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INNOVATIVE GEARBOXES WITH BEVELOID GEARS FOR ROBOTIC
APPLICATIONS

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ALMA MATER STUDIORUM - UNIVERSITÀ DI BOLOGNA

INNOVATIVE GEARBOXES WITH BEVELOID GEARS FOR ROBOTIC APPLICATIONS

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in Mechanics and Advanced Engineering Sciences

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*To my family,
who have loved
and supported me
throughout this journey.*

Abstract

Gearboxes are transmission systems that use a set of gears to change speed, direction of rotation, and torque between a motor (the input) and its load (the output). Gearboxes in industrial robot joints must be designed to meet stringent requirements: high torque density, low weight, compact design, minimal torsional backlash, small transmission ratios and high bending and torsional stiffness. The most commonly used transmissions in serial robots nowadays are harmonic, cycloidal and planetary gearboxes. Harmonic gearboxes, in particular, are employed in joints near the end-effector (EE) thanks to their zero backlash and high torque-to-weight ratio. Cycloidal gearboxes are generally used in the first three joints of a robot for their stiffness and robustness. Planetary gearboxes can have a simple or a compound architecture: the former is used when simplicity and low cost are required, while the latter is preferred when better, more balanced performances are required. Only a limited number of companies worldwide have the capability to achieve the level of accuracy in design, manufacturing, and assembly required for long-term durability, precision, and robustness. Product validation and verification of theoretical design data are essential for guaranteeing expected performance over time.

This thesis investigates the Omni planetary compound gearbox, for which Bonfiglioli Riduttori holds a patent and carried out validation tests. The design adopts a Wolfrom planetary architecture combined with preloaded axial beveloid gears to achieve a backlash of under 1 arcminute. The Omni gearbox is available in two versions—OH (with central hole for cable routing) and OS (solid shaft). It is designed in ten sizes (010–100, by payload capacity) and four ratios (1:80, 1:100, 1:125, 1:160).

The validation process for the Omni began with the development of preliminary sizing tools. These included an analytical kinetostatic model, a KISSsoft model for beveloid gear pairs based on their midsection geometry, and a model of the main bearing based on the standard ISO 281. A more detailed design was then created using multi-body, finite-element and KISSsys models. These models enabled the optimisation of the gearbox design and the definition of validation experiments. The latter, focused on endurance and characterisation, were performed on OH prototypes, size 060, with a 1:125 ratio. Results were generally positive, with minor issues related to efficiency and service life. These are currently being addressed by optimising lubrication, refining gear flank geometry, and adjusting spring preload.

The beveloid gear design adds complexity to parameter optimisation due to the absence of standard sizing procedures. Two analytical models were built to address this issue: the first used the minimum potential energy principle to evaluate contact-stresses in parallel-axis straight beveloid gears; the second predicted pitting in parallel-axis helical beveloid gears in accordance with ISO 6336-1 (Annex E) and ISO 6336-2. These models support the optimisation of design parameters to minimise contact pressure. A simple (non-planetary) parallel-axis gearbox including helical beveloid gears was chosen as a case study to validate the second model through endurance tests. Preliminary

test results confirm the expected lifetime. Ongoing work focuses on extending the minimum potential energy method to helical beveloid gears and planetary gear systems.

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Chapter 1

Introduction

In industrial serial robots, transmissions (such as gearboxes) have a central role. A gearbox can be defined as a mechanical device that transfers motion and torque from the motor to the driven part of the system, adapting speed and torque to the specific requirements of the application. Gearboxes are fundamental components within the actuation chain, strongly influencing the overall performance of the robot. However, standard gearboxes often have limitations that restrict their use in industrial robot joints, due to challenging operating conditions. Unlike traditional gearbox applications, which involve the continuous transmission of torque and speed in a single direction under relatively stable loads, robotic joints are subject to frequent changes in motion. This results in variable speeds and rotation direction, rapid accelerations and decelerations, and fluctuating torques. Given the critical nature of the applications and the need to prevent machine breakdowns, gearboxes requires precise sizing procedures and thorough analysis. *Precision gearboxes* are required, to satisfy stringent requirements, such as:

- high torque density;
- low weight and compact design;
- low torsional backlash;
- positioning accuracy and repeatability;
- high bending and torsional stiffness;
- small transmission ratios¹;
- long operating life and reliable performance over time.

The design of precision gearboxes is a complex task, since all parameters discussed above are strongly interrelated. To achieve high positioning accuracy and repeatability, it is essential to ensure low torsional backlash and a highly rigid structure (high stiffness). Moreover, using lightweight reducers helps reduce inertia, which in turn minimises deformations during acceleration transients, ultimately resulting in better positioning accuracy.

¹In this work, the transmission ratio is defined as $\tau = \dot{\theta}_{out}/\dot{\theta}_{in}$, so that it is always between 0 and 1 for a gear reducer. Note that, in other works, it is defined as $i = \dot{\theta}_{in}/\dot{\theta}_{out}$, so that $\tau = 1/i$.

On the other hand, achieving high stiffness and low weight within a compact design requires careful optimisation of all components. This includes improving the quality of both manufacturing processes and materials, and optimally designing each element, avoiding any unnecessary oversizing. All mechanical clearances must be minimised to increase accuracy. However, reducing backlash inevitably increases friction between meshing tooth flanks. This leads to higher operating temperatures, undesirable for gear reducers, as well as reduced efficiency and premature wear of some components, which in turn negatively affect operating life and reliability.

The most widely used architectures today [1–3] are *harmonic*, *cycloidal* and *planetary* gearboxes, since they all have, to varying extent, the aforementioned requirements. Due to their zero backlash and excellent torque-to-weight ratio, harmonic gearboxes are typically used in joints close to the end-effector (EE). In contrast, cycloid and planetary gearboxes are preferred for the first three joints of a robot due to their robustness. In terms of accuracy, compactness and transmission ratios, cycloid and harmonic gearboxes are the best [1, 2]. However, they also present some disadvantages, including high cost, a non-linear hysteresis curve, and torque ripple. In contrast, simple planetary gearboxes are often chosen when efficiency is required with standard transmission ratios ($\tau \leq 1 : 12$), where the main priorities are a low cost, a high torque density, and a straightforward, robust design. For smaller transmission ratios ($\tau \approx 1 : 50$), some solutions employ multiple stages of simple planetary gearboxes, but this leads to bulky and heavy systems that are unsuitable for robotic arms. On the other hand, compound planetary architectures, such as the Wolfrom, achieve the same goal with fewer stages and components, making them particularly attractive.

As there are no official reference standards for sizing precision gearboxes in robotic applications, designers must rely on their experience and the scientific literature. Similar design and sizing approaches are adopted for wind turbine gearboxes [4, 5]. These gearboxes operate under highly variable duty cycles and demanding environmental conditions, characterised by complex load patterns, torque reversals, transient forces during start-ups and emergency stops, as well as wide temperature fluctuations, which can lead to premature component failures.

International standards provide general guidelines for sizing gears and bearings [6–9]. However, failures are more likely to occur in applications involving oscillatory motion (such as in wind turbines and robotic joints). Alternating bending conditions negatively affect bending safety factors, increasing root stresses by up to 30% [10]. In bearings, oscillatory motion results in limited lubrication, leading to scuffing and micropitting on the contact surfaces [11, 12]. Moreover, repeated oscillations around the same angular position can significantly reduce bearing life. Standard methods for sizing bearings in such cases are not yet available, but several non-standard approaches have been proposed in the literature [13, 14].

This PhD thesis was developed in collaboration with Bonfiglioli Riduttori SpA, the patent holder for Omni [15], an innovative Wolfrom planetary gearbox for robotics applications. However, the Wolfrom design is usually improved by using beveloid gears. The main advantage provided by beveloid gears is the ability to achieve torsional backlash close to zero, by pushing the gears axially through springs. The Omni achieves transmission ratios of up to 1:160 and is available in two versions, OH and OS, with the former having a central hole for cable routing. The Omni architecture and the analytical model of the exchanged forces within the OS version were presented in a preliminary conference paper in [16]. The extended paper [17] also discusses the OH

version, as well as the results and methodology of the tests, with a particular focus on endurance and characterisation tests.

Beveloid gears, also known as conical involute gears, are a type of spur or helical gears in which the profile shift coefficient, root diameter and head diameter vary linearly along the width of the gear. Beveloid gears can transmit motion between skew, intersecting, or parallel axes. In the case of parallel axes, the gear pairs are usually mounted with opposite conicity and pressed together axially by springs. This helps to minimise backlash during meshing, even when flanks wears out. The design and manufacturing of beveloid gears were initially investigated by Merritt [18] and Beam [19,20], and subsequently by Purkiss [21], Pfauter [22], Thomas [23], Szekley [24], Smith [25], and Innocenti [26, 27]. A significant contribution was made by Mitome, who developed a comprehensive system for the design and production of this gear type [28, 29]. In [30–32], he examined several gear-cutting methods and later turned his attention to grinding [33, 34], where he proposed four distinct procedures. He also introduced a measurement technique to verify the finished dimensions [35, 36]. Owing to Mitome’s pioneering research and experimental work, beveloid gears found increasing use in various applications, as summarized by Börner et al. [37]: marine transmissions [38–40], low-backlash planetary gearboxes [41], and vehicle transmissions [42, 43].

The Omni planetary gearbox has a Wolfrom architecture, characterised by very compact shape and small transmission ratio. This design, originally proposed in 1912 by U. Wolfrom [44], is commonly referred to as the 3K gear train, particularly in Russian, Japanese, and Chinese literature [45–48]. The Wolfrom configuration is characterised by high backlash and relatively low efficiency [45]. Due to its potential use in robotic transmissions [3], several researchers have explored ways to address these limitations. Crispel et al. [49] introduced a solution in which the planets have three rims². Each rim meshes independently with the fixed ring, the mobile ring, and the sun gear; this enables each meshing to be optimised separately, increasing efficiency. In 1996, Schulz et al. [50] used beveloid gears in a Wolfrom gearbox for the first time by inclining the planet axes slightly relative to the main gearbox axis. To the best of our knowledge, this concept was not pursued any further, most likely because it was not very efficient. Bonfiglioli’s patented solution [15] differs from that proposed by Schulz et al. [50] since the Omni planets are pushed against the rings by springs and the planet axes are parallel to the input and output axes. More recently, Riera et al. [51] presented a modified Wolfrom design in which each planet consists of two rims. Each rim is split into two parts and connected by torsional springs, which reduce meshing backlash. In [52], Höhn et al. proposed a Wolfrom architecture without a carrier, which achieved improved efficiency. Another important aspect of Wolfrom gearboxes is their back-drivability [47, 53], which is a key feature for collaborative robots (cobots). Unlike industrial robots, which operate in controlled environments, cobots interact with humans. They require gearboxes that can detect external forces as torque spikes. Back-drivability is strongly linked to efficiency and gearbox lifespan, and it can be enhanced through careful design [53, 54]. For example, Matsuki et al. [47] optimised a Wolfrom design achieving forward and backward efficiencies of 89.0% and 85.3%, respectively.

The Omni design started with its kinetostatic analysis, needed to evaluate all the forces and the torques exchanged between the different parts for the preliminary design. A more accurate model of the gearbox has also been developed using multibody software, finite-element analysis and with “KISSys”, a commercial software for gear

²The term “rim” refers to the outer, toothed portion of a gear.

train analysis. In the Wolfrom architecture the forces acting on planets are in opposite directions, causing significant stress concentration. This requires special attention during sizing. Therefore, a loaded tooth contact analysis was performed using the finite-element model to evaluate the stress distribution, the contact pressure, and to study the micro-geometrical flank modifications. The main bearing is a custom-designed axial-angular roller bearing, optimised for high bending stiffness and integrated into the housing. Its rated life was evaluated according to ISO 281 [9], and contact analysis considering real load conditions was performed to size it.

Two types of tests were performed on the test bench to evaluate the performance of the current Omni architecture: endurance testing and characterisation testing. The endurance test was designed to assess the product's service life under the most demanding conditions. Accelerated life-testing techniques were employed to reduce the time required for testing [55–57]. The characterisation test was designed to evaluate specific properties, such as hysteresis, transmission error, bending stiffness and efficiency. The overall results are positive, with minor issues in terms of efficiency and service life. These issues can be overcome by making minor variations, such as type and quantity of lubricant, variation of micro modifications on the planet gears' flanks, and different preload of the internal springs. Efficiency and service life are related, as low efficiency tends to reduce gearbox life by increasing contact friction and, consequently, working temperatures. Improving the efficiency of a Wolfrom gearbox requires a careful selection of gear design parameters, such as the number of teeth, face width, and profile shift coefficients. In our case, the use of beveloid gears adds further complexity, as some parameters cannot be optimised independently due to geometric constraints that may lead to tooth undercutting and pointing issues in sections farther from the middle. Furthermore, since no recognised standards exist for sizing beveloid gears and given the need to find optimal geometric parameters, we started developing a first analytical model for predicting pitting on beveloid gears based on ISO 6336-1, Annex E [58]. The beveloid gear pair is divided into n sections of equal width, considered as n cylindrical gear pairs, with different shift coefficients, to simulate the real beveloid gear pair. Through an iterative load distribution analysis, the load among the sections of the gear pair was evaluated. The pitting safety factor for each section was determined, and the section with the lowest safety factor was identified [7]. In collaboration with the Universidad Nacional de Educación a Distancia (UNED, Madrid, Spain) a different model to evaluate the contact stress of straight beveloid gears based on the minimisation of potential energy was developed [59]. The model allows different design parameters to be varied within a defined range, enabling the evaluation of multiple design scenarios. This parametric approach supports optimisation studies aimed at identifying the most suitable configuration, i.e. the one that results in the lowest contact pressures. Currently, the method is only applicable to straight beveloid gears with parallel axes. Future work will involve extending the model to helical gears and planetary gearboxes.

1.1 Thesis goals and outline

The project associated with this thesis aims to develop a new series of coaxial and parallel-shaft precision gearboxes for the actuation of joints in serial robots, covering a torque range approximately from 100 Nm to 3600 Nm. The goal is to create a product with performance characteristics equal to or exceeding current market standards, by using an innovative combination of a Wolfrom architecture with beveloid gears.

The development opportunity arises from the analysis of specific market requirements compared with the technologies available at Bonfiglioli Riduttori. The latter has initiated product marketing activities aimed at identifying the most strategically relevant features for the serial robot drive sector. These features enabled the definition of suitable design requirements. The results of this analysis guided the development phases, with a focus on axes 1, 2, and 3 of robot actuation.

Additional requirements perceived by the market include high modularity in the combination of motor and gearbox, as well as reduced production and testing times. Moreover, in recent years, regulatory frameworks have progressively increased energy-saving requirements, and current drive systems risk failing to comply with these standards. Solutions are therefore envisaged that allow strong integration between the gearbox and the electric motor.

The aim of this PhD thesis is to combine the main technical requirements for this type of application (high torsional and bending stiffness, reduced backlash and hysteresis, low transmission error, high efficiency) with market requirements (modularity and product industrialization to reduce lead times and production costs). These requirements, typical of robotic drive systems, are addressed starting from the solutions currently under development at Bonfiglioli.

Chapter 2 introduces the concept of beveloid gears and discusses their general properties, typical applications, force transmission characteristics, and manufacturing processes. It then presents a detailed contact stress analysis based on the principle of minimal potential energy for straight gears with parallel axes, followed by a pitting resistance evaluation based on ISO 6336-1 (Annex E) and ISO 6336-2, applicable to helical gears with parallel axes. Finally, the chapter describes the design, modeling, and experimental validation of a high-precision parallel-axis gearbox, including pitting resistance calculations and test-rig results. Chapter 3 focuses on gearboxes for industrial robots, highlighting the limitations of standard gearboxes and exploring the Wolf from architecture in terms of kinematics, efficiency, and duty cycles. Finally, Chapter 4 is dedicated to the Omni gearbox, covering its kinetostatic analysis, design considerations, multibody and finite element modeling, and experimental characterisation. Chapter 5 concludes the thesis by summarizing the main findings and outlining open issues for future research.

Chapter 2

Contact-stress analysis and pitting resistance of beveloid gears

This chapter introduces beveloid gears and their properties. Section 2.1 presents the general features, with a focus on beveloid gears' main applications (Section 2.1.1), force transmission (Section 2.1.2), and manufacturing processes (Section 2.1.3). In Section 2.3, a method for calculating pitting resistance is proposed, based on ISO 6336-1 (Annex E) and ISO 6336-2 [6, 7, 58]. Section 2.2 analyses the contact-stress of straight beveloid gears with parallel axes using the minimum potential energy principle, and provides validation through finite-element analysis.

2.1 General properties

Beveloid gears are a special type of spur or helical gears which were first introduced by Merritt in 1954 [18]. These gears can be applied to achieve transmission between parallel, intersecting, or crossed shafts. They are manufactured in the same way as standard gears, but differ in that the rack-cutter (with module m and pressure angle α_n) is inclined at an angle δ with respect to the gear axis, with cutting motion being parallel to this direction (Fig. 2.1). This leads to a gear with variable shift profile coefficient x , outside radius r_o , root radius r_d and thickness along the tooth width and constant module m , number of teeth Z and pressure angle α_n . Figure 2.2 shows the differences between spur, conical and beveloid gears. Spur gears have constant module, profile shift coefficient, and pitch and base diameter (Fig. 2.2a). Standard conical gears have instead a constant profile shift coefficient, with module and pitch and base diameters (Fig. 2.2b) that are variable. Beveloid gears can be considered a combination of the two types of gear. Such gears are interesting because they are easy to manufacture, are insensitive to assembly errors and reduce the backlash of parallel-axis gear pairs mounted with opposite cone angles, as shown in Fig. 2.1b and the rightmost drawing in Fig. 2.2c. During sizing, geometric parameters must be chosen carefully: a large inclination angle ($\delta > 5^\circ$), combined with a large face width b , may lead to pointing (associated with a high profile shift coefficient) and undercutting (associated with a low profile shift coefficient) in the transverse sections near the gear extremities. The meshing analysis of beveloid gears, or *Tooth Contact Analysis* (TCA), provides important information about contact lines, lines of action, backlash, and transmission error during the meshing process. In [60], Mitome investigated tooth contact both analytically and experimentally, while in [26], Innocenti proposed a method to determine backlash in beveloid gear-

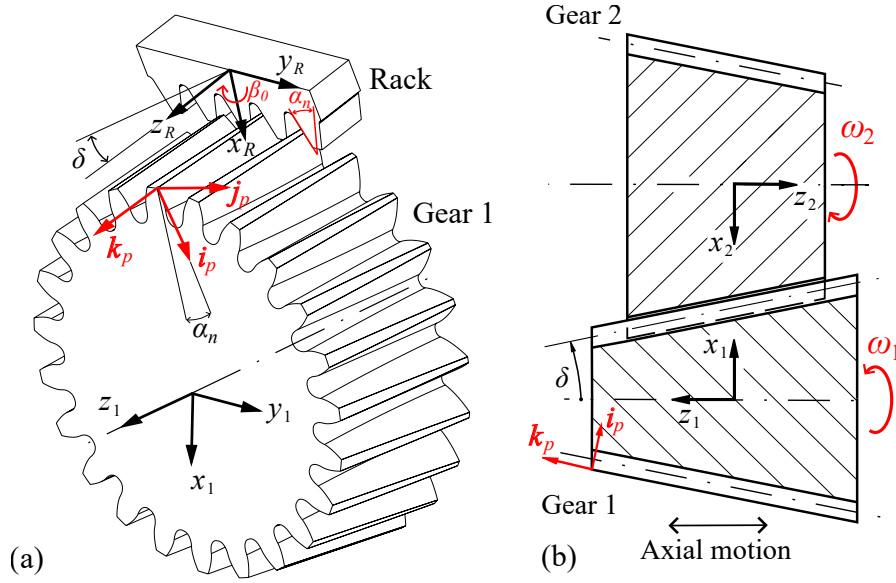


Figure 2.1: (a): Beveloid gear cutting, with the global frame on the gear (x_1, y_1, z_1), the rack frame (x_R, y_R, z_R), and the local frame on the gear tooth ($\mathbf{i}_p, \mathbf{j}_p, \mathbf{k}_p$): $\mathbf{i}_p$ and $\mathbf{k}_p$ are tangent to the tooth flank, while $\mathbf{j}_p$ is normal to it. (b): Section view of the meshing between beveloid gears with parallel axes z_1 and z_2 : axes y_1 and y_2 are normal to the section plane and pointing towards it; axis z_i of gear i points towards the cone tip.

ings. When two beveloid gears mesh with their axes intersecting at an angle $\gamma \neq 0^\circ$ (Fig. 2.3a), the theoretical contact is confined to a single point. In practice, due to the elastic properties of the gear material, the contact spreads into an elliptical area around this point (Fig. 2.3b), with the ellipse becoming smaller as the axis intersection angle increases [60,61]. The localized contact leads to stress concentration, reducing durability and load-carrying capacity. In [62], Mitome introduced the concept of the “concave conical gear”, which improves the contact pattern during meshing, avoiding point contact. When beveloid gears mesh with parallel axes ($\gamma = 0^\circ$, Fig. 2.3c), contact occurs along a line, over the entire tooth flank, as illustrated in Fig. 2.3d [60,61]. It enables the transmission of higher forces and power. For practical applications, crowning is recommended in cases where significant axial misalignments are expected, in order to avoid detrimental edge contacts [61].

Beveloid gears find application in automotive steering transmissions [43], marine gearboxes [39,40], and planetary gear systems for robots [17]; for more details refer to Section 2.1.1. In these moving devices, minimising weight of gear pairs is essential to reduce the inertia of rotating components, and thus an optimised design is always required to limit both weight and overall dimensions. The geometry of beveloid gears is unusual, making them particularly difficult to size. Unlike other gear types [6–8], no recognized standard calculation exist for determining their load capacity. Finite element analysis (FEA) is generally required to evaluate contact-stress, load distribution, and mesh stiffness. Although FEA provides accurate results, the modeling process is highly time-consuming. This discussion highlights the need for more efficient approaches to assess the load capacity of parallel-axis beveloid gears, suitable for optimisation studies.

The two main failure modes for gears, whether beveloid or conventional, are *pitting* and *bending*. If the surface stress on the meshing flanks exceeds an admissible values, pitting occurs, namely, particles start detaching from the gears, forming pits

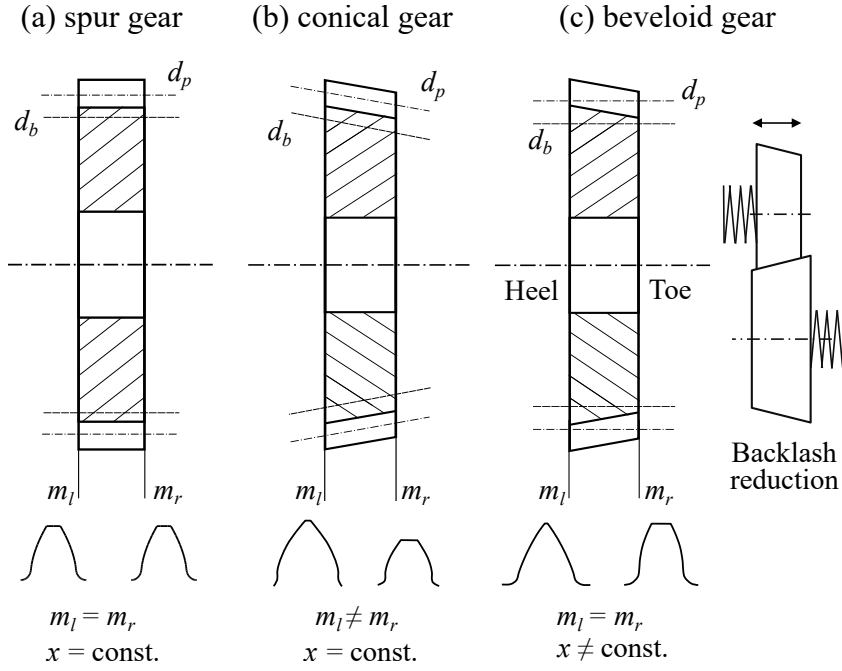


Figure 2.2: Comparison between spur (a), conical (b) and beveloid (c) gears. For each gear, the figure shows a section view (above) and the tooth profiles at the left and right sections (below), which have modules m_l and m_r , respectively. Backlash reduction using spring is shown for beveloid gear meshing in the rightmost drawing.

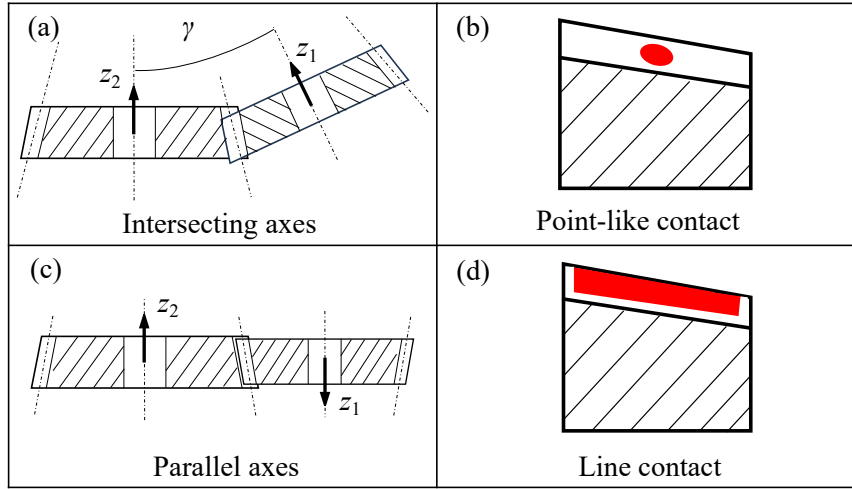


Figure 2.3: Beveloid-gear contact behaviour, depending on the angle γ between the gear axes z_1 and z_2 . (a,c): section views. (b,d): detail views, with contact areas in red.

and leaving debris in the lubricant. The acceptable level of pitting, in terms of pit size and quantity, varies significantly depending on the application. While in some cases a considerable degree of pitting can be tolerated, in others no visible damage can be accepted [7]. For precision gearboxes, the permissible level of pitting is essentially zero, since even small amounts increase angular backlash, which is critical in this type of application.

This thesis proposes two analytical models aimed at calculating the surface durability (pitting) of beveloid gears [58, 59]. Particularly, in [58], an analytical model for predicting pitting in beveloid gears, developed in accordance with ISO 6336-1 (Annex E) [6] and ISO 6336-2 [7]. The beveloid gear pair is segmented into n transversal sec-

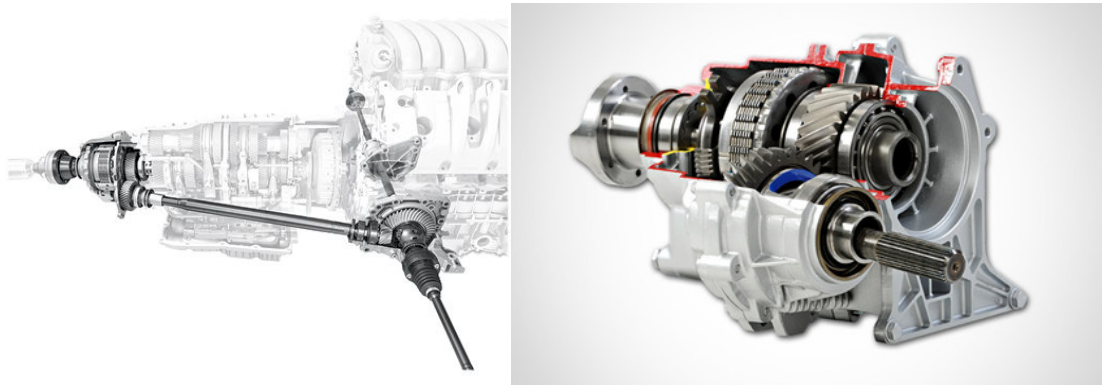


Figure 2.4: All wheel drive systems, respectively: Porsche Panamera Gran Turismo (left) and Magna Transfer Case (right). Image source: [43, 65].

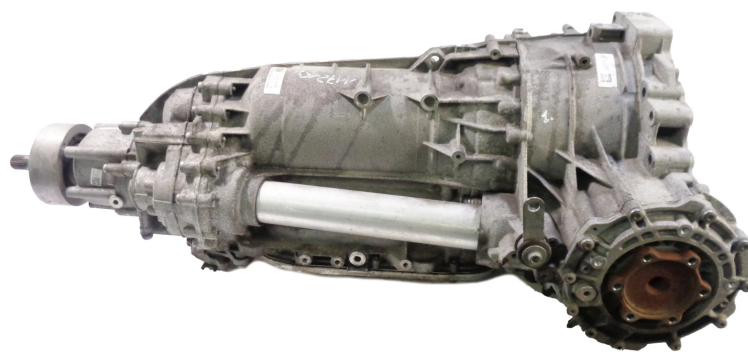


Figure 2.5: Audi A5 automatic transmission (code: “CNCD CNC QJZ 0BK30004”). Image source: [66].

tions with the same width, each treated as an independent cylindrical gear pair with distinct shift coefficients, in order to replicate the behavior of the actual beveloid gear pair. An iterative analysis of load distribution is performed to assess how the load is shared across the sections. For each section, the pitting safety factor is calculated, and the one with the lowest value is identified following the guidelines of [7]. A different model is developed to evaluate the contact-stress in straight beveloid gears, based on the minimisation of potential energy [59]. This model enables variation of design parameters within a specified range, allowing for the assessment of multiple design configurations. The parametric nature of this approach makes it easy to carry out optimisation studies to determine the most effective configuration — specifically, the one that results in the lowest contact pressure. However, this methodology is currently limited to straight beveloid gears with parallel axes.

Even with regard to tooth root load carrying capacity, related to bending stresses, there are no properties of standards; most of the work has been carried out by Brecher et al. in [63, 64].

2.1.1 Applications

The properties of beveloid gears make them particularly suitable for applications where compact design, high precision, easy manufacturing and low cost are required [37].

Beveloid gears are widely used in automotive transmission and differential systems [43, 65, 66] due to their ability to transmit torque between shafts with angles below 15° and

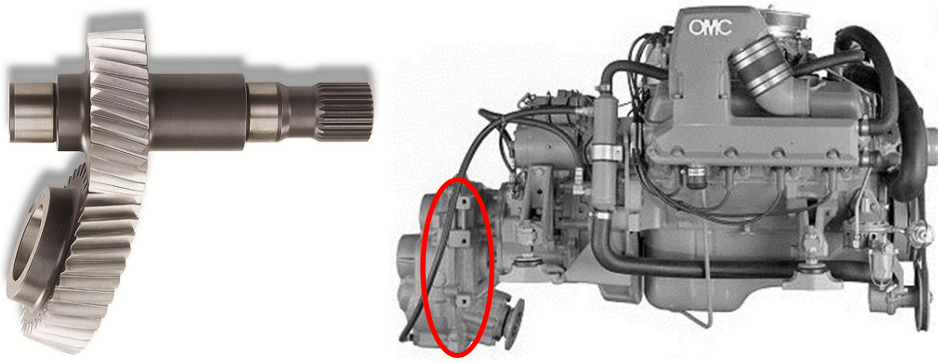


Figure 2.6: Marine transmission beveloid gear pair with skew axes. Image source: [68].

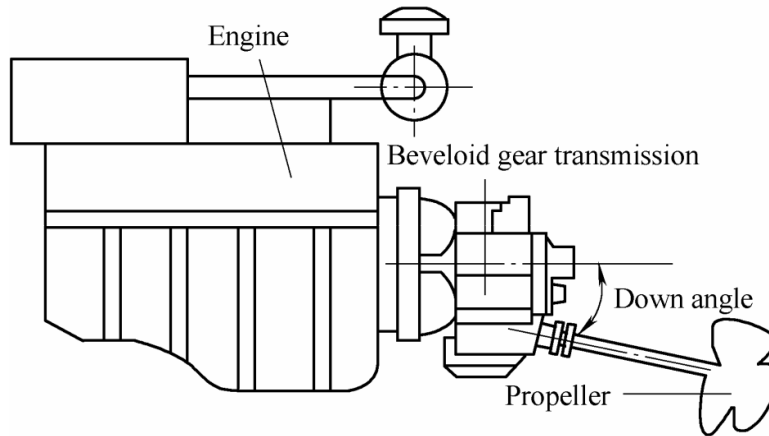


Figure 2.7: Marine transmission with down-angle beveloid gears. Image source: [38].

tolerate mounting inaccuracies. This feature makes them well-suited for modern drive trains, which demand high torque density within compact, lightweight assemblies. Additionally, beveloid gears help reduce noise and vibration, improving both comfort and efficiency. Their application extends to steering transmissions as well [67]. Examples of beveloid gears currently employed in commercial products include the powertrain transmission of the Porsche Panamera Gran Turismo (Fig. 2.4-left) and the Magna Powertrain transfer case (Fig. 2.4-right). According to [43], beveloid gears are also employed to transmit torque between small angle shafts in the Audi A5 automatic transmission (Fig. 2.5).

Beveloid gears are widely used in marine transmissions [38–40,68,70,71]. In [39,40], helical beveloid gears with skew axes are proposed for heavy-duty marine transmissions; to address the well-known issue of point contact, the authors employ a new type of conical gear, the concave conical gear, introduced by Mitome in [62]. The proposed gear pairs were designed and experimentally tested to validate the theoretical calculations. The ZF220A marine gearbox developed by ZF Friedrichshafen [71] also includes a 10° down-angle beveloid gear pair (Fig. 2.7). Such a small angle can be achieved using beveloid gears, ensuring a compact, space-saving design while providing long service life and smooth operation. In [38], the authors propose the geometry design of a crossed beveloid gear pair with small shaft angle for marine transmission. Finally, a practical experimental setup for a marine transmission system is performed, and tooth bearing tests are conducted to demonstrate the proposed design procedure.

In robotics, where precision and low inertia are essential, beveloid gears with par-



Figure 2.8: PSC precision planetary gearbox (section view). Image source: [69].

allel axes provide smooth torque transmission with minimal backlash. Currently, the only commercially available gearbox for robotic applications using beveloid gears is the PSC developed by Schaeffler [72] (Fig. 2.8). It achieves performance comparable to cycloidal technology, its direct competitor (see Sec. 3). Another solution, which represents the main focus of this thesis, is the Omni gearbox [15, 17], developed and patented by Bonfiglioli Riduttori (patent filed in 2017). The Omni gearbox is currently in the validation phase (see Sec. 4). The objective of this thesis is to contribute to the validation of the Omni gearbox for robotic applications. The long-term vision is to create a full family of gearboxes, in different sizes and torque capacities, specifically designed to be mounted on serial robots (see Sec. 3).

2.1.2 Force transmission

This section revises the meshing forces transmitted between beveloid gears mounted on parallel axes [16, 28, 73]. Referring to Fig. 2.1a, two reference systems are considered: a *global* reference system $x_1 y_1 z_1$ with z_1 aligned with the gear axis, and a reference system $x_R y_R z_R$ attached to the cutter-rack. The coordinate system $x_R y_R z_R$ is obtained by rotating $x_1 y_1 z_1$ about the axis y_1 through an angle δ . Helical beveloid gears can be generated by additionally rotating the rack by an angle β_0 around the axis x_R ; naturally, for straight beveloid gears, $\beta_0 = 0$. Finally, a *local* reference system $x_p y_p z_p$ is defined on the gear tooth by applying a rotation of $\pm\alpha_n$ to the coordinate system $x_R y_R z_R$ around the axis z_R , where the sign $+$ holds for the right flank of the gear tooth and the sign $-$ holds for the left flank. The unit vectors of the reference system $x_p y_p z_p$ are the three orthonormal vectors $\mathbf{k}_p$ (parallel to the rack feed direction z_R), $\mathbf{j}_p$ (normal to the tooth flank), and $\mathbf{i}_p = \mathbf{j}_p \times \mathbf{k}_p$. The rotation matrix $\mathbf{R}$ between the global and the local frame is:

$$\begin{aligned} \mathbf{R} &= \mathbf{R}_{y_1}(\delta) \mathbf{R}_{x_R}(\beta_0) \mathbf{R}_{z_R}(\pm\alpha_n) = [\mathbf{i}_p \quad \mathbf{j}_p \quad \mathbf{k}_p] \\ &= \begin{bmatrix} c_\delta c_{\alpha_n} \pm s_\delta s_{\alpha_n} s_{\beta_0} & \mp c_\delta s_{\alpha_n} + s_\delta c_{\alpha_n} s_{\beta_0} & s_\delta c_{\beta_0} \\ \pm s_{\alpha_n} c_{\beta_0} & c_{\alpha_n} c_{\beta_0} & -s_{\beta_0} \\ -s_\delta c_{\alpha_n} \pm c_\delta s_{\alpha_n} s_{\beta_0} & \pm s_\delta s_{\alpha_n} + c_\delta c_{\alpha_n} s_{\beta_0} & c_\delta c_{\beta_0} \end{bmatrix} \end{aligned} \quad (2.1)$$

where $c_\delta = \cos(\delta)$, $s_\delta = \sin(\delta)$, $c_{\beta_0} = \cos(\beta_0)$, $s_{\beta_0} = \sin(\beta_0)$, $c_{\alpha_n} = \cos(\alpha_n)$ and $s_{\alpha_n} = \sin(\alpha_n)$. $\mathbf{R}_w(\gamma)$ denotes the rotation matrix around axis w by an angle γ .

The resultant force is transmitted along the $\mathbf{j}_p$ direction. It consists of tangential (F_t), radial (F_r), and axial (F_a) components, acting respectively along the y_1 , x_1 and z_1 axes. If T is the applied torque and r_p is the pitch working radius, the tangential force F_t is:

$$F_t = \frac{T}{r_p} \quad (2.2)$$

For the right profile, the radial and axial components are given by:

$$\frac{F_r}{F_t} = \frac{j_{px}}{j_{py}} = \frac{\cos(\delta) \tan(\alpha_n)}{\cos(\beta_0)} - \sin(\delta) \tan(\beta_0) = \tan(\alpha_{t,r}) \quad (2.3a)$$

$$\frac{F_a}{F_t} = \frac{j_{pz}}{j_{py}} = \frac{\sin(\delta) \tan(\alpha_n)}{\cos(\beta_0)} + \cos(\delta) \tan(\beta_0) = \tan(\beta_r) \quad (2.3b)$$

and for the left profile by:

$$\frac{F_r}{F_t} = \frac{\cos(\delta) \tan(\alpha_n)}{\cos(\beta_0)} + \sin(\delta) \tan(\beta_0) = \tan(\alpha_{t,l}) \quad (2.4a)$$

$$\frac{F_a}{F_t} = -\frac{\sin(\delta) \tan(\alpha_n)}{\cos(\beta_0)} + \cos(\delta) \tan(\beta_0) = \tan(\beta_l) \quad (2.4b)$$

The angles β_l and $\alpha_{t,l}$ are, respectively, the helix and transverse pressure angles of the left flank profile at the pitch cylinder, and β_r and $\alpha_{t,r}$ are the same angles for the right flank profile. These angles are used to compute the radial and axial forces from the tangential force F_t . It is worth observing that the left and right helix and transverse pressure angles are different (even if, unlike in [28], the flank profiles of the rack tooth are symmetric), unless $\beta_0 = 0$, as in straight beveloid gears. In the latter case, in fact, $|\alpha_{t,l}| = |\alpha_{t,r}| = |\alpha_t|$ and $|\beta_l| = |\beta_r| = |\beta|$. However, even if $\beta_0 = 0$, the tooth flanks still have a nonzero helix angle $\pm\beta$, which is why straight beveloid gears also generate axial forces.

2.1.3 Manufacturing

The procedure for generating *external* beveloid gears is the same to the one used for spur gears. The cutter-rack translates along the y_R axis with velocity $v_r = \omega r_p$ and along the z_R axis to progressively remove material from the blank, while the gear blank rotates about its symmetry axis z_1 at angular speed ω . As shown in Fig. 2.1, the only difference is the inclination of the rack-cutter z_R axis by the angle δ with respect z_1 .

Alternative manufacturing techniques have also been investigated. Mitome [29] identified table sliding taper hobbing as the most practical method. This approach employs a rotating disk cutter to machine a gear blank that rotates about its axis z_1 . The cutter is equipped with two synchronized feed motions, aligned respectively with the x_1 and z_1 axes of the blank. The primary limitation of this method lies in its low production speed, as each tooth must be hobbled individually. In contemporary practice, beveloid gears are manufactured and finished using CNC machines, a technique also commonly applied to spur gears [26].

The fabrication of *internal* beveloid gears—such as the ring gears employed in the Omni system—is considerably more complex. Given the spatial constraints and stringent quality requirements, power skiving is adopted. This continuous gear-cutting process offers high production efficiency and precision [74, 75].

2.2 Contact-stress analysis based on the minimum potential energy principle for straight beveloid gears with parallel axes

2.2.1 Introduction

This Section analyses the contact behaviour of straight beveloid gears with parallel axis [59], taking into consideration tooth pointing and undercutting in the extreme sections. Indeed, one can then have tooth pointing (with high profile shift coefficient) towards the bigger cone base, namely in the extremal transverse section called the *heel*, or tooth undercutting (with low profile shift coefficient) at the opposite end, called the *toe* (see Fig. 2.2c)

To assess load-carrying capacity and develop the corresponding stress calculations, an analysis of how the tooth-pair mesh stiffness (called single mesh stiffness, SMS) and load distribution vary in the meshing cycle is essential [38, 70, 79]. Since each transverse section of the beveloid gear pair has a different geometry, the radii of the inner and outer points of contact of each section are different as well, which influences the contact conditions and load distribution.

Firstly, the start of contact is not simultaneous in all transverse sections. Contact starts at the section with the lowest inner point of contact and progressively extends to the section with the highest inner point of contact. Similarly, contact ends progressively, from the section with the lowest outer point of contact to the section with the highest one. This results in an axial contact ratio, similar to helical gears, even for beveloid gears with straight teeth [80].

Secondly, the length of the contact segment over the tooth surfaces changes during the meshing cycle. In the case of straight teeth, the line of contact is approximately parallel to the gear axis, and all its points are (almost) at an equal distance from the gear axis [61]. Nevertheless, since not all transverse sections are in contact simultaneously, the length of the contact segment increases at the onset of contact and decreases as contact ends.

Finally, the SMS per unit face width is not uniform along the instantaneous contact line. Although the point of contact is almost at the same radius for all transverse sections, the thickness of each section differs, so the mesh stiffness varies accordingly. In addition, tooth undercutting or pointing (if present) at the extremal transverse sections will also influence the contact-segment length and mesh stiffness [59].

The above considerations show that the contact geometry of beveloid gears presents a certain complexity. A detailed analysis of the instantaneous contact segments and mesh stiffness is required to determine meshing conditions and load distribution [81], which are essential for assessing contact and bending stresses, and, consequently, load-carrying capacity. However, analytic methods found in the literature for determining the mesh stiffness of beveloid gears [38, 70, 79] do not consider tooth undercutting and pointing. Calculation methods based on numerical techniques, specifically finite element method approaches [61, 80], are time-consuming and have high computational cost, making them unsuitable for optimisation studies.

An analytical model for determining the contact forces in anti-backlash beveloid gears mounted with parallel axes has been recently presented in [82]. This model utilizes the minimum potential energy principle and accounts for axial preload and back-

side contact. A parametric analysis on the influence of the pressure angle and the cone angle on the contact force is also presented. However, this approach accounts neither for the contact forces generated by power transmission between the two gears nor for possible tooth undercutting and pointing.

This Section investigates the contact geometry, mesh stiffness, load sharing, and contact-stress of beveloid gears with straight teeth and parallel axes. Unlike [82], this work focuses on load sharing and contact-stress caused by the tangential forces induced by power transmission, without accounting for preload axial forces. In addition, the instantaneous contact lines are determined considering tooth undercutting and pointing at the pinion toe and heel [83]. The mesh stiffness and load sharing ratio are found by applying the minimum elastic potential energy method [84–86]. A detailed discussion of the computation of the tooth-pair potential energy, adapted from [87], is provided in Annex A.2 for completeness. Finally, the contact-stress over the tooth face is evaluated according to Hertz equation [88].

Section 2.2.2 presents the geometry of a beveloid-gear pair and the evolution of the contact line along the meshing cycle, considering tooth undercutting and pointing. Section 2.2.3 presents the equation for the determination of the time-varying meshing stiffness and load-sharing ratio, according to the minimum energy method. Section 2.2.4 assesses the contact-stress along the entire meshing cycle according to Hertz's theory. Section 2.2.5 validates the contact-stress results by FEA. Finally, the conclusions are addressed in Section 2.2.6.

2.2.2 Geometry and contact map

Consider a pair of beveloid gears with straight teeth and parallel axes, as shown in Fig. 2.1b. The pinion has Z_1 teeth, while the gear has Z_2 teeth, both feature the same cone angle δ to ensure a correct mesh between parallel axes. The gears share the same module m and pressure angle α_n in all transverse sections, as well as the same pitch r_p and base r_b radii. However, the outside radii r_{o1} and r_{o2} , the root radii r_{d1} and r_{d2} , and the profile shift coefficient x_1 and x_2 vary linearly along the face width with z :

$$r_{o1/o2} = r_{o01/o02} \pm z \tan \delta, \quad r_{d1/d2} = r_{d01/d02} \pm z \tan \delta, \quad x_{1/2} = x_{01/02} \pm \frac{z}{m} \tan \delta \quad (2.5)$$

where the subscripts 1 and 2 denote the pinion and the wheel, respectively. The upper and lower signs correspond to the pinion and the wheel as well. Subscript 0 denotes the values at a reference transverse section, which coincides with the middle section for both gears, where $z = 0$. Clearly:

$$-\frac{b}{2} \leq z \leq \frac{b}{2} \quad (2.6)$$

where b is the face width.

Since module m , pressure angle α_n , and tooth numbers Z_1 and Z_2 , are the same for all transverse sections, the pitch radii r_{p1} and r_{p2} , and the base radii r_{b1} and r_{b2} are also constant. Consequently, the meshing condition is the same for all transverse sections and does not depend on the z -coordinate [84]:

$$\xi_1 + \xi_2 = \frac{Z_1 + Z_2}{2\pi} \tan \alpha'_t = \lambda_\xi \quad (2.7)$$

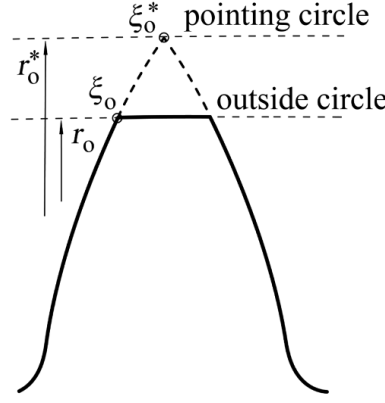


Figure 2.9: Pointing radius for a generic tooth.

where α'_t is the operating transverse pressure angle, and ξ_1 and ξ_2 are the profile parameters of the contact points in the pinion and the wheel. These dimensionless parameters can be expressed as [84]:

$$\xi_{1/2} = \frac{Z_{1/2}}{2\pi} \sqrt{\left(\frac{r_{c1/c2}}{r_{b1/b2}}\right)^2 - 1} = \frac{Z_{1/2}}{2\pi} \frac{\rho_{1/2}}{r_{b1/b2}} \quad (2.8)$$

where r_{c1} and r_{c2} are the (variable) radii at the contact point, and $\rho_{1/2} = \sqrt{r_{c1/c2}^2 - r_{b1/b2}^2}$ is the curvature radius of the transverse profile. ξ_1 and ξ_2 vary linearly along the path of contact, so it is possible to define, along this path, a coordinate ξ that takes the value of ξ_1 when contact occurs at the corresponding point. Accordingly, the contact ratio ε_α at each transverse section is given by:

$$\varepsilon_\alpha(z) = \xi_{\text{out}}(z) - \xi_{\text{inn}}(z) \quad (2.9)$$

with $\xi_{\text{out}}(z)$ and $\xi_{\text{inn}}(z)$ representing the values that ξ_1 assumes at the outer and inner points of contact in the section described by z . Here, $\xi_{\text{out}}(z)$ can be computed from Eq. (2.8), for:

- r_{c1} equal to the pinion outside radius r_{o1} given by Eq. (2.5), for transverse sections in which pointing does not arise;
- r_{c1} equal to the pointing radius r_{o1}^* (see Fig. 2.9 and A.3), for pointed transverse sections.

If ξ_o and ξ_o^* denote the parameters corresponding to r_{o1} and r_{o1}^* , respectively, $\xi_{\text{out}}(z)$ can be computed as:

$$\xi_{\text{out}}(z) = \min(\xi_o(z), \xi_o^*(z)) \quad (2.10)$$

The inner point of contact depends on the location of the inner point of the pinion involute profile (denoted E_1) and the point which theoretically meshes with the outer point of the wheel profile (ξ_{inn1}^*). ξ_{inn1}^* can be computed from Eq. (2.7) as $\xi_{\text{inn1}}^* = \lambda_\xi - \xi_{o2}$ where ξ_{o2} is computed from Eq. (2.8) for $r_{c2} = r_{o2}$. E_1 is the intersection point of the involute and root trochoid of the pinion, whose profile parameter ξ_{E1} can be computed as in [83]. As seen in Fig. 2.10, the effective inner point of contact will be the outermost between the two mentioned above, and therefore:

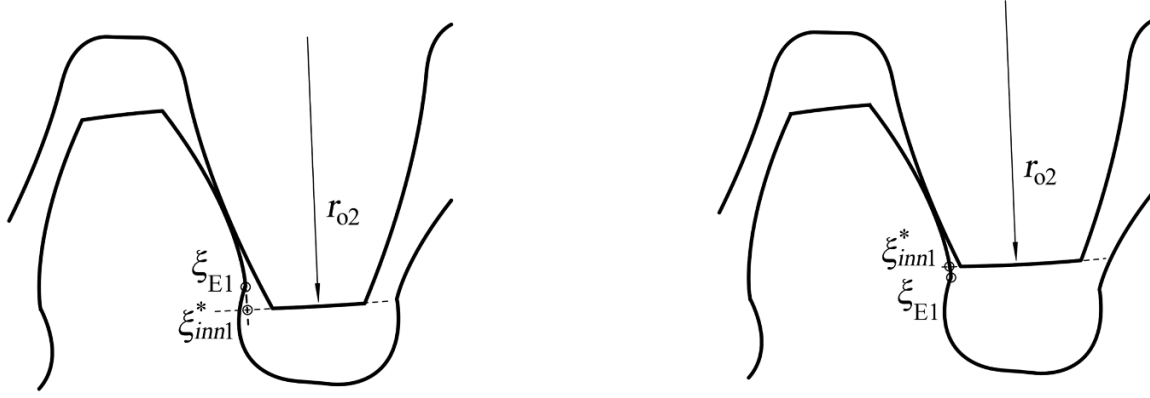


Figure 2.10: Inner point of contact in an undercut tooth.

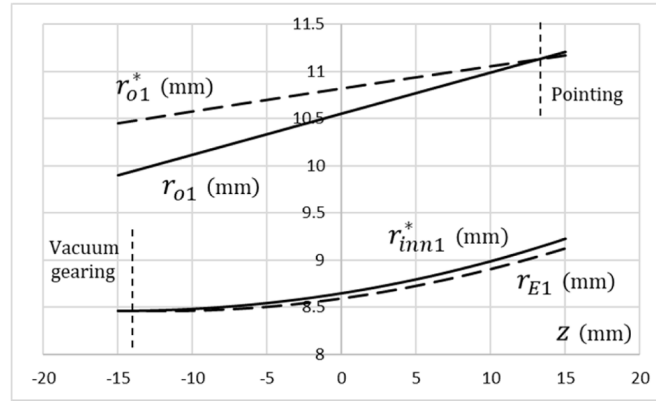


Figure 2.11: Limiting radii for the pinion of the sample beveloid gear pair in Tab. 2.1.

$$\xi_{inn}(z) = \max(\xi_{E1}(z), \xi_{inn1}^*(z)) \quad (2.11)$$

Fig. 2.10 shows an example of both possible cases, $\xi_{inn}(z) = \xi_{E1}(z)$ (Fig. 2.10-left) and $\xi_{inn}(z) = \xi_{inn1}^*(z)$ (Fig. 2.10-right), for an undercut pinion tooth. For non-undercut teeth only the second case, $\xi_{inn}(z) = \xi_{inn1}^*(z)$, is possible, because the first one would result in root interference, which is not admissible and therefore not considered in this study.

Fig. 2.11 shows the plots of the radii r_{o1} , r_{o1}^* , r_{E1} , and r_{inn1}^* along the tooth face of the pinion of the sample beveloid gear pair described in Tab. 2.1. The radius r_{inn1}^* was obtained from ξ_{inn1}^* and Eq. (2.8).

As seen in Fig. 2.10, when $\xi_{E1}(z) > \xi_{inn1}^*(z)$, and thus $\xi_{inn}(z) = \xi_{E1}(z)$, the outer points of the wheel profile, due to undercut, do not have a mating profile to mesh with. This phenomenon, referred to as vacuum gearing, is more likely to occur near the pinion toe, since a smaller shift coefficient leads to greater undercut, as shown in Fig. 2.11 for the sample beveloid gear pair of Tab. 2.1.

Although undercut reduces tooth root strength and, in the case of vacuum gearing, may result in small values of the contact ratio, it does not prevent the gear pair from operating correctly for power transmission. On the contrary, a pointed tooth is unable to transmit load, so it is not acceptable in practice. For instance, AGMA recommends that, for power transmission gears, the tooth thickness at the tip should be at least 0.3 times the module [89]. The easiest way to avoid tooth pointing at the sections close to

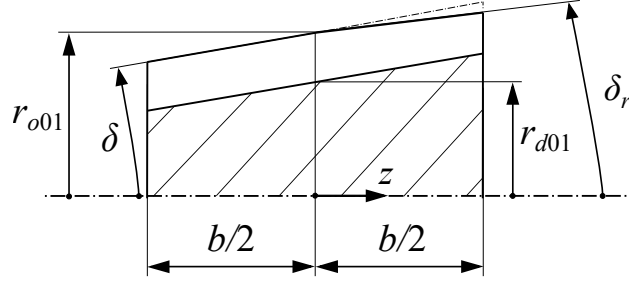


Figure 2.12: Section view of a pinion with reduced tooth height at the heel.

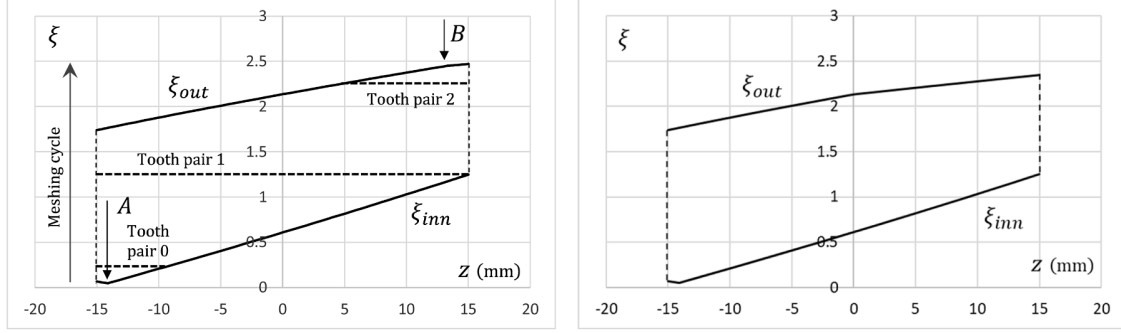


Figure 2.13: Contact maps of an example beveloid gearing, before (left) and after reducing the tooth height (right). Points A and B represent the limits of the vacuum gearing and pointing sections, respectively.

the heel of the pinion¹ is to reduce the outside radius. This can be done by removing the material outside a cone, having the same axis as the pinion, but with cone angle $\delta_r < \delta$. If, for simplicity, the intersection between this cone and the one defined by the outside radius r_{o1} is placed at the reference transverse section (Fig. 2.12), Eq. (2.5) becomes:

$$r_{o1} = \begin{cases} r_{o01} + z \tan \delta, & -\frac{b}{2} \leq z \leq 0 \\ r_{o01} + z \tan \delta_r, & 0 \leq z \leq \frac{b}{2} \end{cases} \quad (2.12)$$

where δ_r is the reduced cone angle.

The angle δ_r is computed from the outside radius at the reference section r_{o01} and the required outside radius at the heel to ensure the required thickness at the tip. An approximate equation to calculate the outside radius for a given tooth tip thickness is presented in A.4.

The contact map is the graphical representation in a diagram $\xi - z$ of the contact points existing in each position of the meshing cycle. Fig. 2.13-left shows the contact map for the beveloid gear having the parameters in Tab. 2.1. Its borders are the curves of $\xi_{inn}(z)$ and $\xi_{out}(z)$, and the limit values of the axial coordinate z (in this case, $z_{min} = -15\text{mm}$ and $z_{max} = 15\text{mm}$). At any gear position, the contact points of all transverse sections are located at the same distance from the gear axis (equal to r_c), and therefore all of them have equal ξ . Consequently, the instantaneous contact lines are represented by horizontal lines in the contact map. In addition, the lines corresponding to adjacent tooth-pairs are at a distance of 1 in terms of ξ .

The contact map provides interesting information for contact analysis. The first

¹Undercut may arise at the wheel root for gear ratio close to 1. The procedure would be exactly the same.

transverse section to get in contact, which is the section with minimum $\xi_{inn}(z)$ and may not coincide with the pinion toe, can be immediately determined. The contact ratio of each transverse section is represented by the difference between the values of $\xi_{out}(z)$ and $\xi_{inn}(z)$. The number of tooth pairs in simultaneous contact at any meshing position can also be easily obtained from the map.

The presence of vacuum gearing can be observed in the sections close to the smallest base of the pinion cone (pinion toe), from $z = -15$ mm to $z = -14.2$ mm (transverse section designated by A). Thus, in this interval, the slope of the ξ_{inn} curve is negative and ε_α decreases for lower values of z , because the smaller the shift coefficient x , the greater the amount of undercut, and thus the greater the value of $\xi_{E1} = \xi_{inn}$ (Fig. 2.10). The tooth meshing starts at the point with the lowest ξ_{inn} , which in this case is not at the pinion toe, but at transverse section A, where there is no longer vacuum gearing.

From Fig. 2.13, one can also see that pointing arises near the pinion heel, beyond section B. The contact ratio decreases in the interval between B and $z = z_{max}$, but the reduction is less pronounced than for the toe, and the value of ξ_{out} still increases with z .

A tooth pair will begin contact at the point corresponding to the minimum of the ξ_{inn} curve, namely, at $\xi_{inn}(z_A)$, and will end contact at the point corresponding to the maximum of the ξ_{out} curve, namely at $\xi_{out}(z_{max})$, as shown in Fig. 2.13-left. The effective contact ratio of the beveloid gear is therefore:

$$\varepsilon_{\alpha,eff} = \xi_{out}(z_{max}) - \xi_{inn}(z_A) \quad (2.13)$$

which, for the case in Tab. 2.1, is equal to 2.423. This means that there are regions along the contact interval where three tooth pairs are simultaneously engaged. Fig. 2.13-left shows an example: tooth pair 0 contacting at $\xi = 0.25$ (with contact line extended in the interval $-15, \text{mm} \leq z \leq -9, \text{mm}$), tooth pair at $\xi = 1.25$ (with contact line extended in the interval $-15, \text{mm} \leq z \leq 15, \text{mm}$), and tooth pair at $\xi = 2.25$ (with contact line extended in the interval $4, \text{mm} \leq z \leq 15, \text{mm}$).

To avoid tooth pointing, the outside radius of the heel should be reduced. According to Eq. (A.83), to ensure a tooth thickness at the tip no smaller than $0.3m$, the outside radius should be:

$$r_{o1}(z_{max}) \leq 10.991 \text{ mm} \quad (2.14)$$

Considering that, from Tab. 2.1, the outside radius of the reference transverse section is 10.55 mm, the cone angle at the sections with $z \geq 0$ should verify:

$$\delta_r \leq \arctan \left(\frac{10.991 - 10.55}{15} \right) = 1.68^\circ \quad (2.15)$$

In the case of Fig. 2.13-right, a value of $\delta_r = 1.5^\circ$ was chosen. Fig. 2.13-right shows the contact map of the considered beveloid gear, after applying the tooth-height reduction. The curve for ξ_{inn} is identical to the one in Fig. 2.13-left; the curve for ξ_{out} is identical for $z \leq 0$ and different for $z > 0$. The maximum value of ξ_{out} , namely $\xi_{out}(z_{max})$ decreases, and accordingly the effective contact ratio $\varepsilon_{\alpha,eff}$ decreases as well, from 2.423 to 2.300.

Symbol	Pinion	Wheel
Module m [mm]	1	1
Pressure angle α_n [°]	20	20
Dedendum mh_{a0} [mm]	1.25	1.25
Addendum mh_a [mm]	1	1
Tool tip radius mr_f [mm]	0.25	0.25
Number of teeth Z	18	120
Rack shift coeff. x_0	0.55	-0.55
Min. shift coeff. $x(z_{min})$	-0.1049	0.1049
Max. shift coeff. $x(z_{max})$	1.2049	-1.2049
Outside radius r_o [mm]	10.55	60.45
Face width b [mm]	30	30
Center distance I [mm]	69	69
Cone angle δ [°]	2.5	2.5
Reduced cone angle δ_r [°]	1.5	2.5

Table 2.1: Geometry of an example beveloid gear transmission

2.2.3 Meshing stiffness and load sharing ratio

The meshing stiffness of a straight beveloid gear pair can be computed from the principle of the minimum elastic potential energy, following the same procedure described in [82] and Annex A.2. However, while [82] focuses on the back-side contact force due to a significant axial load, here a different operating condition are considered: the tangential load induced by the transmitted power is predominant and the axial force is negligible. According to [85, 86], the SMS dK_M of each transverse section, considered as a spur gear having infinitesimal thickness, can be expressed as:

$$dK_M(\xi, z) = K_{M\max}^*(z) \cos(b_0(z)(\xi - \xi_m(z))) dz \quad (2.16)$$

where $K_{M\max}^*$ is the maximum SMS per unit face width for each transverse section, and b_0 and ξ_m are auxiliary parameters. ξ_m is the midpoint of the contact interval of the so-called reference spur gear, which is the gear with the same geometry except for the tooth addendum (equal to the module) and the center distance (nominal value). b_0 can be calculated from the contact ratio of this reference spur gear and the analytical equation presented in [86]. $K_{M\max}^*$ should be computed from the equations of the elastic potential energy (including bending, compressive, and shear components), the contact stiffness, and the gear body stiffness, for teeth in contact at ξ_m (even if ξ_m does not belong to the actual contact interval), as shown in [85]. All these parameters depend on the geometry of the transverse section, therefore they are functions of z . Fig. 2.14 shows the plots of $K_{M\max}^*$, b_0 , ξ_m , z varies, for the numerical example in Tab. 2.1.

When contact occurs at the point having coordinate ξ , the SMS of a specific tooth pair is:

$$K_M(\xi) = \int_{z_{\min}(\xi)}^{z_{\max}(\xi)} dK_M(\xi, z) = \int_{z_{\min}(\xi)}^{z_{\max}(\xi)} K_{M\max}^*(z) \cos(b_0(z)(\xi - \xi_m(z))) dz \quad (2.17)$$

where $z_{\min}(\xi)$ and $z_{\max}(\xi)$ are the limits of the contact segment at the contact position ξ , which can be determined from the curves $\xi_{\text{inn}}(z)$ and $\xi_{\text{out}}(z)$ in Fig. 2.13. These curves

2.2. Contact-stress analysis based on the minimum potential energy principle for straight beveloid gears with parallel axes

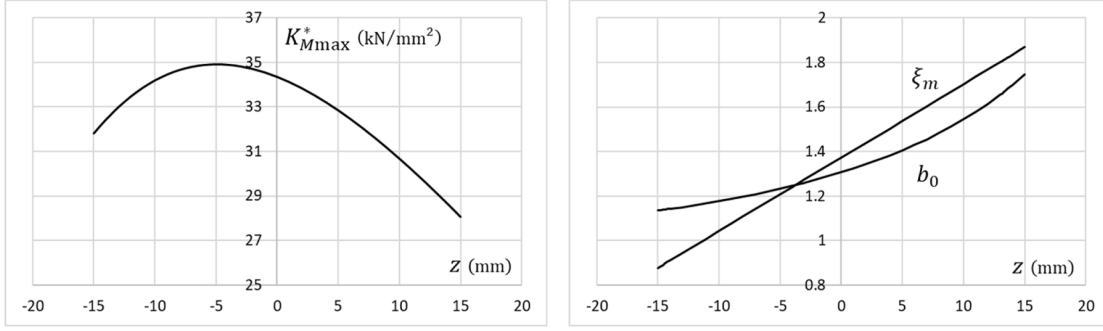


Figure 2.14: Parameters for the calculation of the single mesh stiffness (SMS).

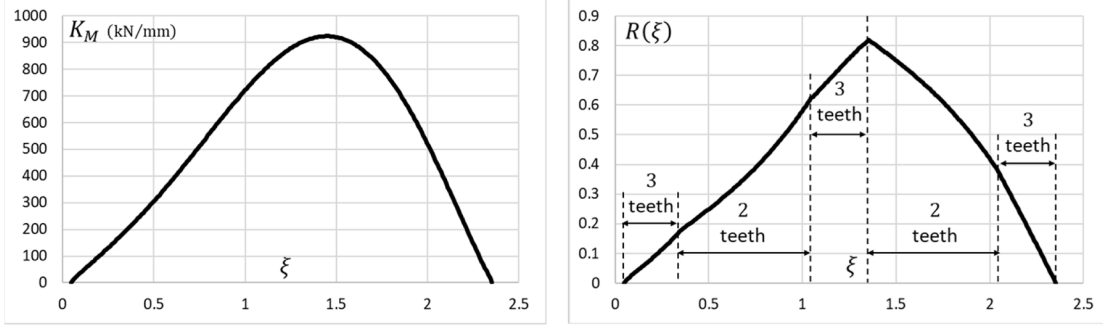


Figure 2.15: SMS (left) and LSR (right) of the beveloid gear pair in Tab. 2.1.

are nearly linear and can be obtained from the coordinates of the singular points, which, for the case of Tab. 2.1 with reduced tooth height, are the following: $(z_{\min}, \xi_{inn}(z_{\min}))$, $(z_A, \xi_{inn}(z_A))$ and $(z_{\max}, \xi_{inn}(z_{\max}))$ for ξ_{inn} , and $(z_{\min}, \xi_{out}(z_{\min}))$, $(0, \xi_{out}(0))$ and $(z_{\max}, \xi_{out}(z_{\max}))$ for $\xi_{out}(z)$. The final result is illustrated in Fig. 2.15.

This curve of SMS is quite different from that of a cylindrical spur gear. Indeed, when a spur pair starts meshing, the contact immediately occurs along the complete face width, and the SMS takes a nonzero finite value. In beveloid spur gears, instead, contact starts at a single contact point, and the length of the contact segment increases progressively. Accordingly, the SMS curve starts from the zero value and, for this reason, it is more similar to that of a helical gear.

The load sharing ratio (LSR) R of a given tooth pair is computed as:

$$R(\xi) = \frac{K_M(\xi)}{\sum_j K_{Mj}(\xi)} \quad (2.18)$$

where the sum extends over all tooth pairs j in simultaneous contact at the meshing position ξ . Index $j = 0$ refers to the considered tooth pair, $j = 1$ to the previous pair in contact, $j = -1$ to the next pair, and so on. Considering that $K_{Mj}(\xi) = K_M(\xi + j)$, the LSR curve obtained from Eqs. 2.17 and 2.18 is shown in Fig. 2.15-right. The maximum LSR is at $\xi(R_{\max}) = 1.352$, that is, when the previous tooth pair finishes its meshing, because $\xi_{\max} = 2.352$, as seen in Fig. 2.13-right. This is not a coincidence: when the tooth-pair $j = 0$ meshes at a point around $\xi = 1.352$, the previous pair ($j = 1$) meshes around $\xi = 2.352$, and the next one ($j = -1$) around $\xi = 0.352$. From Fig. 2.15-left, $K_M(\xi)$ is quite uniform around $\xi = 1.352$, it has a positive slope for $\xi = 0.352$, and a negative (more pronounced) slope for $\xi = 2.352$. Accordingly, the numerator of the fraction at the right-hand side of Eq. (2.18) is almost constant around $\xi = 1.352$, while the denominator decreases with ξ for $\xi < 1.352$ and increases for $\xi > 1.352$ (because beyond $\xi = 1.352$ the previous tooth-pair is no longer in contact). Thus, the LSR increases for $\xi < \xi_{\max} - 1$

and decreases for $\xi > \xi_{\max} - 1$. Consequently, the LSR is maximum at $\xi = \xi_{\max} - 1$, namely:

$$\xi(R_{\max}) = \xi_{\max} - 1 \quad (2.19)$$

The LSR curve (Fig. 2.15-right) differs from the single tooth contact force diagrams reported in [82], since here the focus is on the sharing ratio of the tangential load transmitted by a single tooth pair due to the transmitted power, rather than the contact force due to an axial load applied to remove backlash. Nevertheless, in both cases the maximum load on a single tooth pair occurs when the previous pair finishes the contact.

2.2.4 Contact-stress analysis

The LSR given by Eq. 2.18 is the fraction of the load transmitted by a specific tooth pair at a specific contact position. However, this load is not evenly distributed along the contact line, because the meshing stiffness per unit face width changes along z , as seen in Fig. 2.14-left. Also, the contact-stress depends on the curvature radii of the meshing profiles at the contact point, which vary with the contact-point parameter ξ , as seen in Eq. 2.8. Accordingly, the contact-stress will be different at each contact point (described by z) along the contact line and at each contact position (described by ξ) of the meshing cycle. The contact-stress can therefore be expressed as a function of ξ and z , and its maximum is found by calculating its maximum value along the instantaneous contact line, for all contact lines along a complete meshing cycle.

For beveloid gears, the infinitesimal fraction of the load dR^* at the transverse section having coordinate z and at meshing position ξ is given by:

$$dR^*(\xi, z) = \frac{K_M(\xi)}{\sum_j K_{Mj}(\xi)} \frac{dK_M(\xi, z)}{K_M(\xi)} = \frac{dK_M(\xi, z)}{\sum_j K_{Mj}(\xi)} = \frac{dK_M(\xi, z)}{\sum_j K_M(\xi + j)} \quad (2.20)$$

where $dK_M(\xi, z)$ and $K_M(\xi)$ are computed from Eqs. 2.16 and 2.17, respectively.

The contact-stress $\sigma_H(\xi, z)$ at coordinate z on the contact line and at the meshing position ξ can be computed from Hertz's equation [88], which for a tooth contact can be expressed as:

$$\sigma_H(\xi, z) = Z_E \sqrt{\frac{dF(\xi, z)}{dz} \frac{Z_1}{2\pi r_{b1}} \left(\frac{1}{\xi} + \frac{1}{\lambda_\xi - \xi} \right)} \quad (2.21)$$

where Z_E is the ISO elasticity factor [7], λ_ξ is the parameter described in Eq. 2.7, and the contact load is:

$$dF(\xi, z) = dR^*(\xi, z) F_T = \frac{dK_M(\xi, z)}{\sum_j K_{Mj}(\xi)} F_T \quad (2.22)$$

where F_T is the total transmitted load along the pressure line. For the example in Table 2.1 with reduced cone angle at the heel, the maximum contact-stress of each transverse section along the instantaneous contact line ($\sigma_H(\xi) = \max_z \sigma_H(\xi, z)$), for all contact positions along the meshing cycle of a tooth pair, is shown in Fig. 2.16 (solid line) for a transmitted load of 9940 N.

The maximum contact-stress is very high, around 6050 MPa, and occurs at the beginning of contact, for small values of ξ (close to 0 at the minimum inner point of contact, as shown in Fig. 2.13). From Eq. 2.8, the contact parameter ξ is proportional to

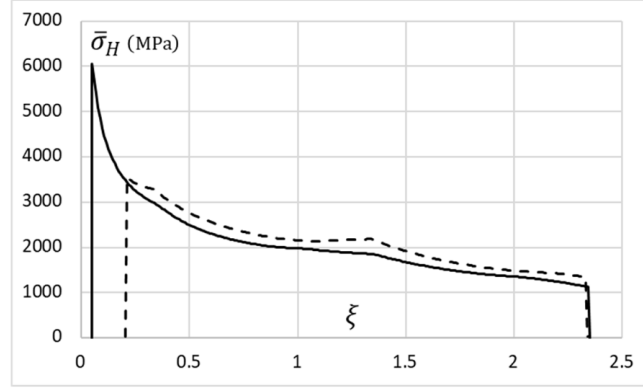


Figure 2.16: Plot of the maximum contact-stress as function of the meshing cycle for a longer (solid line) and shorter (dashed line) face width.

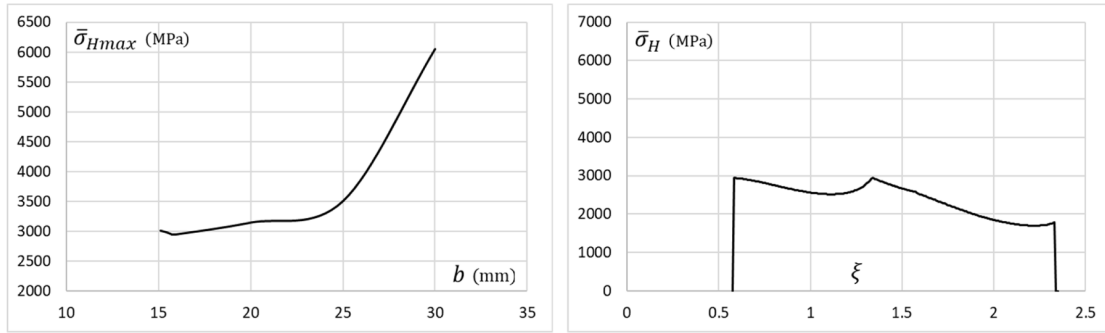


Figure 2.17: Evolution of the maximum contact-stress with the face width (left) and plot of the maximum contact-stress along the meshing cycle for the optimal face width, $b = 15.75$ mm (right).

the curvature radius ρ_1 of the pinion transverse profile. From Hertz's equation, if the curvature radius is close to 0, the contact-stress tends to infinity.

The problem of high contact-stress is not unusual in beveloid gears with parallel axes. As the contact-stress frequently introduces a more restrictive design constraint than the tooth root stress, a possible solution may be to reduce the face width. The load per unit length will increase (since the length of the contact segment will decrease), but the curvature radius at the outermost transverse sections will increase too. In this perspective, a different gearing could be considered, with the same parameters as in Table 2.1 but with a face width 5 mm smaller, thus $-10 \text{ mm} \leq z \leq 15 \text{ mm}$. In this case, the result is the contact-stress represented by the dashed line in Fig. 2.16, with the maximum contact-stress being drastically reduced to 3505 MPa, due to the increase in the curvature radius at the outermost sections. This increases the load-carrying capacity by 198%.

Since reducing the face width decreases the contact-stress at the beginning of contact but increases the stress peak in the middle of the contact path, an optimal face width may exist that balances the contact-stress at both points. Fig. 2.17-left illustrates how the maximum contact-stress varies with the face width for the sample beveloid gear described in Table 2.1, with reduced cone angle at the pinion heel (namely, $\delta = 2.5^\circ$ and $\delta_r = 1.5^\circ$). A local minimum occurs at a face width of 15.75 mm, with a critical contact-stress of 2945 MPa. Fig. 2.17-right shows the plot of the contact-stress along the meshing cycle for this optimal value. As expected, the contact-stress at the beginning of contact equals the peak in the middle of the meshing cycle (correspond-

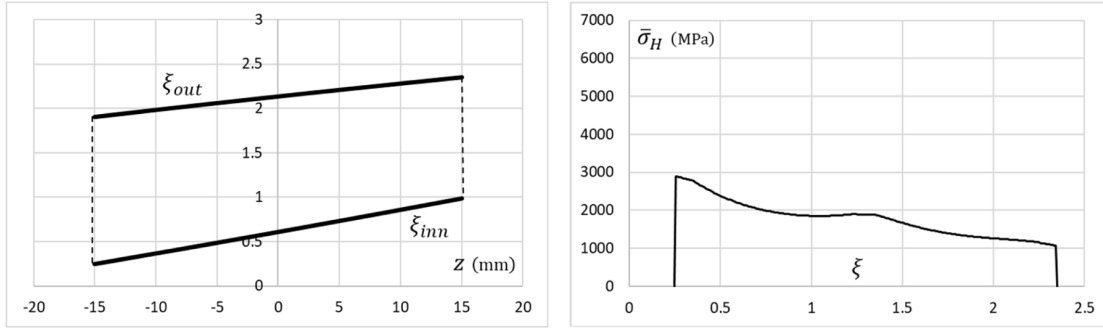


Figure 2.18: Contact map (left) and maximum contact pressure (right) for a beveloid gear with cone angle $\delta = 1.5^\circ$, no reduction of the cone angle at the pinion heel ($\delta = \delta_r$) and face width $b = 30$ mm.

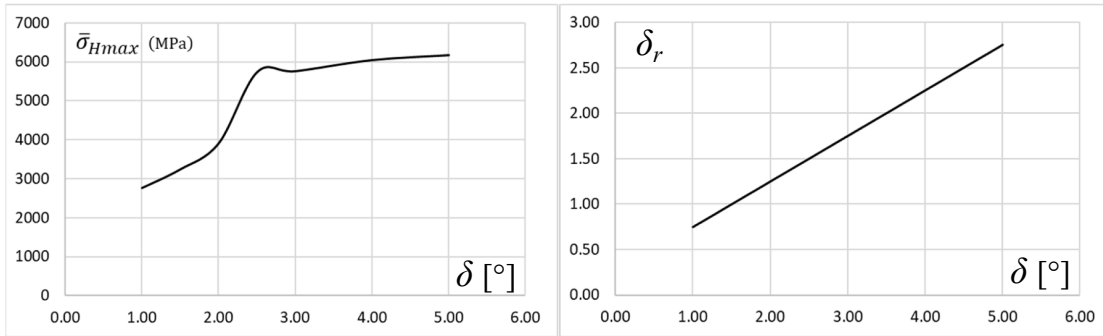


Figure 2.19: Variation of the maximum contact-stress with the cone angle δ (left), when the pinion heel is modified with a reduced cone angle δ_r that varies linearly with δ to prevent tooth pointing (right).

ing to the moment in which the previous tooth pair finishes its contact).

Another interesting insight from Fig. 2.17-left is that, while reducing the face width from 30 mm to 25 mm yields a significant decrease in contact-stress, further reductions from 25 mm to 15 mm result in only modest improvements. Accordingly, if a face width of 15.75 mm is too small to meet other design criteria, a face width between 20 mm and 25 mm could be considered a quasi-optimal solution.

Another possible solution is decreasing the cone angle δ while keeping the same face width. In this case, the root radius at the pinion toe is greater, which increases the curvature radius and lowers the contact-stress at the toe; moreover, reducing the outside radius of the right half of the pinion can be avoided. For example, for a cone angle δ of 1.5° and a 30 mm face width, the maximum contact-stress is 2890 MPa. This increases the load-carrying capacity by 338% with respect to the initial solution and by 47% with respect to the solution with a smaller face width. For this geometry, there is no undercutting or tooth pointing. The contact map is shown in Fig. 2.18-left: notice that the curve representing ξ_{out} is a single straight segment, since the external radius is not reduced. The plot of the maximum contact-stress along the meshing cycle is reported in Fig. 2.18-right. The maximum contact-stress occurs at the beginning of contact, as in the previous cases.

Fig. 2.19-left shows the evolution of the maximum contact-stress as the cone angle δ varies between 1° and 5° , for the sample beveloid gear described in Table 2.1. To prevent tooth pointing (namely, to ensure a tooth thickness at the tip no smaller than $0.3m$), a reduced cone angle δ_r at the pinion heel should be computed for each value

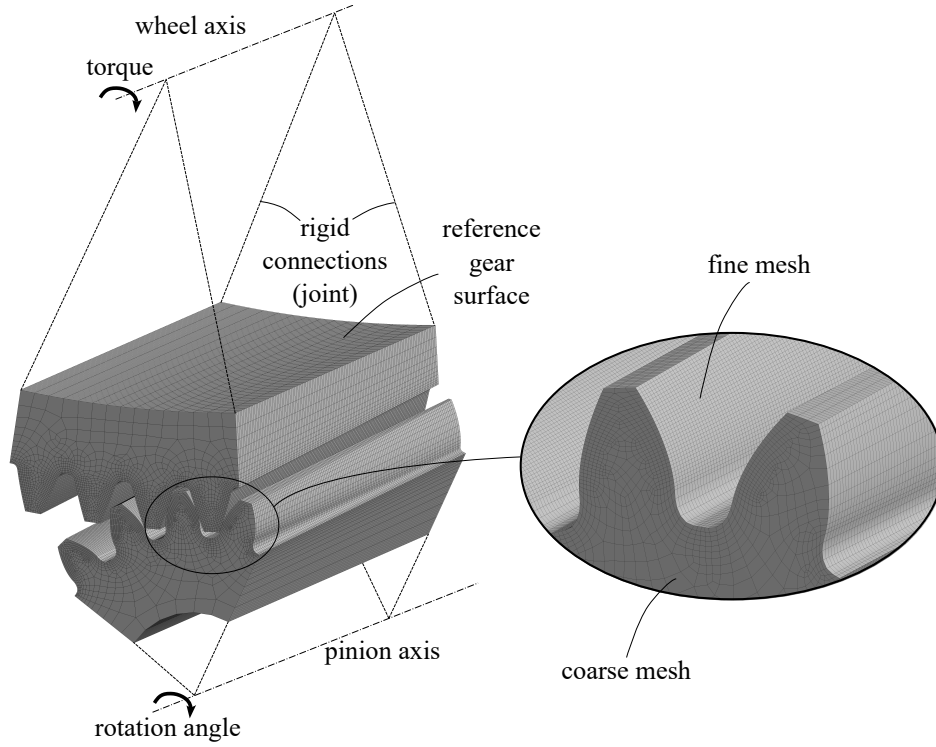


Figure 2.20: FEA mechanical model and pinion mesh detail.

of δ . For the sake of simplicity and to ensure consistency, Fig. 2.19-left was generated assuming a linear decrease in δ_r relative to δ , as shown in Fig. 2.19-right². In this case, no local minimum of stress occurs within the examined range of δ . However, the reduction in critical contact-stress is modest for δ between 1° and 2° and between 2.5° and 5° , but significantly more pronounced in the interval $2^\circ \div 2.5^\circ$, which provides a useful insight for design purposes.

It can be concluded that contact at points with a small curvature radius leads to high contact-stresses and should therefore be avoided. The points with minimum curvature radii are located at the pinion toe, corresponding to the beginning of meshing. The curvature radius at these points can be increased by reducing the cone angle or the face width, although the optimal reduction will depend on the specific case at hand.

2.2.5 Validation by Finite Element Analysis (FEA)

In this Section, the analytical model for computing the load sharing ratio, the contact-stress, the contact line position and the contact map is validated through a FEA performed in ANSYS Workbench. The gearing is the same as in Table 2.1, except for the cone angles, for which $\delta = \delta_r = 1.5^\circ$ has been considered for both the pinion and the wheel. The main features of the model are the following.

1. **Geometry and material:** A beveloid gear profile can be generated by virtually simulating gear manufacturing [90] or by analytically calculating the flank surfaces [61, 79]. Using the beveloid module of the gear analysis software “KISSsoft”, setting 50 sections along the face width and a modeling tolerance of $0.5 \mu\text{m}$. Four

²Notice that, according to Fig. 2.19-right, a reduced cone angle δ_r of 1° is used for the case $\delta = 1.5^\circ$, which is different from the case reported in Fig. 2.18, where $\delta_r = \delta = 1.5^\circ$. This explains why the contact-stress peak in Fig. 2.18-right is slightly smaller than the one reported in Fig. 2.19-left for $\delta = 1.5^\circ$.

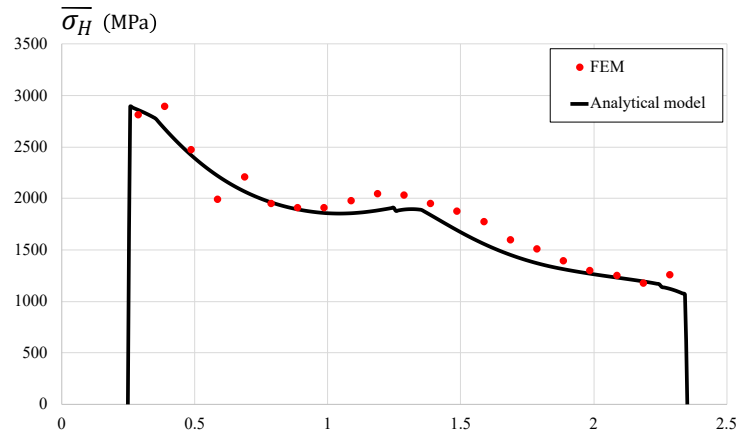


Figure 2.21: Comparison between the FEM and the analytical model results.

teeth were isolated on both the pinion and the wheel (Fig. 2.20-left), since the transverse contact ratio is between 2 and 3. The gears are made of structural steel, defined as an isotropic material with elastic modulus $E_y = 210$ GPa and Poisson's ratio $\nu = 0.3$.

2. **Meshing:** Using the sweep meshing method, both gears are divided into 130 equally spaced transversal slices. In contact areas and at the tooth root, the mesh is refined with elements having a characteristic length of 0.065 mm and having approximately square faces. The model has 655,200 Hex8 elements (8-nodes hexahedral) and 8320 Wedge6 elements (6-nodes, prism-shaped) (Fig. 2.20-right).
3. **Constraints and boundary conditions:** For each gear, the inner cylindrical surface is defined as a rigid reference, linked to the fixed gear axis by a rotational joint. The pinion rotates by 50° , which is the minimum angle such that the two central teeth experience contact conditions across the whole range of ξ . A resistant torque of 560 Nm is applied to the wheel, corresponding to a transmitted load of 9940 N (as in Section 2.2.4).
4. **Contact:** Contact is assumed to be frictionless, since power losses are not considered in the analytical model.
5. **Solver settings:** Large deflections are included in the solution and the analysis is repeated for any 2° of pinion rotation.

The FEA confirmed the contact map in Fig. 2.18: the contact (Fig. 2.20) starts from the pinion toe ($z_{\min} = -15$) and ends at the heel ($z_{\max} = 15$). In Fig. 2.21, the maximum contact-stresses provided by the FEA at each transversal section along the path of contact are compared to those obtained via the analytical model: the two graphs are very close to each other, with an average difference of 5.8%.

2.2.6 Conclusions

This Section studied the contact geometry, meshing stiffness, load sharing and contact-stress of beveloid gears with straight teeth and parallel axes. In particular, beveloid gears for power transmission with negligible axial load were considered; therefore, contact occurs only at one side. The contact map, which is the graphic representation of

the contact lines at each meshing position, was introduced, providing relevant information about the tooth contact, such as the contact ratio or the point at which the contact starts. Tooth undercutting and pointing in the outermost transverse sections of the pinion were also considered. Moreover, a procedure to avoid tooth pointing by linearly reducing the tooth height was presented.

From contact geometry, the principle of minimum potential deformation energy was applied to calculate the stiffness of each transverse section of the tooth pair; then, integrating along the contact line, the total stiffness was obtained. The resulting stiffness curve differs from that of a cylindric spur gear, and it is more similar to that of a cylindrical helical gear. In both cases, tooth contact progressively begins at different transverse sections; however, contact may not start at the extreme transverse section of a beveloid gear if undercut is present.

Based on the stiffness curve, the curve of the load distribution coefficient along the contact path was found. This curve has a maximum at the contact point corresponding to the end of contact of the previous tooth pair. Finally, an analysis of contact-stress was conducted based on load distribution. At the beginning of contact, the pinion has small radii of curvature at contact points; thus, pressure reaches extremely high values, which reduces load-carrying capacity. However, small reductions in the face width, removing the initial sections with the smallest curvature radii, can significantly reduce the contact-stress. An example showed that a 20 % reduction in the face width resulted in a 198 % increase in the load capacity of the transmission. A similar effect may be obtained by crowning the surfaces of the pinion teeth, although this may increase the tooth root stress at the middle transverse sections of the teeth. Similarly, a reduction in the cone angle, which increases the curvature radius at the points where contact begins, also results in lower contact-stress and greater load-carrying capacity. A study was conducted on a sample case to examine the influence of both parameters, revealing optimal ranges for face width and cone angle that ensure acceptable contact-stress levels.

A FEA model produced results consistent with the analytical model, thus confirming the findings obtained from the contact map and contact-stress analyses.

The proposed analytical model provides accurate results while allowing rapid and straightforward modification of gear parameters, unlike time-consuming FEA models, where even minor changes require restarting the analysis from scratch. This flexibility enables optimisation studies, unfeasible with non-analytical methods, focused on minimising contact-stress by adjusting the cone angle and face width, while accounting for other design constraints.

2.3 Contact-stress analysis and pitting resistance based on ISO 6336-1 and ISO 6336-2 for helical beveloid gears with parallel axes

2.3.1 Introduction

The two main types of fatigue damage in a gear pair are pitting and tooth bending. Pitting occurs when contact forces generate high stresses on the flank surface and sub-surface during meshing. In addition, friction and sliding phenomena generate heat, causing the temperature to rise and the lubricant film to reduce in thickness. This re-

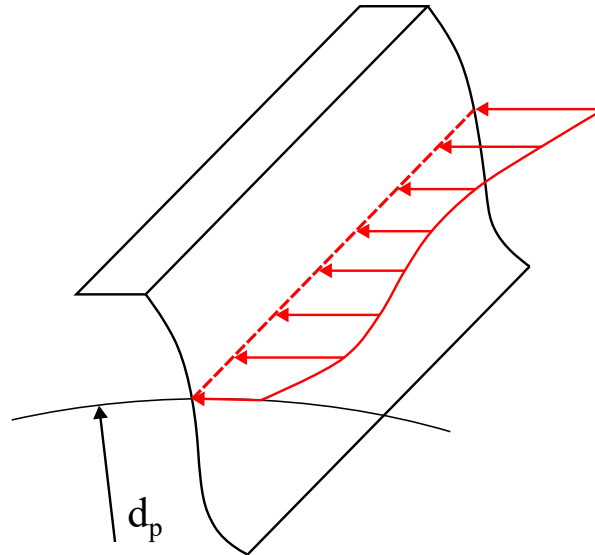


Figure 2.22: One-dimensional load distribution analysis in accordance with ISO 6336-1: Annex E. Image source: [77].

duces the load-carrying capacity and causes pitting to occur more quickly [76]. Pitting can be detected in the early stages as small cracks. This is followed by material detachment, which creates craters. The detached particles then mix with the circulating lubricant, accelerating the damage.

This paragraph presents a simple analytical method for determining the pitting resistance of helical beveloid gears with parallel-axis to accelerate sizing and avoid performing a time-consuming LTCA. This method is based on that described in Annex E of the ISO 6336-1 standard for spur and helical gears [6]. This standard is itself derived from AGMA 927-A01 [78], which is a one-dimensional load distribution analysis along the face width at the operating pitch diameter (Fig. 2.22). Compared to a LTCA, the advantages of the Annex E method are summarized in Table 2.2.

Table 2.2: Comparison between the one-dimensional load distribution analysis (ISO 6336-1: Annex E) and the Loaded Tooth Contact Analysis (LTCA).

One-dimensional load distribution analysis (ISO 6336-1: Annex E)	Standard Loaded Tooth Contact Analysis
Considers gear micro-geometry	Considers gear micro-geometry
Considers shafts, bearings, carrier, gear body deformation	Considers shafts, bearings, carrier, gear body deformation
Faster (does not use LTCA)	Slower but more accurate
Considers manufacturing deviations	Time consuming to include manufacturing deviations
Mesh stiffness considered as constant	Considers non-linear tooth stiffness model
Applied on pitch circle over the face width	Applied over the path of contact and face width
No slice linking effects considered	Slice linking effects considered

In this method the gear pair is divided in n sections of equal width and an iterative *Load Distribution* (LD) analysis is performed to evaluate the load share transmitted by

each section. Uneven LD over the tooth width is due to the deformation of gear teeth and gear supporting elements, such as shafts, bearings, and housing; this effect is accounted for by the $K_{H\beta}$ factor. The real beveloid gear pair is simulated by n “equivalent” spur gear pairs with different shift coefficients. Finally, the contact-stress and pitting resistance for each section is determined according to [7], using the loads found at the last iteration. The section with the lowest safety factor is chosen as representative of the beveloid gear pair. For pitting, it appears reasonable to consider each section separately; this is the same approach used by commercial software for contact analysis. For the bending analysis, this approach is not applicable, since the root strength of each section depends on the sections nearby; the method presented in [63] could then be used instead. Model validation is currently in progress at Bonfiglioli facilities by monitoring of pitting damage during endurance tests on six industrial gearboxes for robotics, each with two parallel-axis stages: a helical gear pair and a beveloid gear pair. The gearboxes under test are designed for fast, intermittent motions, where the direction of rotation changes frequently: to speed up the tests and observe pitting in a reasonable time, however, one-directional continuous duty cycles are performed, with output torque higher than the nominal one, to focus on the pitting of one flank. Details of the gearbox and the test setup are provided in Sec. 2.4, while only the general analytical model is presented here.

2.3.2 Beveloid-gear pitting resistance determination

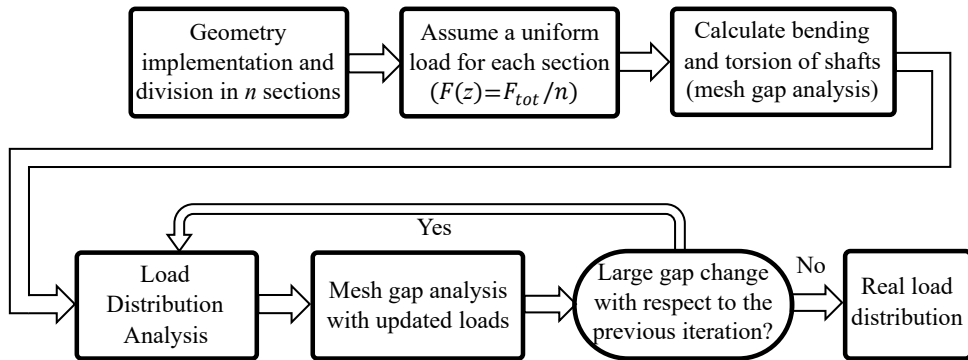


Figure 2.23: Flowchart of the iterative procedure for finding the load distribution and mesh gap.

The method to determine the pitting resistance of a beveloid gear pair follows the procedure in Fig. 2.23. Consider two beveloid gears, namely, gear 1 and gear 2, exchanging a resultant load F_{tot} in the plane of action. The first step is to divide the gear pair in n sections of equal face width $\Delta_z = b/n$ (Fig. 2.24), where $b = \min(b_1, b_2)$ is the shortest face width between gear 1 and gear 2. To simulate the beveloid geometry, each section is assumed to be cylindrical, but with different shift profile coefficient, outside radius and root radius. To locate the n sections, an axial coordinate z is adopted, with origin at the midsections of the gear pair, i.e. $-b/2 < z < b/2$. The higher the number of sections n , the more precise the final geometry. $F_j(z)$ is the partial load (normal to the tooth flank) acting on the section defined by z at the j -th iteration. For the initial iteration ($j = 0$), a uniform load distribution, namely $F_0(z) = F_{tot}/n$, is assumed to obtain reference values for the bending and torsional deflections of the shafts. For each section, a *mesh gap analysis* is performed to evaluate the gap between the meshing flanks

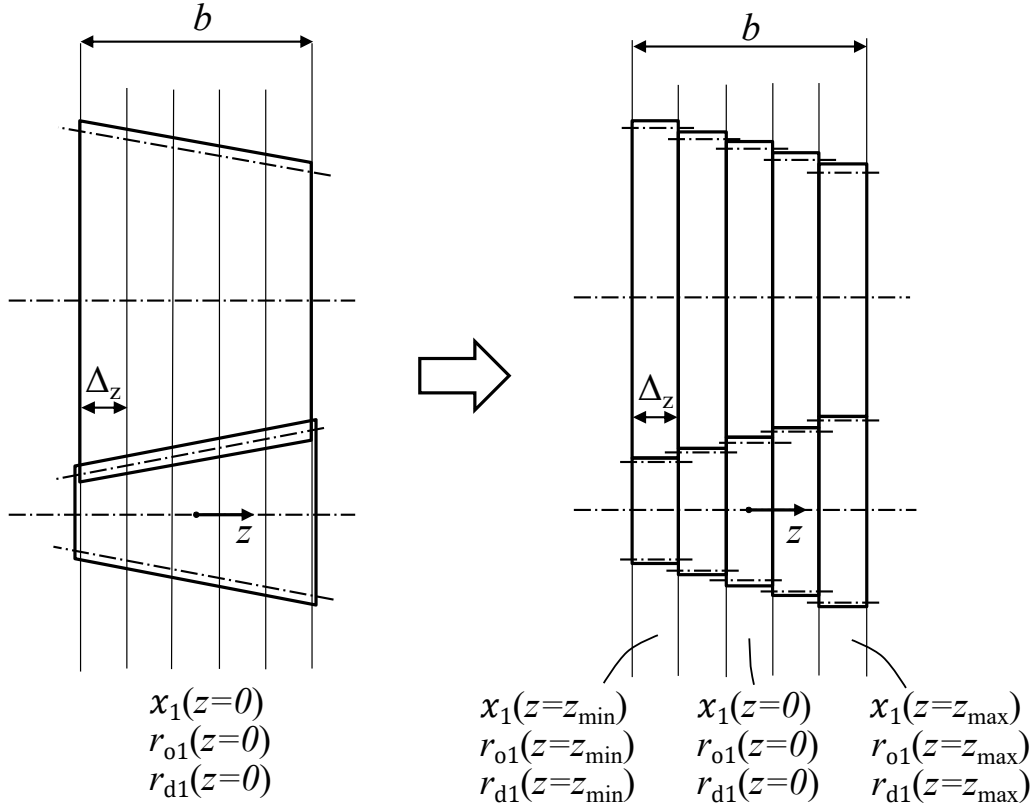
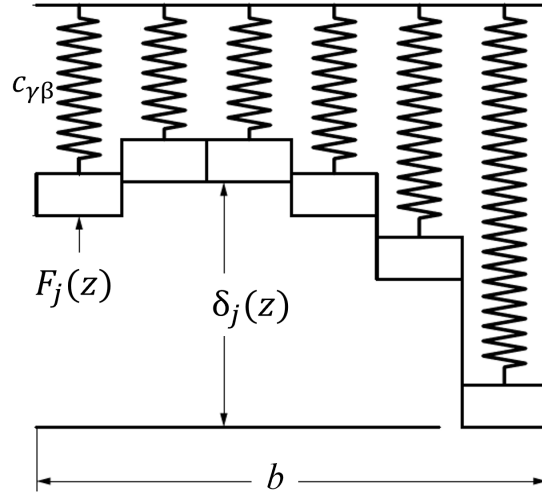


Figure 2.24: Schematic of a beveloid gear pair divided in sections.


 Figure 2.25: Schematic of the tooth pair sections, modeled as springs, with partial load $F_j(z)$ and deformation $\delta_j(z)$. Image source: [6].

at the operating pitch point, considering shaft deflections and tooth modifications; the resulting gap is then projected onto the plane of action. For gear 1, the mesh gap differs for each section and depends on z ; it can be computed considering the contribution of bending deflection, torsional deflection and micro geometry modification along the plane on action, as:

$$f_{1,j}(z) = y_1(z) \cos \alpha_{n,w} + x_1(z) \sin \alpha_{n,w} + \frac{\tau_{1,j}(z) \cdot d_{b1}}{2} + m_1(z) \quad (2.23)$$

where:

- $f_{1,j}(z)$: is the mesh gap of gear 1 in μm ;
- $y_{1,j}(z)$: is the bending deflection in the yz -plane in μm ;
- $x_{1,j}(z)$: is the bending deflection in the xz -plane in μm ;
- $\alpha_{n,w}$: is the working pressure angle [rad];
- $\tau_{1,j}(z)$: is the torsional deflection [rad];
- d_{b1} : the base diameter in μm ;
- $m_1(z)$: the micro-geometry modification in μm .

Please note that not all parameters depend on the iteration j , since some are constant values. The KISSsoft shaft module can be used to evaluate shaft deformations. Bearing clearance and stiffness can be included in the analysis to enhance its accuracy.

The deflection of the tooth pair $\delta_j(z)$ at the j -th iteration, can be computed as:

$$\delta_j(z) = f_{1,j}(z) - f_{2,j}(z) \quad (2.24)$$

where $f_{2,j}(z)$ is the mesh gap of gear 2, obtained by applying Eq. 2.23 with the parameters of gear 2. According to Fig. 2.25, each tooth is considered as a spring with stiffness³ $c_{\gamma\beta}$. The partial load per unit length is:

$$F_j(z)/\Delta_z = \delta_j(z)c_{\gamma\beta} \quad (2.25)$$

while the total force is:

$$F_{tot} = \sum_{z=z_{\min}}^{z_{\max}} F_j(z) \quad (2.26)$$

The model considers that only one tooth pair is in contact and that $F_j(z)$ is applied in the center of each section at the operating pitch diameter. This LD is then used to update the shaft deformation. The actual LD is thus obtained through an iterative process (Fig. 2.23) until the difference in each gap between two consecutive iterations is sufficiently small; two or three iterations are usually sufficient to reach convergence. Finally, knowing the load and the geometry for each section, the safety factor $S_H(z)$ for each section is calculated according to [7]. The section with the smallest safety factor is considered as representative of the entire gear pair. This method also allows the face load factor $K_{H\beta}$ to be determined. This factor is defined in [6] by the ratio of the peak load intensity to the mean load intensity across the width of the gear face. It accounts for the effects of non-uniform load distribution across the face width.

2.3.3 Numerical example

The method described in Sec. 2.3.2 is applied to a case-study helical beveloid gear pair present in a Bonfiglioli gearbox. Details of the gearbox and the experimental procedure used to validate the numerical example are presented in Sec. 2.4. The schematic of the

³ $c_{\gamma\beta}$ is the tooth meshing stiffness in $\text{N/mm}/\mu\text{m}$. All teeth are considered with the same $c_{\gamma\beta}$, equal to the average stiffness between the n gear pairs.

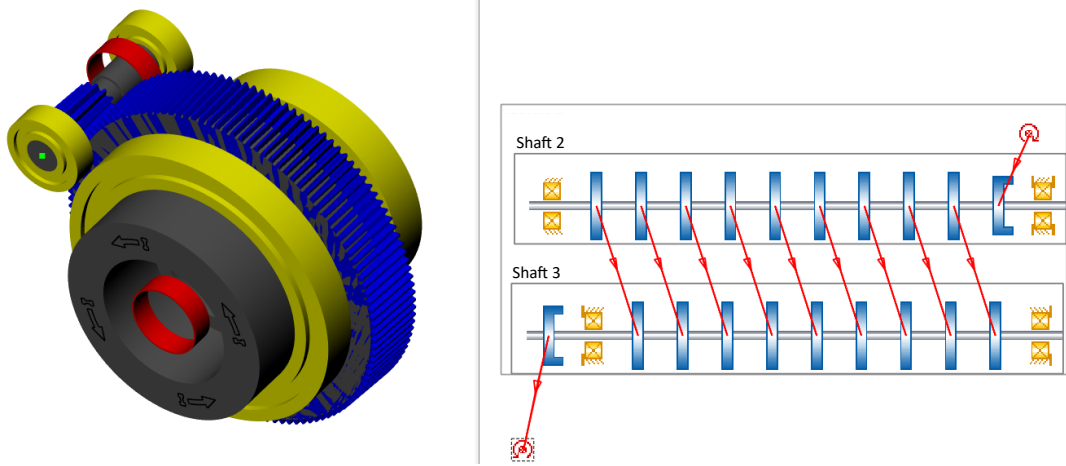


Figure 2.26: KISSsys model of the beveloid gear pair under study, modeled as nine spur gears of different correction factors (linearly varying along the tooth width) fixed on the same axis.

Symbol	Pinion	Wheel
Module m [mm]	1.25	1.25
Pressure angle α_n [°]	20	20
Helix angle β [°]	3	3
Dedendum h_{a0}	1.25	1.25
Addendum h_a	1	1
Tool tip radius mr_f [mm]	0.20	0.20
Number of teeth Z	14	120
Rack shift coefficient x_0	0.5750	0.3773
Min. shift coefficient $x(z_{min})$	0.3654	0.1678
Max. shift coefficient $x(z_{max})$	0.7846	0.5869
Outside radius r_o [mm]	21.36	153.68
Face width b [mm]	35	27
Center distance I [mm]	85	85
Cone angle δ [°]	2.5	2.5

Table 2.3: Geometrical parameters of the beveloid gear pair considered in the numerical example.

Table 2.4: Test parameters. The duration is evaluated with minimal pitting safety coefficient $S_H = 0.9$ and an efficiency $\eta = 0.81$.

	Test 1	Test 2
Planned test duration [hours]	600	100
Motor speed [rpm]	3000	3000
Input torque [Nm]	4	5
Output torque [Nm]	420	525
Power [kW]	1.26	0.85
Work cycles (gear 3)	8.8×10^6	1.47×10^6
Max Hertzian pressure [N/mm ²]	2000	2300

gearbox is proposed in Fig. 2.27 and the geometrical parameters of the beveloid gear pair is reported in Tab. 2.3.

This geometry was implemented in KISSsys (including all gears, bearings, shafts, and supports), which was used to calculate the safety factors and the deformation of the shafts (Fig. 2.26-left). The beveloid pair is split into $n = 9$ separate spur gear pairs (Fig. 2.26-right), as a compromise between accuracy and computational speed. Each section has a different geometry, to account for the variation of the shift profile coefficient (and of all related parameters) along the gear axis. A procedure was implemented in KISSsys [77] to split the power between the sections (see Fig. 2.26-right).

Two test conditions are considered (Tab. 2.4), both at 3000 rpm input speed, with output torques of 420 Nm (test 1) and 525 Nm (test 2); both these torques are close to the maximum allowable values according to the datasheet provided in Tab. 2.5 of Section 2.4. Higher output torques are avoided to limit pitting due to unrealistic loading conditions (such as excessive deflections of shafts and supports).

The pinion and gear are made of case-hardening steel. The model considers the minimum, and effective, value of face width $b = 27$ mm. For the conditions of test 1, the mesh gap obtained after 10 iterations is in Fig. 2.28a, with an error between the last two iterations below $0.01 \mu\text{m}$. The initial calculation of the shaft deflection is performed by assuming uniformly distributed torques. The calculation is repeated using an iterative procedure: the shaft deflection is recalculated using the updated torque distribution until convergence is achieved, i.e., until the deflection does not change between two successive iterations. Using the actual LD and a test duration of 600 hours, the safety factors S_H were calculated for the 9 sections in gear 1 (Fig. 2.28b); the extreme section at $z = 13.5$ is the most critical one, with a safety factor of 0.9, thus pitting was expected there during tests. The results (load distribution and $K_{H\beta}$) are coherent with those from a LTCA performed using the contact analysis module in KISSsoft, in which the beveloid gears are modeled as spur ones (due to the limitations of the software), having constant shift coefficient x (equal to the value of x at the middle section). This LTCA is used as a reference against the approach proposed in Sec. 2.3.2. The LTCA for test 1 (Fig. 2.29) gives a maximum contact pressure of 2000 MPa on the flanks of

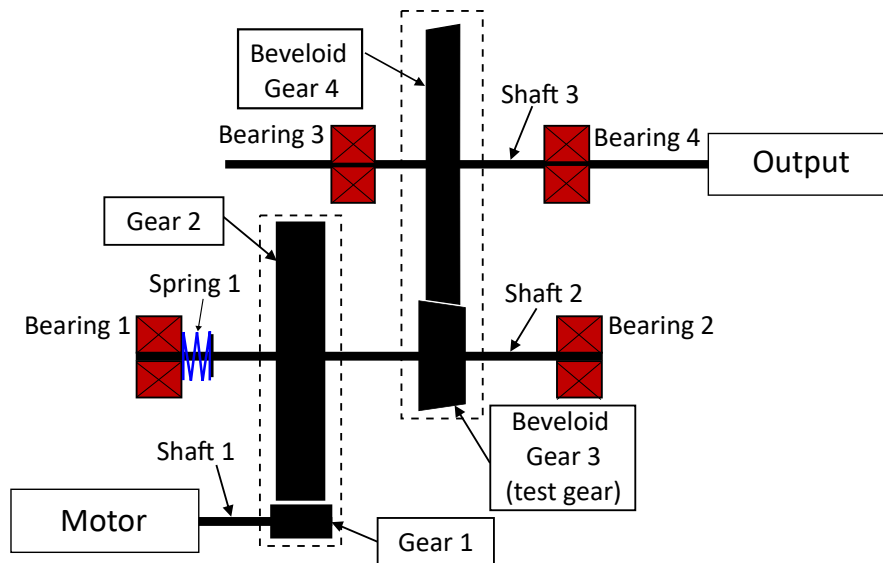


Figure 2.27: Schematic of the analyzed gearbox. Gears 1 and 2 are helical, while gears 3 and 4 are helical beveloid.

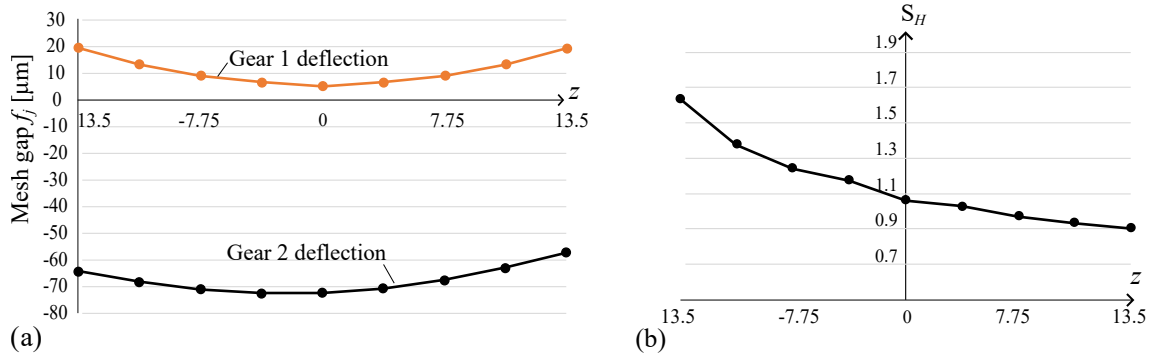


Figure 2.28: (a) final results of the mesh gap analysis with test 1 parameters; (b) S_H of gear 1 for each section.

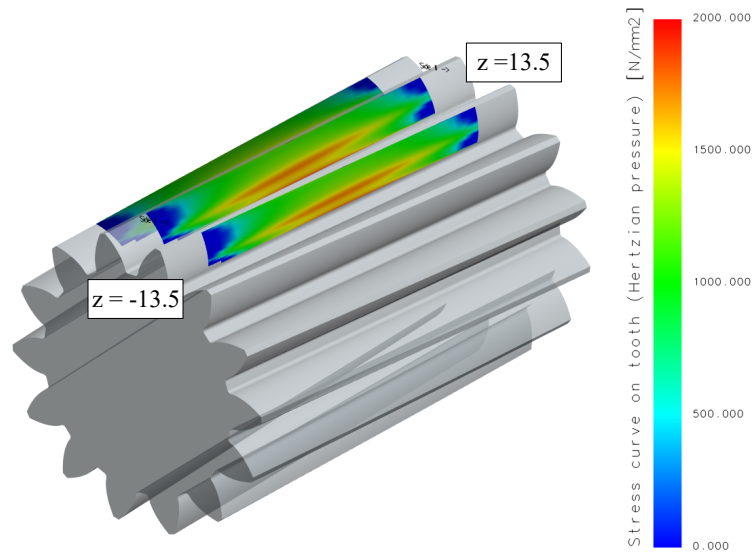


Figure 2.29: Contact pressures from the LTCA performed in KISSsoft (the beveloid gear is approximated by a spur one). The load parameters are those from test 1. Gear 3 is represented in the image.

gear 3. From industrial experience, this is a reasonable value for standard applications (where the gearbox always rotates in the same direction), and pitting can be expected within 1000 hours of testing. In Fig. 2.30, the values of $F(z)$ obtained from the proposed method and the LTCA are compared: the trends are similar, and the maximum and average values are comparable. The first (Fig. 2.30-a) provides a maximum value of 276 N/mm and an average value of 222 N/mm, thus $K_{H\beta} = \frac{276}{222} = 1.24$; while the second (Fig. 2.30-b) provides a maximum value of 263 N/mm and average value of 206 N/mm, thus $K_{H\beta} = \frac{263}{206} = 1.28$.

2.3.4 Conclusions

This section proposed a method to determine the contact-stress and pitting resistance of a beveloid gear pair inspired from the analytical approaches of ISO 6336-1 (Annex E) and ISO 6336-2 standards. A numerical example was provided for an industrial robot gearbox. The results obtained are comparable overall to those obtained through an LTCA conducted using KISSsoft, although they do not coincide perfectly. The observed

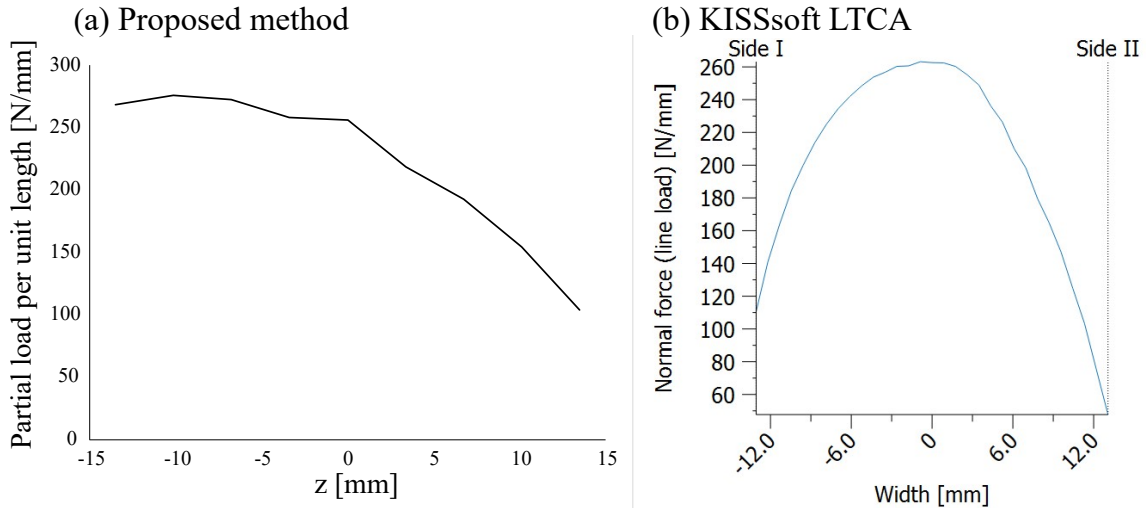


Figure 2.30: $F(z)$ from the proposed method (a) and from the KISSsoft LTCA (b).

differences can be attributed to the fact that the LTCA model implemented in KISSsoft is an approximation based on the beveloid gear mid-section. Therefore, the results provided by the proposed method are expected to be more accurate. This gearbox will also be the object of experimental tests, that will be described in Section 2.4.

2.4 Pitting detection in a parallel axes beveloid precision gearbox

The two main failure modes of gears, whether beveloid or conventional, are bending and pitting (or surface durability). Exceeding the surface durability of the meshing flanks causes particles to detach from the surfaces, leading to the formation of pits and debris. The acceptable level of pitting, in both size and quantity, varies significantly depending on the industry and application. While some fields (such as heavy industrial applications) tolerate a considerable degree of pitting, others (such as aerospace) aim to avoid any visible pitting altogether [7]. In the case of precision gearboxes, the permissible level of pitting is essentially zero, since even small amounts increase angular backlash, which is critical in this type of application, as will be explained in Sec. 3.2. For this reason, the present study focuses on pitting as the primary failure mode of a parallel-axis precision gearbox for robotic applications.

2.4.1 Gearbox description and experimental objectives

Figure 2.27 shows a schematic diagram of the internal gearbox under study. It is manufactured by Bonfiglioli Riduttori and includes a pair of parallel-axis helical gears (gears 1 and 2) and a pair of parallel-axis beveloid gears (gears 3 and 4). The specifications of the entire gearbox are reported in Table 2.5. These specifications were derived using simplified models and conservative safety factors were then applied, based on engineering experience, to ensure safe operation. However, the beveloid stage has never been optimised or tested. The gearbox is installed on the third axis of an industrial robot (the specific model cannot be disclosed for confidentiality reasons). It consists of two parallel-axis stages and has a reduction ratio $\tau = 1 : 104.42$. This gearbox was selected for back-to-back endurance tests, which are aimed at exploring the Wöhler

curve of the beveloid gear pair that it contains. These gears are made of case-hardened steel. There are six gearboxes available for testing. Spare parts are also available in case of any unexpected or accidental breakages.

The first step was to generate a model with all the gearbox components (shafts, bearings, housing, and gear pairs) using standard methods. A simplified beveloid gear pair was first developed to be studied as a classic spur gear pair, where the beveloid gear pair is modelled as a simple spur gear pair having the geometry at the middle section of the beveloid gears. According to the preliminary evaluations using the rated parameters, pitting is expected to occur on gear 1 and gear 3. Therefore, the test campaign was designed to determine the onset of pitting on the tooth flanks of gear 3. The final test parameters are established using the model described in Section 2.3.3.

Table 2.5: Technical specifications of gearbox to be tested.

Parameter	Symbol	Value	Unit
Maximum torque	$M_{\max}$	650	Nm
Acceleration torque	M_{acc}	250	Nm
Continuous torque	M_{const}	180	Nm
Maximum load cycles	$L_{\max}$	5000	-
Acceleration load cycles	L_{acc}	5×10^6	-
Continuous load cycles	L_{const}	2.2×10^7	-
Maximum speed	$n_{\max}$	30	rpm
Reduction ratio	τ	1:104.42	-
Rotational stiffness (125–250 Nm)	-	136–184	Nm/arcmin
Hysteresis	-	0.70	arcmin
Backlash	-	0.35	arcmin
Lost motion	-	0.50	arcmin
L_{10} bearings lifetime	L_{10}	32000	h
Nominal lifetime	L	40000	h
Lubrication (oil type)	-	Shell Omala 320	-
Weight	m	6.5	kg

In order to stop tests as soon as pitting is observed, the gearbox under test must be continuously monitored using the appropriate instrumentation, and it must be periodically disassembled to verify the progression of damage (see Secs. 2.4.3 and 2.4.2). This is because pitting may appear earlier or later than predicted by the model. According to [7], the condition of the flanks should be examined at least three times. The first examination should, take place after at least 10^6 cycles. Further examination should take place after a period of service depending on the results of the previous examination. One potential issue is that debris from gear 1 could cause pitting in gear 3. For this reason, gear 1 is replaced halfway through the test.

Our area of interest in the Wöhler curve is the limited-life region ($1 \times 10^6 < N_L < 5 \times 10^7$ cycles, [91]). Industrial experience and the scientific literature indicate that in this region the Wöhler curve is approximately a straight line, which thus can be defined by two points; therefore, testing a beveloid gear pair at two distinct load levels allows constructing the Wöhler curve for pitting damage. To ensure repeatability, at least three identical tests should be conducted for each point. In conclusion, six gearboxes were chosen for testing.

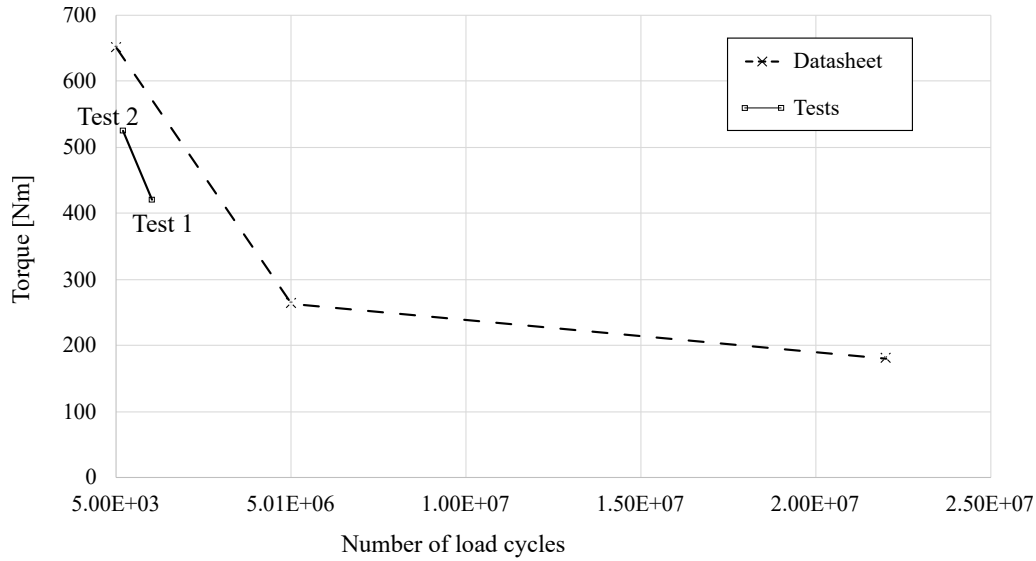


Figure 2.31: Wöhler curve from tests 1 and 2 and from the datasheet. The datasheet curve is obtained from the values in Tab. 2.5, namely $M_{\max}$, M_{peak} , and M_{const} .

Once testing has stopped, it can be difficult to determine whether pitting is present. An evaluation can be based on the general principles set out in [7]:

- Inspection of the tooth flanks to identify the presence, number, and size of pits.
- Assessment of whether the pitting is in its initial or destructive phase. In the first case, assess whether the formation of pits reduces over time (degressive pitting) or is ceasing altogether (arrested pitting); in the second case, assess if the formation is increasing over time. Teeth considered to be at elevated risk of damage should be marked and monitored.
- Evaluation of the amount of worn debris in the lubricant.
- The tooth flank may occasionally exhibit craters resulting from various causes, such as lubricant contamination by debris released from the bearing raceways or other gear pairs. Pitting craters (2 in Fig. 2.35,) are generated by stresses beneath the surface, whereas bigger craters (1 in Fig. 2.35) are mainly caused by the compressive force of debris striking the surface.

2.4.2 Test rig

The test rig is shown in Fig. 2.32. It is composed of an input motor, a test gearbox, a torque sensor, an output gearbox (used as a multiplier), and an output motor (used as a brake). Accelerometers and temperature probes are installed on the gearbox housing to detect possible failures. In particular, the accelerometers are located in the housing, close to the bearings of shaft 2 (where the gear 3 under test is mounted).

The input motor used for the test is the BMD_102_9.6_100_19 from Bonfiglioli Riduttori with keyway on the output shaft. This motor was selected because it is more compact compared to an asynchronous motor of similar power, and it has the same center distance between the holes on the gearbox housing flange. This motor was deemed the most suitable also since torque and speed must be varied between different tests.

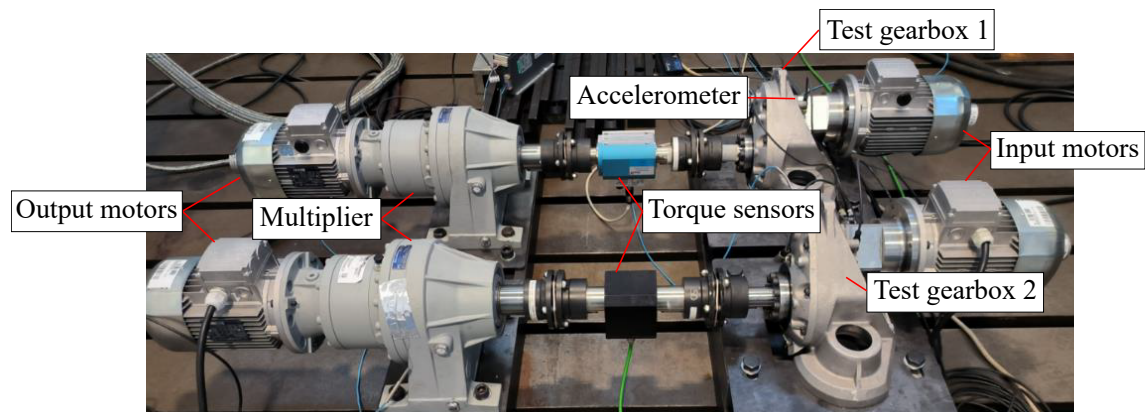


Figure 2.32: Test rig.

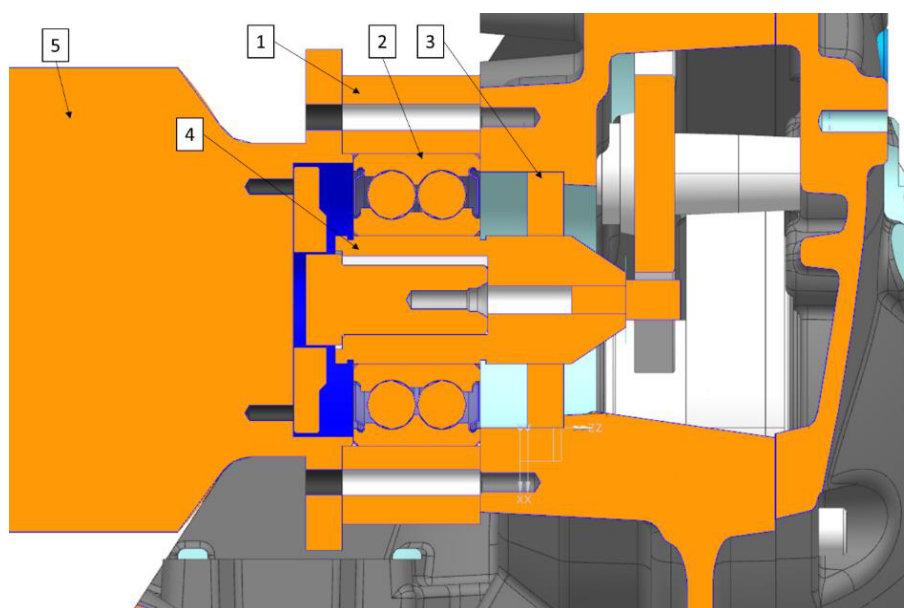


Figure 2.33: Study developed to connect the motor to the gearbox from the input side.

Referring to Fig. 2.33, a study was conducted on the equipment required to connect the motor (element 5) to the gearbox input:

1. Mounting flange: designed in two parts, since in its original form it could not be mounted due to bolt length.
2. SKF angular contact ball bearing 3307 A-2Z.
3. Sealing ring 35x70x10 with double lip for dust and fluid protection.
4. Coupling joint: it has been designed with two grooves for circlips. Motion transmission is provided by a key between the motor shaft and the joint, and by interference fit between the joint and the pinion.

A standard torque sensor with maximum torque of 1500 Nm connects the gearbox under test and the multiplier, for which a Bonfiglioli gearbox (series 300) is used. This is a robust planetary drive train, used as multiplier in a back-to-back configuration due to its back-drivability. The reduction ratio (1:105) was selected to be very close to the one of the gearbox under test. A standard asynchronous motor (BN 90 L 2) is adopted

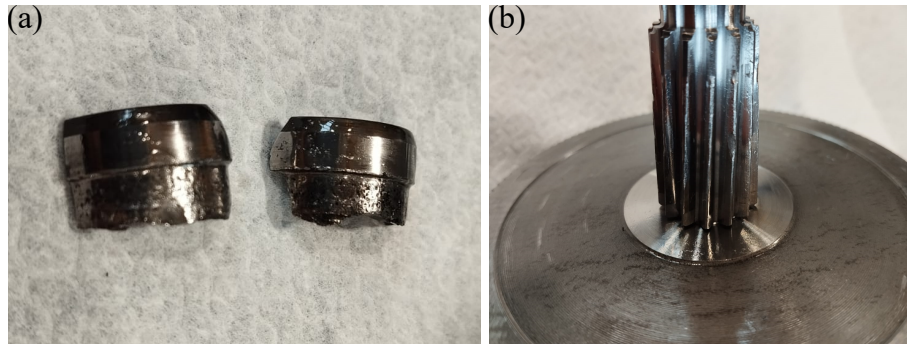


Figure 2.34: (a) The raceways of bearing 2 in gearbox A failed due to incorrect axial positioning of shaft 2. (b) Beveloid gear 3 subsequently failed due to operating without support and with contaminated lubricant.

as output: this motor is very flexible and can be used for a wide range of industrial applications.

2.4.3 Experimental validation

The test rig is shown in Fig. 2.32. Firstly, the parameters from test 1 (Tab. 2.4) are adopted. As the method discussed in Sez. 2.3 has not yet been fully validated, the tests should stop as soon as pitting occurs on the tested gear, rather than at the target operating hours. This may happen before or after the planned testing time. Therefore, in addition to periodic measurements of vibration and temperature, inspections should be intensified around the expected operating hours. Disassembly of the entire gearbox are planned at 60%, 80%, and 100% of the planned test duration. If pitting or micro-pitting is detected during an inspection, the frequency of these scheduled inspections must increase. An inspection port is used for preliminary inspections with an endoscope camera if pitting is suspected. Sampling of the lubricant for laboratory analysis, to evaluate possible contamination, are also performed. All these checks are necessary to properly assess the early onset of surface pitting.

Each test is carried out at constant speed (continuous operation, S1). The operating temperature must be checked, and forced cooling by fan should be considered if necessary.

Before performing each test, a running-in period of 8–12 hours must be carried out at an input speed of 1000 rpm and an output torque of 150 Nm. During this phase, it is important to evaluate the regular operation of the gearbox and of the new test bench by measuring temperatures, torques, and vibrations; the efficiency of the system is also measured.

The lubricant specified for the tests is Shell Omala 320. For a single test, it is not necessary to change the lubricant. The oil quantity must be sufficient to reach all meshing areas and rolling elements. If the gearbox overheats and forced cooling is ineffective, the oil level must be reduced.

2.4.4 Results

Due to unexpected problems that arose during the tests, it was not possible to test all six gearboxes. Only partial results are available for the two gearboxes tested under Test 1 conditions; these are referred to as gearbox A and gearbox B in this paragraph. The

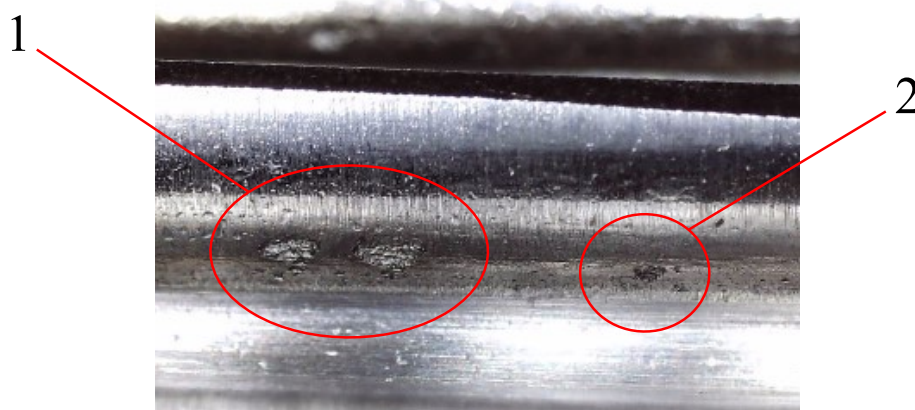


Figure 2.35: The image shows craters along the tooth flank of gear 3 (gearbox B) after completing around 500 hours of testing under test 1 parameters at Bonfiglioli Riduttori. The first two craters (1) are larger and are caused by debris impacting the flank, while the other ones (2) are smaller and are caused by pitting.

assessment of pitting was affected by uncertainties due to the unexpected failure of bearing 2. Additionally, as anticipated, gear 1 released debris into the lubricant, complicating the evaluation of pitting.

The failure of bearing 2 was caused by the axial displacement of shaft 2, which is required to adjust the backlash of the beveloid gear pair. As shown in Fig. 2.27, spring 1 is used to axially preload shaft 2 and continuously reduce the backlash. Since the tests were designed to run always in the same rotational direction and not in alternating conditions (which is the original design mode of the gearbox), and because the applied torques were higher than the rated values, the spring was not able to win the axial forces generated by the beveloid gear pair. As a result, shaft 2 continuously operated shifted to one side (for instance, towards the left side according to Fig. 2.27). This caused bearing 2 to operate only on part of its raceway.

Figure 2.34a shows the raceways of bearing 2 of gearbox A, where it can be observed that only a portion of the raceway was subjected to load (the worn area), which ultimately led to its fracture into two parts. Unfortunately, this failure also caused damage to gear 3 (Fig. 2.34b) earlier than expected, after approximately 400 hours of testing.

Although Gearbox B did not fail at the same time (400 hours), it exhibited very similar behaviour and was close to failure. There were still few craters visible on the tooth flank of gear 3, but no pitting. Once the main issue was identified, the bearing was replaced and shaft 2 was axially fixed. A second inspection of the beveloid gear pair of gearbox B (Fig. 2.35) was carried out after 500 hours of testing. Both craters caused by circulating debris and pitting damage were observed. Although these results are not fully reliable due to the failures that occurred, they still show a good agreement with the model predictions, considering that the target test duration was 600 hours.

Due to time constraints, no further results are reported in this thesis, although testing is still ongoing at Bonfiglioli Riduttori.

Chapter 3

Gearboxes for industrial robots

Industrial robots are widely adopted in manufacturing due to their precision, speed, and ability to perform repetitive tasks. Their mechanical architecture must ensure high reliability under dynamic loads, and the transmission system plays a central role in this context. This chapter focuses on the gearboxes used in industrial robots, paying particular attention to their design.

The chapter is organized as follows: Section 3.1 introduces the general properties of industrial robots, highlighting their evolution, operating principles, and typical applications. Section 3.2 discusses the performance requirements for gearboxes in robotic systems and presents the most commonly used architectures in this field. Section 3.3 presents the Wolfrom architecture, providing information on kinematic constraints (Sec. 3.3.1), transmission ratios (Sec. 3.3.2) and efficiency (Sec. 3.3.3). Finally, Section 3.4 describes the procedure for building a virtual prototype of a serial robot, enabling the estimation of joint dynamics and the corresponding torque.

3.1 Introduction to industrial robots

Industrial robots are automated, reprogrammable manipulators used in manufacturing and production environments. Differently from collaborative robots, these robots typically operate inside dedicated and fenced cells, to ensure safety and isolate robot motion from human operators. Their tasks can include welding, painting, assembly, palletizing, pick-and-place, inspection, and similar. In general, a robot of this kind is a manipulator designed to move materials, parts, tools or specialised devices through variable, programmed motions to perform a variety of tasks. It can also acquire information from the environment and move intelligently in response [92].

The first industrial robots appeared in the 1960s, such as the PUMA from General Motors or the SCARA from Sankyo Seiki. In 1996, KUKA introduced a PC-based robot control system that revolutionized robot control. Modern industrial robots offer exceptional precision, speed, and payload capacity. Leading manufacturers include ABB, KUKA, FANUC and Yaskawa, each driving innovation in automation across industries.

Because these systems often operate under heavy loads, transmitting significant forces, it is essential to ensure their safety, control, and reliability. Industrial robots usually have three or more axes¹. They are very stiff and typically bulky to ensure precision, yet they need to be fast to reduce duty cycle time. Industrial robots are now a

¹A robot axis is a joint with a motor, linear or rotary, controlling an independent Degree-of-Freedom.

mature technology. Collaborative robots, on the other hand, are much younger but are growing rapidly. Thanks to their many advantages, robots can ensure consistent product quality and speed up operations while keeping workers away from dirty, dangerous or demanding tasks.

3.2 Standard gearboxes

Robots require robust and reliable components that can withstand normal operating loads and safety-critical events, such as sudden emergency stops and the resulting inertial forces. The role of the transmission is crucial, and the following aspects are important during its design:

- High torsional and tilting stiffness: to avoid oscillations which can reduce precision and accuracy of positioning.
- Compactness and low moment of inertia: in order to reduce the kinetic energy in moving parts; this helps safety (less energy in crashes), performance (faster accelerations/decelerations) and precision positioning.
- Minimal backlash: it increases precision positioning, especially when changing direction or stopping.
- Reliability under continuous operation: cells often undergo many cycles per hour. The transmission must therefore support numerous start/stop cycles and emergency stops.
- Small transmission ratio and compact overall dimensions contribute to achieving a high torque density.

Robot gearboxes often have special designs with respect to conventional ones. In this Section, we compare the most widespread architectures [1–3], namely *Harmonic Gearboxes* (HG), *Cycloid Gearboxes* (CG), *Simple Planetary Gearboxes* (SPG) and *Compound Planetary Gearboxes* (CPG) (Fig. 3.1), with the Omni gearbox being a particular type of CPG, since it has a *Wolfrom* architecture. Recent reviews [3, 93] show that CGs are used in more than 75% of industrial-robot joints, mostly in the joints closer to the base. HGs account for approximately 20% of robot joints, mainly those closer to the end-effector. SPGs are employed in only 5-7% of the cases. The main design parameters shown in Tab. 3.1 were obtained from the datasheets of leading companies in each category: “Harmonic Drive SE” for HGs [94], “Nabtesco” and “Sumitomo” for CGs [95, 96], Wittenstein for SPGs (RP+ series) [97] and “PSC Schaeffler” for CPGs [72].

HGs, also known as “strain wave gearboxes”, consist of a fixed ring gear (called the *circular spline*), an *elliptical wave generator* (a cam mounted on the central input shaft), and a flexible external gear (the *flex spline*), placed between the wave generator and the circular spline (Fig. 3.1a). The flex spline is deformed by the wave generator and engages with the circular spline. HGs are commonly employed in a single-stage configuration, achieving transmission ratios up to 1:160. Meshing occurs in two zones that are 180 degree apart, and there are multiple teeth in contact at any time. Thus, HGs

Parameter	HG	CG	SPG	CPG	Omni
Min. τ	1:160	1:120	1:12 ^a	1:200	1:160
η [%]	60-80	80-90	95	90	70
Max. φ_b [arcmin]	0	1	5	0.1	0.5
Max. φ_{lm} [arcmin]	1	1	5	0.6	0.6
Stiffness	Low	Excellent	Medium	High	High
Max. speed	Medium	Low	High	High	High
Compactness	Excellent	High	Low	High	High
Weight	Low	High	High	Medium	Medium
Maintenance	High	High	Low	Medium	Medium
Price	High	High	Low	High	Medium
Back-drivability ^b	Medium	Medium	High	High	Medium
Torque capacity	Medium	Excellent	Low	High	High

^a This value holds for a single reduction stage with three planets [45]. ^b Back-drivability is usually evaluated by measuring the torque at the output axis.

Table 3.1: Comparison of common gearbox designs for robotics, with their performance levels (low, medium, high and excellent) for the most relevant practical requirements. τ is the transmission ratio, η is the forward-motion efficiency, φ_b is the circumferential backlash and φ_{lm} is the lost motion (see Sec. 4.4.2.1 for its definition and measure).

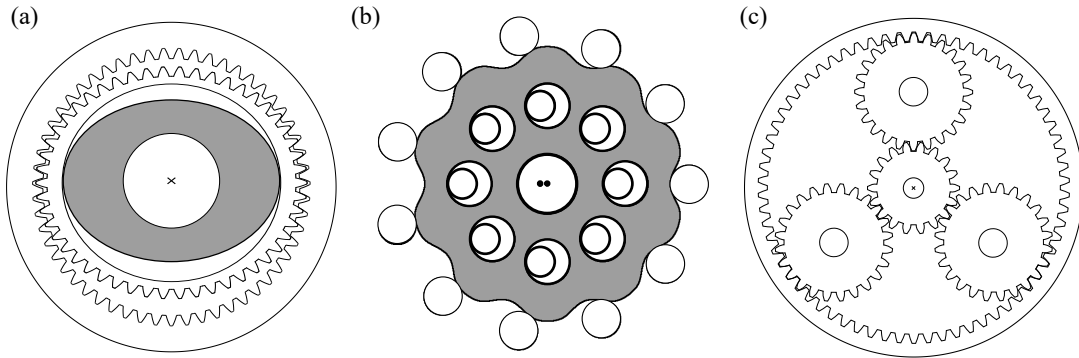


Figure 3.1: Architectures for precision gearboxes: (a) harmonic, (b) cycloid and (c) simple planetary.

provide a reasonable torque capacity (not as high as CGs or CPGs), but in a very compact space. Backlash is usually under 1 arcmin. HGs have medium efficiency and back-drivability. When subjected to large impulse forces (e.g., rapid accelerations, common in industrial robots), there is a significant risk of damaging critical components. For this reason, many HGs are equipped with brakes or clutches for protection against impulsive loads.

CGs have an eccentric input shaft that moves a cycloidal disc (cam) with lobes and holes (Fig. 3.1b). Fixed ring pins are arranged in a circular pattern around the cycloidal disc and engage with its lobes. An output shaft (pin disc) has roller pins also arranged in a circular pattern around its axis, engaging with the holes in the cycloidal disc. The fixed pins exceed by one the number of cam lobes. The main advantage of CGs is that they can handle high torque requirements and impact loads: CGs can withstand acceleration torques up to three times their rated torque, as well as emergency deceleration torques up to five times their rated torque. They are usually used in multiple-stage designs, in combination with planetary gears (like Nabtesco's RV [95] and the Sumitomo's Fine-Cyclo UA [96]), thus reaching values of τ up to 1:200. Although CGs have larger backlash and are less compact than HGs, their torsional stiffness is the highest among the designs considered here. CGs are commonly employed in first, second and third joints of a robot arm.

The most typical SPG architecture has a central sun, a ring, and multiple planets mounted on a carrier that allows planets to freely rotate about their own axes (Fig. 3.1c). Planets mesh with both the sun gear and the ring gear. Different configurations exist, with the most common one having a fixed ring gear, the sun gear acting as the input, and the carrier being the output. Typically, a single stage can achieve transmission ratios up to 1:12. Smaller ratios can be obtained by coupling multiple planetary stages (up to 5) in series, although this negatively affects the overall size and weight.

The field of CPGs is vast, with many known architectures [98]. The Schaeffler's design [72] can achieve τ 's up to 1:200 with efficiencies up to 90%. An alternative concept, called *reduced* CPGs by Arnaudov [45], consists of two SPGs with the carriers merged into one. These configurations can achieve very small τ 's in a compact form: one example is the Wolfrom architecture (discussed in detail in Sec. 3.3). While CPGs are currently less common [3] than HGs, CGs and SPGs, they have features of great interest for robotics:

- their versatile configurations enable reaching smaller transmission ratios in compact designs and very good torque density;
- lower starting torques lead to higher η when operating at partial loads.
- high reverse-motion efficiencies, up to 85.3% [47], by optimising the shift profile coefficient and the number of teeth.

HGs and CGs also have important drawbacks, such as high costs, significant maintenance requirements, non-linear behaviour of the hysteresis curve, and torque ripple. The latter issue is due, for CGs, to the high sensitivity of this architecture to manufacturing errors [2] and, for HGs, to the deformation of components, which also cause a periodic output speed variation [99]. Their starting (no-load) torques are about ten times larger than those of CPGs, which are typically below 2% of the rated input torque. This is a significant advantage of CPGs in robotics, especially for frequent operations at partial loads [3]. CPGs can achieve a backlash lower than 1 arcmin, which is slightly worse than HGs, which have the lowest backlash thanks to flexible components (which, however, decrease stiffness). The Schaeffler's CPG shares many aspects with the Omni gearbox. The main difference is in that Omni uses beveloid gears coupled with springs, which ensure a constant and low backlash throughout the lifespan of the gearbox, even after component wear. This means that manufacturing precision can be lower than for Schaeffler CPG, making Omni a cost-effective solution.

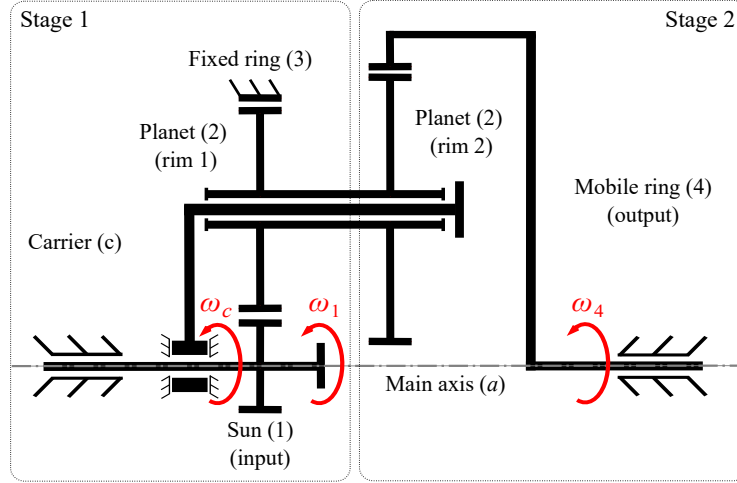


Figure 3.2: Schematic of the typical Wolfrom architecture. From here onwards, all gearbox schematics are shown by their top half-sections, as they are axially symmetrical.

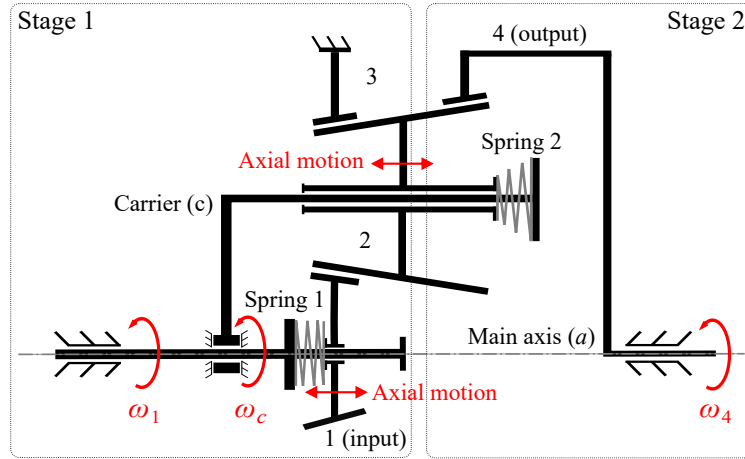


Figure 3.3: Schematic of the Wolfrom architecture for the Omni OS gearbox.

3.3 Wolfrom architecture

In this Section, we revise the features of the Wolfrom architecture, which was adopted for the Omni gearbox. The conventional Wolfrom architecture comprises two-rim planets (Fig. 3.2): each planet (2) is made of two rims with different numbers of teeth, respectively meshing with a fixed ring (3) and a mobile ring (4), which have different numbers of teeth too. The input link is the sun gear (1), which meshes with the planets on the carrier, while the output link is the mobile ring (4). As opposed to typical planetary gearboxes, the size of the Wolfrom gearbox mainly depends on the output torque and is not significantly influenced by the gear ratio. This leads to a better torque density, resulting in a corresponding decrease in the weight and size of the electrical motor [53]. A more advantageous variant of the Wolfrom architecture contains one-rim planets [45]: each planet is composed of a single rim with same number of teeth along the tooth width. The contact can be ensured by selecting a proper shift profile coefficient in each meshing or by using beveloid gears as for the Omni gearbox (Figs. 3.3 and 3.4).

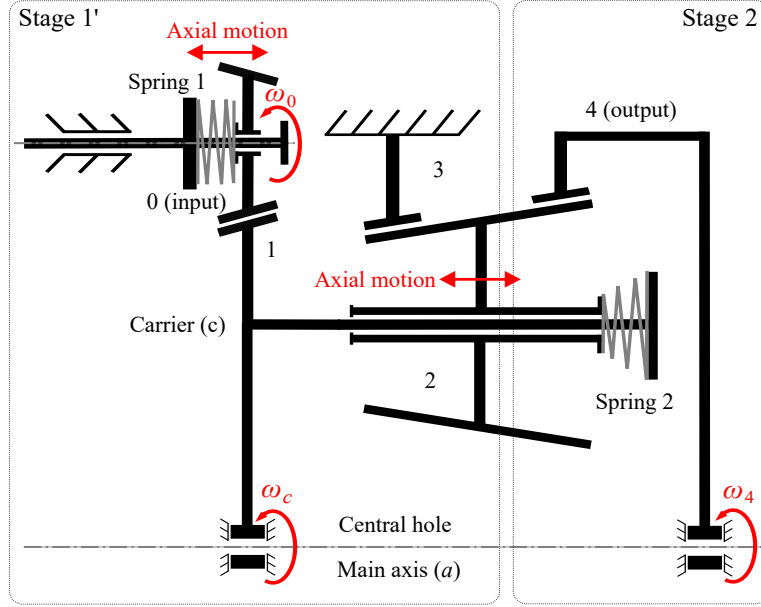


Figure 3.4: Schematic of the Wolfrom architecture for the Omni OH gearbox.

3.3.1 Kinematic constraints

Kinematic constraints emerge from the application of the Willis formula [45]. The number of planets is n_p , $\alpha_p = 2\pi/n_p$ is the angle (measured around the main axis a) between two consecutive planets, and ω_c is the carrier speed. For the i -th gear, ω_i is the speed, Z_i is the number of teeth, k_i is an integer, and $\alpha_i = k_i 2\pi/Z_i$ is the rotation of the gear corresponding to a carrier rotation equal to α_p ². Considering ring 3 as fixed ($\omega_3 = 0$), the basic transmission ratio τ_{10} of the first stage is (Fig. 3.2):

$$\tau_{10} = \frac{-\omega_c}{\omega_1 - \omega_c} = \frac{-\alpha_p}{\alpha_1 - \alpha_p} = \frac{-\frac{2\pi}{n_p}}{\frac{2\pi}{Z_1} k_1 - \frac{2\pi}{n_p}} = -\frac{Z_1}{Z_3} \quad (3.1)$$

which yields:

$$k_1 = \frac{Z_3 + Z_1}{n_p} \in \mathbb{N} \quad (3.2)$$

For the second stage, the Willis formula gives:

$$\tau_{20} = \frac{\omega_4 - \omega_c}{\omega_1 - \omega_c} = \frac{\alpha_4 - \alpha_p}{\alpha_1 - \alpha_p} = \frac{\frac{2\pi}{Z_4} k_4 - \frac{2\pi}{n_p}}{\frac{2\pi}{Z_1} k_1 - \frac{2\pi}{n_p}} = -\frac{Z_1}{Z_4} \quad (3.3)$$

which provides:

$$k_1 + k_4 = \frac{Z_1 + Z_4}{n_p} \quad (3.4)$$

Substituting Eq. (3.2) into Eq. (3.4) provides a condition for k_4 :

$$k_4 = \frac{Z_4 - Z_3}{n_p} \in \mathbb{N} \quad (3.5)$$

²When the carrier rotates by α_p , the i -th gear must rotate k_i times the gear angular pitch to ensure that all gears return to the same meshing condition, with no interference.

3.3.2 Transmission ratio

Considering the schematic of Fig. 3.3, the transmission ratio τ_{OS} of the Omni OS gearbox is the same as the one of a Wolfrom gearbox, and it can be obtained by multiplying the transmission ratios τ_1 and τ_2 of the two stages. Indeed, Eqs. (3.1) and (3.3) can be rewritten considering that $\tau_1 = \omega_c/\omega_1$ and $\tau_2 = \omega_4/\omega_c$:

$$\tau_{10} = \frac{-\omega_c}{\omega_1 - \omega_c} = \frac{1}{1 - \frac{1}{\tau_1}} = -\frac{Z_1}{Z_3} \quad (3.6a)$$

$$\tau_{20} = \frac{\omega_4 - \omega_c}{\omega_1 - \omega_c} = -(\tau_2 - 1) \frac{1}{(1 - \frac{1}{\tau_1})} = (\tau_2 - 1) \frac{Z_1}{Z_3} = -\frac{Z_1}{Z_4} \quad (3.6b)$$

Manipulating the last two terms at the right-hand side of each equation leads to:

$$\tau_1 = \frac{1}{1 + \frac{Z_3}{Z_1}}, \quad \tau_2 = 1 - \frac{Z_3}{Z_4} \quad (3.7)$$

and thus:

$$\tau_{OS} = \frac{\omega_4}{\omega_1} = \tau_1 \tau_2 = \left(\frac{1}{1 + \frac{Z_3}{Z_1}} \right) \left(1 - \frac{Z_3}{Z_4} \right) = \frac{Z_1 (Z_4 - Z_3)}{Z_4 (Z_1 + Z_3)} = \frac{Z_1}{Z_4} \frac{k_4}{k_1} \quad (3.8)$$

where, in the last term at the right-hand side, we substituted $Z_1 + Z_3 = k_1 n_p$ from Eq. (3.2) and $Z_4 - Z_3 = k_4 n_p$ from Eq. (3.5). In the Omni gearbox, to minimise τ_{OS} , k_4 is set equal to 1 by choosing $Z_4 - Z_3 = n_p$. Since Z_1 is much smaller than Z_4 and k_1 is on the order of several tens, τ_{OS} can easily be smaller than 1:100.

In the OH version (Fig. 3.4), in order to provide space for cable routing along the main axis, the sun gear is replaced by two helical beveloid gears (0 and 1) with parallel axes, having transmission ratio $\tau'_1 = Z_0/Z_1$, and gear 1 is fixed to the carrier. Thus, the transmission ratio τ_{OH} becomes:

$$\tau_{OH} = \tau'_1 \left(1 - \frac{Z_3}{Z_4} \right) \quad (3.9)$$

With this design, we can easily reach transmission ratios similar to those of the Omni OS architecture.

3.3.3 Efficiency

The efficiency η of a Wolfrom gearbox is typically around $0.7 \div 0.8$. In general, η depends on the friction coefficient μ in teeth meshing, and in turn, μ depends on the specific tooth geometry (especially on the profile shift x and the contact ratio ϵ). Thus, the manufacturing quality of the tooth surfaces has a strong influence on efficiency. According to [45], due to the intrinsic architecture of the Wolfrom mechanism, the meshing losses between the external gears 1 and 2 are much higher than the losses in the meshing with ring gears.

3.4 Virtual robot modelling

This section shows how to create a virtual prototype of a serial industrial robot. The model allows us to estimate the actual joint speeds and accelerations and to determine the corresponding torques, which are the key parameter of interest in this study.

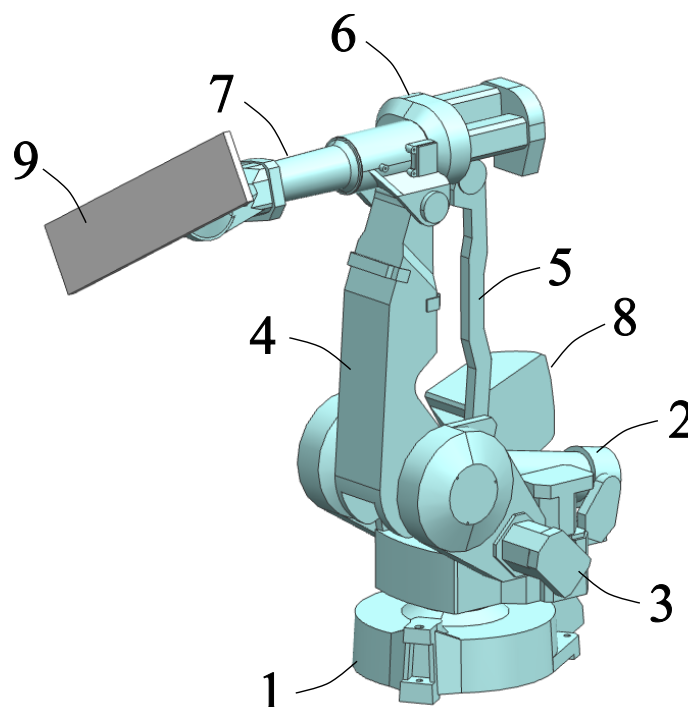


Figure 3.5: 3D rendering of the ABB IRB4400 robot created from STEP files provided by the manufacturer, used for virtual prototyping and torque analysis. The lists of components is provided in Tab. 3.2.

Component number	Name	Weight [kg]
1	Base	350
2	Piston	50
3	Link 1	230
4	Link 2	65
5	Link Rod	18
6	Link 3	135
7	End Effector	83.5
8	Counterweight	100
9	Payload	58

Table 3.2: List of the IRB4400 components used to create the virtual prototype and their respective weights. The ABB catalogue does not provide the weight of all members, so for some of them the weight has been estimated from the STEP files. Moments of inertia and centers of gravity were also estimated using STEP files.

A robot manufacturer was contacted to ask which robot model and axis would be best for mounting the Omni OH size 060 gearbox. They proposed the first axis of a robot with payload capacity of 60 kg and a reach of 1.95 meters.

An external robot provider was consulted to verify the availability of the same robot model for testing under specific configurations (motion angle, payload, and reach). An experimental campaign was designed with the aim of obtaining the speed and acceleration profiles of the first axis under known duty cycles and payload conditions. Several tests were carried out, to represent different operating conditions of the robot.

The controller of the industrial robot does not accept speed and acceleration inputs

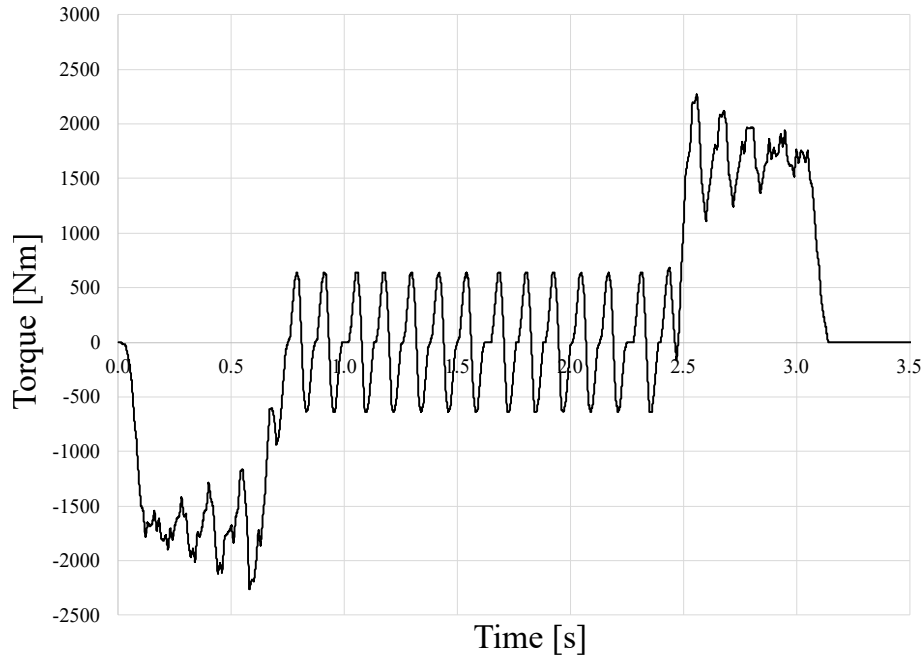


Figure 3.6: Torque curve of axis 1 of the IRB4400 robot, obtained from the multibody simulation using the measured acceleration input.

in SI units (rad/s and rad/s^2); instead, it uses dimensionless parameters ranging from 1 (minimum) to 10 (maximum). However, the controller allows recording the speed and acceleration over time. This information proved useful for determining both the maximum values and the corresponding motion profiles.

These limitations exist because industrial robots may be used by inexperienced users. The manufacturer ensures safe operation by restricting input parameters and limiting maximum speed and acceleration. However, as a gearbox manufacturer, Bonfiglioli needs to know the actual acceleration and speed limits imposed by the robot controller. These values are not available in the catalogue, as they represent proprietary information of the robot manufacturer.

The most critical conditions occur when all parameters are set to 10. We then selected a payload of 58 kg and positioned the robot at its maximum reach for the corresponding load³, and measured the speed and acceleration of axis 1. This measurement is later used to define a duty cycle in the multibody software.

The next step was to determine the torque. To do so, it was necessary to account for the inertia and weight of the robot links. This was done using catalogue data and STEP files obtained from the robot manufacturer official website, and are summarized in Table 3.2. A multibody model of the robot was then created, as shown in Fig. 3.5. The multibody model was developed by replicating the kinematic constraints of the real system. Component 1 is rigidly fixed to the frame, while Component 3 is connected to Component 1 through a revolute joint. Component 8 is rigidly attached to Component 3. Components 4 and 5 are connected to Components 3 and 6 via revolute joints, forming a parallelogram linkage. Finally, Component 6 is connected to Component 7 through a revolute joint. The payload is rigidly fixed to the end effector (EE). By applying the measured profiles of acceleration and position to axis 1, and fixing the robot with the same reach of the actual tests, the torque curve over time can be obtained

³The maximum payload of a robot varies depending on its reach. As the arm extends, the maximum allowable payload decreases. Further details can be found in the robot manufacturer's catalogues.

(Fig. 3.6).

The torque is proportional to the acceleration and exhibits a trapezoidal pattern. The acceleration/deceleration torque oscillates around ± 1680 Nm, a value very close to the acceleration torque of the Omni OH 060, the gearbox we intended to apply.

Chapter 4

Omni

In this chapter, after recalling the general properties and architectural features of the Omni gearbox (Sec. 4.1), the kinetostatic analysis (Sec. 4.2) of both the Omni OS and OH is presented. The objective is to provide a theoretical framework for the preliminary sizing of key components such as gears, springs, and bearings, and to establish the basis for the subsequent design considerations (Sec. 4.3) and experimental validation (Sec. 4.4).

4.1 General properties

As seen in Section 3, the Wolfrom architecture offers features useful for robotic gearboxes, such as compactness and very low transmission ratio. The Omni gearbox employs this architecture and comes in two variants: the Omni OH, which features a central hole for cable routing, and the Omni OS, which does not. The Omni gearbox is designed in ten sizes (denoted as 010, 020 and so on) and four different transmission ratios (1:80, 1:100, 1:125 and 1:160). Section 4.4 reports experimental results from tests on prototypes of the Omni OH gearbox, with size 060 and $\tau = 1:125$. Table 4.1 provides some of the rated parameters of the tested gearbox.

The meshing backlash, which is critical for robot gearboxes, is reduced by using parallel axis beveloid gears preloaded through axial springs. As seen in Figs. 3.3 and 3.4 of Section 3.3, spring 2 pushes the planets axially against the external ring gears, while spring 1 (which is only present with the Omni OS) pushes the input gear against the planets. The use of springs, which continuously minimise meshing backlash, ensures consistent performance throughout the gearbox life. Springs 1 and 2 have different types (cylindrical and conical, respectively) for space reasons. During assembly, additional adjustments are made to further minimise backlash in both gear meshings. Gears have a symmetrical involute tooth profile, similar to the one from ISO 53 [100], with manufacturing pressure angle equal to $\alpha_n = 20^\circ$. All beveloid gears have helix angle $\beta_0 = 0$, except the pinion of the OH version and the corresponding gear on the carrier, for which $\beta_0 \neq 0$. To enhance efficiency, tooth surfaces are machined with high precision, and micro-geometric modifications are applied to the tooth surface of planets to ensure an even distribution of the load along the tooth width¹. Micro-geometric modifications can also be applied to rings for superior performance; however, due to their internal teeth, their production and hobbing processes would appreciably in-

¹Details are omitted as they constitute confidential information of Bonfiglioli.

T_N	T_A	T_E	M_b	φ_{lm}	W	L_h	$\dot{\theta}_{out}$
[Nm]	[Nm]	[Nm]	[Nm]	[arcmin]	[kg]	[h]	[rpm]
1150	1610	3830	2410	<0.6	26.3	12000	15

Table 4.1: Technical parameters for OS and OH gearboxes of size 060 and $\tau = 1:125$. T_N is the rated torque (used for sizing) corresponding to the rated service life L_h and the rated output speed $\dot{\theta}_{out}$; T_A is the maximum allowable torque during accelerations and decelerations; T_E is the maximum allowable torque for emergency stops or external impacts; M_b is the maximum allowable bending moment. Finally, φ_{lm} is the lost motion, and W is the gearbox weight excluding lubricant.

crease manufacturing costs. By using a custom-designed main bearing between the fixed frame and the mobile ring, the gearbox is capable of withstanding large forces and bending torques in all directions. The rolling elements are preloaded to achieve high bending stiffness.

4.2 Kinetostatic analysis

4.2.1 Omni OS

Figure 4.1 presents a schematic representation of the forces and torques acting on Omni OS gears during steady-state operation at constant speed. Although this is not a common service in robotic applications, it serves as a basis for a first analysis and sizing. All beveloid gears have the same cone angle δ and parallel axes. The sun and the planet axes are mounted so that their axes z_1 and z_2 point in opposite directions. In the following figures, for clarity, δ is exaggerated, and the gears are represented in a simplified way; this also applies to all bearings, including the main one and those supporting the carrier. The resultant contact force between each pair of meshing gears is assumed to be transmitted at a single contact point located at the pitch working radius, to simplify the analysis.

Each planet meshes simultaneously with three other gears (namely, the sun and the two rings), resulting in three distinct pitch working radii, due to different profile shift coefficients at each section, as derived from the beveloid geometry. The pitch working radius of planet 2 when meshing with gear j is $r_{2,j}$, for $j \in \{1, 3, 4\}$. Each one of the remaining gears has a single-pitch working radius r_i . These radii can be determined based on the number of teeth Z_i 's and the center distance I (Fig. 4.1), namely:

$$\begin{cases} \frac{r_{2,1}}{r_1} = \frac{Z_2}{Z_1} \\ r_{2,1} + r_1 = I \end{cases} \Rightarrow \begin{cases} r_1 = \frac{Z_1}{Z_1 + Z_2} I \\ r_{2,1} = \frac{Z_2}{Z_1 + Z_2} I \end{cases} \quad (4.1)$$

and

$$\begin{cases} \frac{r_{2,3}}{r_3} = \frac{Z_2}{Z_3} \\ r_3 - r_{2,3} = I \end{cases} \Rightarrow \begin{cases} r_3 = \frac{Z_3}{Z_3 - Z_2} I \\ r_{2,3} = \frac{Z_2}{Z_3 - Z_2} I \end{cases} \quad (4.2)$$

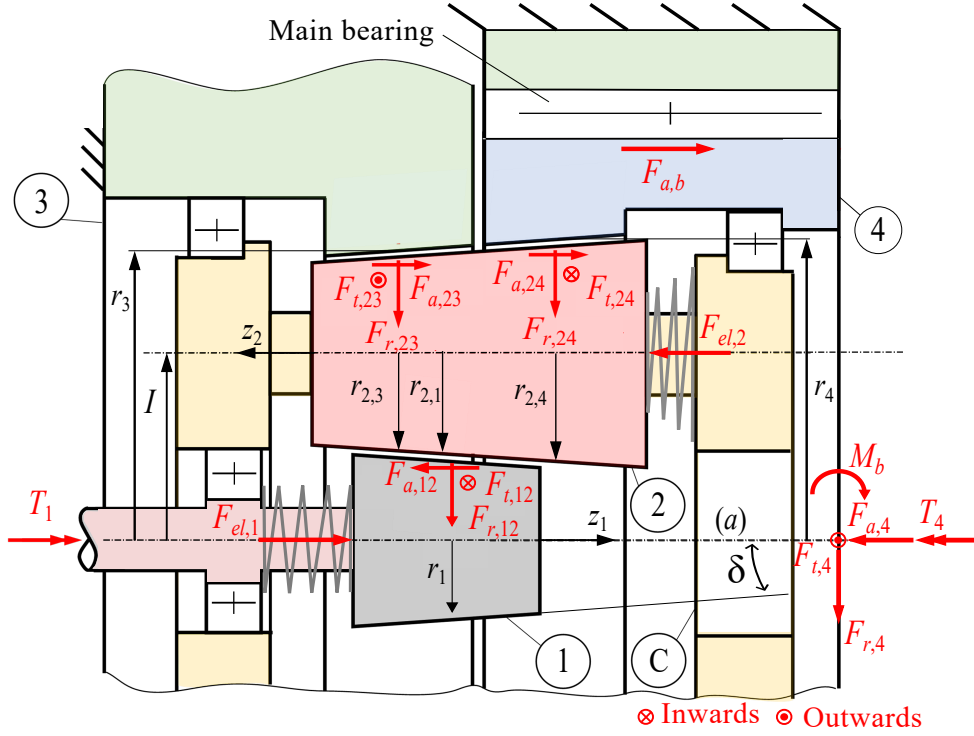


Figure 4.1: Simplified section view of the Omni OS gearbox, with the sun (grey), the planet (red), the carrier (orange), and the fixed (green) and mobile (blue) ring gears. Forces F_i and torques T_i are represented as single- and double-headed arrows, respectively. Subscripts indicate which gears the forces and torques are acting on. For simplicity, the angle δ is represented only for gear 1 (for the sake of clarity, δ is exaggerated). The tangent forces $F_{t,ij}$ are normal to the view plane, pointing inwards or outwards as indicated. Only the component of the bending moment (M_b) in the section plane is represented, for the sake of simplicity.

r_4 and $r_{2,4}$ may be found by a similar procedure:

$$\begin{cases} \frac{r_{2,4}}{r_4} = \frac{Z_2}{Z_4} \\ r_4 - r_{2,4} = I \end{cases} \Rightarrow \begin{cases} r_4 = \frac{Z_4}{Z_4 - Z_2} I \\ r_{2,4} = \frac{Z_2}{Z_4 - Z_2} I \end{cases} \quad (4.3)$$

The relationships between the radial, axial, and tangential force components are provided by Eqs. (2.3) or (2.4), depending on which pairs of tooth flanks are in contact. A force component F_x transmitted from gear b to gear a is denoted as $F_{x,ab}$. When a force component F_x (or torque T) is applied externally to gear a , it is labelled as $F_{x,a}$ (or T_a). $F_{el,1}$ is the force exerted by spring 1, pushing the sun gear against the planets, while $F_{el,2}$ is the force applied by spring 2, pushing each planet against the outer rings. Forces and torques are considered positive when aligned with the directions indicated in Fig. 4.1, and negative otherwise. The output loads, specifically the torque T_4 (along the main axis a), the bending moment M_b , the axial force $F_{a,4}$, and the radial force $F_{r,4}$ are assumed to be known. M_b and $F_{r,4}$ are fully supported by the main bearing. For simplicity, the load is assumed to be evenly distributed among all planets; load concentration factors could be introduced in future work for a more precise force and torque estimation during sizing.

The torque equilibrium of the output ring and the carrier yield, respectively:

$$F_{t,24} r_4 n_p = T_4 \quad (4.4a)$$

$$F_{t,2C} In_p = 0 \quad (4.4b)$$

which provide the tangential forces exchanged between the planet and the output ring ($F_{t,24}$) and between the planet and the carrier ($F_{t,2C}$), with the latter being nought. The torque and force equilibriums along the tangential direction of each planet give:

$$F_{t,12}r_{2,1} - F_{t,23}r_{2,3} + F_{t,24}r_{2,4} = 0 \quad (4.5a)$$

$$F_{t,12} + F_{t,23} - F_{t,24} = F_{t,2C} = 0 \quad (4.5b)$$

which allow the tangential forces $F_{t,12}$ and $F_{t,23}$ on the planet to be computed.

Once the tangential forces are known, the axial and radial forces acting on each planet in each meshing can be computed from Eqs. (2.3) and (2.4).

The equilibrium of the axial forces on the planet provides the force exerted by spring 2:

$$F_{el,2} = F_{a,12} + F_{a,23} + F_{a,24} \quad (4.6)$$

whereas the axial force equilibrium of the output ring provides the axial force withstood by the main bearing:

$$F_{a,b} = F_{a,4} + F_{a,24} \quad (4.7)$$

Finally, the torque and force equilibriums of the sun gear provide the input torque and the axial force exerted by spring 1:

$$T_1 = F_{t,12}r_1n_p \quad (4.8a)$$

$$F_{el,1} = F_{a,12}n_p \quad (4.8b)$$

The equilibrium of the radial forces is not considered in the above relations because they always balance out, as they have the same magnitude and are oriented at uniformly spaced angles.

4.2.2 Omni OH

In the Omni OH version (Fig. 3.4), the sun gear is replaced by a parallel-axis beveloid gear pair coupled with a Wolfrom gearbox. This architecture, unlike the OS one, has non-coincident input and output axes. Since there is no sun gear, the OH has an internal space for cables to pass through, which is a common requirement in robotic applications.

Considering the schematic drawing in Fig. 4.2, equilibrium equations can be derived for the entire OH gearbox, assuming the output torque T_4 (along the main axis a), the bending moment M_b , the axial force $F_{a,4}$ and the radial force $F_{r,4}$ to be known. As for the Omni OS, M_b and $F_{r,4}$ are supported by the main bearing, so that they are not considered in the following. The moment equilibrium of the output ring (4) provides $F_{t,24}$ as:

$$F_{t,24}r_4n_p = T_4 \quad (4.9)$$

while the moment and tangential force equilibrium equations of each planet, namely:

$$F_{t,23}r_{2,3} - F_{t,24}r_{2,4} = 0 \quad (4.10a)$$

$$F_{t,23} - F_{t,24} + F_{t,2C} = 0 \quad (4.10b)$$

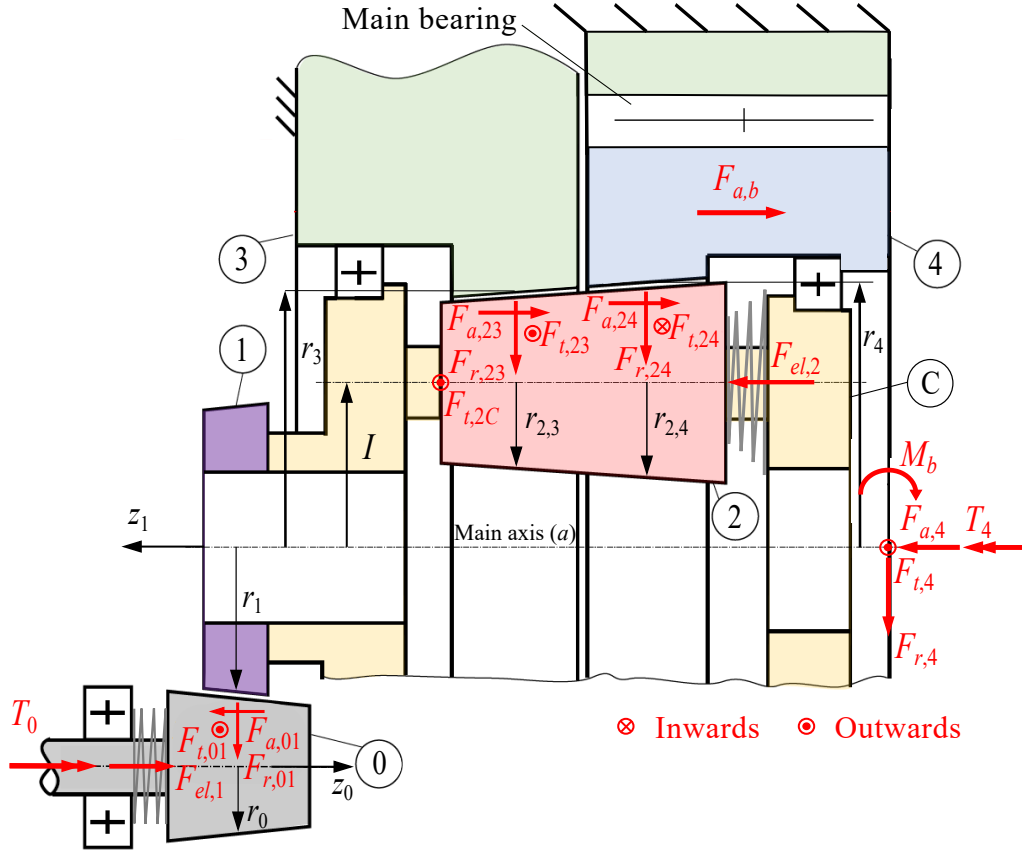


Figure 4.2: Section view with the schematic of the forces acting on the Omni OH gearbox; compare with Fig. 4.1. Here, 0 is the input gear (in grey), while 1 (purple) is a gear fixed on the carrier.

give the tangential forces $F_{t,23}$ and $F_{t,2C}$ (notice that the latter is not zero, unlike in the OS architecture). After solving Eq. (4.10), the axial and radial forces can be determined by using Eqs. (2.3) and (2.4). The elastic force on the planet can be computed as:

$$F_{el,2} = F_{a,23} + F_{a,24} \quad (4.11)$$

In this case, too, the radial forces on the ring gears balance out.

The moment equilibrium of the carrier provides the tangential force $F_{t,01}$ exchanged between the input (gear 0) and the gear on the carrier (1):

$$F_{t,2C} I n_p = F_{t,01} r_1 \quad (4.12)$$

Finally, the equilibrium of the input gear gives:

$$F_{t,01} r_0 = T_0 \quad (4.13)$$

After finding $F_{t,01}$, the radial and axial forces $F_{r,01}$ and $F_{a,01} = F_{el,1}$ can be computed using Eqs. (2.3) or (2.4). In this case, $F_{r,01}$ from the first stage gearing is *not* balanced, but is transferred to the bearings of gear 0 and the carrier. Finally, the total axial force on the main bearing is:

$$F_{a,b} = F_{a,4} + F_{a,24} - F_{a,01} \quad (4.14)$$

The axial force $F_{a,01}$ on gear 1 is transmitted to the carrier, and from there (through the right-side bearing in Fig. 4.2) to the mobile gear.

4.3 Design considerations

The kinetostatic analysis in Sec. 4.2 is essential for the gearbox preliminary design, guiding the initial sizing of key components such as springs, bearings, and gears. In order to achieve a more accurate design, a multibody model (Sec. 4.3.1), a finite-element model (Sec. 4.3.2) and a KISSsys model (Sec. 4.3.3) were also developed. The validation of force analysis and the support of the design of the experimental tests presented in Section 4.4 were supported by these models.

An important aspect to consider when designing a Wolfrom planetary gearbox is the direction of the tangential forces on the planets when meshing with the two rings. Figure 4.3 shows a schematic of the resultant forces exerted by the fixed and output rings on one tooth of the planet: these forces cause the tooth to bend. Design experience [45] indicates that stresses are higher at the mid-section of the planet: a qualitative pressure distribution of the load over the tooth face is shown in Fig. 4.3b. To distribute the load more evenly, in addition to increasing the machining accuracy and surface quality of the components, the planet teeth are designed with a groove in the middle (see gear 2 of Fig. 4.5), thus reducing the stress concentration in that zone (as suggested by [45]). To validate the design, a finite-element model was developed to simulate tooth contact, mainly between gears 2, 3 and 4. The purpose of this model was twofold: first, to confirm that the highest stress concentration occurs at the mid-section of the planet teeth, thus the utility of the groove and the relief micro modification in this area (according to [45]); and second, to perform a detailed contact analysis in order to identify potential critical issues in the gearbox under realistic loading conditions. The technical details of the model implementation are presented in Sec. 4.3.2. The model was built for the Omni OH and is represented in Figure 4.5. The contact analysis for nine different meshing configurations is reported Fig. 4.6, showing high pressure and stress concentrations near the mid-section of gear-2 tooth (on the flank and at the tooth root) which is consistent with previous studies [45].

Another essential step during the design stage is the choice of the gear geometric parameters. In the Wolfrom architecture, each planet meshes simultaneously with three other gears, and these parameters cannot be chosen separately [47, 54]. In our case, the minimum tooth face width is constrained by the design load. Due to the use

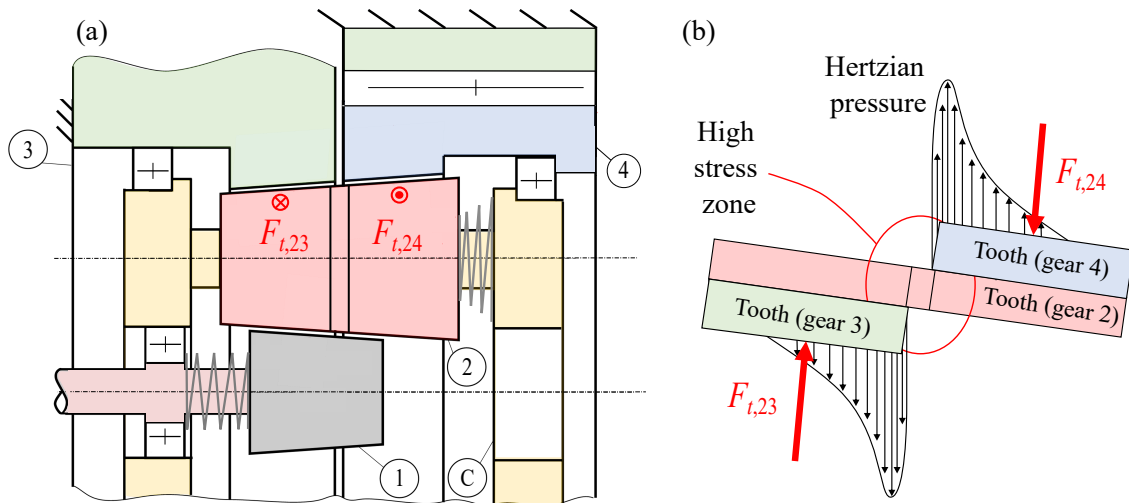


Figure 4.3: Schematic of the forces $F_{t,23}$ and $F_{t,24}$ acting (in opposite directions) on the meshing tooth of the planet, in section view (a) and top view (b).

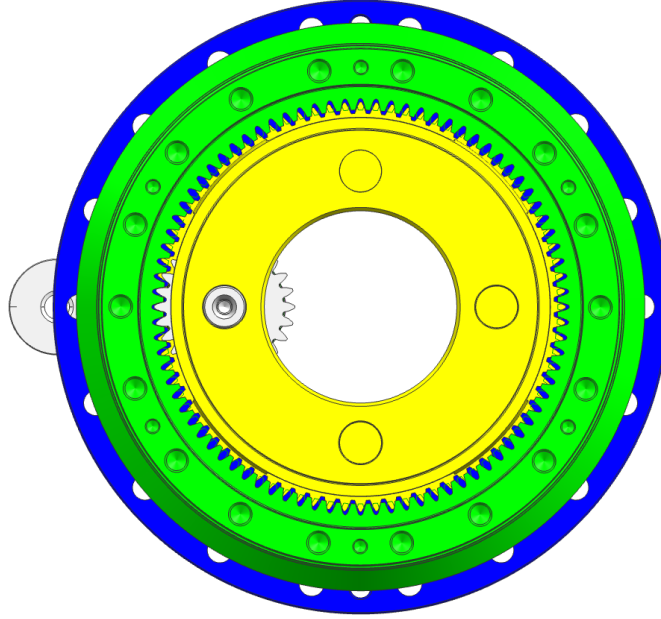


Figure 4.4: Multibody model developed with NX motion software.

of beveloid gears, the value of the profile shift coefficient x at the mid-section and angle δ are chosen to avoid undercut and pointed tooth tips at the heel and toe [59]. Z_i and α_n are selected to respect design constraints (Sec. 3.3.1), the target transmission ratio, and constraints due to the production method. The optimisation of these parameters, which leads to a nonlinear problem with several constraints, will be the object of future work.

The main bearing is integrated into the housing to minimise the overall volume. The two rolling surfaces of this bearing are directly machined on the mobile ring and on the frame with high accuracy, then heat-treated to ensure long fatigue life. The rolling elements are preloaded to ensure high bending stiffness and efficiency. More details are provided in Sec. 4.3.4.

4.3.1 Multibody model

A multibody model was developed for the Omni OH gearbox, with size 060 and $\tau = 1:125$ (Fig. 4.4), which is the prototype currently under test. The objectives of this model were, firstly, to validate the results obtained from the kinetostatic analysis and, secondly, to investigate the contact conditions within the kinematic chain (for example, to quantify the number of contacts occurring during a complete revolution of the output shaft). This tool was used as support in the design of both the overall gearbox and the experimental tests.

To reduce computational effort, a single planet was modeled, since all planets engage simultaneously and therefore experience the same contact conditions. The following kinematic constraints were imposed:

- The fixed ring gear (gear 3) was constrained to the frame.
- The output ring gear (gear 4) was allowed to rotate only around the main axis of the gearbox.

- The carrier was also constrained to rotate about the same axis, and a gear rigidly attached to it was driven by the input pinion (gear 0); the distance between the axes of the carrier and of the input is $r_0 + r_1$.
- The planets (gears 2) were constrained to rotate about their own axes, which are fixed to the carrier.

All gear contacts were defined according to the schematic in Fig. 4.2, thus preserving the correct kinematic relationships between the elements. Once the model was assembled, its validity was verified by applying an input speed and torque to the driving pinion (gear 0) and checking that the output ring gear (gear 4) exhibited the expected speed and torque values according to the theoretical transmission ratio.

The model was particularly useful for the interpretation of the experimental switching tests (Sec. 4.4.1.1). The same motion law applied to the motors of the test bench was applied to the gearbox input, replicating the dynamic conditions of the switching cycles. During these tests, the gearbox is fully assembled, so direct information about the internal contacts is unavailable. If the test fails, the multibody simulation provides a valuable tool for understanding the internal gear interactions and identifying possible causes of failure. The switching tests are characterised by frequent reversals of motion; thus, the gear teeth are subjected to alternating bending, a phenomenon that requires particular attention during sizing [8]. The model made it possible to optimise the oscillation angle, ensuring that the same tooth was repeatedly subjected to stress during the reversal of the rotational direction, where the load increases due to acceleration and deceleration phases (see Fig. 4.17, right). This approach enabled the acceleration of the test process.

4.3.2 Finite-element model

A finite-element model was developed to simulate tooth contact, mainly between gears 2, 3 and 4. The procedure followed is similar to that described in Tab. 4.2 and used in [59]. However, as the Omni gearbox is more complex, additional precautions must be taken. Figure 4.5 shows the model developed for the Omni OH gearbox whose rated parameters are listed in Tab. 4.1. Since the Omni OS prototype was not yet available for testing and the development of a finite-element model is time-consuming, the focus was exclusively on the Omni OH, with size 060 and $\tau = 1:125$. To reduce computational time, only one planet was considered, and three teeth of gear 2 were selected to mesh with three teeth of rings 3 and 4. The mesh mostly consists of hexahedral elements, with finer near-cubic elements being used close to contact areas, at the tooth root and near the groove, to enhance accuracy, whereas coarser elements are used elsewhere. All gears are constrained to rotate about their own axes, except for gear 3, which is fixed. The boundary conditions are as follows: the torque T_N/n_p is applied to gear 4, while the axis z_2 of gear 2 (and thus the carrier) is rotated by $\alpha_C = 10^\circ$ around the Omni main axis a . Applying Willis' formula similarly to Eqs. (3.1) and (3.6a) provides the rotation of gear 2 around z_2 , namely $\alpha_2 = \frac{Z_2 - Z_3}{Z_3} \alpha_C$; this way, the central tooth of gear 2 performs a full meshing cycle. The contact analysis is shown in Fig. 4.6 for nine different meshing configurations denoted by a dimensionless parameter ξ , ranging from 0.1 (soon after the contact starts, at $\xi = 0$) to 0.9 (soon before the contact ends, at $\xi = 1$). All gears are made of the same steel, modelled as an isotropic, linearly elastic material; lubrication and thermal effects are disregarded to simplify the model. In the initial contact phase

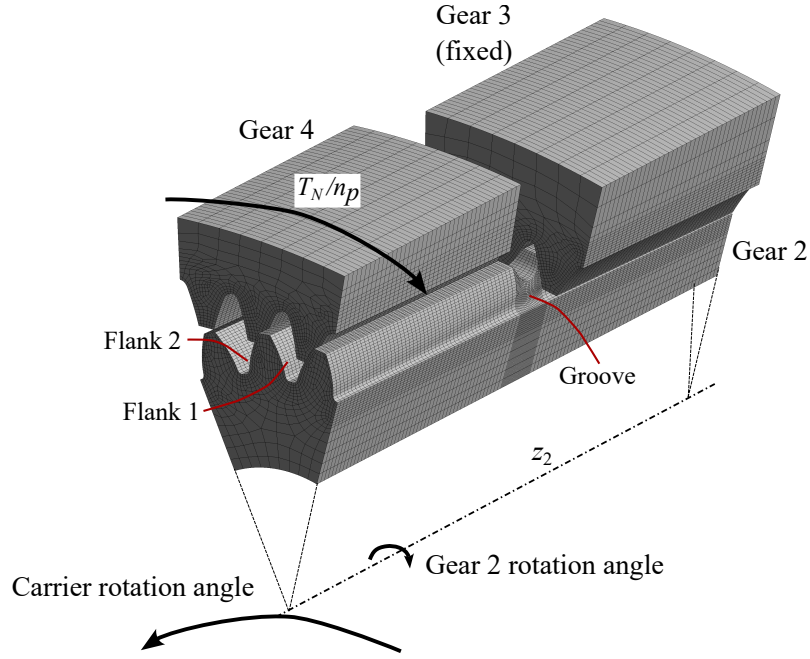


Figure 4.5: Finite-element model of the meshing between gears 2, 3, and 4 of the Omni OH gearbox tested in Sec. 4.4 and whose rated parameters are reported in Tab. 4.1. The nominal torque T_N , divided by the number of planets n_p , is applied to gear 4. Gear 3 is fixed, while the axis of gear 2 (the carrier) is rotated around the Omni main axis a to perform the contact analysis at various meshing positions. Each planet tooth meshes with gear 3 on flank 1, and with gear 4 on flank 2. The model mesh is refined close to the groove, in the contact areas and at the tooth root to achieve more accurate results.

(Fig. 4.6a), for $\xi = 0.1$ and $\xi = 0.2$, only flank 1 (meshing with gear 3) is in contact, while flank 2 (meshing with gear 4) is not yet engaged; for $\xi = 0.3$, flank 2 starts meshing. For $\xi = 0.4, 0.5, 0.6$ both flanks are loaded along their entire width (Fig. 4.6b). In the final contact phase (Fig. 4.6c), the contact area is gradually reducing (more rapidly on flank 1). The equivalent von Mises stress at the tooth root of flank 2, where the highest stress concentration occurs, is shown in Fig. 4.6d. The results in Fig. 4.6 clearly show high pressure and stress concentrations near the mid-section of gear-2 tooth (on the flank and at the tooth root) in agreement with previous works [45]. Even with the groove, the highest pressure is still at the mid-section.

4.3.3 KISSsys model

4.3.3.1 Model definition

This Section describes the step-by-step procedure used to build the KISSsys model of the Omni OH, specifically a planetary Wolfrom gearbox without a sun gear. Since beveloid gears are not available in KISSsoft, spur gears with an equivalent mid-section geometry are used as substitutes. The model is useful because, although the gear strength calculations are not fully accurate due to the simplified modeling of beveloid gears, it is still suitable for kinematic calculations, load cycle counting, and the sizing of shafts and bearings. Moreover, beveloid gear sizing based on the mid-section is a commonly adopted procedure, as it is simple and leads to reasonably accurate results.

The elements required to build the gearbox are available from the KISSsys *Element*

Table 4.2: Step-by-step procedure for building a finite-element model of beveloid gear pairs with parallel-axis in Ansys Workbench (the same adopted in [59]).

No.	Step	Description
1	Geometry generation	Model the beveloid geometry using the KISSsoft beveloid module. Export the geometry as a STEP file with at least 50 sections along the face width and minimum modelling tolerance of $0.5 \mu\text{m}$.
2	Geometry manipulation	Import the gear pair into CAD software. Define the initial contact position and isolate at least 3 teeth per gear (the central one is the one under analysis, the others are for realism). Check for surface discontinuities, especially in high-stress areas.
3	Material definition	Assign the material, its elastic modulus E_y and the Poisson's ratio ν to both gears.
4	Meshing	Use sweep method with at least $n = 100$ transversal slices. Refine mesh in contact and root areas with element size of at least 0.05 mm . Final mesh should include a majority of Hex8 elements.
5	Constraints	Define a rigid reference surface for each gear, connected to the gear axis via rotational joint. Allow only rotation about the axis.
6	Boundary conditions	Impose a rotation on the pinion to cover the full contact range. Apply a torque to the wheel. Ensure a consistent sign convention for rotation and torque.
7	Contact	Model gear flank contact as frictionless and asymmetrical. Define wheel teeth as contact target for the pinion. Include both flanks per tooth, for modeling the zero-backlash condition. Use the normal Lagrange formulation for greater accuracy or the augmented Lagrange formulation for a faster solution.
8	Solver settings	Enable large deflections. Discretize pinion rotation into at least 25 sub-steps (depending on the rotation selected at point 6) to capture all contact positions along the meshing cycle.

box (using administrator mode). The model developed in KISSsys (Fig. 4.7) was built as follows.

1. Create the main group at the root of the modelling tree and name it *GroupBox*².
2. Create the coaxial shaft groups *CarrierShaft*, *RingShaft1* and *RingShaft2*, which correspond to carrier, ring gear 3 and ring gear 4, respectively. Coaxial shafts must be organised in individual subgroups, to keep the tree structure clear.
3. Add the planet subgroup: create a subgroup named *Planet* under *CarrierShaft*. Add the coaxial shafts *PlanetGearBody* and *PinShaft* to the *Planet* subgroup. This accounts for the planet gears 2 and their shafts.
4. Add gears: add *PlanetGear1* and *PlanetGear2* to the coaxial group *PlanetGearBody*, to represent the two-rim planets. Add *RingGear1* and *RingGear2* to *Ring-*

²In this Section, all the italic words are referring to the element used in the “Modelling tree” of Fig. 4.7-left.

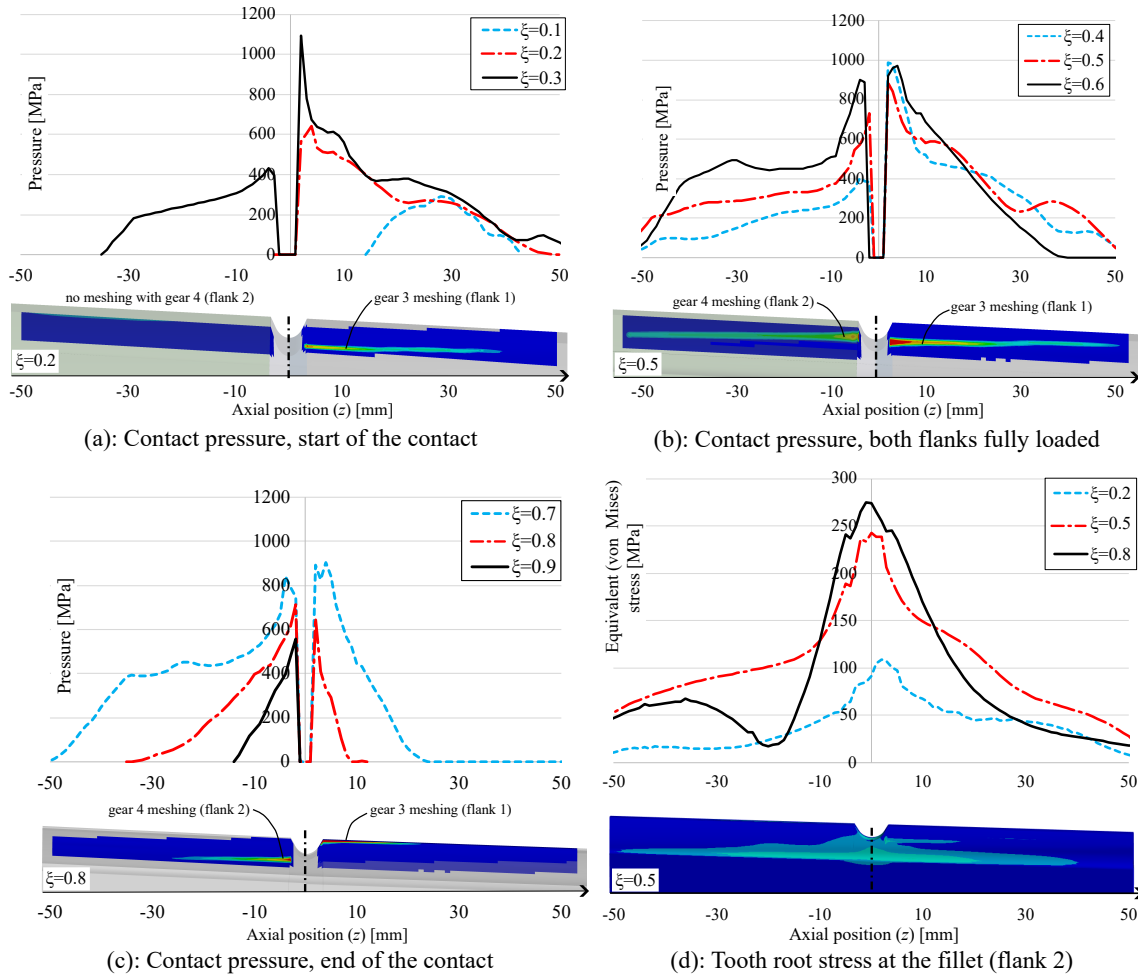


Figure 4.6: Contact analysis showing the contact pressure (subfigures a, b, c) and the equivalent (von Mises) stress distribution (subfigure d) on one tooth of gear 2, for 9 different meshing configurations, denoted by the dimensionless parameter ξ . (a) Initial contact phase with partial flank engagement. (b) Full flank engagement. (c) Final contact phase. The contact lines are at a constant radial distance from the axis of the gear; however, they are not aligned with each other on the two flanks depending on the contact configuration. Under each figure, an image from the finite-element analysis of the contact pressure distribution on the two flanks is shown (for a representative configuration): flank 2 is shown on the right-hand side, while flank 1 is shown on the left-hand side. (d) Equivalent (von Mises) stress at the tooth root of flank 2, where the highest stress concentration occurs (particularly around the mid-section, at $z = 0$ mm), for $\xi = 0.2, 0.5, 0.8$.

Shaft1 and *RingShaft2*, respectively.

5. Add coupling elements into *CarrierShaft*, *RingShaft1* and *RingShaft2*. Couplings will be used later to define boundary conditions.
6. Add the carrier element: add *CarrierElement* to *CarrierShaft*. Once the carrier element is present, KISSsys recognises the configuration as a planetary gearbox.
7. Add supports to the groups *CarrierShaft*, *PinShaft*, *RingShaft1* and *RingShaft2* as required.

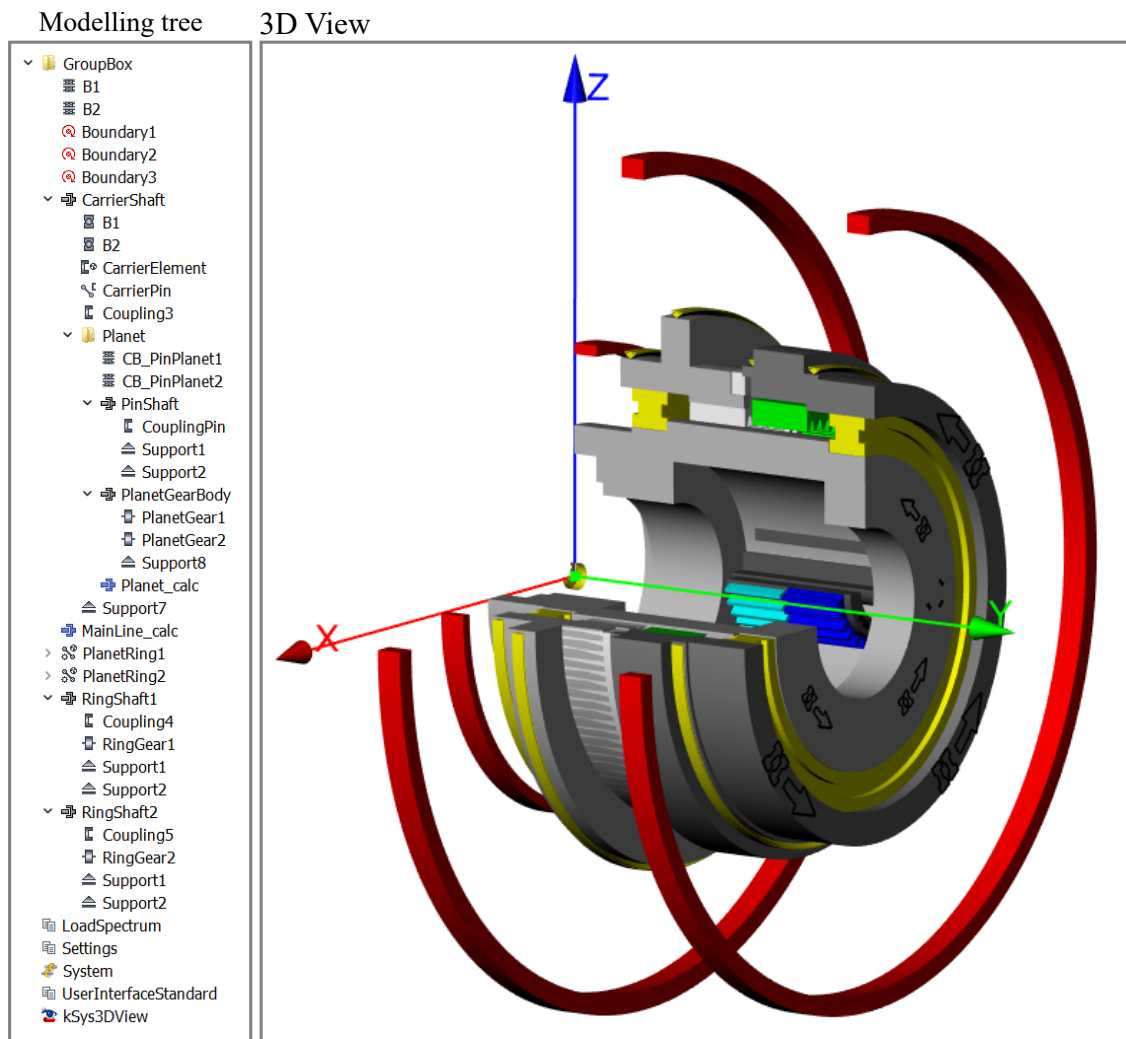


Figure 4.7: Model developed using KISSsys of the Omni OH. On the left, the modelling tree displays all elements used to define the gearbox architecture. On the right, a cut view of the Omni gearbox as implemented in KISSsys is shown.

8. Add bearings: insert two bearings named *B1* and *B2* with inner race on *CarrierShaft* and outer race on *RingShaft1* and *RingShaft2*. In the *Planet* group, connect *PinShaft* and *PlanetGearBody* using the connecting bearings *CB_PinPlanet1* and *CB_PinPlanet2*; define inner and outer races for these bearings.
9. Add constraints to define gear meshing and to create coupling connections between *CarrierShaft* and *PinShaft*. Define the planet-to-ring meshing with constraints *PlanetRing1* and *PlanetRing2*. Use the *CouplingConstraint* option to connect *PinShaft* and *CarrierShaft*; here, one can set the number of planets and the meshing efficiency (which is set to 1 in this work for simplicity's sake).
10. Attach the shaft and gear calculation files. This introduces the geometry of the shaft and gear pair to the model. Define the shaft calculations *MainLine_calc* and *Planet_calc*, and the gear calculations *PlanetRing1_calc* and *PlanetRing2_calc* for the two main meshings.
11. Assign boundary conditions (input, fixed ring, output ring) to the appropriate coupling elements, namely *CarrierShaft.Coupling3*, *RingShaft1.Coupling4* and

RingShaft2.Coupling5. Set the prescribed torque and rotational speed for each boundary condition.

The model is now ready for the kinematic calculation, the first step in identifying potential errors and ensuring the model converges correctly.

At this stage the model can be used to size any Wolfrom gearbox by means of the built-in rough sizing tools (e.g., for the gear pairs), giving the user the possibility to set the element dimensions within a given range. The same procedure must then be applied to bearings and shafts. Alternatively, if the geometry is already known, as in the present case, the user should open each KISSsoft calculation and adjust the relevant geometric parameters directly. These parameters include the center distances, the number of teeth, the bearing positioning and types, the shaft positions and diameters, and so on.

The user can also choose to set up different tables to obtain a quick general overview; for example, the input and output data (*UserInterfaceStandard* table) and the global settings defined for the model (*Settings* table). The *LoadSpectrum* table was also included for the calculation of safety factors not only under a single load case but over an entire load spectrum. In applications where fatigue is a relevant failure mode, the loading conditions typically vary throughout the service life, and this variability must be taken into account. Therefore, instead of relying solely on standard Wöhler curves developed for constant stress amplitudes, this method allows for the evaluation of safety factors under realistic, variable loading scenarios.

4.3.3.2 Service life calculation based on load spectrum definition

Failure mode	Bin No.	Load cycles n_i [mln]	Output speed $\dot{\theta}_{out,i}$ [rpm]	Time t_i [h]	Output torque $T_{out,i}$ [Nm]	PlanetRing1		PlanetRing2	
						Y_{M1}	Y_{M2}	Y_{M1}	Y_{M2}
Pitting and bending	1	1	15	1111.11	800	1	1	1	1
	2	2.0	15	2222.22	1150	1	1	1	1
	3	0.30	15	333.33	1610	1	1	1	1
Bending	4	1	15	1111.11	-800	0.7	0.7	0.7	0.7
	5	2.0	15	2222.22	-1150	0.7	0.7	0.7	0.7
	6	0.30	15	333.33	-1610	0.7	0.7	0.7	0.7

Table 4.3: For each bin i , considering the number of load cycles n_i and the output speed $\dot{\theta}_{out,i}$, the time is calculated as $t_i = n_i / (60\dot{\theta}_{out,i})$.

After the full geometry has been implemented and the kinematic analysis completed without errors, all corresponding KISSsoft calculations can be performed. The process concludes with the generation of a detailed report containing comprehensive results for the entire gearbox.

As mentioned earlier, in the case of the Omni gearbox, the KISSsoft calculation is significantly approximated, since no calculation method for beveloid gears is implemented. The same limitation applies to the analysis of bearings and shafts, since the

model cannot include load components such as the axial forces generated by beveloid gears. On the other hand, the kinematic analysis is fully valid and directly applicable to the Omni OH gearbox. Overall, this tool proved valuable in supporting the entire design process.

Calculations are based on the real duty cycle used during the switching test described in Section 4.4, as described in Tab. 4.8. The data used for the KISSsys load spectrum, defined according to the guidelines in [10, 101], are reported in Tab. 4.3 using six bins³. The software gives also the possibility to manually set other parameters, such as the face load factor $K_{h\beta}$ and the load distribution factor K_{γ} . These corrective factors are defined in [6] and account for the deviation of the actual load distribution from the ideal assumption of uniform load. In this case, $K_{h\beta} = 1$ and $K_{\gamma} = 1$ are used for all bins, since the components are designed with high production quality. The alternating bending factors Y_{M1} and Y_{M2} refer to the pinion (1) and to the wheel (2), respectively. It should be noted that bin pairs 1 and 4, 2 and 5, and 3 and 6 correspond to identical test conditions. These sets represent the low, rated, and high torque levels defined in Tab. 4.8, respectively. This approach was adopted due to the complexity of the duty cycle and the gearbox itself. During the switching test, the output load undergoes symmetrical oscillations at a fixed angle; thus, the flank under load varies periodically. To accelerate testing, the test angle was set so that the same teeth were consistently subjected to load when direction is reversed, i.e., when the torque is higher. Therefore, only these critical teeth need to be sized, as the others are under low or zero load. The load spectrum in Tab. 4.3 is defined differently for the two failure modes, as follows.

Pitting Pitting failure should only be evaluated for one flank. This flank is in contact for only half of the oscillation time, and only during the accelerating phase, when changes of direction occur. In other words, as seen in Fig. 4.17-b, only the two positive torque peaks are damaging for that flank; thus, only such accelerating time should be taken into account for the analysis. Since the torque varies continuously, it is complicated to estimate an accelerating time; on the other hand, it is obvious that there are two peaks over one complete oscillation. Thus, to build Tab. 4.3, the number of cycles N_C indicated in Tab. 4.8 were doubled and the maximum speed for all bins (the most unfavorable condition) was considered, thus allowing the calculation of the corresponding time⁴ t_i . In that way, the software calculates the gears with the correct number of cycles, which is the important parameter for fatigue calculations. This procedure was applied to the three load conditions of Tab. 4.8, obtaining bins 1, 2 and 3.

Bending The focus is always on the same teeth that experience loads (in particular, on the root fillets of the flanks checked for pitting) due to accelerations and decelerations at the reversal points of the switching test. Unlike in pitting, bending stresses reach their maximum values in both oscillation directions, albeit differently, as the load conditions change. In bins. 4, 5 and 6, the side which is not loaded is still subject to a compressive stress (Fig. 4.8), which has to be considered during sizing. According

³To obtain the load spectra required for fatigue damage calculations, the range of measured or calculated loads is divided into bins. Each bin contains the number of load occurrences recorded in its load range [101].

⁴The calculated time, i.e., the service life, is based on the theory that each load cycle (each revolution) causes damage to the gear [101].

to $[10, 101]$, stress should be reduced by a factor $Y_M = 0.7$. The other parameters are the same, except for the torque, which is negative⁵.

The time t_i found with this method is much larger than the testing hours t_{op} in Tab. 4.8, since the experimental tests are designed to be accelerated. On the other hand, KISSsoft considers a complete, one-directional rotation of the shaft, then verifies all the teeth. In our testing, only the critical teeth (subject to the maximum load at each cycle) of the planet gear are verified.

From the results reported in Tab. 4.4, it can be observed that none of the calculated safety factors fall below the threshold values, which are set to $S_H = 1.0$ for the flank safety and $S_F = 1.2$ for the root safety. However, changing the value of the application factor⁶ from $K_A = 1$ to 1.25 (see Tab. 4.5), obtaining a bending safety values below 1.2 for the planet gear in the meshing with both ring gears, which is precisely the gear that failed due to bending during the tests that will be discussed in Sec. 4.4.1.1. This is also consistent with the fact that the switching test is highly demanding, involving alternating loads and significant wear. Moreover, the gearbox is mounted with zero backlash, which leads to particularly high stresses on gear meshes.

The results appear realistic, also considering that the gears used in the KISSsys model are not beveloid gears but spur gears, which likely introduces a non-negligible error. In addition, the KISSsys model does not consider that the planet gear tooth is in contact on both flanks and is subject to a complex stress condition, uncommon in most gear applications (see Sec. 4.3).

Calculation file	PlanetRing1		PlanetRing2	
	Gear 1	Gear 2	Gear 1	Gear 2
Root safety [S_F]	1.352	1.829	1.459	1.382
Flank safety [S_H]	1.738	2.346	1.459	1.459

Table 4.4: Safety factors calculated with the load spectrum in Tab. 4.3, considering an application factor $K_A = 1$.

Calculation file	PlanetRing1		PlanetRing2	
	Gear 1	Gear 2	Gear 1	Gear 2
Root safety [S_F]	1.131	1.530	1.183	1.120
Flank safety [S_H]	1.587	2.142	1.314	1.314

Table 4.5: Safety factors calculated with the load spectrum in Tab. 4.3, considering an application factor $K_A = 1.25$.

⁵In the KISSsys settings, it is specified that pitting is evaluated only for positive load bins; therefore, the software skips the calculation for negative torques.

⁶The application factor, K_A , is used to adjust the nominal load to account for increases in load on the gears due to external sources. The magnitude of these additional loads largely depends on the characteristics of the drive and driven machines, as well as the masses and stiffness of the system, including the shafts and couplings used in the application [6].

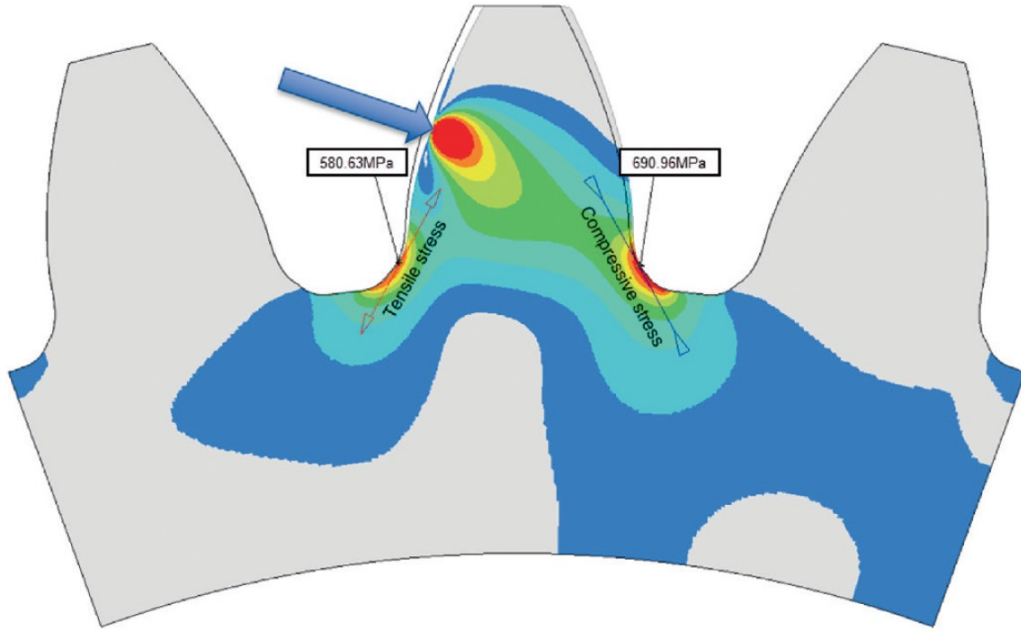


Figure 4.8: Stresses during alternate bending. The unloaded flank is subjected to a higher root stress than the loaded one. Thus, in bins 4, 5, and 6, all the alternating bending factors are set to 0.7, as indicated by [101]. Image source: [10].

4.3.4 Main bearing

While well-established and precise standards exist for designing gearbox elements in normal operation, with unidirectional rotating loads, these approaches are less applicable to bearings used in robotics and wind turbines. Indeed, in these cases, bearings are subject to loads fluctuating in magnitude and direction, alternating motions, and continuously changing operating conditions (such as temperatures and lubrication). This Section discusses the sizing of the main bearings of the Omni gearbox.

The international standard DIN ISO 281 [9] established the calculation of the fatigue life for rotating bearings. The standard is quite accurate for point contacts (e.g., ball bearings) but less for linear contacts (e.g., roller bearings), due to the inflection of the related shaft, which creates different contact conditions. Contact-stress analysis must therefore be performed in the latter case to achieve more accurate results. The general formula for estimating (with 90% reliability) the bearing rating life in millions of revolutions is given by

$$L_{10} = \left(\frac{C}{P} \right)^p \quad (4.15)$$

Here, C is the dynamic load, P is the equivalent load, and p is an exponent depending on the geometry of the rolling elements (3 for ball bearings and 10/3 for roller bearings). If the rotational speed n (in rpm) is constant, according to [102] it is preferable to calculate the life in operating hours:

$$L_{10h} = \frac{10^6}{60n} L_{10} \quad (4.16)$$

This approach is valid for standard bearings under standard conditions, but service life can vary in actual applications. Factors such as the amount of lubricant (and

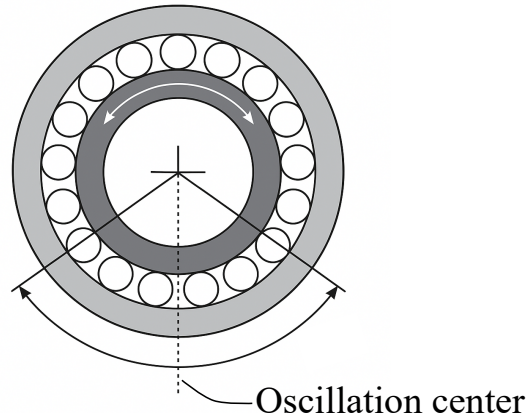


Figure 4.9: Oscillating motions around the oscillation center. Image source: [11].

its degree of contamination), incorrect mounting, and type of motion (with continuous rotation or oscillation) can have a significant impact on the service life [102]. In our case, the focus is on the motion type, as it is the most influential and least documented factor, making it particularly interesting to study. Furthermore, other effects can reasonably be neglected, since the high quality of all components, together with the mounting of each prototype, minimises the probability of incorrect assembly. The main bearing raceways are case-hardened to minimise pitting and wear; consequently, contamination by debris in the lubricant is minimal. Finally, the used grease ensures full-film lubrication, which further reduces friction.

For oscillating bearings, different assumptions must be made, because only some parts of the rollers and of the raceway are loaded. Different approaches are known in the literature, with varying degrees of complexity. Schwack [103] and Wöll [11] provide exhaustive reviews of the different methods; the most used one is by Houppert and Harris [12, 104].

Under radial load and pure oscillatory motion, only a portion of the rollers and limited areas of the raceways are subjected to stress. Thus, one might expect a longer fatigue life in oscillating applications. According to [104], a bearing oscillating n times per minute has a longer fatigue life than that one rotating continuously at n rotations per minute under the same load, due to the lower mean equivalent load acting on each rolling element. In this case, the bearing lifetime increases if, throughout the service life, the oscillation center (Fig. 4.9) changes along the entire raceway. This way, all the rolling elements and all parts of the raceway surface are periodically subjected to stress. If, instead, the bearing experiences, for its whole life, oscillations with the same angle around the same oscillation center, the bearing life is governed by the rollers and the raceway area under stress, which reach failure first. In this case, the mean equivalent load defined in [104] can no longer be applied.

A novel approach must then be adopted, since the currently available calculation methods (such as ISO 281) do not distinguish [11] whether a bearing oscillates around a single central position or changes its oscillation center during its service life. As noted in [105], an effective practice for manual applications is to periodically dismount and remount a bearing, changing its orientation. This allows the load to be redistributed over different areas of the raceway, thereby extending the service life.

Another aspect that must be considered when calculating the bearing life under oscillating motions is lubrication. Back-and-forth movements cause lack of lubrication

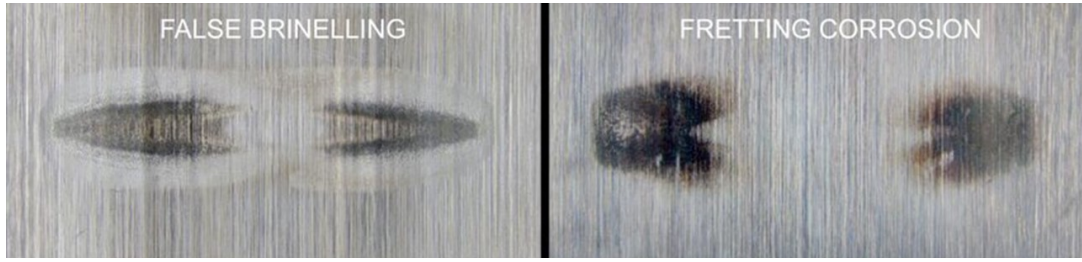


Figure 4.10: Examples of false brinelling and fretting corrosion. Image source: [106].

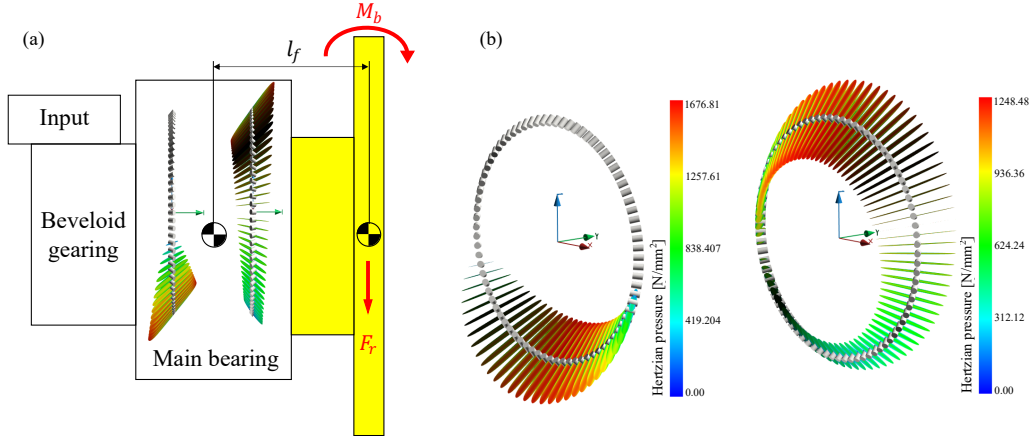


Figure 4.11: (a): Schematic of the loading on the main bearing. Simplified contact pressures are shown on the main bearing rolling element, from red (high values) to green (low values). (b): A detail of the contact analysis for the main bearing.

in the contact area, since the grease is not pushed around enough. This phenomenon worsens as the angle of oscillation decreases, resulting in two types of wear that affect the bearing raceway: *false brinelling* (lubricated condition) and *fretting corrosion* (dry conditions) [106]. They both create clearly recognizable grooves on the bearing raceway (see Fig. 4.10). The Schaeffler catalogue [69] introduces a factor (pag. 79, tab. 10) which reduces the grease operating life (not the bearing life) depending on the angle of oscillation. The catalogue suggests changing the lubricant more often in such cases. Similarly, the THK bearings catalogue [107] suggests contacting the manufacturer for applications with oscillating motions.

In conclusion, an analytical approach is necessary, supported by experimental validation, to dimension a bearing undergoing oscillating motions, taking into account the limited lubrication and the repeated loading of the same areas of the bearing.

4.3.4.1 Bearing calculation method for the Omni

The bearing calculation method presented in this Section was developed specifically for the switching test described in Sec. 4.4.1.1 and illustrated in Fig. 4.11. The kinematic model is outlined in Appendix A.1.

The main bearing used in the Omni is custom-designed, developed internally by Bonfiglioli Riduttori. For this reason, its internal geometry is not disclosed in this work. The bearing belongs to the family of axial angular cylindrical roller bearings. Two identical bearings are placed in a “X” arrangement, enabling the gearbox to withstand extremely high combined loads, including axial, radial, and bending forces. The preloaded cylindrical rollers ensure high accuracy and rigidity, and are guided by a cage

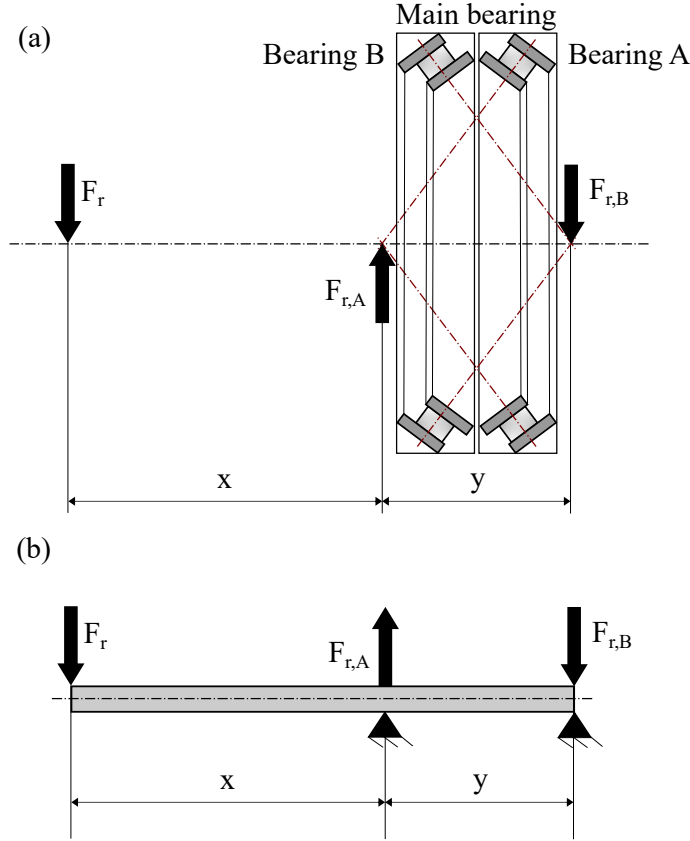


Figure 4.12: (a): Bearing schematic, showing the thrust centers (at the intersection of the lines perpendicular to the lines of contact between raceway and rollers). (b) Schematic of an equivalent beam model: the thrust centers correspond to the two supports.

made of wear-resistant plastic.

The main bearing and the corresponding radial forces can be modeled as shown in Fig. 4.12-a, and then further schematized as in Fig. 4.12-b, representing a loaded shaft.

The reaction forces are found from standard beam theory [108] are:

$$F_r = mg \quad F_{r,A} = \frac{x+y}{x} F_r \quad F_{r,B} = \frac{y}{x} F_r \quad (4.17)$$

where F_r is the radial force due to the flywheel (having mass m), g is the gravitational acceleration, and $F_{r,A}$ and $F_{r,B}$ are the radial bearing reaction forces.

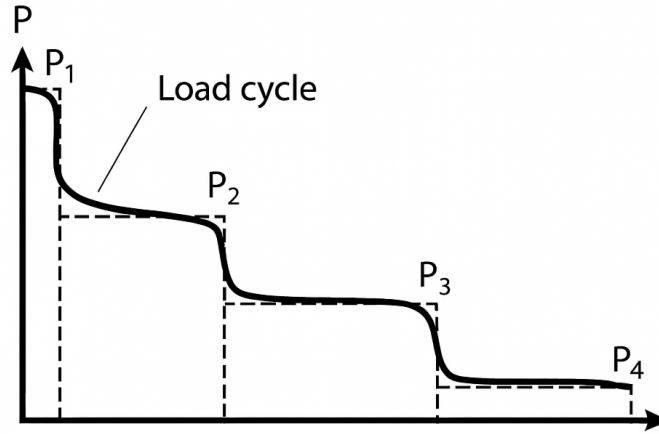
When an angular roller bearing is subjected to radial loads, an axial load component is generated due to the inclination of the raceways [102]. The greater axial component is generated by the larger radial force, namely, $F_{r,A}$. The axial components must balance each other out:

$$F_{a,A} = F_{a,B} = 0.5 \frac{F_{r,A}}{Y} \quad (4.18)$$

where Y is the dynamic axial load factor. According to [9, 69], the radial (X) and axial (Y) dynamic load factors for radial roller bearings are determined as shown in Tab. 4.6. In this case, the axial component is not externally applied, but depends on the factor Y (see Eq. 4.18). According to Tab. 4.6, this study sets $e = 1.5$ and $\frac{F_a}{F_r} = e$, which were substituted in Eq. 4.18; thereby, considering a double-row bearing, the resulting values are $X = 1$ and $Y = 0.33$. This evaluation assumes that the axial components are lower than the radial ones.

Table 4.6: Values of X and Y for radial roller bearings with $\alpha \neq 0^\circ$. Source [9].

Bearing type	$\frac{F_a}{F_r} \leq e$		$\frac{F_a}{F_r} > e$		e
	X	Y	X	Y	
Single-row	1	0	0.4	$0.4 \cot \alpha$	$1.5 \tan \alpha$
Double-row	1	$0.45 \cot \alpha$	0.67	$0.67 \cot \alpha$	$1.5 \tan \alpha$

Figure 4.13: Load cycle with constant loads P_i .

It is now possible to calculate the equivalent dynamic bearing load P , as:

$$P_A = XF_{r,A} + YF_{a,A} \quad (4.19)$$

$$P_B = XF_{r,B} + YF_{a,B} \quad (4.20)$$

When the bearing operates under variable conditions (fluctuating load, as shown in Fig. 4.13), a more detailed analysis is required. The bearing rated life is evaluated by summing the contributions of the partial equivalent dynamic loads P_i , assuming a constant load within each time interval. For each load level i , the rated life expressed in hours is:

$$L_{10h,i} = a_{osc,i} \frac{10^6}{60 n_i} \left(\frac{C_r}{P_i} \right)^p \quad (4.21)$$

with

- P_i : equivalent dynamic load at load level i [N];
- C_r : basic dynamic load rating [N];
- n_i : rotational speed at load level i [rpm];
- p : load exponent ($p = 3.33$ for roller bearings);
- $a_{osc,i} = \frac{180}{2\gamma_i}$: oscillation factor at load level i .

Factor $a_{osc,i}$ [102] is introduced to account for oscillating loads. As explained at the beginning of this Section, this factor applies only under specific conditions: the oscillation angles must be large, and the oscillation center must vary periodically during

Torque level	$\max T_{out}$ [Nm]	l_i [h]	$D_{i,A}$	$D_{i,B}$
Low	800	323	0.145	0.04
Rated (T_N)	1150	558	0.286	0.088
High (T_A)	1610	74	0.04	0.01
Total			0.471	0.138

Table 4.7: Evaluation of main bearing damage during the switching test at three torque levels. $D_{i,A}$ and $D_{i,B}$ denote the damage values for bearings A and B, respectively, while l_i corresponds to the operating time t_{op} listed in Table 4.8.

the tests [105]. The latter condition implies that the gearbox is periodically remounted. This way, the load is redistributed over different areas of the raceway, and standard dimensioning procedures can be applied. In this case, the factor $a_{osc,i}$ increases the bearing life, as expected [104].

The cumulative damage (Palmgren–Miner rule) is calculated as:

$$D = \sum_i D_i = \sum_i \frac{l_i}{L_{10h,i}} \quad (4.22)$$

where l_i is the time spent at load level i , in hours. A bearing is considered to have reached its fatigue life when $D = 1$.

The last step is the evaluation of the basic dynamic load rating C_r according to [9], which depends on the internal geometry (roller diameter and length, number of rollers, and bearing diameters). For confidentiality reasons, detailed calculations are omitted, but its value is $C_r = 33623$ N.

Once the model is ready, the objective is to evaluate the damage caused during the switching test of Sec. 4.4.1.1, using data from Tab. 4.8. The table provides the maximum torque for each load level, but said torque is only achieved for a short period during the full cycle. To obtain a precise calculation, the concept of cumulative damage should be applied twice, as follows.

- Firstly, for each torque level in Tab. 4.8, the output torque is considered over the oscillation cycle (Fig. 4.17-b) and divided into several equidistant torque levels. Then, Eq. (4.21) is applied to all load levels P_i , setting the rotational speed n_i equal to the average speed at a given load level. Finally, Eq. 4.22 is applied for each torque level (i.e., 800 Nm, 1150 Nm, and 1610 Nm), taking into account the corresponding time spent at that load level, obtaining three separate damage values.
- Secondly, the same concept is applied to combine the three torque levels: the total damage is obtained by summing the three individual damage values.

The results are reported in Table 4.7. The damage values for both bearings are below 50%. It is worth noting that the radial load during the switching test is constant, since the flywheel weight does not change; only the instantaneous rotational speed varies.

To summarize, recent studies [109, 110] show that small-angle oscillations are particularly detrimental to the rolling elements and raceways of a bearing. These failure



Figure 4.14: Main bearing failure.

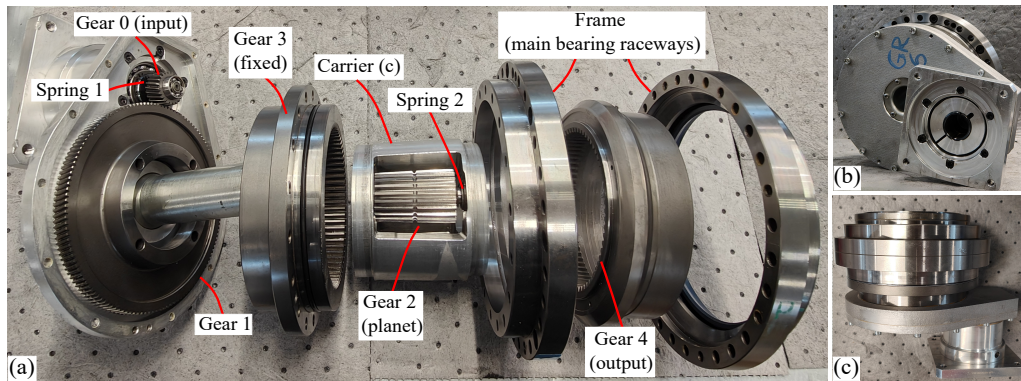


Figure 4.15: (a): Prototype of the Omni OH, size 060 (disassembled view). (b, c): External views after assembly.

modes are hard to predict accurately using conventional bearing-life estimation methods, thus the results of Tab. 4.7 should be taken with caution. Therefore, contact analysis and experimental tests become necessary.

In the Omni first prototypes, the raceways for the main bearing were not heat treated: in tests, pitting failure was found at the points of maximum pressure and limited lubrication (Fig. 4.14, upper-right side of the main bearing). The current prototypes, with heat-treated raceways, yielded instead excellent results. A contact analysis in KISSsoft, considering the internal geometry of the main bearing and the shaft deformations, confirmed the experimental results. Indeed, the maximum contact pressure on the rolling surfaces is 1676 MPa (Fig. 4.11b), which is acceptable for heat-treated case-hardening steel.

4.4 Experimental results

This Section will report experimental results only for prototypes of the Omni OH gear-box, with size 060 and $\tau = 1:125$ (Fig. 4.15), whose rated parameters are provided in Table 4.1. The same tests were performed on three Omni OH prototypes, finding very little variability in results, which thus showed very good repeatability. The results were obtained on a prototype without micro-modifications of the planets. Prototypes of the Omni OS version were also tested (Fig. 4.16): the corresponding results are not reported since they are very similar to those obtained for the OH architecture.

In particular, results are reported from *endurance tests*, in which reliability is assessed over time, and from *characterisation tests*, aimed at evaluating specific gear-

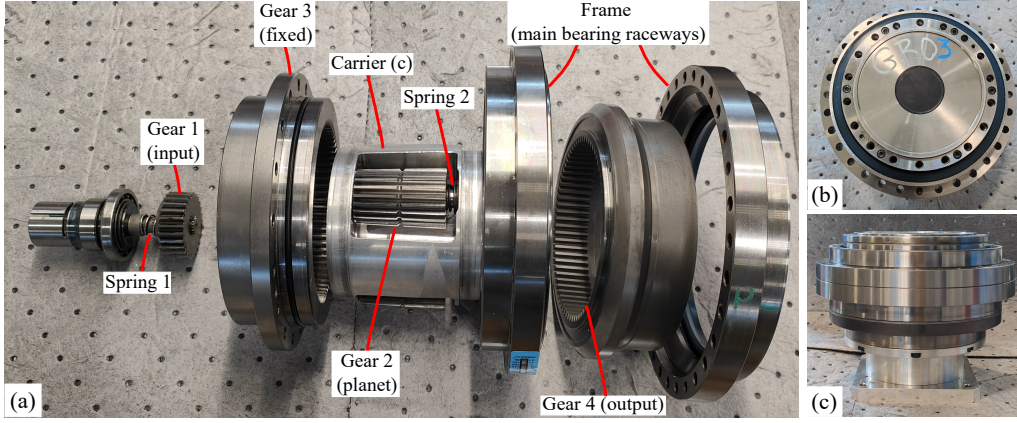


Figure 4.16: (a): Prototype of the Omni OS, size 060 (disassembled view). (b, c): External views after assembly.

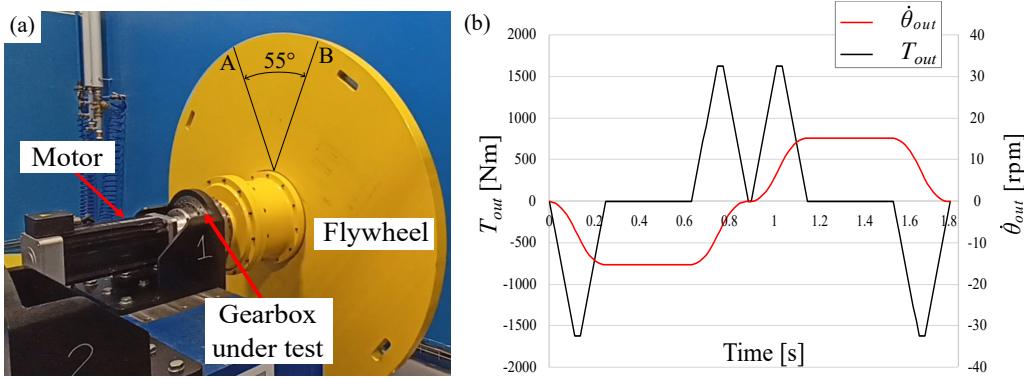


Figure 4.17: (a): Experimental bench for the switching test. (b): The output torque and speed during the switching test at the highest torque level. The speed is defined with a trapezoidal trajectory, including an acceleration phase, a constant-speed phase, a deceleration phase, and a change of direction.

box properties. Preliminary contact pressure analyses and multibody simulations were also conducted to forecast the expected behaviour and identify potential failure causes.

In this Section, the rotation angle and the torque at the output shaft (gear 4) are θ_{out} and T_{out} , respectively. The input angle and torque, relative to gear 0 (for the OH) are θ_{in} and T_{in} .

4.4.1 Endurance tests

Since the actual use of the gearbox in a robot arm cannot be predicted, it is necessary to conduct tests in the most demanding conditions. Thus, during endurance tests, the gearbox was subjected to extreme conditions that are unlikely to occur in its operational lifetime.

4.4.1.1 Switching

This test involves periodic oscillations of fixed amplitude, with an inertial load due to a 662 kg flywheel fixed to the output flange (Fig. 4.17a). The flywheel is at an offset $l_f = 300$ mm from the main bearing: due to the flywheel weight $W_b = 6.49$ kN, the bearing is subject to a bending moment $M_b = 1950$ Nm and to a radial force $F_r = W_b$ (Fig. 4.11a). The test is carried out in three steps, with increasing levels of torque

Torque level	max T_{out} [Nm]	max $\ddot{\theta}_{out}$ [rad/s ²]	t_C [s]	N_C [millions]	t_{op} [h]	t_{op} [%]
Low	800	5.0	2.33	0.5	323	33.8
Rated (T_N)	1150	7.5	2.01	1.0	558	58.4
High (T_A)	1610	10.0	1.78	0.15	74	7.8
Total				1.65	955	100

Table 4.8: Switching-test torque levels. For all torque levels, the maximum output speed $\dot{\theta}_{out}$ is at the rated value of 15 rpm (Tab. 4.1). t_C is the cycle time, N_C is the number of cycles performed, while $t_{op} = t_C N_C / (3600)$ is the total operational time at each level.

(Tab. 4.8). The torque is modified by increasing the angular acceleration while keeping all other parameters constant (oscillation angle, maximum speed, and flywheel inertia). The output torque over time for the highest torque level is given in Fig. 4.17b (the lower torque curves are just scaled down and cycle times are slightly higher). The cycle consists of a repetitive 55° oscillation from position “A” to position “B” and return. This angle was chosen to have the peak torque on the same circular sector of each planet (containing about three teeth), since planets are the gears with the lowest safety factors. To this end, the Omni kinematic analysis was solved to calculate the position, speed, and acceleration of each gear, and, thus, of each tooth, finding that a certain oscillating angle exists in which the acceleration peak is always supported by the same circular sector of the planets. A multibody analysis of the internal components of the gearbox was performed as a verification. In the multibody model, created using NX Motion Simulation by Siemens, the components were rigid and constrained to move according to Figs. 3.3 and 3.4. The kinematic parameters were set to replicate those of the actual test. The goal of this simulation was to validate the kinetostatic analysis in Sec. 4.2 and verify that, in each oscillation, the planet teeth being loaded are always the same ones.

The Omni OH was tested first at the “low” torque level, followed by the “rated” and “high” ones (Tab. 4.8), until bending failure occurred on a planet tooth. The testing cycle represented in Table 4.8 was defined by taking a routine operation of a real robot as a reference. Considering the 1st joint (equipped with a gearbox similar to Omni OH, size 060) of a palletizing robot having a rated payload of 50 kg and performing an average cycle composed of roughly 2 s of under-load operation and 18 s of rest. The torque spectrum of the under-load operation was discretized according to [101], thus originating the three torque levels listed in Table 4.8. However, our tests did not comprise any rest period, in order to reach failure in a shorter time, thus leading to a more severe condition than the real operation. If a rest phase of 18 s at any motion inversion of the flywheel is considered, namely a rest time of 36 s per cycle, the total rest time associated with the cycles listed in Table 4.8 would be $t_{rest,tot} = 36 N_{C,tot} / 3600 = 16500$ h. Assuming a continuous operation of 12 hours a day for 365 days a year, the theoretical duration of the tested gearbox in a more realistic scenario would be $(t_{op,tot} + t_{rest,tot}) / (12 \cdot 365) \approx 4$ years. In our experience, a robot gearbox has

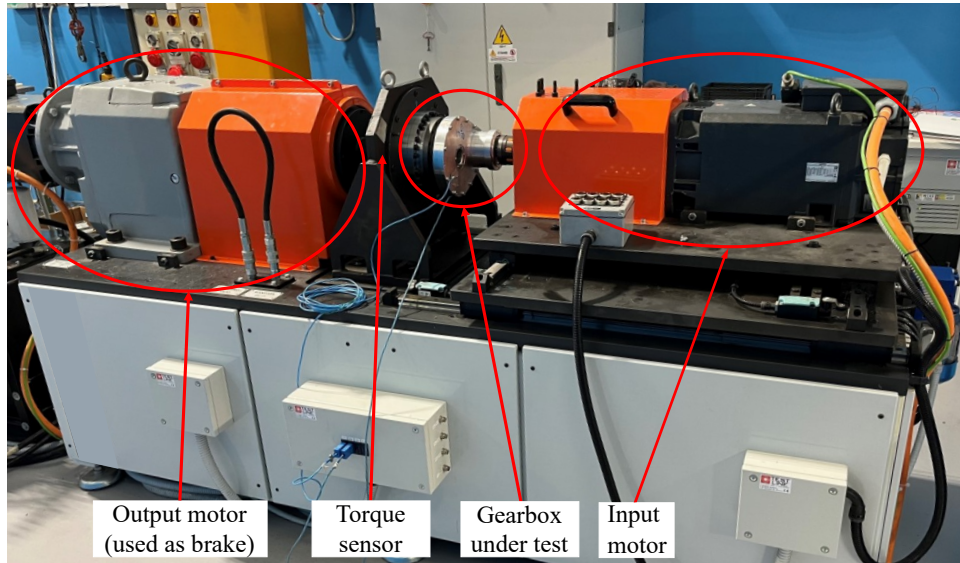


Figure 4.18: The test bench used for continuous, monodirectional testing.

a lifespan of around 5 years. Thus, the performance of the Omni gearbox appears acceptable, considering that it is still at a prototype stage. A robot lifespan is strongly influenced by its use: for example, the occurrence of emergency stops has a significant impact. For this reason, in future tests, more time will be spent running the gearbox at the highest torque level.

The switching test is designed to stress both the teeth, through alternating movements, and the main bearing, due to the bending moment and the radial force generated by the flywheel. This test is particularly stressful for teeth, because the gears undergo alternated bending. For this condition, the allowable bending stress of the material is reduced 0.7 times [8] relative to the standard case (for unidirectional tooth loading).

4.4.1.2 Continuous duty cycle

Figure 4.18 illustrates the test bench used for tests under a continuous, monodirectional duty cycle (S1). While the switching test (Sec. 4.4.1.1) focuses on bending failure (which always occurs before pitting), with a peak torque reached only for about 0.06 s four times per cycle (Fig. 4.17b), in this case, the load torque is kept constant. This test is focused on identifying pitting on one flank (depending on the rotation direction), as the torque never changes sign: thus, the opposite flank is never loaded. This particularly unfavourable condition is not very realistic since it does not occur during actual use of precision gearboxes, but it is convenient for accelerated fatigue testing. According to [91], considering the Wöhler curve of a case-carburized steel, the fatigue limit for pitting is reached at 50 million cycles, while for bending it is at only 3 million cycles, which explains why focusing on one flank accelerates the test. The number of cycles N_{pl} of the planets are considered, since they have the lowest safety factors. From our multibody model, a given tooth of the planet experiences the peak load 64.5 times in a complete revolution of the output ring gear. The Omni OH 060 gearbox was tested at an output speed of 15 rpm, with 6 levels of constant torque, gradually increasing from the lowest level to the highest (Tab. 4.9). After 505 hours (corresponding to about 29 million load cycles on each planet tooth) the test was stopped since abnormal vibrations were

Torque level [Nm]	N_C	N_{pl} [mln]	t_T [h]	t_T [%]
800	48600	3.13	55	11
900	41400	2.67	45	9
1000	17100	1.1	20	4
1200	48600	3.13	55	11
1300	34200	2.2	40	8
1600	261000	16.8	290	57
Total	450900	29	505	100

Table 4.9: Torques levels of the continuous duty cycle. N_C is the number of cycles of the output ring gear, whereas N_{pl} is the number of cycles performed by a planet. t_T is the total time at each torque range.

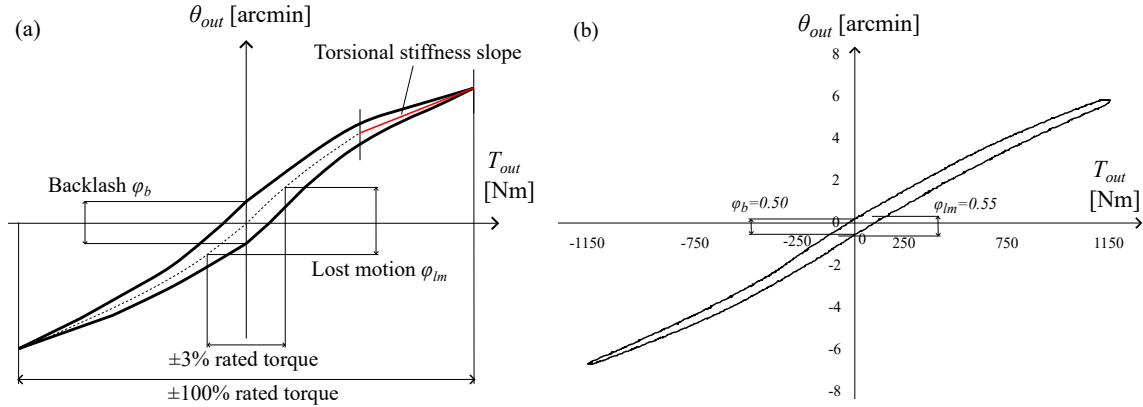


Figure 4.19: (a): Example of a generic hysteresis curve with the main related definitions. The interval $\pm 3\%$ of the rated torque is not drawn to scale, for the sake of clarity. (b): The hysteresis curve for the Omni OH gearbox, with size 060, and transmission ratio $\tau_{OH} = 1:125$.

detected by the accelerometers mounted on the gearbox: pitting was indeed detected on the planet flank. The number of cycles supported by the planets before pitting is in line with theoretical calculations performed through our KISSsys model.

4.4.2 Characterisation tests

4.4.2.1 Hysteresis

This test is performed by applying a certain torque T_{out} to the output shaft, both in the clockwise and counter-clockwise directions, and measuring the corresponding rotation θ_{out} while the input shaft is fixed. The torque-angle curve thus derived, exemplified in Fig. 4.19a, allows the estimation of several parameters. The average slope (in red) in the interval between 50% and 100% of the rated torque is the *torsional stiff-*

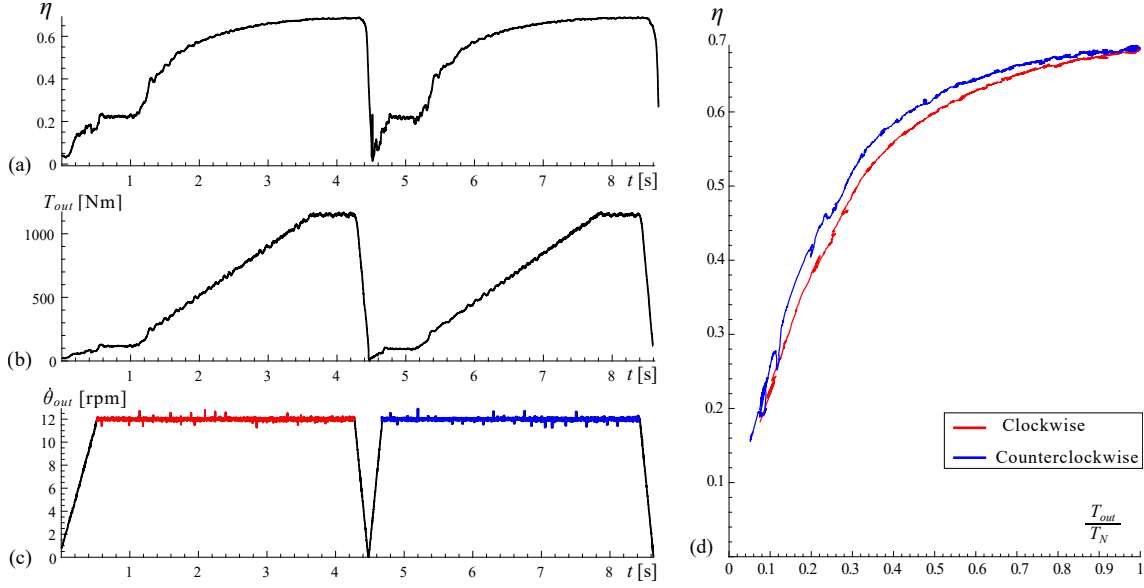


Figure 4.20: (a,b,c): Efficiency η , output torque T_{out} and output speed $\dot{\theta}_{out}$ as a function of time, for clockwise and counter-clockwise rotations of the output ring, for the Omni OH 060, transmission ratio 1:125. (d): Efficiency curve as a function of the output torque. During these tests, the oil temperature was constant within $\pm 0.05^\circ\text{C}$.

ness. The *backlash* is the maximum angular displacement of the output shaft when no torque is applied. Finally, the *lost motion*, which integrates the effects of torsional stiffness and backlash, is the maximum angular displacement of the output shaft, under an external torque T_{out} no greater than 3% of the rated torque.

Figure 4.19b shows the hysteresis curve of the Omni OH 060 gearbox, which has a torsional stiffness of 226 Nm/arcmin, when loaded between 600 Nm and 1200 Nm, and of 224 Nm/arcmin, between -600 Nm and -1200 Nm. The lost motion and the backlash are 0.55 arcmin and 0.5 arcmin, respectively. As expected, the hysteresis curve is nearly linear, a characteristic highly valued by robot manufacturers, as it simplifies control. The overall results are satisfactory: the torsional stiffness, with a target of around 320 Nm/arcmin, requires some improvement, while the lost-motion and backlash requirements, with a target of < 0.6 arcmin, are fully met.

4.4.2.2 Efficiency

To measure efficiency, the input and output shafts are connected respectively to a motor and a brake, which apply known torques T_{in} and T_{out} . The efficiency is then found as $\eta = (\tau T_{out})/T_{in}$. The bench is the same one used for endurance tests (Fig. 4.18). In the leftmost graphs of Fig. 4.20, the efficiency of the Omni OH 060 is shown as a function of time, together with the output torque T_{out} and speed $\dot{\theta}_{out}$; in the rightmost graph, efficiency is plotted against the output torque T_{out} , reaching a steady-state value of 70%. The variation of the efficiency with the output torque is mainly due to meshing losses that are load-dependent, reaching smaller values close to the rated torque. An advantage of the Omni architecture is that its efficiency quickly stabilizes at a steady-state value. Since robot gearboxes typically operate at lower torques than the rated one, the average efficiency (which is around 57.5 and 58.5 % for clockwise and counter-clockwise rotations) is a more meaningful parameter.

Although steady-state efficiency is lower than competitor designs (see Tab. 3.1), the

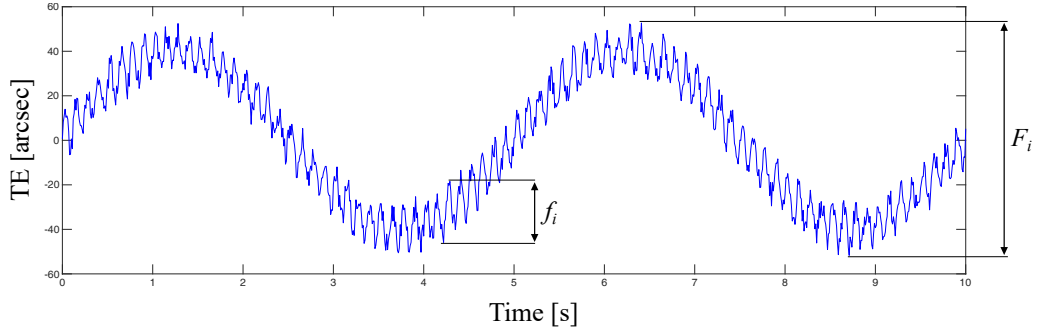


Figure 4.21: Schematic of a measurement of the TE over time, with the first- and second-order contributions (F_i and f_i , respectively).

Omni is still a prototype, whose efficiency can be significantly improved. To this end, possible improvements are changes in the seal type, the type and quantity of lubricant, the number of planets, quality of tooth surfaces, the main bearing preload, and the micro-geometry of the planets. With higher values of η , the expectation is to conduct endurance tests over longer periods due to the reduction in temperature and friction.

4.4.2.3 Transmission error

The *transmission error* (TE) is often used in gear diagnostics [111] or to control the quality of gearbox components. The TE is defined as the difference between the theoretical angular position $\theta_{out,t}$ of the output shaft (for a given position $\theta_{in,r}$ of the input) and its actual position $\theta_{out,r}$:

$$TE = \theta_{out,r} - \theta_{out,t} = \theta_{out,r} - \theta_{in,r}/\tau \quad (4.23)$$

The angles $\theta_{in,r}$ and $\theta_{out,r}$ are measured by encoders on the respective shafts.

The TE is caused by geometrical errors and deformations of the components of the gearbox, mainly the shafts and the gear teeth. According to [112], there are three components of the TE:

1. *Geometric* TE: it is due to deviations of the real tooth profile from the ideal one and is measured by performing a test at minimum load and speed.
2. *Static* TE: it is due to elastic tooth deformations and is measured at the nominal load and at the minimum speed;
3. *Dynamic* TE: it is due to elastic deformations combined with inertial and stiffness effects; it is measured at nominal speed and nominal load.

The measurement of the TE is not trivial, and calculating its spectrum is usually required to properly filter the raw signal [113]. This way, the first-order contributions F_i , due to macroscopic errors (such as the misalignment of shafts or gears), can be separated from second-order effects f_i , typically due to errors in the tooth profile (such as tooth shape, pitch, and so on); the latter also depends on the gear mesh frequency (Fig. 4.21). First-order contributions may also arise from errors during the mounting of the gearbox on the bench or from the bench itself, particularly when long input and output shafts are used. For robotic gearboxes, both F_i and f_i are important: the former is significant during *intermittent* motions (i.e., pick-and-place operations), where a good accuracy between initial and final positions is required, while the latter is more

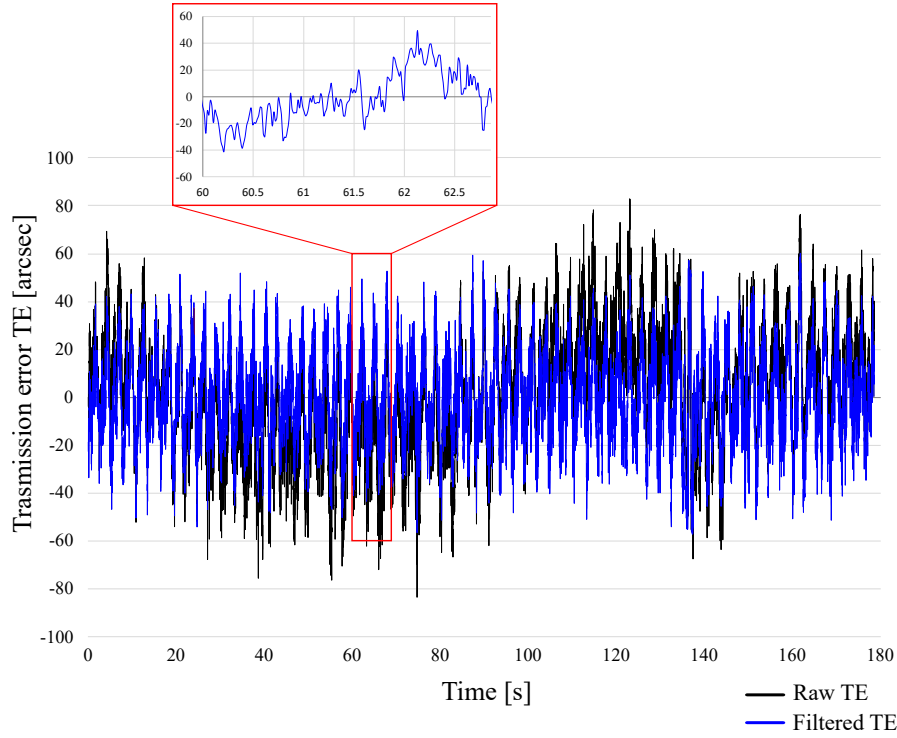


Figure 4.22: Transmission error for the Omni OH 060. A zoom plot is reported in red. Note that here, unlike in Fig. 4.19b, the angular unit of measurement is 1 arcsec = 1/60 arcmin.

relevant in *continuous* operations (i.e., welding), where a good accuracy is required along the entire path. In accordance with other gearbox producers, the larger of the two values is reported, which is generally F_i .

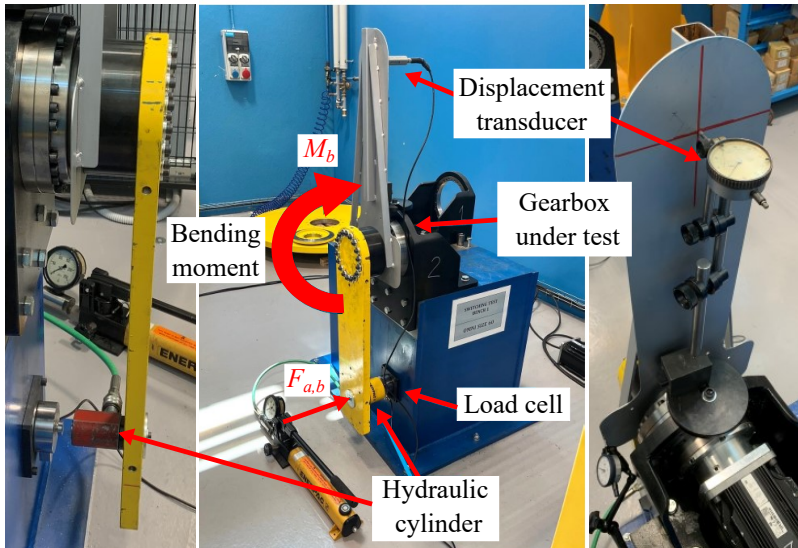
In this study, only geometric TE was measured due to our bench limitations (measuring static and dynamic TE's requires high-resolution encoders due to the higher speeds). The graph in Fig. 4.22 was obtained by rotating the output shaft for 360° at an input speed $\dot{\theta}_{in} = 42$ rpm, and filtering the raw signal to remove low-frequency components caused by misalignments in the bench shafts. Given the real transmission ratio $\tau_{OH} = 1 : 125.087$ of the Omni OH 060, and considering that, for each rotation of the output shaft, the planets mesh $N_m = 64.5$ times with the ring gears, the meshing frequency f_m in Hertz (Hz) is

$$f_m = \frac{\dot{\theta}_{in}}{60\tau_{OH}} N_m = 0.361 \quad (4.24)$$

Thus, high-pass filter is applied to the encoder signals, removing the components at frequencies lower than f_m . The filtered TE is within a range of ± 60 arcsec (namely ± 1 arcmin), which is comparable to the results reported by competitors [72, 95, 96].

4.4.2.4 Bending stiffness

The setup for testing bending stiffness is shown in Fig. 4.23. The gearbox output flange is pushed by a force $F_{a,b}$ exerted by a hydraulic piston at a distance $L = 600$ mm from the main axis a , using the yellow link. This way, a bending moment $M_b = F_{a,b}L$ is applied to the gearbox output. A linear transducer is used to measure the displacement Δ of a point on the gray link (opposite the piston) at a distance of 600 mm from the main axis a ; this way, the rotation of the output flange can be easily derived from the



Test #	Orientation [°]	$F_{a,b}$ [N]	M_b [Nm]	Δ [mm]	ϕ [arcmin]	K_ϕ [Nm/arcmin]
1	0	3862	2317	0.038	0.218	10.63
2	90	3870	2322	0.039	0.223	10.41

Table 4.10: Results of bending-stiffness tests for two orientations of the housing, reporting the axial force $F_{a,b}$, the bending moment $M_b = F_{a,b}L$, the axial displacement Δ , the rotation $\phi = \Delta/L$, and the bending stiffness $K_\phi = M_b/\phi$. The axial stiffness of the gearbox is considered almost infinite, thus Δ is assumed not to be transferred to the gearbox output shaft.

linear displacement measured by the transducer. Table 4.10 summarizes the results of two tests conducted at different orientations of the housing (0° and 90°) to assess possible non-uniform preload in the main bearing. The data show minimal variation between tests. The average value of the bending stiffness K_ϕ of the gearbox is 10.52 kNm/arcmin, which positively exceeds the target stiffness of 2300 Nm/arcmin. Given that an excessively high stiffness is not necessary, the preload on the main bearing could be reduced to further enhance efficiency.

Chapter 5

Conclusions

This thesis focused on the design, modelling, and testing of precision gearboxes for robotic joints. The work centered on the innovative and patented Omni gearbox — a Wolfrom planetary gearbox with beveloid gears — developed by Bonfiglioli Riduttori SpA. The project successfully combined academic research and industrial testing, bridging theory and real product validation.

The project contributed to the scientific understanding of beveloid gears, for which there are currently no recognised sizing standards. The development of analytical models for contact analysis and pitting resistance determination provides a valuable framework for optimisation and design studies.

The Omni gearbox was designed to meet the demanding needs of robotic applications: minimal backlash, high stiffness, compact size, and long service life. It is available in two versions, with a solid (OS) or hollow (OH) shaft, and designed in ten sizes from 010 to 100. The study focused on the Omni 060 prototype, with a 1:125 reduction ratio, as the first step toward full product validation.

A detailed understanding of beveloid gears and Wolfrom gear architectures was essential to achieve these goals. Since there are no official standards for sizing beveloid gears, this work developed new analytical models to support their design and optimisation.

The analytical model based on ISO 6336-1 (Annex E) and ISO 6336-2, was created to predict pitting on parallel-axis helical beveloid gears. The method divides the beveloid gear into sections, each treated as a spur gear, and calculates load distribution across them through an iterative process. The model considers both shaft deflections and profile modifications, providing a pitting safety factor for each section and identifying the most critical one. Its predictions were investigated by experimental tests.

An improved model was developed during a research stay at UNED (Madrid). This version, based on elastic potential energy minimisation, can analyse straight beveloid gears with parallel axes under negligible axial load. It allows the study of contact geometry, stiffness, and load sharing while accounting for undercutting and pointing effects. A contact map was introduced to visualise meshing behaviour and identify key parameters, such as the contact ratio, as well as the start and end contact points. Stiffness was calculated for each transverse section and integrated along the contact line. The contact behaviour and stiffness curve are similar to those of helical gears. Finally, the load distribution was derived, followed by a contact-stress analysis. A finite-element model confirmed the analytical results.

Two experimental case studies were considered, as follows:

- **Parallel-axis beveloid gearbox:** it is a parallel-axis gearbox manufactured by Bonfiglioli, used to validate the analytical model based on ISO 6336-1 (Annex E) and ISO 6336-2 through back-to-back endurance tests. Its relative simplicity and availability made it ideal for this purpose. The objective was to investigate the Wöhler curve of the beveloid gear pair. The model was used to determine the test parameters, namely to predict the test duration at a given output speed and torque. To prevent unexpected failures of the remaining gearbox components — including shafts, bearings, housing, and additional gear pairs — a KISSsys model was created using standard methods. Complete results are not available due to unexpected failures. Only one beveloid gear pair was fully tested, with pitting observed after 500 hours versus the expected 600 hours. However, the debris circulating in the lubricant may have influenced these results, meaning they cannot be considered exhaustive.
- **Omni gearbox:** it is the main object of this research. The architecture, general properties, and kinetostatic analysis were first discussed. Particular attention was given to the practical design considerations employed in sizing the Omni, especially with regard to its main bearing and to the meshing between planets and ring gears. The step-by-step procedures for building multibody, finite-element, and KISSsys models were detailed. The combination of a Wolfrom planetary architecture with beveloid gears allowed the gearbox to achieve very low torsional backlash (< 0.6 arcmin) and small transmission ratios (up to 1:160) while maintaining a compact design. Endurance and characterisation tests confirmed that the Omni gearbox can withstand highly demanding operating conditions, delivering performance comparable to other market products. Most measured parameters, such as torsional and bending stiffness, transmission error, and backlash, met or exceeded the target values.

The tests on the Omni gearbox produced some very interesting results. However, there is still room for improvement in terms of efficiency and durability. Future work will focus on optimising lubrication, gear flank modifications and preloads. Another important goal will be to study efficiency in reverse motion, which is essential for good back-drivability in robotic joints.

Research and experimental testing of helical beveloid gears is currently ongoing. Ultimately, the methodology will be extended to complete planetary gear systems, especially Wolfrom architectures to support the design of next-generation precision transmissions for robotics.

Appendix A

A.1 Kinematic model for the switching test

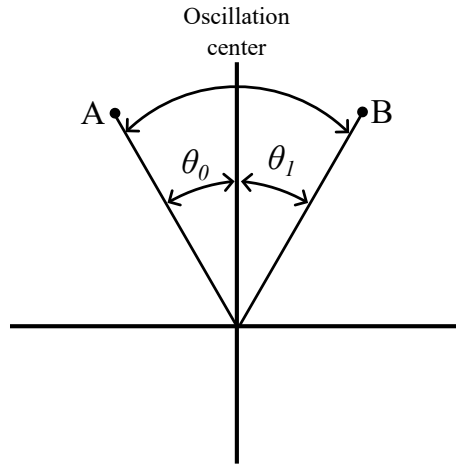


Figure A.1: Switching test schematic.

Here we provide the formulas used to develop the kinematic model of the switching test presented in Sec. 4.4.1.1. The test involves oscillations around a fixed center as in Fig. A.1. The cycle starts in “A”, reaches the maximum velocity at the oscillation center, stops in “B” to change direction, and finally goes back to “A”. Each oscillation is modeled with a trapezoidal speed profile, which includes three phases:

1. constant acceleration (linear increase in speed);
2. zero acceleration (constant speed);
3. constant deceleration (linear decrease in speed).

For each oscillation, the following parameters are defined

1. initial θ_0 and final $\theta_1 = -\theta_0$ angular positions, measured with respect to the oscillation center;
2. initial velocity $\dot{\theta}_0$ and maximum velocity $\dot{\theta}_{\max}$ to be reached during the zero acceleration phase;
3. maximum acceleration reached $\ddot{\theta}_{\max}$.

The following quantities are derived:

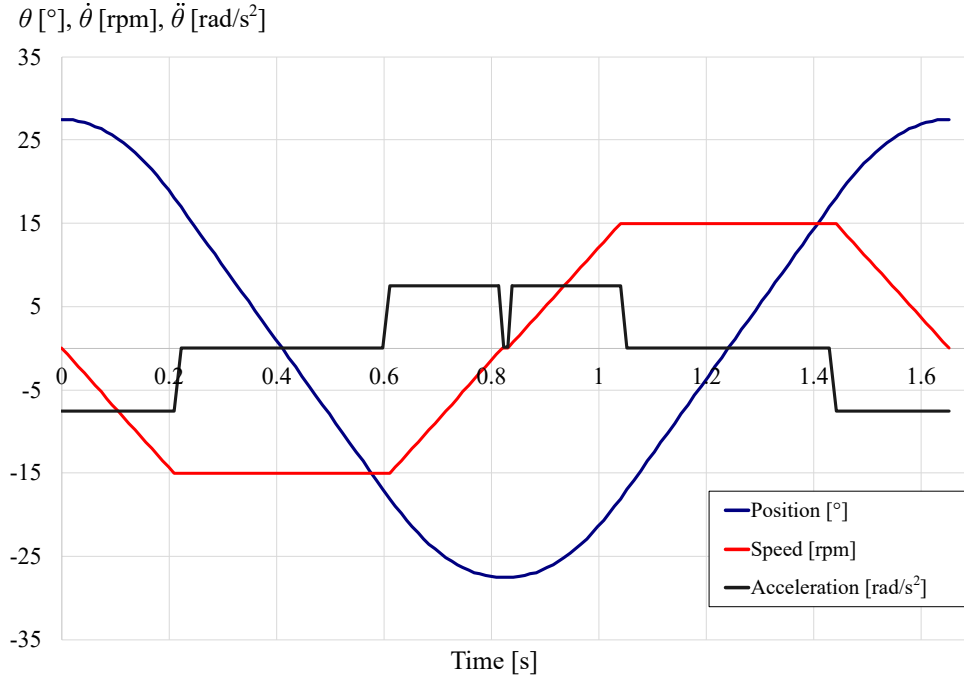


Figure A.2: Duty cycle with maximum acceleration $\ddot{\theta}_{\max} = 7.5 \text{ rad/s}^2$, maximum speed $\dot{\theta}_{\max} = 15 \text{ rpm}$, and $\theta_0 = -\theta_1 = 27.5^\circ$.

1. Acceleration time:

$$t_{\text{acc}} = \frac{\dot{\theta}_{\max} - \dot{\theta}_0}{\ddot{\theta}_{\max}} \quad (\text{A.1})$$

2. Distance covered during acceleration:

$$\Delta\theta_{\text{acc}} = \frac{\ddot{\theta}_{\max}}{2} t_{\text{acc}}^2 + \dot{\theta}_0 t_{\text{acc}} + \theta_0 \quad (\text{A.2})$$

3. Distance and time at constant velocity:

$$\Delta\theta_{\text{vel,c}} = (\theta_1 - \theta_0) - 2\Delta\theta_{\text{acc}}, \quad t_{\text{vel,c}} = \frac{\Delta\theta_{\text{vel,c}}}{\dot{\theta}_{\max}} \quad (\text{A.3})$$

The motion law during the acceleration phase, is built as follows.

1. Define θ_0 , $\dot{\theta}_0$ and $\ddot{\theta}_{\max}$.
2. At each time t_i , the acceleration is constant and equal to $\ddot{\theta}_{\max}$.
3. At each t_i , the velocity is:

$$\dot{\theta}_i = \dot{\theta}_{i-1} + \ddot{\theta}_{\max}(t_i - t_{i-1}) \quad (\text{A.4})$$

4. At each t_i , the position is:

$$\theta_i = \theta_{i-1} + \frac{\dot{\theta}_i + \dot{\theta}_{i-1}}{2}(t_i - t_{i-1}) \quad (\text{A.5})$$

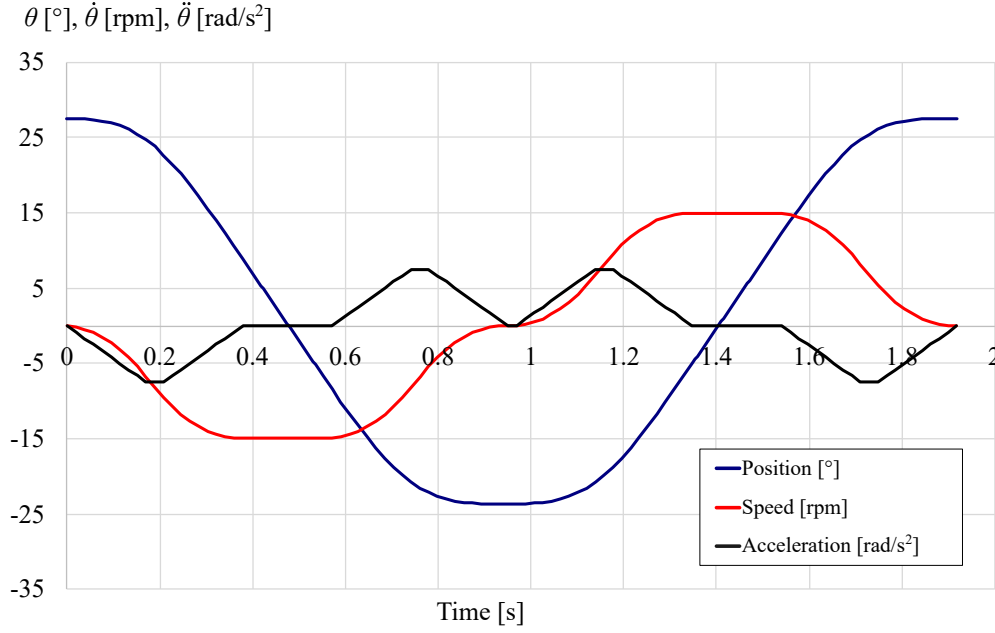


Figure A.3: Duty cycle with triangular acceleration profile: $\ddot{\theta}_{\max} = 7.5 \text{ rad/s}^2$, $\dot{\theta}_{\max} = 15 \text{ rpm}$, $\theta_0 = -\theta_1 = 27.5^\circ$.

To build the motion law during the constant velocity phase, at each time t_i the following holds:

1. the acceleration is zero: $\ddot{\theta}_i = 0$
2. the velocity is constant: $\dot{\theta}_i = \dot{\theta}_{\max}$
3. the position is:

$$\theta_i = \theta_{i-1} + \dot{\theta}_{\max}(t_i - t_{i-1}) \quad (\text{A.6})$$

For the deceleration phase, formulas similar to those for the first phase can be derived using $-\ddot{\theta}_{\max}$. The curves of position, speed and acceleration are provided in Fig. A.2.

A more realistic law of motion can be achieved by replacing the step acceleration with a *triangular* acceleration profile. All the considerations made so far are still valid, except the definition of the acceleration. A *triangular* acceleration profile can be defined over the time interval $0 < t_i < t_{acc}$, characterised by a maximum value $\ddot{\theta}_{\max}$ reached at the central instant $t_{acc}/2$ of the acceleration phase. The instantaneous acceleration at time t_i is defined as follows:

$$\ddot{\theta}_i = \begin{cases} \frac{2\ddot{\theta}_{\max}}{t_{acc}} t, & \text{if } 0 \leq t \leq \frac{t_{acc}}{2} \\ \frac{2\ddot{\theta}_{\max}}{t_{acc}} (t_{acc} - t), & \text{if } \frac{t_{acc}}{2} < t \leq t_{acc} \end{cases} \quad (\text{A.7})$$

The resulting profile has a triangular shape, with a linearly increasing acceleration phase from 0 to $t_{acc}/2$, followed by a symmetric decreasing phase from $t_{acc}/2$ to t_{acc} . The obtained motion law is more similar to the one provided by the switching test motor. The acceleration profile, with the related speed and position, is shown in Fig. A.3,

which are describing the duty cycle at rated torque level $T_N = 1150$ Nm. Comparing the profiles in Figs. A.2 and A.3, it can be observed that the speed profile in the second case is smoother, particularly near the transition points.

A.2 Deformation potential energy principle

A.2.1 Introduction

Previous studies [84] developed a new model for load distribution along the line of contact for involute external gears, based on the minimum potential energy principle. Analytical expressions were derived for the elastic energy potentials associated with bending, compression, and shear, as functions of the contact point and of the applied force. Subsequently, the load distribution that minimises this potential was determined, resulting in a non-uniform load distribution that varies both along the line of contact and along the line of action. This approach was then applied to helical gears, considered as being composed of many spur gears of infinitesimal face width, whose geometry is defined by the transverse section tooth profile, and extending the integrals along the line of contact. This approach allows determining the load per unit length at any point along the line of contact and for any position within the meshing cycle.

Initially, numerical results were obtained through the numerical integration of the elasticity equations. Later, an approximate analytical equation for the meshing stiffness and the load sharing ratio was developed [86], allowing the same analysis to be performed in a shorter time.

In the deformation potential energy approach, the spur gear tooth is modeled as a cantilever beam with rectangular cross-section. The beam has a constant thickness, corresponding to the tooth face width, and a variable width (equal to the tooth thickness) along the involute profile. The material is assumed to be homogeneous and isotropic, and the stiffness varies along the tooth profile. Similarly, the total potential of a pair of meshing teeth is equal to the sum of their individual potentials at the point of contact.

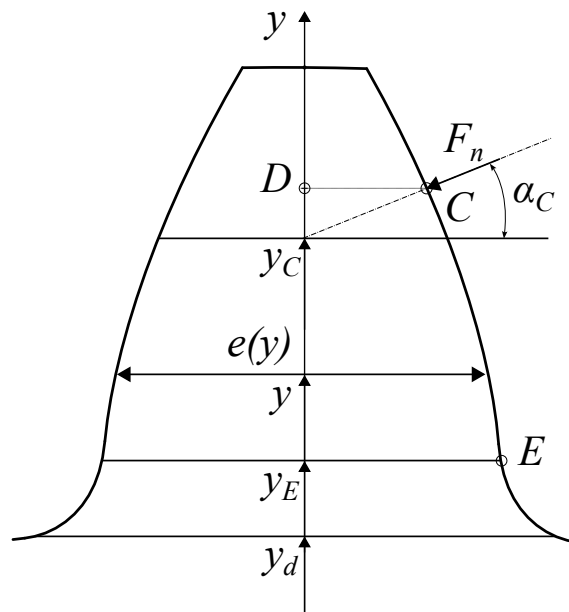


Figure A.4: Geometrical parameters of an involute tooth. Image source: [87].

A.2.2 Tooth deformation potential energy

The method reported here is derived from the thesis work in [87]. The deformation potential energy U_D of a spur gear tooth subjected to a normal force F_n on its flank can be expressed as the sum of bending (U_b), shear (U_s) and compression (U_n) potential energies:

$$U_D = U_b + U_s + U_n \quad (\text{A.8})$$

These components can be evaluated [114] according to the theory of elasticity:

$$U_b = \int_{y_d}^{y_c} \frac{M(y)^2}{2E_y I(y)} dy \quad (\text{A.9})$$

$$U_s = \int_{y_d}^{y_c} \frac{C_s T^2}{2G\Omega(y)} dy \quad (\text{A.10})$$

$$U_n = \int_{y_d}^{y_c} \frac{N^2}{2E_y \Omega(y)} dy \quad (\text{A.11})$$

Defining the geometrical parameters of an involute tooth as in Fig. A.4, the variable y is measured along the symmetry axis of the tooth from the center of the gear; y_d is the value of y at the root section, and y_c is the value of y at the point C where the load is applied. The latter is defined by the intersection of the line of action of F_n (which is normal to the tooth profile at C) and the symmetry axis of the tooth. The coefficients E_y and G are, respectively, the Young's and the shear moduli of the material. The coefficient C_s is a correction factor for the shear potential which, according to Collignon's theorem, accounts for the non-uniform distribution of shear stresses across the section. For rectangular sections, this coefficient is $C_s = 1.5$. M , T and N are the bending moment, the shear force and the compression force due to F_n . T and N are perpendicular and parallel to the y -axis, respectively. The aforementioned components are computed as:

$$M = F_n \cos \alpha_C (y_C - y) \quad (\text{A.12})$$

$$T = F_n \cos \alpha_C \quad (\text{A.13})$$

$$N = F_n \sin \alpha_C \quad (\text{A.14})$$

Note that the bending moment depends on position of C , unlike forces. The contact angle α_C can be defined according to Fig. A.5 as:

$$\alpha_C = \theta_C - \gamma_b/2 = \theta_C - \frac{1}{2} \left[\frac{\pi}{Z} + \frac{4x}{Z} \tan \alpha_n + 2(\tan \alpha_t - \alpha_t) \right] \quad (\text{A.15})$$

where γ_b is the angular thickness evaluated at the base radius r_b , Z is the number of teeth, x is the shift profile coefficient, α_n is the pressure angle, and α_t is the transverse pressure angle. α_n and α_t are constants and depend only on the teeth geometry (and, for beveloid gears, α_t can be found from α_n , δ and β). θ_C defines the position of the contact point:

$$\theta_C = \sqrt{\frac{r_C^2}{r_{b1}^2} - 1} \quad (\text{A.16})$$

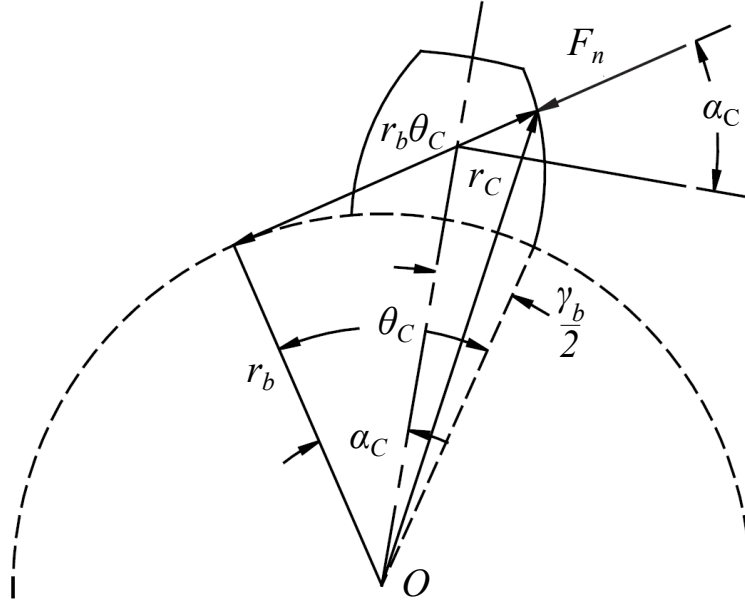


Figure A.5: Contact angle α_C . Image source: [87].

The moment of inertia I and the area Ω of a rectangular section are computed as:

$$I = \frac{b e^3(y)}{12} \quad (\text{A.17})$$

$$\Omega = b e(y) \quad (\text{A.18})$$

where b is the face width. The tooth thickness $e(y)$ at the section having coordinate y depends on the flank profile and is found as

$$e(y) = 2r(y) \sin \frac{\gamma(y)}{2} \quad (\text{A.19})$$

Here, $r(y)$ is the radius from the center of the gear to the point of the involute profile at coordinate y , while $\gamma(y)$ is the tooth angular thickness, given by

$$\gamma(y) = \frac{\pi}{Z} + \frac{4x}{Z} \tan \alpha_n + 2(\tan \alpha_t - \alpha_t) - 2 \left[\sqrt{\frac{r(y)^2}{r_b^2} - 1} - \arctan \left(\sqrt{\frac{r(y)^2}{r_b^2} - 1} \right) \right] \quad (\text{A.20})$$

Substituting Eqs. (A.12) to (A.14), (A.17) and (A.18) in Eqs. (A.9) to (A.11), we find:

$$U_b = 6 \frac{F^2 \cos^2 \alpha_C}{E_y b} \int_{y_d}^{y_c} \frac{(y_C - y)^2}{e(y)^3} dy \quad (\text{A.21})$$

$$U_s = \frac{1}{2} \frac{F^2 \cos^2 \alpha_C}{G b} \int_{y_d}^{y_c} \frac{1}{e(y)} dy \quad (\text{A.22})$$

$$U_n = \frac{1}{2} \frac{F^2 \sin^2 \alpha_C}{E_y b} \int_{y_d}^{y_c} \frac{1}{e(y)} dy \quad (\text{A.23})$$

To simplify the integrals, we express M depending on point D , at the intersection of axis y with the line perpendicular to said axis and passing through C (Fig. A.4). Equation (A.12) becomes:

$$M = F \cos \alpha_C (y_D - y) - F \sin \alpha_C \frac{e(y_D)}{2} \quad (\text{A.24})$$

To further simplify the problem, the integrals can be split by defining two regions with distinct geometries, one for the part of the tooth flank having involute profile and the other for the trochoid part of the profile. The boundary between these regions is at point E (having coordinate y_E , see Fig. A.4), which can be analytically determined [87]. The bending potential energy is thus computed as follows:

$$U_b = U_{b,tr} + U_{b,in} \quad (A.25)$$

$$U_{b,tr} = \frac{6F^2}{E_y b} \int_{y_d}^{y_E} \frac{\left[\cos \alpha_C (y_D - y) - \sin \alpha_C \frac{e(y_D)}{2} \right]^2}{e(y)^3} dy \quad (A.26)$$

$$U_{b,in} = \frac{6F^2}{E_y b} \int_{y_E}^{y_D} \frac{\left[\cos \alpha_C (y_D - y) - \sin \alpha_C \frac{e(y_D)}{2} \right]^2}{e(y)^3} dy \quad (A.27)$$

The shear potential energy is found as:

$$U_s = U_{s,tr} + U_{s,in} \quad (A.28)$$

$$U_{s,tr} = \frac{1}{2} \frac{F^2 \cos^2 \alpha_C}{Gb} \int_{y_d}^{y_E} \frac{1}{e(y)} dy \quad (A.29)$$

$$U_{s,in} = \frac{1}{2} \frac{F^2 \cos^2 \alpha_C}{Gb} \int_{y_E}^{y_D} \frac{1}{e(y)} dy \quad (A.30)$$

Finally, the compression potential energy is:

$$U_n = U_{n,tr} + U_{n,in} \quad (A.31)$$

$$U_{n,tr} = \frac{1}{2} \frac{F^2 \sin^2 \alpha_C}{E_y b} \int_{y_d}^{y_E} \frac{1}{e(y)} dy \quad (A.32)$$

$$U_{n,in} = \frac{1}{2} \frac{F^2 \sin^2 \alpha_C}{E_y b} \int_{y_E}^{y_D} \frac{1}{e(y)} dy \quad (A.33)$$

A.2.3 Tooth pair deformation potential energy

In a pair of involute gears (pinion and wheel, denoted by subscripts 1 and 2, respectively), the curvature radii of the transverse profile at the contact point obey the following [115]:

$$\rho_1 + \rho_2 = \sqrt{r_{C1}^2 - r_{b1}^2} + \sqrt{r_{C2}^2 - r_{b2}^2} = (r_{b1} + r_{b2}) \tan \alpha'_t \quad (A.34)$$

where α'_t is the working transverse pressure angle, and r_{C1} and r_{C2} are the radii of C with respect to the axes of the two gears. α'_t is constant along the gear wheel and is equal to α_t if the center distance is equal to the theoretical value. α_C is equal to α'_t at the moment when C is on the pitch circle, at the midpoint of the contact segment. In general, however, α_C varies during contact.

A variable ξ is introduced to represent the contact position of the pinion along the meshing cycle:

$$\xi_1 = \frac{Z_1}{2\pi} \theta_{C1} = \frac{Z_1}{2\pi} \sqrt{\frac{r_{C1}^2}{r_{b1}^2} - 1} \quad (A.35)$$

This parameter is introduced to simplify the calculation of the potential energy and serves the same purpose as θ_C (namely, defining the actual contact point), but is dimensionless. The contact limits are:

- θ_{inn1} and ξ_{inn1} , at the tooth head of the wheel (where the contact starts);
- θ_{out1} and ξ_{out1} , at the tooth head of the pinion (where the contact ends).

It holds $\xi_{out} = \xi_{inn} + \varepsilon_\alpha$. Equation (A.34) can be rewritten as a function of ξ_1 and ξ_2 , as follows:

$$\xi_1 + \xi_2 = \lambda = \frac{Z_1 + Z_2}{2\pi} \tan \alpha'_t \quad (\text{A.36})$$

The deformation potential energy U of a pair of meshing teeth is equal to the sum of the deformation potential energies of the pinion and wheel. From Eqs. A.34 and A.36, U can be rewritten as function of ξ_1 , as:

$$U = U_1(r_{C1}) + U_2(r_{C2}) = U_1(r_{C1}) + U_2(r_{C1}) = U(r_{C1}) = U(\xi_1) \quad (\text{A.37})$$

Therefore, a change of variable (from y to ξ) is required to calculate the total potential energy. Also, the integration limits must be updated accordingly:

$$U_{b,in} = \frac{6F^2}{E_y b} \int_{\xi_E}^{\xi_D} \frac{\left[\cos \alpha_C (y_D - y(\xi)) - \sin \alpha_C \frac{e(y_D)}{2} \right]^2}{e(\xi)^3} \frac{dy}{d\xi} d\xi \quad (\text{A.38})$$

$$U_{s,in} = \frac{1}{2} \frac{F^2 \cos^2 \alpha_C}{Gb} \int_{\xi_E}^{\xi_D} \frac{1}{e(\xi)} \frac{dy}{d\xi} d\xi \quad (\text{A.39})$$

$$U_{n,in} = \frac{1}{2} \frac{F^2 \sin^2 \alpha_C}{E_y b} \int_{\xi_E}^{\xi_D} \frac{1}{e(\xi)} \frac{dy}{d\xi} d\xi \quad (\text{A.40})$$

Note that the definition of ξ only applies to the points on the involute profile. For the trochoid curve, the change of variable is different, and y becomes function of φ , as will be shown in the next page (Eq. (A.45)):

$$U_{b,tr} = \frac{6F^2}{E_y b} \int_{\varphi_d}^{\varphi_E} \frac{\left[\cos \alpha_C (y_D - y(\varphi)) - \sin \alpha_C \frac{e(y_D)}{2} \right]^2}{e(\varphi)^3} \frac{dy}{d\varphi} d\varphi \quad (\text{A.41})$$

$$U_{s,tr} = \frac{1}{2} \frac{F^2 \cos^2 \alpha_C}{Gb} \int_{\varphi_d}^{\varphi_E} \frac{1}{e(\varphi)} \frac{dy}{d\varphi} d\varphi \quad (\text{A.42})$$

$$U_{n,tr} = \frac{1}{2} \frac{F^2 \sin^2 \alpha_C}{E_y b} \int_{\varphi_d}^{\varphi_E} \frac{1}{e(\varphi)} \frac{dy}{d\varphi} d\varphi \quad (\text{A.43})$$

The equations up to this point are known from [87], where however the integrals were not solved explicitly. This step is not trivial and is detailed here for completeness.

The variable ξ is made explicit in the parameters needed to compute the involute integrals in Eqs. (A.38), (A.39) and (A.40):

$$e(\xi) = 2r(\xi) \sin \left(\frac{\gamma(\xi)}{2} \right) \quad (\text{A.44})$$

$$y(\xi) = r(\xi) \cos \left(\frac{\gamma(\xi)}{2} \right) \quad (\text{A.45})$$

$$r(\xi) = r_b \sqrt{\left(\frac{2\pi}{Z}\xi\right)^2 + 1} \quad (\text{A.46})$$

$$\gamma(\xi) = \frac{\pi}{Z} + \frac{4x}{Z} \tan \alpha_n + 2(\tan \alpha_t - \alpha_t) - 2 \left[\frac{2\pi}{Z}\xi - \arctan\left(\frac{2\pi}{Z}\xi\right) \right] \quad (\text{A.47})$$

$$\alpha_C(\xi) = \theta_C(\xi) - \gamma_b/2 = \frac{2\pi}{Z}\xi - \gamma_b/2 \quad (\text{A.48})$$

The derivative of y with respect to ξ follows from the chain rule:

$$\frac{dy}{d\xi} = \frac{d}{d\xi} \left[r(\xi) \cos\left(\frac{\gamma(\xi)}{2}\right) \right] = r'(\xi) \cos\left(\frac{\gamma(\xi)}{2}\right) - r(\xi) \sin\left(\frac{\gamma(\xi)}{2}\right) \left(\frac{1}{2}\gamma'(\xi)\right) \quad (\text{A.49})$$

Evaluating the derivatives r' and γ' leads to:

$$\frac{dy}{d\xi} = \frac{4r_b\pi^2\xi \cos\left(\frac{\gamma(\xi)}{2}\right)}{Z^2\sqrt{\frac{4\pi^2\xi^2}{Z^2} + 1}} + \left[r_b\sqrt{\frac{4\pi^2\xi^2}{Z^2} + 1} \right] \sin\left(\frac{\gamma(\xi)}{2}\right) \left(\frac{2\pi}{Z} - \frac{2\pi}{Z\left(\frac{4\pi^2\xi^2}{Z^2} + 1\right)} \right) \quad (\text{A.50})$$

Now, the integrals in Eqs. (A.38), (A.39) and (A.40) can be evaluated numerically; their lower and upper limits are

$$\xi_{E1} = \frac{Z_1}{2\pi}\theta_{E1}, \quad \xi_{E2} = \frac{Z_2}{2\pi}\theta_{E2} \quad (\text{A.51})$$

where θ_{E1} and θ_{E2} can be computed with Eq. (A.16).

While the lower limits ξ_{E1} and ξ_{E2} are fixed, the upper limits depend on the point of contact (in the interval $\theta_{inn} < \theta_D < \theta_{out}$). Thus, the same integral must be solved for all the contact points within this interval (using the chosen discretisation). Finally, the upper limits of the integrals can be computed as follows:

$$\xi_{D1} = \frac{Z_1}{2\pi}\theta_{D1}, \quad \xi_{D2} = \frac{Z_1 + Z_2}{2\pi} \tan(\alpha'_t) - \xi_{D1} \quad (\text{A.52})$$

where the latter equation is derived from Eq. (A.36).

Now the potential energy has to be evaluated for the trochoid profile, using Eqs. (A.41) to (A.43). In this case the integral limits are fixed, and their values are $\varphi_P = \pi/2$ and $\varphi_E = \alpha_n$ [87]. The parameters can be written as functions of φ , as follows:

$$r(\varphi) = \sqrt{(r_p - \delta_x(\varphi))^2 + \left(\frac{\delta_x(\varphi) \cos \beta}{\tan \varphi}\right)^2} \quad (\text{A.53})$$

$$\gamma(\varphi) = \gamma_b - 2(\eta(\varphi) + \varepsilon(\varphi) - \alpha_t) \quad (\text{A.54})$$

$$\eta(\varphi) = \tan \alpha_t + \frac{1}{r_p} \left(mr_f \frac{\cos \varphi}{\cos \beta} - \delta_y(\varphi) - \delta_x(\varphi) \frac{\cos \beta}{\tan \varphi} \right) \quad (\text{A.55})$$

$$\varepsilon(\varphi) = \text{atan} \left(\frac{\delta_x(\varphi) \cos \beta}{(r_p - \delta_x(\varphi)) \tan \varphi} \right) \quad (\text{A.56})$$

$$\delta_x(\varphi) = m(h_{a0} - x - r_f + r_f \sin \varphi), \quad \delta_y = \frac{mr_f}{\cos \alpha_n \cos \beta} + m(h_{a0} - x - r_f) \tan \alpha_t \quad (\text{A.57})$$

The definition of these parameters can be found in [87]; they will not be detailed here, as it is outside the scope of this work.

From the above, the derivative of y with respect to φ can be written as

$$\frac{dy}{d\varphi} = \frac{d \left[r(\varphi) \cos \left(\frac{\gamma(\varphi)}{2} \right) \right]}{d\varphi} = f'(\varphi)g(\varphi) + f(\varphi)g'(\varphi) \quad (\text{A.58})$$

$$f(\varphi) = r(\varphi) = \sqrt{(r_p - \delta_x(\varphi))^2 + \left(\frac{\delta_x(\varphi) \cos \beta}{\tan \varphi} \right)^2} \quad (\text{A.59})$$

$$f'(\varphi) = \frac{-mr_f \cos \varphi (r_p - \delta_x(\varphi)) - \frac{\cos^2 \beta (\tan^2 \varphi + 1) \delta_x(\varphi)^2}{\tan^3 \varphi} + \frac{m_n r_f \cos^2 \beta \cos \varphi \delta_x(\varphi)}{\tan^2 \varphi}}{f(\varphi)} \quad (\text{A.60})$$

$$g(\varphi) = \cos \left(\frac{\gamma(\varphi)}{2} \right) = \cos \left(\frac{\gamma_b - 2[\eta(\varphi) + \varepsilon(\varphi) - \alpha_t]}{2} \right) \quad (\text{A.61})$$

$$g'(\varphi) = \sin \left(\frac{\gamma_b - 2[\eta(\varphi) + \varepsilon(\varphi) - \alpha_t]}{2} \right) [\eta'(\varphi) + \varepsilon'(\varphi)] \quad (\text{A.62})$$

$$\eta'(\varphi) = - \frac{\left(\frac{mr_f \sin \varphi}{\cos \beta} \right) + \left(\frac{mr_f \cos \beta \cos \varphi}{\tan \varphi} \right) - \left(\frac{\cos \beta (\tan^2 \varphi + 1) \delta_x(\varphi)}{\tan^2 \varphi} \right)}{r_p} \quad (\text{A.63})$$

$$\varepsilon'(\varphi) = \frac{\frac{mr_f \cos \beta \cos \varphi}{\tan \varphi (r - \delta_x(\varphi))} - \frac{\cos \beta (\tan^2 \varphi + 1) \delta_x(\varphi)}{\tan^2 \varphi (r_p - \delta_x(\varphi))} + \frac{mr_f \cos \beta \cos \varphi \delta_x(\varphi)}{\tan \varphi (r_p - \delta_x(\varphi))^2}}{1 + \left(\frac{\delta_x(\varphi) \cos \beta}{(r_p - \delta_x(\varphi)) \tan \varphi} \right)^2} \quad (\text{A.64})$$

We can now evaluate the total potential elastic energy of the gear pair by summing all components:

$$U(\xi) = U_{in,1}(\xi) + U_{in,2}(\xi) + U_{tr,1}(\xi) + U_{tr,2}(\xi) \quad (\text{A.65})$$

For each contact point, the *single mesh stiffness* (SMS), namely, the stiffness of a single tooth pair in contact, is then calculated as:

$$K_M(\xi) = \frac{F_n^2}{2U(\xi)} \quad (\text{A.66})$$

A.2.4 Multiple teeth in contact

The number of tooth pairs in contact varies along the meshing cycle, and in general there will be multiple contacts, depending on the value of ε_α . Thus, the total potential is the sum of the potentials for all pairs in contact. To account for this, the *Load Sharing Ratio* (LSR) is introduced:

$$R_i(\xi) = \frac{F_i(\xi_i)}{F} = \frac{K_M(\xi)}{\sum_j K_{M,j}(\xi)} \quad (\text{A.67})$$

where $F_i(\xi_i)$ is the force acting on the i -th tooth pair. When only one tooth-pair is in contact, then LSR is 1. The sum $\sum_j K_{M,j}(\xi)$ in Eq. (A.67) considers all the tooth pairs (denoted by index j) in simultaneous contact at the meshing position defined by ξ ; for example, $j = 0$ is for the tooth pair under consideration, $j = 1$ is for the previous tooth pair, $j = -1$ is for the next tooth pair, and so on. The limits of index j are given by integer closest to the effective contact ratio $\varepsilon_{\alpha,eff}$. This parameter defines how the total load F_n is distributed among all the teeth in contact at the contact position defined by ξ .

A.2.5 Final considerations

Note that the load distribution $F_i(\xi_i)$ defined in Eq. (A.67) is constant along the tooth width, if all contact points along the contact line are defined by the same ξ (as it is the case for spur gears).

For helical gears the load distribution is not constant along the face width. We can instead view the helical tooth as divided into an infinite number of slices, each perpendicular to the gear axis. This way, a tooth is a sequence of infinitesimally thin spur teeth, each rotated by the helix angle relative to the others. For each section, the approach shown in this Section can be applied.

The load distribution of beveloid spur and helical gears is not constant across the width of the tooth due to the varying geometry of the sections. Their behaviour is similar to that of standard helical gears.

A.3 Approximate computation of the pointing radius

The angular thickness at the tooth tip is:

$$\gamma_o = \frac{\pi}{Z} + \frac{4x}{Z} \tan \alpha_n^* + 2(\tan \alpha_n^* - \alpha_n^*) - 2(\theta_o - \arctan \theta_o) \quad (\text{A.68})$$

where:

$$\theta_o = \sqrt{\left(\frac{r_o}{r_b}\right)^2 - 1} \quad (\text{A.69})$$

$$\alpha_n^* = \arctan(\tan \alpha_n \cos \delta) \quad (\text{A.70})$$

α_n^* being the pressure angle of the transverse section of the beveloid gear and δ the cone angle. Accordingly, for the pointed teeth the angular thickness is null:

$$0 = \frac{\pi}{Z} + \frac{4x}{Z} \tan \alpha_n^* + 2(\tan \alpha_n^* - \alpha_n^*) - 2(\theta_o^* - \arctan \theta_o^*) \quad (\text{A.71})$$

Subtracting Eqs. (A.68) and (A.71) term by term yields:

$$(\theta_o^* - \arctan \theta_o^*) - (\theta_o - \arctan \theta_o) = \frac{\gamma_o}{2} \quad (\text{A.72})$$

Function $f(\theta_o^*) = (\theta_o^* - \arctan \theta_o^*)$ can be approximated by the first three terms of the Taylor series around θ_o , which results in:

$$f(\theta_o^*) = f(\theta_o) + f'(\theta_o)(\theta_o^* - \theta_o) + \frac{1}{2}f''(\theta_o)(\theta_o^* - \theta_o)^2 \quad (\text{A.73})$$

$$f(\theta_o^*) - f(\theta_o) = \frac{\gamma_o}{2} = \frac{\theta_o^2}{1 + \theta_o^2}(\theta_o^* - \theta_o) + \frac{1}{2} \frac{2\theta_o}{(1 + \theta_o^2)^2}(\theta_o^* - \theta_o)^2 \quad (\text{A.74})$$

The last equation can be directly solved, resulting in:

$$(\theta_o^* - \theta_o) = \frac{1}{2}(1 + \theta_o^2) \left[-\theta_o \pm \sqrt{\theta_o^2 + 2\frac{\gamma_o}{\theta_o}} \right] \quad (\text{A.75})$$

If the initial value of γ_o is greater than 0 (the initial tooth is not pointed), θ_o^* should be greater than θ_o , and therefore $(\theta_o^* - \theta_o) > 0$, so the real solution is given by the “+” sign:

$$\theta_o^* = \theta_o + \frac{(1 + \theta_o^2)}{2} \left[\sqrt{\theta_o^2 + 2 \frac{\gamma_o}{\theta_o}} - \theta_o \right] \quad (\text{A.76})$$

Finally, according to Eq. (A.69), the pointing radius is:

$$r_o^* = r_b \sqrt{1 + \theta_o^{*2}} \quad (\text{A.77})$$

A.4 Approximate computation of the outside radius for a specific tip thickness

To ensure a specific tooth thickness at the tip (some Standards recommend a minimum thickness of $0.3m$ [89]), the thickness function $e(\theta_o)$ can be developed in Taylor series around the pointing value θ_o^* computed in A.3 and approximated by the first three terms.

The tooth thickness at the tip can be expressed as:

$$e(\theta_o) = r_o \gamma_o = r_b \sqrt{1 + \theta_o^2} [\gamma_b - 2(\theta_o - \arctan \theta_o)] \quad (\text{A.78})$$

where γ_b is the angular thickness between the involute profiles of both flanks at the base circle:

$$\gamma_b = \frac{\pi}{Z} + \frac{4x}{Z} \tan \alpha_n^* + 2(\tan \alpha_n^* - \alpha_n^*) \quad (\text{A.79})$$

Approximating $e(\theta_o)$ by the first three terms of the Taylor series gives:

$$e(\theta_o) = -2r_b \frac{\theta_o^{*2}}{\sqrt{1 + \theta_o^{*2}}} (\theta_o - \theta_o^*) - \frac{1}{2} 4r_b \frac{\theta_o^*}{\sqrt{1 + \theta_o^{*2}}} (\theta_o - \theta_o^*)^2 \quad (\text{A.80})$$

which results in:

$$\theta_o = \frac{1}{2} \left(\theta_o^* + \sqrt{\theta_o^{*2} - 2 \frac{\sqrt{1 + \theta_o^{*2}}}{\theta_o^*} \frac{e(\theta_o)}{r_b}} \right) \quad (\text{A.81})$$

Specifically, to meet the minimum tip thickness recommended by the standards [89], the tip point should verify the condition:

$$\theta_o \leq \frac{1}{2} \left(\theta_o^* + \sqrt{\theta_o^{*2} - 2 \frac{\sqrt{1 + \theta_o^{*2}}}{\theta_o^*} \frac{0.3m}{r_b}} \right) \quad (\text{A.82})$$

and the outside radius should verify:

$$r_o \leq r_b \sqrt{1 + \theta_o^2} \quad (\text{A.83})$$

List of Author Publications

1. F. Magri, G. Sciarra, G. Mottola, G. Lelli, and M. Carricato, "Design and Modeling of a Dosing Unit in a Capsule-Filling Machine," in *Proceedings of IFIT 2024*, Turin, Italy, 11–13 September 2024. https://doi.org/10.1007/978-3-031-64553-2_21
2. G. Sciarra, G. Mottola, G. Casamenti, and M. Carricato, "An Innovative Low-Backlash Wolfrom Gearbox with Beveloid Gears for Robotic Applications," in *Proceedings of EuCoMeS 2024*, Padova, Italy, 18–20 September 2024. https://doi.org/10.1007/978-3-031-67295-8_38
3. G. Sciarra, G. Mottola, L. Pezzuolo, G. Casamenti, and M. Carricato, "Pitting resistance determination for beveloid gears," in *Book of Abstracts of MMT Symposium 2024*, Guimarães, Portugal, pp. 317–318, 26–28 June 2024. <https://hdl.handle.net/11585/973198>
4. G. Sciarra, G. Mottola, G. Casamenti, and M. Carricato, "Omni: A low-backlash planetary Wolfrom gearbox with beveloid gears for robotic applications," *Mechanism and Machine Theory*, vol. 214, 106098, 2025. <https://doi.org/10.1016/j.mechmachtheory.2025.106098>
5. G. Sciarra, M. Carricato, M. Pleguezuelos, M. B. Sánchez, and J. I. Pedrero, "Determinación del dominio de contacto y reparto de carga en engranajes cónicos de evolvente de ejes paralelos y dientes rectos (in Spanish)," presented at XXV Congreso Nacional de Ingeniería Mecánica, Santander, Spain, 2025; published in: *Revista Iberoamericana de Ingeniería Mecánica*, vol. 29, no. 1, pp. 03–16, 2025.
6. G. Sciarra, G. Mottola, M. Pleguezuelos, M. B. Sánchez, M. Carricato, and J. I. Pedrero, "Analysis of the contact stress of beveloid gears with straight teeth and parallel axes," *Mechanism and Machine Theory*, vol. 214, 106156, 2025. <https://doi.org/10.1016/j.mechmachtheory.2025.106156>

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