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ON THE GEOMETRY OF SOME FANO AND HYPERKÄHLER VARIETIES VIA HOMOGENEOUS SPACES

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Introduction

Algebraic geometry is the study of algebraic varieties, which, roughly speaking, can be described as zero loci of polynomials. Although the goal of the subject can be stated in such simple terms, one can quickly fall into a deep and intricate abyss. As in every field of human inquiry, where there is a tendency to bring order to the objects of study by producing classifications and encyclopedias, the same desire arises in algebraic geometry.

The attributes used to produce such classifications are many, but perhaps one of the most historically relevant is the ampleness of the anticanonical bundle. This property provides a broad subdivision, in each dimension, into varieties with ample, trivial, or non-ample anticanonical bundle. Varieties with ample anticanonical bundle are the Fano varieties, while those with ample canonical bundle are said to be of general type. Varieties with trivial canonical bundle are built from abelian varieties, strict Calabi–Yau varieties, and hyperkähler varieties.

In this thesis, we study aspects of the Fano and hyperkähler worlds, with the long-term goal of understanding the interplay between these two realms.

We now outline the state of the art of the different facets of this work. For a more detailed discussion, we refer the reader to the introductions of the individual chapters.

We begin by contextualizing Fano varieties. The study of Fano varieties is a central topic in birational geometry. Current research directions include the Minimal Model Program (MMP), K-stability, and the program led by C. Casagrande.

Fano varieties are formally defined as varieties with ample anticanonical bundles, and they are, in some sense, the simplest possible shapes. Kollár, Miyaoka, and Mori proved that, in each dimension, there exists only a finite number of deformation classes of smooth Fano manifolds. There is a single Fano manifold in dimension one, the projective line. In dimension two, there are ten families of Fano manifolds, the Del Pezzo surfaces, already known to Del Pezzo. The classification of smooth Fano threefolds began with the work of Fano himself and was continued by Iskovskikh. Eventually, thanks to the work of S. Mori and S. Mukai, it was proven that there exist 105 families of smooth Fano threefolds, explicitly described.

In dimension four and higher, the classification problem remains widely open. In dimension four, there are several different approaches to tackle the problem. For example, V. Batyrev classified toric Fano fourfolds.

Moreover, there are various programs that aim to generalize the birational and biregular methods used in dimension three, but each still seems far from achieving a complete classification. These approaches have produced a large number of examples of Fano manifolds through *computer algebra*. These families still need to be compared in order to obtain a clearer picture of the landscape. In particular, we recall three databases of Fano fourfolds produced in different ways:

- T. Coates, A. Kasprzyk, and T. Prince produced a database of 738 deformation families of Fano fourfolds that are complete intersections in toric varieties;
- E. Kalashnikov produced a database of 749 deformation families of Fano fourfolds realized as quiver flag zero loci (zero loci of sections in special quiver moduli spaces);
- M. Bernardara, E. Fatighenti, L. Manivel, and F. Tanturri produced a database of at least 634 deformation families of Fano fourfolds that are zero loci of sections in products of Grassmannians.

Each database is constructed differently and has distinct aims. The first two are oriented toward the computation of the quantum period, a conjectural invariant of deformation families. The third one is better suited for computing Hodge numbers as deformation invariants and, more importantly, for understanding biregular and birational descriptions. A natural question that arises is how these databases intersect. Using the quantum period, the first two databases can be compared. In particular, in the second one there are 141 Fano manifolds with previously unknown quantum periods.

It is more complicated to compare the third database with the others. The first step consists in establishing a correspondence between quiver flag zero loci and section zero loci in products of Grassmannians. E. Fatighenti, F. Tanturri, and the author successfully constructed such a correspondence between these two settings, inspired by the work of A. Craw and E. Kalashnikov. We also obtained the full birational description of the 141 Fano varieties mentioned above, which led to Chapter 2.

This opens the possibility of extending this correspondence to better understand the geometry of quiver moduli spaces, which are in general more difficult to study.

The third database also provides a natural framework for identifying Fano varieties of K3-type, which form a link between the two main themes of this thesis. Fano varieties of K3-type are Fano manifolds whose Hodge diamond exhibits a K3-structure. They were introduced by A. Iliev and L. Manivel in [**IM15**], and later studied by E. Fatighenti and G. Mongardi in [**FM21**], and by M. Bernardara, E. Fatighenti, L. Manivel, and F. Tanturri in [**BFMT21**].

Such varieties play a key role in the construction of explicit examples of hyperkähler manifolds, starting from the Fano variety of lines on a smooth cubic fourfold, as shown by Beauville and Donagi. The cubic fourfold is the archetypal example of a Fano variety of K3-type. Many other Fano varieties of K3-type have given rise to examples of hyperkähler manifolds, including the Gushel–Mukai fourfold, the Debarre–Voisin twentyfold, the Verra fourfold, and several others described by E. Fatighenti and G. Mongardi.

The general principle is that a Fano of K3-type admits a semiorthogonal component which is a (possibly non-commutative) K3 category, and suitable moduli spaces of objects in this category often give rise to hyperkähler manifolds. These moduli spaces can be constructed either by classical geometric methods or via moduli of Bridgeland-stable objects in these K3 subcategories.

This idea motivates, in Chapter 4, the study of the mysterious family of Fano fourfolds of K3-type known as the Küchle $c5$.

We have mentioned hyperkähler manifolds without introducing them formally. Hyperkähler manifolds are compact, simply connected Kähler manifolds endowed with a unique up to scalar, holomorphic, non-degenerate symplectic two-form. Beauville and Bogomolov proved that hyperkähler manifolds, together with complex tori and Calabi–Yau varieties, constitute the building blocks of manifolds with trivial canonical bundle. Despite this, only a handful of examples are known: Beauville constructed two deformation families in each even dimension (the $K3^{[n]}$-type and the generalized Kummer varieties), while O'Grady found two additional sporadic families in dimensions six and ten (OG6 and OG10).

The lack of examples has directed research along two main lines: the search for new classes and explicit examples, and the study of general properties under the assumption that other examples exist. In this thesis we focus on the first direction.

As discussed above, the study of Fano varieties of K3-type is deeply intertwined with that of hyperkähler manifolds, although they are not the only source of such examples. Another approach arises from moduli spaces of (semi)stable sheaves on K3 or abelian surfaces. This method is particularly powerful, as, possibly after desingularization, it encompasses all known examples.

A natural next step is to study the moduli spaces of sheaves on hyperkähler manifolds of dimension greater than two. In this setting, a notion that replaces that of stable coherent sheaves is that of stable modular sheaves. Introduced by O'Grady, these sheaves serve as analogues of stable coherent sheaves on K3 and abelian surfaces.

The first examples of modular sheaves on the Beauville–Donagi and Debarre–Voisin hyperkähler manifolds, found by O'Grady, were rigid, i.e. their moduli spaces have dimension zero. More recently, additional examples were found by A. Bottini, E. Fatighenti, and C. Onorati, and later by K. O'Grady. In particular, E. Fatighenti and C. Onorati, in [**FO24**], constructed infinite families of stable modular sheaves on the Beauville–Donagi hyperkähler manifold using representation-theoretic methods, finding examples whose moduli spaces have dimension or expected dimensions 20 and 40.

Discovering further examples and understanding their properties represents a new and exciting direction of research. This motivates the search for modular sheaves also on the Debarre–Voisin fourfold, which is the subject of Chapter 3.

Content of the thesis. This thesis is organized into four chapters. Chapter 1 provides a concise overview of the background material and revisits classical tools in the study of homogeneous varieties. It introduces the main techniques employed to compute Hodge numbers and explains the philosophy behind choosing homogeneous varieties (and in particular Grassmannians) as our main setting.

Chapter 2 and Chapter 3 are self-contained and originate from author's previous works.

In particular, Chapter 2, based on the preprint [**FTT24**] coauthored with Enrico Fatighenti and Fabio Tanturri, presents a systematic analysis of the families first introduced by Kalashnikov in [**Kal19**]. In that work, Kalashnikov classified 141 families of Fano fourfolds with new quantum periods (among a database of 749), realized as zero loci in quiver flag manifolds and not obtainable as toric complete intersections. She further identified 29 additional families that do not appear in toric quiver flag varieties, but their quantum period corresponds to the one of some toric complete intersections.

In this chapter, we revisit these 170 families and reinterpret them as zero loci of sections on products of Grassmannians, making use of Corollary 2.2.24. Their birational structures are summarized in Theorem 2.1.1 and analyzed in detail in Sections 2.4 to 2.7. Specifically, the 170 Fano fourfold families can be grouped as follows:

- 9 have Picard rank $\rho = 1$ and are already known;
- 124 are blow-ups of smooth Fano fourfolds (from the list of 749) along smooth centers;
- 8 are products of lower-dimensional Fano varieties;

- Fano 2-43, Fano 2-48, and Fano 3-50 are projective bundles over Fano threefolds, none of which can be realized as blow-ups;
- 8 are conic bundles, not realizable as blow-ups or projective bundles;
- Fano 2-1 and Fano 2-35 are stratified projective bundles over Fano threefolds;
- 16 are small resolutions of singular fourfolds, not realizable as blow-ups, conic bundles, projective bundles, or stratified projective bundles.

Chapter 3 is based on the preprint [FT25], coauthored with Alessandro Frassineti. Here we prove that all the Schur functors of the restriction of the quotient bundle of $\mathrm{Gr}(6, 10)$ to X are modular and polystable vector bundles. We focused on applying the techniques developed in [FO24], for the Beauville–Donagi fourfold, to the Debarre–Voisin fourfold. Since both the Beauville–Donagi fourfold and Debarre–Voisin fourfolds are locally complete families of $\mathrm{K3}^{[2]}$-type, our goal is to study modular sheaves on a very general Debarre–Voisin fourfold $X_{DV} \subseteq \mathrm{Gr}(6, 10)$.

We further compute the Ext-groups of various modular vector bundles on X, collecting the resulting dimensions in Table 1. Notably, we identify examples whose first Ext-groups have dimensions 20 and 40, matching the dimensions found in [FO24] for Schur functors of $\mathcal{Q}$ with the same weight. Finally, we demonstrate in Proposition 3.7.5 that these bundles are atomic if and only if they correspond to symmetric powers of the restriction of the quotient bundle.

Chapter 4 is different. It presents an ongoing project with Marcello Bernardara and Enrico Fatighenti, which will soon appear as a preprint on arXiv. In this chapter, we use the techniques and results developed in [FTT24] to study the geometry of a mysterious Fano fourfold of K3-type, the Küchle $c5$ fourfold.

The geometry of this fourfold has been investigated for years, and many questions remain open. The two main problems we aim to address concern its K3-structure and its rationality. To this end, we have constructed sixteen varieties related to a general Küchle $c5$ fourfold and analyzed each of them. During this process, we identified three fourfolds that are birational to the $c5$, that might provide new insights into its geometry.

We then decided to approach the problem by studying degenerations of the Küchle $c5$, focusing on possible degenerations of the three birational models we had found. In Theorem 4.1.2 we prove the rationality of three singular degenerations of the Küchle $c5$ fourfold.

Notation. Throughout this thesis we work over $\mathbb{C}$. We denote by V_n an n-dimensional vector space, and by $\mathrm{Gr}(k, n)$ the Grassmannian parametrizing its k-dimensional subspaces; similarly for the flag varieties $\mathrm{Fl}(k_1, k_2, \dots, k_t, n)$.

We denote by $\mathcal{U}$ and $\mathcal{Q}$ the tautological and quotient bundles on $\mathrm{Gr}(k, n)$, of ranks k and $n-k$, respectively. We follow the convention that $\det \mathcal{Q} \cong \mathcal{O}(1)$.

If α is a partition, we denote by $\Sigma_\alpha \mathcal{U}$, $\Sigma_\alpha \mathcal{Q}$, and $\Sigma_\alpha V_n$ the Schur functors applied to $\mathcal{U}$, $\mathcal{Q}$, and V_n, respectively, associated with α. Similarly, on flag varieties, $\mathcal{U}_i$ and $\mathcal{Q}_i$ denote the tautological and quotient bundles pulled back from the corresponding base Grassmannians, and $\mathcal{R}_i$ the intermediate tautological bundles defined by the exact sequence

$$0 \to \mathcal{U}_{i-1} \to \mathcal{U}_i \to \mathcal{R}_i \to 0.$$

Moreover, Grassmann bundles also admit a relative Euler sequence; we denote all relative bundles with the index $\mathcal{R}$, for instance, the relative tautological bundle is $\mathcal{U}_{\mathcal{R}}$.

We also adopt the following conventions: if X is a complex variety and $\mathcal{F}$ is a vector bundle on X, we denote by $\mathscr{Z}(\mathcal{F}) \subset X$ the zero locus of a generic section of $\mathcal{F}$. If $X = \mathbb{P}^{n+1}$ and $\mathcal{F} = \mathcal{O}(2)$, hence defining an n-dimensional quadric, we denote it by Q_n. Given a morphism φ between vector bundles on X, we denote by $\mathrm{D}_k(\varphi) \subset X$ the k-th degeneracy locus, i.e., the locus of points in X where the morphism φ has rank at most k.

Finally, we denote del Pezzo surfaces by dP_d, where d is the degree of the surface.

Additionally, in Chapter 2, Chapter 3, and Chapter 4, we introduce some specific notation used only within those chapters:

- In Chapter 2, if $X = \mathrm{Gr}(k, n)$ and $\mathcal{F} = \mathcal{O}(1)^{\oplus k(n-k)-d}$, we denote the d-fold $\mathscr{Z}(\mathcal{F}) \subset X$ by X_n^d. Moreover, we refer to K_n as the n-th Fano fourfold in [**Kal19**, Table 3].
- In Chapter 3, for simplicity of notation in computations, we denote by $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{Q}}$ the tautological and quotient bundles on $\mathrm{Gr}(k, n)$, respectively, while $\mathcal{U}$ and $\mathcal{Q}$ denote their restrictions to a subvariety $\mathscr{Z}(\mathcal{F}) \subset \mathrm{Gr}(k, n)$.
- In Chapter 4, we denote the general Küchle $c5$ fourfold by X_{c5}^4.

CHAPTER 1

Homogeneous varieties as setting

1.1. Homogeneous varieties and Grassmannians

One of the main problem in algebraic geometry is to construct explicit examples of algebraic varieties. There are several approaches to producing such examples. A fruitful strategy is to obtain algebraic variety from algebraic construction, such as toric varieties or quiver moduli spaces. The former are associated with polytopes, while the latter are associated with quivers.

In this chapter we introduce the method we use to construct our examples, i.e. homogeneous varieties. For background and further details, we refer to the standard references [**Ott95**], [**Sno89**] and [**Wey03**].

Definition 1.1.1. *Let G be a connected, simply connected semi-simple complex Lie group and let P be a parabolic subgroup of G. The quotient G/P carries a structure of smooth projective variety and is called homogeneous variety.*

Definition 1.1.2. *A vector bundle E on G/P is called homogeneous if the natural action of G by multiplication on the base lifts to E. In other words, the following diagram commutes:*

$$\begin{tikzcd} G \times E \arrow[r] \arrow[d] & E \arrow[d] \\ G \times G/P \arrow[r] & G/P \end{tikzcd}$$

We now introduce the main source of examples: the Grassmannians.

Definition 1.1.3. *The Grassmannian $\mathrm{Gr}(k,n)$ is the variety whose points parametrizes the k-dimensional subspaces of the n-dimensional vector space V_n.*

This definition fixes our notation. In fact, in the literature one sometimes finds the dual notation, where $\mathrm{Gr}(k,n)$ denotes the Grassmannian of quotient spaces of dimension k. With our notation $\mathbb{P}^n$ is $\mathrm{Gr}(1, n+1)$.

Remark 1.1.4. *Grassmannians are homogeneous varieties. The group $\mathrm{SL}(n)$ acts transitively on $\mathrm{Gr}(k,n)$. So if we define the subgroup of $\mathrm{SL}(n)$*

$$P := \left\{ \begin{pmatrix} A & * \\ 0 & B \end{pmatrix} \in \mathrm{SL}(n) \;\middle|\; A \in \mathrm{SL}(k),\ B \in \mathrm{SL}(n-k) \right\}.$$

then $Gr(k,n) = \mathrm{SL}(n)/P$.

Grassmannians carry two natural bundles: the tautological bundle and the quotient bundle.

Definition 1.1.5. *Let us consider $\mathrm{Gr}(k,n)$. We define the tautological bundle to be the vector bundle $\pi : \mathcal{U}_{\mathrm{Gr}(k,n)} \to \mathrm{Gr}(k,n)$ s.t. $\forall p \in \mathrm{Gr}(k,n)$, $\pi^{-1}(p) = V_k$, i.e. the k-dimensional subspace* *of V_n parametrized by p. We define the quotient bundle to be the vector bundle $\pi : \mathcal{Q}_{\mathrm{Gr}(k,n)} \to \mathrm{Gr}(k,n)$ s.t. $\forall p \in \mathrm{Gr}(k,n)$, $\pi^{-1}(p) = V_n/V_k$, i.e. the (n-k)-dimensional quotient space of V_n, with V_k as above.*

These bundles fit an analogue of the Euler sequence for projective spaces:

$$0 \to \mathcal{U}_{\mathrm{Gr}(k,n)} \to \mathcal{O}_{\mathrm{Gr}(k,n)} \otimes V_n \to \mathcal{Q}_{\mathrm{Gr}(k,n)} \to 0.$$

Throughout this work, we refer to this sequence as Euler sequence.

Remark 1.1.6. *The tautological and the quotient bundle are homogeneous by definition. In particular bundles on $\mathrm{Gr}(k,n)$ can be associated to the representations of $\mathrm{SL}(n)$ and $\mathrm{SL}(k)\times\mathrm{SL}(n-k)$.*

Recall that the irreducible representations of $\mathrm{SL}(n)$ are parametrized by partitions $\lambda \in \mathbb{Z}^n$ of length n, as it is shown in [**Wey03**, Theorem 2.2.9]. These partitions are in bijection with Young diagrams, and the corresponding representations are given by the Schur functors $\Sigma_\lambda V_n$, as shown in [**Wey03**, Chapter 2]. One may view Schur functors as a natural generalization of exterior and symmetric powers. In particular:

- if $\lambda = (1,0,\dots,0)$ then $\Sigma_\lambda V_n = V_n$;
- if $\lambda = (1,1,\dots,1,0,..,0)$ with the last 1 in the t-th position, then $\Sigma_\lambda V_n = \bigwedge^t V_n$;
- if $\lambda = (d,0,\dots,0)$ then $\Sigma_\lambda V_n = \mathrm{Sym}^d V_n$;
- if $\lambda + d = (\lambda_1 + d,\dots,\lambda_n + d)$ then $\Sigma_{\lambda+d} V_n = \Sigma_\lambda V_n \otimes (\bigwedge^n V_n)^{\otimes d}$;
- if $\lambda = (\lambda_1,\dots,\lambda_n)$ then the dual Schur functor $\Sigma_\lambda V_n^\vee$ is $\Sigma_{\lambda^*} V_n$, with $\lambda^* = (-\lambda_n,\dots,-\lambda_1)$.

Now, if we have two partitions λ and μ, of length k and $n-k$ respectively, we set

$$\Sigma_{\mu|\lambda} = \Sigma_\mu V_{n-k} \otimes \Sigma_\lambda V_k$$

which denotes the irreducible representation of $\mathrm{SL}(k)\times\mathrm{SL}(n-k)$ indexed by (μ,λ). In this way, the vector bundle $\Sigma_\mu\mathcal{Q}\otimes\Sigma_\lambda\mathcal{U}$ is represented by $\Sigma_{\mu|\lambda} = \Sigma_\mu V_{n-k}\otimes\Sigma_\lambda V_k$. Moreover, we have the following correspondeces.

Remark 1.1.7. *With the above notation it follows that:*

- *$\mathcal{U}$ corresponds to $\Sigma_{\mu|\lambda}$ with $\mu = (0,\dots,0)$ and $\lambda = (1,0,\dots,0)$;*
- *$\mathcal{Q}$ corresponds to $\Sigma_{\mu|\lambda}$ with $\mu = (1,0,\dots,0)$ and $\lambda = (0,\dots,0)$;*
- *$\mathcal{O}(-1) = det\mathcal{U}$ corresponds to $\Sigma_{\mu|\lambda}$ with $\mu = (0,\dots,0)$ and $\lambda = (1,\dots,1)$;*
- *$\mathcal{O}(1) = det\mathcal{Q}$ corresponds to $\Sigma_{\mu|\lambda}$ with $\mu = (1,\dots,1)$ and $\lambda = (0,\dots,0)$;*
- *$\mathcal{O} = \mathcal{O}(d)\otimes\mathcal{O}(-d)$ hence $\Sigma_{\mu|\lambda} = \Sigma_{\mu+d|\lambda+d}$;*
- *$\Sigma_\lambda\mathcal{U}^\vee$ corresponds to $\Sigma_{\mu|\lambda^*}$ with $\mu = (0,\dots,0)$ and $\lambda^* = (-\lambda_k,\dots,-\lambda_1)$, which can be rewritten tensoring with $\mathcal{O} = \mathcal{O}(\lambda_1)\otimes\mathcal{O}(-\lambda_1)$. This corresponds to consider $\Sigma_{\mu+\lambda_1|\lambda^*+\lambda_1}$;*
- *$\Sigma_\mu\mathcal{Q}^\vee$ corresponds to $\Sigma_{\mu^*|\lambda}$ with $\mu = (-\mu_{n-k},\dots,-\mu_1)$ and $\lambda = (0,\dots,0)$, which can be rewritten tensoring with $\mathcal{O} = \mathcal{O}(\mu_1)\otimes\mathcal{O}(-\mu_1)$. This corresponds to consider $\Sigma_{\mu^*+\mu_1|\lambda+\mu_1}$;*

This produces infinitely many homogeneous vector bundles on $\mathrm{Gr}(k,n)$ of the form $\Sigma_{\mu|\lambda}$.

What we have discussed so far can be generalized to products of Grassmannians. For instance, consider

$$G = \mathrm{Gr}(k_1,n_1)\times\mathrm{Gr}(k_2,n_2).$$

This is again a homogeneous variety. If $\mathcal{E}_1$ and $\mathcal{E}_2$ tare homogeneous vector bundles on $\mathrm{Gr}(k_1,n_1)$ and $\mathrm{Gr}(k_2,n_2)$ respectively, then $\mathcal{E} = \mathcal{E}_1\boxtimes\mathcal{E}_2$ is an homogeneous vector bundle on G. Moreover, if $\mathcal{E}$ and $\mathcal{F}$ are homogeneous, then so is $\mathcal{E}\oplus\mathcal{F}$. In this way, the construction extends to arbitrary finite products of Grassmannians:

$$G = \prod_{i=1}^{p} \mathrm{Gr}(k_i, n_i) \text{ and } \mathcal{E} = \bigoplus_{j=1}^{t} (\mathcal{E}_{1,j} \boxtimes \mathcal{E}_{2,j} \boxtimes \cdots \boxtimes \mathcal{E}_{t,j})$$

where $\mathcal{E}$ is again a homogeneous vector bundle.

With this notation, let $\mathcal{E}$ be globally generated and let σ be a global section of $\mathcal{E}$. We define $\mathscr{Z}(\mathcal{E}) \subset G$ to be the zero locus of σ in G. If we denote this variety by

$$X = \mathscr{Z}(\mathcal{E}) \subset G,$$

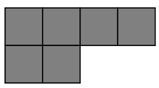
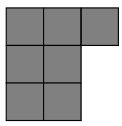
then, if σ is general, X is smooth by Bertini-like Theorem [**Laz**, Section 7.2]. Such varieties constitute the main subject of this thesis. In the next section we describe in detail the general results available in this framework.

1.2. Some practical tools

In the previous section we introduced the setting of our study. In the present one we describe the main tools that can be exploited in this context. A natural first step when investigating the geometry of a variety is to compute its cohomology. This task is not always straightforward, nor is there a single method to achieve it. In our setting, however, we can rely on several combinatorial techniques.

1.2.1. Young diagrams, Borel–Weil–Bott and Littlewood–Richardson. Let us begin with the simplest case, a single Grassmannian $\mathrm{Gr}(k,n)$ and a single bundle of the form $\Sigma_\gamma \mathcal{Q}$ or $\Sigma_\gamma \mathcal{U}$, where $\gamma = (\mu, \lambda)$. As recalled above, the partition γ is $(\mu, 0, \dots, 0)$ in the case of $\Sigma_\gamma \mathcal{Q}$, and $(0, \dots, 0, \lambda)$ in the case of $\Sigma_\gamma \mathcal{U}$. Such partitions can be represented by Young diagrams (or tableaux). More precisely, if $\gamma = (\gamma_1, \dots, \gamma_n)$then the Young diagram has exactly γ_i boxes in the i-th row. Let us illustrate this with two examples in $\mathrm{Gr}(2,5)$:

Example 1.2.1.

$\Sigma_{4,2}\mathcal{U}^\vee \leftrightarrow$  $\Sigma_{3,2,2}\mathcal{Q} \leftrightarrow$ 

Where the Young diagrams are the one associated to the partitions $(4,2)$ *and* $(3,2,2)$ *respectively.*

Our next goal is to compute the cohomology of the bundles $\Sigma_\gamma \mathcal{Q}$ and $\Sigma_\gamma \mathcal{U}$. This reduces to a purely combinatorial procedure:

Lemma 1.2.2. [**Wey03**, Corollary 4.1.9] *Let* $G = \mathrm{Gr}(k,n)$*, let* μ *be a partition of length* $n-k$*, let* λ *be a partition of length* k*. Let* $\gamma = (\mu, \lambda)$ *and let* $\mathcal{E} = \Sigma_\gamma$*. Then one of the following mutually exclusive possibilities occours:*

- *there exists a permutation* $\sigma \in S_n$ *not equal to the identity such that* $\sigma(\gamma + (n-1, \dots, 1, 0)) - (n-1, \dots, 1, 0) = \gamma$*, then* $H^i(G, \mathcal{E}) = 0$ *for all* $i \geq 0$
- *there exists a permutation* $\sigma \in S_n$ *such that* $\sigma(\gamma + (n-1, \dots, 1, 0)) - (n-1, \dots, 1, 0) = \tau$ *is a partition, then*

$$H^i(G, \mathcal{E}) = \begin{cases} 0 \ \forall i \neq l(\sigma) \\ \Sigma_\tau V_n \text{ for } i = l(\sigma) \end{cases}$$

with $l(\sigma)$ *the number of disorders in* $\gamma + (n-1, \dots, 1, 0)$*.*

Let us clarify this statement with an explicit example.

Example 1.2.3. *Consider* $\mathrm{Gr}(2,5)$ *and* $\mathrm{Sym}^2\,\mathcal{U}$ *corresponding to* $\gamma = (0,0,0|2,0)$. *Let us compute* $H^i(\mathrm{Gr}(2,5), \mathrm{Sym}^2\,\mathcal{U})$. *Consider the weight* $\gamma = (0,0,0,2,0)$. *Adding the partition* $(4,3,2,1,0)$ *gives* $(4,3,2,3,0)$. *Note that if we apply the permutation* $\sigma = (2,4)$ *then we obtain again* $(4,3,2,3,0)$. *This tells us that the cohomology is trivial.*

Now consider $\mathrm{Gr}(2,5)$ *and* $\mathrm{Sym}^4\,\mathcal{U}$, *i.e.* $\gamma = (0,0,0,|,4,0)$. *Adding* $(4,3,2,1,0)$ *gives* $(4,3,2,5,0)$. *The permutation* $\sigma = (1,2,3,4)$ *gives* $\sigma(\gamma + (4,3,2,1,0)) - (4,3,2,1,0) = (1,1,1,1,0)$, *which is a partition. Since* $l(\sigma) = 3$, *the only non-vanishing cohomology group is*

$$H^3(\mathrm{Gr}(2,5), \mathrm{Sym}^5\,\mathcal{U}) = \Sigma_{1,1,1,1,0} V_5.$$

Note that $\Sigma_{1,1,1,1,0} V_5 = \bigwedge^4 V_5 = V_5^\vee$.

Finally, to compute the dimension of $\Sigma_\tau V_n$. we recall the following classical result, known as the Weyl character formula:

Lemma 1.2.4. [FH04, Corollary 24.6] *Let* V_n *be a vector bundle of dimension* n, *and let* $\lambda = (\lambda_1, \dots, \lambda_n)$ *be a partition of length* n. *Then the following formula holds true:*

$$dim \Sigma_\lambda V_n = \Pi_{1 \le i < j \le n} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

The combination of Theorem 1.2.2 and Theorem 1.2.4 yields the Borel–Weil–Bott theorem(BWB), which provides a powerful and explicit method to compute the cohomology of all bundles of the form $\Sigma_{\mu|\lambda}$ on the Grassmannian $\mathrm{Gr}(k,n)$.

Next, let us consider the case of two bundles on the same Grassmannian $\mathrm{Gr}(k,n)$. With the above notation, let these be $\Sigma_{\mu|\lambda}$ and $\Sigma_{\mu'|\lambda'}$. To compute

$$H^i(\mathrm{Gr}(k,n), \Sigma_{\mu|\lambda} \oplus \Sigma_{\mu'|\lambda'}),$$

it suffices to apply the additivity of the cohomology. We obtain

$$H^i(\mathrm{Gr}(k,n), \Sigma_{\mu|\lambda} \oplus \Sigma_{\mu'|\lambda'}) = H^i(\mathrm{Gr}(k,n), \Sigma_{\mu|\lambda}) \oplus H^i(\mathrm{Gr}(k,n), \Sigma_{\mu'|\lambda'}).$$

The case $\Sigma_{\mu|\lambda} \otimes \Sigma_{\mu'|\lambda'}$ is different. In this case the bundle is not irreducible anymore, but there is a way to decompose it as a direct sum of irreducible bundles, after which we can again apply the additivity of cohomology. First, observe that the following factorization holds:

$$\Sigma_{\mu|\lambda} \otimes \Sigma_{\mu'|\lambda'} = (\Sigma_\mu \mathcal{Q} \otimes \Sigma_{\mu'} \mathcal{Q}) \otimes (\Sigma_\lambda \mathcal{U} \otimes \Sigma_{\lambda'} \mathcal{U}).$$

Hence we can treat separately the factors $\Sigma_\mu \mathcal{Q} \otimes \Sigma_{\mu'} \mathcal{Q}$ and $\Sigma_\lambda \mathcal{U} \otimes \Sigma_{\lambda'} \mathcal{U}$, and then combine the results.

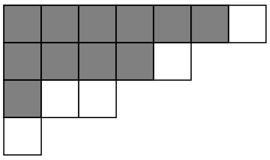
The key fact is the following classical theorem.

THEOREM 1.2.5. [Wey03, Theorem 2.3.4] *Let* λ, λ' *be two partitions of the same length* n. *The tensor product of the associated Schur functors decomposes as*

$$\Sigma_\lambda \otimes \Sigma_{\lambda'} = \bigoplus_\nu N^\nu_{\lambda,\lambda'} \Sigma_\nu.$$

Here the sum runs over all the partitions ν *of* $|\lambda| + |\lambda'|$ *of length* n *and* $N^\nu_{\lambda,\lambda'}$ *are the Littlewood–Richardson's numbers, which can be computed with the following algorithm.*

Algorithm 1.2.6. *[Littlewood–Richardson's rule] Start with the Young diagram corresponding to the partition λ (in gray), and add $|\lambda'|$ boxes so that the result is again a Young diagram corresponding to some partition ν:*



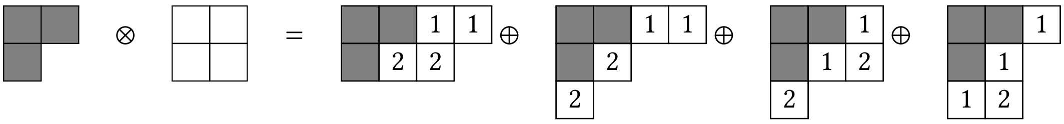
Fill the added boxes (in white) with the content of λ', i.e., with λ'_1 "ones", λ'_2 "twos" and so on, subject to the following three conditions: rules:

- *for each row, the row-word has to be weakly increasing (left to right);*
- *for each column, the column-word has to be strictly increasing (up to down);*
- *starting from top-right, if we list the entries row by row, from right to left, at each step the number of "ones" has to be greater or equal to the number of "twos", the number of "twos" has to be greater or equal to the number of "threes" and so on.*

The number of admissible fillings of the shape ν corresponds to $N^{\nu}_{\lambda,\lambda'}$.

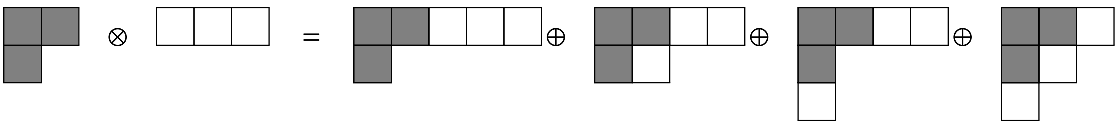
Theorem 1.2.5 is known as the Littlewood–Richardson Theorem, for which we will refear as LR. We now illustrate the algorithm with some examples.

Example 1.2.7. *Let $\lambda = (2,1,0)$ and $\lambda' = (2,2,0)$ then Littlewood–Richardson's rule returns*



Remark 1.2.8. *A special case of Littlewood–Richardson's rule occurs when $\lambda' = (m, 0, \ldots, 0)$, i.e. when we tensor with a symmetric power of a vector bundle. This case is known as Pieri's rule. Here, since all the added boxes must be labeled with m ones, the Littlewood–Richardson's numbers are either zeros or ones. The admissible partitions ν are exactly those obtained from λ by adding m blocks with at most one per column.*

Example 1.2.9. *Let $\lambda = (2,1,0)$ and $m = 3$ then Pieri's rule gives*



What remains is to consider the case of a product of two Grassmannians. Let us consider

$$G = \mathrm{Gr}(k_1, n_1) \times \mathrm{Gr}(k_2, n_2),$$

and let $\mathcal{E}$ be $\Sigma_\gamma \boxtimes \Sigma_{\gamma'}$, where Σ_γ and $\Sigma_{\gamma'}$ are irreducible vector bundles on $\mathrm{Gr}(k_1, n_1)$ and $\mathrm{Gr}(k_2, n_2)$ respectively. In this situation we can directly apply the Künneth formula [**Kem93**, Section 9.2]. We obtain:

$$H^i(G, \mathcal{E}) = \bigoplus_{l+t=i} H^l(\mathrm{Gr}(k_1, n_1), \Sigma_\gamma) \otimes H^t(\mathrm{Gr}(k_2, n_2), \Sigma_{\gamma'}).$$

Thus, the cohomology of such vector bundles can be computed explicitly. In the following we show how this tools help us to compute the cohomology of bundles of varieties obtained as $\mathscr{Z}(\mathcal{E}) \subset G$.

1.2.2. Koszul complex and Hodge numbers. We have explained how to compute cohomology of homogeneous vector bundles on products of Grassmannians G. We now turn to the cohomology of vector bundles on varieties that can be realized as zero loci

$$\mathscr{Z}(\mathcal{E}) \subset G,$$

where $\mathcal{E}$ is a rank r globally generated sum of irreducible homogeneous vector bundles on G.

Let $X := \mathcal{Z}(\mathcal{E}) \subset G$. Then X is a smooth variety equipped with a natural embedding

$$\iota : X \hookrightarrow G.$$

Lemma 1.2.10. [**Eis**, Chapter 17] *In the above setting we have the following resolution:*

$$0 \to \det\left(\mathcal{E}^\vee\right) \to \bigwedge^{r-1} \mathcal{E}^\vee \to \cdots \to \bigwedge^{2} \mathcal{E}^\vee \to \mathcal{E}^\vee \to \mathcal{O}_G \to \iota_*\mathcal{O}_X \to 0.$$

This result is known as Koszul complex, provides a tool to compute the cohomology of vector bundles on X obtained by restricting vector bundles from G.

Example 1.2.11. *Let $\mathcal{F}$ be an homogeneous vector bundle of G. To compute the cohomology of $\mathcal{F}_{|X}$ we twist the complex by $\mathcal{F}$, obtaining:*

$$0 \to \det\left(\mathcal{E}^\vee\right) \otimes \mathcal{F} \to \bigwedge^{r-1} \mathcal{E}^\vee \otimes \mathcal{F} \to \cdots \to \bigwedge^{2} \mathcal{E}^\vee \otimes \mathcal{F} \to \mathcal{E}^\vee \otimes \mathcal{F} \to \mathcal{F} \to \mathcal{F}_{|X} \to 0.$$

This sequence can be decomposed into short exact sequences of the form

$$0 \to \det\left(\mathcal{E}^\vee\right) \otimes \mathcal{F} \to \bigwedge^{r-1} \mathcal{E}^\vee \otimes \mathcal{F} \to K_1 \to 0$$

$$0 \to K_1 \to \bigwedge^{r-2} \mathcal{E}^\vee \otimes \mathcal{F} \to K_2 \to 0$$

$$\cdots$$

$$0 \to K_{i-1} \to \bigwedge^{r-i+)} \mathcal{E}^\vee \otimes \mathcal{F} \to K_i \to 0$$

$$\cdots$$

$$0 \to K_{r-2} \to \mathcal{E}^\vee \otimes \mathcal{F} \to K_r \to 0$$

$$0 \to K_{r-1} \to \mathcal{F} \to \mathcal{F}_{|X} \to 0.$$

Since all tensor products here involve homogeneous vector bundles, the LR rule decomposes them into sums of irreducible bundles. Each short exact sequence gives rise to a long exact sequence in cohomology, and the BWB theorem allows us to compute all cohomologies, except for those of the intermediate bundles K_i. For these, one generally needs to perform diagram chasing. In certain cases ambiguities may occur, which must be resolved either by inspecting the maps directly or by using alternative models of X.

Example 1.2.12. *Consider*

$$X = \mathscr{Z}(\mathcal{O}(2) \oplus \mathcal{O}(1)) \subset \mathrm{Gr}(2,4),$$

which is a Fano surface of degree 4, i.e. a dP_4*. We wish to compute the cohomology of* $\mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(1)_{|X}$.

The Koszul complex in this case reads

$$0 \to \mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(-2) \to \mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(-1) \oplus \mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)} \to \mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(1) \to \mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(1)_{|X} \to 0,$$

which splits into

$$0 \to \mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(-2) \to \mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(-1) \oplus \mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)} \to K_1 \to 0$$

$$0 \to K_1 \to \mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(1) \to \mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(1)_{|X} \to 0.$$

Using BWB we find

$$H^i(\mathrm{Gr}(2,4), \mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}) = \begin{cases} 0 \ \forall i \neq 0 \\ V_4 \ \mathit{for}\ i = 0 \end{cases}, \ H^i(\mathrm{Gr}(2,4), \mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}) = \begin{cases} 0 \ \forall i \neq 0 \\ \Sigma_{2,2,1,0} V_4 \ \mathit{for} i = 0, \end{cases}$$

$$H^i(\mathrm{Gr}(2,4), \mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(-2)) = 0 \ \forall i, \ \mathit{and}\ H^i(\mathrm{Gr}(2,4), \mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(-1)) = 0 \ \forall i$$

From the associated long exact sequences we deduce

$$0 \to V_4 \cong H^0(\mathrm{Gr}(2,4), K_1) \to 0,$$

$$0 \to H^0(\mathrm{Gr}(2,4), K_1) \to \Sigma_{2,2,1,0} V_4 \to H^0(\mathrm{Gr}(2,4), \mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(1)_{|X}) \to 0.$$

Using LR we obtain that $\dim \Sigma_{2,2,1,0} V_4 = 20$*, hence we obtain*

$$h^i(\mathrm{Gr}(2,4), \mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(1)_{|X}) = \begin{cases} 0 \ \forall i \neq 0 \\ 16 \ \mathit{for}\ i = 0. \end{cases}$$

These are the basic tools that we will use throughout the following chapters. In each chapter we enrich this toolkit depending on the problem at hand, thereby highlighting how the framework of homogeneous varieties can be exploited to study the geometry of diverse families of varieties.

CHAPTER 2

Fano varieties, quiver moduli spaces and Grassmannians

2.1. About Fano varieties

Fano varieties and their classification are one of the most studied topics in algebraic geometry. It is well known that Fano varieties are *bounded*, thanks to a famous and motivating result by Kollár, Miyaoka, and Mori [**KMM92**]: in each dimension, there is a finite number of deformation classes of Fano varieties. Up to dimension three, we have a total of 116 families of smooth Fano varieties, but from dimension four onwards, the classification problem is widely open.

When the dimension is sufficiently low, we understand the full picture: there is only one Fano variety in dimension one, namely the projective line. In dimension two, there are ten families of Fano varieties, the Del Pezzo surfaces, already known since [**DP87**].

The classification of Fano threefolds began with the work of Fano himself in [**Fan29**], and was carried on by Iskovskikh [**Isk77, Isk78**]. In particular, Iskovskikh completed the classification of the seventeen prime Fano threefolds, i.e., the ones with Picard rank equal to one. Subsequently, Mori and Mukai classified all 105 families of Fano threefolds using Mori's birational theory of extremal rays, see [**MM86, MM03**].

Mukai's contribution did not stop with the classification. In fact, in [**Muk89**], he revisited the work of Iskovskikh using the biregular vector bundle method. In this way, he could describe the prime ones as zero loci of appropriate vector bundles, and almost all of them as linear sections in homogeneous or quasi-homogeneous varieties. A very recent update on the vector bundle method was given in [**BKM24**], with a complete proof of Mukai's Theorem on the existence of certain exceptional vector bundles on prime Fano threefolds.

A natural question arises: do all the 105 deformation families of Fano threefolds admit a general member described this way? A first answer can be found in [**CCGK16**]. There, all the families of Fano threefolds are described as zero loci of a section of homogeneous vector bundles over GIT quotients $V /\!\!/ G$ where G is a product of groups $\mathrm{GL}_n(\mathbb{C})$, and V is a representation of G. This framework is particularly efficient for computing quantum periods. Within this program, the Fano threefolds can be written as complete intersections in toric varieties or subvarieties in products of Grassmannians.

In the fourfold case, the program highlighted in [**CCG$^+$13**] was further developed through a series of works employing the techniques introduced in [**CCGK16**]; for example [**CGKS20, CKP17, Kal19**]. In particular, in [**CGKS20**], the authors constructed an extensive database of known Fano fourfolds. This database includes the 35 well-known and classified families of Fano fourfolds with index $\iota_X > 1$, as detailed in [**IP99**], and also many other fourfolds obtained in [**Bat99, Øbr07, Sat00, Str14**]. They were able to compute the quantum period of all toric complete intersections, products of lower-dimension Fano varieties, and certain complete intersections in projective bundles.

Conversely, in [**Kal19**] the method was developed in a slightly different way. The aim was to produce a list of Fano fourfolds realized as zero loci of global sections of certain homogeneous vector bundles over quiver flag manifolds. Consequently, the database in [**Kal19**] encompasses the non-toric fourfolds from [**CGKS20**] and introduces 141 families of Fano fourfolds with quantum periods not previously recorded in [**CGKS20**].

To some extent, the above technique can be seen as a partial, yet far-reaching, generalization of the work of Küchle [**Kü95**]. There, Küchle showed that there exist precisely 18 Fano fourfolds of index one that can be realized as zero loci of sections of completely reducible homogeneous vector bundles over a single Grassmannian $\mathrm{Gr}(k, n)$, $k \geq 2$. In particular, 17 of them have different deformation invariants, hence they give rise to 17 distinct deformation families of Fano fourfolds. The remaining two, which share the same invariants, were subsequently identified by Manivel in [**Man15**].

One of the main advantages of Küchle's method is that the computation of invariants, such as Hodge numbers and volume, relies on combinatorial data. As a result, these invariants can be computed explicitly using computer algebra techniques. For this reason, as explained in [**DBFT22**], the next step is to consider zero loci of sections of completely reducible homogeneous vector bundles over products of Grassmannians.

This strategy was first carried out in dimension three in [**DBFT22**], and subsequently extended to dimension four in [**BFMT21**]. In the latter work, the authors constructed a database of at least 634 Fano fourfolds and extracted a subset with prescribed invariants. In particular, they focused on the so-called K3-type fourfolds and identified 64 deformation families exhibiting this property. These families were then described in detail from both birational and biregular perspectives. The exploration of the remaining examples of the database has barely started; see, for example, [**BFK+24, Man23**] for the first few extra results. It will also be interesting to compare this database with other lists of Fano described by birational terms, see for example [**Cas23**].

Both of the previously described approaches yield databases of Fano fourfolds. The one constructed in [**Kal19**] aims to study the quantum period of Fano fourfolds. On the other hand, the one used in [**BFMT21**] is particularly suited to compute invariants such as Hodge numbers, volumes, and allows a relatively easy birational description.

There is a nontrivial intersection between the two databases; however, neither is a subset of the other. For instance, the fourfold K_{743} is in [**Kal19**] database, but it is absent in one of [**BFMT21**]. Vice versa, the fourfold c2 in [**Kü95**] is in the second database but not in the first.

In this chapter, we translate the language of section zero loci in quiver flag manifolds into the framework of section zero loci in products of Grassmannians. Although the process can be technically subtle, it reveals interesting geometric features of the associated fourfolds. As an illustrative case, we point out one of the Fano varieties described in [**CKP17**], which was later understood by Casagrande using a very natural yet interesting birational description.

The translation process is detailed in Section 2.2.5, and it is a direct application of the formalism developed in [**Cra11**]. An algorithmic formulation is provided in Corollary 2.2.24, which enables the process to be automated using as input the adjacency matrix of the quiver flag manifold, the dimension vector, and the representation of the homogeneous vector bundles.

One of our main advances is that we can distinguish certain Fano fourfolds admitting a blow-down to another Fano fourfold from those that do not. Following Mukai's terminology, we call the latter *primitive*. In this paper, we carry out a case-by-case analysis of the 170 Fano fourfolds, providing a benchmark for future investigation in this direction. Our idea was to gain the most comprehensive possible understanding of the network of birational maps between Fano fourfolds.

2.1.1. Main results. This chapter wants to systematically analyze the families first described by Kalashnikov in [**Kal19**]. In the aforementioned paper, Kalashnikov describes 141 families of Fano fourfolds with a new quantum period (from a database of 749 families) realized as zero loci in quiver flag manifolds and not obtainable as toric complete intersections. Additionally, she identifies 29 families that are not found in toric quiver flag varieties, although they might appear as toric complete intersections via other constructions. We focus on these 170 families and translated them as zero loci of sections in products of Grassmannians, using the machinery described in Section 2.2.5. Their birational structure can be summarized in the following result:

THEOREM 2.1.1. *The 170 families of Fano fourfolds mentioned above can be subdivided into subsets as follows:*

- *9 have Picard rank* $\rho = 1$ *and are classical, appearing for example in* [**IP99, Kü95**]*;*
- *124 are blow-ups of smooth Fano fourfolds (from the list of 749) in smooth centers. Of these, 19 also appear in* [**BFMT21**] *and Fano 2-12 is described in* [**Kuz19**]*;*
- *8 are products of lower-dimensional Fano varieties;*
- *Fano 2-43, Fano 2-48 and Fano 3-50 are projective bundles over Fano threefolds. None of them can be realized as a blow-up;*
- *8 are conic bundles, of which one appears in* [**BFMT21**]*. None of them can be realized as a blow-up or conic bundle;*
- *Fano 2-1 and Fano 2-35 are stratified projective bundles over Fano threefolds, described in* [**Lang98**]*. None of them can be realized as a blow-up, conic bundle, or projective bundle;*
- *16 are small resolutions of singular fourfolds. None of them can be realized as a blow-up, conic bundle, projective bundle, or stratified projective bundle.*

To clarify the above statement, we note that it is based on the analysis of the canonical projections onto the factors of the ambient product of Grassmannians. For example, when we state that 124 families are blow-ups, we mean that, for each such family, there exists at least one canonical projection under which the fourfold is realized as the blow-up of a smooth Fano variety along a smooth center.

Similarly, the 11 families listed as projective or conic bundles are such that none of the canonical projections realizes the fourfold as a blow-up, while at least one projection exhibits a structure of the desired type. The same interpretation applies to the other subsets in the classification.

Fano fourfolds arising as blow-ups of others are presented solely in this form, to highlight their birational relationships. The 16 Fano fourfolds described only as small resolution are:

- Fano 2-4, Fano 2-14, Fano 2-15, Fano 2-37, Fano 2-50, Fano 2-51, Fano 2-60, Fano 2-73, Fano 3-12, Fano 3-14, Fano 3-43 from the list of 141 families.
- Fano 3-11, Fano 3-17, Fano 3-20, Fano 3-34, Fano 4-5 in the extra 29 families.

Looking at the projections to the factors of the ambient spaces of these 16 families of fourfolds, we were able to describe them only as small resolutions of singular fourfolds. Thus, these families have a more interesting and complicated geometry and deserve a deeper study.

During the systematic study of the 170 families of fourfolds, we were also able to give some information about their rationality. In particular, we obtained that out of the 170, 129 are rational fourfolds. Unfortunately, we are not able to say more about the rationality of the remaining ones.

We refer to Section 2.8 for the summarizing table with all the information about the deformation invariants and the rationality of these 170 families of Fano fourfolds.

2.1.2. Computational methods and a database. As part of the results of this paper, we provide a computational method to determine, for a given Fano variety realized as the zero locus of a general global section of a completely reducible homogeneous vector bundle over a product of Grassmannians, its Hodge numbers, the number of sections of the anticanonical bundle, and the Euler characteristic of the tangent bundle. These methods, based on the content of Section 2.2.1, have been implemented in the computer algebra software [**GS**] and made publicly available at [**FTT**], which is to be considered the ancillary repository for this manuscript. There, the interested reader will also find the complete database of invariants for each of the 170 families of Fano fourfolds under investigation. In setting up this repository, we have followed the FAIR principles of findability, accessibility, interoperability, and reusability of datasets.

2.2. Working tools

In this section, we will recall some useful results that help to speed up the birational description of zero loci in products of Grassmannians. Several of these results are classical and can be found extensively in literature, e.g. , [**Wey03**], [**DBFT22**], or [**BFMT21**].

2.2.1. Computing Hodge Numbers. In [**DBFT22, BFMT21**], an automatable method is presented for computing the Hodge numbers for zero loci of a general global section of a globally generated completely reducible homogeneous vector bundle on products of Grassmannians.

Summarizing what is explained in [**DBFT22**, Section 3], it suffices to use a combination of the Koszul complex and the cotangent sequence to obtain an appropriate resolution of the $k-th$ exterior power of the cotangent bundle. In particular, if we have $X = \mathscr{Z}(\mathcal{F}) \subset Y$, from the co-normal sequence, we will obtain:

$$0 \to \operatorname{Sym}^j \mathcal{F}^\vee_{|X} \to (\operatorname{Sym}^{j-1} \mathcal{F}^\vee \otimes \Omega^1_Y)_{|X} \to \cdots \to (\operatorname{Sym}^{j-k} \mathcal{F}^\vee \otimes \Omega^k_Y)_{|X} \to \cdots \to \Omega^j_{Y|X} \to \Omega^j_X \to 0$$

and each term can be resolved by a Koszul complex of the following type:

$$0 \to \bigwedge^r \mathcal{F}^\vee \otimes \operatorname{Sym}^{j-k} \mathcal{F}^\vee \otimes \Omega^k_Y \to \cdots \to \mathcal{F}^\vee \otimes \operatorname{Sym}^{j-k} \mathcal{F}^\vee \otimes \Omega^k_Y$$

Since we are dealing with globally generated, completely reducible homogeneous vector bundles, we can apply the Borel–Bott–Weil and Littlewood–Richardson theorems to compute the dimension of the cohomology groups of each resolution. Even if those computations can become quickly cumbersome, we can use some computer algebra software, such as [**GS**], to automate and speed up computations. This is the core of the computational methods presented in [**FTT**].

Other important invariants that are computed in this paper are $h^0(-K)$, $(-K)^4$, and $\chi(T_X)$. Since we are dealing with products of Grassmannians, integration is explicit, and then the Hirzebruch–Riemann–Roch Theorem provides a method to compute $\chi(\mathcal{E})$ for any vector bundle $\mathcal{E}$ with prescribed Chern classes. The [GS] package [GSS+] provides an environment to perform such computations, which have been incorporated in our repository [FTT].

2.2.2. Degeneracy Loci. In our setting, we consider zero loci of global sections of globally generated homogeneous vector bundles in products of Grassmannians. These are the varieties of the form

$$\mathscr{Z}(\mathcal{F}) \subset \prod_{i=1}^{m} \mathrm{Gr}(k_i, n_i),$$

where the bundle $\mathcal{F}$ is expressed as

$$\mathcal{F} = \bigoplus_{j=1}^{s} \mathcal{F}_{1,j} \boxtimes \cdots \boxtimes \mathcal{F}_{m,j},$$

and each factor $\mathcal{F}_{i,j}$ is an irreducible homogeneous vector bundle on $\mathrm{Gr}(k_i, n_i)$.

We study these varieties by looking at the projections on the factors of $\prod_{i=1}^{m} \mathrm{Gr}(k_i, n_i)$, using the following result:

Proposition 2.2.1. *Let* $\mathrm{Gr}_1 = \mathrm{Gr}(k_1, V_{n_1})$ *and* $\mathrm{Gr}_2 = \mathrm{Gr}(k_2, V_{n_2})$ *be Grassmannians, and let* $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *be globally generated irreducible homogeneous vector bundles on* $\mathrm{Gr}_1, \mathrm{Gr}_2$*, respectively. Set* $\mathcal{E}_1 \boxtimes \mathcal{E}_2$ *with* $H^0(\mathrm{Gr}_1 \times \mathrm{Gr}_2, \mathcal{E}_1 \boxtimes \mathcal{E}_2) = \Sigma_{\alpha_1} V_{n_1} \otimes \Sigma_{\alpha_2} V_{n_2}$*, and define* $\delta := min\{rk\mathcal{E}_1, h^0(\mathrm{Gr}_2, \mathcal{E}_2)\}$*. Let* $X = \mathscr{Z}(\mathcal{E}_1 \boxtimes \mathcal{E}_2) \subset \mathrm{Gr}_1 \times \mathrm{Gr}_2$*, then the projection* $\pi_1 : X \to \mathrm{Gr}_1$ *induces one of the following birational relations between* X *and* Gr_1*:*

(1) if $rk\mathcal{E}_1\ \mathrm{rk}\,\mathcal{E}_2 \leq k_2(n_2 - k_2)$*, the general fiber of* π_1 *is* $Z_2 = \mathscr{Z}(\mathcal{E}_2^{\oplus \mathrm{rk}\,\mathcal{E}_1}) \subset \mathrm{Gr}_2$*. Moreover, the exceptional loci of* π_1*, if not empty, are the nested loci where the map*

$$\varphi : \mathcal{E}_1^\vee \to \Sigma_{\alpha_2} V_{n_2} \otimes \mathcal{O}_{\mathrm{Gr}_1}$$

which is induced by the section of $\mathcal{E}_1 \boxtimes \mathcal{E}_2$ *degenerates to a* $\delta - k$ *map, in other words* $\mathrm{D}_{\delta-k}(\varphi)$*, for* $k > 0$*;*

(2) if $rk\mathcal{E}_1\ \mathrm{rk}\,\mathcal{E}_2 > k_2(n_2 - k_2)$*, the general fiber of* π_1 *is empty. Moreover, exists a* $l \leq \delta - 1$ *such that over* $\mathrm{D}_l(\varphi) \subset X$ *the fiber is given by the zero loci* $Z_2 = \mathscr{Z}(\mathcal{E}_2^{\oplus l}) \subset \mathrm{Gr}_2$*, and each further exceptional locus, if not empty, is given by* $\mathrm{D}_{l-k}(\varphi)$*,* $k > 0$*, with fiber* $Z_{2,k} = \mathscr{Z}(\mathcal{E}_2^{\oplus l-k}) \subset \mathrm{Gr}_2$*.*

Proof. Let us start proving (1). By Künneth Theorem a global section σ in $H^0(\mathrm{Gr}_1 \times \mathrm{Gr}_2, \mathcal{E}_1 \boxtimes \mathcal{E}_2)$ can be seen as

$$\sigma = \pi_1^* \sigma_1 \otimes \pi_2^* \sigma_2 \in H^0(\mathrm{Gr}_1, \mathcal{E}_1) \otimes H^0(\mathrm{Gr}_2, \mathcal{E}_2).$$

By definition, if $\Pi_i \in \mathrm{Gr}_i$, the evaluation satisfies $\sigma(\Pi_1, \Pi_2) = \sigma_1(\Pi_1) \otimes \sigma_2(\Pi_2)$. By adjunction, one has

$$H^0(\mathrm{Gr}_1 \times \mathrm{Gr}_2, \mathcal{E}_1 \boxtimes \mathcal{E}_2) = \mathrm{Hom}(\pi_1^* \mathcal{E}_1^\vee, \pi_2^* \mathcal{E}_2) = \mathrm{Hom}(\mathcal{E}_1^\vee, H^0(\mathrm{Gr}_2, \mathcal{E}_2))$$

so that σ can be regarded as a morphism

$$\varphi : \mathcal{E}_1^\vee \to \Sigma_{\alpha_2} V_{n_2} \otimes \mathcal{O}_{\mathrm{Gr}_1}.$$

For any $\Pi_1 \in \mathrm{Gr}_1$, its preimage via π_1 is given by all points (Π_1, Π_2) such that all elements of the image of $\varphi(\Pi_1)$ vanish in Π_2, when regarded as sections of $\mathcal{E}_2$. Over a general Π_1 we have

$rk\mathcal{E}_1$ sections of $\mathcal{E}_2$, which give a non-empty fiber if $rk\mathcal{E}_1\ \mathrm{rk}\,\mathcal{E}_2 \leq k_2(n_2 - k_2)$. The exceptional locus is given by the locus where the fibers have higher dimension, and this is exactly where the image of φ does not have maximal dimension. This concludes the proof of the first part. Notice that (2) is proved essentially as the first one, with the only difference being that the projection π_1 is not surjective, because the general fiber is empty. This implies that to obtain the image of π_1 we have to consider where the morphism φ loses enough rank to obtain at least a finite number of points as general fiber. We can consider the first degeneracy locus $\mathrm{D} \subset \mathrm{Gr}_1$, for which the general fiber is non-empty and smooth. We can replicate the procedure of the first part by taking into account the smaller degeneracy locus.

□

Another useful tool often used with the above Proposition 2.2.1 is the following Lemma, which is a consequence of the one proved by Kuznetsov in [**Kuz16**, Lemma 2.1]:

Lemma 2.2.2. *Let $\varphi : \mathcal{E} \to \mathcal{F}$ be a morphism of vector bundles of ranks $rk\mathcal{F} = f < rk\mathcal{E} = e$ on a Cohen–Macaulay scheme $\mathcal{S}$. Consider the projectivization $p : \mathbb{P}_S(\mathcal{E}) \to S$, then φ gives a global section of the vector bundle $p^*\mathcal{F} \otimes \mathcal{O}_{\mathcal{R}}(1)$. Moreover, let $k < f$ be the greatest integer such that $\mathrm{D}_k(\varphi)$ is not empty, and for all $h < k$, $\mathrm{D}_h(\varphi)$ is empty. Then $\mathrm{D}_k(\varphi) \cong \mathscr{Z}(p^*\mathcal{E}^\vee \otimes \mathcal{O}_{\mathcal{R}}(1)) \subset \mathbb{P}_S(\mathcal{F}^\vee)$.*

Example 2.2.3. *Let us consider the threefold $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^3} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \oplus \mathcal{O}(0,1)^{\oplus 2}) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,6)$. Notice that $X = \mathscr{Z}(\mathcal{O}(0,1)^{\oplus 2}) \subset X'$, with $X' = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^3} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)}) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,6)$. So if we want to describe X, we can start by studying X'. By the first point of Proposition 2.2.1 we obtain that*

$$\pi_1 : X' \to \mathbb{P}^3$$

has a general fiber equal to $\mathbb{P}^2$. With the same computation, we have that the first degeneracy locus is empty. Hence X' is $\mathbb{P}_{\mathbb{P}^2}(\mathcal{E})$, with $\mathcal{E}$ a rank-3 bundle on $\mathbb{P}^3$.

Now we can study X as $\mathscr{Z}(\mathcal{O}(0,1)^{\oplus 2}) \subset X'$, which can be also reinterpreted as $\mathscr{Z}(\mathcal{O}_R(1)^{\oplus 2}) \subset \mathbb{P}_{\mathbb{P}^2}(\mathcal{E})$. The general fiber of

$$\pi_1 : X \to \mathbb{P}^3$$

is now $\mathbb{P}^2$ cut by two hyperplanes, hence a point. We have proved that X is birational to $\mathbb{P}^3$. Is there an exceptional locus of π_1? The answer is positive. The fiber becomes a $\mathbb{P}^1$ along a $\mathbb{P}^1$. This happens because the bundle $\mathcal{Q}_{\mathbb{P}^3} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)}$ induces a map

$$\varphi : V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \to \mathcal{Q}_{\mathbb{P}^3}.$$

Since φ does not have a degeneracy locus, we get the rank three bundle $\mathcal{K} = ker\varphi$, which can be verified to be isomorphic to $\mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}^{\oplus 2}$. By taking the second exterior power of the exact sequence

$$0 \to \mathcal{K} \xrightarrow{i} V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\varphi} \mathcal{Q}_{\mathbb{P}^3},$$

we get the induced injective map

$$\phi : \bigwedge^2 \mathcal{K} \to \bigwedge^2 V_6 \otimes \mathcal{O}_{\mathbb{P}^3}.$$

Recall that $\sigma \in H^0(\mathbb{P}^3 \times \mathrm{Gr}(2,6), \mathcal{O}(0,1)^{\oplus 2})$, induces a map

$$\tau : \bigwedge^2 V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \to V_2 \otimes \mathcal{O}_{\mathbb{P}^3},$$

obtained by considering the two copies of $\mathcal{O}(0,1)$*. Since* $X \subset X'$ *we have a further map:*

$$\Phi : \bigwedge^2 \mathcal{K} \xrightarrow{\bigwedge^2 i} \bigwedge^2 V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\tau} V_2 \otimes \mathcal{O}_{\mathbb{P}^3}.$$

This map exists because we have the bundle $\mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \boxtimes \mathcal{O}_{\mathrm{Gr}(2,6)}(1)$*. Thus* $\mathrm{D}_1(\tau_{|X'}) = \mathrm{D}_1(\Phi)$ *and corresponds to the exceptional locus of the projection to* $\mathbb{P}^3$*. Since* $\mathcal{K} = \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}^{\oplus 2}$*, then* $\bigwedge^2 \mathcal{K} = \mathcal{O}(-1)^{\oplus 2} \oplus \mathcal{O}$*. By Lemma 2.2.2,* $\mathrm{D}_1(\Phi) = \mathscr{Z}(\mathcal{O}(1,1)^{\oplus 2} \oplus \mathcal{O}(1,0)) \subset \mathbb{P}^1 \times \mathbb{P}^3$*, which is* $\mathbb{P}^1$*.*

X *is birational to* $\mathbb{P}^3$ *through* π_1*. Its exceptional locus is a* $\mathbb{P}^1 \subset \mathbb{P}^3$*, and the fiber of* π_1 *on* $\mathbb{P}^1$ *is a* $\mathbb{P}^1$*. For this* $X = \mathrm{Bl}_{\mathbb{P}^1} \mathbb{P}^3$*.*

Remark 2.2.4. *Note that Proposition 2.2.1 cannot easily be generalized. In particular, all hypotheses are essential. Consider Example 2.2.3. We can write* X *as* $\mathscr{Z}(\mathcal{Q}_{\mathbb{P}^3} \boxtimes \mathcal{U}^\vee_{\mathrm{S}_2\,\mathrm{Gr}(2,6)}) \subset \mathbb{P}^3 \times \mathrm{S}_2\,\mathrm{Gr}(2,6)$*, with* $\mathrm{S}_2\,\mathrm{Gr}(2,6)$ *the bisymplectic Grassmannian in* $\mathrm{Gr}(2,6)$*.*

If we apply Proposition 2.2.1 to X *written in this way, ignoring the hypothesis that the ambient space has to be a product of type A Grassmannians, we would conclude that* X *is* $\mathbb{P}^3$*, which is not the case.*

Proposition 2.2.1 allows us to describe the variety X using the stratification of the degeneracy loci, with the control of the fiber of the projection on each stratum. Typical examples of this phenomenon include stratified projective bundles (for example, blow-ups), where we have local triviality on each stratum.

However, this is not the only possible situation: conic bundles, for instance, naturally arise in this framework. It is important to emphasize that Proposition 2.2.1 should be viewed as a general guiding principle; to understand the geometric picture requires a proper case-by-case analysis.

The next step is to study degeneracy loci of morphisms φ between vector bundles on $\mathrm{Gr}(k,n)$, hence several classical tools (see [**Wey03**]) come in handy:

Theorem 2.2.5. *Let* X *be a smooth projective variety, let* $\mathcal{E}$ *and* $\mathcal{F}$ *vector bundles on* X *such that* $\mathcal{E} \otimes \mathcal{F}$ *is globally generated, and let* $\varphi \in H^0(X, \mathcal{E} \otimes \mathcal{F})$ *a general global section. If* $\mathrm{D}_k(\varphi) \neq \emptyset$*, then the dimension of* $\mathrm{D}_k(\varphi)$ *is* $m_k = \dim X - (rk\mathcal{E} - k)(rk\mathcal{F} - k)$*.*

Theorem 2.2.6. *Let* $\varphi : \mathcal{E} \to \mathcal{F}$ *a morphism between vector bundles on* M*. Suppose* $e = rk\mathcal{E} > f = rk\mathcal{F}$*, and suppose* $\operatorname{codim} \mathrm{D}_f(\varphi) = e - f + 1$*. Then the* $\mathcal{O}_M$*-module* $\mathcal{O}_{\mathrm{D}_f(\varphi)}$ *admits a locally free resolution given by the Eagon–Northcott complex*

$$0 \to \bigwedge^e \mathcal{E} \otimes \mathrm{Sym}^{e-f} \mathcal{F}^\vee \otimes det\mathcal{F}^\vee \to \cdots \to \bigwedge^{f+1} \mathcal{E} \otimes \mathcal{F}^\vee \otimes det\mathcal{F}^\vee \to \bigwedge^{f} \mathcal{E} \otimes det\mathcal{F}^\vee \to \mathcal{O}_M \to \mathcal{O}_{\mathrm{D}_{f-1}}(\varphi) \to 0$$

For example, if we consider a degeneracy locus $\mathrm{D}_f(\varphi)$ of dimension at most two, we can compute its Hilbert polynomial using the associated resolution.

The combination of Proposition 2.2.1, Lemma 2.2.2, Theorem 2.2.5, and Theorem 2.2.6 significantly facilitates the description of section zero loci of products of Grassmannians. Since we are working with varieties of dimension 4, computing the invariants of the degeneracy loci often leads to the complete identification of the exceptional locus of the projection.

Remark 2.2.7. *Another crucial hypothesis in Proposition 2.2.1 is the non-emptiness of the exceptional locus of the projection. Consider, for example, the zero locus* $\mathscr{Z}(\mathcal{Q} \boxtimes \mathcal{U}^\vee) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,6)$*.*

Recall that in this case X *is* $\mathbb{P}^4$*. From a linear algebra viewpoint, a global section of* $\mathcal{Q} \boxtimes \mathcal{U}^\vee$ *corresponds to an element* $\alpha \in V_6^\vee \otimes V_5$*, i.e.* $\alpha : V_6 \to V_5$*, which is of maximal rank by generality.*

Let $K \subset V_6$ denote its 1-dimensional kernel. The zero locus of α corresponds to the pairs $(l, \Pi) \in \mathbb{P}^4 \times \mathrm{Gr}(2,6)$ with $\alpha(\Pi) \subset l$. For a fixed l, the fiber is thus $\Pi = \langle l, K\rangle$, hence a point.

The first degeneracy loci $\mathrm{D}_3(\varphi)$ corresponds to the pairs (l, Π) such that $\alpha(\Pi) = 0$. However, this cannot occur, as a general α has a 1-dimensional kernel.

This fact can also be confirmed by computing the invariants of $\mathrm{D}_3(\varphi)$ by using the Eagon–Northcott complex, as described in Theorem 2.2.6.

If, however, one were to apply Proposition 2.2.1 without verifying the non-emptiness of the exceptional locus, one would, incorrectly, conclude that the first projection $\pi_1 : X \to \mathbb{P}^4$ is a birational map, with $\mathrm{D}_3(\varphi)$ as a 1-dimensional exceptional locus.

2.2.3. Birational tricks. Here we recollect some classical results and variations that naturally arise when we deal with zero loci in products of Grassmannians.

Throughout the paper, the concept of Grassmann bundle plays a central role. In particular, $\mathrm{Gr}(k_1, n_1) \times \mathrm{Gr}(k_2, n_2)$ can be also seen as $\mathbb{G}\mathrm{r}_{\mathrm{Gr}(k_1,n_1)}(k_2, \mathcal{O}^{\oplus n_2}_{\mathrm{Gr}(k_1,n_1)})$.

As consequence, if we have a vector bundle $\mathcal{D}$ on $\mathrm{Gr}(k_1, n_1)$, the behavior of varieties like

$$\mathscr{Z}(\mathcal{D} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(k_2,n_2)}) \subset \mathrm{Gr}(k_1, n_1) \times \mathrm{Gr}(k_2, n_2)$$

is important itself:

Theorem 2.2.8. *Let $\mathcal{F}$, $\mathcal{E}$ and $\mathcal{D}$ vector bundles on $\mathrm{Gr}(k, n)$, such that:*

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{D} \to 0$$

is a short exact sequence of vector bundles. Let $X = \mathbb{G}\mathrm{r}_{\mathrm{Gr}(k,n)}(h, \mathcal{F})$, then $X = \mathscr{Z}(\mathcal{U}^\vee_{\mathcal{R}} \boxtimes \mathcal{D}) \subset \mathbb{G}\mathrm{r}_{\mathrm{Gr}(k,n)}(h, \mathcal{E})$.

The next lemma, on the other hand, is in some sense more general and gives a precise description of the geometry of X, a complete proof of this can be found in [**BFMT21**, Lemma 3.1]:

Lemma 2.2.9. *Let X be a smooth projective variety with a globally generated line bundle $\mathcal{L}$, and consider $Y = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^n} \boxtimes \mathcal{L}) \subset \mathbb{P}^n \times X$. Then $Y = \mathrm{Bl}_Z X$, where $Z = \mathscr{Z}(\mathcal{L}^{\oplus n+1}) \subset X$.*

The results as Lemma 2.2.9 are often referred to as Cayley tricks, and a generalization of those in the Grassmannian world are the following, which are proved in [**DBFT22**, Lemma 2.2] and [**BFMT21**, Prop. 3.3]:

Lemma 2.2.10. *Let us consider $\mathrm{Gr}_1 = \mathrm{Gr}(k_1, n_1)$, $\mathrm{Gr}_2 = \mathrm{Gr}(k_2, n_2)$, $M = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(k_1,n_1)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(k_2,n_2)}) \subset \mathrm{Gr}_1 \times \mathrm{Gr}_2$, then we have:*

1. *if $n_2 = n_1$ and $k_2 < k_1$ then $M = \mathrm{Fl}(k_2, k_1, n)$;*
2. *if $k_1 = k_2$ and $n_1 = n_2 - 1$ then $M = \mathrm{Bl}_{\mathrm{Gr}(k_1-1,n_1)} \mathrm{Gr}_2$, where the center of the blow-up $\mathrm{Gr}(k_1 - 1, n_1)$ is identified with $\mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}_2}) \subset \mathrm{Gr}_2$;*
3. *if $k_2 = k_1$ and $n_2 = n_1 + 2$ then outside a point p, M is isomorphic to $\mathrm{Bl}_{\tilde{D}} \mathrm{Gr}_2$, with $\tilde{D} \cong \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^{n-1}} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}_2}) \subset \mathbb{P}^{n-1} \times \mathrm{Gr}_2$*

The first point of the previous lemma is the key to obtaining a chain of results that we summarize in the following:

Corollary 2.2.11. *Let us consider $\mathrm{Gr}_1 = \mathrm{Gr}(k_1, n_1)$, $\mathrm{Gr}_2 = \mathrm{Gr}(k_2, n_2)$, $M = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(k_1,n_1)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(k_2,n_2)}) \subset \mathrm{Gr}_1 \times \mathrm{Gr}_2$, then we have:*

1. *if $k_2 < k_1 < n_1$ and $n_2 = n_1 - 1$ then $M = \mathscr{Z}(\mathcal{U}_1^\vee) \subset \mathrm{Fl}(k_2, k_1, n_1)$;*

(2) if $k_2 < k_1 < n_1$ and $n_1 = n_2 - 1$ then $M = \mathscr{Z}(\mathcal{Q}_2) \subset \mathrm{Fl}(k_2, k_1 + 1, n_2)$;
(3) if $k_2 \geq k_1$ and $n_2 = n_1 + c$ for every $c \geq 0$ then $M = \mathbb{G}\mathrm{r}_{\mathrm{Gr}_1}(k_2, \mathcal{U}_{\mathrm{Gr}_1} \oplus \mathcal{O}^{\oplus c}_{\mathrm{Gr}_1})$.

Proof. The proof of these results is very straightforward:

(1) recall that $\mathrm{Gr}(k,n) \supset \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(k,n)}) \cong \mathrm{Gr}(k, n-1)$, hence $\mathrm{Gr}(k_1, n-1) \times \mathrm{Gr}(k_2, n) \supset \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(k_1,n-1)} \boxtimes \mathcal{Q}_{\mathrm{Gr}(k_2,n)}) \cong \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(k_1,n)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(k_1,n)} \boxtimes \mathcal{Q}_{\mathrm{Gr}(k_2,n)}) \subset \mathrm{Gr}(k_1, n) \times \mathrm{Gr}(k_2, n)$. Now we apply Lemma 2.2.10 (1) and we are done;
(2) Recall that $\mathrm{Gr}(k,n) \supset \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(k,n)}) \cong \mathrm{Gr}(k-1, n-1)$, hence arguing as in Corollary 2.2.11 (1) we are done;
(3) Note that $\mathrm{Gr}(k_1, n+c) \times \mathrm{Gr}(k_2, n)$ can be written as $\mathbb{G}\mathrm{r}_{\mathrm{Gr}(k_2,n)}(k_1, \mathcal{O}^{\oplus n+c}_{\mathrm{Gr}(k_2,n)})$, hence we can rewrite $\mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(k_1,n+c)} \boxtimes \mathcal{Q}_{\mathrm{Gr}(k_2,n)}) \subset \mathrm{Gr}(k_1, n+c) \times \mathrm{Gr}(k_2, n)$ as $\mathscr{Z}(\mathcal{U}^\vee_{\mathcal{R}} \boxtimes \mathcal{Q}_{\mathrm{Gr}(k_2,n)}) \subset \mathbb{G}\mathrm{r}_{\mathrm{Gr}(k_2,n)}(k_1, \mathcal{O}^{\oplus n+c}_{\mathrm{Gr}(k_2,n)})$, and by applying Theorem 2.2.8 we complete the proof. □

The last result of this flavor we apply frequently throughout the paper is [**DBFT22**, Lemma 2.5].

Lemma 2.2.12. *Let $\mathrm{Fl}(k_1, k_2, n)$ be a two-step flag variety. We have the following identifications:*

- $\mathrm{Fl}(k_1, k_2, n) \cong \mathbb{G}\mathrm{r}_{\mathrm{Gr}(k_2,n)}(k_1, \mathcal{U}) \cong \mathbb{G}\mathrm{r}_{\mathrm{Gr}(k_1,n)}(k_2 - k_1, \mathcal{Q}(-1))$;
- *let $X = \mathscr{Z}(\mathcal{Q}_2) \subset \mathrm{Fl}(k_1, k_2, n)$, then $X \cong \mathbb{G}\mathrm{r}_{\mathrm{Gr}(k_2-1,n-1)}(k_1, \mathcal{U}_{\mathrm{Gr}(k_2-1,n-1)} \oplus \mathcal{O})$*
- *let $\mathscr{Z}(\mathcal{U}_1^\vee) \subset \mathrm{Fl}(k_1, k_2, n)$, then $X \cong \mathbb{G}\mathrm{r}_{\mathrm{Gr}(k_1,n-1)}(k_2 - k_1, \mathcal{Q}_{\mathrm{Gr}(k_1,n-1)} \oplus \mathcal{O}(-1))$*

2.2.4. Conic Bundles. Some of the Fano fourfolds we have studied have a natural structure of conic bundles. Hence to describe them in the most detailed way we analyzed also their determinant locus, for this, we refer to [**BFMT21**, Lemma 3.5]:

Lemma 2.2.13. *Let $f : X \to B$ be a conic bundle, given by a line bundle $\mathcal{K}$, a rank three bundle $\mathcal{E}$ on B and a bilinear map $\mathcal{K}^\vee \to \mathrm{Sym}^2 \mathcal{E}^\vee$. Then the degeneracy loci of the conic bundle are given by the discriminant divisor Δ and its codimension two singular locus Δ_{sing}. In particular, let $k := c_1(\mathcal{K})$ and $c_i := c_i(\mathcal{E})$, then*

$$[\Delta] = 2c_1 + 3k \text{ and } [\Delta_{sing}] = 4(k^3 + 2k^2c_1 + kc_1^2 + kc_2 + c_1c_2 - c_3).$$

2.2.5. From zero loci of quiver flag varieties to zero loci of products of flag manifolds. In this subsection we present a method for translating quiver flag zero loci into zero loci of general global sections of irreducible globally generated homogeneous vector bundles in the product of Grassmannians.

Let us briefly recall what a quiver flag variety is, for more details, we refer to [**Cra11**]. To define such a variety, we have to fix some notation.

- Let Q be a finite, acyclic quiver with a unique source vertex, denoted by 0.
- Let $Q_0 = \{0, 1, 2, \ldots, \rho\}$ be the finite set of vertices and Q_1 the finite set of arrows between vertices of Q.
- Let $a \in Q_1$ such that $a : i \to j$. We define $t,\ h$ to be the tail and head map, i.e. $t(a) = i$ and $h(a) = j$.
- Let us call $r = (1, r_1, \ldots, r_\rho) \in \mathbb{Z}^{\rho+1}$ the dimension vector.
- Let W be a representations of the quiver Q, i.e., a collection $((W_i)_{i\in Q_0}, (\omega_a)_{a\in Q_1})$, with W_i a $\mathbb{C}$-vector space of dimension r_i and ω_a a linear map between $W_{t(a)}$ and $W_{h(a)}$.
- Let $\theta \in \mathbb{Q}^{\rho+1}$ be the special weight $(-\sum_{i=1}^{\rho} r_i, 1, \ldots, 1)$. Let us define $\theta(W)$ as $\sum_{i\in Q_0} \theta_i r_i$.

- We say that a representation W is θ-semistable if $\theta(W) = 0$ and for each subrepresentation $W' \subset W$, $\theta(W') \geq 0$. It is stable if it is semistable and $\theta(W') = 0$ if and only if W' is 0 or W.

Definition 2.2.14. *A quiver flag variety $\mathcal{M}_\theta(Q, r)$ is the moduli space of the θ-stable representation of a quiver Q, with Q, θ and r as above.*

Remark 2.2.15. *The quiver flag variety $\mathcal{M}_\theta(Q, r)$ carries a universal local free sheaf $\bigoplus_{i \in Q_0} \mathcal{W}_i$ with $\mathrm{rk}(\mathcal{W}_i) = r_i$ and $\mathcal{W}_0 = \mathcal{O}_{\mathcal{M}_\theta(Q,)}$.*

Definition 2.2.16. *A quiver zero loci variety is a variety defined as $\mathscr{Z}(E) \subset \mathcal{M}_\theta(Q, r)$, where $E = \bigoplus \Sigma_\alpha \mathcal{W}_i$.*

In [Kal19] they build a database of quiver zero loci Fano fourfolds. They described them using the data:

- the adjacency matrix $A = (n_{i,j})_{i,j}$;
- the dimension vector r;
- the vector bundle E.

Remark 2.2.17. *In [Kal19], the vector bundle E is described through a list of vectors whose entries are Young diagrams. More explicitly,*

$$E = (t_1^1, \dots, t_\rho^1), \dots, (t_1^n, \dots, t_\rho^n),$$

which denotes

$$E = \bigoplus_{j=1}^{n} \bigotimes_{i=1}^{\rho} \Sigma_{t_i^j} \mathcal{W}_i.$$

Each Young diagram t_i^j is either black or white, indicating whether we are considering $\mathcal{W}_i$ or its dual $\mathcal{W}_i^\vee$, respectively.

Quiver flag varieties, along with their universal locally free sheaves, can be reinterpreted as iterated Grassmannian bundles equipped with their corresponding relative universal bundles. Before presenting the structural theorem for quiver flag varieties, we introduce some notation.

Definition 2.2.18. *Consider the quiver Q, the vector r and the special weight θ as above. For $0 \leq i \leq \rho$, let $Q(i)$ denote the subquiver of Q obtained by removing all vertices $k > i$ and all arrows with head at any of these vertices. In particular, the vertex set of $Q(i)_0 = \{0, \dots, i\}$ and the arrow set is $Q(i)_1 = \{a \in Q_1 \,:\, 1 \leq h(a) \leq i\}$. For the quiver $Q(i)$ and for the dimension vector $r(i) = [1, r_1, \dots, r_i]$, we let*

$$Y_i := \mathcal{M}_{\theta(i)}(Q(i), r(i))$$

denote the quiver flag variety, where $\theta(i) = (-\sum_{j=1}^{i} r_j, 1, \dots, 1) \in \mathbb{Z}^{i+1}$ is the special weight. Write $\mathcal{W}_0^{(i)}$, $\mathcal{W}_1^{(i)}$,…, $\mathcal{W}_i^{(i)}$ for the tautological bundles on Y_i.

We state now the structure theorem for quiver flag varieties, as proved in [Cra11].

Theorem 2.2.19. [Cra11, Theorem 3.3] *For any quiver Q and vector r as above, $\mathcal{M}_\theta(Q, r)$ is a smooth projective Mori dream space that gives rise to a tower of Grassmann bundles*

$$\mathcal{M}_\theta(Q, r) = Y_\rho \to Y_{\rho-1} \to \cdots \to Y_1 \to Y_0 = \operatorname{Spec} \mathbb{C},$$

where for $1 \leq i \leq \rho$, the variety Y_i is isomorphic to the Grassmann bundle $\mathbb{G}\mathrm{r}_{Y_{i-1}}(r_i, \mathcal{F}_i)$, where $\mathcal{F}_i$ is the locally free sheaf on Y_{i-1} given by

$$\mathcal{F}_i := \bigoplus_{\{a \in Q_1 \,:\, h(a)=i\}} \mathcal{W}^{i-1}_{t(a)}.$$

Remark 2.2.20. *In* [**Cra11**]*, they consider the Grassmannian of quotient spaces rather than the Grassmannian of subspaces. In particular, the relation between the two notations is the following:*

Let $A(k,n)$ denote the variety parametrizing the k-dimensional quotients of V_n, i.e. V_n/V_{n-k}, and let $B(k,n)$ denote the variety parametrizing the k-dimesnional subspaces $V_k^\vee \subset V_n^\vee$. These varieties are isomorphic up to dualizing.

Note that Theorem 2.2.19 remains unchanged under this correspondence. However, in our notation, the bundles $\mathcal{W}_i$ are the tautological vector bundles of the Grassmannian of the subspaces, whereas in [**Cra11**] *they are referred to as the tautological quotient bundles of the Grassmannian of the quotient spaces.*

Remark 2.2.21. *Note that $\mathrm{rk}(\mathcal{F}_i) = \sum_{\{a \in Q_1 \,:\, h(a)=i\}} r_{t(a)}$. Moreover, since $Y_i = \mathbb{G}\mathrm{r}_{Y_{i-1}}(r_i, \mathcal{F}_i)$ for each i, the bundle $\mathcal{W}_i^{(i)}$ is the relative tautological vector bundle of Y_i, that is $\mathcal{U}_{\mathcal{R}_i}$.*

Recall the sequence

$$Y_0 \subset Y_1 \subset \cdots \subset Y_{i-1} \subset Y_i$$

is a filtration of Grassmann bundles. Consequently, for $1 < j < i$, the bundle $\mathcal{W}_j^{(i)}$ is a relative tautological bundle on the j-th Grassmann bundle Y_j, namely $\mathcal{U}_{\mathcal{R}_j}$.

If $j = 1$ then $\mathcal{W}_1^{(i)}$ is the tautological bundle $\mathcal{U}$, since $\mathcal{W}_0^{(i)} = \mathcal{O}_{Y_i}$. Therefore, we have

$$Y_1 = \mathbb{G}\mathrm{r}_{\mathrm{Spec}\,\mathbb{C}}(r_1, \mathcal{O}_{Y_i}^{\oplus n_{01}}),$$

which implies $Y_1 = \mathrm{Gr}(r_1, n_{01})$, and hence $\mathcal{W}_1^{(i)} = \mathcal{U}_{\mathrm{Gr}(r_1,n_{01})}$.

Proposition 2.2.22. *With the above notation, the quiver flag variety Y_i can be realized as the zero locus*

$$Y_i = \mathscr{Z}\Big(\bigoplus_{\{a \in Q(i)_1 | t(a) \neq 0\}} \mathcal{Q}_{t(a)} \boxtimes \mathcal{U}^\vee_{h(a)}\Big) \subset \Pi_{l=1}^{i} \mathrm{Gr}(r_l, v_l),$$

where v_i denotes number of paths from the source vertex 0 to the vertex i.

Proof. We proceed by induction. As noted in Remark 2.2.21, Y_1 is the Grassmannian $\mathrm{Gr}(r_i, n_{01})$. Note that Y_2 is the Grassmann bundle

$$\mathbb{G}\mathrm{r}_{\mathrm{Gr}(r_i,n_{01})}(r_2, \mathcal{F}_2)$$

where

$$\mathcal{F}_2 = (\mathcal{W}_0^{(1)})^{\oplus n_{02}} \oplus (\mathcal{W}_1^{(1)})^{\oplus n_{12}}.$$

By Remark 2.2.21, we have $\mathcal{W}_0^{(1)} = \mathcal{O}_{Y_1}$ and $\mathcal{W}_1^{(1)} = \mathcal{U}_{Y_1}$, so we can rewrite Y_2 as

$$\mathbb{G}\mathrm{r}_{Y_1}(r_2, \mathcal{O}_{Y_1}^{\oplus n_{02}} \oplus \mathcal{U}_{Y_1}^{\oplus n_{12}}).$$

We now apply Theorem 2.2.8 using n_{12} copies of the Euler sequence. This gives:

$$Y_2 = \mathscr{Z}((\mathcal{Q}_{\mathrm{Gr}(r_1,n_{01})} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(r_2,n_{0,2}+n_{12}n_{01})})^{\oplus n_{12}}) \subset \mathrm{Gr}(r_1, n_{01}) \times \mathrm{Gr}(r_2, n_{02} + n_{12}n_{01}),$$

where n_{01} is the number of directed paths from 0 to 1, and $n_{02} + n_{12}n_{01}$ is the number of directed paths from 0 to 2.

Now, assume that for some $j < i$, the following expression holds:

$$Y_j = \mathscr{Z}(\bigoplus_{\{a \in Q(j)_1 | t(a) \neq 0\}} \mathcal{Q}_{t(a)} \boxtimes \mathcal{U}^{\vee}_{h(a)}) \subset \Pi_{l=1}^{j} \operatorname{Gr}(r_j, v_j)$$

Let us prove this for i.

Recall that

$$Y_i = \mathbb{G}\mathrm{r}_{Y_{i-1}}(r_i, \mathcal{F}_i),$$

where

$$\mathcal{F}_i = \bigoplus_{\{a \in Q_1 : h(a) = i\}} \mathcal{W}^{i-1}_{t(a)}.$$

By inductive hypothesis:

- if $t(a) = 0$, then $\mathcal{W}^{i-1}_0 = \mathcal{O}_{Y_{i-1}}$;
- if $t(a) \neq 0$ then$\mathcal{W}^{i-1}_{t(a)} = \mathcal{U}_{Y_{t(a)}}$.

Therefore

$$\mathcal{F}_i = \mathcal{O}^{\oplus n_{0i}}_{Y_{i-1}} \oplus \bigoplus_{0<j<i} \mathcal{U}_j^{\oplus n_{ji}}.$$

It follows that

$$Y_i = \mathbb{G}\mathrm{r}_{Y_{i-1}}(r_i, \mathcal{O}^{\oplus n_{0i}}_{Y_{i-1}} \oplus \bigoplus_{0<j<i} \mathcal{U}_j^{\oplus n_{ji}}).$$

Applying Theorem 2.2.8 again, we obtain

$$Y_i = \mathscr{Z}(\bigoplus_{0<j<1} (\mathcal{Q}_{\operatorname{Gr}(r_j, v_j)} \boxtimes \mathcal{U}^{\vee}_{\operatorname{Gr}(r_i, n_{0i} + \sum_{0<j<i} v_j n_{ji})})^{n_{ji}}) \subset Y_{i-1} \times \operatorname{Gr}(r_i, n_{0i} + \sum_{0<j<i} v_j n_{ji}),$$

where v_j is the number of paths between 0 and j. Thus, the total number of paths from 0 to i is:

$$n_{0i} + \sum_{0<j<i} v_j n_{ji},$$

because each path from 0 to i is either a direct path, counted by n_{0i}, or a path through an intermediate vertex j, in which case there are v_j paths from 0 to j and n_{ji} direct paths from j to i.

We can rewrite Y as:

$$Y_i = \mathscr{Z}(\bigoplus_{\{a \in Q(j)_1 | t(a) \neq 0\}} \mathcal{Q}_{t(a)} \boxtimes \mathcal{U}^{\vee}_{h(a)} \oplus \bigoplus_{0<j<1} (\mathcal{Q}_{\operatorname{Gr}(r_j, v_j)} \boxtimes \mathcal{U}^{\vee}_{\operatorname{Gr}(r_i, v_i)})^{n_{ji}}) \subset \Pi_{l=1}^{i} \operatorname{Gr}(r_l, v_l).$$

Observe that

$$\bigoplus_{0<j<1} (\mathcal{Q}_{\operatorname{Gr}(r_j, v_j)} \boxtimes \mathcal{U}^{\vee}_{\operatorname{Gr}(r_i, v_i)})^{n_{ji}}$$

can be rearranged as

$$\bigoplus_{\{a \in Q(i)_1 | t(a) \in Q(j)_0 \setminus \{0\},\ h(a) = i\}} \mathcal{Q}_{\operatorname{Gr}(r_{t(a)}, v_{t(a)})} \boxtimes \mathcal{U}^{\vee}_{\operatorname{Gr}(r_i, v_i)}.$$

Since $\{a \in Q(i)_1 | t(a) \in Q(j)_0 \setminus \{0\},\ h(a) = i\}$ is the set of arrows from the verteces of $Q(j)$ to the vertex i, we can reindex the full expression with the set $\{a \in Q(i)_1 | t(a) \neq 0\}$. This completes the inductive step.

□

Corollary 2.2.23. *With the above notation, a quiver flag variety $Y = \mathcal{M}_\theta(Q, r)$ admits the description*

$$Y = \mathscr{Z}\Big(\bigoplus_{0<j<i\leq\rho} (\mathcal{Q}_j \boxtimes \mathcal{U}_i^\vee)^{\oplus n_{ji}}\Big) \subset \Pi_{l=1}^{\rho} \operatorname{Gr}\Big(r_l, \sum_{k=1}^{\rho} (A^k)_{0,l}\Big)$$

Proof. By Proposition 2.2.22, we identify Y with:

$$Y = \mathscr{Z}\Big(\bigoplus_{\{a\in Q_1 | t(a)\neq 0\}} \mathcal{Q}_{t(a)} \boxtimes \mathcal{U}_{h(a)}^\vee\Big) \subset \Pi_{l=1}^{\rho} \operatorname{Gr}(r_l, v_l)$$

Since we are considering the full quiver Q, we can group the direct summands as follows:

$$\bigoplus_{\{a\in Q_1|t(a)\neq 0\}} \mathcal{Q}_{t(a)} \boxtimes \mathcal{U}_{h(a)}^\vee = \bigoplus_{\{i,j\in Q_0\setminus\{0\}\}} \bigoplus_{\{a\in Q_1|t(a)=j\wedge h(a)=i\}} \mathcal{Q}_{t(a)} \boxtimes \mathcal{U}_{h(a)}^\vee = \bigoplus_{\{i,j\in Q_0\setminus\{0\}\}} (\mathcal{Q}_j \boxtimes \mathcal{U}_i^\vee)^{n_{ji}}.$$

This yields the description:

$$Y = \mathscr{Z}\Big(\bigoplus_{0<j<i\leq\rho} (\mathcal{Q}_j \boxtimes \mathcal{U}_i^\vee)^{\oplus n_{ji}}\Big) \subset \Pi_{l=1}^{\rho} \operatorname{Gr}(r_l, v_l).$$

To compute the v_l's, we exploit the adjacency matrix of the quiver, A. By [FH69, Theorem 13.1] the number of paths between two vertices, say $j < i$, in a quiver Q as above, is:

$$N_{j,i} = \sum_{k=1}^{\rho} (A^k)_{j,i}.$$

In particular, $v_l = N_{0,l}$. This concludes the argument. □

Corollary 2.2.24. *With the above notation, let X be the quiver zero loci $\mathscr{Z}(E) \subset \mathcal{M}_\theta(Q, r)$ where*

$$E = (t_1^1, \dots, t_\rho^1), \dots, (t_1^n, \dots, t_\rho^n).$$

Let α_i^j be the partition associated with the Young diagram t_i^j. X can be described as

$$X = \mathscr{Z}\Big(\Big(\bigoplus_{j=1}^{n} \boxtimes_{i=1}^{\rho} \Sigma_{\alpha_i^j} \mathcal{U}_i\Big) \oplus \bigoplus_{0<j<i\leq\rho} (\mathcal{Q}_j \boxtimes \mathcal{U}_i^\vee)^{\oplus n_{ji}}\Big) \subset \Pi_{l=1}^{\rho} \operatorname{Gr}\Big(r_l, \sum_{k=1}^{\rho} (A^k)_{0,l}\Big).$$

Proof. Thanks to Corollary 2.2.23 it remains to study how to translate the vector bundle E. As noted in Remark 2.2.17, we have that E is

$$\bigoplus_{j=1}^{n} \bigotimes_{i=1}^{\rho} \Sigma_{t_i^j} \mathcal{W}_i.$$

However, as observed in Remark 2.2.21, the bundles $\mathcal{W}_i$ are precisely the relative tautological bundles of the Grassmann bundles Y_i. Therefore, E is

$$E = \bigoplus_{j=1}^{n} \bigotimes_{i=1}^{\rho} \Sigma_{t_i^j} \mathcal{U}_{\mathcal{R}_i}.$$

Finally, by Proposition 2.2.22, $\mathcal{M}_\theta(Q, r)$ is a section zero locus in a product of Grassmannians, so we can rewrite E as:

$$E = \bigoplus_{j=1}^{n} \boxtimes_{i=1}^{\rho} \Sigma_{t_i^j} \mathcal{U}_{\operatorname{Gr}(r_i, v_i)}$$

Since $X = \mathscr{Z}(E) \subset \mathcal{M}_\theta(Q, r)$, by Corollary 2.2.23 it can be expressed as

$$X = \mathscr{Z}\Big(\big(\bigoplus_{j=1}^{n} \boxtimes_{i=1}^{\rho} \Sigma_{\alpha_i^j} \mathcal{U}_i\big) \oplus \bigoplus_{0<j<i\leq\rho} (\mathcal{Q}_j \boxtimes \mathcal{U}_i^\vee)^{\oplus n_{ji}}\Big) \subset \Pi_{l=1}^{\rho}\, \mathrm{Gr}\big(r_l, \sum_{k=1}^{\rho} (A^k)_{0,l}\big).$$

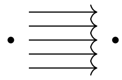
□

The advantage of expressing quiver flag varieties and quiver zero loci as subvarieties of products of Grassmannians lies in their computational tractability. When working with zero loci of general global sections of irreducible, globally generated homogeneous vector bundles on products of Grassmannians, we can apply the full suite of tools developed in Section 2.2. This allows us to compute their Hodge numbers and to provide a biregular and birational description. Let us give a couple of examples of translation:

Example 2.2.25. *We want to show that K_1 is $\mathbb{P}^4$ using Proposition 2.2.22. Indeed K_1 is described in* [Kal19, Table 3] *using:*

$$A = \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}, \quad r = (1, 1).$$

A is the adjacency matrix associated with the quiver Q:

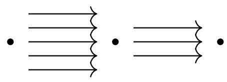


Since $r \in \mathbb{Z}^2$, then $\mathcal{M}_\theta(Q, r) = Y_1$. Applying Corollary 2.2.23 we directly obtain that $\mathcal{M}_\theta(Q, r) = \mathrm{Gr}(1, 5) = \mathbb{P}^4$.

Example 2.2.26. *We show that Fano variety K_2 is $\mathscr{Z}(\mathcal{O}(1, 1)) \subset \mathbb{P}^4 \times \mathbb{P}^1$. In* [Kal19, Table 3] *$K_2$ is described via the following data:*

$$A = \begin{bmatrix} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}, \quad r = (1, 1, 1), \quad E = (\square, \blacksquare) \oplus (\square, \varnothing).$$

Thus the quiver Q associated to A is:



Since $r \in \mathbb{Z}^3$, we have a tower $\mathcal{M}_\theta(Q, r) = Y_2 \supset Y_1$. By applying Corollary 2.2.24, we directly obtain

$$X = \mathscr{Z}(\mathcal{O}(-1, 1) \oplus \mathcal{O}(0, 1) \oplus \mathcal{Q}_{\mathbb{P}^4}(0, 1)^{\oplus 3}) \subset \mathbb{P}^4 \times \mathbb{P}^{14}.$$

However, a priori, $\mathcal{O}(-1, 1)$ is not globally generated on $\mathbb{P}^4 \times \mathbb{P}^{14}$, and therefore the tools described in Section 2.2 cannot be directly applied. Note that by avoiding the direct application of Corollary 2.2.24 and instead analyzing the Grassmann bundle structure, we can simplify the construction.

Recall that $\mathcal{M}_\theta(Q, r) = Y_2 \supset Y_1$, and, by Remark 2.2.21, we have $Y_2 = \mathbb{P}_{\mathbb{P}^4}(\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 3})$. Therefore,

$$X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(1) \boxtimes (\mathcal{O}_{\mathbb{P}^4}(-1) \oplus \mathcal{O}_{\mathbb{P}^4})) \subset \mathbb{P}_{\mathbb{P}^4}(\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 3}).$$

We now twist $\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 3}$ with $\mathcal{O}_{\mathbb{P}^4}(1)$, so the ambient space becomes $\mathbb{P}_{\mathbb{P}^4}(\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4}$, and the new relative tautological bundle satisfies $\mathcal{O}_{\mathcal{R}'}(-1) = \mathcal{O}_{\mathcal{R}}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^4}(1)$. Consequently, $\mathcal{O}_{\mathcal{R}}(1) = \mathcal{O}_{\mathcal{R}'}(1) \boxtimes \mathcal{O}_{\mathbb{P}^4}(1)$, yielding:

$$X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}'}(1) \boxtimes (\mathcal{O}_{\mathbb{P}^4}(1-1) \oplus \mathcal{O}_{\mathbb{P}^4}(1))) \subset \mathbb{P}_{\mathbb{P}^4}(\mathcal{O}_{\mathbb{P}^4}^{\oplus 3}).$$

Since $\mathbb{P}_{\mathbb{P}^4}(\mathcal{O}_{\mathbb{P}^4}^{\oplus 3}) = \mathbb{P}^4 \times \mathbb{P}^2$, and the space $\mathbb{P}^4 \times \mathbb{P}^1$ can be realized as $\mathscr{Z}(\mathcal{O}(0,1)) \subset \mathbb{P}^4 \times \mathbb{P}^2$, then we conclude that

$$X = \mathscr{Z}(\mathcal{O}(1,1)) \subset \mathbb{P}^4 \times \mathbb{P}^1.$$

Example 2.2.27. *We show that Fano variety K_{299} is given by the zero locus*

$$K_{299} = \mathscr{Z}\left(\mathcal{Q}_{\mathbb{P}^2_2}(0,0,1) \oplus \mathcal{O}(1,0,1)^{\oplus 2}\right) \subset \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathbb{P}^4.$$

In [**Kal19**, Table 3], *K_{299} is described by the data*

$$A = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 \end{bmatrix}, \quad r = (1,1,1,1), \quad E = (\varnothing, \square, \varnothing)^{\oplus 2}.$$

Thus the quiver Q associated to A is:

$$\begin{tikzcd}
\bullet \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \bullet \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[dr, shift left] \arrow[dr, shift right] & \bullet \arrow[d] \\
& & \bullet
\end{tikzcd}$$

Since $r \in \mathbb{Z}^4$, the quiver flag variety $\mathcal{M}_\theta(Q, r)$ admits a tower structure

$$Y_3 \supset Y_2 \supset Y_1.$$

Applying Corollary 2.2.24, we obtain the embedding

$$X = \mathscr{Z}(\mathcal{O}(0,0,1)^{\oplus 2} \oplus \mathcal{Q}_{\mathbb{P}^2}(0,1,0)^{\oplus 3} \oplus \mathcal{Q}_{\mathbb{P}^2}(0,0,1)^{\oplus 2} \oplus \mathcal{Q}_{\mathbb{P}^4}(0,0,1)) \subset \mathbb{P}^2 \times \mathbb{P}^8 \times \mathbb{P}^{14}$$

which is directly suitable for the tools developed in Section 2.2. However, a simplified description arises by inspecting the tower of Grassmann bundles.

By Remark 2.2.21, the tower reads:

$$\begin{aligned}
Y_1 &= \mathbb{P}^2, \\
Y_2 &= \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}(-1)^{\oplus 3}), \\
Y_3 &= \mathbb{P}_{Y_1}(\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathcal{R}_{Y_2}}(-1)).
\end{aligned}$$

Moreover

$$X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}_{Y_3}}(1)^{\oplus 2}) \subset \mathbb{P}_{\mathbb{P}_{\mathbb{P}^2}(\mathcal{O}(-1)^{\oplus 3})}(\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathcal{R}_{Y_2}}(-1))$$

Twisting $\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3}$ by $\mathcal{O}_{\mathbb{P}^2}(1)$, the bundle becomes trivial, and hence

$$Y_2 = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}^{\oplus 3}) = \mathbb{P}^2_1 \times \mathbb{P}^2_2.$$

The relative tautological bundle transforms as

$$\mathcal{O}_{\mathcal{R}_{Y_2}}(-1) = \mathcal{O}_{\mathcal{R}'_{Y_2}}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(-1),$$

so over $\mathbb{P}^2_1 \times \mathbb{P}^2_2$, we interpret this as $\mathcal{O}(-1,-1)$. Thus,

$$X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}_{Y_3}}(1)^{\oplus 2}) \subset \mathbb{P}_{\mathbb{P}^2 \times \mathbb{P}^2}(\mathcal{O}(-1,0)^{\oplus 2} \oplus \mathcal{O}(-1,-1)).$$

We now twist the bundle $\mathcal{O}(-1,0)^{\oplus 2} \oplus \mathcal{O}(-1,-1)$ *by* $\mathcal{O}(1,0)$. *This yields:*

$$\mathbb{P}_{\mathbb{P}^2_1 \times \mathbb{P}^2_2}\left(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(0,-1)\right),$$

and the tautological bundle becomes

$$\mathcal{O}_{\mathcal{R}'_{Y_3}}(-1) = \mathcal{O}_{\mathcal{R}_{Y_3}}(-1) \boxtimes \mathcal{O}(1,0),$$

so that

$$\mathcal{O}_{\mathcal{R}_{Y_3}}(1) = \mathcal{O}_{\mathcal{R}'_{Y_3}}(1) \boxtimes \mathcal{O}(1,0).$$

Substituting, we obtain:

$$X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}'_{Y_3}}(1) \boxtimes \mathcal{O}(1,0)^{\oplus 2}) \subset \mathbb{P}_{\mathbb{P}^2_1 \times \mathbb{P}^2_2}(\mathcal{O}^{\oplus 2}_{\mathbb{P}^2 \times \mathbb{P}^2} \oplus \mathcal{O}(0,-1))$$

Finally, applying Theorem 2.2.8 (the Euler sequence for $\mathbb{P}^2_2$*), we recover:*

$$X = \mathscr{Z}\left(\mathcal{O}(1,0,1)^{\oplus 2} \oplus \mathcal{Q}_{\mathbb{P}^2_2}(0,0,1)\right) \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^4,$$

2.3. A few worked examples

In this section, we want to give four examples of the 170 varieties we studied. These are chosen to explain in detail the different methods applied throughout the systematic description of these manifolds.

Example 2.3.1. ***Fano 2-1***

- $X = \mathscr{Z}(\mathcal{O}(1,1) \oplus \mathcal{O}(0,1)^{\oplus 3}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,5)$;
- *Invariants:* $h^0(-K) = 39$, $(-K)^4 = 160$, $\chi(T_X) = -9$, $h^{1,1} = 2$, $h^{3,1} = 0$, $h^{2,2} = 7$;
- *Description:* X *is generically a* $\mathbb{P}^1$*-bundle on* X^3_5 *which degenerates to a* $\mathbb{P}^2$*-bundle over five points.*

Identification. In this example, we present all the explicit computations involving degeneracy loci.

Consider the Fano fourfold K_{508}, *which can be described as a* $\mathbb{P}^1$*-bundle over* X^3_5, *whose fibers jump to a* $\mathbb{P}^2$ *on five points. In* [**Kal19**] *this Fano fourfold corresponds to the following data:*

$$A = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}, \quad r = (1,1,2), \quad E = (\mathrm{Sym}^2 \blacksquare, \bigwedge^2 \square)^{\oplus 3} \oplus (\blacksquare, \bigwedge^2 \square)$$

This means that we have a quiver Q *of this form:*

$$\begin{tikzcd} \bullet \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \bullet \arrow[r, shift left=4] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[r, shift right=4] & \bullet \end{tikzcd}$$

The fourfold K_{508} *is realized as zero locus* $\mathscr{Z}(E) \subset \mathcal{M}_\theta(Q,r)$, *where* E *is expressed in terms of universal bundles arising from the quiver moduli interpretation, see Section 2.2.5. In this case,* $E = (\mathrm{Sym}^2 \mathcal{W}_1^\vee \otimes \mathcal{W}_2)^{\oplus 3} \oplus (\mathcal{W}_1^\vee \otimes \bigwedge^2 \mathcal{W}_2)$.

Applying Corollary 2.2.24, we see that $\mathcal{M}_\theta(Q,r)$ *defines a Grassmann bundle over* $\mathbb{P}^2$, *namely* $\mathrm{Gr}_{\mathbb{P}^2}(2, \mathcal{O}(-1)^{\oplus 5})$.

In this way K_{508} *can also be written as*

$$X = \mathscr{Z}(\mathcal{O}(-1) \boxtimes \mathcal{O}_{\mathcal{R}}(1) \oplus \mathcal{O}(-2) \boxtimes \mathcal{O}_{\mathcal{R}}(1)^{\oplus 3} \subset \mathrm{Gr}_{\mathbb{P}^2}(2, \mathcal{O}(-1)^{\oplus 5}).$$

Twisting $\mathcal{O}(-1)^{\oplus 5}$ with $\mathcal{O}(1)$, we obtain the equivalent description $X = \mathscr{Z}(\mathcal{O}(1,1)\oplus\mathcal{O}(0,1)^{\oplus 3}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,5)$.

This puts us in the appropriate setting to apply a combination of Koszul, co-normal sequence, Borel–Bott–Weil, and Littlewood–Richardson theorems to compute Hodge numbers.

Since we are dealing with Fano fourfolds, the only interesting Hodge numbers are $h^{0,0}(X)$, $h^{1,1}(X)$, $h^{2,1}(X)$, $h^{3,1}(X)$, and $h^{2,2}(X)$. Using Borel–Bott–Weil and Littlewood–Richardson Theorems with some diagram chasing, we obtain the following numbers:

$$\begin{array}{ccccccccc}
0 & & 0 & & 7 & & 0 & & 0 \\
& 0 & & 0 & & 0 & & 0 & \\
& & 0 & & 2 & & 0 & & \\
& & & 0 & & 0 & & & \\
& & & & 1 & & & &
\end{array}$$

Other key invariants to compute are he dimension of global sections of the anticanonical bundle, the anticanonical volume, $h^0(-K)$, $vol(X) = (-K)^4$, and the holomorphic Euler characteristic of the tangent bundle $\chi(T_X)$.

- *For $h^0(-K)$, applying the adjunction formula, Koszul complex and Borel–Bott–Weil, yields $h^0(-K) = 39$.*
- *The volume can be computed using the Hirzebruch–Riemann–Roch Theorem and Chern classes. In this case, we obtain $vol(X) = 160$.*
- *As for the Euler characteristic $\chi(T_X)$, it can be computed again using the Hirzebruch—Riemann—Roch theorem. Alternatively, since X is a Fano fourfold, we can apply Kodaira vanishing to reduce computations to $\chi(T_X) = h^0(X,T_X) - h^1(X,T_X)$. In this way, we obtain $\chi(T_X) = -9$.*

Those computations can become quite cumbersome very quickly, for this reason, and for the large number of computations we have to make we intensively use computer algebra software as [**GS**].

We can now proceed with the birational description of X. Let us consider the natural projection π_2 on $\mathrm{Gr}(2,5)$

$$\pi_{2|X} : X \to \mathrm{Gr}(2,5).$$

Note that the fiber of $\pi_{2|X}$ is generically empty. This is because, when X is described as the zero locus of a global section in $\mathbb{P}^2 \times \mathrm{Gr}(2,5)$, the ambient is cut out by three general sections of $\mathcal{O}(0,1)$, which corresponds to $\pi_2^*\mathcal{O}_{\mathrm{Gr}(2,5)}(1)$.

Therefore X can also be described as

$$\mathscr{Z}(\mathcal{O}(1,1)) \subset \mathbb{P}^2 \times X_5^3,$$

where $X_5^3 = \mathscr{Z}(\mathcal{O}(1)^{\oplus 3}) \subset \mathrm{Gr}(2,5)$ the Fano threefold 1-15 in [**DBFT22**, Table 1].

In this setting, the projection π_2 can be reinterpreted as

$$\pi_{2|X} : X \to X_5^3.$$

For each point $p \in X_5^3$ the general fiber $\pi_{2|X}^{-1}(p)$ is $\mathbb{P}^2$ cut by a section of $\mathcal{O}_{\mathbb{P}^2}(1)\otimes\pi_2^*\mathcal{O}_{\mathrm{Gr}(2,5)}(1)(p)$. Since $\pi_2^*\mathcal{O}_{\mathrm{Gr}(2,5)}(1)(p) = \mathbb{C}$, $\mathcal{O}_{\mathbb{P}^2}(1) \otimes \pi_2^*\mathcal{O}_{\mathrm{Gr}(2,5)}(1)(p) = \mathcal{O}_{\mathbb{P}^2}(1)$. Therefore, under the projection π_2, X is generically a $\mathbb{P}^1$-fibration over the Fano threefold X_3^3.

We can arrive at the same description by rewriting $\mathbb{P}^2 \times \mathrm{Gr}(2,5)$ as $\mathbb{P}_{\mathrm{Gr}(2,5)}(\mathcal{O}^{\oplus 3})$, so that

$$X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(1) \boxtimes \mathcal{O}_{\mathrm{Gr}(2,5)}(1)_{X_5^3}) \subset \mathbb{P}_{X_5^3}(\mathcal{O}_{X_5^3}^{\oplus 3}).$$

Twisting $\mathcal{O}_{X_5^3}^{\oplus 3}$ by $\mathcal{O}_{X_5^3}(-1)$, we obtain

$$X \subset \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(1)) \subset \mathbb{P}_{X_5^3}(\mathcal{O}_{X_5^3}(-1)^{\oplus 3}),$$

which is generically a $\mathbb{P}^2$ bundle over X_5^3, whose general fiber is cut by a section of the relative bundle $\mathcal{O}_{\mathcal{R}}(1)$. Hence, this yields again a birational model of X as $\mathbb{P}^1$-fibration over the Fano threefold X_3^3.

This description is made with the generality assumption, and thus is valid only birationally. Up to this point, we know that X is birational to a $\mathbb{P}^1$-fibration over X_5^3, namely

$$\hat{X} = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(1)) \subset \mathbb{P}_{\mathrm{Gr}(2,5)}(\mathcal{O}(-1)^{\oplus 3}),$$

via the restriction of the natural projection π_2. However, we can provide a more refined description by examining the exceptional locus $\pi_{2|X}$.

Note that $X = \mathscr{Z}(\sigma) \subset \mathbb{P}^2 \times X_5^3$, where $\sigma \in H^0(\mathbb{P}^2 \times X_5^3, \mathcal{O}(1,1)) = V_3^\vee \otimes \bigwedge^2 V_5^\vee$. To study the exceptional locus of $\pi_{2|X}$, we consider the set of points $p \in X_5^3$, for which $\sigma(l,p) = 0$ for all $l \in \mathbb{P}^2$.

This condition can be translated into the language of degeneracy loci. Since $\sigma \in V_3^\vee \otimes \bigwedge^2 V_5^\vee$, we may write it as $v \otimes (w_1 \wedge w_2)$. The condition $\sigma(l,p) = 0$ for all $l \in \mathbb{P}^2$ then means we are looking for $p \in X_5^3$, such that $v \otimes (w_1 \wedge w_2)(p) = 0$. But $w_1 \wedge w_2$ is a global section of $\mathcal{O}_{X_5^3}(1)$, and v can be seen as a global section of $V_3^\vee \otimes \mathcal{O}_{X_5^3}$, since it can not vanish on points of X_5^3.

Thus the problem can be translated into finding points of X_5^3 where the global section $\gamma \in H^0(X_5^3, V_3^\vee \otimes \mathcal{O}_1)$ is zero or where the morphism

$$\varphi : \mathcal{O}(-1) \to V_3^\vee \otimes \mathcal{O}$$

is of rank 0.

This is exactly a degeneracy locus of a morphism between vector bundles on X_5^3. In particular, we identify the exceptional locus of $\pi_{2|X}$ with $\mathrm{D}_0(\varphi)$. *By Theorem 2.2.5* $\mathrm{D}_0(\varphi)$ *is a finite number of points, and we can apply Eagon–Northcott Theorem to compute the exact number:*

$$0 \to \mathcal{O}_{X_5^3}(-2) \to \mathcal{O}_{X_5^3}(-2)^{\oplus 2} \to \mathcal{O}_{X_5^3}(-1)^{\oplus 3} \to \mathcal{O}_{X_5^3} \to \mathcal{O}_{\mathrm{D}_0(\varphi)} \to 0.$$

Since the alternating sum of the Euler characteristic of vector bundles in an exact sequence is equal to zero, we obtain that

$$\chi(\mathcal{O}_{\mathrm{D}_0(\varphi)}) = \chi(\mathcal{O}_{X_5^3}) - 3\chi(\mathcal{O}_{X_5^3}(-1)) + \chi(\mathcal{O}_{X_5^3}(-2)).$$

The right-hand side is straightforward to compute, since $\chi(\mathcal{O}_{X_5^3}) = 4$, and the others can be deduced through a combination of diagram chasing and Borel–Bott–Weil and Littlewood–Richardson theorems. In particular we find $\chi(\mathcal{O}_{X_5^3}(-1)) = 0$ and $\chi(\mathcal{O}_{X_5^3}(-2) = 1$. These computations imply that $\mathrm{D}_0(\varphi) = \{5 \text{ points}\}$. Therefore, we have a complete biregular description of X as a $\mathbb{P}^1$-fibration over X_5^3, which jumps to a $\mathbb{P}^2$ over five points.

An alternative approach to identifying the exceptional locus, sometimes more computationally convenient. Instead of using the Eagon–Northcott complex, we could have observed that the degeneracy locus of φ is:

$$\mathscr{Z}(\mathcal{O}(1)^{\oplus 3}) \subset \mathbb{P}_{X_5^3}(\mathcal{O}),$$

which corresponds to the intersection of three general hyperplanes in X_5^3, that is:

$$X_5^3 \cap H \cap H' \cap H'' = \{5 \text{ points}\}.$$

This gives another route to the biregular description used throughout the paper.

We can also address the rationality of this manifold. Consider

$$\pi_1 : \mathbb{P}^2 \times \mathrm{Gr}(2,5) \to \mathbb{P}^2.$$

When restricted to X, the projection π_1 is surjective and the general fiber is X_5^3 cut by a general section of $\mathcal{O}_{\mathrm{Gr}(2,5)}(1)$. This defines a dP_5. *Hence $\pi_{1|X}$ induces a* dP_5 *fibration on $\mathbb{P}^2$. X is birational to a* dP_5*-fibration, and by* [**Isk96**]*, this suffices to conclude that X is rational.*

As we have already seen X can be written as $\mathscr{Z}(\mathcal{O}_{X_5^3} \boxtimes \mathcal{O}_{\mathcal{R}}(1)) \subset \mathbb{P}_{X_5^3}(\mathcal{O}^{\oplus 2})$. If we consider the sequence of sheaves:

$$0 \to \mathcal{K} \to \mathcal{O}_{X_5^3}^{\oplus 3} \to \mathcal{O}_{X_5^3}(1)$$

and we apply Theorem 2.2.8, we can write X as $\mathbb{P}_{X_5^3}(\mathcal{K})$. If we dualize the previous sequence, we obtain $\mathcal{E} = \mathcal{K}^\vee$, defined by

$$0 \to \mathcal{O}_{X_5^3}(-1) \to \mathcal{O}_{X_5^3}^{\oplus 3} \to \mathcal{E} \to 0.$$

This is the model of Fano fourfold described in [**Lang98**, Theorem 8.2.3].

Example 2.3.2. ***Fano 2.13***

- *$X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,5)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 3} \oplus \mathcal{O}(2,0)) \subset \mathbb{P}^4 \times \mathrm{Gr}(3,5)$;*
- *Invariants: $h^0(-K) = 30,\ (-K)^4 = 116,\ \chi(T_X) = -11,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 10$;*
- *Description: The projection $\pi_1 : X \to Q_3$ is a conic bundle with discriminant a sextic surface in Q_3, singular in 40 points. The second projection $\pi : X \to X_5^3$ is a conic bundle discriminant a K3 surface singular in eight points.*

Identification. In this example, we write the computation for the conic bundles described in the paper.

Consider the Fano fourfold K_{582}. We can describe it as a conic bundle over X_5^3, with discriminant a K3 surface singular in eight points.

In [**Kal19**]*, K_{582} is given as the data of:*

$$A = \begin{bmatrix} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad r = (1,1,3), \quad E = (\varnothing, \bigwedge^3 \square)^{\oplus 3} \oplus (\mathrm{Sym}^2 \square, \varnothing).$$

This means that we have a quiver of the following form:

$$\begin{tikzcd} \bullet \arrow[r, shift left=4] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[r, shift right=4] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

The fourfold K_{582} is realized as zero locus $\mathscr{Z}(E) \subset \mathcal{M}_\theta(Q,r)$. In this case, $E = ((\bigwedge^3 \mathcal{W}_2^\vee)^{\oplus 3} \oplus \mathrm{Sym}^2 \mathcal{W}_1^\vee)$.

Applying Corollary 2.2.24 we obtain that $\mathcal{M}_\theta(Q,r)$ defines a projective bundle over that $\mathrm{Gr}(2,5)$ *as $\mathbb{P}_{\mathrm{Gr}(3,5)}(\mathcal{U})$. Hence K_{582} can be rewritten as*

$$X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(2) \oplus \mathcal{O}_{\mathrm{Gr}(3,5)}(1)^{\oplus 3}) \subset \mathbb{P}_{\mathrm{Gr}(3,5)}(\mathcal{U}).$$

If we use Theorem 2.2.8 with the sequence:

$$0 \to \mathcal{Q} \to V_5 \to \mathcal{U} \to 0$$

we obtain

$$X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,5)}(1,0) \oplus \mathcal{O}(2,0) \oplus \mathcal{O}(0,1)^{\oplus 3}) \subset \mathbb{P}^4 \times \mathrm{Gr}(3,5).$$

If we dualize $\mathrm{Gr}(3,5)$*, the we can describe* X *as*

$$X = \mathscr{Z}(\mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{O}(2,0) \oplus \mathcal{O}(0,1)^{\oplus 3}) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,5).$$

We apply the same strategy as in Example 2.3.1, and we compute the following invariants:

$$h^0(-K) = 30,\ (-K)^4 = 116,\ \chi(T_X) = -11;$$

and the following Hodge numbers:

$$\begin{array}{ccccccccc} 0 & & 0 & & 10 & & 0 & & 0 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & 2 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

To provide a birational description let us consider $X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(2) \oplus \mathcal{O}_{\mathrm{Gr}(3,5)}(1)^{\oplus 3}) \subset \mathbb{P}_{\mathrm{Gr}(3,5)}(\mathcal{U})$*. The three sections of* $\mathcal{O}(0,1)$ *cut on* $\mathrm{Gr}(3,5)$ *a Fano threefold* X_5^3*, while the section* $\mathcal{O}_{\mathcal{R}}(2)$ *cuts on the fibers of the* $\mathbb{P}^2$ *fibration a conic. Consequently,* X *is a conic bundle on* X_5^3*. To analyze the discriminant locus* Δ *of this conic bundle, we apply Lemma 2.2.13. In this setting, the rank three bundle is* $\mathcal{U}_{\mathrm{Gr}(3,5)}$*, and the line bundle is the trivial one.*

By computation of Chern classes, we obtain:

$$[\Delta] = 2c_1(\mathcal{U}_{\mathrm{Gr}(3,5)|X_5^3}) + 3c_1(\mathcal{O}_{\mathrm{Gr}(3,5)|X_5^3}) = 2c_1(\mathcal{U}_{\mathrm{Gr}(3,5)|X_5^3}) = 2H.$$

Which is a surface $S \subset X_5^3$ *of degree two. In particular* S *can be described as*

$$S = \mathscr{Z}(\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus 3}) \subset \mathrm{Gr}(2,5),$$

which is a $K3$ *surface, singular along*

$$\begin{aligned}[\Delta_{sing}] = 4(&c_1(\mathcal{O}_{\mathrm{Gr}(3,5)|X_5^3})^3 + 2c_1(\mathcal{O}_{\mathrm{Gr}(3,5)|X_5^3})^2 c_1(\mathcal{U}_{\mathrm{Gr}(3,5)|X_5^3}) + c_1(\mathcal{O}_{\mathrm{Gr}(3,5)|X_5^3})c_1(\mathcal{U}_{\mathrm{Gr}(3,5)|X_5^3})^2 + \\ &+ c_1(\mathcal{O}_{\mathrm{Gr}(3,5)|X_5^3})c_2(\mathcal{U}_{\mathrm{Gr}(3,5)|X_5^3}) + c_1(\mathcal{U}_{\mathrm{Gr}(3,5)|X_5^3})c_2(\mathcal{U}_{\mathrm{Gr}(3,5)|X_5^3}) - c_3(\mathcal{U}_{\mathrm{Gr}(3,5)|X_5^3})) = \\ &= 4(c_1(\mathcal{U}_{\mathrm{Gr}(3,5)|X_5^3})c_2(\mathcal{U}_{\mathrm{Gr}(3,5)|X_5^3}) - c_3(\mathcal{U}_{\mathrm{Gr}(3,5)|X_5^3})) = 8H^2.\end{aligned}$$

In conclusion, X *is a conic bundle over* X_5^3 *with discriminant* Δ*, a* $K3$ *surface singular at eight points.*

The other projection induces a second conic bundle structure over Q_3*. Specifically,* X *can be described as*

$$\mathscr{Z}(\mathcal{O}_{\mathcal{R}}(1)^{\oplus 3}) \subset \mathrm{Gr}_{Q_3}(2, \mathcal{Q}_{\mathbb{P}^4|Q_3}(-1))$$

where the rank three bundle $\mathcal{E}$ *is obtained as the kernel in:*

$$0 \to \mathcal{E} \to \bigwedge^2 \mathcal{Q}^{\vee}(1) \to \mathcal{O}_{Q_3}^{\oplus 3}.$$

In this case $\mathcal{K} = \mathcal{O}_{Q_3}$*. Applying Lemma 2.2.13 we conclude that the discriminant* Δ *is a sextic surface in* Q_3*, singular in* $\Delta_{sing} = \{40 \text{ points}\}$*.*

Example 2.3.3. ***Fano 3.24***

- $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0,0) \oplus \mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)}(1,0,0)) \subset \mathbb{P}^5 \times \mathbb{P}^2 \times \mathrm{Gr}(2,5)$*;*
- *Invariants:* $h^0(-K) = 53,\ (-K)^4 = 235,\ \chi(T_X) = -1,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 7$*;*
- *Description:* $X = \mathrm{Bl}_{\mathrm{dP}_6} Y$*, with* Y *the Fano 6 in* [Kal19, Table 3].

Identification. In this last example, we want to show another useful method, which combines techniques from quiver flag manifolds and degeneracy loci methods.

Consider the Fano fourfold K_{212}*. We describe it as* $\mathrm{Bl}_{\mathrm{dP}\,6}\, Y$*, where* Y *is the Fano fourfold*

$$\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(-1)^{\oplus 2}).$$

In [**Kal19**], K_{212} *is described with the following data:*

$$A = \begin{bmatrix} 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad r = (1,1,1,2), \quad E = (\square, \varnothing, \square).$$

Then the matrix A *is associated with the quiver of the following form:*

$$\begin{tikzcd} \bullet \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[dr, shift left] \arrow[dr, shift right] \arrow[ddr, bend right] & \bullet \arrow[d] \\ & \bullet \arrow[d] \\ & \bullet \end{tikzcd}$$

The fourfold K_{212} *is realized as zero locus* $\mathscr{Z}(E) \subset \mathcal{M}_\theta(Q, r)$*. In this case,* $E = (\mathcal{W}_1^\vee \otimes \mathcal{W}_3^\vee)$*.*

Applying Corollary 2.2.24, we can describe X *as*

$$X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0,0)) \subset \mathbb{P}^5 \times \mathbb{P}^2 \times \mathrm{Gr}(2,5).$$

We now follow the standard strategy to carry out the computation of invariants:

$$h^0(-K) = 53, \ (-K)^4 = 235, \ \chi(T_X) = -1, \ h^{0,0} = 1;$$

and Hodge numbers:

$$\begin{array}{ccccccccc} 0 & & 0 & & 7 & & 0 & & 0 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & 3 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

We want to give the birational description of X*. Consider the projection:*

$$\pi_{2,3|X} \ : \ X \to \mathbb{P}^2 \times \mathrm{Gr}(2,5).$$

The image of this map is

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,5).$$

For a general $y \in Y$*, the fiber* $\pi_{2,3}^{-1}(y)$ *is* $\mathbb{P}^5$ *cut by*

$$\mathscr{Z}(\mathcal{O}_{\mathbb{P}^5} \otimes \pi_{2,3}^*(\pi_3^* \mathcal{Q}_{\mathrm{Gr}(2,5)})(y)) \ \text{ and } \ \mathscr{Z}(\mathcal{O}_{\mathbb{P}^5} \otimes \pi_{2,3}^*(\pi_3^* \mathcal{U}^\vee_{\mathrm{Gr}(2,5)})(y))).$$

Since $\pi_3^* \mathcal{Q}_{\mathrm{Gr}(2,5)}$ *is a rank three bundle over* Y *while* $\pi_3^* \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}$ *is a rank two bundle over* Y*.*

Therefore it follows that the general fiber $\pi_{2,3}^{-1}(y)$ *is* $\mathbb{P}^5$ *cut by five sections of* $\mathcal{O}_{\mathbb{P}^5}(1)$*, hence a point. This shows that* X *is birational, via* $\pi_{2,3|X}$*, to*

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,5).$$

Furthermore, Y can be rewritten as $\mathrm{Gr}_{\mathbb{P}^2}(2, \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus 2})$, and, dualizing the fiber of this Grassmannian bundle, we find that

$$Y = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}(1) \oplus \mathcal{O}^{\oplus 2}).$$

If we twist with $\mathcal{O}(-1)$ we obtain $Y = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(-1)^{\oplus 2})$.

Recall that, using the identification in Proposition 2.2.22, we get that Y is the quiver flag variety that can be described via the data of

$$A = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \quad r = [1, 1, 1],$$

which corresponds to K_6.

We have recovered that X is birational to an already known Fano fourfold. Indeed, $Y = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(-1)^{\oplus 2})$ appears as the fifth case in the main theorem in [SW902]. In particular, Y is described as the blow-up of the cone in $\mathbb{P}^6$ over $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ along its vertex.

To complete the biregular description, it suffices to compute the exceptional locus of the map:

$$\pi_{2,3|X} : X \to Y.$$

We proceed as in Example 2.3.1. We compute the invariants of the exceptional locus, and by applying Theorem 2.2.5, we deduce that it is a surface. Moreover, using Proposition 2.2.1, we see that it can be described as

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(1, 0, 0) \oplus \mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)}(1, 0, 0)) \subset \mathbb{P}^3 \times Y.$$

This description allows us to compute the canonical bundle K_S via the adjunction formula. The other usual invariants are:

$$K_S = \mathcal{O}(-1, -1)_{|S}, \ h^0(-K) = 7, \ (-K)^2 = 6; .$$

and the Hodge numbers:

$$\begin{matrix} 0 & & 4 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

These invariants coincide with those of a dP_6, and since the anticanonical is ample, the classification of surfaces concludes the argument.

Finally, since the fibers over S are a $\mathbb{P}^1$, the variety X can be identified as $X = \mathrm{Bl}_{\mathrm{dP}_6} Y$.

Remark 2.3.4. *This Fano fourfold has the same invariants as the Fano fourfold $X^5_{0,1}$ in [Sec23,* Table 2]. *Recall that $X^5_{0,1}$ can be also written as $\mathrm{Bl}_{\mathrm{dP}_5} \mathbb{P}_{X^3_5}(\mathcal{O}_{X^3_5} \oplus \mathcal{O}_{X^3_5}(-1))$, while we have described K_{212} as $\mathrm{Bl}_{\mathrm{dP}_6} \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2})$. These fourfolds are possibly isomorphic, but we could not find a proof for it.*

2.4. Fano varieties with Picard rank 1

Fano 1–1. [#K_{742}] $X = \mathscr{Z}(\mathcal{O}(1) \oplus \mathcal{O}(3)) \subset \mathrm{Gr}(2, 5)$

Invariants. $h^0(-K) = 9, \ (-K)^4 = 15, \ \chi(T_X) = -109, \ h^{1,1} = 1, \ h^{3,1} = 41, \ h^{2,2} = 232.$

Description. X is the Fano fourfold b1 in [Kü95]. •

Fano 1–2. [#K_{740}] $X = \mathscr{Z}(\mathcal{O}(2)^{\oplus 2}) \subset \mathrm{Gr}(2, 5)$

Invariants. $h^0(-K) = 10,\ (-K)^4 = 20,\ \chi(T_X) = -72,\ h^{1,1} = 1,\ h^{3,1} = 20,\ h^{2,2} = 132.$
Description. X is the Fano b2 in [**Kü95**]. •

Fano 1–3. [# K_{715}] $X = \mathscr{Z}(\mathcal{Q}(1)) \subset \mathrm{Gr}(2,6)$
Invariants. $h^0(-K) = 15,\ (-K)^4 = 42,\ \chi(T_X) = -42,\ h^{1,1} = 1,\ h^{3,1} = 6,\ h^{2,2} = 57.$
Description. X is the Fano b3 in [**Kü95**]. •

Fano 1–4. [# K_{730}] $X = \mathscr{Z}(\mathcal{U}^\vee(1) \oplus \mathcal{O}(1)^{\oplus 2}) \subset \mathrm{Gr}(2,6)$
Invariants. $h^0(-K) = 13,\ (-K)^4 = 33,\ \chi(T_X) = -48,\ h^{1,1} = 1,\ h^{3,1} = 8,\ h^{2,2} = 70.$
Description. X is the Fano b5 in [**Kü95**]. •

Fano 1–5. [# K_{734}] $X = \mathscr{Z}(\mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}(2)) \subset \mathrm{Gr}(2,6)$
Invariants. $h^0(-K) = 12,\ (-K)^4 = 28,\ \chi(T_X) = -63,\ h^{1,1} = 1,\ h^{3,1} = 15,\ h^{2,2} = 106.$
Description. X is the Fano b6 in [**Kü95**]. •

Fano 1–6. [# K_{24}] $X = \mathscr{Z}(\mathcal{O}(1)^{\oplus 2}) \subset \mathrm{Gr}(2,5)$
Invariants. $h^0(-K) = 85,\ (-K)^4 = 405,\ \chi(T_X) = 8,\ h^{1,1} = 1,\ h^{3,1} = 0,\ h^{2,2} = 2.$
Description. X is the Fano X_5^4. •

Fano 1–7. [# K_{365}] $\mathscr{Z}(\mathcal{O}(1)^{\oplus 4}) \subset \mathrm{Gr}(2,6)$
Invariants. $h^0(-K) = 51,\ (-K)^4 = 224,\ \chi(T_X) = -9,\ h^{1,1} = 1,\ h^{3,1} = 0,\ h^{2,2} = 8.$
Description. X is the Fano X_6^4. •

Fano 1–8. [# K_{443}] $\mathscr{Z}(\mathcal{U}^\vee(1)) \subset \mathrm{Gr}(2,5)$
Invariants. $h^0(-K) = 45,\ (-K)^4 = 192,\ \chi(T_X) = -15,\ h^{1,1} = 1,\ h^{3,1} = 0,\ h^{2,2} = 12.$
Description. X is the Fano V_{12}^4 in [**CGKS20**]. •

Fano 1–9. [# K_{496}] $\mathscr{Z}(\mathcal{O}(1) \oplus \mathcal{O}(2)) \subset \mathrm{Gr}(2,5)$
Invariants. $h^0(-K) = 39,\ (-K)^4 = 160,\ \chi(T_X) = -24,\ h^{1,1} = 1,\ h^{3,1} = 1,\ h^{2,2} = 22.$
Description. X is the Fano fourfold X_{10} in [**BFMT21**, Table 3]. •

2.5. Fano varieties with Picard rank 2

Fano 2–1. [# K_{508}] $X = \mathscr{Z}(\mathcal{O}(1,1) \oplus \mathcal{O}(0,1)^{\oplus 3}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,5)$
Invariants. $h^0(-K) = 39,\ (-K)^4 = 160,\ \chi(T_X) = -9,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 7.$
Description. X is a $\mathbb{P}^1$-bundle over X_5^3 that degenerates to a $\mathbb{P}^2$-bundle over five points.

Identification. See Example 2.3.1. •

Fano 2–2. [#K_{509}] $\mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{O}(0,1) \oplus \mathcal{O}(1,1)) \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 33,\ (-K)^4 = 130,\ \chi(T_X) = -22,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 23.$

Description. X is the Fano fourfold GM-22 in [BFMT21, Table 5]. •

Fano 2–3. [#K_{517}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 2}) \subset \mathbb{P}^5 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 33,\ (-K)^4 = 131,\ \chi(T_X) = -13,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 12.$

Description. $X = \mathrm{Bl}_S X_5^4$, with $S = \mathrm{Bl}_{9\mathrm{pts}} \mathbb{P}^2$.

Identification. The restriction of the projection π to $\mathrm{Gr}(2,5)$ induces a birational map between X and X_5^4, since the general fiber is a fixed codimension five linear section of $\mathbb{P}^5$. To determine where the dimension of the fiber increases we apply Proposition 2.2.1 following the approach in Example 2.3.1.

Recall that X is the zero locus of $\sigma_1 \in V_6^\vee \otimes V_5$, $\sigma_2 \in V_6^\vee \otimes V_5$ and $\omega_1,\ \omega_2 \in \bigwedge^2 V_5^\vee$.

If we consider only σ_1 and σ_2 and project to $\mathrm{Gr}(2,5)$ we are fixing a plane $\Pi \in V_5$. We then look for $l \in V_6$ such that the relations $\sigma_1(l,\Pi) = 0$ and $\sigma_2(l,\Pi) = 0$ hold simultaneously. This fact is equivalent to studying the map

$$\varphi : \mathcal{Q}_{\mathrm{Gr}(2,5)|X_5^4} \oplus \mathcal{U}_{\mathrm{Gr}(2,5)|X_5^4} \to \mathcal{O}_{X_5^4}^{\oplus 6}.$$

The exceptional locus of π is where φ degenerates to a rank-4 map. We can apply Theorem 2.2.6 to compute the invariants. Moreover, $\mathrm{D}_4(\varphi)$ can be described as

$$S = \mathscr{Z}(\mathbb{Q}_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 2}) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,5).$$

This is a surface S with $e(S) = 12$, $(-K_S)^2 = 0$, $h^{1,0} = h^{2,0} = 0$, and $K_S = \mathcal{O}(1,-1)_{|S}$. In particular, $K_S^2 = 0$ and $h^0(S, K_S^{\otimes 2}) = 0$, hence by classification, it is the $\mathrm{Bl}_{9\mathrm{pts}} \mathbb{P}^2$.

Since X is birational to X_5^4, then it is rational. •

Fano 2–4. [#K_{527}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{O}(0,1)^{\oplus 4}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,7)$

Invariants. $h^0(-K) = 30,\ (-K)^4 = 114,\ \chi(T_X) = -20,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 17.$

Description. X is a small resolution of Y, a fourfold singular in two points with $e(Y) = 21$ and $\deg(Y) = 28$.

Identification. This fourfold is the zero locus of $\sigma \in V_3 \otimes V_7^\vee$ and $\omega_1,\ \omega_2,\ \omega_3,\ \omega_4 \in \bigwedge^2 V_7^\vee$. We study the projection π to $\mathrm{Gr}(2,7)$ of $\mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)})$.

After projecting to $\mathrm{Gr}(2,7)$, we look for lines $l \subset V_3$ such that, for a point $\Pi \in \mathrm{Gr}(2,7)$, the relation $\sigma(l,\Pi) = 0$ holds.

Equivalently, with the same technique in Example 2.3.1, we apply Proposition 2.2.1 and study where the map

$$\varphi : \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \to \mathcal{O}^{\oplus 3}_{\mathrm{Gr}(2,7)}$$

degenerates.

Using Theorems 2.2.5 and 2.2.6, that φ drops rank to 1 on an eightfold $Z = \mathrm{D}_1(\varphi)$ of degree 28.

The variety Z is singular on a fourfold $K = \mathrm{D}_0(\varphi) \subset Z$ of degree 2.

To complete the description, we cut Z with 4 linear sections. In this way, we obtain that π is a birational map between X and a fourfold

$$Y = Z \cap H_1 \cap H_2 \cap H_3 \cap H_4$$

with $e(Y) = 21$, degree 28.

Y is singular in two points, and π contracts two $\mathbb{P}^2$'s to these points. Thus, π corresponds to a small resolution of Y.

The projection to $\mathbb{P}^2$ gives a dP_5 fibration. Indeed, the general fiber is $\mathrm{Gr}(2,5) \cong \mathscr{Z}(\mathcal{U}^{\vee \oplus 2}) \subset \mathrm{Gr}(2,7)$, cut by 4 hyperplane sections. By [Isk96], this implies that X is rational. •

Fano 2–5. [#K_{558}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)} \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(0,2)) \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 30,\ (-K)^4 = 114,\ \chi(T_X) = -24,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 24.$

Description. X is the Fano fourfold GM-21 in [BFMT21, Table 5]. •

Fano 2–6. [#K_{559}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 2} \oplus \mathcal{O}(1,1)) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 29,\ (-K)^4 = 110,\ \chi(T_X) = -21,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 22.$

Description. X is the Fano fourfold in [BFMT21, Table 6]. •

Fano 2–7. [#K_{566}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{O}(0,1) \oplus \mathcal{O}(0,2)) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 26,\ (-K)^4 = 94,\ \chi(T_X) = -28,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 28.$

Description. X is the Fano fourfold GM-20 in [BFMT21, Table 5]. •

Fano 2–8. [#K_{547}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1) \oplus \mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(0,1)^{\oplus 2}) \subset \mathrm{Gr}(2,4) \times \mathbb{P}^6$

Invariants. $h^0(-K) = 30,\ (-K)^4 = 116,\ \chi(T_X) = -16,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 15.$

Description. $X = \mathrm{Bl}_S \mathrm{Gr}(2,4)$, with $S = \mathrm{Bl}_{12\mathrm{pts}} \mathbb{P}^2$

Identification. The projection π to $\mathrm{Gr}(2,4)$ inducrs a birational map between X and $\mathrm{Gr}(2,4)$. The fiber is a codimension six linear subspace of $\mathbb{P}^6$.

To describe the exceptional locus of π, note that X is the zero locus of a section

$$\sigma \in V_7^{\vee} \otimes (V_4 \oplus (V_4^{\vee})^{\oplus 2}).$$

We seek lines $l \subset V_7$ such that, for a fixed plane $\Pi \subset V_4$, the relation $\sigma(l, \Pi) = 0$ holds.

Projecting to $\mathrm{Gr}(2,4)$ and applying Proposition 2.2.1, this is equivalent to studying the map

$$\varphi : \mathcal{Q}^{\vee}_{\mathrm{Gr}(2,4)} \oplus \mathcal{U}^{\oplus 2}_{\mathrm{Gr}(2,4)} \to \mathcal{O}^{\oplus 7}_{\mathrm{Gr}(2,4)}.$$

The projection π is not a isomorphism along $\mathrm{D}_3(\varphi)$.

By Theorems 2.2.5 and 2.2.6, $\mathrm{D}_3(\varphi)$ is a surface

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1)^{\oplus 2} \oplus \mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(0,1)) \subset \mathrm{Gr}(2,4) \times \mathbb{P}^4$$

In particular, S has $e(S) = 15$, $\deg(S) = 9$, $K_S = \mathcal{O}(-1,1)_{|S}$, $h^{1,0} = h^{2,0} = 0$ and $h^0(S, K_S^{\otimes 2}) = 0$. Thus $S \cong \mathrm{Bl}_{12\mathrm{pts}}\,\mathbb{P}^2$.

Since X is birational to $\mathrm{Gr}(2,4)$ in particular is rational. •

Fano 2–9. [#K_{549}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 3} \oplus \mathcal{O}(2,0)) \subset \mathbb{P}^5 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 32$, $(-K)^4 = 126$, $\chi(T_X) = -11$, $h^{1,1} = 2$, $h^{3,1} = 0$, $h^{2,2} = 10$.

Description. X is a conic bundle which discriminates on a $K3$ surface S singular in eight points.

Identification. Using Corollary 2.2.11 (2), we can write

$$X = \mathscr{Z}(\mathcal{O}(0,1)^{\oplus 3} \oplus \mathcal{O}(2,0) \oplus \mathcal{Q}_2) \subset \mathrm{Fl}(1,3,6).$$

By Lemma 2.2.12, this is

$$\mathscr{Z}(\mathcal{O}_{\mathcal{R}}(2)) \subset \mathbb{P}_{X_5^3}(\mathcal{U}_{\mathrm{Gr}(2,5)|X_5^3} \oplus \mathcal{O}_{X_5^3})$$

a conic bundle over X_5^3.

Applying Lemma 2.2.13 with $\mathcal{E} = \mathcal{U}^\vee \oplus \mathcal{O}$ and $\mathcal{K} = \mathcal{O}$, the discriminant locus is a degree 2 surface $S \subset X_5^3$, with eight singular points, which is a $K3$ surface singular in 8 points.

We analyze the other projection. Let

$$X' = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 3}) \subset \mathbb{P}^5 \times \mathrm{Gr}(2,5).$$

The first projection makes X' birational to $\mathbb{P}^5$.

Intersecting $\mathbb{P}^5$ with $\mathscr{Z}(\mathcal{O}(2))$, we see that X is birational to Q_4.

To study the exceptional locus note that X' is defined as the vanishing of $\alpha \in V_6^\vee \otimes V_5$ and $\beta \in \bigwedge^3 V_5 \otimes V3$.

By Proposition 2.2.1 we examine the maps:

$$\varphi : V_5^\vee \otimes \mathcal{O}_{Q_4} \to \mathcal{O}_{Q_4}(1);$$

$$\phi : \bigwedge^3 V_5^\vee \otimes \mathcal{O}_{Q_4} \to \mathcal{O}_{Q_4}^{\oplus 3}.$$

From Theorems 2.2.5 and 2.2.6, we find $\mathrm{D}_0(\varphi) = \{p\}$, with fiber $\mathbb{P}^1 \times \mathbb{P}^1$ over p under π_1.

The map φ fits into

$$0 \to \mathcal{K} \xrightarrow{i} V_5^\vee \otimes \mathcal{O}_{Q_4} \xrightarrow{\varphi} \mathcal{O}_{Q_4}(1).$$

Taking the third exterior power gives

$$\bigwedge^3 i : \bigwedge^3 \mathcal{K} \to \bigwedge^3 V_5.$$

Composing with ϕ yiekds

$$\Phi : \bigwedge^3 \mathcal{K} \to \mathcal{O}_{Q_4}^{\oplus 3}.$$

In particular, $\mathrm{D}_2(\phi_{|\alpha=0}) = \mathrm{D}_2(\Phi)$, where S iis a dP_2.

For general $p \in S$, $\pi_1^{-1}(p) = \mathbb{P}^1$. Thus X is birational to Q_4 and, away from one point, X is $\mathrm{Bl}_{\mathrm{dP}_2} Q_4$.

The final picture is X as small resolution of $Y = \mathrm{Bl}_S\, Q_4$, where S is a dP_2 singular in a point.

This gives also the rationality of X. •

Fano 2–10. [#K_{552}] $X = \mathscr{Z}(\mathcal{O}(0,1)^{\oplus 2} \oplus \mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)}(1,0)^{\oplus 2}) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 30,\ (-K)^4 = 116,\ \chi(T_X) = -14,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 13.$

Description. $X = \mathrm{Bl}_{\mathrm{Bl}_{10\mathrm{pts}}\,\mathbb{P}^2} X_5^4$.

Identification. The projection π to $\mathrm{Gr}(2,5)$ induces a birational map between X and X_5^4. The general fiber is $\mathbb{P}^4$ cut by 4 hyperplane sections.

To stuudy the fiber degenerates, recall that X is the zero locus of

$$\sigma \in V_5 \otimes (V_5^{\vee \oplus 2}),\ \text{and}\ \omega_1, \omega_2 \in \bigwedge^2 V_5.$$

By Proposition 2.2.1, the projection of X to X_5^4 is controlled by

$$\varphi : \mathcal{U}^{\oplus 2}_{\mathrm{Gr}(2,5)|X_5^4} \to \mathcal{O}^{\oplus 5}_{X_5^4}$$

The map π degenerates along $\mathrm{D}_1(\varphi)$.

From Theorems 2.2.5 and 2.2.6, $\mathrm{D}_1(\varphi)$ is

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0)^{\oplus 2}) \subset \mathbb{P}^4 \times X_5^4.$$

The surface S has $e(S) = 13$, $\deg(S) = 11$, $K_S = \mathcal{O}(1,-1)_{|S}$, $(-K_S)^2 = 1$, $h^{1,0} = h^{2,0} = 0$, and $h^0(S, K_S^{\otimes 2}) = 0$.

By the classification of surfaces we get that S is $\mathrm{Bl}_{10\mathrm{pts}}\,\mathbb{P}^2$.

Since X is birational to X_5^4, it is rational. •

Fano 2–11. [#K_{25}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)} \oplus \mathcal{O}(0,1)^{\oplus 2}) \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 70,\ (-K)^4 = 325,\ \chi(T_X) = 4,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 4.$

Description. $X = \mathrm{Bl}_{\mathbb{P}^1} X_5^4$.

Identification. Consider

$$X' = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)}) \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5).$$

By Lemma 2.2.10 (2), we have

$$X' = \mathrm{Bl}_{D'} \mathrm{Gr}(2,5)$$

where $D' = \mathbb{P}^3 = \mathscr{Z}(\mathcal{Q}) \subset \mathrm{Gr}(2,5)$.

Since $X = X' \cap H_1 \cap H_2$, with H_1, H_2, general hyperplane sections, it follows that

$$X = \mathrm{Bl}_D X_5^4,$$

where $D = D' \cap H_1 \cap H_2$ which is $\mathbb{P}^1$.

Because X is birational to X_5^4, it is rational. •

Fano 2–12. [#K_{554}] $X = \mathscr{Z}(\mathcal{O}(1) \boxtimes Sym^2 \mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 24,\ (-K)^4 = 90,\ \chi(T_X) = -9,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 12.$

Description. $X = \mathrm{Bl}_S \mathrm{Gr}(2,4)$, with S an Enriques surface.

Identification. The projection

$$\pi : X \to \mathrm{Gr}(2,4)$$

is birational. Indeed, $\mathrm{Sym}^2 \mathcal{U}^\vee_{\mathrm{Gr}(2,4)}$ has rank 3, so the general fiber is a codimension three linear subspace of $\mathbb{P}^3$.

The variety X is the zero locus of

$$\sigma \in V_4 \otimes \mathrm{Sym}^2 V_4^\vee.$$

Projecting to $\mathrm{Gr}(2,4)$ means once fixing a plane, $\Pi \subset V_4$ and looking for lines $l \subset \mathbb{P}^3$ such that, $\sigma(l,\Pi) = 0$.

By Proposition 2.2.1, this is equivalent to studying

$$\varphi : \mathrm{Sym}^2 \mathcal{U}_{\mathrm{Gr}(2,4)} \to \mathcal{O}^{\oplus 4}_{\mathrm{Gr}(2,4)}.$$

The fiber π has higher dimension where φ degenerates, i.e. on $\mathrm{D}_2(\varphi)$.

Using Theorems 2.2.5 and 2.2.6 we find that $\mathrm{D}_2(\varphi)$ is a surface S with $e(S) = 12$ and $\deg(S) = 10$.

Note that X is described in[**Kuz19**, Theorem 2], hence S is an Enriques surface.

Since X is birational to $\mathrm{Gr}(2,4)$, it is rational.

•

Fano 2–13. [#K_{582}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,5)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 3} \oplus \mathcal{O}(2,0)) \subset \mathbb{P}^4 \times \mathrm{Gr}(3,5)$

Invariants. $h^0(-K) = 30,\ (-K)^4 = 116,\ \chi(T_X) = -11,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 10.$

Description. The projection $\pi_1 : X \to Q_3$ is a conic bundle. Its discriminant is a sextic surface in Q_3, singular in 40 points. The second projection $\pi : X \to X_5^3$ is a conic bundle. Its discriminant is a $K3$ surface singular in eight points.

Identification. See Example 2.3.2.

•

Fano 2–14. [#K_{583}] $\mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}_1(2,4)}(0,1) \oplus \mathcal{U}^\vee_{\mathrm{Gr}_2(2,4)}(1,0)) \subset \mathrm{Gr}_1(2,4) \times \mathrm{Gr}_2(2,4)$.

Invariants. $h^0(-K) = 28,\ (-K)^4 = 106,\ \chi(T_X) = -16,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 15.$

Description. Outside two points X is isomorphic to $\mathrm{Bl}_{\mathrm{dP}_3} \mathrm{Gr}(2,4)$ via the projection π on one of the $\mathrm{Gr}(2,4)$. At those two points, the fiber of π is $\mathbb{P}^1 \times \mathbb{P}^1$.

Identification. We note that X is defined by $\alpha_1 \in V_4^\vee \otimes \bigwedge^2 W_4^\vee$ and $\alpha_2 \in \bigwedge^2 V_4^\vee \otimes W_4^\vee$.

We study the projection π to $\mathrm{Gr}(2, W_4) = \mathrm{Gr}_2(2,4)$. By Proposition 2.2.1, this amounts to analyzing the degeneracy loci of the following maps:

$$\varphi : V_4 \otimes \mathcal{O}_{\mathrm{Gr}_2(2,4)} \to \mathcal{O}_{\mathrm{Gr}_2(2,4)}(1),$$

$$\phi : \bigwedge^2 V_4 \otimes \mathcal{O}_{\mathrm{Gr}_2(2,4)} \to \mathcal{U}^\vee_{\mathrm{Gr}_2(2,4)}.$$

By Theorems 2.2.5 and 2.2.6, we find that $D = \mathrm{D}_0(\varphi) = \{2 \text{ points}\}$ and the fiber of π over each point is a quadric surface Q_2.

The first vector bundles of φ and ϕ are related by the following sequences:

$$0 \to \mathcal{K} \to V_4 \otimes \mathcal{O}_{\mathrm{Gr}_2(2,4)} \xrightarrow{\varphi} \mathcal{O}_{\mathrm{Gr}_2(2,4)}(1) \to \mathcal{F} \to 0$$

where $\mathcal{K} = ker(\varphi)$ and $\mathcal{F} = coker(\varphi)$.

Taking the second exterior power yields

$$0 \to \bigwedge^2 \mathcal{K} \xrightarrow{\tau} \bigwedge^2 V_4 \otimes \mathcal{O}_{\mathrm{Gr}_2(2,4)} \to \mathcal{K}(1) \to \mathcal{K} \otimes \mathcal{F} \to 0.$$

Composing τ with ϕ

$$\Phi : \bigwedge^2 \mathcal{K} \xrightarrow{\phi\circ\tau} \mathcal{U}^\vee_{\mathrm{Gr}_2(2,4)}.$$

We have $D' = \mathrm{D}_1(\phi_{|D}) = \mathrm{D}_1(\Phi)$. By Theorems 2.2.5 and 2.2.6, D' is a dP_3. Therefore, outside two points, X is isomorphic to $\mathrm{Bl}_{\mathrm{dP}_3}\mathrm{Gr}(2,4)$ via π. Over those two points, the fiber of π is Q_2.

Hence X is a small resolution of $Y = \mathrm{Bl}_S \mathrm{Gr}(2,4)$, where S is a dP_3 singular in two points.

Since X is birational to $\mathrm{Gr}(2,4)$, it is rational.

•

Fano 2–15. [#K_{589}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2}(0,1)^{\oplus 2}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 24,\ (-K)^4 = 85,\ \chi(T_X) = -24,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 22.$

Description. X is a small resolution of a fourfold Y singular in five points with $e(Y) = 23$ and $\deg(Y) = 15$.

Identification. We have X as the zero locus of $\sigma \in V_3 \otimes (\bigwedge^2 V_5^\vee)^{\oplus 2}$. Consider the projection π to $\mathrm{Gr}(2,5)$.

Geometrically, this corresponds to lines $l \subset \mathbb{P}^2$ such that, for a fixed the plane $\Pi \subset \mathrm{Gr}(2,5)$, we have $\sigma(l,\Pi) = 0$.

By Proposition 2.2.1, this is equivalent to studying the map

$$\varphi : \mathcal{O}_{\mathrm{Gr}(2,5)}(-1)^{\oplus 2} \to \mathcal{O}^{\oplus 3}_{\mathrm{Gr}(2,5)}.$$

The projection π has image $Y = \mathrm{D}_1(\varphi) \subset \mathrm{Gr}(2,5)$. Since the fiber over a general point of $\mathrm{D}_1(\varphi)$ is $\mathbb{P}^2$ cut by a section of $\mathcal{Q}_{\mathbb{P}^2}$, hence a point, π is biratinal.

The map π degenerates over $\mathrm{D}_2(\varphi)$, which is also the singular locus of Y. Over $\mathrm{D}_2(\varphi) \subset \mathrm{D}_1(\varphi)$ the fiber is $\mathbb{P}^2$.

Using Theorems 2.2.5 and 2.2.6, we find that Y is a fourfold with $e(Y) = 23$ and $\deg(Y) = 15$, and $\mathrm{D}_2(\varphi)$ consists of 5 points. Thus X is a small resolution of the singular fourfold Y.

The projection to $\mathbb{P}^2$ is a dP_5-fibration: the general fiber is $\mathrm{Gr}(2,5)$ cut by 4 hyperplane sections. By [**Isk96**], X is rational. •

Fano 2–16. [#K_{579}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0)^{\oplus 2} \oplus \mathcal{O}(1,1)) \subset \mathbb{P}^5 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 27,\ (-K)^4 = 101,\ \chi(T_X) = -21,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 23.$

Description. X is the Fano fourfold M-9 in [**BFMT21**, Table 4]. •

Fano 2–17. [#K_{597}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(1,1)) \subset \mathbb{P}^5 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 27,\ (-K)^4 = 100,\ \chi(T_X) = -23,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 24.$

Description. X is the Fano fourfold K3-33 in [BFMT21, Table 6]. •

Fano 2–18. [#K_{600}] $X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(0,1) \oplus \mathcal{O}(2,0)) \subset \mathbb{P}^6 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 27,\ (-K)^4 = 100,\ \chi(T_X) = -20,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 18.$

Description. $X = \mathrm{Bl}_{\mathrm{Bl}_{9pts}\mathbb{P}^2} Q_5 \cap Q_5'$.

Identification. By Theorem 2.2.8, we can rewrite

$$X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(1) \oplus \mathcal{O}_{\mathcal{R}}(2)) \subset \mathbb{P}_{Q_3}(\mathcal{U}_{\mathrm{Gr}(2,4)|Q_3} \oplus \mathcal{Q}^\vee_{\mathrm{Gr}(2,4)|Q_3}).$$

The general fiber of the projection π to Q_3 is a $\mathbb{P}^3$ cut by a hyperplane and a quadric. Thus X is a conic bundle over Q_3.

The rank three bundle defining this fibration is not explicit. To recover it we notice that $\mathcal{O}_{\mathcal{R}}(1)$ induces

$$\varphi : \mathcal{U}_{\mathrm{Gr}(2,4)|Q_3} \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)|Q_3} \to \mathcal{O}_{Q_3},$$

and hence an exact sequence of vector bundles on Q_3:

$$0 \to \mathcal{E} \to \mathcal{U}_{\mathrm{Gr}(2,4)|Q_3} \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)|Q_3} \to \mathcal{O}_{Q_3} \to 0$$

where $\mathcal{E}$ has rank three and is determined by the Chern classes of $\mathcal{U}_{\mathrm{Gr}(2,4)|Q_3} \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)|Q_3}$ and $\mathcal{O}_{Q_3}$.

Applying Theorem 2.2.8 again,

$$X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(2)) \subset \mathbb{P}_{Q_3}(\mathcal{E}).$$

We can then use Lemma 2.2.13 with $\mathcal{K} = \mathcal{O}_{Q_3}$, obtaining that the discriminant Δ is a quartic surface and $|\Delta_{sing}| = 12$.

We study the projection π_1 to $\mathbb{P}^6$. Recall that X is obtained as the vanishing of the sections

$$\alpha \in V_6^\vee \otimes V_4^\vee,\ \beta \in V_6^\vee \otimes V_4,\ \omega \in \mathrm{Sym}^2 V_6^\vee,\ \text{and},\ \tau \in \bigwedge^2 V_4.$$

First, consider $\mathscr{Z}(\alpha) \cap \mathscr{Z}(\beta) \subset \mathbb{P}^6 \times \mathrm{Gr}(2,4)$. By Proposition 2.2.1, to understand the projection to $\mathbb{P}^6$ we study:

$$\varphi : \mathcal{O}(-1) \to V_4^\vee,$$
$$\phi : \mathcal{O}(-1) \to V_4$$

Dualizing ϕ and composing with φ gives

$$\Phi : \mathcal{O}(-1) \xrightarrow{\varphi} V_4^\vee \xrightarrow{\phi^\vee} \mathcal{O}(1)$$

Thus $\Phi(l) = 0$ for $l \in \mathbb{P}^6$, characterizes the image of π_1.

After twisting the bundles we can rewrite Φ as :

$$\Phi' : \mathcal{O}(-2) \to \mathcal{O}$$

and $\Phi(l) = 0$ if and only if $\Phi'(l) = 0$, which is $Q_5 \subset \mathbb{P}^6$.

We analyze the fiber of π_1 over Q_5.

For $l \subset Q_5$, $\varphi(l) \subset V_4^\vee$ satisfies $\phi^\vee(\varphi(l)) = 0$, hence $\varphi(l) \in \mathcal{K}_l = Ker\phi^\vee$. The fiber over l is $\mathbb{P}(\mathcal{K}_l/\varphi(l))$.

Generically $dim\mathcal{K}_l = 3$ and $dim\varphi(l) = 1$, so the fiber over Q_5 is a $\mathbb{P}^1$.

Intersecting $\mathbb{P}^6$ with $Q_5' = \mathscr{Z}(\omega)$ and the fiber with $\mathscr{Z}(\tau)$, yields $\pi_1(X) = Q_5 \cap Q_5'$ and the general fiber a point.

Thus π_1 is a birational map between X and the intersection of two quadrics in $\mathbb{P}^6$.

We study the exceptional locus of π_1. By [Ott95, Theorem 1.4], $\mathcal{U}^\vee_{\mathrm{Gr}(2,4)|Q_3} = \mathcal{Q}_{\mathrm{Gr}(2,4)|Q_3}$ is the spinor bundle on Q_3. We can then rewrite

$$X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,4)|Q_3}(1,0)^{\oplus 2} \oplus \mathcal{O}(2,0)) \subset \mathbb{P}^6 \times Q_3.$$

By Proposition 2.2.1, the exceptional π_1 is $\mathrm{D}_1(\theta)$ for

$$\theta : \mathcal{O}(-1)^{\oplus 2} \to V_4^\vee.$$

From Theorems 2.2.5 and 2.2.6, $\mathrm{D}_1(\theta)$ is a surface S with $h^{0,0} = 1$ and $h^{1,1} = 10$, hence $S = \mathrm{Bl}_{9\mathrm{pts}}$. Therefore $X = \mathrm{Bl}_{\mathrm{Bl}_{9\mathrm{pts}}\mathbb{P}^2} Q_5 \cap Q_5'$. •

Fano 2–19. [#K_{603}] $X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,6)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 4}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,6)$

Invariants. $h^0(-K) = 27,\ (-K)^4 = 100,\ \chi(T_X) = -18,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 16.$

Description. $X = \mathrm{Bl}_S X_6^4$ with S a dP_2.

Identification. The varieties X and X_6^4 are birational. Indeed, the general fiber of the projection

$$\pi : X \to X_6^4 \subset \mathrm{Gr}(2,6)$$

is a codimension-two linear subspace in $\mathbb{P}^2$.

We study the exceptional locus of π. The variety X is zero locus of

$$\sigma \in V_3 \otimes V_6^\vee,\ \text{and} \omega_1,\ \omega_2, \omega_3, \omega_4 \in \bigwedge^2 V_6^\vee.$$

Consider first the zero locus of σ and we project to $\mathrm{Gr}(2,6)$. We are looking for lines $l \subset \mathbb{P}^3$ such that $\sigma(l,\Pi) = 0$ for a fixed plane $\Pi \in \mathrm{Gr}(2,6)$.

Restricting to $X_6^4 \subset \mathrm{Gr}(2,6)$, Proposition 2.2.1, reduces the problem to the map

$$\varphi : \mathcal{U}_{\mathrm{Gr}(2,6)|X_6^4} \to \mathcal{O}_{X_6^4}^{\oplus 3}$$

The fiber is a $\mathbb{P}^2$ where φ drops rank, i.e. over $\mathrm{D}_1(\varphi)$.

By Theorems 2.2.5 and 2.2.6, $\mathrm{D}_1(\varphi_{|X_6^4})$ is a surface

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,6)}(1,0)) \subset \mathbb{P}^2 \times X_6^4.$$

The surface satisfies: $e(S) = 10$, $K_S = \mathcal{O}(1,-1)_{|S}$, $(-K_S)^2 = 2$, $h^{1,0} = h^{2,0} = 0$ and $h^0(S, K_S^{\otimes 2}) = 0$. By the classification of surfaces $S = \mathrm{dP}_2$.

In the end $X = \mathrm{Bl}_{\mathrm{dP}_2} X_6^4$. Since X_6^4 is rational, X is rational as well. •

Fano 2–20. [#K_{604}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,5)}(1,0) \oplus \mathcal{O}(1,1) \oplus \mathcal{O}(0,1)^{\oplus 2}) \subset \mathbb{P}^3 \times \mathrm{Gr}(3,5)$

Invariants. $h^0(-K) = 26,\ (-K)^4 = 95,\ \chi(T_X) = -22,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 23.$

Description. X is the Fano fourfold K3-34 in [BFMT21, Table 6]. •

Fano 2–21. [#K_{29}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0)^{\oplus 2}) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 60,\ (-K)^4 = 273,\ \chi(T_X) = 3,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 3.$

Description. $X = \mathrm{Bl}_{\mathbb{P}^2} \mathrm{Gr}(2,4)$.

Identification. The projection

$$\pi : X \to \mathrm{Gr}(2,4)$$

is birational. Indeed, the general fiber is a codimension-four linear subspace in $\mathbb{P}^4$.

We study the exceptional locus of π. The variety X is the zero locus of

$$\sigma \in V_5 \otimes (V_4^\vee)^{\oplus 2}.$$

Following the strategy in Example 2.3.1 and applying Proposition 2.2.1, the problem reduces to studying the map

$$\varphi : \mathcal{U}^{\oplus 2}_{\mathrm{Gr}(2,4)} \to \mathcal{O}^{\oplus 5}_{\mathrm{Gr}(2,4)}.$$

The exceptional locus of π corresponds to the locus where φ drops rank to 3, namely $\mathrm{D}_3(\varphi)$.

By Theorems 2.2.5 and 2.2.6, $\mathrm{D}_3(\varphi)$ has the same invariants as the surface $S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0)^{\oplus 2}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,4)$. The surfaces S has $e(S) = 3$, $K_S = \mathcal{O}(1,-2)_{|S}$, $(-K_S)^2 = 9$, $h^{1,0} = h^{2,0} = 0$ and $h^0(S, K_S^{\otimes 2}) = 0$. By the classification of surfaces, $S = \mathbb{P}^2$.

We obtain $X = \mathrm{Bl}_{\mathbb{P}^2} \mathrm{Gr}(2,4)$. Since X is birational to $\mathrm{Gr}(2,4)$, it is rational. •

Fano 2–22. [$\#K_{612}$] $X = \mathscr{Z}(\mathcal{Q}(0,2)) \subset \mathbb{P}^2 \times \mathbb{P}^4$

Invariants. $h^0(-K) = 30,\ (-K)^4 = 113,\ \chi(T_X) = -12,\ h^{1,1} = 2,\ h^{2,1} = 5,\ h^{3,1} = 0\ h^{2,2} = 3.$

Description. $X = \mathrm{Bl}_{Q_2 \cap Q_2' \cap Q_2''} \mathbb{P}^4$.

Identification. We can simply apply Lemma 2.2.9 and obtain that $X = \mathrm{Bl}_{Q_2 \cap Q_2' \cap Q_2''} \mathbb{P}^4$. X is rational by definition. •

Fano 2–23. [$\#K_{614}$] $X = \mathscr{Z}(\mathcal{O}(1,1) \oplus \bigwedge^3 \mathcal{Q}_{\mathbb{P}^4}(0,1)) \subset \mathbb{P}^4 \times \mathbb{P}^5$

Invariants. $h^0(-K) = 24,\ (-K)^4 = 86,\ \chi(T_X) = -24,\ h^{1,1} = 2,\ h^{3,1} = 1\ h^{2,2} = 26.$

Description. X is the Fano fourfold C-7 in [**BFMT21**, Table 4]. •

Fano 2–24. [$\#K_{616}$] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(0,2) \oplus \mathcal{O}(2,0) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 26,\ (-K)^4 = 96,\ \chi(T_X) = -13,\ h^{1,1} = 2,\ h^{2,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 10.$

Description. X, is a conic fibration over $Z = \mathscr{Z}(\mathcal{O}(2)) \subset \mathrm{Gr}(2,4)$ which discriminates on a $K3$ surface singular in 8 points.

Identification. By Corollary 2.2.11 (2) and [**DBFT22**, Lemma 5], we have

$$X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(2))) \subset \mathbb{P}_Z(\mathcal{U}_{\mathrm{Gr}(2,4)|Z} \oplus \mathcal{O}_Z),$$

where

$$Z = \mathscr{Z}(\mathcal{O}(2)) \subset \mathrm{Gr}(2,4),$$

is the Fano threefold 1-14 in [**DBFT22**, Table 1].

Applying Lemma 2.2.13, with $\mathcal{E} = \mathcal{U}^\vee \oplus \mathcal{O}$ and $\mathcal{K} = \mathcal{O}$, the discriminant locus of the conic bundle is a degree-2 surface in Z, singular in 8 points. By the adjunction formula, a $K3$ surface singular in 8 points.

•

Fano 2–25. [#K_{617}] $X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(0,1) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0)) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,5)$.

Invariants. $h^0(-K) = 25,\ (-K)^4 = 90,\ \chi(T_X) = -21,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 19.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_3} Y$, with Y the Fano 1–8.

Identification. The projection

$$\pi : X \to \mathrm{Gr}(2,5)$$

is a birational map between X and

$$Y = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1)) \subset \mathrm{Gr}(2,5),$$

i.e., the Fano 1–8, since the general fiber is $\mathbb{P}^2$ cut by two hyperplanes.

By Proposition 2.2.1, the exceptional locus of π is where the map

$$\varphi : \mathcal{U}_{\mathrm{Gr}(2,5)|Y} \to \mathcal{O}^{\oplus 3}_{\mathrm{Gr}(2,5)|Y}$$

has rank 1 map, i.e. $\mathrm{D}_3(\varphi)$.

From Theorems 2.2.5 and 2.2.6, $\mathrm{D}_1(\varphi_{|Y})$ is a surface

$$S = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0)) \subset \mathbb{P}^1 \times Y,$$

with $e(S) = 9$, $K_S = \mathcal{O}(1,-1)_{|S}$, $(-K_S)^2 = 3$, $h^{1,0} = h^{2,0} = 0$ and $h^0(S, K_S^{\otimes 2}) = 0$. By classification $S = \mathrm{dP}_3$.

The projection to $\mathbb{P}^2$ is a dP_5 fibration: the general fiber is $\mathrm{Gr}(2,5)$ cut by a section of $\mathcal{U}^\vee(1)$ and a section of $\mathcal{U}^\vee$, giving a rational surface of degree 5, and hence a dP_5. By [**Isk96**], X is rational. •

Fano 2–26. [#K_{32}] $X = \mathscr{Z}(\mathcal{Q}^\vee_{\mathbb{P}^4_1}(1,1)) \subset \mathbb{P}^4_1 \times \mathbb{P}^4_2$

Invariants. $h^0(-K) = 51,\ (-K)^4 = 225,\ \chi(T_X) = -1,\ h^{1,1} = 2,\ h^{2,1} = 1,\ h^{3,1} = 0,\ h^{2,2} = 3.$

Description. $X = \mathrm{Bl}_S \mathbb{P}^4$, with $S = \mathbb{P}_E(\mathcal{F})$ a $\mathbb{P}^1$-bundle over a quintic elliptic curve.

Identification. The projection π defines a birational map between X and $\mathbb{P}^4_1$; indeed, the general fiber is a linear subspace of $\mathbb{P}^4_2$ of codimension four. To determine the exceptional locus of π, we apply Proposition 2.2.1 and identify the locus where

$$\varphi : \mathcal{Q}_{\mathbb{P}^4}(-1) \to \mathcal{O}^{\oplus 5}_{\mathbb{P}^4}$$

drops rank to three. In other words, we study $\mathrm{D}_3(\varphi)$. By Theorems 2.2.5 and 2.2.6, $\mathrm{D}_3(\varphi)$ is a surface S that can be described as

$$\mathscr{Z}(\mathcal{O}_{\mathcal{R}}(1)^{\oplus 5}) \subset \mathbb{P}_{\mathbb{P}^4_1}(\mathcal{Q}(-1)).$$

Applying [**DBFT22**, Lemma 5], we find that S is equivalent also to

$$\mathscr{Z}(\mathcal{O}(0,1)^{\oplus 5}) \subset \mathrm{Fl}(1,2,5).$$

Using Lemma 2.2.12, S can be rewritten as $\mathbb{P}_E(\mathcal{U}|_E)$, where E is a quintic elliptic curve obtained as a linear complete intersection in $\mathrm{Gr}(2,5)$.

We note that X is rational by definition. •

Fano 2–27. [#K_{109}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 3}) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 60,\ (-K)^4 = 272,\ \chi(T_X) = 3,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 2.$

Description. $X = \mathrm{Bl}_{v_2(\mathbb{P}^2)} \mathbb{P}^4$

Identification. By applying Lemma 2.2.10 (1) together with Lemma 2.2.12, we obtain the identification

$$X = \mathbb{P}_{X_5^3}(\mathcal{U}_{\mathrm{Gr}(2,5)|X_5^3}).$$

We now study the projection π_1 to $\mathbb{P}^4$. Generically the fiber is a point, so π_1 induces a birational morphism between X and $\mathbb{P}^4$. Combining Proposition 2.2.1,Theorem 2.2.5, and Theorem 2.2.6, we find that the exceptional locus of π_1 is a surface S given by

$$\mathscr{Z}(\mathcal{Q}^\vee_{\mathbb{P}^4}(1,1)) \subset \mathbb{P}^2 \times \mathbb{P}^4$$

In particular, S is isomorphic to $\mathbb{P}^2$ embedded in $\mathbb{P}^4$ in a non-degenerate way; hence it can be identified with the projection of the Veronese surface $v_2(\mathbb{P}^2)$. The fibers of π_1 along this $\mathbb{P}^2$ are a $\mathbb{P}^1$, hence $X = \mathrm{Bl}_S \mathbb{P}^4$. •

Fano 2–28. [#K_{101}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{O}(0,1) \oplus \mathcal{O}(1,0)) \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 75,\ (-K)^4 = 352,\ \chi(T_X) = 7,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 3.$

Description. $X = \mathrm{Bl}_{\mathbb{P}^2} X_5^4$.

Identification. Recall that, by Lemma 2.2.10 (2),

$$\mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}) \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$$

is $\mathrm{Bl}_{\mathbb{P}^3} \mathrm{Gr}(2,5)$, where $\mathbb{P}^3 = \mathscr{Z}(\mathcal{Q}) \subset \mathrm{Gr}(2,5)$. The remaining sections cut $\mathrm{Gr}(2,5)$ as hyperplane sections.

More precisely,

$$\mathscr{Z}(\mathcal{O}(1,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}) \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$$

contains the $\mathbb{P}^3$ in $\mathrm{Bl}_{\mathbb{P}^3} \mathrm{Gr}(2,5)$, while

$$\mathscr{Z}(\mathcal{O}(0,1) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}) \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$$

intersects both $\mathbb{P}^3$ and $\mathrm{Gr}(2,5)$.

The final description is

$$X = \mathrm{Bl}_{\mathbb{P}^2} X_5^4.$$

Since X is birational to X_5^4, it is, in particular, rational.

•

Fano 2–29. [#K_{102}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 2}) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 69, (-K)^4 = 320,\ \chi(T_X) = 4,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 4.$

Description. $X = \mathrm{Bl}_{Q_2} X_5^4$.

Identification. The variety X is defined by $\alpha_1, \alpha_2 \in \bigwedge^2 V_5^\vee$ and by $\beta \in V_4^\vee \otimes V_5 = \mathrm{Hom}(V_4, V_5)$. We Consider the projection to $\mathrm{Gr}(2,5)$, whose image is the locus of 2-planes isotropic with respect to both α_i, namely X_5^4.

The fiber of π over a general point p of X_5^4 is the zero locus on $\mathbb{P}^3$ of a three-dimensional space of linear sections. Hence, the projection to X_5^4 is generically birational.

By Proposition 2.2.1, to determine the exceptional locus, we analyze the locus where the (generically rank three) morphism

$$\varphi : \mathcal{Q}_{\mathrm{Gr}(2,5)}|_{X_5^4} \to \mathcal{O}_{X_5^4} \otimes V_4$$

drops ranks to 2. This locus is, by definition, $\mathrm{D}_2(\varphi_{|X_5^4})$.By dimension reasons, no further rank degeneration occurs.

Applying Theorems 2.2.5 and 2.2.6, we find that $\mathrm{D}_2(\varphi)$ is a surface given by

$$\mathscr{Z}(\mathcal{O}(1)^{\oplus 2}) \subset \mathrm{Gr}(2,4),$$

which is the quadric surface Q_2. Therefore,

$$X = \mathrm{Bl}_{Q_2} X_5^4.$$

In particular, X is rational. •

Fano 2–30. [# K_{625}] $\mathscr{Z}(\mathcal{O}(1,1)^{\oplus 2} \oplus \mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(1,0)) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 24,\ (-K)^4 = 85,\ \chi(T_X) = -28,\ h^{1,1} = 2,\ h^{3,1} = 2,\ h^{2,2} = 32.$

Description. $X = \mathrm{Bl}_{\mathcal{E}} \mathrm{Gr}(2,4)$, with $\mathcal{E}$ the elliptic fibration with base $\mathbb{P}^1$ and general fiber an elliptic curve E of degree five.

Identification. The projection

$$\pi : X \to \mathrm{Gr}(2,4)$$

is a birational map between X and $\mathrm{Gr}(2,4)$. Moreover, using the self-duality of $\mathrm{Gr}(2,4)$, we can rewrite

$$X = \mathscr{Z}(\mathcal{O}(1,1)^{\oplus 2} \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0)) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,4).$$

By Corollary 2.2.11 (2) and Lemma 2.2.12, this is equivalent to

$$X = \mathscr{Z}(\mathcal{O}(1,1)^{\oplus 2}) \subset \mathbb{P}_{\mathrm{Gr}(2,4)}(\mathcal{U}_{\mathrm{Gr}(2,4)} \oplus \mathcal{O}_{\mathrm{Gr}(2,4)}).$$

Twisting $\mathcal{U}_{\mathrm{Gr}(2,4)} \oplus \mathcal{O}_{\mathrm{Gr}(2,4)}$ with $\mathcal{O}_{\mathrm{Gr}(2,4)}(-1)$, we obtain

$$X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(1)^{\oplus 2}) \subset \mathbb{P}_{\mathrm{Gr}(2,4)}(\mathcal{U}_{\mathrm{Gr}(2,4)}(-1) \oplus \mathcal{O}_{\mathrm{Gr}(2,4)}(-1)).$$

To determine the exceptional locus of π, we apply the standard combination of Proposition 2.2.1, Theorem 2.2.5, and Theorem 2.2.6, and find that it is a surface

$$S = \mathscr{Z}(\mathcal{U}^{\vee}(1,1) \oplus \mathcal{O}(1,1)) \subset \mathbb{P}^1 \times \mathrm{Gr}(2,4).$$

Projecting S onto $\mathbb{P}^1$, the general fiber is the curve

$$E = \mathscr{Z}(\mathcal{U}^{\vee}(1) \oplus \mathcal{O}(1)) \subset \mathrm{Gr}(2,4)$$

which satisfies $K_E = \mathcal{O}_E$ and therefore, by classification, is an elliptic curve of degree five. Thus S is an elliptic fibration $\mathcal{E}$, and

$$X = \mathrm{Bl}_{\mathcal{E}} \mathrm{Gr}(2,4).$$

In particular, since X is birational to $\mathrm{Gr}(2,4)$, it is rational. •

Fano 2–31. [# K_{626}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(0,1) \oplus \mathcal{O}(1,1) \oplus \mathcal{O}(2,0)) \subset \mathbb{P}^5 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 25,\ (-K)^4 = 90,\ \chi(T_X) = -23,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 24.$

Description. X is the Fano fourfold K3-23 in [**BFMT21**, Table 6]. •

Fano 2–32. [#K_{628}] $X = \mathscr{Z}(\bigwedge^2 \mathcal{Q}_{\mathbb{P}^3}(0,1) \oplus \mathcal{O}(1,1) \oplus \mathcal{O}(0,2)) \subset \mathbb{P}^3 \times \mathbb{P}^6$
Invariants. $h^0(-K) = 23,\ (-K)^4 = 80,\ \chi(T_X) = -27,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 28.$
Description. X is the Fano fourfold K3-31 in [**BFMT21**, Table 6]. •

Fano 2–33. [#K_{629}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2}(0,1) \oplus \mathcal{O}(1,1) \oplus \mathcal{O}(0,1)) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,5)$
Invariants. $h^0(-K) = 23,\ (-K)^4 = 80,\ \chi(T_X) = -26,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 27$
Description. Description X is the Fano fourfold GM-19 in [**BFMT21**, Table 5]. •

Fano 2–34. [#K_{115}] $X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0)) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,4)$
Invariants. $h^0(-K) = 60,\ (-K)^4 = 272,\ \chi(T_X) = 1,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 4.$
Description. $X = \mathrm{Bl}_{Q_2} \mathrm{Gr}(2,4)$
Identification. The projection to $\mathrm{Gr}(2,4)$ gives a birational map between X and $\mathrm{Gr}(2,4)$. By Proposition 2.2.1, the fiber becomes a $\mathbb{P}^1$ along the locus where the morphism

$$\varphi : \mathcal{U}_{\mathrm{Gr}(2,4)} \oplus \mathcal{Q}^\vee_{\mathrm{Gr}(2,4)} \to \mathcal{O}^{\oplus 5}_{\mathrm{Gr}(2,4)}$$

drops rank to 3 map, on $\mathrm{D}_3(\varphi)$.

Using Theorems 2.2.5 and 2.2.6, we find that $\mathrm{D}_3(\varphi)$ is a surface

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0)) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,4).$$

In particular, S is such that $e(S) = 4$, $K_S = \mathcal{O}(1,-2)_{|S}$, $(-K_S)^2 = 8$, $h^{1,0} = h^{2,0} = 0$ and $h^0(S, K_S^{\otimes 2}) = 0$. Hence S is a dP_8. Since S is birational to $\mathbb{P}^1 \times \mathbb{P}^1$ then it is a quadric surface Q_2.

Since X is birational to $\mathrm{Gr}(2,4)$, it is, in particular, rational. •

Fano 2–35. [#K_{126}] $\mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(1,1) \oplus \mathcal{O}(0,1)) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,4)$
Invariants. $h^0(-K) = 51,\ (-K)^4 = 224,\ \chi(T_X) = -5,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 7.$
Description. X is generically a $\mathbb{P}^1$-bundle with fiber jumping to a $\mathbb{P}^2$ over five points. X is a small resolution of $Y = \mathrm{Bl}_S \mathbb{P}^4$ with S a dP_7 singular in a point.
Identification. By Corollary 2.2.11 (2) and Lemma 2.2.12, we can write

$$X = \mathscr{Z}(\mathcal{O}(1,1)) \subset \mathbb{P}_{Q_3}(\mathcal{U}_{\mathrm{Gr}(2,4)|Q_3} \oplus \mathcal{O}_{Q_3}).$$

Twisting $\mathcal{U}_{\mathrm{Gr}(2,4)|Q_3} \oplus \mathcal{O}_{Q_3}$ with $\mathcal{O}_{\mathrm{Gr}(2,4)}(-1)_{|Q_3}$ yields X is the $\mathbb{P}^1$-bundle

$$X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(1)) \subset \mathbb{P}_{Q_3}(\mathcal{U}_{\mathrm{Gr}(2,4)|Q_3}(-1) \oplus \mathcal{O}_{Q_3}(-1)).$$

Applying Proposition 2.2.1, Theorems 2.2.5 and 2.2.6 we find that the fiber of the natural projection π to a $\mathbb{P}^2$ over five points.

As observed above, we can also view

$$X = \mathbb{P}_{Q_3}(\mathcal{O}_{Q_3}(-1) \oplus \mathcal{U}_{\mathrm{Gr}(2,4)|Q_3}(-1)).$$

Here, $\mathcal{U}_{\mathrm{Gr}(2,4)|Q_3}$ is the dual of the spinor bundle on Q_3, so

$$X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(1)) \subset \mathbb{P}_{Q_3}(\mathcal{O}_{Q_3}(-1) \oplus \mathcal{S}^\vee(-1)).$$

Consider the exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_{Q_3}(-1) \oplus \mathcal{S}^\vee(-1) \to \mathcal{O}_{Q_3} \to 0$$

and we define $\mathcal{E} = \mathcal{K}^\vee(-1)$. Dualizing the above sequence and applying Theorem 2.2.8, we obtian

$$X = \mathbb{P}_{Q_3}(\mathcal{E}^\vee),$$

which is a model for the Fano fourfold described in [**Lang98**, Theorem 7.2.12].

We now study the projection π_1 on $\mathbb{P}^4$. The general fiber is a point, hence π_1 induces a birational map between X and $\mathbb{P}^4$.

Moreover, X is cut out by $\alpha \in V_5^\vee \otimes V_4$ and $\beta \in (V_5^\vee \oplus V_1) \otimes V_4$. To analyze the degeneracy locus of π_1, we apply Proposition 2.2.1 to the morphisms:

$$\varphi : V_4^\vee \otimes \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_{\mathbb{P}^4}(-1),$$

$$\phi : \bigwedge^2 V_4^\vee \otimes \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_{\mathbb{P}^4}(-1) \oplus \mathcal{O}_{\mathbb{P}^4}.$$

We find that $\mathrm{D}_0(\varphi) = \{p\}$ and that $\pi_1^{-1}(p) = \mathbb{P}^1 \times \mathbb{P}^1$. The morphism φ induces the sequence

$$0 \to \mathcal{K} \xrightarrow{i} V_4^\vee \otimes \mathcal{O}_{\mathbb{P}^4} \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}^4}(-1)$$

whose second exterior power gives a map:

$$\bigwedge^2 i : \bigwedge^2 \mathcal{K} \to \bigwedge^2 V_4^\vee.$$

Composing with ϕ we obtain

$$\Phi = \phi \circ \bigwedge^2 i = \bigwedge^2 \mathcal{K} \to \mathcal{O}_{\mathbb{P}^4}(-1) \oplus \mathcal{O}_{\mathbb{P}^4}.$$

The locus $\mathrm{D}_1(\phi_{|\alpha=0}) = \mathrm{D}_1(\Phi)$ is a surface $S = \mathrm{dP}_7$. The general fiber of π_1^{-1} over S is a $\mathbb{P}^1$, so outside a point we have

$$X = \mathrm{Bl}_{\mathrm{dP}_7} \mathbb{P}^4.$$

In conclusion X is a small resolution of $Y = \mathrm{Bl}_S \mathbb{P}^4$, where S is a dP_7 singular at a point. •

Fano 2–36. [#K_{124}] $X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(1,1)) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 54,\ (-K)^4 = 240,\ \chi(T_X) = -4,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 7.$

Description. $X = \mathrm{Bl}_S \mathrm{Gr}(2,4)$, with $S = \mathrm{dP}_5$.

Identification. Recall that, by applying Lemma 2.2.10 (1), the self-duality of $\mathrm{Gr}(2,4)$ and Lemma 2.2.12, we can rewrite

$$X = \mathscr{Z}(\mathcal{O}(1,1)) \subset \mathbb{P}_{\mathrm{Gr}(2,4)}(\mathcal{U}_{\mathrm{Gr}(2,4)}).$$

Twisting $\mathcal{U}_{\mathrm{Gr}(2,4)}$, we obtain

$$X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(1)) \subset \mathbb{P}_{\mathrm{Gr}(2,4)}(\mathcal{U}_{\mathrm{Gr}(2,4)}(-1)).$$

Here X is a $\mathbb{P}^1$-bundle cut by a relative hyperplane section, and is therefore birational to its base $\mathrm{Gr}(2,4)$.

Applying Proposition 2.2.1, Theorems 2.2.5 and 2.2.6, we find that the exceptional locus of the projection π to $\mathrm{Gr}(2,4)$, is a surface

$$S = \mathscr{Z}(\mathcal{U}^\vee(1)) \subset \mathrm{Gr}(2,4),$$

which is as Del Pezzo surface of degree 5, dP_5.

Thus

$$X = \mathrm{Bl}_{\mathrm{dP}_5} \mathrm{Gr}(2,4),$$

and, in particular X is rational. •

Fano 2–37. [#K_{642}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \oplus \mathcal{O}(0,1) \oplus \mathcal{O}(0,2)) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,6)$

Invariants. $h^0(-K) = 24,\ (-K)^4 = 82,\ \chi(T_X) = -38,\ h^{1,1} = 2,\ h^{3,1} = 3,\ h^{2,2} = 42.$

Description. X is a small resolution of Y, a fourfold singular in two points with $e(Y) = 52$.

Identification. The variety X is the zero locus of $\sigma \in V_3 \otimes V_6^\vee$ intersected with a 1-dimensional linear subspace of $\bigwedge^2 V_6^\vee$ and a quadric in $\bigwedge^2 V_6^\vee$. We first study X', the zero locus of σ.

Projecting to $\mathrm{Gr}(2,6)$ amounts to fixing a plane $\Pi \in \mathrm{Gr}(2,6)$ and seeking for the lines $l \in \mathbb{P}^2$ such that the condition $\sigma(l,\Pi) = 0$ holds.

By Proposition 2.2.1, for a general Π no such line l exists, so we have to restrict to a subvariety $Y' \subset \mathrm{Gr}(2,6)$. This Y' coincides to the degeneracy locus where

$$\varphi : \mathcal{U}_{\mathrm{Gr}(2,6)} \to \mathcal{O}^{\oplus 3}_{\mathrm{Gr}(2,6)}$$

drops rank to 1, i.e., $\mathrm{D}_1(\varphi)$.

Using Theorem 2.2.5 we have that Y' is a sixfold, singular on $\mathrm{D}_0(\varphi)$. The latter is nonempty; indeed, by Theorem 2.2.5, it is a surface S with $e(S) = 3$ and $\deg(S) = 1$.

Thus X' is birational to Y' via the projection π to $\mathrm{Gr}(2,6)$, whose fiber degenerates to a $\mathbb{P}^2$ along $S \subset Y'$.

The variety X is obtained by cutting X' with a hyperplane H and a quadric Q, so $Y = Y' \cap H \cap Q$ is a fourfold with $e(Y) = 52$, singular on 2 points corresponding to $S \cap H \cap Q$. Consequently, X is a small resolution of Y along its singular locus.

The other projection realizes X as a dP_4-fibration: the fiber over a general point of $\mathbb{P}^2$ is $\mathrm{Gr}(2,6)$ cut by two sections of $\mathcal{U}^\vee_{\mathrm{Gr}(2,6)}$, a linear subspace of codimension one and a quadric subspace of codimension one. •

Fano 2–38. [#K_{653}] $X = \mathscr{Z}(\mathcal{O}(0,1)^{\oplus 2} \oplus \mathcal{O}(1,1)^{\oplus 2}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,5)$.

Invariants. $h^0(-K) = 22,\ (-K)^4 = 75,\ \chi(T_X) = -28,\ h^{1,1} = 2,\ h^{3,1} = 2,\ h^{2,2} = 32.$

Description. $X = \mathrm{Bl}_{\mathcal{E}} X_5^4$ with $\mathcal{E}$ elliptic fibration over $\mathbb{P}^1$ with general fiber a quintic elliptic curve E.

Identification. We can view X as

$$X = \mathscr{Z}(\mathcal{O}(1,1)^{\oplus 2}) \subset \mathbb{P}_{X_5^4}(\mathcal{O}_{X_5^4}^{\oplus 3}).$$

Twisting $\mathcal{O}_{X_5^4}^{\oplus 3}$ by $\mathcal{O}_{X_5^4}(-1)$, we can rewrite X as

$$X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(1)^{\oplus 2}) \subset \mathbb{P}_{X_5^4}(\mathcal{O}_{X_5^4}(-1)^{\oplus 3}),$$

which is a $\mathbb{P}^2$-bundle over X_5^4 cut by 2 relative hyperplane sections. Hence X is generically isomorphic to X_5^4.

By Proposition 2.2.1, to understand the exceptional locus of the natural projection $\pi : X \to X_5^4$, we study where the map

$$\varphi : \mathcal{O}_{\mathrm{Gr}(2,5)}(-1)^{\oplus 2} \to \mathcal{O}_{\mathrm{Gr}(2,5)}^{\oplus 3}$$

restricted to X_5^4 degenerates to a rank one, i.e., $\mathrm{D}_1(\varphi_{X_5^4})$.

Using Theorems 2.2.5 and 2.2.6 we get that $\mathrm{D}_1(\varphi)$ is a surface

$$S = \mathscr{Z}(\mathcal{O}(1,1)^{\oplus 3}) \subset \mathbb{P}^1 \times X_5^4.$$

The projection of S to $\mathbb{P}^1$ defines an elliptic fibration $\mathcal{E}$, whose fiber is the elliptic curve

$$E = \mathscr{Z}(\mathcal{O}(1)^{\oplus 2}) \subset X_5^4,$$

a quintic curve.

Therefore we have

$$X = \mathrm{Bl}_{\mathcal{E}} X_5^4,$$

which also shows that X is rational. •

Fano 2–39. [#K_{663}] $X = \mathscr{Z}(\mathcal{O}(2,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,5)}(0,1)) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 30,\ (-K)^4 = 112,\ \chi(T_X) = -12,\ h^{1,1} = 2,\ h^{2,1} = 5,\ h^{3,1} = 0,\ h^{2,2} = 2.$

Description. X is $Q_1 \times Y$, with Y the Fano threefold 1-7 in [DBFT22, Table 1].

Identification. Since the bundles cut each factor of the product separately, we see that X is simply the product

$$X = Q_1 \times Y$$

where

$$Y = \mathscr{Z}(\mathcal{Q}(1)) \subset \mathrm{Gr}(2,5).$$

The threefold Y is Fan, since $K_Y = \mathcal{O}_{\mathrm{Gr}(2,5)}(-1)_{|Y}$. Moreover, Y has the following invariants: $e(Y) = -6$, $vol(Y) = 14$, $h^0(-K_Y) = 10$ and $h^{2,1} = 5$. By the classification of the Fano threefolds, these invariants identify Y as the Fano threefold 1-7 in [DBFT22, Table 1], which can also be realized as

$$\mathscr{Z}(\mathcal{O}(1)^{\oplus 5}) \subset \mathrm{Gr}(2,6).$$

•

Fano 2–40. [#K_{664}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(0,1) \oplus (2,1)) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 21,\ (-K)^4 = 69,\ \chi(T_X) = -40,\ h^{1,1} = 2,\ h^{3,1} = 5,\ h^{2,2} = 52.$

Description. X is a conic bundle which discriminates on a surface $S \subset Q_3$ of degree 10 with 40 singular points. X is a small resolution of $Y = \mathrm{Bl}_S \mathbb{P}^4$ with S a surface singular in a point with $h^{0,0} = 1$, $h^{1,0} = 0$, $h^{2,0} = 5$ and $h^{1,1} = 48$

Identification. Applying Corollary 2.2.11 (2) and Lemma 2.2.12, we can rewrite

$$X = \mathscr{Z}(\mathcal{O}(1,2)) \subset \mathbb{P}_{Q_3}(\mathcal{U}_{\mathrm{Gr}(2,4)|Q_3} \oplus \mathcal{O}_{\mathrm{Gr}(2,4)}).$$

Twisting $\mathcal{U}_{\mathrm{Gr}(2,4)|Q_3} \oplus \mathcal{O}_{\mathrm{Gr}(2,4)}$ by $\mathcal{O}_{Q_3}(-1)$, we obtain

$$X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(2)) \subset \mathbb{P}_{Q_3}(\mathcal{U}_{\mathrm{Gr}(2,4)}(-1)_{|Q_3} \oplus \mathcal{O}_{Q_3}(-1))$$

which realizes X as a conic bundle on Q_3.

Using Lemma 2.2.13 with $\mathcal{E} = \mathcal{U}^\vee \oplus \mathcal{O}$ and $\mathcal{K} = \mathcal{O}(1)$, the discriminant locus is a surface $S \subset Q_3$ of degree 5 with 40 singular points.

The projection to $\mathbb{P}^4$ gives a birational map: the general fiber is $\mathrm{Gr}(2,4)$ cut with Q, hence $\mathbb{P}^2$, and then 2 hyperplane sections cut $\mathbb{P}^2$, so X is rational.

To understand the exceptional locus of π_1, note that X is defined by

$$\alpha \in V_5^\vee \otimes V_4,\ \beta \in (\mathrm{Sym}^2 V_5^\vee \oplus V_1) \otimes \bigwedge^2 V_4$$

By Proposition 2.2.1, this is equivalent to studying where the morphisms:

$$\varphi : V_4^\vee \otimes \mathcal{O} \to \mathcal{O}(-1),$$

$$\phi : \bigwedge^2 V_4^\vee \otimes \mathcal{O} \to \mathcal{O}(-2) \oplus \mathcal{O}.$$

degenerate.

Using Theorems 2.2.5 and 2.2.6, we find that $\mathrm{D}_0(\varphi) = \{p\}$ and $\pi_1^{-1}(p) = \mathbb{P}^1 \times \mathbb{P}^1$. The map φ induces a sequence:

$$0 \to \mathcal{K} \xrightarrow{i} V_4^\vee \otimes \mathcal{O} \xrightarrow{\varphi} \mathcal{O}(-1)$$

and taking the second exterior power gives

$$\bigwedge^2 i : \bigwedge^2 \mathcal{K} \to \bigwedge^2 V_4^\vee.$$

If we compose $\phi \circ \bigwedge^2 i$, we obtain

$$\Phi : \bigwedge^2 \mathcal{K} \to \mathcal{O}(-2) \oplus \mathcal{O}.$$

By Theorems 2.2.5 and 2.2.6, $\mathrm{D}_1(\phi_{|\alpha=0}) = \mathrm{D}_1(\Phi)$ is a surface S with $h^{0,0} = 1$, $h^{1,0} = 0$, $h^{2,0} = 5$ and $h^{1,1} = 48$. Hence, outside a point p, X is $\mathrm{Bl}_S \mathbb{P}^4$ and over p the fiber is $\mathbb{P}^1 \times \mathbb{P}^1$. This gives the interpretation of X as a small resolution of $\mathrm{Bl}_{S'} \mathbb{P}^4$, where S' singular p and whose resolution is S. •

Fano 2–41. [#K_{665}] $\mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(1,2)) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,4)$.

Invariants. $h^0(-K) = 20,\ (-K)^4 = 64,\ \chi(T_X) = -44,\ h^{1,1} = 2,\ h^{3,1} = 6,\ h^{2,2} = 59$

Description. $X = \mathrm{Bl}_S \mathrm{Gr}(2,4)$, with S a canonical surface with $e(S) = 71$, $K_S = \mathcal{O}(1)_S$, and $(-K_S)^2 = 13$.

Identification. Applying Lemma 2.2.10 (1), we can rewrite

$$X = \mathscr{Z}(\mathcal{O}(2,1)) \subset \mathbb{P}_{\mathrm{Gr}(2,4)}(\mathcal{U}_{\mathrm{Gr}(2,4)}).$$

Twisting $\mathcal{U}_{\mathrm{Gr}(2,4)}$ by $\mathcal{O}(-2)$, we obtain

$$X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(1)) \subset \mathbb{P}_{\mathrm{Gr}(2,4)}(\mathcal{U}_{\mathrm{Gr}(2,4)}(-2)).$$

Thus, X is realized as a $\mathbb{P}^1$-bundle over $\mathrm{Gr}(2,4)$ cut with a relative hyperplane section, hence X is birational to $\mathrm{Gr}(2,4)$.

To understand the exceptional locus of The projection $\pi : X \mathrm{Gr}(2,4)$, we apply the usual combination of Proposition 2.2.1, Theorems 2.2.5 and 2.2.6. This shows that the exceptional locus is a surface

$$S = \mathscr{Z}(\mathcal{U}^{\vee}(2)) \subset \mathrm{Gr}(2,4).$$

The surface S has the following invariants: $e(S) = 71$, $K_S = \mathcal{O}(1)_S$, $(-K_S)^2 = 13$, $h^{1,0} = 0$, $h^{2,0} = 6$ and $h^{1,1} = 57$.

Hence X can be described as

$$X = \mathrm{Bl}_S \mathrm{Gr}(2,4),$$

which in particular implies that X is rational. •

Fano 2–42. [#K_{669}] $X = \mathscr{Z}(\mathcal{O}(2,1) \oplus \mathcal{O}(0,1)^{\oplus 3}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,5)$.

Invariants. $h^0(-K) = 21,\ (-K)^4 = 70,\ \chi(T_X) = -30,\ h^{1,1} = 2,\ h^{3,1} = 3,\ h^{2,2} = 37.$

Description. X is a conic bundle on X_5^3 which discriminates along a degree three surface singular in three points.

Identification. We can rewrite

$$X = mZ(\mathcal{O}(1,2)) \subset \mathbb{P}_{X_5^3}(\mathcal{O}_{X_5^3}^{\oplus 3}).$$

Twisting $\mathcal{O}_{X_5^3}^{\oplus 3}$ by $\mathcal{O}_{X_5^3}(-1)$, we obtain

$$X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(2)) \subset \mathbb{P}_{X_5^3}(\mathcal{O}_{X_5^3}(-1) \oplus 3).$$

Applying Lemma 2.2.13 with $\mathcal{E} = \mathcal{O}^{\oplus 3}$ and $\mathcal{K} = \mathcal{O}(1)$, the discriminant locus of the resulting conic bundle is a degree 3 surface with 20 singular points.

Furthermore, the projection to $\mathbb{P}^2$ gives a dP_5-fibration, since the fiber is $\mathrm{Gr}(2,5)$ cut with 4 hyperplane sections. By [**Isk96**], this structure implies that X is rational.

•

Fano 2–43. [#K_{136}] $\mathscr{Z}(\mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(0,1)) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 81,\ (-K)^4 = 384,\ \chi(T_X) = 10,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 2.$

Description. $X = \mathbb{P}_{Q_3}(\mathcal{U}_{\mathrm{Gr}(2,4)|Q_3})$.

Identification. We simply apply the self-duality of $\mathrm{Gr}(2,4)$, Lemma 2.2.10 (1) and Lemma 2.2.12 and obtain that X is $\mathbb{P}_{Q_3}(\mathcal{U}_{\mathrm{Gr}(2,4)|Q_3})$. •

Fano 2–44. [#K_{723}] $X = \mathscr{Z}(\mathcal{O}(0,1)^{\oplus 2} \oplus \mathcal{O}(1,2)) \subset \mathbb{P}^1 \times \mathrm{Gr}(2,5)$.

Invariants. $h^0(-K) = 16,\ (-K)^4 = 45,\ \chi(T_X) = -50,\ h^{1,1} = 2,\ h^{3,1} = 8,\ h^{2,2} = 72.$

Description. $X = \mathrm{Bl}_S X_5^4$, with $S = Q_3 \cap Q_3' \subset X_5^4$.

Identification. We simply apply Lemma 2.2.9, and obtain that $X = \mathrm{Bl}_S X_5^4$, with $S = \mathscr{Z}(\mathcal{O}(2)^{\oplus 2}) \subset X_5^4$, in this way we have that $e(S) = 88$, $K_S = \mathcal{O}_{\mathrm{Gr}(2,5)}(1)_{|S}$, $(-K_S)^2 = 20$, $h^{1,0} = 0$, $h^{2,0} = 8$. Since X is birational to X_5^4, it is rational.

•

Fano 2–45. [#K_{219}] $X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 3}) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 51$, $(-K)^4 = 224$, $\chi(T_X) = -1$, $h^{1,1} = 2$, $h^{3,1} = 0$, $h^{2,2} = 3$.

Description. X a $\mathbb{P}^1$-bundle whose fiber jumps to a $\mathbb{P}^2$ over one point.

Identification. We can write

$$X = \mathscr{Z}(\mathcal{O}(0,1) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(0,1)) \subset \mathbb{P}_{X_5^3}(\mathcal{O}^{\oplus 5}_{X_5^3}).$$

Using Theorem 2.2.8, this can be rewritten as

$$X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(1)) \subset \mathbb{P}_{X_5^3}(\mathcal{Q}^\vee_{X_5^3})$$

which exhibits X as a $\mathbb{P}^1$-bundle over X_5^3.

To understand where the fiber of the projection $\pi : X \to X_5^3$ becomes a $\mathbb{P}^2$, we apply Proposition 2.2.1, Theorems 2.2.5 and 2.2.6 and find that this occurs at a single point of X_5^3.

The projection $\pi_1 : X \to \mathbb{P}^3$ is a conic bundle over $\mathbb{P}^3$. Indeed, the general fiber of π_1 is $\mathrm{Gr}(2,5)$ cut by a section of $\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}$ and a codimension three linear subspace, hence a conic.

By Theorem 2.2.8 we can write

$$X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(1)^{\oplus 3}) \subset \mathrm{Gr}_{\mathbb{P}^3}(2, \mathcal{Q}^\vee_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}),$$

which gives the morphism

$$\varphi : \bigwedge^2 (\mathcal{Q}^\vee \oplus \mathcal{O}) \to \mathcal{O}^{\oplus 3}.$$

If we denote $\mathcal{E} = Ker\varphi$, then it is the rank three bundle defining the conic bundle structure.

By Lemma 2.2.13 with $\mathcal{K} = \mathcal{O}$, the conic bundle degenerates along Δ which is a sextic surface in $\mathbb{P}^3$ with 40 singular points. •

Fano 2–46. [#K_{195}] $X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 2}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 63$, $(-K)^4 = 288$, $\chi(T_X) = 2$, $h^{1,1} = 2$, $h^{3,1} = 0$, $h^{2,2} = 4$.

Description. $X = \mathrm{Bl}_{Q_2} X_5^4$.

Identification. The projection

$$\pi : X \to \mathrm{Gr}(2,5)$$

is a birational map between X and X_5^4: indeed, the fiber is $\mathbb{P}^2$ cut with a codimension-two linear subspace.

By combining Proposition 2.2.1, Theorems 2.2.5 and 2.2.6 we deduce that the exceptional locus of π is a surface

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0)) \subset \mathbb{P}^1 \times X_5^4.$$

The surface S has the following invariants: $e(S) = 4$, $K_S = \mathcal{O}(1,-2)_{|S}$, $(-K_S)^2 = 8$, $h^{1,0} = h^{2,0} = 0$ and $h^0(S, K_S^{\otimes 2}) = 0$. Hence, by the classification, S is a dP_8.

The projection to $\mathbb{P}^1$ has as general fiber given by $\mathrm{Gr}(2,5)$ intersected with a section of $\mathcal{Q}_{\mathrm{Gr}(2,5)}$ and a with codimension-two linear subspace. Thus, the fiber is a $\mathbb{P}^1$.

This implies that S is birational to $\mathbb{P}^1 \times \mathbb{P}^1$. Since S is a dP_8, it follows that $S = \mathbb{P}^1 \times \mathbb{P}^1$.

In conclusion, $X = \mathrm{Bl}_S X_5^4$, and, in particular, is rational. •

Fano 2–47. [#K_{230}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(0,1) \oplus \mathcal{O}(2,0)) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 54$, $(-K)^4 = 240$, $\chi(T_X) = 1$, $h^{1,1} = 2$, $h^{3,1} = 0$, $h^{2,2} = 2$.

Description. X is a conic bundle that discriminates on a degree-two surface $S \subset Q_3$ singular in 4 points.

Identification. Using Corollary 2.2.11 (2), we can rewrite

$$X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(2)) \subset \mathbb{P}_{Q_3}(\mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)|Q_3} \oplus \mathcal{O}_{Q_3}),$$

which is a conic bundle. By Lemma 2.2.13, let $\mathcal{E} = \mathcal{U}^{\vee} \oplus \mathcal{O}$ and $\mathcal{K} = \mathcal{O}$, then the discriminant is a degree-two surface singular at 4 points.

The other projection π_2 over $Q_3 \subset \mathbb{P}^4$, has a general fiber $\mathbb{P}^1$. It is given by $\mathrm{Gr}(2,4)$ intersected with one global section of $\mathcal{Q}_{\mathrm{Gr}(2,4)}$ and a global section of $\mathcal{O}_{\mathrm{Gr}(2,4)}(1)$.

By Proposition 2.2.1 and Theorem 2.2.5, the fiber dimension does not jump; hence X is a $\mathbb{P}^1$-bundle on Q_3.

To recover the rank-two bundle on Q_3 that we are projectivizing, note that by self-duality we can rewrite

$$X = \mathscr{Z}(\mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(2,0) \oplus \mathcal{O}(0,1)) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,4).$$

Consider the map

$$\varphi : V_4 \otimes \mathcal{O}_{Q_3} \to \mathcal{O}_{Q_3}(1).$$

By Theorem 2.2.5, φ does not degenerate, so we can define $\mathcal{K}' = ker\varphi$ as a rank-three vector bundle over Q_3; the fiber dimension is constant. In particular, $\mathcal{K}'$ fits into the exact sequence

$$\mathcal{K}' \to V_4 \otimes \mathcal{O}_{Q_3} \to \mathcal{O}_{Q_3}(1).$$

By Theorem 2.2.8,

$$X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(1)) \subset \mathrm{Gr}_{Q_3}(2, \mathcal{K}').$$

Using the Plücker embedding, we obtain

$$X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(1)) \subset \mathbb{P}_{Q_3}(\bigwedge^2 \mathcal{K}').$$

Arguing as before, there is an exact sequence

$$0 \to \mathcal{K} \to \bigwedge^2 \mathcal{K}' \xrightarrow{\phi} \mathcal{O}_{Q_3}.$$

so that X can be written as $\mathbb{P}_{Q_3}(\mathcal{K})$.

•

Fano 2–48. [#K_{238}] $X = \mathscr{Z}(\mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(0,2)) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 45$, $(-K)^4 = 192$, $\chi(T_X) = -4$, $h^{1,1} = 2$, $h^{2,1} = 2$, $h^{3,1} = 0$, $h^{2,2} = 2$.

Description. $X = \mathbb{P}_Y(\mathcal{U}_{\mathrm{Gr}(2,4)|Y})$, with Y the Fano threefold 1-14 in [DBFT22, Table 1].

Identification. We apply Lemma 2.2.10 (1) and Lemma 2.2.12 to rewrite

$$X = \mathbb{P}_Y(\mathcal{U}_{\mathrm{Gr}(2,4)|Y}),$$

where

$$Y = \mathscr{Z}(\mathcal{O}(2)) \subset \mathrm{Gr}(2,4),$$

which can also be expressed as

$$\mathscr{Z}(\mathcal{O}(2)^{\oplus 2}) \subset \mathbb{P}^5,$$

and is therefore the Fano threefold 1-14 in [DBFT22, Table 1].

The projection π_1 on $\mathbb{P}^3$ is a conic bundle: indeed, the general fiber of π_1 is $\mathrm{Gr}(2,4)$ intersected with a global section of $\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}$ and a global section of $\mathcal{O}_{\mathrm{Gr}(2,4)}(2)$.

Using the self-duality of $\mathrm{Gr}(2,4)$ and Lemma 2.2.10, we can write

$$X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(2)) \subset \mathbb{P}_{\mathbb{P}^3}(\mathcal{Q}_{\mathbb{P}^3}(-1)).$$

Applying Lemma 2.2.13 with $\mathcal{E} = \mathcal{Q}^\vee_{\mathbb{P}^3}(1)$ and $\mathcal{K} = \mathcal{O}_{\mathbb{P}^3}$ we find that the discriminant Δ is a degree four surface in $\mathbb{P}^3$, hence a $K3$ surface, singular along $\Delta_{sing} = \{8 \text{ points}\}$. •

Fano 2–49. [#K_{679}] $X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,1)) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 18,\ (-K)^4 = 56,\ \chi(T_X) = -36,\ h^{1,1} = 2,\ h^{3,1} = 4,\ h^{2,2} = 47.$

Description. $X = \mathrm{Bl}_S \mathrm{Gr}(2,4)$, with S a surface with $e(S) = 55$, $K_S = \mathcal{O}_{\mathbb{P}^3\times\mathrm{Gr}(2,4)}(1,0)_{|S}$ and $(-K_S)^2 = 5$.

Identification. The projection

$$\pi : X \to \mathrm{Gr}(2,4)$$

is a birational map between X and $\mathrm{Gr}(2,4)$. Indeed, we can rewrite

$$X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,1)) \subset \mathbb{P}_{\mathrm{Gr}(2,4)}(\mathcal{O}^{\oplus 3}_{\mathrm{Gr}(2,4)}).$$

Twisting $\mathcal{O}^{\oplus 3}_{\mathrm{Gr}(2,4)}$ by $\mathcal{O}_{\mathrm{Gr}(2,4)}(-1)$ we obtain

$$X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0)) \subset \mathbb{P}_{\mathrm{Gr}(2,4)}(\mathcal{O}_{\mathrm{Gr}(2,4)}(-1)^{\oplus 3}).$$

This is a $\mathbb{P}^2$-bundle cut by a codimension-two linear subspace in the fibers. Thus, the projection π gives a birational map.

To describe the exceptional locus of π we apply Proposition 2.2.1, Theorems 2.2.5 and 2.2.6. We find that it is a surface

$$S \subset \mathrm{Gr}(2,4)$$

given by

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(1,1)^{\oplus 3}) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,4).$$

The invarinat of S are: $e(S) = 55$, $K_S = \mathcal{O}_{\mathbb{P}^3\times\mathrm{Gr}(2,4)}(1,0)_{|S}$, $(-K_S)^2 = 5$, $h^0(S, -K_S) = 10$, $h^{1,0} = 0$ and $h^{2,0} = 4$.

Since $X = \mathrm{Bl}_S \mathrm{Gr}(2,4)$, it follows that X is rational.

•

Fano 2–50. [#K_{253}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^4} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,8)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,8)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 2}) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,8)$

Invariants. $h^0(-K) = 42,\ (-K)^4 = 177,\ \chi(T_X) = -11,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 11.$

Description. X is a small resolution of $Y = \mathrm{Bl}_S \mathbb{P}^4$, with S a dP_6 singular in two points.

Identification. Consider the projection

$$\pi : X \to \mathbb{P}^4.$$

The general fiber is $\mathrm{Gr}(2,8)$ intersected with 5 sections of $\mathcal{U}^\vee$ and 2 hyperplane sections, hence it is a point.

To determine the exceptional loci, we examine the two maps between vector bundles obtained by applying Proposition 2.2.1 on π:

$$\varphi : V_8 \otimes \mathcal{O}_{\mathbb{P}^4} \to \mathcal{Q}_{\mathbb{P}^4} \oplus \mathcal{O}_{\mathbb{P}^4}(1),$$

$$\phi : \bigwedge^2 V_8 \otimes \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_{\mathbb{P}^4}^{\oplus 2}.$$

By Theorems 2.2.5 and 2.2.6, we have $\mathrm{D}_4(\varphi) = \{2 \text{ points}\}$, and the fiber of π over each of these points is a smooth quadric surface Q_2.

To analyze $\mathrm{D}_1(\phi)$, note that φ and ϕ are related via the sequence

$$0 \to \mathcal{K} \to V_8 \otimes \mathcal{O}_{\mathbb{P}^4} \xrightarrow{\varphi} \mathcal{Q}_{\mathbb{P}^4} \oplus \mathcal{O}_{\mathbb{P}^4}(1) \to \mathcal{F} \to 0$$

where $\mathcal{K} = ker(\varphi)$ and $\mathcal{F} = coker(\varphi)$.

Taking the second exterior power of this sequence yields a map

$$\tau : \bigwedge^2 \mathcal{K} \to \bigwedge^2 V_8.$$

Composing τ and ϕ gives

$$\Phi : \bigwedge^2 \mathcal{K} \to \mathcal{O}_{\mathbb{P}^4}^{\oplus 2}.$$

Thus by Theorems 2.2.5 and 2.2.6, $D' := \mathrm{D}_1(\phi_{|D}) = \mathrm{D}_1(\Phi)$, is a dP_6.

The resulting geometry is as follows:

$$X \setminus \{2 \text{ pts}\} = \mathrm{Bl}_{\mathrm{dP}_6} \mathbb{P}^4,$$

and over each of the two points the fiber is Q_2.

The projection to $\mathrm{Gr}(2,8)$ gives a small resolution of a singular fourfold Y of degree 28.

•

Fano 2–51. [#K_{142}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \oplus \mathcal{O}(0,1)^{\oplus 2}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,6)$

Invariants. $h^0(-K) = 75,\ (-K)^4 = 352,\ \chi(T_X) = 6,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 4.$

Description. X is a small resolution of Y, a fourfold singular in a point with $e(Y) = 9$.

Identification. The variety X is the zero locus of

$$\sigma \in V_3 \otimes V_6^\vee \text{ and } \omega_1, \omega_2 \in \bigwedge^2 V_6^\vee.$$

We first study $X' = \mathscr{Z}(\sigma)$.

Projecting to $\mathrm{Gr}(2,6)$, and applying Proposition 2.2.1, we are concerned with the locus where the map

$$\varphi : \mathcal{U}_{\mathrm{Gr}(2,6)} \to \mathcal{O}_{\mathrm{Gr}(2,6)}^{\oplus 3}$$

drops rank to one; that is, we seek $\mathrm{D}_1(\varphi)$.

By Theorem 2.2.5, $\mathrm{D}_1(\varphi)$ is a sixfold. Since $\mathrm{D}_0(\varphi) \subset Y'$ is nonempty, it follows that $\mathrm{D}_1(\varphi)$ is singular along $\mathrm{D}_0(\varphi)$, which is a surface S.

Intersecting with the 2 remaining linear sections yields X is birational to

$$Y := Y' \cap H_1 \cap H_2,$$

where Y is a fourfold, singular at a point

$$p = S \cap H_1 \cap H_2.$$

The projection $\pi : X \to \mathrm{Gr}(2,6)$ gives a birational map between X and Y. Over the singular point $p \in Y$, the fiber is a $\mathbb{P}^2$. Thus, X is a small resolution of Y at its singular point.

The projection π_1 to $\mathbb{P}^2$ defines a quadric bundle: indeed, the general fiber of π_1 is $\mathrm{Gr}(2,6)$ intersected with two global sections of $\mathcal{U}^\vee_{\mathrm{Gr}(2,6)}$ and a codimension-two linear subspace.

•

Fano 2–52. [#K_{686}] $X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(0,1) \oplus \mathcal{O}(0,1) \oplus \mathcal{O}(2,0) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 27,\ (-K)^4 = 96,\ \chi(T_X) = -15,\ h^{1,1} = 2,\ h^{2,1} = 7,\ h^{3,1} = 0,\ h^{2,2} = 2.$

Description. X is $Q_1 \times Y$, with Y the Fano threefold 1-6 in [**DBFT22**, Table 1].

Identification. The bundles cut the factors as $X = Q_1 \times Y$, with $Y = \mathscr{Z}(\mathcal{O}(1) \oplus \mathcal{U}_{\mathrm{Gr}(2,5)}(1)) \subset \mathrm{Gr}(2,5)$, which is exactly the Fano threefold 1-6 in [**DBFT22**, Table 1]. •

Fano 2–53. [#K_{689}] $\mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2}(0,2) \oplus \mathcal{O}(1,1)) \subset \mathbb{P}^2 \times \mathbb{P}^5$

Invariants. $h^0(-K) = 17,\ (-K)^4 = 51,\ \chi(T_X) = -36,\ h^{1,1} = 2,\ h^{3,1} = 2,\ h^{2,2} = 41.$

Description. X is the Fano fourfold A-67 in [**BFMT21**, Table 8] •

Fano 2–54. [#K_{699}] $X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(0,1) \oplus \mathcal{O}(1,1)) \subset \mathbb{P}^1 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 19,\ (-K)^4 = 60,\ \chi(T_X) = -31,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 32.$

Description. X is the Fano K3-24 in [**BFMT21**, Table 6]. •

Fano 2–55. [#K_{288}] $X = \mathscr{Z}(\mathcal{O}(1,1) \oplus (0,1)^{\oplus 2}) \subset \mathbb{P}^1 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 54,\ (-K)^4 = 240,\ \chi(T_X) = -4,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 7.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_5} X_5^4$.

Identification. X can be rewritten as $\mathscr{Z}(\mathcal{O}(1,1)) \subset \mathbb{P}^1 \times X_5^4$. In this way, we simply apply Lemma 2.2.9 and obtain that $X = \mathrm{Bl}_S X_5^4$, with $S = X_5^4 \cap H_1 \cap H_2$ and this is a dP_5.
Since $X = \mathrm{Bl}_{\mathrm{dP}_5} X_5^4$ is in particular rational. •

Fano 2–56. [#K_{464}] $X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \boxtimes \mathcal{Q}_{\mathrm{Gr}(3,5)} \oplus \mathcal{O}(1,0)^{\oplus 2} \oplus \mathcal{O}(0,1)^{\oplus 2}) \subset \mathrm{Gr}(2,5) \times \mathrm{Gr}(3,5)$

Invariants. $h^0(-K) = 39,\ (-K)^4 = 162,\ \chi(T_X) = -8,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 8.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_6} X_5^4$.

Identification. Using Lemma 2.2.10 (1) and Lemma 2.2.12 we rewrite

$$X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(1)^{\oplus 2}) \subset \mathbb{P}_{X_5^4}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(-1)_{|X_5^4}).$$

The natural projection to X_5^4 defines a birational map between the X and X_5^4. Indeed, X is a $\mathbb{P}^2$-bundle over X_5^4, with general fiber cut by two relative hyperplane sections.

Applying Proposition 2.2.1, Theorems 2.2.5 and 2.2.6, we see that the fiber degenerates to a $\mathbb{P}^1$ along a surface

$$S = \mathscr{Z}(\bigwedge^2 \mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 2}) \subset \mathbb{P}^1 \times \mathrm{Gr}(2,5).$$

The surface S has the following invariants: $e(S) = 8$, $K_S = \mathcal{O}(1,-1)_{|S}$, $(-K_S)^2 = 4$, $h^{1,0} = h^{2,0} = 0$ and $h^0(S, K_S^{\otimes 2}) = 0$. By the classification of surfaces, it follows that S is a dP_4.

Since $X = \mathrm{Bl}_{\mathrm{dP}_4} X_5^4$, the variety X is rational. •

Fano 2–57. [#K_{350}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,6)} \oplus \mathcal{O}(0,1)^{\oplus 3}) \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(3,6)$

Invariants. $h^0(-K) = 42,\ (-K)^4 = 177,\ \chi(T_X) = -12,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 12.$

Description. $X = \mathrm{Bl}_{\mathrm{Bl}_{9\mathrm{pts}}\mathbb{P}^2} \mathrm{Gr}(2,4)$.

Identification. Using the self-duality of both $\mathrm{Gr}(2,4)$ and $\mathrm{Gr}(3,6)$, we can write

$$X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{Q}_{\mathrm{Gr}(3,6)} \oplus \mathcal{O}(0,1)^{\oplus 3}) \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(3,6).$$

Applying Corollary 2.2.11 (2), we obtain

$$X = \mathscr{Z}(\mathcal{U}_1^\vee \oplus \mathcal{O}(0,1)^{\oplus 3}) \subset \mathrm{Fl}(2,3,6),$$

and by Lemma 2.2.12 we can rewrite

$$X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(1)^{\oplus 3}) \subset \mathbb{P}_{\mathrm{Gr}(2,4)}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(-1) \oplus \mathcal{O}(-1)^{\oplus 2}).$$

This exhibits X as a $\mathbb{P}^3$-bundle over $\mathrm{Gr}(2,4)$ with the general fiber cut by 3 relative-hyperplane sections. Hence, the natural projection to $\mathrm{Gr}(2,4)$ defines a birational map between X and $\mathrm{Gr}(2,4)$.

To determine where the fiber degenerates, we apply Proposition 2.2.1, Theorems 2.2.5 and 2.2.6, which shows that the degeneracy locus is a surface

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(1,1)^{\oplus 2}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,4).$$

The surface S has invariants: $e(S) = 12$, $K_S = \mathcal{O}(1,-1)_{|S}$, $K_S^2 = 0$, $h^{1,0} = h^{2,0} = 0$ and $h^0(S, K_S^{\otimes 2}) = 0$. By the classification of surfaces, S is the blow-up of $\mathbb{P}^2$ in 9 points.

Since $X = \mathrm{Bl}_S \mathrm{Gr}(2,4)$, the variety X is rational. •

Fano 2–58. [#K_{361}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^3} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{O}(0,1)^{\oplus 2} \oplus \mathcal{O}(1,1)) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,7)$

Invariants. $h^0(-K) = 33,\ (-K)^4 = 129,\ \chi(T_X) = -23,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 23.$

Description. X is the Fano fourfold R-62 in [**BFMT21**, Table 7]. •

Fano 2–59. [#K_{364}] $X = \mathscr{Z}(\mathcal{O}(2,0) \oplus \mathcal{O}(0,1)^{\oplus 3}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 69,\ (-K)^4 = 320,\ \chi(T_X) = 6,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 2.$

Description. X is $Q_1 \times Y$ with Y the Fano threefold 1-15 in [DBFT22, Table 1]

Identification. The bundles cut the factors of the ambient space separately, hence X is $Q_1 \times Y$ with $Y = \mathscr{Z}(\mathcal{O}(1)^{\oplus 3}) \subset \mathrm{Gr}(2,5)$, hence the Fano threefold 1-15 in [DBFT22, Table 1]. •

Fano 2–60. [#K_{374}] $\mathscr{Z}(\mathcal{Q}_{\mathbb{P}^6} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,9)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,9)}(1,0)^{\oplus 2}) \subset \mathbb{P}^6 \times \mathrm{Gr}(2,9)$

Invariants. $h^0(-K) = 45,\ (-K)^4 = 193,\ \chi(T_X) = -14,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 15.$

Description. X is a small resolution of Y, a fourfold with $e(Y) = 17$, $\deg(Y) = 5$ and singular in four points.

Identification. Let X be the zero locus of

$$\sigma \in (V_7 \oplus V_7^\vee \oplus V_7^\vee) \otimes V_9^\vee.$$

Projecting to $\mathbb{P}^6$, we are interested in planes $\Pi \in \mathrm{Gr}(2,9)$ such that, for a fixed line $l \in \mathbb{P}^6$ the relation $\sigma(l,\Pi) = 0$ holds. By Proposition 2.2.1, this is equivalent to studying the map

$$\varphi : \mathcal{Q}^\vee_{\mathbb{P}^6} \oplus \mathcal{O}(-1)^{\oplus 2} \to \mathcal{O}^{\oplus 9}_{\mathbb{P}^6}.$$

For dimensional reason, X is birational to the fourfold $Y = \mathrm{D}_7(\varphi)$.

Applying Theorem 2.2.6, we compute $e(Y) = 17$ and $\deg(Y) = 5$.

Since $\mathrm{D}_6(\varphi)$ is not empty, Y is singular along $\mathrm{D}_6(\varphi)$. By Theorems 2.2.5 and 2.2.6, $\mathrm{D}_6(\varphi)$ consists of four points. Over each of these points, the projection to $\mathbb{P}^2$ has fiber $\mathbb{P}^2$. Hence, X is a small resolution of Y. •

Fano 2–61. [#K_{385}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \oplus \mathcal{O}(1,0)^{\oplus 2} \oplus \mathcal{O}(0,1)^{\oplus 2}) \subset \mathrm{Gr}(2,5) \times \mathrm{Gr}(2,6)$

Invariants. $h^0(-K) = 45,\ (-K)^4 = 193,\ \chi(T_X) = -8,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 9.$

Description. $X = \mathrm{Bl}_{\mathbb{P}^2} X_6^4$.

Identification. Consider

$$X' = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)}) \subset \mathrm{Gr}(2,5) \times \mathrm{Gr}(2,6).$$

By Lemma 2.2.10 (2)

$$X' = \mathrm{Bl}_{\mathbb{P}^4} \mathrm{Gr}(2,6).$$

Consider the bundles $\mathcal{O}(1,0)^{\oplus 2}$ and $\mathcal{O}(0,1)^{\oplus 2}$, corresponding to four hyperplane sections of $\mathrm{Gr}(2,6)$. The first two hyperplanes are chosen not intersect the exceptional locus $\mathbb{P}^4 = \mathscr{Z}(\mathcal{Q}) \subset \mathrm{Gr}(2,6)$.

Then X is obtained from X' via

$$X = \mathrm{Bl}_{\mathbb{P}^4 \cap H_3 \cap H_4} \mathrm{Gr}(2,6) \cap H_1 \cap H_2 \cap H_3 \cap H_4,$$

which is $\mathrm{Bl}_{\mathbb{P}^2} X_6^4$.

Since X is birational to X_6^4, the variety X is rational. •

Fano 2–62. [#K_{15}] $\mathscr{Z}(\mathcal{Q}_{\mathbb{P}^4} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,7)}(1,0)) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,7)$

Invariants. $h^0(-K) = 90,\ (-K)^4 = 433,\ \chi(T_X) = 13,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 3.$

Description. $X = \mathrm{Bl}_C \mathbb{P}^4$, with C a rational curve.

Identification. The projection

$$\pi : X \to \mathbb{P}^4$$

is a birational map between X and $\mathbb{P}^4$.

To understand the exceptional locus of π, we apply Proposition 2.2.1, Theorems 2.2.5 and 2.2.6. We find that the locus where the fiber of π degenerates to a $\mathbb{P}^2$ is a rational curve C. Computing its degree, we see that C is a conic.

•

Fano 2–63. [#K_{407}] $\mathscr{Z}(\mathcal{Q}_{\mathbb{P}^5} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,8)} \oplus \mathcal{U}_{\mathrm{Gr}(2,5)^{\vee}(1,0)} \oplus \mathcal{O}(1,1)) \subset \mathbb{P}^5 \times \mathrm{Gr}(2,8)$

Invariants. $h^0(-K) = 39,\ (-K)^4 = 161,\ \chi(T_X) = -21,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 23.$

Description. X is the Fano fourfold C-5 in [**BFMT21**, Table 4] •

Fano 2–64. [#K_{473}] $\mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)} \oplus \mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(0,1)) \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 36,\ (-K)^4 = 146,\ \chi(T_X) = -15,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 14.$

Description. $X = \mathrm{Bl}_{\mathbb{P}^2} Y$ with Y the Fano fourfold Fano 1–8.

Identification. Let us first consider

$$X' = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)}) \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5).$$

Using Lemma 2.2.10 (2), we have

$$X' = \mathrm{Bl}_{\mathbb{P}^3} \mathrm{Gr}(2,5).$$

For the remaining section, notice that cutting X' with the zero locus of a section of $\mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(0,1)$ is equivalent to cutting $\mathrm{Gr}(2,5)$ with $\mathscr{Z}(\mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)}(1))$, and cutting $\mathbb{P}^3 = \mathscr{Z}(\mathcal{Q}) \subset \mathrm{Gr}(2,5)$ with a hyperplane section, since $\mathscr{Z}(\mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)})$ contains $\mathbb{P}^3$.

In this way

$$X = \mathrm{Bl}_{\mathbb{P}^2} Y,$$

where $Y = \mathscr{Z}(\mathcal{U}^{\vee}(1)) \subset \mathrm{Gr}(2,5)$, which corresponds to Fano 1–8. •

Fano 2–65. [#K_{439}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,6)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 4}) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,6)$

Invariants. $h^0(-K) = 35,\ (-K)^4 = 141,\ \chi(T_X) = -14,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 13.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_5} X^4_6$.

Identification. The projection

$$\pi : X \to \mathrm{Gr}(2,6)$$

gives a birational map between X and X^4_6.

To understand the exceptional locus of π, note that X is the zero locus of

$$\sigma \in V_5^{\vee} \otimes V_6 \text{ and } \omega_1, \omega_2, \omega_3, \omega_4 \in \bigwedge^2 V_6^{\vee}.$$

First, consider

$$X' = \mathscr{Z}(\sigma).$$

Projecting X' to $\mathrm{Gr}(2,6)$ is equivalent to finding the lines $l \in \mathbb{P}^4$ such that, for fixed a plane $\Pi \in \mathrm{Gr}(2,6)$, we have $\sigma(l,\Pi) = 0$.

By Proposition 2.2.1, this corresponds to studying the map

$$\varphi : \mathcal{Q}^{\vee}_{\mathrm{Gr}(2,6)|X_6^4} \to \mathcal{O}^{\oplus 5}_{X_6^4}$$

The exceptional locus of π coincides to the locus where φ restricted to X_6^4 drops rank ti 3, i.e., $\mathrm{D}_1(\varphi)$.

Using Theorems 2.2.5 and 2.2.6, we have that $\mathrm{D}_1(\varphi_{X_6^4})$ is a surface S with invariants: $e(S) = 7$, $K_S = \mathcal{O}_{\mathrm{Gr}(2,6)}(-1)_{|S}$, $(-K_S)^2 = 5$, $h^{1,0} = h^{2,0} = 0$ and $h^0(S, K_S^{\otimes 2}) = 0$.

By the classification of surfaces we have that S is dP_5.

Since X is birational to X_6^4, it follows that X is rational. •

Fano 2–66. [#K_{717}] $X = \mathscr{Z}(\mathcal{O}(0,1) \oplus \mathcal{O}(0,2) \oplus \mathcal{O}(1,1)) \subset \mathbb{P}^1 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 17,\ (-K)^4 = 50,\ \chi(T_X) = -38,\ h^{1,1} = 2,\ h^{3,1} = 2,\ h^{2,2} = 42.$

Description. X is the Fano fourfold A-69 [**BFMT21**, Table 8]. •

Fano 2–67. [#K_{474}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2}(0,1) \oplus \mathcal{O}(0,1)^{\oplus 2}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 40,\ (-K)^4 = 165,\ \chi(T_X) = -7,\ h^{1,1} = 2,\ h^{2,1} = 1,\ h^{3,1} = 0,\ h^{2,2} = 4.$

Description. $X = \mathrm{Bl}_E X_5^4$, with E a quintic elliptic curve.

Identification. We can rewrite

$$X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2}(0,1)) \subset \mathbb{P}^2 \times X_5^4.$$

By Lemma 2.2.9, this is

$$X = \mathrm{Bl}_C X_5^4,$$

where

$$C = \mathscr{Z}(\mathcal{O}(1)^{\oplus 3}) \subset X_5^4.$$

using the adjunction formula and the classification of curves, C is an elliptic curve E .

Since X is birational to X_5^4, it follows that X is rational. •

Fano 2–68. [#K_{483}] $\mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,6)} \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 3}) \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,6)$

Invariants. $h^0(-K) = 36,\ (-K)^4 = 146,\ \chi(T_X) = -11,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 10.$

Description. $X = \mathrm{Bl}_C X_6^4$ with C a rational curve.

Identification. Consider

$$X' = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,6)}) \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,6).$$

By Lemma 2.2.10 (3), outside a point p, we have

$$X' = \mathrm{Bl}_{\tilde{D}} \mathrm{Gr}(2,6) \text{ with } \tilde{D} = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^3} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,6)}) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,6).$$

The remaining sections cut $\mathrm{Gr}(2,6)$ by 4 hyperplanes, but $\tilde{D}$ is cut only by 3 of them, since $\mathscr{Z}(\mathcal{O}(1,0)) \supset \tilde{D}$.

Hence, the final description is

$$X = \mathrm{Bl}_{\tilde{D}\cap\mathscr{Z}(\mathcal{O}(0,1)^{\oplus 3})} X_6^4$$

where

$$C = \tilde{D} \cap \mathscr{Z}(\mathcal{O}(0,1)^{\oplus 3}) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,6).$$

The curve C is rational and

$$K_C = \mathcal{O}(-2,0)_{|C},$$

hence it is a Q_1.

Since X is birational to X_6^4, it follows that X is rational.

•

Fano 2–69. [# K_{529}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(0,1)) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 29,\ (-K)^4 = 110,\ \chi(T_X) = -19,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 17.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_5} Y$, with Y the Fano fourfold Fano 1–8

Identification. The projection $\pi : X \to \mathrm{Gr}(2,5)$ gives a birational map between X and

$$Y = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1)) \subset \mathrm{Gr}(2,5),$$

which is Fano 1–8.

To understand where the fiber of π degenerates, we use the combination of Proposition 2.2.1, Theorems 2.2.5 and 2.2.6. The exceptional locus is a surface

$$S = \mathscr{Z}(\mathcal{U}^\vee(1)) \subset \mathrm{Gr}(2,4)$$

which is a dP_5.

•

Fano 2–70. [# K_{714}] $X = \mathscr{Z}(\mathcal{O}(2,0) \oplus \mathcal{O}(0,2) \oplus \mathcal{O}(0,1)^{\oplus 2}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 24,\ (-K)^4 = 80,\ \chi(T_X) = -19,\ h^{1,1} = 2,\ h^{2,1} = 10,\ h^{3,1} = 0,\ h^{2,2} = 2.$

Description. $X = Q_1 \times Y$ with Y the Fano threefold 1-5 in [DBFT22, Table 1].

Identification. The bundles cut the two factors separately, so we have $X = Q_1 \times Y$, with $Y = \mathscr{Z}(\mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2))$. Recall that Y is the Fano threefold 1-5 in [DBFT22, Table 1]. •

Fano 2–71. [# K_{426}] $X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{Q}_{\mathrm{Gr}(3,5)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(0,1)) \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(3,5)$

Invariants. $h^0(-K) = 36,\ (-K)^4 = 147,\ \chi(T_X) = -13,\ h^{1,1} = 2,\ h^{3,1} = 0,\ h^{2,2} = 13.$

Description. $X = \mathrm{Bl}_{\mathrm{Bl}_{10\mathrm{pts}}\mathbb{P}?2} \mathrm{Gr}(2,4)$.

Identification. By applying Corollary 2.2.11 (1) and Lemma 2.2.12, the variety X can be rewritten as

$$\mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{O}_{\mathcal{R}}(1)) \subset \mathbb{P}_{\mathrm{Gr}(2,4)}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(-1) \oplus \mathcal{O}_{\mathrm{Gr}(2,4)}(-1)),$$

realizing X as a $\mathbb{P}^2$-bundle over $\mathrm{Gr}(2,4)$. The general fibers are cut by two relative hyperplane sections, hence the projection map to $\mathrm{Gr}(2,4)$ is a birational map between X and $\mathrm{Gr}(2,4)$.

To identify the locus where the fiber has higher dimensions we can apply Proposition 2.2.1. We study where the map

$$\varphi : \mathcal{U}_{\mathrm{Gr}(2,4)} \to \mathcal{Q}_{\mathrm{Gr}(2,4)}(-1) \oplus \mathcal{O}_{\mathrm{Gr}(2,4)}(-1)$$

drops rank to one, i.e., $\mathrm{D}_1(\varphi)$.

By Theorems 2.2.5 and 2.2.6, we find that $\mathrm{D}_1(\varphi)$ is a surface

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0)^{\oplus 2} \oplus \mathcal{O}(1,1)) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,4).$$

The surface S has the following invarinats: $e(S) = 13$, $K_S = \mathcal{O}(1,-1)_S$, $(-K_S)^2 = -1$, $h^{1,0} = h^{2,0} = 0$. Recall that $h^0(S, K_S^{\otimes 2}) = 0$, hence S is rational. Thus, by classification of surfaces S is $\mathrm{Bl}_{10\mathrm{pts}}\,\mathbb{P}^2$.

Since X is birational to $\mathrm{Gr}(2,4)$, it follows that X is rational. •

Fano 2–72. [#K_{662}] $\mathscr{Z}(\mathcal{U}^\vee(1,0) \oplus \mathcal{O}(0,1) \oplus \mathcal{O}(0,2)) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 22,\ (-K)^4 = 74,\ \chi(T_X) = -30,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 30.$

Description. X is the Fano fourfold GM-18 in [BFMT21, Table 5]. •

Fano 2–73. [#K_{671}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2}(0,1) \oplus \mathcal{Q}_{\mathbb{P}^2}(0,2)) \subset \mathbb{P}^2 \times \mathbb{P}^6$

Invariants. $h^0(-K) = 18,\ (-K)^4 = 55,\ \chi(T_X) = -40,\ h^{1,1} = 2,\ h^{3,1} = 3,\ h^{2,2} = 47.$

Description. X is a small resolution of Y, a fourfold with $e(Y) = 51$ and $\deg(Y) = 7$ singular in 8 points.

Identification. Let X be the zero locus of

$$\sigma \in V_3 \otimes (V_7^\vee \oplus \mathrm{Sym}^2 V_7^\vee),$$

and π denote the projection to $\mathbb{P}^6$.

Studying π is equivalent to determining the lines $l \in \mathbb{P}^2$ such that, for a fixed line $\lambda \in \mathbb{P}^6$, the relation $\sigma(l,\lambda) = 0$ holds.

By dimension count, this relation is generically empty; hence, the image of π coincides with the locus where the map

$$\varphi : \mathcal{O}_{\mathbb{P}^6}(-1) \oplus \mathcal{O}_{\mathbb{P}^6}(-2) \to \mathcal{O}_{\mathbb{P}^6}^{\oplus 3}$$

drops rank to 1, i.e. $\mathrm{D}_1(\varphi)$.

Applying Theorems 2.2.5 and 2.2.6, we obtain that the image of π is a fourfold Y with $e(Y) = 51$ and $\deg(Y) = 7$.

Since the general fiber over Y is a point, the projection π defines a birational map between X and Y.

Furthermore, the locus $\mathrm{D}_2(\varphi) \subset \mathrm{D}_1(\varphi)$ is non-empty, so Y is singular along $\mathrm{D}_2(\varphi)$. Using Theorems 2.2.5 and 2.2.6 again, we compute that $\mathrm{D}_2(\varphi)$ is a set of 8 points.

Consequently X is a small resolution of Y, with $\mathbb{P}^2$'s over the singular points.

Remark 2.5.1. *Note that the fourfold can be also directed described as the complete intersection between* $\mathrm{Bl}_{Q_2 \cap Q_2' \cap Q_2''}\,\mathbb{P}^6 \cap \mathrm{Bl}_{\mathbb{P}^3}\,\mathbb{P}^6$.

•

Fano 2–74. [#K_{562}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,6)}(0,1)) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,6)$

Invariants. $h^0(-K) = 27,\ (-K)^4 = 99,\ \chi(T_X) = -25,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 25$

Description. X is the Fano fourfold GM-20 in [BFMT21, Table 5]. •

2.6. Fano varieties with Picard rank 3

Fano 3–1. [#K_{503}] $X = \mathscr{Z}(Q_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(0,1,1) \oplus \mathcal{O}(2,0,0) \oplus \mathcal{O}(0,0,1)) \subset \mathbb{P}^2 \times \mathbb{P}^3 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 45,\ (-K)^4 = 192,\ \chi(T_X) = -2,\ h^{1,1} = 3,\ h^{2,1} = 1,\ h^{3,1} = 0,\ h^{2,2} = 4.$

Description. X is $Q_1 \times Y$ with Y the Fano threefold 2-17 in [DBFT22, Table 1].

Identification. Applying Lemma 2.2.10 (1) we can rewrite

$$X = \mathscr{Z}(\mathcal{O}(2,0,0) \oplus \mathcal{O}(0,1,1) \oplus \mathcal{O}(0,0,1)) \subset \mathbb{P}^2 \times \mathrm{Fl}(1,2,4).$$

We obtain $X = Q_1 \times Y$, where

$$Y = \mathscr{Z}(\mathcal{O}(1,1) \oplus \mathcal{O}(0,1)) \subset \mathrm{Fl}(1,2,4),$$

which is the Fano threefold 2-17 in [DBFT22, Table 1]. •

Fano 3–2. [#K_{20}] $X = \mathscr{Z}(Q_{\mathrm{Gr}(2,4)}(1,0,0) \oplus Q_{\mathrm{Gr}(2,4)}(0,1,0)) \subset \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 76,\ (-K)^4 = 358,\ \chi(T_X) = 9,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 5.$

Description. $X = \mathrm{Bl}_{\mathrm{Bl}_{2\mathrm{pts}}\mathbb{P}^2} \mathrm{Bl}_{\mathbb{P}^2} \mathrm{Gr}(2,4).$

Identification. Consider the projection

$$\pi_1 : X \to \mathbb{P}^2_2 \times \mathrm{Gr}(2,4).$$

This induces a birational map between X and

$$Y = \mathscr{Z}(Q_{\mathrm{Gr}(2,4)}(1,0)) \subset \mathbb{P}^2_2 \times \mathrm{Gr}(2,4).$$

Applying Proposition 2.2.1 and Theorem 2.2.5, we deduce that, along a surface S, the fiber of π_1 is $\mathbb{P}^1$.

By Theorem 2.2.6, we have that

$$S = \mathscr{Z}(\mathcal{O}(1,1)) \subset \mathbb{P}^1 \times \mathbb{P}^2.$$

Thus S is $\mathrm{Bl}_{\mathrm{pt}} \mathbb{P}^2$.

We analyze the projection

$$\pi_2 : Y \to \mathrm{Gr}(2,4)$$

This is a birational map between Y and $\mathrm{Gr}(2,4)$. Applying, again, the above combination of results, we finds that, along a surface S' the fiber of π_2 is a $\mathbb{P}^1$.

By the same argument, we have that $S' = \mathbb{P}^2$.

Hence, X can be described as

$$X = \mathrm{Bl}_{\mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^2} \mathrm{Bl}_{\mathbb{P}^2} \mathrm{Gr}(2,4),$$

from which it follows that X is rational. •

Fano 3–3. [#K_{521}] $X = \mathscr{Z}(\mathcal{O}(1,0,1) \oplus \mathcal{O}(0,1,1) \oplus Q_{\mathrm{Gr}(2,4)}(0,1,0)) \subset \mathbb{P}^1 \times \mathbb{P}^3 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 34,\ (-K)^4 = 136,\ \chi(T_X) = -12,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 14.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_3} Y$, with Y the Fano fourfold Fano 2–36.

Identification. Applying Lemma 2.2.9, we have that

$$X = \mathrm{Bl}_S\, Y$$

, where

$$Y = \mathscr{Z}(\mathcal{O}(1,1) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0)),$$

is the Fano 2–36 and

$$S = \mathscr{Z}(\mathcal{O}(0,1)^{\oplus 2}) \subset Y.$$

The surface S has the following invariants: $e(S) = 9$, $K_S = \mathcal{O}(-1,0)_{|S}$, $(-K_S)^2 = 3$, $h^{1,0} = h^{2,0} = 0$ and $h^0(S, 2K_S) = 0$. By the classification of surfaces S is a dP_3.

Hence, $X = \mathrm{Bl}_{\mathrm{dP}_3} Y$, from which it follows that X is rational. •

Fano 3–4. [#K_{524}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(0,0,2) \oplus \mathcal{O}(1,1,0) \subset \mathbb{P}^1 \times \mathbb{P}^3 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 34$, $(-K)^4 = 136$, $\chi(T_X) = -8$, $h^{1,1} = 3$, $h^{2,1} = 2$, $h^{3,1} = 0$, $h^{2,2} = 8$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_4} Y$ with Y the Fano 2–48.

Identification. Applying Lemma 2.2.9, we have that

$$X = \mathrm{Bl}_S\, Y$$

where

$$Y = \mathscr{Z}(\mathcal{O}(0,2) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0))$$

is Fano 2–48, and

$$S = \mathscr{Z}(\mathcal{O}(1,0)^{\oplus 2}) \subset Y.$$

The surface S has invariants: $e(S) = 8$, $K_S = \mathcal{O}(0,-1)_{|S}$, $(-K_S)^2 = 4$, $h^{1,0} = h^{2,0} = 0$ and $h^0(S, 2K_S) = 0$. By the classification of surfaces S is a dP_4. •

Fano 3–5. [#K_{21}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^4} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,7)}(1,0,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,7)}(0,1,0)) \subset \mathbb{P}^5 \times \mathbb{P}^4 \times \mathrm{Gr}(2,7)$

Invariants. $h^0(-K) = 74$, $(-K)^4 = 347$, $\chi(T_X) = 8$, $h^{1,1} = 3$, $h^{3,1} = 0$, $h^{2,2} = 5$

Description. $X = \mathrm{Bl}_{Q_2} Y$, with Y the fourfold Fano 2–62.

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^4} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,7)}(1,0)) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,7)$$

is the fourfold Fano 2–62.

Hence

$$X = \mathscr{Z}(\mathcal{O}(1,0,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,7)}(1,0,0)) \subset \mathbb{P}^6 \times Y.$$

Applying Theorem 2.2.8, we can rewrite X as

$$X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(1)) \subset \mathbb{P}_Y(\mathcal{U}_{\mathrm{Gr}(2,7)|Y}),$$

so that X is birational to Y via the natural projection

$$\pi : \mathbb{P}_Y(\mathcal{U}_{\mathrm{Gr}(2,7)|Y}) \to Y$$

onto the base of the projective bundle.

To determine where the fiber of π has higher dimension we apply Theorems 2.2.5 and 2.2.6. This yields a surface S over which the fiber is a $\mathbb{P}^1$. Moreover, S can be described as

$$S = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^4} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,6)}(1,0)) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,6).$$

The invariants of S are: $e(S) = 4$, $K_S = \mathcal{O}(-2,0)_{|S}$, $(-K_S)^2 = 8$, $h^{1,0} = h^{2,0} = 0$ and $h^0(S, 2K_S) = 0$. By the classification of surfaces S is a dP_8, and since the degree is two it is a quadric surface Q_2.

Since Y is rational, it follows that X is also rational. •

Fano 3–6. [# K_{526}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2_1}(0,1,1) \oplus \mathcal{O}(1,0,1)) \subset \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathbb{P}^3$

Invariants. $h^0(-K) = 30,\ (-K)^4 = 116,\ \chi(T_X) = -15,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 17$

Description. $X = \mathrm{Bl}_{13\mathrm{pts}}\, \mathbb{P}^2$.

Identification. The variety X is birational to $\mathbb{P}^2 \times \mathbb{P}^2$ via the projection

$$\pi : X \to \mathbb{P}^2 \times \mathbb{P}^2.$$

To determine the exceptional locus, we consider the first degeneracy locus of the map

$$\varphi : V_4 \to \mathcal{Q}_{\mathbb{P}^2_1}(0,1) \oplus \mathcal{O}(1,0)$$

Applying Proposition 2.2.1, Theorems 2.2.5 and 2.2.6 we obtain that

$$\mathrm{D}_2(\varphi) = \mathrm{Bl}_{13\mathrm{pts}}\, \mathbb{P}^2.$$

•

Fano 3–7. [# K_{555}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(0,0,1) \oplus \mathcal{O}(1,1,1)) \subset \mathbb{P}^1 \times \mathbb{P}^3 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 31,\ (-K)^4 = 120,\ \chi(T_X) = -18,\ h^{1,1} = 3,\ h^{3,1} = 1,\ h^{2,2} = 22$

Description. X is the Fano fourfold K3-48 in [**BFMT21**, Table 6]. •

Fano 3–8. [# K_{623}] $X = \mathscr{Z}(\mathcal{O}(0,0,1)^{\oplus 2} \oplus \mathcal{O}(1,0,1) \oplus \mathcal{O}(0,1,1)) \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 28,\ (-K)^4 = 105,\ \chi(T_X) = -16,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 17$

Description. $X = \mathrm{Bl}_{\mathrm{Bl}_{9\mathrm{pts}}\mathbb{P}^2}\, Y$ with Y the fourfold Fano 2–55.

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{O}(1,1)) \subset \mathbb{P}^1 \times X^4_5$$

is the fourfold Fano 2–55. We rewrite

$$X = \mathscr{Z}(\mathcal{O}(1,0,1)) \subset \mathbb{P}^1 \times Y.$$

Applying Lemma 2.2.9, we obtain

$$X = \mathrm{Bl}_S\, Y$$

where S is the surface

$$S = \mathscr{Z}(\mathcal{O}(0,1)^{\oplus 2}) \subset Y.$$

The invariants of S $e(S) = 12$, $K_S = \mathcal{O}(-1,0)_{|S}$, $(-K_S)^2 = 0$, $h^{1,0} = h^{2,0} = 0$. Moreover, since $h^0(S, K_S^{\otimes 2}) = 0$, it follows that S is rational. By the classification of surfaces, we conclude that $S = \mathrm{Bl}_{9\mathrm{pts}}\, \mathbb{P}^2$. •

Fano 3–9. [#K_{294}] $X = \mathscr{Z}(\mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)}(1,0,0) \oplus \mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)} \boxtimes \mathcal{Q}_{\mathrm{Gr}(2,4)} \oplus \mathcal{O}(0,1,0) \oplus \mathcal{O}(0,0,1)) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,5) \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 54,\ (-K)^4 = 241,\ \chi(T_X) = 1,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 6$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_7} Y$, with Y the fourfold Fano 2–28

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)} \boxtimes \mathcal{Q}_{\mathrm{Gr}(2,4)} \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(0,1)) \subset \mathrm{Gr}(2,5)\,\mathrm{Gr}(2,4)$$

is the fourfold Fano 2–28. Hence

$$X = \mathscr{Z}(\mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)}(1,0,0)) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,5).$$

The projection

$$\pi : X \to Y$$

is birational. Applying Proposition 2.2.1, we get that the fiber of π degenerates where the map

$$\varphi : \mathcal{U}_{\mathrm{Gr}(2,5)|Y} \to \mathcal{O}_Y^{\oplus 3}$$

drops rank to 1 map, that is, along $\mathrm{D}_1(\varphi)$.

By Theorems 2.2.5 and 2.2.6, we find that $\mathrm{D}_1(\varphi)$ is a surface

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)|Y}(1,0,0)) \subset \mathbb{P}^1 \times Y.$$

The invariant of S are: $e(S) = 5$, $K_S = \mathcal{O}(1,-1,-1)_{|S}$, $(-K_S)^2 = 7$, $h^{1,0} = h^{2,0} = 0$ and $h^0(S, K_S^{\otimes 2}) = 0$. By the classification of surfaces $S = \mathrm{dP}_7$.

Since $X = \mathrm{Bl}_{\mathrm{dP}_7} Y$, it follows that X is rational. •

Fano 3–10. [#K_{637}] $X = \mathscr{Z}(\mathcal{O}(0,0,1)^{\oplus 3} \oplus \mathcal{O}(1,1,1)) \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 27,\ (-K)^4 = 100,\ \chi(T_X) = -18,\ h^{1,1} = 3,\ h^{3,1} = 1,\ h^{2,2} = 22$

Description. X is the Fano fourfold K3-38 in [**BFMT21**, Table 6]. •

Fano 3–11. [#K_{158}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,6)} \oplus \mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,6)}) \subset \mathrm{Gr}(2,5) \times \mathrm{Gr}(2,6)$

Invariants. $h^0(-K) = 60,\ (-K)^4 = 274,\ \chi(T_X) = 2,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 8$

Description. X is the small contraction of Y, a fourfold singular in 4 points with $e(Y) = 12$ and $\deg(Y) = 10$.

Identification. The variety X is the zero locus of

$$\sigma \in V_6^{\vee} \otimes (V_5 \oplus V_5^{\vee}),$$

and, upon projection to $\mathrm{Gr}(2,5)$, we may describe X as the planes $\Pi_1 \in \mathrm{Gr}(2,6)$ such that, for fixed a plane $\Pi_2 \in \mathrm{Gr}(2,5)$, the relation $\sigma(\Pi_1, \Pi_2) = 0$ holds.

By Proposition 2.2.1, we have that X is birational to the locus where the map

$$\varphi : \mathcal{Q}^{\vee}_{\mathrm{Gr}(2,5)} \oplus \mathcal{U}_{\mathrm{Gr}(2,5)} \to \mathcal{O}^{\oplus 6}_{\mathrm{Gr}(2,5)}$$

drops to rank 4 map; hence X is birational to $Y = \mathrm{D}_1(\varphi)$.

Applying Theorems 2.2.5 and 2.2.6, Y is a fourfold with $e(Y) = 12$ and $\deg(Y) = 10$.

Moreover, since there are points in Y where the map φ drops to rank 3 map, Y is singular along $\mathrm{D}_2(\varphi)$. By Theorems 2.2.5 and 2.2.6, $\mathrm{D}_2(\varphi)$ is a set of 4 points.

Over each of these four points, the fiber of the projection is a $\mathbb{P}^2$. Therefore, X is a small resolution of Y along its singular locus. •

Fano 3–12. [#K_{398}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{O}(0,1) \oplus \mathrm{Sym}^2\, \mathcal{U}^\vee_{\mathrm{Gr}(2,7)}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,7)$

Invariants. $h^0(-K) = 39,\ (-K)^4 = 164,\ \chi(T_X) = -7,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 12.$

Description. X is a small resolution of Y, a fourfold singular in 4 points with $e(Y) = 16$ and $\deg(Y) = 36$.

Identification. The variety X is the zero locus of

$$\sigma \in V_3 \otimes V_7^\vee \text{ and } \alpha \in \bigwedge^2 V_7^\vee \oplus \mathrm{Sym}^2\, V_7^\vee.$$

Consider

$$X' = \mathscr{Z}(\sigma)$$

and then restrict to

$$X = X' \cap \mathscr{Z}(\alpha).$$

Projecting to $\mathrm{Gr}(2,7)$, the fiber consists of lines $l \in \mathbb{P}^2$ such that, for fixed a plane $\Pi \in \mathrm{Gr}(2,7)$, the relation $\sigma(l, \Pi) = 0$ holds.

By Theorem 2.2.6, in order for the fiber to be non-empty, we have Π must lie in the locus where

$$\varphi : \mathcal{U}_{\mathrm{Gr}(2,7)} \to \mathcal{O}^{\oplus 3}_{\mathrm{Gr}(2,7)}$$

drops to rank 1. Since $X = X' \cap \mathscr{Z}(\alpha)$, we restrict φ on $\mathscr{Z}(\alpha)$, obtaining that X is birational to

$$Y = \mathrm{D}_1(\varphi_{|\mathscr{Z}(\alpha)}).$$

By Theorems 2.2.5 and 2.2.6, Y is a fourfold with $e(Y) = 16$ and $\deg(Y) = 36$. Furthermore, there are points in Y where φ drops to rank 0, so Y is singular along $\mathrm{D}_0(\varphi_{\mathscr{Z}(\alpha)})$. Again, by Theorems 2.2.5 and 2.2.6, $\mathrm{D}_0(\varphi_{\mathscr{Z}(\alpha)})$ is a set of 4 points. Over each of these four points, the fiber of the projection is a $\mathbb{P}^2$. Hence, X is a small resolution of Y.

The projection

$$\pi_1 : X \to \mathbb{P}^2$$

is a Q_2 fibration. •

Fano 3–13. [#K_{431}] $X = \mathscr{Z}(\mathcal{R}_1(1,0,0) \oplus \mathcal{O}(0,1,1)) \subset \mathbb{P}^2 \times \mathrm{Fl}(1,3,4)$

Invariants. $h^0(-K) = 36,\ (-K)^4 = 148,\ \chi(T_X) = -8,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 12.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_4} Y$ with Y the Fano fourfold obtained as a linear section in $\mathrm{Fl}(1,3,4)$.

Identification. Note that the projection

$$\pi : X \to \mathrm{Fl}(1,3,4)$$

induces a birational map between X and

$$Y = \mathscr{Z}(\mathcal{O}(1,1)) \subset \mathrm{Fl}(1,3,4),$$

since the general fiber is a point.

By Lemma 2.2.10 (1) and, upon dualizing $\mathrm{Gr}(3,4)$, we obtain

$$Y = \mathscr{Z}(\mathcal{O}(1,1)^{\oplus 2}) \subset \mathbb{P}^3 \times \mathbb{P}^3.$$

By [FMMR23, Lemma 1. 1], this is a $\mathbb{P}^1$-bundle over $\mathbb{P}^3$, whose fiber jumps to $\mathbb{P}^2$ over four points. This fourfold appears in MW_7^4 in [CGKS20].

To study the exceptional locus Δ of the projection π, we proceed as usual, applying Proposition 2.2.1, Theorems 2.2.5 and 2.2.6. In this way, we find that Δ is the surface

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,6)}(1,0) \oplus \mathcal{O}(0,1) \oplus \mathrm{Sym}^2\, \mathcal{U}^\vee_{\mathrm{Gr}(2,6)}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,6).$$

Its invariants are $e(S) = 8$, $K_S = \mathcal{O}(1,-1)_{|S}$, $(-K_S)^2 = 4$, $h^{1,0} = h^{2,0} = 0$ and $h^0(S, K_S^{\otimes 2}) = 0$. By the classification of surfaces $S = \mathrm{dP}_4$. Hence $X = \mathrm{Bl}_{\mathrm{dP}_4} Y$.

Since Y is rational, X is also rational.

The original model of this Fano is

$$\mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,6)}(1,0) \oplus \mathcal{O}(0,1) \oplus \mathrm{Sym}^2\, \mathcal{U}^\vee_{\mathrm{Gr}(2,6)}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,6),$$

but using the identification between

$$\mathrm{Fl}(1,3,4) = \mathscr{Z}(\mathrm{Sym}^2\, \mathcal{U}^\vee) \subset \mathrm{Gr}(2,6)$$

we obtain the above description.

•

Fano 3–14. [#K_{76}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^4} \boxtimes \mathcal{U}_{\mathrm{Gr}(2,7)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,4)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0)) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,7) \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 72,\ (-K)^4 = 337,\ \chi(T_X) = 8,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 5.$

Description. X is a small resolution of Y, a fourfold singular in a point with $e(Y) = 12$.

Identification. By Corollary 2.2.11 (1), we can rewrite

$$X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \boxtimes \mathcal{Q}_{\mathcal{R}} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,4)}) \subset \mathrm{Gr}(2,7) \times \mathbb{P}_{\mathrm{Gr}(2,4)}(\mathcal{U}_{\mathrm{Gr}(2,4)} \oplus \mathcal{O}_{\mathrm{Gr}(2,4)}).$$

By Proposition 2.2.1, X is birational to the first degeneracy locus of the map

$$\varphi : p^*\mathcal{U}_{\mathrm{Gr}(2,4)} \oplus \mathcal{Q}^\vee_{\mathcal{R}} \to \mathcal{O}_{\mathcal{R}}^{\oplus 7}$$

where p is

$$p : Z = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{O}_{\mathcal{R}}(1)) \subset \mathbb{P}_{\mathrm{Gr}(2,4)}(\mathcal{U}_{\mathrm{Gr}(2,4)} \oplus \mathcal{O}_{\mathrm{Gr}(2,4)}) \to \mathrm{Gr}(2,4)$$

Hence X is birational to $Y = \mathrm{D}_3(\varphi_{|Z})$.

Applying Proposition 2.2.1, Theorems 2.2.5 and 2.2.6, we find that Y is a fourfold with $e(12)$.

Moreover, since $\mathrm{D}_2(\varphi_{|Z}) = \{1 \text{ point}\}$, Y is singular at a single point. The variety X is a small resolution of Y, with the exceptional fiber a $\mathbb{P}^2$.

The projection to $\mathbb{P}^4$ gives a birational map to X, so X is rational. However, even though X is birational to $\mathbb{P}^4$, it be cannot recovered simply as an iterated blow-up of $\mathbb{P}^4$. •

Fano 3–15. [#K_{116}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,0,1) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{O}(0,1,0)^{\oplus 2}) \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5) \times \mathbb{P}^2$

Invariants. $h^0(-K) = 57,\ (-K)^4 = 257,\ \chi(T_X) = 1,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 7.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_7} Y$ with Y the fourfold Fano 2–11.

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)} \oplus \mathcal{O}(0,1)^{\oplus 2}) \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$$

is the fourfold Fano 2–11. Then

$$X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)|Y}(1,0)) \subset \mathbb{P}^2 \times Y.$$

In particular the projection

$$\pi : X \to Y$$

is birational.

The invariants of the exceptional locus of π, Δ, can be computed using Proposition 2.2.1, Theorems 2.2.5 and 2.2.6. We finds that Δ is a surface

$$S = \mathscr{Z}(\mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 2}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,5),$$

with invariants $e(S) = 5$, $K_S = \mathcal{O}(-1,-2)_S$, $(-K_S)^2 = 7$, $h^{1,0} = h^{2,0} = 0$. Hence, by the classification of surfaces, we find that $S = \mathrm{dP}_7$.

Therefore, $X = \mathrm{Bl}_{\mathrm{dP}_7} Y$. Since Y is rational, it follows that X is rational as well. •

Fano 3–16. [# K_{125}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0)^{\oplus 2}) \subset \mathbb{P}^2 \times \mathbb{P}^4 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 48$, $(-K)^4 = 211$, $\chi(T_X) = 0$, $h^{1,1} = 3$, $h^{3,1} = 0$, $h^{2,2} = 7$.

Description. $X = \mathrm{Bl}_{\mathrm{dP}_6} Y$, with Y the fourfold Fano 2–21

Identification. Consider the projection

$$\pi : X \to \mathbb{P}^4 \times \mathrm{Gr}(2,4).$$

Then X is birational to

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0)^{\oplus 2}) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,4),$$

which is the fourfold Fano 2–21.

To determine where the fiber degenerates to a $\mathbb{P}^1$ we apply Proposition 2.2.1. The invariants of the exceptional locus Δ can be computed using Theorems 2.2.5 and 2.2.6.

We find that Δ is a surface S with invariants $e(S) = 6$, $(-K_S)^2 = 6$, $h^{1,0} = h^{2,0} = 0$ and $h^0(S, K_S^{\otimes 2}) = 0$. By the classification of surfaces, S is a dP_6.

Therefore $X = \mathrm{Bl}_{\mathrm{dP}_6} Y$. Since Y is rational, it follows that X is rstional. •

Fano 3–17. [# K_{48}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,11)} \oplus (\mathcal{Q}_{\mathbb{P}^4} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,11)})^{\oplus 2}) \subset \mathbb{P}^2 \times \mathbb{P}^4 \times \mathrm{Gr}(2,11)$

Invariants. $h^0(-K) = 90$, $(-K)^4 = 433$, $\chi(T_X) = 14$, $h^{1,1} = 3$, $h^{3,1} = 0$, $h^{2,2} = 5$.

Description. X is a small resolution of Y, a fourfold singular in a point with $e(Y) = 12$.

Identification. Let us first consider

$$\mathscr{Z}((\mathcal{Q}_{\mathbb{P}^4} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,11)})^{\oplus 2}) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,11).$$

We can rewrite this as

$$Z = \mathscr{Z}(\mathcal{U}^{\vee}_{\mathcal{R}} \boxtimes (\mathcal{Q}^{\oplus 2}_{\mathbb{P}^4})) \subset \mathrm{Gr}_{\mathbb{P}^4}(2, \mathcal{O}^{\oplus 5}_{\mathbb{P}^4} \oplus \mathcal{O}^{\oplus 5}_{\mathbb{P}^4} \oplus \mathcal{O}_{\mathbb{P}^4}).$$

Using Theorem 2.2.8 backwards, we obtain

$$Z = \mathrm{Gr}_{\mathbb{P}^4}(2, \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^4}),$$

hence

$$X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}_{\mathcal{R}}^{\vee}) \subset \mathbb{P}^2 \times \mathrm{Gr}_{\mathbb{P}^4}(2, \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^4}).$$

Twisting the bundle $\mathcal{O}(-1)^{\oplus 2} \oplus \mathcal{O}$ by $\mathcal{O}(1)$, we get

$$X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}_{\mathcal{R}}^{\vee}(1,0)) \subset \mathbb{P}^2 \times \mathrm{Gr}_{\mathbb{P}^4}(2, \mathcal{O}_{\mathbb{P}^4}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^4}(1)).$$

Dualizing the Grassmann bundle, we obtain:

$$X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{Q}_{\mathcal{R}}(1,0)) \subset \mathbb{P}^2 \times \mathbb{P}_{\mathbb{P}^4}(\mathcal{O}(-1) \oplus \mathcal{O}^{\oplus 2})$$

Iterating twice Lemma 2.2.12, we finally get

$$X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{Q}_{\mathcal{R}}(1,0) \oplus \mathcal{Q}_{\mathrm{Gr}(3,7)}^{\oplus 2}) \subset \mathbb{P}^2 \times \mathrm{Gr}(3,7)$$

and using Borel–Bott–Weil Theorem we finally get

$$\mathscr{Z}(\mathcal{Q}_{\mathbb{P}^4} \boxtimes \mathcal{R}_1 \oplus \mathcal{Q}_2^{\oplus 2}) \subset \mathbb{P}^2 \times \mathrm{Fl}(1,3,7)$$

By Proposition 2.2.1, X can be described as the restriction to

$$Y' = \mathscr{Z}(\mathcal{Q}_2^{\oplus 2}) \subset \mathrm{Fl}(1,3,7)$$

of the map

$$\phi : \mathcal{R}_1^{\vee} \to \mathcal{O}_{\mathrm{Fl}(1,3,7)}^{\oplus 3}$$

giving

$$Y = \mathrm{D}_1(\phi_{|Y'})$$

a fourfold with $e(Y) = 12$. Since $\mathrm{D}_0(\phi_{|Y'}) = \{1pt\}$, Y is singular. Hence X which is a small resolution of Y.

•

Fano 3–18. [# K_{188}] $X = \mathscr{Z}(\mathcal{O}(1,0,1) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(0,0,1)) \subset \mathbb{P}^1 \times \mathbb{P}^3 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 60,\ (-K)^4 = 272,\ \chi(T_X) = 4,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 4.$

Description. $X = \mathrm{Bl}_{Q_2} Y$ with Y the fourfold Fano 2–43.

Identification. Note that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(0,1)) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,4)$$

is the fourfold Fano 2–43. Then

$$X = \mathscr{Z}(\mathcal{O}(1,0,1)) \subset \mathbb{P}^1 \times Y.$$

Applying Lemma 2.2.9, we obtain $X = \mathrm{Bl}_S Y$, where

$$S = \mathscr{Z}(\mathcal{O}(0,1)^{\oplus 2}) \subset Y.$$

Equivalently, S can be described as

$$S = \mathbb{P}_{Q_3 \cap H \cap H'}(\mathcal{U}_{\mathrm{Gr}(2,4)})$$

which is a dP_8.

Since S is birational to $\mathbb{P}^1 \times \mathbb{P}^1$, it is identified with a quadric surface Q_2.

Finally, since Y is rational X as well. •

Fano 3–19. [# K_{190}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}_{\mathrm{Gr}(2,6)}^{\vee} \oplus \mathcal{Q}_{\mathrm{Gr}(2,6)}(1,0,0) \oplus \mathcal{O}(0,0,1)^{\oplus 2}) \subset \mathbb{P}^4 \times \mathbb{P}^2 \times \mathrm{Gr}(2,6)$

Invariants. $h^0(-K) = 57,\ (-K)^4 = 256,\ \chi(T_X) = 1,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 6.$

Description. $X = \mathrm{Bl}_{\mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^2} Y$ with Y the fourfold Fano 2–51.

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \oplus \mathcal{O}(0,1)^{\oplus 2}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,6)$$

is the fourfold Fano 2–51. Then

$$X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,6)}(1,0,0)) \subset \mathbb{P}^4 \times Y.$$

The projection

$$\pi : X \to Y$$

is birational.

To understand its exceptional locus, we apply Proposition 2.2.1, Theorems 2.2.5 and 2.2.6. This shows π has fiber a $\mathbb{P}^1$ over a surface

$$S = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \oplus \mathcal{O}(0,1)^{\oplus 2}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,5).$$

The invariants of S are: $e(S) = 4$, $K_S = \mathcal{O}(-1,-1)_{|S}$ and $(-K_S)^2 = 8$. Hence S is a dP_8. Moreover, S is birational to $\mathbb{P}^2$, hence it is a $\mathrm{Bl}_{\mathrm{pt}}\,\mathbb{P}^2$. •

Fano 3–20. $[\#K_{191}]$ $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2_1} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{Q}_{\mathbb{P}^2_2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{O}(0,0,1)^{\oplus 2}) \subset \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathrm{Gr}(2,7)$

Invariants. $h^0(-K) = 54,\ (-K)^4 = 241,\ \chi(T_X) = -2,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 9.$

Description. X is a small resolution of Y, a fourfold singular in 2 points with $e(Y) = 15$ and $\deg(Y) = 19$.

Identification. Consider

$$Y' = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2_1} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{O}(0,1)^{\oplus 2}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,7).$$

By Proposition 2.2.1, Y' can be described as $Y' = \mathrm{D}_1(\varphi'_{|Z})$ where

$$\varphi' : \mathcal{U}_{\mathrm{Gr}(2,7)} \to \mathcal{O}^{\oplus 3}_{\mathrm{Gr}(2,7)} \text{ and} Z = \mathscr{Z}(\mathcal{O}(1)^{\oplus 2}) \subset \mathrm{Gr}(2,7).$$

Applying Theorems 2.2.5 and 2.2.6, we compute that the further degeneracy locus is a surface $S = \mathrm{D}_0(\varphi'_{|Z})$. Hence Y' is singular along S.

Consider, now, the second $\mathbb{P}^2$, and the map

$$\varphi : \mathcal{U}_{\mathrm{Gr}(2,7)} \to \mathcal{O}^{\oplus 3}_{\mathrm{Gr}(2,7)}$$

restricted to Y'. Then, using Theorems 2.2.5 and 2.2.6 we obtain

$$Y = \mathrm{D}_1(\phi_{|Y'}) \subset Y$$

which is a singular fourfold with $e(Y) = 15$, $\deg(Y) = 19$.

The singular locus of Y consists of two points:

$$\mathrm{D}_0(\phi_{|Y'}) = \{1 \text{ point}\}, \text{ and } Y \cap S = \{1 \text{ point}\},$$

which are generically distinct. Over each of these points, the fiber of the projection to $\mathrm{Gr}(2,7)$ is a $\mathbb{P}^2$.

Hence X, is obtained as a small resolution of Y. •

Fano 3–21. [#K_{173}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \boxtimes \mathcal{Q}_{\mathrm{Gr}(2,5)} \oplus \mathcal{O}(0,1,0)^{\oplus 2}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,6) \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 65,\ (-K)^4 = 298,\ \chi(T_X) = 4,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 5.$

Description. $X = \mathrm{Bl}_{\mathbb{P}^2} Y$, with Y the fourfold Fano 2–51.

Identification. Consider

$$X' = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \boxtimes \mathcal{Q}_{\mathrm{Gr}(2,5)} \oplus \mathcal{O}(1,0)^{\oplus 2}) \subset \mathrm{Gr}(2,6) \times \mathrm{Gr}(2,5).$$

By Lemma 2.2.10 (2) and using the fact that the 2 hyperplane sections cut $\mathrm{Gr}(2,6)$ generically, we obtain

$$X' = \mathrm{Bl}_{\mathbb{P}^2} X^6_6,$$

where

$$\mathbb{P}^2 = \mathscr{Z}(\mathcal{Q} \oplus \mathcal{O}(1)^{\oplus 2}) \subset \mathrm{Gr}(2,6).$$

Consequently

$$X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}) \subset \mathbb{P}^2 \times X'.$$

The section $\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}$ in X' cuts X^6_6 and contains $\mathbb{P}^2$, hence $X = \mathrm{Bl}_{\mathbb{P}^2} Y$, where

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)}) \subset \mathbb{P}^2 \times X^6_6,$$

is the fourfold Fano 2–51. •

Fano 3–22. [#K_{178}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(0,0,1) \oplus \mathcal{O}(1,1,0)) \subset \mathbb{P}^1 \times \mathbb{P}^3 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 66,\ (-K)^4 = 304,\ \chi(T_X) = 6,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 4.$

Description. $X = \mathrm{Bl}_{Q_2} Y$, with Y the fourfold Fano 2–43

Identification. Recall that

$$Y = \mathscr{Z}((\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(0,1)) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,4),$$

which is the fourfold Fano 2–43. Hence

$$X = \mathscr{Z}(\mathcal{O}(1,1,0)) \subset \mathbb{P}^1 \times Y.$$

Applying Lemma 2.2.9, we obtain $X = \mathrm{Bl}_S Y$, where

$$S = \mathscr{Z}(\mathcal{O}(1,0)^{\oplus 2}) \subset Y.$$

In this case $S = Q_1 \times \mathbb{P}^1$, i.e. Q_2.

Since Y is rational, it follows that X is rational as well. •

Fano 3–23. [#K_{211}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,5)}(0,1,0) \oplus \mathcal{O}(0,0,1)^{\oplus 2}) \subset \mathbb{P}^3 \times \mathbb{P}^3 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 54,\ (-K)^4 = 241,\ \chi(T_X) = 0,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 7.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_7} Y$ with Y the fourfold Fano 2–29.

Identification. Note that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 2}) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,5)$$

is the fourfold Fano 2–29. Hence

$$X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0,0)) \subset \mathbb{P}^3 \times Y.$$

The variety X is birational to Y via the projection

$$\pi : X \to Y.$$

To determine the exceptional locus Δ of π, we use Proposition 2.2.1, Theorems 2.2.5 and 2.2.6, and obtain that Δ is a surface

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 2}) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,4).$$

We compute $e(S) = 5$, $K_S = \mathcal{O}(-2,-2)_{|S}$, $(-K_S)^2 = 7$, $h^{1,0} = h^{2,0} = 0$ and $h^0(S, K_S^{\otimes 2}) = 0$. By the classification of surfaces, S is a dP_7.

Since $X = \mathrm{Bl}_{\mathrm{dP}_7} Y$, and Y is rational, we conclude that X is rational as well. •

Fano 3–24. [#K_{212}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0,0) \oplus \mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)}(1,0,0)) \subset \mathbb{P}^5 \times \mathbb{P}^2 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 53,\ (-K)^4 = 235,\ \chi(T_X) = -1,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 7.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_6} Y$, with Y the Fano 6 in [Kal19, Table 3]

Identification. See Example 2.3.3. •

Fano 3–25. [#K_{202}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0,0) \oplus \mathcal{O}(0,1,0) \oplus \mathcal{O}(0,0,1)) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 59,\ (-K)^4 = 267,\ \chi(T_X) = 3,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 5.$

Description. $X = \mathrm{Bl}_{Q_2} Y$ with Y the fourfold Fano 2–28.

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)} \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(0,1)) \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5),$$

which is the fourfold Fano 2–28. Hence

$$X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0)) \subset \mathbb{P}^3 \times Y,$$

so X is birational to Y via the projection

$$\pi : X \to Y.$$

Applying Proposition 2.2.1, Theorems 2.2.5 and 2.2.6 we compute some invariants of the exceptional locus of π, which is a surface

$$S = \mathscr{Z}(\mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)}) \subset Y.$$

In particular, we find that $e(S) = 4$, $K_S = \mathcal{O}(-1,-1)_{|S}$, $(-K_S)^2 = 8$ and $h^0(S, K_S^{\otimes 2}) = 0$. Hence S is a dP_8. Moreover, since S is birational to $\mathbb{P}^1 \times \mathbb{P}^1$, it is Q_2.

Since X is birational to Y, we conclude that X is rational. •

Fano 3–26. [#K_{232}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^3}(0,1,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(0,0,1)) \subset \mathbb{P}^3 \times \mathbb{P}^5 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 52,\ (-K)^4 = 230,\ \chi(T_X) = 0,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 6.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_6} Y$ with Y the fourfold Fano 2–43

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(0,1)) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,4),$$

which is the fourfold Fano 2–43. If we instead write

$$Y = \mathscr{Z}(\mathcal{O}(0,1)) \subset \mathrm{Fl}(1,2,4),$$

then

$$X = \mathscr{Z}(\mathcal{U}_2(1,0,0) \oplus \mathcal{Q}_1(1,0,0)) \subset \mathbb{P}^5 \times Y.$$

In particular, the projection

$$\pi : X \to Y$$

is birational.

To determine where the fiber of π degenerates to a $\mathbb{P}^1$, we apply Proposition 2.2.1, Theorems 2.2.5 and 2.2.6. In this way, we find that the exceptional locus of π is a surface

$$S = \mathscr{Z}(\mathcal{Q}_2(1,0,0) \oplus \mathcal{U}_1^\vee(1,0,0)) \subset \mathbb{P}^1 \times Y.$$

We compute the invariants: $e(S) = 6$, $K_S = \mathcal{O}(1,-1,-1)_{|S}$, $(-K_S)^2 = 6$ and $h^0(S, K_S^{\otimes 2}) = 0$. Hence, by classification, S is a dP_6.

It follows that $X = \mathrm{Bl}_{\mathrm{dP}_6} Y$, which also implies the rationality of X.

•

Fano 3–27. [#K_{231}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^4} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,7)}(0,1,0) \oplus \mathcal{O}(1,0,1)) \subset \mathbb{P}^1 \times \mathbb{P}^4 \times \mathrm{Gr}(2,7)$

Invariants. $h^0(-K) = 51,\ (-K)^4 = 225,\ \chi(T_X) = -3,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 9.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_4} Y$, with Y the fourfold Fano 2–62.

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^4} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,7)}(1,0)) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,7)$$

which is the fourfold Fano 2–62. Hence

$$X = \mathscr{Z}(\mathcal{O}(1,0,1)) \subset \mathbb{P}^1 \times Y.$$

Applying Lemma 2.2.9, we obtain that $X = \mathrm{Bl}_S Y$, where

$$S = \mathscr{Z}(\mathcal{O}(0,1)^{\oplus 2}) \subset Y.$$

We compute: $e(S) = 8$, $K_S = \mathcal{O}(1,-1,-1)_{|S}$, $(-K_S)^2 = 4$. Hence S is a dP_4.

Since X is birational to Y, it follows that X is rational as well. •

Fano 3–28. [#K_{235}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2}(0,1,0) \oplus \mathcal{Q}_{\mathbb{P}^2}(0,0,1) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0)) \subset \mathbb{P}^2 \times \mathbb{P}^4 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 46,\ (-K)^4 = 199,\ \chi(T_X) = -3,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 8.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_6} \mathrm{Bl}_{Q_3 \cap Q_3' \cap Q_3''} \mathrm{Gr}(2,4).$

Identification. Consider the projection

$$\pi_{13} : X \to \mathbb{P}^2 \times \mathrm{Gr}(2,4),$$

we get that the image $\pi_{13}(X)$ is

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2}(0,1)) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,4).$$

By Lemma 2.2.9, we obtain that

$$Y = \mathrm{Bl}_{Q_3 \cap Q_3' \cap Q_3''} \mathrm{Gr}(2,4).$$

Since the general fiber of π_{13} over its image is one point, we conclude that X is birational to Y.

To determine the exceptional locus of π_{13}, we apply Proposition 2.2.1. We study the first degeneracy locus of

$$\varphi : \mathcal{Q}^{\vee}_{\mathbb{P}^2|Y} \oplus \mathcal{Q}^{\vee}_{\mathrm{Gr}(2,4)} \to \mathcal{O}_Y^{\oplus 5}.$$

Using Theorems 2.2.5 and 2.2.6, we obtain that $\mathrm{D}_3(\varphi)$ is a dP_6.

Hence $X = \mathrm{Bl}_{\mathrm{dP}_6} \mathrm{Bl}_{Q_3 \cap Q_3' \cap Q_3''} \mathrm{Gr}(2,4)$, which also implies the rationality of X. •

Fano 3–29. [#K_{226}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)} \oplus \mathcal{U}_{\mathrm{Gr}(2,5)}(0,1,0)^{\oplus 2}) \subset \mathbb{P}^2 \times \mathbb{P}^4 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 51,\ (-K)^4 = 225,\ \chi(T_X) = -4,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 10.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_3} Y$, with Y the Fano fourfold 6 in [**Kal19**, Table 3]

Identification. As we have argued in Fano 3–24,

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,5)$$

is the Fano fourfold 6 in [**Kal19**, Table 3]. Hence,

$$X = \mathscr{Z}(\mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)}(1,0,0)^{\oplus 2}) \subset \mathbb{P}^4 \times Y.$$

The projection onto Y induces a birational map between the X and Y.

Applying Proposition 2.2.1, Theorems 2.2.5 and 2.2.6, we find that the fiber of π degenerates to a $\mathbb{P}^1$ along a surface

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0,0)^{\oplus 2}) \subset \mathbb{P}^4 \times Y.$$

We compute: $e(S) = 9$, $K_S = \mathcal{O}(1,-1,-1)_{|S}$, $(-K_S)^2 = 3$ and $h^0(S, K_S^{\otimes 2}) = 0$. Hence, by classification of surfaces, S is a dP_3.

It follows that $X = \mathrm{Bl}_{\mathrm{dP}_3} Y$, which also implies the rationality of X. •

Fano 3–30. [#K_{224}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)} \oplus \mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(1,0,0) \oplus \mathcal{O}(0,0,1)^{\oplus 2}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 56,\ (-K)^4 = 251,\ \chi(T_X) = 1,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 6.$

Description. $X = \mathrm{Bl}_{\mathrm{Bl}_{\mathrm{pt}} \mathbb{P}^2} Y$ with Y the fourfold Fano 2–11

Identification. Recall that

$$Y = \mathscr{Z}((\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)} \oplus \mathcal{O}(0,1)^{\oplus 2}) \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5),$$

which is the fourfold Fano 2–11. Since

$$X = \mathscr{Z}(\mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(1,0,0)) \subset \mathbb{P}^2 \times Y,$$

the projection π to Y is birational.

Using Proposition 2.2.1, Theorems 2.2.5 and 2.2.6, we compute some invariants of the exceptional locus of π. In particular, the exceptinal locus is a surface S with $e(S) = 4$, $K_S = \mathcal{O}(-1,-1)_{|S}$, $(-K_S)^2 = 8$. Hence, by classification, S is a dP_8. Since S is birational to $\mathbb{P}^2$, it is $\mathrm{Bl}_{\mathrm{pt}}\,\mathbb{P}^2$. •

Fano 3–31. [#K_{242}] $X = \mathscr{Z}(\mathcal{Q}^{\vee}_{\mathbb{P}^3}(1,1,0) \oplus \mathcal{Q}_{\mathbb{P}^4}(0,0,1) \oplus \mathcal{O}(1,0,1)) \subset \mathbb{P}^3 \times \mathbb{P}^4 \times \mathbb{P}^5$

Invariants. $h^0(-K) = 51$, $(-K)^4 = 224$, $\chi(T_X) = -3$, $h^{1,1} = 3$, $h^{3,1} = 0$, $h^{2,2} = 8$.

Description. $X = \mathrm{Bl}_{\mathrm{dP}_4} Y$, with Y the projectivization of the null correlation bundles over $\mathbb{P}^3$.

Identification. Let us note that

$$Y = \mathscr{Z}(\mathcal{Q}^{\vee}_{\mathbb{P}^3}(1,1)) \subset \mathbb{P}^3 \times \mathbb{P}^4$$

is a Fano fourfold with $\rho(Y) = 2$, $\iota(Y) = 2$ and $vol(Y) = 384$. In particular, if we denote by $\mathcal{E}^{\vee}$ the null correlation bundle on $\mathbb{P}^3$, then

$$Y = \mathbb{P}_{\mathbb{P}^3}(\mathcal{E}),$$

as described in [**SW90**]. This fourfold appears as MW^4_{11} in [**CGKS20**].

We can rewrite

$$X = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(1)) \subset \mathbb{P}_Y(\mathcal{O}_{\mathbb{P}^4}(-1) \oplus \mathcal{O}_{\mathbb{P}^4}),$$

which is birational to Y via the natural projection π.

To analyze the locus where the fiber degenerates, we twist by $\mathcal{O}_{\mathbb{P}^3}(-1)$, obtaining

$$X = \mathbb{P}_Y(\mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^4}(-1,-1) \oplus \mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^4}(-1,0)),$$

and we apply Proposition 2.2.1, Theorems 2.2.5 and 2.2.6 to compute some invariants of the degeneracy locus.

In this way, we have that the exceptional locus is a surface

$$S = \mathscr{Z}(\mathcal{O}(1,0) \oplus \mathcal{O}(1,1)) \subset Y$$

with $e(S) = 8$, $K_S = \mathcal{O}(0,-1)_{|S}$, $(-K_S)^2 = 4$ and $h^0(S, K_S^{\otimes 2}) = 0$. Hence by classification S is a dP_4.

It follows that $X = \mathrm{Bl}_{\mathrm{dP}_4} Y$, and since Y is rational, so is X.

•

Fano 3–32. [#K_{244}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^3}(0,1,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0) \oplus \mathcal{O}(0,1,1) \oplus \mathcal{O}(0,0,1)) \subset \mathbb{P}^3 \times \mathbb{P}^4 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 47$, $(-K)^4 = 203$, $\chi(T_X) = -5$, $h^{1,1} = 3$, $h^{3,1} = 0$, $h^{2,2} = 9$.

Description. $X = \mathrm{Bl}_{\mathrm{dP}_3} Y$,with Y the fourfold Fano 2–43.

Identification. Let us recall that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(0,1)) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,4),$$

which is the fourfold Fano 2–43. Hence

$$X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^3}(1,0,0) \oplus \mathcal{O}(1,0,1)) \subset \mathbb{P}^4 \times Y,$$

and the projection π onto Y induces a birational map between X and Y.

To compute the exceptional locus Δ of π, we proceed as usual. Applying Proposition 2.2.1, Theorems 2.2.5 and 2.2.6, we find that Δ is a surface

$$S = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(1)^{\oplus 5}) \subset \mathbb{P}_Y(\mathcal{Q}^\vee_{\mathbb{P}^3|Y} \oplus \mathcal{O}_{\mathrm{Gr}(2,4)}(-1)_{|Y}).$$

Moreover, we have that $K_S = \mathcal{O}(-1,-1,2)_{|S}$, and by classification S is a dP_3.

Since Y is rational, it follows that X is rational as well. •

Fano 3–33. [$\#K_{450}$] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0)^{\oplus 2} \oplus \mathcal{O}(0,0,1)) \subset \mathbb{P}^1 \times \mathbb{P}^4 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 51,\ (-K)^4 = 224,\ \chi(T_X) = 1,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 4.$

Description. $X = \mathbb{P}^1 \times Y$, with Y the Fano threefold 2-21 in [**DBFT22**, Table 1]

Identification. Since the bundles cut $\mathbb{P}^1$ and $\mathbb{P}^4 \times \mathrm{Gr}(2,4)$ separately, we have

$$X = \mathbb{P}^1 \times Y$$

with

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0)^{\oplus 4} \oplus \mathcal{O}(0,1)) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,5),$$

which is the Fano threefold 2-21 in [**DBFT22**, Table 1].

Moreover, since X is product of rational manifolds, it is rational. •

Fano 3–34. [$\#K_{145}$] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^4} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0)) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,7) \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 75,\ (-K)^4 = 353,\ \chi(T_X) = 8,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 6.$

Description. X is the a small resolution of Y a fourfold with $e(Y) = 13$ singular in a point.

Identification. Recall that, by Corollary 2.2.11 (2)

$$\mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0)) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,4)$$

can be identified with $\mathscr{Z}(\mathcal{Q}_2) \subset \mathrm{Fl}(1,3,5)$. In particular, we have

$$X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \boxtimes \mathcal{Q}_1 \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \boxtimes \mathcal{Q}_2 \oplus \mathcal{Q}_2) \subset \mathrm{Gr}(2,7) \times \mathrm{Fl}(1,3,5)$$

Using Proposition 2.2.1 is clear that X is birational to $Y = \mathrm{D}_5(\varphi)$, with

$$\varphi : \mathcal{Q}^\vee_{1|Z} \oplus \mathcal{Q}^\vee_{2|Z} \to \mathcal{O}_Z^{\oplus 7}, \text{ and} Z = \mathscr{Z}(\mathcal{Q}_2) \subset \mathrm{Fl}(1,3,5).$$

Moreover, applying Theorems 2.2.5 and 2.2.6, we find that Y is a fourfold with $e(Y) = 13$.

Since $\mathrm{D}_4(\varphi_{|Z}) = \{1 \text{ point}\}$, we conclude that Y is singular at a single point, and X provides a small resolution of Y along this singularity. •

Fano 3–35. [#K_{295}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0,0)^{\oplus 2}) \subset \mathbb{P}^6 \times \mathbb{P}^2 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 55,\ (-K)^4 = 244,\ \chi(T_X) = 0,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 5.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_8} Y$, with Y the Fano 6 in [Kal19, Table 3].

Identification. As argued in Fano 3–24,

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,5)$$

is the Fano 6 in [Kal19, Table 3]. Consequently

$$X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0,0)^{\oplus 2}) \subset \mathbb{P}^6 \times Y,$$

and the projection $\pi : X \to Y$ is a birational map between X and Y.

Moreover, applying Proposition 2.2.1, Theorems 2.2.5 and 2.2.6 we see that the fiber of π degenerates to a $\mathbb{P}^1$ along a surface

$$S = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0,0)^{\oplus 2}) \subset \mathbb{P}^2 \times Y,$$

for which we have $e(S) = 4$, $K_S = \mathcal{O}(1,-1,-1)_{|S}$, $(-K_S)^2 = 8$ and $h^0(S, K_S^{\otimes 2}) = 0$. Hence, by classification, S is a dP_8. n particular, the projection onto $\mathbb{P}^2$, is a birational map via a single blow-up. Thus S is $\mathrm{Bl}_{\mathrm{pt}} \mathbb{P}^2$.

Since Y is rational, it follows that X is also rational. •

Fano 3–36. [#K_{320}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(0,1,0) \oplus \mathcal{O}(0,1,1)) \subset \mathbb{P}^2 \times \mathbb{P}^3 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 45,\ (-K)^4 = 193,\ \chi(T_X) = -8,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 12.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_1} Y$, with Y the Fano 6 in [Kal19, Table 3].

Identification. As argued in Fano 3–24,

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,5)$$

is the Fano 6 in [Kal19, Table 3]. Consequently,

$$X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0,0) \oplus \mathcal{O}(1,0,1)) \subset \mathbb{P}^3 \times Y,$$

and the projection $\pi : X \to Y$ gives a birational map between X and Y.

To analyze where the fiber π degenerates to a $\mathbb{P}^1$, we consider

$$\varphi : \mathcal{U}_{\mathrm{Gr}(2,5)|Y} \oplus \mathcal{O}_Y(-1) \to \mathcal{O}_Y^{\oplus 4},$$

and by Proposition 2.2.1, we study $\mathrm{D}_2(\varphi)$.

Applying Theorems 2.2.5 and 2.2.6, we obtain that $\mathrm{D}_2(\varphi_{|Y})$ is a surface S, which can be described as

$$\mathscr{Z}(\mathcal{O}_{\mathcal{R}}(1)^{\oplus 4}) \subset \mathbb{P}_Y(\mathcal{U}_{\mathrm{Gr}(2,5)|Y} \oplus \mathcal{O}_{\mathrm{Gr}(2,5)}(-1)_{|Y}),$$

with $e(S) = 11$ and $K_S = (\mathcal{O}_{\mathcal{R}}(1) \otimes \pi^* \mathcal{O}_Y(-1,-1))_{|S}$. By classification of surfaces, this identifies S as dP_1. •

Fano 3–37. [#K_{321}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,5)}(0,1,0) \oplus \mathcal{O}(0,1,1)) \subset \mathbb{P}^2 \times \mathbb{P}^4 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 47,\ (-K)^4 = 202,\ \chi(T_X) = -7,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 10.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_3} Y$, with Y the Fano 6 in [Kal19, Table 3].

Identification. As argued in Fano 3–24,

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,5)$$

is the Fano 6 in [Kal19, Table 3]. Consequently,

$$X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0,0) \oplus \mathcal{O}(1,0,1)) \subset \mathbb{P}^4 \times Y,$$

and the projection $\pi : X \to Y$ gives a birational map between X and Y.

To determine where the fiber π degenerates a $\mathbb{P}^1$, we apply Proposition 2.2.1, and we consider $\mathrm{D}_3(\varphi)$, with

$$\varphi : \mathcal{Q}^\vee_{\mathrm{Gr}(2,5)|Y} \oplus \mathcal{O}_{\mathrm{Gr}(2,5)|Y}(-1) \to \mathcal{O}_Y^{\oplus 5}.$$

By Theorems 2.2.5 and 2.2.6, we find that $\mathrm{D}_3(\varphi_{|Y})$ is a surface

$$S = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(1)^{\oplus 5}) \subset \mathbb{P}_Y(\mathcal{Q}^\vee_{\mathrm{Gr}(2,5)|Y} \oplus \mathcal{O}_{\mathrm{Gr}(2,5)}(-1)_{|Y})$$

with $e(S) = 9$ and $K_S = \mathcal{O}(-1,-2,1)_{|S}$. By the classification of surfaces, S is a dP_3.

Since X is birational to Y, it follows that X is rational as well. •

Fano 3–38. [#K_{322}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2_1}(0,1,1)) \subset \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathbb{P}^2_3$

Invariants. $h^0(-K) = 46,\ (-K)^4 = 198,\ \chi(T_X) = -2,\ h^{1,1} = 3,\ h^{2,1} = 1,\ h^{3,1} = 0,\ h^{2,2} = 5.$

Description. $X = \mathrm{Bl}_E\, \mathbb{P}^2_2 \times \mathbb{P}^2_3$ with E a sextic elliptic curve.

Identification. By Lemma 2.2.9, we have

$$X = \mathrm{Bl}_C\, \mathbb{P}^2 \times \mathbb{P}^2,$$

where

$$C = \mathscr{Z}(\mathcal{O}(1,1)^{\oplus 3}).$$

Since C is a curve with $K_C = \mathcal{O}_C$, it follows, by classification of curves, that X is an elliptic curve E of degree six.

Hence, this Fano fourfold is rational, as it is birational to $\mathbb{P}^2 \times \mathbb{P}^2$. •

Fano 3–39. [#K_{341}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(0,1,1) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0)) \subset \mathbb{P}^2 \times \mathbb{P}^3 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 42,\ (-K)^4 = 178,\ \chi(T_X) = -7,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 11.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_6} Y$, with Y the fourfold Fano 2–36

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(0,1,1)) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,4)$$

is the fourfold Fano 2–36. Then

$$X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0)) \subset \mathbb{P}^2 \times Y,$$

the projection $\pi : X \to Y$ gives a birational map between X and Y.

Using Proposition 2.2.1, Theorems 2.2.5 and 2.2.6, we see that the fiber of π degenerates to a $\mathbb{P}^1$ along a surface

$$S = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}) \subset Y,$$

which can be identified with

$$\mathscr{Z}(\mathcal{O}(1,1)^{\oplus 2}) \subset \mathbb{P}^2 \times \mathbb{P}^2,$$

hence S is a dP_6.

Therefore, $X = \mathrm{Bl}_{\mathrm{dP}_6} Y$. Since Y is rational, it follows that X is also rational. •

Fano 3–40. [#K_{342}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(1,1,0) \oplus \mathcal{O}(0,0,1)) \subset \mathbb{P}^3_1 \times \mathbb{P}^3_2 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 44,\ (-K)^4 = 188,\ \chi(T_X) = -4,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 8.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_4} Y$, with Y the fourfold Fano 2–43

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(0,0,1)) \subset \mathbb{P}^3_2 \times \mathrm{Gr}(2,4)$$

is the fourfold Fano 2–43. Then

$$X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0) \oplus \mathcal{O}(1,1,0)) \subset \mathbb{P}^3_1 \times Y,$$

so the projection $\pi : X \to Y$ gives a birational map between X and Y.

Applying Proposition 2.2.1, Theorems 2.2.5 and 2.2.6, the fiber of π degenerates to a $\mathbb{P}^1$ along a surface

$$S = \mathscr{Z}(\mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(1,0,0) \oplus \mathcal{Q}_{\mathbb{P}^3_2}(1,0,0)) \subset \mathbb{P}^3 \times Y.$$

Computing the invariants, we have: $e(S) = 8$, $K_S = \mathcal{O}(1,1,-1)_{|S}$, $(-K_S)^2 = 4$ and $h^0(S, K_S^{\otimes 2}) = 0$. Hence, by classification, S is a dP_4. •

Fano 3–41. [#K_{343}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2}(0,1,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(1,0,1) \oplus \mathcal{O}(0,0,1)) \subset \mathbb{P}^2 \times \mathbb{P}^4 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 43,\ (-K)^4 = 183,\ \chi(T_X) = -5,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 9.$

Description. $X = \mathrm{Bl}_{Q_2} Y$, with Y the fourfold Fano 2–35.

Identification. Let us start with the fivefold

$$X' = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2}(0,1,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(0,0,1)) \subset \mathbb{P}^2 \times \mathbb{P}^4 \times \mathrm{Gr}(2,4).$$

By Corollary 2.2.11 (2), we can rewrite

$$X' = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2}(0,1,0) \oplus \mathcal{O}(0,0,1) \oplus \mathcal{Q}_2) \subset \mathbb{P}^2 \times \mathrm{Fl}(1,3,5).$$

Using Lemma 2.2.9, we have that $X' = \mathrm{Bl}_S Y'$, with

$$Y' = \mathscr{Z}(\mathcal{Q}_2 \oplus \mathcal{O}(0,1)) \subset \mathrm{Fl}(1,3,5).$$

By Lemma 2.2.12, Y' can be written as

$$Y' = \mathbb{P}_{Q_3}(\mathcal{U}_{\mathrm{Gr}(2,4)} \oplus \mathcal{O}_{\mathrm{Gr}(2,4)}).$$

The center of the blow-up is

$$S = \mathscr{Z}(\mathcal{O}_{\mathcal{R}}(1)^{\oplus 3}) \subset \mathbb{P}_{Q_3}(\mathcal{U}_{\mathrm{Gr}(2,4)} \oplus \mathcal{O}_{\mathrm{Gr}(2,4)}),$$

which, by Theorem 2.2.8, can also be written as

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(0,1) \oplus \mathcal{O}(1,0)^{\oplus 3}) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,4).$$

Thus,

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)|Q_3}(1,0)) \subset \mathbb{P}^1 \times Q_3.$$

Projecting onto $\mathbb{P}^1$, the general fiber is $\mathbb{P}^1$. Computing the invariants, we obtain: $e(S) = 4$, $K_S = \mathcal{O}(0,-2)_{|S}$, $(-K_S)^2 = 8$ and $h^0(S, K_S^{\otimes 2}) = 0$. Hence, by classification, S is Q_2.

The variety X' can be expressed as

$$X' = \mathscr{Z}(\mathcal{O}_{Y'}(1) \boxtimes \mathcal{O}_{\mathcal{R}}(1)) \subset \mathrm{Bl}_{Q_2} Y'.$$

The bundle $\mathcal{O}_{Y'}(1) \boxtimes \mathcal{O}_{\mathcal{R}}(1)$ fixes S, hence $X = \mathrm{Bl}_S Y$, with

$$Y = \mathscr{Z}(o\,f_{Y'}(1) \boxtimes \mathcal{O}_{\mathcal{R}}(1)) \subset Y',$$

which is exactly the fourfold Fano 2–35.

Since Y is rational, X is rational as well. •

Fano 3–42. [#K_{346}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2_1} \boxtimes \mathcal{Q}_{\mathbb{P}^2_2} \boxtimes \mathcal{O}_{\mathbb{P}^4}(1)) \subset \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathbb{P}^4$

Invariants. $h^0(-K) = 36$, $(-K)^4 = 150$, $\chi(T_X) = -4$, $h^{1,1} = 3$, $h^{3,1} = 0$, $h^{2,2} = 10$.

Description. $X = \mathrm{Bl}_{\mathrm{dP}_3} \mathbb{P}^2 \times \mathbb{P}^2$.

Identification. Let us denote

$$Y = \mathbb{P}^2_1 \times \mathbb{P}^2_2.$$

Projecting everything onto Y, Proposition 2.2.1 shows that this projection is birational, degenerating when the map

$$\varphi : \mathcal{Q}^\vee_{\mathbb{P}^2_1} \boxtimes \mathcal{Q}^\vee_{\mathbb{P}^2_2} \to \mathcal{O}_Y^{\oplus 5}$$

derops rank to 3.

Using Theorems 2.2.5 and 2.2.6, the degeneracy locus $\mathrm{D}_3(\varphi)$ is a surface

$$S = \mathscr{Z}(\mathcal{Q}^\vee_{\mathbb{P}^2_2}(1,0,1) \oplus \mathcal{O}(0,1,1)^{\oplus 3}) \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3.$$

Computing the invariants, we obtain: $e(S) = 6$ and $K_S = \mathcal{O}(-1,-1,1)_S$, $(-K_S)^2 = 3$, $h^0(S, K_S^{\otimes 2}) = 0$ and $h^0(S, (-K_S)^{\otimes 2}) = 10$. By classification, S is a dP_3.

Since X is birational to $\mathbb{P}^4$ via the projection, it follows that X is rational. •

Fano 3–43. [#K_{265}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \oplus \mathcal{O}(0,1,0) \oplus \mathcal{O}(0,0,1)) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,6)$

Invariants. $h^0(-K) = 60$, $(-K)^4 = 273$, $\chi(T_X) = 3$, $h^{1,1} = 3$, $h^{3,1} = 0$, $h^{2,2} = 6$.

Description. $\pi : X \to \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,6)$ is a small resolution of Y a fourfold with $e(Y) = 13$ singular in a point.

Identification. Consider

$$Z = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(0,1)) \subset \mathrm{Gr}(2,6) \times \mathrm{Gr}(2,4).$$

By Corollary 2.2.11 (2), Z can also be written as

$$\mathscr{Z}(\mathcal{Q}_2^{\oplus 2} \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(0,1)) \subset \mathrm{Fl}(2,4,6).$$

We rewrite X as

$$X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}_1^\vee) \subset \mathbb{P}^2 \times Z.$$

Applying Proposition 2.2.1, X is birational to $\mathrm{D}_1(\varphi)$, with

$$\varphi : \mathcal{U}_{1|Z} \to \mathcal{O}^{\oplus 3}_{\mathrm{Fl}(2,4,6)|Z}.$$

Using Theorems 2.2.5 and 2.2.6, $\mathrm{D}_1(\varphi)$ corresponds to a fourfold Y with $e(Y) = 13$. Since $\mathrm{D}_2(\phi) = \{1 \text{ point}\}$ the fourfold Y is singular at a single point. In particular, the projection $\pi : X \to \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,6)$ degenerates to a $\mathbb{P}^2$ over the singular point of Y, so X is a small resolution of Y. •

Fano 3–44. [#K_{372}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)} \oplus \mathcal{O}(0,1,0) \oplus \mathcal{O}(0,0,1)^{\oplus 2}) \subset \mathbb{P}^1 \times \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 60,\ (-K)^4 = 272,\ \chi(T_X) = 4,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 4.$

Description. X is $\mathbb{P}^1 \times Y$, with Y the Fano threefold 2-26 in [DBFT22, Table 1].

Identification. The bundles cut the factors $\mathbb{P}^1$ and $\mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$ separately, hence $X = \mathbb{P}^1 \times Y$, with

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)} \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 2}) \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5),$$

which is the Fano threefold 2-26 in [DBFT22, Table 1].

Since X is a product of rational manifolds, it is rational. •

Fano 3–45. [#K_{388}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(0,1,0) \oplus \mathcal{O}(0,0,1)^{\oplus 3}) \subset \mathbb{P}^1 \times \mathbb{P}^3 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 54,\ (-K)^4 = 240,\ \chi(T_X) = 2,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 4.$

Description. X is $\mathbb{P}^1 \times Y$, with with Y the Fano threefold 2-22 in [DBFT22, Table 1].

Identification. The bundles cut the factors $\mathbb{P}^1$ and $\mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$ separately, so we have $X = \mathbb{P}^1 \times Y$, with

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 3}) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,5),$$

which is the Fano threefold 2-22 in [DBFT22, Table 1].

Since X is a product of rational manifolds, it is rational. •

Fano 3–46. [#K_{421}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(0,1,0) \oplus \mathcal{O}(1,1,0) \oplus \mathcal{O}(0,0,1)^{\oplus 3}) \subset \mathbb{P}^1 \times \mathbb{P}^4 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 43,\ (-K)^4 = 183,\ \chi(T_X) = -3,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 7.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_5} Y$, with Y the fourfold Fano 2–27

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 3}) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,5)$$

is the fourfold Fano 2–27. Hence,

$$X = \mathscr{Z}(\mathcal{O}(1,1,0)) \subset \mathbb{P}^1 \times Y.$$

Applying Lemma 2.2.9, we obtain $X = \mathrm{Bl}_S Y$, with

$$S = \mathscr{Z}(\mathcal{O}(0,1)^{\oplus 3} \oplus \mathcal{O}(1,0)^{\oplus 2}) \subset \mathrm{Fl}(1,2,5).$$

Note that $e(S) = 7$, $K_S = \mathcal{O}(0,-1)_{|S}$, $(-K_S)^2 = 5$ and $h^0(S, K_S^{\otimes 2}) = 0$. Thus, S is a dP_5.

Since Y is rational, it follows that X is rational as well. •

Fano 3–47. [#K_{422}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(0,1,1) \oplus \mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(1,0,0)) \subset \mathbb{P}^2 \times \mathbb{P}^3 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 41,\ (-K)^4 = 172,\ \chi(T_X) = -7,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 10.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_7} Y$, with Y the fourfold Fano 2–36.
Identification. Recall that

$$Y = \mathscr{Z}((\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(1,1)) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,4)$$

is the fourfold Fano 2–36. Hence,

$$X = \mathscr{Z}(\mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(1,0,0)) \subset \mathbb{P}^2 \times Y,$$

so X is birational to Y via the projection $\pi : X \to Y$.

By Theorems 2.2.5 and 2.2.6, we see that the exceptional locus of π is a surface

$$S = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2}(0,1) \oplus \mathcal{O}(1,1)) \subset \mathbb{P}^2 \times \mathbb{P}^3.$$

In particular, $e(S) = 5$, $K_S = \mathcal{O}(-1,-1)_{|S}$, $(-K_S)^2 = 7$ and $h^0(S, K_S^{\otimes 2}) = 0$. Hence S is a dP_7.

Since Y is rational, it follows that X is rational as well. •

Fano 3–48. [$\#K_{17}$] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2_1} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)} \oplus \mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)}(0,1,0)) \subset \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 85$, $(-K)^4 = 406$, $\chi(T_X) = 12$, $h^{1,1} = 3$, $h^{3,1} = 0$, $h^{2,2} = 5$.

Description. $X = \mathrm{Bl}_{Q_2} Y$, with Y the Fano fourfold 6 in [**Kal19**, Table 3].

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2_1} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)}) \subset \mathbb{P}^2_1 \times \mathrm{Gr}(2,5)$$

is the Fano fourfold 6 in [**Kal19**, Table 3], as proved in Fano 2–24. Hence,

$$X = \mathscr{Z}(\mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)}(1,0,0)) \subset \mathbb{P}^2 \times Y,$$

and it is birational to Y via the projection $\pi : X \to Y$.

By Theorems 2.2.5 and 2.2.6, the fiber of π degenerates to a $\mathbb{P}^1$ along a surface

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0,0)) \subset \mathbb{P}^1 \times Y.$$

Moreover, $e(S) = 4$, $K_S = \mathcal{O}(1,-1,-2)_{|S}$, $(-K_S)^2 = 8$ and $h^0(S, K_S^{\otimes 2}) = 0$. Hence, by classification, S is a dP_8.

Projecting onto $\mathbb{P}^1$, the general fiber is the intersection of Y with a section of $\mathcal{Q}_{\mathrm{Gr}(2,5)}$, which is a $\mathbb{P}^1$. Thus S is birational to $\mathbb{P}^1 \times \mathbb{P}^1$, which means that is Q_2.

In conclusion, $X = \mathrm{Bl}_{Q_2} Y$, and since Y is rational, it follows that X is rational as well. •

Fano 3–49. [$\#K_{472}$] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0)^{\oplus 2} \oplus \mathcal{O}(1,1,0) \oplus \mathcal{O}(0,0,1)) \subset \mathbb{P}^1 \times \mathbb{P}^5 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 39$, $(-K)^4 = 162$, $\chi(T_X) = -7$, $h^{1,1} = 3$, $h^{3,1} = 0$, $h^{2,2} = 10$.

Description. $X = \mathrm{Bl}_{\mathrm{dP}_2} \mathbb{P}^1 \times Q_3$

Identification. Applying Theorem 2.2.8 to

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0)^{\oplus 2} \oplus \mathcal{O}(0,1)) \subset \mathbb{P}^5 \times \mathrm{Gr}(2,4)$$

we obtain

$$\mathscr{Z}(\mathcal{O}_{\mathcal{R}}(1)^{\oplus 2}) \subset \mathbb{P}_{Q_3}(\mathcal{U}^{\oplus 2}_{\mathrm{Gr}(2,4)|Q_3}).$$

Hence,

$$X = \mathscr{Z}(\mathcal{O}(1,1,0)) \subset \mathbb{P}^1 \times Y.$$

By Lemma 2.2.9, we deduce that $X = \mathrm{Bl}_S Y$, with

$$S = \mathscr{Z}(\mathcal{O}(1,0)^{\oplus 2}) \subset Y.$$

Since $e(S) = 8$, $K_S = \mathcal{O}(0,-1)_{|S}$, $(-K_S)^2 = 4$ and $h^0(S, K_S^{\otimes 2}) = 0$, by classification S is a dP_4.

Note that Y is a $\mathbb{P}^1$-bundle over Q_3, and by Theorems 2.2.5 and 2.2.6 the fiber of the projection onto the base degenerates to a $\mathbb{P}^2$ over 2 points.

Considering the projection π_{13} onto $\mathbb{P}^1 \times Q_3$, we obtain a birational map between X and $\mathbb{P}^1 \times Q_3$.

By Proposition 2.2.1, the exceptional locus of π_{13} can be described as the first degeneracy locus of the morphism

$$\varphi : \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{Q}^{vee}_{\mathrm{Gr}_{2,4}\,|Q_3} \to \mathcal{O}^{\oplus 4}_{\mathbb{P}^2 \times Q_3}.$$

Applying Theorems 2.2.5 and 2.2.6, we find that $\mathrm{D}_2(\varphi)$ is a dP_2. Hence $X = \mathrm{Bl}_{\mathrm{dP}_2} \mathbb{P}^1 \times Q_3$.

Since X is birational to the product of two rational manifolds, it is rational.

•

Fano 3–50. [#K_{512}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(1,0,1) \oplus \mathcal{O}(0,2,0) \oplus \mathcal{O}(0,0,1)) \subset \mathbb{P}^1 \times \mathbb{P}^4 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 37,\ (-K)^4 = 152,\ \chi(T_X) = -5,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 8.$

Description. X is the blow-up of the base of the conic bundle Fano 2–47 along a Q_1.

Identification. Consider

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(2,0) \oplus \mathcal{O}(0,1)) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,4),$$

which is the Fano Fano 2–47. Thus,

$$X = \mathscr{Z}(\mathcal{O}(1,0,1)) \subset \mathbb{P}^1 \times Y.$$

By Lemma 2.2.9, we obtain $X = \mathrm{Bl}_C Y$ with

$$C = Q_3 \cap H \cap H' = Q_1.$$

Since the fiber of the conic bundle is not involved in the blow-up, we conclude

$$X = \mathbb{P}_{\mathrm{Bl}_{Q_1} Q_3}(\mathcal{U}_{\mathrm{Gr}(2,4)|Q_3} \oplus \mathcal{O}_{Q_3}).$$

•

Fano 3–51. [#K_{516}] $X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(0,1,0) \oplus \mathcal{O}(1,0,1) \oplus \mathcal{O}(0,0,1)^{\oplus 2}) \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 35,\ (-K)^4 = 141,\ \chi(T_X) = -10,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 12.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_2} Y$, with Y the fourfold Fano 2–46.

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 2}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,5)$$

is the fourfold Fano 2–46. Hence,

$$X = \mathscr{Z}(\mathcal{O}(1,0,1)) \subset \mathbb{P}^1 \times Y.$$

By Lemma 2.2.9, we have $X = \mathrm{Bl}_S Y$, with

$$S = \mathscr{Z}(\mathcal{O}(0,1)^{\oplus 2}) \subset Y.$$

Since $e(S) = 10$, $K_S = \mathcal{O}(-1,0)_{|S}$, $(-K_S)^2 = 2$ and $h^0(S, K_S^{\otimes 2}) = 0$, by classification, S is a dP_2.

Since X is birational to Y, it follows that X is rational. •

Fano 3–52. [$\#K_{519}$] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(0,0,1) \oplus \mathcal{O}(0,1,1) \oplus \mathcal{O}(1,1,0)) \subset \mathbb{P}^1 \times \mathbb{P}^4 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 35,\ (-K)^4 = 141,\ \chi(T_X) = -11,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 13.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_4} Y$, with Y the fourfold Fano 2–35.

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(0,1) \oplus \mathcal{O}(1,1)) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,4)$$

is the fourfold Fano 2–35. Hence,

$$X = \mathscr{Z}(\mathcal{O}(1,0,1)) \subset \mathbb{P}^1 \times Y.$$

By Lemma 2.2.9, we obtain $X = \mathrm{Bl}_S Y$, with

$$S = \mathscr{Z}(\mathcal{O}(1,0)^{\oplus 2}) \subset Y.$$

Since $e(S) = 8$, $K_S = \mathcal{O}(0,-1)_{|S}$, $(-K_S)^2 = 4$ and $h^0(S, K_S^{\otimes 2}) = 0$, by classification, S is a dP_4.

Since Y is rational, it follows that X is rational as well. •

Fano 3–53. [$\#K_{593}$] $X = \mathscr{Z}(\mathcal{O}(1,1,0) \oplus \mathcal{O}(1,0,1) \oplus \mathcal{O}(0,0,1)^{\oplus 3}) \subset \mathbb{P}^2 \times \mathbb{P}^1 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 33,\ (-K)^4 = 130,\ \chi(T_X) = -11,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 12.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_5} Y$ with Y the Fano fourfold Fano 2–1.

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{O}(1,1) \oplus \mathcal{O}(0,1)^{\oplus 3}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,5)$$

is the fourfold Fano 2–1. Hence,

$$X = \mathscr{Z}(\mathcal{O}(1,1,0)) \subset \mathbb{P}^1 \times Y.$$

Applying, Lemma 2.2.9, we obtain $X = \mathrm{Bl}_S Y$, with

$$S = \mathscr{Z}(\mathcal{O}(1,0)^{\oplus 2}) \subset Y.$$

The invariants of S are $e(S) = 7$, $K_S = \mathcal{O}(0,-1)_{|S}$, $(-K_S)^2 = 5$ and $h^0(S, K_S^{\otimes 2}) = 0$. Hence, by classification, S is a dP_5.

Since Y is rational, it follows that X is rational as well. •

Fano 3–54. [$\#K_{591}$] $X = \mathscr{Z}(\mathcal{O}(0,1,1) \oplus \mathcal{O}(0,0,1)^{\oplus 3}) \subset \mathbb{P}^1_1 \times \mathbb{P}^1_2 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 39,\ (-K)^4 = 160,\ \chi(T_X) = -4,\ h^{1,1} = 3,\ h^{2,1} = 1,\ h^{3,1} = 0,\ h^{2,2} = 4$

Description. $X = \mathbb{P}^1 \times Y$, with Y the Fano threefold 2-14 in [**DBFT22**, Table 1]

Identification. Since the bundles restrict separately to the factors $\mathbb{P}^1_1$ and $\mathbb{P}^1_2 \times \mathrm{Gr}(2,5)$, we can write $X = \mathbb{P}^1 \times Y$, with

$$Y = \mathscr{Z}(\mathcal{O}(1,1) \oplus \mathcal{O}(0,1)^{\oplus 3}) \subset \mathbb{P}^1 \times \mathrm{Gr}(2,5),$$

which corresponds to the Fano threefold 2-14 in [**DBFT22**, Table 1].

Since X is the product of two rational varieties, it is itself rational. •

Fano 3–55. [#K_{668}] $X = \mathscr{Z}(\mathcal{O}(1,1,0) \oplus \mathcal{Q}_{\mathbb{P}^2}(0,0,2)) \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^4$

Invariants. $h^0(-K) = 25,\ (-K)^4 = 89,\ \chi(T_X) = -14,\ h^{1,1} = 3,\ h^{2,1} = 5,\ h^{3,1} = 0,\ h^{2,2} = 9.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_4} Y$, with Y the fourfold Fano 2–22

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2}(0,2)) \subset \mathbb{P}^2 \times \mathbb{P}^4,$$

which is the fourfold Fano 2–22. Hence,

$$X = \mathscr{Z}(\mathcal{O}(1,1,0)) \subset \mathbb{P}^1 \times Y.$$

Applying Lemma 2.2.9, we obtain $X = \mathrm{Bl}_S Y$, with

$$S = \mathscr{Z}(\mathcal{O}(1,0)^{\oplus 2}) \subset Y.$$

The invariants of S are: $e(S) = 8$, $K_S = \mathcal{O}(0,-1)_{|S}$, $(-K_S)^2 = 4$ and $h^0(S, K_S^{\otimes 2}) = 0$. By classification, S is a dP_4.

Since Y is rational, it follows that X is rational as well. •

Fano 3–56. [#K_{182}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^4} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,7)}(1,0,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,7)}(0,1,0)) \subset \mathbb{P}^2 \times \mathbb{P}^4 \times \mathrm{Gr}(2,7)$

Invariants. $h^0(-K) = 60,\ (-K)^4 = 273,\ \chi(T_X) = 2,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 7.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_6} Y$, with Y the fourfold Fano 2–62

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^4} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,7)}(1,0)) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,7)$$

is the fourfold Fano 2–62. Hence,

$$X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,7)}(1,0,0)) \subset \mathbb{P}^2 \times Y,$$

so that X is birational to Y via the projection $\pi : X \to Y$.

Applying Theorems 2.2.5 and 2.2.6, we see that the fiber of π degenerates to a $\mathbb{P}^1$ on a surface

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,7)}(1,0,0)) \subset \mathbb{P}^3 \times Y.$$

The invariants of S are: $e(S) = 6$, $K_S = \mathcal{O}(1,-1,-1)_{|S}$, $(-K_S)^2 = 6$ and $h^0(S, -K_S) = 7$, $h^0(S, K_S^{\otimes 2}) = 0$. Hence, by classification, S is a dP_6.

Since Y is rational, it follows that X is rational as well. •

Fano 3–57. [#K_{282}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^3} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,6)}(1,0,0) \oplus \mathcal{O}(0,1,1)) \subset \mathbb{P}^2 \times \mathbb{P}^3 \times \mathrm{Gr}(2,6)$

Invariants. $h^0(-K) = 51,\ (-K)^4 = 225,\ \chi(T_X) = -5,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 11$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_5} Y$, with Y the Fano fourfold 70 in [Kal19, Table 3].

Identification. Note that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^3} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \oplus \mathcal{O}(1,1)) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,6)$$

can be rewritten, by Theorem 2.2.8, as

$$\mathscr{Z}(\mathcal{O}(1,1)) \subset \mathrm{Gr}_{\mathbb{P}^3}(2, \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus 2}).$$

By duality and twisting with $\mathcal{O}_{\mathbb{P}^3}(-1)$, we also have

$$Y = \mathscr{Z}(\mathcal{O}(1,1)) \subset \mathbb{P}_{\mathbb{P}^3}(\mathcal{O} \oplus \mathcal{O}(-1)^{\oplus 2}).$$

Equivalently, Y can be realized as zero loci of homogeneous bundles on a quiver flag variety as the variety with adjacency matrix:

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

dimension vector $(1, 1, 1)$ and cut by the vector $(\square, \square)$. This is the Fano fourfold 70 in [Kal19, Table 3].

Now, X can be written as

$$\mathscr{Z}(\mathcal{U}^{\vee}_{\mathrm{Gr}(2,6)}(1, 0, 0)) \subset \mathbb{P}^2 \times Y,$$

which gives a birational map between X and Y via the projection $\pi : X \to Y$.

Applying Theorems 2.2.5 and 2.2.6, the fibers of π degenerate to a $\mathbb{P}^1$ along a surface

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,6)}(1, 0, 0)) \subset \mathbb{P}^2 \times Y.$$

The invariants of S are: $e(S) = 7$, $K_S = \mathcal{O}(1, -1, -1)_{|S}$, $(-K_S)^2 = 5$ and $h^0(S, 2K_S) = 0$. Hence, by classification, S is a dP_5.

Note that Y is birational to a product of rational manifolds, hence it is rational, and also X is as such. •

Fano 3–58. [#K_{303}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2_2} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,6)} \oplus \mathcal{O}(0, 0, 1)^{\oplus 2} \oplus \mathcal{U}^{\vee}_{\mathrm{Gr}(2,6)}(1, 0, 0)) \subset \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathrm{Gr}(2, 6)$

Invariants. $h^0(-K) = 48,\ (-K)^4 = 209,\ \chi(T_X) = -3,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 8.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_6} Y$, with Y the fourfold Fano 2–51.

Identification. Note that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,6)} \oplus \mathcal{O}(0, 1)^{\oplus 2}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2, 6)$$

is the Fano fourfold Fano 2–51. Hence,

$$X = \mathscr{Z}(\mathcal{U}^{\vee}_{\mathrm{Gr}(2,6)}) \subset \mathbb{P}^2 \times Y.$$

The projection $\pi : X \to Y$ induces a birational map between X and Y.

By Theorems 2.2.5 and 2.2.6, the fiber degenerates to $\mathbb{P}^1$ over a surface

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,6)}(1, 0, 0)) \subset \mathbb{P}^2 \times Y.$$

The invariants of S are: $e(S) = 6$, $K_S = \mathcal{O}(1, -1, -1)_{|S}$, $(-K_S)^2 = 6$ and $h^0(S, K_S^{\otimes 2}) = 0$. Hence, by classification, S is a dP_6. •

Fano 3–59. [#K_{460}] $X = \mathscr{Z}(\mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)}(0, 1, 0) \oplus \mathcal{O}(2, 0, 0) \oplus \mathcal{O}(0, 0, 1)^{\oplus 3}) \subset \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathrm{Gr}(2, 5)$

Invariants. $h^0(-K) = 48,\ (-K)^4 = 208,\ \chi(T_X) = 0,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 4$

Description. X is the product $Q_1 \times Y$, with Y the Fano threefold 2-20 in [DBFT22, Table 1].

Identification. Note that the bundles cut $\mathbb{P}^2_1$ and $\mathbb{P}^2_2 \times \mathrm{Gr}(2, 5)$ separately. Hence, $X = Q_1 \times Y$, with

$$Y = \mathscr{Z}(\mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)}(1, 0) \oplus \mathcal{O}(0, 1)^{\oplus 3}) \subset \mathbb{P}^2_2 \times \mathrm{Gr}(2, 5),$$

which is the Fano threefold 2-20 in [DBFT22, Table 1]. •

Fano 3–60. [#K_{318}] $X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,5)}(0,1,0) \oplus \mathcal{O}(0,0,1)^{\oplus 2}) \subset \mathbb{P}^2 \times \mathbb{P}^3 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 48,\ (-K)^4 = 209,\ \chi(T_X) = -2,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 7.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_7} Y$, with Y the fourfold Fano 2–29.

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 2}) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,5)$$

is the fourfold Fano 2–29, and

$$X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0,0)) \subset \mathbb{P}^2 \times Y.$$

The projection $\pi : X \to Y$ is birational. Applying Theorems 2.2.5 and 2.2.6, the fibers of π degenerate to a $\mathbb{P}^1$ along a surface

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0,0)) \subset \mathbb{P}^1 \times Y.$$

For S we have $e(S) = 5$, $K_S = \mathcal{O}(1,-1,-1)_{|S}$, $(-K_S)^2 = 7$ and $h^0(S, K_S^{\otimes 2}) = 0$, so by classification, S is a dP_7.

Thus, $X = \mathrm{Bl}_{\mathrm{dP}_7} Y$, and it is rational because Y is rational. •

Fano 3–61. [#K_{326}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{O}(1,1,0) \oplus \mathcal{O}(0,0,1)^{\oplus 2}) \subset \mathbb{P}^1 \times \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 48,\ (-K)^4 = 209,\ \chi(T_X) = -4,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 9$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_5} Y$, with Y the fourfold Fano 2–11

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{O}(0,0,1)^{\oplus 2}) \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$$

is the fourfold Fano 2–11. Hence, X can be rewritten as

$$X = \mathscr{Z}(\mathcal{O}(1,1,0)) \subset \mathbb{P}^1 \times Y.$$

Applying Lemma 2.2.9, we obtain $X = \mathrm{Bl}_S Y$, with

$$S = \mathscr{Z}(\mathcal{O}(1,0)^{\oplus 2}) \subset Y.$$

The inviariant of S are: $e(S) = 7$, $K_S = \mathcal{O}(0,-1)_{|S}$, $(-K_S)^2 = 5$, $h^0(S, K_S^{\otimes 2}) = 0$. Hence, by classification, S is a dP_5.

Since Y is rational, it follows that X is rational as well. •

Fano 3–62. [#K_{327}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0,0)) \subset \mathbb{P}^2 \times \mathbb{P}^4 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 47,\ (-K)^4 = 204,\ \chi(T_X) = -2,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 7.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_7} Y$, with Y the fourfold Fano 2–34

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0)) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,4)$$

is the fourfold Fano 2–34, and

$$X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0,0)) \subset \mathbb{P}^2 \times Y.$$

The projection $\pi : X \to Y$ is birational. Applying Theorems 2.2.5 and 2.2.6, we get that the fibers of π degenerate to a $\mathbb{P}^1$ along a surface

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}) \subset \times Y.$$

For S, we have $e(S) = 5$, $K_S = \mathcal{O}(-1,-1)_{|S}$, $(-K_S)^2$, so by classification S is a dP_7.

Since Y is rational, it follows that X is rational as well. •

Fano 3–63. [#K_{339}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0)^{\oplus 2} \oplus \mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(0,1,0)) \subset \mathbb{P}^4 \times \mathbb{P}^2 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 46$, $(-K)^4 = 199$, $\chi(T_X) = 0$, $h^{1,1} = 3$, $h^{3,1} = 0$, $h^{2,2} = 5$.

Description. $X = \mathrm{Bl}_{\mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^2} Y$, with Y the fourfold Fano 2–21.

Identification. Note that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0)^{\oplus 2}) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,4)$$

is the fourfoldFano 2–34. Hence, X can be rewritten as

$$X = \mathscr{Z}(\mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(1,0,0)) \subset \mathbb{P}^2 \times Y.$$

The projection $\pi : X \to Y$ is birational.

Applying Proposition 2.2.1, Theorems 2.2.5 and 2.2.6, the fibers of π degenerate to a $\mathbb{P}^1$ along a surface

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}) \subset Y.$$

Now, $\mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)})$ can be identified with

$$\mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2}(1,0)^{\oplus 2}) \subset \mathbb{P}^4 \times \mathbb{P}^2,$$

and, by Theorem 2.2.8, this becomes

$$\mathscr{Z}(\mathcal{O}_{\mathcal{R}}(1)) \subset \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}(-1)^{\oplus 2}).$$

Twisting $\mathcal{O}(-1)^{\oplus 2}$ by $\mathcal{O}(1)$, we can describe S as $\mathscr{Z}(\mathcal{O}(1,1)) \subset \mathbb{P}^1 \times \mathbb{P}^2$. By Lemma 2.2.9 we have that S is $\mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^2$.

Since Y is rational, it follows that X is rational as well. •

Fano 3–64. [#K_{344}] $X = \mathscr{Z}(\mathcal{O}(1,0,1) \oplus \mathcal{Q}_{\mathbb{P}^2}(0,1,0) \oplus \mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(1,0,0)) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,4) \times \mathbb{P}^2$

Invariants. $h^0(-K) = 42$, $(-K)^4 =, 178$ $\chi(T_X) = -5$, $h^{1,1} = 3$, $h^{3,1} = 0$, $h^{2,2} = 9$.

Description. $X = \mathrm{Bl}_{\mathrm{dP}_5}\mathrm{Bl}_{Q_1}\mathrm{Gr}(2,4)$

Identification. Consider

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2}(0,1)) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,4),$$

which, by Lemma 2.2.9, is $\mathrm{Bl}_{Q_1}\mathrm{Gr}(2,4)$. Then

$$X = \mathscr{Z}(\mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(1,1)) \subset \mathbb{P}^3 \times Y,$$

and the projection $\pi : X \to Y$ is birational.

Applying Theorems 2.2.5 and 2.2.6, the fibers of π degenerate to a $\mathbb{P}^1$ along a surface

$$S = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2}(1,0,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0)) \subset \mathbb{P}^2_1 \times Y.$$

Now, S satisfies $e(S) = 7$, $K_S = \mathcal{O}(1,-1,-1)_{|S}$, $(-K_S)^2 = 5$ and $h^0(S, K_S^{\otimes 2}) = 0$, so by classification, S is a dP_5.

Finally, X is rational because it is birational to $\mathrm{Gr}(2,4)$. •

Fano 3–65. [# K_{413}] $X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(0,0,1) \oplus \mathcal{O}(0,2,0)) \subset \mathbb{P}^2 \times \mathbb{P}^4 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 44,\ (-K)^4 = 188,\ \chi(T_X) = -2,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 6.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_6} Y$, with Y the fourfold Fano 2–47.

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(0,1) \oplus \mathcal{O}(2,0)) \subset \mathbb{P}^4 \times \mathrm{Gr}(2,4)$$

is the fourfold Fano 2–34, and

$$X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0,0)) \subset \mathbb{P}^2 \times Y.$$

The projection $\pi : X \to Y$ is birational.

By Proposition 2.2.1, Theorems 2.2.5 and 2.2.6, the fibers of π jump to a $\mathbb{P}^1$ along a surface

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}) \subset Y.$$

We have $e(S) = 6$, $K_S = \mathcal{O}(-1,-1)_{|S}$, $(-K_S)^2 = 4$ and $h^0(S, K_S^{\otimes 2}) = 0$, so by classification S is a dP_4. •

Fano 3–66. [# K_{423}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(0,1,0) \oplus \mathcal{O}(1,0,1) \oplus \mathcal{O}(0,0,1)^{\oplus 2}) \subset \mathbb{P}^1 \times \mathbb{P}^3 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 40,\ (-K)^4 = 167,\ \chi(T_X) = -8,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 11.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_3} Y$, with Y the fourfold Fano 2–29.

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 2}) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,5)$$

is the fourfold Fano 2–29, and

$$X = \mathscr{Z}(\mathcal{O}(1,0,1)) \subset \mathbb{P}^1 \times Y.$$

By Lemma 2.2.9, we have $X = \mathrm{Bl}_S Y$, with

$$S = \mathscr{Z}(\mathcal{O}(0,1)^{\oplus 2}) \subset Y.$$

Moreover, $e(S) = 9$, $K_S = \mathcal{O}(-1,0)_{|S}$, $(-K_S)^2 = 3$ and $h^0(S, K_S^{\otimes 2}) = 0$, so by classification S is a dP_3.

Since $X = \mathrm{Bl}_{\mathrm{dP}_3} Y$, and Y is rational, it follows that X is rational as well. •

Fano 3–67. [# K_{424}] $X = \mathscr{Z}(\mathcal{O}(1,0,1) \oplus \mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \oplus \mathcal{O}(0,0,0)^{\oplus 2}) \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathrm{Gr}(2,6)$

Invariants. $h^0(-K) = 39,\ (-K)^4 = 161,\ \chi(T_X) = -10,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 12.$

Description. $\pi : X \to \mathbb{P}^2 \times \mathrm{Gr}(2,6)$ is the map associated to $\mathrm{Bl}_{\mathrm{dP}_2} Y$, with Y the fourfold Fano 2–61

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \oplus \mathcal{O}(0,1)^{\oplus 2}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,6)$$

is the fourfold Fano 2–61, and

$$X = \mathscr{Z}(\mathcal{O}(1,0,1)) \subset \mathbb{P}^1 \times Y.$$

By Lemma 2.2.9, we have $X = \mathrm{Bl}_S Y$, with

$$S = \mathscr{Z}(\mathcal{O}(0,1)^{\oplus 2}) \subset Y.$$

Moreover, $e(S) = 10$, $K_S = \mathcal{O}(-1,0)_{|S}$, $(-K_S)^2 = 2$ and $h^0(S, K_S^{\otimes 2}) = 0$, so by classification S is a dP_2.

Since $X = \mathrm{Bl}_{\mathrm{dP}_2} Y$, and Y is rational, so it follows that X is rational as well. •

Fano 3–68. [#K_{428}] $X = \mathscr{Z}(\mathcal{O}(1,0,1) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0)^{\oplus 2}) \subset \mathbb{P}^1 \times \mathbb{P}^4 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 39,\ (-K)^4 = 163,\ \chi(T_X) = -5,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 9.$

Description. $\pi : X \to \mathrm{Gr}(2,4)$ is the map associated to $\mathrm{Bl}_{\mathrm{dP}_4} Y$ with Y the Fano fourfold Fano 2–21

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0)^{\oplus 2}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,6)$$

is the Fano fourfold Fano 2–21, and

$$X = \mathscr{Z}(\mathcal{O}(1,0,1)) \subset \mathbb{P}^1 \times Y.$$

By Lemma 2.2.9, we have $X = \mathrm{Bl}_S Y$, with

$$S = \mathscr{Z}(\mathcal{O}(0,1)^{\oplus 2}) \subset Y.$$

Moreover, $e(S) = 8$, $K_S = \mathcal{O}(-1,0)_{|S}$, $(-K_S)^2 = 4$ and $h^0(S, K_S^{\otimes 2}) = 0$, so by classification S is a dP_2.

Since $X = \mathrm{Bl}_{\mathrm{dP}_4} Y$, and Y is rational, it follows that X is rational as well. •

Fano 3–69. [#K_{430}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(1,0,1) \oplus \mathcal{O}(1,1,0) \oplus \mathcal{O}(0,0,1)) \subset \mathbb{P}^2 \times \mathbb{P}^3\ \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 39,\ (-K)^4 = 162,\ \chi(T_X) = -7,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 10.$

Description. $\pi : X \to \mathbb{P}^3 \times \mathrm{Gr}(2,4)$ is the map associated to $\mathrm{Bl}_{Q_2} Y$, with Y the fourfold Fano 2–43

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(0,1)) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,4)$$

is the fourfold Fano 2–43, and

$$X = \mathscr{Z}(\mathcal{O}(1,0,1)) \subset \mathbb{P}^1 \times Y.$$

By Lemma 2.2.9, we have $X = \mathrm{Bl}_S Y$, with

$$S = \mathscr{Z}(\mathcal{O}(0,1)^{\oplus 2}) \subset Y.$$

Projecting S onto $\mathrm{Gr}(2,4)$ gives a birationality between S and Q_2.

Moreover $e(S) = 4$, $K_S = \mathcal{O}(-2,0)_{|S}$, $(-K_S)^2 = 8$ and $h^0(S, 2K_S) = 0$, so S is a Q_2.

Since Y is rational, it follows that X is rational as well. •

Fano 3–70. [# K_{438}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2_1}(0,0,1) \oplus \mathcal{Q}_{\mathbb{P}^2_1}(0,1,1)) \subset \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathbb{P}^4$

Invariants. $h^0(-K) = 33$, $(-K)^4 = 131$, $\chi(T_X) = -15$, $h^{1,1} = 3$, $h^{3,1} = 0$, $h^{2,2} = 17$.

Description. $X = \mathrm{Bl}_{\mathrm{Bl}_{13\mathrm{pts}}\mathbb{P}^2} \mathbb{P}^2 \times \mathbb{P}^2$.

Identification. Note that X is birational to $\mathbb{P}^2_1 \times \mathbb{P}^2_2$ via the projection $\pi : X \to Y$.

To determine where the fiber of π degenerates to a $\mathbb{P}^1$, we argue as usual and, using Theorems 2.2.5 and 2.2.6, we find that the exceptional locus of π is a surface

$$S = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2_1}(1,0,0)^{\oplus 3} \oplus \mathcal{O}(1,0,1) \oplus \mathcal{O}(1,-1,1)) \subset \mathbb{P}^6 \times \mathbb{P}^2_1 \times \mathbb{P}^2_2.$$

Moreover, since $e(S) = 16$, $K_S = \mathcal{O}(1,-1,-1)_{|S}$, $(-K_S)^2 = -4$ and $h^0(S, K_S^{\otimes 2}) = 0$, we conclude that S is rational. Thus, $S = \mathrm{Bl}_{13\mathrm{pts}}\mathbb{P}^2$ and the final picture of X is

$$X = \mathrm{Bl}_{\mathrm{Bl}_{13\mathrm{pts}}\mathbb{P}^2} \mathbb{P}^2 \times \mathbb{P}^2.$$

Since X is birational to $\mathbb{P}^2 \times \mathbb{P}^2$, it follows that X is rational. •

Fano 3–71. [# K_{478}] $X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(0,1,0) \oplus \mathcal{O}(0,0,1)^{\oplus 3} \oplus \mathcal{O}(1,1,0)) \subset \mathbb{P}^1 \times \mathbb{P}^3 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 39$, $(-K)^4 = 162$, $\chi(T_X) = -5$, $h^{1,1} = 3$, $h^{3,1} = 0$, $h^{2,2} = 8$.

Description. $X = \mathrm{Bl}_{\mathrm{dP}_5} Y$, with Y the fourfold Fano 2–45

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 3}) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,5)$$

is the fourfold Fano 2–45, and

$$X = \mathscr{Z}(\mathcal{O}(1,1,0)) \subset \mathbb{P}^1 \times Y.$$

By Lemma 2.2.9, we obtain $X = \mathrm{Bl}_S Y$, with

$$S = \mathscr{Z}(\mathcal{O}(1,0)^{\oplus 2}) \subset Y.$$

Moreover, since $e(S) = 7$, $K_S = \mathcal{O}(1,-1)_{|S}$, $(-K_S)^2 = 5$ and $h^0(S, K_S^{\otimes 2}) = 0$, by classification, S is a dP_5. •

Fano 3–72. [# K_{154}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(0,0,1)) \subset \mathbb{P}^2 \times \mathbb{P}^3 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 69$, $(-K)^4 = 320$, $\chi(T_X) = 7$, $h^{1,1} = 3$, $h^{3,1} = 0$, $h^{2,2} = 4$.

Description. $X = \mathrm{Bl}_{\mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^2} Y$, with Y the fourfold Fano 2–43.

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(0,1)) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,4)$$

is the fourfold Fano 2–43, and

$$X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0)) \subset \mathbb{P}^2 \times Y.$$

The projection $\pi : X \to Y$ is birational.

By Theorems 2.2.5 and 2.2.6, the exceptional locus of π is a surface

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}) \subset Y.$$

Moreover, $\mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}) \subset Y$ can be identified with

$$\mathscr{Z}(\mathcal{Q}_2 \oplus \mathcal{O}(0,1)) \subset \mathrm{Fl}(1,2,4),$$

which, by Lemma 2.2.12 and Theorem 2.2.8, is $\mathscr{Z}(\mathcal{O}(1,1)) \subset \mathbb{P}^1 \times \mathbb{P}^2$, i.e. $\mathrm{Bl}_{\mathrm{pt}}\,\mathbb{P}^2$. In the end $X = \mathrm{Bl}_{\mathrm{Bl}_{\mathrm{pt}}\,\mathbb{P}^2}\, Y$.

Since Y is rational, it follows that X is rational as well. •

Fano 3–73. [#K_{404}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)} \oplus \mathcal{O}(0,1,0) \oplus \mathcal{O}(0,0,1) \oplus \mathcal{O}(1,0,1)) \subset \mathbb{P}^1 \times \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 45,\ (-K)^4 = 193,\ \chi(T_X) = -5,\ h^{1,1} =,\ h^{3,1} = 0,\ h^{2,2} = 9.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_4}\, Y$, with Y the fourfold Fano 2–28

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)} \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(0,1)) \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$$

is the fourfold Fano 2–28, and

$$X = \mathscr{Z}(\mathcal{O}(1,0,1)) \subset \mathbb{P}^1 \times Y.$$

Applying Lemma 2.2.9, we obtain $X = \mathrm{Bl}_S\, Y$, with

$$S = \mathscr{Z}(\mathcal{O}(0,1)^{\oplus 2}) \subset Y.$$

We have $e(S) = 8$, $K_S = \mathcal{O}(-1,0)_{|S}$, $(-K_S)^2 = 4$ and $h^0(S, K_S^{\otimes 2}) = 0$,so S is a dP_4.

Since $X = \mathrm{Bl}_{\mathrm{dP}_4}\, Y$, and Y is rational, it follows that $X$is rational. •

Fano 3–74. [#K_{406}] $X = \mathscr{Z}(\mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)}(1,0,0) \oplus \mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)}(0,1,0) \oplus \mathcal{O}(0,0,1)^{\oplus 2}) \subset \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 43,\ (-K)^4 = 183,\ \chi(T_X) = -4,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 8.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_6}\, Y$, with Y the fourfold Fano 2–46

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 2}) \subset \mathbb{P}^2 \times \mathrm{Gr}(2,5)$$

is the fourfold Fano 2–46, and

$$X = \mathscr{Z}(\mathcal{U}^{\vee}_{\mathrm{Gr}(2,5)}(1,0,0)) \subset \mathbb{P}^2 \times Y.$$

In particular, X is birational to Y via the projection $\pi : X \to Y$. By Theorems 2.2.5 and 2.2.6, the exceptional locus of π is a surface

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0,0)) \subset \mathbb{P}^1 \times Y.$$

We have $e(S) = 6$, $K_S = \mathcal{O}(1,-1,-1)_{|S}$, $(-K_S)^2 = 6$ and $h^0(S, K_S^{\otimes 2}) = 0$, so S is a dP_6.

Since $X = \mathrm{Bl}_{\mathrm{dP}_6}\, Y$, and Y is rational, it follows that X is rational as well. •

2.7. Fano varieties with Picard rank 4

Fano 4–1. [#K_{280}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,6)} \oplus \mathcal{O}(1,0) \oplus \mathrm{Sym}^2\, \mathcal{U}^{\vee}_{\mathrm{Gr}(2,6)}) \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,6)$

Invariants. $h^0(-K) = 51,\ (-K)^4 = 228,\ \chi(T_X) = 2,\ h^{1,1} = 4,\ h^{3,1} = 0,\ h^{2,2} = 10$

Description. $\pi : X \to \mathrm{Gr}(2,6)$ is the map associated to $\mathrm{Bl}_{\mathbb{P}^1 \bigsqcup \mathbb{P}^1}\, Y$ with Y the Fano fourfold obtained as a linear section in $\mathrm{Fl}(1,3,4)$.

Identification. Let

$$T = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,6)}) \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,6).$$

Applying Lemma 2.2.10 (3), we obtain that, away from a point, $T = \mathrm{Bl}_{Z'} \mathrm{Gr}(2,6)$, where

$$Z' = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^3} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,6)}) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,6).$$

Note that

$$X = \mathscr{Z}(\mathscr{Z}(\mathcal{O}_{\mathrm{Gr}(2,4)}(1) \oplus \mathrm{Sym}^2 \mathcal{U}^{\vee}_{\mathrm{Gr}(2,6)}) \subset T,$$

so the sections of $\mathcal{O}_{\mathrm{Gr}(2,4)}(1)$ and $\mathrm{Sym}^2 \mathcal{U}^{\vee}_{\mathrm{Gr}(2,6)}$ cut both Z' and $\mathrm{Gr}(2,6)$. In this way $X = \mathrm{Bl}_Z Y$, with

$$Z = \mathscr{Z}(\mathcal{O}_{\mathrm{Gr}(2,6)}(1) \oplus \mathrm{Sym}^2 \mathcal{U}^{\vee}_{\mathrm{Gr}(2,6)}) \subset Z'$$

and

$$Y = \mathscr{Z}(\mathrm{Sym}^2 \mathcal{U}^{\vee}_{\mathrm{Gr}(2,6)} \oplus \mathcal{O}_{\mathrm{Gr}(2,6)}(1)) \subset \mathrm{Gr}(2,6).$$

Observe that

$$\mathscr{Z}(\mathrm{Sym}^2 \mathcal{U}^{\vee}_{\mathrm{Gr}(2,6)}) \subset \mathrm{Gr}(2,6)$$

is $\mathrm{Fl}(1,3,4)$. Hence,

$$Y = \mathscr{Z}(\mathcal{O}(1,1)) \subset \mathrm{Fl}(1,3,4),$$

which us already described in Fano 3–13.

By Lemma 2.2.10 (1) and dualizing, we find that

$$Y = \mathscr{Z}(\mathcal{O}(1,1)^{\oplus 2}) \subset \mathbb{P}^3 \times \mathbb{P}^3,$$

which is exactly the fourfold described in Fano 3–23.

Finally, Z is simply the disjoint union of 2 $\mathbb{P}^1$s, and therefore $X = \mathrm{Bl}_{\mathbb{P}^1 \bigsqcup \mathbb{P}^1} Y$. •

Fano 4–2. [#K_{26}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,0,1,0)) \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 63,\ (-K)^4 = 290,\ \chi(T_X) = 6,\ h^{1,1} = 4,\ h^{3,1} = 0,\ h^{2,2} = 8.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_7} Y$, with Y the fourfold Fano 3–2

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0)) \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathrm{Gr}(2,4)$$

is the fourfold Fano 3–2. Hence,

$$X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0,0)) \subset \mathbb{P}^2 \times Y,$$

so in particular X is birational to Y via the projection $\pi : X \to Y$.

To determine where the fiber of π degenerates to a $\mathbb{P}^1$, we apply Theorems 2.2.5 and 2.2.6 and find that the exceptional locus of π is a surface

$$S = \mathscr{Z}(\mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}) \subset Y.$$

Note that $\mathscr{Z}(\mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)}) \subset Y$ is

$$\mathscr{Z}(\mathcal{O}(1,0,1) \oplus \mathcal{O}(0,1,1)) \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2,$$

which is a dP_7.

Since X is $\mathrm{Bl}_{\mathrm{dP}_7} Y$, it follows that X is rational, as Y is rational as well. •

Fano 4–3. [#K_{73}] $X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0,0,0) \oplus \mathcal{Q}_{\mathbb{P}^2_2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}) \subset \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5))$

Invariants. $h^0(-K) = 78,\ (-K)^4 = 369,\ \chi(T_X) = 11,\ h^{1,1} = 4,\ h^{3,1} = 0,\ h^{2,2} = 7.$

Description. $X = \mathrm{Bl}_{\mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^2} \mathrm{Bl}_{\mathbb{P}^1} Y$, with Y the Fano fourfold 6 in [Kal19, Table 3].

Identification. Using the same argument as in Fano 3–24, we obtain

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2_2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}) \subset \mathbb{P}^2_2 \times \mathrm{Gr}(2,5),$$

which is the Fano fourfold 6 in [Kal19, Table 3].

Let us consider

$$Z = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}) \subset \mathrm{Gr}(2,4) \times Y.$$

By Lemma 2.2.10 (2), we have $Z = \mathrm{Bl}_C Y$, with

$$C = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}) \subset Y.$$

In particular, $e(C) = 2$, $h^{1,0} = 0$, and has degree 1, hence, by classification, it is $\mathbb{P}^1$.

Next consider,

$$X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0,0,0)) \subset \mathbb{P}^2 \times Z,$$

and denote by

$$\pi' : X \to Z$$

the natural projection. This yields a birational map between X and Z.

By Theorems 2.2.5 and 2.2.6, the fibers of π' degenerate to $\mathbb{P}^1$ along the surface

$$S = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}) \subset Z.$$

We compute $e(S) = 4$, $K_S = \mathcal{O}(-1,-1,-1)_{|S}$, $(-K_S)^2 = 8$, hence S is a dP_8.

Explicitly,

$$S = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,4)} \oplus \mathcal{Q}_{\mathbb{P}^2_2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}) \subset \mathrm{Gr}(2,4) \times \mathbb{P}^2 \times \mathrm{Gr}(2,5).$$

Recall that $\mathrm{Gr}(2,3) = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}) \subset \mathrm{Gr}(2,4)$, and $\mathcal{Q}_{\mathrm{Gr}(2,4)|\,\mathrm{Gr}(2,3)} = \mathcal{Q}_{\mathrm{Gr}(2,3)} \oplus \mathcal{O}_{\mathrm{Gr}(2,3)}$. Therefore

$$S = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,4)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,3)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,4)}) \subset \mathrm{Gr}(2,3) \times \mathbb{P}^2_2 \times \mathrm{Gr}(2,4).$$

By Lemma 2.2.10 (2), the surface S can also be expressed as

$$\mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,4)}) \subset \mathbb{P}^2 \times \mathrm{Bl}_{\mathbb{P}^2} \mathrm{Gr}(2,4).$$

Consider the projection

$$\pi'' : S \to \mathbb{P}^2.$$

The general fiber is a point; hence S is birational to $\mathbb{P}^2$, and by classification it follows that $S = \mathrm{Bl}_{\mathrm{pt}} \mathbb{P}^2$.

Consequently, we obtain that X is $\mathrm{Bl}_{\mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^2} \mathrm{Bl}_{\mathbb{P}^1} Y$. Since Y is rational, it follows that X is rational as well. •

Fano 4–4. [#K_{144}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0,0) \oplus \mathcal{Q}_{\mathbb{P}^2_2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}) \subset \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5))$

Invariants. $h^0(-K) = 78,\ (-K)^4 = 369,\ \chi(T_X) = 11,\ h^{1,1} = 4,\ h^{3,1} = 0,\ h^{2,2} = 7.$

Description. $X = \mathrm{Bl}_{Q_2} \mathrm{Bl}_{\mathbb{P}^1} Y$, with Y the Fano fourfold 6 in [Kal19, Table 3].

Identification. The description is almost the same as before, except that

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}) \subset Z.$$

Consider the projection

$$\pi'' : S \to \mathbb{P}^2 \times \mathbb{P}^2_2.$$

By Proposition 2.2.1 π'' is surjective onto $\mathbb{P}^1 \times \mathbb{P}^1$, and its general fiber over $\mathbb{P}^1 \times \mathbb{P}^1$ is a point. Thus, S is birational to $\mathbb{P}^1 \times \mathbb{P}^1$, and therefore $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

Consequently, we obtain that X is $\mathrm{Bl}_{\mathbb{P}^1 \times \mathbb{P}^1} \mathrm{Bl}_{\mathbb{P}^1} Y$. As in the previous case, this Fano fourfold is rational. •

Fano 4–5. [#K_{256}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2_1} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{Q}_{\mathbb{P}^2_2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{Q}_{\mathbb{P}^2_3} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)}) \subset \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathbb{P}^2_3 \times \mathrm{Gr}(2,7)$

Invariants. $h^0(-K) = 81$, $(-K)^4 = 385$, $\chi(T_X) = 12$, $h^{1,1} = 4$, $h^{3,1} = 0$, $h^{2,2} = 7$.

Description. X is a small resolution of Y, 4 fold with $e(Y) = 16$, $\deg(Y) = 13$ and singular in a point.

Identification. Consider

$$Z = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2_2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{Q}_{\mathbb{P}^2_3} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)}) \subset \mathbb{P}^2_2 \times \mathbb{P}^2_3 \times \mathrm{Gr}(2,7).$$

Since $\mathbb{P}^2 \times \mathbb{P}^2$ can be also seen as

$$\mathscr{Z}(\mathcal{Q}^\vee_{\mathrm{Gr}(2,6)}(1)) \subset \mathrm{Gr}(2,6)$$

we can rewrite everything as

$$\mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,6)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{Q}^\vee_{\mathrm{Gr}(2,6)}(1)) \subset \mathrm{Gr}(2,6) \times \mathrm{Gr}(2,7).$$

Thus, by Corollary 2.2.11 (2),

$$Z = \mathscr{Z}(\mathcal{Q}_2 \oplus \mathcal{Q}^\vee_2(1)) \subset \mathrm{Fl}(2,3,7).$$

Hence,

$$X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2_1} \boxtimes \mathcal{U}^\vee_1) \subset \mathbb{P}^2_1 \times Z.$$

The fiber of the projection $\pi : X \to Z$ is generically empty; therefore, by Proposition 2.2.1, X is birational to the fourfold $Y = \mathrm{D}_1(\varphi)$, with

$$\varphi : \mathcal{U}_{1|Z} \to \mathcal{O}^{\oplus 3}_{\mathrm{Fl}(2,3,7)|Z}.$$

By Theorems 2.2.5 and 2.2.6, Y is a fourfold with $e(Y) = 16$ and $\deg(Y) = 13$.

Moreover, since $\mathrm{D}_2(\varphi_{|Z})$ consists of a point, Y is singular there, and the projection to Z contractsf a $\mathbb{P}^2$. Equivalently, X is a small resolution of Y.

Now consider the projection $\pi_{123} : X \to \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathbb{P}^2_3$. By Proposition 2.2.1, the fiber is a point along the $\mathrm{D}_5(\varphi)$, with

$$\varphi : \mathcal{Q}^\vee_{\mathbb{P}^2_1} \oplus \mathcal{Q}^\vee_{\mathbb{P}^2_2} \oplus \mathcal{Q}^\vee_{\mathbb{P}^2_3} \to V^\vee_7 \otimes \mathcal{O}_{\mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathbb{P}^2_3}.$$

By Theorems 2.2.5 and 2.2.6, $\mathrm{D}_5(\varphi)$ is a fourfold Y' with $e(Y') = 16$. Furthermore, again by Theorems 2.2.5 and 2.2.6, the fiber of π_{123} jumps to a $\mathbb{P}^2$ over $\mathrm{D}_4(\varphi) = \{p\}$. Thus, X is a small resolution of Y'. •

Fano 4–6. [#K_{78}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2_2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0,0,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(0,1,0,0)) \subset \mathbb{P}^3 \times \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 71,\ (-K)^4 = 331,\ \chi(T_X) = 8,\ h^{1,1} = 4,\ h^{3,1} = 0,\ h^{2,2} = 7.$

Description. $X = \mathrm{Bl}_{\mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^2} Y$, with Y the fourfold Fano 3–48.

Identification. Let

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2_2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0,0)) \subset \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathrm{Gr}(2,5).$$

Note that Y is the fourfold Fano 3–48. Hence,

$$X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0,0,0)) \subset \mathbb{P}^3 \times Y,$$

so X is birational to Y via the projection $\pi : X \to Y$.

By Theorems 2.2.5 and 2.2.6, the fiber of π degenerates to a $\mathbb{P}^1$ along a surface

$$S = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}) \subset Y.$$

We compute $e(S) = 4$, $K_S = \mathcal{O}(-1,-1,-1)_{|S}$, $(-K_S)^2 = 8$, hence S is a dP_8. Projecting S onto $\mathbb{P}^2_2$, the general fiber consists of a single point. Thus S is birational to $\mathbb{P}^2$, and consequently $S = \mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^2$. •

Fano 4–7. [#K_{104}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2_3} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0,0,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(0,1,0,0)) \subset \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathbb{P}^2_3 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 66,\ (-K)^4 = 305,\ \chi(T_X) = 6,\ h^{1,1} = 4,\ h^{3,1} = 0,\ h^{2,2} = 8.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_7} Y$, with Y the fourfold Fano 3–48.

Identification. Note that the argument is exactly the same as in Fano 4–6, but in this case the exceptional locus is a surface

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0,0,0)) \subset \mathbb{P}^1 \times Y.$$

We compute $e(S) = 4$, $K_S = \mathcal{O}(1,-1,-1,-1)_{|S}$, $(-K_S)^2 = 7$ and $h^0(S, K_S^{\otimes 2}) = 0$. Hence S is a dP_7. •

Fano 4–8. [#K_{201}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(0,0,1,0)) \subset \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathbb{P}^2_3 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 61,\ (-K)^4 = 278,\ \chi(T_X) = 6,\ h^{1,1} = 4,\ h^{3,1} = 0,\ h^{2,2} = 6.$

Description. $X = \mathrm{Bl}_{\mathbb{P}^2}$, with Y the fourfold Fano 3–2.

Identification. Recall that

$$Y = \mathscr{Z}((\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0)) \subset \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathrm{Gr}(2,4)$$

is the fourfold Fano 3–2. Hence,

$$X = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0,0,0)) \subset \mathbb{P}^2 \times Y,$$

so X is birational to Y via the projection $\pi : X \to Y$.

By Theorems 2.2.5 and 2.2.6, the fiber of π degenerates to a $\mathbb{P}^1$ along a surface

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}) \subset Y.$$

We compute $e(S) = 3$, $K_S = \mathcal{O}(-1,-1,-1)_{|S}$, $(-K_S)^2 = 9$, hence S is $\mathbb{P}^2$.

Since Y is rational, it follows that X is rational as well. •

Fano 4–9. [#K_{204}] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2_2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(0,1,0,0) \oplus \mathcal{O}(1,0,0,1)) \subset \mathbb{P}^1 \times \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathrm{Gr}(2,5)$

Invariants. $h^0(-K) = 57,\ (-K)^4 = 257,\ \chi(T_X) = 2,\ h^{1,1} = 4,\ h^{3,1} = 0,\ h^{2,2} = 9.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_6} Y$, with Y the fourfold Fano 3–48.

Identification. Recall that

$$Y = \mathscr{Z}((\mathcal{Q}_{\mathbb{P}^2_2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0,0)) \subset \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathrm{Gr}(2,5)$$

is, by the same argument in Fano 4–6, the Fano fourfold 17 in [Kal19, Table 3]. Hence,

$$X = \mathscr{Z}(\mathcal{O}(1,0,0,1)) \subset \mathbb{P}^1 \times Y.$$

By Lemma 2.2.9, we have $X = \mathrm{Bl}_S Y$, where

$$S = \mathscr{Z}(\mathcal{O}(0,0,1)^{\oplus 2}) \subset Y.$$

In particular, $e(S) = 6$, $K_S = \mathcal{O}(-1,-1,0)_{|S}$, $(-K_S)^2 = 6$ and $h^0(S, K_S^{\otimes 2}) = 0$, so S is a dP_6. •

Fano 4–10. [#K_{227}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,0,1,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0,0) \oplus \mathcal{O}(1,0,0,1)) \subset \mathbb{P}^1 \times \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 53,\ (-K)^4 = 236,\ \chi(T_X) = 1,\ h^{1,1} = 4,\ h^{3,1} = 0,\ h^{2,2} = 9.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_6}$, with Y the fourfold Fano 3–2.

Identification. Recall that

$$Y = \mathscr{Z}((\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0)) \subset \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathrm{Gr}(2,4)$$

is the fourfold Fano 3–2. Hence,

$$X = \mathscr{Z}(\mathcal{O}(1,0,0,1)) \subset \mathbb{P}^1 \times Y.$$

By Lemma 2.2.9, we have $X = \mathrm{Bl}_S Y$, where

$$S = \mathscr{Z}(\mathcal{O}(0,0,1)^{\oplus 2}) \subset Y.$$

In particular, $e(S) = 6$, $K_S = \mathcal{O}(-1,-1,0)_{|S}$, $(-K_S)^2 = 6$ and $h^0(S, K_S^{\otimes 2}) = 0$, so S is a dP_4.

Since Y is rational, it follows that X is rational as well. •

Fano 4–11. [#K_{297}] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,0,1,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0,0) \oplus \mathcal{O}(1,0,1,0) \oplus \mathcal{O}(0,0,0,1)) \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^3 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 55,\ (-K)^4 = 246,\ \chi(T_X) = 3,\ h^{1,1} = 4,\ h^{3,1} = 0,\ h^{2,2} = 7.$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_7} Y$, with Y the fourfold Fano 3–72.

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0) \oplus \mathcal{O}(0,0,1)) \subset \mathbb{P}^2 \times \mathbb{P}^3 \times \mathrm{Gr}(2,4)$$

is the fourfold Fano 3–2. Hence

$$X = \mathscr{Z}(\mathcal{O}(1,0,1,0)) \subset \mathbb{P}^1 \times Y.$$

By Lemma 2.2.9, we have $X = \mathrm{Bl}_S Y$, where

$$S = \mathscr{Z}(\mathcal{O}(0,1,0)^{\oplus 2}) \subset Y.$$

In particular, $e(S) = 5$, $K_S = \mathcal{O}(-1,0,-1)_{|S}$, $(-K_S)^2 = 7$ and $h^0(S, K_S^{\otimes 2}) = 0$, so S is a dP_7. •

Fano 4–12. [$\#K_{403}$] $X = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,0,1,0) \oplus \mathcal{O}(1,0,1,0) \oplus \mathcal{O}(0,1,0,1) \oplus \mathcal{O}(0,0,0,1)) \subset \mathbb{P}^1_1 \times \mathbb{P}^1_2 \times \mathbb{P}^3 \times \mathrm{Gr}(2,4)$

Invariants. $h^0(-K) = 47$, $(-K)^4 = 204$, $\chi(T_X) = 0$, $h^{1,1} = 4$, $h^{3,1} = 0$, $h^{2,2} = 8$.

Description. $X = \mathrm{Bl}_{\mathrm{dP}_6} Y$, with Y the fourfold Fano 3–18.

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(1,0,1) \oplus \mathcal{O}(0,0,1)) \subset \mathbb{P}^1_2 \times \mathbb{P}^3 \times \mathrm{Gr}(2,4)$$

is the fourfold Fano 3–2. Hence,

$$X = \mathscr{Z}(\mathcal{O}(1,0,1,0)) \subset \mathbb{P}^1 \times Y.$$

By Lemma 2.2.9, we have $X = \mathrm{Bl}_S Y$, where

$$S = \mathscr{Z}(\mathcal{O}(0,1,0)^{\oplus 2}) \subset Y.$$

In particular, $e(S) = 6$, $K_S = \mathcal{O}(-1,0,-1)_{|S}$, $(-K_S)^2 = 6$ and $h^0(S, K_S^{\otimes 2}) = 0$, so, by classification, S is a dP_6. •

Fano 4–13. [$\#K_{482}$] $X = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2_1}(0,0,1,1) \oplus \mathcal{O}(1,1,0,0)) \subset \mathbb{P}^1 \times \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathbb{P}^2_3$

Invariants. $h^0(-K) = 39$, $(-K)^4 = 162$, $\chi(T_X) = -4$, $h^{1,1} = 4$, $h^{2,1} = 1$, $h^{3,1} = 0$, $h^{2,2} = 9$

Description. $X = \mathrm{Bl}_{\mathrm{dP}_6} Y$, with Y the fourfold Fano 3–38.

Identification. Recall that

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2_1}(0,1,1)) \subset \mathbb{P}^2 \times \mathbb{P}^2_2 \times \mathbb{P}2_3$$

is the Fano fourfold Fano 3–38. Hence,

$$X = \mathscr{Z}(\mathcal{O}(1,1,0,0)) \subset \mathbb{P}^1 \times Y.$$

By Lemma 2.2.9, we have $X = \mathrm{Bl}_S Y$, with

$$S = \mathscr{Z}(\mathcal{O}(1,0,0)^{\oplus 2}) \subset Y.$$

Note that

$$\mathscr{Z}(\mathcal{O}(1,0,0)^{\oplus 2}) \subset Y \text{ is } \mathscr{Z}(\mathcal{O}(1,1)) \subset \mathbb{P}^2 \times \mathbb{P}^2,$$

hence it is a dP_6. •

2.8. Summary table

In the table, we adopt the following notation:

- K_n is the family of Fano fourfolds in the n-th entry of [**Kal19**, Table 3];
- X_n^d are the d-folds obtained as complete intersections of linear sections in Grassmannians $\mathrm{Gr}(2, n)$;
- Str. p. bdl. on T, with T a threefold, is a fourfold obtained as a general $\mathbb{P}^1$-bundle, whose fiber jumps to a projective space of higher dimension along a certain number of points;
- Conic bdl. on T, is a conic bundle over a threefol d T;
- $F_{\rho-n}$ is the Fano threefold $\rho - n$ in [**Bel**];
- Q_n is the n-th dimensional quadrics;
- dP_n is a Del Pezzo surface of volume n;
- E is an elliptic curve;
- $\mathcal{E}$ is an elliptic fibration of dimension two;
- Small Res., is a Fano fourfold obtained as a small resolution of a singular fourfold.

Table 1. Fano fourfolds.

X	Id	$h^0(-K)$	$(-K)^4$	$h^{1,1}$	$h^{2,1}$	$h^{3,1}$	$h^{2,2}$	Amb.	$\mathcal{F}$	Desc.	Rat.
Fano 1-1	K_{742}	9	15	1	0	41	232	$\mathrm{Gr}(2,5)$	$\mathcal{O}(1)\oplus\mathcal{O}(3)$	[**Kü95**, b1]	?
Fano 1-2	K_{740}	10	20	1	0	20	132	$\mathrm{Gr}(2,5)$	$\mathcal{O}(2)^{\oplus 2}$	[**Kü95**, b2]	?
Fano 1-3	K_{715}	15	42	1	0	6	57	$\mathrm{Gr}(2,6)$	$\mathcal{Q}(1)$	[**Kü95**, b3]	?
Fano 1-4	K_{730}	13	33	1	0	8	70	$\mathrm{Gr}(2,6)$	$\mathcal{U}^\vee(1)\mathcal{O}(1)^{\oplus 2}$	[**Kü95**, b5]	?
Fano 1-5	K_{734}	12	28	1	0	15	106	$\mathrm{Gr}(2,6)$	$\mathcal{O}(1)^{\oplus 3}\oplus\mathcal{O}(2)$	[**Kü95**, b6]	?
Fano 1-6	K_{24}	85	405	1	0	0	2	$\mathrm{Gr}(2,5)$	$\mathcal{O}(1)^{\oplus 2}$	X_5^4	+
Fano 1-7	K_{365}	51	224	1	0	0	8	$\mathrm{Gr}(2,6)$	$\mathcal{O}(1)^{\oplus 4}$	X_6^4	+
Fano 1-8	K_{433}	45	192	1	0	0	12	$\mathrm{Gr}(2,5)$	$\mathcal{U}^\vee(1)$	[**CGKS20**, V_{12}^4]	?
Fano 1-9	K_{496}	39	160	1	0	1	22	$\mathrm{Gr}(2,5)$	$\mathcal{O}(1)\oplus\mathcal{O}(2)$	[**BFMT21**, X_{10}]	?
Fano 2-1	K_{508}	39	160	2	0	0	7	$\mathbb{P}^2\times\mathrm{Gr}(2,5)$	$\mathcal{O}(1,1)\oplus\mathcal{O}(0,1)^{\oplus 3}$	Str. p. bdl. on X_5^3	+
Fano 2-2	K_{509}	33	130	2	0	1	23	$\mathrm{Gr}(2,4)\times\mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}\boxtimes\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}\oplus\mathcal{O}(0,1)\oplus\mathcal{O}(1,1)$	[**BFMT21**, GM-22]	+
Fano 2-3	K_{517}	33	131	2	0	0	12	$\mathbb{P}^5\times\mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0)\oplus\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0)\oplus\mathcal{O}(0,1)^{\oplus 2}$	$\mathrm{Bl}_{\mathrm{Bl}_{9\mathrm{pts}}\mathbb{P}^2}X_5^4$	+
Fano 2-4	K_{527}	30	114	2	0	0	17	$\mathbb{P}^2\times\mathrm{Gr}(2,7)$	$\mathcal{Q}_{\mathbb{P}^2}\boxtimes\mathcal{U}^\vee_{\mathrm{Gr}(2,7)}\oplus\mathcal{O}(0,1)^{\oplus 4}$	Small res.	+
Fano 2-5	K_{558}	30	114	2	0	1	24	$\mathrm{Gr}(2,4)\times\mathrm{Gr}(2,5$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}\boxtimes\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}\oplus\mathcal{O}(1,0)\oplus\mathcal{O}(0,2)$	[**BFMT21**, GM-21]	?
Fano 2-6	K_{559}	29	110	2	0	1	22	$\mathbb{P}^4\times\mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0)\oplus\mathcal{O}(0,1)^{\oplus 2}\oplus\mathcal{O}(1,1)$	[**BFMT21**, K3-32]	+
Fano 2-7	K_{566}	26	94	2	0	1	28	$\mathbb{P}^3\times\mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0)\oplus\mathcal{O}(0,1)\oplus\mathcal{O}(0,2)$	[**BFMT21**, GM-20]	?
Fano 2-8	K_{547}	30	116	2	0	0	15	$\mathrm{Gr}(2,4)\times\mathbb{P}^6$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1)\oplus\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(0,1)^{\oplus 2}$	$\mathrm{Bl}_{\mathrm{Bl}_{12\mathrm{pts}}\mathbb{P}^2}\mathrm{Gr}(2,4)$	+
Fano 2-9	K_{549}	32	126	2	0	0	10	$\mathbb{P}^5\times\mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0)\oplus\mathcal{O}(0,1)^{\oplus 3}\oplus\mathcal{O}(2,0)$	Conic bdl. on X_5^3	+
Fano 2-10	K_{552}	30	116	2	0	0	13	$\mathbb{P}^4\times\mathrm{Gr}(2,5)$	$\mathcal{O}(0,1)^{\oplus 2}\oplus\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0)^{\oplus 2}$	$\mathrm{Bl}_{\mathrm{Bl}_{10\mathrm{pts}}\mathbb{P}^2}X_5^4$	+
Fano 2-11	K_{25}	70	325	2	0	0	4	$\mathrm{Gr}(2,4)\times\mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}\boxtimes\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}\oplus\mathcal{O}(0,1)^{\oplus 2}$	$\mathrm{Bl}_{\mathbb{P}^1}X_5^4$	+
Fano 2-12	K_{554}	24	90	2	0	0	12	$\mathbb{P}^3\times\mathrm{Gr}(2,4)$	$\mathcal{O}(1)\boxtimes Sym^2\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}$	[**Kuz19**]	+
Fano 2-13	K_{582}	30	116	2	0	0	10	$\mathbb{P}^4\times\mathrm{Gr}(3,5)$	$\mathcal{Q}_{\mathrm{Gr}(3,5)}(1,0)\oplus\mathcal{O}(0,1)^{\oplus 3}\oplus\mathcal{O}(2,0)$	Double conic bdl.	+
Fano 2-14	K_{583}	28	106	2	0	0	15	$\mathrm{Gr}_1(2,4)\times\mathrm{Gr}_2(2,4)$	$\mathcal{U}^\vee_{\mathrm{Gr}_1(2,4)}(0,1)\oplus\mathcal{U}^\vee_{\mathrm{Gr}_2(2,4)}(1,0)$	Small res.	+
Fano 2-15	K_{589}	24	85	2	0	0	22	$\mathbb{P}^2\times\mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathbb{P}^2}(0,1)^{\oplus 2}$	Small res.	+
Fano 2-16	K_{579}	27	101	2	0	1	23	$\mathbb{P}^5\times\mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0)^{\oplus 2}\oplus\mathcal{O}(1,1)$	[**BFMT21**, C-9]	+
Fano 2-17	K_{597}	27	101	2	0	1	24	$\mathbb{P}^5\times\mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0)\oplus\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0)\oplus\mathcal{O}(1,1)$	[**BFMT21**, K3-33]	+
Fano 2-18	K_{600}	27	100	2	0	0	18	$\mathbb{P}^6\times\mathrm{Gr}(2,4)$	$\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0)\oplus\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0)\oplus\mathcal{O}(0,1)\oplus\mathcal{O}(2,0)$	$\mathrm{Bl}_{\mathrm{Bl}_{9\mathrm{pts}}\mathbb{P}^2}Q_5\cap Q_5'$	?
Fano 2-19	K_{603}	27	100	2	0	0	16	$\mathbb{P}^2\times\mathrm{Gr}(2,6)$	$\mathcal{U}^\vee_{\mathrm{Gr}(2,6)}(1,0)\oplus\mathcal{O}(0,1)^{\oplus 4}$	$\mathrm{Bl}_{\mathrm{dP}_2}X_6^4$	+
Fano 2-20	K_{604}	26	95	2	0	1	23	$\mathbb{P}^3\times\mathrm{Gr}(3,5)$	$\mathcal{Q}_{\mathrm{Gr}(3,5)}(1,0)\oplus\mathcal{O}(1,1)\oplus\mathcal{O}(0,1)^{\oplus 2}$	[**BFMT21**, K3-34]	+
Fano 2-21	K_{29}	60	273	2	0	0	3	$\mathbb{P}^4\times\mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0)^{\oplus 2}$	$\mathrm{Bl}_{\mathbb{P}^2}\mathrm{Gr}(2,4)$	+
Fano 2-22	K_{612}	30	113	2	5	0	3	$\mathbb{P}^2\times\mathbb{P}^4$	$\mathcal{Q}(0,2)$	$\mathrm{Bl}_{Q_2\cap Q_2'\cap Q_2''}\mathbb{P}^4$	+
Fano 2-23	K_{614}	24	86	2	0	1	26	$\mathbb{P}^4\times\mathbb{P}^5$	$\mathcal{O}(1,1)\oplus\bigwedge^3\mathcal{Q}_{\mathbb{P}^4}(0,1)$	[**BFMT21**, C-7]	+
Fano 2-24	K_{616}	26	96	2	2	0	10	$\mathbb{P}^4\times\mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0)\oplus\mathcal{O}(0,2)\oplus\mathcal{O}(2,0)$	Conic bdl. on $\mathscr{Z}(\mathcal{O}(2))\subset\mathrm{Gr}(2,4)$	?
Fano 2-25	K_{617}	25	90	2	0	0	19	$\mathbb{P}^2\times\mathrm{Gr}(2,5)$	$\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(0,1)\oplus\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0)$	$\mathrm{Bl}_{\mathrm{dP}_3}K_{433}$	+
Fano 2-26	K_{32}	51	225	2	1	0	3	$\mathbb{P}^4_1\times\mathbb{P}^4_2$	$\mathcal{Q}^\vee_{\mathbb{P}^4_1}(1,1)$	$\mathrm{Bl}_S\mathbb{P}^4$	+

Continued on next page

Table 1 continued from previous page

X	Id	$h^0(-K)$	$(-K)^4$	$h^{1,1}$	$h^{2,1}$	$h^{3,1}$	$h^{2,2}$	Amb.	$\mathcal{F}$	Desc.	Rat.
Fano 2-27	K_{109}	60	272	2	0	0	2	$\mathbb{P}^4 \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 3}$	$\mathrm{Bl}_{\nu_2(\mathbb{P}^2)} \mathbb{P}^4$	+
Fano 2-28	K_{101}	75	352	2	0	0	3	$\mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{O}(0,1) \oplus \mathcal{O}(1,0)$	$\mathrm{Bl}_{\mathbb{P}^2} X_5^4$	+
Fano 2-29	K_{102}	69	320	2	0	0	4	$\mathbb{P}^3 \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 2}$	$\mathrm{Bl}_{Q_2} X_5^4$	+
Fano 2-30	K_{625}	24	85	2	0	2	32	$\mathbb{P}^4 \times \mathrm{Gr}(2,4)$	$\mathcal{O}(1,1)^{\oplus 2} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0)$	$\mathrm{Bl}_{\mathcal{E}} \mathrm{Gr}(2,4)$	+
Fano 2-31	K_{626}	25	90	2	0	1	24	$\mathbb{P}^5 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(0,1) \oplus \mathcal{O}(1,1) \oplus \mathcal{O}(2,0)$	[**BFMT21**, K3-35]	+
Fano 2-32	K_{628}	23	80	2	0	1	28	$\mathbb{P}^3 \times \mathbb{P}^6$	$\bigwedge^2 \mathcal{Q}_{\mathbb{P}^3}(0,1) \oplus \mathcal{O}(1,1) \oplus \mathcal{O}(0,2)$	[**BFMT21**, K3-31]	+
Fano 2-33	K_{629}	23	80	2	0	1	27	$\mathbb{P}^2 \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathbb{P}^2}(0,1) \oplus \mathcal{O}(1,1) \oplus \mathcal{O}(0,1)$	[**BFMT21**, GM-19]	+
Fano 2-34	K_{115}	60	272	2	0	0	4	$\mathbb{P}^4 \times \mathrm{Gr}(2,4)$	$\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0)$	$\mathrm{Bl}_{Q_2} \mathrm{Gr}(2,4)$	+
Fano 2-35	K_{126}	51	224	2	0	0	7	$\mathbb{P}^4 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(1,1) \oplus \mathcal{O}(0,1)$	Str. p. bdl. on Q_3	+
Fano 2-36	K_{124}	54	240	2	0	0	7	$\mathbb{P}^3 \times \mathrm{Gr}(2,4)$	$\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(1,1)$	$\mathrm{Bl}_{\mathrm{dP}_5} \mathrm{Gr}(2,4)$	+
Fano 2-37	K_{642}	24	82	2	0	3	42	$\mathbb{P}^2 \times \mathrm{Gr}(2,6)$	$\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \oplus \mathcal{O}(0,1) \oplus \mathcal{O}(0,2)$	Small res.	?
Fano 2-38	K_{653}	22	75	2	0	2	32	$\mathbb{P}^2 \times \mathrm{Gr}(2,5)$	$\mathcal{O}(0,1)^{\oplus 2} \oplus \mathcal{O}(1,1)^{\oplus 2}$	$\mathrm{Bl}_{\mathcal{E}} X_5^4$	+
Fano 2-39	K_{663}	30	112	2	5	0	2	$\mathbb{P}^2 \times \mathrm{Gr}(2,5)$	$\mathcal{O}(2,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,5)}(0,1)$	$Q_1 \times F_{1-7}$	?
Fano 2-40	K_{664}	21	69	2	0	5	52	$\mathbb{P}^4 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(0,1) \oplus (2,1)$	Conic bdl. on Q_3	+
Fano 2-41	K_{665}	20	64	2	0	6	59	$\mathbb{P}^3 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(1,2)$	$\mathrm{Bl}_S \mathrm{Gr}(2,4)$	+
Fano 2-42	K_{669}	21	70	2	0	3	37	$\mathbb{P}^2 \times \mathrm{Gr}(2,5)$	$\mathcal{O}(2,1) \oplus \mathcal{O}(0,1)^{\oplus 3}$	Conic bdl. on X_5^3	+
Fano 2-43	K_{136}	81	384	2	0	0	2	$\mathbb{P}^3 \times \mathrm{Gr}(2,4)$	$\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(0,1)$	$\mathbb{P}_{Q_3}(\mathcal{U}_{\mathrm{Gr}(2,4)\vert Q_3})$	+
Fano 2-44	K_{723}	16	45	2	0	8	72	$\mathbb{P}^1 \times \mathrm{Gr}(2,5)$	$\mathcal{O}(0,1)^{\oplus 2} \oplus \mathcal{O}(1,2)$	$\mathrm{Bl}_{Q_3 \cap Q_3'} X_5^4$	+
Fano 2-45	K_{219}	51	244	2	0	0	3	$\mathbb{P}^3 \times \mathrm{Gr}(2,5)$	$\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 3}$	Conic bdl. on $\mathbb{P}^3$	?
Fano 2-46	K_{195}	63	288	2	0	0	4	$\mathbb{P}^2 \times \mathrm{Gr}(2,5)$	$\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 2}$	$\mathrm{Bl}_{Q_2} X_5^4$	+
Fano 2-47	K_{230}	54	240	2	0	0	2	$\mathbb{P}^4 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(0,1) \oplus \mathcal{O}(2,0)$	Conic bdl. on Q_3	?
Fano 2-48	K_{238}	45	192	2	2	0	2	$\mathbb{P}^3 \times \mathrm{Gr}(2,4)$	$\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(0,2)$	$\mathbb{P}_{F_{1-4}}(\mathcal{U}_{\mathrm{Gr}(2,4)\vert F_{1-4}})$	+
Fano 2-49	K_{679}	18	56	2	0	4	47	$\mathbb{P}^2 \times \mathrm{Gr}(2,4)$	$\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,1)$	$\mathrm{Bl}_S \mathrm{Gr}(2,4)$	+
Fano 2-50	K_{253}	42	177	2	0	0	11	$\mathbb{P}^4 \times \mathrm{Gr}(2,8)$	$\mathcal{Q}_{\mathbb{P}^4} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,8)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,8)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 2}$	Small res.	+
Fano 2-51	K_{142}	75	352	2	0	0	4	$\mathbb{P}^2 \times \mathrm{Gr}(2,6)$	$\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \oplus \mathcal{O}(0,1)^{\oplus 2}$	Small res.	?
Fano 2-52	K_{686}	27	96	2	7	0	2	$\mathbb{P}^2 \times \mathrm{Gr}(2,5)$	$\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(0,1) \oplus \mathcal{O}(0,1) \oplus \mathcal{O}(2,0)$	$Q_1 \times F_{1-6}$	+
Fano 2-53	K_{689}	17	51	2	0	2	41	$\mathbb{P}^2 \times \mathbb{P}^5$	$\mathcal{Q}_{\mathbb{P}^2}(0,2) \oplus \mathcal{O}(1,1)$	[**BFMT21**, A-67]	?
Fano 2-54	K_{699}	19	60	2	0	1	32	$\mathbb{P}^1 \times \mathrm{Gr}(2,5)$	$\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(0,1) \oplus \mathcal{O}(1,1)$	[**BFMT21**, K3-24]	+
Fano 2-55	K_{288}	54	240	2	0	0	7	$\mathbb{P}^1 \times \mathrm{Gr}(2,5)$	$\mathcal{O}(1,1) \oplus (0,1)^{\oplus 2}$	$\mathrm{Bl}_{\mathrm{dP}_5} X_5^4$	+
Fano 2-56	K_{464}	39	162	2	0	0	8	$\mathrm{Gr}(2,5) \times \mathrm{Gr}(3,5)$	$\mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \boxtimes \mathcal{Q}_{\mathrm{Gr}(3,5)} \oplus \mathcal{O}(1,0)^{\oplus 2} \oplus \mathcal{O}(0,1)^{\oplus 2}$	$\mathrm{Bl}_{\mathrm{dP}_6} X_5^4$	+
Fano 2-57	K_{350}	42	177	2	0	0	12	$\mathrm{Gr}(2,4) \times \mathrm{Gr}(3,6)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,6)} \oplus \mathcal{O}(0,1)^{\oplus 3}$	$\mathrm{Bl}_{\mathrm{Bl}_{9\mathrm{pts}} \mathbb{P}^2} \mathrm{Gr}(2,4)$	+
Fano 2-58	K_{361}	33	129	2	0	1	23	$\mathbb{P}^3 \times \mathrm{Gr}(2,7)$	$\mathcal{Q}_{\mathbb{P}^3} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{O}(0,1)^{\oplus 2} \oplus \mathcal{O}(1,1)$	[**BFMT21**, R-62]	?
Fano 2-59	K_{364}	69	320	2	0	0	2	$\mathbb{P}^2 \times \mathrm{Gr}(2,5)$	$\mathcal{O}(2,0) \oplus \mathcal{O}(0,1)^{\oplus 3}$	$Q_1 \times F_{1-15}$	+
Fano 2-60	K_{374}	45	193	2	0	0	15	$\mathbb{P}^6 \times \mathrm{Gr}(2,9)$	$\mathcal{Q}_{\mathbb{P}^6} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,9)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,9)}(1,0)^{\oplus 2}$	Small res.	?
Fano 2-61	K_{385}	45	193	2	0	0	9	$\mathrm{Gr}(2,5) \times \mathrm{Gr}(2,6)$	$mQ_{\mathrm{Gr}(2,5)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \oplus \mathcal{O}(1,0)^{\oplus 2} \oplus \mathcal{O}(0,1)^{\oplus 2}$	$\mathrm{Bl}_{\mathbb{P}^2} X_6^4$	+
Fano 2-62	K_{15}	90	433	2	0	0	3	$\mathbb{P}^4 \times \mathrm{Gr}(2,7)$	$\mathcal{Q}_{\mathbb{P}^4} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,7)}(1,0)$	$\mathrm{Bl}_{Q_1} \mathbb{P}^4$	+

Continued on next page

Table 1 continued from previous page

X	Id	$h^0(-K)$	$(-K)^4$	$h^{1,1}$	$h^{2,1}$	$h^{3,1}$	$h^{2,2}$	Amb.	$\mathcal{F}$	Desc.	Rat.
Fano 2-63	K_{407}	39	161	2	0	1	23	$\mathbb{P}^5 \times \mathrm{Gr}(2,8)$	$\mathcal{Q}_{\mathbb{P}^5} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,8)} \oplus \mathcal{U}_{\mathrm{Gr}(2,5)^\vee(1,0)} \oplus \mathcal{O}(1,1)$	[**BFMT21**, C-5]	?
Fano 2-64	K_{473}	36	146	2	0	0	14	$\mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(0,1)$	$\mathrm{Bl}_{\mathbb{P}^2} K_{433}$	?
Fano 2-65	K_{439}	35	141	2	0	0	13	$\mathbb{P}^4 \times \mathrm{Gr}(2,6)$	$\mathcal{Q}_{\mathrm{Gr}(2,6)}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 4}$	$\mathrm{Bl}_{\mathrm{dP}_5} X_6^4$	+
Fano 2-66	K_{717}	17	50	2	0	2	42	$\mathbb{P}^1 \times \mathrm{Gr}(2,5)$	$\mathcal{O}(0,1) \oplus \mathcal{O}(0,2) \oplus \mathcal{O}(1,1)$	[**BFMT21**, A-69]	?
Fano 2-67	K_{474}	40	165	2	1	0	4	$\mathbb{P}^2 \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathbb{P}^2}(0,1) \oplus \mathcal{O}(0,1)^{\oplus 2}$	$\mathrm{Bl}_E X_5^4$	+
Fano 2-68	K_{483}	36	146	2	0	0	10	$\mathrm{Gr}(2,4) \times \mathrm{Gr}(2,6)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 3}$	$\mathrm{Bl}_{Q_1} X_6^4$	+
Fano 2-69	K_{529}	29	110	2	0	0	17	$\mathbb{P}^3 \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(0,1)$	$\mathrm{Bl}_{\mathrm{dP}_5} K_{433}$	?
Fano 2-70	K_{714}	24	80	2	10	0	2	$\mathbb{P}^2 \times \mathrm{Gr}(2,5)$	$\mathcal{O}(2,0) \oplus \mathcal{O}(0,2) \oplus \mathcal{O}(0,1)^{\oplus 2}$	$Q_1 \times F_{1-5}$	?
Fano 2-71	K_{426}	36	147	2	0	0	13	$\mathrm{Gr}(2,4) \times \mathrm{Gr}(3,5)$	$\mathcal{U}^\vee_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{Q}_{\mathrm{Gr}(3,5)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(0,1)$	$\mathrm{Bl}_{\mathrm{Bl}_{10\mathrm{pts}}\,\mathbb{P}^2} \mathrm{Gr}(2,4)$	+
Fano 2-72	K_{662}	22	74	2	0	1	30	$\mathbb{P}^2 \times \mathrm{Gr}(2,5)$	$\mathcal{U}^\vee(1,0) \oplus \mathcal{O}(0,1) \oplus \mathcal{O}(0,2)$	[**BFMT21**, GM-18]	?
Fano 2-73	K_{671}	18	55	2	0	3	47	$\mathbb{P}^2 \times \mathbb{P}^6$	$\mathcal{Q}_{\mathbb{P}^2}(0,1) \oplus \mathcal{Q}_{\mathbb{P}^2}(0,2)$	Small res.	?
Fano 2-74	K_{562}	27	99	2	0	1	25	$\mathbb{P}^2 \times \mathrm{Gr}(2,6)$	$\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,6)}(0,1)$	[**BFMT21**, GM-20]	?
Fano 3-1	K_{503}	45	192	3	1	0	4	$\mathbb{P}^2 \times \mathbb{P}^3 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(0,1,1) \oplus \mathcal{O}(2,0,0) \oplus \mathcal{O}(0,0,1)$	$Q_1 \times F_{2-17}$	+
Fano 3-2	K_{20}	76	358	3	0	0	5	$\mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0)$	$\mathrm{Bl}_{\mathrm{dP}_8} \mathrm{Bl}_{\mathbb{P}^2} \mathrm{Gr}(2,4)$	+
Fano 3-3	K_{521}	34	136	3	0	0	14	$\mathbb{P}^1 \times \mathbb{P}^3 \times \mathrm{Gr}(2,4)$	$\mathcal{O}(1,0,1) \oplus \mathcal{O}(0,1,1) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0)$	$\mathrm{Bl}_{\mathrm{dP}_3} K_{124}$	+
Fano 3-4	K_{524}	34	136	3	2	0	8	$\mathbb{P}^1 \times \mathbb{P}^3 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(0,0,2) \oplus \mathcal{O}(1,1,0)$	$\mathrm{Bl}_{\mathrm{dP}_4} K_{238}$	+
Fano 3-5	K_{21}	74	347	3	0	0	5	$\mathbb{P}^5 \times \mathbb{P}^4 \times \mathrm{Gr}(2,7)$	$\mathcal{Q}_{\mathbb{P}^4} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,7)}(1,0,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,7)}(0,1,0)$	$\mathrm{Bl}_S K_{15}$	+
Fano 3-6	K_{526}	30	116	3	0	0	17	$\mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathbb{P}^3$	$\mathcal{Q}_{\mathbb{P}^2_1}(0,1,1) \oplus \mathcal{O}(1,0,1)$	$\mathrm{Bl}_{\mathrm{Bl}_{13\mathrm{pts}}\,\mathbb{P}^2} \mathbb{P}^2 \times \mathbb{P}^2$	+
Fano 3-7	K_{555}	31	120	3	0	1	22	$\mathbb{P}^1 \times \mathbb{P}^3 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(0,0,1) \oplus \mathcal{O}(1,1,1)$	[**BFMT21**, K3-48]	+
Fano 3-8	K_{623}	28	105	3	0	0	17	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathrm{Gr}(2,5)$	$\mathcal{O}(0,0,1)^{\oplus 2} \oplus \mathcal{O}(1,0,1) \oplus \mathcal{O}(0,1,1)$	$\mathrm{Bl}_{\mathrm{Bl}_{9\mathrm{pts}}\,\mathbb{P}^2} K_{288}$	+
Fano 3-9	K_{294}	54	241	3	0	0	6	$\mathbb{P}^2 \times \mathrm{Gr}(2,5)\,\mathrm{Gr}(2,4)$	$\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \boxtimes \mathcal{Q}_{\mathrm{Gr}(2,4)} \oplus \mathcal{O}(0,1,0) \oplus \mathcal{O}(0,0,1)$	$\mathrm{Bl}_{\mathrm{dP}_7} K_{101}$	+
Fano 3-10	K_{637}	27	100	3	0	1	22	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathrm{Gr}(2,5)$	$\mathcal{O}(0,0,1)^{\oplus 3} \oplus \mathcal{O}(1,1,1)$	[**BFMT21**, K3-38]	+
Fano 3-11	K_{158}	60	274	3	0	0	8	$\mathrm{Gr}(2,5) \times \mathrm{Gr}(2,6)$	$\mathcal{Q}_{\mathrm{Gr}(2,5)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)}$	Small res.	?
Fano 3-12	K_{398}	39	164	3	0	0	12	$\mathbb{P}^2 \times \mathrm{Gr}(2,7)$	$\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{O}(0,1) \oplus \mathrm{Sym}^2\, \mathcal{U}^\vee_{\mathrm{Gr}(2,7)}$	Small res.	?
Fano 3-13	K_{431}	36	148	3	0	0	12	$\mathbb{P}^2 \times \mathrm{Fl}(1,3,4)$	$\mathcal{R}_1(1,0,0) \oplus \mathcal{O}(0,1,1)$	$\mathrm{Bl}_{\mathrm{dP}_4} Y$	+
Fano 3-14	K_{76}	72	337	3	0	0	5	$\mathbb{P}^4 \times \mathrm{Gr}(2,7) \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathbb{P}^4} \boxtimes \mathcal{U}_{\mathrm{Gr}(2,7)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,4)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0)$	Small res.	+
Fano 3-15	K_{116}	57	257	3	0	0	7	$\mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5) \times \mathbb{P}^2$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,0,1) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{O}(0,1,0)^{\oplus 2}$	$\mathrm{Bl}_{\mathrm{dP}_7} K_{25}$	+
Fano 3-16	K_{125}	48	211	3	0	0	7	$\mathbb{P}^2 \times \mathbb{P}^4 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0)^{\oplus 2}$	$\mathrm{Bl}_{\mathrm{dP}_6} K_{29}$	+
Fano 3-17	K_{48}	90	433	3	0	0	5	$\mathbb{P}^2 \times \mathbb{P}^4 \times \mathrm{Gr}(2,11)$	$\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,11)} \oplus (\mathcal{Q}_{\mathbb{P}^4} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,11)})^{\oplus 2}$	Small res.	?
Fano 3-18	K_{188}	60	272	3	0	0	4	$\mathbb{P}^1 \times \mathbb{P}^3 \times \mathrm{Gr}(2,4)$	$\mathcal{O}(1,0,1) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(0,0,1)$	$\mathrm{Bl}_{Q_1} K_{136}$	+
Fano 3-19	K_{190}	57	256	3	0	0	6	$\mathbb{P}^4 \times \mathbb{P}^2 \times \mathrm{Gr}(2,6)$	$\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,6)}(1,0,0) \oplus \mathcal{O}(0,0,1)^{\oplus 2}$	$\mathrm{Bl}_{\mathrm{Bl}_{\mathrm{pt}}\,\mathbb{P}^2} K_{142}$	?
Fano 3-20	K_{191}	54	241	3	0	0	9	$\mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathrm{Gr}(2,7)$	$\mathcal{Q}_{\mathbb{P}^2_1} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{Q}_{\mathbb{P}^2_2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{O}(0,0,1)^{\oplus 2}$	Small res.	?
Fano 3-21	K_{173}	65	298	3	0	0	5	$\mathbb{P}^2 \times \mathrm{Gr}(2,6) \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \boxtimes \mathcal{Q}_{\mathrm{Gr}(2,5)} \oplus \mathcal{O}(0,1,0)^{\oplus 2}$	$\mathrm{Bl}_{\mathbb{P}^2} K_{142}$	?
Fano 3-22	K_{178}	66	304	3	0	0	4	$\mathbb{P}^1 \times \mathbb{P}^3 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(0,0,1) \oplus \mathcal{O}(1,1,0)$	$\mathrm{Bl}_{Q_2} K_{136}$	+
Fano 3-23	K_{211}	54	241	3	0	0	7	$\mathbb{P}^3 \times \mathbb{P}^3 \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,5)}(0,1,0) \oplus \mathcal{O}(0,0,1)^{\oplus 2}$	$\mathrm{Bl}_{\mathrm{dP}_7} K_{102}$	+
Fano 3-24	K_{212}	53	235	3	0	0	7	$\mathbb{P}^5 \times \mathbb{P}^2 \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0,0)$	$\mathrm{Bl}_{\mathrm{dP}_6} K_6$	+

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Table 1 continued from previous page

X	Id	$h^0(-K)$	$(-K)^4$	$h^{1,1}$	$h^{2,1}$	$h^{3,1}$	$h^{2,2}$	Amb.	$\mathcal{F}$	Desc.	Rat.
Fano 3-25	K_{202}	59	267	3	0	0	5	$\mathbb{P}^3 \times \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0,0) \oplus \mathcal{O}(0,1,0) \oplus \mathcal{O}(0,0,1)$	$\mathrm{Bl}_{Q_2} K_{101}$	+
Fano 3-26	K_{232}	52	230	3	0	0	6	$\mathbb{P}^3 \times \mathbb{P}^5 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathbb{P}^3}(0,1,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(0,0,1)$	$\mathrm{Bl}_{\mathrm{dP}_6} K_{136}$	+
Fano 3-27	K_{231}	51	225	3	0	0	9	$\mathbb{P}^1 \times \mathbb{P}^4 \times \mathrm{Gr}(2,7)$	$\mathcal{Q}_{\mathbb{P}^4} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,7)}(0,1,0) \oplus \mathcal{O}(1,0,1)$	$\mathrm{Bl}_{\mathrm{dP}_4} K_{15}$	+
Fano 3-28	K_{235}	46	199	3	0	0	8	$\mathbb{P}^2 \times \mathbb{P}^4 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathbb{P}^2}(0,1,0) \oplus \mathcal{Q}_{\mathbb{P}^2}(0,0,1) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0)$	Small res.	+
Fano 3-29	K_{226}	51	225	3	0	0	10	$\mathbb{P}^2 \times \mathbb{P}^4 \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{U}_{\mathrm{Gr}(2,5)}(0,1,0)^{\oplus 2}$	$\mathrm{Bl}_{\mathrm{dP}_3} K_6$	+
Fano 3-30	K_{224}	56	251	3	0	0	6	$\mathbb{P}^2 \times \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0,0) \oplus \mathcal{O}(0,0,1)^{\oplus 2}$	$\mathrm{Bl}_{\mathrm{Bl}_{\mathrm{pt}} \mathbb{P}^2} K_{25}$	+
Fano 3-31	K_{242}	51	224	3	0	0	8	$\mathbb{P}^3 \times \mathbb{P}^4 \times \mathbb{P}^5$	$\mathcal{Q}^\vee_{\mathbb{P}^3}(1,1,0) \oplus \mathcal{Q}_{\mathbb{P}^4}(0,0,1) \oplus \mathcal{O}(1,0,1)$	$\mathrm{Bl}_{\mathrm{dP}_4} Y$	+
Fano 3-32	K_{244}	47	203	3	0	0	9	$\mathbb{P}^3 \times \mathbb{P}^4 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathbb{P}^3}(0,1,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0) \oplus \mathcal{O}(0,1,1) \oplus \mathcal{O}(0,0,1)$	$\mathrm{Bl}_{\mathrm{dP}_3} K_{136}$	+
Fano 3-33	K_{450}	51	224	3	0	0	4	$\mathbb{P}^1 \times \mathbb{P}^4 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0)^{\oplus 2} \oplus \mathcal{O}(0,0,1)$	$\mathbb{P}^1 \times F_{2-21}$	+
Fano 3-34	K_{145}	75	353	3	0	0	6	$\mathbb{P}^4 \times \mathrm{Gr}(2,7) \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathbb{P}^4} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0)$	Small res.	?
Fano 3-35	K_{295}	55	244	3	0	0	5	$\mathbb{P}^6 \times \mathbb{P}^2 \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0,0)^{\oplus 2}$	$\mathrm{Bl}_{\mathrm{dP}_8} K_6$	+
Fano 3-36	K_{320}	45	193	3	0	0	12	$\mathbb{P}^2 \times \mathbb{P}^3 \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(0,1,0) \oplus \mathcal{O}(0,1,1)$	$\mathrm{Bl}_{\mathrm{dP}_1} K_6$	+
Fano 3-37	K_{321}	47	202	3	0	0	10	$\mathbb{P}^2 \times \mathbb{P}^4 \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,5)}(0,1,0) \oplus \mathcal{O}(0,1,1)$	$\mathrm{Bl}_{\mathrm{dP}_3} K_6$	+
Fano 3-38	K_{322}	46	198	3	1	0	5	$\mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathbb{P}^2_3$	$\mathcal{Q}_{\mathbb{P}^2_1}(0,1,1)$	$\mathrm{Bl}_E \mathbb{P}^2_2 \times \mathbb{P}^2_3$	+
Fano 3-39	K_{341}	42	178	3	0	0	11	$\mathbb{P}^2 \times \mathbb{P}^3 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(0,1,1) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0)$	$\mathrm{Bl}_{\mathrm{dP}_6} K_{124}$	+
Fano 3-40	K_{342}	44	188	3	0	0	8	$\mathbb{P}^3_1 \times \mathbb{P}^3_2 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(1,1,0) \oplus \mathcal{O}(0,0,1)$	$\mathrm{Bl}_{\mathrm{dP}_4} K_{136}$	+
Fano 3-41	K_{343}	43	183	3	0	0	9	$\mathbb{P}^2 \times \mathbb{P}^4 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathbb{P}^2}(0,1,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(1,0,1) \oplus \mathcal{O}(0,0,1)$	$\mathrm{Bl}_{Q_2} K_{126}$	+
Fano 3-42	K_{346}	36	150	3	0	0	10	$\mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathbb{P}^4$	$\mathcal{Q}_{\mathbb{P}^2_1} \boxtimes \mathcal{Q}_{\mathbb{P}^2_2} \boxtimes \mathcal{O}_{\mathbb{P}^4}(1)$	$\mathrm{Bl}_{\mathrm{dP}_3} \mathbb{P}^2 \times \mathbb{P}^2$	+
Fano 3-43	K_{265}	60	273	3	0	0	6	$\mathbb{P}^2 \times \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,6)$	$\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \oplus \mathcal{O}(0,1,0) \oplus \mathcal{O}(0,0,1)$	Small res.	?
Fano 3-44	K_{372}	60	272	3	0	0	4	$\mathbb{P}^1 \times \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{O}(0,1,0) \oplus \mathcal{O}(0,0,1)^{\oplus 2}$	$\mathbb{P}^1 \times F_{2-26}$	+
Fano 3-45	K_{388}	54	240	3	0	0	4	$\mathbb{P}^1 \times \mathbb{P}^3 \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathrm{Gr}(2,5)}(0,1,0) \oplus \mathcal{O}(0,0,1)^{\oplus 3}$	$\mathbb{P}^1 \times F_{2-22}$	+
Fano 3-46	K_{421}	43	183	3	0	0	7	$\mathbb{P}^1 \times \mathbb{P}^4 \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathrm{Gr}(2,5)}(0,1,0) \oplus \mathcal{O}(1,1,0) \oplus \mathcal{O}(0,0,1)^{\oplus 3}$	$\mathrm{Bl}_{\mathrm{dP}_5} \mathrm{Bl}_{v_2(\mathbb{P}^2)} \mathbb{P}^4$	+
Fano 3-47	K_{422}	41	172	3	0	0	10	$\mathbb{P}^2 \times \mathbb{P}^3 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(0,1,1) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0)$	$\mathrm{Bl}_{\mathrm{dP}_7} K_{124}$	+
Fano 3-48	K_{17}	85	406	3	0	0	5	$\mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathbb{P}^2_1} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(0,1,0)$	$\mathrm{Bl}_{\mathrm{dP}_8} K_6$	+
Fano 3-49	K_{472}	39	162	3	0	0	10	$\mathbb{P}^1 \times \mathbb{P}^5 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0)^{\oplus 2} \oplus \mathcal{O}(1,1,0) \oplus \mathcal{O}(0,0,1)$	$\mathrm{Bl}_{\mathrm{dP}_2} \mathbb{P}^1 \times Q_3$	+
Fano 3-50	K_{512}	37	152	3	0	0	8	$\mathbb{P}^1 \times \mathbb{P}^4 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(1,0,1) \oplus \mathcal{O}(0,2,0) \oplus \mathcal{O}(0,0,1)$	$\mathbb{P}_{\mathrm{Bl}_{Q_1} Q_3}(\mathcal{U}_{\mathrm{Gr}(2,4)\|Q_3} \oplus \mathcal{O}_{Q_3})$	?
Fano 3-51	K_{516}	35	141	3	0	0	12	$\mathbb{P}^1 \times \mathbb{P}^2 \times \mathrm{Gr}(2,5)$	$\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(0,1,0) \oplus \mathcal{O}(1,0,1) \oplus \mathcal{O}(0,0,1)^{\oplus 2}$	$\mathrm{Bl}_{\mathrm{dP}_2} K_{195}$	+
Fano 3-52	K_{519}	35	141	3	0	0	13	$\mathbb{P}^1 \times \mathbb{P}^4 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(0,0,1) \oplus \mathcal{O}(0,1,1) \oplus \mathcal{O}(1,1,0)$	$\mathrm{Bl}_{\mathrm{dP}_4} K_{126}$	+
Fano 3-53	K_{593}	33	130	3	0	0	12	$\mathbb{P}^2 \times \mathbb{P}^1 \times \mathrm{Gr}(2,5)$	$\mathcal{O}(1,1,0) \oplus \mathcal{O}(1,0,1) \oplus \mathcal{O}(0,0,1)^{\oplus 3}$	$\mathrm{Bl}_{\mathrm{dP}_5} K_{508}$	+
Fano 3-54	K_{591}	39	160	3	1	0	4	$\mathbb{P}^1_1 \times \mathbb{P}^1_2 \times \mathrm{Gr}(2,5)$	$\mathcal{O}(0,1,1) \oplus \mathcal{O}(0,0,1)^{\oplus 3}$	$\mathbb{P}^1 \times F_{2-14}$	+
Fano 3-55	K_{668}	25	89	3	5	0	9	$\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^4$	$\mathcal{O}(1,1,0) \oplus \mathcal{Q}_{\mathbb{P}^2}(0,0,2)$	$\mathrm{Bl}_{\mathrm{dP}_4} K_{612}$	+
Fano 3-56	K_{182}	60	273	3	0	0	7	$\mathbb{P}^2 \times \mathbb{P}^4 \times \mathrm{Gr}(2,7)$	$\mathcal{Q}_{\mathbb{P}^4} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,7)}(1,0,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,7)}(0,1,0)$	$\mathrm{Bl}_{\mathrm{dP}_6} K_{15}$	+
Fano 3-57	K_{282}	51	225	3	0	0	11	$\mathbb{P}^2 \times \mathbb{P}^3 \times \mathrm{Gr}(2,6)$	$\mathcal{Q}_{\mathbb{P}^3} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,6)}(1,0,0) \oplus \mathcal{O}(0,1,1)$	$\mathrm{Bl}_{\mathrm{dP}_5} K_{70}$	+
Fano 3-58	K_{303}	48	209	3	0	0	8	$\mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathrm{Gr}(2,6)$	$\mathcal{Q}_{\mathbb{P}^2_2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \oplus \mathcal{O}(0,0,1)^{\oplus 2} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,6)}(1,0,0)$	$\mathrm{Bl}_{\mathrm{dP}_6} K_{142}$	?
Fano 3-59	K_{460}	48	208	3	0	0	4	$\mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathrm{Gr}(2,5)$	$\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(0,1,0) \oplus \mathcal{O}(2,0,0) \oplus \mathcal{O}(0,0,1)^{\oplus 3}$	$Q_1 \times F_{2-20}$	+
Fano 3-60	K_{318}	48	209	3	0	0	7	$\mathbb{P}^2 \times \mathbb{P}^3 \times \mathrm{Gr}(2,5)$	$\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,5)}(0,1,0) \oplus \mathcal{O}(0,0,1)^{\oplus 2}$	$\mathrm{Bl}_{\mathrm{dP}_7} K_{102}$	+

Continued on next page

Table 1 continued from previous page

X	Id	$h^0(-K)$	$(-K)^4$	$h^{1,1}$	$h^{2,1}$	$h^{3,1}$	$h^{2,2}$	Amb.	$\mathcal{F}$	Desc.	Rat.
Fano 3-61	K_{326}	48	209	3	0	0	9	$\mathbb{P}^1 \times \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{O}(1,1,0) \oplus \mathcal{O}(0,0,1)^{\oplus 2}$	$\mathrm{Bl}_{\mathrm{dP}_5} K_{25}$	+
Fano 3-62	K_{327}	47	204	3	0	0	7	$\mathbb{P}^2 \times \mathbb{P}^4 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0,0)$	$\mathrm{Bl}_{\mathrm{dP}_7} K_{115}$	+
Fano 3-63	K_{339}	46	199	3	0	0	5	$\mathbb{P}^4 \times \mathbb{P}^2 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0)^{\oplus 2} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(0,1,0)$	$\mathrm{Bl}_{\mathrm{dP}_8} K_{29}$	+
Fano 3-64	K_{344}	42	178	3	0	0	9	$\mathbb{P}^3 \times \mathrm{Gr}(2,4) \times \mathbb{P}^2$	$\mathcal{O}(1,0,1) \oplus \mathcal{Q}_{\mathbb{P}^2}(0,1,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0,0)$	$\mathrm{Bl}_{\mathrm{dP}_5} \mathrm{Bl}_{Q_1} \mathrm{Gr}(2,4)$	+
Fano 3-65	K_{413}	44	188	3	0	0	6	$\mathbb{P}^2 \times \mathbb{P}^4 \times \mathrm{Gr}(2,4)$	$\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(0,0,1) \oplus \mathcal{O}(0,2,0)$	$\mathrm{Bl}_{\mathrm{dP}_6} K_{230}$	?
Fano 3-66	K_{423}	40	167	3	0	0	11	$\mathbb{P}^1 \times \mathbb{P}^3 \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathrm{Gr}(2,5)}(0,1,0) \oplus \mathcal{O}(1,0,1) \oplus \mathcal{O}(0,0,1)^{\oplus 2}$	$\mathrm{Bl}_{\mathrm{dP}_3} K_{102}$	+
Fano 3-67	K_{424}	39	161	3	0	0	12	$\mathbb{P}^1 \times \mathbb{P}^2 \times \mathrm{Gr}(2,6)$	$\mathcal{O}(1,0,1) \oplus \mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \oplus \mathcal{O}(0,0,1)^{\oplus 2}$	$\mathrm{Bl}_{\mathrm{dP}_2} K_{385}$	+
Fano 3-68	K_{428}	39	163	3	0	0	9	$\mathbb{P}^1 \times \mathbb{P}^4 \times \mathrm{Gr}(2,4)$	$\mathcal{O}(1,0,1) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0)^{\oplus 2}$	$\mathrm{Bl}_{\mathrm{dP}_4} K_{29}$	+
Fano 3-69	K_{430}	39	162	3	0	0	10	$\mathbb{P}^2 \times \mathbb{P}^3 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(1,0,1) \oplus \mathcal{O}(1,1,0) \oplus \mathcal{O}(0,0,1)$	$\mathrm{Bl}_{\mathrm{dP}_8} K_{136}$	+
Fano 3-70	K_{438}	33	131	3	0	0	17	$\mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathbb{P}^4$	$\mathcal{Q}_{\mathbb{P}^2_1}(0,0,1) \oplus \mathcal{Q}_{\mathbb{P}^2_1}(0,1,1)$	$\mathrm{Bl}_{\mathrm{Bl}_{13\mathrm{pts}} \mathbb{P}^2} \mathbb{P}^2 \times \mathbb{P}^2$	+
Fano 3-71	K_{478}	39	162	3	0	0	8	$\mathbb{P}^1 \times \mathbb{P}^3 \times \mathrm{Gr}(2,5)$	$\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(0,1,0) \oplus \mathcal{O}(0,0,1)^{\oplus 3} \oplus \mathcal{O}(1,1,0)$	$\mathrm{Bl}_{\mathrm{dP}_5} K_{219}$	?
Fano 3-72	K_{154}	69	320	3	0	0	4	$\mathbb{P}^2 \times \mathbb{P}^3 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(0,1,0) \oplus \mathcal{O}(0,0,1)$	$\mathrm{Bl}_{\mathrm{dP}_8} K_{136}$	+
Fano 3-73	K_{404}	45	193	3	0	0	9	$\mathbb{P}^1 \times \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{O}(0,1,0) \oplus \mathcal{O}(0,0,1) \oplus \mathcal{O}(1,0,1)$	$\mathrm{Bl}_{\mathrm{dP}_4} K_{101}$	+
Fano 3-74	K_{406}	43	183	3	0	0	8	$\mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathrm{Gr}(2,5)$	$\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(0,1,0) \oplus \mathcal{O}(0,0,1)^{\oplus 2}$	$\mathrm{Bl}_{\mathrm{dP}_6} K_{195}$	+
Fano 4-1	K_{280}	51	228	4	0	0	10	$\mathrm{Gr}(2,4) \times \mathrm{Gr}(2,6)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \oplus \mathcal{O}(1,0) \oplus \mathrm{Sym}^2 \mathcal{U}^\vee_{\mathrm{Gr}(2,6)}$	$\mathrm{Bl}_{\mathbb{P}^1 \sqcup \mathbb{P}^1} Y$	?
Fano 4-2	K_{26}	63	290	4	0	0	8	$\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,0,1,0)$	$\mathrm{Bl}_{\mathrm{dP}_7} K_{20}$	+
Fano 4-3	K_{73}	78	369	4	0	0	7	$\mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$	$\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0,0,0) \oplus \mathcal{Q}_{\mathbb{P}^2_2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}$	$\mathrm{Bl}_{\mathrm{dP}_8} \mathrm{Bl}_{\mathbb{P}^1} K_6$	+
Fano 4-4	K_{144}	78	369	4	0	0	7	$\mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0,0) \oplus \mathcal{Q}_{\mathbb{P}^2_2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}$	$\mathrm{Bl}_{Q_2} \mathrm{Bl}_{\mathbb{P}^1} K_6$	+
Fano 4-5	K_{256}	81	385	4	0	0	7	$\mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathbb{P}^2_3 \times \mathrm{Gr}(2,7)$	$\mathcal{Q}_{\mathbb{P}^2_1} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{Q}_{\mathbb{P}^2_2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)} \oplus \mathcal{Q}_{\mathbb{P}^2_3} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)}$	Small res.	?
Fano 4-6	K_{78}	71	331	4	0	0	7	$\mathbb{P}^3 \times \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathbb{P}^2_2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{Q}_{\mathrm{Gr}(2,5)}(1,0,0,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(0,1,0,0)$	$\mathrm{Bl}_{\mathrm{dP}_8} K_{17}$	+
Fano 4-7	K_{104}	66	305	4	0	0	8	$\mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathbb{P}^2_3 \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathbb{P}^2_3} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(0,1,0)$	$\mathrm{Bl}_{\mathrm{dP}_7} K_{17}$	+
Fano 4-8	K_{201}	61	278	4	0	0	6	$\mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathbb{P}^2_3 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0,0,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(0,0,1,0)$	$\mathrm{Bl}_{\mathbb{P}^2} K_{20}$	+
Fano 4-9	K_{204}	57	257	4	0	0	9	$\mathbb{P}^1 \times \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathbb{P}^2_2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(0,1,0,0) \oplus \mathcal{O}(1,0,0,1)$	$\mathrm{Bl}_{\mathrm{dP}_6} K_{17}$	+
Fano 4-10	K_{227}	53	236	4	0	0	9	$\mathbb{P}^1 \times \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,0,1,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0,0) \oplus \mathcal{O}(1,0,0,1)$	$\mathrm{Bl}_{\mathrm{dP}_6} K_{20}$	+
Fano 4-11	K_{297}	55	246	4	0	0	7	$\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^3 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,0,1,0) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(0,1,0,0) \oplus \mathcal{O}(1,0,1,0) \oplus \mathcal{O}(0,0,0,1)$	$\mathrm{Bl}_{\mathrm{dP}_7} K_{154}$	+
Fano 4-12	K_{403}	47	204	4	0	0	8	$\mathbb{P}^1_1 \times \mathbb{P}^1_2 \times \mathbb{P}^3 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)}(0,0,1,0) \oplus \mathcal{O}(1,0,1,0) \oplus \mathcal{O}(0,1,0,1) \oplus \mathcal{O}(0,0,0,1)$	$\mathrm{Bl}_{\mathrm{dP}_6} K_{188}$	+
Fano 4-13	K_{482}	39	162	4	1	0	9	$\mathbb{P}^1 \times \mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathbb{P}^2_3$	$\mathcal{Q}_{\mathbb{P}^2_1}(0,0,1,1) \oplus \mathcal{O}(1,1,0,0)$	$\mathrm{Bl}_{\mathrm{dP}_6} K_{322}$	+

CHAPTER 3

Hyperkähler manifolds and modular vector bundles

3.1. About hyperkähler manifolds

Hyperkähler manifolds, or irreducible holomorphic symplectic manifolds, are one of the most studied topics in modern algebraic geometry.

There are a few known examples of such varieties. In every even complex dimension, two families have constructed: the deformations of the Hilbert scheme of n-points on K3 surfaces, known as $\mathrm{K3}^{[n]}$-type, and the deformations of the generalized Kummer, known as Kum_n-type. In addition, there exist two sporadic families in dimensions ten and six, OG10 and OG6, which are desingularizations of special moduli spaces of semistable sheaves on K3 and abelian surfaces, respectively.

All these families, a posteriori, can be reinterpreted as moduli spaces of sheaves. It is therefore natural to ask whether one can systematically construct families of hyperkähler manifolds as moduli spaces of suitable chosen stable sheaves.

For this reason, the study of sheaves on high dimensional hyperkähler manifolds is rapidly gaining interest. One of the first approaches is to generalize the rich geometry of moduli spaces of sheaves on K3 surfaces, initiated by Mukai in [**Muk84**]. In particular, one key property that must be preserved is their nice behavior in families. To achieve this, it appears necessary to consider sheaves with special features; so far, three different but related notions have been introduced.

Verbitsky in [**Ver93**] was the first to introduce such ideas. In that work, he considered *projectively hyper–holomorphic* vector bundles on hyperkähler manifolds in order to control the deformations along twistor families. In this way, he proved the existence of a hyperkähler structure on the irreducible component of the moduli space of simple sheaves on hyperkähler manifolds containing a fixed hyper–holomorphic vector bundle. The study of such objects for hyperkähler manifolds was recently done by Meazzini and Onorati in [**MO23**].

Subsequently, O'Grady, in [**OG19**], introduced the notion of *modular* sheaves. In this case, the variation of stability in the ample cone (and later in the Kähler cone) of X resembles the one of the K3 surface case.

Finally, Beckmann in [**Bec22**], and independently Markman in [**Mar24**], introduced the concept of *atomic* sheaves. They were defined by having a Mukai vector in the extended Mukai lattice, a condition that holds trivially in the case of K3 surfaces.

As previously mentioned, these notions are intertwined. Every atomic torsion-free sheaf is modular, and every atomic polystable vector bundle is projectively hyper–holomorphic. The converse, however, does not hold in general: for instance, the tangent bundle is always modular but fails to be atomic for the known families of hyperkähler manifolds of dimension greater than two.

In the $K3^{[2]}$ case we can say even more. Combining results of [**Ver93**], one finds that any polystable modular vector bundle on hyperkähler manifold of $K3^{[2]}$-type is projectively hyper–holomorphic.

In this chapter, we focus on one of these types of sheaves, namely modular sheaves. In [**OG19**], modular sheaves were introduced, together with explicit examples. In particular, O'Grady provided a method to construct modular sheaves on hyperkähler manifolds of $K3^{[2]}$-type. Each example of a modular sheaf in [**OG19**] is rigid, in the sense that its moduli space consists of a single point; thus, it is not suitable for producing new hyperkähler manifolds.

The first examples of non-rigid modular sheaves were given by Markman in [**Mar24**]. Moreover, Bottini in [**Bot24**] constructed a modular sheaf whose moduli space is birational with a hyperkähler manifold of OG10-type. Later, using other techniques, Fatighenti in [**Fat24**] constructed another example of non-rigid modular sheaf, later generalized by Fatighenti and Onorati in [**FO24**]. They worked on an explicit model of hyperkähler manifold of $K3^{[2]}$-type, the Beauville–Donagi fourfold and obtained the following result:

Example 3.1.1 (Fatighenti–Onorati)**.** *Let $X_{\mathrm{BD}} \subseteq \mathrm{Gr}(2,6)$ be the Fano variety of lines of a very general cubic fourfold, which is a hyperkähler manifold of $K3^{[2]}$-type. Let $\widetilde{\mathcal{Q}}$ be the tautological quotient bundle on the Grassmannian $\mathrm{Gr}(2,V_6)$ and let $\mathcal{Q}$ be its restriction on X_{BD}. Let $\lambda = (\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ be a partition and consider $\Sigma_\lambda\mathcal{Q}$ the associated Schur functor. Then:*

(1) $\Sigma_\lambda\mathcal{Q}$ is modular for every λ;
(2) $\Sigma_\lambda\mathcal{Q}$ is polystable for every λ;
(3) $\dim \mathrm{Ext}^1(\Sigma_\lambda\mathcal{Q},\Sigma_\lambda\mathcal{Q}) \in \{0,20,40\}$.

The Fano variety of lines is one of the few hyperkähler manifolds that can be realized as a subvariety of a Grassmannian. Another hyperkähler manifold of $K3^{[2]}$-type that can be realized inside a Grassmannian is the Debarre–Voisin fourfold.

In this chapter, we focused on applying the techniques developed in [**FO24**] to the Debarre–Voisin fourfold. In particular, since both the Beauville–Donagi fourfold and Debarre–Voisin fourfolds are locally complete families of $K3^{[2]}$-type, we aim to study modular sheaves on a very general Debarre–Voisin fourfold $X_{\mathrm{DV}} \subseteq \mathrm{Gr}(6,10)$.

Our results can be summarized as follows:

Theorem 3.1.2. *Let $X_{\mathrm{DV}} \subseteq \mathrm{Gr}(6,10)$ be the zero locus of a general global section of $\bigwedge^3 \widetilde{\mathcal{U}}^\vee$, i.e., a very general Debarre–Voisin hyperkähler manifold. Let $\widetilde{\mathcal{Q}}$ be the tautological quotient bundle on the Grassmannian $\mathrm{Gr}(6,10)$ and let $\mathcal{Q}$ be its restriction on X_{DV}. Let $\lambda = (\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ be a partition and consider $\Sigma_\lambda\mathcal{Q}$ the associated Schur functor. Then:*

(1) $\Sigma_\lambda\mathcal{Q}$ is modular for every λ;
(2) $\Sigma_\lambda\mathcal{Q}$ is polystable for every λ;
(3) the dimensions of $\mathrm{Ext}^*(\Sigma_\lambda\mathcal{Q},\Sigma_\lambda\mathcal{Q})$ are given in Table 1.

Remark 3.1.3. *Note that the values appearing in Table 1 coincide with those computed in* [**FO24**, Theorem C] *for X_{BD}. This seems not to be a coincidence; rather, it suggests the existence of a "deformation link" between the corresponding Schur functors on the two models of hyperkähler manifolds, analogous to the one stated in* [**OG24**, Prop. 3.8].

Such a correspondence is also expected to hold for every Schur functor of the restricted quotient bundle, but so far, there is no proof of such a result. In particular, for $\bigwedge^2 \mathcal{Q}$, this relation was already

TABLE 1. Cohomological values for $\lambda_1 < 5$.

λ	$\hom(\Sigma_\lambda Q, \Sigma_\lambda Q)$	$\mathrm{ext}^1(\Sigma_\lambda Q, \Sigma_\lambda Q)$	$\mathrm{ext}^2(\Sigma_\lambda Q, \Sigma_\lambda Q)$
$(1,0,0,0)$	1	0	1
$(1,1,0,0)$	1	20	2
$(2,0,0,0)$	1	0	191
$(2,1,0,0)$	1	20	401
$(2,1,1,0)$	1	20	191
$(2,2,0,0)$	1	20	590
$(3,0,0,0)$	1	0	5545
$(3,1,0,0)$	1	20	21419
$(3,1,1,0)$	1	20	10649
$(3,2,0,0)$	1	–	–
$(3,2,1,0)$	1	40	35406
$(3,3,0,0)$	–	–	–
$(4,0,0,0)$	1	0	53065
$(4,1,0,0)$	1	–	–
$(4,1,1,0)$	1	20	141746
$(4,2,0,0)$	–	–	–
$(4,2,1,0)$	1	–	–
$(4,2,2,0)$	1	20	172910
$(4,3,0,0)$	–	–	–
$(4,3,1,0)$	–	–	–
$(4,4,0,0)$	–	–	–

noted in [**Fat24**, Remark 2.11]. *The expected reason is that we have that the projectivization of the restricted quotient bundle on the Beauville–Donagi fourfold deforms to the projectivization of the restricted quotient bundle on the Debarre–Voisin fourfold.*

Remark 3.1.4. *The computations on the Debarre–Voisin fourfold are harder than those on the Beauville–Donagi fourfold, since the codimension of this variety is* 20 *(compared with* 4 *in the other case). This leads to a longer Koszul complex, which is more difficult to handle in practice. In fact, in our situation, the indeterminacy of the cohomology values arises earlier, and we can compute the exact values only for a finite number of partitions.*

In particular, in Table 1, we restrict to partitions λ satisfying $\lambda_4 = 0$ and $\lambda_1 \geq \lambda_2 + \lambda_3$, since the graded algebra $\mathrm{Ext}^*(\Sigma_\lambda Q, \Sigma_\lambda Q)$ *is invariant under tensoring $\Sigma_\lambda Q$ by a line bundle or under dualization. Moreover, we impose $\lambda_1 < 5$, because for $\lambda \geq 5$, the computation of cohomology via the Koszul complex becomes unfeasible due to the presence of too many non-trivial terms.*

Finally, the symbol "–" in Table 1 indicates that the corresponding value could not be computed.

This method of producing modular sheaves with a potentially non-trivial moduli space is not the only one. Indeed, O'Grady, in [**OG24**] exhibited examples of a non-rigid modular sheaves on hyperkähler manifolds of $\mathrm{K3}^{[2]}$-type, whose tangent space in the moduli space has dimension $2(a^2+1)$ for $a > 1$. In the future, we aim to translate the techniques used in [**Fat24**], [**FO24**] and in this chapter to the framework used in [**OG24**].

3.2. Working tools

This section introduces all the shortcuts in the computations for the very general Debarre–Voisin fourfold. In particular, since it can be described as zero locus of a section inside a Grassmannian, here we write explicitly some useful computations that are described in a more general setting in Section 1.2.

Remark 3.2.1. *We apologise for the abrupt change of notation. From this point onward, we adopt the notation of* [FT25] *to ensure consistency with the original source and to avoid confusion for the reader. The tautological bundle $\widetilde{\mathcal{U}}$ on the Grassmannian $\mathrm{Gr}(6,10)$ is a homogeneous vector bundle of rank 6. It corresponds to the standard representation of $\mathrm{SL}(6)$, while the quotient bundle $\widetilde{\mathcal{Q}}$ is a homogeneous vector bundle of rank 4. It corresponds to the standard representation of $\mathrm{SL}(4)$.*

Lemma 3.2.2. *With the above notation, for $\mathrm{Gr}(6,10)$ the following isomorphism holds:*

$$\Sigma_{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}\widetilde{\mathcal{Q}} \cong \Sigma_{(\lambda_1-\lambda_4,\lambda_2-\lambda_4,\lambda_3-\lambda_4,0)}\widetilde{\mathcal{Q}} \otimes \mathcal{O}_X(-\lambda_4). \tag{1}$$

Moreover, all the possible bundles of the form $\Sigma_\lambda\widetilde{\mathcal{Q}} \otimes \Sigma_\lambda\widetilde{\mathcal{Q}}^\vee$ are determined by partitions of the form $\lambda = (m,t,s,0)$ with $m \geq t+s$.

Proof. Combining the identities in Remark 1.1.7, we obtain (1). Notice that, by (1), considering the partition $(m,t,s,0)$ is equivalent to considering its dual $(m, m-s, m-t, 0)$. Now, if $m < s+t$, then $m = s+t-k$ for some $k > 0$, and the dual partition is

$$(m', t', s', 0) = (s+t-k, t-k, s-k, 0),$$

which satisfies $m' \geq s' + t'$. $\square$

Notation 3.2.3. *For any non–increasing sequence $\lambda \in \mathbb{Z}^4$ we consider the Schur functor $\Sigma_\lambda\widetilde{\mathcal{Q}}$ and we denote by*

$$\widetilde{\mathcal{E}}_\lambda := \mathcal{E}nd(\Sigma_\lambda\widetilde{\mathcal{Q}}) = \Sigma_\lambda\widetilde{\mathcal{Q}} \otimes \Sigma_\lambda\widetilde{\mathcal{Q}}^\vee$$

the associated endomorphisms bundle. When we restrict it to the Debarre–Voisin fourfold X, we put

$$\mathcal{Q} := \widetilde{\mathcal{Q}}|_X, \quad \Sigma_\lambda\mathcal{Q} := (\Sigma_\lambda\widetilde{\mathcal{Q}})|_X \text{ and } \mathcal{E}_\lambda := (\widetilde{\mathcal{E}}_\lambda)|_X.$$

3.2.1. Koszul complex for Debarre–Voisin fourfold.

Let X be the Debarre–Voisin fourfold, hence $X = \mathscr{Z}\left(\bigwedge^3 \mathcal{U}^\vee_{\mathrm{Gr}(6,10)}\right) \subset \mathrm{Gr}(6,10)$. Let us denote by $\widetilde{\mathcal{U}}$ the tautological bundle $\mathcal{U}_{\mathrm{Gr}(6,10)}$ and $\widetilde{\mathcal{Q}}$ the quotient bundle $\mathcal{Q}_{\mathrm{Gr}(6,10)}$.

Since X is the zero locus of $\bigwedge^3 \widetilde{\mathcal{U}}^\vee$ in $\mathrm{Gr}(6,10)$, the Koszul complex of $\iota : X \hookrightarrow \mathrm{Gr}(6,10)$ is:

$$0 \to \det\left(\bigwedge^3 \widetilde{\mathcal{U}}\right) \to \bigwedge^{19}\bigwedge^3 \widetilde{\mathcal{U}} \to \cdots \to \bigwedge^2\bigwedge^3 \widetilde{\mathcal{U}} \to \bigwedge^3 \widetilde{\mathcal{U}} \to \mathcal{O}_{\mathrm{Gr}(6,10)} \to \iota_*\mathcal{O}_X \to 0.$$

Using the tricks described in the previous subsections, we decompose all the bundles appearing in the above complex into sums of irreducible factors. We stored all the information in Table 2.

Note that the columns correspond to the decomposition of the pth exterior power of $\bigwedge^3 \widetilde{\mathcal{U}}$. We did not write the factors of $\bigwedge^p \bigwedge^3 \widetilde{\mathcal{U}}$ for $p \geq 11$ because of the following lemma:

Lemma 3.2.4. *If $\widetilde{\mathcal{U}}$ is the tautological bundle of $\mathrm{Gr}(6,10)$ and $k \in \mathbb{Z}_{\geq 1}$ then $\bigwedge^{10+k}\bigwedge^3 \widetilde{\mathcal{U}} \cong A_{10-k} \otimes \mathcal{O}(-k)$, where A_{10-k} denotes the decomposition in irreducible factors of $\bigwedge^{10-k}\bigwedge^3 \widetilde{\mathcal{U}}$.*

TABLE 2. Irreducible factors in the Koszul complex of X_{DV}.

$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$	$p=9$	$p=10$
(0,0,0,0,0,0)	(1,1,1,0,0,0)	(2,2,1,1,0,0)	(3,3,1,1,1,0)	(4,4,1,1,1,1)	(5,4,2,2,1,1)	(6,4,3,2,2,1)	(7,4,4,2,2,2)	(8,4,4,3,3,2)	(9,4,4,4,3,3)	(10,4,4,4,4,4)
-	-	(1,1,1,1,1,1)	(3,2,2,2,0,0)	(4,3,2,2,1,0)	(5,3,3,2,2,0)	(6,3,3,3,3,0)	(7,4,3,3,3,1)	(7,5,5,3,2,2)	(8,5,5,4,3,2)	(9,5,5,5,3,3)
-	-	-	(2,2,2,1,1,1)	(3,3,3,3,0,0)	(4,4,3,3,1,0)	(5,5,3,3,1,1)	(6,5,4,3,2,1)	(7,5,4,4,3,1)	(8,5,5,3,3,3)	(9,5,5,4,4,3)
-	-	-	-	(3,3,2,2,1,1)	(4,4,2,2,2,1)	(5,5,2,2,2,2)	(6,5,3,3,2,2)	(7,5,4,3,3,2)	(8,5,4,4,4,2)	(8,6,6,4,4,2)
-	-	-	-	(2,2,2,2,2,2)	(4,3,3,3,1,1)	(5,4,4,3,2,0)	(6,4,4,4,3,0)	(7,4,4,4,4,1)	(7,6,6,3,3,2)	(8,6,6,4,3,3)
-	-	-	-	-	(3,3,3,2,2,2)	(5,4,3,3,2,1)	(6,4,4,3,3,1)	(6,6,5,3,3,1)	(7,6,5,4,4,1)	(8,6,5,5,4,2)
-	-	-	-	-	-	(4,4,4,4,1,1)	(5,5,5,3,3,0)	(6,6,4,4,2,2)	(7,6,5,4,3,2)	(8,6,5,4,4,3)
-	-	-	-	-	-	(4,4,3,3,2,2)	(5,5,4,4,2,1)	(6,6,3,3,3,3)	(7,6,4,4,3,3)	(8,5,5,5,5,2)
-	-	-	-	-	-	(3,3,3,3,3,3)	(5,5,3,3,3,2)	(6,5,5,4,4,0)	(7,5,5,5,4,1)	(7,7,7,3,3,3)
-	-	-	-	-	-	-	(5,4,4,4,2,2)	(6,5,5,4,3,1)	(7,5,5,4,4,2)	(7,7,6,4,4,2)
-	-	-	-	-	-	-	(4,4,4,3,3,3)	(6,5,4,4,3,2)	(6,6,6,4,4,1)	(7,7,5,5,5,1)
-	-	-	-	-	-	-	-	(5,5,5,5,2,2)	(6,6,5,5,5,0)	(7,7,5,5,3,3)
-	-	-	-	-	-	-	-	(5,5,4,4,3,3)	(6,6,5,5,3,2)	(7,7,4,4,4,4)
-	-	-	-	-	-	-	-	(4,4,4,4,4,4)	(6,6,4,4,4,3)	(7,6,6,5,5,1)
-	-	-	-	-	-	-	-	-	(6,5,5,5,3,3)	(7,6,6,5,4,2)
-	-	-	-	-	-	-	-	-	(5,5,5,4,4,4)	(7,6,5,5,4,3)
-	-	-	-	-	-	-	-	-	-	(6,6,6,6,6,0)
-	-	-	-	-	-	-	-	-	-	(6,6,6,6,3,3)
-	-	-	-	-	-	-	-	-	-	(6,6,5,5,4,4)
-	-	-	-	-	-	-	-	-	-	(5,5,5,5,5,5)

The Koszul complex is longer than the one for the Beauville–Donagi fourfold. For this reason, the computations are more cumbersome than the ones in [FO24]. The Koszul complex is the key to computing the cohomologies of the vector bundles on X that are restrictions of vector bundles on the ambient space $\mathrm{Gr}(6,10)$.
For any vector bundle $\widetilde{\mathcal{F}}$ on $\mathrm{Gr}(6,10)$ that we want to restrict on X, we have to split the long exact sequence into the following list of short exact sequences:

$$0 \to \widetilde{\mathcal{F}} \otimes \bigwedge^{20} \bigwedge^{3} \widetilde{\mathcal{U}} \to \widetilde{\mathcal{F}} \otimes \bigwedge^{19} \bigwedge^{3} \widetilde{\mathcal{U}} \to K_1 \to 0$$

$$0 \to K_1 \to \widetilde{\mathcal{F}} \otimes \bigwedge^{18} \bigwedge^{3} \widetilde{\mathcal{U}} \to K_2 \to 0$$

$$\cdots$$

$$0 \to K_{i-1} \to \widetilde{\mathcal{F}} \otimes \bigwedge^{20-i} \bigwedge^{3} \widetilde{\mathcal{U}} \to K_i \to 0$$

$$\cdots$$

$$0 \to K_{18} \to \widetilde{\mathcal{F}} \otimes \bigwedge^{3} \widetilde{\mathcal{U}} \to K_{19} \to 0$$

$$0 \to K_{19} \to \widetilde{\mathcal{F}} \to \widetilde{\mathcal{F}}|_X \to 0.$$

If $\widetilde{\mathcal{F}}$ is a homogeneous vector bundle then we use the Borel–Weil–Bott theorem to compute every cohomology except for the cohomology of the K_i's. To recover them, we need to study the short exact sequences, but it is not always possible to determine such cohomologies just via diagram chasing. We will denote these cases as *indeterminate*. We discuss the cohomology computations in more detail in Section 3.4.

3.3. Chern classes

In this section we explicitly compute the Chern classes of any Schur functor $\Sigma_{m,t,s,0}\mathcal{Q}$ and hence its Bogomolov discriminant

$$\Delta(\Sigma_{m,t,s,0}\mathcal{Q}) := c_1(\Sigma_{m,t,s,0}\mathcal{Q})^2 - 2\,\mathrm{rk}(\Sigma_{m,t,s,0}\mathcal{Q})ch_2(\Sigma_{m,t,s,0}\mathcal{Q}).$$

This explicit computation will prove item (1) of Theorem 3.1.2.

We start by recalling some well-known facts for hyperkähler manifolds of K3$^{[2]}$-type (see e.g. [FO24, Lemma 1.6]).

Lemma 3.3.1. *Let X be a hyperkähler manifold of K3$^{[2]}$-type. Denote by $\mathsf{p} \in H^8(X, \mathbb{C})$ the class of a point. Then the following relations hold:*

(1) $\int_X \mathsf{c}_2(X)^2 = 828$;
(2) $\mathrm{td}_X = 1 + \frac{1}{12}\mathsf{c}_2(X) + 3\mathsf{p}$;
(3) $\sqrt{\mathrm{td}_X} = 1 + \frac{1}{24}\mathsf{c}_2(X) + \frac{25}{32}\mathsf{p}$.

The following Lemma tells us the relations between the Chern characters of $\mathcal{Q}$ coming from the intersection ring of a very general Debarre–Voisin fourfold X. Notice that in degree 2 (that is, in $H^{2,2}(X, \mathbb{Q})$) the intersection ring is the same as the one of the Grassmannian $\mathrm{Gr}(6, 10)$. These results will allow us to express the Chern characters of the Schur functor $\Sigma_{m,t,s,0}\mathcal{Q}$ in terms of the Chern characters of $\mathcal{Q}$.

Lemma 3.3.2. *Let $X \subseteq \mathrm{Gr}(6, 10)$ be a very general Debarre-Voisin hyperkähler manifold, then:*

(1) $\Delta(\mathcal{Q}) = \mathsf{ch}_1(\mathcal{Q})^2 - 8\,\mathsf{ch}_2(\mathcal{Q}) = \mathsf{c}_2(X)$.
(2) The space $\mathrm{H}^{3,3}(X, \mathbb{Q})$ *has dimension* 1 *and the following relations hold:*
  (a) $\mathsf{ch}_1(\mathcal{Q})^3 = -264\,\mathsf{ch}_3(\mathcal{Q})$;
  (b) $\mathsf{ch}_1(\mathcal{Q}) \cdot \mathsf{ch}_2(\mathcal{Q}) = -18\,\mathsf{ch}_3(\mathcal{Q})$.
(3) The space $\mathrm{H}^8(X, \mathbb{Q})$ *has dimension* 1 *and the following relations hold:*
  (a) $\mathsf{ch}_1(\mathcal{Q})^4 = -5808\,\mathsf{ch}_4(\mathcal{Q})$;
  (b) $\mathsf{ch}_1(\mathcal{Q})^2 \cdot \mathsf{ch}_2(\mathcal{Q}) = -396\,\mathsf{ch}_4(\mathcal{Q})$;
  (c) $\mathsf{ch}_1(\mathcal{Q}) \cdot \mathsf{ch}_3(\mathcal{Q}) = 22\,\mathsf{ch}_4(\mathcal{Q})$;
  (d) $\mathsf{ch}_2(\mathcal{Q})^2 = -60\,\mathsf{ch}_4(\mathcal{Q})$;
  (e) $\int_X \mathsf{ch}_4(\mathcal{Q}) = -\frac{1}{4}$.

PROOF. These relations follow from the *normal bundle sequence*

$$0 \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{U}^\vee \otimes \mathcal{Q} \longrightarrow \bigwedge^3 \mathcal{U}^\vee \longrightarrow 0.$$

Following the notations in [DV10], we denote $\mathsf{c}_i(\mathcal{U}^\vee)$ by c_i. Using the tautological sequence of the Grassmannian $\mathrm{Gr}(6, 10)$ we get

$$\mathsf{ch}_1(\mathcal{Q}) = \mathsf{c}_1,$$

$$\mathsf{ch}_2(\mathcal{Q}) = \mathsf{c}_2 - \frac{1}{2}\mathsf{c}_1^2,$$

$$\mathsf{ch}_3(\mathcal{Q}) = \frac{1}{2}\left(\mathsf{c}_3 - \mathsf{c}_1\mathsf{c}_2 + \frac{1}{3}\mathsf{c}_1^3\right),$$

$$\mathsf{ch}_4(\mathcal{Q}) = \frac{1}{6}\left(\mathsf{c}_4 - \mathsf{c}_1\mathsf{c}_3 + \mathsf{c}_1^2\mathsf{c}_2 - \frac{1}{2}\mathsf{c}_2^2 - \frac{1}{4}\mathsf{c}_1^4\right),$$

and

$$\mathsf{ch}(\mathcal{U}^\vee \otimes \mathcal{Q}) = 24 + 10\mathsf{ch}_1(\mathcal{Q}) + \left(\mathsf{ch}_1(\mathcal{Q})^2 + 2\mathsf{ch}_2(\mathcal{Q})\right) + 10\mathsf{ch}_3(\mathcal{Q}) + \left(2\mathsf{ch}_4(\mathcal{Q}) + 2\mathsf{ch}_1(\mathcal{Q})\mathsf{ch}_3(\mathcal{Q}) - \mathsf{ch}_2(\mathcal{Q})^2\right).$$

By applying the *splitting principle* to a rank 6 vector bundle we obtain

$$\begin{aligned}\mathsf{ch}\left(\bigwedge^3 \mathcal{U}^\vee\right) = 20 + 10\mathsf{ch}_1(\mathcal{Q}) + \left(2\mathsf{ch}_1(\mathcal{Q})^2 - 6\mathsf{ch}_2(\mathcal{Q})\right) + \left(\frac{1}{6}\mathsf{ch}_1(\mathcal{Q})^3 - 3\mathsf{ch}_1(\mathcal{Q})\mathsf{ch}_2(\mathcal{Q})\right) + \\ + \left(6\mathsf{ch}_4(\mathcal{Q}) + \mathsf{ch}_1(\mathcal{Q})\mathsf{ch}_3(\mathcal{Q}) + \mathsf{ch}_2(\mathcal{Q})^2 - \frac{1}{2}\mathsf{ch}_1(\mathcal{Q})^2\mathsf{ch}_2(\mathcal{Q})\right).\end{aligned}$$

On the other hand, from the general intersection theory of a hyperkähler manifold of type $\mathrm{K3}^{[2]}$ we deduce

$$\mathsf{ch}(\mathcal{T}_X) = 4 + 0 - \mathsf{c}_2(X) + 0 + 15\mathsf{p}.$$

Hence we have the following equalities

$$\begin{aligned}\mathsf{c}_2(X) &= \mathsf{ch}_1(\mathcal{Q})^2 - 8\mathsf{ch}_2(X),\\ 10\mathsf{ch}_3(\mathcal{Q}) &= \frac{1}{6}\mathsf{ch}_1(\mathcal{Q})^3 - 3\mathsf{ch}_1(\mathcal{Q})\mathsf{ch}_2(\mathcal{Q}),\\ 15\mathsf{p} &= -4\mathsf{ch}_4(\mathcal{Q}) + \mathsf{ch}_1(\mathcal{Q})\mathsf{ch}_3(\mathcal{Q}) - 2\mathsf{ch}_2(\mathcal{Q})^2 + \frac{1}{2}\mathsf{ch}_1(\mathcal{Q})^2\mathsf{ch}_2(\mathcal{Q}).\end{aligned}$$

The first one yields to (1). From the intersection numbers

$$\mathsf{c}_1\mathsf{c}_3 = 330\mathsf{p} \quad \mathsf{c}_4 = 105\mathsf{p} \quad \mathsf{c}_1^2\mathsf{c}_2 = 825\mathsf{p} \quad \mathsf{c}_2^2 = 477\mathsf{p} \quad \mathsf{c}_1^4 = 1452\mathsf{p},$$

(see the proof of [**DV10**, Lemma 4.5]) we get

$$\mathsf{ch}_1(\mathcal{Q})\mathsf{ch}_3(\mathcal{Q}) = -\frac{11}{2}\mathsf{p} \quad \mathsf{ch}_4(\mathcal{Q}) = -\frac{1}{4}\mathsf{p} \quad \mathsf{ch}_1(\mathcal{Q})^2\mathsf{ch}_2(\mathcal{Q}) = 99\mathsf{p} \quad \mathsf{ch}_2(\mathcal{Q})^2 = 15\mathsf{p} \quad \mathsf{ch}_1(\mathcal{Q})^4 = 1452\mathsf{p}.$$

(3) follows from this.
Since X is a very general projective hyperkähler manifold of $\mathrm{K3}^{[2]}$–type, its Picard number is 1 and the group $H^{1,1}(X,\mathbb{Q})$ is generated by $\mathsf{ch}_1(\mathcal{Q})$. Dually, the group $H^{3,3}(X,\mathbb{Q})$ is generated by $\mathsf{ch}_1(\mathcal{Q})^\vee$, i.e., the class satisfying $\int_X \mathsf{ch}_1(\mathcal{Q})^\vee \cdot \alpha = q_X(\mathsf{ch}_1(\mathcal{Q}),\alpha)$ for every $\alpha \in H^2(X,\mathbb{Q})$. Let q_X be the Beauville–Bogomolov–Fujiki form, then by the Beauville–Bogomolov–Fujiki relation applied to $\mathsf{ch}_1(\mathcal{Q})$, we have $q_X(\mathsf{ch}_1(\mathcal{Q})) = 22$ and

$$\mathsf{ch}_1(\mathcal{Q})^\vee\mathsf{ch}_1(\mathcal{Q}) = 22\mathsf{p}.$$

It follows that

$$\mathsf{ch}_1(\mathcal{Q})^3 = 66\mathsf{ch}_1(\mathcal{Q})^\vee \quad \mathsf{ch}_1(\mathcal{Q})\mathsf{ch}_2(\mathcal{Q}) = \frac{9}{2}\mathsf{ch}_1(\mathcal{Q})^\vee \quad \mathsf{ch}_3(\mathcal{Q}) = -\frac{1}{4}\mathsf{ch}_1(\mathcal{Q})^\vee,$$

from which we get (2). □

We are now ready to compute the Chern characters of the general Schur functor $\Sigma_{m,t,s,0}\mathcal{Q}$. In degrees 2, 3, and 4, these formulae were computed by interpolation of several values of the tuple (m, t, s). To state the next proposition, we need the following notations:

$$
\begin{aligned}
r(m,t,s) &:= \frac{(m+3)(t+2)(s+1)(m-t+1)(m-s+2)(t-s+1)}{12},\\
\ell(m,t,s) &:= \frac{m+t+s}{4},\\
\delta(m,t,s) &:= \frac{3m^2-2mt-2ms+3t^2+3s^2-2ts+12m+4t-4s}{60},\\
\tau(m,t,s) &:= 15\delta(m,t,s)-44\ell(m,t,s)^2,\\
\alpha_3(t,s) &:= -60t-60s+30,\\
\alpha_2(t,s) &:= -109t^2-241ts-109s^2+103t+108s-21,\\
\alpha_1(t,s) &:= -60t^3-241t^2s-241ts^2-60s^3+65t^2+78ts+4s^2-t+8s+6,\\
\alpha_0(t,s) &:= -10t^4-60t^3s-109t^2s^2-60ts^3-10s^4+10t^3+19t^2s-19ts^2+\\
&\quad -10s^3+3t^2-13ts+3s^2+14t-14s,\\
\xi(m,t,s) &:= \frac{-10m^4+\alpha_3(t,s)m^3+\alpha_2(t,s)m^2+\alpha_1(t,s)m+\alpha_0(t,s)}{20}.
\end{aligned}
$$

Proposition 3.3.3. *Let $X \subseteq \mathrm{Gr}(6,10)$ be a very general Debarre–Voisin hyperkähler manifold, and let $\lambda = (m, t, s, 0)$ with $m \geq t + s$, then:*

(1) $\mathsf{ch}_0(\Sigma_\lambda \mathcal{Q}) = r(m,t,s)$, which is the rank of $\Sigma_{m,t,s,0}\mathcal{Q}$.;
(2) $\mathsf{ch}_1(\Sigma_\lambda \mathcal{Q}) = \ell(m,t,s)r(m,t,s)\mathsf{ch}_1(\mathcal{Q})$;
(3) $\mathsf{ch}_2(\Sigma_\lambda \mathcal{Q}) = \delta(m,t,s)r(m,t,s)\mathsf{ch}_2(\mathcal{Q}) + \frac{1}{2}\left(\ell(m,t,s)^2 - \frac{\delta(m,t,s)}{4}\right) r(m,t,s)\mathsf{ch}_1(\mathcal{Q})^2$;
(4) $\mathsf{ch}_3(\Sigma_\lambda \mathcal{Q}) = \tau(m,t,s)\ell(m,t,s)r(m,t,s)\mathsf{ch}_3(\mathcal{Q})$;
(5) $\mathsf{ch}_4(\Sigma_\lambda \mathcal{Q}) = \xi(m,t,s)r(m,t,s)\mathsf{ch}_4(\mathcal{Q})$.

Proof. The proofs of (1) and (2) are straightforward.

(1) It is an application of Weyl's dimension formula for representations of Lie groups of type A_3.
(2) It follows directly from the main result in [**Rub13**].

From degree 2 on, there is no formula for the Chern character of a general Schur functor. Therefore, all the polynomials in the statement are obtained by interpolation of several values of m, t, and s. To check that they work in general, we need to conduct an induction on the partition entries.

- Case $t = 0$ This corresponds to the case of the symmetric power and follows from Lemma 3.6.1.

- Case $t > 0$, $s = 0$ Here we use Pieri's rule on

$$\mathrm{Sym}^m\mathcal{Q} \otimes \mathrm{Sym}^t\mathcal{Q} = \bigoplus_{i=0}^{t} \Sigma_{m+t-i,i}\mathcal{Q}$$

in order to obtain the Chern character of $\Sigma_{m,t}\mathcal{Q}$ as

$$\mathsf{ch}\left(\Sigma_{m,t}\mathcal{Q}\right) = \mathsf{ch}\left(\mathrm{Sym}^m\mathcal{Q}\right)\mathsf{ch}\left(\mathrm{Sym}^t\mathcal{Q}\right) - \sum_{i=0}^{t-1}\mathsf{ch}\left(\Sigma_{m+t-i,i}\mathcal{Q}\right).$$

If we isolate the degree $d = 2$, 3 and 4 we can use the inductive hypothesis to get $\mathrm{ch}_d\left(\Sigma_{m,t}\mathcal{Q}\right)$.

- Case $s > 0$ Here we use Pieri's rule on $\Sigma_{m,t}\mathcal{Q} \otimes \mathrm{Sym}^s\mathcal{Q}$, which gives

$$\begin{aligned}\Sigma_{m,t}\mathcal{Q} \otimes \mathrm{Sym}^s\mathcal{Q} = \Sigma_{m+s,t}\mathcal{Q} \oplus \Sigma_{m+s-1,t+1}\mathcal{Q} \oplus \Sigma_{m+s-2,t+2}\mathcal{Q} \oplus \ldots \oplus \Sigma_{m,t+s}\mathcal{Q} \oplus \\ \oplus \Sigma_{m+s-1,t,1}\mathcal{Q} \oplus \Sigma_{m+s-2,t+1,1}\mathcal{Q} \oplus \ldots \oplus \Sigma_{m,t+s-1,1}\mathcal{Q} \oplus \\ \oplus \Sigma_{m+s-2,t,2}\mathcal{Q} \oplus \ldots \oplus \Sigma_{m,t+s-2,2}\mathcal{Q} \oplus \\ \vdots \\ \oplus \Sigma_{m+1,t,s-1}\mathcal{Q} \oplus \Sigma_{m,t+1,s-1}\mathcal{Q} \\ \oplus \Sigma_{m,t,s}\mathcal{Q}.\end{aligned}$$

  To perform induction, one applies the Chern character on both sides and extracts the term $\mathrm{ch}\left(\Sigma_{m,t,s}\mathcal{Q}\right)$. As before, for $d = 2$, 3 and 4 we can use the inductive hypothesis to get $\mathrm{ch}_d\left(\Sigma_{m,t}\mathcal{Q}\right)$.

Then one has to check several equalities between long polynomials in several variables, which is straightforward but quite cumbersome. For this reason, we made the computations using computer algebra software, such as Macaulay2 and Wolfram Alpha. □

Remark 3.3.4. *The polynomial appearing in the fourth Chern character is quite complicated. Unlike the previous ones, it is not possible to express it as a polynomial in δ and ℓ. More precisely, it is not a linear combination of ℓ^4, $\ell^2\delta$ and δ^2.*

We obtain two key results from Proposition 3.3.3.

Corollary 3.3.5. *Let $X \subseteq \mathrm{Gr}(6, 10)$ be a very general Debarre–Voisin hyperkähler manifold, and let $\lambda = (m, t, s, 0)$ with $m \geq t + s$, then*

$$\Delta(\Sigma_\lambda\mathcal{Q}) = \frac{\delta(m, t, s)}{4} r(m, t, s)^2 \mathrm{c}_2(X).$$

In particular, $\Sigma_\lambda\mathcal{Q}$ is modular, i.e. $\Delta(\Sigma_{m,t,s,0}\mathcal{Q}) \in \mathbb{Q}q_X$, where q_X is the Beauville–Fujiki–Bogomolov form on X.

Now, to simplify the notation, for a vector bundle $\mathcal{F}$ let us put

$$\Xi(\mathcal{F}) := \mathrm{ch}_2(\mathcal{F})^2 - 2\mathrm{c}_1(\mathcal{F})\mathrm{ch}_3(\mathcal{F}) + 2\,\mathrm{rk}\,\mathcal{F}\mathrm{ch}_4(\mathcal{F}).$$

Corollary 3.3.6. *Under the hypothesis of Proposition 3.3.3, we have:*

(1) $\Xi(\Sigma_\lambda\mathcal{Q}) = -\frac{1}{3312}\left(-m^4 + \frac{\alpha_3}{10}m^3 + \frac{\alpha_2}{10}m^2 + \frac{\alpha_1}{10}m + \frac{\alpha_0}{10} + 484\ell^4 - 330\ell^2\delta - \frac{207}{4}\delta^2\right) r^2\mathrm{c}_2(X)^2$;

(2) $\int_X \Xi(\Sigma_\lambda\mathcal{Q}) = -\frac{1}{4}\left(-m^4 + \frac{\alpha_3}{10}m^3 + \frac{\alpha_2}{10}m^2 + \frac{\alpha_1}{10}m + \frac{\alpha_0}{10} + 484\ell^4 - 330\ell^2\delta - \frac{207}{4}\delta^2\right) r^2$;

(3) $\chi(\Sigma_\lambda\mathcal{Q}) = 3\left(1 + \frac{-276\delta - 1936\ell^4 + 1320\delta\ell^2 + 207\delta^2 - 8\xi}{48}\right) r^2$.

Remark 3.3.7. *We notice that the Chern characters of degree 0, 1 and 2 are given by the same polynomials as in [FO24, Theorem 4.1], while they differ in degree 3 and 4. More precisely, the polynomials r, ℓ and δ are the same, while τ and ξ are different. This is a consequence of the fact that the relations of the intersection ring of X appear from degree 3 onward. Nevertheless, the Euler characteristic is given by the same polynomial as in the case of [FO24].*

3.4. Cohomological computations

In this section, we demonstrate the computation of $\mathrm{Ext}(\Sigma_\lambda\mathcal{Q}, \Sigma_\lambda\mathcal{Q})$ for a given partition λ. As the entries of λ grow larger, performing these computations in practice becomes increasingly challenging. However, the general theory provides an algorithmic approach to these calculations.

3.4.1. The computation. Fix a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. Following Lemma 3.2.2, we can assume that $\lambda = (m, s, t, 0)$, with $m \geq t + s$. Recall that

$$\mathrm{Ext}^k\left(\Sigma_{m,t,s}\mathcal{Q}, \Sigma_{m,t,s}\mathcal{Q}\right) \cong H^k(\Sigma_{m,t,s}\mathcal{Q} \otimes \Sigma_{m,t,s}\mathcal{Q}^\vee) \cong H^k\left(\Sigma_{m,t,s}\mathcal{Q} \otimes \Sigma_{m,m-s,m-t}\mathcal{Q} \otimes \mathcal{O}_X(-m)\right),$$

hence, using Littlewood–Richardson's rule (Algorithm 1.2.6), we need to compute the cohomology of irreducible factors of the form

$$E := \Sigma_\mu\mathcal{Q} \otimes \mathcal{O}_X(-m).$$

To do so, we twist the Koszul complex Equation 3.2.1 by $\Sigma_\mu\widetilde{\mathcal{Q}} \otimes \mathcal{O}_{\mathrm{Gr}}(-m)$, obtaining a long exact sequence

$$0 \to \mathcal{O}_{\mathrm{Gr}(6,10)}(-10-m) \to \bigwedge^{19}\bigwedge^{3}\widetilde{\mathcal{U}} \otimes \Sigma_\mu\widetilde{\mathcal{Q}} \otimes \mathcal{O}_{\mathrm{Gr}}(-m) \to \dots$$
$$\dots \to \bigwedge^{2}\bigwedge^{3}\widetilde{\mathcal{U}} \otimes \Sigma_\mu\widetilde{\mathcal{Q}} \otimes \mathcal{O}_{\mathrm{Gr}}(-m) \to \bigwedge^{3}\widetilde{\mathcal{U}} \otimes \Sigma_\mu\widetilde{\mathcal{Q}} \otimes \mathcal{O}_{\mathrm{Gr}}(-m) \to \Sigma_\mu\widetilde{\mathcal{Q}} \otimes \mathcal{O}_{\mathrm{Gr}}(-m) \to E \to 0.$$

As explained in Section 3.2.1, each exterior power decomposes into irreducible pieces. For instance, in degree-3 we have

$$\Sigma_\mu\widetilde{\mathcal{Q}}\otimes\Sigma_{3+m,3+m,1+m,1+m,1+m,m}\widetilde{\mathcal{U}}\oplus\Sigma_\mu\widetilde{\mathcal{Q}}\otimes\Sigma_{3+m,2+m,2+m,2+m,m,m}\widetilde{\mathcal{U}}\oplus\Sigma_\mu\widetilde{\mathcal{Q}}\otimes\Sigma_{2+m,2+m,2+m,1+m,1+m,1+m}\widetilde{\mathcal{U}}.$$

For each irreducible factor appearing in the twisted Koszul complex, it is possible to compute its cohomology using Borel–Weil–Bott Theorem.

Denote by $(d_0^p, d_1^p, \dots, d_{24}^p)$ the tuple containing the dimensions of the cohomology vector spaces of

$$\bigwedge^{p}\bigwedge^{3}\widetilde{\mathcal{U}} \otimes \Sigma_\mu\widetilde{\mathcal{Q}} \otimes \mathcal{O}_{\mathrm{Gr}}(-m).$$

Recall that, for an irreducible vector bundle, at most one cohomology group is non-zero. In particular, lots of the d_i^p could be zero. To compute the cohomology of E we split the long exact sequence into short exact sequences as explained in Equation 3.2.1.

$$0 \to \bigwedge^{20}\bigwedge^{3}\widetilde{\mathcal{U}} \otimes \Sigma_\mu\widetilde{\mathcal{Q}} \otimes \mathcal{O}_{\mathrm{Gr}}(-m) \to \bigwedge^{19}\bigwedge^{3}\widetilde{\mathcal{U}} \otimes \Sigma_\mu\widetilde{\mathcal{Q}} \otimes \mathcal{O}_{\mathrm{Gr}}(-m) \to K_1 \to 0$$
$$0 \to K_1 \to \bigwedge^{18}\bigwedge^{3}\widetilde{\mathcal{U}} \otimes \Sigma_\mu\widetilde{\mathcal{Q}} \otimes \mathcal{O}_{\mathrm{Gr}}(-m) \to K_2 \to 0$$
$$\dots$$
$$0 \to K_{p-1} \to \bigwedge^{20-p}\bigwedge^{3}\widetilde{\mathcal{U}} \otimes \Sigma_\mu\widetilde{\mathcal{Q}} \otimes \mathcal{O}_{\mathrm{Gr}}(-m) \to K_p \to 0$$
$$\dots$$
$$0 \to K_{18} \to \bigwedge^{3}\widetilde{\mathcal{U}} \otimes \Sigma_\mu\widetilde{\mathcal{Q}} \otimes \mathcal{O}_{\mathrm{Gr}}(-m) \to K_{19} \to 0$$
$$0 \to K_{19} \to \Sigma_\mu\widetilde{\mathcal{Q}} \otimes \mathcal{O}_{\mathrm{Gr}}(-m) \to E \to 0.$$

In principle, one can compute the cohomology of K_1 using the first short exact sequence and use this information to compute the cohomology of K_2 and so on. Eventually, from the last short exact sequence one obtains the cohomology of E, as wanted. In practice, this is possible only for low values of m, t and s, where the cohomological maps must be trivial. In some cases, one can still argue to solve the indeterminacy as in Section 3.4.2.

Summing up all the cohomology of the irreducible factors E appearing (with multiplicities) in Littlewood–Richardson's formula (if determinate), one obtains $H^*(\Sigma_{m,t,s}\mathcal{Q} \otimes \Sigma_{m,t,s}\mathcal{Q}^\vee)$. Following this process, we get

Proposition 3.4.1. *The dimensions of the* Ext*-group of Schur functors $\Sigma_{(m,t,s,0)}\mathcal{Q}$ with $m < 5$ are the ones stated in Table 1. Moreover, any Schur functor of $\mathcal{Q}$ with $m \geq 5$ is indeterminate.*

Proof. The first part is obtained with a case-by-case analysis following the previous algorithm. In Section 3.5, we treat two examples in detail.
For the second part, we need the next lemma (Lemma 3.4.2), which tells us that if a factor appears in the Littlewood–Richardson's decomposition of the endomorphisms bundle of a given Schur functor, it keeps appearing as we increase the entries of the partition. We use this lemma to ensure that we did not miss any "computable" Schur functors. Take any $\lambda = (m, t, s, 0)$ with $m \geq t + s$, we have the following cases:

- Case $m - t \geq 5$
  The indeterminacy follows from the one of $(5, 0, 0, 0)$ by adding s-times $(1, 1, 1, 0)$, $(t - s)$-times $(1, 1, 0, 0)$ and $(m - t - 5)$-times $(1, 0, 0, 0)$;
- Case $m - t = 4$
  If $t - s \geq 1$ then the indeterminacy follows from the one of $(5, 1, 0, 0)$, otherwise we check a finite number of cases with $t = s$, $s \leq 4$;
- Case $m - t = 3$
  If $t - s \geq 2$ then the indeterminacy follows from the one of $(5, 2, 0, 0)$, otherwise we check a finite number of cases with $t = s + 1$, $s \leq 3$ and $t = s$, $s \leq 3$;
- Case $m - t = 2$
  If $t - s \geq 2$ then the indeterminacy follows from the one of $(4, 2, 0, 0)$, otherwise we check a finite number of cases with $t = s + 1$, $s \leq 2$ and $t = s$, $s \leq 2$;
- Case $m - t = 1$
  If $t - s \geq 3$ then the indeterminacy follows from the one of $(4, 3, 0, 0)$, otherwise we check a finite number of cases with $t = s + 2$, $s \leq 1$ and $t = s + 1$, $s \leq 1$ and $t = s$, $s \leq 1$;
- Case $m - t = 0$
  In this case s must be zero and the indeterminacy follows from the one of $(3, 3, 0, 0)$ (if $m = t \geq 3$).

□

Lemma 3.4.2. *If $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$ is a partition appearing in the Littlewood–Richardson decomposition of $\mathcal{E}nd\left(\Sigma_{(m,t,s,0)}\mathcal{Q}\right)$, then $(\mu_1 + 1, \mu_2 + 1, \mu_3 + 1, \mu_4 + 1)$ appears in the decomposition of*

$$\mathcal{E}nd\left(\Sigma_{(m+1,t,s,0)}\mathcal{Q}\right), \mathcal{E}nd\left(\Sigma_{(m+1,t+1,s,0)}\mathcal{Q}\right) \text{ and } \mathcal{E}nd\left(\Sigma_{(m+1,t+1,s+1,0)}\mathcal{Q}\right).$$

Proof. This is a generalization of [FO24, Proposition 8.3], where it is proved for

$$\mathcal{E}nd\left(\Sigma_{(m+1,t,s,0)}\mathcal{Q}\right).$$

The other two cases are obtained with the same ideas, following the Littlewood–Richardson's rule. ☐

3.4.2. An example of solved indeterminacy. Unfortunately, solving the short exact sequences is not an algorithmic process, but in some cases, the cohomology of the irreducible factors forces all the cohomology maps to be trivial. This happens for most of the determined cases in Table 1. However, the cases $(3,2,1,0)$, $(4,1,1,0)$ and $(4,2,2,0)$, present a "small" indeterminacy which can be solved thanks to the following argument.

- $\underline{\lambda = (3,2,1,0)}$
  In the Littlewood–Richardson decomposition of $\mathcal{E}nd\,(\Sigma_\lambda \mathcal{Q})$ appears the indeterminate factor
  $$\Sigma_{5,5,2,0|3,3,3,3,3,3}.$$
  Recall that if λ and μ are two partitions of lengths four and six, respectively, then $\Sigma_{\lambda|\mu}$ denotes $\Sigma_\lambda V_4 \otimes \Sigma_\mu V_6$, whereas if ν is a partition of length ten, then Σ_ν denotes $\Sigma_\nu V_{10}$.
  We follow the argument in Section 3.4.1. By Borel–Weil–Bott's theorem, the only non-vanishing cohomologies in the Koszul complex are:
  - $H^{15}\left(\bigwedge^{13}\bigwedge^{3}\widetilde{\mathcal{U}}\otimes\Sigma_{5,5,2,0}\widetilde{\mathcal{Q}}\otimes\mathcal{O}_{\mathrm{Gr}}(-3)\right) = \bigwedge^{9} V_{10}$;
  - $H^{14}\left(\bigwedge^{12}\bigwedge^{3}\widetilde{\mathcal{U}}\otimes\Sigma_{5,5,2,0}\widetilde{\mathcal{Q}}\otimes\mathcal{O}_{\mathrm{Gr}}(-3)\right) = \bigwedge^{6} V_{10}$;
  - $H^{12}\left(\bigwedge^{11}\bigwedge^{3}\widetilde{\mathcal{U}}\otimes\Sigma_{5,5,2,0}\widetilde{\mathcal{Q}}\otimes\mathcal{O}_{\mathrm{Gr}}(-3)\right) = \mathrm{Sym}^3 V_{10}$;
  - $H^{11}\left(\bigwedge^{9}\bigwedge^{3}\widetilde{\mathcal{U}}\otimes\Sigma_{5,5,2,0}\widetilde{\mathcal{Q}}\otimes\mathcal{O}_{\mathrm{Gr}}(-3)\right) = \Sigma_{2,2,2,2,2,2,2,2,1,0} V_{10}$;
  - $H^{10}\left(\bigwedge^{8}\bigwedge^{3}\widetilde{\mathcal{U}}\otimes\Sigma_{5,5,2,0}\widetilde{\mathcal{Q}}\otimes\mathcal{O}_{\mathrm{Gr}}(-3)\right) = \bigwedge^{4} V_{10}\oplus\Sigma_{2,1,1} V_{10}$;
  - $H^{6}\left(\bigwedge^{4}\bigwedge^{3}\widetilde{\mathcal{U}}\otimes\Sigma_{5,5,2,0}\widetilde{\mathcal{Q}}\otimes\mathcal{O}_{\mathrm{Gr}}(-3)\right) = \bigwedge^{2} V_{10}\oplus\Sigma_{2,2,2,1,1,1,1,1,1,0} V_{10}$.

  The first non-trivial short exact sequence is
  $$0\to K_6\to\bigwedge^{13}\bigwedge^{3}\widetilde{\mathcal{U}}\otimes\Sigma_{5,5,2,0}\widetilde{\mathcal{Q}}\otimes\mathcal{O}_{\mathrm{Gr}}(-3)\to K_7\to 0,$$
  which tells us that $H^{15}(K_7) = \bigwedge^9 V_{10}$. Then we have
  $$0\to K_7\to\bigwedge^{12}\bigwedge^{3}\widetilde{\mathcal{U}}\otimes\Sigma_{5,5,2,0}\widetilde{\mathcal{Q}}\otimes\mathcal{O}_{\mathrm{Gr}}(-3)\to K_8\to 0,$$
  which tells us that $H^{14}(K_8) = \bigwedge^9 V_{10}\oplus\bigwedge^6 V_{10}$. Then we have
  $$0\to K_8\to\bigwedge^{11}\bigwedge^{3}\widetilde{\mathcal{U}}\otimes\Sigma_{5,5,2,0}\widetilde{\mathcal{Q}}\otimes\mathcal{O}_{\mathrm{Gr}}(-3)\to K_9\to 0,$$
  $$0\to K_9\to\bigwedge^{10}\bigwedge^{3}\widetilde{\mathcal{U}}\otimes\Sigma_{5,5,2,0}\widetilde{\mathcal{Q}}\otimes\mathcal{O}_{\mathrm{Gr}}(-3)\to K_{10}\to 0,$$
  from which we get that $H^{12}(K_{10}) = H^{13}(K_9) = \bigwedge^9 V_{10}\oplus\bigwedge^6 V_{10}$ and $H^{11}(K_{10}) = H^{12}(K_9) = \mathrm{Sym}^3 V_{10}$.
  Here comes the first non-trivial short exact sequence
  $$0\to K_{10}\to\bigwedge^{9}\bigwedge^{3}\widetilde{\mathcal{U}}\otimes\Sigma_{5,5,2,0}\widetilde{\mathcal{Q}}\otimes\mathcal{O}_{\mathrm{Gr}}(-3)\to K_{11}\to 0.$$
  In cohomology, we have
  $$0\to H^{10}(K_{11})\to\mathrm{Sym}^3 V_{10}\xrightarrow{\varphi}\Sigma_{2,2,2,2,2,2,2,2,1,0} V_{10}\to H^{11}(K_{11})\to\bigwedge^{9} V_{10}\oplus\bigwedge^{6} V_{10}\to 0.$$

Using Weyl's formula, we compute $\dim\left(\mathrm{Sym}^3 V_{10}\right) = 220$ and $\dim\left(\Sigma_{2,2,2,2,2,2,2,2,1,0} V_{10}\right) = 330$, hence we want to prove that the map φ is injective. Arguing as in [**KMM10**, Appendix B], by generality of X_{DV}, we have the injectivity of the map

$$\mathrm{Sym}^3 V_{10} \hookrightarrow \mathrm{Sym}^3 V_{10} \otimes \bigwedge^7 V_{10} \otimes \bigwedge^3 V_{10}^\vee$$

given by the multiplication of a symmetric tensor with the equation defining X_{DV}. Our map corresponds to the composition of this map and

$$\mathrm{Sym}^3 V_{10} \otimes \bigwedge^7 V_{10} \otimes \bigwedge^3 V_{10}^\vee \longrightarrow \Sigma_{3,2,1,1,1,1,1,1,1,1} V_{10} \cong \Sigma_{2,1} V_{10}^\vee \cong \Sigma_{2,2,2,2,2,2,2,2,1,0} V_{10}$$

give by the Littlewood–Richardson's decomposition. Since the multiplicity of $\Sigma_{2,2,2,2,2,2,2,2,1,0}$ in the decomposition is 1, we obtain that φ is injective.

This yields to $H^{10}(K_{11}) = 0$ and

$$H^{11}(K_{11}) \cong \bigwedge^9 V_{10} \oplus \bigwedge^6 V_{10} \oplus {}^{\Sigma_{2,2,2,2,2,2,2,2,1,0} V_{10}}\big/_{\mathrm{Sym}^3 V_{10}}.$$

The remaining short exact sequences are all trivial, thus we get that the only non-vanishing cohomology of $\Sigma_{5,5,2,0|3,3,3,3,3,3}$ is $H^2\left(\Sigma_{5,5,2,0|3,3,3,3,3,3}\right)$, given by

$$\bigwedge^9 V_{10} \oplus \bigwedge^6 V_{10} \oplus {}^{\Sigma_{2,2,2,2,2,2,2,2,1,0} V_{10}}\big/_{\mathrm{Sym}^3 V_{10}} \oplus \bigwedge^4 V_{10} \oplus \Sigma_{2,1,1} V_{10} \oplus \bigwedge^2 V_{10} \oplus \Sigma_{2,2,2,1,1,1,1,1,1,0} V_{10},$$

whose dimension is 2730.

- $\underline{\lambda = (4,1,1,0)}$
  In the Littlewood–Richardson decomposition of $\mathcal{E}nd\,(\Sigma_\lambda \mathcal{Q})$ appears the indeterminate factor
  $$\Sigma_{7,5,4,0|4,4,4,4,4,4}.$$
  By Borel–Weil–Bott's theorem, the only non-vanishing cohomologies in the Koszul complex are:
  - $H^{17}\left(\bigwedge^{16}\bigwedge^3 \widetilde{\mathcal{U}} \otimes \Sigma_{7,5,4,0}\widetilde{\mathcal{Q}} \otimes \mathcal{O}_{\mathrm{Gr}}(-4)\right) = \Sigma_{4,3,3,3,3,3,3,3,3,0} V_{10}$;
  - $H^{14}\left(\bigwedge^{12}\bigwedge^3 \widetilde{\mathcal{U}} \otimes \Sigma_{7,5,4,0}\widetilde{\mathcal{Q}} \otimes \mathcal{O}_{\mathrm{Gr}}(-4)\right) = \Sigma_{2,2,2,2,2,2,2,2,0,0} V_{10}$;
  - $H^{13}\left(\bigwedge^{11}\bigwedge^3 \widetilde{\mathcal{U}} \otimes \Sigma_{7,5,4,0}\widetilde{\mathcal{Q}} \otimes \mathcal{O}_{\mathrm{Gr}}(-4)\right) = \Sigma_{2,2,2,2,2,1,1,1,0,0} V_{10}$;
  - $H^{12}\left(\bigwedge^{10}\bigwedge^3 \widetilde{\mathcal{U}} \otimes \Sigma_{7,5,4,0}\widetilde{\mathcal{Q}} \otimes \mathcal{O}_{\mathrm{Gr}}(-4)\right) = \Sigma_{2,2,1,1,1,1,1,1,0,0} V_{10}$;
  - $H^{10}\left(\bigwedge^{8}\bigwedge^3 \widetilde{\mathcal{U}} \otimes \Sigma_{7,5,4,0}\widetilde{\mathcal{Q}} \otimes \mathcal{O}_{\mathrm{Gr}}(-4)\right) = \Sigma_{2,2,2,2,2,2,1,1,0,0} V_{10} \oplus \bigwedge^4 V_{10}$;
  - $H^{8}\left(\bigwedge^{6}\bigwedge^3 \widetilde{\mathcal{U}} \otimes \Sigma_{7,5,4,0}\widetilde{\mathcal{Q}} \otimes \mathcal{O}_{\mathrm{Gr}}(-4)\right) = \Sigma_{2,2,1,1,1,1} V_{10} \oplus \Sigma_{2,1,1,1,1,1,1,0,0,0} V_{10}$;
  - $H^{5}\left(\bigwedge^{3}\bigwedge^3 \widetilde{\mathcal{U}} \otimes \Sigma_{7,5,4,0}\widetilde{\mathcal{Q}} \otimes \mathcal{O}_{\mathrm{Gr}}(-4)\right) = \Sigma_{3,1,1,1,1,1,1,0,0,0} V_{10}$.

  Arguing as before, we obtain that the only non-vanishing cohomology of $\Sigma_{7,5,4,0|4,4,4,4,4,4}$ is in degree 2. Its dimension is 32550 and it is given by

$$\begin{aligned}{}^{\Sigma_{2,2,2,2,2,1,1,1,0,0} V_{10}}\big/_{H^{13}(K_8)} &\oplus \Sigma_{2,2,1,1,1,1,1,1,0,0} V_{10} \oplus \Sigma_{2,2,2,2,2,2,1,1,0,0} V_{10} \oplus \bigwedge^4 V_{10} \oplus \\ &\oplus \Sigma_{2,2,1,1,1,1} V_{10} \oplus \Sigma_{2,1,1,1,1,1,1,0,0,0} V_{10} \oplus \Sigma_{3,1,1,1,1,1,1,0,0,0} V_{10},\end{aligned}$$

  where $H^{13}(K_8)^\vee \cong {}^{\Sigma_{4,1,1,1,1,1,1,1,1,0} V_{10}}\big/_{\Sigma_{2,2} V_{10}}$.

- $\underline{\lambda = (4, 2, 2, 0)}$
  In the Littlewood–Richardson decomposition of $\mathcal{E}nd\,(\Sigma_\lambda Q)$, besides both of the previous ones, it appears the indeterminate factor
  $$\Sigma_{6,6,4,0|4,4,4,4,4,4}.$$
  By Borel–Weil–Bott's theorem, the only non-vanishing cohomologies in the Koszul complex are:
  - $H^{14}\left(\bigwedge^{13}\bigwedge^3 \widetilde{\mathcal{U}} \otimes \Sigma_{6,6,4,0}\widetilde{\mathcal{Q}} \otimes \mathcal{O}_{\mathrm{Gr}}(-4)\right) = \Sigma_{3,2,2,2,2,2,2,2,2,0} V_{10}$;
  - $H^{14}\left(\bigwedge^{12}\bigwedge^3 \widetilde{\mathcal{U}} \otimes \Sigma_{6,6,4,0}\widetilde{\mathcal{Q}} \otimes \mathcal{O}_{\mathrm{Gr}}(-4)\right) = \Sigma_{2,2,2,2,2,2,2,1,1,0} V_{10}$;
  - $H^{11}\left(\bigwedge^{10}\bigwedge^3 \widetilde{\mathcal{U}} \otimes \Sigma_{6,6,4,0}\widetilde{\mathcal{Q}} \otimes \mathcal{O}_{\mathrm{Gr}}(-4)\right) = \Sigma_{3,1,1,1,1,1,1,1,0,0} V_{10}$;
  - $H^{11}\left(\bigwedge^{9}\bigwedge^3 \widetilde{\mathcal{U}} \otimes \Sigma_{6,6,4,0}\widetilde{\mathcal{Q}} \otimes \mathcal{O}_{\mathrm{Gr}}(-4)\right) = \Sigma_{2,1,1,1,1,1,0,0,0,0} V_{10}$;
  - $H^{9}\left(\bigwedge^{7}\bigwedge^3 \widetilde{\mathcal{U}} \otimes \Sigma_{6,6,4,0}\widetilde{\mathcal{Q}} \otimes \mathcal{O}_{\mathrm{Gr}}(-4)\right) = \Sigma_{2,2,2,1,1,1,1,1,0,0} V_{10}$;
  - $H^{7}\left(\bigwedge^{5}\bigwedge^3 \widetilde{\mathcal{U}} \otimes \Sigma_{6,6,4,0}\widetilde{\mathcal{Q}} \otimes \mathcal{O}_{\mathrm{Gr}}(-4)\right) = \bigwedge^5 V_{10}$;
  - $H^{5}\left(\bigwedge^{3}\bigwedge^3 \widetilde{\mathcal{U}} \otimes \Sigma_{6,6,4,0}\widetilde{\mathcal{Q}} \otimes \mathcal{O}_{\mathrm{Gr}}(-4)\right) = \Sigma_{2,2,1,1,1,1,1,0,0,0} V_{10}$.

  Arguing as before, we obtain that the only non-vanishing cohomology of $\Sigma_{6,6,4,0|4,4,4,4,4,4}$ is in degree 2 and it is of dimension 10206.

3.4.3. Combinatorial results. Here we deal with some combinatorial aspects of the cohomological dimensions of the sheaf of endomorphisms of Schur functors. The goal is to follow the Littlewood–Richardson's rule to predict which Schur functors appear in the decomposition of $\mathcal{E}nd\,(\Sigma_\lambda Q)$. The main result is Corollary 3.4.4.

Lemma 3.4.3. *Let $\lambda = (m, t, s, 0)$ be a partition with $m \geq t + s$, then, among the irreducible factors of*
$$\Sigma_{(m,t,s,0)} \otimes \Sigma_{(m,m-s,m-t,0)},$$
we have:

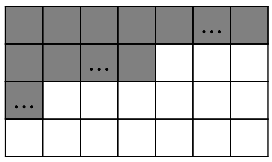
(1) $\Sigma_{(m,m,m,m)}$ *with multiplicity* 1;

(2) $\Sigma_{(m+1,m+1,m-1,m-1)}$ *with multiplicity* $\begin{cases} 0 & \textit{if } t = s = 0 \\ 2 & \textit{if } m > t > s \\ 1 & \textit{otherwise} \end{cases}$

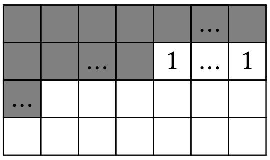
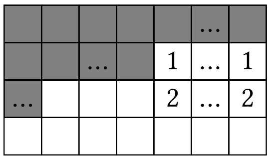
(3) $\Sigma_{(m+2,m,m,m-2)}$ *with multiplicity* $\begin{cases} 0 & \textit{if } m = 1 \\ \geq 1 & \textit{otherwise} \end{cases}$

Proof. This is an application of Littlewood–Richardson's formula.

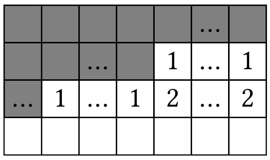
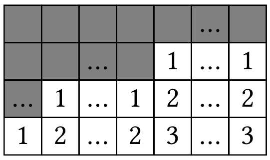
For (1), we need to show that there is exactly one way to fill the diagram on the right with content $(m, m-s, m-t, 0)$ following the rules of Algorithm 1.2.6.



Since we have to start with a 1 and the columns have to be strictly increasing we get

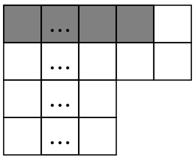
Now, the number of 2 equals the number of 1, and hence we complete the third and fourth rows as

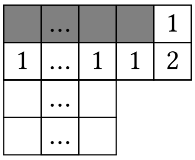
One can check that the content of this semi–standard Young tableaux is $(m, m-s, m-t, 0)$. Since we had not made any choice during the filling process, the multiplicity of $\Sigma_{(m,m,m,m)}$ in $\Sigma_{(m,t,s,0)} \otimes \Sigma_{(m,m-s,m-t,0)}$ is 1.
In order to prove (2), we consider four separate cases.

- Case 1.

  Assume that $t = s = 0$.
  We want to show that there is no admissible filling for the diagram on the right.



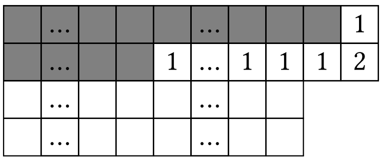
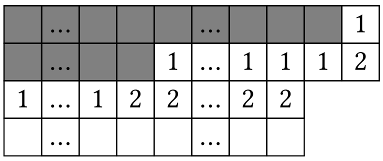
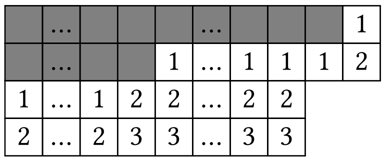
  We have a 1 in the first row and then 2 under it. Hence we get the tableaux on the right, which is not admissible since we have $m + 1$ times the number 1.



- Case 2.

  Assume that $t > s = 0$.
  We want to show that there is exactly one way to fill the diagram on the right.

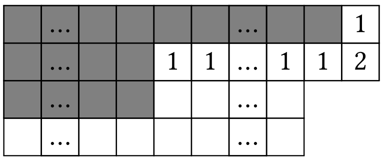
  We start as before. For the third row, we put the 2s under all the 1s, along with exactly one other 2: this is the last admissible 2 at this step and if we don't put it then we will have too much 1s.

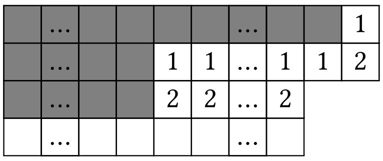




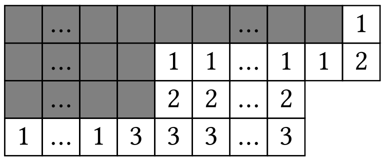
- Case 3.

  Assume that $t = s > 0$. We want to show that there is exactly one way to fill the diagram on the right.

  As before, we get the following filling.

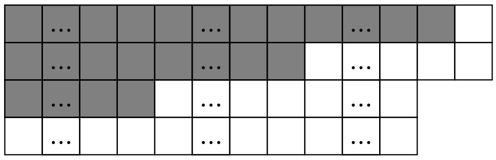


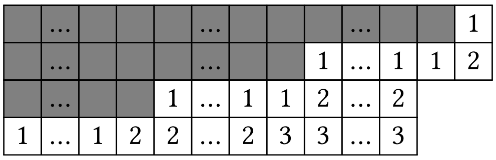


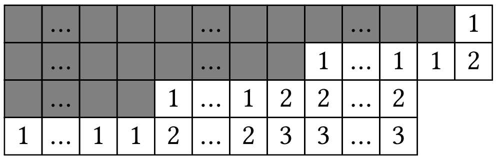


- Case 4.

  Assume that $t > s > 0$. There are two possible ways to fill the diagram on the right.

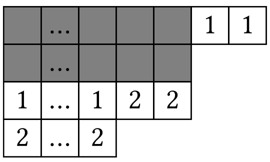






It remains to prove (3). Notice that if $m = 1$ then we have only the cases $\lambda = (1, 0, 0, 0)$, $(1, 1, 0, 0)$ and the result trivially holds. Assume that $m > 1$. We divided the proof into three cases.

- Case 1. Assume that $m - t = 0$ then we can fill the diagram (at least) in the following way



- <u>Case 2.</u> Assume that $m - t = 1$ then we can fill the diagram (at least) in the following ways (respectively if $s = 1$ and if $s = 0$).

$$\begin{array}{cccccccc} \blacksquare & \blacksquare & \cdots & \blacksquare & \blacksquare & \blacksquare & 1 & 1 \\ \blacksquare & \blacksquare & \cdots & \blacksquare & \blacksquare & 2 & & \\ \blacksquare & 1 & \cdots & 1 & 2 & 3 & & \\ 1 & 2 & \cdots & 2 & & & & \end{array} \qquad \begin{array}{ccccccc} \blacksquare & \cdots & \blacksquare & \blacksquare & \blacksquare & 1 & 1 \\ \blacksquare & \cdots & \blacksquare & \blacksquare & 2 & & \\ 1 & \cdots & 1 & 2 & 3 & & \\ 2 & \cdots & 2 & & & & \end{array}$$

- <u>Case 3.</u> Assume that $m - t \geq 2$ then we can fill the diagram (at least) in the following way

$$\begin{array}{ccccccccccccc} \blacksquare & \cdots & \blacksquare & \blacksquare & \cdots & \blacksquare & \blacksquare & \cdots & \blacksquare & \blacksquare & \blacksquare & 1 & 1 \\ \blacksquare & \cdots & \blacksquare & \blacksquare & \cdots & \blacksquare & 1 & \cdots & 1 & 2 & 2 & & \\ \blacksquare & \cdots & \blacksquare & 1 & \cdots & 1 & 2 & \cdots & 2 & 3 & 3 & & \\ 1 & \cdots & 1 & 2 & \cdots & 2 & 3 & \cdots & 3 & & & & \end{array}$$

Note that there are other admissible fillings besides these so that the multiplicity can be bigger than 1. $\square$

Corollary 3.4.4. *Let $X \subsetneq \mathrm{Gr}(6, 10)$ be a Debarre–Voisin hyperkähler manifold, and let $\lambda = (m, t, s, 0)$ with $m \geq t + s$. Among the factors of the Littlewood–Richardson decomposition of $\Sigma_\lambda Q \otimes \Sigma_\lambda Q^\vee$ we have*

- *$\mathcal{O}_X$ with multiplicity 1;*
- *$\Sigma_{2,2,0,0}Q \otimes \mathcal{O}_X(-1)$ with multiplicity* $\begin{cases} 0 & \text{if } t = s = 0 \\ 2 & \text{if } m > t > s \\ 1 & \text{otherwise} \end{cases}$
- *$\Sigma_{4,2,2,0}Q \otimes \mathcal{O}_X(-2)$ with multiplicity* $\begin{cases} 0 & \text{if } m = 1 \\ \geq 1 & \text{otherwise} \end{cases}$

Proof. The result follows directly from the Lemma 3.4.3. In fact, we need to twist each irreducible factor by $\mathcal{O}_X(-m)$ in order to get an irreducible piece of $\Sigma_\lambda Q \otimes \Sigma_\lambda Q^\vee$. $\square$

Remark 3.4.5. *As a consequence of the computation in Section 3.4.1, we have that*

$$H^1\left(\Sigma_{2,2,0,0}Q \otimes \mathcal{O}_X(-1)\right) \cong V_{10} \oplus V_{10}^\vee \ \text{ and } \ H^2\left(\Sigma_{2,2,0,0}Q \otimes \mathcal{O}_X(-1)\right) \cong \mathbb{C}.$$

One can also notice that, differently from Beauville–Donagi cases, in these cases the Ext^1's are not irreducible representations. This can possibly lead to vector bundles with $\mathrm{ext}^1 = 10, 30$.

Remark 3.4.6. *As a representation of $\mathrm{SL}(10)$, the second wedge power of the first cohomology corresponds to*

$$\bigwedge^2 V_{10} \oplus \Sigma_{2,1,1,1,1,1,1,1,1}V_{10} \oplus \bigwedge^2 V_{10}^\vee \oplus \mathbb{C}.$$

Similarly, one notices that

$$H^2\left(\Sigma_{4,2,2,0}Q \otimes \mathcal{O}_X(-1)\right) \cong \bigwedge^2 V_{10} \oplus \Sigma_{2,1,1,1,1,1,1,1,1}V_{10} \oplus \bigwedge^2 V_{10}^\vee.$$

After an analysis of our computations, we shall give the following

Conjecture 3.4.7. *Let $X \subseteq \mathrm{Gr}(6,10)$ be a Debarre-Voisin hyperkähler manifold, and let $\lambda = (m,t,s,0)$ with $m \geq t+s$, then:*

- *among the factors of the Littlewood–Richardson decomposition of $\Sigma_\lambda Q \bigotimes \Sigma_\lambda Q^\vee$, the vector bundle*
$$\Sigma_{2,2,0,0}Q \otimes \mathcal{O}_X(-1)$$
is the only one which gives a non-trivial contribution to $\mathrm{ext}^1(\Sigma_\lambda Q, \Sigma_\lambda Q)$. In particular, from Lemma 3.4.3, it follows that
$$\mathrm{ext}^1(\Sigma_\lambda Q, \Sigma_\lambda Q) \in \{0, 20, 40\}.$$
- *among the factors of the Littlewood–Richardson decomposition of $\Sigma_\lambda Q \bigotimes \Sigma_\lambda Q^\vee$, the vector bundle $\mathcal{O}_X$ is the only which gives a non-trivial contribution to $\mathrm{ext}^0(\Sigma_\lambda Q, \Sigma_\lambda Q)$. In particular, from Lemma 3.4.3, it follows that $\Sigma_\lambda Q$ is simple.*

Remark 3.4.8. *If the conjecture holds, from the above discussion we get*
$$\bigwedge^2 \mathrm{Ext}^1(\Sigma_\lambda Q, \Sigma_\lambda Q) \subseteq \mathrm{Ext}^2_0(\Sigma_\lambda Q, \Sigma_\lambda Q),$$
at least in the case where $\mathrm{ext}^1(\Sigma_\lambda Q, \Sigma_\lambda Q) = 20$. The 40−dimensional case still works but it is more complicated, see below. This should lead to the smoothness of the moduli space at the point $[\Sigma_\lambda Q]$, as outlined at the end of Section 3.7.2.

3.4.4. The 40–dimensional case. Let $\lambda = (m,t,s,0)$ with $m > t > s > 0$ (and $m \geq t+s$), then if Conjecture 3.4.7 holds, we have

$$\mathrm{Ext}^1\left(\Sigma_{(m,t,s,0)}Q \otimes \Sigma_{(m,t,s,0)}Q^\vee\right) \cong \left(V_{10} \oplus V_{10}^\vee\right)^{\oplus 2}, \text{ and}$$
$$\bigwedge^2 \mathrm{Ext}^1\left(\Sigma_{(m,t,s,0)}Q \otimes \Sigma_{(m,t,s,0)}Q^\vee\right) \cong \mathrm{Sym}^2 V_{10}^\vee \oplus \left(\Sigma_{2,1,1,1,1,1,1,1,1}V_{10}\right)^{\oplus 4} \oplus \left(\bigwedge^2 V_{10}^\vee\right)^{\oplus 3} \oplus$$
$$\oplus \mathrm{Sym}^2 V_{10} \oplus \left(\bigwedge^2 V_{10}\right)^{\oplus 3} \oplus \mathbb{C}^{\oplus 4}.$$

As a consequence of the computation in Section 3.4.1, we get
$$H^2\left(\Sigma_{4,3,1,0} \otimes \mathcal{O}_X(-2)\right) \cong \mathrm{Sym}^2 V_{10}^\vee \oplus \Sigma_{2,1,1,1,1,1,1,1,1}V_{10} \oplus \mathrm{Sym}^2 V_{10} \oplus \mathbb{C}.$$

Arguing as in the proof of Lemma 3.4.3, we can show that there are at least two occurrences of both $\Sigma_{(m+2,m,m,m-2)}$ and $\Sigma_{(m+2,m+1,m-1,m-2)}$ in $\Sigma_{(m,t,s,0)} \otimes \Sigma_{(m,m-s,m-t,0)}$. It follows that
$$\bigwedge^2 \mathrm{Ext}^1(\Sigma_\lambda Q, \Sigma_\lambda Q) \subseteq \mathrm{Ext}^2_0(\Sigma_\lambda Q, \Sigma_\lambda Q),$$
as claimed in Remark 3.4.8.

3.5. A worked example

This section shows the computations for two examples of modular sheaves with non-trivial Ext^1.

3.5.1. First example $\mathrm{ext}^1 = 20$, $\bigwedge^2 \mathcal{Q}$. Following notation 3.2.3, consider $\mathcal{E}_{1,1}$, which, by Littlewood–Richardson's decomposition, is equal to

$$\Sigma_{1,1,0,0}\mathcal{Q} \otimes \Sigma_{1,1,0,0}\mathcal{Q} \otimes \mathcal{O}_X(-1) = \Sigma_{2,2,0,0}\mathcal{Q}(-1) \oplus \Sigma_{2,1,1,0}\mathcal{Q}(-1) \oplus \mathcal{O}_X.$$

In order to compute $h^p(X, \mathcal{E}_{1,1})$, we need to twist the Koszul complex with $\widetilde{\mathcal{E}}_{1,1}$. In this way, we can split it into three Koszul complexes and sum the cohomologies. In particular:

$$0 \to \bigwedge^{20}\bigwedge^{3}\widetilde{\mathcal{U}} \otimes \Sigma_{2,2,0,0}\widetilde{\mathcal{Q}}(-1) \to \cdots \to \bigwedge^{3}\widetilde{\mathcal{U}} \otimes \Sigma_{2,2,0,0}\widetilde{\mathcal{Q}}(-1) \to \Sigma_{2,2,0,0}\widetilde{\mathcal{Q}}(-1) \to \Sigma_{2,2,0,0}\mathcal{Q}(-1) \to 0$$

$$0 \to \bigwedge^{20}\bigwedge^{3}\widetilde{\mathcal{U}} \otimes \Sigma_{2,1,1,0}\widetilde{\mathcal{Q}}(-1) \to \cdots \to \bigwedge^{3}\widetilde{\mathcal{U}} \otimes \Sigma_{2,1,1,0}\widetilde{\mathcal{Q}}(-1) \to \Sigma_{2,1,1,0}\widetilde{\mathcal{Q}}(-1) \to \Sigma_{2,1,1,0}\mathcal{Q}(-1) \to 0$$

$$0 \to \bigwedge^{20}\bigwedge^{3}\widetilde{\mathcal{U}} \to \cdots \to \bigwedge^{2}\bigwedge^{3}\widetilde{\mathcal{U}} \otimes \to \bigwedge^{3}\widetilde{\mathcal{U}} \to \mathcal{O}_{\mathrm{Gr}(6,10)} \to \mathcal{O}_X \to 0$$

Now,notice that, by Borel–Weil–Bott's theorem, the complex twisted by $\Sigma_{2,2,0,0}\widetilde{\mathcal{Q}}(-1)$ has five factors that are not acyclic, namely:

$$\begin{aligned}
\Sigma_{2,2,0,0|4,4,2,2,2,1} &\text{ in } \bigwedge^{3}\bigwedge^{3}\widetilde{\mathcal{U}} \otimes \Sigma_{2,2,0,0}\widetilde{\mathcal{Q}}(-1),\\
\Sigma_{2,2,0,0|8,5,5,3,3,3} &\text{ in } \bigwedge^{7}\bigwedge^{3}\widetilde{\mathcal{U}} \otimes \Sigma_{2,2,0,0}\widetilde{\mathcal{Q}}(-1),\\
\Sigma_{2,2,0,0|8,5,5,3,3,3} &\text{ in } \bigwedge^{7}\bigwedge^{3}\widetilde{\mathcal{U}} \otimes \Sigma_{2,2,0,0}\widetilde{\mathcal{Q}}(-1),\\
\Sigma_{2,2,0,0|8,8,6,6,4,4} &\text{ in } \bigwedge^{10}\bigwedge^{3}\widetilde{\mathcal{U}} \otimes \Sigma_{2,2,0,0}\widetilde{\mathcal{Q}}(-1),\\
\Sigma_{2,2,0,0|9,9,9,7,7,4} &\text{ in } \bigwedge^{13}\bigwedge^{3}\widetilde{\mathcal{U}} \otimes \Sigma_{2,2,0,0}\widetilde{\mathcal{Q}}(-1),\\
\Sigma_{2,2,0,0|11,10,10,10,8,8} &\text{ in } \bigwedge^{17}\bigwedge^{3}\widetilde{\mathcal{U}} \otimes \Sigma_{2,2,0,0}\widetilde{\mathcal{Q}}(-1).
\end{aligned}$$

Again, by Borel–Weil–Bott's theorem, all the factors of the complex twisted by $\Sigma_{2,1,1,0}\widetilde{\mathcal{Q}}(-1)$ are acyclic, and hence do not contribute to the cohomology of $\mathcal{E}_{1,1}$. The third complex has three factors that are not acyclic:

$$\bigwedge^{20}\bigwedge^{3}\widetilde{\mathcal{U}} = \mathcal{O}_{\mathrm{Gr}(6,10)}(-10),$$

$$\Sigma_{0,0,0,0|7,7,7,3,3,3} \text{ in } \bigwedge^{10}\bigwedge^{3}\widetilde{\mathcal{U}} \text{ and } \mathcal{O}_{\mathrm{Gr}(6,10)}.$$

In this way, we obtain, by diagram chasing:

- $H^0(X, \mathcal{E}_{1,1}) = H^0\left(\mathrm{Gr}(6,10), \mathcal{O}_{\mathrm{Gr}(6,10)}\right) = \mathbb{C}$;
- $H^1(X, \mathcal{E}_{1,1}) = H^4\left(\mathrm{Gr}(6,10), \Sigma_{2,2,0,0|4,4,2,2,2,1}\right) \oplus H^8\left(\mathrm{Gr}(6,10), \Sigma_{2,2,0,0|8,5,5,3,3,3}\right) = V_{10}^\vee \oplus V_{10}$;
- $H^2(X, \mathcal{E}_{1,1}) = H^{12}\left(\mathrm{Gr}(6,10), \Sigma_{2,2,0,0|8,8,6,6,4,4}\right) \oplus H^{12}\left(\mathrm{Gr}(6,10), \Sigma_{0,0,0,0|7,7,7,3,3,3}\right) = \mathbb{C} \oplus \mathbb{C}$;
- $H^3(X, \mathcal{E}_{1,1}) = H^{16}\left(\mathrm{Gr}(6,10), \Sigma_{2,2,0,0|9,9,9,7,7,4}\right) \oplus H^{20}\left(\mathrm{Gr}(6,10), \Sigma_{2,2,0,0|11,10,10,10,8,8}\right) = V_{10} \oplus V_{10}^\vee$;

- $H^4(X, \mathcal{E}_{1,1}) = H^{24}\left(\mathrm{Gr}(6,10), \mathcal{O}_{\mathrm{Gr}(6,10)}(-10)\right) = \mathbb{C}$.

From Corollary 3.3.5, we get that $\bigwedge^2 \mathcal{Q}$ is modular, hence we obtained an example of a twenty-moduli modular sheaf on the Debarre–Voisin fourfold, which was announced in [Fat24], but without a proof. Notice that here $\mathrm{Ext}^2 = \mathbb{C}^2$, with $\mathrm{Ext}^2_0 = \mathbb{C}$, which is a promising indication that $\bigwedge^2 \mathcal{Q}$ could be unobstructed (see Section 3.7.2).

Remark 3.5.1. *In* [OG24, section 3.3], *O'Grady described the vector bundle $\bigwedge^2 \mathcal{Q}$ on the Beauville–Donagi fourfold in terms of deformations of a modular sheaf on a Hilbert scheme of couples of points on a K3 surface. It seems that the same techniques can be adapted also for the Debarre–Voisin case. This should be a starting point to show that the Schur functors of $\mathcal{Q}$ on the Beauville–Donagi fourfold deforms to Schur functors of $\mathcal{Q}$ on the Debarre–Voisin foufold, passing through some modular sheaves on the Hilbert scheme of points on a K3 surface.*

3.5.2. Second example $\mathrm{ext}^1 = 40$, $\Sigma_{3,2,1,0}\mathcal{Q}$. The second example we would like to show is the vector bundle $\Sigma_{3,2,1,0}\mathcal{Q}$, for which we have the following decomposition:

$$\begin{aligned}\mathcal{E}_{3,2,1} = &\Sigma_{6,4,2,0|3,3,3,3,3,3} \oplus \Sigma_{5,3,0,0|2,2,2,2,2,2} \oplus \Sigma_{6,3,3,0|3,3,3,3,3,3} \oplus 2\Sigma_{5,2,1,0|2,2,2,2,2,2} \oplus \\ &\oplus \Sigma_{4,0,0,0|1,1,1,1,1,1} \oplus \Sigma_{5,5,2,0|3,3,3,3,3,3} \oplus \Sigma_{4,4,0,0|2,2,2,2,2,2} \oplus 2\Sigma_{5,4,3,0|3,3,3,3,3,3} \oplus \\ &\oplus 4\Sigma_{4,3,1,0|2,2,2,2,2,2} \oplus 3\Sigma_{4,2,2,0|2,2,2,2,2,2} \oplus 3\Sigma_{3,1,0,0|1,1,1,1,1,1} \oplus \Sigma_{4,4,4,0|3,3,3,3,3,3} \oplus \\ &\oplus 3\Sigma_{3,3,2,0|2,2,2,2,2,2} \oplus 2\Sigma_{2,2,0,0|1,1,1,1,1,1} \oplus 3\Sigma_{2,1,1,0|1,1,1,1,1,1} \oplus \mathcal{O}_X.\end{aligned}$$

In this case, we need sixteen sequences to compute the cohomology. We do not write the computations, but, by Borel–Weil–Bott's theorem, we have that $\Sigma_{3,1,0,0|1,1,1,1,1,1}$, $\Sigma_{3,3,2,0|2,2,2,2,2,2}$ and $\Sigma_{2,1,1,0|1,1,1,1,1,1}$ are acyclic. We have already computed the cohomologies of $\mathcal{O}_X$ and $\Sigma_{2,2,0,0|1,1,1,1,1,1}$. In particular, since $\Sigma_{2,2,0,0|1,1,1,1,1,1}$ appears two times in the decomposition we have that $H^1(X, \mathcal{E}_{3,2,1}) = H^1(\mathcal{E}_{1,1})^{\oplus 2} = (V_{10} \oplus V_{10}^\vee)^{\oplus 2}$, as predicted by Corollary 3.4.4. By Borel–Weil–Bott's theorem, the other factors only contribute to $H^2(X, \mathcal{E}_{3,2,1})$, whose dimension is 35406.

3.6. About the symmetric power

In this section, we will restrict ourselves to the case of $\mathrm{Sym}^m(\mathcal{Q})$, which is a particular Schur functor given by $\lambda = (m, 0, 0, 0)$. We perform some of the computations of the previous sections in this specific case to make them clearer and concrete.

We start by computing the Chern classes of $\mathrm{Sym}^m(\mathcal{Q})$.

Lemma 3.6.1. *Let $X \subseteq \mathrm{Gr}(6,10)$ a very general Debarre–Voisin fourfold, then:*

(1) $\mathrm{rk}(\mathrm{Sym}^m \mathcal{Q}) = \frac{(m+3)(m+2)(m+1)}{6} =: r_m$;

(2) $\mathrm{ch}(\mathrm{Sym}^m \mathcal{Q}) = r_m + \left(\frac{m}{4} r_m \mathrm{ch}_1(\mathcal{Q})\right) + \left(\frac{m^2-m}{40} r_m \mathrm{ch}_1(\mathcal{Q})^2 + \frac{m^2+4m}{20} r_m \mathrm{ch}_2(\mathcal{Q})\right) + \left(-\frac{2m^3-3m^2}{4} r_m \mathrm{ch}_3(\mathcal{Q})\right) + \left(-\frac{10m^4-30m^3+21m^2-6m}{20} r_m \mathrm{ch}_4(\mathcal{Q})\right)$;

(3) $\Delta(\mathrm{Sym}^m \mathcal{Q}) = \frac{m^2+4m}{80} r_m^2 \Delta(\mathcal{Q}) = \frac{m^2+4m}{80} r_m^2 \mathrm{c}_2(X)$;

(4) $\Xi(\mathrm{Sym}^m \mathcal{Q}) = \frac{m(m+4)(9m^2+36m-5)}{110400} r_m^2 \mathrm{c}_2(X)^2$;

(5) $\int_X \Xi(\mathrm{Sym}^m \mathcal{Q}) = \frac{3m(m+4)(9m^2+36m-5)}{400} r_m^2$.

(6) $\chi(\mathrm{Sym}^m \mathcal{Q}, \mathrm{Sym}^m \mathcal{Q}) = 3\left(\frac{3m^2+12m-20}{20}\right)^2 r_m^2$.

Proof. We proceed in order:

(1) it follows from the dimension of the m-symmetric power of a 4 dimensional vector space;
(2) it follows from a more general result in [Svr93]. One needs to apply to our case [Svr93, Theorem 4.8];
(3) it follows directly from the definitions;
(4) it follows from the definitions too;
(5) it is a combination of (1) and (4) of Lemma 3.3.1;
(6) it is an application of Hirzebruch–Riemann–Roch theorem:

$$\chi(\operatorname{Sym}^m \mathcal{Q}, \operatorname{Sym}^m \mathcal{Q}) = \chi(\operatorname{Sym}^m \mathcal{Q} \otimes \operatorname{Sym}^m \mathcal{Q}^\vee) = \int_X \mathsf{ch}\,(\mathcal{E}nd(\operatorname{Sym}^m \mathcal{Q}))\,\mathrm{td}_X$$

Recall that $\mathsf{ch}\,(\mathcal{E}nd(\operatorname{Sym}^m \mathcal{Q})) = r_m^2 + 0 - \Delta(\operatorname{Sym}^m \mathcal{Q}) + 0 + \Xi(\operatorname{Sym}^m \mathcal{Q})$. Thus,

$$\chi(\operatorname{Sym}^m \mathcal{Q}, \operatorname{Sym}^m \mathcal{Q}) = \int_X \left(3r_m^2\mathsf{p} - \frac{1}{12}\Delta(\operatorname{Sym}^m \mathcal{Q})\mathsf{c}_2(X) + \Xi(\operatorname{Sym}^m \mathcal{Q})\right) =$$

$$= 3r_m^2 - \frac{828}{12}\frac{m^2+4m}{80}r_m^2 + \frac{3m(m+4)(9m^2+36m-5)}{400}r_m^2 = 3\left(\frac{3m^2+12m-20}{20}\right)^2 r_m^2.$$

□

Let us compute the cohomology of the Ext-group of $\operatorname{Sym}^m \mathcal{Q}$. Notice that we denoted the sheaf of endomorphisms $\mathcal{E}nd\,(\operatorname{Sym}^m\mathcal{Q}, \operatorname{Sym}^m\mathcal{Q})$ by $\mathcal{E}_m$.

Remark 3.6.2. *By Pieri's rule, we get the following decomposition:*

$$\mathcal{E}_m = \bigoplus_{i=0}^{m} \Sigma_{2m-i,m,m,i}\mathcal{Q} \otimes \mathcal{O}_X(-m),$$

which leads to $\mathcal{E}_m = \Sigma_{2m,m,m,0}\mathcal{Q} \otimes \mathcal{O}_X(-m) \oplus \mathcal{E}_{m-1}$. *Thus, it suffices to compute* $E_m := \Sigma_{2m,m,m,0}\mathcal{Q} \otimes \mathcal{O}_X(-m)$ *for each* m.

We now state a useful lemma that cuts some computations.

Lemma 3.6.3. *Let be* E_m *as above. If we twist the Koszul complex Equation 3.2.1 by* $\widetilde{E}_m := \Sigma_{2m,m,m,0}\widetilde{\mathcal{Q}} \otimes \mathcal{O}_{\mathrm{Gr}(6,10)}(-m)$, *we have the resolution:*

$$0 \to \widetilde{E}_m \otimes \bigwedge^{20}\bigwedge^{3}\widetilde{\mathcal{U}} \to \cdots \to \widetilde{E}_m \otimes \bigwedge^{2}\bigwedge^{3}\widetilde{\mathcal{U}} \to \widetilde{E}_m \otimes \bigwedge^{3}\widetilde{\mathcal{U}} \to \widetilde{E}_m \to E_m \to 0.$$

Denote $\widetilde{E}_m \otimes \bigwedge^q \bigwedge^3 \widetilde{\mathcal{U}}$ *by* E_m^q, *for* $0 \leq q \leq 10$. *Then, among the irreducible factors of* $E_m^q = \bigoplus_i \Sigma_{(2m,m,m,0|\mu^q_{m,i})}$, *the following partitions* μ_m^q *are the ones which are not always acyclic:*

	$\mu^q_{m,1}$	$\mu^q_{m,2}$	$\mu^q_{m,3}$	$\mu^q_{m,4}$
$q=0$	(m,m,m,m,m,m)	-	-	-
$q=1$	$(m+1,m+1,m+1,m,m,m)$	-	-	-
$q=2$	$(m+1,m+1,m+1,m+1,m+1,m+1)$	-	-	-
$q=6$	$(m+5,m+5,m+3,m+3,m+1,m+1)$	$(m+5,m+5,m+2,m+2,m+2,m+2)$	-	-
$q=7$	$(m+6,m+5,m+3,m+3,m+2,m+2)$	$(m+5,m+5,m+3,m+3,m+3,m+2)$	-	-
$q=8$	$(m+6,m+6,m+3,m+3,m+3,m+3)$	-	-	-
$q=9$	$(m+7,m+6,m+6,m+3,m+3,m+2)$	$(m+6,m+6,m+6,m+4,m+4,m+1)$	-	-
$q=10$	$(m+8,m+6,m+6,m+4,m+4,m+2)$	$(m+8,m+6,m+6,m+4,m+3,m+3)$	$(m+7,m+7,m+7,m+3,m+3,m+3)$	$(m+7,m+7,m+6,m+4,m+4,m+2)$

For $11 \leq q \leq 20$ *we have a symmetric situation. In particular, if* $q = 10+k$ *then* E_m^q *has the same number of irreducible factors that are not always acyclic as in* E_m^{10-k}.

Proof. Let μ_m^q be a partition appearing in Table 2 twisted by $\mathcal{O}(-m)$. Consider the partition $(2m, m, m, 0|\mu_m^q)$, if we sum the partition $\delta = (9, 8, 7, 6, 5, 4, 3, 2, 1, 0)$, then we have $(2m, m, m, 0|\mu_m^q)+\delta = (2m+9, m+8, m+7, 6|\mu_m^q+\delta')$ with $\delta' = (5, 4, 3, 2, 1, 0)$. By confronting $(2m+9, m+8, m+7, 6)$ with $\mu_m^q + \delta'$ we recollect only the partitions with no repeated entries. By Lemma 3.2.4 we can restrict ourselves for $0 \le q \le 10$. □

Now that we know which factors are involved in the cohomologies we can make m running in $\mathbb{N}$ and recover a result analogous to [FO24, Proposition 2.3]:

Proposition 3.6.4. *For $m \le 4$ we have*

$$\dim \operatorname{Ext}^p(\operatorname{Sym}^m \mathcal{Q}, \operatorname{Sym}^m \mathcal{Q}) = \begin{cases} 1 & p = 0, 4 \\ 0 & p = 1, 3 \\ 3\left(\frac{3m^2+12m-20}{20}\right)^2 r_m^2 - 2 & p = 2. \end{cases}$$

More precisely, for $0 \le m \le 4$, $\operatorname{Ext}^2(\operatorname{Sym}^m \mathcal{Q}, \operatorname{Sym}^m \mathcal{Q})$ is the direct sum of vector spaces in the columns of Table 3.6.4.

$m=0$	$m=1$	$m=2$	$m=3$	$m=4$
$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$
-	-	$\Sigma_{1,1,1,1,1,1,1,1,0,0}(V_{10} \oplus V_{10}^\vee)$	$\Sigma_{1,1,1,1,1,1,1,1,0,0}(V_{10} \oplus V_{10}^\vee)$	$\Sigma_{1,1,1,1,1,1,1,1,0,0}(V_{10} \oplus V_{10}^\vee)$
-	-	$\Sigma_{2,1,1,1,1,1,1,1,1,0} V_{10}$	$\Sigma_{2,1,1,1,1,1,1,1,1,0} V_{10}$	$\Sigma_{2,1,1,1,1,1,1,1,1,0} V_{10}$
-	-	-	$\Sigma_{3,0,0,0,0,0,0,0,0,0}(V_{10} \oplus V_{10}^\vee)$	$\Sigma_{3,0,0,0,0,0,0,0,0,0}(V_{10} \oplus V_{10}^\vee)$
-	-	-	$\Sigma_{2,1,1,1,1,1,1,0,0,0}(V_{10} \oplus V_{10}^\vee)$	$\Sigma_{2,1,1,1,1,1,1,0,0,0}(V_{10} \oplus V_{10}^\vee)$
-	-	-	$V_{10}^{\oplus 2}$	$V_{10}^{\oplus 2}$
-	-	-	$\Sigma_{2,2,2,2,2,2,2,2,1,0}(V_{10} \oplus V_{10}^\vee)$	$\Sigma_{2,2,2,2,2,2,2,2,1,0}(V_{10} \oplus V_{10}^\vee)$
-	-	-	$\Sigma_{2,2,1,1,1,1,1,1,0,0} V_{10}$	$\Sigma_{2,2,1,1,1,1,1,1,0,0} V_{10}$
-	-	-	-	$\Sigma_{2,1,1,0,0,0,0,0,0,0}(V_{10} \oplus V_{10}^\vee)$
-	-	-	-	$\Sigma_{3,2,2,2,2,2,2,2,2,0}(V_{10} \oplus V_{10}^\vee)$
-	-	-	-	$\Sigma_{2,2,2,2,1,1,0,0,0,0} V_{10}$

Proof. Here we produced explicitly the vector space using Borel–Weil–Bott's theorem. In particular:

- ($m = 0$) By looking at the table Table 3.6.3, we have that the partitions associated to non-acyclic bundles for $m = 0$ are $\sigma_1 = (0, 0, 0, 0|0, 0, 0, 0, 0, 0)$ and $\sigma_2 = (0, 0, 0, 0|7, 7, 7, 3, 3, 3)$, which corresponds to $\mathcal{O}_{\mathrm{Gr}(6,10)}$ and to a factor of $\bigwedge^{10} \bigwedge^3 \widetilde{\mathcal{U}}$. We recover $\operatorname{Ext}^0(\mathcal{O}_X, \mathcal{O}_X) = H^0(X, \mathcal{O}_{\mathrm{Gr}(6,10)}) = \mathbb{C}$, and by symmetry also $\operatorname{Ext}^4(\mathcal{O}_X, \mathcal{O}_X)$. Moreover $\operatorname{Ext}^2(\mathcal{O}_X, \mathcal{O}_X) = H^{12}(\mathrm{Gr}(6, 10), \Sigma_{0,0,0,0|7,7,7,3,3,3}) = \mathbb{C}$.

- ($m = 1$) All the partitions in the table are associated with acyclic bundles. By Remark 3.6.2 we get that $\operatorname{Ext}^p(\mathcal{Q}, \mathcal{Q}) = \operatorname{Ext}^p(\mathcal{O}_X, \mathcal{O}_X)$. This is also a case already known.

- ($m = 2$) The partitions in Table 3.6.3 associated to a non-acyclic bundle for $m = 2$ are $\sigma_1 = (4, 2, 2, 0|7, 7, 5, 5, 3, 3)$, factor of E_2^6, and $(4, 2, 2, 0|10, 8, 8, 6, 6, 4)$, factor of E_2^{10}. They only contribute to the Ext-groups in the Ext^2. In particular:

$$\begin{aligned}
\mathrm{Ext}^0(\mathrm{Sym}^2\mathcal{Q},\mathrm{Sym}^2\mathcal{Q}) &= \mathrm{Ext}^4(\mathrm{Sym}^2\mathcal{Q},\mathrm{Sym}^2\mathcal{Q}) = \mathrm{Ext}^0(\mathcal{Q},\mathcal{Q}) = \mathbb{C},\\
\mathrm{Ext}^1(\mathrm{Sym}^2\mathcal{Q},\mathrm{Sym}^2\mathcal{Q}) &= \mathrm{Ext}^3(\mathrm{Sym}^2\mathcal{Q},\mathrm{Sym}^2\mathcal{Q}) = 0,\\
\mathrm{Ext}^2(\mathrm{Sym}^2\mathcal{Q},\mathrm{Sym}^2\mathcal{Q}) &= H^8(\mathrm{Gr}(6,10),\Sigma_{4,2,2,0|7,7,5,5,3,3})^{\oplus 2}\oplus H^{12}(\mathrm{Gr}(6,10),\Sigma_{4,2,2,0|10,8,8,6,6,4})\oplus \mathrm{Ext}^2(\mathcal{Q},\mathcal{Q}) =\\
&= (\Sigma_{1,1,1,1,1,1,1,1,0,0}V_{10})^{\oplus 2}\oplus \Sigma_{2,1,1,1,1,1,1,1,1,0}V_{10}\oplus \mathrm{Ext}^2(\mathcal{Q},\mathcal{Q}).
\end{aligned}$$

The second copy $\Sigma_{1,1,1,1,1,1,1,1,0,0}V_{10}^{\vee}$ comes from the symmetric of $\Sigma_{4,2,2,0|7,7,5,5,3,3}$ in $\bigwedge^{16}\bigwedge^{3}\widetilde{\mathcal{U}}$.

- ($m = 3$) In this case we have five partitions in Table 3.6.3 associated with non-acyclic bundles: $\sigma_1 = (6,3,3,0|4,4,4,3,3,3)$ in E_3^1, $\sigma_2 = (6,3,3,0|8,8,6,6,4,4)$ in E_3^6, $\sigma_3 = (6,3,3,0|8,8,6,6,6,5)$ in E_3^7, $\sigma_4 = (6,3,3,0|9,9,9,7,7,4)$ in E_3^9, and $\sigma_5 = (6,3,3,0|11,9,9,7,7,5)$ in E_3^{10}. Using Borel–Weil–Bott's theorem, we notice that they contribute only to the Ext^2-group. In particular,

$$\begin{aligned}
\mathrm{Ext}^0(\mathrm{Sym}^3\mathcal{Q},\mathrm{Sym}^3\mathcal{Q}) &= \mathrm{Ext}^4(\mathrm{Sym}^3\mathcal{Q},\mathrm{Sym}^3\mathcal{Q}) = \mathrm{Ext}^0(\mathrm{Sym}^2\mathcal{Q},\mathrm{Sym}^2\mathcal{Q}) = \mathbb{C},\\
\mathrm{Ext}^1(\mathrm{Sym}^3\mathcal{Q},\mathrm{Sym}^3\mathcal{Q}) &= \mathrm{Ext}^3(\mathrm{Sym}^3\mathcal{Q},\mathrm{Sym}^3\mathcal{Q}) = \mathrm{Ext}^1(\mathrm{Sym}^2\mathcal{Q},\mathrm{Sym}^2\mathcal{Q}) = 0,\\
\mathrm{Ext}^2(\mathrm{Sym}^3\mathcal{Q},\mathrm{Sym}^3\mathcal{Q}) &= H^3(\mathrm{Gr}(6,10),\Sigma_{6,3,3,0|4,4,4,3,3,3})^{\oplus 2}\oplus H^8(\mathrm{Gr}(6,10),\Sigma_{6,3,3,0|8,8,6,6,4,4})^{\oplus 2}\oplus\\
&\oplus H^9(\mathrm{Gr}(6,10),\Sigma_{6,3,3,0|8,8,6,6,6,5})^{\oplus 2}\oplus H^{11}(\mathrm{Gr}(6,10),\Sigma_{6,3,3,0|9,9,9,7,7,4})^{\oplus 2}\oplus\\
&\oplus H^{12}(\mathrm{Gr}(6,10),\Sigma_{6,3,3,0|11,9,9,7,7,5})\oplus \mathrm{Ext}^2(\mathrm{Sym}^2\mathcal{Q},\mathrm{Sym}^2\mathcal{Q}) =\\
&= (\mathrm{Sym}^3 V_{10})^{\oplus 2}\oplus(\Sigma_{2,1,1,1,1,1,1,0,0,0}V_{10})^{\oplus 2}\oplus V_{10}^{\oplus 2}\oplus(\Sigma_{2,2,2,2,2,2,2,2,1,0}V_{10})^{\oplus 2}\oplus\\
&\oplus \Sigma_{2,2,1,1,1,1,1,1,0,0}V_{10}\oplus \mathrm{Ext}^2(\mathrm{Sym}^2\mathcal{Q},\mathrm{Sym}^2\mathcal{Q}).
\end{aligned}$$

The double copies of cohomology space come from the symmetric component in the Koszul Complex as explained in Lemma 3.2.4.

- ($m = 4$) We have three partitions associated to non-acyclic bundles: $\sigma_1 = (8,4,4,0|10,10,7,7,7,7)$ in E_4^8, $\sigma_2 = (8,4,4,0|10,10,10,8,8,5)$ in E_4^9, and $\sigma_3 = (8,4,4,0|11,11,11,7,7,7)$ in E_4^{10}. This again, by Borel–Weil–Bott's theorem, gives:

$$\begin{aligned}
\mathrm{Ext}^0(\mathrm{Sym}^4\mathcal{Q},\mathrm{Sym}^4\mathcal{Q}) &= \mathrm{Ext}^4(\mathrm{Sym}^4\mathcal{Q},\mathrm{Sym}^4\mathcal{Q}) = \mathrm{Ext}^0(\mathrm{Sym}^3\mathcal{Q},\mathrm{Sym}^3\mathcal{Q}) = \mathbb{C},\\
\mathrm{Ext}^1(\mathrm{Sym}^4\mathcal{Q},\mathrm{Sym}^4\mathcal{Q}) &= \mathrm{Ext}^3(\mathrm{Sym}^4\mathcal{Q},\mathrm{Sym}^4\mathcal{Q}) = \mathrm{Ext}^1(\mathrm{Sym}^2\mathcal{Q},\mathrm{Sym}^2\mathcal{Q}) = 0,\\
\mathrm{Ext}^2(\mathrm{Sym}^4\mathcal{Q},\mathrm{Sym}^4\mathcal{Q}) &= H^{10}(\mathrm{Gr}(6,10),\Sigma_{8,4,4,0|10,10,7,7,7,7})^{\oplus 2}\oplus H^{11}(\mathrm{Gr}(6,10),\Sigma_{8,4,4,0|10,10,10,8,8,5})^{\oplus 2}\oplus\\
&\oplus H^{12}(\mathrm{Gr}(6,10),\Sigma_{8,4,4,0|11,11,11,7,7,7})\oplus \mathrm{Ext}^2(\mathrm{Sym}^3\mathcal{Q},\mathrm{Sym}^3\mathcal{Q}) =\\
&= (\Sigma_{2,1,1,0,0,0,0,0,0,0}V_{10})^{\oplus 2}\oplus(\Sigma_{3,2,2,2,2,2,2,2,2,0}V_{10})^{\oplus 2}\oplus\\
&\oplus \Sigma_{2,2,2,2,1,1,0,0,0,0}V_{10}\oplus \mathrm{Ext}^2(\mathrm{Sym}^3\mathcal{Q},\mathrm{Sym}^3\mathcal{Q})
\end{aligned}$$

We obtain the dimensions of Ext-spaces by computing the dimensions of the vector spaces we found. In this way, we could describe explicitly the Ext-spaces, but we could also use Lemma 3.6.1. If we restrict ourselves to $0 \le m \le 4$, then we can notice that we do not have any contribution for

$$\mathrm{Ext}^1(\mathrm{Sym}^m\mathcal{Q},\mathrm{Sym}^m\mathcal{Q}) = \mathrm{Ext}^3(\mathrm{Sym}^m\mathcal{Q},\mathrm{Sym}^m\mathcal{Q}),$$

hence it is always 0 in these cases. Moreover the only contribution to $\mathrm{Ext}^0(\mathrm{Sym}^m \mathcal{Q}, \mathrm{Sym}^m \mathcal{Q})$ and $\mathrm{Ext}^4(\mathrm{Sym}^m \mathcal{Q}, \mathrm{Sym}^m \mathcal{Q})$ is from $H^0(X, \mathcal{O}_X) = \mathbb{C}$. Hence, by Lemma 3.6.1:

$$\dim \mathrm{Ext}^2(\mathrm{Sym}^m \mathcal{Q}, \mathrm{Sym}^m \mathcal{Q}) = \chi(\mathrm{Sym}^m \mathcal{Q}, \mathrm{Sym}^m \mathcal{Q}) - 2 = 3\left(\frac{3m^2 + 12m - 20}{20}\right)^2 r_m^2 - 2.$$

□

Remark 3.6.5. *Note that for $m \geq 7$ all the partitions in Table 3.6.3 are associated with non-acyclic bundles. In particular, one can try to mimic the method used in* [**FO24**, Prop. 2.3], *to obtain a stronger result. The problem is that from $m \geq 5$ the Koszul complex of the E_5^q's is indeterminate, as we show in the following example, so it is impossible to conclude, even if it is reasonable that Proposition 3.6.4 holds even for $m > 4$.*

Example 3.6.6. $\mathrm{Ext}(\mathrm{Sym}^5 \mathcal{Q}, \mathrm{Sym}^5 \mathcal{Q})$**.** *We proceed as before. First notice:*

$$\mathrm{Ext}^p(\mathrm{Sym}^5 \mathcal{Q}, \mathrm{Sym}^5 \mathcal{Q}) = H^p(X, E_5) \oplus \mathrm{Ext}^p(\mathrm{Sym}^4 \mathcal{Q}, \mathrm{Sym}^4 \mathcal{Q})$$

So we study, as before, only the cohomology of E_5. We recover the Koszul complex:

$$0 \to E_5^{20} \to \cdots \to E_5^2 \to E_5^1 \to E_5^0 \to E_5 \to 0.$$

Let $0 \leq q \leq 10$, then we have nine partitions in Table 3.6.3 that are associated to non-acyclic factors of the E_5^q's, we recollect them in the following table:

q=6	-	$(10,5,5,0 \mid 10,10,7,7,7,7)$	-	-
q=7	$(10,5,5,0 \mid 11,10,8,8,7,7)$	$(10,5,5,0 \mid 10,10,8,8,8,7)$	-	-
q=8	$(10,5,5,0 \mid 11,11,8,8,8,8)$	-	-	-
q=9	$(10,5,5,0 \mid 12,11,11,8,8,7)$	-	-	-
q=10	$(10,5,5,0 \mid 13,11,11,9,9,7)$	$(10,5,5,0 \mid 13,11,11,9,8,8)$	$(10,5,5,0 \mid 12,12,12,8,8,8)$	$(10,5,5,0 \mid 12,12,11,9,9,7)$

By Borel–Weil–Bott's theorem and Lemma 3.2.4 we obtain the following non-trivial cohomologies:

$$H^{10}(\mathrm{Gr}(6,10), \bigwedge^6 \bigwedge^3 \widetilde{U}) = H^{14}(\mathrm{Gr}(6,10), \bigwedge^{14} \bigwedge^3 \widetilde{U}) = \Sigma_{4,2,1,1,1,0,0,0,0,0} V_{10}$$

$$H^{10}(\mathrm{Gr}(6,10), \bigwedge^7 \bigwedge^3 \widetilde{U}) = H^{14}(\mathrm{Gr}(6,10), \bigwedge^{13} \bigwedge^3 \widetilde{U}) = \Sigma_{4,2,1,1,1,1,0,0,0,0} V_{10} \oplus \Sigma_{4,1,1,1,1,1,1,1,0,0} V_{10}$$

$$H^{10}(\mathrm{Gr}(6,10), \bigwedge^8 \bigwedge^3 \widetilde{U}) = H^{14}(\mathrm{Gr}(6,10), \bigwedge^{12} \bigwedge^3 \widetilde{U}) = \Sigma_{4,2,2,1,1,1,1,1,1,0} V_{10}$$

$$H^{12}(\mathrm{Gr}(6,10), \bigwedge^9 \bigwedge^3 \widetilde{U}) = H^{12}(\mathrm{Gr}(6,10), \bigwedge^{11} \bigwedge^3 \widetilde{U}) = \Sigma_{4,3,2,2,2,2,1,1,0,0} V_{10}$$

$$H^{12}(\mathrm{Gr}(6,10), \bigwedge^{10} \bigwedge^3 \widetilde{U}) = \Sigma_{4,4,2,2,2,2,2,2,0,0} V_{10} \oplus \Sigma_{4,4,2,2,2,2,2,1,1,0} V_{10} \oplus \Sigma_{4,3,3,3,2,2,1,1,1,0} V_{10} \oplus \Sigma_{4,3,3,2,2,2,2,2,0,0} V_{10}$$

Note that when we do the usual diagram chasing to obtain the cohomology we have problems because we have non-trivial cohomologies on the same degree. This gives short exact sequences that are not trivially solved. This is the reason we can not conclude the argument for $m > 4$.

3.7. Comparison with atomicity

Another notion introduced to study the modularity properties of sheaves on hyperkähler manifolds was given by Beckmann in [**Bec22**].

Definition 3.7.1. [**Bec22**, Def.1.1] *Let* $\widetilde{H}(X,\mathbb{Q}) := (\mathbb{Q}\alpha \oplus H^2(X,\mathbb{Q}) \oplus \mathbb{Q}\beta, \widetilde{q})$ *be the extended Mukai lattice, where*

$$\widetilde{q}(\alpha) = \widetilde{q}(\beta), \text{ and } \widetilde{q}(\alpha,\beta) = -1$$

and $\widetilde{q}_{|H^2(X,\mathbb{Q})} = q_X$ *is the Beauville–Bogomolov–Fujiki form. A coherent sheaf* $\mathcal{E}$ *on a hyperkähler manifold* X *is called atomic if there exists a non-zero vector* $\tilde{v} \in \widetilde{H}(X,\mathbb{Q})$ *such that the annihilator Lie subalgebra* $Ann(v(\mathcal{E})) \subset \mathfrak{g}(X)$ *of the representation of* $\mathfrak{g}(X)$ *on* $H^*(X,\mathbb{Q})$ *equals the annihilator Lie subalgebra* $Ann(\tilde{v}) \subset \mathfrak{g}(X) \cong \mathfrak{so}(\widetilde{H}(X,\mathbb{Q}))$.

Moreover, the notion of atomicity is deeply intertwined with the Verbitsky component.

Definition 3.7.2. *The Verbitsky component* $SH(X,\mathbb{Q})$ *is the irreducible* $\mathfrak{g}(X)$*-submodule of* $H^*(X,\mathbb{Q})$ *generated by* $H^2(X,\mathbb{Q})$.

Remark 3.7.3. *Notice that it is possible to define a a bilinear non-degenerate pairing on* $\mathrm{Sym}^2\,\widetilde{H}(X,\mathbb{Q})$ *by the rule*

$$\tilde{q}^{(2)}((v_1,v_2),(w_1,w_2)) = \widetilde{q}(v_1,w_1)\widetilde{q}(v_2,w_2) + \widetilde{q}(v_1,w_2)\widetilde{q}(v_2,w_1).$$

There is a short exact sequence of $\mathfrak{g}(X)$*-moduli*

$$0 \to SH(X,\mathbb{Q}) \xrightarrow{\psi} \mathrm{Sym}^2\,\widetilde{H}(X,\mathbb{Q}) \to \mathbb{Q} \to 0.$$

The map ψ *is an isomtry with respect to* $\tilde{q}^{(2)}$*, and we denote by*

$$T:\ \mathrm{Sym}^2\,\widetilde{H}^2(X,\mathbb{Q}) \to SH(X,\mathbb{Q})$$

the corresponding orthogonal projection.

Definition 3.7.4. *An extended Mukai vector of a coherent sheaf* $\mathcal{E}$ *is an element* $\tilde{v} \in \widetilde{H}(X,Q)$ *such that*

$$T(\tilde{v}^{(2)}) \in \mathbb{Q}\bar{v}(\mathcal{E}),$$

where $\bar{v}(\mathcal{E})$ *is the projection of the Mukai vector* $v(\mathcal{E})$ *to the irreducible component* $SH(X,\mathbb{Q})$.

As shown in [**FO24**, Proposition 1.15], for hyperkähler of type $\mathrm{K3}^{[2]}$, being atomic for a coherent sheaf $\mathcal{E}$ is equivalent to having an extended Mukai vector.

In this section, we will prove that the only atomic bundles among the Schur functors of Q are given by the symmetric powers $\mathrm{Sym}^m(Q)$.Notice that, in general, any atomic torsion-free sheaf is modular by [**Bec22**, Proposition 1.5]. Our main result is the following:

Proposition 3.7.5. *Let* $X \subseteq \mathrm{Gr}(6,10)$ *be a very general Debarre-Voisin hyperkähler manifold, and let* $\lambda = (m,t,s,0)$ *with* $m \geq t+s$*, then* $\Sigma_\lambda Q$ *is atomic if and only if* $\lambda = (m,0,0,0)$.

We divide the proof into steps.

3.7.1. Proof of Proposition 3.7.5.

Lemma 3.7.6. *Following the notations of Section 3.6, let*

$$\widetilde{v} := \left(r_m, \frac{m}{4} r_m \mathsf{ch}_1(Q), \frac{2m^2 - 3m + 5}{4} r_m \right) \in \widetilde{H}(X, \mathbb{Q}),$$

then $\widetilde{v}$ is an extended Mukai vector for $\mathrm{Sym}^m(Q)$. In particular, $\mathrm{Sym}^m(Q)$ is atomic for every $m \geq 1$.

Proof. We first compute the Mukai vector of $\mathrm{Sym}^m(Q)$, $v(\mathrm{Sym}^m(Q)) := \mathsf{ch}(\mathrm{Sym}^m(Q))\sqrt{\mathrm{td}_X}$, using item (2) of Lemma 3.6.1, item (3) of Lemma 3.3.1 and Lemma 3.3.2.

$$\begin{aligned} v(\mathrm{Sym}^m(Q)) = \Big(r_m, &\frac{m}{4} r_m \mathsf{ch}_1(Q), \\ &\frac{m^2 - m}{40} r_m \mathsf{ch}_1(Q)^2 + \frac{m^2 + 4m}{20} r_m \mathsf{ch}_2(Q) + \frac{1}{24} r_m \mathsf{ch}_1(Q)^2 - \frac{1}{3} r_m \mathsf{ch}_2(Q), \\ &- \frac{2m^3 - 3m^2}{4} r_m \mathsf{ch}_3(Q) - \frac{11m}{4} r_m \mathsf{ch}_3(Q) + \frac{3}{2} r_m \mathsf{ch}_3(Q), \\ &- \frac{10m^4 - 30m^3 + 21m^2 - 6m}{20} r_m \mathsf{ch}_4(Q) - \frac{121}{20}(m^2 - m) r_m \mathsf{ch}_4(Q) + \\ &- \frac{33}{40}(m^2 + 4m) r_m \mathsf{ch}_4(Q) + \frac{33}{10}(m^2 - m) r_m \mathsf{ch}_4(Q) + (m^2 + 4m) r_m \mathsf{ch}_4(Q) + \\ &- \frac{25}{8} r_m \mathsf{ch}_4(Q) \Big). \end{aligned}$$

If we denote with $\mathsf{ch}_1(Q)^\vee \in H^6(X, \mathbb{Q})$ the dual class as in the proof of Lemma 3.3.2, it follows that

$$\begin{aligned} v(\mathrm{Sym}^m(Q)) = \Big(r_m, &\frac{m}{4} r_m \mathsf{ch}_1(Q), \\ &\frac{3m^2 - 3m + 5}{120} r_m \mathsf{ch}_1(Q)^2 + \frac{3m^2 + 12m - 20}{60} r_m \mathsf{ch}_2(Q), \\ &\frac{2m^3 - 3m^2 + 5m}{16} r_m \mathsf{ch}_1(Q)^\vee, \\ &\frac{4m^4 - 12m^3 + 29m^2 - 30m + 25}{32} r_m \Big). \end{aligned}$$

On the other hand, we can compute the projection to the Verbitsky component of $\widetilde{v}$ using the formula

$$T((r, \ell, s) \cdot (r, \ell, s)) = \left(r, \ell, \frac{1}{2r} \left(\ell^2 - \frac{\widetilde{q}((r, \ell, s))}{30} \mathsf{c}_2(X) \right), \frac{s}{r} \ell^\vee, \frac{s^2}{2r} \right).$$

For $r = r_m$, $\ell = \mathsf{ch}_1(\mathrm{Sym}^m(Q))$ and $s = \frac{2m^2 - 3m + 5}{4} r_m$, the formula yields

$$\begin{aligned} T(\widetilde{v} \cdot \widetilde{v}) = \Big(r_m, \frac{m}{4} r_m \mathsf{ch}_1(Q), \frac{1}{2r_m} \left(\frac{m^2}{12} r_m^2 \mathsf{ch}_1(Q)^2 - \frac{\widetilde{q}(\widetilde{v})}{30} \mathsf{c}_2(X) \right), \\ \frac{2m^3 - 3m^2 + 5m}{16} r_m \mathsf{ch}_1(Q)^\vee, \frac{4m^4 - 12m^3 + 29m^2 - 30m + 25}{32} r_m \Big). \end{aligned}$$

Moreover, $\widetilde{q}(\widetilde{v}) = \frac{11m^2}{8}r_m^2 - 2\frac{2m^2-3m+5}{4}r_m^2 = \frac{3m^2+12m-20}{8}r_m^2$. After a short computation, one obtains that

$$T(\widetilde{v}\cdot\widetilde{v}) = v(\mathrm{Sym}^m(Q)).$$

Thus, $\widetilde{v}$ is an extended Mukai vector for $\mathrm{Sym}^m(Q)$ and, by [**FO24**, Proposition 1.15], we have that $\mathrm{Sym}^m(Q)$ is atomic. □

In order to prove the converse, we state as a Lemma the content of Remark 3.3.7.

Lemma 3.7.7. *With the notations of Section 3.3, let* $P(m,t,s) := 1 + \frac{-276\delta - 1936\ell^4 + 1320\delta\ell^2 + 207\delta^2 - 8\xi}{48}$. *Then* $P(m,t,s)$ *is the same polynomial computed in the proof of* [**FO24**, Proposition 6.1].

Proof. This is just a lengthy but easy computation. □

This analogy is enough to prove that, if $\Sigma_\lambda Q$ is atomic, then we get $\Sigma_\lambda Q = \mathrm{Sym}^m(Q)$. In fact, for any atomic sheaf, we have that the Euler characteristic of the endomorphisms sheaf divided by $3r^2$ must be the square of a rational number (see [**Bot24**, Theorem 3.17]). Following the computations in [**FO24**, Proposition 6.4], we obtain that, up to duality and tensor product with line bundles, this forces the partition λ to be $(m,0,0,0)$.

The proof of Proposition 3.7.5 is therefore complete.

3.7.2. Comparison with the notion of hyper-holomorphic vector bundles. In [**Ver93**], Verbitsky introduced the notion of hyper-holomorphic connection for vector bundles and of projectively hyper-holomorphic vector bundle on a hyperkähler manifold X. The idea behind the definition is to select those holomorphic bundles that are holomorphic for every complex structure induced by the hyperkähler structure on X. Moreover, in [**Ver93**, Theorem 3.11], he managed to prove that the smooth locus of an irreducible component of the moduli space parametrizing projectively hyper-holomorphic vector bundles with fixed numerical invariants is hyperkähler.

More recently, the study of deformations of hyper-holomorphic vector bundles has been carried on by Meazzini and Onorati in [**MO23**].

Remark 3.7.8. *For X of type $K3^{[2]}$, if E is a stable modular vector bundle on X then E is projectively hyper-holomorphic. This follows from* [**Ver93**, Proposition 1.2 and Theorem 2.5], *see also* [**Bec22**, Proposition 5.6].

In order to determine which Schur functors of Q are projectively hyper-holomorphic we need a stability result.

Proposition 3.7.9. *Let $X \subseteq \mathrm{Gr}(6,10)$ be a very general Debarre-Voisin hyperkähler manifold, then $\Sigma_\lambda Q$ is polystable for every partition λ.*

Proof. Recall that a vector bundle is polystable if it is isomorphic to a direct sum of stable bundles. By [**OG19**, Proposition 8.4], Q is stable and by [**HL10**, Theorem 3.2.11] we have that $Q^{\otimes k}$ is polystable for every $k \geq 1$. Since every Schur functor $\Sigma_\lambda Q$ is a direct summand of $Q^{\otimes|\lambda|}$, it follows that it is polystable. □

Corollary 3.7.10. *Let $X \subseteq \mathrm{Gr}(6,10)$ be a very general Debarre-Voisin hyperkähler manifold, if the Conjecture 3.4.7 holds, then $\Sigma_\lambda Q$ is a stable projectively hyper-holomorphic vector bundle for every partition λ.*

Proof. If Conjecture 3.4.7 holds then $\Sigma_\lambda Q$ is simple and hence stable by the previous Proposition. Since $\Sigma_\lambda Q$ is modular for every partition λ, by Remark 3.7.8, it is projectively hyper-holomorphic. □

We end by recalling the following result of Verbitsky (see also [**Bec22**, Corollary 6.3] or [**MO23**, Theorem 4.8]), which deals with the local behavior of the moduli space.

Proposition 3.7.11. [**Ver93**, Theorem 6.2] *Let E a stable hyper–holomorphic vector bundle on a hyperkähler manifold X and let*

$$\mathrm{Ext}^1(E,E)\times \mathrm{Ext}^1(E,E)\longrightarrow \mathrm{Ext}^2(E,E)$$

be the Yoneda pairing. If the pairing is skew-symmetric, then the moduli space of stable sheaves on X is smooth around the point $[E]$.

This result, along with Remark 3.4.8, is a promising indication that the moduli space is smooth around the point $[\Sigma_\lambda Q]$ for every partition λ.

CHAPTER 4

Rational degenerations of the general Küchle $c5$ fourfold

4.1. About Fano of K3 type

Fano varieties of K3-type serve as a bridge between the world of Fano varieties and that of hyperkähler manifolds. One of the most famous examples of this phenomenon is the cubic fourfold. Indeed, as shown in [**BD85**], the Fano variety of lines on a cubic fourfold is a hyperkähler manifold of $\mathrm{K3}^{[2]}$-type. Another remarkable example is the Debarre–Voisin twentyfold, which, as proved in [**DV10**], is related to the Debarre—Voisin fourfold, itself a hyperkähler manifold of $\mathrm{K3}^{[2]}$-type.

The common feature of all Fano varieties of K3-type is the presence of the Hodge diamond of a K3 surface inside their own Hodge diamond. This can be expressed by the following definition.

Definition 4.1.1. *Let X be a smooth projective n-dimensional Fano variety and j be a non negative integer. The cohomology group*

$$H^j(X,\mathbb{C}) \simeq \bigoplus_{p+q=j} H^{p,q}(X), \text{ with } j \geq k$$

is said to be of k Calabi–Yau type if

- $h^{\frac{k+j}{2},\frac{j-k}{2}} = 1$;
- $h^{p,q} = 0$, *for all* $p+q=j$, $p < \frac{k+j}{2}$.

X is said to be Fano variety of K3*-type if there exists at least a positive j such that $H^j(X,\mathbb{C})$ is of 2 Calabi–Yau type.*

These varieties are primarily studied because they may provide new models of hyperkähler manifolds, but this is not the only motivation. They also constitute an interesting testing ground for conjectures in derived category theory. Although the definition is purely Hodge-theoretic, there are deep connections with the notion of Calabi–Yau subcategories, as shown, for instance, in [**Kuz17**].

For these reasons, it is important to investigate their rationality and their geometric relations with K3 surfaces. A fruitful approach consists in constructing explicit examples and analyzing them case by case, as in [**BFMT21**].

Some Fano fourfolds of K3-type remain only partially understood, both with respect to rationality and to the origin of their K3-type structure. Some such examples are discussed in [**BFMT21**], but one of the earliest is the deformation family of Fano fourfoulds called Küchle $c5$, described in [**Kü95**].

Following the notation in [**Kuz16**], this family is the deformation of the variety realized as the variety of the subspaces $V_3 \subset V_7$ that are isotropic with respect to a general 2-form $\alpha \in \bigwedge^2 V_7^\vee$, and simultaneously annihilated by a general 3-form $\beta \in \bigwedge^3 V_7^\vee$, and a general

4-form $\gamma \in \bigwedge^4 V_7^\vee$. It has Picard rank one, and its Hodge diamond is

$$\begin{array}{ccccccccc} 0 & & 1 & & 24 & & 1 & & 0 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & 1 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

The general Küchle $c5$ fourfold remains rather mysterious, as several fundamental questions are still open:

- What is the origin of its K3 structure?
- Is it rational?
- Is it possible to associate a hyperkähler manifold to it?

In this chapter we present partial results addressing the first two questions. In particular, here we collect some results of a still ongoing project. To investigate rationality problems for the Küchle $c5$ fourfold, we focus on birational models of $c5$ whose birational and biregular descriptions are not overly complicated.

To this end, we consider the projection

$$\phi_i : V_7 \twoheadrightarrow V_i,$$

and we study varieties arising from the degeneracy loci of the maps

$$\varphi_i : \mathcal{U}_{\mathrm{Gr}(3,7)} \to V_7 \otimes \mathcal{O}_{\mathrm{Gr}(3,7)} \xrightarrow{\phi_i \otimes id} V_i \otimes \mathcal{O}_{\mathrm{Gr}(3,7)}, \text{ for all } 1 < i < 7.$$

From this construction, we obtain in Section 4.2 a list of varieties related to the general Küchle $c5$ fourfold, including three fourfolds birational to $c5$. In Section 4.3, we provide explicit descriptions of all of them; however, none yields new insights into the rationality problem.

Adopting an approach inspired by the study of nodal cubic varieties, in Section 4.4 we construct degenerations the general Küchle $c5$ fourfolds.

In particular, note that we can decompose V_7 as $V_i \oplus V_{7-i}$. Then the forms α and β, are chosen from the following spaces:

$$\alpha \in \bigwedge^2 (V_i^\vee \oplus V_{7-i}^\vee) = \bigwedge^2 V_i^\vee \oplus V_i^\vee \otimes V_{7-i}^\vee \oplus \bigwedge^2 V_{7-i}^\vee;$$
$$\beta \in \bigwedge^3 (V_i^\vee \oplus V_{7-i}^\vee) = \bigwedge^3 V_i^\vee \oplus \bigwedge^2 V_i^\vee \otimes V_{7-i}^\vee \oplus V_i^\vee \otimes \bigwedge^2 V_{7-i}^\vee \oplus \bigwedge^3 V_{7-i}^\vee.$$

Defining $\alpha_i \in \bigwedge^2 V_i^\vee$, $\beta_i \in \bigwedge^3 V_i^\vee$, and $\gamma_i \in \bigwedge^4 V_i^\vee$, we recollect all these degenerations in Table 1.

To each such degenerations we provide a smooth resolution. In particular, by studying the geometry of these resolutions we obtain the main result of this chapter.

Theorem 4.1.2. *(Corollary 4.4.4, Corollary 4.4.8, Corollary 4.4.10) The following singular degenerations of the general Küchle $c5$ fourfold are rational:*

- $K_4^1 = \mathscr{Z}(\alpha, \beta_4, \gamma) \subset \mathrm{Gr}(3,7)$ *singular along an elliptic curve of degree eight;*
- $K_5^2 = \mathscr{Z}(\alpha_5, \beta, \gamma) \subset \mathrm{Gr}(3,7)$*, singular in a point;*
- $K_5^3 = \mathscr{Z}(\alpha_5, \beta_5, \gamma) \subset \mathrm{Gr}(3,7)$*, singular at two points.*

This provides the first three examples of rational singular degenerations of the general Küchle $c5$ fourfold.

TABLE 1. Degenerations of the general Küchle $c5$ fourfold

Variety	2-form	3-form	4-form	Useful birationalities
K_4^1	α	β_4	γ	dP_6 fibr. on $\mathbb{P}^2$
K_5^1	α	β_5	γ	
K_5^2	α_5	β	γ	Conic fibr. on Q_3
K_5^3	α_5	β_5	γ	Quadric fibr. on Q_2
K_6^1	α	β_6	γ	
K_6^2	α_6	β	γ	
K_6^3	α_6	β_6	γ	

4.2. Models

This section introduces the method used to produce a list of varieties related to the general Küchle $c5$. In particular, according to our notation, the general Küchle $c5$ fourfold is defined as

$$X^4_{c5} = \mathscr{Z}(\bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1) \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)}) \subset \mathrm{Gr}(3,7).$$

Considering the global sections of $\bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1) \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)}$ we obtain the following sum of vector spaces

$$W := \Gamma(\mathrm{Gr}(3,7), \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1) \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)}) = \bigwedge^2 V_7^\vee \oplus \bigwedge^3 V_7^\vee \oplus \bigwedge^3 V_7.$$

Since $\bigwedge^3 V_7 \cong \bigwedge^4 V_7^\vee$, we can rewrite W as

$$W = \bigwedge^2 V_7^\vee \oplus \bigwedge^3 V_7^\vee \oplus \bigwedge^4 V_7^\vee.$$

Thus, X^4_{c5} can be described as the locus of 3-dimensional subspaces of V_7 isotropic with respect to a 2-form, and simultaneusly annihilated by a 3-form, and a 4-form.

To study the geometry of X^4_{c5}, we construct related varieties using Borel–Weil–Bott theorem, following the approach in [Man23]. We consider $\mathrm{Gr}(k_1, V_i) \times \mathrm{Gr}(k_2, V_7)$ with $i \leq 6$, and define

$$X_{i,k_1,k_2} = \mathscr{Z}(\sigma) \subset \mathrm{Gr}(k_1, V_i) \times \mathrm{Gr}(k_2, V_7)$$

where $\sigma \in V_i \otimes V_7^\vee \oplus W$.

In this setting, we may write $\sigma = (\sigma_1, \sigma_W)$ where $\sigma_1 \in V_i \otimes V_7^\vee \cong Hom(V_7, V_i)$, and $\sigma_W \in W$. In particular, as shown in the proof of Proposition 2.2.1, σ_1 can be regarded as a morphism

$$\varphi_i^T : V_i^\vee \otimes \mathcal{O}_{\mathrm{Gr}(3,7)} \to \mathcal{U}^\vee_{\mathrm{Gr}(3,7)},$$

and the geometry of X_{i,k_1,k_2} can then be studied through the degeneracy loci of φ_i^T, as explained in Section 2.2.

The condition for X_{i,k_1,k_2} to have positive dimension, yields sixteen admissible choices of values i, k_1, and k_2. Indeed, we collect the possibilities in Table 2.

As summarized in Table 2, this construction yields three fourfolds, namely $X_{4,3,3}$, $X_{5,3,3}$, and $X_{6,3,3}$. In the next section we show that these three fourfolds are birational to the general Küchle $c5$ fourfold, and we dedicate the last section to study degenerations of them.

TABLE 2. Possibilities

i	k_1	k_2	Bndls ass. to $V_i \otimes V_7^\vee$	Bndls ass. to W	Dimension of $X_{(i,k_1,k_2)}$
2	1	3	$\mathcal{Q}_{\mathbb{P}^1} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}$	$\bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1)$	2
3	1	3	$\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}$	$\bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1)$	0
3	2	3	$\mathcal{Q}_{\mathrm{Gr}(2,3)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}$	$\bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1)$	3
4	2	3	$\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}$	$\bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1)$	2
4	3	3	$\mathcal{Q}_{\mathrm{Gr}(3,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}$	$\bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1)$	4
4	3	4	$\mathcal{Q}_{\mathrm{Gr}(3,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(4,7)}$	$\bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(4,7)} \oplus \mathcal{O}_{\mathrm{Gr}(4,7)}(1) \oplus \bigwedge^3 \mathcal{U}^\vee_{\mathrm{Gr}(4,7)}$	0
5	2	3	$\mathcal{Q}_{\mathrm{Gr}(2,5)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}$	$\bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1)$	1
5	3	3	$\mathcal{Q}_{\mathrm{Gr}(3,5)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}$	$\bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1)$	4
5	4	3	$\mathcal{Q}_{\mathrm{Gr}(4,5)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}$	$\bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1)$	5
5	4	4	$\mathcal{Q}_{\mathrm{Gr}(4,5)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(4,7)}$	$\bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(4,7)} \oplus \mathcal{O}_{\mathrm{Gr}(4,7)}(1) \oplus \bigwedge^3 \mathcal{U}^\vee_{\mathrm{Gr}(4,7)}$	1
6	2	3	$\mathcal{Q}_{\mathrm{Gr}(2,6)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}$	$\bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1)$	0
6	3	3	$\mathcal{Q}_{\mathrm{Gr}(3,6)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}$	$\bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1)$	4
6	4	3	$\mathcal{Q}_{\mathrm{Gr}(4,6)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}$	$\bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1)$	6
6	4	4	$\mathcal{Q}_{\mathrm{Gr}(4,6)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(4,7)}$	$\bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(4,7)} \oplus \mathcal{O}_{\mathrm{Gr}(4,7)}(1) \oplus \bigwedge^3 \mathcal{U}^\vee_{\mathrm{Gr}(4,7)}$	1
6	5	3	$\mathcal{Q}_{\mathrm{Gr}(5,6)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}$	$\bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1)$	6
6	5	4	$\mathcal{Q}_{\mathrm{Gr}(5,6)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(4,7)}$	$\bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(4,7)} \oplus \mathcal{O}_{\mathrm{Gr}(4,7)}(1) \oplus \bigwedge^3 \mathcal{U}^\vee_{\mathrm{Gr}(4,7)}$	2

4.3. Birational descriptions of the varieties

In this section, we provide a birational description of the sixteen varieties listed in Table 2. We compute their Hodge numbers using the same method described in Section 2.2.1.

4.3.1. Variety $X_{2,1,3}$. Elliptic fibration on $\mathbb{P}^1$.

We can describe $X_{2,1,3}$ as

$$X_{2,1,3} = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(3,7)}(1,0) \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1)) \subset \mathbb{P}^1 \times \mathrm{Gr}(3,7).$$

A dimension count gives $\dim X_{2,1,3} = 2$, and by the adjunction formula we find that its canonical bundle is $K_{X_{2,1,3}} = \mathcal{O}(1,0)_{|X_{2,1,3}}$. The Hodge diamond of $X_{2,1,3}$ is

$$\begin{array}{ccccc} 2 & & 30 & & 2 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Thus, $X_{2,1,3}$ is not a Fano surface.

We study the projection

$$\pi : X_{2,1,3} \to \mathbb{P}^1.$$

It is surjective, and its general fiber is

$$\pi^{-1}(x) = E_x = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1)) \subset \mathrm{Gr}(3,7)$$

which is an elliptic curve of degree six. Hence, $X_{2,1,3}$ is an elliptic fibration on $\mathbb{P}^1$.

4.3.2. Variety $X_{3,1,3}$. Eight points.
We describe $X_{3,1,3}$ as

$$X_{3,1,3} = \mathscr{Z}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1)) \subset \mathbb{P}^2 \times \mathrm{Gr}(3,7).$$

By a dimension count, we deduce that $X_{3,1,3}$ is a finite set of points. In particular, computing $h^{0,0}(X_{3,1,3})$, we obtain $X_{3,1,3} = \{8 \text{ points}\}$.

4.3.3. Variety $X_{3,2,3}$. Calabi–Yau threefold.
We decribe $X_{3,2,3}$ as

$$X_{3,2,3} = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(3,7)}(1,0) \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1)) \subset \mathbb{P}^2 \times \mathrm{Gr}(3,7).$$

A dimension count shows that it is a threefold. By the adjunction formula, it has a trivial canonical bundle; hence it is a Calabi—Yau threefold.

The Hodge diamond of $X_{3,2,3}$ is

$$\begin{array}{ccccccc} 1 & & 36 & & 36 & & 1 \\ & 0 & & 2 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Consider the projection

$$\pi : X_{3,2,3} \to \mathbb{P}^2.$$

The general fiber is

$$\pi^{-1}(x) = E_x = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1)) \subset \mathrm{Gr}(3,7)$$

which is an elliptic curve of degree six. In particular,

$$X_{2,1,3} = \mathscr{Z}(\mathcal{O}(1,0)) \subset X_{3,2,3}.$$

In the databases that we have checked ([**KS**]; and [**AGHJN**]) we could not find this threefold, so it can be possibly a new one

4.3.4. Variety $X_{4,2,3}$. Double cover of $\mathbb{P}^2$ ramified along an octic curve.
We describe $X_{4,2,3}$ as

$$X_{4,2,3} = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1)) \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(3,7).$$

A dimension count shows that it is a surface, and by the adjunction formula we obtain $K_{X_{4,2,3}} = \mathcal{O}(-1,1)_{|X_{4,2,3}}$.

The Hodge diamond of $X_{4,2,3}$ is

$$\begin{array}{ccccc} 3 & & 38 & & 3 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Consider the projection

$$\pi : X_{4,2,3} \to \mathrm{Gr}(3,7),$$

which is surjective onto the degeneracy locus $D_2(\varphi_4)$ of

$$\varphi_4 : \mathcal{U}_{\mathrm{Gr}(3,7)} \to V_4 \otimes \mathcal{O}_{\mathrm{Gr}(3,7)}.$$

This locus coincides with the exceptional locus of the projection

$$\pi : X_{4,3,3} \to \mathrm{Gr}(3,7),$$

and a detailed description is postponed to the discussion of $X_{4,3,3}$.

4.3.5. Variety $X_{4,3,3}$. Blow-up of the general Küchle $c5$ along $X_{4,2,3}$.
We describe $X_{4,3,3}$ as

$$X_{4,3,3} = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(3,7)}(1,0) \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1)) \subset \mathbb{P}^3 \times \mathrm{Gr}(3,7).$$

By a dimension count, we find that it is a fourfold, and by the adjunction formula we obtain $K_{X_{4,3,3}} = \mathcal{O}(-1,0)_{|X_{4,3,3}}$.

The Hodge diamond of $X_{4,3,3}$ is

$$\begin{array}{ccccccccc}
0 & & 4 & & 62 & & 4 & & 0 \\
& 0 & & 0 & & 0 & & 0 & \\
& & 0 & & 2 & & 0 & & \\
& & & 0 & & 0 & & & \\
& & & & 1 & & & &
\end{array}$$

Consider the projection

$$\pi : X_{4,3,3} \to \mathrm{Gr}(3,7).$$

It is surjective onto the general Küchle $c5$ fourfold X^4_{c5}. In particular, for a general $p \in X^4_{c5}$ the fiber $\pi^{-1}(p)$ consists of a single point; hence $X_{4,3,3}$ is birational to X^4_{c5}.

The exceptional locus of π is $D_2(\varphi_4)$, where

$$\varphi_4 : \mathcal{U}_{\mathrm{Gr}(3,7)} \to V_4 \otimes \mathcal{O}_{\mathrm{Gr}(3,7)}.$$

In particular, this locus is a surface S. Since there are no further degeneracy loci, we can apply Lemma 2.2.2, obtaining that can be described as

$$S = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,7)}(1,0) \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1)) \subset \mathbb{P}^2 \times \mathrm{Gr}(3,7).$$

Note that S is defined in the same way as $X_{4,2,3}$, this provides an alternative model for the latter.

The projection $S \to \mathbb{P}^2$ has general fiber consists of two points, so S is a double cover of $\mathbb{P}^2$. By comparing the Hodge diamond of S with the ones of double covers of $\mathbb{P}^2$, the only possibility is that S is a double cover of $\mathbb{P}^2$ ramified along an octic curve.

For every $p \in S$, the fiber $\pi^{-1}(p)$ is a $\mathbb{P}^1$; therefore

$$X_{4,3,3} = \mathrm{Bl}_{X_{4,2,3}} X^4_{c5}.$$

The projection onto $\mathbb{P}^3$ defines an elliptic fibration, and

$$X_{3,2,3} = \mathscr{Z}(\mathcal{O}(1,0)) \subset X_{4,3,3}.$$

4.3.6. Variety $X_{4,3,4}$. Six points.
We describe $X_{4,3,4}$ as

$$X_{4,3,4} = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(4,7)}(1,0) \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(4,7)} \oplus \mathcal{O}_{\mathrm{Gr}(4,7)}(1) \bigwedge^3 \mathcal{U}^\vee_{\mathrm{Gr}(4,7)}) \subset \mathbb{P}^3 \times \mathrm{Gr}(4,7).$$

A dimension count shows that it is a finite set of points; in particular, computind $h^{0,0}$, we get $X_{4,3,4} = \{6 \text{ points}\}$.

4.3.7. Variety $X_{5,2,3}$. Genus three curve.
We describe $X_{5,2,3}$ as

$$X_{5,2,3} = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,5)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1)) \subset \mathrm{Gr}(2,5) \times \mathrm{Gr}(3,7).$$

A dimension count shows that it is a curve, with Hodge diamond

$$\begin{array}{ccc} 3 & & 3 \\ & 1 & \end{array}$$

It is a genus 3 curve.

4.3.8. Variety $X_{5,3,3}$. Blow-up of the general Küchle $c5$ along $X_{5,2,3}$.
We describe $X_{5,3,3}$ as

$$X_{5,3,3} = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1)) \subset \mathrm{Gr}(2,5) \times \mathrm{Gr}(3,7).$$

It is a fourfold with canonical bundle $K_{X_{5,3,3}} = \mathcal{O}(-3,1)_{|X_{5,3,3}}$, and its Hodge diamond is

$$\begin{array}{ccccccccc} 0 & & 1 & & 26 & & 1 & & 0 \\ & 0 & & 3 & & 3 & & 0 & \\ & & 0 & & 2 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

Consider the projection

$$\pi : X_{5,3,3} \to \mathrm{Gr}(3,7).$$

It is surjective onto the general Küchle $c5$ fourfold X^4_{c5}. In particular, on a general point $p \in X^4_{c5}$, the fiber $\pi^{-1}(p)$ is a single point; hence $X_{5,3,3}$ is birational to the general Küchle $c5$ fourfold.

The fiber of π becomes a $\mathbb{P}^2$ along a curve $C = D_2(\varphi_5)$, where

$$\varphi_5 : \mathcal{U}_{\mathrm{Gr}(3,7)} \to V_5 \otimes \mathcal{O}_{\mathrm{Gr}(3,7)}.$$

Note that this curve coincides with the curve $X_{5,2,3}$; hence we have $X_{5,3,3} = \mathrm{Bl}_{X_{5,2,3}} X^4_{c5}$.

The projection to $\mathrm{Gr}(2,5)$ would provide additional information, but we leave it for future work.

4.3.9. Variety $X_{5,4,3}$. Generically $\mathbb{P}^1$-bundle on the general Küchle $c5$ which jumps to a $\mathbb{P}^2$ along $X_{5,2,3}$.

We define the variety $X_{5,4,3}$ as

$$X_{5,4,3} = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(3,7)}(1,0) \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1)) \subset \mathbb{P}^4 \times \mathrm{Gr}(3,7).$$

It is a fivefold with canonical bundle $K_{X_{5,4,3}} = \mathcal{O}(-1,0)_{|X_{5,4,3}}$.

The Hodge diamond of $X_{5,4,3}$ is

$$\begin{array}{ccccccccccc}
0 & & 0 & & 3 & & 3 & & 0 & & 0 \\
& 0 & & 1 & & 26 & & 1 & & 0 & \\
& & 0 & & 0 & & 0 & & 0 & & \\
& & & 0 & & 2 & & 0 & & & \\
& & & & 0 & & 0 & & & & \\
& & & & & 1 & & & & &
\end{array}$$

Consider the the projection

$$\pi : X_{5,4,3} \to X^4_{c5}.$$

This is a projective bundle over X^4_{c5}, in particular $\mathbb{P}_{X^4_{c5}}(\mathcal{K})$, $\mathcal{K}$ is the kernel sheaf of the map

$$\varphi_5^T : V_5^\vee \otimes \mathcal{O}_{\mathrm{Gr}(3,7)} \to \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}.$$

The genera fiber of π is a $\mathbb{P}^1$, and it degenerates to a $\mathbb{P}^2$ along $D_2(\varphi_5)$, which is the curve $X_{5,2,3}$. Moreover, we have

$$X_{4,3,3} = \mathscr{Z}(\mathcal{O}(1,0)) \subset X_{5,4,3}.$$

Recall that the rationality of $X_{5,4,3}$, would imply the stable rationality of X^4_{c5}. Unfortunately the projection on $\mathbb{P}^4$ gives an elliptic fibration, hence we can not conclude anything.

4.3.10. Variety $X_{5,4,4}$. Elliptic curve of degree six.

We write the variety $X_{5,4,4}$ as

$$X_{5,4,4} = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(4,7)}(1,0) \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(4,7)} \oplus \mathcal{O}_{\mathrm{Gr}(4,7)}(1) \bigwedge^3 \mathcal{U}^\vee_{\mathrm{Gr}(4,7)}) \subset \mathbb{P}^4 \times \mathrm{Gr}(4,7).$$

It is a curve with trivial canonical bundle; hence it is an elliptic curve. Intersecting it with an additional section of $\mathcal{O}(0,1)$ yields six points, so the curve is a degree six elliptic curve in $\mathrm{Gr}(4,7)$.

4.3.11. Variety $X_{6,2,3}$. Two points.

We write the variety $X_{6,2,3}$ as

$$X_{6,2,3} = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,6)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1)) \subset \mathrm{Gr}(2,6) \times \mathrm{Gr}(3,7).$$

By a dimension count, we deduce that $X_{3,1,3}$ is a finite set of points, in particular $X_{6,2,3} = \{2 \text{ points}\}$

4.3.12. Variety $X_{6,3,3}$. Blow-up of the general Küchle $c5$ along $X_{6,2,3}$.
We describe $X_{6,3,3}$ as

$$X_{6,3,3} = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(3,6)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1)) \subset \mathrm{Gr}(3,6) \times \mathrm{Gr}(3,7).$$

It is a fourfold with canonical bundle $K_{X_{6,3,3}} = \mathcal{O}(-3,2)_{|X_{6,3,3}}$, and Hodge diamond

$$\begin{array}{ccccccccc} 0 & & 1 & & 26 & & 1 & & 0 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & 3 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

The projection

$$\pi \,:\, X_{6,3,3} \to \mathrm{Gr}(3,7)$$

is surjective onto X^4_{c5}. For a general point $p \in X^4_{c5}$, the fiber $\pi^{-1}(p)$ consists of a single point; hence $X_{6,3,3}$ is birational to the general Küchle $c5$.

The fiber of π becomes a $\mathbb{P}^3$ along $D_2(\varphi_6)$, with

$$\varphi_6 \,:\, \mathcal{U}_{\mathrm{Gr}(3,7)} \to V_6 \otimes \mathcal{O}_{\mathrm{Gr}(3,7)}.$$

This coincides with $X_{6,2,3}$; hence

$$X_{6,3,3} = \mathrm{Bl}_{2\text{ pts}} X^4_{c5}$$

The projection onto $\mathrm{Gr}(3,6)$ would provide additional information, but we leave it for future works.

4.3.13. Variety $X_{6,4,3}$. $\mathbb{P}^2$-bundle on the general Küchle $c5$ which jumps to a $\mathrm{Gr}(2,4)$ along $X_{6,2,3}$.

We describe the variety $X_{6,4,3}$ as

$$X_{6,4,3} = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1)) \subset \mathrm{Gr}(2,6) \times \mathrm{Gr}(3,7).$$

It is a sixfold with canonical bundle $K_{X_{6,4,3}} = \mathcal{O}(-3,1)_{|X_{6,4,3}}$ and Hodge diamond

$$\begin{array}{ccccccccccccc} 0 & & 0 & & 1 & & 28 & & 1 & & 0 & & 0 \\ & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & 1 & & 28 & & 1 & & 0 & & \\ & & & 0 & & 0 & & 0 & & 0 & & & \\ & & & & 0 & & 2 & & 0 & & & & \\ & & & & & 0 & & 0 & & & & & \\ & & & & & & 1 & & & & & & \end{array}$$

The projection

$$\pi \,:\, X_{6,4,3} \to \mathrm{Gr}(3,7).$$

is a $\mathbb{P}^2$ bundle on X^4_{c5}, whose fiber jumps to a $\mathrm{Gr}(2,4)$ over the degeneracy locus $D_2(\varphi_6)$ of

$$\varphi_6 \,:\, \mathcal{U}_{\mathrm{Gr}(3,7)} \to V_6 \otimes \mathcal{O}_{\mathrm{Gr}(3,7)}.$$

Recall that $D_2(\varphi_6)$ is $X_{6,2,3}$, hence it consists of two points.

Moreover

$$X_{6,5,3} = \mathbb{P}_{X^4_{c5}}(\mathcal{K}),$$

where $\mathcal{K} = ker(\varphi_6^T)$ and

$$\varphi_6^T : V_6^\vee \otimes \mathcal{O}_{\mathrm{Gr}(3,7)} \to \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}.$$

The projection to Gr(2, 6) could provide information about the stable rationality of X^4_{c5}, but we leave it for future works.

4.3.14. Variety $X_{6,4,4}$. Elliptic curve of degree six.

We describe the variety $X_{6,4,4}$ as

$$X_{6,4,4} = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(2,6)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(4,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(4,7)} \oplus \mathcal{O}_{\mathrm{Gr}(4,7)}(1) \bigwedge^3 \mathcal{U}^\vee_{\mathrm{Gr}(4,7)}) \subset \mathrm{Gr}(2,6) \times \mathrm{Gr}(4,7).$$

It is a curve, and by the adjunction formula we obtain $K_{X_{6,4,4}} = \mathcal{O}(-2,1)_{|X_{6,4,4}}$.

The Hodge diamond of $X_{6,4,4}$ is

$$\begin{array}{ccc} 1 & & 1 \\ & 1 & \end{array}$$

Hence, by the classification of curves, it is an elliptic curve. Intersecting it with an additional hyperplane section of Gr(3, 7) yields six points, so the curve is a degree-six elliptic curve.

4.3.15. Variety $X_{6,5,3}$. $\mathbb{P}^2$-bundle on the general Küchle $c5$ which jumps to a $\mathbb{P}^3$ along $X_{6,2,3}$

The variety $X_{6,5,3}$ can be written as

$$X_{6,5,3} = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(3,7)}(1,0) \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1)) \subset \mathbb{P}^5 \times \mathrm{Gr}(3,7).$$

It is a sixfold with canonical bundle $K_{X_{6,5,3}} = \mathcal{O}(-3,0)_{|X_{6,5,3}}$.

The Hodge diamond of $X_{6,5,3}$ is

$$\begin{array}{ccccccccccccc}
0 & & 0 & & 1 & & 28 & & 0 & & 0 & & 0 \\
& 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
& & 0 & & 1 & & 26 & & 1 & & 0 & & \\
& & & 0 & & 0 & & 0 & & 0 & & & \\
& & & & 0 & & 2 & & 0 & & & & \\
& & & & & 0 & & 0 & & & & & \\
& & & & & & 1 & & & & & &
\end{array}$$

Tthe projection

$$\pi : X_{6,5,3} \to \mathrm{Gr}(3,7).$$

is a $\mathbb{P}^2$-bundle on X^4_{c5}; in particular,

$$X_{6,5,3} = \mathbb{P}_{X^4_{c5}}(\mathcal{K}),$$

where $\mathcal{K} = ker(\varphi_6^T)$ and

$$\varphi_6^T : V_6^\vee \to \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}.$$

The fiber of π degenerates to a $\mathbb{P}^3$ along $D_2(\varphi_6)$, which coincides with $X_{6,2,3}$, thus 2 points.

Moreover,

$$X_{5,4,3} = \mathscr{Z}(\mathcal{O}(1,0)) \subset X_{6,5,3}.$$

Recall that the rationality of $X_{6,5,3}$, would imply the stable rationality of X^4_{c5}. Unfortunately the projection on $\mathbb{P}^5$ gives an elliptic fibration, hence we can not conclude anything.

4.3.16. Variety $X_{6,5,4}$. $\mathbb{P}^1$-bundle on $X_{6,4,4}$.
We describe the variety $X_{6,5,4}$ as

$$X_{6,5,4} = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(3,7)}(1,0) \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(4,7)} \oplus \mathcal{O}_{\mathrm{Gr}(4,7)}(1) \bigwedge^3 \mathcal{U}^\vee_{\mathrm{Gr}(4,7)}) \subset \mathbb{P}^5 \times \mathrm{Gr}(4,7).$$

It is a surface with canonical bundle $K_{X_{6,5,4}} = \mathcal{O}(-1,0)_{|X_{6,5,4}}$ and Hodge diamond

$$\begin{array}{ccccc} 0 & & 2 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array}$$

Recall that

$$\mathscr{Z}(\bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathrm{Gr}(3,7)}(1)) \subset \mathrm{Gr}(4,7)$$

is an elliptic curve E of degree six, which is $X_{6,4,4}$.

The projection

$$\pi : X_{6,5,4} \to \mathrm{Gr}(4,7).$$

is surjective on E and the fiber does not degenerate. The general fiber is $\mathbb{P}^1$; hence $X_{6,5,4} = \mathbb{P}_E(\mathcal{E})$, with $\mathcal{E}$ rank two vector bundle on E.

4.4. Degenerating the general Küchle $c5$ fourfold

We have identified three fourfolds that are birational to the general Küchle $c5$ fourfolds, although their rationality remains unclear.

The goal of this section is to degenerate these fourfolds to varieties birational to singular Küchle $c5$ fourfolds. This approach also provides examples of rational singular Küchle $c5$ fourfolds, in analogy with the classical situation for nodal cubics.

As already mentioned, the general Küchle $c5$ fourfold can be described as the locus of 3-dimensional subspaces $V_3 \subset V_7$ that are simultaneously isotropic with respect to a general 2-form $\alpha \in \bigwedge^2 V_7^\vee$, and annihilated by a general 3-form $\beta \in \bigwedge^3 V_7^\vee$ and a general $\gamma \in \bigwedge^4 V_7^\vee$. In the following, we focus on degenerating the 2-form and the 3-form.

Recall from Section 4.2 that $X_{4,3,3}$, $X_{5,3,3}$ and $X_{6,3,3}$ are obtained by studying the surjective map:

$$\phi_i : V_7 \to V_i, \text{ for } i = 4,5,6,$$

which yields a decomposition of V_7 as $V_i \oplus \ker \phi_i \cong V_i \oplus V_{7-i}$. Then the forms α and β belong to the spaces:

$$\alpha \in \bigwedge^2 (V_i^\vee \oplus V_{7-i}^\vee) = \bigwedge^2 V_i^\vee \oplus V_i^\vee \otimes V_{7-i}^\vee \oplus \bigwedge^2 V_{7-i}^\vee;$$
$$\beta \in \bigwedge^3 (V_i^\vee \oplus V_{7-i}^\vee) = \bigwedge^3 V_i^\vee \oplus \bigwedge^2 V_i^\vee \otimes V_{7-i}^\vee \oplus V_i^\vee \otimes \bigwedge^2 V_{7-i}^\vee \oplus \bigwedge^3 V_{7-i}^\vee.$$

In particular, we can degenerate the varieties $X_{i,3,3}$ by choosing $\alpha_i \in \bigwedge^2 V_i^\vee$ or $\beta_i \in \bigwedge^3 V_i^\vee$ instead of a general α or β.

Lemma 4.4.1. *For $i \in \{4,5,6\}$ the varieties*

- $X_i^1 = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,i)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}(1,0)) \subset \mathrm{Gr}(3,i) \times \mathrm{Gr}(3,7)$,
- $X_i^2 = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,i)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,i)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}(0,1)) \subset \mathrm{Gr}(3,i) \times \mathrm{Gr}(3,7)$,

- $X_i^3 = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,i)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,i)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}(1,0)) \subset \mathrm{Gr}(3,i) \times \mathrm{Gr}(3,7)$,

are smooth degenerations of $X_{i,3,3}$.

Proof. Notice that a general global section of $\mathcal{Q}_{\mathrm{Gr}(3,i)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}$ is a projection $\phi_i : V_7 \twoheadrightarrow V_i$. Let Y_i be the variety defined by

$$Y_i = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,i)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}) \subset \mathrm{Gr}(3,i) \times \mathrm{Gr}(3,7).$$

Denote by $\pi_2 : Y_i \to \mathrm{Gr}(3,7)$ the natural projection. Then

$$H^0(Y_i, \pi_2^* \bigwedge^k \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}) = \bigwedge^k V_i^\vee \oplus W_k, \qquad \text{for } k = 1,2,3,$$

where W_k is the orthogonal complement of $\bigwedge^k V_i^\vee$ in $\bigwedge^k V_7^\vee$.

In particular, a general global section λ of $\pi_2^* \bigwedge^k \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}$ on Y_i can be written as

$$\lambda = (\lambda_i, t_\lambda w_\lambda),$$

where $t_\lambda \in \mathbb{C}$, λ_i is a general k-form in $\bigwedge^k V_i^\vee$, and w_λ is a general k-form in W_k.

The variety $X_{i,3,3}$ is defined as

$$X_{i,3,3} = \mathscr{Z}(\alpha, \beta, \gamma) \subset Y_i,$$

where

$$\alpha \in H^0(Y_i, \pi_2^* \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}), \quad \beta \in H^0(Y_i, \pi_2^* \bigwedge^3 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}), \quad \gamma \in H^0(Y_i, \pi_2^* \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)})$$

are general sections.

We can write $\alpha = (\alpha_i, t_\alpha w_\alpha)$ and $\beta = (\beta_i, t_\beta w_\beta)$. Varying t_α and t_β in $\mathbb{C}$ induces deformations of $X_{i,3,3}$.

If $t_\alpha = 0$, we obtain a degeneration of $X_{i,3,3}$ given by

$$\mathscr{Z}(\alpha_i, \beta, \gamma) \subset Y_i.$$

In particular,

$$\alpha_i \in \bigwedge^2 V_i^\vee \cong H^0(Y_i, \pi_1^* \bigwedge^2 \mathcal{U}_{\mathrm{Gr}(3,i)}),$$

where $\pi_1 : Y_i \to \mathrm{Gr}(3,i)$ is the natural projection. The same argument applies to β_i.

Therefore, for each i we obtain three degenerations of $X_{i,3,3}$:

- $X_i^1 = \mathscr{Z}(\pi_2^* \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \pi_1^* \bigwedge^3 \mathcal{U}^\vee_{\mathrm{Gr}(3,i)} \oplus \pi_2^* \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)}) \subset Y_i$;
- $X_i^2 = \mathscr{Z}(\pi_1^* \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,i)} \oplus \pi_2^* \bigwedge^3 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \pi_2^* \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)}) \subset Y_i$;
- $X_i^3 = \mathscr{Z}(\pi_1^* \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,i)} \oplus \pi_1^* \bigwedge^3 \mathcal{U}^\vee_{\mathrm{Gr}(3,i)} \oplus \pi_2^* \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)}) \subset Y_i$.

They can be rewritten in $\mathrm{Gr}(3,i) \times \mathrm{Gr}(3,7)$ as

- $X_i^1 = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,i)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}(1,0)) \subset \mathrm{Gr}(3,i) \times \mathrm{Gr}(3,7)$;
- $X_i^2 = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,i)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,i)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}(0,1)) \subset \mathrm{Gr}(3,i) \times \mathrm{Gr}(3,7)$;
- $X_i^3 = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,i)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,i)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}(1,0)) \subset \mathrm{Gr}(3,i) \times \mathrm{Gr}(3,7)$.

By Bertini Theorem we have that X_i^j is smooth. □

Notice that the restriction of the projection $\pi_{2|X_i^j} \to \mathrm{Gr}(3,7)$ gives a birational map to a degeneration of the general Küchle $c5$ fourfold.

Corollary 4.4.2. *With the above notation, for each $i \in \{4,5,6\}$, we define the folowing three degenerations of the general Küchle $c5$ fourfold:*

- *K_i^1 is the zero locus of α, β_i and γ in $\mathrm{Gr}(3,7)$;*
- *K_i^2 is the zero locus of α_i, β and γ in $\mathrm{Gr}(3,7)$;*
- *K_i^3 is the zero locus of α_i, β_i and γ in $\mathrm{Gr}(3,7)$.*

The variety X_i^j is birational to K_i^j for $j = 1,2,3$.

Proof. The variety

$$Y_i = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,i)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}) \subset \mathrm{Gr}(3,i) \times \mathrm{Gr}(3,7)$$

is birational to $\mathrm{Gr}(3,7)$ via the natural projection $\pi_2 : Y_i \to \mathrm{Gr}(3,7)$. The variety $X_{i,3,3} = \mathscr{Z}(\alpha,\beta,\gamma) \subset Y_i$ is birational to the general Küchle $c5$ fourfold $\mathscr{Z}(\alpha,\beta,\gamma) \subset \mathrm{Gr}(3,7)$ as we have discussed in Section 4.3. The general global section $\phi_i \in H^0(\mathrm{Gr}(3,i) \times \mathrm{Gr}(3,7), \mathcal{Q}_{\mathrm{Gr}(3,i)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)})$ induces the following inclusion

$$H^0(\mathrm{Gr}(3,i), \bigwedge^k \mathcal{U}^\vee_{\mathrm{Gr}(3,i)}) \cong H^0(Y_i, \pi_1^* \bigwedge^k \mathcal{U}^\vee_{\mathrm{Gr}(3,i)}) \subset H^0(Y_i, \pi_2^* \bigwedge^k \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}) \cong H^0(\mathrm{Gr}(3,7), \bigwedge^k \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}).$$

In particular, degenerating $X_{i,3,3}$ with the method described in the proof of Lemma 4.4.1, induces a corresponding degeneration of the birational Küchle $c5$ fourfold. □

4.4.1. Degenerations of $X_{4,3,3}$. We have the following degenerations of $X_{4,3,3}$:

- $X_4^1 = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(3,7)}(1,0) \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}(1,0)) \subset \mathrm{Gr}(3,4) \times \mathrm{Gr}(3,7)$;
- $X_4^2 = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(3,7)}(1,0) \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,4)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}(0,1)) \subset \mathrm{Gr}(3,4) \times \mathrm{Gr}(3,7)$;
- $X_4^3 = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(3,7)}(1,0) \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,4)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}(1,0)) \subset \mathrm{Gr}(3,4) \times \mathrm{Gr}(3,7)$.

Among these, only X_4^1 is nonempty. Indeed, cutting $\mathrm{Gr}(3,4)$ with a general section of $\bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,4)}$ yields the empty set.

Lemma 4.4.3. *The variety*

$$X_4^1 = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}(1,0)) \subset \mathrm{Gr}(3,4) \times \mathrm{Gr}(3,7)$$

is a small resolution of K_4^1, and a dP_6 fibration over $\mathbb{P}^2$. Moreover, $K_{X_4^1} = \mathcal{O}(0,-1)_{|X_4^1}$, and its Hodge diamond is

$$\begin{array}{ccccccccc}
0 & & 0 & & 16 & & 0 & & 0 \\
& 0 & & 0 & & 0 & & 0 & \\
& & 0 & & 2 & & 0 & & \\
& & & 0 & & 0 & & & \\
& & & & 1 & & & &
\end{array}$$

Proof. The Hodge numbers are computed using the techniques explained in Section 2.2.1. By Corollary 4.4.2, the restriction $\pi_{2|X_4^1} : X_4^1 \to \mathrm{Gr}(3,7)$ is a birational map between X_4^1 and K_4^1. It remains to prove that X_4^1 is a resolution of K_4^1 and that it is a dP_6 fibration over $\mathbb{P}^2$.

Notice that X_4^1 can be rewritten as

$$X_4^1 = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(3,7)}(1,0) \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)}) \subset \mathbb{P}^2 \times \mathrm{Gr}(3,7).$$

Let Z^1 be the fivefold

$$Z^1 = \mathscr{Z}(\bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)}) \subset \mathrm{Gr}(3,7).$$

Moreover, let us denote by $\phi : V_7 \twoheadrightarrow V_3$ the projection induced by a general global section of $\mathcal{U}^\vee_{\mathrm{Gr}(3,7)}(1,0)$ on $\mathbb{P}^2 \times \mathrm{Gr}(3,7)$, and let $\varphi : \mathcal{U}_{\mathrm{Gr}(3,7)} \to V_3 \otimes \mathcal{O}_{\mathrm{Gr}(3,7)}$ be the map associated with the pushforward of the global section ϕ via $(\pi_{2|X_4^1})_*$.

The variety X_4^1 admits a stratification $\bigsqcup_{k=0}^3 X_k$, where $X_k = \pi^{-1}_{2|X_4^1}(\mathrm{D}_k(\varphi) \setminus \mathrm{D}_{k-1}(\varphi))$. In particular:

- $\mathrm{D}_3(\varphi) \setminus \mathrm{D}_2(\varphi)$ is Z^1, and the fiber of $\pi_{2|X_4^1}$ is empty; thus it is not a stratum of X_4^1;
- $\mathrm{D}_2(\varphi) \setminus \mathrm{D}_1(\varphi)$, by Theorem 2.2.5, has codimension 1 in Z^1, and the fiber of $\pi_{2|X_4^1}$ is a point;
- $\mathrm{D}_1(\varphi) \setminus \mathrm{D}_0(\varphi)$, by Theorem 2.2.5, has codimension 4 in Z^1, and the fiber of $\pi_{2|X_4^1}$ is a $\mathbb{P}^1$;
- $\mathrm{D}_0(\varphi)$ has codimension 9 in Z^1, hence it is empty.

From this, we obtain that X_4^1 is birational to $\mathrm{D}_2(\varphi)$, which is K_4^1. Moreover, $\mathrm{D}_2(\varphi)$ is singular along $\mathrm{D}_1(\varphi)$, which is, by Theorem 2.2.6, a smooth elliptic curve of degree eight. This describes X_4^1 as the small resolution of K_4^1. By Proposition 2.2.1, the map $\pi_{1|X_4^1} : X_4^1 \to \mathbb{P}^2$ is a dP_6 fibration over $\mathbb{P}^2$.

□

Corollary 4.4.4. *The degeneration K_4^1 is rational.*

Proof. Notice that, by Corollary 4.4.2 and Lemma 4.4.3, K_4^1 is birational to X_4^1 which is a dP_6 fibration on $\mathbb{P}^2$. By [**Isk96**, Page 624] or [**AHTV16**, Prop. 9], a dP_6-fibration is rational provided it admits a rational section. We write X_4^1 as

$$X_4^1 = \mathscr{Z}(\mathcal{U}^\vee_{\mathrm{Gr}(3,7)}(1,0) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)}) \subset \mathrm{Gr}(2,3) \times \mathrm{Gr}(3,7),$$

and we consider π to be the projection onto $\mathrm{Gr}(2,3)$. Thus, to establish rationality, it suffices to find a rational section of $\pi_{|X_4^1}$.

Consider

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,3)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}) \subset \mathrm{Gr}(2,3) \times \mathrm{Gr}(3,7),$$

and let

$$\phi : V_7 \to V_3$$

be a general global section

$$\phi \in H^0(\mathrm{Gr}(2,3) \times \mathrm{Gr}(3,7), \mathcal{Q}_{\mathrm{Gr}(2,3)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}) = V_3 \otimes V_7^\vee.$$

The variety Y parametrizes pairs (U_2, W_3), where $U_2 \subset V_3$, $W_3 \subset V_7$, and $\phi(W_3) \subset U_2$. This condition is equivalent to $\dim(W_3 \cap \ker \phi) \geq 1$. Under this identification, the projection π sends (U_2, W_3) to U_2.

Moreover, Y can be described in terms of the stratification by degeneracy loci of the map

$$\varphi' : \mathcal{U}_{\mathrm{Gr}(3,7)} \to V_3 \otimes \mathcal{O}_{\mathrm{Gr}(3,7)}.$$

In particular, Y is birational to $\mathrm{D}_2(\varphi')$, a codimension-1 subvariety of $\mathrm{Gr}(3,7)$, with further degeneration along $\mathrm{D}_1(\varphi')$ (codimension 4 in $\mathrm{Gr}(3,7)$) and $\mathrm{D}_0(\varphi')$ (codimension 9 in $\mathrm{Gr}(3,7)$).

The kernel sheaf $\mathcal{K} = \ker \varphi'$ admits the following stratification:

- for all $p \in \mathrm{D}_2(\varphi') \smallsetminus D_1(\varphi')$, we have $\operatorname{rk} \mathcal{K} = 1$;
- for all $p \in \mathrm{D}_1(\varphi') \smallsetminus D_0(\varphi')$, we have $\operatorname{rk} \mathcal{K} = 2$;
- for all $p \in \mathrm{D}_0(\varphi')$, we have $\operatorname{rk} \mathcal{K} = 3$.

To construct a rational section, it suffices to restrict to

$$D = \mathrm{D}_2(\varphi') \smallsetminus \mathrm{D}_1(\varphi'),$$

where $\operatorname{rk} \mathcal{K} = 1$.

In this setting, for $W_3 = \langle w_1, w_2, w_3 \rangle$ corresponding to $p = w_1 \wedge w_2 \wedge w_3 \in D$, we have

$$\dim\langle \phi(w_1), \phi(w_2), \phi(w_3) \rangle = 2,$$

that is, $\phi(W_3) = U_2$ and $\dim(W_3 \cap \ker \phi) = 1$. In particular, for each U_2 there exists a unique W_3.

Thus, the stratum D parametrizes pairs (U_2, W_3) such that $W_3 = \phi^{-1}(U_2)$.

Adding the extra sections imposes additional conditions on W_3, but within D this does not obstruct the existence of a section. Therefore, we obtain a rational section

$$s : U_2 \longmapsto (U_2, \phi^{-1}(U_2)).$$

This proves that X_4^1 is rational. Since X_4^1 is birational to the singular Küchle $c5$ fourfold K_4^1, the latter is also rational.

$\square$

4.4.2. Degenerations of $X_{5,3,3}$. We have the following degenerations of $X_{5,3,3}$:

- $X_5^1 = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,5)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}(1,0)) \subset \mathrm{Gr}(3,5) \times \mathrm{Gr}(3,7)$
- $X_5^2 = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,5)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,5)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}(0,1)) \subset \mathrm{Gr}(3,5) \times \mathrm{Gr}(3,7)$
- $X_5^3 = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,5)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,5)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}(1,0)) \subset \mathrm{Gr}(3,5) \times \mathrm{Gr}(3,7)$

We study each of these cases.

Lemma 4.4.5. *The variety*

$$X_5^1 = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,5)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}(1,0)) \subset \mathrm{Gr}(3,5) \times \mathrm{Gr}(3,7)$$

is a resolution of K_5^1. Moreover, $K_{X_5^1} = \mathcal{O}(-1,0)_{|X_5^1}$, and its Hodge diamond is

$$\begin{array}{ccccccccc} 0 & & 1 & & 30 & & 1 & & 0 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & 2 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

Proof. The Hodge numbers are computed using the techniques explained in Section 2.2.1. By Corollary 4.4.2, the restriction $\pi_{2|X_5^1} : X_5^1 \to \mathrm{Gr}(3,7)$ is a birational map between X_5^1 and K_5^1. It remains to prove that X_5^1 is a resolution of K_5^1.

Let Z^1 be the fivefold

$$Z^1 = \mathscr{Z}(\bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)}) \subset \mathrm{Gr}(3,7).$$

Let $\phi_5 : V_7 \twoheadrightarrow V_5$ be the projection induced by a general global section of $\mathcal{Q}_{\mathrm{Gr}(3,5)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}$, and let

$$\varphi_5^T : V_5^\vee \otimes \mathcal{O}_{\mathrm{Gr}(3,7)} \to \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}$$

be the transpose morphism associated with the pushforward of ϕ_5 via $(\pi_{2|X_5^1})_*$.

Denote by Y_5 the fivefold

$$Y_5 = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,5)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)|Z^1}) \subset \mathrm{Gr}(3,5) \times Z^1.$$

It admits a stratification $\bigsqcup_{k=0}^3 Y_k$, where $Y_k = \pi_2^{-1}(\mathrm{D}_k(\varphi_5^T) \setminus \mathrm{D}_{k-1}(\varphi_5^T))$.

Let $\mathcal{K} := \ker \varphi_5^T$. Then:

- on $Z^1 \setminus \mathrm{D}_2(\varphi_5^T)$, the fiber of π_2 is a point and $\mathrm{rk}(\mathcal{K}) = 2$;
- $\mathrm{D}_2(\varphi_5^T) \setminus \mathrm{D}_1(\varphi_5^T)$ has codimension 3 in Z^1, the fiber of π_2 is a $\mathbb{P}^2$, and $\mathrm{rk}(\mathcal{K}) = 3$;
- $\mathrm{D}_1(\varphi_5^T)$ has codimension 8 in Z^1, hence it is empty.

By Theorem 2.2.6, $\mathrm{D}_2(\varphi_5^T)$ is a dP_4, which we denote by S. In particular, $Y_5 = \mathrm{Bl}_S Z^1$, with exceptional divisor $E = \mathbb{P}_S(\mathcal{K}_{|S})$.

The variety X_5^1 is given by $\mathscr{Z}(\mathcal{O}_{Y_5}(1,0)) \subset Y_5$. Hence the image of $\pi_{2|X_5^1} : X_5^1 \to Z^1$ is determined by $(\pi_2)_*\mathcal{O}_{Y_5}(1,0)$. A general global section of $\mathcal{O}_{Y_5}(1,0)$ induces a morphism

$$\psi : \bigwedge^2 \mathcal{K} \longrightarrow \bigwedge^2 V_5^\vee \cong \bigwedge^3 V_5 \longrightarrow \mathcal{O}_{\mathrm{Gr}(3,7)}.$$

Since $\mathcal{K}$ is not a vector bundle, we cannot stratify X_5^1 via degeneracy loci as in the proof of Lemma 4.4.3. Therefore, following [IM16, Page 8], we extend $\mathcal{K}$ to a vector bundle on $Y_5 = \mathrm{Bl}_S Z^1$.

On the exceptional divisor $E = \mathbb{P}_S(\mathcal{K}_{|S})$, the sheaf $\mathcal{K}_{|S}$ fits into the following exact sequence

$$0 \to \mathcal{E} \to \mathcal{K}_{|S}(-1) \to \mathcal{O}_S \to 0.$$

Define a vector bundle $\mathcal{F}$ on Y_5 by

$$\mathcal{F}_{|Z^1\setminus S} = \mathcal{K}_{|Z^1\setminus S}, \qquad \mathcal{F}_{|S} = \mathcal{E}.$$

Thus, $\pi_2(X_5^1)$ is obtained as the zero locus of a section of $\bigwedge^2 \mathcal{F}^\vee$ inside $\mathrm{Bl}_S Z^1$. Since $\pi_2 : Y_5 = \mathrm{Bl}_S Z^1 \to Z^1$ is the blow-up contraction, we obtain

$$\pi_2(X_5^1) = K_5^1,$$

which describes X_5^1 as resolution of K_5^1. □

Before doing the next case we prove a useful identification.

Lemma 4.4.6. *The variety*

$$M = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,5)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,5)}) \subset \mathrm{Gr}(3,5) \times \mathrm{Gr}(3,7)$$

can be rewritten as

$$M = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}(1,0)) \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(3,7).$$

Proof. Take

$$a \in H^0(\mathrm{Gr}(3,5) \times \mathrm{Gr}(3,7), \mathcal{Q}_{\mathrm{Gr}(3,5)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}) \cong \mathrm{Hom}(V_7, V_5)$$

and

$$b \in H^0(\mathrm{Gr}(3,5) \times \mathrm{Gr}(3,7), \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,5)}) = \bigwedge^2 V_5^\vee.$$

The variety M parametrizes pairs (U_3, W_3), where $U_3 \subset V_5$ and $W_3 \subset V_7$ are 3-dimensional subspaces such that

$$a(W_3) \subset U_3 \qquad \text{and} \qquad b_{|U_3} = 0.$$

The condition $b_{|U_3} = 0$ implies that there exists a nonzero vector $k \in U_3$ such that

$$\langle k \rangle = \ker(b_{|U_3}).$$

Set

$$U_4 := V_5 / \langle k \rangle.$$

Then we obtain a decomposition

$$V_5 = \langle k \rangle \oplus U_4,$$

together with the natural projections

$$\pi : V_5 \to U_4, \qquad \pi' : \bigwedge^2 V_5^\vee \to \bigwedge^2 U_4^\vee.$$

Define

$$\tilde{a} := \pi \circ a, \qquad \tilde{b} := \pi' \circ b, \qquad \widetilde{U} := \pi(U_3).$$

Since $k \in U_3$, we have $\dim \widetilde{U} = 2$. Hence

$$\tilde{a}(W_3) \subset \widetilde{U} \qquad \text{and} \qquad \tilde{b}_{|\widetilde{U}} = 0.$$

This shows that M parametrizes pairs $(\widetilde{U}, W_3)$, where $\widetilde{U} \subset U_4$ is a 2-dimensional subspace and $W_3 \subset V_7$ is a 3-dimensional subspace such that

$$\tilde{a}(W_3) \subset \widetilde{U} \qquad \text{and} \qquad \tilde{b}_{|\widetilde{U}} = 0.$$

□

Lemma 4.4.7. *The variety*

$$X_5^2 = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,5)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,5)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}(0,1)) \subset \mathrm{Gr}(3,5) \times \mathrm{Gr}(3,7)$$

is a small resolution of K_5^2, and it is a conic bundle on a smooth quadric threefold. Moreover, the canonical bundle of X_5^2 is $K_{X_5^2} = \mathcal{O}(0,-1)_{X_5^2}$, and its Hodge diamond is

$$\begin{array}{ccccccccc}
0 & & 1 & & 24 & & 1 & & 0 \\
& 0 & & 0 & & 0 & & 0 & \\
& & 0 & & 2 & & 0 & & \\
& & & 0 & & 0 & & & \\
& & & & 1 & & & &
\end{array}$$

Proof. The Hodge numbers are computed using the techniques explained in Section 2.2.1. By Corollary 4.4.2, the restriction $\pi_{2|X_5^2} : X_5^2 \to \mathrm{Gr}(3,7)$ is a birational map between X_5^2 and K_5^2. It remains to prove that X_5^2 is a small resolution of K_5^2 and that it is a conic fibration on a smooth quadric threefold.

By Lemma 4.4.6, X_5^2 can be rewritten as

$$X_5^2 = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(0,1)) \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(3,7).$$

Let Z^2 be the sevenfold

$$Z^2 = \mathscr{Z}(\mathcal{O}_{\mathrm{Gr}(3,7)}(1) \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)}) \subset \mathrm{Gr}(3,7).$$

Let $\phi : V_7 \twoheadrightarrow V_4$ be the projection induced by a general global section of $\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(3,7)}$, and let

$$\varphi^T : V_4^{\vee} \otimes \mathcal{O}_{\mathrm{Gr}(3,7)} \to \mathcal{U}^{\vee}_{\mathrm{Gr}(3,7)}$$

be the transpose morphism associated with the pushforward of ϕ via $(\pi_{2|X_5^2})_*$.

Denote by Y_5 the fivefold

$$Y_5 = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(3,7)|Z^2}) \subset \mathrm{Gr}(2,4) \times Z^2.$$

It admits a stratification $\bigsqcup_{k=0}^{3} Y_k$, where $Y_k = \pi_2^{-1}(\mathrm{D}_k(\varphi_5^T) \setminus \mathrm{D}_{k-1}(\varphi_5^T))$.

Let $\mathcal{K} := \ker \varphi^T$. Then:

- on $Z^2 \setminus \mathrm{D}_2(\varphi^T)$, the fiber of π_2 is empty, thus it is not a stratum of Y_5;
- $\mathrm{D}_2(\varphi^T) \setminus \mathrm{D}_1(\varphi^T)$ has codimension 2 in Z^2, the fiber of π_2 is a point, and $\mathrm{rk}(\mathcal{K}) = 2$;
- $\mathrm{D}_1(\varphi^T) \setminus \mathrm{D}_0(\varphi^T)$ has codimension 6 in Z^2, $\mathrm{rk}(\mathcal{K}) = 3$;
- $\mathrm{D}_0(\varphi^T)$ has codimension 12 in Z^2, thus it is empty.

By Theorem 2.2.6, $\mathrm{D}_1(\varphi^T)$ is a $\mathbb{P}^1$, which we denote by C. In particular, Y_5 is a small resolution of $D_2(\varphi^T)$, with exceptional locus $E = \mathbb{P}_C(\mathcal{K}_{|C})$.

The variety X_5^2 is given by $\mathscr{Z}(\mathcal{O}_{Y_5}(1,0)) \subset Y_5$. Hence the image of $\pi_{2|X_5^2} : X_5^2 \to Z^2$ is determined by $(\pi_2)_*\mathcal{O}_{Y_5}(1,0)$. A general global section of $\mathcal{O}_{Y_5}(1,0)$ induces a morphism

$$\psi : \bigwedge^2 \mathcal{K} \longrightarrow \bigwedge^2 V_4^{\vee} \longrightarrow \mathcal{O}_{\mathrm{Gr}(3,7)}.$$

Since $\mathcal{K}$ is not a vector bundle, we cannot study the stratification of X_5^2 as in the proof of Lemma 4.4.3. Therefore, we consider $\bigwedge^2 \mathcal{K}$ on $\mathrm{D}_2(\varphi^) \setminus \mathrm{D}_1(\varphi^T)$ and by [?], extend it to all of $\mathrm{D}_2(\varphi^T)$ via the double dual $\mathcal{L} = (\bigwedge^2 \mathcal{K})^{\vee\vee}$.

Thus, $\pi_2(X_5^2)$ is obtained as the zero locus of a section of $\mathcal{L}$ inside $\mathrm{D}_2(\varphi^T)$. Since $\pi_2 : Y_5 \to \mathrm{D}_2(\varphi^T)$ is a small contraction, we conclude that

$$\pi_2(X_5^2) = K_5^2,$$

which describes X_5^2 as a small resolution of K_5^2. Observe that the singular locus of K_5^2 is $\mathscr{Z}(\mathcal{L}) \subset \mathrm{D}_1(\varphi^T)$, which consists of a single point.

By Proposition 2.2.1, the projection $\pi_{1|X_5^2} : X_5^2 \to \mathrm{Gr}(2,4)$, is a conic fibration on a smooth quadric threefolds $Q_3 = \mathscr{Z}(\mathcal{O}_{\mathrm{Gr}(2,4)}(1)) \subset \mathrm{Gr}(2,4)$.

□

Corollary 4.4.8. *The degeneration K_5^2 is rational.*

Proof. Notice that, by Corollary 4.4.2 and Lemma 4.4.7 the degeneration K_5^2 is birational to X_5^2, which is conic bundle on a smooth quadric threefold. By [**Isk96'**] a conic bundle on a rational variety is rational if and only if it admits a rational section. We write X_5^2 as

$$X_4^2 = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(0,1) \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)}) \subset \mathrm{Gr}(2,3) \times \mathrm{Gr}(3,7),$$

and we consider π to be the projection onto $\mathrm{Gr}(2,4)$. Thus, to establish rationality, it suffices to find a rational section of $\pi_{|X_5^2}$.

Consider

$$Y = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(3,7)}) \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(3,7),$$

and let

$$\phi : V_7 \to V_4$$

be a general global section

$$\phi \in H^0(\mathrm{Gr}(2,4) \times \mathrm{Gr}(3,7), \mathcal{Q}_{\mathrm{Gr}(4,3)} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(3,7)}) = V_4 \otimes V_7^{\vee}.$$

The variety Y parametrizes pairs (U_2, W_3), where $U_2 \subset V_4$, $W_3 \subset V_7$, and $\phi(W_3) \subset U_2$. This condition is equivalent to $\dim(W_3 \cap \ker \phi) \geq 1$. Under this identification, the projection π sends (U_2, W_3) to U_2.

Moreover, Y can be described in terms of the stratification by degeneracy loci of the map

$$\varphi' : \mathcal{U}_{\mathrm{Gr}(3,7)} \to V_4 \otimes \mathcal{O}_{\mathrm{Gr}(3,7)}.$$

In particular, Y is birational to $\mathrm{D}_2(\varphi')$, a codimension-2 subvariety of $\mathrm{Gr}(3,7)$, with further degeneration along $\mathrm{D}_1(\varphi')$ (codimension 6 in $\mathrm{Gr}(3,7)$) and $\mathrm{D}_0(\varphi')$ (codimension 12 in $\mathrm{Gr}(3,7)$).

The kernel sheaf $\mathcal{K} = \ker \varphi'$ admits the following stratification:

- for all $p \in \mathrm{D}_2(\varphi') \setminus D_1(\varphi')$, we have $\operatorname{rk} \mathcal{K} = 1$;
- for all $p \in \mathrm{D}_1(\varphi') \setminus D_0(\varphi')$, we have $\operatorname{rk} \mathcal{K} = 2$;
- for all $p \in \mathrm{D}_0(\varphi')$, we have $\operatorname{rk} \mathcal{K} = 3$.

To construct a rational section, it suffices to restrict to

$$D = \mathrm{D}_2(\varphi') \setminus \mathrm{D}_1(\varphi'),$$

where $\operatorname{rk} \mathcal{K} = 1$.

In this setting, for $W_3 = \langle w_1, w_2, w_3 \rangle$ corresponding to $p = w_1 \wedge w_2 \wedge w_3 \in D$, we have

$$\dim\langle \phi(w_1), \phi(w_2), \phi(w_3) \rangle = 2,$$

that is, $\phi(W_3) = U_2$ and $\dim(W_3 \cap \ker \phi) = 1$. In particular, for each U_2 there exists a unique W_3.

Thus, the stratum D parametrizes pairs (U_2, W_3) such that $W_3 = \phi^{-1}(U_2)$.

Adding the extra sections imposes additional conditions on W_3, but within D this does not obstruct the existence of a section. Therefore, we obtain a rational section

$$s : U_2 \longmapsto (U_2,\ \phi^{-1}(U_2)).$$

This proves that X_5^2 is rational. Since X_5^2 is birational to the singular Küchle $c5$ fourfold K_5^2, the latter is also rational. $\square$

Lemma 4.4.9. *The variety*

$$X_5^3 = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,5)} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^{\vee}_{\mathrm{Gr}(3,5)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}(1,0)) \subset \mathrm{Gr}(3,5) \times \mathrm{Gr}(3,7)$$

is a small resolution of K_5^3 and is a quadric bundle on $\mathbb{P}^1 \times \mathbb{P}^1$. Moreover, $K_{X_5^3} = \mathcal{O}(1,-2)_{|X_5^3}$, and its Hodge diamond is

$$\begin{array}{ccccccccc} 0 & & 0 & & 8 & & 0 & & 0 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & 3 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

Proof. The Hodge numbers are computed using the techniques explained in Section 2.2.1. By Corollary 4.4.2, the restriction $\pi_{2|X_5^3} : X_5^3 \to \mathrm{Gr}(3,7)$ is a birational map between X_5^3 and K_5^3. It remains to prove that X_5^3 is a small resolution of K_5^3 and that it is a quadric fibration on $\mathbb{P}^1 \times \mathbb{P}^1$.

Notice that by Lemma 4.4.6, X_5^3 can be rewritten as

$$X_5^3 = \mathscr{Z}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1) \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0) \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)}) \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathrm{Gr}(3,7).$$

Let Z^3 be the eightfold

$$Z^3 = \mathscr{Z}(\bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)}) \subset \mathrm{Gr}(3,7).$$

Let $\phi : V_7 \twoheadrightarrow V_2$ be the projection induced by a general global section of $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1) \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}$, and $\phi' : V_7 \twoheadrightarrow V_2$ be the projection induced by a general global section of $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0) \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}$. Let

$$\varphi : \mathcal{U}_{\mathrm{Gr}(3,7)} \to V_2 \otimes \mathcal{O}_{\mathrm{Gr}(3,7)}$$

be the morphism associated with the pushforward of ϕ via $(\pi_{2|X_5^3})_*$, and φ' the one associated to ϕ'.

If we define $\mathrm{D}_{s,t} := (\mathrm{D}_s(\varphi) \setminus \mathrm{D}_{s-1}(\varphi)) \cap (\mathrm{D}_t(\varphi) \setminus \mathrm{D}_{t-1}(\varphi))$, the variety X_5^3 admits a stratification $\bigsqcup_{s,t=0}^2 X_{s,t}$, where $X_{s,t} = \pi^{-1}_{2|X_4^1}(\mathrm{D}_{s,t})$. In particular:

- $\mathrm{D}_{2,2}$ has codimension zero in Z^3, and the fiber of $\pi_{2|X_5^3}$ is empty; thus it is not a stratum of X_5^3;
- $\mathrm{D}_{2,1}$ and $\mathrm{D}_{1,2}$ have codimension two in Z^3, and the fiber of $\pi_{2|X_5^3}$ is empty; thus it is not a stratum of X_5^3;
- $\mathrm{D}_{1,1}$, by Theorem 2.2.5, has codimension 4 in Z^3, and the fiber of $\pi_{2|X_4^1}$ is a point;
- $\mathrm{D}_{1,0}$ and $\mathrm{D}_{0,1}$ have codimension 8 in Z^3, and the fiber $\pi_{2|X_4^1}$ is $\mathbb{P}^1$;
- $\mathrm{D}_{0,0}$ has codimension 12 in Z^3, thus it is empty.

From this, we obtain that X_5^3 is birational to $\mathrm{D}_{1,1}$, which is K_5^3. Moreover, $\mathrm{D}_{1,1}$ is singular along $\mathrm{D}_{1,0} \sqcup \mathrm{D}_{0,1}$, which is, by Theorem 2.2.6, a set of two points. This describes X_4^1 as the small resolution of K_5^3. By Proposition 2.2.1, the map $\pi_{1|X_5^3} : X_5^3 \to \mathbb{P}^1 \times \mathbb{P}^1$ is a quadric fibration on $\mathbb{P}^1 \times \mathbb{P}^1$.

□

Corollary 4.4.10. *The degeneration K_5^3 is rational.*

Proof. Notice that, by Corollary 4.4.2 and Lemma 4.4.3, K_5^3 is birational to X_5^3 which is a quadric fibration on $\mathbb{P}^2$. By [HPT18, Prop. 5] a quadric fibration on a rational variety is rational if and only if it admits a rational section. We write X_5^3 as

$$X_5^3 = \mathscr{Z}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0) \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1) \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)}) \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathrm{Gr}(3,7),$$

and we consider π to be the projection onto $\mathbb{P}^1 \times \mathbb{P}^1$. Thus, to establish rationality, it suffices to find a rational section of $\pi_{|X_5^3}$.

Consider

$$Y = \mathscr{Z}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0) \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}) \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathrm{Gr}(3,7),$$

and let

$$\phi : V_7 \to V_2, \text{ and } \phi' : V_7 \to V_2'$$

be general global sections

$\phi \in H^0(\mathbb{P}^1\times\mathbb{P}^1\times\mathrm{Gr}(3,7), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)\boxtimes\mathcal{U}^\vee_{\mathrm{Gr}(3,7)})$, and $\phi' \in H^0(\mathbb{P}^1\times\mathbb{P}^1\times\mathrm{Gr}(3,7), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1)\boxtimes\mathcal{U}^\vee_{\mathrm{Gr}(3,7)})$

The variety Y parametrizes pairs (U_1, U_1', W_3), where $U_1 \subset V_2$, $U_1' \subset V_2'$, $W_3 \subset V_7$, and $\phi(W_3) \subset U_1'$, $\phi'(W_3) \subset U_1'$. This condition is equivalent to $\dim(W_3 \cap \ker\phi) \geq 2$ and $\dim(W_3 \cap \ker\phi') \geq 2$. Under this identification, the projection π sends (U_1, U_1', W_3) to (U_1, U_1').

Moreover, Y can be described in terms of the stratification by degeneracy loci of the maps

$$\varphi : \mathcal{U}_{\mathrm{Gr}(3,7)} \to V_2 \otimes \mathcal{O}_{\mathrm{Gr}(3,7)}, \text{ and } \varphi' : \mathcal{U}_{\mathrm{Gr}(3,7)} \to V_2 \otimes \mathcal{O}_{\mathrm{Gr}(3,7)}$$

In particular, with the notation used in the proof of Lemma 4.4.9, Y is birational to $D_{1,1}$, a codimension-4 subvariety of $\mathrm{Gr}(3,7)$, with further degeneration along $\mathrm{D}_{1,0} \sqcup \mathrm{D}_{0,1}$ (codimension 8 in $\mathrm{Gr}(3,7)$) and $\mathrm{D}_{0,0}$ (codimension 12 in $\mathrm{Gr}(3,7)$).

To construct a rational section, it suffices to restrict to $D_{1,1}$, where both $\ker\phi$ and $\ker\phi'$ have rank 2.

In this setting, for $W_3 = \langle w_1, w_2, w_3\rangle$ corresponding to $p = w_1 \wedge w_2 \wedge w_3 \in D$, we have

$$\dim\langle\phi(w_1), \phi(w_2), \phi(w_3)\rangle = 1, \text{ and } \dim\langle\phi'(w_1), \phi'(w_2), \phi'(w_3)\rangle = 1$$

that is, $\phi(W_3) = U_1$ and $\phi'(W_3) = U_1'$. In particular, for each (U_1, U_1') there exists a unique W_3.

Thus, the stratum D parametrizes pairs (U_2, W_3) such that $W_3 = \langle\phi^{-1}(U_1), \phi'^{-1}(U_1'),\rangle$.

Adding the extra sections imposes additional conditions on W_3, but within $D_{1,1}$ this does not obstruct the existence of a section. Therefore, we obtain a rational section

$$s : (U_1, U_1') \longmapsto (U_1, U_1', \langle\phi^{-1}(U_1), \phi'^{-1}(U_1')\rangle).$$

This proves that X_5^3 is rational. Since X_5^3 is birational to the singular Küchle $c5$ fourfold K_5^3, the latter is also rational. □

4.4.3. Degenerations of $X_{6,3,3}$. We have the following degenerations of $X_{6,3,3}$:

- $X_6^1 = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,6)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}(1,0)) \subset \mathrm{Gr}(3,6)\times\mathrm{Gr}(3,7)$
- $X_6^2 = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,6)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,6)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}(0,1)) \subset \mathrm{Gr}(3,6)\times\mathrm{Gr}(3,7)$
- $X_6^3 = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,6)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,6)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}(1,0)) \subset \mathrm{Gr}(3,6)\times\mathrm{Gr}(3,7)$

These cases are more involved, in particular for no one of them we have a description of the projection to $\mathrm{Gr}(3,6)$.

Lemma 4.4.11. *The variety*

$$X_6^1 = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,6)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}(1,0)) \subset \mathrm{Gr}(3,6)\times\mathrm{Gr}(3,7)$$

is $\mathrm{Bl}_{Q_1} K_6^1$. *Moreover,* $\mathcal{K} = \mathcal{O}(-2,1)_{|X_6^1}$, *and its Hodge diamond is*

$$\begin{array}{ccccccccc} 0 & & 1 & & 26 & & 1 & & 0 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & 2 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

Proof. The Hodge numbers are computed using the techniques explained in Section 2.2.1. By Corollary 4.4.2, the restriction $\pi_{2|X_6^1} : X_6^1 \to \mathrm{Gr}(3,7)$ is a birational map between X_6^1 and K_6^1. It remains to prove that X_6^1 is a blow-up of K_6^1.

Let Z^1 be the fivefold

$$Z^1 = \mathscr{Z}(\bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)}) \subset \mathrm{Gr}(3,7),$$

and denote by Y_6 the fivefold

$$Y_6 = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,6)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)|Z^1}) \subset \mathrm{Gr}(3,6) \times Z^1.$$

By Lemma 2.2.10 we have $Y_6 = \mathrm{Bl}_{Q_1} Z^1$, where Q_1 is a smooth conic obtained as $\mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)}) \subset \mathrm{Gr}(3,7)$. The variety X_6^1 is defined as $\mathscr{Z}(\mathcal{O}_{Y_6}(1,0)) \subset Y_6$. The hyperplane section fixes Q_1, thus X_6^1 is $\mathrm{Bl}_{Q_1} K_6^1$. □

Lemma 4.4.12. *The variety*

$$X_6^2 = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,6)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,6)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}(0,1)) \subset \mathrm{Gr}(3,6) \times \mathrm{Gr}(3,7)$$

is a resolution of K_6^2. Moreover, $K_{X_6^2} = \mathcal{O}(-1,0)_{|X_6^2}$, and its Hodge diamond is

$$\begin{array}{ccccccccc} 0 & & 1 & & 28 & & 1 & & 0 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & 2 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

Proof. The Hodge numbers are computed using the techniques explained in Section 2.2.1. By Corollary 4.4.2, the restriction $\pi_{2|X_6^2} : X_6^2 \to \mathrm{Gr}(3,7)$ is a birational map between X_6^2 and K_6^2. It remains to prove that X_6^2 is a resolution of K_6^2.

Let Z^2 be the sevenfold

$$Z^2 = \mathscr{Z}(\mathcal{O}_{\mathrm{Gr}(3,7)}(1) \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)}) \subset \mathrm{Gr}(3,7).$$

Let $\phi_6 : V_6 \twoheadrightarrow V_6$ be the projection induced by a general global section of $\mathcal{Q}_{\mathrm{Gr}(3,6)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}$, and let

$$\varphi_6^T : V_6^\vee \otimes \mathcal{O}_{\mathrm{Gr}(3,7)} \to \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}$$

be the transpose morphism associated with the pushforward of ϕ_6 via $(\pi_{2|X_6^2})_*$.

Denote by Y_6 the sevenfold

$$Y_6 = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,6)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)|Z^2}) \subset \mathrm{Gr}(3,6) \times Z^2.$$

It admits a stratification $\bigsqcup_{k=0}^3 Y_k$, where $Y_k = \pi_2^{-1}(\mathrm{D}_k(\varphi_5^T) \setminus \mathrm{D}_{k-1}(\varphi_6^T))$.

Let $\mathcal{K} := \ker \varphi_5^T$. Then:

- on $Z^2 \setminus \mathrm{D}_2(\varphi_6^T)$, the fiber of π_2 is a point and $\mathrm{rk}(\mathcal{K}) = 3$;
- $\mathrm{D}_2(\varphi_6^T) \setminus \mathrm{D}_1(\varphi_6^T)$ has codimension 4 in Z^2, the fiber of π_2 is a $\mathbb{P}^3$, and $\mathrm{rk}(\mathcal{K}) = 4$;
- $\mathrm{D}_1(\varphi_6^T)$ has codimension 10 in Z^2, hence it is empty.

Notice that $\mathrm{D}_2(\varphi_6^T)$ is a threefold, which we denote by T. In particular, $Y_6 = \mathrm{Bl}_T Z^2$, with exceptional divisor $E = \mathbb{P}_T(\mathcal{K}_{|T})$.

The variety X_6^2 is given by $\mathscr{Z}((\pi_1)_* \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,6)}) \subset Y_6$. Hence the image of $\pi_{2|X_6^2} : X_6^2 \to Z^2$ is determined by $(\pi_2)_*(\pi_1)_* \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,6)}$. A general global section of $(\pi_1)_* \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,6)}$ induces a morphism

$$\psi : \bigwedge^2 \mathcal{K} \longrightarrow \bigwedge^2 V_6^\vee \longrightarrow \mathcal{O}_{\mathrm{Gr}(3,7)}.$$

Since $\mathcal{K}$ is not a vector bundle, we cannot stratify X_6^2 via degeneracy loci as in the proof of Lemma 4.4.3. Therefore, following [IM16, Page 8], we extend $\mathcal{K}$ to a vector bundle on $Y_6 = \mathrm{Bl}_T Z^2$.

On the exceptional divisor $E = \mathbb{P}_T(\mathcal{K}_{|T})$, the sheaf $\mathcal{K}_{|T}$ fits into the following exact sequence

$$0 \to \mathcal{E} \to \mathcal{K}_{|T}(-1) \to \mathcal{O}_T \to 0.$$

Define a vector bundle $\mathcal{F}$ on Y_6 by

$$\mathcal{F}_{|Z^2 \setminus T} = \mathcal{K}_{|Z^2 \setminus T}, \qquad \mathcal{F}_{|T} = \mathcal{E}.$$

Thus, $\pi_2(X_6^2)$ is obtained as the zero locus of a section of $\bigwedge^2 \mathcal{F}^\vee$ inside $\mathrm{Bl}_T Z^2$. Since $\pi_2 : Y_6 = \mathrm{Bl}_T Z^2 \to Z^2$ is the blow-up contraction, we obtain

$$\pi_2(X_6^2) = K_6^2,$$

which describes X_6^2 as resolution of K_6^2. Moreover, the exceptional locus of X_6^2 is $\mathscr{Z}(\bigwedge^2 \mathcal{F}^\vee) \subset E$.

$\square$

Lemma 4.4.13. *The variety*

$$X_6^3 = \mathscr{Z}(\mathcal{Q}_{\mathrm{Gr}(3,6)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,6)} \oplus \bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)} \oplus \mathcal{O}(1,0)) \subset \mathrm{Gr}(3,6) \times \mathrm{Gr}(3,7)$$

is a small resolution of K_6^3. Moreover, $K_{X_6^3} = \mathcal{O}(0,-1)_{|X_6^3}$, and its Hodge diamond is

$$\begin{array}{ccccccccc} 0 & & 0 & & 16 & & 0 & & 0 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & 2 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

Proof. The Hodge numbers are computed using the techniques explained in Section 2.2.1. By Corollary 4.4.2, the restriction $\pi_{2|X_6^3} : X_6^3 \to \mathrm{Gr}(3,7)$ is a birational map between X_6^3 and K_6^3. It remains to prove that X_6^3 is a resolution of K_6^3.

Let Z^3 be the eightfold

$$Z^3 = \mathscr{Z}(\bigwedge^3 \mathcal{Q}_{\mathrm{Gr}(3,7)}) \subset \mathrm{Gr}(3,7).$$

Let $\phi_6 : V_6 \twoheadrightarrow V_6$ be the projection induced by a general global section of $\mathcal{Q}_{\mathrm{Gr}(3,6)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}$, and let

$$\varphi_6^T : V_6^\vee \otimes \mathcal{O}_{\mathrm{Gr}(3,7)} \to \mathcal{U}^\vee_{\mathrm{Gr}(3,7)}$$

be the transpose morphism associated with the pushforward of ϕ_6 via $(\pi_{2|X_6^2})_*$.

Denote by Y_6 the sevenfold

$$Y_6 = \mathscr{Z}(Q_{\mathrm{Gr}(3,6)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(3,7)|Z^3}) \subset \mathrm{Gr}(3,6) \times Z^3.$$

It admits a stratification $\bigsqcup_{k=0}^3 Y_k$, where $Y_k = \pi_2^{-1}(\mathrm{D}_k(\varphi_5^T) \setminus \mathrm{D}_{k-1}(\varphi_6^T))$.

Let $\mathcal{K} := \ker \varphi_5^T$. Then:

- on $Z^3 \setminus \mathrm{D}_2(\varphi_6^T)$, the fiber of π_2 is a point and $\mathrm{rk}(\mathcal{K}) = 3$;
- $\mathrm{D}_2(\varphi_6^T) \setminus \mathrm{D}_1(\varphi_6^T)$ has codimension 4 in Z^3, the fiber of π_2 is a $\mathbb{P}^3$, and $\mathrm{rk}(\mathcal{K}) = 4$;
- $\mathrm{D}_1(\varphi_6^T)$ has codimension 10 in Z^3, hence it is empty.

Notice that $\mathrm{D}_2(\varphi_6^T)$ is a fourfold, which we denote by F. In particular, $Y_6 = \mathrm{Bl}_T Z^3$, with exceptional divisor $E = \mathbb{P}_F(\mathcal{K}_{|F})$.

The variety X_6^3 is given by $\mathscr{Z}((\pi_1)_* \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,6)} \oplus \mathcal{O}_{Y_6}(1,0)) \subset Y_6$. Hence the image of $\pi_{2|X_6^3} : X_6^3 \to Z^3$ is determined by $(\pi_2)_*((\pi_1)_* \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,6)})$ and $(\pi_2)_*\mathcal{O}_{Y_6}(1,0)$. As argued in the proof Lemma 4.4.11, the hyperplane section fixes F, thus if we denote with Z the sevenfold $Z = \mathscr{Z}(\mathcal{O}_{Z^3}(1)) \subset Z^3$, then the sevenfold $\mathscr{Z}(\mathcal{O}_{Y_6}(1,0)) \subset Y_6$ is $\mathrm{Bl}_F Z$. A general global section of $(\pi_1)_* \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,6)}$ induces a morphism

$$\psi : \bigwedge^2 \mathcal{K} \longrightarrow \bigwedge^2 V_6^\vee \longrightarrow \mathcal{O}_{\mathrm{Gr}(3,7)}.$$

Since $\mathcal{K}$ is not a vector bundle, we cannot stratify X_6^2 via degeneracy loci as in the proof of Lemma 4.4.3. Therefore, following [IM16, Page 8], we extend $\mathcal{K}$ to a vector bundle on $\mathrm{Bl}_F Z$.

On the exceptional divisor $E = \mathbb{P}_F(\mathcal{K}_{|F})$, the sheaf $\mathcal{K}_{|F}$ fits into the following exact sequence

$$0 \to \mathcal{E} \to \mathcal{K}_{|F}(-1) \to \mathcal{O}_F \to 0.$$

Define a vector bundle $\mathcal{F}$ on Y_6 by

$$\mathcal{F}_{|Z\setminus F} = \mathcal{K}_{|Z\setminus 3}, \qquad \mathcal{F}_{|F} = \mathcal{E}.$$

Thus, $\pi_2(X_6^3)$ is obtained as the zero locus of a section of $\bigwedge^2 \mathcal{F}^\vee$ inside $\mathrm{Bl}_F Z$. Since $\pi_2 : \mathrm{Bl}_F Z \to F$ is the blow-up contraction, we obtain

$$\pi_2(X_6^3) = K_6^3,$$

which describes X_6^3 as resolution of K_6^3. Moreover, the exceptional locus of X_6^3 is $\mathscr{Z}(\bigwedge^2 \mathcal{F}^\vee) \subset E$. ☐

Remark 4.4.14. *This variety has the same Hodge diamond as X_4^1, but is not clear yet if the two varieties are isomorphic. In particular, in this case there is no evident dP_6-fibration on $\mathbb{P}^2$.*

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