w \cap \{\overline{z}: |z - \overline{z}| < \varepsilon_0\}} d_M(z)^{1-2s} \frac{|w(z) - w(\overline{z})|^2}{|z - \overline{z}|^{n+1}} d\sigma_z d\sigma_{\overline{z}} \\
&\quad + \int\int_{B_w \cap \{\overline{z}: |z - \overline{z}| > \varepsilon_0\}} d_M(z)^{1-2s} \frac{|w(z) - w(\overline{z})|^2}{|z - \overline{z}|^{n+1}} d\sigma_z d\sigma_{\overline{z}}.
\end{aligned}
$$

Since w is bounded and $\int_{B_w} d_M(z)^{1-2s} dz < C$ for every $0 < s < 1$, we have that

$$
\begin{aligned}
&\int\int_{B_s \cap \{\overline{z}: |z - \overline{z}| > \varepsilon_0\}} \frac{|w(z) - w(\overline{z})|^2}{|z - \overline{z}|^{n+2s}} d\sigma_z d\sigma_{\overline{z}} \\
&\qquad + \int\int_{B_w \cap \{\overline{z}: |z - \overline{z}| > \varepsilon_0\}} d_M(z)^{1-2s} \frac{|w(z) - w(\overline{z})|^2}{|z - \overline{z}|^{n+1}} d\sigma_z d\sigma_{\overline{z}} \le C.
\end{aligned}
$$

On the other hand, for what said before

$$\begin{aligned}
&\int\int_{B_s\cap\{\overline{z}:|z-\overline{z}|<\varepsilon_0\}} \frac{|w(z)-w(\overline{z})|^2}{|z-\overline{z}|^{n+2s}} d\sigma_z d\sigma_{\overline{z}} \\
&\qquad + \int\int_{B_w\cap\{\overline{z}:|z-\overline{z}|<\varepsilon_0\}} d_M(z)^{1-2s} \frac{|w(z)-w(\overline{z})|^2}{|z-\overline{z}|^{n+1}} d\sigma_z d\sigma_{\overline{z}} \\
&\qquad \le \sum_{i=1}^{m} \int\int_{B_s\cap(A_{2r_i}\times A_{2r_i})} \frac{|w(z)-w(\overline{z})|^2}{|z-\overline{z}|^{n+2s}} d\sigma_z d\sigma_{\overline{z}} \\
&\qquad + \sum_{i=1}^{m} \int\int_{B_w\cap(A_{2r_i}\times A_{2r_i})} d_M(z)^{1-2s} \frac{|w(z)-w(\overline{z})|^2}{|z-\overline{z}|^{n+1}} d\sigma_z d\sigma_{\overline{z}}.
\end{aligned}$$

If $A_{r_i}\cap\Gamma\neq\emptyset$ then, by the construction of the open covering $\{A_{r_i}\}$, there exists a bilipschitz map $\varphi_i : B_{2r_i}\to Q_1\times(-1,1)$ such that $\varphi_i(A_{2r_i}) = Q_1$ and $\varphi_i(A_{2r_i}\cap M) = \{x\in Q_1 : x_n<0\}$. We use the bilipschitz map φ_i to flatten the sets $B_s\cap(A_{2r_i}\times A_{2r_i})$ and $B_w\cap(A_{2r_i}\times A_{2r_i})$, and we set $w_0 = w\circ\varphi^{-1}$. Given $x\in Q_1$, let $y=\varphi_i^{-1}(x)\in A_{r_i}$. Recalling the definition of the Lipschitz constant L above, we have $|x_n|\le L d_\Gamma(y)$ and hence

$$\begin{aligned}
|Dw_0(x)| &\le L \quad |Dw(y)| \le L\frac{c_0}{d_\Gamma(y)} \min\left\{1, \left(\frac{d_\Gamma(y)}{\varepsilon}\right)^{\min\{1,2s\}}\right\} \\
&\le \quad L^2\frac{c_0}{|x_n|} \min\left\{1, \left(\frac{|x_n|}{L\varepsilon}\right)^{\min\{1,2s\}}\right\}.
\end{aligned}$$

Thus we can apply the result proved in Step 1, with ε replaced by εL (note that we have $\varepsilon L\le 1/2$, as in Step 1), to the function $w_0/[(1+L^2)c_0]$. Using the Lipschitz property of φ^{-1}, we restate the conclusion for w and we get

$$\begin{aligned}
&\int\int_{B_s\cap(A_{2r_i}\times A_{2r_i})} \frac{|w(z)-w(\overline{z})|^2}{|z-\overline{z}|^{n+2s}} d\sigma_z d\sigma_{\overline{z}} \\
&+ \int\int_{B_w\cap(A_{2r_i}\times A_{2r_i})} d_M(z)^{1-2s} \frac{|w(z)-w(\overline{z})|^2}{|z-\overline{z}|^{n+1}} d\sigma_z d\sigma_{\overline{z}} \le C \begin{cases} 1 & \text{if } 0<s<1/2 \\ \varepsilon^{1-2s} & \text{if } 1/2<s<1. \end{cases}
\end{aligned}$$

Finally, we consider the case $A_{r_i}\cap\Gamma=\emptyset$. We recall that, in this case $r_i = \frac{1}{3}d_M(p_i)$, where r_i and p_i are respectively the radius and the center of the ball B_{r_i}. Then, for every $z\in A_{2r_i}$, we have that $d_M(z)\ge r_i\ge\varepsilon_0$. Thus by hypothesis (4.1.13), we have that $|\nabla w(z)|\le\dfrac{1}{\varepsilon_0}$.

Using this gradient bound and spherical coordinates, we get

$$\begin{aligned}\int\int_{B_s\cap(A_{2r_i}\times A_{2r_i})} &\frac{|w(z)-w(\overline{z})|^2}{|z-\overline{z}|^{n+2s}}d\sigma_z d\sigma_{\overline{z}}\\ &+\int\int_{B_w\cap(A_{2r_i}\times A_{2r_i})} d_M(z)^{1-2s}\frac{|w(z)-w(\overline{z})|^2}{|z-\overline{z}|^{n+1}}d\sigma_z d\sigma_{\overline{z}}\\ &\leq C\int_{A_{2r_i}}\int_0^{2r_i}\frac{|\nabla w|^2}{r^{2s-1}}+\int_{A_{2r_i}}\int_0^{2r_i} d_M(z)^{1-2s}|\nabla w|^2\leq C.\end{aligned}$$

Summing over $i=1,...,m$, we conclude the proof in case $\varepsilon\leq 1/(2L)$.

Finally given $\varepsilon\in(0,1/2)$ with $\varepsilon>1/(2L)$, since (4.1.13) holds with such ε, it also holds with ε replaced by $1/(2L)$. By the previous proof with ε taken to be $1/(2L)$, the energy is bounded by a constant if $0<s<1/2$ and by $C/(2L)^{1-2s}\leq C\varepsilon^{1-2s}$ if $1/2<s<1$. □

4.4 Energy estimate for global minimizers

In this section we give the proof of Theorem 4.1.2, which is based on a comparison argument. Let v be a global minimizer of (4.1.2). The proof can be resumed in 3 steps:

i) construct the comparison function $\overline{w}$, which takes the same values as v on $\partial C_R\cap\{\lambda>0\}$ and thus, by minimality of v

$$\mathcal{E}_{s,C_R}(v)\leq\mathcal{E}_{s,C_R}(\overline{w});$$

ii) apply the extension Theorem 4.1.6 in the cylinder of radius R and height R:

$$\int_{C_R}\lambda^{1-2s}|\nabla\widetilde{w}|^2 dxd\lambda \;\leq\; C\int\int_{B_s}\frac{|w(z)-w(\overline{z})|^2}{|z-\overline{z}|^{n+2s}}d\sigma_z d\sigma_{\overline{z}} \tag{4.4.1}$$

$$+\; C\int\int_{B_w}\lambda^{1-2s}\frac{|w(z)-w(\overline{z})|^2}{|z-\overline{z}|^{n+1}}d\sigma_z d\sigma_{\overline{z}}, \tag{4.4.2}$$

where $z\in\partial C_R$, w is the trace of $\overline{w}$ on ∂C_R and B_s and B_w are defined as in (4.1.9) and (4.1.10), with $A=\partial C_R$ and $M=\partial B_R\times\{0\}$.

iii) prove that the quantity in the right-hand side of (4.4.1) is bounded by CR^{n-2s} if $0<s<1/2$ and is bounded by CR^{n-1} if $1/2<s<1$.

Proof of Theorem 4.1.2. Let v be a bounded global minimizer of (4.1.2). Let u be its trace on $\partial\mathbb{R}^{n+1}_+$. Recall the definition (4.1.5) of the constant c_u. Let $t\in[\inf u,\sup u]$ be such that $G(t)=c_u$.

Throughout the proof, C will denote positive constants depending only on n, s, $||f||_{C^1}$ and $||u||_{L^\infty(\mathbb{R}^n)}$. As explained in (4.1.15), v satisfies the following bounds:

$$|\nabla_x v(x,\lambda)| \le \frac{C}{1+\lambda} \text{ for every } (x,\lambda) \in \mathbb{R}^{n+1}_+ \tag{4.4.3}$$

$$|\partial_\lambda v(x,\lambda)| \le \frac{C}{\lambda} \text{ for every } x \in \mathbb{R}^n \text{ and } \lambda > 1 \tag{4.4.4}$$

$$|\lambda^{1-2s}\partial_\lambda v| \le C \text{ for every } x \in \mathbb{R}^n \text{ and } 0 < \lambda < 1. \tag{4.4.5}$$

We estimate the energy $\mathcal{E}_{s,C_R}(v)$ of v using a comparison argument. We define a function $\overline{w} = \overline{w}(x,\lambda)$ on C_R in the following way. First we define $\overline{w}(x,0)$ on the base of the cylinder to be equal to a smooth function $g(x)$ which is identically equal to s in B_{R-1} and $g(x) = v(x,0)$ for $|x| = R$. The function g is defined as follows:

$$g = s\eta_R + (1-\eta_R)v, \tag{4.4.6}$$

where η_R is a smooth function depending only on $r = |x|$ such that $\eta_R \equiv 1$ in B_{R-1} and $\eta_R \equiv 0$ outside B_R. Then we define $\overline{w}(x,\lambda)$ as the unique solution of the Dirichlet problem

$$\begin{cases} \operatorname{div}(\lambda^{1-2s}\nabla\overline{w}) = 0 & \text{in } C_R \\ \overline{w}(x,0) = g(x) & \text{on } B_R \times \{\lambda = 0\} \\ \overline{w}(x,\lambda) = v(x,\lambda) & \text{on } \partial C_R \cap \{\lambda > 0\}. \end{cases} \tag{4.4.7}$$

Since v is a global minimizer of $\mathcal{E}_{s,C_R}$ and $\overline{w} = v$ on $\partial C_R \times \{\lambda > 0\}$, then

$$\begin{aligned} &\int_{C_R} \frac{1}{2}\lambda^{1-2s}|\nabla v|^2 dx d\lambda + \int_{B_R} \{G(u) - c_u\} dx \\ &\quad \le \int_{C_R} \frac{1}{2}\lambda^{1-2s}|\nabla \overline{w}|^2 dx d\lambda + \int_{B_R} \{G(\overline{w}(x,0)) - c_u\} dx. \end{aligned}$$

We prove now that if $0 < s < 1/2$, then

$$\int_{C_R} \frac{1}{2}\lambda^{1-2s}|\nabla \overline{w}|^2 dx d\lambda + \int_{B_R} \{G(\overline{w}(x,0)) - c_u\} dx \le CR^{n-2s}.$$

While, if $1/2 < s < 1$, then

$$\int_{C_R} \frac{1}{2}\lambda^{1-2s}|\nabla \overline{w}|^2 dx d\lambda + \int_{B_R} \{G(\overline{w}(x,0)) - c_u\} dx \le CR^{n-1}.$$

Observe that, in both cases, the potential energy is bounded by CR^{n-1}, indeed by definition $\overline{w}(x,0) = t$ on B_{R-1}, then

$$\int_{B_R} \{G(\overline{w}(x,0)) - c_u\}dx = \int_{B_R\setminus B_{R-1}} \{G(\overline{w}(x,0)) - c_u\}dx$$
$$\leq C|B_R \setminus B_{R-1}| \leq CR^{n-1}. \tag{4.4.8}$$

Thus, we need to bound the Dirichlet energy. First of all, rescaling, we set

$$w_1(x,\lambda) = \overline{w}(Rx, R\lambda),$$

for $(x,\lambda) \in C_1 = B_1 \times (0,1)$. Moreover, if we set $\varepsilon = 1/R$ then

$$w_1(x,0) = \begin{cases} t & \text{for } |x| < 1-\varepsilon \\ v(Rx,0) & \text{for } |x| = 1. \end{cases}$$

We observe that

$$\int_{C_R} \lambda^{1-2s}|\nabla \overline{w}|^2 dxd\lambda = CR^{n-2s}\int_{C_1} \lambda^{1-2s}|\nabla w_1|^2 dxd\lambda.$$

Thus, it is enough to prove that

$$\begin{cases} \int_{C_1} \lambda^{1-2s}|\nabla w_1|^2 dxd\lambda \leq C & \text{if } 0 < s < 1/2, \\ \int_{C_1} \lambda^{1-2s}|\nabla w_1|^2 dxd\lambda \leq CR^{2s-1} = C\varepsilon^{1-2s} & \text{if } 1/2 < s < 1. \end{cases} \tag{4.4.9}$$

Remind that, by Remark 4.1.7, we have

$$\begin{aligned} \int_{C_1} \lambda^{1-2s}|\nabla w_1|^2 &\leq C||w_1||_{L^2(\partial C_1)} + C\int\int_{B_s} \frac{|w_1(z) - w_1(\overline{z})|^2}{|z-\overline{z}|^{n+2s}} d\sigma_z d\sigma_{\overline{z}} \\ &+ C\int\int_{B_w} d_M(z)^{1-2s}\frac{|w_1(z) - w_1(\overline{z})|^2}{|z-\overline{z}|^{n+1}} d\sigma_z d\sigma_{\overline{z}}, \end{aligned}$$

where B_s and B_w are defined as in (4.1.9) and (4.1.10), with $A = \partial C_1$ and $M = \partial B_1 \times \{0\}$. To bound the two double integral above, we apply Theorem 4.1.8 to $w_{1|\partial C_1}$ in $A = \partial C_1$, taking $\Gamma = \partial B_1 \times \{\lambda = 0\}$. Since $|w_1| \leq C$, we only need to check (4.1.13) in ∂C_1. In the bottom boundary, $B_1 \times \{0\}$, this is simple. Indeed $w_1 \equiv s$ in $B_{1-\varepsilon}$, and thus we need only to control $|\nabla w_1(x,0)| = \varepsilon^{-1}|\nabla g(Rx)| \leq C\varepsilon^{-1}$ for $|x| > 1-\varepsilon$, where g is defined in (4.4.6). Here $d_M(x) < \varepsilon$, and thus (4.1.13) holds here.

Next, to verify (4.1.13) in $\partial C_1 \cap \{\lambda > 0\}$ we use that $\overline{w} = v$ here and we know that v satisfies (4.4.3), (4.4.4), and (4.4.5). Thus the tangential derivatives of w_1 in $\partial C_1 \cap \{\lambda > 0\}$ satisfies

$$|\nabla_x w_1(x,\lambda)| \leq \frac{CR}{1+R\lambda} = \frac{C}{\varepsilon+\lambda} \text{ for every } (x,\lambda) \in C^1$$

$$|\partial_\lambda w_1(x,\lambda)| \le \frac{CR}{R\lambda} = \frac{C}{\lambda} \text{ for every } x \in B_1 \text{ and } \lambda > 1/\varepsilon$$
$$|\lambda^{1-2s}\partial_\lambda w_1| \le \frac{CR}{R^{1-2s}} = \frac{C}{\varepsilon^{2s}} \text{ for every } \overline{x} \in B_1 \text{ and } 0 < \lambda < 1/\varepsilon.$$

Thus,

$$|Dw_1(x,\lambda)| \le \frac{C}{\lambda} \min\left\{1, \left(\frac{\lambda}{\varepsilon}\right)^{\min\{1,2s\}}\right\}.$$

Since $d_\Gamma((x,\lambda)) \ge \lambda$ on $\partial C_1 \cap \{\lambda > 0\}$, $w_{1|_{\partial C_1}}$ satisfies the hypothesis of Theorem 4.1.8. We conclude that (4.4.9) holds.

Thus $w_1 : \partial C_1 \to \mathbb{R}$, satisfies the hypothesis of Theorem 4.1.8 and then (4.4.9) follows. □

4.5 Energy estimate for monotone solutions in $\mathbb{R}^3$

In section 5 of Chapter 3, we gave two technical lemmas which led to the energy estimate for monotone solutions (without limit assumption) in dimension $n = 3$. Here we give analog results but for every fractional power $0 < s < 1$ of the Laplacian.

The first lemma concerns the stability property of the limit functions

$$\underline{v}(x_1,x_2,\lambda) := \lim_{x_3\to-\infty} v(x,\lambda) \text{ and } \overline{v}(x_1,x_2,\lambda) := \lim_{x_3\to+\infty} v(x,\lambda),$$

and some properties of the potential G in relation with these functions. The second proposition establishes that monotone solutions are global minimizers among a suitable class of functions, and allows us to apply a comparison argument, to obtain energy estimates.

Lemma 4.5.1. *Let f be a $C^{1,\beta}$ function, for some $\beta > \max\{0, 1-2s\}$, and u a bounded solution of equation (4.1.1) in $\mathbb{R}^3$, such that $u_{x_3} > 0$. Let v be the s-extension of u in $\mathbb{R}^4_+$.*

Set

$$\underline{v}(x_1,x_2,\lambda) := \lim_{x_3\to-\infty} v(x,\lambda) \text{ and } \overline{v}(x_1,x_2,\lambda) := \lim_{x_3\to+\infty} v(x,\lambda).$$

Then, $\underline{v}$ and $\overline{v}$ are solutions of (4.1.2) in $\mathbb{R}^3_+$, and each of them is either constant or it depends only on λ and one Euclidian variable in the $(x_1,x_2)-$plane. As a consequence, each $\underline{u} = \underline{v}(\cdot,0)$ and $\overline{u} = \overline{v}(\cdot,0)$ is either constant or 1-D.

Moreover, set $m = \inf \underline{u} \le \widetilde{m} = \sup \underline{u}$ and $\widetilde{M} = \inf \overline{u} \le M = \sup \overline{u}$.

Then, $G > G(\widetilde{m}) = G(m)$ in $(m,\widetilde{m})$, $G'(\widetilde{m}) = G'(m) = 0$ and $G > G(\widetilde{M}) = G(M)$ in $(\widetilde{M}, M)$, $G'(\widetilde{M}) = G'(M) = 0$.

Proof. The proof is the same as in the case of the half-Laplacian (see chapter 3). We do not supply all details and just recall the two main steps:

1. show that the functions $\underline{v}$ and $\overline{v}$ are stable solutions of problem (4.1.2) in $\mathbb{R}^3_+$ and thus their trace in $\mathbb{R}^2$ is 1-D;

2. apply Theorem 2.4 of Cabré and Sire [8], which characterizes the nonlinearities f for which there exists a layer solutions for problem (4.1.2) in dimension $n = 1$.

□

Proposition 4.5.2. *Let f be any $C^{1,\beta}$ nonlinearity, for some $\beta > \max\{0, 1-2s\}$. Let u be a bounded solution of (4.1.1) in $\mathbb{R}^n$, such that $u_{x_n} > 0$ and let v be the s-extension of u in $\mathbb{R}^{n+1}_+$.*
Then,

$$\int_{C_R} \frac{1}{2}\lambda^a |\nabla v(x,\lambda)|^2 dx d\lambda \ + \ \int_{B_R} G(v(x,0))dx \\ \leq \int_{C_R} \frac{1}{2}\lambda^a |\nabla w(x,\lambda)|^2 dx d\lambda + \int_{B_R} G(w(x,0))dx,$$

for every $w \in C^1(\overline{\mathbb{R}^{n+1}_+})$ such that $w = v$ on $\partial^+ C_R = \partial C_R \cap \{\lambda > 0\}$ and $\underline{v} \leq w \leq \overline{v}$ in C_R, where $\underline{v}$ and $\overline{v}$ are defined by

$$\underline{v}(x', \lambda) := \lim_{x_n \to -\infty} v(x', x_n, \lambda) \ \ \textit{and} \ \ \overline{v}(x', \lambda) := \lim_{x_n \to +\infty} v(x', x_n, \lambda).$$

Proof. As in the case of the half-Laplacian, this property of local minimality of monotone solutions w such that $\underline{v} \leq w \leq \overline{v}$ follows from the following two results:

i) uniqueness of the solution v of the problem

$$\begin{cases} \operatorname{div}(\lambda^a \nabla w) = 0 & \text{in } C_R, \\ w = v & \text{on } \partial^+ C_R, \\ -\lambda^a \partial_\lambda w = f(w) & \text{on } \partial^0 C_R, \\ \underline{v} \leq w \leq \overline{v} & \text{in } C_R, \end{cases} ; \tag{4.5.1}$$

Thus, the solution must be $w \equiv v$. This is the analog of Lemma 3.1 of [10], and below we comment on its proof.

ii) existence of an absolute minimizer for $\mathcal{E}_{s,C_R}$ in the set

$$C_v = \{w \in H^1_{\lambda^a}(C_R) | w \equiv v \text{ on } \partial^+ C_R, \ \underline{v} \leq w \leq \overline{v}\}.$$

The statement of the proposition follows from the fact that by i) and ii), the monotone solution v, by uniqueness, must agree with the absolute minimizer in C_R.

To prove points i) and ii), we proceed exactly as in [9], For this, it is important that $\underline{v}$ and $\overline{v}$ are respectively, a strict subsolution and a strict supersolution of the Dirichlet- Neumann mixed problem (4.5.1). We make a short comment about these proofs.

i) The proof of uniqueness is based on sliding the function $v(x, \lambda)$ in the direction x_n. We set

$$v^t(x_1, ..., x_n, \lambda) = v(x_1, ..., x_n + t, \lambda) \quad \text{for every } (x, \lambda) \in \overline{C}_R.$$

Since $v^t \to \overline{v}$ as $t \to +\infty$ uniformly in $\overline{C}_R$ and $\underline{v} < w < \overline{v}$, then $w < v^t$ in $\overline{C}_R$, for t large enough. We want to prove that $w < v^t$ in $\overline{C}_R$ for every $t > 0$. Suppose that $s > 0$ is the infimum of those $t > 0$ such that $w < v^t$ in $\overline{C}_R$. Then by applying maximum principle and Hopf's lemma we get a contradiction, since one would have $w \le v^s$ in $\overline{C}_R$ and $w = v^s$ at some point in $\overline{C}_R \setminus \partial^+ C_R$.

ii) To prove the existence of an absolute minimizer for $\mathcal{E}_{C_R}$ in the convex set C_v, we proceed exactly as in [9], substituting -1 and $+1$ by the subsolutions and supersolution $\underline{v}$ and $\overline{v}$, respectively.

□

We give now the proof of the energy estimate in dimension 3 for monotone solutions without the limit assumptions.

Proof of Theorem 4.1.3. We follow the proof of Theorem 5.2 of [1]. We need to prove that the comparison function $\overline{w}$, used in the proof of Theorem 4.1.2, satisfies $\underline{v} \le \overline{w} \le \overline{v}$. Then we can apply Proposition 4.5.2 to make the comparison argument with the function $\overline{w}$ (as for global minimizers). We recall that $\overline{w}$ is the solution of

$$\begin{cases} \operatorname{div}(\lambda^{1-2s} \nabla \overline{w}) = 0 & \text{in } C_R \\ \overline{w}(x, 0) = g(x) & \text{on } B_R \times \{\lambda = 0\} \\ \overline{w}(x, \lambda) = v(x, \lambda) & \text{on } \partial C_R \cap \{\lambda > 0\}, \end{cases} \tag{4.5.2}$$

where $g = s\eta_R + (1 - \eta_R)v$. Thus, if we prove that $\sup \underline{v} \le s \le \inf \overline{v}$, then $\underline{v} \le g \le \overline{v}$ and hence $\underline{v}$ and $\overline{v}$ are respectively, subsolution and supersolutions of (4.5.2). It follows that $\underline{v} \le \overline{w} \le \overline{v}$, as desired.

To show that $\sup \underline{v} \le s \le \inf \overline{v}$, let $m = \inf u = \inf \underline{u}$ and $M = \sup u = \sup \overline{u}$, where $\underline{u}$ and $\overline{u}$ are defined in Lemma 4.5.1. Set $\widetilde{m} = \sup \underline{u}$ and $\widetilde{M} = \inf \overline{u}$, obviously

$\widetilde{m}$ and $\widetilde{M}$ belong to $[m, M]$. By Lemma 4.5.1, $\underline{u}$ and $\overline{u}$ are either constant or monotone 1-D solutions, moreover

$$G > G(m) = G(\widetilde{m}) \quad \text{in } (m, \widetilde{m}) \tag{4.5.3}$$

in case $m < \widetilde{m}$ (i.e. $\underline{u}$ not constant), and

$$G > G(M) = G(\widetilde{M}) \quad \text{in } (\widetilde{M}, M) \tag{4.5.4}$$

in case $\widetilde{M} < M$ (i.e. $\overline{u}$ not constant).

In all four possible cases (that is, each $\underline{u}$ and $\overline{u}$ is constant or one-dimensional), we deduce from (4.5.3) and (4.5.4) that $\widetilde{m} \leq \widetilde{M}$ and that there exists $s \in [\widetilde{m}, \widetilde{M}]$ such that $G(s) = c_u$ (recall that c_u is the infimum of G in the range of u). We conclude that

$$\sup \underline{u} = \sup \underline{v} \leq \widetilde{m} \leq s \leq \widetilde{M} \leq \inf \overline{v} = \inf \overline{u}.$$

Hence we can apply Proposition 4.5.2 to make comparison argument with the function $\overline{w}$ and obtain the desired energy estimate. □

4.6 1-D symmetry in $\mathbb{R}^3$

To prove Theorem 4.1.4 we follow the argument, used by Ambrosio and Cabré [3] in their proof of the conjecture of De Giorgi in dimension $n = 3$. It relies on a Liouville type theorem. We recall an adapted version of this result for the fractional case, given by Cabré and Sire (Theorem 4.9 in [8]).

Theorem 4.6.1. *Let $\varphi \in L^\infty_{loc}(\overline{\mathbb{R}^{n+1}_+})$ be a positive function and suppose that $\sigma \in H^1_{loc}(\mathbb{R}^{n+1}_+)$ is a solution of*

$$\begin{cases} \operatorname{div}(\lambda^a \varphi^2 \nabla \sigma) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ -\lambda^a \frac{\partial \sigma}{\partial \lambda} = 0 & \text{on } \partial \mathbb{R}^{n+1}_+ \end{cases} \tag{4.6.1}$$

in the weak sense. Moreover suppose that $\lambda^a |\nabla \sigma|^2 \in L^1_{loc}(\mathbb{R}^{n+1}_+)$ and assume that for every $R > 1$,

$$\int_{C_R} \lambda^a (\varphi \sigma)^2 \leq C R^2, \tag{4.6.2}$$

for some constant C independent of R.

Then, σ is constant.

We can now give the proof of our one-dimensional symmetry result.

Proof of Theorem 4.1.4. Without loss of generality we can suppose $e = (0, 0, 1)$. We follow the proof of Theorem 1.3 in chapter 3. First of all observe that both global minimizers and monotone solutions are stable. Thus, in both cases (see [8]), there exists a function $\varphi \in C^1_{loc}(\overline{\mathbb{R}^4_+}) \cap C^2(\mathbb{R}^4_+)$ such that $\varphi > 0$ in $\overline{\mathbb{R}^4_+}$ and

$$\begin{cases} \operatorname{div}(\lambda^a \nabla\varphi) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ -\lambda^a \partial_\lambda \varphi = f'(v)\varphi & \text{on } \partial\mathbb{R}^{n+1}_+. \end{cases}$$

Note that, if u is a monotone solution in the direction x_3, then we can choose $\varphi = v_{x_3}$, where v is the s-extension of u in the half space. For $i = 1, 2, 3$ fixed, consider the function:

$$\sigma_i = \frac{v_{x_i}}{\varphi}.$$

We prove that σ_i is constant in $\mathbb{R}^4_+$, using the Liouville type Theorem 4.6.1 and our energy estimate.

We have that

$$\operatorname{div}(\lambda^a \varphi^2 \nabla \sigma_i) = 0 \quad \text{in } \mathbb{R}^4_+.$$

Moreover $-\lambda^a \partial_\lambda \sigma_i$ is zero on $\partial\mathbb{R}^4_+$. Indeed

$$\lambda^a \varphi^2 \partial_\lambda \sigma_i = \lambda^a \varphi v_{\lambda x_i} - \lambda^a v_{x_i} \varphi_\lambda = 0$$

since both v_{x_i} and φ satisfies the same boundary condition

$$-\lambda^a \frac{\partial v_{x_i}}{\partial \lambda} - f'(v) v_{x_1} = 0, \quad -\lambda^a \frac{\partial \varphi}{\partial \lambda} - f'(v)\varphi = 0.$$

Using the energy estimate (4.1.6) for $n = 3$, we have

$$\int_{C_R} (\lambda^{1-2s} (\varphi\sigma_i)^2 \leq \int_{C_R} \lambda^{1-2s} |\nabla v|^2 \leq CR^2, \quad \text{for every } R > 2 \ \textit{and} \ 1/2 < s < 1.$$

Thus, using Theorem 4.6.1, we deduce that σ_i is constant for every $i = 1, 2, 3$. We get

$$v_{x_i} = c_i \varphi \quad \text{for some constant } c_i, \quad \text{with } i = 1, 2, 3.$$

We conclude the proof observing that if $c_1 = c_2 = c_3 = 0$ then v is constant. Otherwise we have

$$c_i v_{x_j} - c_j v_{x_i} = 0 \quad \text{for every } i \neq j,$$

and we deduce that v depends only on λ and on the variable parallel to the vector (c_1, c_2, c_3). Thus $u(x) = v(x, 0)$ is 1-D. □

Bibliography

[1] G. Alberti, L. Ambrosio and X. Cabré, *On a long-standing conjecture of E. De Giorgi: symmetry in 3D for general nonlinearities and a local minimality property,* Acta Appl. Math. **65** (2001), 9–33.

[2] G. Alberti, G. Bouchitté, and S. Seppecher, *Phase transition with the line-tension effect*, Arch. Rational Mech. Anal., **144** (1998), 1–46.

[3] L. Ambrosio and X. Cabré, *Entire solutions of semilinear elliptic equations in $\mathbb{R}^3$ and a Conjecture of De Giorgi,* Journal Amer. Math. Soc. **13** (2000), 725–739.

[4] D. Applebaum, *Lévy processes—from probability to finance and quantum groups*, Notices Amer. Math. Soc. **51** (2004), 1336-1347.

[5] H. Berestycki, L. Caffarelli and L. Nirenberg, *Further qualitative properties for elliptic equations in unbounded domains,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. **25** (1997), 69–94.

[6] X. Cabré and E. Cinti, *Energy estimates and 1-D symmetry for nonlinear equations involving the half-Laplacian*, forthcoming.

[7] X. Cabré and E. Cinti, *Fractional diffusion equations: energy estimates and 1-D symmetry in dimension 3*, forthcoming.

[8] X. Cabré and Y. Sire, *Nonlinear equations for fractional Laplacians I: regularity, maximum principles, and Hamiltonian estimates*, forthcoming.

[9] X. Cabré and Y. Sire, *Nonlinear equations for fractional Laplacians II: existence, uniqueness, and qualitative properties of solutions*, forthcoming.

[10] X. Cabré and J. Solà-Morales, *Layer Solutions in a Halph-Space for Boundary reactions*, Comm. Pure and Appl. Math. **58**, (2005), 1678–1732.

[11] X. Cabré and J. Tan, *Positive solutions of nonlinear problems involving the square root of the Laplacian*, to appear in Advances in Math. arXiv: 09051257.

[12] X. Cabré and J. Terra, *Saddle-shaped solutions of bistable diffusion equations in all of* $\mathbb{R}^{2m}$, J. Eur. Math. Soc. **11** (2009), 819-843.

[13] X. Cabré and J. Terra, *Qualitative properties of saddle-shaped solutions to bistable diffusion equations*, arXiv: 0907.3008.

[14] L. Caffarelli, J-M. Roquejoffre, and O. Savin, *Nonlocal minimal surfaces*, arXiv: 0905.1183.

[15] L. Caffarelli and L. Silvestre, *An extension related to the fractional Laplacian*, Comm. Part. Diff. Eq. **32** (2007), 1245–1260.

[16] L. Caffarelli and P.E. Souganidis, *Convergence of nonlocal threshold dynamics approximations to front propagation*, Arch. Ration. Mech. Anal. **195** (2010), 1–23.

[17] L. Caffarelli and E. Valdinoci, *Regularity properties of nonlocal minimal surfaces via limiting arguments*, forthcoming.

[18] L. Caffarelli and A. Vasseur, *Drift diffusion equations equations with fractional diffusion and the quasi-geostrophic equation*, arXiv: 0608447.

[19] E. Cinti, *Saddle-shaped solutions for nonlinear equations involving the half-Laplacian*, forthcoming.

[20] H. Dang, P. C. Fife, and L. A. Peletier, *Saddle solutions of the bistable diffusion equation,* Z. Angew Math. Phys. **43** (1992), 984–998.

[21] E. De Giorgi, *Convergence problems for functionals and operators*, Proc. Int. Meeting on Recent Methods in Nonlinear Analysis (Rome 1978). Pitagora, Bologna (1979), 131–188.

[22] M. del Pino, M. Kowalczyk and J. Wei, *On De Giorgi Conjecture in dimension* $N \geq 9$, arXiv: 0806.3141.

[23] E. B. Fabes, C. E. Kenig, and R. P. Serapioni, *The local regularity of solutions of degenerate elliptic equations*, Comm. Partial Differential Equations, 7 **1**, 77–116. (1982)

[24] N. Ghoussoub and C. Gui, *On a conjecture of De Giorgi and some related problems,* Math. Ann. **311** (1998), 481–491.

[25] D. Jerison and R. Monneau, *Towards a counter-example to a conjecture of De Giorgi in high dimensions,* Ann. Mat. Pura Appl. **183** (2004), 439–467.

[26] Y.Y. Li and L. Zhang, *Liouville-type theorems and Harnack-type inequalities for semilinear elliptic equations*, J. Anal. Math. **90**, 27–87.

[27] J.L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications I*, Springer-Verlag (1972).

[28] M.d.M. Gonzalez, *Gamma convergence of an energy functional related to the fractional Laplacian*, Calc. Var. Part. Diff. Eq. **36** (2009), 173–210.

[29] L. Modica and S. Mortola, *Un esempio di* Γ^-*-convergenza,* Boll. Un. Mat. Ital. B **14** (1977), 285–299.

[30] L. Modica, *A gradient bound and a Liouville theorem for nonlinear Poisson equations*, Comm. Pure Appl. Math. **38** (1985), 679–684.

[31] L. Moschini, *New Liouville theorems for linear second order degenerate elliptic equations in divergence form*, Ann. I. H. Poincaré **22** (2005), 11–23.

[32] A. Nekvinda, *Characterization of traces of the weighted Sobolev space* $H^{1,p}_{\varepsilon,M}$, Funct. Approx. Comment. Math **20** (1992), 143–151

[33] A. Nekvinda, *Characterization of traces of the weighted Sobolev space* $W^{1,p}(\Omega, d^{\varepsilon}_M)$ *on* M, Czechoslovak Mth. J. **43** (118) (1993), n 4, 713–722.

[34] O. Savin, *Phase transitions: regularity of flat level sets,* Ann. of Math. **169** (2009), 41–78.

[35] M. Schatzman, *On the stability of the saddle solution of Allen-Cahn's equation,* Proc. Roy. Soc. Edinburgh Sect. A **125** (1995), 1241–1275.

[36] L. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Comm. Pure Appl. Math. 60, **1**, 67–112

[37] Y. Sire and E. Valdinoci, *Fractional Laplacian phase transitions and boundary reactions: A geometric inequality and a symmetry result*, Jour. Functional Analysis 256 (2009) **6**, 1842–1864.

[38] J.F. Toland, *The Peierls-Nabarro and Benjamin-Ono equations*, J. Funct. Anal. **145** (1997), no. 1, 136–150.