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DIAGRAM CALCULUS FOR AFFINE COXETER SYSTEMS AND PROPERTIES OF THEIR ASSOCIATED ROOT SYSTEMS

Presentata da: Elisa Sasso

Coordinatore Dottorato

Giovanni Mongardi

Supervisore

Riccardo Biagioli

Co-supervisore

Jacopo Gandini

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Contents

Introduction

The core of the present thesis is devoted to a detailed study of some algebraic structures associated with affine Coxeter systems, with a particular focus on types $\widetilde{D}$ and $\widetilde{B}$. Coxeter groups have been studied from several perspectives such as algebraic, geometric, and combinatorial, and play a fundamental role in many well-known mathematical areas, such as Kazhdan–Lusztig theory. Classical examples of Coxeter groups are the symmetric group S_n and the dihedral group $I_2(m)$, that is the group of symmetries of a regular m-gon. In general, finite Coxeter groups can be realized as groups of geometric transformations generated by reflections. If we relax the notion of reflection, this realization extends to certain infinite Coxeter groups, namely the affine and hyperbolic ones. Finite and affine Coxeter groups have been completely classified (see [8, Appendix A1]).

One of the main objects of investigation in this thesis is the *Temperley–Lieb algebra*, which is a very classical mathematical object studied in algebra, combinatorics, statistical mechanics and mathematical physics, introduced by Temperley and Lieb in 1971 [46]. Kauffman [36] and Penrose [41] showed that the Temperley–Lieb algebra can be realized as a diagram algebra, that is, an associative algebra with a basis given by certain diagrams on the plane. On the other hand, Jones presented the Temperley–Lieb algebra in terms of abstract generators and relations. In [34], he also showed that this algebra occurs naturally as a quotient of the Hecke algebra of type A. The realization of the Temperley–Lieb algebra as a Hecke algebra quotient was generalized by Graham in [24]. He defined the so-called *generalized Temperley–Lieb algebra* $\mathrm{TL}(\Gamma)$ for a Coxeter system of arbitrary type Γ and showed that it has a monomial basis indexed by the fully commutative elements $\mathrm{FC}(\Gamma)$ of the underlying Coxeter group. Over the years, some diagrammatic representations for $\mathrm{TL}(\Gamma)$ have been found. More precisely, Green defined a diagram calculus in finite Coxeter types B, D, E and H, respectively in [26], [29] and [25]. For affine types, in [21] Fan and Green provided a realization of $\mathrm{TL}(\widetilde{A})$ as a diagram algebra on a cylinder and in [18, 19], Ernst represented $\mathrm{TL}(\widetilde{C})$ as an algebra of decorated diagrams. In order to complete the classification of diagrammatic representations, we focused on the affine remaining cases, namely $\widetilde{D}$ and $\widetilde{B}$, and provided two algebras made of decorated diagrams which are isomorphic to the corresponding Temperley–Lieb algebras.

Several strategies have been used to prove the faithfulness of the known diagrammatic representations of Temperley–Lieb algebras of arbitrary type. For instance, in type $\widetilde{C}$ two different proofs of injectivity exist: the original one due to Ernst [19] uses non-cancellable elements in Coxeter groups, while more recently in [3], an algorithmic proof based on a decomposition of the heaps associated with fully commutative elements is proposed. We also provided two ways to prove the faithfulness of these two diagrammatic representations: one uses a subset of fully commutative elements that are irreducible with respect to a well-known operation in Coxeter groups called the *star operation*, and another one a more combinatorial approach from which an algorithm for factorizing decorated diagrams can be derived.

For the first way, we needed to classify such star irreducible fully commutative elements in types $\widetilde{D}$ and $\widetilde{B}$. The notion of the star operation was first introduced by Kazhdan and Lusztig in [37] for simply laced Coxeter systems and later extended to arbitrary Coxeter systems in [38]. Star operations provide a powerful tool to study the decomposition of a Coxeter group into Kazhdan–Lusztig cells. These cells play a fundamental role in representation theory. Here, we consider a special case of the star operation that strictly decreases the length of the element to which it is applied [27], and following [17], we focus on the star and weak star reducibility of fully commutative elements. Over the years, star irreducible elements have been classified. Green [27] defined a Coxeter group to be *star reducible* if and only if all of its star irreducible elements are products of commuting generators. In particular, he proved [27, Theorem 6.3] that finite types A, B, D, E, F, H, I and affine types $\widetilde{A}_n$ (with n even), $\widetilde{C}_n$ (with n odd), $\widetilde{E}_6$, and $\widetilde{F}_5$ are star reducible Coxeter systems. Types $\widetilde{D}$ and $\widetilde{B}$ are not star reducible and their star irreducible elements were not classified, so our first result is a complete characterization of them.

Another important aspect we discussed in this thesis concerns the geometric representations of affine Coxeter groups. It is a standard approach to view Coxeter systems as groups generated by special linear transformations generalizing orthogonal reflections in Euclidean space. Given a Coxeter group W and a real vector space V equipped with a symmetric W-invariant bilinear form B, a *simple system* is a basis of V in bijection with the generators of W, and the orbit of this simple system is the set of *roots* of W. In [31], the authors define the *2-roots* in the symmetric tensor space $S^2(V)$ as

$$\alpha \vee \beta := \alpha \otimes \beta + \beta \otimes \alpha$$

where α and β are orthogonal roots. They focus on particular families of simply laced Weyl groups, and construct a basis $\mathcal{B}$ for the module of 2-roots $\alpha \vee \beta$. In particular, in types A and D, each 2-root of this basis can be depicted as the top portion of a diagram of the Temperley–Lieb algebra associated to the Weyl group. Finally, the authors show that $\mathcal{B}$ behaves like a simple system for a representation of W on $V \otimes V$. In [30], Green works on a Coxeter group of type D and generalizes the notion of 2-roots to k-roots for $k \geq 2$. He then constructs a basis for the module of the k-roots, which turns out to be useful for showing that the spherical functions of the Gelfand pair $(S_n, S_k \times S_{n-k})$ are non-negative linear combinations of this basis. Similarly, we worked to define a representation of an affine group W of type $\widetilde{D}$ on the k-th symmetric tensor product of V. For this, we use the k-roots and define a special subset of the k-roots, called *canonical*, which is in bijection with a subset of decorated diagrams that arise from our diagrammatic representation of the Temperley–Lieb algebra of type $\widetilde{D}$.

The last part of the thesis was motivated by a problem concerning the Poincaré series $T(q)$ of the set of reflections T of a Coxeter system. As reported by Brenti [10], Stembridge, during an open problem session at the Mathematical Sciences Research Institute at Berkeley in 1997, proposed the following problem:

Is it true that the Poincaré series $T(q)$ for a given Coxeter system is rational?

It turns out that a solution to Stembridge's problem was provided in 1999 by de Man [12]. However, the author seemed unaware of Stembridge's question and the Combinatorics community was not aware of the solution. This might be explained by the title of de Man's article *"The generating function for the number of roots of a Coxeter group"*, which address counting reflections by length at the end of the article in [12, §5.2]. His solution is based on the Brink–Howlett automaton [11] that recognized the language of *lexicographically ordered reduced words*. In [12, Theorem 4.1], de Man shows that the generating function $\Phi^+(q)$ of the depth of positive roots is a sum of rational functions that are obtained from a subset of states of the Brink–Howlett automaton and from the enumeration of the vertices of a family of graphs parameterized by T. So the generating function $\Phi^+(q)$ is rational. Since $T(q) = q\,\Phi^+(q^2)$, the Poincaré series of the reflections $T(q)$ is rational as well. However, the Brink–Howlett automaton contains a lot of states, most of them "dead states", which make this automaton difficult to use in practice. For instance, in affine type $\widetilde{A}_2$, the number of states of the Brink–Howlett automaton is bounded by 2^6; de Man provides some example of $T(q)$ using a computer to handle his formula.

We provide a simple explicit formula for $T(q)$ in the case of affine Coxeter systems as

$$T(q) = \sum_{t\in T} q^{\ell(t)} = q\frac{P(q^2)}{1-q^{2M}},$$

where $M \in \mathbb{N}$ and $P(q)$ is a palindromic polynomial.

These considerations, led us to study the language of palindromic reduced words. In fact, it is well-known that reflections admit palindromic reduced words. So, besides finding an explicit formula for a rational function for $T(q)$, another natural approach to this question is to find an automaton that recognizes one palindromic reduced word for each reflection. So such a language would be regular and the associated generating functions of paths in the automaton would be rational. However, it is well-known that even if a language $\mathcal{L}$ is regular, the associated language of palindromic words is not necessary regular. The question of the regularity of the language of palindromic reduced words, which are necessarily reduced words for reflections, is still open. In a recent article, Millićević provides palindromic reduced words for any reflections in finite Weyl groups, but underlines the question of finding a general algorithm. That question leads us to prove that the language of reduced words for *reflection-prefixes* is regular.

Now we briefly outline the structure of the thesis, which is divided into five chapters.

Chapter 1 provides a review of Coxeter system theory and introduces the main tools used throughout the work.

In Chapter 2, we study the star-irreducible fully commutative elements of affine types $\widetilde{D}_{n+2}$ and $\widetilde{B}_{n+1}$. More precisely, in Section 2.3, we classify the star irreducible elements of type $\widetilde{D}_{n+2}$ (Theorem 2.3.20). In Section 2.4, building on this classification and employing an injective map between the sets of fully commutative elements of types $\widetilde{D}_{n+2}$ and $\widetilde{B}_{n+1}$ (Definition 2.4.6), we also characterize the star irreducible elements (Theorem 2.4.9) and weak star irreducible elements (Theorem 2.4.14) of type $\widetilde{B}_{n+1}$.

In Chapter 3, we provide diagrammatic representations for the Temperley–Lieb algebras of types $\widetilde{D}_{n+2}$ and $\widetilde{B}_{n+1}$. We first prove the injectivity of the representation in type $\widetilde{D}_{n+2}$ by using the classification of star irreducible fully commutative elements of type $\widetilde{D}_{n+2}$ introduced in Chapter 2 (Theorem 3.4.8). Then, we define ad hoc injections between $\mathrm{FC}(\widetilde{B}_{n+1})$ and $\mathrm{FC}(\widetilde{D}_{n+2})$ and between $\mathbb{D}(\widetilde{B}_{n+1})$ and $\mathbb{D}(\widetilde{D}_{n+2})$, from which we deduce that $\mathbb{D}(\widetilde{B}_{n+1})$ is a faithful diagrammatic representation of $\mathrm{TL}(\widetilde{B}_{n+1})$ (Theorem 3.9.3).

In Chapter 4, we extend the notion of k-roots to Coxeter systems of affine type $\widetilde{D}_n$. We also generalize the notion of admissible Temperley–Lieb diagram and introduce the n, k-semidiagrams which are made of n horizontally aligned nodes joined by k edges, possibly decorated. Those n, k-semidiagrams are in bijection with the k-roots, and we show that the W-action on the k-roots matches with the W-action on the n, k-semidiagrams. Then we show that each k-root can be expressed as a linear combination of canonical k-roots, with non-negative coefficients, using the properties of n, k-semidiagrams (Theorem 4.4.10). However, the set of canonical k-roots is not a basis for the module of k-roots. A deeper reason lies in the rank of the stabilizer of a simple root of W, as described in the third example of [1, §4], therefore [31, Theorem 2.7] does not hold in this affine case.

Finally, in Chapter 5, we address two problems related to the rationality of the Poincaré series of the reflections. On one hand, we construct an automaton, based on the Garside shadow automaton, which recognizes the language of the so-called *reflection prefixes* (Theorem 5.2.20). Unfortunately, this automaton recognizes all the reduced expressions for each reflection. In future work, we plan to refine this automaton to obtain one that recognizes exactly one expression per reflection. On the other hand, restricting to affine Coxeter systems, we provide explicit formulas for $T(q)$ (Theorem 5.3.17).

Chapter 1

Preliminaries

1.1 Fully commutative elements and generalized Temperley–Lieb algebras

Definition 1.1.1. Let S be a finite set. A matrix $M : S \times S \to \{1, 2, \dots\} \cup \{\infty\}$ such that $(s, t) \mapsto m_{st}$, is called a *Coxeter matrix* if $m_{ss} = 1$ and, for $s \neq t$, $m_{st} = m_{ts}$. The group W defined via the presentation

$$W = \langle S \mid (st)^{m_{st}} = e, m_{st} < \infty \rangle$$

is called the *Coxeter group* associated with the Coxeter matrix M.

The defining relations of W can be written more explicitly as

$$s^2 = e, \quad \text{for all } s \in S; \tag{1.1}$$

$$st = ts, \quad \text{if } m_{st} = 2; \tag{1.2}$$

$$\underbrace{sts\cdots}_{m_{st}} = \underbrace{tst\cdots}_{m_{st}}, \quad \text{if } 3 \leq m_{st} < \infty, \tag{1.3}$$

where (1.2) are called *commutation relations* and (1.3) *braid relations*. The pair (W, S) is called a *Coxeter system*. We call *Coxeter graph* Γ associated to (W, S) the graph with set of vertices S and where the edges are the unordered pairs $\{s, t\}$ such that $m_{st} \geq 3$. The edges with $m_{st} \geq 4$ are labeled by m_{st}, otherwise they are usually left unlabeled.

Note that two vertices labeled by s and t are not adjacent if and only if the generators s and t commute, i.e. $m_{st} = 2$.

Example 1.1.2. The graph associated to the Coxeter system of type A_{n-1} is depicted in Figure 1.1.

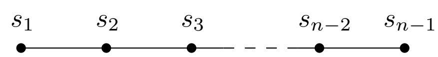


Figure 1.1: Coxeter graph of type A_{n-1}

The associated Coxeter group denoted by $W(A_{n-1})$ is generated by $S = \{s_1, s_2, \ldots, s_{n-1}\}$ and has defining relations

(1) $s_i^2 = e$ for all i;

(2) $s_i s_j = s_j s_1$ if $|i - j| > 1$;

(3) $s_i s_j s_i = s_j s_i s_j$ if $|i - j| = 1$.

In particular, the Coxeter group $W(A_{n-1})$ is isomorphic to the symmetric group S_n via the group isomorphism that maps s_i to the adjacent transposition $(i, i + 1)$ [8, Proposition 1.5.4].

Example 1.1.3. The Coxeter groups defined by the Coxeter graphs in Figure 1.2 are denoted by $W(\widetilde{D}_{n+2})$ and $W(\widetilde{B}_{n+1})$, and will play a central role throughout the thesis. They appear in the classification of the affine Coxeter systems in [8, Appendix A1].

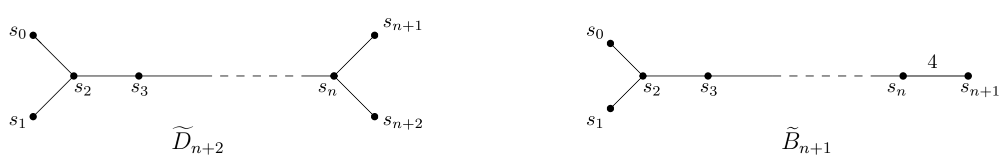


Figure 1.2: Coxeter graphs of type $\widetilde{D}_{n+2}$ and $\widetilde{B}_{n+1}$.

Every element $w \in W$ can be written as a word $\mathbf{w} = s_{i_1} s_{i_2} \cdots s_{i_r}$, called an *expression* for w. If r is minimal among all such expressions for w, then r is the *length* of w, denoted by $\ell(w)$, and the word $\mathbf{w} = s_{i_1} s_{i_2} \cdots s_{i_r}$ is called a *reduced expression* for w. We denote by $\mathcal{R}(w)$ the set of all the reduced expressions for w; we will usually denote the reduced expressions by bold letters.

If $A \subseteq W$, we define the *Poincaré series of A* as

$$A(q) := \sum_{a \in A} q^{\ell(a)}. \tag{1.4}$$

Sometimes, by the symbol $[ab]_k$ we will denote the expression of k letters starting with a and alternating a and b, where $a, b \in S$ and $k \in \mathbb{N}\backslash\{0\}$. For instance, $(st)^{m_{st}} = [st]_{2m_{st}}$.

We define the *left descents set* and the *right descents set* of $w \in W$ respectively by

$$D_L(w) := \{s \in S \mid \ell(sw) < \ell(w)\} \text{ and } D_R(w) := \{s \in S \mid \ell(ws) < \ell(w)\}.$$

We have that $s \in D_L(w)$ ($s \in D_R(w)$) if and only if there exists a reduced expression for w that starts with s (finishes with s). For more details, see [8, §1.4].

Let $u, v, w \in W$. We say that $w = uv$ is a *reduced product* if $\ell(w) = \ell(u) + \ell(v)$. In this case, u is called a *prefix* of w, and v is called a *suffix* of w. In general, u is a *factor* of w if there exist $x, y \in W$ such that $w = xuy$ and $\ell(w) = \ell(x) + \ell(u) + \ell(y)$. We denote by $\mathrm{Pref}(w)$ (respectively, $\mathrm{Suff}(w)$) the set of prefixes (respectively, suffixes) of w.

The following is sometimes called Matsumoto's Theorem (see for instance [33]).

Proposition 1.1.4. *In a Coxeter group W, any two reduced expressions in $\mathcal{R}(w)$, with $w \in W$, differ by a finite sequence of commutations and braid moves.*

We define an equivalence relation in $\mathcal{R}(w)$ as the reflexive transitive closure of the relation: given $\mathbf{w}, \mathbf{w}' \in \mathcal{R}(w)$, $\mathbf{w} \sim \mathbf{w}'$ if and only if $\mathbf{w}'$ can be obtained from $\mathbf{w}$ by a single commutation relation. Equivalence classes of this relation are called *commutation classes*.

Definition 1.1.5. An element $w \in W$ is *fully commutative* if w has exactly one commutation class. The set of all fully commutative elements of W is denoted by $\mathrm{FC}(W)$.

Example 1.1.6. Let $\mathbf{w} = s_4s_3s_1s_2s_1$ be a reduced expression for $w \in W(A_4)$. Then $\mathcal{R}(w) = \{s_4s_3s_1s_2s_1, s_4s_1s_3s_2s_1, s_1s_4s_3s_2s_1, s_4s_3s_2s_1s_2\}$. There are two commutation classes, that is $\{s_1s_4s_3s_2s_1, s_4s_1s_3s_2s_1, s_4s_3s_1s_2s_1\}$ and $\{s_4s_3s_2s_1s_2\}$, hence w is not fully commutative.

Example 1.1.7. The element $w = s_3s_5s_2s_4s_0s_1s_3s_2$ is in $\mathrm{FC}(\widetilde{D}_6)$ and $u = s_4s_2s_1s_3s_2$ in $\mathrm{FC}(\widetilde{D}_4)$, while the element $v = s_1s_2s_1s_3$ in $W(\widetilde{D}_4)$ is not fully commutative.

The next proposition, due to Stembridge [45, Proposition 2.1], characterizes the FC elements of type Γ and it is useful to test whether a given element is FC or not.

Proposition 1.1.8. *An element $w \in W$ is fully commutative if and only if for all $s, t \in S$ such that $3 \le m_{st} < \infty$, there is no expression of $\mathcal{R}(w)$ that contains the factor $[st]_{m_{st}}$.*

In Example 1.1.7, the element $v \in W(\widetilde{D}_4)$ is not fully commutative since its reduced expression $s_1 s_2 s_1 s_3$ contains the factor $s_1 s_2 s_1$ and $m_{s_1 s_2} = 3$.

Let $\mathbf{w} = s_{i_1} \cdots s_{i_r}$ be a reduced expression of $w \in W$. Set $[r] := \{1, 2, \ldots, r\}$; we define the *support* of $w \in W$ to be the set

$$supp(w) := \{s \in S \mid \exists k \in [r],\, s = s_{i_k}\}.$$

We say that $supp(w)$ is *complete* if $supp(w) = S$.

Note that by Matsumoto's Theorem, this set does not depend on the choice of $\mathbf{w}$. In particular, if $w \in \mathrm{FC}(W)$ and $\mathbf{w}' = s_{j_1} \cdots s_{j_r}$ is another reduced expression of w, it follows that

$$|\{k \in [r] \mid s_{i_k} = s\}| = |\{h \in [r] \mid s_{j_h} = s\}|,$$

for all $s \in supp(w)$.

Definition 1.1.9. We say that $w \in \mathrm{FC}(W)$ is *trivially commutative* if it is a product of commuting generators. We denote the set of trivially commutative elements by $\mathrm{TC}(W)$ and we assume that $e \in \mathrm{TC}(W)$.

We now introduce two important closely related tools for studying FC elements: the *Cartier–Foata normal form* and the *heap of an element*, introduced in [20] and [47], respectively.

Theorem 1.1.10 (Cartier–Foata normal form)**.** *Every $w \in \mathrm{FC}(W)$ admits a unique factorization*

$$w = u_0 \cdots u_m \tag{1.5}$$

called the Cartier-Foata normal form (CFNF), satisfying the following conditions:

(a) $supp(u_0) = D_L(w)$;

(b) $u_i \in \mathrm{TC}(W)$ *for all* $i \in \{0, 1, \ldots, m\}$*;*

(c) $\ell(w) = \ell(u_0) + \cdots + \ell(u_m)$*;*

(d) if $t \in supp(u_{j+1})$, $j \in \{0, 1 \ldots, m-1\}$*, then there exists* $s \in supp(u_j)$ *such that* $m_{s,t} \geq 3$*.*

Example 1.1.11. Consider the following reduced expressions of $w_1 \in \mathrm{FC}(\widetilde{D}_7)$ and $w_2 \in \mathrm{FC}(\widetilde{B}_6)$, respectively:

$$\mathbf{w_1} = (s_0 s_4)(s_3 s_5)(s_2 s_4 s_6 s_7) s_1;$$
$$\mathbf{w_2} = (s_3)(s_2 s_4)(s_1 s_3 s_5)(s_2 s_4 s_6)(s_0 s_3 s_5)(s_2 s_6).$$

The factors in parenthesis correspond to the factors of their CFNF.

Lemma 1.1.12. *Let* $w \in \mathrm{FC}(W)$ *and* $w = u_0 \cdots u_m$ *be its CFNF. Let* $z = s_{i_0} \cdots s_{i_k}$ *be a reduced expression of a prefix of* w *satisfying the following conditions:*

(a) if $k \geq 1$, s_{i_j} *and* $s_{i_{j+1}}$ *are not commuting generators for all* $0 \leq j \leq k-1$*;*

(b) $s_{i_k} \in supp(u_m)$*;*

(c) $w = zv$ *is a reduced product and* $s_{i_k} \notin supp(v)$*.*

Then $k = m$*.*

Proof. We have that $s_{i_0} \in supp(u_0)$, since $s_{i_0} \in D_L(w)$. Since $s_{i_0} \cdots s_{i_j}$ is prefix of w for each $1 \leq j \leq k$, by (a) and Theorem 1.1.10(d), we have that $s_{i_j} \in supp(u_j)$ and z is a prefix of $u_0 \cdots u_k$. Furthermore, by (b) and (c) we can conclude that $k = m$. Otherwise if $k < m$, then $w = zv$ is a reduced product and $supp(u_m) \subseteq supp(v)$, so $s_{i_k} \in supp(v)$, that is a contradiction. □

1.1.1 Heaps

The concept of heap helps to capture the notion of full commutativity. We briefly describe a way to define the above mentioned heap and its relations with full commutativity; for more details see for instance [7] and the references cited there.

Let (W, S) be a Coxeter system with Coxeter graph Γ, and fix a reduced expression $\mathbf{w} = s_{a_1} \cdots s_{a_l}$ with $s_{a_j} \in S$. Define a partial ordering $\prec$ on the index set $\{1, \ldots, l\}$ as follows: set $i \prec j$ if $i < j$ and s_{a_i}, s_{a_j} do not commute, and extend it by transitivity. Observe that if $i < j$ and $s_{a_i} = s_{a_j}$, then $i \prec j$ by transitivity and the fact that $\mathbf{w}$ is reduced. We denote by $H(\mathbf{w})$ the triple $(\{1, \dots, l\}, \prec, \varepsilon)$, where $\varepsilon : i \mapsto s_{a_i}$ is the labeling map and we call $H(\mathbf{w})$ a *labeled heap of type* Γ or simply a *heap*. Moreover, if F is a subset of $\{1, \dots, l\}$ and $\prec'$ and ε' are the restrictions of $\prec$ and ε to F, then $(F, \prec', \varepsilon')$ is called a *subheap* of $H(\mathbf{w})$. In particular, given a heap $H = H(\mathbf{w})$ and a subset $I \subset S$, we denote by $H_I = (F, \prec', \varepsilon')$ the subheap of H where F is the set of all elements of $\{1, \dots, l\}$ with labels in I (see [47, §2]).

Heaps are well-defined up to commutation classes [47]; that is, if $\mathbf{w}, \mathbf{w}' \in \mathcal{R}(w)$ belong to the same commutation class, then the corresponding labeled heaps are isomorphic. This means that there exists a poset isomorphism between the heaps that preserves the labels. Therefore, when w is FC we can define $H(w) := H(\mathbf{w})$, where $\mathbf{w}$ is any reduced expression for w. Heaps of this form will be called *FC heaps*. Moreover, if $H(w)$ is a FC heap then its linear extensions are in bjiection with $\mathcal{R}(w)$; see [45, Proposition 2.2].

In the Hasse diagram of $H(\mathbf{w})$, vertices with the same labels are drawn in the same column. Moreover, as in [18], we draw heaps from top to bottom, namely the vertices on the top of $H(\mathbf{w})$ correspond to generators occurring on the left of $\mathbf{w}$. We often identify vertices with their labels.

Example 1.1.13. Consider $w \in W(A_4)$ as in Example 1.1.6. In Figure 1.3, the heaps of two reduced expressions of w belonging to different commutation classes are depicted.

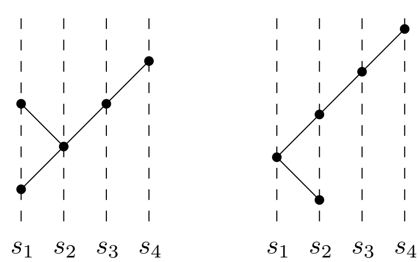


Figure 1.3: Heaps of two reduced expressions $\mathbf{w_1} = s_4 s_3 s_1 s_2 s_1$ and $\mathbf{w_2} = s_4 s_3 s_2 s_1 s_2$ for the same $w \in W(A_4)$.

Example 1.1.14. The heaps of $\mathbf{w} = s_3 s_5 s_2 s_4 s_0 s_1 s_3 s_2$, $\mathbf{u} = s_4 s_2 s_1 s_3 s_2$ and $\mathbf{v} = s_1 s_2 s_1 s_3$ of Example 1.1.7 are represented in Figure 1.4. Recall that w and u are fully commutative elements, while v is not. The subheap of $H(w)$ corresponding to the subset $\{s_2, s_3\}$

is dashed.

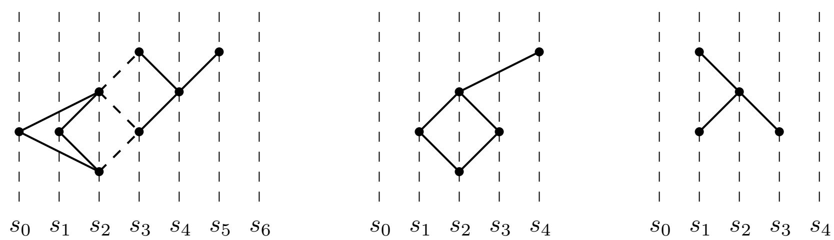


Figure 1.4: The heaps $H(w)$, $H(u)$ and $H(\mathbf{v})$, with the subheap $H(w)_{\{s_2,s_3\}}$ dashed.

Definition 1.1.15. Let (W, S) be a Coxeter system, $w \in \mathrm{FC}(W)$, and $H := H(w)$. We say that H is *alternating* if for each pair of non-commuting generators s, t in S, the chain $H_{\{s,t\}}$ has alternating labels s and t from top to bottom.

Both examples in Figure 1.4 are alternating. Note that if $H(w)$ is alternating, then any reduced expression $\mathbf{w}$ of w is *alternating* in the sense that for each non-commuting generators $s, t \in S$, the occurrences of s alternate with those of t in $\mathbf{w}$. In this case we also say that $w \in \mathrm{FC}(W)$ is alternating.

In the next chapters, we focus mainly on the Coxeter systems of types $\widetilde{D}_{n+2}$ and $\widetilde{B}_{n+1}$. Their Coxeter graphs are depicted in Figure 1.2, and they completely determine the corresponding Coxeter systems. For more details and combinatorial characterization of such groups we refer to [8, §8].

1.1.2 Generalized Temperley–Lieb algebras

Given a Coxeter system (W, S), the Iwahori–Hecke algebra $\mathcal{H} = \mathcal{H}(W)$ is the unital associative algebra over the ring of Laurent polynomials $\mathbb{Z}[q, q^{-1}]$, having a basis $\{T_w \mid w \in W\}$ whose elements satisfy the following multiplication rule:

$$T_s T_w = \begin{cases} T_{sw}, & \text{if } \ell(sw) > \ell(w), \\ qT_{sw} + (q-1)T_w, & \text{if } \ell(sw) < \ell(w). \end{cases}$$

with $s \in S$ and $w \in W$.

Let $J(W)$ be the two-sided ideal of $\mathcal{H}$ generated by the elements

$$\sum_{w \in \langle s,t \rangle} T_w$$

for all the pairs $s, t \in S$ such that $3 \leq m_{st} < \infty$, where $\langle s, t \rangle$ is the parabolic group generated by s and t. Following Graham [24, Definition 6.1], we present the following algebra.

Definition 1.1.16. The *generalized Temperley–Lieb algebra*, $\mathrm{TL}(W)$, is the quotient $\mathbb{Z}[q, q^{-1}]$-algebra $\mathcal{H}(W)/J(W)$.

Let τ be the algebra epimorphism $\tau : \mathcal{H}(W) \to \mathcal{H}(W)/J(W)$, and set $t_w := \tau(T_w)$ for all $w \in W$. Then we have the following result, due to Graham [24, Theorem 6.2], which involves the fully commutative elements of W.

Theorem 1.1.17. *The set* $\{t_w \mid w \in \mathrm{FC}(W)\}$ *is a* $\mathbb{Z}[q, q^{-1}]$*-basis for* $\mathrm{TL}(W)$.

For each $s \in S$, let $b_s := v^{-1}t_s + v^{-1}t_e$, where $v^2 = q$. Consider $w \in \mathrm{FC}(W)$ and let $s_{i_1} \cdots s_{i_k}$ be a reduced expression for w, then define

$$b_w := b_{s_{i_1}} \cdots b_{s_{i_k}}.$$

Observe that b_w does not depend on the choice of the reduced expression for w. Following [27, Proposition 2.4], the set $\{b_w \mid w \in \mathrm{FC}(W)\}$ is a basis for $\mathrm{TL}(W)$, called *monomial basis*.

In [27], Green provides a convenient presentation of $\mathrm{TL}(W)$ in terms of the generators b_s and defining relations. First he introduces the following family of polynomials [27, Definition 6.1].

Definition 1.1.18. The *Chebyshev polynomials of the second kind* are the elements of $\mathbb{Z}[x]$ given by the conditions

$$\begin{cases} P_0(x) &= 1; \\ P_1(x) &= x; \\ P_n(x) &= xP_{n-1}(x) - P_{n-2}(x), \text{ for } n \geq 2. \end{cases}$$

If $f(x) \in \mathbb{Z}[x]$, we define $f_b^{s,t}(x)$ to be the element of $\mathrm{TL}(W)$ given by the linear extension of the map sending

$$x^n \mapsto \underbrace{b_s b_t \cdots}_{n \text{ factors}},$$

where the latter is a product of alternating factors starting with b_s.

Finally, the following statement is [27, Proposition 2.6].

Proposition 1.1.19. *A presentation of the algebra* $\mathrm{TL}(W)$ *is given by the set of generators* $\{b_s \mid s \in S\}$ *and relations:*

(1) $b_s^2 = \delta b_s$*, for each* $s \in S$*,*

(2) $b_s b_t = b_t b_s$*, if* $m_{st} = 2$*,*

(3) $(xP_{m_{st}-1})_b^{s,t}(x) = 0$*, if* $2 < m_{st} < \infty$*,*

where $\delta := v + v^{-1}$.

By Proposition 1.1.19, we derive the presentations of the generalized Temperley–Lieb algebras $\mathrm{TL}(\widetilde{D}_{n+2})$ and $\mathrm{TL}(\widetilde{B}_{n+1})$, which from now on will be used as definitions. Recall that their associated Coxeter graphs are depicted in Figure 1.2.

Definition 1.1.20. The *Temperley–Lieb algebra of type* $\widetilde{D}_{n+2}$, denoted by $\mathrm{TL}(\widetilde{D}_{n+2})$, is the $\mathbb{Z}[\delta]$-algebra generated by $\{b_0, b_1, \ldots, b_{n+2}\}$ with defining relations:

(d1) $b_i^2 = \delta b_i$ for all $i \in \{0, \ldots, n+2\}$;

(d2) $b_i b_j = b_j b_i$ if s_i and s_j are not adjacent nodes in the Coxeter graph of type $\widetilde{D}_{n+2}$;

(d3) $b_i b_j b_i = b_i$ if s_i and s_j are adjacent nodes in the Coxeter graph of type $\widetilde{D}_{n+2}$.

Definition 1.1.21. The *Temperley–Lieb algebra of type* $\widetilde{B}_{n+1}$, denoted by $\mathrm{TL}(\widetilde{B}_{n+1})$, is the $\mathbb{Z}[\delta]$-algebra generated by $\{b_0, b_1, \ldots, b_{n+1}\}$ with defining relations:

(b1) $b_i^2 = \delta b_i$ for all $i \in \{0, \ldots, n+1\}$;

(b2) $b_i b_j = b_j b_i$ if s_i and s_j are not adjacent nodes in the Coxeter graph of type $\widetilde{B}_{n+1}$;

(b3) $b_i b_j b_i = b_i$ if s_i and s_j are adjacent nodes in the Coxeter graph of type $\widetilde{B}_{n+1}$, and $\{i, j\} \neq \{n, n+1\}$;

(b4) $b_i b_j b_i b_j = 2 b_i b_j$ if $\{i, j\} = \{n, n+1\}$.

1.2 Geometric representation of a Coxeter group

In this section, given a Coxeter system (W, S), we present the standard geometric representation of W and the crystallographic root system when dealing with an affine Coxeter graph. Here we follow mostly [33, Ch. 5, §3] and [8, Ch. 4, §2, 4].

Let (W, S) be a Coxeter system and consider a real vector space V having basis $\Delta = \{\alpha_s \mid s \in S\}$. Define a symmetric bilinear form B on V by

$$B(\alpha_s, \alpha_{s'}) = \begin{cases} -\cos\frac{\pi}{m_{ss'}}, & \text{if } m_{ss'} < \infty; \\ -1, & \text{otherwise.} \end{cases}$$

Clearly we have $B(\alpha_s, \alpha_s) = 1$ for each $s \in S$. For each $s \in S$ we define a reflection $\sigma_s : V \to V$ as follows:

$$\sigma_s(\alpha) = \alpha - 2B(\alpha_s, \alpha)\alpha_s. \tag{1.6}$$

Note that $\sigma_s(\alpha_s) = -\alpha_s$ and $\sigma_s^2 = id$ in $GL(V)$. Moreover, σ_s preserves the bilinear form B, i.e. $B(\sigma_s(\alpha), \sigma_s(\beta)) = B(\alpha, \beta)$ for all $\alpha, \beta \in V$.

Theorem 1.2.1. *The homomorphism $\sigma : W \to GL(V)$ such that $s \mapsto \sigma_s$ is injective, hence the representation is faithful (see [8, Theorem 4.2.7]).*

The mapping $\sigma : W \to GL(V)$ is called *the standard geometric representation* of W. From now on, we will write $w(\alpha_s)$ in place of $\sigma(w)(\alpha_s)$, for $w \in W$ and $s \in S$. The *root system* of W is the set $\Phi := \{w(\alpha_s) \mid w \in W, s \in S\}$, where its elements are called *roots*. The elements of $\Delta = \{\alpha_s \mid s \in S\}$ are called *simple roots*. Each root α can be expressed uniquely in the form $\alpha = \sum_{s\in S} c_s\alpha_s$, $c_s \in \mathbb{R}$. We denote by Φ^+ (respectively, Φ^-) the sets of all *positive* (respectively, *negative*) *roots*, which are those roots α such that $c_s \geq 0$ (respectively, $c_s \leq 0$) for all $s \in S$. In particular we have that $\Phi^- = -\Phi^+$ and $\Phi = \Phi^+ \sqcup \Phi^-$.

Extending the definition in (1.6), for every $\gamma \in \Phi$ define an element $s_\gamma \in GL(V)$ by

$$s_\gamma(\beta) = \beta - 2B(\gamma, \beta)\gamma$$

for all $\beta \in V$. Clearly, $s_\gamma^2 = e$ and $s_{-\gamma} = s_\gamma$.

Let $T = \{wsw^{-1} \mid w \in W, s \in S\}$ be the *set of reflections* of W. Let $\gamma = w(\alpha_s) \in \Phi$

and consider $\beta \in V$, then

$$wsw^{-1}(\beta) = w(w^{-1}(\beta) - 2B(\alpha_s, w^{-1}(\beta))\alpha_s) = \beta - 2B(w(\alpha_s), \beta)w(\alpha_s) = s_\gamma(\beta).$$

Hence, $s_\gamma = wsw^{-1}$ in the geometric representation $\sigma(W)$, for any $w \in W$, $s \in S$ such that $\gamma = w(\alpha_s)$. Then we have the following result ([8, Proposition 4.4.5]).

Proposition 1.2.2. *The map $\rho : \Phi^+ \to T$ such that $\gamma \mapsto s_\gamma$ is a bijection.*

1.2.1 The affine root system

Let (W, S) be an affine Coxeter system. When dealing with an affine Weyl group W, one usually considers its *crystallographic root system*; we refer the reader to [15, Example 3.9], [16, § 3], or [35] for details.

Consider a finite irreducible Coxeter group W_0 generated by S_0 acting as a finite reflection group on an Euclidean vector space (V_0, B_0) with basis the *crystallographic root system* $\Delta_0 = \{\alpha_s \mid s \in S_0\}$. The norm of $\alpha \in V$ is denoted by $\|\alpha\| = B_0(\alpha, \alpha)^{1/2}$. The scalar product B_0 is W_0-invariant and is defined for $s, t \in S_0$ as follows:

$$B_0(\alpha_s, \alpha_t) = -\|\alpha_s\|\|\alpha_t\| \cos \frac{\pi}{m_{st}},$$

if $m_{st} < \infty$ and $B_0(\alpha_s, \alpha_t) = -1$ otherwise. The *crystallographic finite root system* $\Phi_0 = W_0(\Delta_0)$ is the W_0-orbit of the simple system Δ_0 with $\Delta_0 = \{\alpha_s \mid s \in S_0\}$ its simple system.

We consider now the vector space $V = V_0 \oplus \mathbb{R}\delta$ and extend B_0 to a symmetric bilinear form B on V by setting $B(\delta, v) = 0$ for all $v \in V$. Then

$$\Phi = \{k\delta + \alpha \mid k \in \mathbb{Z}, \alpha \in \Phi_0\}$$

is a *crystallographic (infinite) root system* for (W, S) with simple system

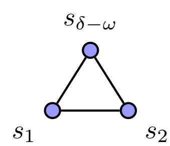
$$\Delta = \Delta_0 \sqcup \{\delta - \omega\},$$

where ω is the highest root of W_0 and $S = S_0 \cup \{\delta - \omega\}$; δ is usually known as the *imaginary root* of W. Finally, the set of positive roots is:

$$\Phi^+ = \{k\delta + \alpha, (k+1)\delta - \alpha \mid k \in \mathbb{N}, \alpha \in \Phi_0^+\} = (\mathbb{N}\delta + \Phi_0^+) \sqcup (\mathbb{N}^*\delta - \Phi_0^+).$$

By normalizing the roots in a crystallographic root system via the map $\alpha \mapsto \frac{\alpha}{\|\alpha\|}$ we obtain a *unitary* root system. It is well-known that all the results valid for a unitary root system hold also for a crystallographic one; see for instance [15, Example 3.9].

Example 1.2.3. Let (W, S) be a Coxeter system of type $\widetilde{A}_2$,



then the underlying finite Coxeter system (W_0, S_0) is of type A_2. Then $S = \{s_1, s_2, s_{\delta-\omega}\}$, where $s_1, s_2 \in S_0$ and $\omega = \alpha_1 + \alpha_2$ is the highest root of W_0. The finite positive root system is $\Phi_0^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$, so the affine simple system is $\Delta = \{\alpha_1, \alpha_2, \delta - \omega\}$ and the affine positive root system becomes

$$\Phi^+ = \{k\delta+\alpha_1, k\delta+\alpha_2, k\delta+\alpha_1+\alpha_2, (k+1)\delta-\alpha_1, (k+1)\delta-\alpha_2, (k+1)\delta-(\alpha_1+\alpha_2) \mid k \in \mathbb{N}\}.$$

Chapter 2

Star and weak star irreducible FC elements in affine types $\widetilde{B}$ and $\widetilde{D}$

The main goal of this chapter is to classify the star irreducible fully commutative elements of affine types $\widetilde{D}_{n+2}$ and $\widetilde{B}_{n+1}$. In [17], Ernst relaxed the notion of star irreducibility by introducing *weak star reducibility* (also referred to as *cancellability*, following Fan [20]), which coincides with star reducibility in the simply laced cases. He classified the weak star irreducible (or non-cancellable) elements of Coxeter systems of types B and $\widetilde{C}$ in [17, Theorem 4.2.1, Theorem 5.1.1]. Since type $\widetilde{B}_{n+1}$ is not simply laced, we also provide the classification of weak star irreducible elements in this case.

In this chapter, we first recall the notions star and weak star reducibility and show that the number of star reductions to apply to a fully commutative element w in order to obtain a star irreducible element is constant (Lemma 2.1.5). In particular, such irreducible element is unique when $supp(w)$ is complete (Theorem 2.1.6). Then we consider a Coxeter system of affine type $\widetilde{D}_{n+2}$ and in Section 2.3, we classify the irreducible fully commutative elements of type $\widetilde{D}_{n+2}$ (Theorem 2.3.20). Finally, thanks to an injective map between the sets of fully commutative elements of types $\widetilde{D}_{n+2}$ and $\widetilde{B}_{n+1}$ (Definition 2.4.6), in Section 2.4, we also characterize the star irreducible elements (Theorem 2.4.9) and weak star irreducible elements (Theorem 2.4.14) of type $\widetilde{B}_{n+1}$.

This chapter, as well as part of the next one, was inspired by a joint work with Riccardo Biagioli and Luca Costantini, whose preprint is available in [4].

2.1 Star and weak star irreducible fully commutative elements

In this section we fix an arbitrary irreducible Coxeter system (W, S). Now, we recall the notions of star and weak star reducibility in (W, S) and we refer the reader to [17, 27, 28] for more details.

Definition 2.1.1. Let $w \in \mathrm{FC}(W)$ and suppose that $s \in D_L(w)$ (respectively, $s \in D_R(w)$). We say that w is *left (respectively, right) star reducible by s with respect to t* to sw (respectively, ws) if there exists $t \in D_L(sw)$ (respectively, $t \in D_R(ws)$) with $m_{s,t} \geq 3$.

Observe that if $m_{st} \geq 3$, then w is left (respectively, right) star reducible by s with respect to t if and only if $w = stv$ (respectively, $w = vts$), with $\ell(w) = \ell(v) + 2$.

We now define the concept of weak star reducible elements, first introduced in [17], which is related the notion of cancellable elements introduced by Fan in [20].

Definition 2.1.2. Let $w \in \mathrm{FC}(W)$ and $s, t \in S$, we say that w is *left (respectively, right) weak star reducible by s with respect to t to sw (respectively, ws)*, if the following hold:

(a) w is left (respectively, right) star reducible by s with respect to t to sw (respectively, ws);

(b) $tw \notin \mathrm{FC}(W)$ (respectively, $wt \notin \mathrm{FC}(W)$).

Let $w \in \mathrm{FC}(W)$, we say that w is *left (respectively, right) (weak) star irreducible* if does not exist $s, t \in S$ such that w is left (respectively, right) (weak) star reducible by s with respect to t. Moreover, we say that w is *(weak) star irreducible* if it is both left and right (weak) star irreducible. We set also

$$\mathrm{I}(W) = \{w \in \mathrm{FC}(W) \mid w \text{ is star irreducible}\},$$
$$\mathrm{I_w}(W) = \{w \in \mathrm{FC}(W) \mid w \text{ is weak star irreducible}\}.$$

Clearly, $\mathrm{I}(W) \subseteq \mathrm{I_w}(W)$ and denote by $\mathrm{I'_w}(W) := \mathrm{I_w}(W) \setminus \mathrm{I}(W)$. Observe that when W is simply laced (i.e., $m_{st} \leq 3$ for all $s, t \in S$) the definitions of star reducible and weak star reducible are equivalent, therefore we will distinguish these two definitions only in Section 2.4 when dealing with type $\widetilde{B}_{n+1}$.

From now on, to simplify the writing, we will say *reducible* (*irreducible*) instead of *star reducible* (*star irreducible*). Moreover, we use the notation

$$w \rightsquigarrow_t sw \quad \text{(respectively, } w \rightsquigarrow_t ws\text{)}$$

to denote that w is (weak) left (respectively, right) reducible by s with respect to t; we omit the t when it is not necessary. In general, by $w \rightsquigarrow v$ we mean that either $w \rightsquigarrow_t sw$ or $w \rightsquigarrow_t ws$ for some $s, t \in S$ with $m_{st} \geq 3$. We will explicitly write "weak" when we consider a weak star reduction.

Remark 2.1.3.

(1) If $w \rightsquigarrow_t sw$ (respectively, $w \rightsquigarrow_t ws$), then $\ell(tw) > \ell(w)$ (respectively, $\ell(wt) > \ell(w)$).

(2) We have that w is left (weak) irreducible if and only if w^{-1} is right (weak) irreducible. Thus, w is (weak) irreducible if and only if w^{-1} is also (weak) irreducible.

(3) If $3 \leq m_{st} < \infty$, define $\xi_{st} := [st]_{m_{st}-1}$. We have that $w \rightsquigarrow_t sw$ (respectively, $w \rightsquigarrow_t ws$) is a weak reduction if and only if ξ_{st} is a prefix (respectively, ξ_{st}^{-1} is a suffix) of w.

(4) Assume $m_{st} = 3$ and $w = u_0 \cdots u_m$ be its CFNF, $w \rightsquigarrow_t sw$ if and only if $s \in supp(u_0)$, $t \in supp(u_1)$ and for every $s' \in S$ such that $m_{s't} = 3$, $s' \notin supp(u_0)$.

(5) Assume $s \in D_L(w)$ (resp. $s \in D_R(w)$). If $t \in S$ such that $m_{s,t} \geq 3$ and $tw \notin \mathrm{FC}(W)$ (respectively, $wt \notin \mathrm{FC}(W)$), then w admits st as a prefix (respectively, ts as a suffix), so $w \rightsquigarrow_t sw$ (respectively, $w \rightsquigarrow_t ws$) is a weak reduction.

(6) Assume $s \neq s'$, if $w \rightsquigarrow_t sw$ and $w \rightsquigarrow_{t'} s'w$ (respectively, $w \rightsquigarrow_t ws$ and $w \rightsquigarrow_{t'} ws'$) are (weak) reductions, then $m_{xx'} = 2$ for all $x \in \{s, t\}$ and $x' \in \{s', t'\}$, therefore $sw \rightsquigarrow_{t'} s'sw$ (respectively, $ws \rightsquigarrow_{t'} wss'$) is a (weak) reduction.

(7) Let $w \rightsquigarrow_t sw$ and $w \rightsquigarrow_{t'} ws'$ be (weak) reductions. Then $ws' \rightsquigarrow_t sws'$ is a (weak) reduction if and only if $sw \rightsquigarrow_{t'} sws'$ is a (weak) reduction.

Definition 2.1.4. Let $w, v \in \mathrm{FC}(W)$. We say that w is *(weak) reducible to* v if there is a sequence (also trivial) of (weak) reductions

$$w_0 = w \rightsquigarrow w_1 \rightsquigarrow w_2 \rightsquigarrow \cdots \rightsquigarrow w_l = v. \tag{2.1}$$

In every step of (2.1), the length of the elements decreases by 1, therefore, any $w \in \mathrm{FC}(W)$ can be reduced to a (weak) irreducible element.

Lemma 2.1.5. *Let $w \in \mathrm{FC}(W)$ be (weak) reducible to v (weak) irreducible with a sequence*

$$w_0 = w \rightsquigarrow w_1 \rightsquigarrow w_2 \rightsquigarrow \cdots \rightsquigarrow w_l = v. \tag{2.2}$$

If there exists a (weak) reduction $w \rightsquigarrow w_1'$, with $w_1' \neq w_1$, then there exists v' (weak) irreducible such that w is (weak) reducible to v' by a sequence

$$w_0' = w \rightsquigarrow w_1' \rightsquigarrow w_2' \rightsquigarrow \cdots \rightsquigarrow w_l' = v'.$$

Moreover, if $|S| > 2$ and supp(v) is complete, then $v = v'$.

Proof. We first prove the statement for star reducibility. We assume without loss of generality that $w \rightsquigarrow_t sw =: w_1'$, with $w_1' \neq w_1$. By hypothesis, $z := st$ is a prefix of w, hence in sequence (2.2), there exists $1 \leq j \leq l-1$ such that z is a prefix of w_i for all $1 \leq i \leq j$ but z is not a prefix of w_{j+1}. Then, either $w_j \rightsquigarrow_t sw_j = w_{j+1}$, or $w_j \rightsquigarrow_s w_j t = w_{j+1}$. Moreover, since $w_i \rightsquigarrow w_{i+1}$, by Remark 2.1.3(6) and (7), sw_i is reducible to sw_{i+1} for all $0 \leq i \leq j-1$. Now, we distinguish the two cases:

- If $w_{j+1} = sw_j$, we define

  $$w_i' := \begin{cases} sw_{i-1}, & \text{for } 2 \leq i \leq j; \\ w_i, & \text{for } j+1 \leq i \leq l. \end{cases}$$

  We note that if $w_{j-1} \rightsquigarrow w_j$ by the pair $\{s_{j-1}, t_{j-1}\}$, then by Remark 2.1.3(6) and (7) $w_j' = sw_{j-1} \rightsquigarrow sw_j = w_{j+1}$ by the same pair. Hence, it follows that w is reducible to v with sequence

  $$w_0' = w \rightsquigarrow w_1' \rightsquigarrow w_2' \rightsquigarrow \cdots \rightsquigarrow w_l' = v.$$

- If $w_{j+1} = w_j t$, we define

$$w'_i := \begin{cases} sw_{i-1}, \text{ for } 2 \leq i \leq j; \\ sw_i t, \text{ for } j+1 \leq i \leq l. \end{cases}$$

  Note that, for $i \leq j-1$, $w'_i \rightsquigarrow w'_{i+1}$ as above. Recall that z is a prefix of w_j but not of w_{j+1}. Hence, $w_j = zu_j = u_j z$, thus for all $x \in supp(u_j)$, x commute with both s and t. Therefore, $w_{j+1} = u_j s = su_j$ and there exists a sequence

$$u_j \rightsquigarrow u_{j+1} \rightsquigarrow \cdots \rightsquigarrow u_{l-1} \in \mathrm{I}(W),$$

  such that $w_i = su_{i-1}$, so u_{i-1} commutes with both s and t for all $j+1 \leq i \leq l$. Moreover, if $w_{j-1} \rightsquigarrow w_j = stu_j$ by the pair $\{s_{j-1}, t_{j-1}\}$, then $w'_j = sw_{j-1} \rightsquigarrow w'_{j+1} = sw_{j+1}t$, since in the case $s_{j-1}t_{j-1}$ is a prefix of w_{j-1} then $sw_{j-1} \rightsquigarrow sw_j = tu_j = sw_{j+1}t$ by Remark 2.1.3(6). Otherwise, if $s_{j-1}t_{j-1}$ is a suffix of w_{j-1}, then $sw_{j-1} \rightsquigarrow sw_j = tu_j = sw_{j+1}t$ by Remark 2.1.3(7). Furthermore, $w'_i = sw_i t = u_{i-1}t \rightsquigarrow u_i t = sw_{i+1}t = w'_{i+1}$ By calling $v' := w'_l = tu_{l-1} \in \mathrm{I}(W)$, we have that w is star reducible to v' with sequence:

$$w'_0 = w \rightsquigarrow w'_1 \rightsquigarrow w'_2 \rightsquigarrow \cdots \rightsquigarrow w'_l = v'.$$

Assume now $|S| > 2$, if $supp(v)$ is complete, then necessarily we have the first case since in the second one $v = su_{l-1} = u_{l-1}s$ which is impossible, therefore $v = v'$.

The assertions for weak star reducibility can be proved similarly, considering the prefix $z = \xi_{s,t}$ instead of $z = st$. □

In the first point of the following result, we consider sequences consisting only of left or only of right reductions.

Theorem 2.1.6.

(a) If $v \in \mathrm{FC}(W)$ is left (respectively, right) (weak) irreducible and obtained from w by a series of left (respectively, right) (weak) reductions, then v is unique.

(b) Let $w, v, v' \in \mathrm{FC}(W)$ with v, v' (weak) irreducible. If w is (weak) reducible to v

and v' with respective sequences

$$w_0 = w \rightsquigarrow w_1 \rightsquigarrow w_2 \rightsquigarrow \cdots \rightsquigarrow w_l = v, \tag{2.3}$$
$$w'_0 = w \rightsquigarrow w'_1 \rightsquigarrow w'_2 \rightsquigarrow \cdots \rightsquigarrow w'_k = v'; \tag{2.4}$$

then $k = l$.

(c) *Assume $|S| > 2$. If w is (weak) reducible to v (weak) irreducible and supp(v) is complete, then v is unique.*

Proof. We first prove the statement for star reducibility.

(a) Suppose w reducible to v by a series of left reductions with sequence

$$w_0 = w \rightsquigarrow w_1 \rightsquigarrow \cdots \rightsquigarrow w_l = v.$$

First we show the following claim by induction on l:

$$\text{if } x \in \mathrm{Suff}(w) \text{ is left irreducible, then } x \in \mathrm{Suff}(v). \tag{2.5}$$

If $l = 0$, then w is left irreducible, so the claim trivially holds. So suppose $l \geq 1$ and $w_1 = sw$ with st a prefix of w with $m_{st} \geq 3$. We have that $x \in \mathrm{Suff}(w_1)$, otherwise st would be a prefix of x which is not possible since x is left irreducible. Therefore, since w_1 is reducible to v by a series of left reductions, by the inductive hypothesis $x \in \mathrm{Suff}(v)$.

Now, we observe that if w is reducible to v' left irreducible by a series of left reductions, we have that $v' \in \mathrm{Suff}(w)$, so by (2.5) $v' \in \mathrm{Suff}(v)$. So, the same argument, switching the role of v and v', shows that $v \in \mathrm{Suff}(v')$, thus $v = v'$, hence statement (a) holds.

(b) We proceed by induction on l. If $l = 0$, then w is irreducible and the thesis trivially holds. So suppose $l \geq 1$. We have that w is left or right reducible to w_1 with respect to a pair $\{s_1, t_1\}$, then by applying Lemma 2.1.5 to the sequence in (2.4), there exists $v'' \in \mathrm{I}(W)$ such that w is reducible to v'' with a sequence

$$w''_0 = w \rightsquigarrow w''_1 = w_1 \rightsquigarrow \cdots \rightsquigarrow w''_k = v''.$$

Therefore, w_1 is reducible to v'' and v, so by inductive hypothesis we can conclude that $k = l$.

(c) Assume $|S| > 2$, let w be reducible to $v \in \mathrm{I}(W)$, such that *supp*(v) is complete. We proceed by induction on l. If $l = 0$, then $w = v$ and the claim trivially holds. On the other hand, assume $l > 0$ and suppose w is reducible to $v' \in \mathrm{I}(W)$ with sequence

$$w'_0 = w \rightsquigarrow w'_1 \rightsquigarrow \cdots \rightsquigarrow w'_l = v'.$$

Applying Lemma 2.1.5, we have that w is reducible to v by a sequence

$$w''_0 = w \rightsquigarrow w''_1 = w'_1 \rightsquigarrow \cdots \rightsquigarrow w''_l = v.$$

Then, w'_1 is reducible to v' and v, so by the inductive hypothesis, $v = v'$.

The statements for weak star reducibility can be proved similarly, in particular in the proof of (a) it is sufficient to work with prefix $\xi_{s,t}$ instead of prefix st. On the other hand, for proofs (b) and (c) one has only to apply the weak star version of Lemma 2.1.5. $\square$

Example 2.1.7.

(1) From Theorem 2.1.6, the irreducible element obtained from w may depend on the choice of the reduction sequence when both left and right reductions are used. For instance, consider $w_1 = s_0s_4s_3s_5s_2s_4s_6s_7s_1 \in \mathrm{FC}(\widetilde{D}_7)$. Then w_1 is reducible to both $v_1 = s_0s_3s_6s_7$ and $v_2 = s_1s_4s_6s_7$ in $\mathrm{I}(\widetilde{D}_7)$ via the following sequences:

$$w_1 \rightsquigarrow s_4w_1 \rightsquigarrow s_4w_1s_1 \rightsquigarrow s_5s_4w_1s_1 \rightsquigarrow s_5s_4w_1s_1s_2 \rightsquigarrow v_1 = s_5s_4w_1s_1s_2s_4 = s_0s_3s_6s_7;$$
$$w_1 \rightsquigarrow s_0w_1 \rightsquigarrow s_4s_0w_1 \rightsquigarrow s_3s_4s_0w_1 \rightsquigarrow s_5s_3s_4s_0w_1 \rightsquigarrow v_2 = s_2s_5s_3s_4s_0w_1 = s_1s_4s_6s_7.$$

(2) Recall that irreducibility and weak irreducibility are different in non-simply laced Coxeter groups. Consider $w_2 = s_3s_2s_4s_1s_3s_5s_2s_4s_6s_0s_3s_5s_2s_6 \in \mathrm{FC}(\widetilde{B}_6)$ then w_2 is weak reducible to $v_2 = s_1s_3s_5s_2s_4s_6s_0s_3s_5 \in \mathrm{I}'_{\mathrm{w}}(\widetilde{B}_6)$ by a sequence:

$$w_2 \rightsquigarrow s_3w_2 \rightsquigarrow s_4s_3w_2 \rightsquigarrow s_2s_4s_3w_2 \rightsquigarrow s_2s_4s_3w_2s_6 \rightsquigarrow v_2 = s_2s_4s_3w_2s_6s_2.$$

Moreover, v_2 is reducible to $s_2s_4s_6 \in \mathrm{I}(\widetilde{B}_6)$. Thus, w_2 is also reducible to $s_2s_4s_6$.

2.2 Classification of fully commutative heaps of type $\widetilde{D}_{n+2}$

From now on we will focus on the affine Coxeter group of type $\widetilde{D}_{n+2}$ whose Coxeter graph is depicted in Figure 1.2. Fully commutative heaps of type $\widetilde{D}_{n+2}$ have been classified in [7, §3.2]. We present now a slightly modified version of the original classification [7, Definition 3.9].

Notation 2.2.1. By the description given above, a heap of $w \in \mathrm{FC}(\widetilde{D}_{n+2})$ should be represented in a Hasse diagram with $n+3$ columns labeled by $s_0, s_1, s_2, \ldots, s_{n+2}$, as we did in Figure 1.4. However, when dealing with Coxeter systems of type $\widetilde{D}_{n+2}$, we will represent heaps in $n+1$ columns with the following criterion. Both vertices with label s_0 and s_1 (respectively s_{n+1} and s_{n+2}) will be drawn in the first (respectively last) column of $H(w)$. Moreover, when s_0s_1 (respectively $s_{n+1}s_{n+2}$) occurs in a reduced expression of w, we represent the vertices associated to these s_0 and s_1 (respectively s_{n+1} and s_{n+2}) with a single mark point labeled s_0s_1 (respectively $s_{n+1}s_{n+2}$). Vertices labeled by s_0, s_1 and s_0s_1 will be called *1-elements* while vertices labeled by s_{n+1}, s_{n+2} and $s_{n+1}s_{n+2}$ will be called *(n+1)-elements* and we will explicitly write their labels in the heap representation. For instance from now on, the two heaps in Figure 1.4 will be depicted as in Figure 2.1.

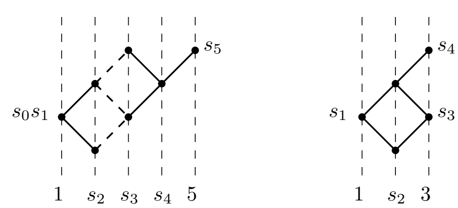


Figure 2.1: The new representations of $H(w)$ and $H(u)$ of Figure 1.4.

If $i \in \{2, \ldots, n\}$, a *peak* is a heap of the form:

$$P_{\rightarrow}(s_i) := H(s_is_{i+1}\cdots s_ns_{n+1}s_{n+2}s_n\cdots s_{i+1}s_i) \text{ or}$$
$$P_{\leftarrow}(s_i) := H(s_is_{i-1}\cdots s_2s_0s_1s_2\cdots s_{i-1}s_i).$$

If H is a heap of type $\widetilde{D}_{n+2}$ and $i \in \{2, \ldots, n\}$, then for brevity, we denote by $H_{\{\leftarrow s_i\}}$ (respectively, $H_{\{\rightarrow s_i\}}$) the subheap of H with labels in the set $\{s_0s_1, s_0, s_1, s_2, \ldots, s_{i-1}, s_i\}$ (respectively, $\{s_i, s_{i+1}, \ldots, s_{n+1}, s_{n+2}, s_{n+1}s_{n+2}\}$).

Definition 2.2.2. We define the following five families of FC heaps of type $\widetilde{D}_{n+2}$.

(PZZ) *Pseudo Zigzags.* $H \in$ (PZZ) if $H = H(w)$ where w has a reduced expression with at least one occurrence of both $s_0 s_1$ and $s_{n+1} s_{n+2}$ and that is a factor of an expression in the commutation class of the element $(s_0 s_1 s_2 \cdots s_n s_{n+1} s_{n+2} s_n \cdots s_2)^k$ for some integer $k \geq 1$.

(ALT) *Alternating.* $H \in$ (ALT) if it is alternating in the sense of Definition 1.1.15, where Γ is of type $\widetilde{D}_{n+2}$, but it is different from H_{Z_1} and H_{Z_2} in Figure 2.3. Moreover, if H contains two or more 1-elements (respectively $(n+1)$-elements), those must be s_0 and s_1 (respectively s_{n+1} and s_{n+2}) and have to alternate.

(LP) *Left-Peaks.* $H \in$ (LP) if there exists $j_\ell \in \{2, \ldots, n\}$ such that:

(1) $H_{\{\leftarrow s_{j_\ell}\}} = P_{\leftarrow}(s_{j_\ell})$;

(2) There is no $s_{j_\ell+1}$-element between the two s_{j_ℓ}-elements;

(3) $H_{\{\rightarrow s_{j_\ell}\}}$ is alternating when the two s_{j_ℓ}-elements are identified.

(4) H is not a pseudo zigzag.

(RP) *Right-Peaks.* $H \in$ (RP) if there exists $j_r \in \{2, \ldots, n\}$ such that:

(1) $H_{\{\rightarrow s_{j_r}\}} = P_{\rightarrow}(s_{j_r})$;

(2) There is no s_{j_r-1}-element between the two s_{j_r}-elements;

(3) $H_{\{\leftarrow s_{j_r}\}}$ is alternating when the two s_{j_r}-elements are identified.

(4) H is not a pseudo zigzag.

(LRP) *Left-Right-Peaks.* $H \in$ (LRP) if there exist $2 \leq j_\ell < j_r \leq n$ such that:

(1) LP(1), LP(2), RP(1) and RP(2) hold;

(2) $H_{\{s_{j_\ell}, \ldots, s_{j_r}\}}$ is alternating when both the two s_{j_ℓ}- and the two s_{j_r}-elements are identified.

(3) H is not a pseudo zigzag.

The following result is proved in [7, Theorem 3.10].

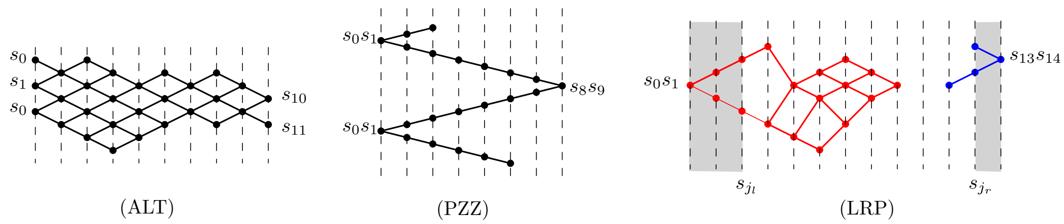


Figure 2.2: Example of FC heaps in different families. The subheap of the (LRP) heap colored in red is in (LP), while the one in blue is in (RP).

Theorem 2.2.3 (Classification of FC heaps of type $\widetilde{D}_{n+2}$)**.** *A heap of type $\widetilde{D}_{n+2}$ is fully commutative if and only if it belongs to one of the families of Definition 2.2.2.*

From now on, we say that $w \in \mathrm{FC}(\widetilde{D}_{n+2})$ is in (PZZ), (ALT), (LP), (RP), or (LRP) if and only if the corresponding heap $H(w)$ is.

Remark 2.2.4. With respect to [7, §3.2], we introduced a new family (PZZ) made of zigzags of the original classification, along with some additional heaps that previously belonged to different families. In particular, the heaps $H_{Z_1} := H(s_0s_1s_2\cdots s_{n+1}s_{n+2})$ and $H_{Z_2} := H(s_{n+2}s_{n+1}s_n\cdots s_1s_0)$ depicted in Figure 2.3 are now in (PZZ) while they were alternating heaps in the previous classification. Analogously, the left and right peaks having at least one initial or final endpoint situated in column 1 or $n+1$, and labeled by s_0s_1 or $s_{n+1}s_{n+2}$, respectively, are now in (PZZ); see Figure 2.3, right. The reasons for these choices will be clarified in Lemmas 3.8.3 and 3.8.5.

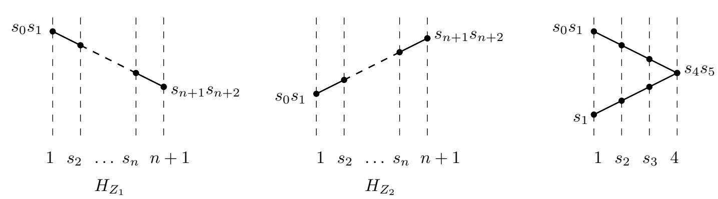


Figure 2.3: Examples of FC heaps in (PZZ).

2.3 Irreducibility in $\mathrm{FC}(\widetilde{D}_{n+2})$

In this section, we classify the irreducible elements in a Coxeter system of type $\widetilde{D}_{n+2}$. Recall that, since $\widetilde{D}_{n+2}$ is simply laced, star reducibility and weak star reducibility coincide.

Definition 2.3.1. Let $w \in \mathrm{FC}(\widetilde{D}_{n+2})$. We denote by

$$f_\bullet(w) := \max\{\#\text{ of occurrences of the factor } s_0s_1 \text{ in } \mathbf{w} \mid \mathbf{w} \in \mathcal{R}(w)\};$$
$$f_\circ(w) := \max\{\#\text{ of occurrences of the factor } s_{n+1}s_{n+2} \text{ in } \mathbf{w} \mid \mathbf{w} \in \mathcal{R}(w)\}.$$

Lemma 2.3.2. *Let* $w \in \mathrm{FC}(\widetilde{D}_{n+2})$, *then*

(a) $f_\bullet(w) = f_\bullet(w^{-1})$ *and* $f_\circ(w) = f_\circ(w^{-1})$;

(b) if $w \rightsquigarrow v$, *then* $f_\bullet(v) = f_\bullet(w)$ *and* $f_\circ(v) = f_\circ(w)$.

Proof. Point (a) follows immediately. Now, assume that $w \rightsquigarrow_t sw$, with $m_{st} \geq 3$. Clearly $\{s,t\} \neq \{s_0, s_1\}$, and if $s \in \{s_0, s_1\}$, then it must be the case that $t = s_2$. Hence s is not part of a factor of type s_0s_1 in any reduced expression of w, and so the occurrences of the factors s_0s_1 in w and sw are the same. The argument is analogous when $w \rightsquigarrow_t ws$. $\square$

Example 2.3.3. Let $w, v \in (\widetilde{D}_{10})$, $w = s_0s_3s_5s_9s_1s_2s_4s_6s_8s_3s_5s_7s_{10}$ and $v = s_0s_5s_9s_1s_{10}$. Then we can choose

$$\mathbf{w} = s_3s_5s_9(s_0s_1)s_2s_4s_6s_8s_3s_5s_7s_{10} \quad \text{and} \quad \mathbf{v} = (s_0s_1)s_5(s_9s_{10}).$$

Hence, $f_\bullet(w) = 1$, $f_\circ(w) = 0$ and $f_\bullet(v) = f_\circ(v) = 1$. In particular, by Lemma 2.3.2, note that v is not a reduction of w.

Notation 2.3.4. Since the generators s_0 and s_1 play equivalent roles, we introduce the notation $s_\bullet$ to denote either s_0 or s_1, and write $\{s_\bullet, t_\bullet\} = \{s_0, s_1\}$. In the same way, we denote by $s_\circ$ either s_{n+1} or s_{n+2}, and $\{s_\circ, t_\circ\} = \{s_{n+1}, s_{n+2}\}$.

We denote by A, B the elements in $\mathrm{FC}(\widetilde{D}_{n+2})$

$$A = s_2s_3\cdots s_ns_{n+1}s_{n+2} \quad \text{and} \quad B = s_ns_{n-1}\cdots s_2s_1s_0. \tag{2.6}$$

Definition 2.3.5. An element $w \in \mathrm{FC}(\widetilde{D}_{n+2})$ is called a *complete zigzag* if it admits a reduced expression of one of the following forms:

1. $s_0s_1(AB)^kA^h$,

2. $s_{n+2}s_{n+1}(BA)^k B^h$,

where $k \geq 0$, $h \in \{0,1\}$ and $k+h>0$.

In other words, $w \in \mathrm{FC}(\widetilde{D}_{n+2})$ is a complete zigzag if $w \in$ (PZZ) and $D_L(w), D_R(w) \in \{\{s_0,s_1\},\{s_{n+1},s_{n+2}\}\}$. Denote by $\mathrm{CZ}(\widetilde{D}_{n+2}) \subset \mathrm{FC}(\widetilde{D}_{n+2})$ the set of complete zigzag elements, and note that they are irreducible (see some examples in Figure 2.4).

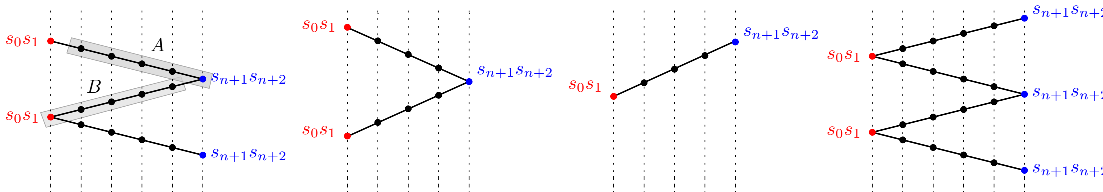


Figure 2.4: Examples of heaps of complete zigzags.

We observe that, if w is a complete zigzag, then we have that $f_\bullet(w), f_\circ(w) \geq 1$. In particular, $\ell(w) = (2k+h)(n+1)+2$, and w is uniquely determined by $\ell(w)$ and $D_L(w)$. Note that $D_L(w) = D_R(w)$ if and only if $h=0$. Moreover, if $w \in \mathrm{CZ}(\widetilde{D}_{n+2})$, then

$$f_\bullet(w) + f_\circ(w) = 2k+h+1 \quad \text{and} \quad |f_\bullet(w) - f_\circ(w)| \leq 1.$$

Observe also that $w \in \mathrm{CZ}(\widetilde{D}_{n+2})$ if and only if $w^{-1} \in \mathrm{CZ}(\widetilde{D}_{n+2})$.

Definition 2.3.6. An element $w \in \mathrm{FC}(\widetilde{D}_{n+2})$ is called a *weak zigzag* if it is a factor of a complete zigzag and $w \notin \mathrm{TC}(\widetilde{D}_{n+2})$.

Recall that the set $\mathrm{TC}(W)$ is introduced in Definition 1.1.9. Note that the family of weak zigzags given in Definition 2.3.6 is not a subset neither of the zigzags introduced in [7, Definition 3.1], nor of the pseudo zigzags defined in Definition 2.2.2. For instance, $s_0s_2s_3s_4$ is a weak zigzag which is in (ALT) according to Definition 2.2.2. Note that w is a weak zigzag if and only if w^{-1} is a weak zigzag. Moreover, a weak zigzag w is irreducible if and only if w is a complete zigzag, since by Definition 2.3.5, any non-complete weak zigzag does not start or end with s_0s_1 or $s_{n+1}s_{n+2}$, then it contains a prefix or suffix s_is_j with $m_{s_is_j}=3$.

Example 2.3.7. Some complete zigzags in $\mathrm{FC}(\widetilde{D}_6)$ are:

1. $w_1 = (s_0s_1)s_2s_3s_4(s_5s_6)$;

2. $w_2 = (s_5 s_6) s_4 s_3 s_2 (s_0 s_1) s_2 s_3 s_4 (s_5 s_6)$;

3. $w_3 = (s_0 s_1) s_2 s_3 s_4 (s_5 s_6) s_4 s_3 s_2 (s_0 s_1) s_2 s_3 s_4 (s_5 s_6)$.

Moreover, some weak zigzags in $\mathrm{FC}(\widetilde{D}_6)$ are:

1. $v_1 = (s_0 s_1) s_2 s_3$, which is a prefix of w_1;

2. $v_2 = s_2 (s_0 s_1) s_2 s_3 s_4 (s_5 s_6)$, which is a suffix of w_2;

3. $v_3 = s_3 s_4 (s_5 s_6) s_4 s_3 s_2 (s_0 s_1) s_2 s_3 s_4 s_6$, which is a factor of w_3.

Lemma 2.3.8. *Let $w \in \mathrm{FC}(\widetilde{D}_{n+2})$ be such that:*

$$D_L(w) = \{s_0, s_1\} \text{ or } D_L(w) = \{s_{n+2}, s_{n+1}\}$$
$$(\textit{respectively, } D_R(w) = \{s_0, s_1\} \text{ or } D_R(w) = \{s_{n+2}, s_{n+1}\}).$$

Then either $w \in \mathrm{TC}(\widetilde{D}_{n+2})$ or w is a weak zigzag. Moreover, w is left (respectively, right) irreducible.

Proof. Without loss of generality suppose that $D_L(w) = \{s_0, s_1\}$. Suppose $w \notin \mathrm{TC}(\widetilde{D}_{n+2})$ and let $w = u_0 \cdots u_m$ be its CFNF, so $m \geq 1$. If $supp(u_i) \neq \emptyset$ then $u_i = s_{i+1}$ for $1 \leq i \leq n-1$ or $supp(u_i) \subseteq \{s_{n+1}, s_{n+2}\}$ for $i = n$. Therefore by CFNF and full commutativity we have:

$$w = \begin{cases} s_0 s_1 s_2 \cdots s_m s_{m+1}, & m < n, \\ s_0 s_1 s_2 \cdots s_n u_n, & m = n \text{ and } u_n \in \{s_{n+1}, s_{n+2}\}, \\ s_0 s_1 s_2 \cdots s_n s_{n+1} s_{n+2} P, & m \geq n \text{ and } P \in \mathrm{Pref}((BA)^k),\ k \geq 1 \end{cases}$$

where A and B are defined in (2.6). In conclusion, either $w \in \mathrm{TC}(\widetilde{D}_{n+2})$ or w is a weak zigzag, and in both cases w is left irreducible. □

Corollary 2.3.9. *Let $w \in \mathrm{FC}(\widetilde{D}_{n+2})$.*

(a) If $x \in \mathrm{Pref}(w) \cup \mathrm{Suff}(w)$ such that $x \in \mathrm{CZ}(\widetilde{D}_{n+2})$, then w is a weak zigzag.

(b) If w is reducible to $v \in \mathrm{CZ}(\widetilde{D}_{n+2})$, then w is a weak zigzag.

Proof. Without loss of generality, we assume $w = xy$ reduced product and $D_R(x) = \{s_0, s_1\}$. Since $x \in \mathrm{CZ}(\widetilde{D}_{n+2})$, $D_L(y) \subseteq \{s_2\}$ by full commutativity. Hence, $D_L(x) = D_L(w)$ and w is a weak zigzag since $w \notin \mathrm{TC}(\widetilde{D}_{n+2})$, and by Lemma 2.3.8, so (a) is proved.

Now assume that w is reducible to $v \in \mathrm{CZ}(\widetilde{D}_{n+2})$, then we can factorize w as $w = v'vv''$ with $\ell(w) = \ell(v') + \ell(v) + \ell(v'')$. By part (a), both $v'v$ and vv'' are weak zigzags. Since $w \in \mathrm{FC}(\widetilde{D}_{n+2})$ and the factorization is length-additive, it follows that w itself is a weak zigzag. □

Lemma 2.3.10. *Let $w \in \mathrm{FC}(\widetilde{D}_{n+2})$ be left irreducible and not a weak zigzag.*

(a) If $f_\bullet(w) \neq 0$, then $s_0, s_1 \in D_L(w)$ and $f_\bullet(w) = 1$;

(b) If $f_\circ(w) \neq 0$, then $s_{n+1}, s_{n+2} \in D_L(w)$ and $f_\circ(w) = 1$.

Proof. We prove (a), since (b) is similar. Since $f_\bullet(w) \neq 0$, we have that w can be factorized as a reduced product $w = vv'$ with $D_R(v) = \{s_0, s_1\}$ and $f_\bullet(v') = 0$. Assume $v \notin \mathrm{TC}(\widetilde{D}_{n+2})$, then by Lemma 2.3.8 we have that v is a weak zigzag and it is right irreducible. Moreover, since w is left irreducible and $v \in \mathrm{Pref}(w)$, it follows that v is a complete zigzag. But this is a contradiction by Corollary 2.3.9, because w is not a weak zigzag. Therefore, $v \in \mathrm{TC}(\widetilde{D}_{n+2})$, so $v = s_0s_1$ and $s_0, s_1 \in D_L(w)$. Moreover, since $f_\bullet(v') = 0$, it follows that $f_\bullet(w) = 1$. □

Lemma 2.3.11. *Let $w \in \mathrm{FC}(\widetilde{D}_{n+2})$, $w = u_0 \cdots u_m$ be its CFNF, and $2 \leq i \leq n$ such that $s_i \in supp(w)$. Let $k \geq 1$ be the minimum index such that $s_i \in supp(u_k)$. If $f_\bullet(w) = 0$, then*

(a) if $i = 2$ and $s_\bullet \in supp(u_{k-1})$, then $k = 1$;

(b) if $i \geq 3$ and $s_{i-1} \in supp(u_{k-1})$, then there exists $z \in \mathrm{Pref}(w)$ with $z \in \mathrm{Suff}(s_\bullet s_2 \cdots s_{i-1})$ and $\ell(z) = k$.

Similarly, if $f_\circ(w) = 0$, then

(c) if $i \leq n-1$ and $s_{i+1} \in supp(u_{k-1})$, then there exists $z \in \mathrm{Pref}(w)$ with $z \in \mathrm{Suff}(s_\circ s_n \cdots s_{i+1})$ and $\ell(z) = k$;

(d) if $i = n$ and $s_\circ \in supp(u_{k-1})$, then $k = 1$.

Proof. We prove (a) and (b), since (c) and (d) are analogous.

Set $i = 2$ and suppose $s_\bullet \in supp(u_{k-1})$. If $k \geq 2$, then $s_2 \in supp(u_{k-2})$ by Theorem 1.1.10(d), but this contradicts the minimality of k. Thus, $k = 1$.

Set $i \geq 3$, assume $s_{i-1} \in supp(u_{k-1})$, and let $w_{k-1} := u_0 \cdots u_{k-1}$. Since $s_{i-1} \in D_R(w_{k-1})$, it is possible to factorize w_{k-1} as a reduced product $w_{k-1} = zv$, with $D_R(z) = \{s_{i-1}\}$, $s_i \notin supp(z)$, and $s_{i-1} \notin supp(v)$. Let $z^{-1} = u'_0 \cdots u'_l$ be its CFNF, then $u'_0 = s_{i-1}$. If $supp(u'_1) \neq \emptyset$, then $u'_1 = s_{i-2}$ by $s_i \notin supp(z)$ and Theorem 1.1.10(d). In general, if $supp(u'_j) \neq \emptyset$ with $i - 1 - j \geq 1$, we have that:

$$u'_j = \begin{cases} s_{i-1-j}, & i-1-j \geq 2; \\ s_\bullet, & i-1-j = 1. \end{cases}$$

If $l \geq i-1$, then $u'_{i-2} = s_\bullet$ and $u'_{i-3} = u'_{i-1} = s_2$, but this contradicts full commutativity since $s_2 s_\bullet s_2$ is a factor of w. Thus, $l \leq i-2$. Therefore, z^{-1} is a prefix of $s_{i-1} \cdots s_2 s_\bullet$, thus z is a suffix of $s_\bullet s_2 \cdots s_{i-1}$. Furthermore, since z is prefix of w_{k-1}, $s_{i-1} \in supp(u_{k-1})$ and $s_{i-1} \notin supp(v)$, by Lemma 1.1.12, $\ell(z) = k$.

$\square$

Example 2.3.12. Let $w = u_0u_1u_2u_3 \in \mathrm{FC}(\widetilde{D}_{11})$ with $u_0 = s_0s_6s_8s_{10}s_{11}$, $u_1 = s_2s_5s_7s_9$, $u_2 = s_3s_6s_8$, and $u_3 = s_4s_7$ its CFNF. Then $f_\bullet(w) = 0$, $s_4 \in supp(u_3)$, $s_4 \notin supp(u_0u_1u_2)$, and $s_3 \in supp(u_2)$. Note that there exists a prefix z of w such that $D_R(z) = \{s_3\}$:

$$w = \underbrace{s_0s_2s_3}_{z} s_6s_8s_{10}s_{11}s_5s_7s_9s_6s_8s_4s_7.$$

Observe that z is of the form described in Lemma 2.3.11 and $\ell(z) = 3$.

Lemma 2.3.13. *Let $w \in \mathrm{FC}(\widetilde{D}_{n+2})$ and $w = u_0 \cdots u_m$ be its CFNF, the following hold.*

(a) Assume that $supp(w) \cap \{s_0, s_1\} \neq \emptyset$ and let k be the minimum index such that $supp(u_k) \cap \{s_0, s_1\} \neq \emptyset$. If $f_\circ(w) = 0$, then there exists $z \in \mathrm{Pref}(w)$ with $z \in \mathrm{Suff}(s_\circ s_n \cdots s_2 s_\bullet)$ and $\ell(z) = k + 1$.

(b) Assume that $supp(w) \cap \{s_{n+1}, s_{n+2}\} \neq \emptyset$ and let k be the minimum index with $supp(u_k) \cap \{s_{n+1}, s_{n+2}\} \neq \emptyset$. If $f_\bullet(w) = 0$, then there exists $z \in \mathrm{Pref}(w)$ with $z \in \mathrm{Suff}(s_\bullet s_2 \cdots s_n s_\circ)$ and $\ell(z) = k + 1$.

Proof. We prove only (a) since (b) is analogous. If $k = 0$, then $s_\bullet$ is a prefix of w. Otherwise, assume $k \geq 1$ and let $w_k := u_0 \cdots u_k$. Since $s_\bullet \in supp(u_k)$ and $supp(u_j) \cap \{s_0, s_1\} = \emptyset$ for all $0 \leq j \leq k-1$, it is possible to factorize w_k as a reduced product $w_k = zv$, with $D_R(z) = \{s_\bullet\}$ and $s_\bullet \notin supp(v)$. Let $z^{-1} = u'_0 \cdots u'_l$ be its CFNF, then necessarily $u'_0 = s_\bullet$. Moreover, if $supp(u'_i) \neq \emptyset$, by Theorem 1.1.10 and full commutativity, we have that

$$u'_i = \begin{cases} s_{i+1}, & 1 \leq i \leq n-1; \\ s_\circ, & i = n. \end{cases}$$

Since $f_\circ(w) = 0$, it follows that $l \leq n$. Otherwise if $l > n$, then $u'_{n+1} = s_n$ by Theorem 1.1.10(d), but this cannot be because $u_n = s_\circ$ and $u_{n-1} = s_n$, so $s_n s_\circ s_n$ would be a factor of w. In conclusion, z^{-1} is a prefix of $s_\bullet s_2 \cdots s_n s_\circ$, thus $z \in \mathrm{Pref}(w_k) \subseteq \mathrm{Pref}(w)$ is a suffix of $s_\circ s_n \cdots s_2 s_\bullet$. Furthermore, by Lemma 1.1.12, $\ell(z) = k+1$. □

Example 2.3.14. Let $w = u_0 u_1 u_2 u_3 \in (\widetilde{D}_{10})$ such that $u_0 = s_4 s_6 s_9$, $u_1 = s_3 s_5 s_8$, $u_2 = s_2 s_4 s_7 s_{10}$, $u_3 = s_0 s_1 s_3 s_8$. Then $f_\circ(w) = 0$ and $supp(w) \cap \{s_0, s_1\} \neq \emptyset$. Hence there exists a prefix z of w such that $D_R(z) = \{s_0\}$:

$$w = \underbrace{s_4 s_3 s_2 s_0}_{z} s_6 s_9 s_5 s_8 s_4 s_7 s_{10} s_1 s_3 s_8.$$

Observe that z is a prefix of w of the form described in Lemma 2.3.13 and $\ell(z) = 4$.

Lemma 2.3.15. *Let $w \in \mathrm{I}(\widetilde{D}_{n+2}) \setminus \mathrm{CZ}(\widetilde{D}_{n+2})$ with complete support and $w = u_0 \cdots u_m$ be its CFNF. The following hold:*

(a) $m \geq 1$;

(b) $f_\bullet(w) = f_\circ(w) = 0$;

(c) $|supp(u_0) \cap \{s_0, s_1\}| = 1$;

(d) $|supp(u_0) \cap \{s_{n+1}, s_{n+2}\}| = 1$.

Proof. If $m = 0$, then $w = u_0$ would be a product of commuting generators and it could not have complete support. Hence, $m \geq 1$ and (a) is proved.

We prove by contradiction that $f_\bullet(w) = 0$; the case with $f_\circ(w)$ is analogous. Assume $f_\bullet(w) \neq 0$. Note that w is not a weak zigzag since it is irreducible and $w \notin \mathrm{CZ}(\widetilde{D}_{n+2})$. Then by applying Lemma 2.3.10 to w and w^{-1}, we have that $s_0, s_1 \in D_L(w) \cap D_R(w)$ and $f_\bullet(w) = 1$. This implies that $s_2 \notin supp(w)$. But this a contradiction since $supp(w)$ is complete.

To prove (c), by point (b) it is sufficient to show the following claim:

$$supp(u_0) \cap \{s_0, s_1\} \neq \emptyset.$$

We have that $f_\circ(w) = 0$, and since $s_0, s_1 \in supp(w)$ by complete support, there exists k minimum such that $supp(u_k) \cap \{s_0, s_1\} \neq \emptyset$. Hence by Lemma 2.3.13, there exists $z \in \mathrm{Pref}(w)$ with

$$z \in \mathrm{Suff}(s_\circ s_n \cdots s_2 s_\bullet)$$

and $\ell(z) \geq 1$. If $\ell(z) \geq 2$, then z is a left reducible prefix of w, but this is a contradiction, since $w \in \mathrm{I}(\widetilde{D}_{n+2})$. Therefore, $z = s_\bullet$, so $supp(u_0) \cap \{s_0, s_1\} \neq \emptyset$. In conclusion, by $f_\bullet(w) = 0$, $|supp(u_0) \cap \{s_0, s_1\}| = 1$. The proof of (d) is analogous of (c).

$\square$

Lemma 2.3.16. *Let $w \in \mathrm{I}(\widetilde{D}_{n+2}) \setminus \mathrm{CZ}(\widetilde{D}_{n+2})$ with complete support, and let $w = u_0 \cdots u_m$ be its CFNF. Then, the following hold:*

(a) $supp(u_0) = \{s_\bullet\} \cup \{s_{2j+1} \mid 3 \leq 2j+1 \leq n\} \cup \{s_\circ\}$;

(b) $supp(u_1) = \{s_{2j} \mid 2 \leq 2j \leq n\}$.

In particular, n is even.

Proof. By Lemma 2.3.15, we have $m \geq 1$, $s_\bullet, s_\circ \in supp(u_0)$, and $f_\bullet(w) = f_\circ(w) = 0$. Consider the sets

$$U_0 := \{2j+1 \mid s_{2j+1} \notin supp(u_0),\ 3 \leq 2j+1 \leq n\} \text{ and}$$
$$U_1 := \{2j \mid s_{2j} \notin supp(u_1),\ 2 \leq 2j \leq n\}.$$

We will show by contradiction that U_0 and U_1 are empty. Set $h := \min(U_0 \cup U_1)$. First, observe that $h \in U_1$, otherwise if $h \in U_0$, then by minimality, $s_{h-2}s_{h-1}$ or $s_\bullet s_2$, for $h = 3$, would be a prefix of w, which is not possible by irreducibility. Also, note that

$s_\bullet \in supp(u_0)$ and $s_2 \notin supp(u_0)$. So if $h = 2$, then $s_2 \notin supp(u_0)$ and $s_2 \notin supp(u_1)$. Furthermore, if $h > 2$ then $s_{h-1} \in supp(u_0)$ by the minimality of h, and so $s_h \notin supp(u_0)$ and $s_h \notin supp(u_1)$. In both cases we have $s_h \notin supp(u_0)$ and $s_h \notin supp(u_1)$. But $s_h \in supp(u_k)$ for some $k \geq 2$ because w has complete support, so assume k is the minimum index with this property. By Lemma 2.3.11, there exists $z \in \mathrm{Pref}(w)$ with

$$z \in \mathrm{Suff}(s_\bullet s_2 \cdots s_{h-1}) \quad \text{or} \quad z \in \mathrm{Suff}(s_\circ s_n \cdots s_{h+1})$$

and $\ell(z) = k \geq 2$. In both cases we have a contradiction. In the first one, z cannot be a prefix since $s_{h-1} \in D_L(w)$ by minimality of h; in the second, z cannot be a prefix since $w \in \mathrm{I}(\widetilde{D}_{n+2})$. Hence $U_0 \cup U_1 = \emptyset$ from which the result follows.

Finally, since $s_\circ \in supp(u_0)$, if n is odd then $s_n \in supp(u_0)$, which is a contradiction. □

Remark 2.3.17. In [27, Theorem 6.3], Green shows that Coxeter groups of finite type D are star reducible. This means that $w \in W(D_n)$ is irreducible if and only if w is a product of commuting generators. In our setting, if $w \in \mathrm{FC}(\widetilde{D}_{n+2})$ and $supp(w)$ is not complete, then w can be considered as a FC element or a product of FC elements in a Coxeter group of finite type D. Hence, if $w \in \mathrm{I}(\widetilde{D}_{n+2})$ with not complete support, then $w \in \mathrm{TC}(\widetilde{D}_{n+2})$.

Definition 2.3.18. Let $w \in \mathrm{FC}(\widetilde{D}_{n+2})$, n even, and let $w = u_0 \cdots u_m$ be its CFNF with $m \geq 2$. We say that w is a *candy* if m is even and the following conditions hold:

- $supp(u_{2i}) = \{s_3, s_5, \ldots, s_{n-1}\} \cup \{x_i, y_i\}$, $x_i \in \{s_0, s_1\}$ and $y_i \in \{s_{n+2}, s_{n+1}\}$ such that $x_i \neq x_{i+1}$ and $y_i \neq y_{i+1}$, for every $0 \leq i \leq \frac{m}{2}$;
- $supp(u_{2i+1}) = \{s_2, s_4, \ldots, s_n\}$, for every $0 \leq i < \frac{m}{2}$.

Denote by $\mathrm{K}(\widetilde{D}_{n+2}) \subset \mathrm{FC}(\widetilde{D}_{n+2})$ the set of candy elements. We set $\mathrm{K}(\widetilde{D}_{n+2}) = \emptyset$ when n is odd. All candy elements are irreducible. In fact, by construction and by the definition of CFNF (Theorem 1.1.10), $supp(w)$ is complete, and for each $t \in supp(u_1)$ there exist $s_i, s_j \in supp(u_0)$ such that $m_{s_i t} = m_{s_j t} = 3$, so there are no reducible prefixes. Analogously, for each $t' \in supp(u_{m-1})$ there exist $s'_i, s'_j \in supp(u_m)$ such that $m_{s'_i t'} = m_{s'_j t'} = 3$, hence there are no reducible suffixes.

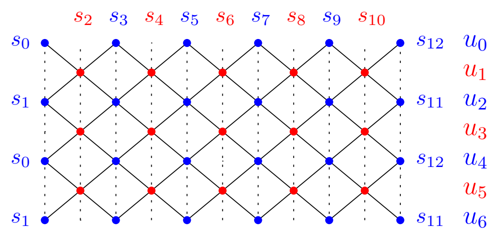


Figure 2.5: Example of a heap of a candy element in $(\widetilde{D}_{12})$.

Any $w \in \mathrm{K}(\widetilde{D}_{n+2})$ is uniquely determined by x_0, y_0 and m; furthermore we have that $D_L(w) = D_R(w)$ if and only if $m/2$ is even. Moreover, $f_\bullet(w) = f_\circ(w) = 0$. Observe that $w \in \mathrm{K}(\widetilde{D}_{n+2})$ if and only if $w^{-1} \in \mathrm{K}(\widetilde{D}_{n+2})$.

Proposition 2.3.19. *Let $w \in \mathrm{I}(\widetilde{D}_{n+2}) \setminus \mathrm{CZ}(\widetilde{D}_{n+2})$ with complete support. Then $w \in \mathrm{K}(\widetilde{D}_{n+2})$.*

Proof. Let $w = u_0 \cdots u_m$ be its CFNF. By Lemma 2.3.16 we know that $m \geq 1$, n is even and $supp(u_0)$, $supp(u_1)$ are described in points (a) and (b). This implies $m \geq 2$, otherwise w would admit $s_\bullet s_2$ as a suffix, which is not possible since $w \in \mathrm{I}(\widetilde{D}_{n+2})$. Note that $s_\bullet \notin supp(u_2)$, otherwise $s_\bullet s_2 s_\bullet$ would be a factor of w, contradicting full commutativity. Analogously, we have that $s_\circ \notin supp(u_2)$. Now, by definition of CFNF, we have that

$$supp(u_2) \subseteq \{t_\bullet\} \cup \{s_3, s_5, \ldots, s_{n-1}\} \cup \{t_\circ\} := \mathcal{S}. \tag{2.7}$$

Define $w_2 := u_2 \cdots u_m$ and note that is in $\mathrm{I}(\widetilde{D}_{n+2})$. In fact, it is right irreducible since it is a suffix of $w \in \mathrm{I}(\widetilde{D}_{n+2})$. Moreover, it is also left irreducible because if st with $m_{st} = 3$ is a prefix of w_2, then $s \in supp(u_2)$ and $t \in supp(u_3)$, but by CFNF we have that $supp(u_3) \subseteq supp(u_1)$, so tst would be a factor of w, that contradicts the hypothesis of fully commutativity.

To summarize, we have $w = u_0 u_1 w_2$, with $w_2 = u_2 \cdots u_m \in \mathrm{I}(\widetilde{D}_{n+2})$ and $m \geq 2$. Now we show by induction on $m \geq 2$ that $w \in \mathrm{K}(\widetilde{D}_{n+2})$.

Suppose $m = 2$, then $t_\bullet, t_\circ \in supp(u_2)$ since w has complete support. Moreover, the inclusion in (2.7) is an equality. Otherwise there would exist $s' \in \mathcal{S} \setminus supp(u_2)$, $s \in supp(u_1)$, $t \in supp(u_2)$ with $m_{st} = m_{s's} = 3$, and the suffix st would appear in w, that is not possible by irreducibility. Hence $w = u_0 u_1 u_2 \in \mathrm{K}(\widetilde{D}_{n+2})$; its heap is depicted in Figure 2.6.

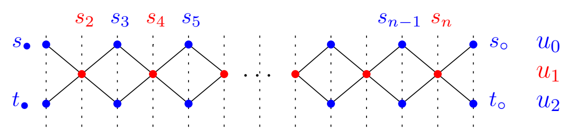


Figure 2.6: Heap of $u_0u_1u_2$ in the proof of Proposition 2.3.19.

If $m > 2$, since w_2 is irreducible, we know that $supp(w_2)$ is complete, otherwise by Remark 2.3.17, $w_2 \in \mathrm{TC}(\widetilde{D}_{n+2})$, but this cannot be since $m > 2$. Moreover, since $w_2 \in \mathrm{Suff}(w)$ and $f_\bullet(w) = f_\circ(w) = 0$ by Lemma 2.3.15, then $f_\bullet(w_2) = f_\circ(w_2) = 0$, so w_2 is not a complete zigzag. Therefore, by inductive hypothesis, w_2 is a candy element, and we can conclude that w is also a candy element, since $w = u_0u_1w_2$. $\square$

Theorem 2.3.20 (Classification of irreducible elements of type $\widetilde{D}_{n+2}$)**.**
The set of irreducible elements in $\mathrm{FC}(\widetilde{D}_{n+2})$ *is partitioned in three families as follows:*

$$\mathrm{I}(\widetilde{D}_{n+2}) = \mathrm{K}(\widetilde{D}_{n+2}) \sqcup \mathrm{CZ}(\widetilde{D}_{n+2}) \sqcup \mathrm{TC}(\widetilde{D}_{n+2}).$$

If n is odd, $\mathrm{K}(\widetilde{D}_{n+2}) = \emptyset$.

Proof. Let $w \in \mathrm{I}(\widetilde{D}_{n+2})$. If $supp(w)$ is not complete then $w \in \mathrm{TC}(\widetilde{D}_{n+2})$, by Remark 2.3.17. If $supp(w)$ is complete and w is not a complete zigzag, then n is even and $w \in \mathrm{K}(\widetilde{D}_{n+2})$, by Proposition 2.3.19. $\square$

Corollary 2.3.21. *If $w \in \mathrm{FC}(\widetilde{D}_{n+2})$ is reducible to $v, v' \in \mathrm{I}(\widetilde{D}_{n+2})$, then v, v' belong to the same family of irreducible elements. In particular, if $v, v' \notin \mathrm{TC}(\widetilde{D}_{n+2})$, then $v = v'$.*

Proof. If $v \notin \mathrm{TC}(\widetilde{D}_{n+2})$, then $supp(v)$ is complete by Theorem 2.3.20, so $v = v'$ by Theorem 2.1.6(c). $\square$

2.4 Irreducibility and weak irreducibility in $\mathrm{FC}(\widetilde{B}_{n+1})$

As we mentioned above, in the non-simply laced cases the notions of reducibility and weak reducibility do not coincide, so we have that $\mathrm{I}'_{\mathrm{w}}(\widetilde{B}_{n+1}) = \mathrm{I}_{\mathrm{w}}(\widetilde{B}_{n+1}) \setminus \mathrm{I}(\widetilde{B}_{n+1})$ is not empty. In this section we classify both the irreducible and weak irreducible elements of a Coxeter group of type $\widetilde{B}_{n+1}$. The families of irreducible elements that we now list are analogous to those arising in type $\widetilde{D}_{n+2}$, although certain differences occur in the present setting.

Similarly to Notation 2.2.1, we represent a heap of $w \in \mathrm{FC}(\widetilde{B}_{n+1})$ with $n+1$ columns, where the criterion for the placement of double vertices is applied only to s_0 and s_1.

Definition 2.4.1. We define the set of *weak trivially commutative* elements $\mathrm{TC_w}(\widetilde{B}_{n+1}) \subseteq \mathrm{FC}(\widetilde{B}_{n+1})$ by

$$\mathrm{TC_w}(\widetilde{B}_{n+1}) := \mathrm{TC}(\widetilde{B}_{n+1}) \cup \{s_n s_{n+1} v, s_{n+1} s_n v \mid v \in \mathrm{TC}(\widetilde{B}_{n+1}), s_{n-1}, s_n, s_{n+1} \notin supp(v)\}.$$

Note that $w \in \mathrm{TC_w}(\widetilde{B}_{n+1})$ is weak irreducible. In particular, $w \in \mathrm{I}(\widetilde{B}_{n+1})$ if and only if $w \in \mathrm{TC}(\widetilde{B}_{n+1})$.

We denote by $\mathcal{A}, \mathcal{B}$ the elements in $\mathrm{FC}(\widetilde{B}_{n+1})$

$$\mathcal{A} = s_2 s_3 \cdots s_n s_{n+1} \quad \text{and} \quad \mathcal{B} = s_n s_{n-1} \cdots s_2 s_1 s_0.$$

Definition 2.4.2. An element $w \in \mathrm{FC}(\widetilde{B}_{n+1})$ is called a *complete zigzag* if it admits a reduced expression of one of the following forms:

1. $s_0 s_1 (\mathcal{A}\mathcal{B})^k \mathcal{A}^h$,
2. $s_{n+1} (\mathcal{B}\mathcal{A})^k \mathcal{B}^h$,

where $k \geq 0$, $h \in \{0, 1\}$ and $k + h > 0$.

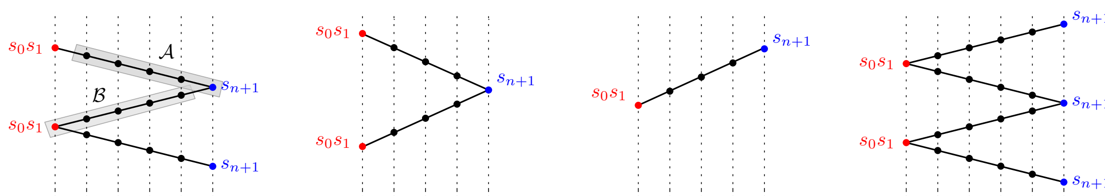


Figure 2.7: Examples of heaps of complete zigzag elements in type $\widetilde{B}_{n+1}$.

Denote by $\mathrm{CZ}(\widetilde{B}_{n+1}) \subset \mathrm{FC}(\widetilde{B}_{n+1})$ the set of complete zigzag elements and note that they are weak irreducible. In particular, $w \in \mathrm{CZ}(\widetilde{B}_{n+1}) \cap \mathrm{I}(\widetilde{B}_{n+1})$ if and only if w is of the form $s_0 s_1 (\mathcal{A}\mathcal{B})^k \mathcal{A}^h$ with $h > 0$.

Definition 2.4.3. An element $w \in \mathrm{FC}(\widetilde{B}_{n+1})$ is called a *weak zigzag* if it is a factor of a complete zigzag and $w \notin \mathrm{TC}(\widetilde{B}_{n+1})$.

Note that a weak zigzag w is weak irreducible if and only if w is a complete zigzag.

Definition 2.4.4. Let $w \in \mathrm{FC}(\widetilde{B}_{n+1})$, n even, and let $w = u_0 \cdots u_m$ be its normal form with $m \geq 2$. We say that w is a *candy* if m is even and:

- $supp(u_{2i}) = \{s_3, s_5, \dots, s_{n+1}\} \cup \{x_i\}$, $x_i \in \{s_0, s_1\}$ such that $x_i \neq x_{i+1}$, for every $0 \leq i \leq \frac{m}{2}$;
- $supp(u_{2i+1}) = \{s_2, s_4, \dots, s_n\}$, for every $0 \leq i < \frac{m}{2}$.

Denote by $\mathrm{K}(\widetilde{B}_{n+1}) \subset \mathrm{FC}(\widetilde{B}_{n+1})$ the set of candy elements and note that they are irreducible since by construction they have complete support, and by Theorem 1.1.10. If n is odd, we set $\mathrm{K}(\widetilde{B}_{n+1}) = \emptyset$.

Definition 2.4.5. Let $w \in \mathrm{FC}(\widetilde{B}_{n+1})$, n odd, and let $w = u_0 \cdots u_m$ be its normal form with $m \geq 2$. We say that w is a *left candy* if m is even and:

- $supp(u_{2i}) = \{s_3, s_5, \dots, s_n\} \cup \{x_i\}$, $x_i \in \{s_0, s_1\}$ such that $x_i \neq x_{i+1}$, for every $0 \leq i \leq \frac{m}{2}$;
- $supp(u_{2i+1}) = \{s_2, s_4, \dots, s_{n+1}\}$, for every $0 \leq i < \frac{m}{2}$.

Denote by $\mathrm{K_w}(\widetilde{B}_{n+1}) \subset \mathrm{FC}(\widetilde{B}_{n+1})$ the set of left candy elements, in particular $\mathrm{K_w}(\widetilde{B}_{n+1}) \subset \mathrm{I}'_{\mathrm{w}}(\widetilde{B}_{n+1})$, since an element $w \in \mathrm{K_w}(\widetilde{B}_{n+1})$ has a prefix $s_n s_{n+1}$, hence $w \notin \mathrm{I}(\widetilde{B}_{n+1})$, but a prefix or suffix of the form $s_n s_{n+1} s_n$ or $s_{n+1} s_n s_{n+1}$ never occurs by complete support and Theorem 1.1.10. If n is even, we set $\mathrm{K_w}(\widetilde{B}_{n+1}) = \emptyset$.

In the next definition, we introduce an injective map

$$\varphi : \mathrm{FC}(\widetilde{B}_{n+1}) \to \mathrm{FC}(\widetilde{D}_{n+2}).$$

This map will serve as a tool for classifying the irreducible and weakly irreducible elements of type $\widetilde{B}_{n+1}$, based on the corresponding classification for type $\widetilde{D}_{n+2}$ developed in Section 2.3.

Definition 2.4.6. Let $w \in \mathrm{FC}(\widetilde{B}_{n+1})$. We define the map $\varphi : \mathrm{FC}(\widetilde{B}_{n+1}) \to \mathrm{FC}(\widetilde{D}_{n+2})$ by the following rule:

(1) If no reduced expression of w contains the factor $s_{n+1}s_ns_{n+1}$ or $s_ns_{n+1}s_n$, consider any $\mathbf{w} \in \mathcal{R}(w)$. Then $\varphi(w)$ is the element in $\mathrm{FC}(\widetilde{D}_{n+2})$ with reduced expression $\mathbf{w}$.

(2) Otherwise, choose a reduced expression $\mathbf{w} \in \mathcal{R}(w)$ that represents

$$\max\{\#\text{ of occurences of the factors } s_{n+1}s_ns_{n+1} \text{ and } s_ns_{n+1}s_n \text{ in } \mathbf{w} \mid \mathbf{w} \in \mathcal{R}(w)\}.$$

Then $\varphi(w)$ is the element whose reduced expression is obtained from $\mathbf{w}$ by replacing each occurrence of $s_{n+1}s_ns_{n+1}$ with $s_{n+1}s_ns_{n+2}$, and each occurrence of $s_ns_{n+1}s_n$ with $s_ns_{n+1}s_{n+2}s_n$.

The map φ is well-defined. Point (1) is immediate, and since in $\mathrm{FC}(\widetilde{B}_{n+1})$ there are no elements containing both factors $s_{n+1}s_ns_{n+1}$ and $s_ns_{n+1}s_n$ (see e.g. [7, §3.2]), point (2) is also well-posed.

Lemma 2.4.7. *The map φ is injective, and $w \in \mathrm{I}(\widetilde{B}_{n+1})$ if and only if $\varphi(w) \in \mathrm{I}(\widetilde{D}_{n+2})$.*

Proof. Note that if there exists a reducible prefix or suffix in w, then it will be transformed by φ into a reducible prefix or suffix in $\varphi(w)$. On the other hand, the only reducible prefixes and suffixes that need to be analyzed are $s_\circ s_n$ and $s_n s_\circ$. If $s_\circ s_n$ or $s_n s_\circ$ is a prefix of $\varphi(w)$, then $s_\circ = s_{n+1}$ by construction of φ, so $s_{n+1}s_n$ or s_ns_{n+1} is a prefix of w. The analogous holds for the suffix, with the only difference that if $s_ns_\circ$ is a suffix of $\varphi(w)$, $s_\circ$ could be equal to s_{n+1} or s_{n+2}. □

Remark 2.4.8. Let $w \in \mathrm{FC}(\widetilde{B}_{n+1})$, we have that w is a weak zigzag if and only if $\varphi(w)$ is a weak zigzag. Moreover, if $w \in \mathrm{I_w}(\widetilde{B}_{n+1})$ and $\varphi(w)$ is a weak zigzag, then $w \in \mathrm{CZ}(\widetilde{B}_{n+1})$. Finally, w is a candy in $\mathrm{FC}(\widetilde{B}_{n+1})$ if and only if $\varphi(w)$ is a candy in $\mathrm{FC}(\widetilde{D}_{n+2})$.

The next theorem follows directly by Theorem 2.3.20, Lemma 2.4.7 and Remark 2.4.8.

Theorem 2.4.9. *The set of irreducible elements in $\mathrm{FC}(\widetilde{B}_{n+1})$ is partitioned as follows*

$$\mathrm{I}(\widetilde{B}_{n+1}) = \mathrm{TC}(\widetilde{B}_{n+1}) \sqcup \mathrm{K}(\widetilde{B}_{n+1}) \sqcup \mathrm{CZ}_\bullet(\widetilde{B}_{n+1}),$$

where $\mathrm{CZ}_\bullet(\widetilde{B}_{n+1}) := \{w \in \mathrm{CZ}(\widetilde{B}_{n+1}) \mid D_L(w) = D_R(w) = \{s_0, s_1\}\}$ *and if* n *is odd,* $\mathrm{K}(\widetilde{B}_{n+1}) = \emptyset$.

Remark 2.4.10. By Definitions 2.1.1 and 2.1.2, if $w \in \mathrm{I}'_{\mathrm{w}}(\widetilde{B}_{n+1})$, then w has a prefix or suffix st, with $m_{st} \geq 3$, such that $tst \in \mathrm{FC}(\widetilde{B}_{n+1})$. This necessarily implies that $\{s,t\} = \{s_n, s_{n+1}\}$. Indeed, if w has a different prefix or suffix of the form st, with $m_{st} \geq 3$, then $m_{st} = 3$ and $sts \notin \mathrm{FC}(\widetilde{B}_{n+1})$.

Lemma 2.4.11. *Let* $w \in \mathrm{I}_{\mathrm{w}}(\widetilde{B}_{n+1}) \setminus \mathrm{CZ}(\widetilde{B}_{n+1})$ *with complete support. Then we have:*

(a) if $w \in \mathrm{I}'_{\mathrm{w}}(\widetilde{B}_{n+1})$*, then* $s_n s_{n+1} \in \mathrm{Pref}(w) \cup \mathrm{Pref}(w^{-1})$*;*

(b) $f_\bullet(\varphi(w)) = f_\circ(\varphi(w)) = 0$*;*

(c) $|D_L(w) \cap \{s_0, s_1\}| = 1$.

Proof. For proving (a) we proceed by contradiction, so assume that $s_n s_{n+1} \notin \mathrm{Pref}(w) \cup \mathrm{Pref}(w^{-1})$. Therefore, by Remark 2.4.10, w has a prefix of the form $s_{n+1}s_n$ or a suffix of the form $s_n s_{n+1}$. Without loss of generality, we assume $s_{n+1}s_n \in \mathrm{Pref}(w)$. We define $v := s_{n+2}\varphi(w)$ if $s_n s_{n+1} \notin \mathrm{Suff}(w)$ or $v := s_{n+2}\varphi(w)s_{n+2}$ if $s_n s_{n+1} \in \mathrm{Suff}(w)$. In both cases, $v \in \mathrm{FC}(\widetilde{D}_{n+2})$ and note that $v \in \mathrm{I}(\widetilde{D}_{n+2})$ and $s_{n+1}, s_{n+2} \in D_L(v)$. Therefore, since $supp(w)$ is complete we have that $supp(v)$ is also complete, so by the classification in Theorem 2.3.20, it follows that $v \in \mathrm{CZ}(\widetilde{D}_{n+2})$. But in all the cases this means that $\varphi(w)$ is a weak zigzag, so by Remark 2.4.8 it follows that $w \in \mathrm{CZ}(\widetilde{B}_{n+1})$, which is a contradiction.

Assume $f_\bullet(\varphi(w)) \neq 0$, we will show that $s_0, s_1 \in D_L(\varphi(w)) \cap D_R(\varphi(w))$. We can write $\varphi(w) = vv'$ reduced, with $D_R(v) = \{s_0, s_1\}$ and $f_\bullet(v) = 1$. If $v \neq s_0 s_1$, hence v is a weak zigzag by Lemma 2.3.8. Moreover, since $w \in \mathrm{I}_{\mathrm{w}}(\widetilde{B}_{n+1})$, $f_\bullet(v) = 1$ and $s_{n+2} \notin D_L(\varphi(w))$, we have that $v = s_{n+1}s_n \cdots s_2 s_1 s_0$. We observe that $s_{n+2} \notin D_L(v')$, otherwise $s_{n+1}s_n s_{n+1}$ is a prefix of w. In particular, $s_{n+2}\varphi(w) \in \mathrm{FC}(\widetilde{D}_{n+2})$ is a reduced product, so $s_{n+2}\varphi(w)$ admits $s_{n+2}v \in \mathrm{CZ}(\widetilde{D}_{n+2})$ as a prefix, thus $s_{n+2}\varphi(w)$ is a weak zigzag by Corollary 2.3.9. Therefore, $\varphi(w)$ is a weak zigzag, and $w \in \mathrm{CZ}(\widetilde{B}_{n+1})$ by Remark 2.4.8, but this is a contradiction. So, $v = s_0 s_1$ and $s_0, s_1 \in D_L(\varphi(w))$. To prove that $s_0, s_1 \in D_R(\varphi(w))$, we factorize $\varphi(w) = y'y$ with $D_L(y) = \{s_0, s_1\}$ and $f_\bullet(y) = 1$ and we argue as above. Therefore, $s_0, s_1 \in D_L(\varphi(w)) \cap D_R(\varphi(w))$, thus $f_\bullet(\varphi(w)) \geq 2$

since $s_2 \in supp(\varphi(w))$. This implies that $\varphi(w)$ is reducible to $x \in \mathrm{CZ}(\widetilde{D}_{n+2})$ by Lemma 2.3.2 and Theorem 2.3.20. Hence $\varphi(w)$ is a weak zigzag by Corollary 2.3.9, in particular $\varphi(w) = x \in \mathrm{CZ}(\widetilde{D}_{n+2})$ since $s_0, s_1 \in D_L(\varphi(w)) \cap D_R(\varphi(w))$. But this is a contradiction because w is not a complete zigzag. In conclusion, $f_\bullet(\varphi(w)) = 0$.

Now we prove that $f_\circ(\varphi(w)) = 0$, as above we factorize $\varphi(w) = vv'$ with $D_R(v) = \{s_{n+1}, s_{n+2}\}$ and $f_\circ(v) = 1$. If $v \neq s_{n+1}s_{n+2}$, then v is a weak zigzag by Lemma 2.3.8. But this cannot be since $f_\bullet(\varphi(w)) = 0$, $f_\circ(v) = 1$ and $w \in \mathrm{I_w}(\widetilde{B}_{n+1})$. Therefore, $v = s_{n+1}s_{n+2}$ and $s_{n+1}, s_{n+2} \in D_L(\varphi(w))$, but this is a contradiction by the definition of φ.

Assume now $s_0, s_1 \notin D_L(w)$ and let $\varphi(w) = u_0 \cdots u_m$ be its CFNF. It follows that $s_0, s_1 \notin supp(u_0)$, so by applying Lemma 2.3.13 there exists $z \in \mathrm{Pref}(\varphi(w))$ with

$$z \in \mathrm{Suff}(s_{n+1}s_n \cdots s_2 s_\bullet)$$

and $\ell(z) > 1$. We observe that in all the cases z is a prefix of w too, by the definition of the map. So the only possibility is that $z = s_{n+1}s_n \cdots s_2 s_\bullet$ and we factorize w as $w = zw'$ reduced product. We observe that $D_L(w') \subseteq \{t_\bullet\}$ by $w \in \mathrm{FC}(\widetilde{B}_{n+1})$ and $w \in \mathrm{I_w}(\widetilde{B}_{n+1})$. But if $t_\bullet \in D_L(w')$, then $f_\bullet(\varphi(w)) \geq 1$, that contradict point (b). Therefore, $w' = e$ and $w = z$ but this a contradiction because $w \in \mathrm{I_w}(\widetilde{B}_{n+1})$.

□

Lemma 2.4.12. *Let $w \in \mathrm{I}'_{\mathrm{w}}(\widetilde{B}_{n+1}) \setminus \mathrm{CZ}(\widetilde{B}_{n+1})$ with complete support, and let $w = u_0 \cdots u_m$ be its CFNF. If $s_n s_{n+1} \in \mathrm{Pref}(w)$, then the following hold:*

(a) $supp(u_0) = \{s_\bullet\} \cup \{s_{2j+1} \mid 3 \leq 2j+1 \leq n+1\}$*;*

(b) $supp(u_1) = \{s_{2j} \mid 2 \leq 2j \leq n+1\}$.

In particular, n is odd.

Proof. It suffices to show that, given $\varphi(w) = v_0 \cdots v_m$ to be its CFNF we have

$$supp(v_0) = \{s_\bullet\} \cup \{s_3, s_5, \ldots, s_{n-2}, s_n\}, \quad \text{and} \quad supp(v_1) = \{s_2, s_4, \ldots, s_{n-1}, s_{n+1}\}.$$

We consider the following sets

$$V_0 := \{2j+1 \mid s_{2j+1} \notin supp(v_0), 1 \leq 2j+1 \leq n-1\}$$
$$V_1 := \{2j \mid s_{2j} \notin supp(v_1), 1 \leq 2j \leq n-1\}.$$

We need to prove that V_0 and V_1 are both empty. Following verbatim the steps of the proof of Lemma 2.3.16, we obtain that there exists $z \in \mathrm{Pref}(w)$ with

$$z \in \mathrm{Suff}(s_\bullet s_2 \cdots s_{h-1}) \quad \text{or} \quad z \in \mathrm{Suff}(s_{n+1}s_n \cdots s_{h+1})$$

and $h \geq 2$ and $\ell(z) \geq 2$. Both cases leads to a contradiction since $w \in \mathrm{I}'_\mathrm{w}(\widetilde{B}_{n+1})$ and $s_n s_{n+1} \in \mathrm{Pref}(w)$. In particular, it follows that n is odd, otherwise $s_{n-1}, s_n \in D_L(w)$.

□

Proposition 2.4.13. *Let n be odd and $w \in \mathrm{I}_\mathrm{w}(\widetilde{B}_{n+1}) \backslash \mathrm{CZ}(\widetilde{B}_{n+1})$ with complete support. Then $w \in \mathrm{K}_\mathrm{w}(\widetilde{B}_{n+1})$.*

Proof. We observe that $w \in \mathrm{I}'_\mathrm{w}(\widetilde{B}_{n+1})$, otherwise $w \in \mathrm{CZ}_\bullet(\widetilde{B}_{n+1})$ by Theorem 2.4.9. Without loss of generality, by Lemma 2.4.11, assume $s_n s_{n+1} \in \mathrm{Pref}(w)$. Let $w = u_0 \cdots u_m$ be its CFNF, then we know that $supp(u_0)$ and $supp(u_1)$ are described in Lemma 2.4.12 (a) and (b). This implies $m \geq 2$, otherwise $s_\bullet s_2$ would be a suffix of w, which is not possible since $w \in \mathrm{I}_\mathrm{w}(B)$. Note that $s_\bullet \notin supp(u_2)$, otherwise $s_\bullet s_2 s_\bullet$ is a factor of w, contradicting full commutativity. By definition of CFNF, we have that

$$supp(u_2) \subseteq \{t_\bullet\} \cup \{s_3, s_5, \ldots, s_n\} := \mathcal{S}. \tag{2.8}$$

Define $w_2 := u_2 \cdots u_m$ and note that is in $\mathrm{I}_\mathrm{w}(\widetilde{B}_{n+1})$. In fact, it is right weak reducible since it is a suffix of $w \in \mathrm{I}_\mathrm{w}(\widetilde{B}_{n+1})$. Moreover, it is also left weak irreducible because if st with $m_{st} = 3$ is a prefix of w_2, then $s \in supp(u_2)$ and $t \in supp(u_3)$, but by CFNF we have that $supp(u_3) \subseteq supp(u_1)$, so tst would be a factor of w, that contradicts the hypothesis of fully commutativity. On the other hand, if sts with $m_{st} = 4$ is a prefix of w_2, then necessarily $s = s_n$ and $t = s_{n+1}$, but in this way we have that $s_{n+1}s_n s_{n+1} s_n$ is a factor of w.

To summarize, we have $w = u_0 u_1 w_2$, with $w_2 = u_2 \cdots u_m \in \mathrm{I}_\mathrm{w}(\widetilde{B}_{n+1})$ and $m \geq 2$. Now we show by induction on $m \geq 2$ that $w \in \mathrm{K}_\mathrm{w}(\widetilde{B}_{n+1})$.

Suppose $m = 2$, then $t_\bullet \in supp(u_2)$ since w has sull support. Moreover, the inclusion in (2.8) is an equality. Otherwise there would exist $s' \in \mathcal{S} \setminus supp(u_2)$, $s \in supp(u_1)$, $t \in supp(u_2)$ with $m_{st} = m_{s's} = 3$, and the suffix st would appear in w, that is not possible. Hence $w = u_0u_1u_2 \in \mathrm{K_w}(\widetilde{B}_{n+1})$; its heap is depicted in Figure 2.8.

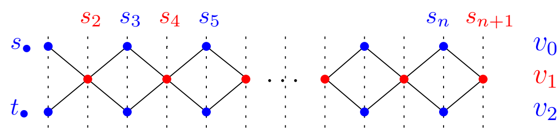


Figure 2.8: Heap of $u_0u_1u_2$ in the proof of Proposition 2.4.13.

If $m > 2$, we first show that $supp(w_2)$ is complete. In fact, if $supp(w_2)$ is not complete, then w_2 is of finite type D, B or a product of elements of type D and B. Hence $w_2 \in \mathrm{TC_w}(\widetilde{B}_{n+1})$ by Remark 2.3.17 and [17, Theorem 4.2.1] because $w_2 \in \mathrm{I_w}(\widetilde{B}_{n+1})$. Since $m > 2$ and $s_{n+1} \in supp(u_1)$, the unique possibility is that $w_2 = s_n s_{n+1} w' = w' s_n s_{n+1}$ reduced product with $w' \in \mathrm{TC}(\widetilde{B}_{n+1})$. But in this case, since $s_{n+1}w' = w's_{n+1}$ and $s_{n+1} \in supp(u_1)$, we have that $s_{n+1}s_ns_{n+1} \in \mathrm{Suff}(w)$ which contradicts the weak irreducibility. Therefore, $supp(w_2)$ is complete and $w_2 \in \mathrm{Suff}(w)$, so w_2 is not a complete zigzag since $f_\bullet(\varphi(w)) = f_\bullet(\varphi(w_2)) = 0$ by Lemma 2.4.11. So, by inductive hypothesis, w_2 is a weak candy element, hence w is a weak candy element as well, since $w = u_0u_1w_2$.

$\square$

Theorem 2.4.14 (Classification of irreducible elements of type $\widetilde{B}_{n+1}$)**.** *The set of irreducible elements in* $(\widetilde{B}_{n+1})$ *is partitioned as follows:*

$$\mathrm{I_w}(\widetilde{B}_{n+1}) = \mathrm{K}(\widetilde{B}_{n+1}) \sqcup \mathrm{K_w}(\widetilde{B}_{n+1}) \sqcup \mathrm{CZ}(\widetilde{B}_{n+1}) \sqcup \mathrm{TC_w}(\widetilde{B}_{n+1}).$$

If n is odd, $\mathrm{K}(\widetilde{B}_{n+1}) = \emptyset$*; if n is even,* $\mathrm{K_w}(\widetilde{B}_{n+1}) = \emptyset$.

Proof. Let $w \in \mathrm{I_w}(\widetilde{B}_{n+1})$, if its support is not complete then it is of type D, B or a product of elements of type D and B. By Remark 2.3.17 and [17, Theorem 4.2.1], we have that $w \in \mathrm{TC_w}(\widetilde{B}_{n+1})$. Assume that $supp(w)$ is complete and w is not a complete zigzag; if n is odd, then $w \in \mathrm{K_w}(\widetilde{B}_{n+1})$, by Proposition 2.4.13. Otherwise, if n is even by Lemmas 2.4.11 and 2.4.12, we have that $w \in \mathrm{I}(\widetilde{B}_{n+1})$ so $w \in \mathrm{K}(\widetilde{D}_{n+2})$ by Theorem 2.4.9 since $w \notin \mathrm{CZ}(\widetilde{B}_{n+1})$.

$\square$

Chapter 3

Diagram Temperley–Lieb algebras of affine types $\widetilde{D}$ and $\widetilde{B}$

The aim of this chapter is to give diagrammatic representations for the Temperley–Lieb algebras of affine types $\widetilde{D}$ and $\widetilde{B}$. Taking inspiration from Ernst's work on type $\widetilde{C}$, we consider two subsets of the decorated diagrams introduced in [18], we endow them with an algebra structure, and then we define, modulo some relations, two new quotient algebras, denoted by $\mathbb{D}(\widetilde{D}_{n+2})$ and $\mathbb{D}(\widetilde{B}_{n+1})$. Our main results show that these algebras are actually isomorphic to $\mathrm{TL}(\widetilde{D}_{n+2})$ and $\mathrm{TL}(\widetilde{B}_{n+1})$, respectively.

We first prove the injectivity of the representation in type $\widetilde{D}_{n+2}$ by using the classification of star irreducible fully commutative elements introduced in Chapter 2. Then, we define ad hoc injections between $\mathrm{FC}(\widetilde{B}_{n+1})$ and $\mathrm{FC}(\widetilde{D}_{n+2})$ and between $\mathbb{D}(\widetilde{B}_{n+1})$ and $\mathbb{D}(\widetilde{D}_{n+2})$, from which we deduce that $\mathbb{D}(\widetilde{B}_{n+1})$ is a faithful diagrammatic representation of $\mathrm{TL}(\widetilde{B}_{n+1})$.

This chapter is organized as follows. In Section 3.1, we recall Ernst's definition of decorated diagrams and introduce our quotient subalgebra $\mathbb{D}(\widetilde{D}_{n+2})$ by defining suitable relations on the set of decorations. In Sections 3.2 and 3.3, we study the decorated diagrams corresponding to star irreducible elements of type $\widetilde{D}_{n+2}$. This allows us, in Section 3.4, to prove our main result, that is the isomorphism between $\mathrm{TL}(\widetilde{D}_{n+2})$ and $\mathbb{D}(\widetilde{D}_{n+2})$ (Theorem 3.4.8), establishing so the faithfulness of the diagrammatic representation. In Section 3.5, we relate Lusztig's **a**-function to the decorated diagrams in our representation. We then investigate the combinatorial properties of the decorated diagrams: in Section 3.6, we define the family of *admissible diagrams* and introduce a notion of length for them. In Section 3.7, we define the *cut-and-paste operation*, which provides an algorithmic strategy to factorize a decorated diagram into a product of so-called *simple diagrams* and in Section 3.8, we prove that the algebra of admissible diagrams is isomorphic to $\mathbb{D}(\widetilde{D}_{n+2})$. Finally, in Section 3.9, we describe how the constructions developed for type $\widetilde{D}_{n+2}$ must be adapted in order to define a diagrammatic representation of $\mathrm{TL}(\widetilde{B}_{n+1})$, and we prove the faithfulness of this representation.

3.1 Decorated diagrams

In this section we consider a subset of the decorated diagrams defined in [18] by Ernst to give his representation of $\mathrm{TL}(\widetilde{C}_{n+1})$. We then introduce a new set of relations to define a quotient subalgebra which will be the main object of our investigations.

A *standard k-box* is a rectangle with $2k$ marked points called *nodes* or *vertices*, labeled as in Figure 3.1. We will refer to the top of the rectangle as the *north face* and to the bottom as the *south face*.

Sometimes, it will be useful for us to think of the standard k-box as being embedded in the xy-plane. In this case, we put the lower left corner of the rectangle at the origin such that each node i (respectively, i') is located at the point $(i, 1)$ (respectively, $(i', 0)$).

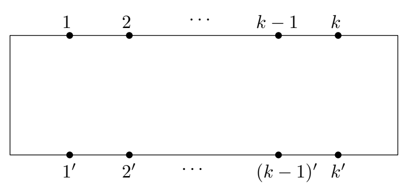


Figure 3.1: The standard k-box.

A *concrete pseudo k-diagram* consists of a finite number of disjoint plane curves, called *edges*, embedded in the standard k-box with the following restrictions. The nodes are endpoints of edges. All other embedded edges must be closed (isotopic to circles) and disjoint from the box. We refer to a closed edge as a *loop*. It follows that there cannot exist isolated nodes and that a single edge starts from each node (an example is given in Figure 3.2). A *non-propagating edge* is an edge that joins two nodes of the same face, while a *propagating edge* is an edge that joins a node of the north face to a node of the south face. If an edge joins the nodes i and i' we call it a *vertical edge*. We denote by $\mathbf{a}(D)$ the number of non-propagating edges on the north face of a concrete pseudo k-diagram D.

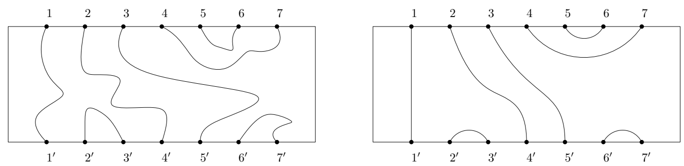


Figure 3.2: A concrete pseudo 7-diagram and a standard representation.

We define an equivalence relation on the set of the concrete pseudo k-diagrams. Two concrete pseudo k-diagrams are *(isotopically) equivalent* if one can be obtained from the other by isotopically deforming the edges while preserving the faces of the rectangle setwise, in such a way that any intermediate diagram is also a concrete pseudo k-diagram. We define a *pseudo k-diagram* as an equivalence class of concrete pseudo k-diagrams with respect to isotopy equivalence, and denote the set of the pseudo k-diagrams by $T_k(\emptyset)$. Moreover, in our analysis it is important to fix a concrete representation in which no edge doubles back on itself, which we call *standard*. What this means is that in such a representation any vertical line intersects any edge, except vertical ones and loops, at most once. In Figure 3.2 a concrete pseudo 7-diagram and one of its standard representations are depicted.

Let D be a concrete pseudo k-diagram. Consider the set $\Omega = \{\bullet, \circ\}$ and the monoid Ω^*. Our goal is to adorn the edges of D with elements of Ω which we call *decorations*. In particular, $\bullet$ is called a *L-decoration* and $\circ$ is called a *R-decoration*. A finite sequence of decorations $\mathbf{b} = x_1 x_2 \cdots x_p$ of Ω^* is called a *block of decorations* of width p. We may adorn an edge with a finite sequence of blocks of decorations $\mathbf{b}_0, \dots, \mathbf{b}_r$. If $\mathbf{b}$ is a block of decorations on an edge e, we read off the sequence of decorations left to right if e is a non-propagating edge, up to down if e is a propagating edge. If e is a loop edge, then the reading of the sequence of decorations depends on an arbitrary choice of the starting point and direction round the loop. We consider two sequences of blocks equivalent if one can be changed to the other or its opposite by any cyclic permutation.

We assign to each decoration x_i a *vertical position* equal to the y-coordinate in the xy-plane.

Definition 3.1.1. Let D be a concrete pseudo k-diagram decorated with at least one decoration in Ω and such that all decorations have different vertical positions. An *L-strip* (respectively *R-strip*) is a part of the diagram between two horizontal lines that contains at least one L-decoration (respectively R-decoration) and no R-decorations (respectively L-decorations). A *strip partition* of D is a partition into *strips* in which L-strips and R-strips alternate.

Now we list the rules by which the edges of D can be decorated.

(D0) If $\mathbf{a}(D) = 0$ then the edges of D are undecorated.

Now suppose $\mathbf{a}(D) \neq 0$.

(D1) All decorated edges can be deformed so as to take L-decorations to the left wall of the standard box and R-decorations to the right wall simultaneously without crossing any other edge.

(D2) If e is a non-propagating or a loop edge, then we allow adjacent blocks on e to be conjoined to form a single block.

(D3) If $\mathbf{a}(D) > 1$ and e is propagating, then as in (D2), we allow adjacent blocks on e to be conjoined to form a single block.

(D4) If $\mathbf{a}(D) = 1$, then we require the following.

  (a) All decorations have different vertical positions.

  (b) All decorations occurring on propagating and loop edges must have vertical position lower (respectively, higher) than the vertical positions of decorations occurring on the (unique) non-propagating edge in the north face (respectively, south face) of D.

  (c) All blocks in the same edge inside the same strip may be conjoined to form a single block.

Remark 3.1.2. To better understand the conditions on the blocks of decorations listed in (D2) and (D4)(c), see Figure 3.3. In diagrams (A) and (B), since the $\mathbf{a}$-value is greater than 1, then the two $\bullet$ on the edge $\{3, 1'\}$ form a single block. In (C) and (D), the two $\bullet$ are in the same strip, so they belong to the same block, while in (E) they form two distinct blocks.

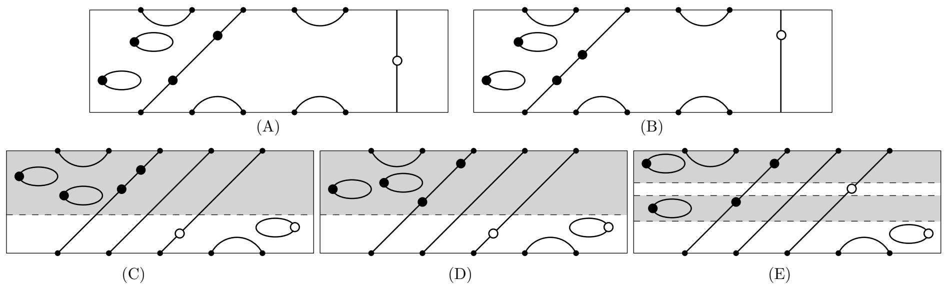


Figure 3.3: Diagrams (A) and (B) are Ω-equivalent. Diagrams (C), (D) and (E) are isotopically equivalent but only (C) and (D) are Ω-equivalent.

We say that two concrete decorated pseudo k-diagrams are Ω-*equivalent* if we can isotopically deform one diagram into the other in a way that any intermediate diagram is also a concrete pseudo k-diagram decorated as above, and the relative vertical positions of the decorations in each block are preserved. In addition, two diagrams with $\mathbf{a}$-value equal to 1 are Ω-*equivalent* if furthermore, in any of their strip partitions, as well as in those of any intermediate diagram, the number of L and R-strips and their relative positions are preserved; see Figure 3.3.

Definition 3.1.3. A *LR-decorated pseudo k-diagram* is defined to be an equivalence class of Ω-equivalent concrete decorated pseudo k-diagrams. We denote the set of LR-decorated pseudo k-diagrams by $T_k^{LR}(\Omega)$.

Definition 3.1.4. We define $\mathcal{P}_k^{LR}(\Omega)$ to be the free $\mathbb{Z}[\delta]$-module having the elements of $T_k^{LR}(\Omega)$ as a basis.

Now we define the product of two elements of $T_k^{LR}(\Omega)$ as their concatenation, and then extend this bilinearly to define a multiplication in $\mathcal{P}_k^{LR}(\Omega)$. To concatenate two diagrams $D, D' \in T_k^{LR}(\Omega)$, we place D' on top of D so that node i of D coincides with node i' of D', conjoin adjacent blocks maintaining Ω-equivalence and then rescale vertically by a factor of $1/2$.

Proposition 3.1.5. *The module $\mathcal{P}_k^{LR}(\Omega)$ with the aforementioned product is a well-defined associative $\mathbb{Z}[\delta]$-algebra, with identity element the pseudo k-diagram I_k made of k undecorated vertical edges.*

Proof. By [39, Proposition 3.3.5], the multiplication by concatenation defined above preserves the condition (D1). Let D and D' be two diagrams with $\mathbf{a}(D) = \mathbf{a}(D') = 1$.

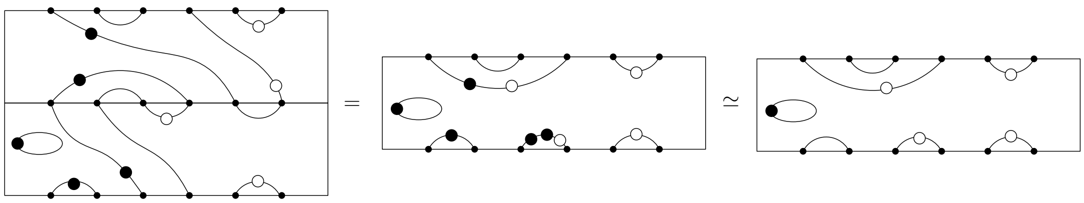


Figure 3.4: Product in $\mathcal{P}_6^{LR}$ and reduction in $\widehat{\mathcal{P}}_6^{LR}$.

If $\mathbf{a}(D'D) > 1$, then there are no concerns, since all the blocks on each edge can be conjoined. Else, if $\mathbf{a}(D'D) = 1$, then (D4) holds. Indeed, if the unique non-propagating edge e' in the south face of D' is $\{i', (i+1)'\}$, then the unique non-propagating edge e in the north face of D is either (a) $\{i-1, i\}$, or (b) $\{i, i+1\}$, or (c) $\{i+1, i+2\}$ (for more details, see [18, Proposition 3.2.7]). If the lowest strip of D' and the highest strip of D are associated to the same decoration, then we conjoin the two strips and the blocks on the edges inside those strips (if any). Otherwise, $D'D$ contains, from the top, the strips of D' followed by those of D and there is no conjoining of blocks. In both cases, (D4) holds; see Figure 3.6. By construction, the multiplication is associative (see also [39, Proposition 3.1.2]) and this concludes the proof. $\square$

Note that $\mathcal{P}_k^{LR}(\Omega)$ is an infinite dimensional algebra since, for instance, when D has a single propagating edge, there is no limit to the number of decorations on such edge, or when D has no propagating edges, there is no limit to the number of loops with both decorations.

We introduce the notation for three types of loop edges in Figure 3.5.

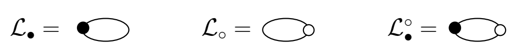
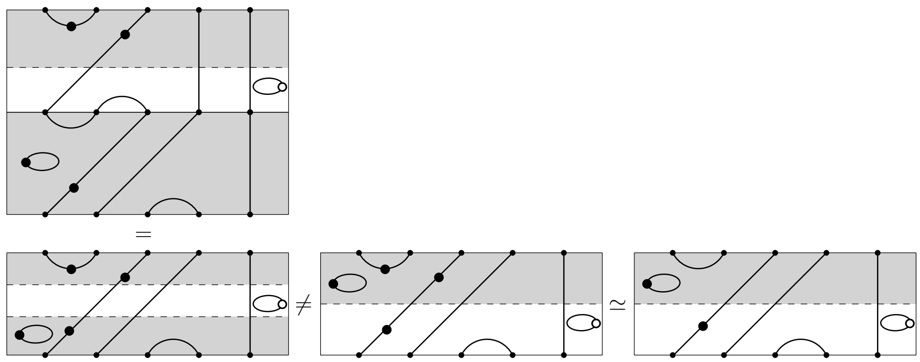


Figure 3.5: Special types of loops.

From now on we follow the setup of Bergman's Diamond Lemma [2, §1]. We define a *reduction system* as depicted in Figure 3.7, where the lines represent a portion of an edge (loops included).

More precisely the reductions r_i described in Figure 3.7 are defined as follows:

1. r_1 and r_2 reduce the number of decorations and they are applied to adjacent decorations in the same block;



(A) (B) (C)

Figure 3.6: Product in $\mathcal{P}_5^{LR}$ with $\mathbf{a}$-value 1. Diagrams (A) and (C) are irreducible while (B) is not; (B) and (C) are equal in $\widehat{\mathcal{P}}_5^{LR}(\Omega)$.

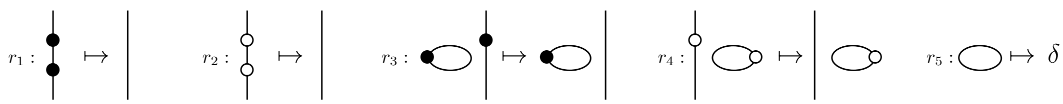


Figure 3.7: The reduction system in $\mathcal{P}_k^{LR}(\Omega)$.

2. if $\mathcal{L}_\bullet$ (respectively, $\mathcal{L}_\circ$) occurs in D, then r_3 (respectively, r_4) removes a $\bullet$ (respectively, $\circ$) on another edge (loop edge included) with the only restriction that if $\mathbf{a}(D) = 1$ then $\mathcal{L}_\bullet$ (respectively, $\mathcal{L}_\circ$) and $\bullet$ (respectively, $\circ$) must belong to the same strip.

3. r_5 removes an undecorated loop edge multiplying D by a factor δ.

Clearly, if $\mathbf{a}(D) > 1$, a finite sequence of applications of r_3 (respectively, r_4) removes all $\bullet$ (respectively, $\circ$) in D except the one on the last remaining loop, while in the case $\mathbf{a}(D) = 1$, it only removes all $\bullet$ (respectively, $\circ$) that belong to the same L-strip (respectively, R-strip).

We say that a LR-decorated diagram is *irreducible* if no reduction can be applied. Note that each block of decorations of an irreducible diagram cannot contain two adjacent decorations of the same type.

Define the function $sh : T_k^{LR}(\Omega) \to T_k(\emptyset)$ such that $sh(D)$ is the *shape* of D, i.e. $sh(D)$ is an element of $T_k(\emptyset)$ which is equal to D with all decorations and loops removed. Also define a function $h : T_k^{LR}(\Omega) \to \mathbb{N}$ where $h(D)$ is the sum of the number of decorations and the number of loops of D.

Define $\leq_{\widehat{\mathcal{P}}}$ on $T_k^{LR}(\Omega)$ via

$$D <_{\widehat{\mathcal{P}}} D' \text{ if and only if } sh(D) = sh(D') \text{ and } h(D) < h(D').$$

Let D, D', A, B be elements of $T_k^{LR}(\Omega)$ with $D <_{\widehat{\mathcal{P}}} D'$. We can easily verify that $sh(ADB) = sh(AD'B)$ and $h(ADB) < h(AD'B)$, so that $ADB <_{\widehat{\mathcal{P}}} AD'B$. Therefore, $\leq_{\widehat{\mathcal{P}}}$ is a semigroup partial order on $T_k^{LR}(\Omega)$. Furthermore, we note that the reductions preserve the shape of the edges and decrease the number of decorations. Hence $\leq_{\widehat{\mathcal{P}}}$ is compatible with the set of reductions. Moreover, $\leq_{\widehat{\mathcal{P}}}$ clearly satisfies the descending chain condition, that is every sequence of reductions ends in a finite number of steps.

Now we want to show that all of the potential ambiguities are resolvable. Note that r_5 commutes with any other reduction. Moreover, if there are two reductions, one involving only $\bullet$-decorations and one involving only $\circ$-decorations which can be applied at the same time, then they commute, so the corresponding ambiguity is resolvable.

So we only need to check ambiguities involving the same type of decorations, but we know that all such ambiguities are resolvable since they coincide with those appearing in the Coxeter type finite D [26, Theorem 4.2].

Let $\widehat{\mathcal{P}}_k^{LR}(\Omega)$ be the quotient of $\mathcal{P}_k^{LR}(\Omega)$ modulo the two-sided ideal generated by the relations determined by the reductions in Figure 3.7. Since all the ambiguities are resolvable, by [2, Theorem 1.2] we have the following result.

Proposition 3.1.6. *The set of LR-decorated irreducible diagrams forms a basis for* $\widehat{\mathcal{P}}_k^{LR}(\Omega)$.

Now define the *simple diagrams* $D_0, \dots, D_{n+2}$ as in Figure 3.8. Since they are irreducible, they are basis elements of $\widehat{\mathcal{P}}_{n+2}^{LR}(\Omega)$.

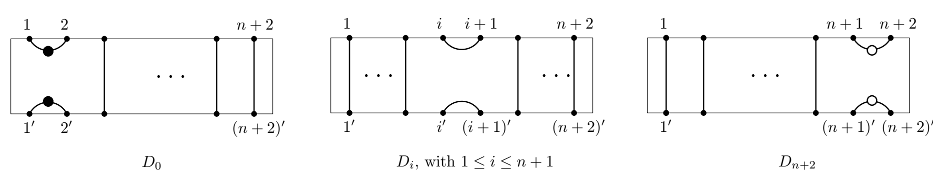


Figure 3.8: The simple diagrams.

It is easy to prove that the simple diagrams satisfy the relations (d1)-(d3) listed in Definition 1.1.20, where b_i's are replaced by D_i's.

Definition 3.1.7. Let $\mathbb{D}(\widetilde{D}_{n+2})$ be the $\mathbb{Z}[\delta]$-subalgebra of $\widehat{\mathcal{P}}^{LR}_{n+2}(\Omega)$ generated as a unital algebra by the simple diagrams with multiplication inherited by $\widehat{\mathcal{P}}^{LR}_{n+2}(\Omega)$.

Let $\mathbf{w} = s_{i_1}s_{i_2}\cdots s_{i_k}$ be a reduced expression of $w \in W$, and set

$$D_{\mathbf{w}} := D_{i_1}D_{i_2}\cdots D_{i_k}.$$

If $w \in \mathrm{FC}(\widetilde{D}_{n+2})$, we can define $D_w := D_{\mathbf{w}}$, where $\mathbf{w}$ is any reduced expression of w, since D_w does not depend on the chosen reduced expression of w by relation (d2) from Definition 1.1.20. We define the $\mathbb{Z}[\delta]$-algebra homomorphism

$$\begin{array}{rcl} \tilde{\theta}_D : \mathrm{TL}(\widetilde{D}_{n+2}) & \longrightarrow & \mathbb{D}(\widetilde{D}_{n+2}) \\ b_i & \mapsto & D_i \end{array} \tag{3.1}$$

for all $i = 0, \dots, n+2$. Clearly, $\tilde{\theta}_D$ is surjective and maps the monomial basis element b_w into the diagram D_w. Our goal is to show that this map is bijective.

3.2 Diagram calculus associated to irreducible FC elements

In this section, we will need the star irreducible fully commutative elements of type $\widetilde{D}_{n+2}$ described and classified in Section 2.3. Now recall some notions and notations introduced in the previous chapters. The sets of left and right descents are respectively:

$$D_L(w) := \{s \in S \mid \ell(sw) < \ell(w)\} \text{ and } D_R(w) := \{s \in S \mid \ell(ws) < \ell(w)\}.$$

The Cartier–Foata normal form (CFNF) is characterized in Theorem 1.1.10. Recall that the quantities $f_\bullet(w)$ and $f_\circ(w)$, for $w \in \mathrm{FC}(\widetilde{D}_{n+2})$, represent the maximun number of occurrences of the factors s_0s_1 and $s_{n+1}s_{n+2}$, respectively, in a reduced expression of w (see Definition 2.3.1). By Theorem 2.3.20, the star irreducible fully commutative elements of type $\widetilde{D}_{n+2}$ are partioned into three families: the trivially commutative elements $\mathrm{TC}(\widetilde{D}_{n+2})$ (Definition 1.1.9), the complete zigzag elements $\mathrm{CZ}(\widetilde{D}_{n+2})$ (Definition 2.3.5), and the candy elements $\mathrm{K}(\widetilde{D}_{n+2})$ (Definition 2.3.18).

Lemma 3.2.1. *Let $w \in \mathrm{FC}(\widetilde{D}_{n+2})$, if $s \in D_L(w)$ (resp. $s \in D_R(w)$) then $D_s D_w = \delta D_w$ (resp. $D_w D_s = \delta D_w$).*

Proof. If $s \in D_L(w)$, then $w = su$, $u \in \mathrm{FC}(\widetilde{D}_{n+2})$ and $\ell(w) = \ell(u) + 1$. Therefore, $D_w = D_s D_u$, so applying (d1) to D_s, it follows that

$$D_s D_w = D_s D_s D_u = \delta D_s D_u = \delta D_w.$$

The same argument can be applied to $s \in D_R(w)$. □

In the following, by $\{i, j\}_d$, with $d \in \Omega^*$, we denote the edge joining the nodes i and j, and decorated with d.

Lemma 3.2.2. *Let $w \in \mathrm{FC}(\widetilde{D}_{n+2})$.*

(a) For $1 \leq i \leq n+1$, if $s_i \in D_L(w)$, then $\{i, i+1\}$ appears in D_w.

(b) If $s_0 \in D_L(w)$, then either $\{1, 2\}_\bullet$ appears in D_w, or $\{1, 2\}$ appears in D_w and the numbers of occurrences of $\mathcal{L}_\bullet$ in D_w and $D_0 D_w$ are the same.

(c) If $s_{n+2} \in D_L(w)$, then either $\{n+1, n+2\}_\circ$ appears in D_w, or $\{n+1, n+2\}$ appears in D_w and the numbers of occurrences of $\mathcal{L}_\circ$ in D_w and $D_{n+2} D_w$ are the same.

Proof. Note that (a) follows directly by the definition of diagram concatenation.

If $w = s_0 u$ is a reduced product, before applying any reductions (if any), the edge $\{1, 2\}_\bullet$ appears in the concatenation of D_0 with D_u. If D_w does not contain a $\mathcal{L}_\bullet$ such that reduction (r3) is applicable, then the thesis follows. Otherwise, after applying (r3) to the edge $\{1, 2\}_\bullet$, it looses its decoration, and $\{1, 2\}$ is in D_w. Moreover, the same occurrence of $\mathcal{L}_\bullet$ gives rise to a reduction (r3) of the new occurrence of $\mathcal{L}_\bullet$, obtained from the concatenation of D_0 with D_w, to an undecorated loop. Note that if $\mathbf{a}(D_w) > 1$, then $\#\mathcal{L}_\bullet = 1$. Hence the numbers of $\mathcal{L}_\bullet$ in D_w and $D_0 D_w$ are the same. This proves (b); (c) is analogous. □

A direct consequence of Lemma 3.2.2 is that $D_w = I_{n+2}$ if and only if $w = e$.

Proposition 3.2.3. *Let $w \in \mathrm{TC}(\widetilde{D}_{n+2})$, then:*

(a) the only types of edges allowed in D_w are simple edges and vertical edges $\{j, j'\}$ with $1 \leq j \leq n+2$;

(b) the only types of loops allowed in D_w are $\mathcal{L}_\bullet$ and $\mathcal{L}_\circ$;

(c) $f_\bullet(w) = \#\mathcal{L}_\bullet \leq 1$ and $f_\circ(w) = \#\mathcal{L}_\circ \leq 1$ in D_w.

Proof. Let $w = s_{i_1} \cdots s_{i_k}$ reduced with $m_{s_{i_j} s_{i_l}} = 2$ for all $j \neq l$, then $D_w = D_{i_1} \cdots D_{i_k}$. Observe that the simple edges associated with two distinct commuting generators, if they share a vertex, then they share both vertices, and the generators must be either s_0 and s_1, or s_{n+1} and s_{n+2}. For this reason, the only allowed edges in D_w are either simple non-propagating edges, or edges of the form $\{j, j'\}$ with $1 \leq j \leq n+2$, and the only allowed loops are those of the form $\mathcal{L}_\bullet$ and $\mathcal{L}_\circ$.

Moreover, it can be observed that a loop $\mathcal{L}_\bullet$ is present in D_w if and only if $s_0, s_1 \in$ *supp*(w), due to the earlier observation on possible shared nodes between simple edges associated with commuting generators. Similarly, the same holds for $\mathcal{L}_\circ$ and the occurrences of s_{n+1} and s_{n+2}. Therefore, we conclude that $f_\bullet(w) = \#\mathcal{L}_\bullet$ and $f_\circ(w) = \#\mathcal{L}_\circ$ in D_w.

$\square$

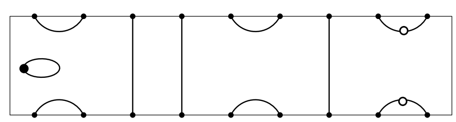


Figure 3.9: D_w with $w = s_0 s_1 s_5 s_9$ in $\mathrm{TC}(\widetilde{D}_9)$.

Recall that if $w \in \mathrm{K}(\widetilde{D}_{n+2})$ and $w = u_0 \cdots u_m$ is its CFNF, then m is even by Definition 2.3.18.

Proposition 3.2.4. *Let n be even and $w = u_0 \cdots u_m \in \mathrm{K}(\widetilde{D}_{n+2})$. Then any loop in D_w is of the form $\mathcal{L}_\bullet^\circ$, and the number of occurrences of $\mathcal{L}_\bullet^\circ$ is $m/2$.*

Proof. Without loss of generality assume $s_0, s_{n+2} \in D_L(w)$ since the other cases are analogous. By definition of candy elements we have that $u_0 = s_0 s_3 \cdots s_{n-1} s_{n+2}$ and for $1 \leq i \leq m-1$,

$$u_i u_{i+1} = \begin{cases} s_2 s_4 \cdots s_{n-2} s_n s_1 s_3 \cdots s_{n-1} s_{n+1}, & \text{for } i \text{ odd;} \\ s_2 s_4 \cdots s_{n-2} s_n s_0 s_3 \cdots s_{n-1} s_{n+2}, & \text{for } i \text{ even.} \end{cases}$$

Then $D_w = D_{u_0} \cdots D_{u_i u_{i+1}} \cdots D_{u_{m-1} u_m}$, and each concatenation gives rise to a loop $\mathcal{L}_\bullet^\circ$, so $\#\mathcal{L}_\bullet^\circ = m/2$; see Figure 3.10, left. No other types of loops are formed through the concatenation. □

Proposition 3.2.5. *Let $w \in \mathrm{CZ}(\widetilde{D}_{n+2})$, then $\mathbf{a}(D_w) = 1$ and any loop in D_w is of the form $\mathcal{L}_\bullet$ or $\mathcal{L}_\circ$, moreover $f_\bullet(w) = \#\mathcal{L}_\bullet$ and $f_\circ(w) = \#\mathcal{L}_\circ$.*

Proof. By Definition 2.3.5, either $D_w = D_0 D_1 (D_A D_B)^k D_A^h$ or $D_w = D_{n+1} D_{n+2} (D_B D_A)^k D_B^h$ with $k \geq 0$, $h \in \{0, 1\}$ and $k + h > 0$. A direct computation shows that $\mathbf{a}(D_w) = 1$ and that each concatenation by D_A gives rise to a $\mathcal{L}_\circ$ and each concatenation by D_B to a $\mathcal{L}_\bullet$. Hence, in the first case, the number of occurrences of $\mathcal{L}_\bullet$ (respectively, $\mathcal{L}_\circ$) in D_w is $k + 1$ (respectively, $k + h$) which is equal to $f_\bullet(w)$ (respectively, $f_\circ(w)$). A similar computation holds for the second case. No other types of loops are formed through these concatenations. An example is given in Figure 3.10, right.

□

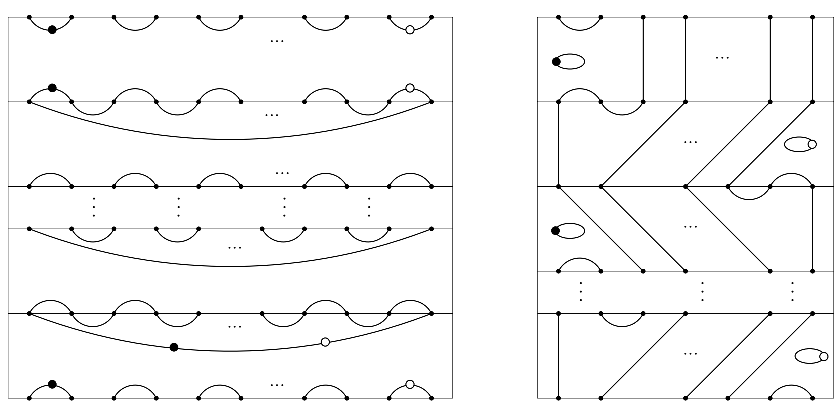


Figure 3.10: On the left, the concatenation $D_w = D_{u_0} D_{u_1 u_2} \cdots D_{u_{m-1} u_m}$ in the proof of Proposition 3.2.4; on the right, the concatenation $D_w = D_0 D_1 (D_A D_B)^k D_A$ in the proof of Proposition 3.2.5.

Proposition 3.2.6. *Let $w \in \mathrm{FC}(\widetilde{D}_{n+2})$. Then*

(a) the diagram D_w does not contain any undecorated loops;

(b) $f_\bullet(w) = \#\mathcal{L}_\bullet$ in D_w;

(c) $f_\circ(w) = \#\mathcal{L}_\circ$ *in* D_w.

Proof. We show that the numbers of undecorated loops, $\mathcal{L}_\bullet$ and $\mathcal{L}_\circ$ are invariant by left and right reductions. The result then follows since (a), (b), and (c) hold for irreducible elements by Propositions 3.2.3, 3.2.4, 3.2.5, and Theorem 2.3.20.

Without loss of generality, assume that $w \rightsquigarrow_t sw =: w_1$. Then

$$D_t D_w = D_t D_s D_{w_1} = D_t D_s D_t D_u = D_t D_u = D_{w_1}$$

where (d3) from Definition 1.1.20 is applied in the third equality, and $w_1 = tu$ is reduced. Since $s \in D_L(w)$, then the simple edge associated to s by Proposition 3.2.2 appears in D_w. Thus, the concatenation $D_t D_w$ does not create new loops, decorated or not, since the simple edge in D_t is adjacent to the one corresponding to s in D_w (see Figure 3.11). Hence, the total number of loops does not change, and (a) is proved. Moreover, by Lemma 2.3.2(b), $f_\bullet(w) = f_\bullet(w_1)$ and $f_\circ(w) = f_\circ(w_1)$, so (b) and (c) also yield. □

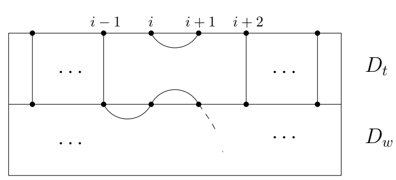


Figure 3.11: $D_t D_w = D_{w_1}$, for Proposition 3.2.6.

3.3 Descents and monomial basis

In Definition 1.1.20, we introduced the algebra $\mathrm{TL}(\widetilde{D}_{n+2})$ generated by $\{b_0, b_1, \ldots, b_{n+2}\}$ and defining relations (d1)-(d3). Recall that $\{b_w \mid w \in \mathrm{FC}(\widetilde{D}_{n+2})\}$ is a basis for $\mathrm{TL}(\widetilde{D}_{n+2})$, called monomial basis. Recall that the symbols $s_\bullet, s_\circ, t_\bullet$ and $t_\circ$ are described in Notation 2.3.4.

Remark 3.3.1. It is not difficult to see that for any finite sequence of indices $\{i_1, \ldots, i_r\}$, $b_{i_1} \cdots b_{i_r} = \delta^k b_x$ with $k \geq 0$ and $x \in \mathrm{FC}(\widetilde{D}_{n+2})$ by applying the relations (d1)-(d3) of $\mathrm{TL}(\widetilde{D}_{n+2})$ given in Definition 1.1.20. In particular, $s_{i_1} \in D_L(x)$ and $s_{i_r} \in D_R(x)$.

Lemma 3.3.2. *Let $w \in \mathrm{FC}(\widetilde{D}_{n+2})$ and suppose there exist $s \in S$ and $s' \in D_L(w)$ with $m_{ss'} = 3$. Then $b_s b_w = b_x$ with $x \in \mathrm{FC}(\widetilde{D}_{n+2})$, $f_\bullet(w) = f_\bullet(x)$ and $f_\circ(w) = f_\circ(x)$.*

Proof. First of all observe that $\ell(sw) = \ell(w) + 1$. If $sw \in \mathrm{FC}(\widetilde{D}_{n+2})$ then $b_s b_w = b_{sw}$ and it follows that $f_\bullet(w) = f_\bullet(sw)$ and $f_\circ(w) = f_\circ(sw)$. If $sw \notin \mathrm{FC}(\widetilde{D}_{n+2})$, then w can be written as $w = s'su$ reduced with $m_{s's} = 3$ by Remark 2.1.3. Hence, by (d3) from Definition 1.1.20, $b_s b_w = b_s b_u = b_{su}$ with $f_\bullet(w) = f_\bullet(su)$ and $f_\circ(w) = f_\circ(su)$ by Lemma 2.3.2. □

Proposition 3.3.3. *Let $w \in \mathrm{FC}(\widetilde{D}_{n+2})$ be reducible to $v \in \mathrm{K}(\widetilde{D}_{n+2}) \cup \mathrm{TC}(\widetilde{D}_{n+2})$, and let $s \notin D_L(w)$ be such that $b_s b_w = \delta^k b_x$ with $x \in \mathrm{FC}(\widetilde{D}_{n+2})$.*

(a) If $k > 0$, then $v \in \mathrm{K}(\widetilde{D}_{n+2})$ and $f_\bullet(x) = f_\circ(x) = 1$.

(b) If $v \in \mathrm{TC}(\widetilde{D}_{n+2})$, then $k = 0$.

Proof. Note that (b) follows by (a). To prove (a) we consider a reduction sequence $w_0 = w \rightsquigarrow w_1 \rightsquigarrow w_2 \rightsquigarrow \cdots \rightsquigarrow w_l = v$ with $v = u_0 \cdots u_m$ its CFNF. We prove the result by induction on l.

If $l = 0$, then $w = v \in \mathrm{K}(\widetilde{D}_{n+2}) \cup \mathrm{TC}(\widetilde{D}_{n+2})$. If $w \in \mathrm{TC}(\widetilde{D}_{n+2})$, then $sw \in \mathrm{FC}(\widetilde{D}_{n+2})$ and $b_s b_w = b_{sw}$. So, we have $w \in \mathrm{K}(\widetilde{D}_{n+2})$ and we assume $\{s_\bullet, s_\circ\} \subset D_L(w)$. If $s = s_j$ with $2 \leq j \leq n$ even, then there exists $s' \in D_L(w)$ such that $m_{ss'} = 3$, so $b_s b_w = b_x$ by Lemma 3.3.2. Thus, $s \in \{t_\bullet, t_\circ\}$, then $b_s b_w = \delta^k b_x$ with $k = \lfloor m/2 \rfloor - 1$ and $x = s_0 s_1 s_3 s_5 \cdots s_{n-1} s_{n+1} s_{n+2} \in \mathrm{TC}(\widetilde{D}_n)$ by direct computation and $f_\bullet(x) = f_\circ(x) = 1$.

Assume $l > 0$. First, we assume that $w \rightsquigarrow_p rw =: w_1$. We have that $m_{sr} = 2$, otherwise if $m_{sr} = 3$, then $b_s b_w = b_y$ with $y \in \mathrm{FC}(\widetilde{D}_{n+2})$ by Lemma 3.3.2. Therefore,

$$\delta^k b_x = b_s b_w = b_s b_r b_{w_1} = b_r b_s b_{w_1}$$

with $k > 0$ and $s \notin D_L(w_1)$. Now we distinguish two cases. Suppose $m_{sp} = 2$. By Remark 3.3.1 we have that $b_s b_{w_1} = \delta^h b_y$ with $h \geq 0$, $y \in \mathrm{FC}(\widetilde{D}_{n+2})$ and $p \in D_L(y)$, so

$$b_r b_s b_{w_1} = \delta^h b_r b_y.$$

Thus by Lemma 3.3.2, $b_r b_y = b_{x'}$ with $x' \in \mathrm{FC}(\widetilde{D}_{n+2})$ and $f_\bullet(x') = f_\bullet(y)$ and $f_\circ(x') = f_\circ(y)$. Therefore, $\delta^k b_x = \delta^h b_{x'}$, hence $h = k > 0$ and $x = x'$. By inductive hypothesis on w_1, we have that $f_\bullet(y) = f_\circ(y) = 1$ and $v \in \mathrm{K}(\widetilde{D}_{n+2})$, in particular

$$f_\bullet(x) = f_\bullet(y) = 1 \text{ and } f_\circ(x) = f_\circ(y) = 1.$$

On the other hand, if $m_{sp} = 3$, we have that $sw_1 \notin \mathrm{FC}(\widetilde{D}_{n+2})$, otherwise if $sw_1 \in \mathrm{FC}(\widetilde{D}_{n+2})$, then $rsw_1 = sw \in \mathrm{FC}(\widetilde{D}_{n+2})$ and $b_s b_w = b_{sw}$. Thus, by Remark 2.1.3(5), it means that $w_1 \rightsquigarrow_s pw_1 := w'$. Therefore, w is star reducible to w', in particular by Theorem 2.1.6(2) there exists a sequence of length l such that

$$w \rightsquigarrow w_1' = w_1 \rightsquigarrow w_2' = w' \rightsquigarrow \cdots \rightsquigarrow w_l' = v' \in \mathrm{K}(\widetilde{D}_n) \cup \mathrm{TC}(\widetilde{D}_n)$$

by Corollary 2.3.21. Thus,

$$\delta^k b_x = b_r b_s b_{w_1} = b_r b_s b_p b_{w'} = b_r b_{w'}$$

with $k > 0$, since $b_s b_p b_{w'} = b_{w'}$ and $s \in D_L(w')$. Moreover, since $w = rpw'$ reduced then $r \notin D_L(w')$. Therefore, by the inductive hypothesis $f_\bullet(x) = f_\circ(x) = 1$ and $v' \in \mathrm{K}(\widetilde{D}_{n+2})$, so we have that $v = v'$ by Corollary 2.3.21.

Now, we suppose that $w \rightsquigarrow_p wr := w_1$. Thus,

$$\delta^k b_x = b_s b_w = b_s b_{w_1} b_r$$

with $k > 0$. By Remark 3.3.1, $b_s b_{w_1} = \delta^h b_y$ and since $p \in D_R(w_1)$, we have that $p \in D_R(y)$. Thus, $b_y b_r = b_{x'}$ with $x' \in \mathrm{FC}(\widetilde{D}_{n+2})$ and $f_\bullet(x') = f_\bullet(y)$ and $f_\circ(x') = f_\circ(y)$ by Lemma 3.3.2. Therefore, $\delta^k b_x = \delta^h b_{x'}$, so $k = h > 0$ and $x = x'$. By inductive hypothesis on w_1, we have that $f_\bullet(y) = f_\circ(y) = 1$ and $v \in \mathrm{K}(\widetilde{D}_{n+2})$, in particular

$$f_\bullet(x) = f_\bullet(y) = 1 \text{ and } f_\circ(x) = f_\circ(y) = 1.$$

□

Remark 3.3.4. Note that the **a**-value is invariant for star reducibility. In fact, assume that $w \rightsquigarrow_t sw =: w'$. Since $m_{st} = 3$, the non-propagating edge on the north face associated to t in $D_{w'}$ has a common node with the simple edge associated to s. Thus, when concatenating D_s and $D_{w'}$, the number of non-propagating edges does not change.

Therefore, since $D_w = D_s D_{w'} = D_{sw'}$, we have $\mathbf{a}(D_w) = \mathbf{a}(D_{w'})$.

Proposition 3.3.5. *Let $w \in \mathrm{FC}(\widetilde{D}_{n+2})$ be reducible to $v \in \mathrm{CZ}(\widetilde{D}_{n+2})$, then $\mathbf{a}(D_w) = 1$. Moreover, if $s \notin D_L(w)$ and $b_s b_w = \delta^k b_x$ with $x \in \mathrm{FC}(\widetilde{D}_{n+2})$ and $k > 0$, then $\mathbf{a}(D_x) \geq 2$.*

Proof. Since $\mathbf{a}(D_v) = 1$ by Proposition 3.2.5, we have that $\mathbf{a}(D_w) = 1$ by Remark 3.3.4.

To conclude the proof, let $s \notin D_L(w)$, $b_s b_w = \delta^k b_x$ with $k > 0$, and $x \in \mathrm{FC}(\widetilde{D}_{n+2})$. We have that $m_{st} = 2$ for all $t \in D_L(w)$, otherwise if there exists $s' \in D_L(w)$ such that $m_{ss'} = 3$, then $b_s b_w = b_y$ with $y \in \mathrm{FC}(\widetilde{D}_{n+2})$ by Lemma 3.3.2. We observe that $D_L(x) \supseteq D_L(w) \sqcup \{s\}$ by Remark 3.3.1, so for this reason $|D_L(x)| \geq 2$ and moreover $D_L(x) \notin \{\{s_0, s_1\}, \{s_{n+1}, s_{n+2}\}\}$. In fact, for instance, if $D_L(x) = \{s_0, s_1\}$, then $s = s_\bullet$ and w is a weak zigzag with $D_L(w) = \{t_\bullet\}$, by Corollary 2.3.9. But this means that $sw \in \mathrm{FC}(\widetilde{D}_{n+2})$ and $b_s b_w = b_{sw}$. Therefore, by Lemma 3.2.2 we have that $\mathbf{a}(D_x) \geq 2$. □

Theorem 3.3.6. *Let $w \in \mathrm{FC}(\widetilde{D}_{n+2})$. Then $s \in D_L(w)$ if and only if $D_s D_w = \delta D_w$. Therefore, if $D_w = D_u$, then $D_L(w) = D_L(u)$ and $D_R(w) = D_R(u)$.*

Proof. Assume $s \in D_L(w)$, then $D_s D_w = \delta D_w$ by Lemma 3.2.1.

Suppose now, $D_s D_w = \delta D_w$ and $s \notin D_L(w)$. We observe that by Lemma 3.3.3 and definition of $\widetilde{\theta}_D$, w is reducible to $v \in \mathrm{K}(\widetilde{D}_{n+2}) \cup \mathrm{CZ}(\widetilde{D}_{n+2})$. By Remark 3.3.1, we have that $b_s b_w = \delta^k b_x$ with $k \geq 0$ and $x \in \mathrm{FC}(\widetilde{D}_{n+2})$. Hence, applying the map $\widetilde{\theta}_D$ we have that $\delta^k D_x = \delta D_w$. Moreover, by Proposition 3.2.6, it means that $k = 1$ and $D_x = D_w$. Now, if $v \in \mathrm{K}(\widetilde{D}_{n+2})$, then $0 = f_\bullet(w) \neq f_\bullet(x) = 1$ by Lemma 2.3.2 and Proposition 3.3.3, but they should be the same by Proposition 3.2.6. On the other hand, if $v \in \mathrm{CZ}(\widetilde{D}_{n+2})$, by Lemma 3.3.5 we have that $2 \leq \mathbf{a}(D_x) \neq \mathbf{a}(D_w) = 1$, which is absurd. Thus, $s \in D_L(w)$.

In conclusion, if $D_w = D_u$, then for all $s \in S$, $D_s D_w = \delta D_w$ if and only if $D_s D_u = \delta D_u$, hence $D_L(w) = D_L(u)$. Moreover, since $D_{w^{-1}} = D_{u^{-1}}$, we have that $D_L(w^{-1}) = D_L(u^{-1})$, so $D_R(w) = D_R(u)$.

□

Corollary 3.3.7. *Let $w \in \mathrm{FC}(\widetilde{D}_{n+2})$.*

(a) For $1 \leq i \leq n+1$, the edge $\{i, i+1\}$ appears in D_w if and only if $s_i \in D_L(w)$.

(b) *The edge $\{1,2\}_\bullet$ appears in D_w or $\{1,2\}$ appears in D_w and the numbers of $\mathcal{L}_\bullet$ in D_w and D_0D_w are the same, if and only if $s_0 \in D_L(w)$.*

(c) *The edge $\{n+1,n+2\}_\circ$ appears in D_w or $\{n+1,n+2\}$ appears in D_w and the numbers of $\mathcal{L}_\circ$ in D_w and $D_{n+2}D_w$ are the same, if and only if $s_{n+2} \in D_L(w)$.*

Proof. The sufficient condition is proved in Lemma 3.2.2. Here we settle the other implication.

First, note that if $\{i,i+1\}$ appears in D_w with $1 \leq i \leq n+1$, then $D_iD_w = \delta D_w$, so $s_i \in D_L(w)$ by Theorem 3.3.6. Hence, (a) is proved.

If $\{1,2\}_\bullet$ is present in D_w, then $D_0D_w = \delta D_w$, thus $s_0 \in D_L(w)$ by Theorem 3.3.6. Suppose now that $\{1,2\}$ appears in D_w. Since the concatenation D_0 with D_w forms a loop $\mathcal{L}_\bullet$ and the numbers of $\mathcal{L}_\bullet$ in D_w and D_0D_w are the same, it means that a $\mathcal{L}_\bullet$ giving rise to a reduction of type (r3) was already present in D_w. Hence $D_0D_w = \delta D_w$ and (b) follows by Theorem 3.3.6. Point (c) is analogous to (b). □

3.4 Injectivity of $\widetilde{\theta}_D$

In this section, we prove that the map $\widetilde{\theta}_D : \mathrm{TL}(\widetilde{D}_{n+2}) \to \mathbb{D}(\widetilde{D}_{n+2})$ such that $b_i \mapsto D_i$ for all i, given in (3.1), is injective. The argument proceeds in three steps: we first show that $\widetilde{\theta}_D$ is injective on the set of irreducible elements of $\mathrm{FC}(\widetilde{D}_{n+2})$. We then prove that if $\widetilde{\theta}_D(b_u) = \widetilde{\theta}_D(b_w)$ and one of u and w is irreducible, then $u = w$. Finally, combining the previous steps we deduce the injectivity of $\widetilde{\theta}_D$.

Proposition 3.4.1. *Let $u, w \in \mathrm{I}(\widetilde{D}_n)$. If $D_u = D_w$, then $u = w$.*

Proof. Since $D_u = D_w$, then $D_L(u) = D_L(w)$ and $D_R(u) = D_R(w)$ by Theorem 3.3.6. Suppose u and w are not in the same family. Note that one of them must be in the family $\mathrm{TC}(\widetilde{D}_n)$ because the cardinalities of the descent sets of a candy and a complete zigzag are never equal. Hence assume $u \in \mathrm{TC}(\widetilde{D}_n)$. If $w \in \mathrm{CZ}(\widetilde{D}_n)$, necessarily $D_L(u) = D_R(u) = D_L(w) = D_R(w)$, hence

$$D_L(w) = D_R(w) = \{s_0, s_1\} \text{ or } D_L(w) = D_R(w) = \{s_{n+1}, s_{n+2}\}.$$

Therefore, $3 \leq f_\bullet(w) + f_\circ(w) \neq f_\bullet(u) + f_\circ(u) = 1$ which is a contradiction by Proposition

3.2.6. If $w \in \mathrm{K}(\widetilde{D}_{n+2})$ then we have a contradiction because D_w has at least one $\mathcal{L}_\bullet^\circ$ while D_u has no occurrence of $\mathcal{L}_\bullet^\circ$ by Proposition 3.2.4 and Proposition 3.2.3(b).

Assume now that u and w are in the same family.

- If $u, w \in \mathrm{TC}(\widetilde{D}_{n+2})$, then $u = w$ trivially.
- If $u, w \in \mathrm{K}(\widetilde{D}_{n+2})$, then consider $u = u_0 u_1 \cdots u_m$ and $w = w_0 w_1 \cdots w_p$ their CFNF. Since $D_L(u) = D_L(w)$ and D_w and D_u have the same number of $\mathcal{L}_\bullet^\circ$, then $u_0 = w_0$ and $m = p$, by Proposition 3.2.4. Therefore, by Definition 2.3.18, $u = w$.
- If $u, w \in \mathrm{CZ}(\widetilde{D}_{n+2})$, then $D_L(u) = D_L(w)$ implies that u and w are of the same form (1) or (2) in Definition 2.3.5. Moreover, the corresponding k and h are equal as well, since $f_\bullet(w) = f_\bullet(u)$ and $f_\circ(w) = f_\circ(u)$ by Proposition 3.2.5.

$\square$

Before proving the following results, we need to introduce a particular representation of a diagram D_w for some $w \in \mathrm{FC}(\widetilde{D}_{n+2})$, by explicitly describing its factorization into simple diagrams geometrically.

Definition 3.4.2. Let $\mathbf{w} = s_{i_1} s_{i_2} \cdots s_{i_m}$ be a reduced expression of $w \in W$. We call the *concrete simple representation* of $D_{\mathbf{w}}$, the concrete diagram $\mathcal{C}(D_{\mathbf{w}})$ that results from stacking standard representations of the simple diagrams $D_{i_1}, \ldots, D_{i_m}$ from top to bottom without deforming any of their edges or applying any relation among the decorations (see Figure 3.12). Recall the description of a standard representation of a diagram given in Section 3.1, after Figure 3.2.

Lemma 3.4.3. *The element $w \in \mathrm{FC}(\widetilde{D}_{n+2})$ if and only if, for any $\mathbf{w} \in \mathcal{R}(w)$, $\mathcal{C}(D_{\mathbf{w}})$ does not contain the patterns in Figure 3.13.*

Proof. Let $w \notin \mathrm{FC}(\widetilde{D}_{n+2})$, then there exists a reduced expression $\mathbf{w}$ of w containing the factor $s_i s_j s_i$ for some s_i, s_j adjacent nodes in the Coxeter graph (Figure 1.2, left). Hence, $D_i D_j D_i$ in $D_{\mathbf{w}}$ gives rise to one of the patterns in Figure 3.13 in $\mathcal{C}(D_{\mathbf{w}})$. Vice versa, assume there exists $\mathbf{w} \in \mathcal{R}(w)$ such that $\mathcal{C}(D_{\mathbf{w}})$ contains the pattern P_i, with $i = 2, \ldots, n+2$. By construction, the diagrams contributing to the top and bottom part of the pattern P_i (respectively, P_{n+2}) are equal to D_i (respectively, D_{n+2}), there

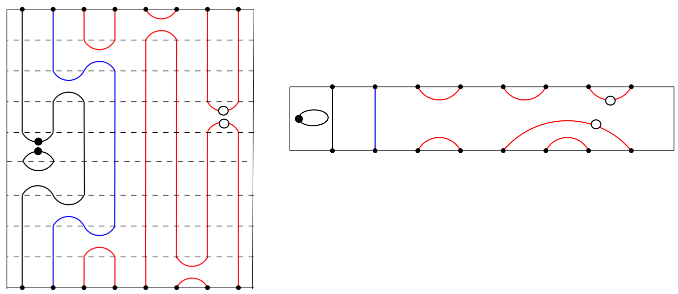


Figure 3.12: The concrete simple representation $\mathcal{C}(D_{\mathbf{w}})$ with $\mathbf{w} = s_5s_3s_2s_8s_0s_1s_2s_3s_6$ in $\mathrm{FC}(\widetilde{D}_8)$ and its associated pseudo diagram D_w.

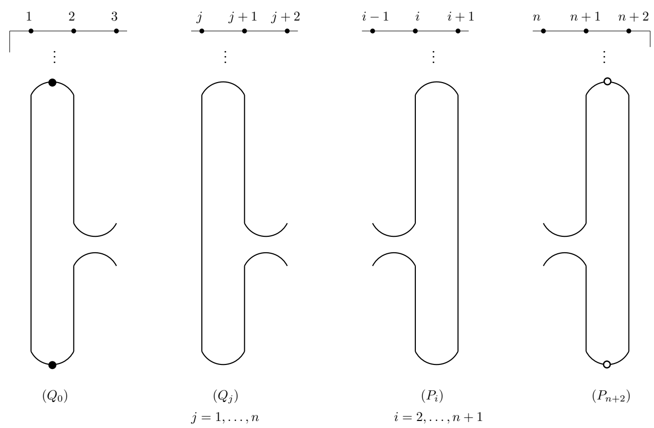


Figure 3.13: Forbidden patterns in a concrete simple representation of a FC element.

is one occurrence of D_{i-1} (respectively, D_n) between the two D_i (respectively, D_{n+2}), and there is no occurrence of D_{i+1} between them. Hence any other simple diagram appearing between these two occurrences of D_i (respectively, D_{n+2}) in the factorization of $D_{\mathbf{w}}$ commutes with them. Therefore there exists a reduced expression $\mathbf{w}'$ in the same commutation class of $\mathbf{w}$ in which the factor $s_is_{i-1}s_i$ (respectively, $s_{n+2}s_ns_{n+2}$) appears and so w is not FC. A similar argument holds for the patterns of the form Q_j, $j = 0, \dots, n$. $\square$

Corollary 3.4.4. *Let $w \in \mathrm{FC}(\widetilde{D}_{n+2})$, then*

(a) D_w contains a $\mathcal{L}_\bullet$ (respectively a $\mathcal{L}_\circ$) if and only if there exists a reduced expression of w containing a factor s_0s_1 (respectively $s_{n+1}s_{n+2}$).

(b) *D_w contains a $\mathcal{L}_\bullet$ and the vertical edge $\{1, 1'\}$ (respectively a $\mathcal{L}_\circ$ and the vertical edge $\{n+2, (n+2)'\}$) if and only if there exists a reduced expression of w containing a factor $s_2s_0s_1s_2$ (respectively $s_ns_{n+1}s_{n+2}s_n$).*

Proof. Note that (a) derives from Proposition 3.2.6. We only prove the direct implication of (b) since the reverse is clear. By the previous point, there exists a reduced expression $\mathbf{w}$ such that $D_{\mathbf{w}}$ contains the factor D_0D_1. The associated $\mathcal{C}(D_{\mathbf{w}})$ has a path from node 1 to node $1'$ if and only if there are two occurrences of D_2, one before and one after D_0D_1, otherwise, for topological reasons, in $D_{\mathbf{w}}$ there would appear a pattern of type (P_i). An analogous argument works for $\mathcal{L}_\circ$ and for the edge $\{n+2, (n+2)'\}$. □

Lemma 3.4.5. *Let $w \in \mathrm{FC}(\widetilde{D}_{n+2})$ such that $f_\bullet(w) = f_\circ(w) = 0$. If the edge $\{i, i'\}$ not decorated appears in D_w, then the following hold.*

(a) *If $i = 1$, then $s_0, s_1 \notin supp(w)$.*

(b) *If $i = 2$, then $s_0, s_1, s_2 \notin supp(w)$.*

(c) *If $3 \leq i \leq n$, then $s_{i-1}, s_i \notin supp(w)$.*

(d) *If $i = n+1$, then $s_n, s_{n+1}, s_{n+2} \notin supp(w)$.*

(e) *If $i = n+2$, then $s_{n+1}, s_{n+2} \notin supp(w)$.*

Proof. We only prove point (c) since the other cases are similar. Then set $3 \leq i \leq n$, and let $w = u_0 \cdots u_m$ be its CFNF. We have $D_w = D_{u_0} \cdots D_{u_m}$ with D_{u_j}, $0 \leq j \leq m$ described in Proposition 3.2.3, since $u_j \in \mathrm{TC}(\widetilde{D}_{n+2})$. First, note that $s_{i-1}, s_i \notin supp(u_0) = D_L(w)$ by Lemma 3.2.2. Now we prove the result by contradiction. Let $1 \leq l \leq m$ be the smallest index such that s_{i-1} or s_i is in $supp(u_l)$, and assume, without loss of generality, that $s_{i-1} \in supp(u_l)$. By Lemma 3.2.2, $\{i-1, i\}$ and $\{(i-1)', i'\}$ are in D_{u_l}, and by Lemma 3.2.3 and Corollary 3.3.7, the undecorated edge $\{i, i'\}$ belongs to D_{u_j}, for all $0 \leq j < l$. Note that the configuration depicted in Figure 3.14(a) never appears by Lemma 3.4.3. Therefore, since $\{i, i'\}$ appears in D_w by hypothesis, the only possibility is that there exist $h_1 < h_2$ such that $s_0 \in supp(u_{h_1})$ and $s_1 \in supp(u_{h_2})$, or vice versa, and $s_0, s_1 \notin supp(u_j)$ for all $h_1 < j < h_2$, see Figure 3.14(b). Suppose h_1 is minimal, then in the diagram $D_{u_0}D_{u_1} \cdots D_{u_l} \cdots D_{u_{h_1}} \cdots D_{u_{h_2}}$ the edge $\{i, 2'\}$ is decorated with a $\bullet$. Now, since D_w contains $\{i, i'\}$, the edge $\{2, i'\}$ in $D_{u_{h_2+1}} \cdots D_{u_m}$

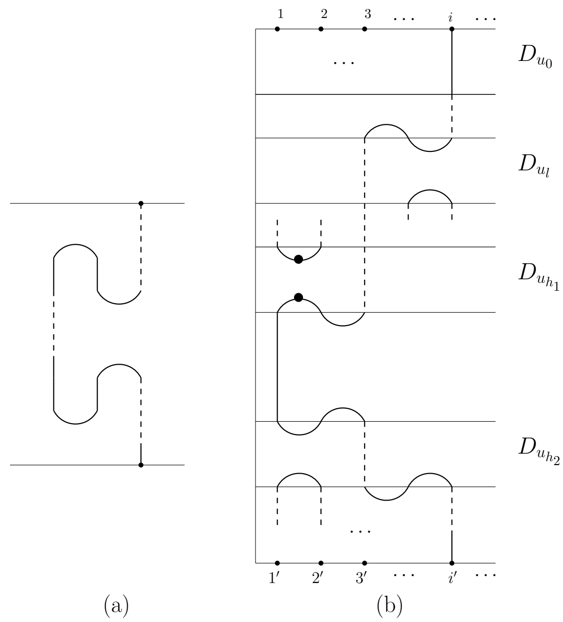


Figure 3.14: Concatenation $D_w = D_{u_0} \cdots D_{u_m}$ for the proof of Lemma 3.4.5.

either is undecorated or its first decoration is a $\circ$. In any case, $\{i, i'\}$ is decorated since $f_\bullet(w) = f_\circ(w) = 0$ by hypothesis, which is a contradiction. The same argument holds if we assume that $s_i \in supp(u_l)$, by exchanging the $\circ$ decorations with the $\bullet$.

$\square$

Proposition 3.4.6. *Let $w \in \mathrm{FC}(\widetilde{D}_{n+2})$. If $u \in \mathrm{TC}(\widetilde{D}_{n+2})$ such that $D_u = D_w$, then $u = w$.*

Proof. We distinguish four cases.

(1) Assume $f_\bullet(w) = f_\circ(w) = 0$. If $w \in \mathrm{I}(\widetilde{D}_n)$, then $u = w$ by Proposition 3.4.1. Otherwise, we suppose that $w \rightsquigarrow_{s_i} s_{i-1}w$, with $3 \leq i \leq n-1$. Therefore since $s_{i-1}s_i$ is a prefix of w, we have that $s_i, s_{i+1} \notin D_L(w)$. Moreover, since $D_u = D_w$, the non-decorated edge $\{i+1, (i+1)'\}$ occurs in D_w by Proposition 3.2.3 and Corollary 3.3.7, which is not possible by Lemma 3.4.5. The cases when $i \in \{2, n, n+1\}$, or $s_0 s_2$, $s_n s_{n+2}$ or $s_j s_{j'}$, with $m_{s_j s_{j'}} = 3$ and $j > j'$, is a prefix of w can be treated in the same way using Lemma 3.4.5. So w is necessarily irreducible and $u = w$.

(2) If $f_\bullet(w) = 1$ and $f_\circ(w) = 0$, then $s_0, s_1 \in D_L(w) \cap D_R(w)$, because $D_L(u) = D_R(u) = D_L(w) = D_R(w)$ by Theorem 3.3.6. Moreover, it is possible to write $w = s_0 s_1 w'$ reduced and $s_0, s_1, s_2 \notin supp(w')$. Let $u = s_0 s_1 u'$ reduced and $u' \in \mathrm{TC}(\widetilde{D}_n)$. Observe that, $\{1, 1'\}$ and $\{2, 2'\}$ appear in $D_{w'}$ and in $D_{u'}$, since $s_0, s_1, s_2 \notin supp(w') \cup supp(u')$. Then, $D_{u'} = D_{w'}$ since the portions of $D_{w'}$ and $D_{u'}$ on the right of $\{2, 2'\}$ are both equal to the corresponding portion of $D_u = D_w$; for an example see Figure 3.15. Therefore, $u' = w'$ by point (1), thus $u = w$.

(3) If $f_\bullet(w) = 0$ and $f_\circ(w) = 1$, then this case is analogous to the previous one.

(4) If $f_\bullet(w) = f_\circ(w) = 1$, then $w = s_0 s_1 w'$ and $u = s_0 s_1 u'$ both reduced, $s_0, s_1, s_2 \notin supp(w') \cup supp(u')$ and $D_{u'} = D_{w'}$ for the same argument in point (2). Moreover, $f_\bullet(w') = 0$ and $f_\circ(w') = 1$, so $u' = w'$ by point (3). Hence, $u = w$.

□

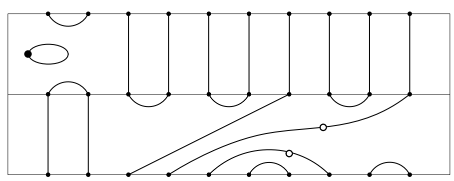


Figure 3.15: $D_w = D_{s_0 s_1} D'_w$, $w = s_0 s_1 w'$ and $w' = s_3 s_5 s_8 s_4 s_6 s_{10} s_5 s_7 s_8 s_6 s_9$.

Proposition 3.4.7. *Let $w \in \mathrm{FC}(\widetilde{D}_{n+2})$. If $u \in \mathrm{CZ}(\widetilde{D}_{n+2}) \cup \mathrm{K}(\widetilde{D}_{n+2})$ such that $D_u = D_w$, then $u = w$.*

Proof. If $u \in \mathrm{CZ}(\widetilde{D}_{n+2})$, then $D_L(w), D_R(w) \in \{\{s_0, s_1\}, \{s_{n+1}, s_{n+2}\}\}$ by Theorem 3.3.6. Hence, w is both left and right irreducible by Lemma 2.3.8, so $w \in \mathrm{I}(\widetilde{D}_{n+2})$. Then, $u = w$ by Proposition 3.4.1.

If $u \in \mathrm{K}(\widetilde{D}_{n+2})$, by Theorem 3.3.6 we can assume $D_L(u) = D_L(w)$ and $D_R(u) = D_R(w)$. Therefore, $w \in \mathrm{I}(\widetilde{D}_{n+2})$ since it cannot admit any prefix or suffix of the form st, $m_{st} = 3$. Then, $u = w$ by Proposition 3.4.1. □

Theorem 3.4.8. *The map $\widetilde{\theta}_D : \mathrm{TL}(\widetilde{D}_{n+2}) \to \mathbb{D}(\widetilde{D}_{n+2})$ is a $\mathbb{Z}[\delta]$-algebra isomorphism.*

Proof. First, we have that $\widetilde{\theta}_D$ is a surjective homomorphism by definition. So we only need to show that the set $\{D_w \in \mathbb{D}(\widetilde{D}_{n+2}) \mid w \in \mathrm{FC}(\widetilde{D}_{n+2})\}$ is a basis for $\mathbb{D}(\widetilde{D}_{n+2})$. Since the diagram D_w, for all $w \in \mathrm{FC}(\widetilde{D}_{n+2})$, does not contain any undecorated loops by Proposition 3.2.6, it is sufficient proving that if $u, w \in \mathrm{FC}(\widetilde{D}_{n+2})$ such that $D_u = D_w$ then $u = w$. Suppose that w is reducible to $v \in \mathrm{I}(\widetilde{D}_{n+2})$ by a sequence $w_0 = w \rightsquigarrow w_1 \rightsquigarrow w_2 \rightsquigarrow \cdots \rightsquigarrow w_l = v$. We proceed by induction on l. If $l = 0$, then $w \in \mathrm{I}(\widetilde{D}_{n+2})$, hence $u = w$ by Propositions 3.4.6 and 3.4.7. Without loss of generality, suppose that $w \rightsquigarrow_t sw =: w_1$. We have that $s \in D_L(u)$, since $D_L(u) = D_L(w)$ by Theorem 3.3.6. Suppose that st is not a prefix of u, then $tu \in \mathrm{FC}(\widetilde{D}_{n+2})$ reduced by Remark 2.1.3(5). Moreover, $D_{w_1} = D_t D_w = D_t D_u = D_{tu}$ so by inductive hypothesis, it means $w_1 = tu$. But this is a contradiction since ts is not a prefix of w_1 but it is a prefix of tu. Therefore, $u \rightsquigarrow_t su =: u_1$. Hence, $D_{w_1} = D_t D_w = D_t D_u = D_{u_1}$ so by inductive hypothesis, $u_1 = w_1$ and $u = w$. ☐

3.5 Some remarks on the Lusztig's a-function

In this section, we want to relate the Lusztig's **a**-function of a fully commutative element to its corresponding TL diagram.

In [38], Lusztig introduced the function $\mathbf{a} : W \to \mathbb{N}_0$ which plays an important role in Kazdhan–Lusztig theory. For instance, this function is constant on the LR-cells of a Coxeter group [38, Theorem 5.4]. In [44], Shi provided a more manageable way to compute the **a**-function of a FC element of a finite or affine Weyl group W through its heap. Let $w \in \mathrm{FC}(W)$, define $n(w)$ as the cardinality of the maximum antichain in the heap $H(w)$. Equivalently, by [44, Lemma 2.7], $n(w)$ is the maximum integer k such that $w = uw'v$ is reduced, $\ell(w') = k$ and $w' \in \mathrm{TC}(W)$. In [44, Theorem 3.1], Shi proved the following important result.

Theorem 3.5.1. *Let $w \in \mathrm{FC}(W)$. Then $\mathbf{a}(w) = n(w)$.*

In [19, Lemma 5.4, 5.5], Ernst proved that when W is an affine Coxeter system of type $\widetilde{C}_n$, then $n(w) = \mathbf{a}(D_w)$, where D_w is the decorated diagram associated to $w \in \mathrm{FC}(\widetilde{C}_n)$ in the diagrammatic realization of $\mathrm{TL}(\widetilde{C}_n)$ he introduced in [18, 19], and $\mathbf{a}(D_w)$ is the number of non-propagating edges on the north face of D_w. However, this property does not hold for the diagrammatic realization of $\mathrm{TL}(\widetilde{D}_n)$ used in this work. For instance, if

$w = s_0 s_1$, we have $2 = n(s_0 s_1) \neq \mathbf{a}(D_w) = 1$. Our goal is to provide a function defined on the *LR*-decorated diagrams of type $\widetilde{D}_{n+2}$, that coincides with Lusztig's **a**-function.

Definition 3.5.2. Let $w \in \mathrm{FC}(\widetilde{D}_{n+2})$. We define the function $\widetilde{\mathbf{a}}(D_w)$ as follows:

$$\widetilde{\mathbf{a}}(D_w) := \begin{cases} \mathbf{a}(D_w) + \#\mathcal{L}_\bullet + \#\mathcal{L}_\circ, & \text{if } \mathbf{a}(D_w) > 1; \\ \mathbf{a}(D_w) + \lambda, & \text{if } \mathbf{a}(D_w) = 1; \\ 0, & \text{if } w = e, \end{cases}$$

where

$$\lambda := \begin{cases} 0, & \text{if } \#\mathcal{L}_\bullet = \#\mathcal{L}_\circ = 0; \\ 1, & \text{otherwise.} \end{cases}$$

Theorem 3.5.3. *If $w \in \mathrm{FC}(\widetilde{D}_{n+2})$, then $n(w) = \widetilde{\mathbf{a}}(D_w)$.*

Proof. By the definitions of irreducible FC elements it is immediate to compute that $n(w) = |D_L(w)|$ for $w \in \mathrm{I}(\widetilde{D}_{n+2})$, that is:

$$n(w) = \begin{cases} \ell(w), & \text{if } w \in \mathrm{TC}(\widetilde{D}_{n+2}); \\ 2, & \text{if } w \in \mathrm{CZ}(\widetilde{D}_{n+2}); \\ n/2 + 1, & \text{if } w \in \mathrm{K}(\widetilde{D}_{n+2}). \end{cases}$$

By Proposition 3.2.5, if $w \in \mathrm{CZ}(\widetilde{D}_{n+2})$ then $\mathbf{a}(D_w) = 1$. By Lemma 3.2.2, if $w \in \mathrm{K}(\widetilde{D}_{n+2})$, then D_w has $n/2 + 1$ non-propagating edges. If $w \in \mathrm{TC}(\widetilde{D}_{n+2})$, by Corollary 3.3.7, $\mathbf{a}(D_w) = \ell(w) - f_\bullet(w) - f_\circ(w)$. Hence, if $w \in \mathrm{I}(\widetilde{D}_{n+2})$, $n(w) = \widetilde{\mathbf{a}}(D_w)$ by Proposition 3.2.6. Moreover, by [44, Lemma 2.9], $n(w)$ is invariant by star reducibility. On the other hand, by Lemma 2.3.2(b) and Remark 3.3.4, and by Proposition 3.2.6, $\mathbf{a}(D_w)$, $\#\mathcal{L}_\bullet$ and $\#\mathcal{L}_\circ$ are invariant by star reducibility as well. Hence $n(w) = \widetilde{\mathbf{a}}(D_w)$ for any $w \in \mathrm{FC}(\widetilde{D}_{n+2})$. □

3.6 Admissible diagrams of type $\widetilde{D}_{n+2}$

In this section, we present a combinatorial description of the TL decorated diagrams of our diagrammatical representation. In particular, we introduce a distinguished family of diagrams, called *admissible diagrams*, and consider the $\mathbb{Z}[\delta]$-module they generate inside $\widehat{\mathcal{P}}^{LR}_{n+2}(\Omega)$. We then show that this module and $\mathbb{D}(\widetilde{D}_{n+2})$ coincide as subalgebras of $\widehat{\mathcal{P}}^{LR}_{n+2}(\Omega)$.

Definition 3.6.1. An irreducible diagram $D \in T^{LR}_{n+2}(\Omega)$ is called a $\widetilde{D}$-*admissible* diagram if it satisfies the following conditions:

(A1) The only possible loops in D are those depicted in Figure 3.5. If a $\mathcal{L}^{\circ}_{\bullet}$ occurs, the other two types of loops cannot appear. Moreover, if $\mathbf{a}(D) > 1$ then there is at most one occurrence of $\mathcal{L}_{\bullet}$ and at most one occurrence of $\mathcal{L}_{\circ}$.

Assume $\mathbf{a}(D) > 1$.

(A2) Both the sum of the number of $\mathcal{L}^{\circ}_{\bullet}$ and $\bullet$ on non-loop edges, and the sum of the number of $\mathcal{L}^{\circ}_{\bullet}$ and $\circ$ on non-loop edges, must be even.

(A3) If e is the vertical edge $\{1, 1'\}$ (respectively $\{n+2, (n+2)'\}$), then exactly one of the following three conditions is met:

(a) e is undecorated;

(b) e is decorated by a single $\circ$ (respectively $\bullet$) and no $\mathcal{L}_{\circ}$ (respectively $\mathcal{L}_{\bullet}$) occurs;

(c) e is decorated by an alternating sequence of $\bullet$ and $\circ$ and there are no loops.

Assume $\mathbf{a}(D) = 1$.

(A4) The western side of D is equal to one of those depicted in Figure 3.16, where the number of $\mathcal{L}_{\bullet}$ might be 0 and $\mathbf{d_1}, \mathbf{d_2} \in \{\emptyset, \bullet\}$. In each $\bullet$-strip, there is a unique $\mathcal{L}_{\bullet}$ and no other decorations, except when $\mathbf{d_1}$ or $\mathbf{d_2}$ of Figures 3.16 (B)-(C)-(D) are not empty. In these cases, the highest (respectively, lowest) strip of the diagram contains two $\bullet$ and no $\mathcal{L}_{\bullet}$. We have similar restrictions for the eastern side of D, where the $\bullet$ are replaced by the $\circ$ and the $\mathcal{L}_{\bullet}$ with the $\mathcal{L}_{\circ}$ (see Remark 3.6.10 for an analogous description).

Remark 3.6.2.

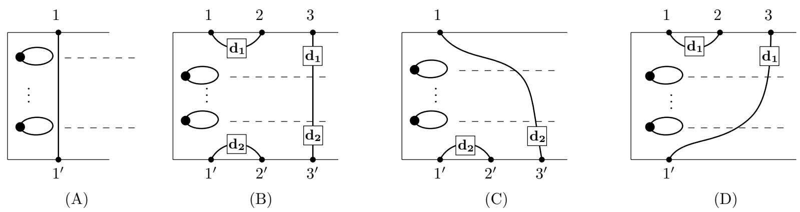


Figure 3.16: Western side of an admissible diagram with $\mathbf{a}(D) = 1$. The number of $\mathcal{L}_\bullet$ in each diagram can be zero.

(a) An edge e of an admissible diagram D can be decorated by both $\bullet$ and $\circ$ only if D has no propagating edges or if e is the unique propagating edge of D.

(b) If $\mathbf{a}(D) = 1$, then the $\bullet$-strips and the $\circ$-strips must alternate.

(c) In a standard representation, all the loops $\mathcal{L}_\bullet$ (respectively $\mathcal{L}_\circ$) will be depicted vertically aligned on the left (respectively right) of the leftmost (respectively rightmost) propagating edge of the diagram (Figure 3.16). When D has no propagating edges, the $\mathcal{L}_\bullet^\circ$ will be drawn horizontally aligned in the middle of the picture (Figure 3.22).

(d) The diagram in Figure 3.6(C) is an example of an irreducible diagram that is not admissible, since it does not satisfy (A2).

Definition 3.6.3. We denote by $\mathtt{ad}(\widetilde{D}_{n+2})$ the set of all $\widetilde{D}$-admissible $(n+2)$-diagrams and by $\mathcal{M}[\mathtt{ad}(\widetilde{D}_{n+2})]$ the $\mathbb{Z}[\delta]$-submodule of $\widehat{\mathcal{P}}^{LR}_{n+2}(\Omega)$ they span.

Since the admissible diagrams $\mathtt{ad}(\widetilde{D}_{n+2})$ are irreducible, they are independent.

Proposition 3.6.4. *The set of admissible diagrams* $\mathtt{ad}(\widetilde{D}_{n+2})$ *is a basis for the module* $\mathcal{M}[\mathtt{ad}(\widetilde{D}_{n+2})]$.

We now classify the admissible diagrams in three main families.

Definition 3.6.5 (PZZ-diagrams)**.** Let $D \in \mathtt{ad}(\widetilde{D}_{n+2})$. We say that D is a *PZZ-diagram* if $\mathbf{a}(D) = 1$ and $\#\mathcal{L}_\bullet, \#\mathcal{L}_\circ \geq 1$. In particular, we say that D is of *type* $\langle a, b\rangle$ where $a, b \in \{\bullet, \circ\}$ if the highest loop in D is $\mathcal{L}_a$ and the lowest loop in D is $\mathcal{L}_b$.

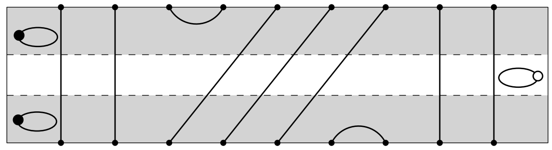
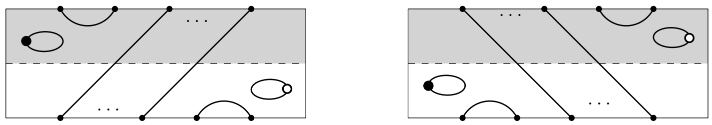


Figure 3.17: A PZZ-diagram of type $\langle \bullet \bullet \rangle$ with $\#\mathcal{L}_\bullet = 2$ and $\#\mathcal{L}_\circ = 1$.



Z_1 Z_2

Figure 3.18: Special PZZ-diagrams.

By (A4) and Remark 3.6.2(b) in a PZZ-diagram we have that $|\#\mathcal{L}_\bullet - \#\mathcal{L}_\circ| \leq 1$.

The two specific PZZ-diagrams depicted in Figure 3.18 will be used later. Note that Z_1 is of type $\langle \bullet \circ \rangle$, Z_2 is of type $\langle \circ \bullet \rangle$ and they both have $\#\mathcal{L}_\bullet = \#\mathcal{L}_\circ = 1$.

Definition 3.6.6 (P-diagrams)**.** Let $D \in \mathtt{ad}(\widetilde{D}_{n+2})$ that is not a PZZ-diagram. We say that:

(LP) D is of *type* LP if it has a single $\mathcal{L}_\bullet$ and there exists $j_\ell \in \{2, \ldots, n+1\}$ such that the leftmost $j_\ell - 1 > 0$ edges are vertical and undecorated, and the one leaving node j_ℓ is not an undecorated vertical edge.

(RP) D is of *type* RP if it has a single $\mathcal{L}_\circ$ and there exists $j_r \in \{2, \ldots, n+1\}$ the rightmost $n + 2 - j_r > 0$ edges are vertical and undecorated, and the one leaving node j_r is not an undecorated vertical edge.

(LRP) D is of *type* LRP if it is of both types LP, RP and if $\mathbf{a}(D) > 1$. The condition $\mathbf{a}(D) > 1$ implies that $j_r - j_\ell > 2$.

If D is of type LP, RP or LRP then we say that D is a *P-diagram*.

Definition 3.6.7 (ALT-diagrams)**.** Let $D \in \mathtt{ad}(\widetilde{D}_{n+2})$. If D is not a P-diagram or a PZZ-diagram, then we say that D is an *ALT-diagram*. In particular, the identity diagram is an ALT-diagram.

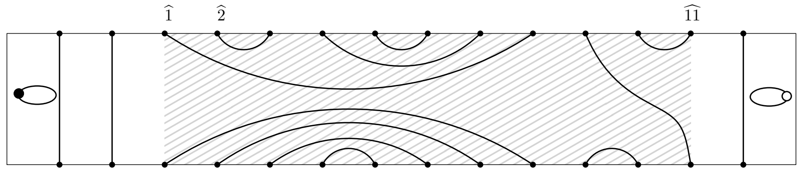


Figure 3.19: A P-diagram of type LRP, with $j_\ell = 3$, $j_r = 13$ and $n = 12$. The shaded part denotes its inner alternating diagram, as defined in Definition 3.6.8.

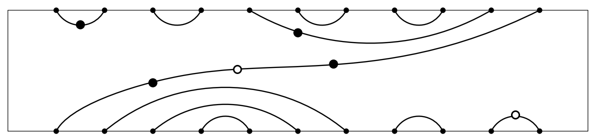


Figure 3.20: An ALT-diagram with $n = 9$.

The conditions required for the ALT, P and PZZ diagrams are consistent with (A1)–(A4); moreover, the three families form a partition of the set of admissible diagrams.

Definition 3.6.8. Given a P-diagram D, we define the *inner alternating diagram*, denoted by $\widehat{D}$, as the diagram embedded in the standard $(j_r - j_\ell + 1)$-box and delimited by the vertical edges $\{j_\ell - 1, (j_\ell - 1)'\}$ and $\{j_r + 1, (j_r + 1)'\}$ of D. We relabel the nodes of $\widehat{D}$ by $\hat{1}, \hat{2}, \hat{3}, \ldots$, where $\hat{k} = k + j_\ell - 1$. If D is of type LP (respectively, RP) we set $j_r = n + 2$ (respectively, $j_\ell = 1$).

Note that $\widehat{D}$ is an ALT-diagram. For example, in Figure 3.19, $\widehat{D}$ is represented by the shaded part.

For our aims, it is useful to introduce the following definition. The notation a^* means that a^* might be either a node a in the north face or a node a' in the south face of D.

Definition 3.6.9. Let $D \in \mathtt{ad}(\widetilde{D}_{n+2})$. The *conjugate diagram* $F(D)$ is the diagram having the set of edges equal to $\{F(e) \mid e \text{ is an edge of } D\}$, where $F(e)$ is defined as follows.
If e is a loop, then

(f1) $F(\mathcal{L}_\bullet) = \mathcal{L}_\circ$, $F(\mathcal{L}_\circ) = \mathcal{L}_\bullet$ and $F(\mathcal{L}_\bullet^\circ) = \mathcal{L}_\bullet^\circ$, with the same vertical positions.

Otherwise,

(f2) if $e = \{i^*, j^*\}$ is a non-propagating edge, then $F(e) = \{(n+3-j)^*, (n+3-i)^*\}$;

(f3) if $e = \{i, j'\}$ is a propagating edge, then $F(e) = \{n+3-i, (n+3-j)'\}$;

(f4) if e is a non-propagating edge decorated with $\bullet\circ$, then $F(e)$ is decorated with $\bullet\circ$. Otherwise, if e has a $\bullet$ (respectively $\circ$) then $F(e)$ has a $\circ$ (respectively $\bullet$) in the same vertical position.

Remark 3.6.10. One can note that this operation is left-right reversal composed with exchanging the two types of decoration. In particular, $F(D) \in \mathtt{ad}(\widetilde{D}_{n+2})$, $F(F(D)) = D$ and $F(DD') = F(D)F(D')$ for $D, D' \in \mathtt{ad}(\widetilde{D}_{n+2})$.

For example $F(Z_1) = Z_2$ and $F(Z_2) = Z_1$, see Figure 3.18. Moreover, in Definition 3.6.1(A4) the eastern side of D can be obtained by applying F to the diagrams depicted in Figure 3.16. Note that $F(D)$ is an ALT (respectively, P, PZZ) diagram if and only if D is an ALT (respectively, P, PZZ) diagram. In particular, a RP-diagram is the conjugate of a LP-diagram, and vice versa, while the conjugate of a LRP-diagram is a LRP-diagram.

3.6.1 Length of an admissible diagram.

In this section we introduce a length function on the set of admissible diagrams. In the diagram algebras of types A and $\widetilde{A}$ the length of a diagram is defined by counting the number of intersections between all the vertical lines $i+1/2$ in the standard box and the edges of the diagram (see [26, Lemma 3.3], [21, Definition 4.4.2]). To define an analogous length function in type $\widetilde{D}$, for instance in the case of an ALT-diagram, one may count the intersections between such vertical lines and the edges of an isotopically equivalent diagram obtained by stretching the decorated edges to touch the left or the right wall of the box without crossing each other, as sketched in Figure 3.21. For the P and PZZ-diagrams the geometric interpretation is more subtle. The algebraic formulation of the

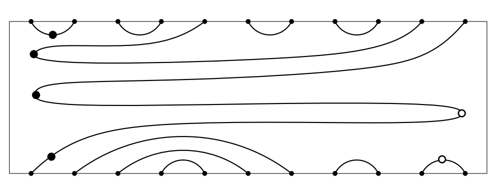


Figure 3.21: Stretched representation of the diagram in Figure 3.20. The length of this diagram is 29.

length function that arises from this approach is the following.

From now on, we consider the standard representation of a diagram (see Section 3.1) and we assign to any edge e of D a *weight* $\mathsf{w}(e)$ whose value depends only on the decorations $\bullet$ and $\circ$. We set $\mathsf{w}(\mathcal{L}_\bullet) = \mathsf{w}(\mathcal{L}_\circ) = 1$ and $\mathsf{w}(\mathcal{L}^\circ_\bullet) = n+1$. If e is undecorated, then $\mathsf{w}(e) = 0$. For $k \in \mathbb{N}_0$, set $(\circ\,\bullet)^k := \underbrace{\circ\,\bullet\cdots\circ\,\bullet}_{2k}$ and $(\bullet\,\circ)^k := \underbrace{\bullet\,\circ\cdots\bullet\,\circ}_{2k}$.

If $i < j$ and $e = \{i, j\}$, $e = \{i', j'\}$ or $e = \{i, j'\}$, then we set

$$\mathsf{w}(e) := \begin{cases} i - 1 + k(n+1), & \text{if } e \text{ is decorated with } \bullet \text{ followed by } (\circ\,\bullet)^k; \\ n + 2 - j + k(n+1), & \text{if } e \text{ is decorated with } \circ \text{ followed by } (\bullet\,\circ)^k; \\ (k+1)(n+1) - j + i, & \text{if } e \text{ is decorated with } (\bullet\,\circ)^{k+1}; \\ (k+1)(n+1), & \text{if } e \text{ is decorated with } (\circ\,\bullet)^{k+1}. \end{cases} \tag{3.2}$$

If $i \geq j$ and $e = \{i, j'\}$,

$$\mathsf{w}(e) := \begin{cases} j - 1 + k(n+1), & \text{if } e \text{ is decorated with } \bullet \text{ followed by } (\circ\,\bullet)^k; \\ n + 2 - i + k(n+1), & \text{if } e \text{ is decorated with } \circ \text{ followed by } (\bullet\,\circ)^k; \\ (k+1)(n+1), & \text{if } e \text{ is decorated with } (\bullet\,\circ)^{k+1}; \\ (k+1)(n+1) - i + j, & \text{if } e \text{ is decorated with } (\circ\,\bullet)^{k+1}. \end{cases} \tag{3.3}$$

Remark 3.6.11. Note that, by definition, the weight of an edge decorated with only a $\bullet$ (respectively, $\circ$) depends only on its leftmost (respectively, rightmost) endpoint. In particular, if the edge $e = \{1^*, j^*\}$ (respectively, $e = \{i^*, (n+2)^*\}$) is decorated only with a $\bullet$ (respectively, $\circ$), then $\mathsf{w}(e) = 0$.

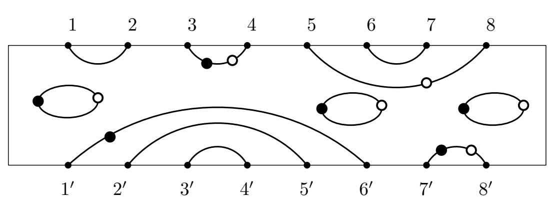


Figure 3.22: The weights of the non-loop edges are all 0 except $\mathsf{w}(3,4) = 6$ and $\mathsf{w}(7',8') = 6$. Moreover, $\mathsf{w}(\mathcal{L}^\circ_\bullet) = 7$ and $\ell(D) = 41$.

Notation 3.6.12. We set $\nu_i(D)$ *the number of non-loop edges of D that intersect the vertical lines* $i + 1/2$, with $i \in \{1, \dots, n+1\}$. Note that $\nu_i(D)$ is well-defined since we work with a standard representation of D and it is always an even number. Also define

$$\nu(D) := \sum_{i=1}^{n+1} \nu_i(D) \qquad \text{and} \qquad \mathsf{w}(D) := \sum \mathsf{w}(e),$$

where the second sum runs over all the edges of D, loops included.

Definition 3.6.13 (Length of a $\widetilde{D}$-admissible diagram)**.** We define the *length* $\ell(D)$ of an admissible diagram $D \in \mathtt{ad}(\widetilde{D}_{n+2})$ as follows.

(a) If D is an ALT-diagram, we define

$$\ell(D) := \frac{1}{2}\nu(D) + \mathsf{w}(D). \tag{3.4}$$

In particular, if D is the identity diagram then $\ell(D) = 0$.

(b) If D is a PZZ-diagram, we set

$$\ell(D) := \begin{cases} 3 - i + n - j + (\#\mathcal{L}_\bullet + \#\mathcal{L}_\circ)(n+1), & \text{if } D \text{ is of type } \langle \circ\, \circ \rangle; \\ 1 - i + j + (\#\mathcal{L}_\bullet + \#\mathcal{L}_\circ)(n+1), & \text{if } D \text{ is of type } \langle \circ\, \bullet \rangle; \\ 1 + i - j + (\#\mathcal{L}_\bullet + \#\mathcal{L}_\circ)(n+1), & \text{if } D \text{ is of type } \langle \bullet\, \circ \rangle; \\ i - n + j - 1 + (\#\mathcal{L}_\bullet + \#\mathcal{L}_\circ)(n+1), & \text{if } D \text{ is of type } \langle \bullet\, \bullet \rangle, \end{cases}$$

where $\{i, i+1\}$ is the unique non-propagating edge on the north face and $\{j', (j+1)'\}$ is the unique non-propagating edge on the south face of D, while $\langle \_\, \_ \rangle$ are as in Definition 3.6.5.

(c) If D is a P-diagram, we set

$$\ell(D) := \begin{cases} 2j_\ell - 1 + \ell(\widehat{D}), & \text{if } D \text{ is of type LP}; \\ 2n - 2j_r + 5 + \ell(\widehat{D}), & \text{if } D \text{ is of type RP}; \\ 2n + 2j_\ell - 2j_r + 4 + \ell(\widehat{D}), & \text{if } D \text{ is of type LRP}, \end{cases}$$

where j_ℓ and j_r are as in Definition 3.6.6, $\widehat{D}$ is the inner alternating diagram, and $\ell(\widehat{D})$ is defined in (3.4).

Note that $\ell(D) = 0$ if and only if D is the identity diagram. In Figures 3.22 and 3.23 the length is computed in some examples.

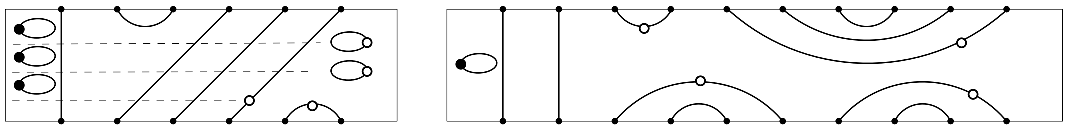


Figure 3.23: A PZZ-diagram of type $\langle \bullet\ \bullet \rangle$ on the left, with $\#\mathcal{L}_\bullet = 3$, $\#\mathcal{L}_\circ = 2$ and $\ell(D) = 27$ and a LP-diagram, right, with $j_\ell = 3$ and $\ell(D) = 24$.

Lemma 3.6.14. *Let $D \in \mathtt{ad}(\widetilde{D}_{n+2})$, then $\ell(F(D)) = \ell(D)$.*

Proof. By Definition 3.6.9,

$$F(e) = \begin{cases} \{n+3-j, n+3-i\}, & \text{if } e = \{i^*, j^*\} \text{ is non-propagating;} \\ \{n+3-i, (n+3-j)'\}, & \text{if } e = \{i, j'\} \text{ is propagating.} \end{cases}$$

Moreover, $F(e)$ is decorated following (f4), so, by Equations (3.2) and (3.3), $\mathtt{w}(F(e)) = \mathtt{w}(e)$. Without considering the decorations, the shape of $F(D)$ is obtained by the shape of D by reflecting through the central vertical axis of D. Hence, the number of intersections with the vertical lines $i + 1/2$ is preserved. These two facts imply that if D is an ALT-diagram, $\ell(F(D)) = \ell(D)$.

If D is a PZZ-diagram, by Definition 3.6.9, the conjugate of a PZZ-diagram of type $\langle \bullet, \bullet \rangle$ is a PZZ-diagram of type $\langle \circ, \circ \rangle$, while the conjugate of a PZZ-diagram of type $\langle \bullet, \circ \rangle$ is of type $\langle \circ, \bullet \rangle$. Hence, when applying the Definition 3.6.13(b), observing that F preserves the quantity $\#\mathcal{L}_\bullet + \#\mathcal{L}_\circ$, it follows that $\ell(F(D)) = \ell(D)$.

Finally, if D is a P-diagram, since F exchanges the values j_ℓ and j_r, and $\widehat{D}$ is an ALT-diagram, then by a quick computation, $\ell(F(D)) = \ell(D)$. □

3.7 The simple edges and the cut and paste operation

Definition 3.7.1. A non-propagating edge on the north face $e = \{i, i+1\}$ of $D \in \mathtt{ad}(\widetilde{D}_{n+2})$ is called a *simple edge of the form*:

($\smile$) if e is undecorated;

($\overset{\bullet}{\smile}$) if $e = \{1, 2\}$ and it is decorated by a single $\bullet$;

($\overset{\circ}{\smile}$) if $e = \{n+1, n+2\}$ and it is decorated by a single $\circ$.

Definition 3.7.2. Let $e = \{i, i+1\}$ be a simple edge in $D \in \mathtt{ad}(\widetilde{D}_{n+2})$ of any of the three forms above. We say that e is of *type*:

(U) U_R (respectively U_L) if D contains the edge $f = \{i+2, k\}$ (respectively $\{j, i-1\}$) with $k > i+2$ (respectively $j < i-1$) decorated with at least one $\bullet$ (respectively $\circ$);

(N) if D contains an edge $f = \{j, k\}$ with $j < i < i+1 < k$;

(P) P_R (respectively P_L) if D contains the edge $f = \{i+2, j'\}$ (respectively $\{i-1, k'\}$) with $j \leq i$ (respectively $k \geq i+1$);

(S) S_R (respectively S_L) if D contains the edge $f = \{i+2, (i+2)'\}$ (respectively $\{i-1, (i-1)'\}$) decorated with at least one $\bullet$ (respectively $\circ$).

Moreover we say that the simple edge $e = \{1, 2\}$ (respectively $e = \{n+1, n+2\}$) is of *type*:

(P^*) $P_R^\bullet$ (respectively P_L°) if e is of type P_R (respectively, P_L), D contains a $\mathcal{L}_\bullet$ (respectively a $\mathcal{L}_\circ$) and the edge $\{3, 1'\}$ (respectively $\{n, (n+2)'\}$) is undecorated.

The various situations are illustrated in Figure 3.24.

Definition 3.7.3. An ALT-diagram D is a *top basic diagram* if it consists of simple edges and a finite number (possibly zero) of undecorated vertical edges and loop edges (see Figure 3.25).

Proposition 3.7.4. *If $D \in \mathtt{ad}(\widetilde{D}_{n+2})$ is an ALT-diagram different from the identity diagram I, then one of the following occurs:*

1. *D is a top basic diagram;*

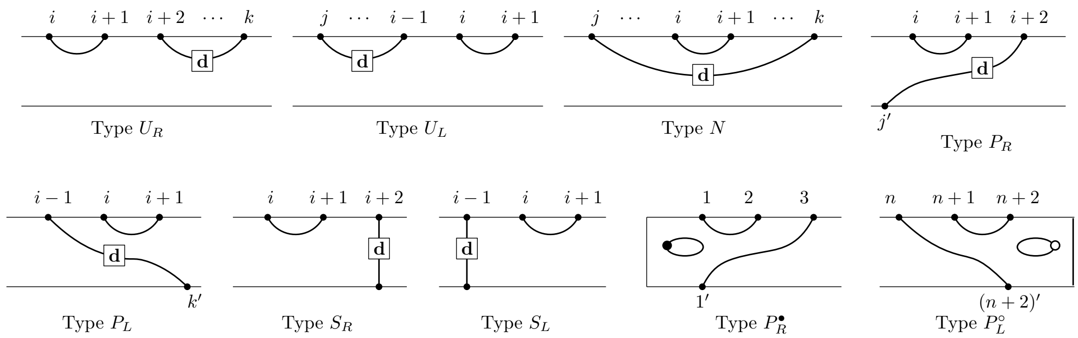


Figure 3.24: Types of simple edges and their neighbors. In cases N and P, the block **d** can be empty.

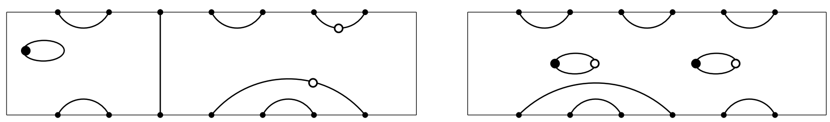


Figure 3.25: Top basic diagrams. The non-propagating edges on the south face can be decorated.

2. there exists a simple edge in D of type U, N, P, S or P^*.

Proof. If $D \neq I$ is not top basic, then there exists at least one non-propagating edge on the north face joining two consecutive nodes. Necessarily one of these is a simple edge $e = \{i, i+1\}$, in fact, by (D1) we cannot have a $\circ$-decoration on a non-propagating edge to the left of an edge with a $\bullet$-decoration. If e is of type U, N, P, S or P^* then (2) is satisfied. Now suppose that i is the maximum index such that e as well as all the simple edges on its left are not of any type. Consider the edge e' leaving node $i+2$.

(a) Suppose $e' = \{i+2, i+k\}$. If $k > 3$ then there exists a simple edge of type N to the right of e, so assume $k = 3$. The edge e' cannot be decorated with a $\bullet$, otherwise e would be of type U_R, and $i \neq n$, otherwise D would be top basic. Then either e' is simple, so by maximality it is of type U, N, P, S or P_L°, or it is decorated with a $\circ$. In the latter case, by (D1), on the right of e' there can be only non-propagating edges. Hence, there exists another simple edge and it has to be of type N or U_L (see Figure 3.26 (A)).

(b) Suppose $e' = \{i+2, (i+2)'\}$. If e' is undecorated, since D is not top basic, there must be another simple edge on its right which, by maximality, has to be of some type. The edge e' cannot be decorated with a $\bullet$ otherwise e would be of type S_R. If e' is decorated with a $\circ$, then $i \neq n$ otherwise D would not be admissible, and by (D1) to the right of e' there can only be non-propagating edges. Then there exists a simple edge of type N, S_L or U_L, see Figure 3.26 (B).

(c) Suppose $e' = \{i+2, m'\}$ is not vertical. Then $m > i+2$ and, since the non-propagating edges right to $i+2$ can be decorated at most with a $\circ$, then there exists a simple edge of type P_L, N, or U_L (see Figure 3.26 (C)).

□

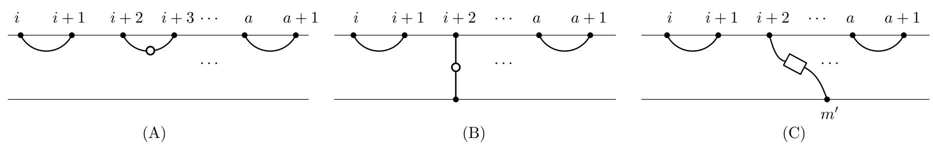


Figure 3.26: Configurations in the proof of Proposition 3.7.4.

Definition 3.7.5 (Suitable edges and neighbors)**.** Let D be an ALT-diagram.

(1) Assume D is top basic:

(a) if D has at least one $\mathcal{L}_\bullet^\circ$, then we say that any simple edge e is *suitable* and the leftmost $\mathcal{L}_\bullet^\circ$ is the *neighbor of* e;

(b) if D has a $\mathcal{L}_\bullet$, then $\{1,2\}$ is *suitable* and the $\mathcal{L}_\bullet$ is its *neighbor*;

(c) if D has a $\mathcal{L}_\circ$, then $\{n+1, n+2\}$ is *suitable* and the $\mathcal{L}_\circ$ is its *neighbor*;

(d) if D has no loops and no decorated edges on the south face, then any simple edge $e = \{i, i+1\}$ is *suitable* and its *neighbor* is the highest edge below e intersected by the vertical line $i + 1/2$;

(e) if D has no loops and at least one decorated edge on the south face, then a simple edge $e = \{i, i+1\}$ is *suitable* if the vertical line $i+1/2$ intersects in the south face either an edge decorated with both a $\bullet$ and a $\circ$, or the rightmost edge decorated with a $\bullet$, or the leftmost edge decorated with a $\circ$. The edge intersected by this vertical line is called a *neighbor of* e.

(2) If D is not top basic, then any simple edge of type U, N, P, S or P^* is *suitable*. Its *neighbor* is the unique edge f defined in Definition 3.7.2, except in type N, where the *neighbor* is the highest non-propagating edge $f = \{j, k\}$ below e, and in type $P_R^\bullet$ (respectively P_L°), where the *neighbor* is the $\mathcal{L}_\bullet$ (respectively $\mathcal{L}_\circ$).

Note that in an ALT-diagram D a suitable edge always exists by definition and Proposition 3.7.4.

Example 3.7.6. Both edges $\{3, 4\}$ and $\{5, 6\}$ in the top basic diagram of Figure 3.27(A) are suitable and their common neighbor is $\{3', 6'\}$, while $\{1, 2\}$ is not suitable. In the diagram in Figure 3.27(B), $\{1, 2\}$ and $\{5, 6\}$ are suitable since they are of type S_R and P_L respectively. In Figure 3.27(C) all the simple edges are suitable with common neighbor the leftmost $\mathcal{L}_\bullet^\circ$.

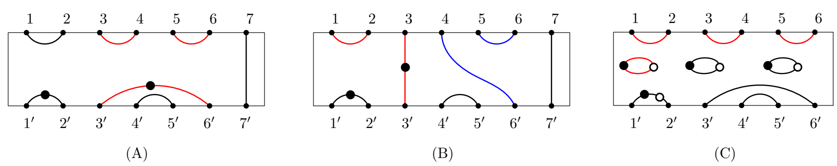


Figure 3.27: Suitable edges and neighbors.

Now, we describe a procedure called *cut and paste operation*, cp-operation for short, that starting from any suitable edge $e = \{i, i + 1\}$ produces an admissible diagram D' such that $D = D_i D'$, or $D = D_0 D'$ if e is of the form $(\overset{\bullet}{\smile})$ or $D = D_{n+2} D'$ if e is of the form $(\overset{\circ}{\smile})$. Recall that D_i, with $i \in \{0, 1 \ldots, n+2\}$ are the simple diagrams described in Figure 3.8. The idea is to replace (except in one case) the suitable edge e with two new non-intersecting edges, as sketched in Figure 3.28. If e is decorated, then we displace the decorations of the neighbor edge of e in an ad hoc way obtaining a new admissible diagram.

Definition 3.7.7 (Cut and paste operation)**.** Let D be an ALT-diagram with a suitable edge $e = \{i, i + 1\}$.

(cp1) Delete the simple edge e.

(cp2) Cut the neighbor of e and join the two free endpoints of the cut edge to the nodes i and $i + 1$ as to obtain two new non-intersecting edges or the edge $\{i, i + 1\}$, see Figure 3.28.

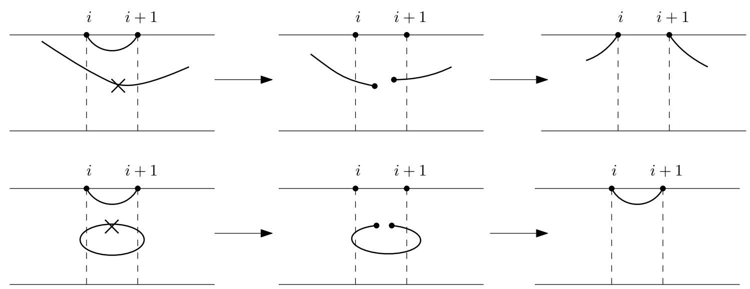


Figure 3.28: Step (cp2) of the cut and paste operation.

(cp3) If the neighbor edge of e is decorated, then:

(1) distribute its decorations on the new edges as in Figure 3.29, if e is one of the depicted cases and denote by $\mathrm{cp}_e(D)$ the diagram obtained from D applying the previous steps;

(2) $\mathrm{cp}_e(D) = F(\mathrm{cp}_{F(e)}(F(D)))$ otherwise.

It is not hard to see that given an ALT-diagram D with a suitable edge e, either e or $F(e)$ appears in Figure 3.29. In Appendix 5.3.2 some detailed examples of applications of the cp-operations are given. In Figure 3.30, we develop an application of the cp-operation for a suitable edge not appearing in Figure 3.29.

Now we show that the newly obtained diagram $\mathrm{cp}_e(D)$ is admissible and enjoys a uniqueness property.

Lemma 3.7.8. *Let $e = \{i, i+1\}$ be a suitable edge of an ALT-diagram D in $\mathtt{ad}(\widetilde{D}_{n+2})$. Then*

1. $\mathrm{cp}_e(D)$ *is admissible;*

2. $D = D_0\,\mathrm{cp}_e(D)$ *if e is of the form* $(\overset{\bullet}{\smile})$, $D = D_{n+2}\,\mathrm{cp}_e(D)$ *if e is of the form* $(\overset{\circ}{\smile})$, $D = D_i\,\mathrm{cp}_e(D)$ *otherwise.*

3. $\ell(\mathrm{cp}_e(D)) = \ell(D) - 1$.

Proof. By (cp2), the edges of $\mathrm{cp}_e(D)$ do not intersect. It is not difficult to verify that all the conditions in Definition 3.6.1 hold. In fact, in $\mathrm{cp}_e(D)$ no new loops occur, so (A1) is satisfied. In Figure 3.29, we describe case by case how the decorations distribute

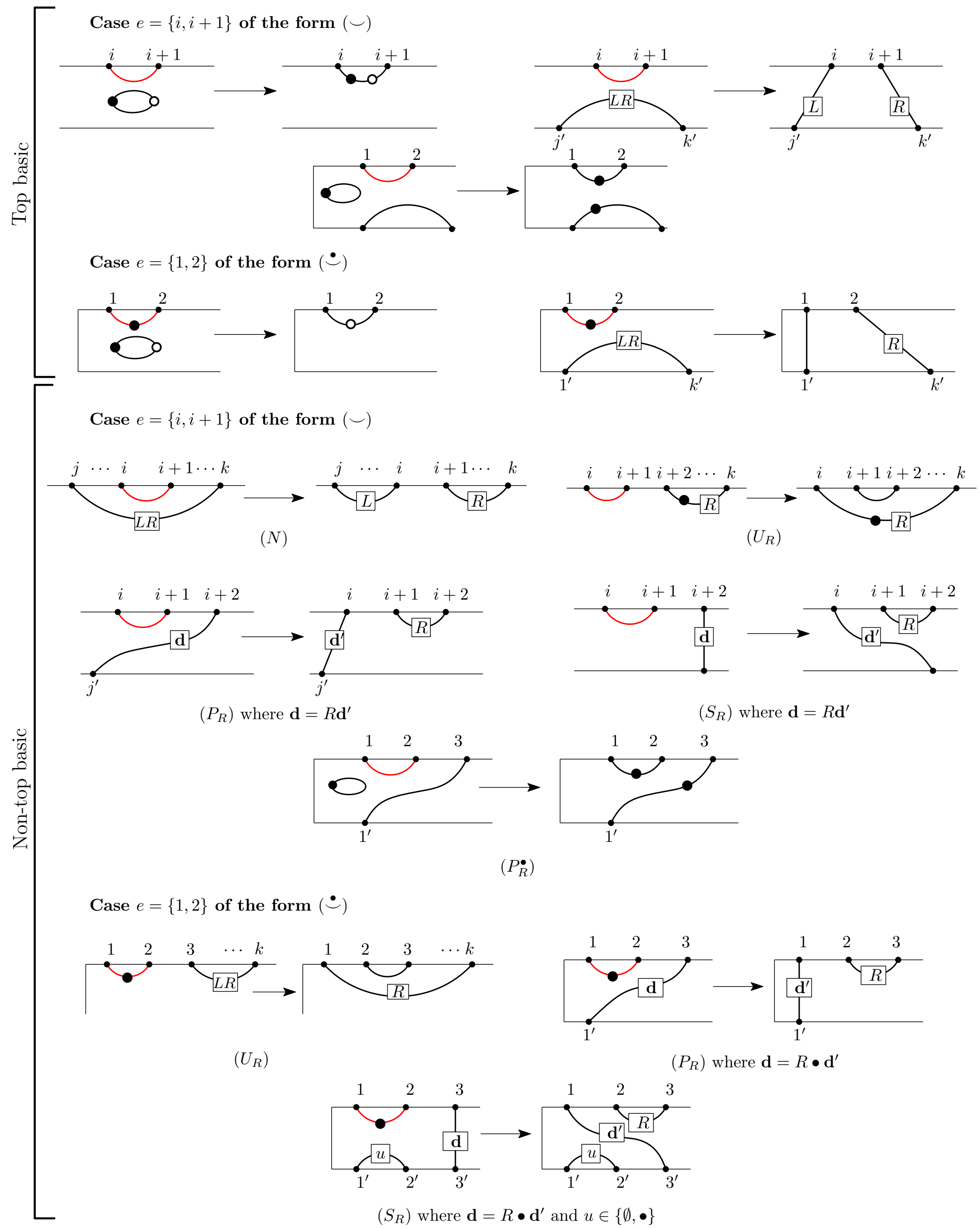


Figure 3.29: Local cp-operations, where $L \in \{\emptyset, \bullet\}$ and $R \in \{\emptyset, \circ\}$.

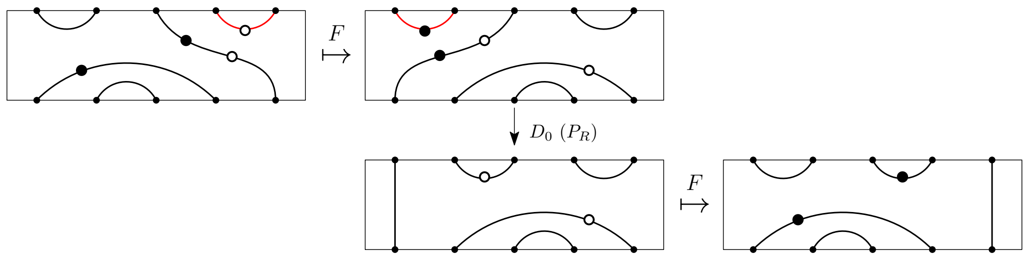


Figure 3.30: Example of a cp-operation for a simple edge e of type P_L. First conjugate the diagram, then apply the cp-operation to $F(e)$ (of type P_R) and finally apply again the conjugate.

in the diagram after the cp-operation. In these procedures, either the number of the decorations is the same, or it varies of 2, so (A2) holds. Finally, in the situations in Figure 3.29 where $\{1, 1'\}$ occurs in $\mathrm{cp}_e(D)$ or $\mathbf{a}(\mathrm{cp}_e(D)) = 1$, (A3) and (A4) hold. By diagram concatenation, (2) easily follows.

By Lemma 3.6.14, $\ell(F(D)) = \ell(D)$. Moreover, by (cp3) in Definition 3.7.7, it suffices to consider the suitable edges in Figure 3.29 since each remaining case is obtained as a conjugate of one of these.

Statement (3) follows from a case-by-case verification based on the computation of the difference between the lengths of D and $\mathrm{cp}_e(D)$. We only need to consider the edges involving the nodes:

(e_1) i and $i+1$ when $i \geq 1$ and e is of the form ($\smile$);

(e_2) 1 and 2 when $i = 1$ and e is of the form ($\overset{\bullet}{\smile}$);

since all the other edges in D and $\mathrm{cp}_e(D)$ coincide. More precisely, one of the following occurs:

(a) $\mathsf{w}(\mathrm{cp}_e(D)) = \mathsf{w}(D)$ and $\nu(\mathrm{cp}_e(D)) = \nu(D) - 2$;

(b) $\mathsf{w}(\mathrm{cp}_e(D)) = \mathsf{w}(D) - 2$ and $\nu(\mathrm{cp}_e(D)) = \nu(D) + 2$;

(c) $\mathsf{w}(\mathrm{cp}_e(D)) = \mathsf{w}(D) - 1$ and $\nu(\mathrm{cp}_e(D)) = \nu(D)$,

from which $\ell(\mathrm{cp}_e(D)) = \ell(D) - 1$, proving the result. For instance, case (c) occurs only when e is of type P^* or when in a top basic diagram there is a loop edge.

The computations for the cases $N, U_R, P_R, P_R^{\bullet}, S_R$ are detailed in Table 5.1 in the Appendix 5.3.2. Now we analyse explicitly the top basic cases referring to Figure 3.29. If D has a $\mathcal{L}_{\bullet}^{\circ}$, then $\mathsf{w}(\mathcal{L}_{\bullet}^{\circ})_D = n+1$ while $\mathsf{w}(i,i+1)_{\mathrm{cp}_e(D)} = n$ and $\nu(\mathrm{cp}_e(D)) = \nu(D)$. If D has a $\mathcal{L}_{\bullet}$, then $\nu(D) = \nu(\mathrm{cp}_e(D))$ and $\mathsf{w}(D) = \mathsf{w}(\mathrm{cp}_e(D)) + 1$ since $\mathsf{w}(\mathcal{L}_{\bullet}) = 1$ and the other edges have weight equal to zero. In the second case of the first row, if $\mathbf{d} = \bullet^{\alpha}\circ^{\beta}$, where $\alpha, \beta \in \{0,1\}$, then $\mathsf{w}(j',k')_D = \alpha(j-1) + \beta(n+2-k)$, while $\mathsf{w}(i,j')_{\mathrm{cp}_e(D)} = \alpha(j-1)$ and $\mathsf{w}(i+1,k')_{\mathrm{cp}_e(D)} = \beta(n+2-k)$. Hence $\mathsf{w}(D) = \mathsf{w}(\mathrm{cp}_e(D))$ and $\nu(D) = \nu(\mathrm{cp}_e(D)) + 2$ since the vertical line $i+1/2$ in $\mathrm{cp}_e(D)$ does not intersect the edges $\{i,i+1\}$ and $\{j',k'\}$ as in D. □

We are able to prove the following.

Proposition 3.7.9. *Let e be a suitable edge of an ALT-diagram D. Then $\mathrm{cp}_e(D)$ is the unique admissible diagram satisfying the conditions (1), (2) and (3) of Lemma 3.7.8.*

Proof. As in the proof of Lemma 3.7.8, by Definitions 3.7.7 and 3.6.9, it is enough to consider the situations described in Figure 3.29. We first assume that $e = \{i,i+1\}$ is undecorated and let D' be an admissible diagram such that $D = D_iD'$ and $\ell(D') = \ell(D) - 1$. Now we show that $sh(D')$ must be equal to $sh(\mathrm{cp}_e(D))$.

If D is a top basic diagram, then we have the following three situations.

(B1) Suppose a $\mathcal{L}_{\bullet}^{\circ}$ occurs in D. If $sh(D') \neq sh(\mathrm{cp}_e(D))$, then the edge $\{i,i+1\}$ is not in D', and since D is top basic with a $\mathcal{L}_{\bullet}^{\circ}$, the two nodes a and b connected respectively to i and $i+1$ in D' are on the north face, and $b < a < i$ or $i+1 < b < a$. In one case, $\{i,a\}$ and $\{b,i+1\}$ intersect the vertical line $(i-1)+1/2$, while in the other, $\{i,a\}$ and $\{i+1,b\}$ intersect the vertical line $(i+1)+1/2$, so $\nu(D') > \nu(D)$. These two edges have weight 0. In fact they are all undecorated, except $\{1,i+1\}$ which is decorated with a $\bullet$ when $\{b,a\} = \{1,2\}$ in D is decorated with a $\bullet$, or $\{i,n+2\}$ which is decorated with a $\circ$ when $\{b,a\} = \{n+1,n+2\}$ in D is decorated with a $\circ$. Note also that the number of $\mathcal{L}_{\bullet}^{\circ}$ is the same in D and D'. It follows that $\mathsf{w}(D') = \mathsf{w}(D)$, so $\ell(D') \geq \ell(D) + 1$, against the hypothesis. Hence $sh(D') = sh(\mathrm{cp}_e(D))$.

(B2) Suppose that a $\mathcal{L}_{\bullet}$ appears in D. If $sh(D') \neq sh(\mathrm{cp}_e(D))$ then the edge $\{1,2\}$ does not occur in D'. Let a^* and b^* be the nodes in D' connected to 1 and 2 respectively. If $a, b > 2$, both the edges $\{1, a^*\}$ and $\{2, b^*\}$ intersect the vertical line $2 + 1/2$. Note that D' necessarily has a $\mathcal{L}_\bullet$ and so it cannot contain any other $\bullet$ decoration. Moreover note that $\{1, a^*\}$ is decorated with a $\circ$ if and only if $\{b^*, a^*\} = \{n+1, n+2\}$ is decorated with $\circ$ in D. The edge $\{2, b^*\}$ is decorated with a $\circ$ if and only if $\{a^*, b^*\} = \{a', b'\}$ in D is decorated with a $\circ$. Therefore, by Remark 3.6.11, $\mathsf{w}(2, b')_{D'} = \mathsf{w}(a', b')_D$. Hence $\ell(D') > \ell(D)$ against the hypothesis and $sh(D') = sh(\mathrm{cp}_e(D))$. If $a^* = 1'$, then D' is a LP-diagram with $j_\ell \geq 2$. It is easy to see that $\ell(\widehat{D'}) = \ell(D) - 2$, hence $\ell(D') = 2j_\ell - 1 + \ell(\widehat{D'}) = 2j_\ell - 1 + \ell(D) - 2 \geq \ell(D) + 1$, against the hypothesis, so also in this case $sh(D') = sh(\mathrm{cp}_e(D))$.

(B3) Now, suppose that there are no loop edges in D; see the first row of Figure 3.29, right. If $sh(D') \neq sh(\mathrm{cp}_e(D))$, then the edge $\{j', k'\}$ occurs in D' and it is decorated as in D. Let a^* and b^* be the nodes in D' connected to i and $i+1$ respectively. As in (B1), $\mathsf{w}(D') = \mathsf{w}(D)$ and $a, b < i$ (respectively $a, b > i+1$), so the edges $\{a^*, i\}$, $\{b^*, i+1\}$ (respectively $\{i, a^*\}$, $\{i+1, b^*\}$) intersect the vertical line $(i-1) + 1/2$ (respectively $(i+1) + 1/2$), then $\ell(D') \geq \ell(D) + 1$, against the hypothesis. Hence again $sh(D') = sh(\mathrm{cp}_e(D))$.

Now assume D is not top basic. There are several cases to consider.

(N) Let e be of type N. Suppose by contradiction that $sh(\mathrm{cp}_e(D)) \neq sh(D')$. Then at least one of the two edges $\{j, i\}$ and $\{i+1, k\}$ is not in D', and suppose $\{j, i\}$ is not in D' (the other case being analogous). Set $t^* \neq i$ the node connected to j in D'. Since j and t^* are not involved in the product $D_i D' = D$, the edge $\{j, t^*\}$ must occur and be decorated as in D and so $t^* = k$. If in D', i is connected to $a \neq i+1$ and $i+1$ is connected to $b \neq i$, then the edge $\{a, b\}$ is in $D = D_i D'$, therefore either $a, b < i$ or $a, b > i+1$ since $\{j, k\}$ is the neighbor of e. In the two situations, either $\{a, i\}$ and $\{b, i+1\}$ intersect the vertical line $(i-1) + 1/2$ or $\{i, a\}$ and $\{i+1, b\}$ intersect the vertical line $(i+1) + 1/2$. Since such edges are undecorated, in both cases, $\ell(D') \geq \ell(D) + 1$ against the hypothesis. Hence $sh(D') = sh(\mathrm{cp}_e(D))$.

(P) Let e be of type P_R. In D', for parity reasons, j' is not connected to $i+1$. As before, suppose by contradiction that $sh(D') \neq sh(\mathrm{cp}_e(D))$, then at least one edge between $\{i+1, i+2\}$ and $\{i, j'\}$ is not in D'. Suppose $\{i+1, i+2\}$ is not in D'

(the other case being analogous), then the edge $\{i+2, j'\}$ occurs in D'. In D', i is connected to a node $a^* \neq i+1$ and $i+1$ is connected to a node $b^* \neq i$, with $a^*, b^* < i$. Then the vertical line $(i-1)+1/2$ does not intersect the edge $\{a^*, b^*\}$ in D while it intersects both edges $\{a^*, i\}$ and $\{b^*, i+1\}$ in D'. Furthermore, the edge $\{a^*, b^*\}$ in D could be decorated with at most a $\bullet$. In this case, the $\bullet$ should appear in $\{a^*, i\}$ or $\{b^*, i+1\}$ in D', and by the definition of weight, one can check that $\mathsf{w}(a^*, b^*)_D \leq \mathsf{w}(a^*, i)_{D'} + \mathsf{w}(b^*, i+1)_{D'}$ leading to $\ell(D') \geq \ell(D)+1$ against the hypothesis. Therefore $sh(\mathrm{cp}_e(D)) = sh(D')$.

(U) Let e be of type U_R. As before, suppose by contradiction that $sh(D') \neq sh(\mathrm{cp}_e(D))$, then necessarily one between $\{i, k\}$ and $\{i+1, i+2\}$ does not occur in D'. Suppose that $\{i, k\}$ is not in D' (the other case being analogous). In D', k must be connected to $i+2$ and decorated with $\bullet R$ as in D. Now assume that i is connected to $a \neq i+1$ and $i+1$ to $b \neq i$. Since $\{i+2, k\}$ contains a $\bullet$ this forces $b < a < i$. Now the edges $\{b, i+1\}$ and $\{a, i\}$ intersect the vertical line $(i-1)+1/2$ in D' while no edges in the north face of D intersect this line. Since $\{b, a\}$ can be decorated with at most a $\bullet$, by Remark 3.6.11, $\mathsf{w}(b, a)_D = \mathsf{w}(b, i+1)_{D'}$. Hence $\ell(D') \geq \ell(D)+1$, against the hypothesis, then $sh(D') = sh(\mathrm{cp}_e(D))$.

(S) Let e be of type S_R. As before, suppose by contradiction that $sh(D') \neq sh(\mathrm{cp}_e(D))$, then at least one between $\{i+1, i+2\}$ and $\{i, (i+2)'\}$ does not occur in D'. Suppose that $\{i+1, i+2\}$ is not in D' (the other case being analogous), then the edge $\{i+2, (i+2)'\}$ must occur and be decorated with $\mathbf{d}$ as in D'. The same argument used in U shows that also in this case $\ell(D') \geq \ell(D)+1$, against the hypothesis. Then $sh(D') = sh(\mathrm{cp}_e(D))$.

($P_R^\bullet$) Let $e = \{1, 2\}$ be of type $P_R^\bullet$. Again, suppose by contradiction that $sh(D') \neq sh(\mathrm{cp}_e(D))$. Then $\{1, 2\}$ is not contained in D', thus necessarily the edges $\{1, 1'\}$, $\{2, 3\}$ and a $\mathcal{L}_\bullet$ occur in D'. Now, D' is a LP-diagram with $j_\ell = 2$ hence $\ell(D') = 3 + \ell(\widehat{D'})$. Moreover as in (B2) we have $\ell(\widehat{D'}) = \ell(D) - 2$, so $\ell(D') = \ell(D)+1$, against the hypothesis. Therefore $sh(D') = sh(\mathrm{cp}_e(D))$.

In Lemma 3.7.8, we showed that $\ell(\mathrm{cp}_e(D)) = \ell(D) - 1$ and we just proved that if $D = D_i D'$ and $\ell(D') < \ell(D)$, then $sh(D') = sh(\mathrm{cp}_e(D))$. Now we have to show that among the ways of splitting the decorations on the edges of D' in an admissible way, the unique for which $\ell(D') = \ell(D) - 1$ yields is the one provided by the cp-operation. In all the top basic cases, U_R and $P_R^{\bullet}$ there is only one admissible way. In cases N, P_R and S_R there are at most three possibilities and in those that differ from the cp-operation way, only one decoration can be displaced. In these cases, the total weight increases and the length of D' is not the right one. Hence the only possibility is $D' = \mathrm{cp}_e(D)$.

To complete the proof one should analyze the suitable edge ($\overset{\bullet}{\smile}$). However, the only differences with respect to the treated cases are extra occurrences of $\bullet$, and the proof is analogous. □

Proposition 3.7.10. *Let D be a P-diagram in* $\mathtt{ad}(\widetilde{D}_{n+2})$*. Then there exist a simple edge $e = \{i, i+1\}$ and a unique admissible diagram D' such that $\ell(D') = \ell(D) - 1$ and $D = D_0 D'$ if e is of the form ($\overset{\bullet}{\smile}$), $D = D_{n+2} D'$ if e is of the form ($\overset{\circ}{\smile}$), $D = D_i D'$ otherwise.*

Proof. Assume that D is of type LP. Let us suppose that

($\star$) D has a simple edge $\{j_\ell, j_\ell + 1\}$ of type P_R or $\widehat{D}$ is top basic.

where $\widehat{D}$ is the inner alternating diagram of Definition 3.6.8 and $j_\ell > 1$. Consider the diagram D' obtained from D by deleting $\{j_\ell, j_\ell + 1\}$, cutting the vertical edge $\{j_\ell - 1, (j_\ell - 1)'\}$ and joining $j_\ell - 1$ to j_ℓ and $(j_\ell - 1)'$ to $j_\ell + 1$ and denote this procedure by *K_L-operation* (see Appendix 5.3.2 for an example of application of the K_L-operation). The diagram D' starts from the left with one $\mathcal{L}_\bullet$, $j_\ell - 2 \geq 0$ undecorated vertical edges and satisfies $D = D_{j_\ell} D'$. Note that $\nu_{j_\ell - 1}(D') = \nu_{j_\ell - 1}(D) + 2$ and $\nu_k(D') = \nu_k(D)$ for all $k \neq j_\ell - 1$. It follows that $\ell(\widehat{D}) = \ell(\widehat{D'}) - 1$. When $j_\ell - 2 = 0$, we set $\widehat{D'}$ as D' without the unique $\mathcal{L}_\bullet$. Hence, $\ell(D') = 2(j_\ell - 1) - 1 + \ell(\widehat{D'}) = 2j_\ell - 3 + \ell(\widehat{D}) + 1 = \ell(D) - 1$. It remains to show that D' is unique.

Let D'' be an admissible diagram such that $D = D_{j_\ell} D''$, $\ell(D'') = \ell(D) - 1$ and suppose by contradiction that $sh(D'') \neq sh(D')$. Since the shapes are different, at least one of the two the edges $\{j_\ell - 1, j_\ell\}$ and $\{j_\ell + 1, (j_\ell - 1)'\}$ does not occur in D''. Since D contains $\{j_\ell - 1, (j_\ell - 1)'\}$, and $D = D_{j_\ell} D''$ then such edge has to be also in D'', that turns out to be a LP-diagram having at least $j_\ell - 1$ consecutive vertical edges from the left. If $j_\ell = n + 1$, then either $D'' = D$, or all the edges of D'' from 1 to n are vertical. Both cases are not possible: the first is clear, while in the second we get a non

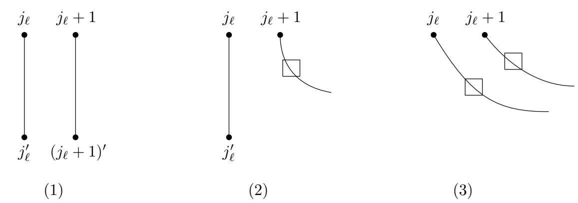


Figure 3.31: Cases in the proof of Proposition 3.7.10. The boxes can be either empty or filled with a $\circ$. In (3), at least one of the two boxes is empty.

admissible diagram since condition (D0) of Section 3.1 is not satisfied. Hence, we have $j_\ell < n+1$.

By Definition 3.6.13, $\ell(D'') = 2k - 1 + \ell(\widehat{D''})$, where $k-1 \geq j_\ell - 1$ is the number of consecutive undecorated vertical edges from the left.

Let a^* be the node connected to j_ℓ and b^* the one connected to $j_\ell + 1$. It is clear that D contains the edge joining a^* and b^*. One of the following three cases can occur in D'', as depicted in Figure 3.31:

(1) $a^* = j'_\ell$, $b^* = (j_\ell + 1)'$ and the edges are undecorated;

(2) $a^* = j'_\ell$ and $b > j_\ell + 1$ or $b^* = (j_\ell + 1)'$ and $\{j_\ell + 1, (j_\ell + 1)'\}$ is decorated with a $\circ$;

(3) $a, b > j_\ell + 1$.

First note that if (1) or (3) occurs, then by $(\star)$, $\widehat{D}$ is top basic, since j'_ℓ is not joined to $j_\ell + 2$. In case (1), D'' has at least $j_\ell + 1$ consecutive undecorated vertical edges from the left. It implies that in D, if $k-1 > j_\ell + 1$, the edges leaving the nodes from $j_\ell + 2$ to $k-1$ are all vertical. Moreover, the portion of D starting from the node k is equal to $\widehat{D''}$, so $\ell(\widehat{D}) = \ell(\widehat{D''}) + 1$. Therefore, $\ell(D'') \geq 2(j_\ell + 2) - 1 + \ell(\widehat{D''}) = 2j_\ell + 3 + \ell(\widehat{D}) - 1 = \ell(D) + 3$, which contradicts the hypothesis. In case (2), comparing the lengths of $\widehat{D''}$ and $\widehat{D}$, it is not difficult to see that $\ell(\widehat{D''}) = \ell(\widehat{D}) - 1$. In fact, note that the size of the box of $\widehat{D''}$ is one less the size of the box of $\widehat{D}$, and if the edge $\{j_\ell + 1, b^*\}$ is decorated with a $\circ$, then $\mathsf{w}_{D''}(\{j_\ell + 1, b^*\}) = \mathsf{w}_D(\{j'_\ell, b^*\})$. Hence, $\ell(D'') = 2j_\ell + 1 + \ell(\widehat{D}) - 1 = \ell(D) + 1$, which again contradicts the hypothesis. In case (3), the standard boxes of $\widehat{D}$ and $\widehat{D''}$ have the same size, so similarly to the arguments in the proof of Proposition 3.7.9 for (B3), we have $\nu(\widehat{D''}) \geq \nu(\widehat{D})$, so $\ell(D'') > \ell(D)$.

Hence $sh(D'') = sh(D')$. This shows that $D'' = D'$ since the edges involving j_ℓ and $j_\ell + 1$ in D are undecorated, being $1 < j_\ell < n + 1$.

If $(\star)$ is not satisfied, then by Proposition 3.7.4, in $\widehat{D}$ we can find a suitable edge $\widehat{e}$ of any type except U_R, S_R and $P_R^\bullet$. It implies that $\widehat{e}$ does not involve the node $\widehat{1}$. Now we can apply the cp-operation to $\widehat{D}$ on $\widehat{e}$. Now, the diagram D' obtained by putting side by side from the left one $\mathcal{L}_\bullet$, $j_\ell - 1$ undecorated vertical edges and $\mathrm{cp}_{\widehat{e}}(\widehat{D})$, satisfies the thesis and it is unique by Proposition 3.7.9.

If D is of type RP, $F(D)$ is a P-diagram of type LP and so we just proved that there exist i and D' with $\ell(D') = \ell(D) - 1$, such that $F(D) = D_i D'$, from which $D = F(F(D)) = F(D_i)F(D')$, where $F(D_i) = D_{n+3-i}$.

Finally, if D is of type LRP, the thesis is obtained combining the two procedures above. $\square$

3.8 Factorizations of admissible diagrams

Lemma 3.8.1. *Let D be a PZZ-diagram or a P-diagram with $\mathbf{a}(D) = 1$. Then D can be written as a finite product of simple diagrams.*

Proof. To prove this result, we provide an explicit factorization for any PZZ-diagram and any P-diagram with $\mathbf{a}(D) = 1$. Let D be a PZZ-diagram, $\{i, i+1\}$ the non-propagating edge on the north face and $\{j', (j+1)'\}$ the non-propagating edge on the south face. Define $z_1 := s_0 s_1 s_2 \cdots s_n$ and $z_2 := s_{n+2} s_{n+1} s_n \cdots s_2$. By direct computations, D can be factorized as one of the following products depending on the number $\mathtt{r} := \#\mathcal{L}_\circ$ and on its type, as defined in Definition 3.6.5:

$$\langle \circ\circ \rangle \; D_0^\alpha (D_i^{1-\alpha} D_{i+1} \cdots D_n)^{1-\delta_{i,n+1}} (D_{z_2} D_{z_1})^{\mathtt{r}-1} D_{n+2} D_{n+1} (D_n \cdots D_j^{1-\beta} D_0^\beta)^{1-\delta_{j,n+1}}$$
$$\langle \bullet\circ \rangle \; D_{n+2}^\eta (D_i^{1-\eta} D_{i-1} \cdots D_2)^{1-\delta_{i,1}} D_{z_1} (D_{z_2} D_{z_1})^{\mathtt{r}-1} D_{n+2} D_{n+1} (D_n \cdots D_j^{1-\beta} D_0^\beta)^{1-\delta_{j,n+1}}$$
$$\langle \circ\bullet \rangle \; D_0^\alpha (D_i^{1-\alpha} D_{i+1} \cdots D_n)^{1-\delta_{i,n+1}} (D_{z_2} D_{z_1})^{\mathtt{r}-1} D_{z_2} D_0 D_1 (D_2 \cdots D_j^{1-\epsilon} D_{n+2}^\epsilon)^{1-\delta_{j,1}}$$
$$\langle \bullet\bullet \rangle \; D_{n+2}^\eta (D_i^{1-\eta} D_{i-1} \cdots D_2)^{1-\delta_{i,1}} (D_{z_1} D_{z_2})^{\mathtt{r}} D_0 D_1 (D_2 \cdots D_j^{1-\epsilon} D_{n+2}^\epsilon)^{1-\delta_{j,1}}$$

where $\delta_{x,y}$ is the Kronecker delta and the parameters $\alpha, \beta, \eta, \epsilon$ are defined as follows:

$\alpha = 1$ if $i = 1$ and $\{1, 2\}$ is decorated with a $\bullet$, $\alpha = 0$ otherwise;

$\beta = 1$ if $j = 1$ and $\{1', 2'\}$ is decorated with a $\bullet$, $\beta = 0$ otherwise;

$\eta = 1$ if $i = n+1$ and $\{n+1, n+2\}$ is decorated with a $\circ$, $\eta = 0$ otherwise;

$\epsilon = 1$ if $j = n+1$ and $\{(n+1)', (n+2)'\}$ is decorated with a $\circ$, $\epsilon = 0$ otherwise.

The case where D is a P-diagram with $\mathbf{a}(D) = 1$ is easier. Using the same notation as above, we have that if D is a LP-diagram, then a factorization is of the form

$$D_{n+2}^{\eta} D_i^{1-\eta} D_{i-1} \cdots D_2 D_0 D_1 D_2 \cdots D_j^{1-\epsilon} D_{n+2}^{\epsilon}.$$

If D is a RP-diagram, it has a factorization of the form

$$D_0^{\alpha} D_i^{1-\alpha} D_{i+1} \cdots D_n D_{n+1} D_{n+2} D_n \cdots D_j^{1-\beta} D_0^{\beta}.$$

□

Example 3.8.2. Let D be the PZZ-diagram of type $\langle \bullet \bullet \rangle$ on the left of Figure 3.23, then $D = D_2(D_{z_1} D_{z_2})^2 D_0 D_1 D_2 D_3 D_4 D_6$, where $D_{z_1} = D_0 D_1 D_2 D_3 D_4$ and $D_{z_2} = D_6 D_5 D_4 D_3 D_2$.

Lemma 3.8.3. *Let $w \in \mathrm{FC}(\widetilde{D}_{n+2})$. If $w \in (\mathrm{PZZ})$ then D_w is a PZZ-diagram.*

Proof. The proof follows from a direct computation. We detail only one specific case since the others are very similar. By Definition 2.2.2, we can assume that w has a reduced expression of the form

$$\mathbf{w} = s_i s_{i+1} \cdots s_n (s_{n+1} s_{n+2} s_n \cdots s_2 s_0 s_1 s_2 \cdots s_n)^h s_{n+1} s_{n+2} s_n \cdots s_j,$$

with $1 \leq i, j < n+1$. Let z_1 and z_2 as defined in Lemma 3.8.1, then

$$D_w = D_i D_{i+1} \cdots D_n (D_{z_2} D_{z_1})^h D_{n+1} D_{n+2} D_n \cdots D_j$$

is a PZZ-diagram of type $\langle \circ \circ \rangle$, with $\{i, i+1\}$ as unique non-propagating edge in the north face, $\{j', (j+1)'\}$ as unique non-propagating edge in the south face, with $h+1$ occurrences of $\mathcal{L}_\circ$ and h of $\mathcal{L}_\bullet$. □

Recall the definitions of the (PZZ) heaps H_{Z_1} and H_{Z_2} given in Remark 2.2.4, and the definitions of the PZZ-diagrams Z_1 and Z_2 given in Figure 3.18.

Corollary 3.8.4. *Let $w \in \mathrm{FC}(\widetilde{D}_{n+2})$ and $H(w)$ be its FC heap, then*

(1) $H(w) = H_{Z_1}$ if and only if $D_w = Z_1$;

(2) $H(w) = H_{Z_2}$ if and only if $D_w = Z_2$.

Lemma 3.8.5. *Let $w \in \mathrm{FC}(\widetilde{D}_{n+2})$. If $w \in$ (LP) (respectively (RP), (LRP)) then D_w is a P-diagram of type LP (respectively RP, LRP).*

Proof. Assume that $w \in$ (LP). By Definition 2.2.2, a reduced expression $\mathbf{w}$ of w contains the factor $s_{j_\ell}s_{j_\ell-1}\cdots s_2s_0s_1s_2\cdots s_{j_\ell-1}s_{j_\ell}$. The corresponding product

$$D_{j_\ell}D_{j_\ell-1}\cdots D_2D_0D_1D_2\cdots D_{j_\ell-1}D_{j_\ell}$$

is a diagram with the leftmost $j_\ell - 1$ edges vertical and undecorated and with a unique $\mathcal{L}_\bullet$. Since there are no other occurrences of s_r with $r \leq j_l$ in $\mathbf{w}$, then such vertical edges appear in D_w too. Furthermore, if $s_{j_\ell+1}$ occurs once in $\mathbf{w}$ then D_w contains either the edge $\{j_\ell, j_\ell+1\}$ or $\{j'_\ell, (j_\ell+1)'\}$. If $\mathbf{w}$ contains two occurrences of $s_{j_\ell+1}$ then, by LP definition, there must be also an occurrence of $s_{j_\ell+2}$ between them, then it can be easily proved that either at least one of the edges leaving j_ℓ and j'_ℓ is a non-propagating edge or D_w has the vertical edge $\{j_\ell, j'_\ell\}$ decorated with a $\circ$. Hence, by Definition 3.6.6, D_w is a P-diagram of type LP. Analogous arguments can be applied for the two other cases. □

Lemma 3.8.6. *Let $w \in \mathrm{FC}(\widetilde{D}_{n+2})$. If $w \in$ (ALT) then D_w is an ALT-diagram.*

Proof. Suppose that D_w is a P-diagram, then by Corollary 3.4.4 there exists a reduced expression of w that contains at least one of the factors $s_2s_0s_1s_2$ and $s_ns_{n+1}s_{n+2}s_n$ and that is not possible since $w \in$ (ALT).

Now suppose D_w is a PZZ-diagram. By the argument right above for the P-diagrams, D_w cannot contain the vertical edges $\{1, 1'\}$ and $\{n+2, (n+2)'\}$. Hence the non-propagating edges of D_w are either $\{1, 2\}$ and $\{(n+1)', (n+2)'\}$ or $\{n+1, n+2\}$ and $\{1', 2'\}$. If $\#\mathcal{L}_\bullet \geq 2$ (respectively $\#\mathcal{L}_\circ \geq 2$), by Corollary 3.4.4 a reduced expression of w has a factor $b = s_0s_1a_1s_{n+1}s_{n+2}a_2s_0s_1$ (respectively $b' = s_{n+1}s_{n+2}a_1s_0s_1a_2s_{n+1}s_{n+2}$), and by the argument in the proof of Corollary 3.8.4, b (respectively b') must contain the factor $s_ns_{n+1}s_{n+2}s_n$ (respectively $s_2s_0s_1s_2$). Both cases cannot occur since $w \in$ (ALT).

Finally, suppose $\#\mathcal{L}_\bullet = \#\mathcal{L}_\circ = 1$. By Corollary 3.8.4, $D_w \neq Z_1, Z_2$, since by Definition 2.2.2, $H_{Z_1}, H_{Z_2} \notin$ (ALT), hence there are two cases left. If $\{1, 2\}$ (respectively

$\{(n+1)',(n+2)'\}$) is the non-propagating edge then D_w is of type $\langle \circ \bullet \rangle$ (respectively $\langle \bullet \circ \rangle$) and w must have both factors $s_2s_0s_1s_2$ and $s_ns_{n+1}s_{n+2}s_n$ which are not allowed.

Therefore D_w has to be an ALT-diagram.

□

Now we prove that the length of D_w as defined in Definition 3.6.13 is consistent with the length of w.

Denote by $\mu_k(e)$ the number of intersections of e with the vertical line $k+1/2$ in $\mathcal{C}(D_{\mathbf{w}})$ and set

$$\mu_k(\mathcal{C}(D_{\mathbf{w}})) := \sum_e \mu_k(e) \quad \text{and} \quad \mu(e) := \sum_{k=1}^{n+1} \mu_k(e), \tag{3.5}$$

where the first sum runs over all the edges of $\mathcal{C}(D_{\mathbf{w}})$. Now, given two different reduced expressions $\mathbf{w}$ and $\mathbf{w}'$ of w we have $\mathcal{C}(D_{\mathbf{w}}) \neq \mathcal{C}(D_{\mathbf{w}'})$, while $\mu_k(\mathcal{C}(D_{\mathbf{w}})) = \mu_k(\mathcal{C}(D_{\mathbf{w}'}))$ for any k. Moreover,

$$\sum_{k=1}^{n+1} \mu_k(\mathcal{C}(D_{\mathbf{w}})) = \sum_e \mu(e) = 2\ell(w). \tag{3.6}$$

Now we are ready to prove the following result.

Proposition 3.8.7. *Let $w \in \mathrm{FC}(\widetilde{D}_{n+2})$, then $\ell(w) = \ell(D_w)$.*

Proof. First assume that D_w is a PZZ-diagram. Then by Lemma 3.8.3, $w \in$ (PZZ). As in the proof of Lemma 3.8.3, we can suppose that

$$\mathbf{w} = s_is_{i+1}\cdots s_n\big(s_{n+1}s_{n+2}s_n\cdots s_2s_0s_1s_2\cdots s_n\big)^h s_{n+1}s_{n+2}s_n\cdots s_j,$$

is a reduced expression of w, with $1 \leq i,j < n+1$. We have that

$$\ell(w) = 2n-i-j+4+(2n+2)h = 3-i+n-j+(2h+1)(n+1) = \ell(D_w)$$

since $\#\mathcal{L}_\bullet = h$ and $\#\mathcal{L}_\circ = h+1$ (Definition 3.6.13).

Now assume D_w is an ALT-diagram and consider a decorated edge e in $\mathcal{C}(D_{\mathbf{w}})$. We now show that

$$\frac{1}{2}(\mu(e)-j+i) = \mathsf{w}(e), \tag{3.7}$$

where $e = \{i^*, j^*\}$ and $i \leq j$.

If e is a non-propagating edge $\{i, j\}$ on the north face decorated with both $\bullet$ and $\circ$ (see Figure 3.32(A)), then $\mathsf{w}(e) = n + 1 - j + i$. Furthermore, $\frac{1}{2}(\mu(e) - j + i) = \frac{1}{2}(\sum_{k=1}^{i-1} \mu_k(e) + \sum_{k=j}^{n+1} \mu_k(e)) = i - 1 + n + 1 - j + 1 = n + 1 - j + i = \mathsf{w}(e)$. Now if e is a propagating edge $\{i, j'\}$ with $i < j$ and is decorated with a $\bullet$ followed by $(\circ\,\bullet)^k$, where $k \in \mathbb{N}_0$ (see Figure 3.32(B)), then, one can verify that $\frac{1}{2}(\mu(e) - j + i) = i - 1 + k(n+1) = \mathsf{w}(e)$. Similar computations show that (3.7) holds if e is decorated in a different way. Moreover, when $e = \{i, j'\}$ with $j < i$, (3.7) holds with i and j exchanged.

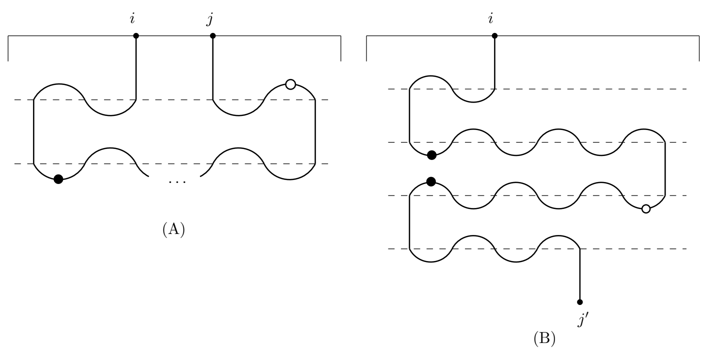


Figure 3.32: Edges in concrete simple representations.

A quick check shows that $\sum_e \mu(e) = 2 \sum_e \mathsf{w}(e) + \sum_{h=1}^{n+1} \nu_h(D_w)$, where the sum on the left hand side runs over all the edges of $\mathcal{C}(D_{\mathbf{w}})$, while the first summand on the right hand side runs over all the edges of D_w, loop edges included. From Definition 3.6.13, Definition 3.4.2 and (3.6), it follows that $\ell(D_w) = \frac{1}{2} \sum_e \mu(e) = \ell(w)$.

Finally, assume that D_w is a LP-diagram. Then a direct computation shows that $\sum_e \mu(e)$ is equal to $4j_\ell - 2$ plus the number of intersections between all the vertical lines and the edges starting and ending on the right of $\{j_\ell - 1, (j_\ell - 1)'\}$ in $\mathcal{C}(D_{\mathbf{w}})$, namely the part of $\mathcal{C}(D_{\mathbf{w}})$ corresponding to $\widehat{D_w}$. Since $D_{\mathbf{w}}$ is an ALT-diagram, this number is equal to $\ell(\widehat{D_w})$. This gives $\ell(w) = \frac{1}{2} \sum_e \mu(e) = 2j_\ell - 1 + \ell(\widehat{D_w}) = \ell(D_w)$ as in Definition 3.6.13(d). A similar computation can be done for the other two cases RP and LRP, and this concludes the proof. $\square$

In Figure 3.12, $4j_\ell - 2 = 10$ counts the intersections of the vertical lines with the black and blue edges of $D_{\mathbf{w}}$ while there are 8 intersections with the red edges, those starting and ending on the right of $\{2, 2'\}$. The total number of intersections is $18 = 2\ell(w)$ as expected.

Proposition 3.8.8. *Every admissible diagram D is of the form D_w where $w \in \mathrm{FC}(\widetilde{D}_{n+2})$.*

Proof. If D is a PZZ or a P-diagram with $\mathbf{a}(D) = 1$, the thesis follows from Lemma 3.8.1, observing that any factorization appearing in the proof of the lemma is associated to a FC element w, as classified in Definition 2.2.2. Hence $D = D_w$.

Assume that D is an ALT-diagram or a P-diagram with $\mathbf{a}(D) > 1$. We proceed by induction on $\ell(D)$. If $\ell(D) = 1$, then it easy to see that it must be $D = D_i$ for some $i \in \{0, 1, \dots, n+2\}$, and trivially $s_i \in \mathrm{FC}(\widetilde{D}_{n+2})$. Suppose the thesis holds for all diagrams of length k and let D be of length $k+1$. By Lemma 3.7.9 and Lemma 3.7.10, there exist an index $i \in \{0, 1, \dots, n+2\}$ and a unique admissible diagram D' such that $D = D_i D'$ and $\ell(D') = k$. Hence $D' = D_{w'}$ with $w' \in \mathrm{FC}(\widetilde{D}_{n+2})$ and $\ell(w') = k$. By letting $w := s_i w'$, we claim that $w \in \mathrm{FC}(\widetilde{D}_{n+2})$ with $\ell(w) = \ell(w') + 1$, from which $D = D_w$ and the result follows. We now prove the claim.

If $\ell(w) = \ell(w') - 1$ then there exists a reduced expression of w' starting with a s_i. Hence $D_i D_{w'}$ contains a δ factor, but D admissible implies D irreducible and so it cannot contain undecorated loops, hence $\ell(w) = \ell(w') + 1$. If $w \notin \mathrm{FC}(\widetilde{D}_{n+2})$ then by Proposition 1.1.8 there exists a reduced expression of w containing a braid relation. After applying all the possible relations between the simple diagrams listed after Figure 3.8, we have

$$D = D_i D_{w'} = \delta^p D_{w''}$$

with $w'' \in \mathrm{FC}(\widetilde{D}_{n+2})$ and $\ell(w'') < \ell(w') + 1$. If $p \neq 0$ we have a contradiction since D is admissible. If $D = D_{w''}$ then $\ell(D) < k + 1$ which is also in contrast with the hypothesis. $\square$

We are now ready to prove that the algebra of admissible diagrams coincides with the algebra generated by the simple diagrams.

Theorem 3.8.9. *The $\mathbb{Z}[\delta]$-algebras $\mathcal{M}[\mathtt{ad}(\widetilde{D}_{n+2})]$ and $\mathbb{D}(\widetilde{D}_{n+2})$ are equal. Moreover, the set of admissible diagrams is a basis for $\mathbb{D}(\widetilde{D}_{n+2})$.*

Proof. We know that $\{b_w \mid w \in \mathrm{FC}(\widetilde{D}_{n+2})\}$ is a basis of $\mathrm{TL}(\widetilde{D}_{n+2})$, hence by Proposition 3.8.8 and the fact that the image of any b_w is an admissible diagram (by Lemmas 3.8.3, 3.8.5 and 3.8.6) it follows that $\mathcal{M}[\mathtt{ad}(\widetilde{D}_{n+2})]$ is a $\mathbb{Z}[\delta]$-subalgebra of $\widehat{\mathcal{P}}^{LR}_{n+2}(\Omega)$. Since the simple diagrams are in $\mathcal{M}[\mathtt{ad}(\widetilde{D}_{n+2})]$ and $\mathbb{D}(\widetilde{D}_{n+2})$ is the smallest algebra containing them, then the two algebras are equal. By Proposition 3.6.4, the set of admissible diagrams is a basis for $\mathcal{M}[\mathtt{ad}(\widetilde{D}_{n+2})]$, then the statement is proved. $\square$

Example 3.8.10. Consider the heaps depicted in Figure 2.2. The image of the (ALT) element via $\tilde{\theta}_D$ is the diagram in Figure 3.20, the image of the (PZZ) element is in Figure 3.17, the image of the (LRP) element is in Figure 3.19.

The results stated in this section can be used to provide an alternative proof of the injectivity of $\widetilde{\theta}_D : \mathrm{TL}(\widetilde{D}_{n+2}) \to \mathbb{D}(\widetilde{D}_{n+2})$, based on a more combinatorial approach. For reasons of length, we have chosen not to include this proof in the present thesis; however, we refer the reader to [6, Theorem 5.14] for further details.

3.9 Type $\widetilde{B}_{n+1}$

We end this chapter by presenting the TL decorated diagrams of type $\widetilde{B}_{n+1}$ and how this representation can be deduced from the one of type $\widetilde{D}_{n+2}$, thanks to an injective map between fully commutative elements. Although this map is very similar to Definition 2.4.6, we need this modified version to manipulate the diagrams. In particular, we describe a new algebra of decorated diagrams called admissible, denoted by $\mathbb{D}(\widetilde{B}_{n+1})$, that turns out to be isomorphic to $\mathrm{TL}(\widetilde{B}_{n+1})$. We list the relevant definitions and we state the main theorems without giving formal proofs, since they are similar to those in type $\widetilde{D}$. For a complete detailed version of this diagrammatic representation, one can refer to [5].

From now on, by abuse of notation we will use same the symbols for elements in $\widetilde{B}_{n+1}$ and $\widetilde{D}_{n+2}$, in $\mathrm{TL}(\widetilde{B}_{n+1})$ and $\mathrm{TL}(\widetilde{D}_{n+2})$, and for diagrams in $\mathbb{D}(\widetilde{B}_{n+1})$ and $\mathbb{D}(\widetilde{D}_{n+2})$.

To define admissible diagrams in type $\widetilde{B}_{n+1}$ we consider the algebra $\mathcal{P}_k^{LR}(\Omega')$ of decorated diagrams introduced in Section 3.1, where the decorations are in the set $\Omega' = \{\bullet, \circ, \vartriangle\}$, with $L = \{\bullet\}$ and $R = \{\circ, \vartriangle\}$. By introducing the following set of reductions, one can show, as in Section 3.1, that the quotient algebra $\widehat{\mathcal{P}}_k^{LR}(\Omega')$ of $\mathcal{P}_k^{LR}(\Omega')$ modulo the relations determined by the reductions in Figure 3.33 is a $\mathbb{Z}[\delta]$-algebra having as a basis the set of irreducible decorated diagrams with decorations in Ω'.

We can now state our main definition.

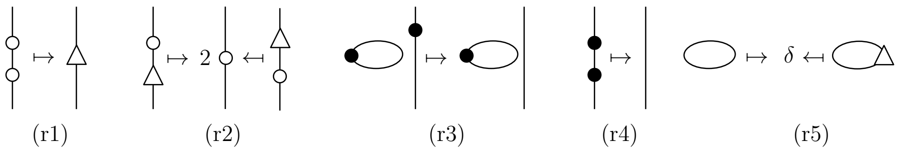


Figure 3.33: The reductions of $\widehat{\mathcal{P}}_k^{LR}(\Omega)$.

Definition 3.9.1. An irreducible diagram $D \in T_{n+2}^{LR}(\Omega')$ is called $\tilde{B}$-*admissible* if it satisfies the following conditions:

(B1) In D loops are all of the same type, either $\mathcal{L}_\bullet$ or $\mathcal{L}_\bullet^\vartriangle$. Moreover, if $\mathbf{a}(D) > 1$ then there is at most one occurrence of $\mathcal{L}_\bullet$.

Figure 3.34: Allowable loops in $\tilde{B}$-admissible diagrams.

Assume $\mathbf{a}(D) > 1$.

(B2) The total number of $\mathcal{L}_\bullet^\vartriangle$ and of $\bullet$ on non-loop edges of D must be even and the total number of $\circ$ is either zero or two.

(B3) If D has no propagating edges, a $\circ$ must appear as last decoration on the non-propagating edges connected to the nodes $n+2$ and $(n+2)'$ respectively.

(B4) If $e = \{1, 1'\}$, then exactly one of the following three conditions holds:

- (a) e is undecorated;
- (b) e is decorated by a single $\vartriangle$;
- (c) e is decorated by an alternating sequence of $\bullet$ and $\vartriangle$ and there are no loops.

(B4)′ If $e = \{n+2, (n+2)'\}$, then exactly one of the following conditions holds:

- (a)′ e is undecorated;
- (b)′ e is decorated by a single $\vartriangle$;

(c)′ e is decorated by an alternating sequence of $\bullet$ and R-decorations such that the first and the last decorations are $\circ$.

Moreover, if $e \neq \{n+2, (n+2)'\}$ is the edge connected to $n+2$ (respectively, $(n+2)'$), then a unique $\circ$ occurs on e.

Assume $\mathbf{a}(D) = 1$.

(B5) The western side of D is equal to one of those depicted in Figure 3.16, as required for type $\widetilde{D}_{n+2}$, see Definition 3.9.1 (A4).

The eastern side of D is equal to one of those depicted in Figure 3.35, where $\mathbf{d}$ represents a sequence of blocks (possibly empty) such that each block is a single $\triangle$. The $\circ$ decoration occurring on the propagating edge on the north (respectively, south) face has the highest (respectively, lowest) relative vertical position of all decorations on any propagating edge and $\mathcal{L}_\bullet$. The situation in Figure 3.35(B) can only occur if no $\bullet$ is in D.

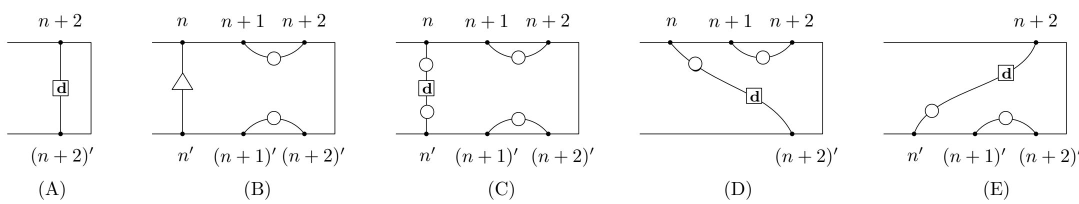


Figure 3.35: Eastern side of an admissible diagram with $\mathbf{a}(D) = 1$.

As in Section 3.1, one can show that the set $\mathtt{ad}(\widetilde{B}_{n+1})$ of admissible diagrams spans the $\mathbb{Z}[\delta]$-submodule $\mathcal{M}[\mathtt{ad}(\widetilde{B}_{n+1})]$, which turns out to be a $\mathbb{Z}[\delta]$-subalgebra of $\widehat{\mathcal{P}}^{LR}_{n+2}(\Omega')$. Moreover as in type $\widetilde{D}_{n+2}$, the set $\mathtt{ad}(\widetilde{B}_{n+1})$ splits in three nonempty disjoint families: the PZZ, the P and the ALT-diagrams.

Definition 3.9.2. Let $D \in \mathtt{ad}(\widetilde{B}_{n+1})$.

(PZZ) We say that D is a *PZZ-diagram* if $\mathbf{a}(D) = 1$ and $\#\mathcal{L}_\bullet, \# \triangle \geq 1$.

(P) D is a *P-diagram* if it satisfies Definition 3.6.6 where any $\mathcal{L}_\circ$ is replaced by a $\triangle$ on the vertical edge $\{n+2, (n+2)'\}$.

(ALT) D is an *ALT-diagram* if D is not a P-diagram or a PZZ-diagram.

To avoid further notation, we denote again by $D_0, \ldots, D_n, D_{n+1}$ the *simple diagrams of type* $\widetilde{B}_{n+1}$, defined as follows. The first $n+1$ diagrams are equal to $D_0, \ldots, D_n$ of type $\widetilde{D}_{n+2}$, while D_{n+1} is equal to D_{n+2} of type $\widetilde{D}_{n+2}$, see Figure 3.8.

Let $\mathbb{D}(\widetilde{B}_{n+1})$ be the $\mathbb{Z}[\delta]$-subalgebra of $\widehat{\mathcal{P}}^{LR}_{n+2}(\Omega')$ generated as a unital algebra by the simple diagrams of type $\widetilde{B}_{n+1}$, then one can show the following result.

Theorem 3.9.3. *The $\mathbb{Z}[\delta]$-algebras $\mathcal{M}[\mathtt{ad}(\widetilde{B}_{n+1})]$ and $\mathbb{D}(\widetilde{B}_{n+1})$ are equal. Moreover, the set of admissible diagrams is a basis for $\mathbb{D}(\widetilde{B}_{n+1})$.*

Consider the $\mathbb{Z}[\delta]$-algebra homomorphism $\tilde{\theta}_B : \mathrm{TL}(\widetilde{B}_{n+1}) \to \mathbb{D}(\widetilde{B}_{n+1})$ defined as usual by $b_i \mapsto D_i$ for all $i = 0, \ldots, n+1$. We will show that the injectivity of $\tilde{\theta}_B$ can be deduced from that of $\tilde{\theta}_D$ using the following maps.

Definition 3.9.4. Let $w \in \mathrm{FC}(\widetilde{B}_{n+1})$, the map $\phi : \mathrm{FC}(\widetilde{B}_{n+1}) \to \mathrm{FC}(\widetilde{D}_{n+2})$ is defined as follows:

a) If no reduced expression of w contains the factor $s_{n+1}s_ns_{n+1}$, consider any $\mathbf{w} = s_{i_1} \cdots s_{i_k} \in \mathcal{R}(w)$, then $\phi(w)$ is the element with a reduced expression obtained by replacing each s_{n+1} in $\mathbf{w}$ by $s_{n+1}s_{n+2}$.

b) If there exists a reduced expression containing a factor $s_{n+1}s_ns_{n+1}$, then choose $\mathbf{w} \in \mathcal{R}(w)$ that contains the maximum number of occurrences of $s_{n+1}s_ns_{n+1}$. Then $\phi(w)$ is the element with a reduced expression obtained by replacing in $\mathbf{w}$ the second occurrence of s_{n+1} by s_{n+2} in any of these factors.

Definition 3.9.5. Let $f_\phi : \mathrm{TL}(\widetilde{B}_{n+1}) \to \mathrm{TL}(\widetilde{D}_{n+2})$ be the map defined by $f_\phi(b_w) = b_{\phi(w)}$ for any $w \in \mathrm{FC}(\widetilde{B}_{n+1})$.

Lemma 3.9.6.

1. *The maps ϕ and f_ϕ are injective.*
2. *If $w \in \mathrm{FC}(\widetilde{B}_{n+1})$ then $sh(D_{\phi(w)}) = sh(D_w)$.*

Proof. The injectivity of ϕ follows directly from its definition, while that of f_ϕ from that of ϕ since $\{b_w \mid w \in \mathrm{FC}(\widetilde{B}_{n+1})\}$ is a basis. Recall that in $sh(D_w)$ loops are not considered, so (2) comes from the definition of ϕ since the first $n+1$ simple diagrams of type $\widetilde{B}_{n+1}$ and $\widetilde{D}_{n+2}$ are all equal and $sh(D_{n+1}) = sh(D_{n+2})$. □

Definition 3.9.7. Let $g : \mathbb{D}(\widetilde{B}_{n+1}) \to \mathbb{D}(\widetilde{D}_{n+2})$ be the map defined as follows. If $D \in \mathbb{D}(\widetilde{B}_{n+1})$, then $g(D)$ has the same shape of D, the same L-decorations of D and the following R-decorations:

(a) none, if D has no R-decorations;

(b) a $\mathcal{L}_\circ$, if D has only two $\circ$ decorations;

(c) a $\circ$ in the same position of each $\triangle$ in D, if D is an ALT or LP-diagram containing an even number of $\triangle$;

(d) a $\circ$ in the same position of each $\triangle$ in D and one $\circ$ on the edge leaving $(n+2)'$ with lowest vertical position, if D is an ALT or LP-diagram containing an odd number of $\triangle$;

(e) a $\mathcal{L}_\circ$ in the same vertical position of the $\triangle$ in D, if D is a RP or LRP-diagram;

(f) a $\mathcal{L}_\circ$ in the same vertical position of each $\triangle$ and an extra $\mathcal{L}_\circ$ with highest (respectively lowest) vertical position if a $\circ$ occurs on the edge leaving $n+2$ (respectively $(n+2)'$) of D, if D is a PZZ-diagram.

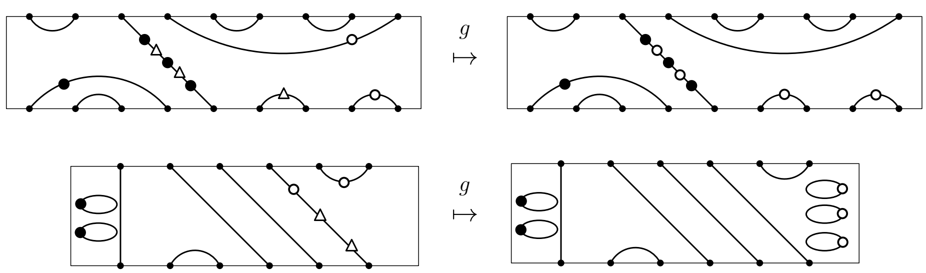


Figure 3.36: Examples of the map g, cases (d) and (f) respectively.

Theorem 3.9.8. *The following diagram*

$$\begin{array}{ccc} \mathrm{TL}(\widetilde{B}_{n+1}) & \xrightarrow{\tilde{\theta}_B} & \mathbb{D}(\widetilde{B}_{n+1}) \\ \big\downarrow{\scriptstyle f_\phi} & & \big\downarrow{\scriptstyle g} \\ \mathrm{TL}(\widetilde{D}_{n+2}) & \xrightarrow{\tilde{\theta}_D} & \mathbb{D}(\widetilde{D}_{n+2}) \end{array}$$

commutes. Furthermore the map $\tilde{\theta}_B$ is an isomorphism.

Proof. Let $w \in \mathrm{FC}(\widetilde{B}_{n+1})$, by Lemma 3.9.6, D_w in $\mathbb{D}(\widetilde{B}_{n+1})$ and $D_{\phi(w)}$ in $\mathbb{D}(\widetilde{D}_{n+2})$ have the same shape. They also have exactly the same number of $\bullet$ decorations placed in the same positions. The differences between the two diagrams concern the loops and the R-decorations. More precisely, if an edge in D_w contains a $\vartriangle$, then the corresponding edge in $D_{\phi(w)}$ contains a $\circ$ in the same position. If the edges leaving $n+2$ and $(n+2)'$ contain a $\circ$ in D_w, then the corresponding edges in $D_{\phi(w)}$ may contain a $\circ$ or not in the same position, depending on the parity of the number of $\vartriangle$. We now detail all the different cases.

(1) If no s_{n+1} occurs in w then $\phi(w) = w$ in $\widetilde{D}_{n+2}$, then $D_{\phi(w)}$ has no $\circ$ decorations since no D_{n+2} occurs. On the other hand, also $\tilde{\theta}_B(b_w) = D_w$ has no R-decorations, so $g(D_w) = D_w = D_{\phi(w)}$.

(2) If a unique s_{n+1} occurs in w, then $D_{n+1}D_{n+2}$ is a factor of $D_{\phi(w)}$ then it is easy to check that $D_{\phi(w)}$ has a single $\mathcal{L}_\circ$ and no other $\circ$ decorations. On the other hand, $\tilde{\theta}_B(b_w) = D_w$ has two $\circ$ decorations, so, by Definition 3.9.7(b), $g(D_w) = D_{\phi(w)}$.

(3) If w is in (ALT) or (LP) and it has $2k$ occurrences of s_{n+1}, then in any factorization of $D_{\phi(w)}$ there are k occurrences of D_{n+1} and D_{n+2} that alternate. In particular, the number of $\circ$ decorations is $2k$ and one of them is the last in the edge leaving $(n+2)'$. Note also that the first decoration in the edge leaving $n+2$ is not a $\circ$. On the other hand, in D_w the first (respectively last) decoration in the edge leaving $n+2$ (respectively $(n+2)'$) is a $\circ$ and the total number of $\vartriangle$ decoration is $2k-1$. Hence, by Definition 3.9.7(d), $g(D_w) = D_{\phi(w)}$.

(4) If w is in (ALT) or (LP) and it has $2k+1$ occurrences of s_{n+1} then in $D_{\phi(w)}$ there are k occurrences of D_{n+2} and $k+1$ of D_{n+1} that alternate. Again, in $D_{\phi(w)}$ the number of $\circ$ decorations is $2k$. On the other hand, D_w has $2k$ occurrences of $\vartriangle$ decorations. Hence, by Definition 3.9.7(c), $g(D_w) = D_{\phi(w)}$.

(5) If $w \in$ (RP) or (LRP), then the factor $D_{j_r} \cdots D_n D_{n+1} D_{n+2} D_n \cdots D_{j_r}$ appears in a factorization of $D_{\phi(w)}$ giving rise to a $\mathcal{L}_\circ$ and to the undecorated vertical edge $\{n+2, (n+2)'\}$, see Corollary 3.4.4. On the other hand, D_w has the vertical edge $\{n+2, (n+2)'\}$ decorated with a $\vartriangle$, then by Definition 3.9.7(e), $g(D_w) = D_{\phi(w)}$.

(6) If $w \in$ (PZZ), then w does not contain factors of the form $s_{n+1}s_ns_{n+1}$. Hence $D_{\phi(w)}$ has as many $D_{n+1}D_{n+2}$ factors as the occurrences of s_{n+1} in w, which means as many $\mathcal{L}_\circ$ as the occurrences of s_{n+1}. At the same time, D_w has as many $\triangle$ as the occurrences of $s_ns_{n+1}s_n$ and a certain number of $\circ$ whenever w starts or ends with a s_{n+1}. Then again, by Definition 3.9.7(f), $g(D_w) = D_{\phi(w)}$.

The map $\tilde{\theta}_B$ is an isomorphism since $g \circ \tilde{\theta}_B = \tilde{\theta}_D \circ f_\phi$, the maps g, f_ϕ are injective and $\tilde{\theta}_D$ is an isomorphism by Theorem 3.4.8. □

Arguments similar to those in this section can be developed and used to describe the diagrammatic representation of $\mathrm{TL}(\widetilde{C}_n)$ introduced by Ernst in [18, 19] and to prove its injectivity.

Chapter 4

A generalized reflection representation of an affine Coxeter group of type $\widetilde{D}$

In the previous chapters, we studied the Coxeter system of type $\widetilde{D}_{n+2}$ in order to remain consistent with the literature on the diagrammatic representation of Temperley–Lieb algebras, ensuring that the decorated diagrams of types $\widetilde{C}_n$, $\widetilde{B}_{n+1}$, and $\widetilde{D}_{n+2}$ can be represented in a box of the same size. In this chapter, we investigate another aspect of the affine Coxeter system of type $\widetilde{D}$. For computational reasons it will be more convenient to work with type $\widetilde{D}_n$ instead of $\widetilde{D}_{n+2}$.

Let (W, S) be a Coxeter system of affine type $\widetilde{D}_n$. The main goal of this chapter is to construct a representation of the affine group W on the k-th symmetric tensor power of V, where V denotes the real vector space of its canonical geometric representation (see Section 1.2). To this end, we introduce the notion of *k-roots* and define a distinguished subset of them, called *canonical k-roots*, which is in bijection with a family of decorated diagrams arising from the diagrammatic representation of the Temperley–Lieb algebra of type $\widetilde{D}_n$ described in Chapter 3. We then extend the concept of admissible diagrams by introducing the *n, k-semidiagrams*, consisting of n horizontally aligned nodes connected by k (possibly decorated) edges. These n, k-semidiagrams are in bijection with the k-roots, and we show that the action of W on the k-roots coincides with its natural action on the corresponding n, k-semidiagrams. Furthermore, we prove that every k-root can be expressed as a non-negative linear combination of canonical k-roots, using the combinatorial properties of n, k-semidiagrams. However, the set of canonical k-roots does not form a basis for the module of k-roots. The underlying reason lies in the rank of the stabilizer of a simple root in W, as discussed in Example 3 of [1, §4]. Consequently, [31, Theorem 2.7] does not hold in the affine case.

This chapter is organized as follows. In Section 4.1, we introduce the *small Temperley–Lieb algebra* as a quotient of the Temperley–Lieb algebra of type $\widetilde{D}_n$. In Section 4.2, we define a W-action on the small Temperley–Lieb algebra and study special reflections that send a basis element of the algebra to its opposite. We give a conjecture to completely characterize such reflections by using the classification of star irreducible fully commutative elements of type $\widetilde{D}_n$ studied in Chapter 2. In Section 4.3, we present the decorated n, k-semidiagrams and the canonical ones, and define the action of W on them. In Section 4.4, we introduce the k-roots and show the bijection with the n, k-semidiagrams (Proposition 4.4.6). In particular, we show that each n, k-semidiagram can be expressed as a linear combination of canonical n, k-semidiagrams with non-negative coefficients by ad hoc substitutions. Thanks to the bijection, this result also holds for the k-roots (Theorem 4.4.10). We conclude this chapter with some remarks about the dimension of the module of n, k-semidiagrams when k is as large as possible (Proposition 4.5.1).

This chapter originates from a project proposed by Professor Richard Green during a visiting period at the University of Colorado Boulder.

4.1 The small Temperley–Lieb algebra of type $\widetilde{D}_n$

From now on, (W, S) is a Coxeter system of type affine $\widetilde{D}_n$, whose Coxeter graph is depicted in Figure 4.1. For more details and combinatorial characterization of this group we refer to [8, §8].

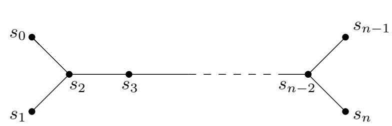


Figure 4.1: Coxeter graph of type $\widetilde{D}_n$.

Consider the group algebra over $\mathbb{Z}$, $\mathcal{H} := \mathbb{Z}[W]$ and the two-sided ideal generated by $J' = \langle sts - st - ts + s + t - 1\rangle$ where $s, t \in S$ and $m_{s,t} = 3$. Define $\mathrm{TL}_n := \mathcal{H}/J'$ the *Temperley–Lieb algebra* of type $\widetilde{D}_n$ over $\mathbb{Z}$, and for all $s \in S$, set

$$b_s := s - 1. \tag{4.1}$$

Then, by [27, Proposition 2.6], TL_n is the $\mathbb{Z}$-algebra generated by $\{b_0, b_1, \ldots, b_n\}$ with defining relations:

(d1) $b_i^2 = -2b_i$ for all $i \in \{0, \ldots, n\}$;

(d2) $b_i b_j = b_j b_i$ if s_i and s_j are not adjacent nodes in the Coxeter graph;

(d3) $b_i b_j b_i = b_i$ if s_i and s_j are adjacent nodes in the Coxeter graph.

If $w \in \mathrm{FC}(W)$ with reduced expression $s_{i_1} s_{i_2} \cdots s_{i_k}$, then $b_w := b_{i_1} b_{i_2} \cdots b_{i_k}$ is well-defined, and $\{b_w \mid w \in \mathrm{FC}(W)\}$ is a basis for TL_n.

Definition 4.1.1. We define the *small Temperley–Lieb algebra of type* $\widetilde{D}_n$, denoted by $\widehat{\mathrm{TL}}_n$, the $\mathbb{Z}$-algebra quotient of TL_n by the following two relations

$$b_0 b_1 = 0 = b_{n-1} b_n. \tag{4.2}$$

Remark 4.1.2. Note that the group algebra $\mathcal{H}$ is the specialization of the Hecke algebra $\mathcal{H}_q(W)$ over the Laurent polynomial ring $\mathbb{Z}[q, q^{-1}]$ in $q = 1$ (see Section 1.1.2). Moreover, TL_n is isomorphic to the generalized Temperley–Lieb algebra $\mathrm{TL}(\widetilde{D}_n)$ in a very specific case. In fact, the Temperley–Lieb algebra of type $\widetilde{D}_n$ is the $\mathbb{Z}[\delta]$-algebra given by the quotient of $\mathcal{H}(W)$ by the two-sided ideal generated by $sts + st + ts + s + t + 1$, for all s, t non-commuting generators, where $\delta = q^{1/2} + q^{-1/2} = 2$ and $b_s = s + 1$. In this work, we consider the twisted-sign version in (4.1), take the quotient ideal J' and set $\delta = -(q^{1/2} + q^{-1/2}) = -2$.

4.2 The inverting reflections

The W-action on $\widehat{\mathrm{TL}}_n$ is defined by

$$s \cdot b_w := (1 + b_s) b_w, \tag{4.3}$$

for $s \in S$, $w \in \mathrm{FC}(W)$. Recall that in Section 1.2, for $\alpha \in \Phi$, we defined $s_\alpha \in T$ as the reflection usu^{-1}, where $s \in S$, $u \in W$, and $\alpha = u(\alpha_s)$. In this section we aim to characterize those reflections $s_\alpha \in T$ such that $s_\alpha \cdot b_w = -b_w$ for some $w \in \mathrm{FC}(W)$.

If $s, t \in S$ are two non-commuting generators, by applying (d1) and (d2), we have

$$\begin{aligned} st \cdot b_s &= (1 + b_s)(1 + b_t)b_s = b_s + b_t b_s + b_s^2 + b_s b_t b_s \\ &= b_s + b_t b_s - 2b_s + b_s \\ &= b_t b_s. \end{aligned}$$

More generally, if $w \in \mathrm{FC}(W)$ and $s \in D_L(w)$, then $b_s b_w = -2b_w$, so from the previous computation it follows that

$$st \cdot b_w = b_t b_w. \tag{4.4}$$

Clearly, if $s \in D_L(w)$, then

$$s \cdot b_w = (1 + b_s)b_w = b_w - 2b_w = -b_w. \tag{4.5}$$

Vice versa, if $s \cdot b_w = -b_w$, then $b_s b_w = -2b_w$ so $s \in D_L(w)$.

Lemma 4.2.1. *Assume $w \in \mathrm{FC}(W)$ such that $s \in D_L(w)$, $t \in D_L(sw)$ and $m_{st} = 3$. If $r_\alpha \cdot b_{sw} = -b_{sw}$ for some $\alpha \in \Phi^+$, then $r_{ts(\alpha)} b_w = -b_w$. If $r_\beta \cdot b_w = -b_w$ for some $\beta \in \Phi^+$, then $r_{st(\beta)} b_{sw} = -b_{sw}$.*

Proof. We know that $st \cdot b_w = b_t b_w$. Now, if we write $w = stw'$ reduced, then $b_t b_w = b_t b_s b_t b_{w'} = b_t b_{w'}$. Hence, $r_\alpha(st \cdot b_w) = r_\alpha b_{tw'} = -b_{tw'}$ by hypothesis. Then, $r_\alpha(st \cdot b_w) = -st \cdot b_w$, which implies $tsr_\alpha st \cdot b_w = -b_w$, where $tsr_\alpha st = r_{ts(\alpha)}$. The computations for the other case are analogous. □

Recall the families of star irreducible fully commutative elements introduced and classified in Section 2.3: $\mathrm{TC}(W), \mathrm{CZ}(W)$, and $\mathrm{K}(W)$. By Theorem 2.3.20, we have that the set of irreducible elements in $\mathrm{FC}(\widetilde{D}_n)$ is partitioned into three families as follows:

$$\mathrm{I}(W) = \mathrm{K}(W) \sqcup \mathrm{CZ}(W) \sqcup \mathrm{TC}(W),$$

where $\mathrm{K}(W) = \emptyset$ if n is odd. Note that by (4.2), if $w \in \mathrm{CZ}(W)$, then $b_w = 0$.

Remark 4.2.2.

(1) Note that if $r_\alpha \cdot b_w = -b_w$, then $supp(r_\alpha) \subseteq supp(w)$. In fact, $b_w = \sum_{u \in W} c_u u$, with $c_u \in \mathbb{Z}$, and $r_\alpha \sum_{u\in W} c_u u = -\sum_{u\in W} c_u u$.

(2) Consider $w \in \mathrm{FC}(W)$ and $t \notin D_R(w)$ such that $wt \in \mathrm{FC}(W)$. If there exists $r_\alpha \cdot b_w = -b_w$, then $r_\alpha \cdot b_{wt} = -b_{wt}$ since $r_\alpha \cdot b_w b_t = (r_\alpha \cdot b_w) b_t$. On the other hand, if there exist $r_\beta \cdot b_{wt} = -b_{wt}$ and $s \in D_R(w)$ such that $m_{s,t} = 3$, then $r_\beta \cdot b_w = r_\beta \cdot b_{wt} b_s = -b_{wt} b_s = -b_w$.

We now state the following conjecture.

Conjecture 4.2.3. *Let $w \in \mathrm{I}(W) \backslash \mathrm{CZ}(W)$, then $r_\alpha \cdot b_w = -b_w$ if and only if $r_\alpha \in D_L(w)$.*

It is clear that if $s \in D_L(w)$, then $s \cdot b_w = -b_w$. Moreover, if $w \in \mathrm{TC}(W)$, then $r_\alpha \cdot b_w = -b_w$ if and only if $r_\alpha \in D_L(w)$. Since $supp(r_\alpha) \subseteq supp(w)$ by Remark 4.2.2(1), and $w \in \mathrm{TC}(W)$, then $\ell(r_\alpha) = 1$ so the thesis yields. However, if $w \in \mathrm{K}(W)$ and $w = u_0 u_1 \cdots u_m$ is its Cartier–Foata normal form, if we prove that $\{\alpha \in \Phi^+ \mid r_\alpha \cdot b_w = -b_w\} = \{\alpha \in \Phi^+ \mid r_\alpha \cdot b_{u_0} = -b_{u_0}\}$, then the thesis would follow by the previous argument on $\mathrm{TC}(W)$.

If Conjecture 4.2.3 is true, the two following statements would follow.

Claim 4.2.4. *Let $w \in \mathrm{FC}(W)$. If w is reducible to $v \in \mathrm{I}(W) \setminus \mathrm{CZ}(W)$, then*

$$|\{\alpha \in \Phi^+ \mid r_\alpha \cdot b_w = -b_w\}| = |D_L(v)|.$$

Proof. By Remark 4.2.2(2), we assume that w is right star irreducible, so v is obtained from w by a series of left star reductions. By Conjecture 4.2.3, $\{r_\alpha \in T \mid r_\alpha \cdot b_v = -b_v\} = D_L(v)$, hence the result follows by Lemma 4.2.1, reversing the left star operations for obtaining v. □

Claim 4.2.5. *Let $w \in \mathrm{FC}(W)$. If s_α and s_β are reflections such that $s_\alpha \cdot b_w = s_\beta \cdot b_w = -b_w$, then α and β are orthogonal.*

Proof. If w is star irreducible, then by Conjecture 4.2.3, the reflections are left descents, hence they are orthogonal. Otherwise, by Lemma 4.4.3 and Theorem 4.2.4, α and β are conjugated to two left descents of a star irreducible element obtained from w by the same element. Finally, since the bilinear form B is W-invariant, then α and β are orthogonal as well. □

4.3 The semidiagrams

Consider the diagrammatic representation of TL_n and let $\mathbb{D}_n$ be the diagram algebra which realizes TL_n (see Definition 3.1.7) . We denote by $\widehat{\mathbb{D}}_n$ the diagrammatic realization of $\widehat{\mathrm{TL}}_n$, obtained as a quotient of $\mathbb{D}_n$ by the following relations

$$\mathcal{L}_\bullet \mapsto 0; \qquad \mathcal{L}_\circ \mapsto 0.$$

Let $\Omega := \{\bullet, \circ\}$ be the decoration set and consider the quotient $\widetilde{\Omega}^* = \Omega^*/\{R_1, R_2\}$ of the free monoid Ω^* by the relations

$$R_1 : \bullet\bullet = \emptyset \quad \text{ and } \quad R_2 : \circ\circ = \emptyset.$$

Definition 4.3.1. Let $k \in \mathbb{N}$ such that $k \leq \lfloor n/2 \rfloor$. We say that a n, k-*semidiagram* is the collection of n horizontally aligned nodes and k edges such that each edge joins two distinct nodes and each node is the endpoint of at most one edge. Moreover, an edge can be decorated with an element of $\widetilde{\Omega}^*$.

Definition 4.3.2. We say that a n, k-semidiagram is *canonical* if it is subjected to the following rules:

(C1) the edges do not cross;

(C2) there is no $\circ$ (respectively, $\bullet$) decoration on the left (respectively, right) of a $\bullet$ (respectively, $\circ$) or of an isolated node, and each edge contains at most two decorations;

(C3) if an edge contains two decorations, then the other edges do not contain more than one decoration;

(C4) if the endpoints of an edge lie between the endpoints of another edge, then the inner edge is undecorated.

Note that a canonical semidiagram might be seen as the top part of some pseudo decorated diagram of the diagrammatic representation of TL_n defined in Chapter 3. Sometimes it happens that we concatenate the diagrams corresponding to b_s and b_w and take the resulting canonical semidiagram multiplied with scalars, if any. When in the diagram of $b_s b_w$ a $\mathcal{L}_{\bullet\circ}$ appears, then we set the associated canonical semidiagram to be the top part of $b_s b_w$ multiplied by 2.

We denote by $\mathbb{SD}_{n,k}$ (respectively, $\mathbb{CSD}_{n,k}$) the $\mathbb{Z}$-module having the n, k-semidiagrams (respectively, the canonical n, k-semidiagrams) as basis, and set

$$\mathbb{SD}_n = \bigoplus_{k \in \mathbb{N}, k \leq \lfloor n/2 \rfloor} \mathbb{SD}_{n,k}.$$

Notation 4.3.3. Let p be a semidiagram, then we denote by $e^d_{i,j}$ the edge in p joining the nodes i and j decorated with $d \in \widetilde{\Omega}^*$. If the edge is undecorated, then we just write $e_{i,j}$. For instance, $e^{\bullet}_{4,7}$ is the edge joining the nodes 4 and 7, decorated with a $\bullet$. Sometimes we regard p as a collection of edges, that is $p = \{e^{d_1}_{j_1,l_1}, \ldots, e^{d_k}_{j_k,l_k}\}$.

Our next goal is to define an action of W on $\mathbb{SD}_n$. Since a semidiagram $p \in \mathbb{SD}_n$ is a collection of (possibly decorated) edges joining distinct nodes, we define the action of $w \in W$ on p by defining the action of a simple reflection s_i over a single edge $e^d_{j,l}$ of p,

$$s_i(p) := \{s_i(e^{d_1}_{j_1,l_1}), \ldots, s_i(e^{d_k}_{j_k,l_k})\},$$

where $s_i(e^d_{j,l})$ is defined as follows. For $i \in [n-1]$, let σ_i be the simple transposition $(i, i+1)$ of the symmetric group S_n. Then we have:

$$s_i(e^d_{j,l}) := \begin{cases} -e_{i,i+1}, & \text{if } j = l-1 = i \text{ and } d = \emptyset; \\ e^{d^{-1}}_{i,i+1}, & \text{if } j = l-1 = i \text{ and } d \neq \emptyset; \\ e^d_{\sigma_i(j),\sigma_i(l)}, & \text{otherwise;} \end{cases}$$

$$s_0(e^d_{j,l}) := \begin{cases} -e^{\bullet}_{1,2}, & \text{if } j = l-1 = 1 \text{ and } d = \bullet; \\ e^{\bullet d^{-1} \bullet}_{1,2}, & \text{if } j = l-1 = 1 \text{ and } d \neq \bullet; \\ e^{\bullet d}_{\sigma_1(j),\sigma_1(l)}, & \text{if } j \in \{1,2\}, l \geq 3 \\ e^d_{\sigma_1(j),\sigma_1(l)}, & \text{otherwise;} \end{cases}$$

$$s_n(e^d_{j,l}) := \begin{cases} -e^{\circ}_{n-1,n}, & \text{if } j = l-1 = n-1 \text{ and } d = \circ; \\ e^{\circ d^{-1}\circ}_{n-1,n}, & \text{if } j = l-1 = n-1 \text{ and } d \neq \circ; \\ e^{d\circ}_{\sigma_{n-1}(j),\sigma_{n-1}(l)}, & \text{if } l \in \{n-1, n\}, j \leq n-2 \\ e^{d}_{\sigma_{n-1}(j),\sigma_{n-1}(l)}, & \text{otherwise.} \end{cases}$$

Note that if $p \in \mathbb{SD}_{n,k}$ for some k, then $w(p) \in \mathbb{SD}_{n,k}$ for all $w \in W$. Note also that from any n, k-semidiagram is possible to obtain a canonical n, k-semidiagram after a series of applications of the action described above.

4.4 The k-roots

As described in Section 1.2.1, let $(W_0, S_0 = \{s_0, s_1, s_2, \ldots, s_{n-1}\})$ be the finite Coxeter system of type D_n underlying the affine Coxeter system (W, S), which acts on a Euclidean space (V_0, B_0), and let Φ_0 be a root system for W_0, with ω its highest root. We extend (V_0, B_0) to a quadratic space (V, B), where $V := V_0 \oplus \mathbb{R}\delta$, and $B(\delta, v) = 0$ for all $v \in V$, where δ is the *imaginary root*. The set of positive roots of W is

$$\Phi^+ = \{k\delta + \alpha, (k+1)\delta - \alpha \mid \alpha \in \Phi_0^+, k \in \mathbb{N}\}$$

and $\Delta := \{\alpha_s \mid s \in S_0\} \cup \{\delta - \omega\}$ is the set of simple roots. We have $S = S_0 \cup \{s_{\delta-\omega}\}$ and $T = \{s_\alpha \mid \alpha \in \Phi^+\}$.

Here we introduce the following notation for all the positive roots in $\Phi_0^+ \setminus \{\alpha_0\}$.

$$\begin{aligned} \alpha_{i,j} &:= \sum_{r=i}^{j-1} \alpha_r, & 1 \leq i < j \leq n, \text{ where } \alpha_{i,i+1} = \alpha_i; \\ \alpha_{0,j} &:= \alpha_0 + \alpha_{2,j}, & 3 \leq j \leq n; \\ \theta_{i,j} &:= \alpha_{1,i} + \alpha_{0,j}, & 2 \leq i < j \leq n. \end{aligned}$$

For instance, $\alpha_{3,7} = \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$; $\alpha_{0,3} = \alpha_0 + \alpha_2$; $\theta_{4,5} = \alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$. The highest root of W_0 is $\omega = \theta_{n-1,n}$ and the imaginary root is $\delta = \theta_{n-1,n} + \alpha_n$. In the next computations, it will be useful to set $\alpha_{0,2} := \alpha_0$.

In particular, a root $\alpha \in \Phi$ is of the form $N\delta \pm \alpha_{i,j}$ or $N\delta \pm \theta_{i,j}$, for $N \in \mathbb{Z}$, and in both cases we say that α is *indexed by* i *and* j.

We now state some technical results about the relations between positive roots of W_0.

Lemma 4.4.1. *If* $0 \leq i < j < m \leq n-1$, *then*

a) $B(\alpha_{i,j}, \alpha_{j,m}) = -1/2$*;*

b) $B(\alpha_{i,j}, \alpha_{i,m}) = B(\alpha_{i,m}, \alpha_{j,m}) = 1/2$, *with* $\{i,j\} \neq \{0,1\}$*;*

c) $B(\alpha_{0,i}, \alpha_{1,i}) = 0$*;*

d) $B(\alpha_{0,i}, \alpha_{1,j}) = -\frac{1}{2}$.

Proof. Point a) follows by observing that $B(\alpha_{i,j}, \alpha_{j,m}) = B(\alpha_{j-1}, \alpha_j) = -\frac{1}{2}$.

Since $\alpha_{i,m} = \alpha_{i,j} + \alpha_{j,m}$, then $B(\alpha_{i,j}, \alpha_{i,m}) = B(\alpha_{i,j}, \alpha_{i,j}) + B(\alpha_{i,j}, \alpha_{j,m}) = 1 - \frac{1}{2} = \frac{1}{2}$. In the same way we show that $B(\alpha_{i,m}, \alpha_{j,m}) = 1/2$, so b) is proved.

Now, recall that $\alpha_{0,2} = \alpha_0$ and $\alpha_{1,2} = \alpha_1$, then if $i = 2$, $B(\alpha_{0,2}, \alpha_{1,2}) = 0$. Suppose $i > 2$, then $B(\alpha_{0,i}, \alpha_{1,i}) = B(\alpha_{0,2}, \alpha_{2,i}) + B(\alpha_{2,i}, \alpha_{2,i}) + B(\alpha_{2,i}, \alpha_{1,2}) = -\frac{1}{2} + 1 - \frac{1}{2} = 0$, by a) and b).

Finally, $B(\alpha_{0,i}, \alpha_{1,j}) = B(\alpha_{0,i}, \alpha_{1,i}) + B(\alpha_{0,i}, \alpha_{i,j}) = -\frac{1}{2} + 0$ by a) and c). □

Lemma 4.4.2. *If* $0 \leq i < j < l < m \leq n-1$, *then* $B(\alpha_{i,j}, \alpha_{l,m}) = B(\alpha_{i,l}, \alpha_{j,m}) = B(\alpha_{i,m}, \alpha_{j,l}) = 0$.

Proof. We just need to make some direct computations. In fact, first we have $B(\alpha_{i,j}, \alpha_{l,m}) = B(\alpha_{j-1}, \alpha_l) = 0$ since $|l - j| \geq 1$.

If we write $\alpha_{i,l} = \alpha_{i,j-1} + \alpha_{j-1} + \alpha_{j,l}$ and $\alpha_{j,m} = \alpha_{j,l} + \alpha_l + \alpha_{l+1,m}$, then by Lemma 4.4.1, $B(\alpha_{i,l}, \alpha_{j,m}) = B(\alpha_{j-1}, \alpha_{j,l}) + B(\alpha_{j,l}, \alpha_{j,l}) + B(\alpha_{j,l}, \alpha_l) = -1/2 + 1 - 1/2 = 0$.

Analogously, $\alpha_{i,m} = \alpha_{i,j-1} + \alpha_{j-1} + \alpha_{j,l} + \alpha_l + \alpha_{l+1,m}$, then $B(\alpha_{i,m}, \alpha_{j,l}) = B(\alpha_{j-1}, \alpha_{j,l}) + B(\alpha_{j,l}, \alpha_{j,l}) + B(\alpha_{j,l}, \alpha_l) = 0$ as above. □

Proposition 4.4.3. *Let* $\alpha, \beta \in \Phi$ *be indexed by* i, j *and* l, m *respectively. If* $|\{i, j, l, m\}| = 4$, *then* $B(\alpha, \beta) = 0$.

Proof. Since $\alpha = N_1\delta \pm \alpha'$ and $\beta = N_2\delta \pm \beta'$, where $\alpha', \beta' \in \Phi_0^+$, then $B(\alpha, \beta) = 0$ if and only if $B(\alpha', \beta') = 0$. Moreover, note that α' is indexed by i, j and β' is indexed by l, m. First suppose that $\alpha' = \alpha_{i,j}$ and $\beta' = \alpha_{l,m}$, then the thesis follows by Lemma 4.4.2.

Assume that $\alpha' = \alpha_{i,j}$ and $\beta' = \theta_{l,m}$. Observe that $\theta_{l,m} = \alpha_{0,l} + \alpha_{1,m}$, hence $B(\alpha_{i,j}, \theta_{l,m}) = B(\alpha_{i,j}, \alpha_{0,l}) + B(\alpha_{i,j}, \alpha_{1,m}) = 0$ by Lemma 4.4.1 if $i = 1$, and by Lemma 4.4.2 otherwise.

Finally, assume that $\alpha' = \theta_{i,j}$ and $\beta' = \theta_{l,m}$. Then $B(\theta_{i,j}, \theta_{l,m}) = B(\alpha_{0,i}, \alpha_{0,j}) + B(\alpha_{0,i}, \alpha_{1,m}) + B(\alpha_{1,j}, \alpha_{0,l}) + B(\alpha_{1,j}, \alpha_{1,m}) = 0$ by Lemma 4.4.1. □

Following [31], let $\alpha, \beta \in V$, then we write $\alpha \vee \beta := \alpha \otimes \beta + \beta \otimes \alpha$ regarded as an element of the symmetric square $S^2(V)$ of V. More generally, let $k \geq 2$ and $\alpha^1, \alpha^2, \ldots, \alpha^k \in V$, then write $\alpha^1 \vee \alpha^2 \vee \ldots \vee \alpha^k$ for the symmetrized k-fold tensor product in $S^k(V)$, defined recursively from the product of two vectors.

Definition 4.4.4. For $k \geq 2$, we say that a *k-root* is a product $\alpha^1 \vee \ldots \vee \alpha^k$, where $\alpha^i \in \Phi$ and for each $i \neq j$, α^i and α^j are indexed by distinct pairs of positive integers in $[n]$.

Remark 4.4.5. By Proposition 4.4.3, if $\alpha^1 \vee \ldots \vee \alpha^k$ is a k-root, then $B(\alpha^i, \alpha^j) = 0$ for each $i \neq j$. Moreover, $\alpha \vee -\beta = -\alpha \vee \beta$, hence each k-root can be written as $\pm\alpha^1 \vee \cdots \vee \alpha^k$, with $\alpha^i \in \Phi^+$ for all $i \in [k]$.

Denote by $C_{n,k}$ the collection of k-roots and $M_{n,k}$ the $\mathbb{Z}$-module having $C_{n,k}$ as a basis. Call $C_{n,k}^+$ the subset of $C_{n,k}$ made of all k-roots of the form $\alpha^1 \vee \cdots \vee \alpha^k$, with $\alpha^i \in \Phi^+$ for all $i \in [k]$.

In particular, W acts on $M_{n,k}$ by

$$s(\alpha^1 \vee \ldots \vee \alpha^k) = s(\alpha^1) \vee \ldots \vee s(\alpha^k), \text{ for all } s \in S.$$

Now we want to show the relation between the k-roots and the n,k-semidiagrams.

Proposition 4.4.6. *There is a bijection between $C_{n,k}^+$ and the set of n,k-semidiagrams.*

Proof. We first assign a (possibly) decorated edge of a n,k-semidiagram to each positive root of W as follows. Consider the map

$$\rho : \Phi^+ \to \{\text{edges of a } n,k\text{-semidiagram}\}$$

defined by

$$\rho(N\delta + \alpha_{i,j}) = e^d_{i,j}, \text{ with } d = (\circ\bullet)^N;$$

$$\rho(N\delta - \alpha_{i,j}) = e^d_{i,j}, \text{ with } d = (\bullet\circ)^N, \ N \geq 1;$$

$$\rho(N\delta + \alpha_{0,j}) = e^d_{1,j}, \text{ with } d = \bullet(\circ\bullet)^N;$$

$$\rho(N\delta - \alpha_{0,j}) = e^d_{1,j}, \text{ with } d = \circ(\bullet\circ)^{N-1}, \ N \geq 1;$$

$$\rho(N\delta + \theta_{i,j}) = e^d_{i,j}, \text{ with } d = \bullet(\circ\bullet)^N;$$

$$\rho(N\delta - \theta_{i,j}) = e^d_{i,j}, \text{ with } d = \circ(\bullet\circ)^{N-1}, \ N \geq 1.$$

Note that ρ is injective and the inverse is well-defined, hence ρ is a bijection. Then the map

$$\rho' : C^+_{n,k} \to \{n, k \text{ -semidiagrams}\}$$
$$\alpha^1 \vee \cdots \vee \alpha^k \mapsto p = \{\rho(\alpha^i)\}_{i=1,\ldots,k}$$

is well-defined. In fact, the k edges of a n, k-semidiagram have distinct endpoints, so the corresponding positive roots are indexed by k distinct pairs of integers. Hence ρ' is a bijection, since ρ is. $\square$

Remark 4.4.7.

(1) Note that if $\alpha \in \Phi^+$ is associated to an edge e, then $s(\alpha)$ is the root associated to $s(e)$ for all $s \in S$. For instance, if $\alpha = N\delta+\alpha_{i,j}$, then by the bijection of Proposition 4.4.6, $e = e^d_{i,j}$, with $d = (\circ\bullet)^N$. Hence, $s_r(\alpha) = \alpha$ if $r \notin \{i, i-1, j, j-1\}$, otherwise, if $j > i+1$, $s_i(\alpha) = \alpha-\alpha_i$, $s_{i-1}(\alpha) = \alpha+\alpha_{i-1}$, $s_j(\alpha) = \alpha+\alpha_j$, $s_{j-1}(\alpha) = \alpha-\alpha_{j-1}$, and in all the cases it is not difficult to prove that each root is associated to $s(e)$ by the definition of the W-action on e.

(2) Let $p = \{e^d_{i,j}\}$ be a canonical semidiagram and let b_w be a diagram with the top part equal to p. Then each root α associated to an edge of p by Proposition 4.4.6 is such that $s_\alpha \cdot b_w = -b_w$.

Example 4.4.8. In Figure 4.2, the $10, 5$-semidiagram is associated to the 5-root: $(\delta - \alpha_{1,4}) \vee \alpha_2 \vee \theta_{5,9} \vee \alpha_6 \vee (\delta + \alpha_{8,10})$.

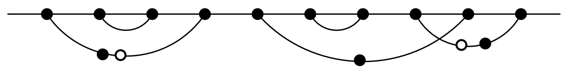


Figure 4.2: A $10, 5$-semidiagram, which is not canonical.

Denote by $B_{n,k}$ the subset of $C^+_{n,k}$ which contains canonical n, k-semidiagrams. We want to show that $B_{n,k}$ is a spanning set for $M_{n,k}$, i.e. we want to express every k-roots as a linear combination of elements in $B_{n,k}$. Equivalently, by Proposition 4.4.6, we aim to express each n, k-semidiagram as a linear combination of canonical n, k-semidiagram.

Proposition 4.4.9. *Let $w \in \mathrm{FC}(W)$ and p be the canonical semidiagram associated to b_w. Then $s(p) = p + \lambda p'$, where $s \in S$, $\lambda \in \mathbb{Z}$, and p' is the canonical semidiagram corresponding to $b_s b_w$.*

Proof. Assume that s exchanges the nodes i and $i+1$, perhaps adding decorations. First we study the cases where just one edge is involved. Suppose that $i+1$ (respectively, i) is an isolated node, and let α be the root associated to the edge joining i (respectively, $i+1$), then $s(\alpha) = \alpha + \alpha_s$. In fact, α is a root of the form $\alpha_{j,i}$ or $\theta_{j,i}$ (respectively, $\alpha_{i+1,h}$ or $\delta - \theta_{i+1,h}$) since p is canonical. Hence, if $\alpha^1 \vee \cdots \vee \alpha \vee \cdots \vee \alpha^k$ is the k-root associated to p, then the k-root associated to $s(p)$ is $\alpha^1 \vee \cdots \vee (\alpha + \alpha_s) \vee \cdots \vee \alpha^k = \alpha^1 \vee \cdots \vee \alpha \vee \cdots \vee \alpha^k + \alpha^1 \vee \cdots \vee \alpha_s \vee \cdots \vee \alpha^k$, which is immediate to prove that corresponds to $p + p'$.

Now suppose that e is an edge in p joining i and $i+1$. If $e = e_{i,i+1}$, then $s(e) = -e$, so $s(p) = -p$, and $b_s b_w = -2b_w$. If $e = e^d_{i,i+1}$, with $d \in \{\bullet, \circ\}$, then $s(e) = e$ and $s \cdot b_w = b_w$, since $\mathcal{L}_\bullet, \mathcal{L}_\circ \mapsto 0$. Finally, if $e = e^{\bullet\circ}_{i,i+1}$, then the associated root is $\delta - \alpha_i$, so $s(\delta - \alpha_i) = \delta + \alpha_i = (\delta - \alpha_i) + 2\alpha_i$. Since we set that $\mathcal{L}_{\bullet\circ}$ gives rise to a factor 2, then it follows that $s(p) = p'$.

Now let $\alpha, \beta \in \Phi^+$ be indexed by $\{j, i\}$ and $\{i+1, h\}$, respectively, and associated to the edges with endpoints i and $i+1$ by Proposition 4.4.6. We first want to show that $s(\alpha \vee \beta) = \alpha \vee \beta + \alpha_i \vee \gamma$, where γ is indexed by $\{j, h\}$. Call $s(\alpha) = \alpha'$ and $s(\beta) = \beta'$, hence α' is indexed by $\{i+1, j\}$ and β' by $\{i, h\}$. Observe that, according to the form of α and β, $\alpha' = \alpha \pm \alpha_s$ and $\beta' = \beta \pm \alpha_s$. Hence we substitute

$$\alpha' \vee \beta' = (\alpha \pm \alpha_s) \vee (\beta \pm \alpha_s) = \alpha \vee \beta + \alpha_s \vee \gamma, \tag{4.6}$$

where $\gamma = \pm\alpha_s \pm \alpha \pm \beta$. By checking few cases, we have that γ is a positive root indexed by j and h, orthogonal to α_s.

In fact, first assume that $j < i < i+1 < h$, so $s = s_i$. If α' is of the form $\alpha_{j,i+1}$ or $\theta_{j,i+1}$ (respectively, $\delta - \alpha_{j,i+1}$ or $\delta - \theta_{j,i+1}$), then $\alpha' = \alpha + \alpha_i$ (respectively, $\alpha' = \alpha - \alpha_i$). If β' is of the form $\alpha_{i,h}$ or $\delta - \theta_{i,h}$ (respectively, $\delta - \alpha_{i,h}$ or $\theta_{i,h}$), then $\beta' = \beta + \alpha_i$ (respectively, $\beta' = \beta - \alpha_i$).

Now assume that $i+1 < h < j$, then either $s = s_i$, or $s = s_0$. Note that necessarily $\beta = \alpha_{i+1,h}$, hence $\beta' = \beta + \alpha_s$. If α' is of the form $\alpha_{i+1,j}$ or $\delta - \theta_{i+1,j}$ (respectively, $\delta - \alpha_{i+1,j}$ or $\theta_{i+1,j}$), then $\alpha' = \alpha - \alpha_s$ (respectively, $\alpha' = \alpha + \alpha_s$). The case $j < h < i$ is analogous.

In all the cases, by combining the admissible possibilities of $\alpha' \vee \beta'$ in (4.6), γ takes one of the following values:

$$\alpha_{j,h}, \theta_{j,h}, \delta - \alpha_{j,h}, \delta - \theta_{j,h}$$

where j might be 0 when it is defined. In particular, by applying the bijection described in Proposition 4.4.6, it is immediate to prove that γ corresponds to the edge joining j and h (or 1 and h, if $j = 0$) of the canonical semidiagram associated to $b_s b_w$. $\square$

Note that Proposition 4.4.9 implies that when the W-action on a canonical semidiagram is defined, then it is equivalent to the W-action on b_w for some $w \in \mathrm{FC}(W)$ described in Section 4.2.

We are now ready to state our main result and prove that $B_{n,k}$ is a spanning set for the module $M_{n,k}$.

Theorem 4.4.10. *Every k-root in $C^+_{n,k}$ can be expressed as a linear combination of $B_{n,k}$ with non-negative coefficients.*

Proof. By Proposition 4.4.6, it suffices to prove that every n,k-semidiagram is a linear combination of canonical n,k-semidiagrams with non-negative coefficients. By Proposition 4.4.9, it follows directly that a semidiagram p can be expressed as $p = \sum c_{p'} p'$, where p' is a canonical semidiagram and $c_{p'} \in \mathbb{Z}$. Now we need to show that $c_{p'} \geq 0$ for each p'.

First, observe that for $\alpha \in \Phi_0^+$, we might write $N\delta \pm \alpha = N(\delta - \alpha) + (N \pm 1)\alpha$.

Hence, for $\alpha, \beta \in \Phi_0^+$ and $N, M \in \mathbb{N}$, we have

$$\begin{aligned} N\delta \pm \alpha \vee M\delta \pm \beta = & NM(\delta - \alpha) \vee (\delta - \beta) + (N \pm 1)M\alpha \vee (\delta - \beta) \\ & + N(M \pm 1)(\delta - \alpha) \vee \beta + (N \pm 1)(M \pm 1)\alpha \vee \beta \end{aligned}$$

This means that by the bijection of Proposition 4.4.6, we are interested only in the cases when the edges of the corresponding semidiagrams are decorated with at most two elements.

If just one edge is involved, then apply the substitutions depicted in Figure 4.3.

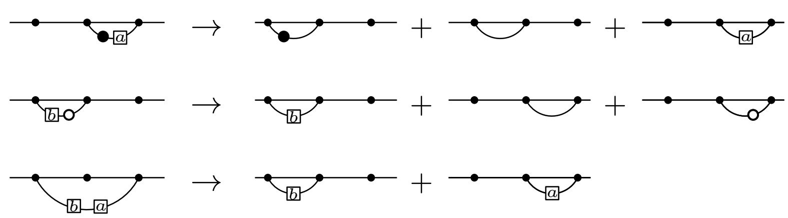


Figure 4.3: Substitutions of Theorem 4.4.10, where $a \in \{\emptyset, \circ\}$ and $b \in \{\emptyset, \bullet\}$.

Otherwise, there are 31 combinations of two edges (or roots) that we need to consider. In Figure 4.7(at the end of the chapter) there are depicted the cases involving both decorations and the substitutions to apply so that the coefficients $c_{p'}$ are all non-negative. The other cases might be treated analogously and are obtained by repeatedly applying Proposition 4.4.9. By a quick check, note that in all the cases, the (2-)roots corresponding to the left hand side are equal to the corresponding linear combinations of (2-)roots on the right hand side.

□

However, the k-roots of $B_{n,k}$ do not form a linearly independent set; for instance, $(\delta - \alpha_j) \vee \alpha_h + \alpha_j \vee \alpha_h = \theta_{j,j+1} \vee \alpha_h + (\delta - \theta_{j,j+1}) \vee \alpha_h$, hence $B_{n,k}$ is a spanning set of $M_{n,k}$ but not a basis.

4.5 Remarks on cardinalities

Denote by $CSD_{n,k}$ the set of canonical n,k-semidiagrams and by $CSD^L_{n,k}$ the subset of $CSD_{n,k}$ made of canonical n,k-semidiagrams which do not contain $\circ$ decorations.

Proposition 4.5.1. *If n is odd, $|CSD_{n,(n-1)/2}| = |B_{n,(n-1)/2}| = 2^{n-1}$. If n is even, then $|CSD_{n,n/2}| = |B_{n,n/2}| = 2^n$.*

Proof. We proceed in steps. First, note that the set of undecorated n, k-semidiagrams is in bijection with $\mathrm{SYT}_{(n-k,k)}$, i.e. the set of standard Young tableaux with shape $\lambda = (n-k, k)$, where in the second row we enter the right endpoints of the edges (see for instance Figure 4.4).

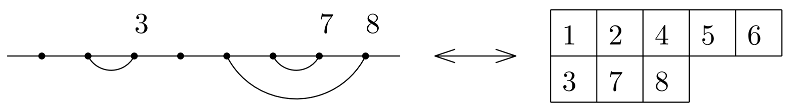


Figure 4.4: The correspondence between an undecorated $8,3$-semidiagram and a Young tableau of shape $(5,3)$.

Hence, by the hook length formula (see [22, Theorem 1]) we get

$$|\,\mathrm{SYT}_{(n-k,k)}\,| = \frac{n!}{\frac{(n-k+1)!}{(n-2k+1)!}(n-2k)!k!} = \frac{n-2k+1}{n-k+1}\binom{n}{k} = \binom{n}{k} - \binom{n}{k-1}.$$

Now observe that the semidiagrams of $CSD^L_{n,k}$ with $h \leq k$ edges decorated with a $\bullet$ are in bijection with the undecorated $n,(k-h)$-semidiagrams, since from an undecorated semidiagram with $k-h$ edges there is only one way to add decorated edges (see Figure 4.5). Hence by the previous step,

$$|CSD^L_{n,k}| = \sum_{h=0}^{k} \binom{n}{h} - \binom{n}{h-1} = \binom{n}{k}.$$

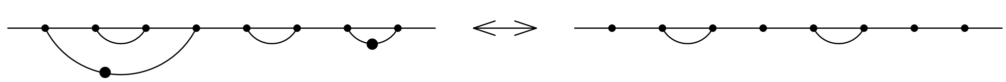


Figure 4.5: Example of the correspondence between a $8,4$-semidiagram with 2 edges decorated with a $\bullet$ and an undecorated $8,2$-semidiagram.

Then, by a similar argument, we have that the canonical n,k-semidiagrams with $h \leq k$ edges decorated only with $\circ$ is in bijection with $CSD^L_{n,(k-h)}$, hence the total number of n,k-semidiagrams whose edges can be decorated with at most one decoration is $\sum_{h=0}^{k}\binom{n}{h}$. In particular, if n is odd and $k=(n-1)/2$, we have

$$|CSD_{n,(n-1)/2}| = \sum_{h=0}^{(n-1)/2}\binom{n}{h} = 2^{n-1}.$$

Note that a semidiagram has at most one edge decorated with both $\bullet$ and $\circ$, and if such an edge appears, it must be n even and $k=n/2$. The diagrams in $CSD_{n,n/2}$ with an edge decorated with both $\bullet$ and $\circ$ are in bijection with $CSD_{n,n/2-1}$ (see Figure 4.6), hence they are $\sum_{h=0}^{n/2-1}\binom{n}{h}$.

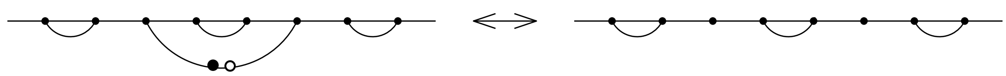


Figure 4.6: Example of the correspondence between a $8,4$-semidiagram with an edge decorated with $\bullet\circ$ and an undecorated $8,3$-semidiagram.

Finally, we sum the total number of semidiagrams whose edges are decorated with at most one decoration, and the total number of semidiagrams with an edge decorated with both decorations:

$$|CSD_{n,n/2}| = \sum_{h=0}^{n/2}\binom{n}{h} + \sum_{h=0}^{n/2-1}\binom{n}{h} = \sum_{h=0}^{n}\binom{n}{k} = 2^{n}.$$

In the end, by Proposition 4.4.6, we have that $|CSD_{n,k}| = |B_{n,k}|$. $\square$

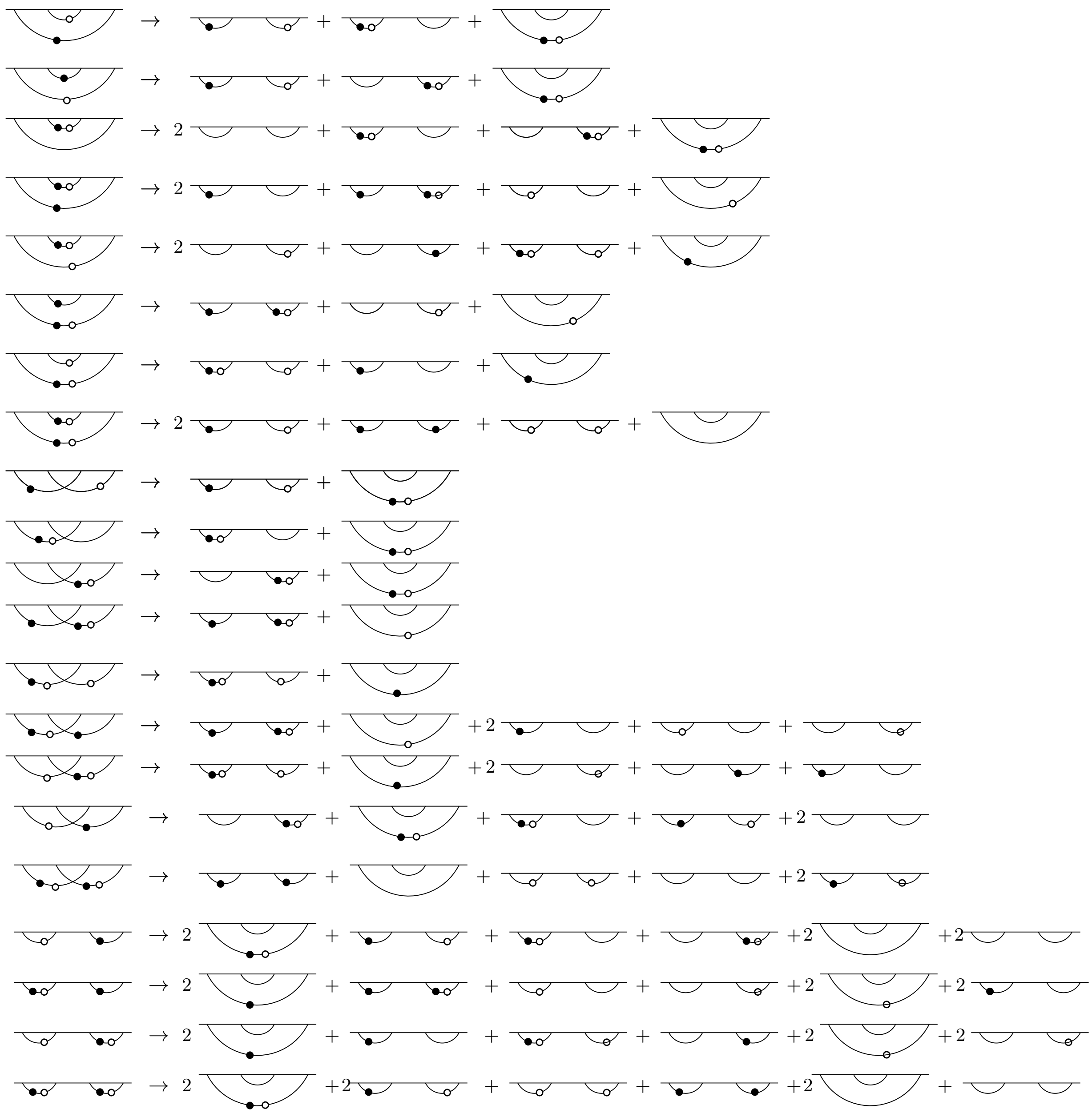


Figure 4.7: Substitutions of 2-roots of Theorem 4.4.10.

Chapter 5

On generating functions associated to reflections and roots in Coxeter systems

In this chapter we discuss two combinatorial problems regarding the *set of reflections* $T = \bigcup_{w \in W} wSw^{-1}$ of a given Coxeter system (W, S):

(1) Is the language of *palindromic reduced words for the reflections* regular?

(2) Are there nice formulas for the Poincaré series of the set T?

For the first question, we introduce *reflection-prefixes*, a class of elements in W arising naturally from palindromic reduced words of reflections, and discuss their properties in relation to the *root poset* and the *dominance order on roots*. Then we show that for any Coxeter systems, that language of reduced words for reflection-prefixes $\mathcal{L}_T$ is regular by using the family of m-canonical automata associated to m-Shi arrangements, see [32, §3.4] and [14], to provide a family of finite deterministic automata that recognized $\mathcal{L}_T$. The problem (1) remains still open.

We give an affirmative answer to (2) in the case of affine Coxeter groups, indeed we provide simple formulas in terms of rational fractions for the Poincaré series of reflections that are obtained from the symmetries of the Hasse diagram of the root poset.

This chapter is organized as follows. In Section 5.1, we recall some facts about the combinatorics of words and roots and, in particular, about two partial orders on the root system of a Coxeter system: the *root poset* and the *dominance order*, which were introduced by Brink and Howlett in their work [11] to show that Coxeter systems are automatic. In Section 5.2, we discuss reflection-prefixes, palindromic reduced words for reflections and automata. In particular, starting from special subsets of W, we build an automa that recognizes the language of reflection-prefixes (Theorem 5.2.20). Finally, in Section 5.3, we focus on affine Coxeter systems and discuss the case of the enumeration of reflections by length. In particular, we provide a nice formula for the generating function of the length over the set of reflections (Theorem 5.3.17).

The results presented in this chapter are part of a joint work in progress with Riccardo Biagioli and Christophe Hohlweg (Université du Québec à Montréal).

5.1 Combinatorics of words and roots in Coxeter systems

Let (W, S) be a Coxeter system. The *(right) weak order* $(W, \leq_R)$ is the poset defined by: $u \leq_R w$ if u is a prefix of w; or in other words, if $\ell(w) = \ell(u^{-1}w) + \ell(u)$. The (right) weak order has for Hasse diagram the (right) Cayley graph for which the edges $\{w, ws\}$ are oriented from w to ws if $\ell(w) < \ell(ws)$.

For $I \subseteq S$, we consider *standard parabolic subgroup* $W_I = \langle I \rangle$. The set of minimal coset representatives of W/W_I is

$$X_I = \{x \in W \mid x \leq_R xs,\ \forall s \in I\}.$$

In particular, any element w has a unique reduced product $w = uv$ with $u \in X_I$ and $v \in W_I$. Moreover, any product uv in W where $u \in X_I$ and $v \in W_I$ is a reduced product; see for instance [8, Proposition 2.4.4]. The following statement is well-known; we include a proof for completeness.

Lemma 5.1.1. *Let $w \in W$ and $I \subseteq D_R(w)$. Then W_I is finite and $w = uw_{\circ,I}$ is a reduced product, where $u \in X_I$ and $w_{\circ,I}$ is the longest element in W_I.*

Proof. We consider the reduced product $w = uv$, where $u \in X_I$ and $v \in W_I$. Since

$I \subseteq D_R(w)$ and by definition of reduced product, we have for all $s \in I$:

$$\ell(w) - 1 = \ell(ws) = \ell(u(vs)) = \ell(u) + \ell(vs).$$

So $\ell(vs) < \ell(v)$ since $\ell(w) = \ell(u) + \ell(v)$. In other words, $D_R(v) = I$, since $v \in W_I$. Therefore, by [8, Proposition 2.3.1 (ii)] applied to W_I, we obtain that W_I is finite and $v = w_{\circ,I}$. □

Björner showed that $(W, \leq_R)$ is a meet-semilattice, meaning that any nonempty subset $X \subseteq W$ admits a *meet*, that is a greatest lower bound denoted by $\bigwedge_R X$; see [8, §3.2]. A subset $X \subseteq W$ is *bounded* in W if there exists $g \in W$ such that $x \leq_R g$ for any $x \in X$. Therefore, any bounded subset $X \subseteq W$ admits a least upper bound $\bigvee_R X$ called the *join* of X:

$$\bigvee_R X = \bigwedge_R \{g \in W \mid x \leq_R g, \forall x \in X\} \in W.$$

A *Garside shadow for* (W, S) is a subset $B \subseteq W$ such that $S \subseteq B$ and

(i) B is closed under join in the right weak order, i.e. if $\emptyset \neq X \subseteq B$ is bounded, then $\bigvee_R X \in B$;

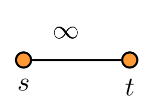
(ii) B is closed under suffixes, i.e. if $w \in B$ and $w = uv$ is reduced, then $v \in B$.

Associated to any Garside shadow B is the *Garside projection*:

$$\begin{aligned} \pi_B : W &\mapsto B \\ w &\mapsto \bigvee_R \{g \in B \mid g \leq_R w\}. \end{aligned}$$

Garside shadows were introduced by Dyer and Hohlweg to show the existence of a finite Garside family in an Artin-Tits monoid, see [15]. In particular, they show that there are finite Garside shadows, whenever S is finite.

Example 5.1.2. Let (W, S) be a Coxeter system of type $I_2(\infty)$.



The subsets $\{e, s, t\}$ and $\{e, s, t, st, sts, ts\}$ are Garside shadows, while $\{e, s, t, sts\}$ is not since it is not closed under suffixes. In Figure 5.1, such subsets are depicted in the bottom portion of the Cayley graph for the weak right order on W.

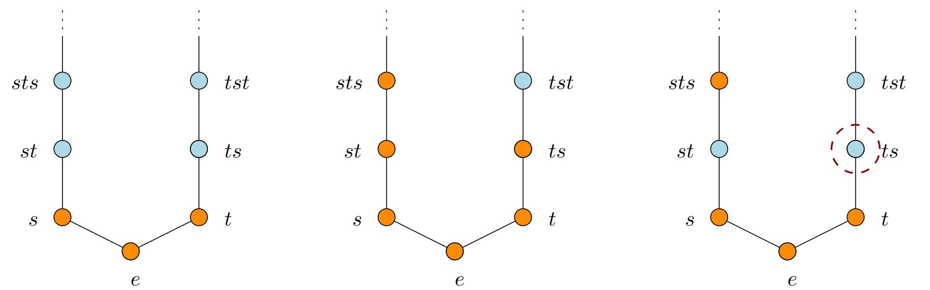


Figure 5.1: Bottom parts of the Cayley graph in Example 5.1.2.

5.1.1 Garside shadows automata

We begin by recalling some basic terminology from automata theory (see [43]). A *finite deterministic automaton* $\mathcal{A}$ over the alphabet S is a quadruple (Q, q_0, F, δ) where:

- Q is a finite set of *states*,
- $q_0 \in Q$ is the *initial state*,
- $F \subseteq Q$ is the set of *final states*,
- δ is a partial function $\delta : Q \times S \to Q$.

If $\delta(q, s) = q'$, we write $q \xrightarrow{s} q'$ and call this a *transition*. An automaton $\mathcal{A}$ can thus be seen as a directed graph on the vertex set Q with edges labeled by elements of S such that for any q, s there is at most one edge with source q and label s. For an automaton $\mathcal{A}$, one naturally extends δ to a partial function $Q \times S^* \to Q$, where S^* denotes the quotient of the free monoid on S by the relations $s^2 = e$ for every $s \in S$. A word $s_1 \cdots s_k \in S^*$ is *accepted* by $\mathcal{A}$ if $\delta(q_0, s_1 \cdots s_k)$ is defined and is in F. The set of all words accepted by $\mathcal{A}$ is the *language recognized* by $\mathcal{A}$ and denoted by $\mathcal{L}(\mathcal{A})$. Languages $L \subseteq S^*$ occurring in this way are called *regular*, and it is a fundamental theorem of Kleene that the class of such languages coincides with the class of rational languages;

see [43, Theorem 2.1].

In [32], Hohlweg, Nadeau and Williams defined for any Garside shadow a deterministic automaton that recognized $\mathcal{R}(W)$, the language of reduced words for (W, S). Let B be a Garside shadow for (W, S). The *Garside shadow automaton* $\mathcal{A}_B$ is the deterministic automaton over the alphabet S defined as follows:

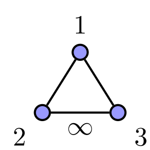
- the set of *states* of $\mathcal{A}_B$ is B;
- the *final states* are the elements in B;
- the *initial state* is the identity e;
- the *transitions* are: $w \in B \xrightarrow{s} \pi_B(sw) \in B$, whenever $\ell(sw) > \ell(w)$.

The automaton $\mathcal{A}_B$ is represented by a directed edge-labeled graph on the vertex set B with edges labeled by elements of S, such that for any $w \in B$ and $s \in S$, there is at most one edge (a transition above) with source w and label s.

Theorem 5.1.3 ([32, Theorem 1.2])**.** *Let B be a finite Garside shadow for (W, S), then $\mathcal{A}_B$ is a finite deterministic automaton that recognizes the language $\mathcal{R}(W)$.*

As a consequence, the authors recover that $\mathcal{R}(W)$ is regular and that the Poincaré series $\mathcal{R}(q)$ is rational. We refer the reader also to [40, 42] for recent developments around Garside shadow automata.

Example 5.1.4. Let (W, S) be a Coxeter system of type



The set $B = W_{\{s_1,s_2\}} \cup W_{\{s_1,s_3\}}$ is a Garside shadow for (W, S) and its associated Garside shadow automaton $\mathcal{A}_B$ is the one represented in Figure 5.2. The transitions by s_1 are colored orange, s_2 in purple, and s_3 in green.

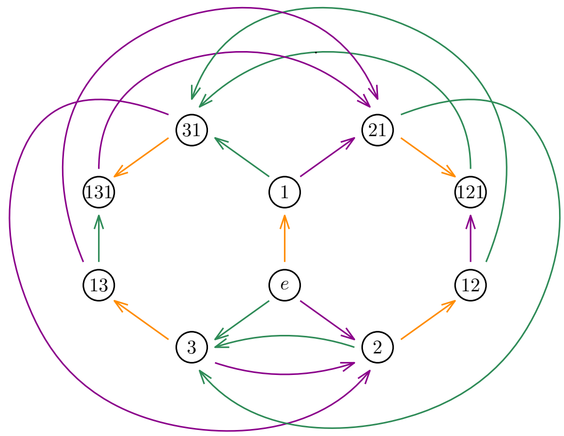


Figure 5.2: The automaton $\mathcal{A}_B$ of Example 5.1.4.

5.1.2 Root systems and inversion sets

Consider a *geometric representation* of (W, S) in a real quadratic space (V, B) in the sense of [15, §2.3] or as described in Section 1.2; the reader may consider for instance the *canonical (or Tits) geometric representation*, as described in [8, §4.1].

The *inversion set of* $w \in W$ is:

$$\Phi(w) = \{\alpha \in \Phi^+ \mid w^{-1}(\alpha) \in \Phi^-\}.$$

It is well-known that if $w = uv$ is a reduced product in W, then

$$\Phi(w) = \Phi(u) \sqcup u(\Phi(v)),$$

see for instance [14, Proposition 2.2]. It follows in particular that:

- $u \leq_R w$ in the weak order if and only if $\Phi(u) \subseteq \Phi(w)$;
- $\Phi(w) = \{\alpha_{s_1}, s_1(\alpha_{s_2}), \ldots, s_1 \cdots s_{k-1}(\alpha_{s_k})\}$ if $w = s_1 \cdots s_k$ is a reduced word for w.

5.1.3 The root poset

We fix a geometric representation for (W, S), with based root system (Φ, Δ). The *depth function* $\mathrm{dp} : \Phi^+ \to \mathbb{N}$ is defined by

$$\mathrm{dp}(\beta) := \min\{\ell(w) \mid w \in W, w(\beta) \in \Delta\}.$$

Remark 5.1.5. In this work we use an analogous definition of the depth of a positive root β with respect to the original one which was given in [11, Definition 1.5], see also [8, Definition 4.6.1], as follows:

$$\mathrm{dp}'(\beta) := \min\{\ell(w) \mid w \in W, w(\beta) \in \Phi^-\}.$$

This is equivalent to $\mathrm{dp}'(\beta) = \min\{\ell(w) \mid w \in W, w(\beta) \in -\Delta\}$. Indeed, if $w = s_1 \cdots s_k$ such that $w(\beta) \in \Phi^-$, then there is $1 \leq i \leq k$ such that $s_i \cdots s_k(\beta) \in \Phi^+$ and $s_{i-1} \cdots s_k(\beta) \in \Phi^-$. Therefore $s_i \cdots s_k(\beta) = \alpha_{s_{i-1}} \in \Delta$, since s_{i-1} is a bijection on $\Phi^+ \setminus \alpha_{s_{i-1}}$. Hence, $\mathrm{dp}'(\beta) = \mathrm{dp}(\beta) + 1$.

The next result is Lemma 4.6.2 in [8] and allows us to understand how the depth of a root changes when acting by a simple reflection $s \in S$.

Lemma 5.1.6. *Let $s \in S$ and $\beta \in \Phi^+ \setminus \{\alpha_s\}$. Then,*

$$\mathrm{dp}(s(\beta)) = \begin{cases} \mathrm{dp}(\beta) - 1, & \textit{if } B(\beta, \alpha_s) > 0; \\ \mathrm{dp}(\beta), & \textit{if } B(\beta, \alpha_s) = 0; \\ \mathrm{dp}(\beta) + 1, & \textit{if } B(\beta, \alpha_s) < 0. \end{cases} \tag{5.1}$$

We equip Φ^+ with a partial order called the *root poset*.

Definition 5.1.7. The *root poset* $(\Phi^+, \leq)$ is defined by: $\alpha \leq \beta$ if and only if there exists $w \in W$ such that $w(\alpha) = \beta$ and $\ell(w) = \mathrm{dp}(\beta) - \mathrm{dp}(\alpha)$.

Here we state well-known properties of the root poset, which are easily deduced from [8, §4.6 & §4.7].

Proposition 5.1.8. *The root poset $(\Phi^+, \leq)$ satisfies:*

(i) $(\Phi^+, \leq)$ *is a graded poset, the rank function is* $\mathrm{dp} : \Phi^+ \to \mathbb{N}$.

(ii) Let $\alpha \in \Phi^+$*, then* $\mathrm{dp}(\alpha) = 0$ *if and only if* $\alpha \in \Delta$*.*

(iii) The cover relations are $\alpha \triangleleft s(\alpha)$ *with* $s \in S$ *and* $\alpha \in \Phi^+ \setminus \{\alpha_s\}$ *such that* $\mathrm{dp}(\alpha) < \mathrm{dp}(s(\alpha))$ *(or equivalently, if* $B(\alpha, \alpha_s) < 0$*).*

(iv) We have $\alpha < \beta$ *if and only if there exist* $s_1, \dots, s_k \in S$ *such that* $\alpha \triangleleft s_1(\alpha) \triangleleft s_2 s_1(\alpha) \triangleleft \dots \triangleleft s_k \cdots s_2 s_1(\alpha) = \beta$*. In this case,* $\mathrm{dp}(\beta) - \mathrm{dp}(\alpha) = k = \ell(s_1 \cdots s_k)$*.*

Definition 5.1.9. Let $s \in S$, $\alpha \in \Phi^+ \setminus \{\alpha_s\}$. A covering edge $\alpha \triangleleft s(\alpha)$ in the root poset $(\Phi^+, \leq)$ is said to be *short* if $-1 < B(\alpha, \alpha_s) < 0$; otherwise, the covering is said to be *long*.

Example 5.1.10. Consider the Coxeter system as in Example 5.1.4. The simple roots are $\Delta = \{\alpha_1, \alpha_2, \alpha_3\}$. In Figure 5.3, the Hasse diagram of the root poset for Φ^+ is depicted up to depth equal to 3. The positive roots are expressed by their coordinates with respect to Δ; for instance, $312 = 3\alpha_1 + \alpha_2 + 2\alpha_3$. Moreover, each edge corresponds to a cover relation, and the colors depend on the simple reflection that is applied: orange corresponds to the simple reflection 1, purple 2 and green 3. Finally, the edges are dashed when the cover is long, otherwise, the cover is short. For instance, $B(\alpha_2 + 2\alpha_3, \alpha_1) = -1/2 - 1 < -1$, hence $012 \triangleleft 312 = s_1(012)$ is a long cover relation, where B is bilinear form of the canonical geometric representation.

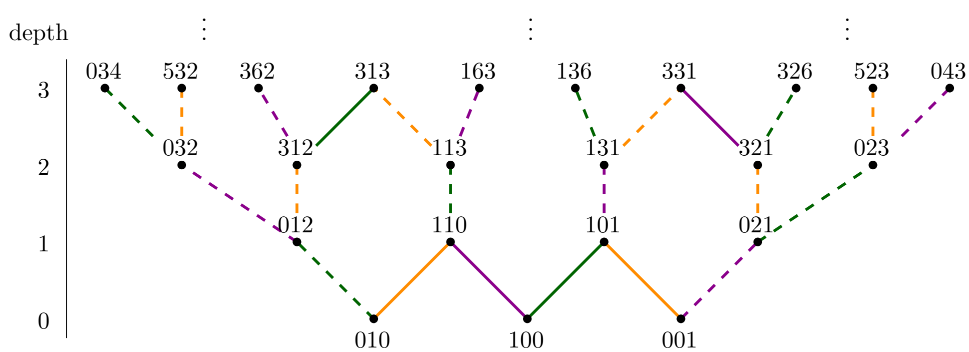


Figure 5.3: Hasse diagram of the root poset of Example 5.1.10.

By [8, Proposition 4.5.4], a dihedral subgroup generated by two reflections $s_\alpha, s_\beta \in T$ is infinite if and only if $|B(\alpha, \beta)| \geq 1$, hence a covering $\alpha \triangleleft s(\alpha)$ is short (respectively, long) if and only if the dihedral group generated by s_α and s is finite (respectively, infinite).

5.1.4 Small roots

A *small root* is a positive root that is reachable from a simple root by a chain of short covering edges in the root poset $(\Phi^+, \leq)$; see for instance [8, §4.7]. We denote by Σ the set of small roots.

Example 5.1.11. Consider (W, S) of Example 5.1.10, then there are only five small roots:

$$\Sigma = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3\}.$$

Remark 5.1.12. In a finite group, all positive roots are small. Indeed, there is no long edge in the root poset of a finite Coxeter system since there is no infinite dihedral subgroups.

Small roots were introduced by Brink and Howlett [11] to show that Coxeter groups are automatic. The key point is the following statement; see also [8, Theorem 4.7.3].

Theorem 5.1.13 (Brink-Howlett [11])**.** *The set Σ is finite whenever S is finite.*

5.1.5 The dominance order

We present yet another partial order $\leq_d$ on the positive roots of W, called the *dominance order*, which was also introduced by Brink and Howlett in [11]. Let $\alpha, \beta \in \Phi^+$, we say that β *dominates* α, denoted by $\alpha \leq_d \beta$, if and only if, for $w \in W$, $\beta \in \Phi(w)$ implies $\alpha \in \Phi(w)$. For $\beta \in \Phi^+$, the *dominance set of* β is:

$$\mathrm{dom}(\beta) = \{\alpha \in \Phi^+ \mid \alpha \leq_d \beta\}.$$

The *dominance-depth (or infinite-depth)* of a positive root $\beta \in \Phi^+$ is defined as follows:

$$\mathrm{dp}_\infty(\beta) := |\{\alpha \in \Phi^+ \setminus \{\beta\} \mid \alpha \leq_d \beta\}| = |\mathrm{dom}(\beta)| - 1.$$

We have an analogous result of Lemma 5.1.6, see for instance [15, Proposition 3.3].

Proposition 5.1.14. *Let $s \in S$ and $\beta \in \Phi^+ \setminus \{\alpha_s\}$, then we have* $\mathrm{dp}_\infty(\alpha_s) = 0$ *and*

$$\mathrm{dp}_\infty(s(\beta)) = \begin{cases} \mathrm{dp}_\infty(\beta), & \text{if } |B(\beta, \alpha_s)| < 1; \\ \mathrm{dp}_\infty(\beta) + 1, & \text{if } B(\beta, \alpha_s) \leq -1; \\ \mathrm{dp}_\infty(\beta) - 1, & \text{if } B(\beta, \alpha_s) \geq 1. \end{cases}$$

Note that if $\alpha \lhd s(\alpha)$ is a long covering in $(\Phi^+, \leq)$, then $\mathrm{dp}_\infty(s(\alpha)) = \mathrm{dp}_\infty(\alpha) + 1$, otherwise $\mathrm{dp}_\infty(s(\alpha)) = \mathrm{dp}_\infty(\alpha)$.

The following characterization of the dominance order, which relates the dominance-depth to the depth of a positive root, is [11, Lemma 2.3].

Proposition 5.1.15. *Let $\alpha, \beta \in \Phi^+$, then $\alpha \leq_d \beta$ if and only if* $\mathrm{dp}(\alpha) \leq \mathrm{dp}(\beta)$ *and* $B(\alpha, \beta) \geq 1$*; there is equality if and only if $\alpha = \beta$.*

It follows that if $\alpha \lhd s(\alpha)$ is a long covering in the root poset, then $\alpha_s \leq_d \alpha$. For instance, in Example 5.1.10, $\alpha_3 \leq_d \alpha_2 + 2\alpha_3 \leq_d \alpha_1 + 3\alpha_2 + 6\alpha_3$.

The next theorem shows that a root is small if it does not dominate any other root but itself; see for instance [8, Proposition 4.7.6] for a proof.

Theorem 5.1.16 (Brink-Howlett)**.** *Let $\alpha \in \Phi^+$, then $\alpha \in \Sigma$ if and only if* $\mathrm{dp}_\infty(\alpha) = 0$.

5.1.6 m-small roots and m-low elements

We say that a positive root β is m-small if $\mathrm{dp}_\infty(\beta) \leq m$, for some $m \in \mathbb{N}$. We denote by Σ_m the set of m-small roots.

Note that $\Sigma_0 = \Sigma$ by Theorem 5.1.16. The following is a generalization of Brink-Howlett's Theorem 5.1.13.

Theorem 5.1.17 (Fu [23])**.** *The set Σ_m is finite for all $m \in \mathbb{N}$.*

For $w \in W$, define $\Sigma_m(w) := \Phi(w) \cap \Sigma_m$ the *m-small inversion set* of w. Moreover, if X is a set, we denote by $\mathrm{cone}(X)$ the set of all linear combinations of elements of X with non-negative coefficients.

Definition 5.1.18. For $m \in \mathbb{N}$, an element $w \in W$ is *m-low* if $\Phi(w) = \mathrm{cone}(\Sigma_m(w)) \cap \Phi$. We denote by $L_m := L_m(W)$ the set of m-low elements in W.

The following theorem is due to Dyer and Hohlweg [15, Theorem 1.1], for $m = 0$ and for affine Coxeter systems, and to Dyer [13, Corollary 1.7] for all $m \in \mathbb{N}$ and any Coxeter system.

Theorem 5.1.19 (Dyer)**.** *For any Coxeter system and any $m \in \mathbb{N}$, the set L_m is a finite Garside shadow.*

We denote $\pi_m : W \mapsto \bigvee_R \{g \in L_m \mid g \leq_R w\}$ the Garside projection on L_m. In [14], the authors showed that the set of m-low elements are precisely the minimal elements (in right weak order) in m-Shi arrangements, that is, the subarrangement of the Coxeter arrangement constituted of the hyperplanes associated to m-small roots. This implies in particular the following useful proposition.

Proposition 5.1.20. *Let $w \in W$, then $\Sigma_m(\pi_m(w)) = \Sigma_m(w)$.*

Proof. Since $\pi_m(w) \leq_R w$, we have $\Sigma_m(\pi_m(w)) \subseteq \Sigma_m(w)$. The converse follows from [14, Theorem 6.4] in which the authors prove that the set $\{g \in W \mid \Sigma_m(g) = \Sigma_m(w)\}$ has a unique element of minimal length which is a m-low element $x \in L_m$ such that $x \leq_R w$. So $\Sigma_m(w) = \Sigma_m(x)$. But $x \leq_R \pi_m(x)$ by definition. Therefore $\Sigma_m(w) = \Sigma_m(x) \subseteq \Sigma_m(\pi_m(w))$. □

5.2 Reflection-prefixes and palindromic reduced words of reflections

In this section, we define and state properties of *reflection prefixes*. In particular, reflection-prefixes have useful properties to study the root poset and the dominance order. Then we produce a family of automata built from the finite Garside shadows L_m ($m \in \mathbb{N}$) that recognize the language of reflection prefixes.

5.2.1 Reflection prefixes

Let $t \in T$ be a reflection of (W, S). It is well known that $\ell(t) = 2k + 1$ for some $k \in \mathbb{N}$ and that if $t = s_1 \dots s_k s_{k+1} s_{k+2} \dots s_{2k+1}$ is a reduced word for t, the word $s_1 \dots s_k s_{k+1} s_k \dots s_1$ is a *palindromic reduced word* for t; see for instance [14, Proposition 2.3].

The following proposition shows in particular that t is uniquely determined by the prefix $s_1 \cdots s_k s_{k+1}$ of its palindromic reduced word.

Proposition 5.2.1. *Let $t \in T$ with corresponding positive root α_t. Let $w \in W$ and assume there is $r \in D_R(w)$ such that $t = wrw^{-1}$ and $\ell(t) = 2\ell(w) - 1$. Then $D_R(w) = \{r\}$. Moreover:*

1. *if $w = s_1 \cdots s_k s_{k+1}$ is a reduced word for w then $s_{k+1} = r$ and $t = s_1 \dots s_k r s_k \dots s_1$ is a palindromic reduced word for t;*
2. *$\alpha_t = w(\alpha_r) = s_1 \dots s_k(\alpha_r)$ and $\mathrm{dp}(\alpha_t) = k = \ell(w) - 1$.*

Proof. Let $w = s_1 \dots s_k r$ be a reduced word. We show that $D_R(w) = \{r\}$. By contradiction, assume that $|D_R(w)| \geq 2$. So there is $s \in D_R(w)$ with $s \neq r$; set $I = \{s, r\} \subseteq D_R(w)$. By Lemma 5.1.1, we know that W_I is finite and there is $u \in X_I$ such that $w = uw_{\circ,I}$ is a reduced product. So $\ell(w) = \ell(u) + m$, where $m = m_{sr} \geq 2$. Now, observe that $w_{\circ,I} r w_{\circ,I} \in I$, since the longest element of a finite Coxeter system acts by conjugation on the set of simple reflections. Therefore

$$t = s_1 \dots s_k s_{k+1} s_k \dots s_1 = s_1 \dots s_k r s_k \dots s_1 = wrw^{-1} = uw_{\circ,I} r w_{\circ,I} u^{-1},$$

implying that $\ell(t) \leq 2\ell(u) + 1 = 2\ell(w) - 2m + 1 < 2\ell(w) - 1 = \ell(t)$, a contradiction. The 'moreover' statement follows from the uniqueness of r, the definition of the depth of roots and from the equalities $\ell(t) = 2\,\mathrm{dp}(\alpha_t) + 1 = 2\ell(w) - 1$. □

Definition 5.2.2 (Reflection-prefixes)**.** A *reflection-prefix* is an element $p \in W$ such that for any reduced word $p = s_1 \dots s_{k+1}$, the word $s_1 \dots s_k s_{k+1} s_k \dots s_1$ is a palindromic reduced word for some reflection $t \in T$. A reflection-prefix for $t \in T$ is called a *t-prefix*.

Thanks to Proposition 5.2.1, the reflection t in the above definition does not depend on the choice of the reduced word for p. Since any $t \in T$ always has a palindromic reduced word, t-prefixes exist. A simple reflection $s \in S$ has a unique s-prefix $p_s = s$.

Example 5.2.3 (Dihedral groups)**.** Let $W = I_2(m)$, with $m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ be the dihedral group generated by the simple reflections $\{r, s\}$ with $m_{rs} = m$. Recall that for two letters $a, b \in S$ and $k \in \mathbb{N}^*$, we denote $[ab]_k = abab\cdots$ the word with k-letters starting by a and alternating the letters a and b. We now distinguish two cases.

(i) Case m is even. Any reflection $t \in T$ is written either as $t = [rs]_{2k+1}$ or $t = [sr]_{2k+1}$, for $0 \leq k < \frac{m}{2}$. In the first case, the unique t-prefix is $p_t = [rs]_{k+1}$, while in the second case we have that the unique t-prefix is $p_t = [sr]_{k+1}$.

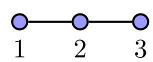
(ii) Case m is odd. Any reflection $t \in T$ is written as:

$$t = \begin{cases} [rs]_{2k+1}, & 0 \leq k < \frac{m-1}{2}; \\ [sr]_{2k+1}, & 0 \leq k < \frac{m-1}{2}; \\ [rs]_m = [sr]_m = \omega_{0,\{r,s\}}. \end{cases}$$

In the first two cases, there is a unique t-prefix as in (i). In the last case, there are two t-prefixes: $p_t = [rs]_{\frac{m-1}{2}}$ and $p'_t = [sr]_{\frac{m-1}{2}}$.

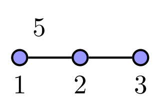
Example 5.2.4 (Universal Coxeter system)**.** Let (U_n, S) be the Universal Coxeter system of rank n, that is $U_n = \langle S \mid s^2 = e \rangle$. Each element has a unique reduced word, so the language $\mathrm{Rpal}(W, S)$ of palindromic reduced words is $\mathrm{Rpal}(T) = \{s_1 \cdots s_{i-1} s_i s_{i-1} \cdots s_1 \mid s_j \in S \text{ and } s_j \neq s_{j+1}\}$, and $\mathcal{L}_T = \mathcal{R}(W)$.

Example 5.2.5. Consider a Coxeter system (W, S) of type A_3,



i.e., W is isomorphic to the symmetric group S_4. Let $t = 12321 = 13231 = 32123$. Then t has three t-prefixes, that is $p_t = 123$, $p'_t = 132 = 312$, and $p''_t = 321$.

Remark 5.2.6. If $t \in T$ and p_t is a t-prefix, it is not true in general that there is an r-prefix p_r with $p_r \leq_R p_t$ for all $\alpha_r \in \Phi(p_t)$. For instance, consider a Coxeter system of type H_3, with associated Coxeter graph:



Take $t = 121232121 \in T$, $p_t = 12123$ and $r = 212$. Then $\alpha_r = 2(\alpha_1) \in \Phi(p_t)$, in fact $p_t^{-1}(2(\alpha_1)) = 32121(2(\alpha_1)) = 312121(\alpha_1) \in \Phi^-$, but $212 \not\leq_R 12123$.

Remark 5.2.7.

1. Proposition 5.2.1 shows that any reflection-prefix is a *Grassmannian element*, that is, an element with at most one right descent. It would be interesting to characterize the subset of Grassmannian elements that are reflection-prefixes.

2. Any t-prefix p_t satisfies by definition $p_t \leq_R t$ in right weak order $(W, \leq_R)$.

3. Since $t^{-1} = t$, the notion of t-suffix exists and all the results valid for t-prefixes are valid for t-suffixes.

The following statement follows from Proposition 5.2.1 and of the fact that, for any $t \in T$, $\ell(t) = 2\,\mathrm{dp}(\alpha_t) + 1$.

Corollary 5.2.8. *Let $w \in W$ and $w = s_1 \cdots s_k r$ be a reduced word. Let $t \in T$, then the following statements are equivalent:*

1. *w is a t-prefix;*

2. *$t = s_1 \cdots s_k r s_k \cdots s_1$ is a palindromic reduced word.*

3. *$\alpha_t = s_1 \cdots s_k(\alpha_r)$ and $\mathrm{dp}(\alpha_t) = \ell(w) - 1$.*

In particular, $\alpha_t \in \Phi(p_t)$ for any t-prefix p_t.

The above corollary implies this useful consequence.

Proposition 5.2.9. *Let $t \in T$ with associated positive root $\alpha_t \in \Phi^+$. For any t-prefix p_t and any $\alpha \in \Phi(p_t)$, we have $\mathrm{dp}(\alpha) \leq \mathrm{dp}(\alpha_t)$; with equality if and only if $\alpha = \alpha_t$.*

Proof. Consider a reduced word $p_t = s_1 \cdots s_k r$ for a t-prefix p_t. By Corollary 5.2.8, we have $\mathrm{dp}(\alpha_t) = k$. Now, let $\alpha \in \Phi(p_t)$, such that $\alpha \neq \alpha_t$. Then there is $1 \leq i \leq k-1$ such that $\alpha = s_1 \cdots s_i(\alpha_{s_{i+1}})$. By definition of the depth we have $\mathrm{dp}(\alpha) \leq i - 1$. Therefore $\mathrm{dp}(\alpha) < \mathrm{dp}(\alpha_t)$, which proves the inequality for all $\alpha \neq \alpha_t$. □

5.2.2 Reflection-prefixes and the root poset

The following statement shows that there is a correspondence between reduced words of reflection-prefixes and saturated chains in the root poset.

Proposition 5.2.10. *Let $t \in T$ and $\alpha_t \in \Phi^+$ be the corresponding positive root.*

1. *Let* $\alpha_r \lhd s_k(\alpha_r) \lhd \cdots \lhd s_1 \cdots s_{k-1}s_k(\alpha_r) = \alpha_t$ *be a saturated chain in* $(\Phi^+, \leq)$, *where* $r \in S$. *Then* $p_t = s_1 \ldots s_k r$ *is a reduced word for some* t*-prefix* p_t.

2. *Let* p_t *be a* t*-prefix and fix a reduced word* $p_t = s_1 \ldots s_k r$, *with* r *the unique right-descent (by Proposition 5.2.1). Then* $\alpha_r \lhd s_k(\alpha_r) \lhd \cdots \lhd s_1 \cdots s_{k-1}s_k(\alpha_r) = \alpha_t$ *is a saturated chain in* $(\Phi^+, \leq)$.

Proof. (1) Let $r \in S$ and $\alpha_r \lhd s_k(\alpha_r) \lhd \cdots \lhd s_1 \cdots s_{k-1}s_k(\alpha_r) = \alpha_t$ be a saturated chain in $(\Phi^+, \leq)$. So $\mathrm{dp}(\alpha_t) = k$. Let $p_t = s_1 \ldots s_k r$ and observe that $\alpha_t = s_1 \cdots s_k(\alpha_r)$. If $s_1 \ldots s_k$ is not reduced, then $\mathrm{dp}(\alpha_t) < k$, which is a contradiction. So $s_1 \ldots s_k$ is reduced. If $s_1 \ldots s_k r$ is not reduced, then $\ell(s_1 \cdots s_k r) < \ell(s_1 \ldots s_k)$. Therefore $\alpha_t = s_1 \ldots s_k(\alpha_r) \in \Phi^-$, again a contradiction. So $s_1 \cdots s_k r$ is a reduced word for p_t. Since $\ell(t) = 2\,\mathrm{dp}(\alpha_t) + 1$, the palindromic word $t = s_{s_1 \cdots s_k(\alpha_r)} = s_1 \cdots s_k r s_k \cdots s_1$ is reduced. Hence p_t is a t-prefix.

(2) Let p_t be a t-prefix and choose a reduced word $p_t = s_1 \ldots s_k r$. By Corollary 5.2.8, the palindromic word

$$s_1 \cdots s_{i-1} s_i \cdots s_k r s_k \cdots s_i s_{i-1} \cdots s_1$$

is reduced. So for any $1 \leq i \leq k$, the palindromic word $s_i \cdots s_k r s_k \cdots s_i$ is also reduced.

Assume now by contradiction that there is $1 \leq i \leq k-1$ such that $s_i \cdots s_k(\alpha_r)$ is not a cover of $s_{i+1} \cdots s_k(\alpha_r)$. In other words, by Proposition 5.1.8, $\mathrm{dp}(s_i \cdots s_k(\alpha_r)) \leq \mathrm{dp}(s_{i+1} \cdots s_k(\alpha_r))$. Take i maximal for that property, that is, $\alpha_r \lhd s_k(\alpha_r) \lhd \cdots \lhd s_{i+1} \cdots s_k(\alpha_r)$. So $\mathrm{dp}(s_{i+1} \cdots s_k(\alpha_r)) = k - i$. Therefore $\mathrm{dp}(s_i \cdots s_k(\alpha_r)) \leq k - i < \ell(s_i \ldots s_k) = k - i + 1$. By Corollary 5.2.8, the element $s_1 \cdots s_k r$ is not a reflection-prefix and the palindromic word $s_i \cdots s_k r s_k \cdots s_i$ is not a reduced word, which is a contradiction. $\square$

Example 5.2.11. Consider the Coxeter system (W, S) of type A_3 and take $t = 12321$. In Example 5.2.5, we have seen that the reflection t admits 3 t-prefixes: $p_t = 123$, $p'_t = 132 = 312$, and $p''_t = 321$. By Proposition 5.2.10, one can individuate these t-prefixes by looking at the saturated chains in the root poset of W, from a simple root to $\alpha_t = \alpha_1 + \alpha_2 + \alpha_3$. In Figure 5.4, the saturated chains associated to the t-prefixes are highlighted. Note that p'_t has two reduced expressions, hence there are two corresponding saturated chains starting from α_2.

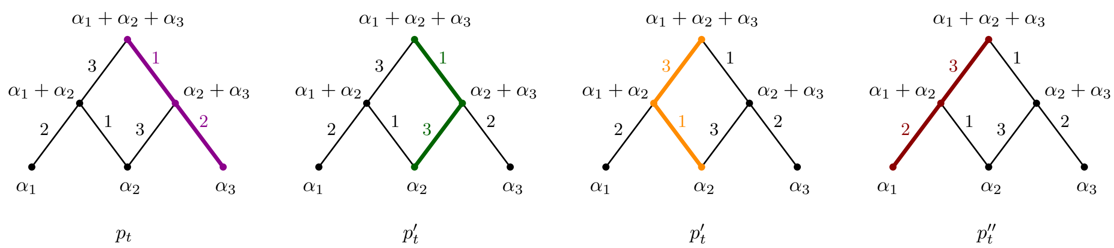


Figure 5.4: The saturated chains corresponding to the t-prefixes of Example 5.2.11.

Recall that the left weak order $(W, \leq_L)$ is defined by $u \leq_L v$ if and only if u is a suffix of v.

Corollary 5.2.12. *Let $t \in T$ with associated root $\alpha_t \in \Phi^+$. Consider p_t be a t-prefix with r its unique right descent. Let $u \in W$ such that $e <_L u \leq_L p_t$ in $(W, \leq_L)$, then $-u(\alpha_r) \leq \alpha_t$ in $(\Phi^+, \leq)$.*

Proof. $e <_L u \leq_L p_t$, there is a reduced word $p_t = s_1 \cdots s_k r$ and $1 \leq i \leq k$ such that $u = s_i \cdots s_k r$. So $-u(\alpha_r) = s_i \cdots s_k(\alpha_r) \leq s_1 \cdots s_k(\alpha_r) = -p_t(\alpha_r) = \alpha_t$ by Proposition 5.2.10. □

The following theorem provides a characterization of reflection-prefixes in term of inversion sets.

Theorem 5.2.13. *Let $w \in W$ and $r \in D_R(w)$. Let $t \in T$ such that $\alpha_t = -w(\alpha_r) \in \Phi(w)$. Then w is a t-prefix if and only if $B(\alpha, \alpha_t) > 0$ for all $\alpha \in \Phi(w)$.*

Proof. Assume first that $w = p_t$ is a t-prefix. By Proposition 5.2.10, we have

$$\alpha_r \lhd s_k(\alpha_r) \lhd \cdots \lhd s_1 \cdots s_{k-1} s_k(\alpha_r) = \alpha_t$$

a saturated chain associated to the reduced word $p_t = s_1 \cdots s_k r$. By the characterization of covers in the root poset, see Proposition 5.1.8, we have:

$$B(s_{i+1} \cdots s_k(\alpha_r), \alpha_{s_i}) < 0, \quad \text{for all } 1 \leq i \leq k-1.$$

Since $p_t = s_1 \cdots s_k r$ is a reduced word, we know that the inversion set of p_t is:

$$\Phi(w) = \{\alpha_{s_1}, s_1(\alpha_{s_2}), \ldots, s_1 \cdots s_{k-1}(\alpha_{s_k}), s_1 \cdots s_k(\alpha_r) = \alpha_t\}$$

Let $\alpha \in \Phi(p_t)$. If $\alpha = \alpha_t$ then $B(\alpha_t, \alpha_t) > 0$. If $\alpha = \alpha_{s_1}$, then

$$B(\alpha_t, \alpha_{s_1}) = B(s_1 \cdots s_k(\alpha_r), \alpha_{s_1}) = -B(s_2 \cdots s_k(\alpha_r), \alpha_{s_1}) > 0.$$

If $\alpha \notin \{\alpha_t, \alpha_{s_1}\}$, then there is $2 \leq i \leq k-1$ such that $\alpha = s_1 \cdots s_{i-1}(\alpha_{s_i})$, then

$$B(\alpha_t, \alpha) = B(s_1 \cdots s_k(\alpha_r), s_1 \cdots s_{i-1}(\alpha_{s_i})) = -B(s_{i+1} \cdots s_k(\alpha_r), \alpha_{s_i}) > 0.$$

Therefore $B(\alpha, \alpha_t) > 0$ for all $\alpha \in \Phi(p_t)$.

Assume now that $B(\alpha, \alpha_t) > 0$ for all $\alpha \in \Phi(w)$. Consider a reduced word $w = s_1 \cdots s_k r$. So $\alpha_t = s_1 \cdots s_k(\alpha_r)$. One just has to show that

$$B(s_{i+1} \cdots s_k(\alpha_r), \alpha_{s_i}) < 0, \quad \text{for all } 1 \leq i \leq k-1.$$

This follows from similar computations done above. $\square$

5.2.3 Reflection-prefixes and the dominance order

Reflection-prefixes allow an efficient way to compute the set of dominance of $\beta \in \Phi^+$: $\mathrm{dom}(\beta) = \{\alpha \in \Phi^+ \mid \alpha \leq_d \beta\}$ and its infinite-depth $\mathrm{dp}_\infty(\beta)$; see §5.1.5.

Proposition 5.2.14. *Let $\beta \in \Phi^+$. Write $t = s_\beta \in T$, i.e., $\beta = \alpha_t$. For any t-prefix p_t we have:*

$$\mathrm{dom}(\beta) = \{\alpha \in \Phi(p_t) \text{ such that } B(\alpha, \beta) \geq 1\} = \{\beta\} \sqcup \{\alpha_s \in \Phi(p_t) \text{ such that } |\langle s, t\rangle| = \infty\}.$$

In particular, $\mathrm{dp}_\infty(\beta) = |\{\alpha_s \in \Phi(p_t) \text{ such that } |\langle s, t\rangle| = \infty\}|$.

Proof. The second equality follows from the well-known fact that a dihedral subgroup containing s, t is infinite if and only if $|B(\alpha_s, \alpha_t)| \geq 1$ and $s \neq t$, see for instance [8, Proposition 4.5.4].

We show the first equality. Set $A = \{\alpha \in \Phi(p_t) \text{ such that } B(\alpha, \beta) \geq 1\}$. Let $\alpha \in A$, then $\mathrm{dp}(\alpha) \leq \mathrm{dp}(\alpha_t) = \mathrm{dp}(\beta)$ by Proposition 5.2.9. By Proposition 5.1.15, we have therefore $\alpha \in \mathrm{dom}(\beta)$. Let $\alpha \in \mathrm{dom}(\beta)$, then $B(\alpha, \beta) \geq 1$ by Proposition 5.1.15 again. Now, $\beta \in \Phi(p_t)$ by Corollary 5.2.8. So $p_t^{-1}(\beta) \in \Phi^-$. Since $\alpha \leq_d \beta$, we have by definition of the dominance order that $p_t^{-1}(\alpha) \in \Phi^-$. In other words $\alpha \in \Phi(p_t)$, which completes the proof. $\square$

Remark 5.2.15. The above proposition provides an algorithm to obtain the set of dominance - and therefore the infinite-depth - that does not require a geometric representation of (W, S). In particular it does not require computing the positive root system (up to a certain depth in the infinite case). Instead of considering the dominance order on positive roots, we consider dominance on the set of reflections T. Let $t \in T$, compute a reduced word for t and take the corresponding t-prefix p_t. Then compute the T-inversion set $T(p_t)$ of p_t, that is $T(p_t) = \{r \in T \mid \ell(rp_t) < \ell(p_t)\}$. Finally we obtain the T-set of dominance:

$$\mathrm{dom}_T(t) = \{t\} \sqcup \{s \in T(p_t) \text{ such that } |\langle s, t\rangle| = \infty\},$$

and, if a geometric representation is considered, we have

$$\mathrm{dom}(\alpha_t) = \{\alpha_s \mid s \in \mathrm{dom}_T(t)\}.$$

Example 5.2.16. Consider the Coxeter system of Example 5.1.4. Let $t = 2321232 \in T$ and $p_t = 2321$, and call $\beta = \alpha_t$. We have $B(\alpha_1, \alpha_2) = B(\alpha_1, \alpha_3) = -1/2$ and $B(\alpha_2, \alpha_3) = -1$. Hence, $\Phi(p_t) = \{\alpha_2, 2\alpha_2 + \alpha_3, 3\alpha_2 + 2\alpha_3, \beta = \alpha_1 + 6\alpha_2 + 3\alpha_3\}$. Finally, to obtain the set of dominance of β, we have to compute the bilinear form B between β and the roots of $\Phi(p_t) \setminus \{\beta\}$ and see when the value is equal or greater than 1. Therefore it is not difficult to check that $\mathrm{dom}(\beta) = \{\alpha_2, 2\alpha_2 + \alpha_3\}$, and $\mathrm{dp}_\infty(\beta) = 2$.

5.2.4 Recognizing the language of reflection-prefixes

Denote by $\mathcal{L}_\mathcal{T}$ the language of reduced words for reflection-prefixes. To end this section, we describe an automaton with states the set L_m of m-low elements (see §5.1.6) that recognizes the language $\mathcal{L}_T$:

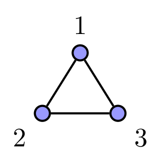
$$\begin{aligned}
\mathcal{L}_T &= \{s_1 \cdots s_k s_{k+1} \in \mathcal{R}(W) \mid p = s_1 \cdots s_k s_{k+1} \text{ is a reflection-prefix}\} \\
&= \{s_1 \cdots s_k s_{k+1} \in \mathcal{R}(W) \mid \mathrm{dp}(s_1 \cdots s_k(\alpha_{s_{k+1}})) = k\} \\
&= \{s_1 \cdots s_k s_{k+1} \in \mathcal{R}(W) \mid \ell(s_1 \cdots s_k s_{k+1} s_k \cdots s_1) = 2k+1\} \\
&= \{s_1 \cdots s_k s_{k+1} \in \mathcal{R}(W) \mid s_1 \cdots s_k s_{k+1} s_k \cdots s_1 \in \mathcal{R}(W)\}.
\end{aligned}$$

This automaton is a directed graph on the vertex set L_m with edges labeled by elements of S, such that for any $w \in L_m$ and $s \in S$, there is at most one edge (*transition*) with source w and label s (see [32, §2]).

Let $\mathcal{A} := \mathcal{A}_m(W,S)$ be an automaton such that

- the set of states is L_m;
- the final states are $\mathcal{F} = \{su \mid \ell(su) > \ell(u) \text{ and } s(\Sigma_m(u)) \cap \Sigma_m(u) = \emptyset\}$;
- the initial state is the identity e;
- the transitions are: $a \in \mathcal{F} \xrightarrow{s} \pi(sa)$, where $\ell(sa) > \ell(a)$ and $\pi_m : W \to L_m$ is the L_m-*projection* defined by: $\pi_m(w) = \bigvee_R \{g \in L_m \mid g \leq_R w\}$.

Example 5.2.17. Consider a Coxeter system (W,S) of type $\widetilde{A}_2$ with Coxeter graph



The 0-low elements of W are

$$L_0 = \{e, 1, 2, 3, 12, 21, 13, 31, 23, 32, 121, 131, 232, 1232, 2313, 3121\}$$

(see [15, Example 3.29]), and the automaton $\mathcal{A}_0(W,S)$ is depicted in Figure 5.5, where the shaded states $131, 121, 232$ are not final states. In fact, for instance, $\Sigma_0(21) = \{\alpha_2, \alpha_1 + \alpha_2\} = s_1(\Sigma_0(21))$, hence $s_1(\Sigma_0(21)) \cap \Sigma_0(21) \neq \emptyset$, so $121 \notin \mathcal{F}$. Note that if the shaded states were final, then we would have obtained the Garside shadow automaton on L_0.

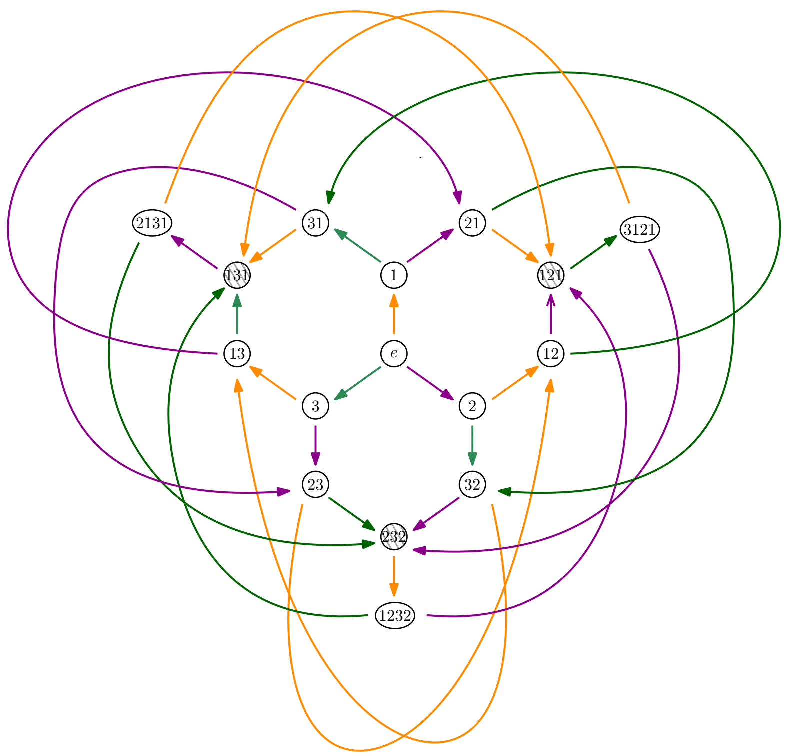


Figure 5.5: The automaton $\mathcal{A}_0(W,S)$ of Example 5.2.17 for type $\widetilde{A}_2$ which recognizes the language of reflection-prefixes of W.

This first lemma is a consequence of results obtained in [32] by Hohlweg, Nadeau and Williams.

Lemma 5.2.18. *Let* $s_1 \cdots s_{k+1} \in \mathcal{L}_T$*. Then*

$$D_L(\pi_m(s_{k+1} \cdots s_1)) = D_L(s_{k+1} \cdots s_1) = \{s_{k+1}\}.$$

Proof. Write $u = s_{k+1} \cdots s_1$. By [32, p.438, proof of Theorem 1.2], The word $\mathbf{s}_1 \cdots \mathbf{s}_{k+1}$ is recognized in the state $\pi_m(u)$ and u^{-1} is a reflection-prefix. By Proposition 5.2.1, $D_L(u) = D_R(u^{-1}) = \{s_{k+1}\}$. Therefore, by [32, Proposition 2.6], we have $D_L(\pi_m(u)) = D_L(u) = \{s_{k+1}\}$. □

Lemma 5.2.19. *Let* $s_1 \cdots s_{k+1} \in \mathcal{R}(W)$*, then* $s_1 \cdots s_{k+1} \in \mathcal{L}_T$ *if and only if* $\pi_m(s_{k+1} \cdots s_1)$ *is a final state.*

Proof. By Proposition 5.1.20, $\Sigma_m(\pi(u)) = \Sigma_m(u)$ for any $u \in W$. By [12, Lemma 2.4], $\mathrm{dp}(s_1 \cdots s_k(\alpha_{k+1})) = k$ if and only if $s_{k+1}(\Sigma_m(s_k \cdots s_1)) \cap \Sigma_m(s_k \cdots s_1) = \emptyset$, hence $s_{k+1}(\Sigma_m(\pi(s_k \cdots s_1))) \cap \Sigma_m(\pi(s_k \cdots s_1)) = \emptyset$ if and only if $\pi(s_{k+1} \cdots s_1) = s_{k+1}\pi(s_k \cdots s_1) \in \mathcal{F}$. □

By Lemma 5.2.18 and Lemma 5.2.19 we proved the following result.

Theorem 5.2.20. *The automaton $\mathcal{A}$ recognizes the language $\mathcal{L}_T$.*

As a consequence of the above theorem and Proposition 5.2.10 we obtain the following corollary.

Corollary 5.2.21. *The language $\mathcal{L}_T$ is regular and the generating function*

$$\begin{aligned} \sum_{u \in \mathcal{L}_T} q^{|u|} &= \sum_{t \in T} |\{_1 \cdots s_k s_{k+1} s_k \cdots s_1 \in \mathcal{L}_T \mid t = s_1 \cdots s_k s_{k+1} s_k \cdots s_1\}| \, q^{\ell(t)} \\ &= \sum_{k \in \mathbb{N}} a_k q^k \end{aligned}$$

is rational, where a_k is the number of saturated chains of length k in the root poset of W.

5.3 Affine Coxeter systems

In this section, we provide an easy formula to compute the Poincaré series $T(q)$ in the case of affine Coxeter systems. The formula is based on a description of the Hasse diagram of the root poset.

The generating function of the depth of positive roots is $\Phi^+(q) = \sum_{\alpha \in \Phi^+} q^{\mathrm{dp}(\alpha)}$.

Lemma 5.3.1. *We have that $T(q) = q\Phi^+(q^2)$.*

Proof. By Proposition 1.2.2, T and Φ^+ are in bijection. Also, it is well-known that $\ell(t_\gamma) = 2\,\mathrm{dp}(\gamma) + 1$, for $\gamma \in \Phi^+$ and $t_\gamma \in T$ is the associated reflection given by the bijection ρ (see [8, §4.6] for instance). Hence,

$$T(q) = \sum_{t_\gamma \in T} q^{\ell(t_\gamma)} = \sum_{\gamma \in \Phi^+} q^{2\,\mathrm{dp}(\gamma)+1} = q\Phi^+(q^2).$$

□

Refer to the crystallographic root system described in Section 1.2 and recall that the set of roots for (W, S) is

$$\Phi = \{k\delta + \alpha \mid k \in \mathbb{Z}, \alpha \in \Phi_0\}$$

with simple system

$$\Delta = \Delta_0 \sqcup \{\delta - \omega\},$$

where ω is the highest root of W_0 and $S = S_0 \cup \{s_{\delta-\omega}\}$. Finally, the set of positive roots is:

$$\Phi^+ = \{k\delta + \alpha, (k+1)\delta - \alpha \mid k \in \mathbb{N}, \alpha \in \Phi_0^+\} = (\mathbb{N}\delta + \Phi_0^+) \sqcup (\mathbb{N}^*\delta - \Phi_0^+).$$

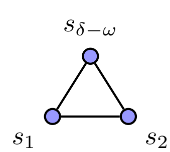
In particular, when one is interested in conditions associated to the sign of the bilinear form B, that condition remains the same for Φ^+ or its “unitary” counterpart. However, for dominance and long edges, conditions on the bilinear form depend on nonzero numbers, so those conditions on crystallographic root systems need to be made precise in the case of a crystallographic root system. Let $\alpha, \beta \in \Phi^+$, we have:

$$B\left(\frac{\alpha}{\|\alpha\|}, \frac{\beta}{\|\beta\|}\right) = \frac{B(\alpha, \beta)}{\|\alpha\|\|\beta\|}.$$

So for instance a condition such as $B(\alpha, \beta) \leq -1$ in Definition 5.1.9 or Proposition 5.1.14 becomes for α, β crystallographic roots the condition $B(\alpha, \beta) \leq -\|\alpha\|\|\beta\|$.

Definition 5.3.2 (Long and short edges in crystallographic root system)**.** A covering $\alpha \lhd s(\alpha)$ in the root poset $(\Phi^+, \leq)$ is *short* if $-\|\alpha\|\|\alpha_s\| < B(\alpha, \alpha_s) < 0$. Otherwise we say that the covering is *long*.

Example 5.3.3. Let (W, S) be a Coxeter system of type $\widetilde{A}_2$.



The underlying finite Coxeter system (W_0, S_0) is of type A_2. Consider $S = \{s_1, s_2, s_{\delta-\omega}\}$, where $s_1, s_2 \in S_0$ and $\omega = \alpha_1 + \alpha_2$ is the highest root of W_0. The finite positive root system is $\Phi_0^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$, so the affine simple system is $\Delta = \{\alpha_1, \alpha_2, \delta - \omega\}$ and the affine positive root system becomes $\Phi^+ = \{k\delta + \alpha_1, k\delta + \alpha_2, k\delta + \alpha_1 + \alpha_2, (k+$

$1)\delta - \alpha_1, (k+1)\delta - \alpha_2, (k+1)\delta - (\alpha_1 + \alpha_2) \mid k \in \mathbb{N}\}$. The Hasse diagram of the root poset is depicted in Figure 5.6, up to depth equal to 3, where each color of the edges corresponds to the simple reflection that is applied: red for s_1, yellow for s_2 and green for $s_3 := s_{\delta-\omega}$. Moreover, the dashed edges correspond to the long coverings.

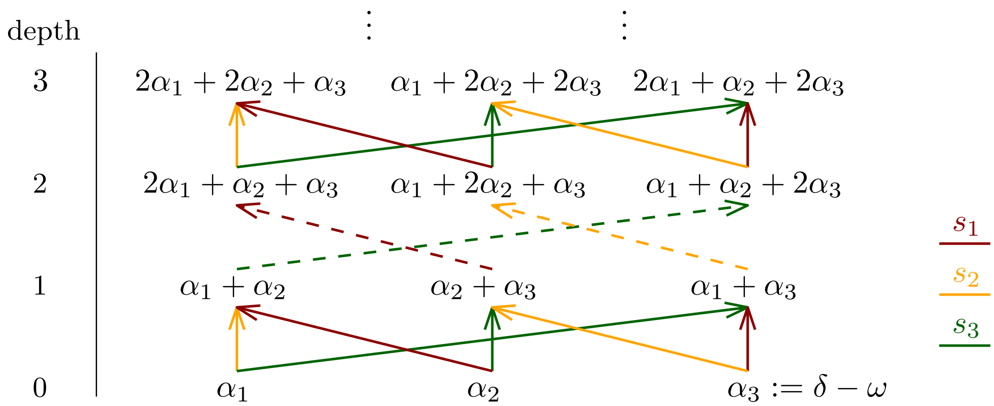
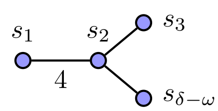


Figure 5.6: Hasse diagram of the root poset of type $\widetilde{A}_2$ up to depth 3.

Example 5.3.4. Let (W, S) be a Coxeter system of type $\widetilde{B}_3$ with the following Coxeter graph



In this case we have $\|\alpha_1\| = 1$ and $\|\alpha_2\| = \|\alpha_3\| = \|\omega\| = \sqrt{2}$, where $\omega = 2\alpha_1 + 2\alpha_2 + \alpha_3$. For instance, $B(\alpha_2, \delta - \omega) = -B(\alpha_2, \omega) = -1 > -\|\alpha_2\|\|\omega\| = -2$, and $B(\delta - \alpha_3, \alpha_3) = -\|\alpha_3\|^2 = -2$. Hence the covering $\alpha_2 \lhd s_{\delta-\omega}(\alpha_2)$ is short, while $\delta - \alpha_3 \lhd s_3(\delta - \alpha_3) = \delta + \alpha_3$ is long. The Hasse diagram of the initial part of the root poset of type $\widetilde{B}_3$ is in Figure 5.7.

5.3.1 Restriction of the root poset on orbits of simple roots in Φ_0

Let (W, S) be an affine Coxeter system with underlying finite system (W_0, S_0) with crystallographic root system Φ_0. Denote by $\mathcal{O}_1, \ldots, \mathcal{O}_h$ the orbits of the action of W_0

on the set Δ_0 of simple roots. Then we have a decomposition

$$\Phi_0^+ = \bigsqcup_{i=1}^{h} \mathcal{O}_i^+,$$

where $\mathcal{O}_i^+ := \mathcal{O}_i \cap \Phi_0^+$. In fact, there are only two possibilities for the number of orbits h: $h = 1$ if W_0 is simply laced, and $h = 2$ otherwise.

Set $\mathcal{O}_i^+ := \mathcal{O}_i \cap \Phi_0^+$, so that

$$\Phi_0^+ = \bigsqcup_{i=1}^{k} \mathcal{O}_i^+.$$

Remark 5.3.5. (1) Note that for each i, the set $\{t \in T_0 \mid \alpha_t \in \mathcal{O}_i^+\}$ is a conjugacy class of reflections in $T_0 = \bigcup_{w \in W_0} wS_0w^{-1}$. In fact, the reflections $t_1 = w_1s_1w_1^{-1}$ and $t_2 = w_2s_2w_2^{-1}$, for some $w_1, w_2 \in W_0$ and $s_1, s_2 \in S_0$, are conjugated if and only if $ut_1u^{-1} = t_2$ for some $u \in W_0$, if and only if $uw_1s_1(uw_1)^{-1} = w_2s_2w_2^{-1}$, if and only if $uw_1(\alpha_{s_1}) = w_2(\alpha_{s_2})$ i.e. $u\alpha_{t_1} = \alpha_{t_2}$ for some $u \in W_0$, where the last equivalence comes from the bijection between Φ_0^+ and T_0 (see Introduction and [8, Proposition 4.4.5]).

(2) If W is of simply laced type, then the W_0-action on Δ_0 has one orbit.

Recall that the set of positive roots of (W, S) is $\Phi^+ = (\mathbb{N}\delta + \Phi_0^+) \sqcup (\mathbb{N}^*\delta - \Phi_0^+)$. Consider the following subsets of Φ^+.

- $\Phi_{\mathcal{O}_i,k} := \{k\delta + \alpha, (k+1)\delta - \alpha \mid \alpha \in \mathcal{O}_i^+\}$ for $k \in \mathbb{N}$, and we set $\Phi_{\mathcal{O}_i,0} =: \Sigma_{\mathcal{O}_i}$;
- $\Phi_{\mathcal{O}_i} := \bigsqcup_{k \in \mathbb{N}} \Phi_{\mathcal{O}_i,k}$.

Our goal is to study in detail the root poset restricted to each subset of Φ^+ introduced above, since we will prove that such subposets have nice symmetrical properties. Note that the Hasse diagram of such subposets in general is not connected.

Example 5.3.6. If (W, S) is of type $\widetilde{B}_3$, then $\Phi_0^+ = \mathcal{O}_1^+ \sqcup \mathcal{O}_2^+$, where $\mathcal{O}_1^+ = \{\alpha_1, s_2(\alpha_1), s_3s_2(\alpha_1)\}$ and $\mathcal{O}_2^+ = \{\alpha_2, \alpha_3, s_1(\alpha_2), s_3(\alpha_2), s_1s_3(\alpha_2), s_2s_1s_3(\alpha_2)\}$. The root poset restricted to $\Sigma_{\mathcal{O}_1} \sqcup \Sigma_{\mathcal{O}_2}$ is depicted in Figure 5.7.

Let $\mathcal{O}$ be an orbit of $\alpha \in \Delta_0$. Here we prove that the root poset restricted to a positive orbit $\mathcal{O}^+$ is isomorphic to the opposite poset of the restriction of the root poset on the positive roots $\delta - \mathcal{O}^+$.

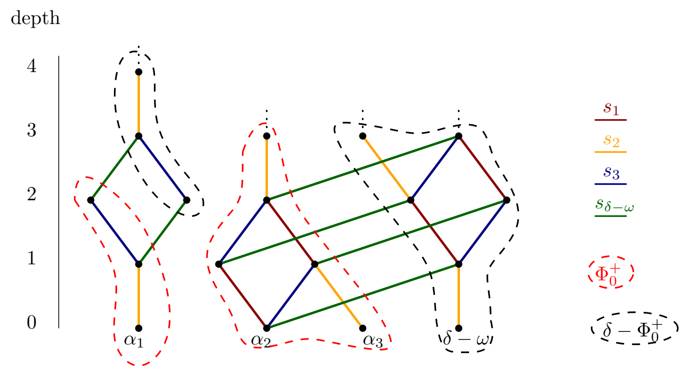


Figure 5.7: Root poset of type $\widetilde{B}_3$ restricted to $\Sigma_{\mathcal{O}_1} \sqcup \Sigma_{\mathcal{O}_2}$. The portion of poset in the red lines is isomorphic to the opposite in the black lines, as stated in Theorem 5.3.7.

Theorem 5.3.7. *We have a poset isomorphism*

$$(\mathcal{O}^+, \leq) \cong (\delta - \mathcal{O}^+, \leq)^{op}$$

i.e. $\alpha \leq \beta$ *if and only if* $\delta - \beta \leq \delta - \alpha$*, for* $\alpha, \beta \in \mathcal{O}^+$*.*

Proof. Let $s \in S_0$ and $\alpha \in \mathcal{O}^+ \setminus \{\alpha_s\}$, then $B(\alpha, \alpha_s) < 0$ if and only if $B(\delta - \alpha, \alpha_s) > 0$, hence there is a covering $\alpha \lhd s(\alpha)$ if and only if $\delta - s(\alpha) \lhd \delta - \alpha$. Also, $B(\alpha, \delta - \omega) < 0$ if and only if $B(\delta - \alpha, \delta - \omega) > 0$, then again, $\alpha \lhd s_{\delta-\omega}(\alpha)$ if and only if $\delta - s_{\delta-\omega}(\alpha) \lhd \delta - \alpha$. □

Following [9, Ch. 6, §1], we introduce the quantity $n(\alpha, \beta) := 2\frac{B(\alpha,\beta)}{B(\beta,\beta)}$ for $\alpha, \beta \in \Phi^+$, which will be useful for the next computations. Then,

$$s_\beta(\alpha) = \alpha - n(\alpha, \beta)\beta.$$

By Proposition 8 (ibidem), if $\alpha \neq \beta$ and $\|\alpha\| \leq \|\beta\|$, then $n(\alpha, \beta) = \{0, \pm 1\}$. It follows that $n(\alpha, \omega) = \{0, \pm 1\}$ when $\alpha \neq \omega$, since $\|\alpha\| \leq \|\omega\|$ for all positive roots α. Finally, note that $n(k\delta + \alpha, \beta) = n(\alpha, \beta) = n(\alpha, m\delta + \beta)$ and $n(k\delta - \alpha, \beta) = -n(\alpha, \beta) = n(\alpha, m\delta - \beta)$ for $k, m \in \mathbb{N}^*$.

Recall that a positive root is called *small* if it is reachable from a simple root by a chain of short covering edges in the root poset (see Section 5.1.4), and denote by Σ the set of small roots.

Lemma 5.3.8. *Let* $\mathcal{O}$ *be an orbit of* $\alpha \in \Delta_0$*. The roots in* $\Sigma_{\mathcal{O}}$ *are small.*

Proof. By Remark 5.1.12, positive roots of a finite group are always small. Hence, the coverings in the root poset restricted to $\mathcal{O}^+$ are short. Also, if $\alpha, s(\alpha) \in \mathcal{O}^+$ and $\alpha \lhd s(\alpha)$ is a covering, then $B(\delta - s(\alpha), \alpha_s) = -B(s(\alpha), \alpha_s) = B(\alpha, \alpha_s)$, hence $\delta - s(\alpha) \lhd \delta - \alpha_s$ is short since $\alpha \lhd s(\alpha)$ is short. Moreover, if $\alpha \lhd s_{\delta-\omega}(\alpha)$ is a covering, with $\alpha \in \mathcal{O}^+$, then such covering is short as well. In fact, $s_{\delta-\omega}(\alpha) = \alpha + n(\alpha,\omega)(\delta - \omega)$, and since $s_{\delta-\omega}(\alpha) \in \delta - \mathcal{O}^+$, then $n(\alpha, \omega) = 1$. Therefore, $B(\alpha, \omega) = \|\omega\|^2/2$. On the other hand, by [9, Proposition 12, Ch. 4 §1], we have that $\|\omega\|^2/\|\alpha\|^2 \in \{1, 2, 3\}$, so it follows that $B(\alpha, \omega) = \|\omega\|^2/2 < \|\omega\|\|\alpha\|$, which implies that $-\|\omega\|\|\alpha\| < B(\alpha, \delta - \omega) < 0$, so that $\alpha \lhd s_{\delta-\omega}(\alpha)$ is short. □

Remark 5.3.9. Let $\alpha \in \Phi_0^+, k \in \mathbb{N}$.

(1) A covering $(k + 1)\delta - \alpha \lhd s_{\delta-\omega}((k + 1)\delta - \alpha)$ never occurs. If such a covering does exist, then $0 > B((k + 1)\delta - \alpha, \delta - \omega) = B(\alpha, \omega)$, so $n(\alpha, \omega) = -1$. Hence, $s_{\delta-\omega}(\alpha) = \alpha - n(\alpha, \delta - \omega)(\delta - \omega) = \alpha + n(\alpha, \omega)(\delta - \omega) = \alpha - \delta + \omega$. Then, $s_{\delta-\omega}((k + 1)\delta - \alpha) = (k + 2)\delta - (\alpha + \omega) \notin \Phi_{\mathcal{O},k}$.

(2) In $\Phi_{\mathcal{O},k}$ there are three types of coverings:

   (i) $k\delta + \alpha \lhd s(k\delta + \alpha) = k\delta + s(\alpha)$, when $B(\alpha, \alpha_s) < 0$;

   (ii) $(k + 1)\delta - \alpha \lhd s((k + 1)\delta - \alpha) = (k + 1)\delta - s(\alpha)$, when $B(\alpha, \alpha_s) > 0$ and $\alpha \neq \alpha_s$ (note that if $\alpha = \alpha_s$, then $(k+1)\delta - s(\alpha) = (k+1)\delta + \alpha_s \in \Phi_{\mathcal{O},k+1}$);

   (iii) $k\delta + \alpha \lhd s_{\delta-\omega}(k\delta + \alpha) = k\delta + s_{\delta-\omega}(\alpha)$, when $B(\alpha, \delta - \omega) < 0$ and $\alpha \neq \omega$, otherwise $s_{\delta-\omega}(k\delta + \omega) = (k + 2)\delta - \omega \in \Phi_{\mathcal{O},k+1}$.

Now we want to show that the restriction of the root poset to each subset $\Phi_{\mathcal{O},k}$ is isomorphic to the restriction of the root poset to $\Sigma_{\mathcal{O}}$.

Theorem 5.3.10. *We have a poset isomorphism*

$$(\Phi_{\mathcal{O},k}, \leq) \cong (\Sigma_{\mathcal{O}}, \leq)$$

for all $k \in \mathbb{N}$. Moreover, a covering in $\Phi_{\mathcal{O},k}$ is short (long) if and only if the corresponding one in $\Sigma_{\mathcal{O}}$ is short (long).

Proof. By Remark 5.3.9, we have to consider three types of coverings. The covering (i) occurs in $\Phi_{\mathcal{O},k}$ if and only if $0 > B(k\delta + \alpha, \alpha_s) = B(\alpha, \alpha_s)$, if and only if $\alpha \lhd s(\alpha)$ is a covering in $\Sigma_{\mathcal{O}}$. The covering (ii) occurs in $\Phi_{\mathcal{O},k}$ if and only if $0 > B((k+1)\delta - \alpha, \alpha_s) = -B(\alpha, \alpha_s)$, if and only if $\delta - \alpha \lhd \delta - s(\alpha)$ is a covering in $\Sigma_{\mathcal{O}}$. The last covering (iii) occurs in $\Phi_{\mathcal{O},k}$ if and only if $0 > B(k\delta + \alpha, \delta - \omega) = B(\alpha, \delta - \omega)$, if and only if $\alpha \lhd s_{\delta-\omega}(\alpha)$ is a covering in $\Sigma_{\mathcal{O}}$. □

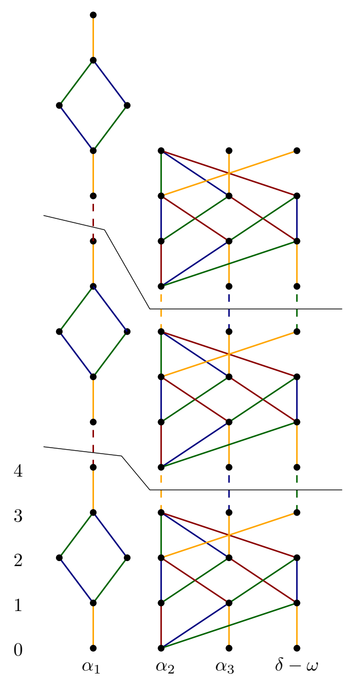


Figure 5.8: The Hasse diagram of the first part of the root poset of type $\widetilde{B}_3$. Note that the bottom part is the same poset of Figure 5.7 but the relations are depicted differently. This is because in this picture, it is possible to better observe the isomorphism stated in Theorem 5.3.10.

Example 5.3.11. Examples of Theorems 5.3.7 and 5.3.10 are depicted respectively in Figures 5.7 and 5.8 in the case of a Coxeter system of type $\widetilde{B}_3$.

Lemma 5.3.12 (Covering relations)**.**

(i) For all $k \in \mathbb{N}, s \in S_0$, $(k+1)\delta - \alpha_s \lhd s((k+1)\delta - \alpha_s)$ is a long covering, where $(k+1)\delta - \alpha_s \in \Phi_{\mathcal{O},k}$ and $s((k+1)\delta - \alpha_s) \in \Phi_{\mathcal{O},k+1}$.

(ii) For all $k \in \mathbb{N}$, $k\delta + \omega \lhd s_{\delta-\omega}(k\delta + \omega)$ is a long covering, where $k\delta + \omega \in \Phi_{\mathcal{O},k}$ and $s_{\delta-\omega}(k\delta + \omega) \in \Phi_{\mathcal{O},k+1}$.

(iii) All the other coverings are short.

Proof. Consider the covering $(k+1)\delta-\alpha_s \lhd s((k+1)\delta-\alpha_s)$, then $B((k+1)\delta-\alpha_s,\alpha_s) = -\|\alpha_s\|^2$, hence the covering is long, and $s((k+1)\delta-\alpha_s) = (k+1)\delta+\alpha_s \in \Phi_{\mathcal{O},k+1}$. Analogously, if we consider the covering $k\delta+\omega \lhd s_{\delta-\omega}(k\delta+\omega)$, $B(k\delta+\omega,\delta-\omega) = -\|\omega\|^2$, so the covering is long. In particular, $s_{\delta-\omega}(k\delta+\omega) = (k+2)\delta-\omega \in \Phi_{\mathcal{O},k+1}$. To prove (iii), we show that the other possible coverings are necessarily short. Note that the types of coverings we need to analyze coincide with the ones listed in Remark 5.3.9(2), hence both roots involved in a covering belong to the same $\Phi_{\mathcal{O},k}$ for some k. Therefore, by Theorem 5.3.10, such coverings are short since the corresponding ones in $\Sigma_{\mathcal{O}}$ are short by Lemma 5.3.8. □

To summarize, by Lemma 5.3.12, the coverings in each subset $\Phi_{\mathcal{O},k}$ are short, while the coverings $\alpha \lhd \beta$ with $\alpha \in \Phi_{\mathcal{O},k}$ and $\beta \in \Phi_{\mathcal{O},k+1}$ for some k, are long. In particular, by Lemma 5.3.8, it follows that $\Sigma = \bigcup_{\mathcal{O}} \Sigma_{\mathcal{O}}$. Moreover, if $\alpha \in \Phi_{\mathcal{O},k}$, then any saturated chain from Δ to α contains k long edges. Therefore, by Proposition 5.1.14 we have just showed the following statement.

Corollary 5.3.13. *If $\alpha \in \Phi_{\mathcal{O},k}$, then $\mathrm{dp}_\infty(\alpha) = k$.*

5.3.2 The generating function of the depth

In this section, we provide a nice formula for the generating function of the depth of positive roots, using the restriction of the root poset to the special subsets $\Sigma_{\mathcal{O}}$ and $\Phi_{\mathcal{O}}$ introduced in the previous section.

Define $M_{\mathcal{O}} := \max\{\mathrm{dp}(\beta) \mid \beta \in \Sigma_{\mathcal{O}}\}$. We want to show that the depth of a positive root depends only on $M_{\mathcal{O}}$ and on the depth of a root in $\mathcal{O}^+$.

Corollary 5.3.14. *Let $\alpha \in \mathcal{O}^+$ and $k \in \mathbb{N}$, then $\mathrm{dp}(k\delta+\alpha) = \mathrm{dp}(\alpha)+k(M_{\mathcal{O}}+1)$ and $\mathrm{dp}((k+1)\delta-\alpha) = \mathrm{dp}(\delta-\alpha)+k(M_{\mathcal{O}}+1)$.*

Proof. By Theorem 5.3.10, the lowest depth in $\Phi_{\mathcal{O},k}$ is $k(M_{\mathcal{O}}+1)$. It suffices to show that the only roots in $\Phi_{\mathcal{O},k}$ that have depth equal to $k(M_{\mathcal{O}}+1)$ are $\{k\delta+\alpha_s, (k+1)\delta-\omega \mid s \in S\}$. Note that $B(k\delta+\alpha_s,\alpha_s) = B(\alpha_s,\alpha_s) > 0$, and $s(k\delta+\alpha_s) = k\delta-\alpha_s \in \Phi_{\mathcal{O},k-1}$. So we have the cover $k\delta-\alpha_s \lhd k\delta+\alpha_s$. Analogously, we also have the cover $k\delta-\omega \lhd (k+1)\delta-\omega$, where $k\delta-\omega \in \Phi_{\mathcal{O},k-1}$. Hence $\{k\delta+\alpha_s, (k+1)\delta-\omega \mid s \in S\}$ are the roots with lowest depth in $\Phi_{\mathcal{O},k}$ and there are no more roots with this property. □

Consider the polynomials

$$P_{\mathcal{O}}(q) := \sum_{\beta\in\mathcal{O}^+} (q^{\mathrm{dp}(\beta)} + q^{M_{\mathcal{O}}-\mathrm{dp}(\beta)}) \quad \text{and} \quad \Phi_{\mathcal{O}}(q) := \sum_{\beta\in\Phi_{\mathcal{O}}} q^{\mathrm{dp}(\beta)},$$

which are generating functions of the depth restricted respectively to $\Sigma_{\mathcal{O}}$ and $\Phi_{\mathcal{O}}$.

Proposition 5.3.15. *We have that $P_{\mathcal{O}}(q)$ is palindromic of degree $M_{\mathcal{O}}$ and*

$$\Phi_{\mathcal{O}}(q) = \frac{P_{\mathcal{O}}(q)}{1-q^{M_{\mathcal{O}}+1}}.$$

Proof. Since $\Sigma_{\mathcal{O}}$ is finite and the coefficients of $P_{\mathcal{O}}$ are non-negative, $\deg(P_{\mathcal{O}})$ is equal to the maximum between the degree of $\sum_{\beta\in\Sigma_{\mathcal{O}}} q^{\mathrm{dp}(\beta)}$ and the degree of $\sum_{\beta\in\Sigma_{\mathcal{O}}} q^{M_{\mathcal{O}}-\mathrm{dp}(\beta)}$, which is equal to $M_{\mathcal{O}}$. Now we need to check that $q^{\deg(P_{\mathcal{O}})}P_{\mathcal{O}}(q^{-1}) = P_{\mathcal{O}}$. So, $q^{M_{\mathcal{O}}}P_{\mathcal{O}}(q^{-1}) = \sum_{\beta\in\Sigma_{\mathcal{O}}} q^{M_{\mathcal{O}}-\mathrm{dp}(\beta)} + \sum_{\beta\in\Sigma_{\mathcal{O}}} q^{M_{\mathcal{O}}-M_{\mathcal{O}}+\mathrm{dp}(\beta)} = P_{\mathcal{O}}(q)$, hence $P_{\mathcal{O}}$ is palindromic.

By Theorem 5.3.10 and Corollary 5.3.2, if we consider the polynomial associated to the root poset restricted to $\Phi_{\mathcal{O},k}$ for some $k \geq 0$, we have

$$\sum_{\beta\in\Phi_{\mathcal{O},k}} q^{\mathrm{dp}(\beta)} = q^{k(M_{\mathcal{O}}+1)}P_{\mathcal{O}}(q).$$

Hence,

$$\Phi_{\mathcal{O}}(q) = \sum_{k\in\mathbb{N}}\sum_{\beta\in\Phi_{\mathcal{O},k}} q^{\mathrm{dp}(\beta)} = \sum_{k\in\mathbb{N}} q^{k(M_{\mathcal{O}}+1)}P_{\mathcal{O}}(q) = \frac{P_{\mathcal{O}}(q)}{1-q^{M_{\mathcal{O}+1}}}.$$

□

Example 5.3.16. Consider (W,S) of type $\widetilde{B}_3$. Then $M_{\mathcal{O}_1} = 4$, $M_{\mathcal{O}_2} = 3$, so $M = lcm(5,4) = 20$. Hence we have the polynomials corresponding to the two orbits: $P_{\mathcal{O}_1}(q) = 1+q+2q^2+q^3+q^4$ and $P_{\mathcal{O}_2}(q) = 3+3q+3q^2+3q^3$.

By combining the polynomials associated to each orbit, we obtain our final result.

Theorem 5.3.17. *The generating function of the depth of positive roots is of the form*

$$\Phi^+(q) = \frac{P(q)}{1-q^M}$$

where $M = lcm(M_{\mathcal{O}} + 1 \mid \mathcal{O}$ *orbit of* $\Phi)$ *and* P *is palindromic.*

Proof. By construction, $\Phi^+(q) = \sum_{i=1}^k \Phi_{\mathcal{O}_i}(q)$, where k is the number of orbits of Φ. For simplicity, set $P_{\mathcal{O}_i} = P_i$ and $M_{\mathcal{O}_i} = M_i$ for all i, and let $M := lcm(M_i+1)_{i=1}^k$. Then we may rewrite $\Phi^+(q)$ as

$$\Phi^+(q) = \frac{\sum_{i=1}^k \frac{1-q^M}{1-q^{M_i+1}} P_i(q)}{1-q^M} =: \frac{P(q)}{1-q^M}.$$

Now we want to show that $P(q)$ is palindromic. Call $D_i(q) := \frac{1-q^M}{1-q^{M_i+1}}$ for every i, then it is not difficult to see that $\deg(D_i(q)) = M - m_i - 1$ and that $D_i(q) = \sum_{j=1}^{M/(m_i+1)} q^{M-j(m_i+1)}$, which is palindromic. In particular, $\deg(D_i(q)P_i(q)) = M-1$ for all i, so $\deg(P(q)) = M-1$. We have

$$\begin{aligned}
q^{M-1}P(q^{-1}) &= \sum_{i=1}^k q^{M-1} D_i(q^{-1})P_i(q^{-1}) \\
&= \sum_{i=1}^k q^{M-m_i-1} D_i(q^{-1}) q^{m_i} P_i(q^{-1}) \\
&= \sum_{i=1}^k D_i(q)P_i(q) = P(q)
\end{aligned}$$

where the third equality holds since $D_i(q)$ and $P_i(q)$ are palindromic for all i (Proposition 5.3.15). Therefore, $P(q)$ is palindromic. □

Example 5.3.18. Now we want to give the explicit generating function of the depth in the situation of Example 5.3.16. In particular, we have $D_{\mathcal{O}_1}(q) = 1 + q^5 + q^{10} + q^{15}$ and $D_{\mathcal{O}_2}(q) = 1 + q^4 + q^8 + q^{12} + q^{16}$. Hence

$$\begin{aligned}
\Phi(q) =& (4q^{19} + 4q^{18} + 5q^{17} + 4q^{16} + 4q^{15} + 4q^{14} + 4q^{13} + 5q^{12} + 4q^{11} + 4q^{10} \\
&+ 4q^9 + 4q^8 + 5q^7 + 4q^6 + 4q^5 + 4q^4 + 4q^3 + 5q^2 + 4q + 4)(1-q^{20})^{-1}.
\end{aligned}$$

Appendix A

Examples of factorizations

In what follows we give explicit applications of the cp-operation and the K_L-operation defined in Definition 3.7.7 and in the proof of Proposition 3.7.10, respectively. By iterating such operations we recover a factorization for each of the three diagrams. The first label in the pictures indicates the simple diagram associated to the simple edge used at each step while the one in brackets the corresponding local operation in Figure 3.29 that is applied.

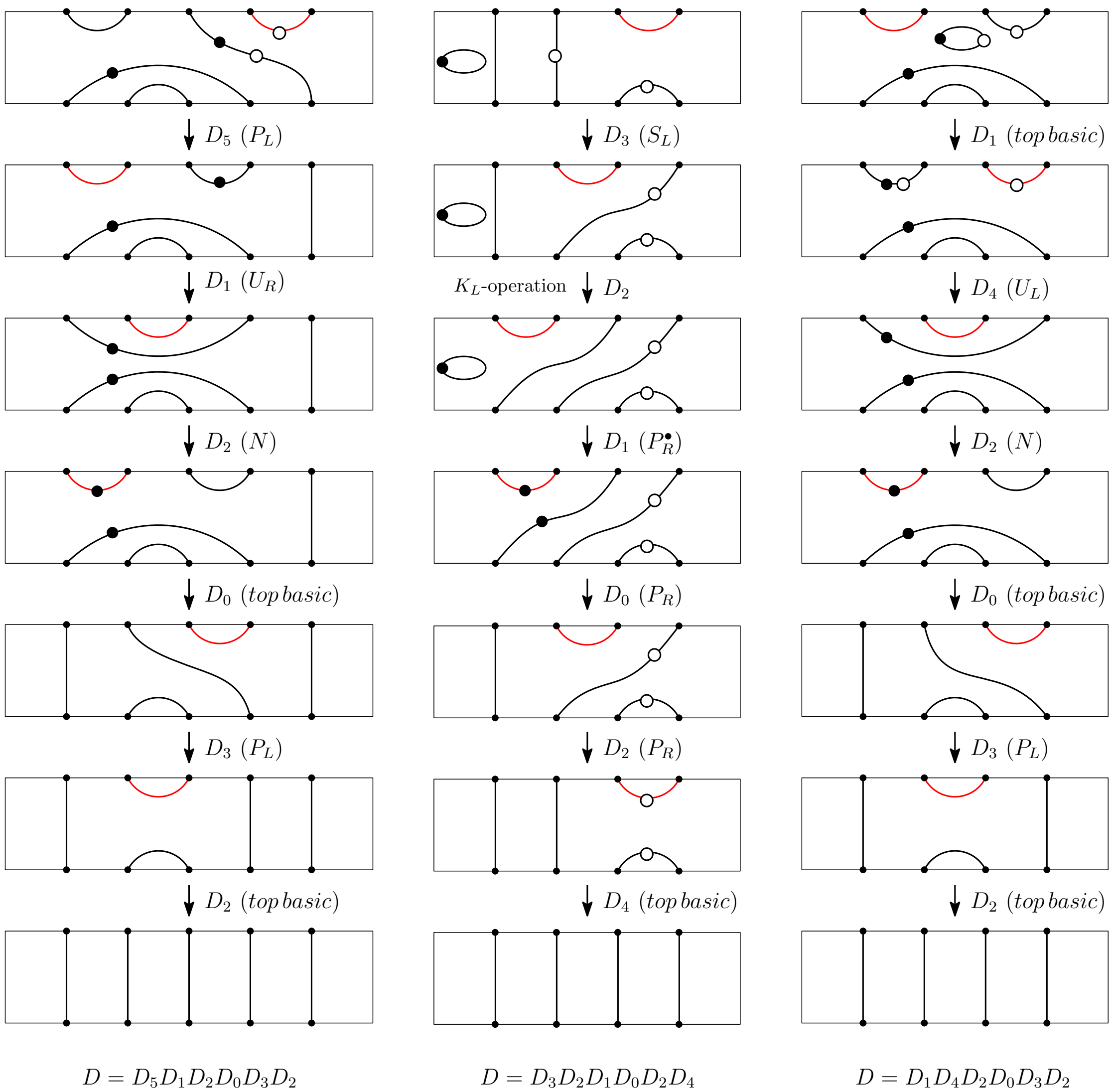


Figure 5.9: Examples of cp and K_L operations.

e	$\mathbf{d}$	w	$\nu(D) - \nu(\mathrm{cp}_e(D))$
N	$\bullet^\alpha \circ^\beta$	$\mathsf{w}(j,k)_D = \alpha(j-1) + \beta(n+2-k)$ $\mathsf{w}(i+1,k)_{\mathrm{cp}_e(D)} = \beta(n+2-k)$ $\mathsf{w}(j,i)_{\mathrm{cp}_e(D)} = \alpha(j-1)$	2
U_R	$\bullet \circ^\beta$	$\mathsf{w}(i+2,k)_D = i+1+\beta(n+2-k)$ $\mathsf{w}(i,k)_{\mathrm{cp}_e(D)} = i-1+\beta(n+2-k)$	-2
P_R			2
	$\bullet(\circ\,\bullet)^p$	$\mathsf{w}(i+2,j')_D = j-1+p(n+1)$ $\mathsf{w}(i,j')_{\mathrm{cp}_e(D)} = j-1+p(n+1)$	
	$\circ(\bullet\,\circ)^p$	$\mathsf{w}(i+2,j')_D = n-i+p(n+1)$ $\mathsf{w}(i,j')_{\mathrm{cp}_e(D)} = p(n+1),$ $\mathsf{w}(i+1,i+2)_{\mathrm{cp}_e(D)} = n-i$	
	$(\bullet\,\circ)^{p+1}$	$\mathsf{w}(i+2,j')_D = (p+1)(n+1)$ $\mathsf{w}(i,j')_{\mathrm{cp}_e(D)} = (p+1)(n+1)$	
	$(\circ\,\bullet)^{p+1}$	$\mathsf{w}(i+2,j')_D = (p+1)(n+1) - i - 2 + j$ $\mathsf{w}(i+1,i+2)_{\mathrm{cp}_e(D)} = n-i$ $\mathsf{w}(i,j')_{\mathrm{cp}_e(D)} = j-1+p(n+1),$	
$P_R^\bullet$		$\mathsf{w}(\mathcal{L}_\bullet)_D = 1$	0
S_R			-2
	$\bullet(\circ\,\bullet)^p$	$\mathsf{w}(i+2,(i+2)')_D = i+1+p(n+1)$ $\mathsf{w}(i,(i+2)')_{\mathrm{cp}_e(D)} = i-1+p(n+1)$	
	$\circ(\bullet\,\circ)^p$	$\mathsf{w}(i+2,(i+2)')_D = n-i+p(n+1)$ $\mathsf{w}(i,(i+2)')_{\mathrm{cp}_e(D)} = p(n+1)-2,$ $\mathsf{w}(i+1,i+2)_{\mathrm{cp}_e(D)} = n-i$	
	$(\bullet\,\circ)^{p+1}$	$\mathsf{w}(i+2,(i+2)')_D = (p+1)(n+1)$ $\mathsf{w}(i,(i+2)')_{\mathrm{cp}_e(D)} = (k+1)(n+1)-2$	
	$(\circ\,\bullet)^{p+1}$	$\mathsf{w}(i+2,(i+2)')_D = (p+1)(n+1)$ $\mathsf{w}(i,(i+2)')_{\mathrm{cp}_e(D)} = i-1+p(n+1),$ $\mathsf{w}(i+1,i+2)_{\mathrm{cp}_e(D)} = n-i$	

Table 5.1: Computations for the proof of statement (3) of Lemma 3.7.8.

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