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POSE ESTIMATION FOR UNDERACTUATED CABLE-DRIVEN PARALLEL
ROBOTS

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POSE ESTIMATION FOR UNDERACTUATED CABLE-DRIVEN PARALLEL ROBOTS

A dissertation submitted in partial fulfillment
of the requirements for the degree of Doctor of Philosophy
in Mechanics and Advanced Engineering Sciences

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*To Aris and Andrea,
for always being by my side,
for the gift of their love.*

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Abstract

This PhD thesis addresses pose estimation for underactuated cable-driven parallel robots (*UACDPRs*) using proprioceptive sensing. In underactuated architectures, the direct geometric mapping from measured cable lengths to end-effector pose is intrinsically underdetermined: geometry alone does not generally yield a unique configuration, and the physically attained pose is typically selected by additional constraints such as static equilibrium and feasibility conditions. While direct geometrico-static formulations can close the problem by enforcing equilibrium, they introduce modeling dependencies on external wrenches and mass/inertia parameters, and they may admit multiple equilibria.

The main objective pursued here is to obtain accurate and reliable pose estimates for *UACDPRs* without explicitly enforcing equilibrium at estimation time, by exploiting redundant proprioceptive measurements and a coherent geometric/differential-kinematic modeling framework. The adopted model covers both spatial and planar cases within a unified formulation and includes routing effects induced by swivel pulleys through explicit swivel and tangency-angle variables. Differential kinematics is derived to obtain analytic Jacobians required by iterative solvers.

Building on these foundations, the thesis develops a sensor-fusion direct-kinematics method cast as a weighted nonlinear least-squares problem. A flexible residual set is constructed by stacking heterogeneous pose-dependent quantities, such as cable lengths, swivel/tangency angles, and directly measured pose components (e.g., orientation from an inclinometer). The resulting estimation problem is solved via Gauss–Newton iterations, with weights selected according to an inverse-error rationale to handle heterogeneous units and sensor reliability. Practical guidelines are provided for residual-set design and Jacobian assembly from analytic differential kinematics.

A central practical contribution concerns iteration termination. Since pose and residual components are heterogeneous (e.g., meters versus radians or degrees), norm-based stopping rules can be difficult to interpret and sensitive to scaling choices. Component-wise termination conditions are therefore formulated to enforce physically meaningful bounds separately on translation and orientation updates and, crucially, on each residual component. These criteria prevent premature “zero-iteration” acceptances and provide a systematic, interpretable stopping mechanism aligned with expected sensor accuracy.

The sensor-fusion estimator is further extended through a tension-aware procedure. In *CDPRs*, the reliability of cable-related measurements can vary significantly with cable tension: low-tension regimes are prone to sagging, backlash, and deviations from ideal straight-line and frictionless-pulley assumptions. This thesis models measurement uncertainty as a function of measured cable tensions via identifiable tension–error relations. Both linear-in-tension and inverse-tension models are considered, and their coefficients are identified offline from calibration data using density-aware fitting strategies. The identified models are then used online to adapt both the weighting matrix

and the component-wise termination thresholds, thereby reshaping the internal solver behavior according to configuration-dependent reliability.

Experimental validations are reported on two platforms: a planar 2-cable *UACDPR* prototype and a spatial 4-cable, 6-DoF *UACDPR* prototype. On the planar system, vision-based ground truth (ChArUco tracking) is used to quantify accuracy and to compare residual sets and termination rules. On the spatial system, Vicon-based ground truth supports an extensive comparison of sensor combinations, stopping criteria, and the proposed tension-aware formulation. The results show that appropriate redundancy choices improve observability and that the component-wise termination strategy provides accuracy comparable to stricter norm-based rules while obtaining lower runtimes. The tension-aware formulation preserves estimation accuracy while markedly reducing average computational cost and iteration counts, often achieving convergence in a single Gauss–Newton update.

Two appendices complement the main contributions. Sensitivity-based uncertainty propagation bounds are summarized to relate bounded measurement errors to pose-estimation uncertainty in underactuated settings. Moreover, a Kantorovich-based convergence certification framework is specialized to the direct geometrico-static problem of 3-cable *UACDPRs*, deriving sufficient conditions for existence/uniqueness of the equilibrium solution and conservative bounds on admissible cable-length measurement errors that guarantee Newton convergence.

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Chapter 1

Introduction

Producing a prescribed motion is one of the core purposes of a machine. Robots push this idea further: they are motion generators whose trajectories can be programmed and adapted to different tasks. Motion capabilities can be described in terms of degrees of freedom, achievable speeds and accelerations, and the accuracy with which a pose can be reached and maintained.

Long before robotics, cables were used to transport and manipulate heavy loads. Construction and industrial cranes are still widely adopted because they are versatile, scalable, and cost-effective. When a suspended load must be positioned accurately, operators often attach additional guide ropes to steer it, damp sway, and counteract disturbances such as wind. Each rope constrains the motion in specific directions, and coordinated pulling by multiple workers provides a form of manual control over the load. Replacing this manual action with motorized winches is a natural evolution; once the winches are synchronized by a controller, the crane concept effectively becomes a cable-driven parallel robot.

Cable-driven parallel robots (*CDPRs*) are parallel mechanisms in which a moving platform is actuated through multiple cables. The idea of cable-suspended, cable-actuated robots was formalized in the 1980s and has since evolved into a broad research area. A key motivation behind their development is the combination of a simple mechanical layout with large workspaces and high dynamic capabilities, since heavy actuators can be placed on or close to the base, while only lightweight cables move with the platform. Beyond scalability, this feature supports high dynamic performance and makes *CDPRs* attractive when coverage area, payload-to-weight ratio, and deployability are important [1].

1.1 Cable-driven parallel-robots

This section briefly surveys representative application domains, with selected examples that illustrate how cable-driven architectures have been tailored to different task requirements.

***CDPR* applications.** The interest in *CDPRs* is largely driven by applications where large workspaces, high payload-to-weight ratios, and low moving inertia are decisive advantages. Early large-scale demonstrations are closely related to cable-suspended transport and camera systems. In [2], the kineto-statics of a SkyCam-type cable-transport

system is analyzed, offering an early and influential perspective on cable-driven motion over large areas.

Since the early 2000s, *CDPRs* have been investigated for an expanding set of applications. In industrial contexts, [3] presents a cable-driven robot for curtain-wall module installation, with an automation framework aimed at safer and faster handling of heavy modules. *CDPRs* have also been proposed for manufacturing tasks such as additive manufacturing [4], and for large-area operations such as ship-side painting [5].

Beyond industrial productivity, creative applications have been proposed: [6] uses human motion-captured information to generate spray-painting trajectories for a cable-driven platform. In [7], a large-scale 3D-printing machine based on a 3-*DoF* *CDPR* is presented and employed for an artistic exhibition. The printing material is a powder made of glass micro-beads, deposited along a prescribed trajectory so that the resulting structure is intentionally transient and progressively collapses over time.

Medical and rehabilitation devices are another active domain: [8] reports the design and experimental validation of a *CDPR* for head-neck movement training with assistance modes tailored to the user. In [9], the *CDPR MARIONET* is introduced as a device to measure human joint motion and to support rehabilitation protocols. Rescue-oriented applications have also been investigated. In [10], the preliminary development of a full-scale, portable, modular, and fully autonomous parallel wire crane intended for rescue operations is reported. More recently, [11] presents a feasibility study and preliminary tests of an underactuated cable-driven parallel robot for automated launch and recovery operations on the sea surface.

CDPR architectures. A distinctive feature of *CDPRs* is that actuation is provided through flexible cables, which can pull but cannot push. This unilateral nature has immediate design implications: admissible configurations and forces are constrained by the requirement of positive tensions and, more generally, by actuator limits and workspace restrictions [1]. For this reason, the choice of the number of cables and of their routing is tightly linked to how the cables constrain the platform motion and to the size of the feasible workspace.

CDPRs have historically been classified by comparing the number of actuated cables n with the *EE DoFs* λ , from a constrain-ability perspective [12]. In this framework:

- if $n \leq \lambda \leq 6$, the robot is *underconstrained*, since the *EE* cannot withstand an arbitrary external wrench with all cables being under tension; conversely, cables cannot remain taut without a counterbalancing external load;
- if $n = \lambda + 1$, the robot is *completely constrained*, since cables are sufficient in number to effectively tension each other even without external counterbalancing loads;
- if $n > \lambda + 1$, the robot is *redundantly constrained*, since the number of cables exceeds the minimum required to constrain the *EE*; additional cables can be exploited to optimize *CDPR* performance.

The classification above is mainly architectural as it depends only on how many cables the robot has relative to the *EE DoFs*. However, in standard mechanics and control terminology, *underconstrained* has a more physical meaning: the *EE* is *underconstrained* if it has more *DoFs* than the dimension of the constraint space spanned by passive and active constraints [13], and thus it retains residual mobility even if the actuators are

locked. In this case, its configuration is not determined by geometry alone, but also depends on mechanical equilibrium. As a consequence, in underconstrained systems, kinematics and statics/dynamics are coupled [14]. For this reason, a different viewpoint in which the constraint status is not only an architectural feature, but an instantaneous one is proposed: it depends on how many cables are actually taut at a given time. With this interpretation:

- $n = \lambda$: when all cables are taut the manipulator is *completely constrained*, since the *EE* configuration is fully determined by cable lengths and, under rigid-body assumptions, the inverse static/dynamic problem yields a unique tension solution;
- $n > \lambda$: when all cables are taut the robot is *redundantly constrained* and the tension distribution is not unique, as it admits infinitely many solutions under rigid-body assumptions;
- even if $n \geq \lambda$, the *EE* becomes *underconstrained* whenever fewer than λ cables are taut [15];
- if $n < \lambda$, the robot is necessarily *underconstrained*.

Depending on the application, the robot is designed accordingly. For instance, redundancy can be used to improve dexterity, avoid workspace singularities, and enhance robustness. A 3-DoF robot driven by eight cables organized in paired parallelogram topologies is presented in [16] to reduce actuator count while avoiding interference and sensitivity issues. Hybrid and reconfigurable concepts further extend the design space. A mixed aerial/ground cable-driven system combining quadrotors and ground robots is modelled in [17], aiming to overcome the workspace limitations of purely fixed-anchor designs.

UACDPRs often adopt simpler hardware, but require careful analysis of feasible motion directions and operational conditions. Moreover, cable lengths alone do not uniquely determine the platform pose: for assigned actuation inputs, the physically attained configuration is selected by additional conditions, most notably mechanical equilibrium and feasibility requirements [18, 19, 20]. Those designs can be exploited to reduce hardware complexity and cost, and they motivate dedicated modelling, estimation, and control strategies tailored to the coupling between geometry, feasibility, and statics [19, 21].

1.2 Pose-Estimation of cable-driven parallel robots

Pose estimation in robotics aims at inferring the *EE* pose, namely its position in a ground-fixed reference frame and its orientation under a chosen parametrization, by combining a system model with sensor data [22]. In parallel robots [23], pose estimation is often intertwined with direct kinematics, since the *EE* pose is constrained by multiple kinematic chains and must be reconstructed from internal measurements [24, 25]. Compared to serial manipulators, this setting typically entails stronger nonlinearities and may admit multiple kinematically admissible poses, so that estimation algorithms must combine accurate modeling with numerical robustness and, when needed, mechanisms to discriminate among competing solutions [26].

Measurements used for pose estimation are commonly categorized into exteroceptive and proprioceptive sources. Exteroceptive sensing, such as motion-capture systems or camera-based tracking, provides direct information on the EE pose but may require external infrastructure, careful calibration of the environment, and favorable visibility conditions [27]. Proprioceptive sensing relies on measurements embedded in the robot, such as winch encoders for cable lengths, pulley or cable-angle sensors, tension sensors, and inertial measurements mounted on the EE [28]. Proprioceptive systems are self-contained and can operate without external setup, which is attractive for field deployment and for large workspaces where external metrology becomes cumbersome [2, 29].

In $CDPR$ s, pose estimation is closely related to the direct-kinematics problem [1], since cable-related measurements and, when available, inertial data are mapped to a platform pose through geometric (and sometimes static [19]) models. This setting introduces challenges that are not as common in conventional rigid-link manipulators, such as unilateral actuation constraints [30], configuration-dependent measurement errors and increased sensitivity of the direct-kinematics solution [31], and the possible coexistence of multiple admissible direct-kinematics solutions. When multiple solutions must be isolated or when existence and uniqueness must be certified over a domain, global numerical methods such as interval analysis can be employed, at a higher computational cost [32].

From an algorithmic viewpoint, pose estimation can be formulated either as a static optimization problem or as a recursive estimation problem. In a static formulation, a residual vector is defined to collect the mismatch between measurements and model-based predictions, and the pose is obtained by minimizing a possibly weighted nonlinear least-squares cost [33]. This cost is typically solved with Newton-type methods such as Gauss–Newton or Levenberg–Marquardt [34, 35]. Weights are commonly used to reflect heterogeneous sensor accuracy and to obtain statistically meaningful estimates under standard noise assumptions [33]. In practice, robust penalties may also be adopted to reduce the influence of occasional outliers in cable-related readings. In a recursive formulation, filtering methods fuse time sequences of proprioceptive measurements and incorporate a process model [36, 37]. In $CDPR$ s, this perspective has been used, for instance, to combine inertial information with cable-related measurements [38]. Depending on the application, equilibrium conditions and non-negativity of cable tensions can be enforced explicitly within the estimation procedure [32], or handled indirectly through suitable measurement selection and modeling choices [33].

A recurring challenge in $CDPR$ pose estimation is that measurement reliability may vary substantially across the workspace. Cable-related readings can be affected by routing imperfections, friction or backlash, sagging, and tension variations, so that the same sensor may not exhibit uniform accuracy in all configurations [39]. Systematic effects, such as geometric miscalibration and encoder offsets, can also introduce biases that are not captured by a constant-variance noise model. These aspects motivate sensor-fusion designs that exploit redundancy through multiple heterogeneous proprioceptive sensing and weighting strategies that modulate the influence of each sensor according to expected reliability [40, 33]. Building on these ideas, the thesis develops a sensor-fusion direct-kinematics framework tailored to $UACDPR$ s, where redundant proprioceptive measurements are leveraged to make the EE pose observable without explicitly imposing equilibrium constraints at estimation time, and where tension-dependent error modeling is used to adapt the estimation weights to configuration-dependent reliability.

1.3 Thesis Goals and Outline

The overall goal of this thesis is to enable accurate and reliable pose-estimation solutions for underactuated cable-driven parallel robots (*UACDPRs*) using proprioceptive sensor measurements. Because underactuation makes the forward geometric mapping intrinsically underdetermined, the core objective is to introduce additional constraints through redundant sensing, so that a unique pose estimate can be obtained in practice.

Within this scope, the thesis pursues the following specific goals:

- establish a coherent geometric and differential-kinematic modeling framework for *UACDPRs*, including routing effects induced by swivel pulleys when relevant;
- review and specialize numerical strategies to solve the nonlinear problems arising in direct kinematics, with emphasis on Newton-type methods and on their convergence properties;
- develop a sensor-fusion direct-kinematics method that exploits redundant proprioceptive measurements within a weighted non-linear least squares (*WNLS*) formulation solved by Gauss–Newton iterations, and provide practical guidelines for residual-set design, weighting, and stopping criteria;
- extend the sensor-fusion estimator with a tension-aware procedure, where measurement uncertainty (and thus both weights and termination thresholds) depend on cable tensions via identifiable tension–error models;
- validate the proposed algorithms experimentally and discuss their performance, robustness, and limitations.

The remainder of the thesis is organized as follows. Chapter 2 introduces the modeling foundations used throughout the manuscript, including the geometric model and the associated differential kinematics for *UACDPRs*. Chapter 3 frames the relevant direct-kinematics problem statements for underactuated *UACDPRs* and surveys numerical solution approaches, from local Newton-type schemes to global techniques, highlighting the aspects that motivate the solver choices adopted in this work. Chapter 4 presents the main outcome of the PhD: the sensor-fusion direct-kinematics framework and its tension-aware extension, formulated consistently with the models of Chapter 2 and the numerical tools recalled in Chapter 3. Chapter 5 reports the experimental results and assesses the proposed methods under representative operating conditions. Finally, Chapter 6 summarizes the main findings and outlines directions for future research.

Two appendices complement the main text by collecting technical material that supports, but would otherwise interrupt, the main narrative. Appendix A summarizes sensitivity-based bounds, adapted from [41], to quantify how bounded measurement uncertainties propagate to pose-estimated variables in *UACDPRs*. Appendix B reports the theoretical developments of [42], which address a direct geometrico-static formulation in the spatial 3-cable case and study the numerical solution of the resulting square nonlinear system via Newton iterations. This part of the thesis stems from the research activity carried out during the candidate’s period abroad within the HEPHAISTOS team at Inria Sophia Antipolis, Université Côte d’Azur, under the supervision of Prof. Jean-Pierre Merlet, and is included here also to highlight the scientific value of that international collaboration. In particular, the appendix derives convergence and uniqueness conditions based on the Kantorovich theorem and uses them to relate

bounded cable-length measurement errors to guaranteed Newton convergence toward the equilibrium solution in the considered DGSP setting.

Chapter 2

Geometric Model and Differential Kinematics of Underactuated Cable-Driven Parallel Robots

This chapter introduces the modelling framework adopted throughout the thesis for underactuated cable-driven parallel-robots [1]. The main developments address spatial architectures evolving in the Special Euclidean space of dimension 3 ($SE(3)$). Planar cases are treated as a constrained specialization of the same formulation, where the end-effector is assumed to translate on a plane and to rotate only about the axis normal to that plane, which allows an equivalent description in the Special Euclidean space of dimension 2 ($SE(2)$) [22]. Cables are modelled as massless and inextensible straight segments, and cable routing at the base is described through swivel pulleys that impose geometric constraints on both the admissible cable directions and the effective cable lengths [1, 39]. Actuator displacements are assumed to be proportional to commanded variations in cable lengths [1].

In the following, a general formulation for spatial *UACDPRs* is introduced, covering the geometric model (Subsec. 2.1), which includes geometric entities and constraint equations, and the corresponding differential kinematics (Subsec. 2.2), which derives first-order kinematic relationships and Jacobian matrices required in subsequent developments.

2.1 Geometric model

This section introduces the geometric model of a generic spatial *UACDPR*, establishing the reference frames, key points, and pulley-related angles used throughout the thesis. Starting from the platform pose in $SE(3)$, the cable routing through swivel pulleys is described via the swivel and tangency angles, leading to the constraint equations that relate cable lengths to the end-effector configuration and clarify the underdetermined nature of the forward problem in the underactuated case [18, 43, 44, 19].

2.1.1 Reference frames and pose parametrization

A spatial underactuated cable-driven parallel-robot (*UACDPR*) consists of a moving platform connected to a fixed base by $n < 6$ cables [1, 19] (Fig. 2.1). An inertial frame $Oxyz$ and a body-fixed frame $Px'y'z'$, attached to the platform, are introduced to

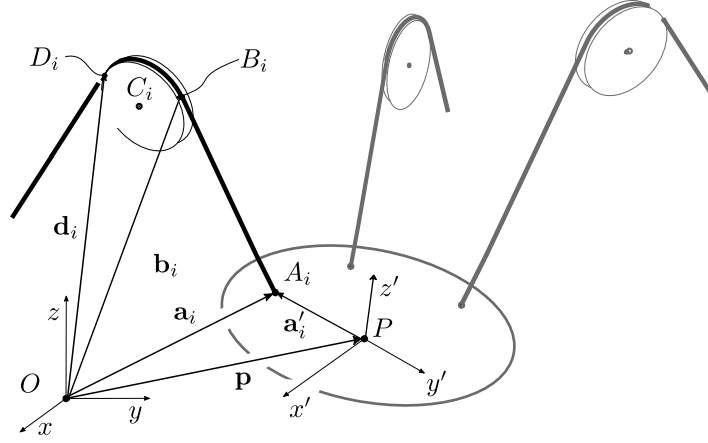


Figure 2.1: Geometric entities for a spatial UACDPR

describe the end-effector (*EE*) pose. The position of the origin of $Px'y'z'$ with respect to $Oxyz$ is denoted by $\mathbf{p} \in \mathbb{R}^3$. The platform orientation is represented by a rotation matrix $\mathbf{R} \in SO(3)$ parameterized through roll–pitch–yaw (RPY) angles [22] $\boldsymbol{\epsilon} = [\psi, \theta, \chi]^T$, i.e.,

$$\mathbf{R} = \mathbf{R}_z(\psi)\mathbf{R}_y(\theta)\mathbf{R}_x(\chi). \quad (2.1)$$

where $\mathbf{R}_z(\cdot)$, $\mathbf{R}_y(\cdot)$ and $\mathbf{R}_x(\cdot)$ are elementary rotation matrices about the z , y and x axes. The generalized coordinates are collected as

$$\boldsymbol{\zeta} = [\mathbf{p}^T \quad \boldsymbol{\epsilon}^T]^T. \quad (2.2)$$

For each cable $i = 1, \dots, n$, the attachment point on the platform is A_i . Its coordinates are constant in $Px'y'z'$ and denoted by ${}^P\mathbf{a}'_i$, whereas its position in $Oxyz$ is

$$\mathbf{a}_i = \mathbf{p} + \mathbf{R}^P\mathbf{a}'_i \quad \text{and} \quad \mathbf{a}'_i = \mathbf{R}^P\mathbf{a}_i. \quad (2.3)$$

2.1.2 Pulley routing model and local pulley frame

Each cable is guided into the workspace by a swivel pulley mounted on a hinged support [39, 1]. The pulley center is denoted by C_i . The pulley rotates about a swivel axis z_i , which is tangent to the pulley groove at the fixed entry point D_i . The coordinates of D_i are constant in $Oxyz$ and its position vector is $\mathbf{d}_i$. The cable exits the groove at point B_i , whose position generally depends on the platform pose and on the pulley geometry [39, 45].

A dedicated reference frame $D_ix_iy_iz_i$ is introduced for each pulley, with origin at D_i . Its orientation with respect to $Oxyz$ is described by unit vectors $\mathbf{i}_i, \mathbf{j}_i, \mathbf{k}_i$ aligned with the axes x_i, y_i, z_i , respectively. In particular, $\mathbf{k}_i$ is directed along the swivel axis z_i .

The vector joining D_i to A_i is defined as

$$\boldsymbol{\rho}_i = \mathbf{a}_i - \mathbf{d}_i. \quad (2.4)$$

Together with $\mathbf{k}_i$, $\boldsymbol{\rho}_i$ defines the *pulley plane*. The swivel angle $\sigma_i \in [-\pi, \pi]$ is introduced as the angle between the coordinate plane x_iz_i and the pulley plane [39] (Fig. 2.2a).

A unit vector normal to the pulley plane is

$$\mathbf{w}_i = -\sin(\sigma_i)\mathbf{i}_i + \cos(\sigma_i)\mathbf{j}_i, \quad (2.5)$$

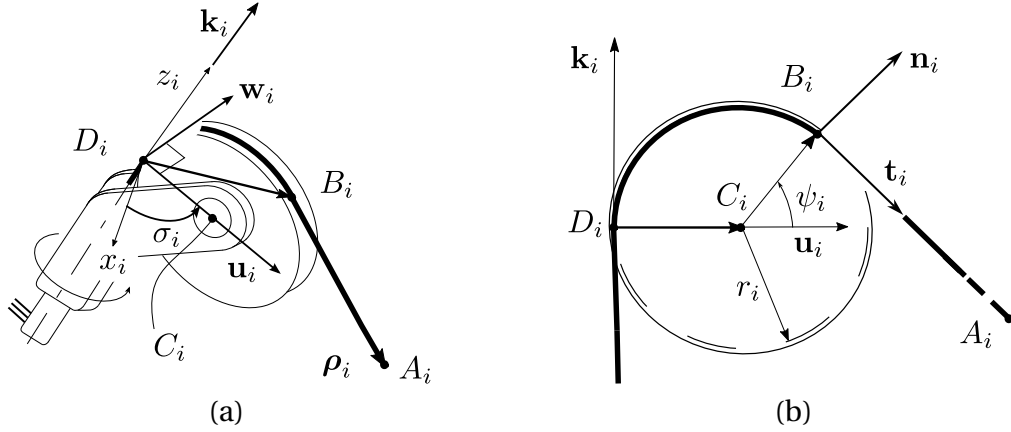


Figure 2.2: Swivel pulley geometric model. Local pulley view (a) and pulley plane representation (b).

and the unit vector along the radial direction in the pulley plane $C_i - D_i$ can be written as

$$\mathbf{u}_i = \mathbf{w}_i \times \mathbf{k}_i = \cos(\sigma_i)\mathbf{i}_i + \sin(\sigma_i)\mathbf{j}_i. \quad (2.6)$$

The tangency angle $\psi_i \in [-\pi, \pi]$ is defined as the angle between $\mathbf{u}_i$ and the direction $B_i - C_i$ [39]. Accordingly, the unit vector $\mathbf{n}_i$ directed as $B_i - C_i$ is

$$\mathbf{n}_i = \cos(\psi_i)\mathbf{u}_i + \sin(\psi_i)\mathbf{k}_i, \quad (2.7)$$

and the unit tangent vector to the cable segment at the pulley exit, directed as $A_i - B_i$, is

$$\mathbf{t}_i = \mathbf{w}_i \times \mathbf{n}_i = \sin(\psi_i)\mathbf{u}_i - \cos(\psi_i)\mathbf{k}_i. \quad (2.8)$$

2.1.3 Pulley constraints and computation of swivel angles

The first constraint imposed by the swivel pulley enforces that $\boldsymbol{\rho}_i$ lies in the pulley plane, i.e.,

$$\mathbf{w}_i \cdot \boldsymbol{\rho}_i = 0. \quad (2.9)$$

By substituting (2.5) into (2.9) and rearranging, the swivel angle can be computed as

$$\sigma_i = \text{atan2}(\mathbf{j}_i \cdot \boldsymbol{\rho}_i, \mathbf{i}_i \cdot \boldsymbol{\rho}_i). \quad (2.10)$$

2.1.4 Pulley constraints and computation of tangency angles

Let $\mathbf{b}_i$ be the position vector of B_i expressed in $Oxyz$, and define the exit-to-attachment cable segment vector $\boldsymbol{\rho}_i$ ¹ as:

$$\boldsymbol{\rho}_i = \mathbf{a}_i - \mathbf{b}_i. \quad (2.11)$$

If the pulley radius is r_i , the exit point B_i can be written as

$$\mathbf{b}_i = \mathbf{d}_i + r_i(\mathbf{u}_i + \mathbf{n}_i), \quad (2.12)$$

¹The symbols $\boldsymbol{\rho}_i$ and $\boldsymbol{\rho}_i$ are purposely chosen to emphasize that they coincide in the limiting case $r_i = 0$.

which yields

$$\boldsymbol{\rho}_i = \mathbf{a}_i - \mathbf{b}_i = \boldsymbol{\varrho}_i - r_i(\mathbf{u}_i + \mathbf{n}_i). \quad (2.13)$$

The second geometric constraint imposed by the pulley enforces tangency between the straight cable segment and the pulley groove, i.e.,

$$\mathbf{n}_i \cdot \boldsymbol{\rho}_i = 0. \quad (2.14)$$

Substituting (2.13) into (2.14) and using (2.7) leads to

$$(\varrho_{u_i} - r_i) \cos(\psi_i) + \varrho_{k_i} \sin(\psi_i) - r_i = 0, \quad (2.15)$$

where $\varrho_{u_i} = \mathbf{u}_i \cdot \boldsymbol{\varrho}_i$ and $\varrho_{k_i} = \mathbf{k}_i \cdot \boldsymbol{\varrho}_i$.

Introducing the half-angle substitution $t_i = \tan(\psi_i/2)$ and

$$\cos(\psi_i) = \frac{1 - t_i^2}{1 + t_i^2}, \quad \sin(\psi_i) = \frac{2t_i}{1 + t_i^2},$$

Eq. (2.15) becomes a quadratic equation in t_i , whose solutions are

$$t_i = \frac{\varrho_{k_i}}{\varrho_{u_i}} \pm \sqrt{\left(\frac{\varrho_{k_i}}{\varrho_{u_i}}\right)^2 + 1 - \frac{2r_i}{\varrho_{u_i}}}. \quad (2.16)$$

For the common routing case in which the cable wraps clockwise on the pulley, the physically admissible solution corresponds to the positive sign. Thus, the tangency angle can be computed as

$$\psi_i = 2 \operatorname{atan} \left[\frac{\varrho_{k_i}}{\varrho_{u_i}} + \sqrt{\left(\frac{\varrho_{k_i}}{\varrho_{u_i}}\right)^2 + 1 - \frac{2r_i}{\varrho_{u_i}}} \right]. \quad (2.17)$$

2.1.5 Cable-length constraint

Let $l_i > 0$ denote the total cable length, accounting for both the straight segment and the portion wrapped on the pulley. The wrapped arc length is $\widehat{D_i B_i} = r_i(\pi - \psi_i)$, and the straight segment length equals $\|\boldsymbol{\rho}_i\|$. Therefore, the cable length model is

$$l_i = r_i(\pi - \psi_i) + \|\boldsymbol{\rho}_i\| = r_i(\pi - \psi_i) + \sqrt{\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i}. \quad (2.18)$$

Consequently, the cable imposes the scalar constraint

$$\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i - [l_i - r_i(\pi - \psi_i)]^2 = 0. \quad (2.19)$$

2.1.6 Geometric inverse and forward problems

Given a platform configuration $\boldsymbol{\zeta}$, Eqs. (2.10)–(2.18) define an inverse geometric mapping from pose to cable variables [1], returning σ_i , ψ_i , l_i for $i = 1, \dots, n$. However, a purely geometric inverse solution does not automatically guarantee physical admissibility, since the unilateral nature of cable actuation may prevent the existence of a strictly positive tension distribution for the prescribed configuration [46].

Conversely, assigning cable lengths yields $n < 6$ scalar equations (2.19) in the 6 unknown components of the pose vector $\boldsymbol{\zeta}$. Here σ_i and ψ_i are not treated as additional

independent unknowns: for a given pose ζ they are uniquely determined by the pulley-geometry relations (2.10) and (2.17) and can be substituted into (2.19), yielding n constraints in the pose variables only. Hence, the associated forward geometric problem is underdetermined [19, 20], and the configuration physically realized for assigned cable lengths is selected among infinitely many kinematically admissible candidates by additional conditions. Most importantly, the pose must be compatible with the robot mechanical equilibrium, and it must satisfy a set of *feasibility constraints*, i.e., physical and operational requirements that rule out geometrically valid but non-realizable solutions. Other common feasibility constraints include:

- *Tension feasibility*: cables can only sustain traction, hence $T_i \geq 0$ for all i , often strengthened to $T_i \geq T_{\min} > 0$ to prevent slack; actuator and structural limitations impose upper bounds $T_i \leq T_{\max}$ [46, 30].
- *Routing and pulley admissibility*: the pulley-related variables must remain within hardware limits, e.g., $\sigma_i \in [\sigma_i^{\min}, \sigma_i^{\max}]$ and $\psi_i \in [\psi_i^{\min}, \psi_i^{\max}]$, and the physically consistent branch must be selected when multiple geometric solutions exist [47] (e.g., for the tangency angle).
- *Cable-length and actuator limits*: feasible solutions must respect winch stroke constraints, $l_i \in [l_i^{\min}, l_i^{\max}]$, and, when relevant, bounds on $\dot{l}_i$ and $\ddot{l}_i$ dictated by actuator capabilities.
- *Workspace and collision constraints*: the end-effector pose must belong to an admissible workspace and avoid collisions and self-intersections.
- *Non-degeneracy conditions*: configurations leading to singular/ill-conditioned geometric relations (e.g., vanishing denominators in the pulley-angle expressions) are excluded to preserve physical consistency and numerical robustness [48].

2.2 Differential kinematics

This section derives the differential kinematic relationships associated with the spatial geometric constraints introduced above. In particular, the time derivatives of the swivel and tangency angles and of the cable lengths are expressed as linear functions of the end-effector twist, leading to the corresponding kinematic Jacobians [1] used later for pose estimation and sensitivity analyses. Further details on the sensitivity matrices and the associated worst-case error bounds used for uncertainty propagation are collected in Appendix A.

2.2.1 Angular velocity and twist representation

Let $\mathbf{\Omega} = \dot{\mathbf{R}}\mathbf{R}^T$ denote the angular velocity matrix. The associated angular velocity vector $\boldsymbol{\omega}$ depends nonlinearly on $\boldsymbol{\epsilon}$ and linearly on $\dot{\boldsymbol{\epsilon}}$ [22]:

$$\boldsymbol{\omega} = \mathbf{H}(\boldsymbol{\epsilon})\dot{\boldsymbol{\epsilon}}, \quad (2.20)$$

with

$$\mathbf{H}(\boldsymbol{\epsilon}) = \begin{bmatrix} -c_\psi s_\theta & -s_\psi & c_\psi s_\theta \\ -s_\psi s_\theta & c_\psi & s_\psi s_\theta \\ 1 - c_\theta & 0 & c_\theta \end{bmatrix}, \quad (2.21)$$

where $c_\psi = \cos \psi$, $s_\psi = \sin \psi$, $c_\theta = \cos \theta$, and $s_\theta = \sin \theta$.

The EE twist is defined as [22] $\mathbf{v} = [\dot{\mathbf{p}}^T \ \boldsymbol{\omega}^T]^T$ and can be related to $\dot{\boldsymbol{\zeta}}$ through

$$\mathbf{v} = \mathbf{D}(\boldsymbol{\epsilon})\dot{\boldsymbol{\zeta}}, \quad \mathbf{D}(\boldsymbol{\epsilon}) = \begin{bmatrix} \mathbf{I}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{H}(\boldsymbol{\epsilon}) \end{bmatrix}. \quad (2.22)$$

2.2.2 Time derivative of the swivel angle

Taking the time derivative of (2.9) gives

$$\dot{\mathbf{w}}_i \cdot \boldsymbol{\rho}_i + \mathbf{w}_i \cdot \dot{\boldsymbol{\rho}}_i = 0. \quad (2.23)$$

Since $\mathbf{d}_i$ is constant, $\dot{\boldsymbol{\rho}}_i = \dot{\mathbf{a}}_i$. Differentiating (2.3) yields

$$\dot{\mathbf{a}}_i = \dot{\mathbf{p}} + \boldsymbol{\omega} \times \mathbf{a}'_i. \quad (2.24)$$

Moreover, differentiating (2.5) gives $\dot{\mathbf{w}}_i = -\dot{\sigma}_i \mathbf{u}_i$. Substitution into (2.23) leads to

$$\rho_{u_i} \dot{\sigma}_i = \dot{\mathbf{p}} \cdot \mathbf{w}_i + \boldsymbol{\omega} \cdot (\mathbf{a}'_i \times \mathbf{w}_i), \quad (2.25)$$

where $\rho_{u_i} = \mathbf{u}_i \cdot \boldsymbol{\rho}_i$. Hence,

$$\dot{\sigma}_i = \boldsymbol{\xi}_{\sigma_i}^T \mathbf{v}, \quad \boldsymbol{\xi}_{\sigma_i} = \frac{1}{\rho_{u_i}} \begin{bmatrix} \mathbf{w}_i \\ \mathbf{a}'_i \times \mathbf{w}_i \end{bmatrix}. \quad (2.26)$$

2.2.3 Time derivative of the tangency angle

Differentiating the tangency condition (2.14) yields

$$\dot{\mathbf{n}}_i \cdot \boldsymbol{\rho}_i + \mathbf{n}_i \cdot \dot{\boldsymbol{\rho}}_i = 0. \quad (2.27)$$

The needed derivatives follow from (2.13), (2.6), and (2.7):

$$\dot{\boldsymbol{\rho}}_i = \dot{\mathbf{a}}_i - r_i (\dot{\mathbf{u}}_i + \dot{\mathbf{n}}_i), \quad (2.28)$$

$$\dot{\mathbf{u}}_i = \dot{\sigma}_i \mathbf{w}_i, \quad (2.29)$$

$$\dot{\mathbf{n}}_i = \cos(\psi_i) \dot{\sigma}_i \mathbf{w}_i - \dot{\psi}_i \mathbf{t}_i. \quad (2.30)$$

Substituting into (2.27) and using $\mathbf{n}_i \cdot \boldsymbol{\rho}_i = 0$ and $\mathbf{t}_i \cdot \boldsymbol{\rho}_i = \|\boldsymbol{\rho}_i\|$ yields

$$\|\boldsymbol{\rho}_i\| \dot{\psi}_i = \dot{\mathbf{p}} \cdot \mathbf{n}_i + \boldsymbol{\omega} \cdot (\mathbf{a}'_i \times \mathbf{n}_i), \quad (2.31)$$

hence

$$\dot{\psi}_i = \boldsymbol{\xi}_{\psi_i}^T \mathbf{v}, \quad \boldsymbol{\xi}_{\psi_i} = \frac{1}{\|\boldsymbol{\rho}_i\|} \begin{bmatrix} \mathbf{n}_i \\ \mathbf{a}'_i \times \mathbf{n}_i \end{bmatrix}. \quad (2.32)$$

2.2.4 Time derivative of the cable length

Differentiating the scalar constraint (2.19) yields

$$2\dot{\boldsymbol{\rho}}_i \cdot \boldsymbol{\rho}_i - 2\|\boldsymbol{\rho}_i\| \frac{d\|\boldsymbol{\rho}_i\|}{dt} = 0. \quad (2.33)$$

From (2.18), $\frac{d\|\boldsymbol{\rho}_i\|}{dt} = \dot{l}_i + r_i \dot{\psi}_i$. Substituting the expressions above and simplifying leads to

$$\dot{l}_i = \dot{\mathbf{p}} \cdot \mathbf{t}_i + \boldsymbol{\omega} \cdot (\mathbf{a}'_i \times \mathbf{t}_i) = \boldsymbol{\xi}_{l_i}^T \mathbf{v}, \quad (2.34)$$

where

$$\boldsymbol{\xi}_{l_i} = \begin{bmatrix} \mathbf{t}_i \\ \mathbf{a}'_i \times \mathbf{t}_i \end{bmatrix}. \quad (2.35)$$

2.2.5 Kinematic Jacobians and proper Jacobians

Stacking (2.26), (2.32), and (2.34) for $i = 1, \dots, n$ yields

$$\dot{\boldsymbol{\sigma}} = \Xi_{\sigma} \mathbf{v}, \quad \dot{\boldsymbol{\psi}} = \Xi_{\psi} \mathbf{v}, \quad \dot{\mathbf{l}} = \Xi_l \mathbf{v}, \quad (2.36)$$

where $\dot{\boldsymbol{\sigma}} = [\dot{\sigma}_1 \dots \dot{\sigma}_n]^T$, $\dot{\boldsymbol{\psi}} = [\dot{\psi}_1 \dots \dot{\psi}_n]^T$, and $\dot{\mathbf{l}} = [\dot{l}_1 \dots \dot{l}_n]^T$. The corresponding Jacobian matrices are defined by

$$\Xi_{\sigma}^T = [\xi_{\sigma_1} \dots \xi_{\sigma_n}], \quad \Xi_{\psi}^T = [\xi_{\psi_1} \dots \xi_{\psi_n}], \quad \Xi_l^T = [\xi_{l_1} \dots \xi_{l_n}]. \quad (2.37)$$

Since $\mathbf{v}$ is not, in general, an integrable coordinate vector, the matrices in (2.36) are *kinematic Jacobians* [22]. A proper Jacobian mapping from generalized coordinate rates can be obtained by combining (2.22) with (2.36), i.e.,

$$\dot{\boldsymbol{\sigma}} = \mathbf{J}_{\sigma} \dot{\boldsymbol{\zeta}}, \quad \dot{\boldsymbol{\psi}} = \mathbf{J}_{\psi} \dot{\boldsymbol{\zeta}}, \quad \dot{\mathbf{l}} = \mathbf{J}_l \dot{\boldsymbol{\zeta}}, \quad (2.38)$$

with

$$\mathbf{J}_{\sigma} = \Xi_{\sigma} \mathbf{D}, \quad \mathbf{J}_{\psi} = \Xi_{\psi} \mathbf{D}, \quad \mathbf{J}_l = \Xi_l \mathbf{D}. \quad (2.39)$$

Modeling assumptions and scope of applicability The geometric and differential-kinematic formulation introduced in this chapter relies on a number of modeling assumptions that define its scope of applicability. In particular, cables are modeled as massless and inextensible straight segments, the moving platform is treated as a rigid body, and the pulley-routing geometry is assumed known through calibrated geometric parameters. In addition, the swivel-pulley model assumes ideal geometric contact between the cable and the pulley groove.

These assumptions are appropriate for the prototypes and operating conditions considered in this thesis, but they may limit generalizability when the real system departs significantly from the ideal geometric description. The main sources of mismatch include cable sagging due to self-weight, elastic cable stretching, friction or backlash at the swivel/pulley interface, and geometric miscalibration of anchor points, pulley parameters, or platform attachment locations. The accuracy of the model may also degrade in near-slack conditions, where cable-related measurements become less representative of the ideal taut-cable geometry. These limitations motivate the weighting strategies and the tension-aware extensions introduced later in the thesis, which are designed to mitigate configuration-dependent losses of measurement reliability without abandoning the geometric formulation.

Chapter 3

Direct Kinematics of Underactuated Cable-Driven Parallel Robots

This chapter reviews the statement of the direct kinematic problem and the numerical strategies commonly adopted to solve it for parallel robots [26, 1], with emphasis on *CDPRs* and, in particular, on underactuated architectures. In *CDPRs*, the forward mapping from actuation variables to end-effector pose is typically implicit and non-linear; it may admit multiple solutions and can be ill-conditioned in portions of the workspace [49, 26, 48, 1]. Accordingly, additional information beyond pure geometry can be incorporated, such as mechanical equilibrium equations or extra proprioceptive measurements, and the resulting numerical solvers range from local iterative schemes (e.g., Newton and Gauss-Newton-type methods) to global techniques [19, 26, 50, 51] (e.g., interval analysis).

3.1 Problem statements for *UACDPR*

3.1.1 Direct geometric problem

Let the end-effector pose be represented by a vector of generalized coordinates ζ (e.g., $\zeta \in \mathbb{R}^6$ as introduced in Section 2.1). Given a set of measured cable lengths $\mathbf{l} = [l_1, \dots, l_n]^T$, the *direct geometric problem* (DGP) consists in finding ζ such that the geometric constraints associated with the cables are satisfied. Using the *CDPR* geometric model (e.g., the one introduced in Section 2.1), these constraints can be written in compact form as

$$\mathbf{r}(\zeta; \mathbf{l}) = \mathbf{0}, \quad \mathbf{r} \in \mathbb{R}^n, \quad (3.1)$$

where the i -th component typically enforces the compatibility between the measured cable length l_i and the model-predicted length (e.g., Eq. (2.18)), possibly including routing effects [39, 1].

For fully constrained parallel robots, solving (3.1) amounts to solving a non-linear system with as many equations as unknowns. For *UACDPRs*, however, $n < \dim(\zeta)$ and (3.1) is underdetermined: geometry alone does not uniquely determine the pose for prescribed cable lengths, and additional information is required [20, 19].

3.1.2 Direct geometrico-static problem

A common way to close the problem for *UACDPRs* is to complement the geometric constraints with equilibrium equations; the resulting formulation is referred to as the *direct geometrico-static problem* (DGSP) [18, 43, 44, 19]. Within this framework, the platform pose is selected by enforcing *static* equilibrium [19], i.e., by requiring the balance of forces and moments under given external loads. Dynamic extensions exist [21], but they are not pursued here, as the thesis focuses on quasi-static conditions and the corresponding static constraint.

Static equilibrium. Let $\zeta \in \mathbb{R}^\lambda$ denote the *EE* pose vector, where $\lambda = 6$ in the spatial case and $\lambda = 3$ in the planar case. Let $\tau = [\tau_1, \dots, \tau_n]^T$ denote the cable tension vector and let $F \in \mathbb{R}^\lambda$ be the external wrench acting on the platform, including gravity and possible additional loads. In static conditions equilibrium can be written in the compact form [18, 1]

$$\Xi_l^T(\zeta) \tau + F = \mathbf{0}_{\lambda \times 1}, \quad (3.2)$$

where $\Xi_l(\zeta)^T$ is the cable-length kinematic Jacobian matrix (Eq. 2.37). To make explicit how the equilibrium pose is constrained by statics, we first introduce a basis for the subspace orthogonal to the columns of $\Xi_l^T(\zeta)$. Assuming that $\Xi_l(\zeta) \in \mathbb{R}^{n \times \lambda}$ with $n < \lambda$ has full row rank (i.e., $\text{rank}(\Xi_l) = n$) in the configurations of interest, the transpose $\Xi_l^T(\zeta) \in \mathbb{R}^{\lambda \times n}$ admits a non-trivial left nullspace of dimension $\lambda - n$ [52]. Let $\Xi_l^\perp(\zeta) \in \mathbb{R}^{(\lambda-n) \times \lambda}$ be any full-row-rank matrix whose rows span such a left nullspace, namely

$$\Xi_l^\perp(\zeta) \Xi_l^T(\zeta) = \mathbf{0}_{(\lambda-n) \times n}. \quad (3.3)$$

Pre-multiplying the static equilibrium equation (3.2) by $\Xi_l^\perp(\zeta)$ eliminates the tension variables and yields the following *static constraint* on the end-effector pose:

$$\Xi_l^\perp(\zeta) F = \mathbf{0}_{(\lambda-n) \times 1}. \quad (3.4)$$

Nonlinear system defining the DGSP. In the *direct* geometrico-static setting, cable lengths are prescribed or measured as $l_1^*, \dots, l_n^*$, and the unknown pose is the one that satisfies: (i) the geometric constraints from the modelling equations in Sec. 2.1, usually length relations (2.18), and (ii) the static compatibility constraint (3.4). This yields a square nonlinear system of $n + (\lambda - n) = \lambda$ equations $\mathbf{r}(\zeta, \mathbf{l})$, in the λ unknown components of ζ :

$$\mathbf{r}(\zeta, \mathbf{l}) = \begin{bmatrix} l_1(\zeta) - l_1^* \\ \vdots \\ l_n(\zeta) - l_n^* \\ \Xi_l^\perp(\zeta) F \end{bmatrix} = \mathbf{0}_\lambda \quad (3.5)$$

where each $l_i(\zeta)$ is computed consistently with the geometric model (including pulley routing when applicable).

The solution ζ can be determined by solving a square system of nonlinear equations (3.5) with one of the methods detailed in Subsec. 3.2.1. Compared with purely geometric formulations, DGSP-based solutions typically require additional modeling of mass and inertia, as well as the external wrench, and may admit multiple equilibria [1, 19]. Nevertheless, DGSP provides a principled way to predict the configuration when actuation inputs alone are insufficient.

Tension recovery, feasibility, and equilibrium stability. Once a solution ζ of (3.5) is found, cable tensions can be obtained by solving the linear equilibrium equation (3.2) and the resulting configuration is accepted only if feasibility requirements are met, most notably $\tau \geq \mathbf{0}$, and, in many applications, $\tau_i \in [\tau_{\min}, \tau_{\max}]$ for all $i = 1, \dots, n$ [1, 20].

In addition, feasibility alone does not guarantee that the equilibrium is physically observable: a stability check is generally required to exclude locally unstable equilibria, i.e., poses for which small perturbations drive the system away instead of generating restoring effects. In practice, stability can be assessed by verifying that the equilibrium corresponds to a local minimum of the potential energy associated with the external loads, within the admissible set defined by the active tension constraints [53].

3.1.3 Sensor-fusion direct kinematics as non-linear least squares

An alternative and complementary approach, central to [41] and [54], is to use additional proprioceptive measurements so that the pose becomes observable [50, 51] without explicitly relying on mechanical equilibrium. Let $\mathbf{h}(\zeta) \in \mathbb{R}^w$ denote a vector of pose-dependent quantities predicted by the model (e.g., cable lengths, Eq. (2.18), swivel angles, Eq. (2.10), tangency angles, Eq. (2.17), or, directly, position $\mathbf{p}$ or orientation ϵ), and let $\mathbf{z} \in \mathbb{R}^w$ be the corresponding measurement vector. By defining the residual array

$$\mathbf{r}(\zeta) = \mathbf{h}(\zeta) - \mathbf{z}, \quad (3.6)$$

the direct kinematics problem can be posed as a non-linear least-squares estimation:

$$\min_{\zeta} f(\zeta), \quad f(\zeta) = \frac{1}{2} \mathbf{r}^T(\zeta) \mathbf{Q}^{-1} \mathbf{r}(\zeta), \quad (3.7)$$

where $\mathbf{Q}$ can be selected as an ad hoc weighting matrix reflecting physical or numerical considerations. When $\mathbf{Q}$ is chosen as the covariance matrix of the residual array $\mathbf{r}(\zeta)$, the solution ζ is the best linear unbiased estimator (BLUE) of the *EE* pose [33, 22]. This formulation naturally yields overdetermined systems ($w \geq \dim(\zeta)$) and motivates Gauss–Newton and Levenberg–Marquardt approaches.

3.2 Families of solution strategies

Methods used in practice can be grouped into the following families.

3.2.1 Local iterative solvers for non-linear square systems

When the problem is stated as a square system $\mathbf{r}(\zeta) = \mathbf{0}$, with $\zeta \in \mathbb{R}^\lambda$ and $\mathbf{r} \in \mathbb{R}^\lambda$, classical local solvers include Newton’s method and quasi-Newton variants [55, 56]. These methods are efficient near a solution, but require an initial guess within a convergence basin and can fail near singular Jacobians or in the presence of multiple solutions [48, 55].

3.2.1.1 Newton’s method for the direct kinematics of CDPRs

Consider a system of λ equations $\mathbf{r}$ in λ unknowns ζ (e.g. Eq. (3.1) for fully constrained parallel robots and Eq. (3.5) for UACDPRs),

$$\mathbf{r}(\zeta) = \mathbf{0}, \quad \mathbf{r} : \mathbb{R}^\lambda \rightarrow \mathbb{R}^\lambda. \quad (3.8)$$

Let $\mathbf{J}(\boldsymbol{\zeta}) = \nabla \mathbf{r}(\boldsymbol{\zeta}) \in \mathbb{R}^{\lambda \times \lambda}$ be the Jacobian matrix.

Given an initial guess $\boldsymbol{\zeta}_0$ such that $\mathbf{J}(\boldsymbol{\zeta}_0)$ is non-singular, Newton's method generates the sequence

$$\mathbf{J}(\boldsymbol{\zeta}_k) \Delta \boldsymbol{\zeta}_k = -\mathbf{r}(\boldsymbol{\zeta}_k), \quad \boldsymbol{\zeta}_{k+1} = \boldsymbol{\zeta}_k + \Delta \boldsymbol{\zeta}_k. \quad (3.9)$$

Under the assumptions stated in Section 3.3, the iterates converge to a solution $\boldsymbol{\zeta}_s$ of (3.8).

In CDPR kinematics, Newton's method must address three practical issues.

Multiple solutions and initialization. Forward kinematics may admit multiple solutions [49, 26], and Newton's method typically converges to a nearby root. Initialization is therefore crucial, especially along trajectories. Starting from the previous neighbouring estimate is effective when the sampling is sufficiently dense and the solution branch is continuous, which is consistent with typical model-based control loops [49]. Beyond this heuristic, the convergence analysis discussed in Section 3.3 provides a formal framework to certify the existence and uniqueness of a solution within a neighbourhood of an initial guess and to bound the corresponding convergence region. A concrete application of this Kantorovich-based certification to the DGSP of 3-3 UACDPRs is reported in Appendix B.

Singularities and ill-conditioning. Near singular configurations, the Jacobian can become ill-conditioned [48], leading to large steps or numerical instability. Regularization strategies can be introduced (e.g., damped Newton) or switching to a trust-region scheme when needed [56].

Stopping rules. Newton's method must be equipped with practical termination criteria. Standard Newton-like methods stops iterations as soon as one of the following conditions is satisfied [55, 56]:

1. *Maximum iteration count.* The iteration counter reaches a prescribed upper bound,

$$k \geq T_k \quad (3.10)$$

This safeguard prevents unbounded runtimes and typically indicates that either the initial guess lies outside the convergence basin or the Jacobian has become ill-conditioned along the iterates.

2. *Step-size criterion.* The Newton correction becomes sufficiently small,

$$\|\Delta \boldsymbol{\zeta}_k\| \leq T_{\Delta \boldsymbol{\zeta}}, \quad (3.11)$$

where $\Delta \boldsymbol{\zeta}_k$ is obtained from (3.9). This condition detects stagnation when subsequent iterates become indistinguishable at the chosen numerical precision. In practice, it is particularly relevant when the method approaches a solution or when the linear solve yields a tiny correction due to ill-conditioning.

3. *Gradient-norm criteria.* Termination can also be based on the norm of the gradient,

$$\|\nabla \mathbf{r}(\boldsymbol{\zeta})|_{\boldsymbol{\zeta}_k}\| \leq T_{\|\nabla \mathbf{r}\|}, \quad (3.12)$$

which indicates that further iterations are unlikely to yield a meaningful decrease in the residual.

4. *Residual norm below a tolerance.* The norm of the residual vector falls below a threshold,

$$\|\mathbf{r}(\boldsymbol{\zeta}_k)\| \leq T_{\|\mathbf{r}\|}. \quad (3.13)$$

This criterion is the most direct indicator that Eq. (3.9) is satisfied to the desired accuracy. The tolerance T_g should be selected consistently with the measurement noise and modeling uncertainty: forcing $\|\mathbf{r}\|$ below the noise floor generally provides limited benefits and may amplify numerical effects [22].

In *CDPR* applications, the pose vector $\boldsymbol{\zeta}$, the residual and the associated gradient may collect quantities with different physical units and typically different magnitudes (e.g., position components in meters and orientation components in radians). As a consequence, tolerances expressed directly on the step norm (Eq. (3.9)), the gradient norm (Eq. (3.12)) and residual norm (Eq. (3.13)) have a primarily numerical meaning and do not admit an immediate physical interpretation unless appropriate scaling or normalization is introduced [56, 22].

3.2.2 Local iterative solvers for non-linear least squares optimization

When the direct kinematic problem of *CDPRs* is posed as a weighted nonlinear least-squares optimization problem (as in Eq. (3.7)), Gauss–Newton provides a natural baseline: it is simple to implement and typically performs well when the residuals are small and the model is only moderately nonlinear. Levenberg–Marquardt (LM) further improves robustness by introducing a damping (trust-region-like) mechanism, at the cost of additional tuning and linear algebra [56].

3.2.2.1 Gauss–Newton for sensor-fusion direct kinematics

Consider the weighted least-squares optimization problem in Eq. (3.7), which is recalled here for convenience:

$$\min_{\boldsymbol{\zeta}} f(\boldsymbol{\zeta}), \quad f(\boldsymbol{\zeta}) = \frac{1}{2} \mathbf{r}^T(\boldsymbol{\zeta}) \mathbf{Q}^{-1} \mathbf{r}(\boldsymbol{\zeta}).$$

The search for a local minimizer can be interpreted as seeking a root of the first-order optimality condition $\nabla f(\boldsymbol{\zeta}) = \mathbf{0}$. Let $\mathbf{J}(\boldsymbol{\zeta}) = \partial \mathbf{r}(\boldsymbol{\zeta}) / \partial \boldsymbol{\zeta}$ denote the residual Jacobian.

To compute the gradient compactly, recall the following identity. Let $\alpha(\boldsymbol{\zeta}) = \mathbf{c}^T(\boldsymbol{\zeta}) \mathbf{A} \mathbf{c}(\boldsymbol{\zeta})$ be a symmetric quadratic form, with $\mathbf{A} \in \mathbb{R}^{q \times q}$ symmetric and constant, and $\mathbf{c}(\boldsymbol{\zeta}) \in \mathbb{R}^q$ differentiable. Then¹

$$\nabla \alpha(\boldsymbol{\zeta}) = 2 (\nabla \mathbf{c}^T(\boldsymbol{\zeta})) \mathbf{A} \mathbf{c}(\boldsymbol{\zeta}), \quad \nabla \mathbf{c}^T(\boldsymbol{\zeta}) \in \mathbb{R}^{\lambda \times q}. \quad (3.14)$$

Applying this result with $\mathbf{c}(\boldsymbol{\zeta}) = \mathbf{r}(\boldsymbol{\zeta})$ and $\mathbf{A} = \frac{1}{2} \mathbf{Q}^{-1}$ yields

$$\nabla f(\boldsymbol{\zeta}) = (\nabla \mathbf{r}^T(\boldsymbol{\zeta})) \mathbf{Q}^{-1} \mathbf{r}(\boldsymbol{\zeta}) = \mathbf{J}^T(\boldsymbol{\zeta}) \mathbf{Q}^{-1} \mathbf{r}(\boldsymbol{\zeta}). \quad (3.15)$$

The exact Hessian $\nabla^2 f(\boldsymbol{\zeta})$ contains a term depending on second-order derivatives of $\mathbf{r}(\boldsymbol{\zeta})$, i.e., derivatives of $\mathbf{J}(\boldsymbol{\zeta})$. Computing such terms requires handling third-order tensors

¹The proof follows from standard matrix-product differentiation rules; for similar derivations, see the appendix of [57].

and is typically expensive². Gauss–Newton therefore adopts the Gauss approximation, neglecting the second-order contribution:

$$\nabla^2 f(\zeta) = \nabla(\mathbf{J}^T(\zeta) \mathbf{Q}^{-1} \mathbf{r}(\zeta)) \approx \mathbf{J}^T(\zeta) \mathbf{Q}^{-1} \mathbf{J}(\zeta). \quad (3.16)$$

Starting from an initial guess ζ_0 , Gauss–Newton generates the sequence $\{\zeta_k\}_{k \geq 0}$ by applying a Newton step to the first-order condition $\nabla f(\zeta) = \mathbf{0}$ with the approximation (3.16). Equivalently, linearizing the residual,

$$\mathbf{r}(\zeta + \Delta\zeta) \approx \mathbf{r}(\zeta) + \mathbf{J}(\zeta) \Delta\zeta, \quad (3.17)$$

and minimizing the resulting quadratic model leads to the normal equations

$$(\mathbf{J}_k^T \mathbf{Q}^{-1} \mathbf{J}_k) \Delta\zeta_k = -\mathbf{J}_k^T \mathbf{Q}^{-1} \mathbf{r}_k, \quad \zeta_{k+1} = \zeta_k + \Delta\zeta_k, \quad (3.18)$$

where $\mathbf{r}_k = \mathbf{r}(\zeta_k)$ and $\mathbf{J}_k = \mathbf{J}(\zeta_k)$. In sensor-fusion settings, $\mathbf{r}$ stacks cable-related residuals (e.g., $l_i(\zeta)$, $\sigma_i(\zeta)$ and $\psi_i(\zeta)$) together with directly-measured pose components-related residuals, producing an overdetermined system ($m > 6$) that benefits from the smoothing effect of least squares [54].

3.2.2.2 Levenberg–Marquardt as a damped Gauss–Newton method.

Gauss–Newton may become unstable when $\mathbf{J}_k^T \mathbf{Q}^{-1} \mathbf{J}_k$ is ill-conditioned or when the local linear model of the residual is inaccurate. Levenberg–Marquardt (LM) addresses these issues by damping the normal equations with a parameter $\mu_k > 0$:

$$(\mathbf{J}_k^T \mathbf{Q}^{-1} \mathbf{J}_k + \mu_k \mathbf{I}) \Delta\zeta_k = -\mathbf{J}_k^T \mathbf{Q}^{-1} \mathbf{r}_k. \quad (3.19)$$

For $\mu_k \rightarrow 0$, the update approaches the Gauss–Newton step (3.18), whereas for large μ_k it becomes closer to a gradient-like direction. In practice, μ_k is adapted according to the achieved decrease of $f(\zeta)$, yielding a trust-region interpretation [56]. LM is therefore preferable when robustness is prioritized over minimal per-iteration cost.

Stopping rules. The number of iterations depends on the adopted termination conditions, which are specified through user-defined tolerances. Typical criteria include the termination conditions discussed in paragraph *Stopping rules.* of Section 3.2.1.1.

3.2.3 Interval analysis and other global techniques

When all admissible solutions are of interest, or when one seeks certified guarantees, global methods provide an alternative to local Newton-type iterations. A prominent family is *interval analysis* combined with branch-and-bound strategies, which can enclose and isolate the roots of the kinematic equations by working with boxes in the pose space and manipulating interval extensions of the residual function [59, 26, 20]. Applied to Eq. (3.1), these methods can deliver guaranteed enclosures of every root within a prescribed domain by iteratively contracting boxes (to discard regions that

²The second derivative of a vector-valued function is a third-order tensor; explicit computations can be found, e.g., in the appendix of [58]. In many pose-estimation settings, this contribution is negligible near the solution compared to the Gauss term, and can be omitted without affecting the estimate in practice.

cannot contain a solution) and bisecting the remaining ones until a desired resolution is reached. Compared with local iterations, the computational cost is typically higher, but the payoff is robustness to multiple solutions and the ability to certify properties such as existence and, in favorable cases, uniqueness within a region.

Beyond interval approaches, *continuation/homotopy* techniques aim at recovering solution sets by tracking a solution branch as parameters vary. In the context of *CDPR* forward kinematics, continuation has been exploited to maintain robustness along workspace trajectories, reducing the need for repeated global initialization and helping to traverse parameter regions where the mapping becomes ill-conditioned [49, 47]. These methods are particularly appealing when the pose must be computed repeatedly along a trajectory, since the solution at one step naturally provides a reliable predictor for the next.

3.3 Convergence analysis: the Kantorovich theorem and its role in *CDPR* kinematics

For Newton-type methods, local quadratic convergence can be established under standard regularity assumptions. In particular, if the initial guess lies in a neighbourhood of a simple root and the Jacobian is nonsingular at the solution, Newton's iterates converge quadratically; sufficient conditions can be derived, e.g., within the Kantorovich framework [60, 61, 55]. In *CDPR* direct kinematics, these conditions are particularly critical, because the initialization is affected by sampling rate, measurement noise, and modelling errors, and cannot be assumed to lie in a reliably safe neighbourhood of the solution. Therefore, a convergence analysis is required to certify that the Newton method converges to a unique solution.

3.3.1 Kantorovich theorem

This section recalls the Kantorovich theorem, which provides a convenient theoretical tool to assess existence/uniqueness of a solution and convergence of Newton iterations when solving square nonlinear systems arising from direct kinematics [26] and, in particular, from DGSP formulations [20].

Consider the square system in (3.8), where λ denotes the dimension of the unknown vector and U is a domain where the analysis is carried out, and $\zeta_0 \in U$ be an initial guess such that the Jacobian matrix

$$\mathbf{J}(\zeta) = \frac{\partial \mathbf{r}}{\partial \zeta}(\zeta) \quad (3.20)$$

is nonsingular at ζ_0 . The constants $A_0, B_0, C \in \mathbb{R}^+$, with $\|\cdot\|$ denoting the infinity norm, are defined as:

$$\begin{aligned} A_0 &= \|\mathbf{J}^{-1}(\zeta_0)\|, \\ 2B_0 &= \|\mathbf{J}^{-1}(\zeta_0) \mathbf{r}(\zeta_0)\|, \\ C &= \sum_{k=1}^N \left| \frac{\partial^2 r_k(\zeta)}{\partial \zeta_i \partial \zeta_j} \right| \quad \text{for } i, j = 1, \dots, \lambda, \quad \zeta \in U, \end{aligned} \quad (3.21)$$

and define the closed set

$$U = \{\zeta : \|\zeta - \zeta_0\| \leq 2B_0\}. \quad (3.22)$$

If the quantities A_0 , B_0 and C satisfy the Kantorovich condition

$$K_0 = 2\lambda A_0 B_0 C \leq 1, \quad (3.23)$$

then the Kantorovich theorem guarantees that: (i) there exists a unique solution $\zeta_s \in U$ to (3.8); (ii) Newton's method initialized at ζ_0 converges to ζ_s ; and (iii) an explicit *a priori* bound on the distance between the solution and the initial guess holds, namely

$$\|\zeta_s - \zeta_0\| \leq \frac{2B_0}{1 + \sqrt{1 - K_0}}, \quad (3.24)$$

and, in addition, the Newton iterates satisfy the corresponding *a posteriori* estimate

$$\|\zeta_s - \zeta_k\| \leq 2B_0, \quad (3.25)$$

which provides a simple and computable error bound for the numerical estimate ζ_k at iteration k .

3.3.2 Comments on Gauss–Newton and LM convergence

Kantorovich-type results are stated for square systems and therefore apply directly to Newton schemes of the form (3.9) when the direct kinematics problem is cast as a root-finding task. A Kantorovich-based analysis is reported in Appendix B, where it is specialized to the DGSP of 3-3 UACDPRs [34] to derive sufficient conditions for existence and uniqueness of the equilibrium solution and to obtain conservative bounds on admissible cable-length measurement errors that guarantee Newton convergence. It is stressed that these bounds are sufficient, but not necessary, and thus may be conservative.

For Gauss–Newton and LM, the direct kinematics is instead formulated as a least-squares problem, so convergence is typically discussed within optimization theory. Local convergence can be established under standard regularity assumptions, in particular when $\mathbf{J}$ has full column rank and the residual is sufficiently small at the solution [56]. From the standpoint of Kantorovich-type certification, extending the same methodology to sensor-fusion formulations is substantially more challenging: one would need to apply the theorem to the stationarity condition of the cost function, i.e., to the root-finding problem $\nabla f(\zeta) = \mathbf{0}$, which would require bounding higher-order derivatives. In the author's opinion, developing a tractable certification framework for such redundant sensor-fusion kinematics constitutes a promising direction, building on the convergence foundations established in Section 3.3 and on the preliminary insights provided in Appendix B.

In practice, for sensor-fusion direct kinematics, the main robustness levels considered in this thesis remain: (i) informative and well-scaled residual components through $\mathbf{Q}$; (ii) problem-specific ITCs tailored to noise levels, as emphasized in [54, 34, 62].

Chapter 4

Sensor-Fusion Direct Kinematics and Tension-Aware Estimation

This Chapter presents the main contribution of the PhD thesis: a sensor-fusion framework to solve the direct kinematics of underactuated cable-driven parallel robots (*UACD-PRs*) by exploiting redundant proprioceptive measurements. The method complements cable-length readings with additional pose-dependent measurements so that the end-effector pose becomes observable without explicitly enforcing mechanical equilibrium. The formulation builds upon the geometric model introduced in Chapter 2 and on the weighted nonlinear least-squares (*WNLS*) viewpoint recalled in Section 3.1.3. For the numerical solver, we rely on the Gauss–Newton algorithm, presented in Section 3.2.2.1, and we focus here on: (i) residual-set design and Jacobian assembly for sensor fusion, (ii) practical aspects of the *WNLS* solution, including the choice of physically meaningful weighting parameters, iteration termination criteria. Lastly, it is shown that residual uncertainties can be modeled as tension dependent, making the solution algorithm faster and more reliable.

4.1 Sensor-fusion formulation for direct kinematics

This section introduces the sensor-fusion direct-kinematics formulation as a weighted estimation problem and details the algorithmic ingredients that make it effective in the *UACDPR* context. Beyond the standard Gauss–Newton machinery, the proposed approach hinges on two practical design choices: (i) a weighting strategy that accounts for heterogeneous residual units and reliability, and (ii) termination criteria formulated component-wise to provide physically interpretable convergence checks for both pose variables and residual components. These elements are presented next, together with the construction of the residual Jacobian from the differential kinematic relations established in Chapter 2.

4.1.1 Sensor-fusion measurement model and *WNLS* formulation

Let the *EE* pose be represented by $\zeta \in \mathbb{R}^\lambda$ (Chapter 2). Assume that, at a given time instant, a set of $m \geq \lambda$ proprioceptive measurements is available and collected in an array $\mathbf{z} \in \mathbb{R}^m$. The sensor-fusion model associates to each measurement a corresponding pose-dependent prediction through a mapping

$$\mathbf{h}(\zeta) \in \mathbb{R}^m, \quad (4.1)$$

built from the geometric relations of Chapter 2¹. Typical components of $\mathbf{h}(\zeta)$ include: cable lengths $l_i(\zeta)$ (Eq. (2.18)); swivel angles $\sigma_i(\zeta)$ (Eq. (2.10)); tangency angles $\psi_i(\zeta)$ (Eq. (2.17)); and, when available, directly measured pose components (e.g., orientation angles or position components). Stacking any subset of such predicted quantities yields a flexible sensor-fusion residual set.

The residual array is defined as in Section 3.1.3,

$$\mathbf{r}(\zeta) = \mathbf{h}(\zeta) - \mathbf{z} \in \mathbb{R}^m. \quad (4.2)$$

The pose estimate is obtained by minimizing the *WNLS* cost (3.7), recalled here for completeness:

$$\min_{\zeta} f(\zeta), \quad f(\zeta) = \frac{1}{2} \mathbf{r}^T(\zeta) \mathbf{Q}^{-1} \mathbf{r}(\zeta), \quad (4.3)$$

where $\mathbf{Q} \in \mathbb{R}^{m \times m}$ is a diagonal weighting matrix, $\mathbf{Q} = \text{diag}(q_1, \dots, q_m)$, introduced to (i) render the *WNLS* objective f dimensionless and computable if heterogeneous residuals are included in the residual array $\mathbf{r}(\zeta)$, and (ii) reflect the expected reliability of the available proprioceptive measurements.

4.1.2 Weight selection from error estimates

In this work, the diagonal weights are selected according to an inverse-error rationale [33],

$$q_i = e_i^2, \quad i = 1, \dots, m, \quad (4.4)$$

where

$$\mathbf{e} = [e_1, \dots, e_m]^T \in \mathbb{R}^m \quad (4.5)$$

collects scalar uncertainty levels associated with the components of the measurement array $\mathbf{z}$ and, equivalently, with the residuals in $\mathbf{r}(\zeta)$. Accordingly, e_i represents an estimate of the typical magnitude of the error affecting the i -th residual component.

In this thesis, these uncertainty levels are chosen according to two complementary criteria. The first is a *nominal* or *conservative* estimate, in which e_i is set to a fixed value derived from a preliminary model–sensor characterization or from manufacturer specifications. This approach is adopted when no systematic calibration dataset is available, and the resulting e_i represents an upper bound on the expected error magnitude rather than a statistically identified quantity (see Section 5.1 where constant uncertainty levels are assigned based on a preliminary characterization of each sensing modality).

The second criterion is *empirical identification* from a calibration dataset: given a set of ground-truth poses ζ_{gt} , measurement-error samples are computed as $|z_i - h_i(\zeta_{gt})|$ and the resulting distribution is characterized through its standard deviation, which is then used as e_i (see Section 5.2, Tables 5.4, 5.5). When cable-related measurements are affected by configuration-dependent reliability (e.g., due to tension variations), the uncertainty levels are further extended to tension-dependent functions $e_i = \hat{e}_i(\tau_i)$, identified offline from calibration data as detailed in Section 5.3.

Depending on the adopted error model, e_i can therefore be set either as a constant value (e.g., a nominal accuracy level provided by sensor specifications or a measured

¹While $\mathbf{h}(\zeta)$ provides a well-defined mapping from pose to modeled geometric quantities, the physical realizability of a pose for given actuation inputs may still depend on feasibility constraints (e.g., positive tensions, tension bounds, workspace limits), which are not enforced by geometry alone. These constraints are discussed in Section 2.1.6.

error standard deviation) or as a configuration-dependent quantity, as discussed in Section 4.2. With the choice (4.4), residual components expected to be less accurate (larger e_i) are down-weighted and therefore contribute less to the descent direction and to the Gauss–Newton update. Moreover, when e_i is selected as the standard deviation of the measurement error associated with the i -th residual (so that e_i^2 is its variance), the rule $q_i = e_i^2$ coincides with the classical inverse-variance weighting. Under the usual assumptions of independent, zero-mean Gaussian errors, this makes the resulting *WNLS* estimator equivalent to a maximum-likelihood estimator.

4.1.3 Gauss–Newton update and Residual Jacobian computation

The Gauss–Newton update (3.18) recalled here for completeness:

$$\left(\mathbf{J}_{r,k}^T \mathbf{Q}^{-1} \mathbf{J}_{r,k}\right) \Delta \boldsymbol{\zeta}_k = -\mathbf{J}_{r,k}^T \mathbf{Q}^{-1} \mathbf{r}_k, \quad \boldsymbol{\zeta}_{k+1} = \boldsymbol{\zeta}_k + \Delta \boldsymbol{\zeta}_k, \quad (4.6)$$

requires the computation of the residual Jacobian

$$\mathbf{J}_r(\boldsymbol{\zeta}) = \frac{\partial \mathbf{r}}{\partial \boldsymbol{\zeta}}(\boldsymbol{\zeta}) = \frac{\partial \mathbf{h}}{\partial \boldsymbol{\zeta}}(\boldsymbol{\zeta}) \in \mathbb{R}^{m \times \lambda}. \quad (4.7)$$

When $\mathbf{h}$ includes cable lengths, swivel angles, and tangency angles, $\mathbf{J}_r$ can be assembled by stacking the corresponding analytic Jacobians derived from the first-order kinematics of Chapter 2. In particular, the matrices $\mathbf{J}_l$, $\mathbf{J}_\sigma$, and $\mathbf{J}_\psi$ are obtained from the kinematic Jacobians $\boldsymbol{\Xi}_l$, $\boldsymbol{\Xi}_\sigma$, and $\boldsymbol{\Xi}_\psi$ via the mapping in Eq. (2.39), and their rows can be selected/stacked consistently with the chosen residual components. If additional residuals correspond to directly measured pose components, their Jacobian rows are simply selection rows from the identity matrix. To avoid repeating derivations, we only recall that the update is computed from the normal equation (3.18) using $\mathbf{J}_r(\boldsymbol{\zeta}_k)$ and $\mathbf{r}(\boldsymbol{\zeta}_k)$.

4.1.4 Iteration termination conditions

The Gauss–Newton iteration termination conditions follow classical stopping rules for Newton-type methods (see the discussion in the paragraph *Stopping rules* in Section 3.2.1.1), but they are written in a component-wise form to account for the heterogeneous physical nature of the pose variables. In particular, $\mathbf{p}$ is usually expressed in meters, whereas the orientation parameters $\boldsymbol{\epsilon}$ are usually expressed in radians: aggregating them in a single norm may lead to thresholds that are difficult to interpret and to tune, and may also bias the stopping decision toward the component with the largest numerical scale. For the same reason, residual components may have different units and error levels, hence they are checked against individual tolerances.

1. *Maximum number of iterations.* The iteration counter reaches a prescribed upper bound T_k , namely

$$k \geq T_k \quad (4.8)$$

This is a safeguard against excessive computation and possible divergence; in this case, the last iterate $\boldsymbol{\zeta}_k$ is returned as the best available estimate, but convergence is not certified.

2. *Negligible step size (stagnation)*. The update becomes small in both translation and orientation, namely

$$\|\Delta \mathbf{p}_k\| \leq T_{\mathbf{p}} \quad \text{and} \quad \|\Delta \boldsymbol{\epsilon}_k\| \leq T_{\boldsymbol{\epsilon}}. \quad (4.9)$$

This typically indicates that the algorithm has reached a fixed point of the iterations (e. g. a local minimizer), or that progress is limited by numerical conditioning and measurement noise.

3. *Small gradient (approximate stationarity)*. The norm of the cost gradient becomes small in both pose blocks, namely

$$\left\| \frac{\partial f}{\partial \mathbf{p}} \right|_{\zeta_k} \leq T_{\left\| \frac{\partial f}{\partial \mathbf{p}} \right\|} \quad \text{and} \quad \left\| \frac{\partial f}{\partial \boldsymbol{\epsilon}} \right|_{\zeta_k} \leq T_{\left\| \frac{\partial f}{\partial \boldsymbol{\epsilon}} \right\|}. \quad (4.10)$$

This corresponds to an (approximately) first-order optimality condition for the WNLS objective: the current iterate is close to a stationary point of f and further descent is expected to be marginal.

4. *Residuals within expected accuracy (data-consistent fit)*. Each residual component satisfies

$$|r_{k,i}| \leq T_{r,i}, \quad \forall i = 1, \dots, m. \quad (4.11)$$

This criterion is the most directly interpretable in terms of measurement quality: the algorithm stops once the model fits each measurement within its expected uncertainty level. A practical choice is $T_{r,i} = e_i$ (or a multiple thereof), consistently with the adopted error model, so that accurate measurements impose stricter requirements while noisier ones do not prevent termination. When this condition triggers, the returned estimate ζ_k can be regarded as consistent with the assumed sensor accuracy.

For practical online use, it is useful to discuss the computational structure of the proposed sensor-fusion estimator. Recalling that m denote the number of residual components and λ the dimension of the pose vector, at each Gauss–Newton iteration, the dominant operations are: (i) evaluation of the residual vector $\mathbf{r}(\zeta)$ and of the residual Jacobian $\mathbf{J}_r(\zeta)$, (ii) construction of the normal-equation matrix $\mathbf{J}_r^T \mathbf{Q}^{-1} \mathbf{J}_r$, and (iii) solution of the resulting linear system (Eq. (3.18)) in λ unknowns.

Assembling the residual Jacobian $\mathbf{J}_r \in \mathbb{R}^{m \times \lambda}$ requires evaluating quantities associated with all residual–pose pairs, which leads to a cost proportional to $m\lambda$. Likewise, forming the matrix $\mathbf{J}_r^T \mathbf{Q}^{-1} \mathbf{J}_r \in \mathbb{R}^{\lambda \times \lambda}$ requires evaluating all its λ^2 entries. Each entry is obtained by summing the contributions of the m residual components, weighted by the corresponding diagonal term of $\mathbf{Q}^{-1}$. Since $\mathbf{Q}^{-1}$ is diagonal, its coefficients can be precomputed and the weighting operation is straightforward; nevertheless, for each of the λ^2 entries of $\mathbf{J}_r^T \mathbf{Q}^{-1} \mathbf{J}_r$, one still has to account for up to m products involving the corresponding columns of $\mathbf{J}_r$. Under the general assumption that the Jacobian is not sparse and that all these contributions may be significant, this leads to a cost proportional to $m\lambda^2$. Finally, solving the resulting linear system in λ unknowns has a cost proportional to λ^3 [55]. In the present context, however, λ is typically small ($\lambda \leq 6$), so the practical computational cost is usually more sensitive to the number and type of residuals included in the sensor-fusion model than to the pose dimension itself.

Consequently, richer residual sets may improve observability and robustness, but they also increase the per-iteration cost approximately linearly with the number of residual components.

This scaling suggests several practical guidelines for online implementation. First, warm-start initialization from the estimate computed at the previous sampling instant usually keeps the current iterate close to the local minimizer and significantly reduces the number of Gauss–Newton updates required for convergence. Second, analytic Jacobians should be preferred whenever available, since they avoid the additional overhead and numerical sensitivity associated with finite-difference approximations. Third, the residual set should be chosen by balancing observability and computational cost: adding proprioceptive measurements is beneficial only when the additional information they provide justifies the extra per-iteration cost.

The stopping criteria should be interpreted accordingly. For online applications, the most meaningful termination condition is the residual-based one, especially in the component-wise form of Eq. (4.11), because it directly reflects the expected accuracy of each measurement and yields thresholds with a clear physical interpretation. By contrast, the maximum-iteration, step-size, and gradient conditions are best used as safeguards: their thresholds should be selected so that they rarely trigger before the residual-based condition is satisfied, while still preventing excessive iterations and numerically unproductive updates. Overall, when low-dimensional pose models are combined with warm-started Gauss–Newton iterations, analytic Jacobians, and engineering-based stopping thresholds, the proposed formulation is well suited to online application.

4.2 Tension-aware sensor fusion

In CDPRs, the reliability of proprioceptive measurements is not necessarily constant across configurations. In particular, the quality of cable-related measurements may degrade when a cable approaches slackness or, more generally, when the tension distribution changes. The tension-aware extension introduced here accounts for this effect by making both the weighting matrix and the iteration termination conditions depend on a configuration-dependent error model (Eq. (4.5)) driven by cable tensions.

4.2.1 Tension-dependent error modeling

The goal of this subsection is to describe how the uncertainty affecting cable-related measurements varies with auxiliary information reflecting their reliability. In this thesis, cable tension is adopted as the primary reliability indicator; accordingly, the modeled error is assumed to increase as tension decreases.

Cable-related measurements include cable length, swivel angle, and tangency angle. These measurements capture different aspects of the cable configuration and rely on specific modeling assumptions:

- *Cable length*: it provides a direct geometric measure; however, it does not account for cable self-weight (sagging) or elasticity.
- *Swivel angle*: it is typically interpreted under the assumption of negligible friction at the swivel/pulley interface.

- *Tangency angle*: similarly to cable length, it is affected by sagging; when sagging occurs, the tangency point location changes and cannot be assumed constant when cable self-weight is non-negligible.

In the following, we use the variable $\mathbf{z}$ to denote the set of cable-related measurements. Depending on the specific formulation, $\mathbf{z}$ may include a selected set (e.g., only cable lengths, $\mathbf{z} = [l_1, \dots, l_n]^T$), a subset, or the full collection of available cable-related measurements.

For each cable and measurement type, a scalar uncertainty level is modeled as a function of the corresponding tension, i.e., $e = e(\tau)$. Two correlation models are considered:

- a *linear-in-tension model*

$$e(\tau) = a + b\tau, \quad b < 0, \quad (4.12)$$

where the negative slope captures the expected increase of error at lower tensions;

- an *inverse-tension model*

$$e(\tau) = \frac{c}{\tau} + d, \quad (4.13)$$

which yields a stronger penalty in the low-tension regime and satisfies $e(\tau) \rightarrow \infty$ as $\tau \rightarrow 0^+$.

In the tension-aware *WNLS* formulation, the uncertainty e_i associated with the i -th residual component determines the corresponding weight parameter through Eq. (4.4). Therefore, if $e(\tau)$ grows unbounded as $\tau \rightarrow 0^+$ (Eq. (4.13)), the weight parameter satisfies $q(\tau) = e(\tau)^2 \rightarrow \infty$. Consequently, the corresponding diagonal element of the diagonal weight matrix $\mathbf{Q}^{-1}$ in the *WNLS* cost (Eq. (3.7)) tends to zero, effectively down-weighting measurements acquired under near-zero tension.

4.2.2 Identification of model coefficients from calibration data

The coefficients in Eqs. (4.12)–(4.13) can be identified from a calibration dataset covering a representative set of robot configurations. For each sampled configuration and for each cable i , the following quantities are collected: the tension τ_i and one or more cable-related measurements, each of which is denoted by z_i throughout. A reference ground-truth pose ζ_{gt} is required to compute the corresponding model-based values $z_i(\zeta_{gt})$ through the inverse geometric relations of Chapter 2 (e.g., $l_i(\zeta_{gt})$ from Eq. (2.18), $\sigma_i(\zeta_{gt})$ from Eq. (2.10) and, $\psi_i(\zeta_{gt})$ from Eq. (2.17)). Measurement-error samples are then defined as

$$e_i^z = |z_i - z_i(\zeta_{gt})| \quad (4.14)$$

After collecting the samples, the goal is to extract a robust relationship between tension and error. In practice, raw couples (τ_i, e_i^z) may be noisy and unevenly distributed over the tension range; moreover, for estimation purposes we are primarily interested in a *conservative* characterization, i.e., one that does not underestimate errors in low-reliability regimes. For this reason, before fitting the parametric models, the data can be processed in two complementary ways, depending on whether one aims at a worst-case envelope or at an unbiased regression over a non-uniform dataset.

Conservative binning (worst-case envelope). A simple worst-case characterization is obtained by partitioning the tension range into fixed-width intervals and retaining, within each interval, only the maximum observed error, discarding the remaining samples. For each tension range, the retained value represents a worst-case sample rather than an average behavior. The subsequent model fit therefore tends to build an upper envelope of the observed errors.

Density-aware fitting (non-uniform sampling). When using all samples directly, one must account for the fact that tension data are rarely uniformly distributed: medium-tension values often appear frequently, whereas low- and high-tension regimes may be sparsely sampled. A standard (unweighted) least-squares fit would then be biased toward densely populated regions, potentially under-representing the behavior precisely where reliability changes most. To mitigate this effect, a weighted regression can be adopted to compensate for uneven sampling density, as described next.

4.2.3 Density-aware weighting strategy

To implement a density-aware regression, each tension sample is assigned a weight inversely proportional to its local sampling density. Let $\delta_i^{(k)}$ denote the estimated density associated with the k -th tension sample $\tau_i^{(k)}$ for cable i . The density $\delta_i^{(k)}$ is estimated via kernel density estimation (KDE), a nonparametric method to approximate the probability density function of sampled data [63]. For a set of N tension samples $\{\tau_i^{(1)}, \dots, \tau_i^{(N)}\}$, the KDE estimate at $\tau_i^{(k)}$ is

$$\delta_i^{(k)} = \frac{1}{Nh} \sum_{m=1}^N K\left(\frac{\tau_i^{(k)} - \tau_i^{(m)}}{h}\right), \quad (4.15)$$

where $K(\cdot)$ is a kernel function and $h > 0$ is the bandwidth controlling the smoothness of the estimate. A Gaussian kernel

$$K(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right), \quad (4.16)$$

is adopted, as it provides a smooth and differentiable density profile. Small bandwidth values yield density estimates that follow the samples closely, whereas larger bandwidths produce smoother estimates. A reasonable bandwidth can be selected using Silverman's rule of thumb, which provides a data-driven choice based on sample spread and size [64]; it can then be fine-tuned after this initial estimate. The corresponding sample weight is then set to

$$w_i^{(k)} = \frac{1}{(\delta_i^{(k)})^2}. \quad (4.17)$$

With Eq. (4.17), samples in sparsely populated regions of the tension domain (typically low and high tensions) receive higher influence in the regression, while densely populated regions contribute proportionally less, yielding a model that better reflects the full tension range.

4.2.4 Correlation models

For each cable and measurement type z_i , we fit a tension-dependent uncertainty function $\hat{e}(\tau)$. The identified functions are then used to compute configuration-dependent

weights via Eq. (4.4) and, consistently, to set tension-aware residual thresholds (e.g., $T_{r,i} = e_i$ in Eq. (4.11); see Section 4.1).

4.2.4.1 Linear-in-tension fit

A first option is the affine model

$$\hat{e}(\tau) = a + b\tau, \quad (4.18)$$

with $b < 0$. For each cable i and each measurement type z_i , the parameters are estimated independently by solving a weighted least-squares problem:

$$[a_i^z, b_i^z] = \min_{a,b} \sum_{k=1}^N w_i^{(k)} \left(e_i^{z(k)} - (a + b\tau_i^{(k)}) \right)^2, \quad (4.19)$$

4.2.4.2 Inverse-tension fit

A second option is the rational model

$$\hat{e}(\tau) = \frac{c}{\tau} + d, \quad (4.20)$$

which enforces $\hat{e}(\tau) \rightarrow \infty$ as $\tau \rightarrow 0^+$ and thus implies $q(\tau) = \hat{e}(\tau)^{-2} \rightarrow 0$ in Eq. (4.4). Parameters are obtained via weighted least squares:

$$[c_i^z, d_i^z] = \min_{c,d} \sum_{k=1}^N w_i^{(k)} \left(e_i^{z(k)} - \left(\frac{c}{\tau_i^{(k)}} + d \right) \right)^2, \quad (4.21)$$

4.2.4.3 Identified functions and use in sensor fusion

For each cable i and each measurement type z_i , the identification yields a fitted function. For the linear model (4.18),

$$\hat{e}_i^z(\tau_i) = a_i^z + b_i^z \tau_i, \quad (4.22)$$

whereas for the inverse model (4.20),

$$\hat{e}_i^z(\tau_i) = \frac{c_i^z}{\tau_i} + d_i^z. \quad (4.23)$$

In the tension-aware sensor-fusion algorithm, the function (4.22) or (4.23) is evaluated at the measured/estimated tensions to build configuration-dependent uncertainty levels $\hat{e}_i^z(\tau_i)$, which are then mapped to weights via Eq. (4.4) and to residual thresholds (e.g., $T_{r,i} = \hat{e}_i^z(\tau_i)$ in Eq. (4.11)). The selection between the two correlation models is guided by the best fit to the observed tension–error trend in the considered experimental setup.

Chapter 5

Experimental Results

This chapter reports experimental results validating the sensor-fusion direct-kinematics method introduced in Chapter 4. The algorithm is implemented in MATLAB to assess (i) the practical observability gained by augmenting cable-length readings with additional proprioceptive measurements, and (ii) the impact of weighting and stopping rules on accuracy and computational running-time.

5.1 Experimental validation on a planar 2-cable UACDPR

This section adapts and extends the experimental validation originally presented in [54]. The method of Chapter 4 is tested on a planar 2-cable UACDPR, where the geometric relations used are recalled in Chapter 2.

5.1.1 Prototype and sensing setup

The testbed is obtained by modifying the planar CDPR prototype in [65]: the two lower cables were removed, resulting in an underactuated architecture (Fig. 5.1). The EE motion is constrained to the vertical xy plane by a parallelogram cable layout. As in other planar CDPR systems based on parallelogram mechanisms [66], the platform is confined to a prescribed plane, while low-amplitude oscillatory motions due to external disturbances are suppressed [67].

The system uses two cables arranged in a closed parallelogram loop to control the platform's 3 DoFs: two in-plane translations along x and y axes, and one rotation about the out-of-plane axis z , parameterized by the angle χ . Each cable loop is driven by a winch equipped with a rotary encoder, which provides the cable length measurement l_i , for $i = 1, 2$. The cable guidance is realized through swivel pulleys whose axes $\mathbf{k}_i$ are parallel to the z axis; their encoders supply additional kinematic information via the measured swivel angles σ_i . Assuming negligible friction at the swivel/pulley interface, the pulley is expected to align with the cable direction. Hence, the tangency angle ψ_i is modeled as $\psi_i = \sigma_i$, for $i = 1, 2$. Moreover, the intersections of the swivel axes with the work plane xy define the cable exit points B_i . Under this modeling choice, each cable is represented by a straight segment connecting B_i to the platform attachment point A_i , i.e., $\mathbf{p}_i = \mathbf{q}_i$, for $i = 1, 2$. Cable tensions are measured by shear-beam load cells integrated into the anchor system. Overall, the setup reduces the number of actuators by exploiting a single cable loop, provides continuous measurements of cable tensions and angles, and enables an accurate definition of cable lengths as distances between

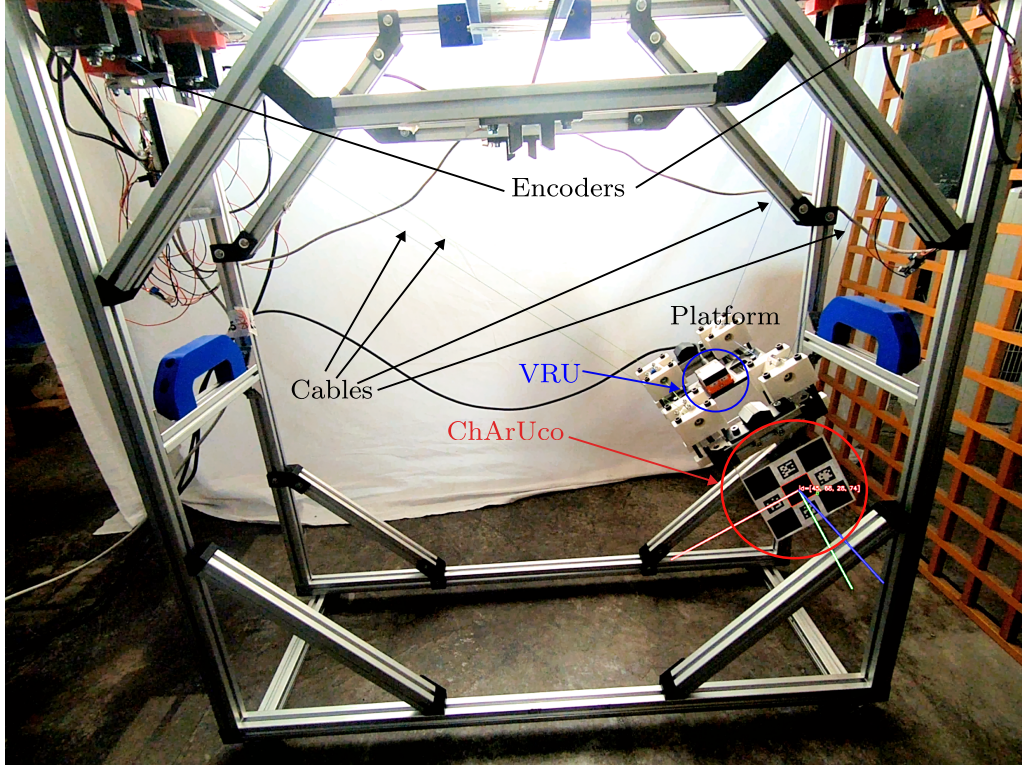


Figure 5.1: Planar 2-cable *UACDPR* prototype and sensing layout used for validation (adapted from [54]).

two well-defined points associated with the swivel-axis intersections with the work plane. The geometric and inertial parameters of the prototype are:

$${}^P\mathbf{a}'_1 = \begin{bmatrix} -6.5 \\ 4 \\ 0 \end{bmatrix} \text{ cm}, \quad {}^P\mathbf{a}'_2 = \begin{bmatrix} 6.5 \\ 4 \\ 0 \end{bmatrix} \text{ cm}, \quad {}^P\mathbf{g}' = \begin{bmatrix} 0.13 \\ 1.53 \\ 0 \end{bmatrix} \text{ cm},$$

$$\mathbf{b}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ cm}, \quad \mathbf{b}_2 = \begin{bmatrix} 95 \\ 0 \\ 0 \end{bmatrix} \text{ cm}, \quad m = 1 \text{ Kg}$$

where ${}^P\mathbf{g}'$ is the constant position vector of the platform center of mass in the platform-attached moving frame $Px'y'z'$, which coincides with the ChArUco board frame shown in Fig. 5.1. The prototype is equipped with redundant proprioceptive sensing.

A trapezoidal motion law was prescribed directly in cable space, since more elaborate Cartesian trajectory planning was unnecessary for this experiment [58]:

$$\mathbf{l}(t) = \mathbf{l}_s - (\mathbf{l}_f - \mathbf{l}_s)u(t), \quad \mathbf{l}_s = \begin{bmatrix} 44.59 \\ 88.50 \end{bmatrix} \text{ cm}, \quad \mathbf{l}_f = \begin{bmatrix} 88.41 \\ 44.60 \end{bmatrix} \text{ cm} \quad (5.1)$$

where $\mathbf{l}_s$ and $\mathbf{l}_f$ denote the cable lengths at the initial and final equilibrium poses, namely $\boldsymbol{\zeta}_s = [15 \text{ cm}, -50 \text{ cm}, -36^\circ]^T$ and $\boldsymbol{\zeta}_f = [80 \text{ cm}, -50 \text{ cm}, 36^\circ]^T$. The scalar function $u(t)$ is the trapezoidal velocity profile in [58], with $u(0) = 0$ and $u(T = 9.29 \text{ s}) = 1$.

At each instant, the acquired proprioceptive array $\mathbf{z}$ in Eq. (4.2) includes cable-length readings, cable direction angle measurements, and the *EE* orientation angle. Cable lengths are estimated by incremental encoders mounted on the winches ($\approx 0.1^\circ$ resolution). Angular sensors located at the cable exit points provide measurements of

the cable direction angles ψ_i , with $i = 1, 2$ ($\approx 0.02^\circ$ resolution). A VRU mounted on the end-effector provides a direct measurement of the in-plane orientation angle χ ($\approx 0.3^\circ$ resolution). Based on a preliminary model-sensor characterization, the error levels used by the sensor-fusion were set to

$$e_l = 0.003 \text{ m} \quad \text{for cable lengths,} \quad (5.2)$$

$$e_\sigma = 0.0523 \text{ rad} \approx 3^\circ \quad \text{for cable angles,} \quad (5.3)$$

$$e_\chi = 0.0087 \text{ rad} \approx 0.5^\circ \quad \text{for platform orientation.} \quad (5.4)$$

Encoders and inclinometer measurements were sampled at 100 Hz; therefore, $N = 929$ measurement sets $\mathbf{z} = [\mathbf{z}_l, \mathbf{z}_\sigma, z_\chi]^T$ were collected along the prescribed trajectory. Ground-truth pose data were obtained via a vision-based system: a ChArUco board was mounted on the platform (Fig. 5.1) and recorded using a GoPro HERO6 camera (2.7K resolution, 4:3 aspect ratio) at 30 fps. The recorded frames were then processed with the OpenCV Python library [68] to reconstruct the platform pose over time. Finally, the resulting ground-truth sequence was temporally interpolated to enable a direct comparison with the higher-rate estimates produced by the sensor-fusion algorithm.

5.1.2 Estimation strategies

All estimators solve the *WNLS* problem in Eq. (4.3) using Gauss–Newton iterations (Section 3.2.2.1). For the first pose of the trajectory, the algorithm initial guess was set to $\zeta_0 = [15 \text{ cm}, -50 \text{ cm}, -36^\circ]$, corresponding to the nominally assigned initial pose. This configuration is a known static-equilibrium pose of the prototype for the assigned initial cable-length vector $\mathbf{l}_s$ and the inertial parameters reported in Section 5.1.1. For the remaining $N - 1$ poses, the initial guess was taken as the solution obtained at the previous trajectory point. The comparison targets two design choices that are central in practice:

Residual sets. Four residual configurations (Eq. (4.2)) are considered, obtained by enabling different subsets of the available proprioceptive measurements:

- *sl*: cable length and swivel angle $\mathbf{z} = [l_1, l_2, \sigma_1, \sigma_2]^T$;
- *sl χ* : cable length, swivel angle, and inclinometer $\mathbf{z} = [l_1, l_2, \sigma_1, \sigma_2, \chi]^T$;
- *li*: cable length and inclinometer $\mathbf{z} = [l_1, l_2, \chi]^T$;
- *si*: swivel angle, and inclinometer $\mathbf{z} = [\sigma_1, \sigma_2, \chi]^T$.

All predicted quantities are generated through the geometric model in Section 2.1 (i.e., $l_i(\zeta)$, for $i = 1, 2$, from Eq. (2.18), $\sigma_i(\zeta)$, for $i = 1, 2$, from Eq. (2.10), $\chi = \chi(\zeta)$ from Eq. (2.2)).

Weights and termination rules. Each diagonal entry of $\mathbf{Q}$ (Eq. (4.4)) is set based on the tolerance assigned to the corresponding measurement in the residual vector $\mathbf{r}$ (Eqs. (5.2)–(5.4)). This choice is commonly adopted when no more reliable estimate of the variance of $\mathbf{r}$ is available [69]. Comparing these residual sets illustrates how redundant information improves pose observability and reduces sensitivity to noise.

The stopping criteria follow the discussion in the paragraph *Stopping rules* of Section 3.2.1.1. In all tests, the iteration-, step-, and gradient-based conditions in Eqs. (3.10)–(3.12) are kept identical and are used only as safeguards, with

$$T_k = 100, \quad T_{\Delta\zeta} = 10^{-4}, \quad T_{\|\nabla\mathbf{r}\|} = 10^{-4}. \quad (5.5)$$

The three case studies differ in the *residual-based* termination rule, which is the condition the algorithm is expected to satisfy at convergence. In particular, we compare:

- *NormGN*: the standard *norm-based* residual condition (3.13), with a nominal threshold set to $T_{\|\mathbf{r}\|} = \|\mathbf{e}\|$, where $\mathbf{e}$ is defined as in (4.5) and its components are selected from Eqs. (5.2)–(5.4) according to the residual set under analysis;
- *StGN*: the same norm-based condition (3.13), but using a purely numerical and intentionally *strict* threshold, $T_{\|\mathbf{r}\|} = 10^{-4}$;
- *EngGN*: the proposed component-wise, uncertainty-consistent condition (4.11), where each residual component is compared against an *engineering meaningful* bound reflecting the expected accuracy of the corresponding measurement, i.e., $T_{\mathbf{r},i} = e_i$, with e_i selected from Eqs. (5.2)–(5.4) according to the residual set under analysis.

The three numerical thresholds in Eq. (5.5) were selected empirically so that convergence is driven primarily by the residual-based stopping conditions, either the residual norm or the component-wise residual condition. At the same time, they preserve explicit control over the maximum number of iterations, the step norm, and the gradient norm. Accordingly, T_k is chosen sufficiently large, whereas the thresholds on the step and gradient norms are chosen sufficiently small, so that these conditions do not trigger premature termination, while still acting as safeguards against excessive iterations and numerically unproductive updates.

5.1.3 Results and discussion

This subsection reports the experimental results obtained along the prescribed trajectory, focusing on both estimation accuracy and computational cost. The compared configurations differ in the selected residual set (Section 5.1.2) and in the residual-based termination rule, allowing the effect of measurement redundancy and stopping criteria to be assessed systematically.

Accuracy. Figure 5.2 compares the camera-based reconstruction (*Cam*) with the pose estimates obtained using the *NormGN* and *EngGN* termination rules, for the four residual sets defined in Section 5.1.2. A distinctive behavior emerges for *NormGN*: the estimated pose often remains constant over several consecutive samples. This occurs because the norm-based residual condition (3.13) can be satisfied before performing the first Gauss–Newton iteration, so that the initial guess is immediately accepted as the solution at that sample. This outcome is plausible under a norm-based rule: even when one or more residual components exceed their expected error bounds, other components may be well below their tolerances, yielding an overall residual norm smaller than the prescribed threshold.

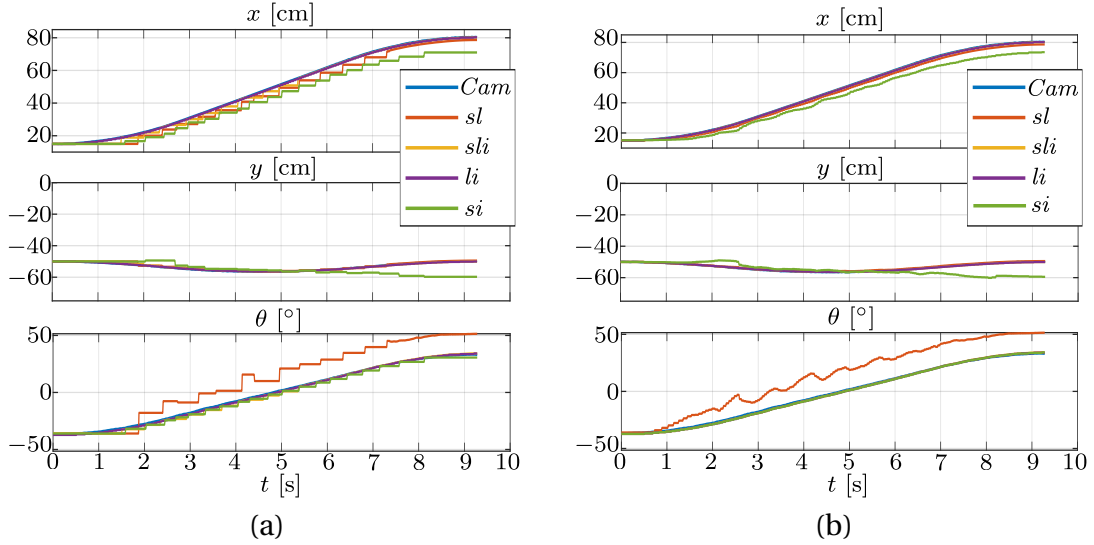


Figure 5.2: Platform poses along the trajectory obtained with the NormGN termination rule (a) and the EngGN rule (b). The camera-based reconstruction is labeled *Cam*, whereas the acronyms denote the estimates obtained using the different residual definitions (adapted from [54]).

The impact of measurement redundancy is highlighted by the error distributions in Fig. 5.3. When using only cable lengths and swivel angles (*sl*), the orientation estimate is markedly inaccurate, with an average absolute orientation error of about 12° . Conversely, when using only swivel angles and the inclinometer (*si*), position accuracy degrades, with average absolute errors of approximately 6 cm and 3.5 cm along the x and y directions, respectively. Comparing *li* and *sli* further shows that adding cable-angle measurements does not necessarily improve performance: *li* is slightly more accurate than *sli*, especially along the x direction (about 0.5 cm versus 1.5 cm average absolute error). This behavior is consistent with the limited accuracy of the cable-angle measurement, whose noise can dominate the fusion when included in the residual.

With residual sets *sli* and *li*, *StGN* improves accuracy by roughly a factor of two compared with *NormGN* (Fig. 5.3). Finally, the *EngGN* termination rule produces a noticeably smoother pose evolution than *NormGN* (Fig. 5.2b). In most samples, the component-wise condition (4.11) is not met without performing at least one iteration, preventing the “zero-iteration” acceptance that leads to piecewise-constant estimates.

In terms of accuracy, *EngGN* improves over *NormGN* by up to 67% for most configurations, with the exception of the orientation estimate using residual set *sl*, where it is about 14% less accurate. Moreover, *EngGN* attains accuracy comparable to *StGN* across all residual choices.

Overall, the norm-based termination rules (*NormGN* and *StGN*) are strongly affected by the scaling of the residual components and, therefore, by the chosen measurement units, making the selection of a meaningful threshold nontrivial. In contrast, *EngGN* enforces physically interpretable bounds on each residual component through Eq. (4.11), thereby providing a practical and systematic way to define the termination condition.

Runtime summary. Table 5.1 reports the average per-sample runtimes for all combinations of termination rules and residual sets. The computational cost of *StGN* is

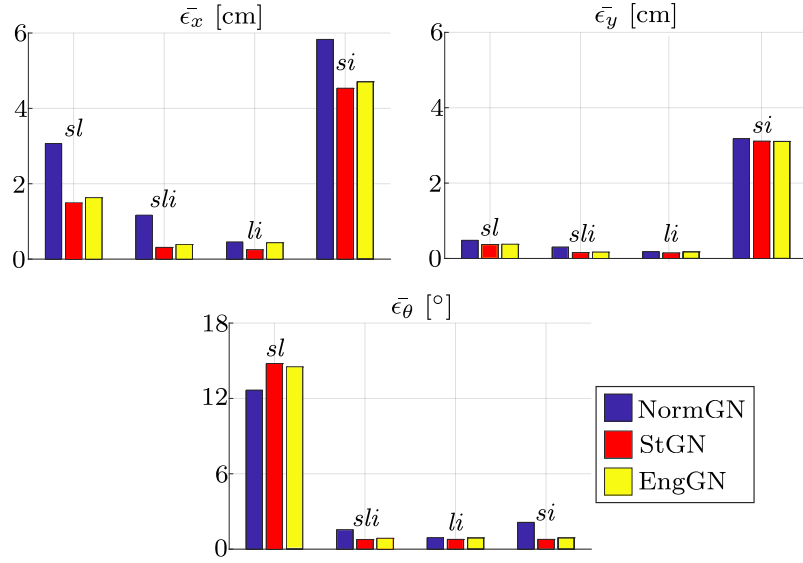


Figure 5.3: Error distributions across the compared configurations (adapted from [54]).

Table 5.1: Average per-sample runtimes for the compared termination rules and residual sets.

Set of residuals	NormGN [ms]	StGN [ms]	EngGN [ms]
<i>sl</i>	0.55	1.10	0.62
<i>sli</i>	0.65	0.99	0.75
<i>li</i>	0.45	0.68	0.45
<i>si</i>	0.38	0.65	0.41

consistently higher than that of *NormGN*, and is close to twice as large in several cases, in line with its stricter termination requirement. By contrast, *EngGN* remains comparable to *NormGN*, being at worst about 15% slower and never faster in this dataset, while achieving accuracy comparable to *StGN*. Since these tests were executed in MATLAB, an embedded implementation is expected to run comfortably within a small fraction of typical real-time cycle periods (e.g., 0.5–2 ms), making *EngGN* an attractive choice for practical deployments.

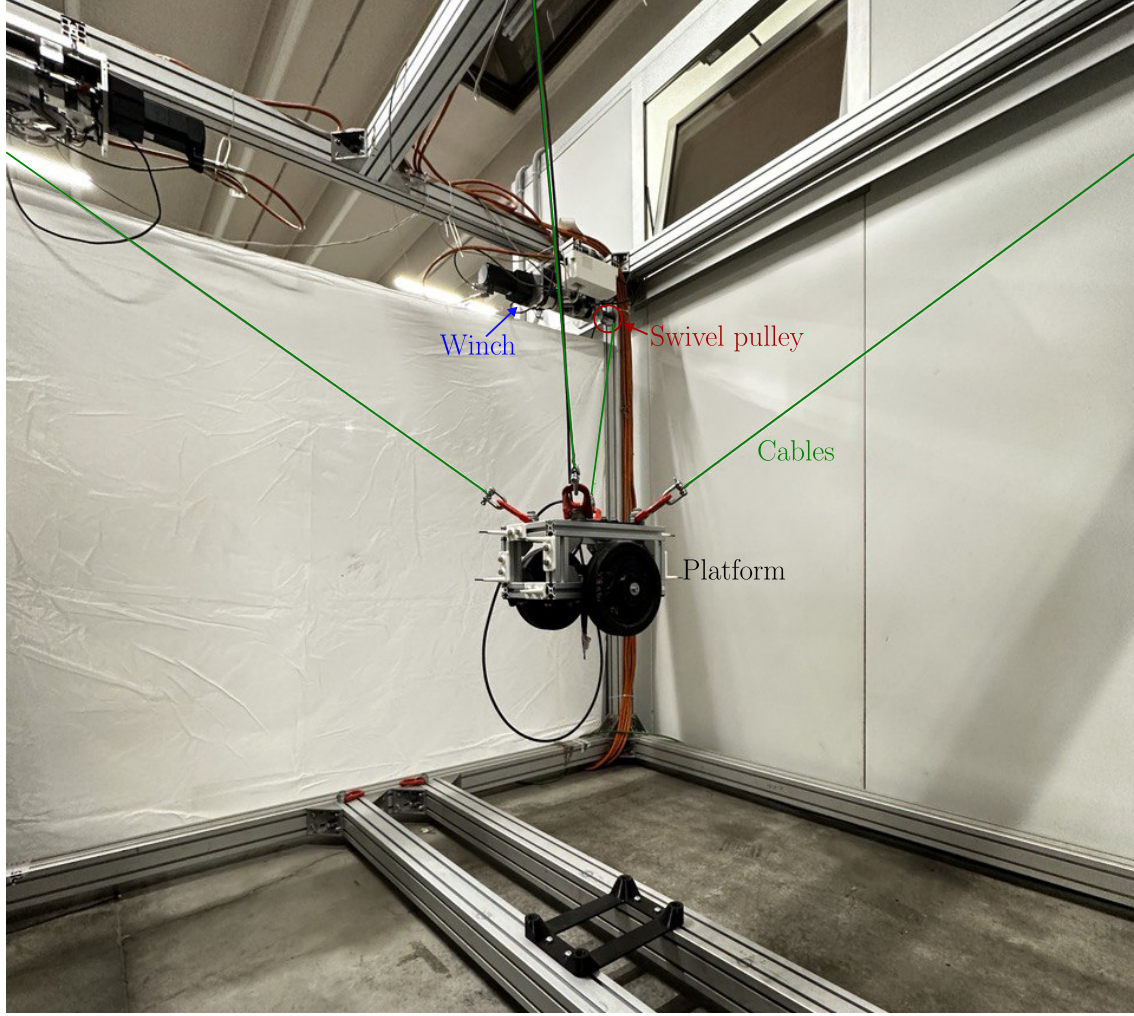


Figure 5.4: Spatial 4-cable *UACDPR* prototype used for experimental validation (adapted from [34]).

5.2 Experimental validation on a spatial 4-cable *UACDPR*

This section adapts and extends the experimental validation originally presented in [54]. The sensor-fusion *WNLS* formulation of Chapter 4 is validated on a spatial 4-cable, 6-*DoF* *UACDPR* prototype developed at the University of Bologna (Fig. 5.4). A 4-cable architecture is considered since, in this architecture, stability and controllability are highly sensitive to how the cables are kept under tension and controlled [46]. The experimental dataset is processed offline in *MATLAB* to:

- assess the influence of different residual sets (sensor combinations);
- compare norm-based and component-wise termination rules, consistently with the stopping-rule discussion in Subsections 3.2.1.1 and 4.1.4.

5.2.1 Prototype, trajectory, and sensing setup

Geometric and inertial parameters were extracted from the prototype CAD model and are summarized in Tables 5.2-5.3. In Table 5.2, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ denote the unit vectors of $Oxyz$; the coordinates of $\mathbf{d}_i$ and ${}^P\mathbf{a}'_i$ are expressed in $Oxyz$ and $Px'y'z'$, respectively.

Table 5.2: Geometric parameters of the spatial 4-cable *UACDPR* prototype.

i	1	2	3	4
$\mathbf{d}_i$ [m]	$\begin{bmatrix} 1.068 \\ -1.155 \\ 1.839 \end{bmatrix}$	$\begin{bmatrix} 1.068 \\ 1.155 \\ 1.839 \end{bmatrix}$	$\begin{bmatrix} -1.068 \\ 1.155 \\ 1.839 \end{bmatrix}$	$\begin{bmatrix} -1.068 \\ -1.155 \\ 1.839 \end{bmatrix}$
r_i [m]	0.03	0.03	0.03	0.03
${}^P\mathbf{a}'_i$ [m]	$\begin{bmatrix} 0.075 \\ -0.075 \\ 0.280 \end{bmatrix}$	$\begin{bmatrix} 0.075 \\ 0.075 \\ 0.280 \end{bmatrix}$	$\begin{bmatrix} -0.075 \\ 0.075 \\ 0.280 \end{bmatrix}$	$\begin{bmatrix} -0.075 \\ -0.075 \\ 0.280 \end{bmatrix}$
x_i	$-\mathbf{k}$	$-\mathbf{k}$	$-\mathbf{k}$	$-\mathbf{k}$
y_i	$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{j}$	$-\mathbf{j}$
z_i	$\mathbf{i}$	$\mathbf{i}$	$-\mathbf{i}$	$-\mathbf{i}$

Table 5.3: Inertial parameters of the *EE* of the spatial 4-cable *UACDPR* prototype.

$m = 12.966$ kg	$\mathbf{I} = \begin{bmatrix} 0.18 & 0 & 0 \\ 0 & 0.12 & 0 \\ 0 & 0 & 0.19 \end{bmatrix}$ kg m ²	$\mathbf{r}_G = \begin{bmatrix} 0 \\ 0 \\ 0.173 \end{bmatrix}$ m
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Workspace-scanning trajectory. To reduce bias due to workspace location, a quasi-static scanning motion was prescribed through a set of linear segments connecting the set-points¹:

$$\mathbf{p}_i = \begin{bmatrix} (-1)^{\lfloor i/22 \rfloor} (0.5 - 0.1 \lfloor i/2 \rfloor) - 1.1 \lfloor i/22 \rfloor + 4.4 \lfloor i/44 \rfloor \\ (-1)^{\lfloor (i+3)/2 \rfloor} 0.5 \\ 0.25(1 + \lfloor i/22 \rfloor) \end{bmatrix} \text{ m}, \quad i = 0, \dots, 65, \quad (5.6)$$

where $\lfloor \cdot \rfloor$ denotes the floor function. The reference point is commanded along the segments at constant speed 1 cm/s. The resulting trajectories are shown as red lines in Fig. 5.5. The dataset covers a broad set of configurations and spans a wide range of cable tensions, from highly tensioned regimes to cases where one or more cables approach slackness. This variability is crucial for Section 5.3 as it enables the evaluation of the proposed tension-dependent error modeling, particularly in configurations where the straight, taut-cable assumption is only weakly satisfied.

Proprioceptive measurements. The measurement array $\mathbf{z}$ in Eq. (4.2) includes cable-length readings, swivel-angle measurements, and *EE* orientation angles.

- *Cable lengths:* incremental encoders on motor axes (resolution $\approx 0.1^\circ$) are combined with the motor kinematic model [45] to estimate the cable lengths l_i , $i = 1, 2, 3, 4$.
- *Swivel angles:* incremental encoders mounted on the swivel-pulley axes (resolution $\approx 0.02^\circ$) provide swivel-angle measurements σ_i , $i = 1, 2, 3, 4$.
- *Platform orientation:* an XSens MTi-630 AHRS inclinometer provides roll ψ and pitch θ (resolution $\approx 0.5^\circ$), and yaw φ (resolution $\approx 3^\circ$) measurements.

¹For safety, the experimental scan is performed at a conservative distance from the workspace boundary. A complementary workspace-wide convergence study is reported in the Appendix of [34].

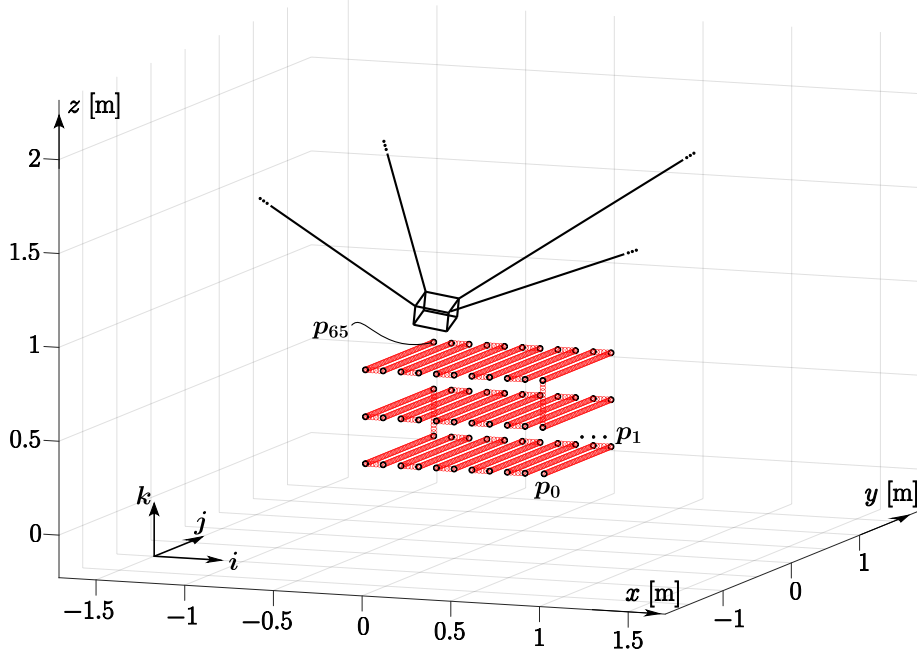


Figure 5.5: Assigned workspace-scanning trajectory for the spatial 4-cable *UACDPR* (adapted from [34]).

All measurements are sampled at 100 Hz, yielding $N = 73891$ measurement sets along the trajectory:

$$\mathbf{z} = [l_1, l_2, l_3, l_4, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \psi, \theta, \psi]^T. \quad (5.7)$$

N sets of cable tension measurements $\boldsymbol{\tau} = [\tau_1, \tau_2, \tau_3, \tau_4]^T$ are also acquired and are exploited in Section 5.3 to evaluate the tension-aware error models introduced in Section 4.2.

Ground truth and measurement-error characterization. Ground-truth poses are acquired using a Vicon photogrammetry system tracking nine reflective markers mounted on the *EE* (marker-position accuracy ± 0.2 mm). To reduce measurement noise, marker trajectories are processed with a zero-phase finite-impulse-response low-pass filter with a stop-band frequency of 4 Hz. Given the quasi-static nature of the experiment and the absence of modeled oscillatory dynamics above this frequency range, the filter removes high-frequency noise without altering the rigid-body motion trend [57].

From the filtered marker dataset, the *EE* reference pose $\boldsymbol{\zeta}_{gt}$ is reconstructed. The same ground-truth pose sequence is then mapped to the corresponding geometric quantities predicted by the model in Section 2.1, namely: cable lengths $l_i(\boldsymbol{\zeta}_{gt})$ via Eq. (2.18), swivel angles $\sigma_i(\boldsymbol{\zeta}_{gt})$ via Eq. (2.10) and *EE* orientation angles via Eq. (2.2). These reconstructions are used as ground truth for error characterization of length and swivel measurements.

Although the empirical error distributions deviate from Gaussianity, they are approximated here through Gaussian statistics to obtain practical uncertainty levels for weight selection (Eq. (4.4)) and residual-wise termination thresholds (Eq. (4.11)). An example histogram is reported in Fig. 5.6. The resulting standard deviations are reported in Table 5.4 (lengths and swivel angles) and Table 5.5 (orientation). When constructing $\mathbf{Q}$ and the termination thresholds, the values are converted to meters and radians to match the units used in the solver.

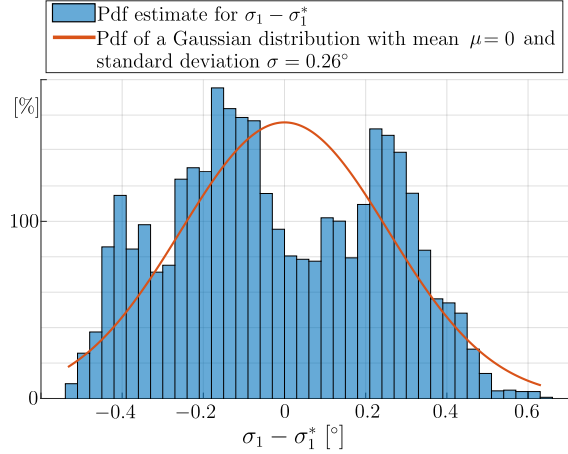


Figure 5.6: Example error distribution for the swivel-angle measurement σ_1 and its Gaussian approximation (adapted from [34]).

Table 5.4: Standard deviations of cable-length and swivel-angle measurement errors.

Cable lengths	$\sigma_{l_1} = 0.39 \text{ cm}$	$\sigma_{l_2} = 0.26 \text{ cm}$	$\sigma_{l_3} = 0.10 \text{ cm}$	$\sigma_{l_4} = 0.15 \text{ cm}$
Pulley swivel angles	$\sigma_{\sigma_1} = 0.22^\circ$	$\sigma_{\sigma_2} = 0.62^\circ$	$\sigma_{\sigma_3} = 0.28^\circ$	$\sigma_{\sigma_4} = 0.38^\circ$

Table 5.5: Standard deviations of inclinometer orientation errors.

Roll, pitch, yaw	$\sigma_\psi = 0.30^\circ$	$\sigma_\theta = 0.16^\circ$	$\sigma_\varphi = 0.35^\circ$
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5.2.2 Estimation strategies

All estimators solve the WNLS problem (4.3), using the same datasets and initialization strategy. For the first pose of the trajectory, the initial guess is set to

$$\zeta_0 = [0.5037, -0.5520, 0.5155, -0.1790, -0.1758, 0.0089]^T, \quad (5.8)$$

This value is not an arbitrary initial guess. Before data acquisition starts, the platform moves from a known homing configuration to the first recorded pose ζ_1 . The intermediate poses traversed along this motion are recursively reconstructed using the same pose-estimation algorithm described in this section, starting from the homing pose. The configuration reported in Eq. (5.8) therefore represents the estimated pose immediately preceding the first recorded sample, and is used to initialize the first solve. For the remaining samples (ζ_i , $i = 2, \dots, N$), the estimator is warm-started with the solution computed at the previous sample (ζ_{i-1}).

The comparison targets two design choices that are central in practice:

Residual sets. Four residual configurations are considered by enabling different subsets of the proprioceptive measurements in $\mathbf{z}$ (Eq. (5.7)):

- *sl*: cable length, swivel angle, and inclinometer $\mathbf{z} = [l_1, l_2, l_3, l_4, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \psi, \theta, \chi]^T$;
- *sl*: cable length and swivel angle $\mathbf{z} = [l_1, l_2, l_3, l_4, \sigma_1, \sigma_2, \sigma_3, \sigma_4]^T$;
- *li*: cable length and inclinometer $\mathbf{z} = [l_1, l_2, l_3, l_4, \psi, \theta, \chi]^T$;
- *si*: swivel angle, and inclinometer $\mathbf{z} = [\sigma_1, \sigma_2, \sigma_3, \sigma_4, \psi, \theta, \chi]^T$.

Predicted quantities are generated through the geometric model of Chapter 2 in particular $l_i(\zeta)$, for $i = 1, 2, 3, 4$, from Eq. (2.18), $\sigma_i(\zeta)$, for $i = 1, 2, 3, 4$, from Eq. (2.10), $\psi = \psi(\zeta)$, $\theta = \theta(\zeta)$ and $\chi = \chi(\zeta)$ from Eq. (2.2).

Weights and termination rules. When comparing different sensor sets, the same uncertainty level is used both to weight the residuals in the *WNLS* cost function and to define the residual-based stopping thresholds. Specifically, for each measurement component with standard deviation σ_i (Tables 5.4–5.5), the corresponding diagonal entry of the weighting matrix is set to $Q_{ii} = (3\sigma_i)^2$, while the component-wise rule (4.11) is applied with $T_{r,i} = 3\sigma_i$. Under the Gaussian approximation of a zero-mean measurement error with standard deviation σ_i , a single realization satisfies $|e_i| \leq 3\sigma_i$ with probability $\approx 99.7\%$; hence, $3\sigma_i$ provides a practical, relatively loose tolerance that favors shorter runtimes while preserving physically interpretable residual bounds.

Additional tests are then performed using the full sensor set *sli* to assess the effect of tighter tolerances. In these tests, the weighting matrix is built from the one-sigma variances, i.e., $Q_{ii} = \sigma_i^2$, and the residual-based termination rule is varied as follows:

- *EngGN* with 1σ component-wise thresholds, $T_{r,i} = \sigma_i$, corresponding to a tighter uncertainty band (under the Gaussian approximation, a single realization satisfies $|e_i| \leq \sigma_i$ with probability $\approx 68.3\%$);
- *StGN* with the strict norm-based condition (3.13) and threshold

$$T_{\|\mathbf{r}\|} = 10^{-8}, \quad (5.9)$$

For completeness, these results are also compared against the looser baseline configuration (*sli* with *EngGN*, $(3\sigma_i)^2$ weights, and $3\sigma_i$ thresholds).

In all tests, the remaining termination conditions are used only as safeguards and are set according to Section 4.1.4:

1. $T_k = 100$;
2. $T_p = 10^{-6} \text{ m}$ and $T_\epsilon = 10^{-6} \text{ rad}$;
3. $T_{\left\| \frac{\partial f}{\partial \mathbf{p}} \right\|} = 10^{-6} \text{ m}^{-1}$ and $T_{\left\| \frac{\partial f}{\partial \epsilon} \right\|} = 10^{-6} \text{ rad}^{-1}$.

5.2.3 Results and discussion

This subsection reports the main outcomes of the spatial experimental campaign. First, the relative impact of each sensor modality is assessed through the four residual sets of Section 5.2.2. Then, the influence of the residual-based termination rule is discussed by comparing *NormGN* and *EngGN* under consistent datasets and weights.

Sensor-set comparison. Table 5.6 shows that *sli* and *li* provide comparable accuracy, whereas *sl* is 90% less accurate in position estimation and 1280% less accurate in determining the orientation. Moreover, *si* fails to converge to a numerically meaningful solution on this dataset, indicating that swivel angles and orientation alone are insufficient to reliably constrain the pose in the tested configurations.

The reduced accuracy of *sl* is consistent with the limited reliability of swivel-angle measurements when cable tensions decrease: under low tension, friction and backlash

Table 5.6: Average and maximum pose errors using *EngGN* with 3σ thresholds for different residual sets.

Average error:	<i>sli</i>	<i>li</i>	<i>sl</i>	<i>si</i>
Position [cm]	0.215	0.208	0.839	NaN
Orientation [°]	0.229	0.235	3.158	NaN
Maximum error:	<i>sli</i>	<i>li</i>	<i>sl</i>	<i>si</i>
Position [cm]	1.21	1.16	3.62	NaN
Orientation [°]	1.19	1.17	11.1	NaN

Table 5.7: Average and maximum per-sample runtimes for different residual sets (*EngGN*, 3σ thresholds).

Average time:	<i>sli</i>	<i>li</i>	<i>sl</i>	<i>si</i>
[ms]	2.06	0.966	0.973	0.879
Maximum time:	<i>sli</i>	<i>li</i>	<i>sl</i>	<i>si</i>
[ms]	26.8	5.36	9.72	11.5

Table 5.8: Average and maximum pose errors with different residual-based stopping rules (*sli* residual set).

Average error:	<i>NormGN</i>	<i>EngGN</i> (1σ)	<i>EngGN</i> (3σ)
Position [cm]	0.159	0.163	0.215
Orientation [°]	0.216	0.217	0.229
Maximum error:	<i>NormGN</i>	<i>EngGN</i> (1σ)	<i>EngGN</i> (3σ)
Position [cm]	0.883	0.884	1.21
Orientation [°]	0.924	0.921	1.19

at the swivel interface may prevent the pulley from aligning with the cable direction, violating the modeling assumptions and injecting bias into the residuals. In contrast, the *li* configuration avoids swivel angles and leverages the inclinometer to directly constrain orientation, yielding accuracy comparable to *sli* while relying on fewer sensors.

Table 5.7 reports the corresponding runtimes. The full residual set *sli* is the most expensive, while *li* and *sl* more than halve the average runtime. Overall, *li* offers the best compromise on this dataset: it remains accurate while reducing both sensing complexity and computational cost.

Termination-rule comparison. The comparison is now restricted to the full residual set *sli* to isolate the effects of weighting strategies and residual-based stopping rules. Table 5.8 reports average and maximum pose errors for: (i) *NormGN* (Eq. (3.13)) with $T_{\parallel\mathbf{r}} = 10^{-8}$ (Eq. (5.9)), and (ii) *EngGN* (Eq. (4.11)) with 1σ and 3σ thresholds.

In terms of accuracy, *NormGN* and *EngGN* with 1σ thresholds are essentially equivalent on this dataset. Using 3σ thresholds accelerates convergence but yields a moderate loss in accuracy, as expected from a more permissive termination rule.

In terms of computational cost, the same trend is observed: *NormGN* and *EngGN* (1σ) require comparable time, whereas *EngGN* (3σ) substantially reduces runtime. In this experiment (meters/radians), the average per-sample runtime is 6.30 ms for

NormGN, 5.20 ms for *EngGN* (1σ), and 2.10 ms for *EngGN* (3σ). Beyond raw speed, *EngGN* offers a clearer tuning rationale: each residual component is compared against a physically interpretable bound, and the thresholds are independent of residual scaling and unit conventions.

5.3 Tension-aware sensor fusion: validation on the spatial 4-cable UACDPR

This section evaluates the tension-aware formulation introduced in Section 4.2 on the same spatial prototype and dataset of Section 5.2. The key idea is to treat the uncertainty levels $\mathbf{e}$ (Eq. (4.5)) as configuration dependent, by modeling cable-related errors as functions of measured cable tensions. The resulting tension-dependent uncertainty estimates are used consistently to build the weighting matrix $\mathbf{Q}$ (Eq. (4.4)) and to set the component-wise termination thresholds in Eq. (4.11).

5.3.1 Offline identification of tension-dependent error models

Three error models are compared:

- *Constant variance*: the cable-length and swivel-angle uncertainties are set to the (constant) standard deviations in Table 5.4.
- *Linear model*: cable-length and swivel-angle uncertainty levels follow the affine form in Eq. (4.22).
- *Inverse model*: cable-length and swivel-angle uncertainty levels follow the rational form in Eq. (4.23).

For the tension-dependent models, the parameters are identified offline from the dataset described in Section 5.2.1 through the density-aware weighted least-squares procedure of Subsection 4.2.2. A conservative peak-selection step is adopted: within each 1 N-wide tension interval centered at integer tension values, only local error peaks are retained to avoid underestimating uncertainty in low-density regions. Then, the coefficients of the linear and inverse correlation models are identified via Eqs. (4.19) and (4.21), for each cable $i = 1, 2, 3, 4$ and for each measurement type l and σ . The bandwidth parameter h is selected using Silverman's rule of thumb [64] and, for the measurement set considered, it yields $h = 1.06$.

The resulting fitted models (Eqs. (4.22) and (4.23)) are:

$$\begin{aligned}
 \hat{e}_1^l(\tau_1) &= 0.9213 \text{ cm} - 0.0038 \text{ cm/N} \tau_1 & \hat{e}_2^l(\tau_2) &= 0.7164 \text{ cm} - 0.0054 \text{ cm/N} \tau_2 \\
 \hat{e}_3^l(\tau_3) &= 0.2090 \text{ cm} - 0.0002 \text{ cm/N} \tau_3 & \hat{e}_4^l(\tau_4) &= 0.3877 \text{ cm} - 0.0014 \text{ cm/N} \tau_4; \\
 \hat{e}_1^\sigma(\tau_1) &= 0.6066^\circ - 0.0024^\circ/\text{N} \tau_1 & \hat{e}_2^\sigma(\tau_2) &= 1.6844^\circ - 0.0106^\circ/\text{N} \tau_2 \\
 \hat{e}_3^\sigma(\tau_3) &= 0.7017^\circ - 0.0026^\circ/\text{N} \tau_3 & \hat{e}_4^\sigma(\tau_4) &= 0.8340^\circ - 0.0050^\circ/\text{N} \tau_4.
 \end{aligned} \tag{5.10}$$

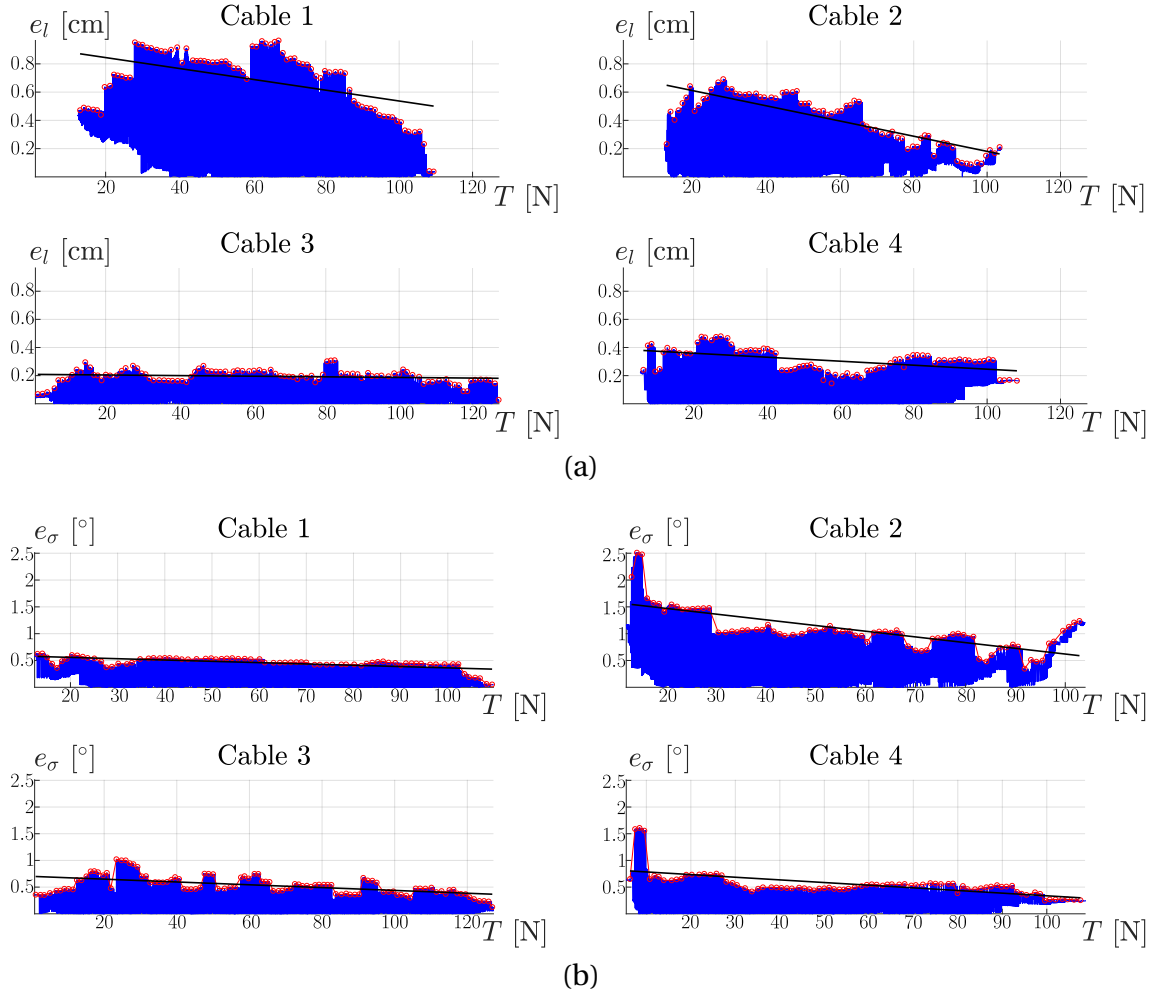


Figure 5.7: Tension–error data and linear-in-tension regression for the four cables: cable-length errors vs. tension (a) and swivel-angle errors vs. tension (b).

$$\begin{aligned}
 \hat{e}_1^l(\tau_1) &= \frac{20.68 \text{ cm/N}}{\tau_1} + 0.30 \text{ cm} & \hat{e}_2^l(\tau_2) &= \frac{17.11 \text{ cm/N}}{\tau_2} + 0.04 \text{ cm} \\
 \hat{e}_3^l(\tau_3) &= \frac{2.22 \text{ cm/N}}{\tau_3} + 0.16 \text{ cm} & \hat{e}_4^l(\tau_4) &= \frac{2.68 \text{ cm/N}}{\tau_4} + 0.24 \text{ cm}; \\
 \hat{e}_1^\sigma(\tau_1) &= \frac{9.2740^\circ/\text{N}}{\tau_1} + 0.2717^\circ & \hat{e}_2^\sigma(\tau_2) &= \frac{19.5473^\circ/\text{N}}{\tau_2} + 0.6011^\circ \\
 \hat{e}_3^\sigma(\tau_3) &= \frac{10.7300^\circ/\text{N}}{\tau_3} + 0.3102^\circ & \hat{e}_4^\sigma(\tau_4) &= \frac{8.6496^\circ/\text{N}}{\tau_4} + 0.3229^\circ.
 \end{aligned} \tag{5.11}$$

The two tension-dependent models capture the expected degradation of measurement reliability as cable tension decreases. In particular, the inverse model naturally diverges as $\tau_i \rightarrow 0$, reflecting loss of reliability under near-slack conditions (e.g., sagging, backlash, and deviations from the straight-line cable model), and therefore tends to suppress the influence of very-low-tension measurements within the estimator.

The resulting datasets and fitted trends are illustrated in Figs. 5.7 and 5.8: the blue curves report the measurement errors sorted by increasing tension, the red markers highlight the selected peak values within each 1 N interval, and the black curve denotes the fitted regression model (linear in Fig. 5.7 and inverse in Fig. 5.8).

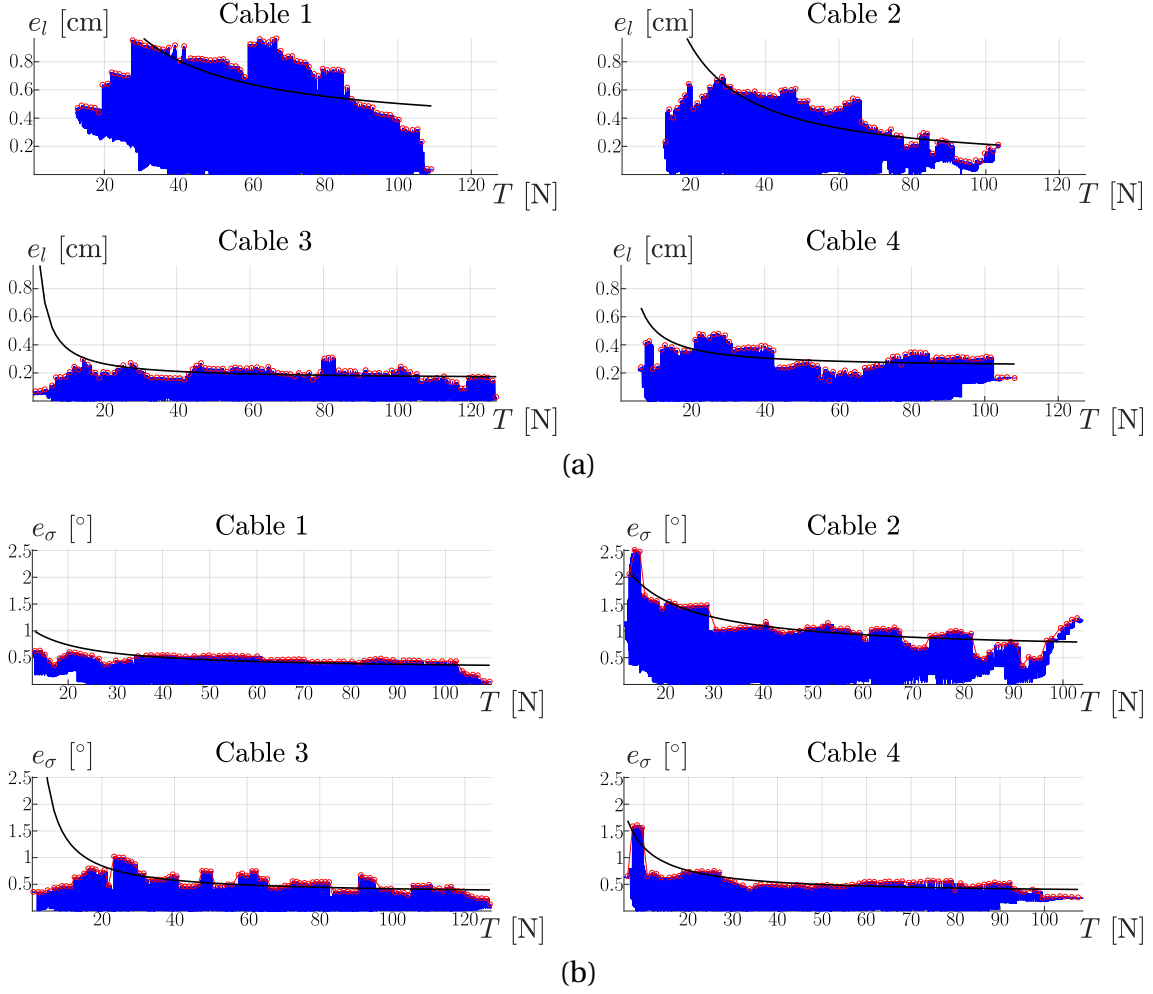


Figure 5.8: Tension–error data and inverse regression for the four cables: cable-length errors vs. tension (a) and swivel-angle errors vs. tension (b).

5.3.2 Impact on pose-estimation performance

The three error models described in Section 5.3.1 are evaluated in terms of performance achieved in estimating the *EE* poses, using the spatial prototype and dataset described in Section 5.2. In all the performed tests, the iteration termination conditions 1 – 3 were set according to the discussion in Section 4.1.4, using strict numerical values:

1. $T_k = 100$;
2. $T_p = 10^{-6} \text{ m}$ and $T_\epsilon = 10^{-6} \text{ rad}$;
3. $T_{\|\frac{\partial f}{\partial \mathbf{p}}\|} = 10^{-6} \text{ m}^{-1}$ and $T_{\|\frac{\partial f}{\partial \boldsymbol{\epsilon}}\|} = 10^{-6} \text{ rad}^{-1}$.

For iteration termination condition 4, *EngGN* rule (4.11) is adopted to highlight the influence of tension-aware error functions on the solver convergence while keeping the discussion concise, since *EngGN* has already been compared against *StGN* in the previous analysis.

The solver is tested with two measurement sets:

- *sl*: cable-length and swivel-angle measurements, which are directly affected by cable tension and are the target of the proposed tension-dependent error modeling;

Table 5.9: Pose estimation errors using residual set *sli* with different error models (*EngGN*).

Average error:	Constant variance	Linear model	Inverse model
Position [cm]	0.571	0.616	0.658
Orientation [°]	0.431	0.465	0.496
Maximum error:	Constant variance	Linear model	Inverse model
Position [cm]	1.260	1.438	1.679
Orientation [°]	0.849	0.907	0.980

- *sli*: the full measurement set available, including cable lengths, swivel angles, and *EE* orientation measurements, to assess the impact of tension-aware modeling when additional, tension-independent information is also provided.

Only these two sensor sets are considered because they include the tension-dependent measurements, and the goal is to isolate how the proposed linear and inverse tension-dependent error models affect the estimation results.

The uncertainty levels are varied consistently across weighting and stopping: for each measurement component, a scalar uncertainty parameter e_i is defined and used both to build the weighting matrix and to set the component-wise termination thresholds: the diagonal entries of $\mathbf{Q}$ (Eq. (4.4)) are set to $(e_i)^2$, and the residual-based stopping rule (4.11) is applied with $T_{r,i} = e_i$. The uncertainty levels e_i are assigned according to the following error models:

- *Constant variance*: $e_i = \sigma_i$, with σ_i taken from Table 5.4 for cable length and swivel angle measurements;
- *Linear model*: $e_i = \hat{e}_i(\tau_i)$, with $\hat{e}_i(\cdot)$ given by the affine fits in (5.10) and evaluated online using the measured tensions τ_i ;
- *Inverse model*: $e_i = \hat{e}_i(\tau_i)$, with $\hat{e}_i(\cdot)$ given by the rational fits in (5.11) and evaluated online using the measured tensions τ_i .

Only the cable-length and swivel-angle uncertainty parameters are made configuration dependent and varied across the three case studies. The inclinometer uncertainties are kept constant, since their error statistics are approximately Gaussian and do not depend on cable tension; the corresponding values are taken from Table 5.5.

Performance is assessed in terms of average and maximum position and orientation errors with respect to ground truth, as well as average and maximum per-sample runtime and iteration count.

Accuracy. Tables 5.9–5.10 show that the three error models yield comparable accuracy. Including the inclinometer (*sli*, Table 5.9) improves both position and orientation accuracy with respect to *sl* (Table 5.10), consistently with the observability arguments of Section 5.2.2.

These results indicate that tension-aware modeling is not primarily aimed at improving accuracy on this dataset. Rather, the comparable accuracy across models confirms that introducing tension dependence preserves estimation quality while reshaping internal weighting and termination behavior. For the *sl* configuration, the inverse model

Table 5.10: Pose estimation errors using residual set sl with different error models ($EngGN$).

Average error:	Constant variance	Linear model	Inverse model
Position [cm]	1.267	1.259	1.254
Orientation [°]	0.954	0.948	0.945
Maximum error:	Constant variance	Linear model	Inverse model
Position [cm]	5.040	5.163	8.180
Orientation [°]	2.887	2.958	4.686

Table 5.11: Runtime using residual set sli with different error models ($EngGN$).

Time:	Constant variance	Linear model	Inverse model
Average [ms]	8.4	3.6	3.7
Maximum [ms]	39.8	37.3	28.3

Table 5.12: Runtime using residual set sl with different error models ($EngGN$).

Time:	Constant variance	Linear model	Inverse model
Average [ms]	3.1	1.8	1.7
Maximum [ms]	97.5	67.6	146.1

Table 5.13: Iteration count using residual set sli with different error models ($EngGN$).

Iteration number:	Constant variance	Linear model	Inverse model
Average	3.047	0.824	0.689
Maximum	5	5	5

Table 5.14: Iteration count using residual set sl with different error models ($EngGN$).

Iteration number:	Constant variance	Linear model	Inverse model
Average	1.663	0.652	0.510
Maximum	59	18	100

produces larger worst-case errors, which is consistent with aggressive down-weighting of low-tension measurements: in occasional samples, the estimator effectively relies on a reduced subset of informative residuals.

Computational efficiency and convergence. A markedly different behavior emerges in terms of runtime and iteration count. Tables 5.11–5.14 show that both tension-dependent models substantially reduce the average computational cost, for both sli and sl . For sli , the average time drops from 8.4 ms (constant variance) to 3.6–3.7 ms (tension-aware), and the average iteration count decreases from about 3.0 to below 1, indicating that convergence is typically achieved in a single Gauss–Newton update. A similar behavior is observed for sl .

Overall, the tension-dependent uncertainty models primarily affect convergence behavior rather than accuracy. By reinforcing the influence of high-tension (more reliable) measurements and down-weighting low-tension (less reliable) ones, the Gauss–Newton

iterations become faster. The inverse model achieves the lowest average iteration count, but it also exhibits larger worst-case variability (notably for sl), which is consistent with its more aggressive attenuation under near-slack conditions.

However, the dataset is currently limited to a central region of the workspace and therefore does not capture the full range of operating conditions. Future work will focus on collecting more representative data that explicitly covers challenging regimes, including configurations close to the workspace boundaries and near-slack cable conditions, to further validate and refine the proposed error-modeling strategies.

Chapter 6

Conclusions

This thesis investigated pose estimation for underactuated cable-driven parallel robots (*UACDPRs*) using proprioceptive sensing. Underactuation makes the direct geometric mapping from actuation-related measurements to end-effector pose intrinsically underdetermined: cable lengths alone are generally insufficient to uniquely determine the configuration, and additional constraints are required. The central goal pursued here was to obtain accurate and reliable pose estimates by introducing such constraints through redundant sensing, rather than by enforcing mechanical equilibrium during estimation. The resulting framework was developed consistently across modeling and experimental validation, with particular attention to practical aspects that govern robustness in real deployments such as weighting, stopping criteria, and configuration-dependent measurement reliability.

6.1 Main results

A coherent modeling framework was established for spatial and planar *UACDPRs*. The formulation explicitly accounts for pulley routing through swivel-pulley variables and introduces the swivel and tangency angles needed to represent cable directions and effective lengths under routing constraints. Within this model, first-order differential kinematics were derived to obtain analytic Jacobians relating pose variations to variations in modeled geometric quantities. These Jacobians form the backbone of the proposed sensor-fusion estimator, enabling systematic residual/Jacobian assembly without resorting to numerical differentiation.

The direct-kinematics formulations for *UACDPRs* are reviewed, emphasizing the limitations of purely geometric approaches and the role of additional information in closing the problem. In particular, direct geometrico-static formulations were discussed as a principled way to select an equilibrium pose by coupling geometry with static constraints, at the cost of additional modeling dependencies. The numerical landscape was framed in terms of local iterative schemes (Newton-type methods, Gauss–Newton for least squares) and global techniques. The role of convergence certification was highlighted through a Kantorovich-based perspective, motivating conservative but computable conditions ensuring existence/uniqueness of solutions and guaranteed Newton convergence in square-system settings.

The main contribution of the thesis is a sensor-fusion framework that renders the pose observable by augmenting cable-length readings with additional pose-dependent proprioceptive measurements. The approach is cast as a weighted nonlinear least-

squares estimation problem, where a flexible residual set stacks heterogeneous modeled quantities (e.g., cable lengths, cable orientation angles such as swivel angles, and directly measured pose components such as orientation). The problem is solved via Gauss–Newton iterations. A practical weighting strategy was proposed based on inverse-error principles, making the objective dimensionless and allowing measurement reliability to be reflected explicitly in the update direction.

A key practical outcome concerns iteration termination. Norm-based stopping rules can be difficult to tune and can be sensitive to the numerical scaling and unit conventions of heterogeneous residual components. Component-wise termination conditions were therefore introduced to enforce physically interpretable bounds separately on translation and orientation increments and on each residual component. This design prevents premature acceptance of the initial guess (notably when norm-based residual thresholds are met without performing iterations) and provides a systematic stopping mechanism directly linked to expected sensor accuracy. Experimental evidence supports the relevance of this choice. In the planar validation, a norm-based residual rule can lead to piecewise-constant estimates over multiple samples due to “zero-iteration” acceptance, whereas the component-wise residual check yields smoother trajectories and improved accuracy in most configurations while preserving low runtime. On the spatial dataset, component-wise rules provide comparable accuracy to norm-based rules under consistent conditions, while enabling clear trade-offs through interpretable thresholds (e.g., 1σ versus 3σ bounds).

The thesis also introduced a tension-aware extension motivated by the observation that cable-related measurement reliability varies with cable tension. Low-tension regimes are more likely to violate ideal modeling assumptions due to sagging, backlash, and friction effects; therefore, treating sensor uncertainty as constant can be suboptimal. The proposed method models measurement uncertainty as a function of measured cable tensions through identifiable tension–error relations. Two correlation models were considered: a linear-in-tension model and an inverse-tension model, the latter naturally diverging as tension approaches zero and thus strongly down-weighting near-slack information. Model parameters were identified offline using conservative peak selection and density-aware fitting procedures to avoid underestimating uncertainty in sparse regimes. When used online, the tension-dependent uncertainty estimates were employed consistently in both the weighting matrix and the component-wise termination thresholds. On the spatial prototype and dataset, the tension-aware models preserved estimation accuracy while markedly improving computational efficiency: average runtimes and iteration counts decreased substantially, often yielding convergence in a single Gauss–Newton update. The inverse model achieved the lowest average iteration count, at the price of larger worst-case variability in some configurations, consistent with its aggressive attenuation under near-slack conditions.

Two experimental campaigns validated the proposed approach and clarified practical design trade-offs.

- On the planar 2-cable prototype, ground truth obtained by vision-based tracking enabled a direct comparison of residual sets and stopping rules. The results highlighted the role of measurement redundancy and sensor reliability: residual sets lacking a direct orientation measurement can produce large orientation errors, whereas including an inclinometer improves observability of rotational motion. The component-wise termination rule achieved accuracy comparable to stricter norm-based criteria while keeping runtimes within sub-millisecond

ranges in the MATLAB implementation.

- On the spatial 4-cable prototype, Vicon-based ground truth supported a systematic evaluation of sensor combinations and stopping rules. Residual sets including a direct orientation measurement (inclinometer) delivered substantially higher accuracy than sets relying heavily on swivel-angle information alone, which can degrade under low-tension conditions. The analysis further showed that a reduced residual set can offer an excellent accuracy–runtime trade-off when it avoids unreliable modalities while still constraining the pose sufficiently. Finally, the tension-aware extension consistently reduced computational cost across tested residual configurations without compromising accuracy, on the considered dataset.

Two appendices supported the main narrative. Sensitivity-based uncertainty propagation bounds were summarized to relate bounded measurement uncertainty to pose-estimation uncertainty in underactuated settings where a subset of degrees of freedom is measured independently. In addition, a Kantorovich-based convergence certification framework was specialized to the DGSP of 3-cable *UACDPRs*, yielding computable sufficient conditions for existence/uniqueness of the equilibrium solution and conservative bounds on admissible cable-length measurement errors that guarantee Newton convergence.

6.2 Open Issues

The results of this thesis point to several directions where the proposed framework can be strengthened and extended.

- *Broader datasets and challenging regimes.* The tension-aware validation indicates clear computational benefits, but the available dataset covers a limited region of operation. A more comprehensive experimental campaign should explicitly include configurations close to workspace boundaries, near-slack conditions, and regimes where routing, friction, and sagging effects become dominant. Such data would improve both the identification of tension–error relations and the assessment of worst-case behavior, particularly for inverse-tension models.
- *Richer reliability indicators and correlation structure.* Tension is a meaningful proxy for reliability, yet additional indicators could be incorporated, such as cable speed/acceleration or pulley friction estimates. Moreover, the current weighting assumes independent residual components (diagonal weighting). Extending the framework to structured (possibly configuration-dependent) covariance models could capture cross-correlations between cable-related measurements, especially when shared mechanisms or common disturbances are present.
- *Convergence certification beyond square systems.* Kantorovich-based certification provides conservative guarantees for square nonlinear systems solved via Newton iterations and was exploited in the DGSP context. Extending comparable certification ideas to redundant sensor-fusion formulations remains challenging, as it requires bounding higher-order derivatives of stationarity conditions. Developing tractable certification tools for Gauss–Newton-based estimators, tailored to *CDPR* kinematics, is, in the author’s opinion, a promising research direction.

In summary, this thesis provided a unified modeling and estimation framework for *UACDPR* pose estimation based on redundant proprioceptive sensing. The sensor-fusion *WNLS* formulation, the component-wise termination strategy, and the tension-aware extension collectively offer a practical route to accurate, reliable, and computationally efficient direct-kinematics estimation, validated on both planar and spatial prototypes, and supported by complementary sensitivity and convergence analyses.

Appendices

Appendix A

Sensitivity-based uncertainty propagation for underactuated pose estimation

This appendix summarizes the sensitivity analysis developed in the [41] and exploits it to provide practical bounds on how measurement uncertainties propagate to pose-estimated variables in *UACDPRs*. The focus is on underactuated configurations in which a subset of the end-effector (*EE*) degrees-of-freedom (*DoFs*) is measured independently (e.g., by an inertial unit), so that the remaining *DoFs* can be recovered from cable-length constraints.

A.1 Setting and partition of measured and estimated variables and differential relationship

Let the *EE* pose be parameterized by the generalized coordinates of Eq. (2.2), which is recalled here for completeness

$$\zeta = [\mathbf{p}^T \quad \mathbf{e}^T]^T. \quad (\text{A.1})$$

The sensitivity analysis reported here consider a 6-*DoF* *EE* suspended by $n = 4$ cables, with an additional sensor providing roll ψ and pitch θ measurements. Accordingly, the pose vector is partitioned into

$$\zeta = [\zeta_u^T \quad \zeta_k^T]^T, \quad \zeta_u = [\mathbf{p}^T \quad \chi]^T, \quad \zeta_k = [\psi \quad \theta]^T, \quad (\text{A.2})$$

where χ is the yaw angle and ψ, θ are the roll and pitch angles. The components collected in ζ_k are regarded as *known* inputs affected by bounded measurement errors, whereas the components in ζ_u are *unknown* and are computed by direct kinematics.

For a fixed configuration ζ , the kinematic relationship between the cable lengths and the pose in Eq. (2.38) can be expressed in differential form as

$$\delta \mathbf{l} = \mathbf{J}_l(\zeta) \delta \zeta, \quad (\text{A.3})$$

where $\mathbf{J}_l$ is the cable-length Jacobian defined in Eq. (2.39) (see also [70]). By reordering the columns of $\mathbf{J}_l$ consistently with the partition (A.2), Eq. (A.3) becomes

$$\mathbf{J}_u \delta \zeta_u + \mathbf{J}_k \delta \zeta_k = \delta \mathbf{l}, \quad (\text{A.4})$$

with $\mathbf{J}_u \in \mathbb{R}^{4 \times 4}$ and $\mathbf{J}_k \in \mathbb{R}^{4 \times 2}$. Outside singular configurations, $\mathbf{J}_u$ is full rank and thus invertible.

A.2 Sensitivities to cable-length measurement errors

Assuming that the roll and pitch measurements are exact, i.e., not affected by measurement noise, it follows that $\delta\zeta_k = \mathbf{0}$. Eq. (A.4) yields

$$\delta\zeta_u = \mathbf{K}\delta\mathbf{l}, \quad \mathbf{K} = \mathbf{J}_u^{-1}. \quad (\text{A.5})$$

To obtain dimensionally consistent quantities, $\mathbf{K}$ is decomposed so that

$$\begin{bmatrix} \delta\mathbf{p} \\ \delta\chi \end{bmatrix} = \begin{bmatrix} \mathbf{K}_p \\ \mathbf{K}_\chi \end{bmatrix} \delta\mathbf{l}, \quad (\text{A.6})$$

where $\mathbf{K}_p \in \mathbb{R}^{3 \times 4}$ and $\mathbf{K}_\chi \in \mathbb{R}^{1 \times 4}$.

Following [71], the point-displacement sensitivity to cable-length errors is defined as

$$\sigma_{\infty,2}^{p,l} = \max_{\|\delta\mathbf{l}\|_\infty=1} \|\delta\mathbf{p}\|_2 = \|\mathbf{K}_p\|_{\infty,2}, \quad (\text{A.7})$$

and the yaw sensitivity is defined as

$$\sigma_{\infty,2}^{\chi,l} = \max_{\|\delta\mathbf{l}\|_\infty=1} \|\delta\chi\|_2 = \|\mathbf{K}_\chi\|_{\infty,2}, \quad (\text{A.8})$$

where $\|\cdot\|_\infty$ and $\|\cdot\|_2$ are the ∞ -norm and the 2-norm of $(\cdot)$ [72].

A.3 Sensitivities to roll and pitch measurement errors

If cable-length measurements are exact, i.e., not affected by measurement noise, it follows that $\delta\mathbf{l} = \mathbf{0}$. Eq. (A.4) becomes

$$\mathbf{J}_u \delta\zeta_u = -\mathbf{J}_k \delta\zeta_k. \quad (\text{A.9})$$

By inverting $\mathbf{J}_u$, the perturbation on the estimated variables is obtained as

$$\delta\zeta_u = -\mathbf{F}\delta\zeta_k, \quad \mathbf{F} = \mathbf{J}_u^{-1}\mathbf{J}_k. \quad (\text{A.10})$$

As in (A.6), $\mathbf{F}$ is partitioned into

$$\begin{bmatrix} \delta\mathbf{p} \\ \delta\chi \end{bmatrix} = -\begin{bmatrix} \mathbf{F}_p \\ \mathbf{F}_\chi \end{bmatrix} \delta\zeta_k, \quad (\text{A.11})$$

where $\mathbf{F}_p \in \mathbb{R}^{3 \times 2}$ and $\mathbf{F}_\chi \in \mathbb{R}^{1 \times 2}$.

The point-displacement sensitivity to roll–pitch errors is defined as

$$\sigma_{\infty,2}^{p,\zeta_k} = \max_{\|\delta\zeta_k\|_\infty=1} \|\delta\mathbf{p}\|_2 = \|\mathbf{F}_p\|_{\infty,2}, \quad (\text{A.12})$$

and the yaw sensitivity to roll–pitch errors is defined as

$$\sigma_{\infty,2}^{\chi,\zeta_k} = \max_{\|\delta\zeta_k\|_\infty=1} \|\delta\chi\|_2 = \|\mathbf{F}_\chi\|_{\infty,2}. \quad (\text{A.13})$$

A.4 Illustrative case study

An illustrative application is presented for a 4-cable *UACDPR* (Fig. A.1), whose *EE* has mass $m = 1$ kg and the following geometrical parameters:

$${}^P\mathbf{a}'_1 = \begin{bmatrix} 0.1 \\ 0.15 \\ 0.15 \end{bmatrix} m, \quad {}^P\mathbf{a}'_2 = \begin{bmatrix} 0.1 \\ -0.15 \\ 0.15 \end{bmatrix} m, \quad {}^P\mathbf{a}'_3 = \begin{bmatrix} -0.1 \\ -0.15 \\ 0.15 \end{bmatrix} m, \quad {}^P\mathbf{a}'_4 = \begin{bmatrix} -0.1 \\ 0.15 \\ 0.15 \end{bmatrix} m,$$

$$\mathbf{b}_1 = \begin{bmatrix} 1.5 \\ 1 \\ 0 \end{bmatrix} m, \quad \mathbf{b}_2 = \begin{bmatrix} 1.5 \\ -1 \\ 0 \end{bmatrix} m, \quad \mathbf{b}_3 = \begin{bmatrix} -1.5 \\ -1 \\ 0 \end{bmatrix} m, \quad \mathbf{b}_4 = \begin{bmatrix} -1.5 \\ 1 \\ 0 \end{bmatrix} m,$$

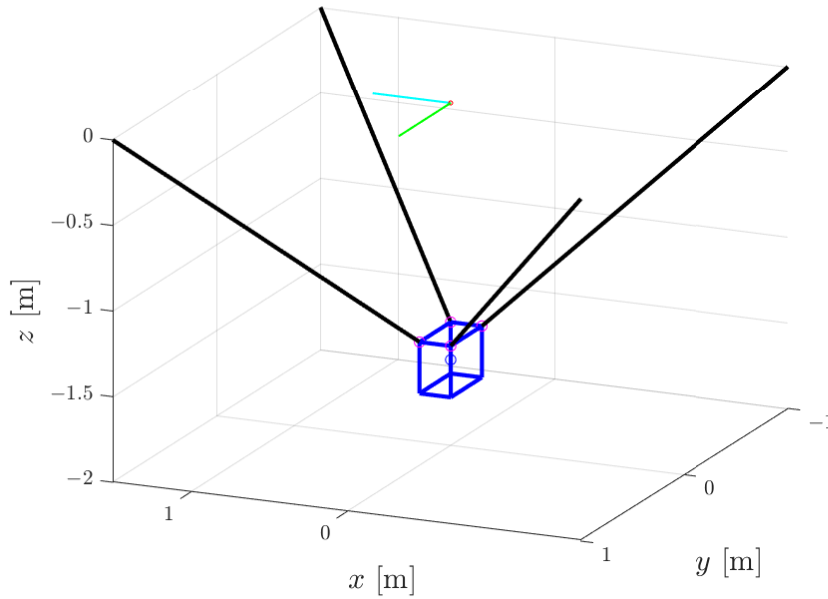


Figure A.1: Prototype used in the illustrative 4-cable example (from [41]).

The sensitivity indices (A.7)–(A.13) are evaluated over the workspace. For the considered example, the indices are reported to range from 1.78 to 6.19 mm/mm for $\sigma_{\infty,2}^{p,l}$ and from 0.77 to 1.51 °/mm for $\sigma_{\infty,2}^{\chi,l}$. Moreover, the indices are reported to range from 0.64 to 3.89 mm/° for $\sigma_{\infty,2}^{p,\zeta_k}$ and from 0.01 to 1.81 °/° for $\sigma_{\infty,2}^{\chi,\zeta_k}$. The corresponding sensitivity maps, highlighting how cable-length errors affect both point displacement and yaw throughout the workspace, are shown in Fig. A.2, whereas the sensitivities induced by roll–pitch errors are reported in Fig. A.3.

If maximum errors in cable lengths dl_M and in roll-and-pitch angles $d\zeta_{k,M}$ are known, the following worst-case bounds are reported:

$$\|\delta\mathbf{p}_{max}\|_2 = \sigma_{\infty,2}^{p,l} dl_M + \sigma_{\infty,2}^{\chi,l} d\zeta_{k,M}, \quad (\text{A.14})$$

$$\|\delta\chi_{max}\|_2 = \sigma_{\infty,2}^{p,\zeta_k} dl_M + \sigma_{\infty,2}^{\chi,\zeta_k} d\zeta_{k,M}. \quad (\text{A.15})$$

For the same 4-cable *UACDPR*, assuming $dl_M = 1$ mm (compatible with incremental encoders accuracies and cable elasticity effects) and $d\zeta_{k,M} = 0.4^\circ$ (compatible with accurate vertical-reference units [73]), the combined point-displacement and yaw errors

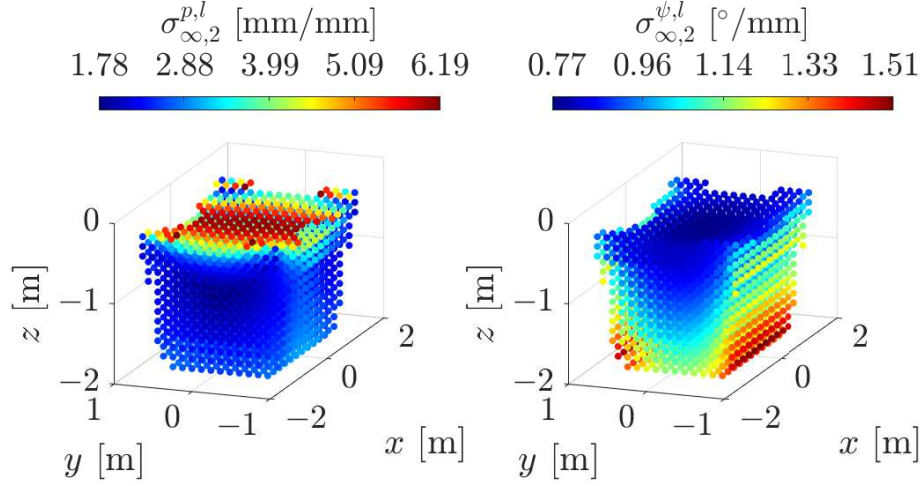


Figure A.2: Point-displacement and yaw sensitivities to cable-length errors (adapted from [41]).

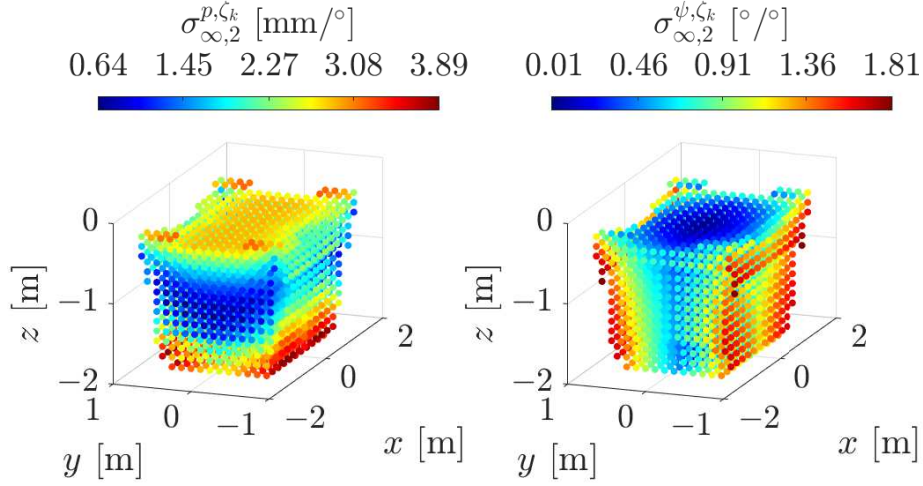


Figure A.3: Point-displacement and yaw sensitivities to roll-pitch errors (adapted from [41]).

computed throughout the workspace are reported to range from 2.13 to 7.46 mm for $\delta \mathbf{p}$ and from 0.77° to 2.12° for $\delta \chi$. The resulting worst-case error distributions are shown in Fig. A.4. Higher accuracies are reported in the central part of the workspace, whereas the largest inaccuracies occur in the upper part for $\mathbf{p}$ and in the side parts for χ .

A.5 Discussion

This appendix is intended to complement the thesis by illustrating how bounded input uncertainties can be propagated to pose variables through sensitivity indices and associated worst-case bounds. It is noted, however, that the indices and maximum errors reported here are derived by casting the kinematic relations into a *square* system of equations. Therefore, the present derivation is not a general result that can be directly transferred to the sensor-fusion formulation introduced in Chapter 4, which typically leads to more complex (e.g., overdetermined) systems. Rather, the analysis should be regarded as a methodology that can inform future extensions, in which analogous indices could be defined for systems arising in sensor-fusion settings. Moreover, the

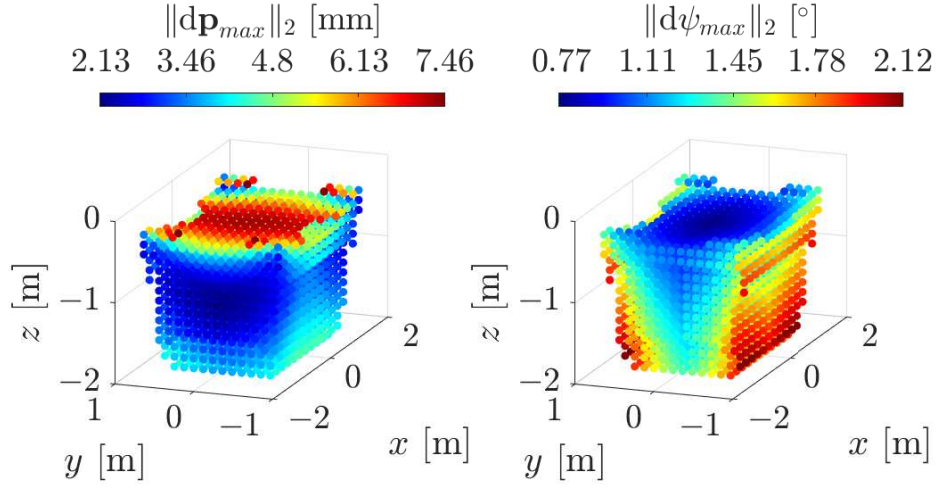


Figure A.4: Combined worst-case point-displacement and yaw errors (adapted from [41]).

numerical study is restricted to the considered 4-cable *UACDPR* and is not meant to provide a general characterization across architectures. Despite these limitations, the reported workspace maps offer useful insight into the spatial distribution of error propagation, indicating which regions are intrinsically more sensitive to inaccuracies in the input measurements and, consequently, where larger pose errors are expected under comparable uncertainty levels.

Appendix B

Kantorovich-based convergence certification for the DGSP of 3-3 UACDPRs

This appendix specializes the Kantorovich framework [60, 61, 32] recalled in Section 3.3.1 to the direct geometrico-static problem (DGSP) of a 3-cable UACDPR [18]. The objective is to obtain computable sufficient conditions ensuring existence and uniqueness of the equilibrium solution in a neighbourhood of a numerical estimate, and to derive a conservative bound on the maximum admissible cable-length measurement error that still guarantees Newton convergence to that unique equilibrium.

B.1 Quadratic DGSP formulation for a 3-cable UACDPR

The 3-cable UACDPR considered consists of a rigid end-effector (*EE*) that is constrained to move in $SE(3)$ by three cables actuated by three winches, according to the geometric model presented in Section 2.1. Zero-radius pulleys are assumed, so the frame exit points are constant, and the cables are modeled as massless, so no sagging occurs, and inextensible, so no elasticity effects are considered. Let A_i , $i = 1, 2, 3$, denote the attachment points on the moving *EE* and let B_i , $i = 1, 2, 3$, denote the corresponding points on the fixed-frame. Their position vectors in $Oxyz$ are denoted by $\mathbf{a}_i$ and $\mathbf{b}_i$, respectively, with $\mathbf{b}_i$ constant.

The *EE* geometry is enforced by the fixed distances between attachment points, denoted here by $a_{21} = \|\mathbf{a}_2 - \mathbf{a}_1\|$, $a_{32} = \|\mathbf{a}_3 - \mathbf{a}_2\|$, and $a_{13} = \|\mathbf{a}_1 - \mathbf{a}_3\|$. Cable-length measurements are denoted by l_i , $i = 1, 2, 3$, and satisfy the geometric relations $\|\mathbf{a}_i - \mathbf{b}_i\| = l_i$, $i = 1, 2, 3$.

Under the quasi-static assumption (Section 3.1.2), equilibrium under gravity can be written by introducing the positive scalars

$$\lambda_i = \frac{\tau_i}{l_i}, \quad i = 1, 2, 3, \quad (\text{B.1})$$

where τ_i denotes the cable tension. Let P be the center of mass of the *EE*; its position $\mathbf{p}$ can be expressed as an affine combination of the three attachment points, if A_1 , A_2 and A_3 are not aligned [74]. With the moment reduction pole chosen at $O \equiv B_1$ [18], the

DGSP can be written as a square system of $N = 12$ scalar equations in N unknowns:

$$\mathbf{r}(\mathbf{X}) = \begin{bmatrix} \|\mathbf{a}_2 - \mathbf{a}_1\|^2 - a_{21}^2 \\ \|\mathbf{a}_3 - \mathbf{a}_2\|^2 - a_{32}^2 \\ \|\mathbf{a}_1 - \mathbf{a}_3\|^2 - a_{13}^2 \\ \|\mathbf{a}_1 - \mathbf{b}_1\|^2 - l_1^2 \\ \|\mathbf{a}_2 - \mathbf{b}_2\|^2 - l_2^2 \\ \|\mathbf{a}_3 - \mathbf{b}_3\|^2 - l_3^2 \\ \sum_{i=1}^3 \begin{bmatrix} \mathbf{b}_i - \mathbf{a}_i \\ \tilde{\mathbf{b}}_i (\mathbf{b}_i - \mathbf{a}_i) \end{bmatrix} \lambda_i + \begin{bmatrix} m\mathbf{g} \\ \tilde{\mathbf{p}}(m\mathbf{g}) \end{bmatrix} \end{bmatrix} = \mathbf{0}_{12 \times 1}, \quad (\text{B.2})$$

where m is the *EE* mass and $\mathbf{g}$ is the gravity vector expressed in $Oxyz$. The unknown vector is

$$\mathbf{X} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \in \mathbb{R}^{12}. \quad (\text{B.3})$$

A key property of (B.2) is that all components of $\mathbf{r}(\mathbf{X})$ are quadratic in $\mathbf{X}$. As a consequence, the second-order derivatives entering the Kantorovich constant are constant (i.e., independent of $\mathbf{X}$) [26].

B.2 Interval modelling of cable-length errors

Let the cable-length measurements be affected by bounded errors. For a prescribed error bound $\Delta l \geq 0$, the true cable lengths are assumed to satisfy

$$L_i \in [l_i - \Delta l, l_i + \Delta l], \quad i = 1, 2, 3. \quad (\text{B.4})$$

Substituting L_i in place of l_i in the length constraints of (B.2) yields an interval-valued residual vector, denoted by $\mathbf{R}(\mathbf{X}, \Delta l)$:

$$\mathbf{R}(\mathbf{X}, \Delta l) = \begin{bmatrix} \|\mathbf{a}_2 - \mathbf{a}_1\|^2 - a_{21}^2 \\ \|\mathbf{a}_3 - \mathbf{a}_2\|^2 - a_{32}^2 \\ \|\mathbf{a}_1 - \mathbf{a}_3\|^2 - a_{13}^2 \\ \|\mathbf{a}_1 - \mathbf{b}_1\|^2 - l_1^2 \\ \|\mathbf{a}_2 - \mathbf{b}_2\|^2 - l_2^2 \\ \|\mathbf{a}_3 - \mathbf{b}_3\|^2 - l_3^2 \\ \sum_{i=1}^3 \begin{bmatrix} \mathbf{b}_i - \mathbf{a}_i \\ \tilde{\mathbf{b}}_i (\mathbf{b}_i - \mathbf{a}_i) \end{bmatrix} \lambda_i + \begin{bmatrix} m\mathbf{g} \\ \tilde{\mathbf{p}}(m\mathbf{g}) \end{bmatrix} \end{bmatrix}. \quad (\text{B.5})$$

When $\Delta l = 0$, the interval degenerates and $\mathbf{R}(\mathbf{X}, 0) = \mathbf{r}(\mathbf{X})$.

B.3 Maximum admissible measurement error via the Kantorovich constant

Let $\mathbf{X}_0$ be a numerical estimate obtained by solving Eq. (B.2) with the nominal cable lengths l_i using the approach proposed in [18]. This approach returns all solutions of Eq. (B.2), and $\mathbf{X}_0$ is selected among them within the workspace region of interest.

Algorithm 1: Bisection search for $\Delta l_{\max}$ based on $\overline{K}_0(\Delta l)$

Input: Nominal estimate $\mathbf{X}_0$; tolerance $t > 0$; bracket $[\Delta l_a, \Delta l_b]$ s.t. $\overline{K}_0(\Delta l_a) < 1$ and $\overline{K}_0(\Delta l_b) > 1$.

Output: $\Delta l_{\max}$ such that $1 - t \leq \overline{K}_0(\Delta l_{\max}) \leq 1 + t$.

while true do

$\Delta l_m \leftarrow (\Delta l_a + \Delta l_b)/2$

if $\overline{K}_0(\Delta l_m) > 1$ **then**

$\Delta l_b \leftarrow \Delta l_m$

else

$\Delta l_a \leftarrow \Delta l_m$

if $1 - t \leq \overline{K}_0(\Delta l_m) \leq 1 + t$ **then**

$\Delta l_{\max} \leftarrow \Delta l_m$

break

The Kantorovich quantities introduced in Section 3.3.1 are evaluated at $\mathbf{X}_0$, for $\mathbf{R}(\mathbf{X}, \Delta l)$ (Eq. (B.5)) using the infinity norm:

$$A_0 = \|\mathbf{J}^{-1}(\mathbf{X}_0)\|, \quad C = \sum_{k=1}^{12} \left| \frac{\partial^2 r_k(\mathbf{X})}{\partial X_i \partial X_j} \right|, \quad i, j = 1, \dots, 12, \quad (\text{B.6})$$

where $\mathbf{J}(\mathbf{X}) = \partial \mathbf{R} / \partial \mathbf{X}$. Since $\mathbf{R}(\mathbf{X}, \Delta l)$ is linear in Δl , the quantity A_0 is independent of Δl . Moreover, owing to the quadratic structure of Eq. (B.2), the quantity C is constant with respect to $\mathbf{X}$ [26]. In contrast, the quantity $2B_0$ depends on the cable-length uncertainty through (B.5):

$$2B_0(\Delta l) = \|\mathbf{J}^{-1}(\mathbf{X}_0) \mathbf{R}(\mathbf{X}_0, \Delta l)\|, \quad (\text{B.7})$$

which is interval-valued when $\Delta l > 0$ due to the interval nature of $\mathbf{R}(\mathbf{X}_0, \Delta l)$. Consequently, the Kantorovich constant (Eq. (3.23)) becomes an interval function

$$K_0(\Delta l) = 24 A_0 B_0(\Delta l) C, \quad (\text{B.8})$$

since $N = 12$ in (B.2). Let $\overline{K}_0(\Delta l)$ denote the upper bound of the interval $K_0(\Delta l)$. The maximum admissible cable-length error $\Delta l_{\max}$ is obtained by enforcing the Kantorovich condition through the interval upper bound,

$$\overline{K}_0(\Delta l_{\max}) \leq 1. \quad (\text{B.9})$$

When (B.9) holds, the Kantorovich theorem ensures (Section 3.3.1) existence and uniqueness of a solution in the neighbourhood of $\mathbf{X}_0$ and convergence of Newton's method initialized at $\mathbf{X}_0$.

In [42], $\Delta l_{\max}$ is computed with a bisection scheme (Algorithm 1) applied to the scalar function $\overline{K}_0(\Delta l)$. An interval $[\Delta l_a, \Delta l_b]$ is first selected such that $\overline{K}_0(\Delta l_a) < 1$ and $\overline{K}_0(\Delta l_b) > 1$, with $\Delta l_a = 0$. Then, the interval is iteratively bisected until a midpoint $\Delta l_m = (\Delta l_a + \Delta l_b)/2$ satisfies $1 - t \leq \overline{K}_0(\Delta l_m) \leq 1 + t$, where $t > 0$ is a prescribed tolerance that sets the stopping condition for the iterative bisection method.

Table B.1: Geometric parameters of the 3-cable *UACDPR* used in the numerical assessment.

i	1	2	3
${}^P\mathbf{a}'_i$ [m]	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1.2 \\ 0 \end{bmatrix}$
$\mathbf{b}_i$ [m]	$\begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0.1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix}$

B.4 Guaranteed bound on the solution error

Once $\Delta l_{\max}$ is obtained, the Kantorovich *a posteriori* estimate (Eq. (3.25)) provides a conservative bound on the distance between the true solution associated with admissible perturbed cable lengths and the nominal estimate $\mathbf{X}_0$. Since $2B_0(\Delta l)$ is interval-valued, the guaranteed bound is given by its upper endpoint:

$$\|\mathbf{X}_s - \mathbf{X}_0\| \leq \overline{2B_0}(\Delta l_{\max}), \quad (\text{B.10})$$

where $\mathbf{X}_s$ denotes the unique solution certified within the Kantorovich ball of radius $2B_0(\Delta l_{\max})$ centered at $\mathbf{X}_0$. Therefore, once the maximum admissible error on the cable-length measurements $\Delta l_{\max}$ is prescribed, the guaranteed accuracy of the estimate $\mathbf{X}_0$, obtained as a solution of Eq. (B.2), is quantified by $\overline{2B_0}(\Delta l_{\max})$.

B.5 Numerical assessment

The convergence certification procedure presented in Section B.3 was applied to quantify the maximum admissible error on the cable-length measurements that still guarantees convergence of Newton's method when solving the DGSP in Eq. (B.2) for a 3-cable *UACDPR*. The geometric parameters adopted in the simulations are reported in Table B.1. In this assessment, the *EE* is subject to a gravitational force of magnitude 1 N applied at point *P*.

A workspace-wide evaluation was performed. The workspace was computed with the method in [46], imposing cable-tension bounds between 0.25 N and 10 N. For a set of $M = 162$ reachable poses, the maximum admissible cable-length error $\Delta l_{\max}$ and the associated guaranteed accuracy bound for the DGSP estimate $\mathbf{X}_0$, given by $\overline{2B_0}(\Delta l_{\max})$, were evaluated. The simulations show that $\Delta l_{\max}$ ranges from $8.4E^{-5}$ m to $5.4E^{-4}$ m. The admissible cable-length errors required to guarantee convergence are therefore very small, especially near the workspace boundaries. Moreover, additional error sources such as friction, cable sagging, and cable elasticity were not modeled here and are typically several orders of magnitude larger than sensor-induced cable-length errors. In this context, enforcing convergence guarantees would require restricting the estimation to workspace regions where $\Delta l_{\max}$ exceeds the accuracy threshold of the adopted sensing system. Even in the central regions of the workspace, where the convergence conditions are most favorable, the required cable-length accuracy remains demanding, being on the order of $5E^{-4}$ m.

It is noted that the present outcome pertains to a specific, non-optimized geometry with fixed design parameters. Different architectures or optimized geometric designs

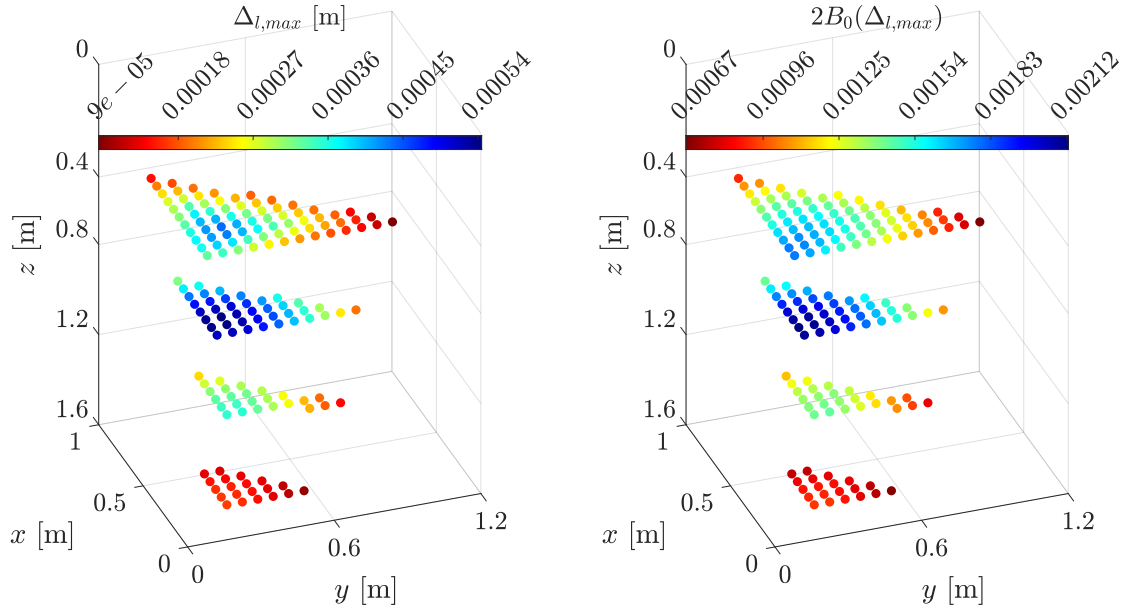


Figure B.1: Maximum admissible cable-length error $\Delta l_{\max}$ and corresponding convergence radius $2\overline{B}_0(\Delta l_{\max})$ across the workspace of the considered 3-cable *UACDPR* (adapted from [42]).

may lead to substantially less stringent accuracy requirements. Hence, the proposed certification procedure remains relevant and potentially practical when applied to appropriately designed setups.

The convergence radius $2\overline{B}_0(\Delta l_{\max})$, which bounds the uncertainty on the certified DGSP solution, ranges from $6.7E^{-6}$ to $2.1E^{-3}$ and generally increases toward the center of the workspace. This trend follows from the larger admissible cable-length errors in those regions: as $\Delta l_{\max}$ increases, the set of admissible measurements enlarges and so does the region in which the true platform pose is guaranteed to lie.

Figure B.1 also indicates that $\Delta l_{\max}$ and $2\overline{B}_0(\Delta l_{\max})$ are not rotationally symmetric with respect to the vertical axis within the analyzed horizontal planes, since the base and platform attachment triangles are neither equilateral nor homothetic.

B.6 Discussion

The present workspace study should be regarded as a preliminary assessment carried out under simplifying assumptions with respect to those adopted in Chapter 4. In particular, the analysis is performed on the geometrico-static equation set rather than on a redundant formulation as in the sensor-fusion problem. This choice is crucial because it allows the direct application of the Kantorovich theorem to a zero-finding problem. Extending the same type of certification to the sensor-fusion formulation would require applying Kantorovich to the first-order optimality condition, i.e., to the equation $\nabla f(\zeta) = \mathbf{0}$ associated with the cost function f (Eq. (4.3)). In turn, this would involve bounding second derivatives of ∇f , hence computing and managing third-order derivatives, which significantly increases the analytical and computational complexity. Moreover, in this preliminar study only a 3-cable *UACDPR* with zero-radius pulleys is considered; as a consequence, the Hessian structure of the DGSP equations is greatly simplified and the quantity C can be treated as constant, which substantially reduces the computational burden of the certification procedure.

Despite these limitations, the obtained results already provide a clear indication of the practical issues arising when the pose estimate relies solely on cable-length measurements: the admissible errors $\Delta l_{\max}$ are extremely small, and for the tested configuration convergence cannot be guaranteed over the entire workspace when considering the accuracy of currently available sensors. It is also recalled that the Kantorovich theorem provides a sufficient, but not necessary, condition for convergence; therefore, lack of certification does not imply that Newton's method fails to converge, but rather that convergence cannot be guaranteed by the adopted bound. In the author's opinion, pursuing the extension to the redundant sensor-fusion setting appears worthwhile from a physical viewpoint: introducing measurement redundancy tends to rule out spurious solutions, since a pose compatible with one subset of measurements may become infeasible once additional measurements are enforced. As a result, substantially less stringent accuracy requirements may be obtained, potentially enabling convergence certification for the sensor-fusion algorithm proposed in this thesis.

List of Author Publications

1. Gabaldo, S., Idà, E., Carricato, M., Sensitivity of the direct kinematics of under-actuated cable-driven parallel robots to redundant sensor-measurement errors. In: Altuzarra, O., Kecskeméthy, A. (eds.) *Advances in Robot Kinematics 2022*, Springer Proceedings in Advanced Robotics, vol 24, pp. 131–138, Springer, 2022, doi: [10.1007/978-3-031-08140-8_15](https://doi.org/10.1007/978-3-031-08140-8_15)
2. Gabaldo, S., Idà, E., Carricato, M., Pose-Estimation Methods for Planar Underactuated Cable-Driven Parallel Robots, *Mechanisms and Machine Science*, 132, pp. 3 – 15, 2023, doi: [10.1007/978-3-031-32322-5_1](https://doi.org/10.1007/978-3-031-32322-5_1)
3. Gabaldo, S., Idà, E., Carricato, M., Pose-estimation methods for underactuated cable-driven parallel robots, *Mechanism and Machine Theory*, 199, pp. 1-13, 2024, doi: [10.1016/j.mechmachtheory.2024.105690](https://doi.org/10.1016/j.mechmachtheory.2024.105690)
4. Gabaldo, S., Idà, E., Merlet, J. P., Carricato, M., Direct geometrico-static problem of underactuated cable-driven parallel robots with 3 cables: convergence analysis. Presented at the 8th International Workshop on Computational Kinematics (CK), 25-28 November, IIT Madras, India, 2025. Book in editing phase.
5. Gabaldo, S., Idà, E., Carricato, M., Tension-Aware Error Modeling for Fast Pose Estimation of Underactuated Cable-Driven Parallel Robots, submitted for presentation at the 20-th International Symposium on Advances in Robot Kinematics, 28 June-2 July, Barcelona, Spain, 2026.

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