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TOWARDS A LOGIC-BASED APPROACH TO LEGAL CASE-BASED REASONING

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Abstract

The thesis develops formal, logic-based models of legal case-based reasoning (CBR). Broadly, CBR is a reasoning paradigm in which decisions or conclusions are drawn by analogy with previously decided cases, collected in a case base. It constitutes a central form of legal reasoning in common law systems, and its formalization is increasingly significant for the theoretical foundations of AI and Law. Although several formal frameworks for legal CBR have been proposed, they typically rest on strong assumptions that limit their applicability to real legal practice. These include: A1) that legal cases can be represented as sets of case features (e.g. factors) favoring one of the parties, where the directionality of the features is known a priori; A2) that factual relevance alone determines whether a precedent constrains a decision; and A3) that no conflicts among precedents arise in the decision of new cases. This thesis systematically explores the consequences of abandoning these assumptions. First, it investigates whether it is possible to identify ex post which features of cases behave as factors and determine their directionality. Second, it examines which precedents are binding for a given case, considering both factual relevance and the jurisdictional relations among the deciding courts. Finally, it acknowledges the possibility of conflicts among binding precedents and analyses how such conflicts can be resolved in a legally grounded way. The proposed account is formalized within the Modal Logic of Binary Input Classifiers (BCL) along with suitable extensions introduced in this work.

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Introduction

The Topic of the Thesis: Legal CBR

This thesis develops formal models of legal Case-Based Reasoning (legal CBR). CBR is a form of reasoning that occurs whenever decisions or conclusions are reached by referring to previously solved cases, collected in a *case base*. This approach assumes significant importance in the legal domain, especially in contexts where judicial reasoning frequently relies on established precedents. The phenomenon is particularly evident in common law systems, where judicial precedents constitute a source of law and bind future decisions in similar cases. The principle of *stare decisis et non quieta movere* — the duty to follow precedents — underlines the central role of CBR in common law systems.

Since the early days of the field, a substantial body of research in AI & Law has sought to formalize legal CBR — that is, to develop computational models capable of reproducing the way decision-makers reason by analogy with precedents. In existing models of legal CBR, a typical task involves determining whether a new case should be decided in favor of one party or the other — typically the *plaintiff* or the *defendant* — based on its similarity to previously decided cases.

Why Study Legal CBR?

Beyond its intrinsic theoretical interest, legal CBR provides a rigorous theoretical foundation for interpreting and validating the results of modern machine learning systems used in so-called *predictive justice*. Many machine learning algorithms employed to

predict the outcomes of legal disputes operate by providing the outcome of a new case based on large datasets of past judicial decisions.¹ In this sense, these systems perform an abstract form of CBR, as they ground their predictions on a *case base* of precedents. Naturally, this is not to claim that such algorithms explicitly implement CBR methods. However, the functional relationship between the input (the case) and the output (the predicted decision) can be interpreted as a form of case-based inference.

This insight is significant because formal models of legal CBR, can serve as a *foundational framework* for verification methods aimed at evaluating the normative soundness of machine learning algorithms in predictive justice.² Evaluating such algorithms requires, first and foremost, verifying that they conform to certain minimum normative constraints. In common law systems, one of the most fundamental of these constraints is the *precedential constraint*—the requirement that judicial decisions respect binding precedents. Notably, some existing formal models of legal CBR—particularly the *reason and result models* developed by Horty [50]—explicitly represent forms of precedential constraint within a rigorous formal framework.

The Problem of Strong Assumptions in Legal CBR Models

Legal CBR models that aim to provide a credible framework for theoretical inquiry and practical verification must rest on realistic assumptions regarding the representation of cases, the structure of precedential relations, and the resolution of normative conflicts.

However, most existing models are based on strong assumptions that are excessively restrictive when compared to actual legal practice. Some of these assumptions concern the problem of how legal cases should be represented, including:

- A1) Legal cases can be represented as sets of *case features* (hereafter referred to as *factors*), each corresponding to a legally relevant factual pattern that favors one

¹For a technical overview of some developments in predictive justice and legal judgment prediction, see for example [42, 68, 29]. Examples of predictive-justice tools include COMPAS [6, 57] in the United States (criminal risk assessment), HART [76] in the United Kingdom (reoffending-risk prediction), and Split-Up in Australia (property division in divorce) [106].

²A similar intuition, with a more specific focus on the justification and explanation of machine learning outcomes, underlies works by Prakken and Ratsma [81], Odekerken et al. [74], and van Woerkom et al. [100].

of the two parties (e.g., the plaintiff or the defendant). The set of *factors* and their *directionality*— that is, which party they favor — are known *a priori*.

For example, within the HYPO model [5], a simple case involving an allegation by the plaintiff that the defendant misappropriated a trade secret might be described in terms of two factors: “the defendant obtained the secret by deceiving the plaintiff” and “the product is reverse-engineerable”. The first factor favors the plaintiff, while the second favors the defendant.

Such a factor-based representation presupposes that cases have been previously annotated in terms of factors by domain experts, thereby giving rise to the well-known *knowledge acquisition bottleneck* problem [40].

Some existing models of legal CBR—most notably the *reason and result models* by Horty —also rely on two additional assumptions:

- A2) Factual similarity (or comparison) *alone* is taken to determine whether a precedent case exerts a constraining effect on the decision in a given case (e.g., by comparing the factors present in each case).
- A3) The case base is constructed so that no conflict of precedents may occur — i.e., there are no situations in which, for the same case to be decided, two precedents force opposite outcomes (e.g., one favoring the plaintiff and another the defendant).

Both assumptions are problematic. In actual legal reasoning, the binding force of precedent depends not only on factual similarity (or comparison) but also on institutional aspects: a court is bound by another court’s decision only when the latter possesses binding authority over the former. For example, in common law systems, higher courts generally bind the decisions of lower courts, but not vice versa. Moreover, conflicts among precedents are endemic to legal practice, and both legal doctrine and jurisprudence have developed principles for their resolution.

Research Questions

Building on these observations, this thesis proposes to abandon the strong assumptions underlying traditional legal CBR models in order to more accurately capture the complexity of legal reasoning.

Specifically, the research is guided by the following research questions, which mirror the assumptions A1–A3:

- Q1)** *Is it possible to identify ex post which features of cases in the case base effectively play the role of factors, and what their directionality is?*
- Q2)** *Which precedents are binding for the case at hand, considering both the factual relevance between the cases and the jurisdictional relations among the courts that decided them?*
- Q3)** *How can conflicts among binding precedents be resolved in a legally grounded way?*

The Framework: The Modal Logic of Classifiers (BCL)

The answers to these questions are developed within the framework of the Modal Logic of Binary Input Classifiers (BCL) introduced by Liu and Lorini [63], along with suitable extensions proposed in this thesis.

BCL belongs to a broader line of research that seeks to develop a logic-based approach to explainable AI (XAI), in particular to explainable machine learning classifiers. It can express some of the most relevant notions of explanation in symbolic XAI, such as *abductive*, *contrastive*, and *counterfactual explanations*.

Our interest in BCL is motivated by two main reasons:

1. **Correspondence with Horty’s Model.** In previous work [64], Liu and others established a correspondence between Horty’s models of *precedential reasoning*

and BCL, providing a strong conceptual foundation. This correspondence makes explicit the intuition outlined above that there exists a connection between legal CBR and machine learning systems (in particular classifiers).

2. **Explanatory Capacity.** BCL offers a framework endowed with a formal notion of *counterfactual explanation*, which will be essential for answering question Q1.

More generally, the choice to operate within a modal-logical framework is theoretically motivated. As observed by Blackburn et al. [18], “modal languages are simple yet expressive languages for talking about relational structures”. Accordingly, a modal-logical framework offers the expressive resources necessary to capture the dependencies and connections among the elements of the domain under investigation. In this setting, individual cases can be represented as states within a relational structure, enabling the formal modeling of various types of relations among cases—such as factual relevance, temporal sequence, or the hierarchical and binding relations between the courts that decide the cases.

This relational perspective proves particularly valuable for addressing questions Q2–Q3, since, as we shall see, determining which cases function as precedents and which possess binding authority depends on the underlying relations of relevance, temporality, and hierarchy among cases. From a different perspective, adopting a modal logic framework allows us to articulate the conditions that render a decision *necessary*—that is, when it ought to be made in virtue of a necessity that is at once temporal, jurisdictional, and grounded in relevance.

Roadmap to the Thesis

The thesis is organized into four main chapters:

- **Chapter 1 — Background.** Provides an overview of factor-based models of legal CBR, with particular attention to Horty’s *Reason and Result Model*, and introduces BCL and its correspondence with Horty’s framework.

- **Chapter 2 — Finding Factors (Q1).** Proposes a model capable of inferring *ex post* which features of cases are genuinely discriminative and in what direction. Through counterfactual analysis (“what would have happened if the feature were absent?”), the model constructs a taxonomy distinguishing *factors*, *irrelevant features*, and *ambiguous features*, thus enabling reasoning even in the absence of pre-labeled information.
- **Chapter 3 — When Precedents Clash (Q2–Q3).** Extends the BCL framework to represent binding relations among cases and to formally model conflicts between binding precedents. The developed modal logic makes relevance, temporal, and jurisdictional relations among cases explicit, and provides principles for conflict resolution that are consistent with legal practice.
- **Chapter 4 — Conditional Judicial Authority.** Refines the model introduced in the previous chapter by elaborating a context-dependent notion of courts’ authority, determined by procedural conditions and subject-matter competence.

Origin of the Material

The chapters of this thesis are based on previously published articles or on work currently in preparation.

- **Chapter 2** is based on the following publications:

Cecilia Di Florio, Xinghan Liu, Emiliano Lorini, Antonino Rotolo, and Giovanni Sartor. *Finding Factors in Legal Case-Based Reasoning*. In *Logics for AI and Law (LAIL-23)*, pp. 175–192. College Publications, 2023.

Cecilia Di Florio, Xinghan Liu, Emiliano Lorini, Antonino Rotolo, and Giovanni Sartor. *Inferring New Classifications in Legal Case-Based Reasoning*. In *Proceedings of JURIX 2023*, pp. 23–32. IOS Press, 2023.

- **Chapter 3** is based on the following publications:

Cecilia Di Florio, Huimin Dong, and Antonino Rotolo. *When Precedents Clash*. In *Proceedings of JURIX 2024*, pp. 37–47. IOS Press, 2024.

Cecilia Di Florio, Huimin Dong, and Antonino Rotolo. *A Modal Logic for Temporal and Jurisdictional Classifier Models*. In *Proceedings of PRIMA 2025*, pp. 251–259. Springer Nature, 2025.

- **Chapter 4** is based on work currently in preparation by Cecilia Di Florio, Huimin Dong, and Antonino Rotolo.

Use of Generative AI. ChatGPT – from August 2025 to February 2026, including GPT-4o and subsequent GPT-5 series models– was used to generate/refine the figures and tables included in this thesis, based on specific prompts and subsequent revisions by the author. It was also employed for English-language editing and, in some passages, for translation into English, with further revisions by the author.

1

Background

In this chapter, we introduce background material that will be used throughout the thesis. Section 1.1 presents factor-based models for legal CBR, and in particular, Section 1.1.1 discusses the reason-and-result model formalized by Horty. Section 1.2 reviews the modal logic BCL, introduced by Liu and Lorini. Section 1.3 examines the relationship between Horty’s framework and the BCL framework. Finally, Section 1.4 clarifies a conceptual assumption that will be adopted throughout the thesis.

1.1 Factor-Based Legal Case Based Reasoning

In this section, we provide a brief overview of factor-based models of legal CBR. In doing so, we recall assumption A1 mentioned in the Introduction.

From the very beginning, much of the work in AI & Law has focused on formal and computational models for legal CBR. A common task addressed across these models concerns the decision of whether a case should be resolved in favor of one of two opposing parties, based on the analogy or similarity between the case at hand and previously adjudicated cases. The two opposing parties typically considered—and which we shall adopt throughout—are those found in civil proceedings, namely the *plaintiff* and the *defendant*.

We emphasize some aspects of legal CBR models, which are tied to broader foundational challenges in AI & Law. These challenges concern both the representation of

legal knowledge and how “to acquire information with which to populate the knowledge representations” [11].

With respect to the representation of legal knowledge—and in particular the representation of legal cases—a widely adopted approach in CBR traces back to the seminal work on the HYPO system for US trade secrets law [9, 87]. Within this tradition,

- A1) legal cases are represented as sets of *case features*, each corresponding to a legally relevant factual pattern that favors one of the two parties. The set of *features* and their *directionality*—that is, which party they favor—are known *a priori*.

Particular emphasis has been placed on Boolean legal features, commonly referred to as *factors*. For example, following the HYPO framework, a simple case concerning an allegation of trade secret misappropriation by the defendant might be described in terms of two factors: “the defendant obtained the secret by deceiving the plaintiff” and “the product is reverse-engineerable.” The first factor favors the plaintiff, while the second favors the defendant.

Regarding the acquisition of legal knowledge, especially in application-oriented models, this assumption leads to the well-known knowledge acquisition bottleneck, since it requires, for example, human experts to manually examine the cases in the input case base and index them according to the relevant factors.

The factor-based approach has been extended in many directions, resulting in a wide variety of models. Some are more application-oriented, while others are more formally oriented. Binary factors have been generalized into dimensions, i.e., multivalued features that support an outcome to varying degrees depending on their value in the case at hand (for a detailed discussion of the relation between factors and dimensions, see [88, 51]). Hierarchies (ontologies) of factors at different levels of abstraction have also been proposed (see e.g., [4, 5]).

Prakken and Sartor [82], while maintaining a factor-based perspective, showed that rule-based and case-based reasoning can be combined, as the set of precedents may be represented as a set of rules together with priorities among them. The use of rules also enabled the introduction of argumentation schemes to support CBR [104, 83].

Moreover, rules made it possible to develop formal accounts of *precedential constraint* in legal CBR, as shown by Horty and Bench-Capon [50, 53] and Rigoni [86].¹

In particular, in the next section we will focus on two well-known factor-based models of the *precedential constraint* proposed by Horty: the *reason model* and the *result model* [50].

1.1.1 Horty's Models of Precedential Constraint

In this section, we draw on Horty's *reason* and *result* model [50], while departing from the original notation in favor of the framework introduced in [64], in order to ensure consistency throughout the work. The *reason* and *result* models formalize a foundational concept in common law—*precedential constraint*—by explicitly defining the conditions under which the decision in a new case is determined by a body of prior cases.

In line with the tradition established by HYPO, we consider civil cases involving two opposing parties: the plaintiff and the defendant. Therefore, we assume that the outcome o of an assessed case can take the value 1 or 0, indicating, respectively, that the case was decided in favor of the plaintiff or the defendant. Hence, $o \in \{0, 1\}$.

Then, we have a finite set of factual features `Facts` used to describe cases.

Assumption A1 in Horty's models

Horty's models adopt assumption A1. The factual features correspond to factors, which we recall are Boolean case features. Also, factors are *a priori* partitioned between pro-plaintiff factors (*Plt*) and pro-defendant factors (*Dfd*). Namely, we have $\text{Facts} = \text{Dfd} \uplus \text{Plt}$.²

From now on we adopt following notation: for $o \in \{0, 1\}$, $\bar{o} = 1 - o$. Also, Facts^o , will denote *Plt* if $o = 1$, and *Dfd* if $o = 0$.

¹The history of legal CBR is extensive. We have outlined only a few developments here. Please refer to [89, 10, 14], for a more extensive discussion. Also, some developments of CBR models have been published over the years in the proceedings of ICAIL and JURIX, as well as in the journal *Artificial Intelligence and Law*.

² $\uplus$ denotes disjoint union.

A *fact situation* s , of the kind presented within legal case, is a particular subset of factors: $s \subseteq \text{Facts} = \text{Plt} \uplus \text{Dfd}$.

The main difference between the reason and result models lies in the role that the factors of the case play in determining the outcome of the case.

Within the result model, *all* factors appearing in a case are assumed to be significant for its outcome. In this context, a *precedent case* can be seen as a triple $k = (s, X, o)$, where s is a fact situation; $o \in \{0, 1\}$ is the outcome; X is the set of all pro-factors for o in s —namely $X = s \cap \text{Facts}^o$. Thus, $X = s \cap \text{Plt}$ if $o = 1$ and $X = s \cap \text{Dfd}$ otherwise.

In the reason model, on the other hand, *a subset* of the factors that favor the outcome of the case may be sufficient for the outcome of the case. In other words, it is possible to identify the *reason* for the case, namely a subset of the pro-factors which the decision makers considers sufficient to support that outcome, relative to all con-factors in the case. In this context, a *precedent case* can be seen as a triple $k = (s, X, o)$, where s is a fact situation; $o \in \{0, 1\}$ is the outcome; X , the *reason* of the case, is a subset set of all pro-factors for o in s , namely $X \subseteq s \cap \text{Facts}^o$. From now on, as in [64], we will consider the result model as a special type of reason model, where $X = s \cap \text{Facts}^o$. We will refer to a set of precedent cases within the reason or result model as a (*Horty*) *case base*.³

Example 1.1 (Running example I [64]). *In this section, we make use of a running example taken from [64], which in turn is adapted from [80]. Following the HYPO framework, the example concerns cases involving trade secret misappropriation. Such cases center on the plaintiff’s claim that the defendant engaged in the misuse of trade secrets. Cases of this kind can be represented in terms of factors, each of which either favors the outcome “misuse of trade secrets” (plaintiff wins) or “no misuse of trade secrets” (defendant wins). For our purposes, we draw on a selected subset of HYPO’s factors: the defendant obtained the secret information by deceiving the plaintiff (π_1) or by bribing an employee of the plaintiff (π_2); the plaintiff had taken security measures to safeguard the secret (π_3); the information is obtainable elsewhere (δ_1); the product is reverse-engineerable (δ_2); and the plaintiff had voluntarily disclosed the secret to outsiders (δ_3). Hence, in this example, $\text{Facts} = \{\pi_1, \pi_2, \pi_3, \delta_1, \delta_2, \delta_3\}$ with*

³In this section, we will simply use the term *case base*; the term *Horty case base* will be used in the following sections.

$Plt = \{\pi_1, \pi_2, \pi_3\}$ and $Dfd = \{\delta_1, \delta_2, \delta_3\}$. Let us consider a case base $CB = \{k_1, k_2\}$ where $k_1 = (\{\pi_1, \pi_3, \delta_1, \delta_3\}, \{\pi_1\}, 1)$; $k_2 = (\{\pi_2, \delta_1, \delta_3\}, \{\pi_1\}, 0)$; namely:

- k_1 has fact situation $s_1 = \{\pi_1, \pi_3, \delta_1, \delta_3\}$, reason $\{\pi_1\}$ and outcome 1;
- k_2 has fact situation $s_2 = \{\pi_2, \delta_1, \delta_3\}$, reason $\{\delta_3\}$ and outcome 0.

We now discuss the notion of precedential constraint, introduced by Horty. This notion is based on the idea, in line with the principle of *stare decisis*, that later courts must respect the decisions of earlier courts, in the sense that later decisions must be *consistent* with previous decision. We observe that, in every decided case, we know not only the outcome but also that the decision-maker has determined that the factors supporting the outcome—namely, the factors constituting the reason of the case (or, within the framework of the result model, all factors favoring the outcome)—prevail over the combination of all the factors against the outcome contained in the case. Each case therefore establishes a preference relation among set of factors.⁴

Definition 1.1 (Preference relation derived from a case). Let $k = (s, X, o)$ be a case. Then the *preference relation* $<_k$ derived from k is s.t. for any two set of factors Y, Y' favoring o and $\bar{o}$ respectively, $Y' <_k Y$ if and only if $Y' \subseteq s \cap \text{Facts}^{\bar{o}}$ and $X \subseteq Y$.

The preference derived from a case can be extended to the entire case base where the case belongs to.

Definition 1.2 (Preference relation derived from a case base). Let CB be a case base. Then the *preference relation* $<_{CB}$ derived from CB is s.t. for any two sets of factors Y, Y' favoring o and $\bar{o}$ respectively, $Y' <_{CB} Y$ if and only if $\exists k \in CB$ s.t. $Y' <_k Y$.

Intuitively, a decision is consistent with previous decisions in the case base if it does not introduce inconsistencies. In this context, from the previous definition we know that $Y' <_{CB} Y$, means that Y in favor of o is preferred to Y' in favor of $\bar{o}$. Hence, in order to preserve consistency in the decisions, we expect this preference relationship to be asymmetric: if $Y' <_{CB} Y$ then not $Y <_{CB} Y'$. Otherwise, the case base would contain at least two cases: one establishing that Y is preferred to Y' and the other establishing that Y' is preferred to Y .

⁴Following Definitions 1.1, 1.2, 1.3. except symbolic difference, are taken from on [64] and in turn based on [50, 80].

Definition 1.3 ((In)consistency). A case base CB is *inconsistent*, if there are two Y, Y' s.t. $Y' <_{CB} Y$ and $Y <_{CB} Y'$. CB is consistent if it is not inconsistent.

Given such notion of consistency, we can introduce the concept of *precedential constraint*. The intuition is that, in deciding a case, a court is required to preserve the consistency of the starting case base.

Definition 1.4 (Precedential constraint). Let CB be a consistent case base. The *precedential constraint* requires s to be decided with outcome o and reason $X \subseteq s \cap \text{Facts}^o$, so that the new case base $CB \cup \{(s, X, o)\}$ is still consistent.

Notice that the notions of consistency and precedential constraint are inherently related to the assumption that factors are partitioned into pro-plaintiff and pro-defendant. Without such a partition of factors, it would not be possible to establish any order of preference from the cases and from the case bases.

Remark 1.5 (A fortiori reasoning). We would like to point out that in [50] the notions of preference relation derived from a case (and case base) and precedential constraint are actually defined in the context of the reason model. The corresponding concepts within the result model are those of strength for a side and a fortiori constraint.⁵ We will use the terms preference relation and precedential constraint when referring to both models or specifically to the reason model. When we refer specifically to the result model, we use the term strength for a side and a fortiori constraint.⁶

Example 1.2. Consider two fact situations in the context of case base CB^{ex} in running Example 1.1.

- Given the new fact situation $s_3 = \{\pi_1, \pi_3, \delta_1\}$, only a decision for 1 is consistent with CB^{ex} , since a decision for 0 would entail that $\{\pi_1\} <_{CB^{ex}} \{\delta_1\}$, contrary to the preference $\{\delta_1\} <_{CB^{ex}} \{\pi_1\}$, which is derivable from k_1 .

⁵See Definition 9 pag. 22 and Definition 10 pag.23 in [50].

⁶Horty's notion of a *fortiori* differs from its ordinary meaning. A common-sense example of a *fortiori* reasoning is: "If I do not go out when it rains, I will certainly not go out when there is a deluge." However, such reasoning typically relies on implicit assumptions—for instance, the inference depends on background knowledge that a deluge entails rain. In Horty's model, by contrast, all relevant elements must be explicitly represented, and a fortiori reasoning is grounded in the formal ordering among these elements. The author thanks Giovanni Sileno for drawing attention to this aspect.

- Given the new fact situation $s_4 = \{\pi_2, \delta_2\}$, both $(s_4, \{\pi_2\}, 1)$ and $(s_4, \{\delta_2\}, 0)$ are consistent with CB^{ex} , since neither $\{\delta_2\} <_{CB^{ex}} \{\pi_2\}$ nor $\{\pi_2\} <_{CB^{ex}} \{\delta_2\}$.

The precedential constraint can serve as a mechanism for inferring (binding) decisions for new cases. To illustrate, in the previous example we saw that the precedential constraint requires s_3 to be decided as 1, because otherwise the preference relation derived from k_1 would be violated. In this sense, we will say that k_1 *forces* a decision as 1 for s_3 . This inferential mechanism can be made explicit in terms of subset-inclusion relations among the factors of cases. Specifically, for a case $k' = (s', X', o)$ to force the decision in a situation s in the direction of o , it is sufficient that its reason X' contain fewer factors supporting o than those present in s , and that s' contain more factors opposing o than s .

Definition 1.6 (Decision forced). Let s a fact situation to be decided. $k' = (s', X', o)$, *forces* a decision for s as $o \in \{0, 1\}$, *on the basis of precedential constraint*, noted $\text{forces}(k', s, o)$, iff $X' \subseteq s \cap \text{Facts}^o$ and $s \cap \text{Facts}^{\bar{o}} \subseteq s' \cap \text{Facts}^{\bar{o}}$.⁷

Remark 1.7 (A fortiori reasoning). Notice that, within the result model, a case is represented as $(s', s' \cap \text{Facts}^o, o)$, with $s' \subseteq \text{Facts}$ and $o \in \{0, 1\}$. The second component is redundant and can be derived directly from the first and third component. For this reason, in the context of the result model we can write directly cases as pairs (s', o) . On this basis, and following Remark 1.5 and Definition 1.6, we say that $k' = (s', o)$, *forces* a decision for s as $o \in \{0, 1\}$, by a fortiori reasoning, still noted $\text{forces}(k', s, o)$, iff $\text{forces}((s', s' \cap \text{Facts}^o, o), s, o)$. Hence, within the result model, $\text{forces}(k', s, o)$ iff $s' \cap \text{Facts}^o \subseteq s \cap \text{Facts}^o$ and $s \cap \text{Facts}^{\bar{o}} \subseteq s' \cap \text{Facts}^{\bar{o}}$.

Example 1.3. Recall from Example 1.2, that the only decision consistent for s_3 is 1, on the basis of k_1 . Indeed, we can verify that $\text{forces}(k_1, s_3, 1)$: $\{\pi_1\} \subseteq s_3 \cap \text{Facts}^1 = \{\pi_1, \pi_3\}$ and $\{\delta_1\} = s_3 \cap \text{Facts}^0 \subseteq s_1 \cap \text{Facts}^0 = \{\delta_1, \delta_3\}$.

⁷The relation *forces* is not explicitly introduced by Horty; it is our own definition. We introduce it here to highlight that the precedential constraint is based on the comparison of factual elements (i.e., factors). Similar definitions can be found in [80, 24].

Assumption A2 in Horty's models

Definition 1.6 and the previous example help clarify A2, one of the strong assumptions underlying the CBR models presented in the Introduction. Recall that we stated:

- A2) Factual similarity/comparison *alone* is taken to determine whether a precedent case exerts a constraining effect on the decision in a given case (e.g., by comparing the factors present in each case).

In Horty's models, the factual comparison used to determine whether a case constrains (or forces) the decision in another case is a comparison based exclusively on factors. This is precisely the relation we expressed through forces, defined solely in terms of subset-inclusion relations among (specific) factors of cases (or of their reasons). No additional elements are considered in determining whether a decision is forced.

We now highlight an aspect that will play a fundamental role in this work: the relation forces, we introduced in Definition 1.6, can be applied to any case base. However, note that, according to Definition 1.4, *in the context of the reason and result models proposed by Horty, the precedential constraint applies only to consistent case bases*. The assumption of consistency is crucial to avoid *conflicts of precedent* for a new case. In other words, the consistency assumption ensures that, for any new case s , it is not possible to retrieve two cases s' and s'' such that one forces the decision of s as 0 and the other forces it as 1.

Proposition 1.8. *Let CB a consistent case base, and s a fact situation to be decided. If $\exists k' \in CB$ s.t. $\text{forces}(k', s, o)$ then $\nexists k'' \in CB$ s.t. $\text{forces}(k'', s, \bar{o})$.*

Proposition 1.8 does not apply to inconsistent case bases. We illustrate this with an example involving an inconsistent case base in the result model.⁸

Example 1.4.⁹

We consider the same set of factors as in Example 1.1. Thus, the set of factors is $\text{Facts} = \{\pi_1, \pi_2, \pi_3, \delta_1, \delta_2, \delta_3\}$ with $\text{Plt} = \{\pi_1, \pi_2, \pi_3\}$ and $\text{Dfd} = \{\delta_1, \delta_2, \delta_3\}$.

⁸Similar examples can be provided for the reason model.

⁹Adapted from Example 5.9 in [25], pag. 154.

The case base is $CB = \{k_6, k_7\}$ where

- k_6 has fact situation $s_6 = \{\pi_1, \delta_1, \delta_2\}$, and outcome 1 ($k_6 = (s_6, 1)$);
- k_7 has fact situation $s_7 = \{\pi_1, \pi_2, \delta_1\}$, and outcome 0 ($k_7 = (s_7, 0)$);
- The new fact situation to be decided is $s_8 = \{\pi_1, \delta_1\}$.

The case base CB is inconsistent. From k_6 , we derive $\{\delta_1\} <_{CB} \{\pi_1\}$. From k_7 we derive $\{\pi_1\} <_{CB} \{\delta_1\}$. Also, on the basis of Definition 1.6, we know that $\text{forces}(k_6, s_8, 1)$ and $\text{forces}(k_7, s_8, 0)$. So, we have a conflict of precedents, on the basis of the a fortiori reasoning.

Assumption A3 in Horty's models

Proposition 1.8 and the previous example help clarify one of the strong assumptions usually adopted in legal CBR —specifically, assumption A3—that we stated in Introduction. In particular, recall that we formulated A3 as follows:

- A3) The case base is constructed so that no conflict of precedents may occur — i.e., there are no situations in which, for the same case to be decided, two precedents force opposite outcomes (e.g., one favoring the plaintiff and another the defendant).

In Horty's reason and result models the precedential constraint applies only to consistent case bases. Proposition 1.8 has shown that the assumption of consistency ensures that, for any given case, it is not possible to find two cases that force opposite decisions. Conversely, Example 1.4 has demonstrated that, in the presence of an inconsistent case base, it is possible to identify two cases that force opposite outcomes, thereby generating a conflict of precedents. In other words, we can instantiate A3 for Horty's reason and result models as follows: *The case base is constructed to be consistent so that no conflict of precedents may occur.*

We conclude this section by noting that some recent works define a precedential constraint in the context of inconsistent case bases. In particular, Canavotto [25] develops a deontic logic that allows consistent obligations for decisions to be derived even from

inconsistent case bases. In broad terms, within inconsistent case bases, it is permissible to decide a fact situation for one outcome whenever doing so does not introduce further inconsistencies. Conversely, a case base entails that it is obligatory to decide a fact situation for a outcome whenever every permitted decision within that case base supports that outcome. We illustrate this through two examples. Canavotto formulates the precedential constraint for the reason model; however, this constraint can be adapted to the result model, recalling that the result model may be viewed as a particular instance of the reason model. Accordingly, the two examples below will refer directly to the result model—a perspective that will be useful in Chapter 3.

Example 1.5. *Let us consider a case base $CB = \{k_9, k_{10}\}$ where*

- *k_9 has fact situation $s_9 = \{\pi_1, \delta_1, \delta_2\}$ and outcome 1;*
- *k_{10} has fact situation $s_{10} = \{\pi_1, \delta_1\}$ and outcome 0;*
- *The new fact situation to be decided is $s_{11} = \{\pi_1, \pi_2, \delta_1, \delta_2\}$.*

The case base CB is inconsistent: from k_9 , we obtain $\{\delta_1, \delta_2\} <_{CB} \{\pi_1\}$; while, from k_{10} we obtain $\{\pi_1\} <_{CB} \{\delta_1, \delta_2\}$.

Notice also that, from k_9 , $\{\delta_1, \delta_2\} <_{CB} \{\pi_1, \pi_2\}$. While, from case k_{10} we cannot derive $\{\pi_1, \pi_2\} <_{CB} \{\delta_1, \delta_2\}$.

Suppose now s_{11} is decided as 1. This would not introduce any further inconsistency in the case base: we would simply re-obtain $\{\delta_1, \delta_2\} <_{CB} \{\pi_1, \pi_2\}$.

Suppose instead s_{11} is decided as 0: we would obtain $\{\pi_1, \pi_2\} < \{\delta_1, \delta_2\}$. In this situation, we add a new inconsistency because of already established $\{\delta_1, \delta_2\} <_{CB} \{\pi_1, \pi_2\}$.

Then, according to [25], s_{11} ought to be decided as 1. Notice also that $\text{forces}(k_9, s_{11}, 1)$ but not $\text{forces}(k_{10}, s_{11}, 0)$.

The previous example illustrated the meaning of an obligatory decision in the context of [25]. We now present an example of opposite decisions, both of which are permitted according to [25].

Example 1.6. *Consider again Example 1.4.*

We know the case base CB is inconsistent. Also from k_6 , we derive $\{\delta_1\} <_{CB} \{\pi_1\}$. From k_7 we derive $\{\pi_1\} <_{CB} \{\delta_1\}$.

Suppose now s_8 is decided as 1. This would not introduce any further inconsistency in the case base: we would simply re obtain $\{\delta_1\} < \{\pi_1\}$.

Suppose now s_8 is decided as 0. This would not introduce any further inconsistency in the case base: we would simply re obtain $\{\pi_1\} < \{\delta_1\}$.

In this context, according to [25] the court is not obliged to find for one particular side but is permitted to find for either one side or the other.

We note from previous example that what we interpreted as a conflict of precedent corresponds, in Canavotto’s framework, to the existence of multiple permitted decisions. In such cases, where more than one decision is permissible, Canavotto suggests resolving the situation by considering the relative support for each option—that is, by weighing the number of cases that endorse or oppose a given decision. We will not enter into details. We emphasize two points. First, this solution assumes that all cases carry equal weight—that each is equally important. Second, the solution does not always suffice to determine which permissible outcome should be chosen. For instance, in Example 1.6, the number of cases supporting or opposing a decision for the plaintiff is the same as for the defendant—one for each side.

1.2 Binary Input Classifiers Logic – BCL

In recent years, research in the field of explainable AI (XAI) has flourished also as a response to interconnected issues in the AI domain such as trustworthiness and transparency of machine learning algorithms. In this direction, a number of machine-learning based, model-agnostic and local techniques have been developed on the basis of [84]. Many of these techniques have been explained by taking classifiers as a case

study.¹⁰ In response to these methods, a local logic-based approach to explainable classifiers has been pursued. In particular, Liu and Lorini in [63, 62] propose BCL, a logic for (binary-input) classifier models. Such logic models a binary-input classifier as a partition of an S5 Kripke model. Within this modal logic perspective the authors can express some of the most relevant notions of classifier explanation in symbolic XAI, i.e. *abductive explanation*, *contrastive explanation*, *counterfactual explanation*, and concepts of *bias* and *fairness*.

BCL is of particular interest for the purposes of this thesis because Liu et al. [64] established a correspondence between a subclass of classifier models introduced within BCL and the reason–result model proposed by Horty. This supports the intuition, presented in the Introduction, that there is a correspondence between machine learning algorithms—particularly classifiers—and case-based reasoning models.¹¹

In this section, we present the language and semantics of BCL (Section 1.2.1), and we also recall the notion of counterfactual conditionals as modeled in BCL (Section 1.2.2).¹²

Remark 1.9. *We assume familiarity with basic notions of modal logic. In particular, the notions of logical consequence, deductibility, consistency, soundness, and completeness are taken as standard. We will use the usual notation for basic normal modal logics, e.g., K, S4, S5 [18].*

1.2.1 BCL Language and Semantics

In this section, we introduce the language for modeling binary input classifiers and the related semantics [62]. Atm_0 is a *finite* set whose elements are used to represent input variables of a classifier. A finite set Val is the set of output values of a classifiers. Since we are interested in classifiers used in the legal domain, we assume that $Val = \{1, 0, ?\}$

¹⁰By classifiers, we mean algorithms/methods that categorize data into specific classes. For a more detailed discussion on classification in ML we refer to [44].

¹¹In particular, in Sections 1.3 and 2.3, we show that there exists a non-bijective mapping between models of case-based reasoning and a subclass of classifiers: every case base can be mapped to a classifier, while a classifier may correspond to *multiple* case bases.

¹²Following Definitions 1.10, 1.11, 1.12, 1.15, 1.16, 1.16 and Propositions 1.13, 1.17, 1.19, 1.20 except symbolic difference, are taken from on [62].

where elements stand for *plaintiff wins*, *defendant wins* and *absence of decision* respectively. For any $o \in Val$, $t(o)$ is a *decision atom*, to be read as “the actual decision (or output) takes value o ”. The set of decision atoms is $Dec = \{t(o) \mid o \in Val\}$. Finally $Atm = Atm_0 \cup Dec$ is the set of atomic propositions. As in previous sections, for $o \in \{0, 1\}$, the “opposite” value $\bar{o}$ denotes $1 - o$.¹³

The modal language $\mathcal{L}(Atm)$ of BCL is therefore defined as

$$\varphi ::= p \mid t(o) \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi$$

where p ranges over Atm_0 , $t(o)$ ranges over Dec . $\Box$ quantifies over all states (universal modality). Modal operator $\Diamond$ is the dual of $\Box$ and is defined as usual: $\Diamond\varphi =_{def} \neg\Box\neg\varphi$.

For any $X \subseteq Y \subseteq Atm_0$, the following definition syntactically specifies a valuation on Y in which all variables in X are assigned true, while all remaining variables in Y are assigned false.

$$cn_{X,Y} =_{def} \bigwedge_{p \in X} p \wedge \bigwedge_{p \in Y \setminus X} \neg p.$$

The language $\mathcal{L}(Atm)$ is interpreted relative to classifier models defined as follows.

Definition 1.10 (Classifier model). A classifier model (CM) is a pair $C = (S, f)$ where:

- $S \subseteq 2^{Atm_0}$ is a set of states and
- $f : S \longrightarrow Val$ is a decision (or classification) function.

The class of classifier models is noted **CM**.

A pointed classifier model is a pair (C, s) with $C = (S, f)$ a classifier model and $s \in S$. Formulas in $\mathcal{L}(Atm)$ are interpreted relative to pointed classifier models.

¹³ Note that the opposite value is defined only for 1 and 0, and not for ?. For example, defining the opposite of 1 as 0 allows us to indicate when a decision has been made for the side opposite to the plaintiff, namely, in favour of the defendant. The same holds for 0. In this sense, it is meaningless to define the opposite of ?, since it denotes an undecided case.

Definition 1.11 (Satisfaction relation). Let (C, s) be a pointed classifier model with $C = (S, f)$ and $s \in S$. Then:

$$\begin{aligned}
(C, s) \models p &\iff p \in s, \\
(C, s) \models \mathbf{t}(o) &\iff f(s) = o, \\
(C, s) \models \neg\varphi &\iff (C, s) \not\models \varphi, \\
(C, s) \models \varphi \wedge \psi &\iff (C, s) \models \varphi \text{ and } (C, s) \models \psi, \\
(C, s) \models \Box\varphi &\iff \forall s' \in S : (C, s') \models \varphi.
\end{aligned}$$

A formula φ of $\mathcal{L}(Atm)$ is satisfiable relative to the class **CM** if there exists a pointed classifier model (C, s) such that $(C, s) \models \varphi$. It is said to be valid if $\neg\varphi$ is not satisfiable relative to **CM** and noted as $\models_{\mathbf{CM}} \varphi$.

Definition 1.12 (Axiomatics of BCL). BCL is the extension of classical propositional logic given by the following axioms and rules of inference:

$$\begin{aligned}
&(\Box\varphi \wedge \Box(\varphi \rightarrow \psi)) \rightarrow \Box\psi && (\mathbf{K}_{\Box}) \\
&\Box\varphi \rightarrow \varphi && (\mathbf{T}_{\Box}) \\
&\Box\varphi \rightarrow \Box\Box\varphi && (\mathbf{4}_{\Box}) \\
&\varphi \rightarrow \Box\Diamond\varphi && (\mathbf{B}_{\Box}) \\
&\bigvee_{o \in Val} \mathbf{t}(o) && (\mathbf{AtLeastValue}) \\
&\mathbf{t}(o) \rightarrow \neg\mathbf{t}(o') \text{ if } o \neq o' && (\mathbf{AtMostValue}) \\
&\bigwedge_{Y \subseteq \text{fin}Atm_0} \left((\text{cn}_{Y, Atm_0} \wedge \mathbf{t}(o)) \rightarrow \Box(\text{cn}_{Y, Atm_0} \rightarrow \mathbf{t}(o)) \right) && (\mathbf{Funct}) \\
&\frac{\varphi \rightarrow \psi, \varphi}{\psi} && (\mathbf{MP}) \\
&\frac{\varphi}{\Box\varphi} && (\mathbf{Nec}_{\Box})
\end{aligned}$$

$\Box$ is a S5 operator. **AtLeastValue**, **AtMostValue**, **Funct**, syntactically represent the decision function [63], so that every expression cn_{Y, Atm_0} is mapped into a unique $\mathbf{t}(o)$.

Proposition 1.13 (Completeness). *BCL is sound and complete with respect to CM.*

Some comments on BCL are stated in order.

First, the logic is called *Binary Input Classifiers Logic* because the inputs of the classifiers are binary, in the sense that for each input variable p of a classifier, we know whether p is true in a given state or not. In other words, we cannot have multi-valued input variables.

Second, what we refer to here as BCL is actually a variant of the logic proposed by Liu and Lorini [63, 62], adapted to better represent legal CBR. The differences between this variant and the original version are as follows:

- the original BCL can have as finite many decision atoms Dec as possible. For the purpose of the work here we specify Dec as cardinality three.¹⁴
- The original BCL may also have a countable set of input atoms Atm_0 . But in this case the functionality axiom **Funct** can not be adopted.
- In BCL is actually introduced a *selectis paribus* operator $[X]\phi$. In this work, however, we require only the universal modality $\Box$, since we will not make use of $[X]\phi$, and $[X]\phi$ can moreover be derived from the universal modality (see the reduction axiom in [63, 62]).

Finally, we point out that the set Atm of propositional variables in the language is assumed to be *finite*. This assumption may appear unusual from the perspective of modal logic. There are two justifications for adopting it.

The first justification arises from the particular application of BCL that concerns us—namely, representing models of legal CBR within BCL. One result in this direction was already established by Liu et al. [64], according to which Horty’s models can be mapped onto a subclass of classifier models. We will return to this result briefly in Section 1.3. Within the framework of legal CBR, it is common to treat (sometimes implicitly) the set of factors/facts as finite. Recall that Atm_0 describes the input variables of the classifiers. These will correspond to the factors of a case when representing Horty’s models. More precisely, in Section 1.3, we will assume that $Atm_0 = \text{Facts} = \text{Plt} \uplus \text{Dfd}$.

¹⁴Moreover, we established that classifiers should allow for *incomplete-knowledge*: not all states are necessarily associated with a decision in favour of the defendant or the plaintiff. Namely, a state, besides 0 and 1, can also be classified as ?, where ? means absence of decision. In this sense, the classifier covers together the meta-level (no decision is taken) and the object level (decision for the plaintiff/decision for the defendant).

Hence, if the assumption of finiteness of factors is accepted, Atm_0 can be assumed finite.¹⁵ Since the set Atm of propositional atoms is given by the union of the finite set of decision atoms (Dec) and Atm_0 , it is natural to regard Atm itself as finite.

The second justification arises from a technical consideration: the finiteness of atoms allows us to introduce pointers for states. More precisely, since Atm_0 is finite, for any $X \subseteq Y \subseteq Atm_0$ we can guarantee that $cn_{X,Y}$ is well defined. This observation is particularly relevant because, in particular, for any $X \subseteq Atm_0$, the formula cn_{X,Atm_0} is satisfied by a unique state s , namely $s = X$. In this sense, cn_{X,Atm_0} serves as a pointer to a specific state. More precisely, we have the following proposition.

Proposition 1.14. *Let $s \in S$. $(C, s) \models cn_{X,Atm_0}$ iff $s = X$.*

1.2.2 A counterfactual conditional

In BCL a simple notion of counterfactual conditional, inspired from Lewis' notion [59], can be formalised.

Definition 1.15 (Counterfactual conditional). We write $\varphi \Rightarrow \psi$ to mean that *if it were φ it would be ψ* and define it as follows:

$$\varphi \Rightarrow \psi =_{def} \bigwedge_{0 \leq k \leq |Atm_0|} \left(\maxSim(\varphi, Atm_0, k) \rightarrow \bigwedge_{Y \subseteq Atm_0: |Y|=k} [Y](\varphi \rightarrow \psi) \right),$$

with

$$\maxSim(\varphi, Atm_0, k) =_{def} \bigvee_{Y \subseteq Atm_0: |Y|=k} \langle Y \rangle \varphi \wedge \bigwedge_{Y \subseteq Atm_0: k < |Y|} [Y] \neg \varphi.$$

To provide the semantic interpretation of $\varphi \Rightarrow \psi$, the following notion of similarity between states is needed.

Definition 1.16 (Similarity between states). Let $C = (S, f)$ be a classifier model, $s, s' \in S$. The degree of similarity between s and s' in S relative to the set of features Atm_0 , noted $sim_C(s, s', Atm_0)$, is defined as follows:

$$sim_C(s, s', Atm_0) = |\{p \in Atm_0 : (C, s) \models p \text{ iff } (C, s') \models p\}|.$$

¹⁵We return to this aspect in Section 2.5.

The following proposition shows that the conditional $\varphi \Rightarrow \psi$ holds in a state if all closest states to the actual state, in which the antecedent is true, satisfy the consequent of the conditional. In this sense $\varphi \Rightarrow \psi$ captures the intended meaning *if it were φ it would be ψ* .

Proposition 1.17. *Let $C = (S, f)$ be a classifier model, $s \in S$. Then, $(C, s) \models \varphi \Rightarrow \psi$ if and only if $\text{closest}_C(s, \varphi, \text{Atm}_0) \subseteq \|\psi\|_C$, where*

$$\text{closest}_C(s, \varphi, \text{Atm}_0) = \arg \max_{s' \in \|\varphi\|_C} \text{sim}_C(s, s', \text{Atm}_0),$$

and for every $\varphi \in \mathcal{L}(\text{Atm})$:

$$\|\varphi\|_C = \{s \in S : (C, s) \models \varphi\}.$$

As argued in [62], it can be shown that $\Rightarrow$ satisfies all the semantic conditions of Lewis' VC logic for the counterfactual conditional [59].

Counterfactuals play a crucial role in the notion of explanation in philosophy in general, and in the field of explainable machine learning (XAI) in particular. In XAI, counterfactual explanations are used to account for the outcomes of machine learning algorithms. A counterfactual explanation describes a causal situation of the form: “If (X) had not occurred, (Y) would not have occurred” [69]. In this setting, “a counterfactual explanation of a prediction describes the smallest change to the feature values that changes the prediction to a predefined output” [69]. In this sense, much research in XAI considers counterfactual explanations to be related [103], or interchangeable [93], with contrastive explanations. The latter highlight the minimal aspect of a situation whose modification would falsify the current decision [62]. The correspondence between counterfactual and contrastive explanations is formally analyzed in [62].¹⁶

1.3 Relation between BCL logic and Horty models

Liu et al. [64] identified a correspondence between Horty's reason and the result models and classifier models for BCL. More precisely, the work in [64] showed that every

¹⁶Thanks to Giovanni Sileno for highlighting the need to introduce a comparison with counterfactual reasoning.

consistent Horty case base can be represented by a *set* of classifiers models.¹⁷ Later, Liu et al. [35, 62] obtained a stronger representation theorem, showing that every consistent Horty case base admits a *unique* classifier representation. This unique classifier is called the *genuine* classifier of the case base. Here we report the definition of genuine classifier and the related representation result presented in [62, 35], referring to [62, 64], for further discussion.

The language of BCL need to be adapted for mapping Horty case based into the framework of classifiers. Specifically, the set of atomic proposition is settled to be finite and partitioned in pro-plaintiff factors and pro-defendant factors. Namely $Atm_0 = Facts = Plt \uplus Dfd$. We will use following notation: $Atm_0^o = Facts^o$, to denote Plt if $o = 1$, and Dfd if $o = 0$. Recall, for $o \in \{0, 1\}$, the “opposite” $\bar{o}$ is noted for the value $1 - o$.¹⁸

It can be shown that every consistent Horty case base can be transformed into a unique *genuine* classifier. The basic intuition behind the notion of a genuine classifier representing a case base is twofold. First, each state s of the classifier corresponds to a factual situation of a case, and the outcome assigned to s by the classification function f is the outcome of that case. Second, the genuine classifier represents the case base *closed under the precedential constraint*: it outputs ? for all, and only those, factual situations that cannot be forced to yield a decision by the case base and the precedential constraint. That is, $f(s) = ?$ if and only if there is no case that forces a decision for s .

Definition 1.18 (Genuine classifier). Let CB be a consistent Horty case base. The *genuine classifier* of CB is the function $f : 2^{Atm_0} \rightarrow \{0, 1, ?\}$, s.t. for any situation s ,

$$f(s) = \begin{cases} o & \text{if } \exists k' = (s', X, o) \in CB \text{ s.t. forces}(k, s, o) \\ ? & \text{otherwise.} \end{cases}$$

The genuine classifier model of CB is therefore $C = (2^{Atm_0}, f)$.¹⁹

We can also define the genuine classifier model for a Horty consistent case base in terms of a formula.

¹⁷Recall that in Section 1.1.1 we stated that a *Horty case base* is any set of precedent cases within the reason or result model.

¹⁸The opposite is defined only for 0, 1, see Footnote 13.

¹⁹We adapted the Definition in [35, 62] to our notation.

Proposition 1.19. *Let CB be consistent Horty case base. Then a classifier model $C = (S, f)$ is the genuine classifier of CB , if $\forall s \in S, (C, s) \models \varphi_{CB}$ where*

$$\varphi_{CB} =_{def} \bigwedge_{X \subseteq Atm_0} \Diamond \text{cn}_{X, Atm_0} \wedge \bigwedge_{o \in \{0,1\}} \Box \left(t(o) \leftrightarrow \bigvee_{k=(s', X, c) \in CB} \text{cn}_{X, X \cup (Atm_0^o \setminus s')} \right).$$

Looking more closely at the formula φ_{CB} , we note that the second conjunct of φ_{CB} , formalizes the closure under precedential constraint. The first conjunct, $\bigwedge_{X \subseteq Atm_0} \Diamond \text{cn}_{X, Atm_0}$, requires that *every possible* situation description be satisfied by the classifier, where a situation description is a conjunction of factors (those present, X) and negations of factors (those absent, $Atm_0 \setminus X$). It is easy to verify that $\bigwedge_{X \subseteq Atm_0} \Diamond \text{cn}_{X, Atm_0}$ holds in a state of a classifier $C = (S, f)$ iff $S = 2^{Atm_0}$. From now on, we will say that in such a case the classifier is *situation-complete*.

Example 1.7. *The genuine classifier model of the case base in Example 1.1 is $C^{ex} = (2^{Atm_0}, f^{ex})$. Where,*

$$f^{ex}(s) = \begin{cases} 1 & \text{if } \text{forces}(k_1, s, 1) \\ 0 & \text{if } \text{forces}(k_2, s, 0) \\ ? & \text{otherwise.} \end{cases}$$

Proposition 1.20. *Every consistent Horty case base induces exactly one genuine classifier model.*

1.4 Issues and Concerns: Conceptual Clarification

We would like to clarify an assumption that we adopt throughout this thesis.

The decision of a legal case may involve the resolution of multiple legal *issues*, and each issue may, in turn, be divided into *sub-issues*. In the legal CBR literature, the sub-issues are typically referred to as *concerns*. To clarify these notions, let us consider some examples.

An issue typically corresponds to a distinct legal question or point of law that the court must determine—for instance, whether there was a misappropriation of a trade secret,

whether a contract was breached, whether negligence occurred, or whether a trademark was infringed.

A concern, on the other hand, refers to a sub-issue or a specific component of reasoning within an issue. Concerns represent intermediate stages of legal analysis, connecting factual assessments to partial conclusions that ultimately contribute to resolving the broader issue. For example, in trade secret law, concerns related to the issue of misappropriation may include whether the information in contention qualifies as a trade secret and whether the plaintiff took reasonable measures to maintain secrecy.

Several approaches to the representation of issues and concerns in legal CBR models have been proposed (see, among many others, [20, 16, 15, 26, 27, 23, 4, 98, 99]). Broadly speaking, these approaches adopt a multi-step reasoning process, in which the resolution of all relevant issues and concerns is necessary to determine the outcome of the case as a whole.

It is worth noting that in Horty's reason and result models, it is implicitly assumed that the cases under consideration involve a single issue and a single concern within that issue. In these models, the decision-making process is single-step, leading directly from the facts of the case to its outcome. In this thesis, we follow such approach and maintain the assumption that *our analysis focuses on a single issue and a single concern within that issue*.

2

Finding Factors

As highlighted in the previous chapter, in factor-based models of CBR, an assumption related to the problem of legal knowledge representation have been adopted:

- A1) legal cases are represented as sets of *case features* (factors) each corresponding to a legally relevant factual pattern that favors one of the two parties. The set of *factors* and their *directionality*— that is, which party they favor — are known *a priori*.

Furthermore, particularly in Horty's reason and result models, a further assumption is made on case bases, also in order to infer new decisions for new cases:

- A2) Factual similarity (or comparison) *alone* is taken to determine whether a precedent case exerts a constraining effect on the decision in a given case (e.g., by comparing the factors present in each case).
- A3) The case base is constructed so that no conflict of precedents may occur — i.e., there are no situations in which, for the same case to be decided, two precedents force opposite outcomes (e.g., one favoring the plaintiff and another the defendant).

In particular, with respect to A2, in Horty's models a comparison of factors determines whether one case forces the decision in another. As for A3, it can be instantiated for

Horty's models as follows: *The case base is constructed to be consistent so that no conflict of precedents may occur.*

In this chapter, we abandon assumption A1, and A3, while retaining A2. We do so by pursuing the idea that case-based reasoners are, in essence, binary input classifiers; accordingly, we adopt the framework of BCL. In particular, in Sections 2.1 and 2.2, we start by abandoning the assumption A1 about case representation. In doing so, we intend to answer the following research question:

Q1) *Is it possible to identify, ex post, which features of the cases within a case base effectively function as factors, and what their directionality is?*

To address this question, we begin by assuming that cases are represented in terms of generic features. We argue that the *ex post* identification of which features qualify as (or, more precisely, *behave as*) factors—that is, directional features—and the determination of their direction can be achieved through *counterfactual reasoning*. Counterfactual reasoning allows us to determine whether a feature *p* is *discriminating* in favor of an outcome: this is the case if there exists a case such that the absence of *p* would have led to the opposite decision. A feature that is never discriminating in any direction is deemed *irrelevant*. A *discriminating* feature, in turn, can behave as follows:

1. *factor*: if it discriminates in favour of an outcome and never discriminates in favour of the opposite outcome;
2. *ambiguous*: it discriminates for an outcome and also discriminates for the opposite outcome.

We wish to emphasize an important point from the outset. As we will see, what we are able to extract is the *directionality* of features. In this respect, the elements we identify as factors through counterfactual reasoning are not necessarily factors in the classical sense. Rather, they are unidirectional features, but they do not necessarily correspond to “legally relevant factual patterns”. Whether they do so depends on the nature of the input features used by the classifier. In our framework, therefore, *factors*

are simply *unidirectional features*; the reader may also think of them as *resolutive* (or *decision-driving*) features with a unique direction.

The taxonomy of features obtained is shown in Figure 2.1.

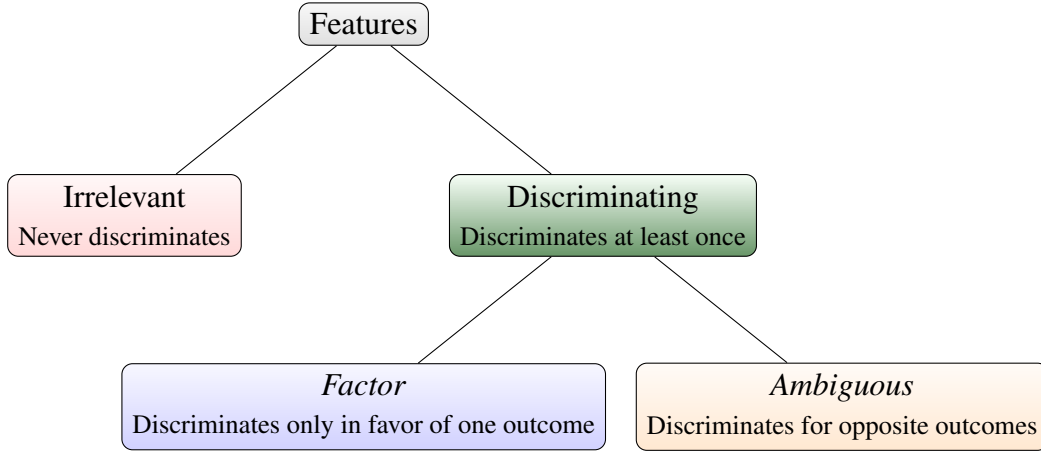


FIGURE 2.1: The taxonomy of features

With such a taxonomy of features in hand, in Section 2.4 we introduce a mechanism for inferring decisions for new cases based on previously decided ones. We do so by modeling a precedential constraint grounded in our taxonomy. This allows us to determine which past cases force a decision for a given case through factual comparison, under assumption A2. However, we do not impose any condition on the case base that guarantees the absence of conflicts among forcing precedents—that is, we do not assume A3.

Instead, our methodology adopts an inference mode that enables case classification even when precedents force opposing outcomes. The underlying intuition is that, according to a preorder relation among cases, some precedents are more authoritative or significant than others. In situations of conflict, the most important cases therefore prevail. To capture this formally, the logic BCL is extended with a modality that expresses the preorder relation.

Roadmap to Chapter 2

- **Section *Setting the Framework* (2.1)** establishes the formal background for our analysis. We restate the language and semantics of the *binary input classifiers logic* (BCL).

- **Section *Factors, Irrelevant and Ambiguous Features* (2.2)** examines the behavior of features within a case base to distinguish among *factors*, *irrelevant features*, and *ambiguous features*.
- **Section *Relation to Horty's Models* (2.3)** explores the connection between our framework and Horty's *reason* and *result* models. In particular, the section discusses how to reconstruct Horty case bases from our classifier models and our analysis of features.
- **Section *Inferring Classifications* (2.4)** develops the inference mechanism for determining the classifications of new cases. Our approach *tolerates ambiguity* and adopts a *liberal* inference strategy, designed to infer decisions even in the presence of conflict. This section is organized into three parts:
 - **Subsection *Ordered Binary Input Classifiers Logic* (2.4.1)** introduces the extended logical language OBCL, which enables explicit reasoning about preorders among states.
 - **Subsection *Static Inference* (2.4.2)** presents a static inference mechanism that derives new classifications based on our taxonomy of features—factors, irrelevant features, and ambiguous features—and on the preorder among cases, without updating the model.
 - **Subsection *Future Work: Dynamic Update* (2.4.3)** outlines a direction for future research, discussing, from a semantic perspective, a dynamic update mechanism.
 - **Section *Discussion and Related Works* (2.5)** concludes the chapter by reviewing relevant literature, summarizing the implications of the proposed framework, and outlining possible directions for future research.

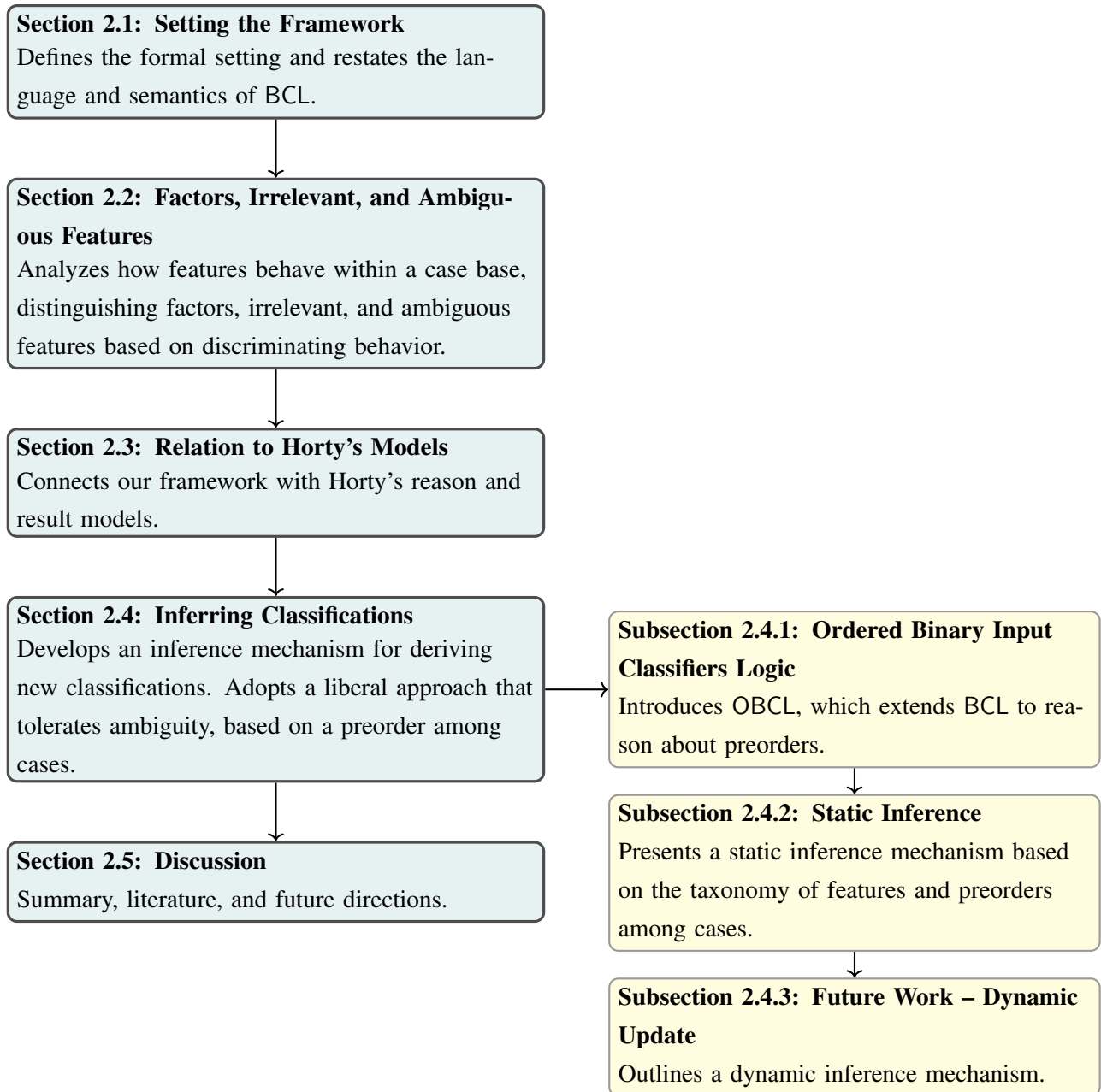


FIGURE 2.2: Roadmap to Chapter 2

2.1 Setting the framework

We want to show how it is possible to determine *which* features of a case base can be considered factors and *how* to extract their direction. We aim to do this in the context of binary input classifiers logic BCL. We refer the reader to Section 1.2.1 for the language and semantics of BCL.

We only recall that the finite set of propositional atoms is $Atm = Atm_0 \cup Dec$, where Atm_0 stands for the finite set of input variables of the classifier and $Dec = \{t(o) : o \in Val = \{0, 1, ?\}\}$ for decision atoms. Moreover, elements in $Val = \{1, 0, ?\}$ stand for *plaintiff wins*, *defendant wins* and *absence of decision* respectively. As usual, for $o \in \{0, 1\}$, the “opposite” $\bar{o}$ is noted for the value $1 - o$.¹

With regard to the input variables of the classifier, we assume that these correspond to factual features, i.e., $Atm_0 = Facts$. However, in contrast to the classifiers employed to model Horty’s case bases, we do not adopt assumption A_1 . Consequently, *Facts* are not factors. As a result, *Facts* is *not* a priori partitioned into pro-plaintiff and pro-defendant factors; that is, the dichotomy $Atm_0 = Facts = Plt \uplus Dfd$ is not imposed.

In order to relate CBR and reasoning about classifiers, we have to suppose that the classifier is situation-complete, that is every possible situation is taken into account by the classifier. So, we will require that the set of states is such that $S = 2^{Atm_0}$, as done for the genuine classifier models (Definition 2.9) in previous chapter.

Then, $\mathbf{CM}^{ic} = \{C = (2^{Atm_0}, f) \in \mathbf{CM}\}$, is the class of *possibly incomplete-knowledge situation-complete classifier models*.² A formula φ of $\mathcal{L}$ is said to be satisfiable relative to the class $\mathbf{CM}^{ic}$ if there exists a pointed classifier model (C, s) with $C \in \mathbf{CM}^{ic}$ such that $(C, s) \models \varphi$. It is said to be valid relative to $\mathbf{CM}^{ic}$, and noted as $\models_{\mathbf{CM}^{ic}} \varphi$, if $\neg\varphi$ is not satisfiable relative to $\mathbf{CM}^{ic}$.

An incomplete-knowledge situation-complete classifier model $C \in \mathbf{CM}^{ic}$, models a situation-complete and possibly incomplete-knowledge case base defined as $CB_C =$

¹The opposite is defined only for 0, 1, see Footnote 13.

²We specify here that the classifiers are *incomplete knowledge* to make it clear that completeness is with respect to fact situations. We changed the notation for the class of classifiers, with respect to the one we used in [35].

$\{k \mid k = (s, f(s)) \text{ with } s \in S\}$. Thus, we follow one of the intuitions already introduced in the context of genuine classifier models for consistent Horty case bases: each s in the classifier corresponds to the factual situation of a case, and the outcome assigned to s by the classification function f is the outcome of that case.³

Henceforth, due to the connection between classifier models and case bases, when referring to classifier models we will also mean the modelled case bases. Similarly when referring to states of classifiers models, we will mean also the modeled cases.

2.2 Factors, Irrelevant and Ambiguous Features

In this section, we aim to determine whether, by observing the behavior of a feature within a case base, we can infer if it is a *factor* and, if so, in which direction. Such observation is necessarily *ex post* and relies solely on the available information. In particular, we expect a feature to qualify as a factor—or, more precisely, *to behave as a factor* within the case base—if it is *discriminating* in *only one* direction. A feature is said to be discriminating in one direction if removing it from a case classified in that direction suffices to obtain the opposite classification. However, as we will show in this section, depending on their discriminating behavior, features may, *within a given classifier/case base*, function not only as factors but also as *ambiguous* or *irrelevant* features.

Example 2.1 (Running example II [35]). *The example of this section deal with hypothetical and simplified cases pertaining compensation claims for trespassing to land. Trespass to land involves the unlawful interference with property rights without permission. In these cases the plaintiffs are the owners of the property. Let us assume that the elements considered in this context are limited to: “the defendant causes property damage” (damage); “the defendant acts in defence of public necessity” (necessity); “the defendant intends to enter the land (intentional)”.*

Let us assume that the judges ruled in favour of plaintiffs’s compensation in the three following hypothetical scenario: 1) the plaintiff claimed his neighbour intentionally

³We note that this chapter considers case bases that are not necessarily Horty case bases. The connection between our framework and Horty’s case bases will be examined in Section 2.3.

occupied part of his land for his own use, causing damage (which we will refer to as case k_2); 2) the plaintiff claimed for damages from the municipality for injuries to land caused unintentionally by a policeman in an attempt to catch a criminal, acting in the public interest (case k_3);⁴ 3) a hunter intentionally entered the plaintiff's property, even if he has not caused damage (case k_5).

Instead, let us assume that the judges ruled in favour of defendant when: 1) the defendant accidentally entered the plaintiff's land while attempting to extinguish a fire in the neighbouring land (case k_7); 2) the defendant, while hiking, accidentally entered the plaintiff's property, without causing damage, because the path was not well signposted (case k_8).

The decisions made by the judges are mapped by the classifier $C_1 = (2^{Atm_0}, f)$, with $Atm_0 = \{intentional, damage, necessity\}$ and $f: 2^{Atm_0} \rightarrow \{0, 1, ?\}$, s.t. $f(s) = 1$ iff $s \in \{\{intentional, damage\}, \{damage, necessity\}, \{intentional\}\}$; $f(s) = 0$ iff $s \in \{\{necessity\}, \emptyset\}$; $f(s) = ?$ otherwise.⁵ The modelled case base is $CB_{C_1} = \{k_i \mid k_i = (s_i, f(s_i)), 1 \leq i \leq 8\}$ represented in following table.

	s_i	$f(s_i)$
k_1 :	$\{intentional, damage, necessity\}$	?
k_2 :	$\{intentional, damage\}$	1
k_3 :	$\{damage, necessity\}$	1
k_4 :	$\{intentional, necessity\}$	?
k_5 :	$\{intentional\}$	1
k_6 :	$\{damage\}$	?
k_7 :	$\{necessity\}$	0
k_8 :	$\emptyset$	0

Intuitively, we expect that in this context *damage* is a factor in favour of the plaintiff. Indeed, it is the only discriminating feature between s_3 -the situation in k_3 , classified as 1-and s_7 -the situation in k_7 -classified as 0; moreover *damage* is not discriminating for any situation classified as 0.

⁴partially inspired by *Wegner v. Milwaukee Mut. Ins. Co.*, 479 N.W.2d 38 (Minn., 1991)

⁵In this context we are implicitly enforcing the "closed-world assumption", assuming that the absence of a feature means that its negation holds.

The discriminating aspect of features can be nicely captured by using the notion of counterfactual explanation for a decision introduced in Section 1.2.2. We just recall here that the counterfactual condition $\varphi \Rightarrow \psi$ reads as “if it were φ it would be ψ ”. The conditional holds in a state of a classifier model iff all the closest (i.e., most similar, according to Definition 1.16) states to the current one, which make φ true, also make ψ true.

Through the notion of counterfactual explanation, we aim to model the discriminating behaviour of features observed in Example 2.1. However, one important aspect requires attention. It might be tempting to claim that a feature is discriminating in a certain direction if removing it from a state classified in that direction results in a state with a *different* classification. This, however, is not correct. The main issue is that the outcome of a case/state can take three possible values: 0, 1, or ?. Consequently, removing a feature from a state classified as 0 or 1 may instead yield a state classified as ?.

Example 2.2. *To make a more concrete example, we refer to Example 2.1. Consider state s_3 , classified as 1. If we remove necessity from s_3 , we obtain s_6 , classified differently from s_3 , i.e., as ?. Is this enough to claim that necessity is discriminating for a decision of 1? We argue this is not the case, since we do not actually know what the decision for s_6 is. Notice that from a counterfactual perspective, s_6 is the closest state to s_3 that satisfies $\neg$ necessity. Hence, $(C, s_3) \models t(1) \wedge (\neg$ necessity $\Rightarrow \neg t(1))$.*

Consider now s_3 and the feature damage. Recall that removing damage from s_3 yields s_7 , classified as 0—that is, in the opposite direction of s_3 . In this sense, damage is discriminating for the decision of s_3 as 1. From a counterfactual perspective, s_7 is the closest state to s_3 that satisfies $\neg$ damage. Hence, $(C, s_3) \models t(1) \wedge (\neg$ damage $\Rightarrow t(0))$.

Keeping in mind what has been discussed so far, we claim that a feature is discriminating in one direction if removing it from a case classified in that direction suffices to obtain the *opposite* classification. This discriminative property of features is captured by the notion of a *strong counterfactual explanation* for a decision, which can be expressed in the language $\mathcal{L}(Atm)$. Specifically, we say that a formula φ of $\mathcal{L}(Atm)$ *strongly counterfactually explains a decision* $o \in \{0, 1\}$ for a given situation s if failing to satisfy φ would cause s to be classified as $\bar{o}$.

We refer to this as a *strong* counterfactual explanation to distinguish it from a *weak* counterfactual explanation, according to which ϕ explains a decision o for a situation s if failing to satisfy ϕ would lead s to be classified *differently* from o (as in the case of necessity and s_3 in the previous example).

Definition 2.1 (Strong counterfactual explanation). We write $\text{SCfXp}(\phi, o)$ to mean that ϕ *strong counterfactually explains a decision for* o and define it as

$$\text{SCfXp}(\phi, o) =_{\text{def}} \mathbf{t}(o) \wedge (\neg\phi \Rightarrow \mathbf{t}(\bar{o})).$$

Example 2.3. Consider C_1 of Example 2.1. As already noted, removing *damage* from the factual situation s_3 , classified as 1, we obtain s_7 , classified as 0. That is, $(C_1, s_3) \models \text{SCfXp}(\text{damage}, 1)$. Similarly, looking at s_5 and s_8 , it can be verified that $(C_1, s_5) \models \text{SCfXp}(\text{intentional}, 1)$.

In the following, we will see that, on the basis of strong counterfactual explanation, we can distinguish three related notions: the notion of *factor* (a unidirectional feature), *ambiguous* (multi-directional) feature, and *irrelevant* (non-directional) feature. We will provide definitions and examples for each of these notions.

A feature p is a *factor* in the direction of o if, as already mentioned, is discriminating only in the direction of o . Formally, this holds if: (i) there exists a fact situation such that p strong counterfactually explains the decision o and (ii) there exists *no* fact situation such that p strong counterfactually explains the opposite decision $\bar{o}$. That is, we have the following definition.

Definition 2.2 (Factor). We write $\text{Factor}(p, o)$ to mean that p is a *factor in the direction of* o and define it as

$$\text{Factor}(p, o) =_{\text{def}} \Diamond \text{SCfXp}(p, o) \wedge \neg \Diamond \text{SCfXp}(p, \bar{o}).$$

Example 2.4. From Example 2.3 and from the fact that *damage* does not strong counterfactually explain a decision for 0, we have $(C_1, s) \models \text{Factor}(\text{damage}, 1)$, for all $s \in S$. Similarly, $(C_1, s) \models \text{Factor}(\text{intentional}, 1)$, for all $s \in S$.

Not all features are factors. Indeed, a feature may either fail to strongly counterfactually explain any decision—in which case it is *irrelevant*—or strongly counterfactually explain opposite decisions, in which case it is *ambiguous*.

Definition 2.3 (Irrelevant feature). We write $\text{Irrelevant}(p)$ to mean that p is irrelevant and define it as

$$\text{Irrelevant}(p) =_{\text{def}} \neg \Diamond \text{SCfXp}(p, 1) \wedge \neg \Diamond \text{SCfXp}(p, 0).$$

Example 2.5. Within C_1 , *necessity* does not explain any decision strong counterfactually, i.e. we have $(C_1, s) \models \text{Irrelevant}(\text{necessity})$ for all $s \in S$.

Definition 2.4 (Ambiguous feature). We write $\text{Amb}(p)$ to mean that p is an ambiguous feature and define it as

$$\text{Amb}(p) =_{\text{def}} \Diamond \text{SCfXp}(p, 1) \wedge \Diamond \text{SCfXp}(p, 0).$$

Example 2.6. In C_1 , there are no ambiguous features. Suppose now that a new case is added to our case base. A private university seeks compensation for protesters who trespassed and caused damage during an event. The protesters invoke the defence of public necessity, arguing that their actions were justified as reasonable efforts to prevent an immediate threat of greater harm posed by the business activities of the event's guests. The judge accepts the public necessity defence and, while acknowledging intentionality, rules in favour of the defendant.⁶ The resulting “updated” case base is modelled by $C_2 = (2^{\text{Atm}_0}, g)$, described in the following table.

	s_i	$g(s_i)$
$k_1 :$	$\{\text{intentional}, \text{damage}, \text{necessity}\}$	0
$k_2 :$	$\{\text{intentional}, \text{damage}\}$	1
$k_3 :$	$\{\text{damage}, \text{necessity}\}$	1
$k_4 :$	$\{\text{intentional}, \text{necessity}\}$	?
$k_5 :$	$\{\text{intentional}\}$	1
$k_6 :$	$\{\text{damage}\}$	?
$k_7 :$	$\{\text{necessity}\}$	0
$k_8 :$	$\emptyset$	0

But then, in this context, we have both $(C_2, s_5) \models \text{SCfXp}(\text{intentional}, 1)$ and $(C_2, s_1) \models \text{SCfXp}(\text{intentional}, 0)$. Namely, $(C_2, s) \models \text{Amb}(\text{intentional})$ for every $s \in S$.⁷

⁶Partially inspired by *Commonwealth v. Carter*, see The New York Times, Apr. 16, 1987.

⁷Notice that, as shown by the examples, the satisfiability of $\text{Factor}(p)$, $\text{Irrelevant}(p)$, and $\text{Amb}(p)$ is independent of the particular state considered; these are global properties.

Some properties concerning the behavior of features are established by the following propositions. First, factors are *unidirectional*: if p is a factor in the direction of o , it cannot simultaneously be a factor in the opposite direction $\bar{o}$.

Proposition 2.5.

$$\models_{\mathbf{CM}^{ic}} \bigwedge_{p \in \text{Atm}_0, o \in \{0,1\}} \left(\text{Factor}(p, o) \rightarrow \neg \text{Factor}(p, \bar{o}) \right).$$

Moreover, the previously and intuitively introduced a four-way partition of features into factors for 0, factors for 1, irrelevant features, and ambiguous features is captured by the following validities.

Proposition 2.6.

$$\begin{aligned} &\models_{\mathbf{CM}^{ic}} \bigwedge_{p \in \text{Atm}_0} \left(\text{Factor}(p, 1) \vee \text{Factor}(p, 0) \vee \text{Irrelevant}(p) \vee \text{Amb}(p) \right). \\ &\models_{\mathbf{CM}^{ic}} \bigwedge_{p \in \text{Atm}_0} \left(\left(\bigvee_{o \in \{0,1\}} \text{Factor}(p, o) \right) \rightarrow \neg \text{Irrelevant}(p) \wedge \neg \text{Amb}(p) \right). \\ &\models_{\mathbf{CM}^{ic}} \bigwedge_{p \in \text{Atm}_0} \left(\text{Irrelevant}(p) \rightarrow \neg \text{Factor}(p, 0) \wedge \neg \text{Factor}(p, 1) \wedge \neg \text{Amb}(p) \right). \\ &\models_{\mathbf{CM}^{ic}} \bigwedge_{p \in \text{Atm}_0} \left(\text{Amb}(p) \rightarrow \neg \text{Factor}(p, 0) \wedge \neg \text{Factor}(p, 1) \wedge \neg \text{Irrelevant}(p) \right). \end{aligned}$$

Example 2.6 highlights an aspect that deserves further attention. Suppose that decisions for previously undecided cases are now provided. In such a situation, features previously identified as ambiguous will remain so; however, it may occur that (a) features previously labelled as irrelevant become ambiguous or factors, and (b) features previously labelled as factors become ambiguous (as is the case for intentional in the “updated” version of Example 2.6).

Thus, our method provides a *pro tanto* analysis of features: we identify *pro tanto* irrelevant features and *pro tanto* factors—that is, features that are irrelevant or factors given the current information. *For this reason, we will refer to our factors as pro tanto factors, particularly when distinguishing them from those in Horty’s models.*

With these considerations in mind, a classifier model that does not admit ambiguous features—at least relative to the current classifications—will be called *consistent*.

Definition 2.7 (Consistency). A classifier $C = (S, f)$ is *consistent* given the current classifications if and only if for every $s \in S$, $(C, s) \models \text{Cons}$, where

$$\text{Cons} =_{\text{def}} \bigwedge_{p \in \text{Atm}_0} \neg \text{Amb}(p).$$

Example 2.7. Classifier C_1 from Example 2.1 is consistent, whereas C_2 from Example 2.6 is inconsistent (and will remain inconsistent regardless of any later classifications of originally undecided cases).

We have introduced a *pro tanto* taxonomy of features, according to which we can identify *pro tanto* factors. We now wish to introduce a form of precedential constraint that, while reminiscent of the “Horty-style” precedential constraint described in Section 1.1.1, differs in important respects.⁸

Intuitively, recall that under the precedential constraint, a case should have the same outcome as a precedent if it includes an equally or more inclusive set of factors supporting the precedent’s outcome, and no additional factor opposing it. In our framework, however, we must also take into account ambiguous and irrelevant features. More precisely, we expect that if a classifier assigns a situation s the outcome o , then it must assign the same outcome to every situation s' such that: (a) s' includes all *pro tanto* factors in the direction of o that are present in s ; (b) s' does *not* include any *pro tanto* factors in the direction of $\bar{o}$ that are absent from s ; and (c) s' includes exactly the same ambiguous and irrelevant features as s .

Definition 2.8 (Monotonicity). We will say that C is *pro tanto* monotone iff $(C, s) \models \text{CIMon}$, where

$$\begin{aligned} \text{CIMon} =_{\text{def}} & \bigwedge_{c \in \{0,1\}, X, Y \subseteq \text{Atm}_0} \left((\text{SetFactor}(X, o) \wedge \text{SetFactor}(Y, \bar{o})) \rightarrow \right. \\ & \bigwedge_{Z \not\subseteq X \cup Y, X' \subseteq X, Y' \subseteq Y} \left(\Diamond (\text{cn}_{X' \cup Y' \cup Z, \text{Atm}_0} \wedge \text{t}(o)) \rightarrow \right. \\ & \left. \left. \bigwedge_{X \supset X'' \supseteq X', Y'' \subseteq Y'} \Box (\text{cn}_{X'' \cup Y'' \cup Z, \text{Atm}_0} \rightarrow \text{t}(o)) \right) \right), \end{aligned}$$

⁸Note that we are considering cases in which the reasons for the decisions are unknown, since we do not know which features count as factors according to the language-level partition. It is therefore more appropriate to speak of an *a fortiori* constraint—typical of the result model—rather than a precedential constraint—typical of the reason model (see Remark 1.5).

where, for every $o \in \{0, 1\}$ and $X \subseteq \text{Atm}_0$:

$$\text{SetFactor}(X, o) =_{\text{def}} \bigwedge_{p \in X} \text{Factor}(p, o) \wedge \bigwedge_{p \in \text{Atm}_0 \setminus X} \neg \text{Factor}(p, o).$$

Example 2.8. Classifier C_1 from Example 2.1 is not monotone, i.e., $(C_1, s) \not\models \text{CIMon}$. Indeed, for CIMon to hold, case s_1 should have been classified as 1, since it contains more factors in favor of 1 (specifically, *intentional*) than s_3 , which is classified as 1.

The preceding example shows that the consistency of a classifier (i.e., the absence of ambiguous features) does not imply its *pro tanto* monotonicity (as observed for C_1). It is worth noting that the reverse is not true either: the *pro tanto* monotonicity of a classifier does not guarantee its consistency.⁹ An “extreme” case of this occurs when all the features of a classifier are ambiguous, so that CIMon is vacuously satisfied.

Example 2.9. By way of example only, consider this new simplified scenario: $C_3 = (2^{\text{Atm}_0}, h)$, where this time $\text{Atm}_0 = \{\text{intentional}, \text{damage}\}$ and the behaviour of h summarized in following table

	s_i	$h(s_i)$
$k_1 :$	$\{\text{damage}, \text{intentional}\}$	0
$k_2 :$	$\{\text{intentional}\}$	1
$k_3 :$	$\{\text{damage}\}$	1
$k_4 :$	$\emptyset$	0

It can be verified that both *damage*, *intentional* are ambiguous features so that vacuously $(C_3, s) \models \text{CIMon}$ for all $s \in S$.

2.3 Relation to Horty’s Models

As mentioned in the Background Chapter, Liu *et al.* [64] showed that every consistent Horty case base can be represented in terms of a set of classifier models. Recall also

⁹Our notion of consistency differs from that proposed by Horty. In the following section, we will see that, to some extent, both *pro tanto* monotonicity and the absence of ambiguous features are required to recover Horty’s notion of consistency.

that this result was further refined in [62, 35], where it was demonstrated that a consistent Horty case base can be mapped onto a *unique* classifier, referred to as a *genuine classifier*. The question that now arises is: *what happens if we start from a classifier model and attempt to reverse-engineer the Horty case bases it can represent?*

To address this question, we must keep some points in mind. First, in Horty's models, features are *a priori* partitioned into pro-plaintiff and pro-defendant factors, whereas in classifiers this division is not presupposed. Instead, through counterfactual reasoning, we can determine whether a feature behaves as a *pro tanto* factor, as *pro tanto* irrelevant, or as ambiguous.

Moreover, this *pro tanto* taxonomy of features is more fine-grained than the simple dichotomy between *Plt* and *Dfd*. For example, a plaintiff factor in Horty's sense may not qualify as a *pro tanto* factor for 1 according to our definition; that is, it may hold that $p \in Plt$ but not $\text{Factor}(p, 1)$.

In this section, we will discuss these aspects in detail.

First, we recall the definition of the *genuine classifier* of a consistent Horty case base. The underlying intuition is that the genuine classifier returns ? for all, and only those, factual situations that cannot be forced to a decision by the precedential constraint.

Definition 2.9 (Genuine classifier). Let CB be a consistent Horty case base. The genuine classifier of CB is the function $f : 2^{Atm_0} \rightarrow \{0, 1, ?\}$, s.t. for any situation s ,

$$f(s) = \begin{cases} o & \text{if } \exists k = (s', X, o) \in CB \text{ s.t. } \text{forces}(k, s, o) \\ ? & \text{otherwise.} \end{cases}$$

The genuine classifier model of CB is therefore $C = (2^{Atm_0}, f)$.

Recall that the following result holds.

Proposition 2.10. *Every consistent Horty case base induces exactly one genuine classifier model.*

The inverse, however, does not hold. One reason for this is that, given a case base, there may exist more than one possible dichotomy, say, $Atm_0 = Plt \uplus Dfd = Plt' \uplus Dfd'$, which make the case base consistent. The simplest example is the following: let $Plt =$

$\{p, q\}$ and $Dfd = \emptyset$ and a singleton case base $CB = \{(\{p, q\}, \{p\}, 1)\}$. It can be verified that by changing to $Plt' = \{p\}$ and $Dfd' = \{q\}$, the case base remains consistent. This shows that reverse-engineering a case base from a classifier generally does not yield a unique dichotomy, but rather several admissible ones.

Definition 2.11 (Admissible dichotomy). Let CB be a Horty case base whose a priori dichotomy is $Atm_0 = Plt \uplus Dfd$. We say that a dichotomy $Atm_0 = Plt' \uplus Dfd'$ is admissible, if applying the precedential constraint under this new partition will not render CB inconsistent.

Plainly speaking, $Atm_0 = Plt' \uplus Dfd'$ is admissible if, upon changing the original partition $Atm_0 = Plt \uplus Dfd$ by replacing Plt with Plt' and Dfd with Dfd' , and then applying the precedential constraint, the case base remains consistent.

We can state necessary and sufficient conditions for the admissibility of a dichotomy in a case base. These conditions are expressed in terms of the ambiguous and *pro tanto* factors extracted from the genuine classifier of the case base.

Proposition 2.12. *Let CB be a consistent Horty case base and C its genuine classifier. Let Atm_0 be the set of features. Then a dichotomy $Atm_0 = Plt' \uplus Dfd'$ is admissible, if and only if for any state s ,*

1. $(C, s) \models \neg \text{Amb}(p)$ for any feature $p \in Atm_0$
2. if $(C, s) \models \text{SetFactor}(X, 1) \wedge \text{SetFactor}(Y, 0)$ for some $X, Y \subseteq Atm_0$, then $X \subseteq Plt', Y \subseteq Dfd'$.

In particular, the previous proposition tells us that, in the context of a consistent case base, no feature can be ambiguous. More importantly, from point 2 we know that every *pro tanto* pro-plaintiff factor (e.g., a p such that $\text{Factor}(p, 1)$) is a pro-plaintiff factor according to any admissible dichotomy, and every *pro tanto* pro-defendant factor (e.g., a q such that $\text{Factor}(q, 0)$) is a pro-defendant factor according to any admissible dichotomy.

The converse does not hold: a feature may be a pro-defendant factor (resp. pro-plaintiff) according to an admissible dichotomy without being a *pro tanto* pro-defendant factor (resp. pro-plaintiff).

Proposition 2.13. *There exists a Horty case base with $Plt \uplus Dfd$ its dichotomy, $C = (2^{Atm_0}, f)$ its genuine classifier model, s.t.: 1) $\exists p \in Plt$ but $(C, s) \models \neg \text{Factor}(p, 1)$ and 2) $\exists q \in Dfd$ but $(C, s) \models \neg \text{Factor}(q, 0)$.*

The previous proposition tells us that the dichotomy Plt / Dfd is not only predetermined but also too coarse from our perspective. Why? Because the dichotomy guarantees only that a factor $p \in Dfd$ is not discriminating for plaintiff (in the sense of counterfactual reasoning), but it does not exclude the possibility that p is actually irrelevant—i.e., that it does not even support the defendant—according to our *pro tanto* analysis. Consider the following example.

Example 2.10. *Let $Plt = \{p\}, Dfd = \{q\}$ and $CB = \{(\{p, q\}, \{p\}, 1)\}$. Then, the genuine classifier model for CB is $C = (2^{Atm_0}, f)$ s.t. $f(s) = 1$ iff $p \in s$; $f(s) = ?$ otherwise. However, $(C, s) \models \neg \text{Factor}(q, 0)$. Moreover, $(C, s) \models \text{Irrelevant}(q)$, since there is no s , s.t. $f(s) \neq f(s \setminus \{q\})$. Notice that, as a consequence, $Plt' = \{p, q\}$ and $Dfd' = \emptyset$ is another admissible dichotomy.*

Previous example reveals that exactly due to irrelevant features we have not a unique, but many admissible dichotomies for the same classifier.

The result of this section establishes the necessary and sufficient condition of being a genuine classifier for some Horty case base regardless of which dichotomy the case base actually has.

Proposition 2.14. *Let $C = (S, f)$ be a classifier model. Then C is the genuine classifier model for some consistent Horty case base with an admissible dichotomy $Atm_0 = Plt \uplus Dfd$, if and only if $C \in \mathbf{CM}^{ci}$, C is pro tanto monotone, C does not have ambiguous features, $f(\emptyset) = ?$ and $\forall p \in Atm_0^o, f(\{p\}) \neq \bar{o}$.*

2.4 Inferring Classifications

From Section 1.1.1, we know that, in the context of Horty's models, the precedential constraint can serve as a criterion for defining an inference mechanism for new cases.

An interesting research question, therefore, is how to infer new factors given our set of features with their various qualifications. We address this issue in this section.

Recall, as discussed in Section 2.2, that our analysis of features is *pro tanto* and may change if new classifications are provided.

We do not require consistency here (in our sense; see Definition 2.7) and instead adopt a rather *liberal* approach, in which the presence of ambiguous features is tolerated. More skeptical approaches are, of course, possible—for example, one might infer classifications for cases but halt the inference process whenever some features turn out to be ambiguous. By contrast, our liberal approach aims to infer as many new classifications as possible without prematurely interrupting the inference. However, this may lead to situations in which not all potential inferences can be drawn consistently—namely, when precedents force opposite decisions for the same case. To resolve such conflicts, we presuppose that some cases may be considered more important than others and associate classifiers with a total preorder on the set of states S . Conceptually, this assumption is reasonable: the order may be constructed, for instance, on the basis of the hierarchy of the courts involved in earlier classifications (e.g., cases decided by higher courts are deemed more important than others), temporal relations among cases (e.g., cases decided later are regarded as more important than earlier ones), or similar considerations.

2.4.1 Ordered Binary Input Classifiers Logic

We aim to enrich the logic BCL so that a preorder between states can be taken into account. Roughly speaking, the preorder tells us which states are better than others. To this end, we first extend the language of BCL to handle preorders. Specifically, we enrich $\mathcal{L}(Atm)$ with a new modal operator $[\leq]\varphi$, to be read as “ φ is true at all states that are at least as good as the current one (with respect to the preorder).” The set of propositional atoms $Atm = Atm_0 \cup Dec$ is still assumed to be finite. The resulting language is defined as follows.

Definition 2.15. The modal language $\mathcal{L}^{\leq}(Atm)$ is defined as:

$$\varphi ::= p \mid t(o) \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid [\leq]\varphi$$

where $p \in \text{Atm}_0$, $t(o) \in \text{Dec}$.

We define the dual operators as usual: $\diamond\varphi =_{\text{def}} \neg\Box\neg\varphi$, $\langle\leq\rangle\varphi =_{\text{def}} \neg[\leq]\neg\varphi$.

Formulas of $\mathcal{L}^{\leq}(\text{Atm})$ are interpreted with respect to ordered classifier models.

Definition 2.16 (Ord. Classifier). An *ordered classifier model* is a triple $C_{\text{or}} = (S, f, \leq)$ where S is a set of states, f is a classification function and $\leq$ is a total preorder on S .¹⁰

We denote $<$ the strict part of $\leq$: $s < s'$ iff $s \leq s'$ and $s' \not\leq s$.

Definition 2.17 (Satisfaction relation). Let $C_{\text{or}} = (S, f, \leq)$ be an ordered classifier model. Let $s \in S$. Then:

$$\begin{aligned} (C_{\text{or}}, s) \models p &\iff p \in s \\ (C_{\text{or}}, s) \models t(o) &\iff f(s) = o \\ (C_{\text{or}}, s) \models \neg\varphi &\iff (C_{\text{or}}, s) \not\models \varphi \\ (C_{\text{or}}, s) \models \varphi \wedge \psi &\iff (C_{\text{or}}, s) \models \varphi \text{ and } (C_{\text{or}}, s) \models \psi \\ (C_{\text{or}}, s) \models \Box\varphi &\iff \forall s' \in S: (C_{\text{or}}, s') \models \varphi \\ (C_{\text{or}}, s) \models [\leq]\varphi &\iff \forall s', s \leq s': (C_{\text{or}}, s') \models \varphi \end{aligned}$$

The class of ordered classifier models is noted $\mathbf{CM}^{\text{or}}$. A pointed ordered classifier model is a pair (C_{or}, s) with $C_{\text{or}} = (S, f, \leq)$ an ordered classifier model and $s \in S$. Satisfiability and validity wrt. $\mathbf{CM}^{\text{or}}$ are defined in the usual way.

Definition 2.18 (Axiomatics of OBCL). We define OBCL to be the extension of BCL (Definition 1.12) generated by the following axioms:

$$\begin{aligned} ([\leq]\varphi \wedge [\leq](\varphi \rightarrow \psi)) &\rightarrow [\leq]\psi & (\mathbf{K}_{[\leq]}) \\ [\leq]\varphi &\rightarrow \varphi & (\mathbf{T}_{[\leq]}) \\ [\leq]\varphi &\rightarrow [\leq][\leq]\varphi & (\mathbf{4}_{[\leq]}) \end{aligned}$$

¹⁰ $\leq$ is a transitive, reflexive relation s.t. for all $s, s' \in S$, $s \leq s'$ or $s' \leq s$.

$$\begin{aligned}
& \bigwedge_{Y \subseteq \text{fin} \text{Atm}_0} \left((\text{cn}_{Y, \text{Atm}_0} \wedge [\leq] \varphi) \rightarrow \Box (\text{cn}_{Y, \text{Atm}_0} \rightarrow [\leq] \varphi) \right) & \textbf{(FunctT)} \\
& \bigwedge_{X, Y \subseteq \text{Atm}_0} \left(\Box \left((\text{cn}_{X, \text{Atm}_0} \wedge [\leq_T] \varphi) \rightarrow \Box (\text{cn}_{Y, \text{Atm}_0} \rightarrow \varphi) \right) \vee \Box \left((\text{cn}_{Y, \text{Atm}_0} \wedge [\leq_T] \varphi) \rightarrow \Box (\text{cn}_{X, \text{Atm}_0} \rightarrow \varphi) \right) \right) \\
& & \textbf{(Total}_{[\leq_T]}\text{)}
\end{aligned}$$

$\mathbf{K}_{[\leq_T]}$, $\mathbf{T}_{[\leq_T]}$, $\mathbf{4}_{[\leq_T]}$ ensures that $[\leq]$ is a **S4** operator. $\mathbf{Total}_{[\leq_T]}$ ensures $[\leq]$ is total. $\mathbf{FunctT}$, will be used in the completeness proof, to guarantee that two copies of the same input instance behave in the same way with respect to the preorder.

Proposition 2.19 (Completeness). *OBCL is sound and complete relative to the class $\mathbf{CM}^{\text{or}}$.*

Recall that, since Atm_0 is a finite set, the set of states $S \subseteq 2^{\text{Atm}_0}$ is also finite. Moreover, since $\leq$ is total, every non-empty subset of S has $\leq$ -maximal elements. In other words, for all $S' \subseteq S$ with $S' \neq \emptyset$, there exists an $s \in S'$ such that, for all $s' \in S'$, $s' \leq s$.

As in the previous sections, we assume that ordered classifier models are situation-complete; therefore, from this point onward, we focus on classifiers such that $S = 2^{\text{Atm}_0}$.

2.4.2 Static inference

In this section, we show how, within the framework of classifiers enriched with a total preorder, new classifications for undecided cases can be inferred based on our *pro tanto* analysis of features. We approach the problem from a static perspective, without updating the model after new decisions are inferred. We illustrate the inference process by means of the following example.¹¹

Example 2.11. *Let $\text{Atm}_0 = \{p_1, p_2, p_3\}$. Consider the ordered classifier model $C_{\text{or}} = (S, f, \leq)$, where: $S = 2^{\text{Atm}_0}$; $f : 2^{\text{Atm}_0} \rightarrow \{0, 1, ?\}$, is such that $f(s) = 0$ iff $s \in \{\{p_1, p_2, p_3\}, \emptyset\}$, $f(s) = ?$ iff $s \in \{\{p_1, p_3\}, \{p_2, p_3\}\}$, $f(s) = 1$ otherwise; $\leq$ is a total preorder on S such that for all $s \in S$, $s \leq s_1$ with $s_1 = \{p_1, p_2, p_3\}$.*

¹¹Notice that all the notions introduced so far (factor, irrelevant feature, ambiguous feature, consistency, and monotonicity) can be readily adapted to ordered classifiers.

	s_i	$f(s_i)$
$k_1 :$	$\{p_1, p_2, p_3\}$	0
$k_2 :$	$\{p_1, p_2\}$	1
$k_3 :$	$\{p_1, p_3\}$	?
$k_4 :$	$\{p_2, p_3\}$	?
$k_5 :$	$\{p_1\}$	1
$k_6 :$	$\{p_2\}$	1
$k_7 :$	$\{p_3\}$	1
$k_8 :$	$\emptyset$	0

We can verify that both p_1 and p_2 are factors in the direction of 1, while p_3 is an ambiguous feature – consider s_7 , s_8 and s_1 , s_2 . Hence, C is not consistent.

We aim to define an inference mechanism for new cases. In doing so, we retain a form of precedential constraint, though not in Horty’s sense. In particular, we adopt the same principle underlying our definition of *pro tanto* monotonicity (see Definition 2.8). While *pro tanto* monotonicity was defined as a property that classifiers may or may not satisfy, here we adopt our version of the precedential constraint as the guiding principle for inference when classifying new cases. More precisely, we claim that if a classifier assigns a situation s the outcome o , we can assign the same outcome to every situation s' such that: (a) s' includes all *pro tanto* factors in the direction of o that are present in s ; (b) s' does not include any *pro tanto* factors in the direction of $\bar{o}$ that are absent from s ; and (c) s' includes exactly the same ambiguous and irrelevant features as s .

Given a new case, based on feature analysis and our precedential constraint, we may retrieve a case from the case base according to which the new case should be classified in the same direction. In this sense—analogously to what we did for Horty’s models, based on Horty’s precedential constraint—we say that the precedent *pro tanto* forces a decision for the new case.¹²

¹²When discussing Horty’s models, we said that one case can *force*, under the precedential constraint, the decision in another case. In our framework, we instead use the term “*pro tanto* forces” to avoid confusion between the two approaches.

However, this inference mechanism may yield conflicting outcomes: two *pro tanto* forcing precedents may support opposite classifications for the new case. In such situations, we maintain that the new case should follow the decision of the best *pro tanto* forcing precedent, as determined by the preorder. To formally capture this notion of “best” we rely on the maximality operator introduced in [96].

Definition 2.20. The notion that *all the best ψ -states are ϕ* is defined as $[\text{Best}(\psi)]\phi =_{\text{def}} \Box(\psi \rightarrow \langle \leq \rangle(\psi \wedge [\leq](\psi \rightarrow \phi)))$

Plainly speaking, according to the previous definition, “the best ψ -states are ϕ if and only if, for all states, either they are not ψ or there is a better ψ -state such that all states above it are either not ψ or ϕ ” [96].

The following proposition shows that $[\text{Best}(\psi)]\phi$ indeed capture the intended meaning *all the best ψ -states are ϕ* .

Proposition 2.21. $(C, s) \models [\text{Best}(\psi)]\phi \Leftrightarrow \forall s' \in S : ((C, s') \models \psi \text{ and } \nexists s'' \in S \text{ s.t. } s' < s'' \text{ and } (C, s'') \models \psi) \Rightarrow (C, s') \models \phi$.

We can now define our inference mechanism. Recall that our analysis of features is *pro tanto*: as new information becomes available, the labeling of features may change. Consequently, once a decision has been inferred on the basis of our precedential constraint, new factors or ambiguous features may be identified, which in turn may enable further decisions to be inferred, and so on. This process gives rise to an iterative inference mechanism.

Definition 2.22. Let $k \geq 0$ and $o \in \{0, 1\}$. We write $\text{Cl}^k(o)$ to indicate that a classification in the direction of o for the current state is *forced at step k* by preference over conflicting forcing precedents, and define it recursively as follows.

1. $\text{Cl}^0(o) =_{\text{def}} \text{t}(o)$;
2. $\text{Cl}^{k+1}(o) =_{\text{def}} \neg \text{Cl}^{j \leq k}(\bar{o}) \wedge \text{SetFactor}^k(F_1^k, 1) \wedge \text{SetFactor}^k(F_0^k, 0) \wedge \bigwedge_{X_1 \subseteq F_1^k, X_0 \subseteq F_0^k, Z \not\subseteq F_1 \cup F_0} \text{cn}_{X_1 \cup X_0 \cup Z, \text{Atm}_0} \wedge [\text{Best}(\text{PtForces}^k(X_1, X_0, Z))] \text{Cl}^{j \leq k}(o)$

where

- $\text{Cl}^{j \leq k}(o) =_{\text{def}} \bigvee_{j \leq k} \text{Cl}^j(o)$;

- for $o \in \{0, 1\}$ and $F_o^k \subseteq \text{Atm}_0$:

$$\text{SetFactor}^k(F_o^k, o) =_{\text{def}} \bigwedge_{p \in F} \text{Factor}^k(p, o) \wedge \bigwedge_{p \in \text{Atm}_0 \setminus F} \neg \text{Factor}^k(p, o).$$

$$\text{and } \text{Factor}^k(p, o) =_{\text{def}} \Diamond \text{SCfXp}^k(p, o) \wedge \neg \Diamond \text{SCfXp}^k(p, \bar{o}),$$

$$\text{SCfXp}^k(p, o) =_{\text{def}} \text{Cl}^{\leq k}(o) \wedge (\neg p \Rightarrow \text{Cl}^{\leq k}(\bar{o}));$$

- for $X_1 \subseteq F_1^k, X_0 \subseteq F_0^k, Z \not\subseteq F_1^k \cup F_0^k$,
 $\text{PtForces}^k(X_1, X_0, Z) =_{\text{def}} \bigvee_{o \in \{0, 1\}} \text{Cl}^{\leq k}(o) \wedge \bigvee_{Y_o \subseteq X_o \subseteq F_o^k, X_{\bar{o}} \subseteq Y_{\bar{o}} \subseteq F_{\bar{o}}^k} \text{cn}_{Y_1 \cup Y_0 \cup Z, \text{Atm}_0}.$

We explain the definition step by step.

- At step 0, a decision for a state in a given direction can be inferred if that state is already decided in that direction ($\text{Cl}^0(o) =_{\text{def}} \text{t}(o)$).
- The inference process then proceeds as follows. Suppose that we have inferred the classifications up to step k ($\text{Cl}^{\leq k}(\cdot)$):
 1. Based on the classifications inferred so far, we extract, using the notion of strong counterfactual explanation at step k ($\text{SCfXp}^k(p, o)$), the *pro tanto* factors at step k ($\text{Factor}^k(p, o)$) in the usual way.
 2. Accordingly, we identify the *sets* of *pro tanto* factors at step k in favor of each $o \in \{0, 1\}$ ($\text{SetFactor}^k(F_o^k, o)$).
 3. For a given state s , we then determine how its features are partitioned—specifically, which are the *pro tanto* factors at step k in favor of $o \in \{0, 1\}$ and which are the non-factors appearing in the state. That is, we identify the sets $X_1 \subseteq F_1^k, X_0 \subseteq F_0^k$, and $Z \not\subseteq F_1^k \cup F_0^k$ such that $(C, s) \models \text{cn}_{X_1 \cup X_0 \cup Z, \text{Atm}_0}$.
 4. Next, we say that, at step k , a state s' *forces* a decision for state s , determined by X_0, X_1 , and Z ($\text{PtForces}^k(X_1, X_0, Z)$), iff its classification has already been inferred as $o \in \{0, 1\}$ and it *pro tanto forces* a classification for o on the basis of the inferred *pro tanto* factors and our precedential constraint—namely, if it includes an equally or more inclusive set of *pro tanto* factors for o and no additional *pro tanto* factors for $\bar{o}$ with respect to s .

5. Finally, we infer a classification for s as o at step $k + 1$ ($\text{Cl}^{k+1}(o)$) if the following hold:
- a) s has not already been inferred in the opposite direction ($\neg \text{Cl}^{j \leq k}(\bar{o})$);
and
 - b) for the best forcing state with respect to X_0 , X_1 , and Z , we have previously inferred a classification as o ($[\text{Best}(\text{PtForces}^k(X_1, X_0, Z))] \text{Cl}^{j \leq k}(o)$).

Example 2.12. Consider Example 2.11. Recall that we know that for all $s \in S$, $s \leq s_1$ with $s_1 = \{p_1, p_2, p_3\}$. Recall also that the set of factors in the direction of 1 at step 0 is $F_1^0 = \{p_1, p_2\}$; while the set of factors in the direction of 0 at step 0 is empty, $F_0^0 = \emptyset$. So, for all $s \in S$, $(C_{\text{or}}, s) \models \text{SetFactor}^0(F_1^0, 1) \wedge \text{SetFactor}^0(F_0^0, 0)$ ($\star$).

We now show that s_3 should be classified as 0, on the basis of s_1 . Indeed:

- $(C_{\text{or}}, s_3) \models \neg \text{Cl}^0(1)$ (since s_3 is classified as ?).
- $(C_{\text{or}}, s_3) \models \text{SetFactor}^0(F_1^0, 1) \wedge \text{SetFactor}^0(F_0^0, 0)$ (because of $\star$).
- $(C, s_3) \models \text{cn}_{X_1 \cup X_0 \cup Z, \text{Atm}_0}$, where $X_1 = s_1 \cap F_1^0 = \{p_1\}$ is the set of factors in direction of 1 in s_3 , $X_0 = s_1 \cap F_0^0 = \emptyset$ is the (empty) set of factors in the direction of 0 in s_3 , $Z = \{p_3\}$ is the set of non factors in s_3 .
- $(C, s_1) \models \text{PtForces}^0(X_1, X_0, Z) \wedge \text{t}(0)$. Indeed, $(C, s_1) \models \text{t}(0) \wedge \text{cn}_{Y_0 \cup Y_1 \cup Z, \text{Atm}_0}$, with $Y_0 = X_0$ and $X_1 \subseteq Y_1 = \{p_1, p_2\}$. s_3 differs from s_1 —which is decided as 0—only in that it contains one fewer pro tanto pro-plaintiff factor (p_2).
- $(C, s_7) \models \text{PtForces}^0(X_1, X_0, Z) \wedge \text{t}(1)$. Indeed, $(C, s_1) \models \text{t}(1) \wedge \text{cn}_{W_0 \cup W_1 \cup Z, \text{Atm}_0}$, with $W_0 = X_0$ and $\emptyset = W_1 \subseteq X_1$. s_3 differs from s_7 —which is decided as 1—only in that it contains one additional pro tanto pro-plaintiff factor (p_1).
- $(C, s_3) \models [\text{Best}(\text{PtForces}^0(X_1, X_0, Z))] \text{Cl}^0(0)$. Indeed, we know that s_1 is “better” than any other state, so $s_7 \leq s_1$. In other words, the best forcing state for s_3 is s_1 , which is classified as 0.
- We can conclude that s_3 is forced at step 1 to be decided as 0: $(C, s_3) \models \text{Cl}^1(0)$.

Similarly, we can verify that $(C, s_4) \models \text{Cl}^1(0)$. We also infer the classification of all other states again, with the exception of s_7 . Indeed, we cannot infer a classification for s_7 as 0, since it has already been classified as 1; nor can we infer it as 1, because s_1 forces a decision for s_7 in the direction of 0. The new inferences are summarized in the following table.

	s_i	Cl^0	Cl^1
$k_1 :$	$\{p_1, p_2, p_3\}$	0	0
$k_2 :$	$\{p_1, p_2\}$	1	1
$k_3 :$	$\{p_1, p_3\}$	?	0
$k_4 :$	$\{p_2, p_3\}$	?	0
$k_5 :$	$\{p_1\}$	1	1
$k_6 :$	$\{p_2\}$	1	1
$k_7 :$	$\{p_3\}$	1	—
$k_8 :$	$\emptyset$	0	0

In the table, the symbol “—” indicates that no classification as 0 or 1 can be inferred.¹³ This aspect deserves further attention. The fact that a classification for s_7 cannot be inferred suggests that s_7 is involved in a form of ambiguity. Recall that p_3 is an ambiguous feature—essentially so by virtue of counterfactual reasoning applied to s_1 and s_7 . We also know that s_1 is stronger than any other state. Accordingly, we may say that s_7 should have been classified in the opposite direction in order to avoid the ambiguity associated with p_3 . This intuition is reflected in the impossibility of inferring the classification of s_7 at step 1.¹⁴

We emphasize that, by definition, a classification in the direction of o can be inferred for a case only if a classification in the opposite direction $\bar{o}$ has not already been inferred for that same case. On this basis, the following result holds.

¹³Recall that, by Definition 9, we can only infer classifications as 0 or 1; we cannot infer them as ?, which denotes the absence of a decision. Instead, the symbol “—” highlights the impossibility of inference.

¹⁴This intuition could form the basis for revising or correcting the case base, but we leave such considerations for future work.

Proposition 2.23. *Let $k \geq 1$ and $s \in S$. If $(C, s) \models \text{Cl}^k(o)$ then, $(C, s) \models \neg \text{Cl}^j(\bar{o})$, for all $j \geq k$.*

Since we cannot infer a classification in a different direction for a case that has already been inferred, and the set of states is finite ($S = 2^{\text{Atm}_0}$), the process can involve at most $2^{|\text{Atm}_0|}$ iterations, inferring at most one new case per iteration. In other words, the inferential process we have defined eventually stabilizes: it reaches a step k at which no new classifications can be inferred for any state.

Proposition 2.24. *It exists $k \leq 2^{|\text{Atm}_0|}$ such that there is no $s \in 2^{\text{Atm}_0}$ such that $(C_{\text{or}}, s) \models \text{Cl}^k(o)$ and $(C_{\text{or}}, s) \not\models \text{Cl}^{j \leq k-1}(o)$.*

2.4.3 Future work: Dynamic Update

In the previous section, we introduced a *static* inference mechanism for deciding new cases—that is, a mechanism that does not require updating the model. In [36], we also proposed a *dynamic* updating mechanism, defined from a semantic perspective, which updates the model as new decisions are inferred. We summarize this approach below, while leaving the study of the corresponding logic for future work.

Let $C_{\text{or}} = (S, f, \leq)$ be an ordered classifier model. This time, we can update the model iteratively as follows. Define

- $C_{\text{or}}^0 =_{\text{def}} (S, h_0, \leq)$ with $h_0 =_{\text{def}} f$.
- For $k \geq 0$ update the model as $C_{\text{or}}^{k+1} =_{\text{def}} (S, h_{k+1}, \leq)$, with

$$h_{k+1}(s) = \begin{cases} 0 & \text{if } (C_{\text{or}}^k, s) \models \text{Cl}^1(0) \\ 1 & \text{if } (C_{\text{or}}^k, s) \models \text{Cl}^1(1) \\ h_k(s) & \text{otherwise.} \end{cases}$$

Note that $\text{Cl}^1(0)$ and $\text{Cl}^1(1)$ are defined in Definition 2.22. We update the model each time new classifications are obtained, following the same reasoning applied in the static inference. However, in this case, it suffices to perform only the “first inference step” described in Definition 2.22.

Example 2.13. *Let us consider the dynamic update of the classifier $C_{\text{or}} = (S, f, \leq)$ from Example 2.11. We report directly the first update step, in which we obtain $C_{\text{or}}^1 = (S, h_1, \leq)$, as described in the table below.*

	s_i	$h_1(s_i)$
$k_1 :$	$\{p_1, p_2, p_3\}$	0
$k_2 :$	$\{p_1, p_2\}$	1
$k_3 :$	$\{p_1, p_3\}$	0
$k_4 :$	$\{p_2, p_3\}$	0
$k_5 :$	$\{p_1\}$	1
$k_6 :$	$\{p_2\}$	1
$k_7 :$	$\{p_3\}$	0
$k_8 :$	$\emptyset$	0

The classifications at step 1 of s_3 and s_4 remain the same as those inferred under the static procedure. It is worth noting that the dynamic update accumulates all the inferences obtained in the previous static steps: for s_7 , the dynamic classification at step 1 coincides with that of the original classifier (and so with the static inference at step 0), whereas in the static model at step 1, no classification could be inferred for s_7 .

The following proposition establishes a form of equivalence between static inference and dynamic updating, at the semantic level, showing that dynamic updating at a given step *accumulates* all that has been inferred statically up to that step.

Proposition 2.25. *For all $s \in S$, $k \geq 0$, $o \in \{0, 1\}$, $(C_{\text{or}}, s) \models \text{Cl}^{j \leq k}(o)$ iff $(C_{\text{or}}^k, s) \models \text{t}(o)$.*

2.5 Discussion

In this chapter, we proposed models of legal CBR that do not rely on the strong assumptions A1, and A3, while retaining assumption A2.

In particular, we began by abandoning assumption A1, and presented a logic-based approach for identifying which features within a case base behave as factors. Relying

on counterfactual reasoning, we offered an analysis of these features. This analysis is *pro tanto*: as new information becomes available, the labeling of features may change. It is also case base–dependent: the same features may behave differently across different case bases. This contrasts with traditional factor-based approaches, in which the partition of features into factors is fixed *a priori*. Indeed, we observed that some factors, in the sense of an *a priori* feature partitioning, do not qualify as *pro tanto* factors according to our analysis.

Building on our *pro tanto* analysis, and relying solely on feature comparison (thus adopting A2), we defined the conditions under which decisions are forced by precedent cases. By admitting the possibility of retrieving conflicting precedents (thereby abandoning A3), our method resolves such conflicts through a preorder relation that ranks cases by their relative authority.

Some aspects deserve further discussion.

Let us first focus on *ambiguous features*. These are features that behave differently depending on the case in which they occur—in other words, features involved in cases whose decisions are inconsistent with one another. Their interpretation varies with context. The first two interpretations we propose correspond to different ways of understanding such inconsistencies.

In a *flat case base*, where all cases are equally important (as in Horty’s reason and result models), ambiguous features may be regarded as the result of an error or a violation of a precedential constraint—that is, as arising from a case inconsistent with the initial case base, one that should have been decided differently.

In contrast, in an *ordered* or *hierarchical case base*, where not all cases carry the same weight, the presence of ambiguous features can be understood as reflecting a shift in jurisprudential approach. Even in common law systems—where consistency and *stare decisis* are fundamental—courts may sometimes change or overturn precedent, for instance when a higher court departs from the decisions of lower courts. In this context, it is reasonable to regard some cases as more important than others, depending on the authority of the deciding court. Thus, what from a flat perspective appears to be an inconsistent decision may, in an ordered case base, represent a legitimate development or modification of the line of case law.

We will return to this issue in detail in the next chapter. For now, note that within the framework of ordered classifiers, we employed a preorder on cases to resolve conflicts between precedents that force opposite outcomes for a new case, giving priority to the “best” precedents. Importantly, in our inference mechanism we fixed ambiguous and irrelevant features at each step and reasoned only with *pro tanto* factors. For future work (as suggested in Footnote 14), if we adopt the view that ambiguity reflects a change in jurisprudential approach, it may be fruitful to use the preference order among cases to revise and refine the existing case base itself, thereby yielding richer information and enabling further inferences.

A different perspective on ambiguous features concerns possible *interactions* among features. The common assumption that factors should have a fixed direction—namely, that every legal factor favors either the plaintiff or the defendant, but never both—can be questioned. Harty [50] notes that, in moral philosophy, there are arguments suggesting that the polarity of a factor may depend on context: a given factor might favor one side when combined with certain features, but the opposite side when combined with others. To illustrate, Harty offers an analogy outside the legal domain: when deciding whether conditions are favorable for a run, both extreme heat and rain are unfavorable separately, yet their combination might be acceptable—or even refreshing. On this interpretation, neither heat nor rain has a fixed polarity: each makes running less favorable in one context (alone) but more favorable in another (together). It would be interesting to investigate whether something similar could occur in the legal domain. A more detailed study of possible interactions among features in legal reasoning is left for future work.

Focusing again on the preorder between cases, we used it to resolve conflicts among multiple forcing cases for a case under examination—specifically, to handle situations in which precedents force opposite outcomes for a new case, giving priority to the decisions of the “best” cases. To this end, we employed the maximality operator ($[\text{Best}(\psi)]\phi$) introduced by van Benthem *et al.* [96]. This operator was originally proposed in the context of dyadic deontic logic—the logic of reasoning with dyadic obligations, where expressions of the form $O(\phi \mid \psi)$ are read as “it ought to be the case that ϕ if it is the case that ψ .”

The earliest semantics for dyadic deontic logic, due to Hansson [46], interpreted $O(\phi \mid \psi)$ as “all the best ψ -states are ϕ .” In [96], the maximality operator captures precisely

this interpretation. For future work, it would be interesting to explore what kind of deontic logic can be derived from our framework, and how it relates to the broader account developed in [96]. In particular, deontic operators would allow us to make explicit when a decision is required in a certain direction because it is constrained by prior cases. Crucially, in this setting deontic operators do not concern the substantive content of decisions: they are not meant to determine the duties or permissions established by a case, but rather whether a decision for a case is obligatory or permitted.¹⁵ This would align with previous work by Canavotto [25].

The finiteness assumption for the states of the model deserves further attention. This follows from our assumption that the input variables of the classifiers are finite. As noted in Chapter 1, this is consistent with a common approach in legal case-based reasoning, where it is often (sometimes implicitly) assumed that the set of factors or case features is finite. However, this is an *ad hoc* assumption, which makes sense in application-oriented models where construction constraints on the reasoner require the finiteness of cases (as in the HYPO system). From a broader perspective, however, whether the features required to describe a case are finite depends on the granularity of the language and on the underlying theory of the world. In the next chapter, we will return to this issue and propose a logic of classifiers with a countably infinite number of input variables/features.

In connection with the previous observation, let us recall that in this chapter we have made extensive use of conjunctive formulas such as $\text{cn}_{X, \text{Atm}_0}$, which—as noted in Chapter 1—serve, in the context of finite models, essentially as pointers to states. That is, $\text{cn}_{X, \text{Atm}_0}$ is satisfied by a single state, namely $s = X$. Having such pointers for states was particularly important for defining the notions of *pro tanto* monotonicity and forcing in our iterative inference mechanism. In the next chapter, we will extend the framework to models with countably many input variables, in which formulas of the form $\text{cn}_{X, \text{Atm}_0}$ will no longer be well-formed. In that setting, specific propositional atoms functioning as names for states will need to be introduced.

In the current framework, each state corresponds to a case described solely in terms of factual situations, and each state is mapped by the classification function to a single

¹⁵The author thanks Giovanni Sileno for highlighting the need to clarify this aspect.

outcome. This excludes the possibility of having two cases with the same factual situation but different decisions. In the enriched framework that we will present in the next chapter, however, it may occur that two distinct states share the same factual situation but have different names. Consequently, it becomes possible to represent two cases with identical factual situations but different decisions.

In our framework, we implicitly rely on the closed-world assumption when treating a factual situation as a conjunction of positive and negative literals. In particular, we assume that the absence of a feature corresponds to the presence of its negation. Horty [50] and Ashley and Aleven [4] suggest that the closed-world assumption should be applied on a case-by-case basis.

As an illustration, Ashley and Aleven—cited by Horty—discuss the application of the closed-world assumption in trade secret law. Consider the factor representing the “existence of an explicit confidentiality agreement between the plaintiff and the defendant.” The absence of positive information establishing such an agreement allows the court to conclude that none exists. By contrast, this reasoning does not apply to the factor representing the “existence of a confidential relationship between the parties”: even in the absence of explicit information, the court cannot infer that no such relationship exists. From this perspective, a negative factor can be interpreted only as the absence of its corresponding positive factor.

It would be worth investigating the implications of distinguishing, within our framework, between negation and absence—a distinction that could lend the framework an intuitionistic character.

We have approached this chapter by emphasizing how classifier logic can be used to extract information about feature behavior within a case base. Yet a more classifier-oriented perspective is also possible. In particular, the case bases under analysis could be taken as the input and output datasets of a binary classifier used in the legal domain to predict case outcomes, independently of the underlying learning model. In this way, the feature analysis we developed for case bases could be applied directly to datasets, allowing us to detect ambiguities and evaluate classifier compliance with precedential constraint. This approach would be in line with other existing works, including those by Prakken and Ratsma [81] and by van Woerkom et al.[100], which aim to develop

post-hoc XAI methods—primarily to justify the outcomes of machine learning algorithms or to verify their consistency by means of existing models of legal case-based reasoning.

Moreover, in this regard, it would also be interesting for future work to explore the relationship between our logic-based approach and previous studies addressing the problem of factor ascription—namely, the extraction of factors from the features that appear in legal cases. To the best of our knowledge, such studies rely either on machine learning methods, as suggested by Bench-Capon and Atkinson [16], or on hybrid techniques (see, among others, [72, 73, 12, 21]). These methods are particularly relevant for tackling the knowledge acquisition bottleneck in application-oriented models, as they support—or even replace—the work of human experts who manually examine the cases in the case base and index them according to the relevant factors.

Last but not least, we have noted that factor-based approaches to legal case-based reasoning are widely adopted. However, they are not the only available approaches. We are not the first to abandon the assumption that cases must be represented in terms of directional features (or factors), nor the first to suggest that cases may instead be ordered according to their relative importance. For example, in previous work by Verheij [102], cases are not modeled through factors but rather in terms of logical properties they satisfy. Within that framework, cases are explicitly ordered according to a preference relation. Such a work is developed with aims that differ from ours: broadly, to show how cases can provide the logical foundation for determining which rules and arguments hold within a given domain. The framework proposed in [102] has subsequently served as the basis for further developments in legal CBR. These include, among others, a formal treatment of different types of hardness in case-based decisions [109], logical comparisons in CBR [108], and a formal demonstration of the validity of legal arguments within CBR [107]. A more systematic investigation of the possible connections between these frameworks and our own is left for future work.

3

When Precedents Clash

In this chapter, we study case-based reasoning models that, in particular, do not satisfy the following strong assumptions:

- A2) Factual similarity/comparison *alone* is taken to determine whether a precedent exerts a constraining effect on the decision in a given case (e.g., by comparing the factors present in each case).
- A3) The case base is constructed so that no conflict of precedents may occur — i.e., there are no situations in which, for the same case to be decided, two precedents force opposite outcomes (e.g., one favoring the plaintiff and another the defendant).

In Section 1.1.1, we discussed how these assumptions are instantiated within Horty's reason and result models. With respect to assumption A2, we noted that, in Horty's models, the factual comparison used to determine whether one case constrains (or forces) the decision in another is based exclusively on factors. For example, in the result model, a new case should be assigned the same outcome as a precedent case if it includes an equally or more comprehensive set of factors supporting the precedent's outcome, and no additional factors opposing it.¹ No other elements are considered in determining whether a decision is forced.

¹This corresponds to the relation denoted by forces; see Definition 1.6.

As for assumption A3, we observed that in Horty's models to prevent conflicts the *consistency* of the starting case base is required. However, requiring a case base to be consistent—that is, requiring all judicial decisions within a corpus to be mutually compatible—can be an excessively demanding constraint. As discussed in Section 1.1.1, Canavotto [25] proposes a notion of precedential constraint that remains applicable even in the context of inconsistent case bases. Nevertheless, not all conflictual situations can be resolved through the method proposed in [25].

To better illustrate the points discussed so far, let us recall Example 1.4.

Example 3.1. *We refer to Example 1.4. Recall we have $\text{Facts} = \{\pi_1, \pi_2, \pi_3, \delta_1, \delta_2, \delta_3\}$ with $\text{Plt} = \{\pi_1, \pi_2, \pi_3\}$ and $\text{Dfd} = \{\delta_1, \delta_2, \delta_3\}$. The case base is $\text{CB} = \{k_6, k_7\}$ where*

- *k_6 has fact situation $s_6 = \{\pi_1, \delta_1, \delta_2\}$, and outcome 1;*
- *k_7 has fact situation $s_7 = \{\pi_1, \pi_2, \delta_1\}$, and outcome 0;*
- *The new fact situation to be decided is $s_8 = \{\pi_1, \delta_1\}$.*

Recall from Example 1.4 that the case base CB is inconsistent. Consequently, the precedential constraint defined in the result model cannot be applied to s_8 , since—as stated in Definition 1.4—Horty's notion of precedential constraint requires the starting case base to be consistent.

In Chapter 1, we also defined the constraining relation forces (Definition 1.6) among cases, which is grounded in the notion of precedential constraint but does not presuppose the consistency of the case base. We introduced this relation primarily to highlight the significance of the consistency assumption in Horty's models. Indeed, we showed that when the case base is inconsistent, it is possible to retrieve two precedent cases that compel opposite outcomes for the same situation at hand. In particular, we demonstrated that in Example 1.4 we obtain both $\text{forces}(k_6, s_8, 1)$ and $\text{forces}(k_7, s_8, 0)$ —meaning that k_6 requires a decision as 1 for s_8 , while k_7 requires a decision as 0 for s_8 . This appears to be a conflict of precedents.

In Example 1.6, we also noted that, according to Canavotto's notion of precedential constraint for inconsistent case bases [25], both decisions 1 and 0 are permissible. Indeed, either outcome would avoid introducing further inconsistencies into the case base.

When multiple decisions are permissible, Canavotto proposes resolving the conflict by considering the relative support for each option—that is, by weighing the number of cases that endorse or oppose a given decision. In this situation, k_6 endorses decision 1 (and opposes 0), while k_7 endorses decision 0 (and opposes 1).

We will not go into further detail regarding the approach in [25], but we emphasize two key points. First, the proposed solution assumes that all cases carry equal weight—that is, each case is equally important. Second, it does not always suffice to disambiguate which permissible outcome should be selected: in this example, the number of cases supporting or opposing a decision for the plaintiff is equal to that for the defendant—one on each side.

In this chapter, we argue that what may appear to be a conflict—or, in Canavotto’s terms, a situation involving multiple permissible decisions pointing in opposite directions—may, in fact, stem from a lack of information. In judicial practice, the precedential constraint depends not only on the facts of the cases but also on other crucial elements. For instance, in this example, we do not know which courts decided the cases under consideration, nor when those decisions were rendered. We will return to this point shortly.

The observations made so far intersect with several of the considerations discussed in the previous chapter. Recall that we proposed a framework of CBR for analyzing the *pro-tanto* behavior of features and, on the basis of this analysis, modeled an inferential mechanism to determine when the decision in a new case is determined by previously decided ones. In doing so, we adopted assumption A2, establishing that precedent cases force the decision for new cases on the basis of feature comparison—taking into account our *pro-tanto* taxonomy of features. However, we rejected assumption A3. Instead, we admitted the possibility of retrieving conflicting precedents and assumed the existence of a preorder among cases, whereby, in the event of a conflict between forcing precedents relevant to the case under examination, the “best” case should be followed.

In this chapter, we aim to further develop the intuition that conflicts between precedents may arise in legal case bases and that, in such situations, certain cases can indeed be regarded as more important than others. In pursuing this line of inquiry, however, we will set aside both our *pro tanto* analysis of features and Horty’s models. Nor will we

assume that a preorder among cases is given a priori. Instead, we shall work within a broader framework, in which neither A2 nor A3 will be assumed, and which we will outline in the following paragraphs.

In light of the fact that assumptions A2 and A3 will not be adopted, we are led to ask the following questions:

- Q2)** *Which precedents are binding for the case at hand, considering both the factual relevance between the cases and the jurisdictional relations among the courts that decided them?*
- Q3)** *How can conflicts among binding precedents be resolved in a legally grounded way?*

As we shall see, Q2 can be broken down into the following sub-questions:

- Q2a)** *What counts as a precedent?*
- Q2b)** *Which precedents are binding—that is, which precedents actually constrain/ force the decision for a given case?*

We will address these questions within the frameworks of classifier models and BCL logic, enriched with the following elements.

Relevance

In Horty's models, as well as in our model with our *pro-tanto* analysis of features, we identified binding cases solely on the basis of factual comparison. That is, we considered as binding those cases that are applicable to the case at hand purely due to factual relevance. This relevance was grounded on different forms of precedential constraint. To better explain what we mean by factual relevance we elaborate on above Example 3.1.

Example 3.2. *In Example 3.1, we know that $\text{forces}(k_6, s_8, 1)$ and $\text{forces}(k_7, s_8, 0)$. Now, recall that $\text{forces}(k_6, s_8, 1)$ and $\text{forces}(k_7, s_8, 0)$ are defined in terms of factual comparison (see Definition 1.6). Recall also that $\text{forces}(k_6, s_8, 1)$ and $\text{forces}(k_7, s_8, 0)$ mean that*

k_6 and k_7 force (or constrain) a decision for s_8 . In this sense, we are considering as binding those cases that are applicable to the case at hand solely on the basis of factual relevance. However, as we will see in the following paragraphs, the binding force of one case with respect to another depends on more than factual relevance alone. For this reason, we refine our terminology: rather than saying that k_6 and k_7 are constraining or forcing for s_8 , we say that k_6 and k_7 are relevant to s_8 .² Determining whether k_6 and k_7 are actually binding for s_8 will require additional information.

In our framework for the *pro tanto* analysis of features, introduced in the previous chapter, we adopted an alternative notion of factual relevance. To illustrate this, we refer to Example 2.12.

Example 3.3. Let us recall the situation in Example 2.12. We had a case base summarised in following table:

	s_i	CI^0	CI^1
$k_1 :$	$\{p_1, p_2, p_3\}$	0	0
$k_2 :$	$\{p_1, p_2\}$	1	1
$k_3 :$	$\{p_1, p_3\}$	?	0
$k_4 :$	$\{p_2, p_3\}$	?	0
$k_5 :$	$\{p_1\}$	1	1
$k_6 :$	$\{p_2\}$	1	1
$k_7 :$	$\{p_3\}$	1	—
$k_8 :$	$\emptyset$	0	0

We established our own notion of precedential constraint based on the *pro tanto* analysis of features. In particular, in this example we identified p_1 and p_2 as *pro tanto* factors, while p_3 was classified as an ambiguous feature. On this basis, we determined that the forcing precedents for s_3 (the factual situation of k_3) were s_1 and s_7 (the factual

²By introducing the terminology of *relevant* cases, we do not intend to capture any cognitive function. Rather, this terminological refinement aims to emphasize that not all cases assumed to constrain the outcome in existing CBR models—such as the result model—are necessarily binding. A case may be factually relevant without being binding, as discussed in the following paragraphs and throughout this chapter. The author thanks Giovanni Sileno for drawing attention to this issue.

situations of k_1 and k_7 respectively). No additional information was used to determine which precedents were constraining. We then applied the preorder among cases to resolve the conflict between these precedents and concluded that the decision of s_1 , namely 0, was to be followed by s_3 .

Our framework differs from Horty's models in that it allows for the possibility of conflicting constraining precedents. However, even within this setting, when determining which precedents were binding, we disregarded certain jurisdictional elements (such as the court that issued the decision) and temporal elements (such as the time at which the decision was rendered).

The relation we identified as a constraining relation among cases can, in fact, be more accurately understood as a relation of relevance: s_1 and s_7 , rather than forcing a decision for s_3 , can be regarded as factually relevant to it. Once again, determining whether they are truly binding requires taking additional elements into account.

The Examples 3.2 and 3.3 just discussed highlight that we are assuming a rather strong notion of relevance: in both cases, we treat as relevant those precedents that are, in a sense, factually constraining—or, more precisely, that leave no room for *distinguishing*. Broadly speaking, a case at hand may be distinguished from a precedent if it can be shown that the precedent may not be applied due to some significant difference, and thus that the case may be decided differently. s_8 in Example 3.2 cannot be distinguished from the relevant cases k_6 and k_7 , since doing so would introduce further inconsistencies.³ The reader may therefore interpret *relevant* as meaning *factually relevant and not distinguishable*.

Other criteria for determining factual relevance are adopted in different approaches to legal CBR. In the *strict rule model* [22], a precedent is considered applicable (and not distinguishable) whenever its underlying reason applies to the new case. In many arguments-based approaches, the similarities and differences between cases can be strategically emphasized or downplayed [83].⁴ In this chapter, we do not commit to any specific model of legal CBR. Our primary interest lies in refining the notions of precedent and binding precedents and in addressing the problem of conflicting binding

³Horty [52] discusses in detail the notion of distinguishing also in the context of the reason model. We will return on this briefly in section 3.6.

⁴Within the argumentation settings, we may again understand relevant cases as those that cannot be distinguished (for example, because any differences can be downplayed).

precedents. Henceforth, when we refer to a case as relevant to the case under consideration, we mean that it is *factually* relevant to the case at hand (and, as we just stated, not distinguishable). Thus, within classifier models, we introduce a relevance relation and its corresponding modal operator, both of which are new with respect to the BCL framework, and which identify the states that are factually relevant to the case at hand. The relevance relation specifies a priori which states are relevant; *the reasons why they are relevant fall outside the scope of this chapter*. Relevant states/cases, depending on specific criteria that we will introduce in this chapter, may qualify as binding precedents, but, as we will see, are not necessarily binding.

Time and Jurisdiction.

As mentioned above, in both Horty's models and in our inferential mechanism, a simple factual (or relevance-based) comparison directly determines whether a case is binding. This approach, however, overlooks some essential aspects of legal reasoning.

First, we ignored so far the *temporal dimension* of cases, which is nevertheless crucial in legal decision-making. Indeed, a case can serve as a precedent for the case at hand only if it precedes it in time. More specifically, a precedent may be defined as a past case that is relevant to the case at hand.

Second, legal cases are decided within specific *jurisdictions*, and the precedential constraint is shaped by the jurisdiction in which decisions are rendered. Each jurisdiction is defined by its own organization and hierarchy of courts. Within a given jurisdiction, a court deciding a new case may be bound by certain courts but not by others. For example, in many common law systems, the principle of *vertical stare decisis* applies: courts are bound by the decisions of higher courts, while the decisions of lower courts are not binding upon them. Consequently, not all precedent cases—that is, previously decided relevant cases—in the case base are binding for future decisions; only those issued by courts with binding authority are.

Moreover, both the hierarchical and temporal dimensions allow us to formalize certain *exceptions* to the binding force of precedent—namely, situations in which a case loses its binding authority over subsequent decisions. In this chapter, we focus on two such exceptions: overruling and *per incuriam*. Broadly speaking, a case is *overruled* when a subsequent decision of a court with the requisite authority departs from it and alters the

established line of law. Cases decided *per incuriam* (literally, “through lack of care”) are those rendered “in ignorance of a relevant statutory provision or a relevant, binding decision of a court” [60]. In this chapter, we restrict attention to *per incuriam* cases that disregarded a binding decision of another court—that is, cases that ignored a binding precedent without having the authority to do so. We will return to these issues in more detail later.

Finally, jurisdictional hierarchy and temporal ordering enable us to formulate *principles* for resolving conflicts between precedents—principles that are, in fact, applied in legal practice and reflect our intuition from the previous chapter that some precedents carry greater weight than others. In particular, we will consider a *Hierarchical Principle* and, as a refinement of it, a *Temporal Hierarchical Principle*. According to the Hierarchical Principle, when precedents conflict, the decision to be followed is the one issued by the highest-ranking court. According to the Temporal Hierarchical Principle, when precedents conflict, the decision to be followed is the *most recent one* issued by the highest-ranking court.

To capture all these considerations, it is necessary to incorporate both the temporal dimension of cases and the hierarchical structure of courts into the classifier models and into the related logic.

Remark 3.1. *The Temporal Hierarchical Principle mentioned above corresponds to the combination of two brocards⁵ governing the resolution of conflicts between norms: lex posterior (lex posterior derogat priori, “later law repeals earlier law”) and lex superior (lex superior derogat inferiori, “higher law repeals lower law”). Legal brocards also include a further principle, lex specialis (lex specialis derogat legi generali, “a law governing a specific matter overrides a law which only governs general matters”). The formalization of lex specialis lies beyond the scope of this thesis. This also follows from the fact that, as we will see, the notion of relevance between cases is here treated as Boolean: a case is either relevant to the case under consideration or it is not. There are no cases that are more relevant than others. The development of a graded notion of relevance is left for future work.⁶*

⁵A brocard is a concise legal maxim, typically rooted in the tradition of Roman law.

⁶Thanks to Giovanni Sileno for drawing attention to this aspect.

Countable set of input features.

In the previous chapter, we assumed that the set of input features Atm_0 of the classifier—that is, the features used to describe cases—was finite. A direct consequence of this assumption is that the set of possible cases, both decided and undecided, is also finite, since the set of states (or cases) S is a subset of 2^{Atm_0} .

In this chapter, however, we separate two levels of analysis, distinguishing between the finiteness of features and the finiteness of decided states/cases.

Specifically, we remove the assumption of finiteness for input features and instead assume that features are countable (possibly infinite). We do this for two main reasons. From a technical perspective, it allows us to define a modal logic (for classifiers) with countably infinite propositional atoms, which is the standard setting in modal logic. From a conceptual perspective, the number of features required to represent a case depends on the level of granularity chosen for the language. Allowing countably many features for case description therefore enables more expressive representations. In any case, note that finite features are also countable. Thus, what we are proposing is a general model that accommodates both finite and countably infinite input features.

Nonetheless, we require that the set of already decided cases remain finite. This assumption is consistent with the finiteness of the institutional world: legal precedents produced so far are, necessarily, finite in number.

Names for states.

We recall that the finiteness assumption on features allowed us to introduce pointers or names for states, through formulas of the form cn_{X,Atm_0} . Once we remove the finiteness assumption, such formulas are no longer well-formed. Nevertheless, we will still need names for states. To achieve this, we introduce dedicated variables serving as names, thereby obtaining a very simple *hybrid logic* [18, 8, 43, 19].

The term hybrid logic refers to a family of logics obtained by extending the expressive power of ordinary modal logics. The simplest hybrid logics arise by enriching the language with so-called *nominals*: propositional symbols of a new kind, each of which is true at exactly one possible world (in our case, a “state”). Since a nominal is true at exactly one point, it can be regarded as a term that refers to that point.

Nominals, moreover, make it possible to define the intersection of two modalities, something not possible within ordinary modal logic [8]. Intuitively, an intersection modality holds when both underlying relations of the intersected modalities hold simultaneously. This is of particular importance for us, since we wish to define the notion of a precedent: a precedent for a given case is a case that was decided earlier and deemed relevant. As mentioned, in our framework we introduce both a relevance modality and a temporal modality. Hence, to capture the notion of a precedent, we must consider the intersection of these modalities.

Naming states is not only useful from a technical standpoint but also meaningful from a legal perspective: in practice, every case is associated with a name, which can be either a unique identifier or a conventional label—typical in common law judgments, such as “A vs. B”.

Roadmap to Chapter 3

- **Section *Example: Cases within the Jurisdiction of England and Wales* (3.1)** introduces two illustrative examples that will be used throughout the chapter.
- **Section *Temporal Jurisdictional Classifiers Models and Logic* (3.2)** presents both the semantics and the logic for modeling legal reasoning while taking into account the elements described so far: the relevance relation, temporal and jurisdictional dimensions, a countable set of input features and names for states.
 - **Subsection *Temporal and Jurisdictional Classifier Models* (3.2.1)** introduces the semantic framework.
 - **Subsection *Temporal and Jurisdictional Classifiers Logic* (3.2.2)** develops the corresponding logic TJCL and provides completeness results.
- **Section *Precedents, Binding Precedents, and Exceptions* (3.3)** builds on the framework introduced in the previous section to formally define the notions of precedent, binding precedent, and exceptions to the binding force of a precedent.
 - **Section *Potentially Binding Precedents* (3.3.1)** specifies the conditions under which a precedent may potentially bind future decisions.

- **Section *Exceptions: Incuriam and Overruling* (3.3.2)** formalizes two situations in which a potentially binding precedent loses its binding force. A precedent can be *overruled* or declared *per incuriam*.
- **Section *Resolving Precedent Conflicts for New Cases* (3.4)** proposes criteria for resolving conflicts between binding precedents without exceptions when applied to new cases. It introduces two central principles:
 - the **Hierarchical Principle**, which resolves conflicts by reference to the hierarchy of courts; and
 - the **Temporal Hierarchical Principle**, which integrates temporal order and hierarchical authority in determining which precedent prevails.
- **Section *More on the Relevance Relation and per incuriam* (3.5)** discusses possible properties of the relevance relation in connection with the notion of *per incuriam*. In particular, it considers a relevance relation grounded in Horty’s *a fortiori* reasoning.
- **Section *Discussion* (3.6)** concludes the chapter by reviewing relevant literature, summarizing the implications of the proposed framework, and outlining possible directions for future research.

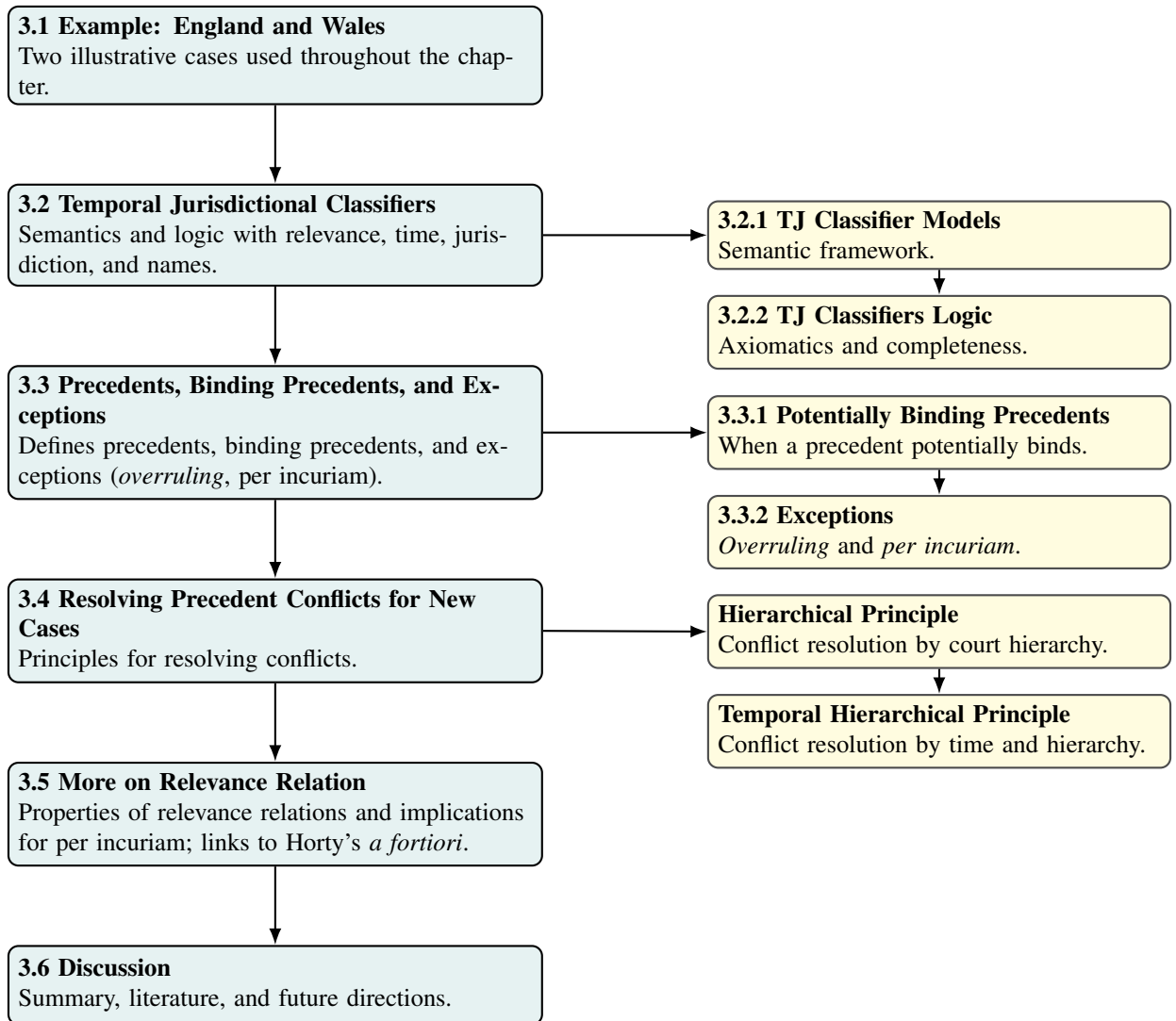


FIGURE 3.1: Roadmap to Chapter 3

3.1 Example: Cases within the Jurisdiction of England and Wales

In this section, we introduce two examples that will be used throughout the remainder of the chapter. The first example provides an overview of the jurisdiction of England and Wales, while the second concerns cases to be decided within that jurisdiction.

Example 3.4 (England and Wales Jurisdiction). *In this chapter, we draw on the civil court system of England and Wales. At the top of the judicial hierarchy stands the Supreme Court of the United Kingdom (denoted c_0). Officially, the Supreme Court is not bound by its own previous decisions.⁷ Immediately below it is the Court of Appeal (c_1), whose decisions are binding on itself. Further down the hierarchy lies the High Court (c_2). For simplicity, we shall assume that the High Court is bound by its own decisions.⁸ At the base of the hierarchy are the County Courts (of which there are approximately 170 [95]), which do not issue binding precedents; for modeling purposes, we include only two of them, c_3 and c_4 .*

In the jurisdiction of England and Wales, vertical stare decisis applies: higher courts always bind lower courts, while lower courts never bind higher courts.

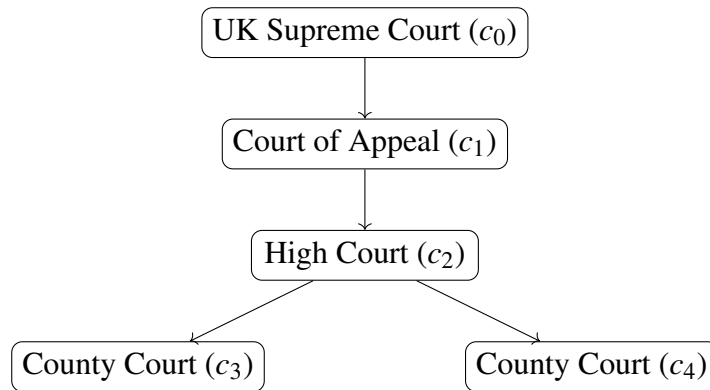


FIGURE 3.2: The structure of the civil courts of England and Wales

⁷In practice, however, the UK Supreme Court treats its own decisions as binding in certain circumstances that depend on the composition of the court—specifically, on the number of judges sitting on the panel deciding the case [65]. We will return to this point in the next chapter.

⁸Some authorities suggest that the High Court is self-bound only when acting in its appellate capacity [45]. We will return to this in the next chapter.

Example 3.5 (Cases within EW Jurisdiction). In Example 3.1, we identified a conflict of precedents for s_8 : both k_6 and k_7 were relevant — based on the *a fortiori* reasoning — to s_8 , but they were decided in opposite directions. From this point onward, we will consider a more complex and abstract example than the one presented above, in order to better illustrate the scenarios in legal practice that we aim to model in this paper. Let s^* represent a new case to be decided based on previous cases $s_9, \dots, s_{16}$. Each case is assigned at least a name: s^* has name n^* and $s_9, \dots, s_{16}$ have names $n_9, \dots, n_{16}$. For example, more intuitively, we may think of n^* as the name “A vs. B” assigned to the case s^* . The relevance relations among the cases, as well as their decisions, are shown in Figure 3.3.

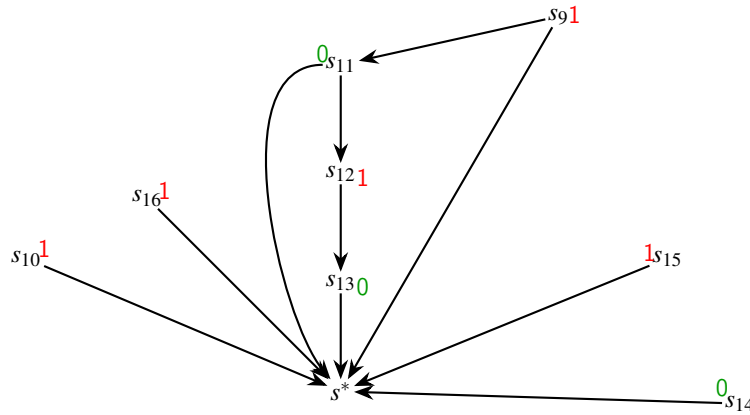


FIGURE 3.3: Relevance relations among cases in Example 3.5.

Legend. 1 denotes "plaintiff wins"; 0 denotes "defendant wins"

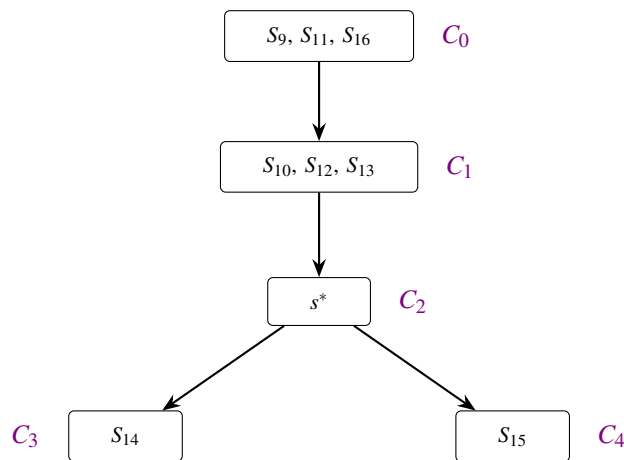


FIGURE 3.4: Hierarchical relation among cases in Example 3.5

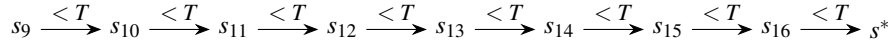


FIGURE 3.5: Temporal relation among cases in Example 3.5

All cases s_i are all relevant to s^* , yet some have been decided as 0 and others as 1. For example, both s_{11} and s_9 are relevant to s^* , but s_{11} is decided for the defendant, while s_9 is decided for the plaintiff – we can think of s^* as analogous to s_8 in Example 3.1, with s_9 corresponding to k_6 and s_{11} to k_7 .

Again, this seems to be a conflict among precedents. But as we said in the introduction to this chapter, we overlooked some foundational elements of legal decision-making: jurisdiction, time and exceptions.

Jurisdiction. Assume the cases are decided or to be decided within the England and Wales jurisdiction reported in Example 3.4. Specifically, there exists a hierarchical relationship between the cases, as illustrated in Figure 3.4. Consider cases s_{14} and s_{15} , which are relevant to our case s^* . However, they were decided by courts c_3 and c_4 , which do not hold binding authority over court c_2 deciding s^* . Therefore, s_{14} and s_{15} are not binding for s^* . We can eliminate s_{14} and s_{15} from binding precedents. But s_9 , s_{10} , s_{11} , s_{12} , s_{13} and s_{16} may still be binding.

Time. Obviously, a case is constrained only by earlier decisions. Assume the timeline for our cases is as in Figure 3.5. Then, case s^* is decided after s_9 , s_{10} , s_{11} , s_{12} , s_{13} and s_{16} , which could therefore be binding for s^* . Time is also useful to model exceptions.

Exceptions. As noted earlier, precedential constraint is subject to exceptions: there may be situations in which earlier cases lose their binding authority over subsequent ones. One such case is overruling. Roughly, overruling occurs when a court decides against a relevant precedent [66, 85] and thereby alters the line of case law. Not all courts, however, possess the authority to overrule. In line with the practice typically adopted in common law systems, in this chapter we assume that a higher court may always overrule the decisions of a lower court, and that a court may overrule its own decisions when it is not self-bound [66].

For example, consider case s_{11} . This case was decided after the relevant precedent s_9 . Both s_{11} and s_9 were issued by the same court—the Supreme Court c_0 . The decision in s_{11} differs from that in s_9 . Since we have assumed that c_0 is not bound by its own prior

decisions, it has the authority to overrule them. Accordingly, s_9 was overruled by s_{11} and therefore lost its binding authority over s^* . Hence, we can eliminate s_9 as a source of conflict.

Now, it seems that cases s_{12} , s_{11} , s_{10} , s_{13} , and s_{16} remain. We will see that some of these cases can be regarded as decided *per incuriam*—that is, in disregard of a binding precedent—and therefore not binding for s^* . Since modeling *per incuriam* requires particular care, we will not go into detail here. This notion will be addressed in Section 3.3.2.

What should be the decision for s^* ? In this chapter, we will formalize the notions introduced so far and attempt to answer this question.

3.2 Temporal Jurisdictional Classifiers Models and Logic

In this section, we aim to enrich the framework of classifier models and BCL in order to formally model a legal decision-making process for new cases that takes into account the considerations discussed so far. In particular, Section 3.2.1 introduces the class of temporal and jurisdictional classifier models; Section 3.2.2 presents the corresponding language for temporal and jurisdictional classifier logic; and Section 3.2.2.1 sets out its axiomatization.

3.2.1 Temporal and Jurisdictional Classifier Models

In this section, we instantiate the jurisdictional and temporal dimensions, as well as the relation of relevance among states, within the framework of classifier models.

First, we define the notion of jurisdiction, which characterises the structure of a specific legal system. This includes the identification of courts, the hierarchical relation among them and which courts issue binding decision.

Definition 3.2 (Jurisdiction). A triple $(\text{Courts}, \mathcal{H}, \mathcal{B})$ is called a *jurisdiction*, denoted as *Jur*, when:

- 1) Courts is a finite and non-empty set of courts;
- 2) $\mathcal{H}, \mathcal{B} \subseteq \text{Courts} \times \text{Courts}$;
- 3) $\mathcal{H}$ is transitive and irreflexive;⁹
- 4) $\mathcal{H}$ is “tree like”: if $(c_k \mathcal{H} c_i \text{ and } c_j \mathcal{H} c_i)$ then $(c_k \mathcal{H} c_j \text{ or } c_j \mathcal{H} c_k \text{ or } c_j = c_k)$.

For $c_i, c_j \in \text{Courts}$, $c_i \mathcal{H} c_j$ is read as “court c_i is hierarchically higher than c_j ”; $c_i \mathcal{B} c_j$ is read as “ c_i has binding authority on c_j ”.

The tree-likeness property of the court hierarchy relation stipulates that, whenever two distinct courts are superior to the same court, one must rank hierarchically above the other.

Example 3.6. *The jurisdiction of Example 3.4, represented in Figure 3.4, can be modeled as $Jur_{ex} = (\text{Courts}_{ex}, \mathcal{H}_{ex}, \mathcal{B}_{ex})$, where: the set of courts is $\text{Courts}_{ex} = \{c_0, c_1, c_2, c_3, c_4\}$ and the hierarchy relation $\mathcal{H}_{ex} \subseteq \text{Courts}_{ex} \times \text{Courts}_{ex}$ is $\mathcal{H}_{ex} = \{(c_i, c_j) \mid i < j, 0 \leq i \leq 2, 1 \leq j \leq 4\}$. The binding relation is $\mathcal{B}_{ex} = \mathcal{H}_{ex} \cup \{(c_1, c_1), (c_2, c_2)\}$.¹⁰*

We introduce now the temporal and jurisdictional classifier models. We start by considering a set of possible input values of the classifiers, Atm_0 . In Atm_0 there are: input variables describing the facts of the case, belonging to the countable set Facts ; the courts that can make the decision, belonging to the finite set Courts ; the name of each case belonging to the countable set Names . Pertaining Names we assume that there is a finite subset of names $\text{Names}_d \subseteq \text{Names}$ for decided cases. The sets Facts , Courts , and Names are non empty and pairwise disjoint to form Atm_0 . In other words, $Atm_0 = \text{Facts} \uplus \text{Names} \uplus \text{Courts}$.¹¹ The classifier outputs have values in $Val = \{1, 0, ?\}$ where elements stand for *plaintiff wins*, *defendant wins* and *absent decision* respectively. As usual, for $o \in \{0, 1\}$, the “opposite” $\bar{o}$ is noted for the value $1 - o$.

Definition 3.3 (Temp. Jur. Classifiers). A *temporal jurisdictional classifier model* (TJCM) is a tuple $C = (S, f, Jur, \leq_T, \mathcal{R})$ which satisfies these conditions:

- $S \subseteq 2^{Atm_0}$ is a non-empty set such that $\forall s \in S \exists! c \in \text{Courts} : c \in s$;¹²

⁹ $\mathcal{H}$ is irreflexive, when: for all $c \in \text{Courts}$, it is *not* the case that $c \mathcal{H} c$.

¹⁰ We note that the jurisdiction of England and Wales also has a root: there exists a court $c_0 \in \text{Courts}$ such that, for any $c_i \in \text{Courts}$, it holds that $c_0 \mathcal{H} c_i$. We did not explicitly assume the existence of a root in Definition 3.2, as it is not technically required. Nevertheless, it represents a realistic condition for most legal jurisdictions.

¹¹ $\uplus$ denotes disjoint union.

¹² $\exists!$ means there exists one and only one.

- $f : S \rightarrow Val$ is called a decision (or classification) function;
- $|S_d|$ is finite, where $S_d = \{s \mid f(s) \neq ?\}$ is the set of decided states;
- Jur is a jurisdiction;
- $\leq_T$ is a total preorder on S ;¹³
- $\mathcal{R} \subseteq S \times S$ is a relation of relevant cases;
- (*name existence*) $\forall s \exists n \in \text{Names} : n \in s$;
- (*name for decided states*) $\forall s \in S \forall n \in \text{Names} \cap s : s \in S_d \Leftrightarrow n \in \text{Names}_d$;
- (*no shared name*) $\forall s, s' \in S : (s \neq s' \Rightarrow s \cap s' \cap \text{Names} = \emptyset)$.

The class of temporal and jurisdictional classifier models is **TJCM**.

Let us look at the definition in more detail.

- S is the set of states of the model, where each state contains a unique court. In this sense, a state $s \in S$ represents a case presented to a specific court $c \in \text{Courts}$.
- The classification function maps each state to a value in the set $\{0, 1, ?\}$, representing the decision associated with the case. As in previous chapter, for each $s \in S$, we either have $f(s) = o$, with $o \in \{0, 1\}$ — indicating that s represents an assessed case — or $f(s) = ?$, meaning s is an unassessed or new case.
- The set S_d of decided cases is finite.
- The structure Jur , as defined in Definition 3.2, specifies the hierarchical and binding relations among the courts.
- The relation $\leq_T$ is a temporal preorder, where $s \leq_T s'$ is interpreted as “ s' was not assessed before s ”. We say that s is simultaneous to s' , denoted $s =_T s'$, if both $s \leq_T s'$ and $s' \leq_T s$. From $\leq_T$, we derive the strict temporal order $<_T$: $s <_T s'$ if and only if $s \leq_T s'$ and $s' \not\leq_T s$. Thus, $s <_T s'$ means “ s was decided strictly before s' ”. We write $\leq_T[s] = \{s' \mid s \leq_T s'\}$ for the set of states that are assessed simultaneous or strictly later than s .

By using a temporal *preorder*, we allow two cases to be considered simultaneous. While it could be argued that, for any two cases, it is always possible to determine which was decided first, the temporal dimension is introduced here primarily to

¹³ $\leq_T$ is a transitive and reflexive relation, with $\forall s, s' \in S$, either $s \leq_T s'$ or $s' \leq_T s$.

determine which precedents may be considered binding. We claim that, given a case s , a relevant case s' decided on the same day or in the same week cannot be considered binding: the court assessing s would not have had the opportunity to take s' into account (e.g., because the reasons of s' had not yet been published). Therefore, depending on the chosen temporal granularity, cases decided on the same day or in the same week may be considered simultaneous.

- The binary relation $\mathcal{R}$ captures the notion of relevance: $s' \mathcal{R} s$ is read as “ s' is relevant to s ”. We write $\mathcal{R}(s) = \{s' \mid s' \mathcal{R} s\}$ for the set of states relevant to s . The relevance relation is crucial in defining the notion of a precedent — roughly, a precedent for a case is a relevant, previously assessed case. No specific definition of relevance is imposed in the model, nor are any particular properties (such as symmetry or transitivity) required for the relevance relation. This flexibility allows the model to accommodate various notions of relevance. For instance, we will show that a relevance relation can be defined within our model on the basis of the a fortiori reasoning introduced in Horty’s result model. We will return to this point at the end of the current section and again in Remark 3.57 and Section 3.5.
- Each state is associated with a name, and no two states share the same name. All and only the decided states have names drawn from the finite subset of names $Names_d$. We say that a state s is n -named if $n \in s$. As it happens in hybrid logic [18], a state can have multiple names. This is not problematic: legal cases may have a common name like “A vs B” alongside a unique identifier.

Terminological note. We stated that a state $s \in S$ of the model represents a case presented to a specific court. For this reason, from now on we will refer interchangeably to the elements of S as either states or cases.

For notational convenience, we extend the relationships between courts to states.

Definition 3.4 (States Hierarchy Sem.). The hierarchy relations over states are defined as:

- $s \prec s'$ iff $c' \mathcal{H} c$ where $c \in s \cap \text{Courts}$ and $c' \in s' \cap \text{Courts}$;
- $s \cong s'$ iff $s \cap \text{Courts} = s' \cap \text{Courts}$.
- $s \preceq s'$ iff $s \prec s'$ or $s \cong s'$

$s \prec s'$ is read as “ s' is decided by a higher court than s ”. $s \cong s'$ is read as “ s' is decided by the same court as s ”. $s \preceq s'$ is read as “ s' is decided by the same court as s , or by a higher court than s ”.

The properties of transitivity, irreflexivity, and tree-likeness required for the hierarchical relation in Definition 3.2 thus extend to the relation between states as well.

Proposition 3.5 (States Hierarchy Properties). *Given $s, s', s'' \in S$:*

- (irreflexive) $s \not\prec s$
- (transitive) if $s \prec s'$ and $s' \prec s''$, then $s \prec s''$
- (tree-like) if $s \prec s'$ and $s \prec s''$, then $s' \prec s''$ or $s'' \prec s'$ or $s' \cong s''$.

Stare Decisis

So far, we have not specified the relationship between hierarchical and binding relations, allowing our framework to accommodate different legal systems. However, in this chapter, we focus on systems where:

- 1) *vertical stare decisis* applies, i.e. decisions of higher courts bind lower courts: if $c_i \mathcal{H} c_j$ then $c_i \mathcal{B} c_j$;
- 2) lower courts have no binding power on higher courts and two non-hierarchically-related courts don't issue binding decisions one for the other: if not $c_i \mathcal{H} c_j$ then not $c_i \mathcal{B} c_j$;
- 3) *horizontal stare decisis* may apply, i.e. some c_i may be self-bound, namely it may be $c_i \mathcal{B} c_i$.

Ultimately from 1), 2) and 3) it follows

$$\mathcal{H} \subseteq \mathcal{B} \subseteq \mathcal{H} \cup \mathcal{I} \quad (\text{SD})$$

where $\mathcal{I} = \{(c, c) \mid c \in \text{Courts}\}$ is the identity relation to identify the self-bound courts.

From now on we focus on $\overline{\text{TJCM}}$, the subclass of TJCMs satisfying (SD).^{14 15}

¹⁴We prefer not to impose condition (SD) directly in the definition of TJCMs, since, as we will see in the next chapter, not all jurisdictions satisfy (SD). In this chapter, however, (SD) will be useful for proving Proposition 3.39.

¹⁵In [30], we imposed also a condition on the temporal dimension of states of classifier models: the new and undecided cases are strictly after already assessed cases.

$$\forall s, s' \in S: (f(s) \neq ? \& f(s') = ?) \Rightarrow s <_T s' \quad (\text{O})$$

Although such a condition is intuitively reasonable, it is not actually required for the present work from a technical standpoint.

Example 3.7. We model the running example through TJCMs. Fix Atm_0 , assuming the finite set of decided names is $Names_d = \{n_9, \dots, n_{16}\}$. Let $S_{ex} = \{s_9, \dots, s_{16}, s^*\} \subseteq 2^{Atm_0}$. Pertaining names, assume $n_i \in s_i$ and $s^* \cap Names = \{n^*\}$.

$C_{ex} = (S_{ex}, f_{ex}, Jur_{ex}, \leq_T^{ex}, \mathcal{R}^{ex})$ is the classifier model, where

- $f_{ex}(s^*) = ?$, $f_{ex}(s_9) = f_{ex}(s_{10}) = f_{ex}(s_{12}) = f_{ex}(s_{15}) = f_{ex}(s_{16}) = 1$ and $f_{ex}(s_{11}) = f_{ex}(s_{13}) = f_{ex}(s_{14}) = 0$;
- the jurisdiction Jur_{ex} is defined in Example 3.6; the courts belonging to each state are shown in the Figure 3.4;
- the relevance relation $\mathcal{R}^{ex}(s^*)$ is represented in Figure 3.3: namely $\mathcal{R}^{ex}(s^*) = \{s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{14}, s_{15}, s_{16}\}$, $\mathcal{R}^{ex}(s_{11}) = \{s_9\}$, $\mathcal{R}^{ex}(s_{12}) = \{s_{11}\}$, $\mathcal{R}^{ex}(s_{13}) = \{s_{12}\}$;
- the temporal relation $\leq_T^{ex}$ is represented in Figure 3.5: $s_9 <_T^{ex} s_{10} <_T^{ex} \dots s_{16} <_T^{ex} s^*$.

It can be verified that C_{ex} satisfies (SD).

An aside: Horty's a fortiori relevance relation

Prior to defining the temporal and jurisdictional logic in the next section, we close this section with an aside. This aside is not essential for understanding the material presented in the following Sections 3.2.2, 3.3, 3.4, and it will be revisited only in Remark 3.57 and Section 3.5. We introduce it here, however, for readers who may prefer a more concrete example of the notion of relevance, which we have so far discussed in abstract terms. As we have already noted, the relevance relation indicates which cases are relevant to a given case, but it does not specify the criterion by which relevance is determined. Different criteria of relevance may be adopted, and we do not, in this context, commit to any particular one. Nevertheless, to provide the reader with a clearer reference point, we illustrate how, within our framework, a specific relevance relation can be defined on the basis of the a fortiori constraint introduced by Horty in the context of the result model, which we formalized in Remark 1.7. More precisely, such a relevance relation can be defined analogously to the a fortiori-based forcing relation forces in Remark 1.7. However, in our framework we must take into account that, unlike in Horty's models, for each case we know not only the facts (or factors) involved, but also the name of the case and the court that decided it.

Recall that in the result model, the facts are called factors. We may have factors for the plaintiff (*Plt*) and factors for the defendant (*Dfd*). So, $\text{Facts} = \text{Plt} \uplus \text{Dfd}$. As usual, we use following notation: for $o \in \{0, 1\}$, $\text{Facts}^o = \text{Plt}$ iff $o = 1$, $\text{Facts}^o = \text{Dfd}$ iff $o = 0$.

Recall that we can represent assessed cases within the result model as couples $k = (s_H, o)$, where $o \in \{0, 1\}$ and $s_H \subseteq \text{Facts}$ (see Remark 1.7).¹⁶

In the result model, according to a fortiori reasoning, a fact situation s'_H is forced to be decided in the same direction as another case (s_H, o) iff s'_H includes an equally or more inclusive set of factors supporting s_H 's outcome, and no additional factors against its outcome. More precisely, we stated that a case $k = (s_H, o)$, within the result model, forces a decision for s'_H as $o \in \{0, 1\}$, by a fortiori reasoning, noted $\text{forces}(k, s'_H, o)$, iff $s_H \cap \text{Facts}^o \subseteq s'_H \cap \text{Facts}^o$ and $s'_H \cap \text{Facts}^{\bar{o}} \subseteq s_H \cap \text{Facts}^{\bar{o}}$.

In our current framework of classifiers a state $s \subseteq \text{Facts} \cup \text{Names} \cup \text{Courts}$, is a case with outcome $f(s)$.

On this basis, we define the following relevance relation.

Definition 3.6 (A fortiori relevance, $\mathcal{R}_F$). Let $s \in S$, such that $f(s) = o$, $o \in \{0, 1\}$. Let $s' \in S$. Then, based on a fortiori constraint, s is relevant to s' , denoted as $s \mathcal{R}_F s'$, iff $s \cap \text{Facts}^o \subseteq s' \cap \text{Facts}^o$ and $s' \cap \text{Facts}^{\bar{o}} \subseteq s \cap \text{Facts}^{\bar{o}}$.

The previous definition resembles indeed the a fortiori based forcing relation forces . But, in our framework of classifiers, for each state we don't have only facts/factors but also names and courts, namely $s \subseteq \text{Facts} \cup \text{Names} \cup \text{Courts}$. Therefore, to retrieve a case in Horty's sense from s and $f(s)$ defined in our enriched classifier model, we have to consider $(s \cap \text{Facts}, f(s))$. On the basis of these considerations we can verify that $s \mathcal{R}_F s'$ iff $\text{forces}((s \cap \text{Facts}, f(s)), (s' \cap \text{Facts}, f(s)))$.

Example 3.8. We propose an example building on Example 3.1. Recall that we considered two cases, k_6 and k_7 , and a new fact situation to be decided, s_8 . The cases k_6 , k_7 , and the situation s_8 were all expressed in terms of factual elements—more precisely, in terms of factors. In that example, we had no information about the courts that decided

¹⁶In this paragraph, the notation s_H and s'_H is used solely to distinguish the factual situations of the cases within the result model, contained in Facts , from the states s in our model, which are instead included in $\text{Facts} \cup \text{Names} \cup \text{Courts}$.

(or will decide) k_6 , k_7 , and s_8 , nor about their temporal relations (in particular, between k_6 and k_7). Here, instead, we provide this information: consider three states $\tilde{s}_6$, $\tilde{s}_7$, and $\tilde{s}_8$, which share the same factual situations as k_6 , k_7 , and s_8 , respectively, but for which we now know the corresponding courts and temporal relation. In particular, $\tilde{s}_6$, $\tilde{s}_7$, and $\tilde{s}_8$ are to be decided in England and Wales Jurisdiction (Example 3.6).

$\tilde{s}_6 = \{\pi_1, \delta_1, \delta_2, c_1, \tilde{n}_6\}$, $f(\tilde{s}_6) = 1$. $\tilde{s}_6$ has the same factual situation and decision as k_6 . We also know that $\tilde{s}_6$ is named $\tilde{n}_6$ and was decided by court c_1 .

$\tilde{s}_7 = \{\pi_1, \pi_2, \delta_1, c_1, \tilde{n}_7\}$, $f(\tilde{s}_7) = 0$. $\tilde{s}_7$ has the same factual situation and decision as k_7 . It is named $\tilde{n}_7$ and was also decided by court c_1 .

$\tilde{s}_8 = \{\pi_1, \delta_1, c_2, \tilde{n}_8\}$, $f(\tilde{s}_8) = ?$. $\tilde{s}_8$ has the same factual situation as s_8 , which is yet to be decided. We know that $\tilde{s}_8$ is named $\tilde{n}_8$ and will be decided by court c_2 .

We assume the temporal relation: $\tilde{s}_6 <_T \tilde{s}_7 <_T \tilde{s}_8$.

It can be verified that $\tilde{s}_6 \mathcal{R}_F \tilde{s}_8$, since $\tilde{s}_6 \cap \text{Facts}^1 \subseteq \tilde{s}_8 \cap \text{Facts}^1$ and $\tilde{s}_8 \cap \text{Facts}^0 \subseteq \tilde{s}_6 \cap \text{Facts}^0$. Notice also that $k_6 = (\tilde{s}_6 \cap \text{Facts}, f(\tilde{s}_6))$ and $s_8 = \tilde{s}_8 \cap \text{Facts}$. Recall that, from Example 3.1, we know $\text{forces}(k_6, s_8, 1)$. This aligns with our earlier observation concerning the connection between $\mathcal{R}_F$ and forces .

Similarly, it can be verified that: $\tilde{s}_7 \mathcal{R}_F \tilde{s}_8$, $\tilde{s}_6 \mathcal{R}_F \tilde{s}_7$, $\tilde{s}_7 \mathcal{R}_F \tilde{s}_6$.

3.2.2 Temporal and Jurisdictional Classifiers Logic

In this section, we present the Temporal and Jurisdictional Classifier Logic TJCL. We begin by introducing the language, which is designed to capture fundamental elements of legal decision-making processes, incorporating temporal, jurisdictional, and relevance operators, as well as names for states.

As usual, for $o \in \text{Val}$, we call $t(o)$ a decision atom, to be read as “the actual decision (or output) takes value o ”. $\text{Dec} = \{t(o) : o \in \text{Val}\}$ is the set of decision atoms. Recall that the set of input variables is $\text{Atm}_0 = \text{Facts} \cup \text{Courts} \cup \text{Names}$. The set of propositional atoms is $\text{Atm} = \text{Atm}_0 \cup \text{Dec}$.

Definition 3.7 (Language). The modal language $\mathcal{L}^{TJ}(\text{Atm})$ is defined as:

$$\varphi ::= p \mid \mathbf{t}(o) \mid \mathbf{H}(c, c) \mid \mathbf{B}(c, c) \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid [\leq_T]\varphi \mid \mathbf{R}^\forall\varphi$$

where $p \in \text{Atm}_0$, $\mathbf{t}(o) \in \text{Dec}$, and $c \in \text{Courts}$.

Let $n \in \text{Names}$. We define the formula $@_n\varphi$ as $\Box(n \rightarrow \varphi)$. The formula has a hybrid logic flavour but is expressed in our language without explicit hybrid operators. The intended meaning of $@_n\varphi$ is that φ holds in one particular state, namely the state to which the name n refers.

$\mathbf{H}(c_i, c_j)$ is read as “the court c_i is hierarchically higher than the court c_j ”. $\mathbf{B}(c_i, c_j)$ is read as “the court c_i issues binding decisions for the court c_j ”. $\Box\varphi$ is read as “ φ is universally true”. $[\leq_T]\varphi$ is read as “ φ is the case and always will be the case”. Finally, $\mathbf{R}^\forall\varphi$ is read “ φ is true at all states that are relevant to the current one”. We define the dual operators as usual: $\Diamond\varphi =_{\text{def}} \neg\Box\neg\varphi$, $\mathbf{R}^\exists\varphi =_{\text{def}} \neg\mathbf{R}^\forall\neg\varphi$, $\langle\leq_T\rangle\varphi =_{\text{def}} \neg[\leq_T]\neg\varphi$. Formulas of $\mathcal{L}^{TJ}(\text{Atm})$ are interpreted with respect to classifier models as follows.

Definition 3.8 (Satisfaction relation). Let $C = (S, f, \text{Jur}, \leq_T, \mathcal{R})$ be a temporal jurisdictional classifier model. Let $s \in S$. Then:

$$\begin{aligned} (C, s) &\models p \iff p \in s; \\ (C, s) &\models \mathbf{t}(o) \iff f(s) = o; \\ (C, s) &\models \mathbf{H}(c_i, c_j) \iff c_i \mathcal{H} c_j; \\ (C, s) &\models \mathbf{B}(c_i, c_j) \iff c_i \mathcal{B} c_j; \\ (C, s) &\models \neg\varphi \iff (C, s) \not\models \varphi \\ (C, s) &\models \varphi \wedge \psi \iff (C, s) \models \varphi \text{ and } (C, s) \models \psi; \\ (C, s) &\models \Box\varphi \iff \forall s' \in S : (C, s') \models \varphi; \\ (C, s) &\models [\leq_T]\varphi \iff \forall s' \in \leq_T[s] : (C, s') \models \varphi; \\ (C, s) &\models \mathbf{R}^\forall\varphi \iff \forall s' \in \mathcal{R}(s) : (C, s') \models \varphi \end{aligned}$$

A formula φ of $\mathcal{L}^{TJ}(\text{Atm})$ is satisfiable relative to the class **TJCM** if there exists a pointed classifier model (C, s) with $C \in \mathbf{TJCM}$ such that $(C, s) \models \varphi$. Validity is defined as usual.

Notice that the satisfiability of the hierarchical and binding operators $(H(c, c), B(c, c))$ is independent of the states of the model; rather, it depends solely on the hierarchical and binding relations $(\mathcal{H}, \mathcal{B})$ defined within the classifier model.

Notational conventions. *In this section, and throughout the remainder of the chapter, we introduce several new definitions, both at the semantic level and at the language (syntactic) level. We also present representation results that connect the syntactic and semantic definitions. When a notion is introduced at the semantic level, the abbreviation Sem. will appear in the title of the Definition; when it is introduced at the language level, the abbreviation Lang. will be used instead. Finally, when addressing representation results, the abbreviation Repr. will be used in the title of the Proposition.*

By using our unary temporal modality $[\leq_T]$ and name formulas, we can define the binary modalities of future and past — “strictly later” and “strictly earlier”. The use of names allows us to capture irreflexivity in the temporal dimension.

Definition 3.9 (Past and Future Lang.).

- The notion that for the n -named state a future state is a φ -state, denoted as $\hat{F}_n\varphi$ is defined as $n \wedge \langle \leq_T \rangle (\neg \langle \leq_T \rangle n \wedge \varphi)$.
- The notion that for the n -named state a past state is m -named and a φ -state, denoted as $\hat{P}_{n,m}\varphi$, is defined as $n \wedge \diamond (\varphi \wedge \hat{F}_m n)$.

$\hat{F}_n$ and $\hat{P}_{n,m}$ act indeed as existential future and past operators — indicating the existence of a strictly later or strictly earlier state — as shown in the Propositions below.

Proposition 3.10 (Future Repr.). $(C, s) \models \hat{F}_n\varphi$ iff $(C, s) \models n$ and $\exists s' \in S$ s.t. $s <_T s'$ and $(C, s') \models \varphi$.

Proposition 3.11 (Past Repr.). $(C, s) \models \hat{P}_{n,m}\varphi$ iff $(C, s) \models n$ and $\exists s' \in S$ s.t. $(C, s') \models m \wedge \varphi$ and $s' <_T s$.

Furthermore, we can define two new operators of hierarchy, which are useful to define binding constraints in Section 3.3.2.

Definition 3.12 (States Hierarchy Lang.).

- The notion that *a lower hierarchical state is a φ -state* denoted as $\text{Lower}\varphi$ is defined as $\bigvee_{c,c' \in \text{Courts}} (c \wedge H(c,c') \wedge \Diamond(c' \wedge \varphi))$.
- The notion that *a higher hierarchical state is a φ -state* denoted as $\text{Higher}\varphi$ is defined as $\bigvee_{c,c' \in \text{Courts}} (c \wedge H(c',c) \wedge \Diamond(c' \wedge \varphi))$.
- The notion that *a state with same court is a φ -state* denoted as $\text{SameCourt}\varphi$ is defined as $\bigvee_{c \in \text{Courts}} (c \wedge \Diamond(c \wedge \varphi))$.

Proposition 3.13 (States Hierarchy Repr.).

- $(C,s) \models \text{Lower}\varphi$ iff there is $s' \in S$ s.t. $s' \prec s$ and $(C,s') \models \varphi$;
- $(C,s) \models \text{Higher}\varphi$ iff there is $s' \in S$ s.t. $s \prec s'$ and $(C,s') \models \varphi$;
- $(C,s) \models \text{SameCourt}\varphi$ iff there is $s' \in S$ s.t. $s \cong s'$ and $(C,s') \models \varphi$.

3.2.2.1 Axiomatics and Completeness

Definition 3.14 (Axiomatics of TJCL). We define TJCL to be the extension of propositional logic with the following axioms and rules:

$\Box\varphi \wedge \Box(\varphi \rightarrow \psi) \rightarrow \Box\psi$	(K$_{\Box}$)
$\Box\varphi \rightarrow \varphi$	(T$_{\Box}$)
$\Box\varphi \rightarrow \Box\Box\varphi$	(4$_{\Box}$)
$\varphi \rightarrow \Box\Diamond\varphi$	(B$_{\Box}$)
$\bigvee_{o \in \text{Val}} \text{t}(o)$	(AtLeastValue)
$\text{t}(o) \rightarrow \neg\text{t}(o')$ if $o \neq o'$	(AtMostValue)
$([\leq_T]\varphi \wedge [\leq_T](\varphi \rightarrow \psi)) \rightarrow [\leq_T]\psi$	(K$_{[\leq_T]}$)
$[\leq_T]\varphi \rightarrow \varphi$	(T$_{[\leq_T]}$)
$[\leq_T]\varphi \rightarrow [\leq_T][\leq_T]\varphi$	(4$_{[\leq_T]}$)
$@_n([\leq_T]\varphi \rightarrow @_m\varphi) \vee @_m([\leq_T]\varphi \rightarrow @_n\varphi)$	(Total$_{[\leq_T]}$)
$\Box\varphi \rightarrow [\leq_T]\varphi$	(MIX$_{\Box, [\leq_T]}$)
$\bigvee_{c_i \in \text{Courts}} c_i$	(AtLeastCourt)

$c_i \rightarrow \neg c_j$ if $c_i \neq c_j, c_i, c_j \in \text{Courts}$	(AtMostCourt)
$\neg H(c_i, c_i)$	(IrrHierarchy)
$H(c_i, c_j) \wedge H(c_j, c_k) \rightarrow H(c_i, c_k)$	(TrHierarchy)
$\bigwedge_{c_j \neq c_k} (H(c_j, c_i) \wedge H(c_k, c_i) \rightarrow H(c_j, c_k) \vee H(c_k, c_j))$	(TreeLikeHier)
$H(c_i, c_j) \rightarrow \Box H(c_i, c_j)$	(GlobHier)
$B(c_i, c_j) \rightarrow \Box B(c_i, c_j)$	(GlobBind)
$(R^\forall \varphi \wedge R^\forall (\varphi \rightarrow \psi)) \rightarrow R^\forall \psi$	($K_{R^\forall}$)
$\Box \varphi \rightarrow R^\forall \varphi$	($MIX_{\Box, R^\forall}$)
$\Diamond(n \wedge \varphi) \rightarrow @_n \varphi$	(nam1)
$\neg t(?) \leftrightarrow \bigvee_{n \in \text{Names}_d} n$	(nam2)
$\frac{n \rightarrow \varphi}{\varphi}$ where n does not occur in φ	(NAME)
$\frac{\varphi \rightarrow \psi, \varphi}{\psi}$	(MP)
$\frac{\varphi}{\Box \varphi}$	(Nec $_{\Box}$)

Axioms $K_{\Box}$, $K_{[\leq T]}$, and $K_{R^\forall}$ ensure the distribution of universal, temporal, and relevance necessity over conditionals. Axioms $T_{\Box}$ and $T_{[\leq T]}$ ensure reflexivity of the universal and temporal operators, while Axioms $4_{\Box}$ and $4_{[\leq T]}$ guarantee their transitivity. Axiom $B_{\Box}$ enforces symmetry of the universal operator. Hence, $\Box$ is an S5 operator, $[\leq T]$ is an S4 operator, and $R^\forall$ is a K operator.

Axiom **Total** $_{[\leq T]}$ guarantees the totality of the temporal relation required in Definition 3.3. Intuitively, consider two states s, s' of the model. States are named: suppose s has name n and s' has name m . The axiom requires that either (i) if all states simultaneous with or later than s are φ -states, then s' is a φ -state, or (ii) if all states simultaneous with or later than s' are φ -states, then s is a φ -state.

Axioms **Total** $_{[\leq T]}$ and **MIX** $_{\Box, R^\forall}$ capture, respectively, the interaction between the universal and temporal modalities, and between the universal and relevance modalities. If φ holds in all states, then it holds in all states simultaneous with or later than the current one; moreover, it also holds in all states relevant to the current one.

Axiom **AtLeastValue** ensures that every state has at least one outcome, while **AtMostValue** ensures that each state has at most one outcome.

Axiom **AtLeastCourt** guarantees that every state has at least one court, and **AtMostCourt** guarantees that each state has at most one court. Axioms **IrrHierarchy**, **TrHierarchy**, and **TreeLikeHier** ensure, respectively, that the court hierarchy is irreflexive, transitive, and tree-like, as required by Definition 3.3. Axioms **GlobHier** and **GlobBind** express the universality of the hierarchical and binding relations: if a court is higher than another court (or binding over another court) in one state, then it is so in all states.

Axiom **nam1**, adapted from [43], requires that if there exists an φ -state named n , then all n -named states are φ -states. This guarantees, in the completeness proof and in the construction of the canonical model, that each name uniquely identifies a single state. Axiom **nam2** ensures that all and only decided states are associated with names in Names_d , and therefore the set of decided states is finite, as required by Definition 3.3. Finally, rule **NAME**, from [18], guarantees the construction of canonical models with named states.

From now on, we focus on TJCL^+ , the extension of TJCL obtained by adding the following axiom, which corresponds to the semantic condition SD.

$$\bigwedge_{c_i \neq c_j} \left((H(c_i, c_j) \rightarrow B(c_i, c_j)) \wedge (\neg H(c_i, c_j) \rightarrow \neg B(c_i, c_j)) \right) \quad (\text{SL})$$

Proposition 3.15. *TJCL is sound and complete with respect to TJCM. TJCL^+ is sound and complete with respect to $\overline{\text{TJCM}}$.*

3.3 Precedents, Binding Precedents and Exceptions

With the new classifier models and the newly introduced logic at hand, we can now formalize some fundamental elements of the legal decision-making process—specifically, of the process that determine when the decision in a given case is constrained by previous cases. In particular, recall that one of the research questions that we want to address in this chapter is:

Q2) *Which precedents are binding for the case at hand, considering both the factual relevance between the cases and the jurisdictional relations among the courts that decided them?*

Answering this question entails addressing the following sub-questions:

Q2a) *What counts as a precedent?*

Q2b) *Which precedents are binding—that is, which precedents actually constrain or determine the decision in a given case?*

In this chapter, we provide answers to these questions.

As we shall see, a *precedent* for a case under consideration is a previously decided case that is relevant to the case at hand.

Not all precedents have equal *force* with respect to the case at hand. Some precedents may have only *persuasive force*—that is, they do not formally bind the court in the present case, yet the court may choose to follow them if it finds their reasoning convincing, for instance to further support its decision [66]. In this work, however, we are primarily interested in the *binding force* of precedents. To model this notion, we begin by considering *potentially binding* precedents. We shall say that a precedent is *potentially binding* only if it was established by a court that holds binding authority over the court deciding the present case. For example, a precedent set by the UK Supreme Court is *potentially binding* on the High Court—but not vice versa.

We use the term *potentially binding precedent* because such precedents are subject to exceptions: under certain circumstances, they may lose their binding force. In what follows, we will consider two types of exceptions, *overruling* and *per incuriam*, which we briefly introduced at the beginning of this chapter.

Finally, *binding precedents* can be defined as those potentially binding precedents that are not subject to exceptions.

Our approach will be as follows: in Section 3.3.1, we define precedent cases and potentially binding cases; in Section 3.3.2, we examine in detail the notion of exceptions

and binding precedents without exceptions. All these notions are modeled both semantically and syntactically.

Note. In the next sections, for $s, s', s'' \in S$, we denote their courts c, c', c'' , (e.g. $c \in s \cap \text{Courts}$).

3.3.1 Potentially Binding precedents

A precedent for a case at hand is a relevant case that was decided earlier. Specifically, s' is a supporting precedent — or simply a precedent — for s if s' is relevant to s and was decided before s .

Definition 3.16 (Precedent Sem). Let $s, s' \in S$ and $o \in \{0, 1\}$. s' is a (*supporting*) precedent for s in the direction of o , noted $\Pi(s', s, o)$, iff $f(s') = o$, $s' \in \mathcal{R}(s)$, and $s' <_T s$.

Supporting precedents are relevant past cases. To represent this notion within the language, we need to intersect the relevance and temporal modalities. To do so, we adopt a hybrid style, using names to define intersection modalities.

Definition 3.17 (Past-Relevance Intersection Lang.). The notion that *for the n -named state a relevant and past state is m -named and a φ -state*, denoted as $(\hat{P} \cap R^\exists)_{n,m} \varphi$ is defined as $\hat{P}_{n,m} \varphi \wedge R^\exists(\varphi \wedge m)$.

Proposition 3.18 (Past-Relevance Intersection Repr.). $(C, s) \models (\hat{P} \cap R^\exists)_{n,m} \varphi$ iff $(C, s) \models n$ and there is $s' \in S$ s.t. $(C, s') \models m$, $s' <_T s$, $s' \in \mathcal{R}(s)$ and $(C, s') \models \varphi$.

Through the intersection modality defined above, we can represent within the language the notion of a supporting precedent, recalling that it is a relevant past state that is also *decided* (i.e., one that satisfies $\neg t(?)$).

Definition 3.19 (Precedent Lang.). The notion that *for the n -named state a precedent supporting state is m -named and a φ -state*, denoted as $\text{Supporting}_{n,m} \varphi$, is defined as $(\hat{P} \cap R^\exists)_{n,m} (\neg t(?) \wedge \varphi)$.

It can be verified that a state s satisfies $\text{Supporting}_{n,m}\varphi$ if and only if s is n -named, and there exists an m -named state s' that is a supporting precedent for s —for a decision equal to $f(s')$ —and that satisfies φ .

Proposition 3.20 (Precedent Repr.). $(C, s) \models \text{Supporting}_{n,m}\varphi$ iff $(C, s) \models n$ and exists $s' \in S$, s.t. $\Pi(s', s, f(s'))$ and $(C, s') \models m \wedge \varphi$.

Example 3.9. We consider Example 3.7. Note that s_9 , decided as 1, is both relevant to and decided before s^* . Hence, s_9 is a (supporting) precedent for s^* : $\Pi(s_9, s^*, 1)$. Within the language, recalling that s_9 is named n_9 and s^* is named n^* , we have $(C_{ex}, s^*) \models \text{Supporting}_{n^*, n_9} t(1)$.

Similarly, it can be verified that all other states $s_{10}, \dots, s_{16}$ are supporting precedent for s^* .

Within the language, $(C_{ex}, s^*) \models \text{Supporting}_{n^*, n_{10}} t(1) \wedge \text{Supporting}_{n^*, n_{11}} t(0) \wedge \text{Supporting}_{n^*, n_{12}} t(1) \wedge \text{Supporting}_{n^*, n_{13}} t(0) \wedge \text{Supporting}_{n^*, n_{14}} t(0) \wedge \text{Supporting}_{n^*, n_{15}} t(1) \wedge \text{Supporting}_{n^*, n_{16}} t(0)$.

Not all supporting precedents are *potentially binding*, i.e. not all of them may force the decision in current case. The potentially binding precedents for a state s , decided by court c , are those precedents s' issued by a court c' that holds *binding authority* over c .

Definition 3.21 (Pot. bind. Sem.). Let $s, s' \in S$ and $o \in \{0, 1\}$. s' is a *potentially binding precedent* for s for a decision as o , denoted as $\beta(s', s, o)$, iff $\Pi(s', s, o)$ and $c' \mathcal{B} c$. We simply write $\beta(s', s)$ iff there is $o \in \{0, 1\}$ s.t. $\beta(s', s, o)$. We define $\beta_s = \{s' \mid \beta(s', s)\}$, the set of *potentially binding precedents* for s .

Definition 3.22 (Pot. bind. Lang.). The notion that for the n -named state a *potentially binding state* is a m -named φ -state, noted as $\text{PBinding}_{n,m}\varphi$, is defined as $\bigvee_{c_i, c_j \in \text{Courts}} (c_j \wedge B(c_i, c_j) \wedge (\text{Supporting}_{n,m}(c_i \wedge \varphi)))$.

It can be verified that a state s satisfies $\text{PBinding}_{n,m}\varphi$ if and only if s is n -named, and there exists an m -named state s' that is potentially binding for s —for a decision equal to $f(s')$ —and that satisfies φ .

Proposition 3.23 (Pot. bind. Repr.). $(C, s) \models \text{PBinding}_{n,m}\varphi$ iff $(C, s) \models n$ and exists $s' \in S$ s.t. $\beta(s', s, f(s'))$ and $(C, s') \models m \wedge \varphi$.

As a direct corollary of the previous proposition, we obtain the following result.

Proposition 3.24. *Let $o \in \{0, 1\}$. $(C, s) \models \text{PBinding}_{m,n}(\text{t}(o) \wedge \varphi)$ iff $(C, s) \models n$ and exists $s' \in S$, s.t. $\beta(s', s, o)$ and $(C, s') \models m \wedge \varphi$.*

Example 3.10. *From Example 3.9, we know that s_9 is a supporting precedent for s^* ($\Pi(s_9, s^*, 1)$). Also, s_9 is decided by court c_0 , which is binding for court c_2 in s^* ($c_0 \mathcal{B} c_2$). Hence, $\beta(s_9, s^*, 1)$ and $s_9 \in \beta_{s^*}$.*

Within the language, $(C_{ex}, s^*) \models \text{PBinding}_{n^*, n_9} \text{t}(1)$.

Same can be verified for s_{10} , s_{12} , s_{16} (with respect to decision 1), s_{11} , s_{13} (with respect to decision 0).

Instead, although s_{14} is a precedent for s^* , it is not binding for s^* . Indeed, s_{14} is decided by c_3 which is not binding for s^* (same can be verified for s_{15}). Within the language, $(C_{ex}, s^*) \not\models \text{PBinding}_{n^*, n_{14}} \text{t}(0)$.

Recall that we are considering a jurisdiction in which vertical stare decisis applies and horizontal stare decisis may apply. More precisely, we assume that SD holds. In this setting, a court may be binding with respect to another court in two cases: either it is a higher court, or it is the same court. Consequently, if a state s' is potentially binding for state s , this is determined either by a higher court ($s \prec s'$) or by the same court ($s \cong s'$).

Proposition 3.25. *If $s' \in \beta_s$ then $s \cong s'$ or $s \prec s'$.*

Actually, SD tells us even more. Namely, according to SD, a higher court is always binding with respect to another lower court. Therefore, a precedent case decided by a higher court is always potentially binding.

Proposition 3.26. *If $\Pi(s', s, f(s'))$ and $s \prec s'$, then $s' \in \beta_s$.*

3.3.2 Exceptions: Incuriam and Overruling

Potentially binding precedents are subject to two exceptions: they may be *overruled* or decided *per incuriam*.

Intuitively, a case s is *overruled* when a subsequent and relevant decision is rendered by a court with the authority to overrule the court that decided s , thereby modifying the governing line of law. What does it mean, in the context of this work, for the line of law

to be altered? It means not only that a new decision departs from the relevant precedent *s*, possibly creating a new constraint on future cases, but also that the precedent *s* loses its *force* with respect to subsequent decisions. Recall that a precedent may have either *persuasive* or *binding* force on subsequent decisions. In this chapter, we are interested in situations in which a precedent loses its *binding force*.¹⁷ Typically, when a precedent loses its binding force as a result of *overruling*, this force is not lifted from the court that issued the overruling decision itself, but from other courts. In common law systems, for instance, when one court overrules another, it does not remove a constraint on itself but rather eliminates that constraint for courts below it.

A distinction is usually drawn between *explicit* and *silent* (or *implicit*) overruling. The former occurs when a court expressly states that it is overruling a previous decision; the latter occurs when the overruling is not explicitly acknowledged. In other words, while in the case of explicit overruling the intention to overturn a precedent is clearly recognized and the overruling can be seen as an act of conscious intervention, this is not necessarily the case with implicit overruling. In this work, however, we do not distinguish between implicit and explicit overruling. Rather, we aim to *derive*, given the relevance relation between cases and the relation among courts, when we can say that one case has been overruled by another and has therefore lost its binding force on subsequent decisions.

Conversely, a case *s* is decided *per incuriam* if it goes against a binding precedent without having the authority to do so. For example, if a case *s* is decided contrary to a binding precedent established by a higher court, this means that the court deciding *s* ignored the binding vertical authority imposed by *vertical stare decisis*. Hence, *s* should have been decided differently and cannot be considered binding.

To formally define these exceptions, we begin by considering the following aspects.

- As already stated, *per incuriam* and overruled cases lose their bindingness on subsequent cases, i.e. we can remove them from binding precedents.

¹⁷Actually, different interpretations of overruling are possible [66]. In the *strict sense*, overruling removes the binding force of a precedent; in the *broader sense*, it may also eliminate its persuasive force. We will briefly revisit the distinction at the beginning of Section 3.3.2, and return to it in greater detail in Chapter 4. In this chapter, our focus is on the removal of binding force.

- The model does not specify which cases are *per incuriam* or overruled. Rather, as shown later in this section, this information is computed through a recursive process involving the following interactions between *per incuriam* and overruling.
- Consider Figure 3.6 below. Suppose we have a case s^* to be decided, for which s_1 is a potentially binding precedent. Now assume that s_2 — for which s_1 was relevant — was decided later and differently from s_1 by a court with overruling authority. Intuitively, this suggests that s_1 has been overruled and should no longer be binding for s^* . However, suppose we then discover that s_2 itself contradicts a binding precedent s_3 , without having the authority to do so — that is, s_2 was decided *per incuriam*. In other words, s_2 should have been decided differently. In that case, the overruling of s_1 by s_2 is not legitimate, and so we can reinstate s_1 as a binding precedent for s^* . Hence, a case loses its overruling power when it is *per incuriam*.

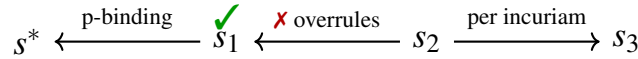


FIGURE 3.6: Interaction between binding and overruling, first example

- Consider Figure 3.7 below. Suppose we have a case s^* to be decided, and a potentially binding precedent s_1 . Now assume that s_1 went against s_2 without the authority to do so. Intuitively, this would suggest that s_1 *per incuriam* and should be excluded from the binding precedents for s^* . However, suppose we later discover that s_2 had already been overruled by s_3 before s_1 was decided. In that case, s_2 was no longer binding at the time of s_1 's decision, and s_1 can be “reinstated” as a valid binding precedent for s^* . A case is not *per incuriam* if it goes against an overruled case.

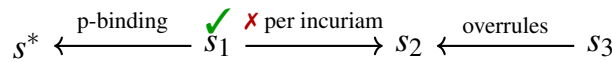


FIGURE 3.7: Interaction between binding and overruling, second example

To capture such interactions, we proceed as follows. In Section 3.3.2.1, we first identify *potential overruling cases*—those that can overrule others unless deemed *per incuriam*. In Section 3.3.2.2, we formally define *per incuriam*. In Section 3.3.2.3, we then define overruled cases—those potentially overruled by a case not *per incuriam*. Finally, in

Section 3.3.2.4, we define binding precedents *without exceptions*, namely those that are neither *per incuriam* nor overruled.

3.3.2.1 Potential Overruling

In this section, we define the notion of *potential overruling*. To this end, we first introduce the concept of *overruling power*, which specifies which courts possess overruling authority over other courts and over themselves. On the basis of this notion, we then define the conditions under which one state can *potentially* overrule another.¹⁸

Not all courts have the authority to overrule prior decisions. Within common law systems, higher courts typically have the power to overrule the decisions of lower courts. Conversely, a lower court cannot overrule a decision of a higher court.¹⁹

Particular attention must be paid to the phenomenon of *self-overruling*. In this chapter, we follow the intuition that a court may overrule its own previous decisions when it is not bound by them.

Accordingly, we assume that a court c' may overrule a decision made by court c when deciding case s' , if and only if: (1) c' is hierarchically higher to c ; or (2) $c' = c$ and not- $c\mathcal{B}c$.

Definition 3.27 (Overruling Power Sem.). *Court c' has the power to overrule (a decision by) court c , denoted $\mathcal{O}(c', c)$, iff $c' \mathcal{H} c$ or $(c' = c \text{ and not-} c\mathcal{B}c)$.*

Definition 3.28 (Overruling Power Lang.). *That a decision made by c_i can be overruled in current state, denoted as $\text{PwOver}(c_i)$, is defined as $\bigvee_{c_j \neq c_i} (c_j \wedge \mathcal{H}(c_j, c_i)) \vee (c_i \wedge \neg \mathcal{B}(c_i, c_i))$.*

Proposition 3.29 (Overruling Power Repr.). *Let $c \in s \cap \text{Courts}$. So, $(C, s) \models \text{PwOver}(c_i)$ iff $\mathcal{O}(c, c_i)$.*

¹⁸We define both the binding power and the overruling power of one case over another as *potential*. We use the term *potential* because, as we will see, there may be situations in which a case that is potentially binding or potentially overruling is subject to exceptions. To some extent, the notion of exception recalls the notion of interference of power discussed in [92]. In [92] power is understood as the reification of a causal mechanism, and an interfering action modifies the power associated with the target of the interference.

¹⁹That is, no *anticipatory overruling* is permitted. By *anticipatory overruling* we refer to a situation in which a lower court refuses to follow a precedent “in anticipation of the likelihood that a higher court will overrule it” [66].

We now define *potential overruling cases*. Intuitively, a case s' may potentially overrule a *precedent* s when the following conditions hold: (1) s' is decided by court c' in the opposite direction to s , which was decided by court c ; and (2) c' possesses overruling authority over c when deciding s' (i.e., $O(c', c)$).

Definition 3.30 (Pot. Over. Sem.). The case s' *potentially overrules* s , denoted as $O(s', s)$, iff $\Pi(s, s', o)$, $f(s') = \bar{o}$, and $O(c', c)$. We write $\omega_s = \{s' \mid O(s', s)\}$ for the set of potentially overruling cases with respect to s . Given a case $\tilde{s} \in S$, we write $\text{Overrule}_T(s, \tilde{s}) = \{s' \mid O(s', s), s' <_T \tilde{s}\}$ to represent the set of cases potentially overruling s and that were assessed before $\tilde{s}$ was decided.

Definition 3.31 (Pot. Over. Lang.). The notion that for the n -named state a potentially overruling state is m -named and a ϕ -state, denoted as $\text{POverruling}_{n,m}\phi$, is defined as $\bigvee_{o \in \{0,1\}, c \in \text{Courts}} \left(n \wedge c \wedge \diamond(\phi \wedge t(\bar{o}) \wedge \text{PwOver}(c) \wedge \text{Supporting}_{m,n}t(o)) \right)$.

Proposition 3.32 (Pot. Over. Repr.). $(C, s) \models \text{POverruling}_{n,m}\phi$ iff $(C, s) \models n$ and exists $s' \in S$ s.t. $O(s', s)$ and $(C, s') \models m \wedge \phi$.

Example 3.11. We refer to Example 3.7. c_0 can overrule itself, i.e. $O(c_0, c_0)$. Indeed, we know that c_0 is not self bound ($\text{not-}c_0\mathcal{B}c_0$).

We can verify that s_9 , decided as 1, was a precedent for s_{11} ($\Pi(s_9, s_{11}, 1)$), indeed s_9 was both relevant and decided before s_9 . But s_{11} was decided as 0, by the same court of s_9 , namely by c_0 , which we know can overrule itself. Hence, s_{11} potentially overrules s_9 . So, $O(s_{11}, s_9)$ and $s_{11} \in \omega_{s_9}$. Also, s_{11} potentially overruled s_9 before a decision for s^* , hence $s_{11} \in \text{Overrule}_T(s_9, s^*)$.

Within the language $(C_{ex}, s_{11}) \models \text{PwOver}(c_0) \wedge t(0)$ and $(C_{ex}, s_9) \models \text{POverruling}_{n_9, n_{11}}t(0)$.

Note that, by Definition 3.27, a court may overrule another either if it is a higher court or if it is the same court (not self-bound). This means that if a case *potentially overrules* another, then it is decided either by a higher court or by the same court.

Proposition 3.33. If $s' \in \omega_s$ then $s \prec s'$ or $s \cong s'$.

We can verify that the relations of potential overruling and potential binding are mutually exclusive. More precisely:

1) if a case s' is potentially binding for another case s , then the court deciding s cannot

overrule the court that decided s' . As a consequence, 2) if a case s' is potentially binding for another case s , then s is not potentially overruling for s' .

Conversely, 3) if a case s' is potentially overruling for another case s , then the court deciding s has no binding authority over the court that decided s' . Hence, 4) if a case s' is potentially overruling for another case s , then s is not potentially binding for s' .

Proposition 3.34. *The following statements hold:*

- 1) Let $c \in s$ and $c' \in s'$. $s' \in \beta_s \Rightarrow \text{not } O(c, c')$.
- 2) $s' \in \beta_s \Rightarrow s \notin \omega_{s'}$.
- 3) Let $c \in s$ and $c' \in s'$. $s' \in \omega_s \Rightarrow \text{not } c\mathcal{B}c'$.
- 4) $s' \in \omega_s \Rightarrow s \notin \beta_{s'}$.

3.3.2.2 Per Incuriam

In this section, we introduce the notion of *per incuriam*. Since this requires several preparatory steps, we begin by outlining the structure of the section to guide the reader. First, we define the notions of *going against* and *being decided according* to a potentially binding precedent (Definitions 3.35 and 3.36). Next, we introduce the notion of *s-graph*, a structure that captures all chains of potentially binding and potentially overruling states originating from the case at hand (Definition 3.38). We then provide a first semantic definition of *per incuriam* (Definition 3.42), formulated on the *s-graph*. Subsequently, we offer two alternative characterizations of *per incuriam* within the semantic (Definitions 3.44 and 3.46). The second characterization will later serve as the basis for the formal definition of *per incuriam* within the language (Definition 3.50).

As mentioned earlier, intuitively, a *per incuriam* case is one that goes *against* a binding precedent. To capture this formally, we introduce the notions of *going against* and *going according to* a binding precedent—more precisely, of going against or according to a *potentially binding* precedent.

More precisely, a state s goes against a potentially binding precedent s' if it is decided in the opposite direction of s' —that is, if $f(s) = o \in \{0, 1\}$ and $f(s') = \bar{o}$. Conversely, a state s is decided according a potentially binding precedent s' if it is decided in the same direction—that is, if $f(s) = f(s') = o \in \{0, 1\}$.

Both of these notions are defined at the semantic and the syntactic levels.

Definition 3.35 (Against/According Sem.). The case s went against s' , denoted as $\text{Against}(s, s')$, iff $f(s) = o \in \{0, 1\}$ and $\beta(s', s, \bar{o})$. The case s was decided according to s' , denoted as $\text{According}(s, s')$, iff $f(s) = o \in \{0, 1\}$ and $\beta(s', s, o)$.

Definition 3.36 (Against/According Lang.). The notion that the n -named state went against a m -named and potentially binding φ -state, denoted as $\text{Against}_{n,m}\varphi$, is defined as $\bigvee_{o \in \{0,1\}} \left(t(o) \wedge \text{PBinding}_{n,m}(\varphi \wedge t(\bar{o})) \right)$. The notion that the n -named state is decided according to a m -named and potentially binding φ -state, denoted as $\text{According}_{n,m}\varphi$, is defined $\bigvee_{o \in \{0,1\}} \left(t(o) \wedge \text{PBinding}_{n,m}(\varphi \wedge t(o)) \right)$.

Proposition 3.37 (Against/According Repr.). The following statements hold:

- I) $(C, s) \models \text{Against}_{n,m}\varphi$ iff $(C, s) \models n$ and $\exists s'$ s.t. $\text{Against}(s, s')$ and $(C, s') \models m \wedge \varphi$;
- II) $(C, s) \models \text{According}_{n,m}\varphi$ iff $(C, s) \models n$ and $\exists s'$ s.t. $\text{According}(s, s')$ and $(C, s') \models m \wedge \varphi$.

Example 3.12. Consider Example 3.5. We can verify that s_{13} went against s_{12} . Indeed, s_{12} was potentially binding for s_{13} : it was decided earlier by the same court c_1 , which is self-bound. However, s_{13} was decided as 0, whereas s_{12} was decided as 1. Hence, $\text{Against}(s_{13}, s_{12})$. Within the language, $(C, s) \models \text{Against}_{n_{13}, n_{12}} t(0)$.

Similarly, we can verify that s_{12} went against s_{11} . In contrast, s_{11} went against no potentially binding precedent (s_9 , as already noted in the previous example, is not potentially binding for s_{11}).

Consider a state s that goes against a potentially binding precedent s' , i.e., $\text{Against}(s, s')$. Since s' was potentially binding for s , by Proposition 3.34, s had no overruling power with respect to s' . This would therefore suggest that s was decided *per incuriam*. However, to properly establish this, four interrelated aspects must be taken into account.

- 1) As mentioned, we infer from the model which cases are decided *per incuriam*, such information is not given. Using the same reasoning done for Figure 3.7, we state that s is not *per incuriam* if s' was overruled by another case $\tilde{s}$. The reason is that, if s' was overruled, then it no longer held binding authority over s . However, we must also require that $\tilde{s}$ was decided *before* s — otherwise, s' had not yet been overruled at the time s was assessed and thus s' still bound s . Moreover, $\tilde{s}$ must itself not be *per incuriam*; otherwise, it would not represent a legitimate overruling.

- 2) Not all courts can disregard a previous decision *per incuriam* [60]. In particular, we will follow the practice adopted in the UK, where the doctrine of *per incuriam* originated, according to which a lower court cannot disregard a precedent decided by a higher court, even if it was rendered *per incuriam*. So, we assume that a lower court cannot disregard a *per incuriam* precedent by a higher court, to which it remains bound.²⁰ But, a court may disregard its own *per incuriam* precedents.²¹
- 3) Returning to our question: if *Against*(s, s'), does it mean s was decided *per incuriam*? It depends. If s' was decided by a higher court (and not overruled, see point 1), then s is necessarily *per incuriam*, since it lacks the authority to disregard s' . If s and s' were decided by the same court, then the court — when deciding s — may have found that s' went against a binding $\bar{s}$. But to confirm that s' was *per incuriam* — so not binding for s — we have to check whether $\bar{s}$ was *per incuriam*, etc. Thus, determining if s is *per incuriam* requires tracing its full chain of binding precedents.
- 4) For a given case s , there may exist conflicting binding precedents: s' and s'' within β_s , such that *Against*(s, s') and *According*(s, s'') hold —namely, s went against the potentially binding precedent s' but was also decided in accordance with the potentially binding precedent s'' . In this scenario, any decision made in s would have contradicted one potentially binding precedent. We argue that s can be considered *per incuriam* if s'' fails to satisfy the previously established criteria — namely, if s'' is itself *per incuriam* and by a court not higher than s , or if it was overruled (prior to s) by a non-*per incuriam* precedent. If s'' fits neither category, there remains another case in which s may still be regarded as *per incuriam*: following a principle for resolving conflicts of precedents (to be detailed later), s should have followed the later precedent from the higher court. Therefore, we examine the temporal and hierarchical relationship between s' and s'' . s qualifies as *per incuriam* if: s'' was decided by a lower court than s' , or by the same court but strictly before.

As observed, determining whether a case was decided *per incuriam* involves a chain-like evaluative process over potentially binding and potentially overruling cases.

Here, a graph $G_s = (V_s, E_s)$ constructed recursively is given to compute whether a case s was decided *per incuriam*. At step 0, the graph ($G_0 = (V_0, E_0)$) has only one

²⁰In *Cassell v Broome* (UK), the Court of Appeal was held bound by *per incuriam* House of Lords decision [66]. Canada follows same principle [78].

²¹Clearly stated for UK Court of App. in *Young v. Bristol Aeroplane Co.*.

node s ($V_0 = \{s\}$) and no edge ($E_0 = \emptyset$). At step 1, we add to the nodes the s' , the potentially binding precedents or the potentially overruling states for s ($V_1 = \{s\} \cup \{s' \mid s' \in \beta_s \text{ or } s' \in \omega_s\}$). Edges joining s to each s' are added ($E_1 = \{(s, s') \mid s' \in \beta_s \text{ or } s' \in \omega_s\}$). We repeat the procedure recursively for each node.

Definition 3.38 (s-Graph). A structure $G_s = (V_s, E_s) = \bigcup_{n \geq 0} G_n$ is called *the s-graph*, when $f(s) \neq ?$ and every G_n satisfies:

- $G_0 = (V_0, E_0)$, where $V_0 = \{s\}$ and $E_0 = \emptyset$;
- $G_{n+1} = (V_{n+1}, E_{n+1})$, where
 - $V_{n+1} = V_n \cup \{s' \mid s' \in \beta_{s''} \text{ or } s' \in \omega_{s''}, \text{ with } s'' \in V_n\}$ and
 - $E_{n+1} = E_n \cup \{(s'', s') \mid s' \in \beta_{s''} \text{ or } s' \in \omega_{s''}, \text{ with } s'' \in V_n\}$.

G_s is a finite directed acyclic graph. Condition SD, which we recall captures vertical and horizontal *stare decisis*, will be crucial in proving this result.

Proposition 3.39. G_s is a finite directed acyclic graph.

We thus can define the *height* of a state s , denoted $\text{height}(s)$, as the number of edges in the longest path from s to a sink node in G_s .

Definition 3.40. (Sink, height of a node/state)

- Let $G = (V, E)$ a finite directed acyclic graph. A node $s \in V$ is a sink if it has no out-going edges. We define $\text{height}_G(s)$ the *height of $s \in V$ in G* as the the number of edges in the longest path connecting that node to a sink node.²²
- For $s \in S$ where $f(s) \neq ?$, the *height of s* , denoted as $\text{height}(s)$, is defined as the height of s in G_s , that is $\text{height}(s) =_{\text{def}} \text{height}_{G_s}(s)$.

In every G_s , there are only decided states; indeed, the elements of G_s are potentially overruling and potentially binding states. We also know that G_s is acyclic and finite. Hence the *height* of a state is atmost the cardinality of S_d , the finite set of decided states.

²²If there are more sink nodes we take the longest paths (the longest paths can be more than one, with same length).

Recall that each decided state has names only in the finite set Names_d (see Definition 3.3). Therefore, $|S_d| \leq |\text{Names}_d|$. From these observations we obtain the following proposition.

Proposition 3.41. *Let $s \in S$. $\text{height}(s) \leq |\text{Names}_d|$.*

By recursively exploring graph G_s , we can compute whether s is *per incuriam* (i.e. $\text{Incuriam}(s)$). Specifically, we claim that s is *per incuriam* if:

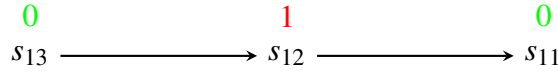
- there is s' adjacent node to s (i.e. $(s, s') \in E_s$), such that:
 - a) s is against s' (i.e. $\text{Against}(s, s')$);
 - b) if s' was decided *per incuriam* then it is hierarchically superior to s (i.e. $\text{Incuriam}(s') \Rightarrow s \prec s'$); and
 - c) every $\tilde{s}$ potentially overruling s' (before s) is itself *per incuriam* (i.e. $\forall \tilde{s} \in \text{Overrule}_T(s', s) : \text{Incuriam}(\tilde{s})$).
- for all s'' , adjacent node to s (i.e. $(s, s'') \in E_s$) if s is according to s'' (i.e. $\text{According}(s, s'')$) we have:
 - a) s'' was decided *per incuriam* by a court that is not higher than the court of s (i.e. $\text{Incuriam}(s'')$ and $s \not\prec s''$); or
 - b) there is $\tilde{s}$ potentially overruling s' (before s) and not *per incuriam* (i.e. $\exists \tilde{s} \in \text{Overrule}_T(s'', s)$ s.t. not $\text{Incuriam}(\tilde{s})$); or
 - c) s'' was decided by a lower court of s' or s'' was decided by the same court of s' but strictly before (i.e. $(s'' <_T s' \text{ and } s' \cong s'') \text{ or } s'' \prec s'$).

Definition 3.42 (Incuriam Sem.). Let $G_s = (V_s, E_s)$ be a s -graph. The state s was decided *per incuriam*, denoted as $\text{Incuriam}(s)$, iff

- $\exists s' \in S$ s.t. $(s, s') \in E_s$, $\text{Against}(s, s')$, $(\text{Incuriam}(s') \Rightarrow s \prec s')$, and $\forall \tilde{s} \in \text{Overrule}_T(s', s) : \text{Incuriam}(\tilde{s})$.
- $\forall s'' \in S$ if $(s, s'') \in E_s$ and $\text{According}(s, s'')$ then either $[\text{Incuriam}(s'') \text{ and } s \not\prec s'']$, or $[\exists \tilde{s} \in \text{Overrule}_T(s'', s) : \text{not } \text{Incuriam}(\tilde{s})]$, or $[(s'' <_T s' \text{ and } s' \cong s'') \text{ or } s'' \prec s']$.

Example 3.13. From Example 3.12 we know that s_{13} went against s_{12} , s_{12} went against s_{11} , while s_{11} did not go against any decision.

We verify that s_{13} is not *per incuriam*, while s_{12} is *per incuriam*. The s_{13} -graph is given in Figure 3.8.

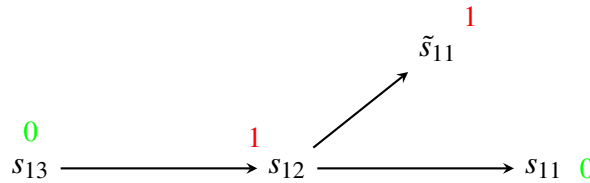
FIGURE 3.8: s_{13} -Graph

Notice that the s_{13} -graph contains the s_{12} -graph (comprising s_{12} , s_{11} , and the edge connecting them), which in turn contains the s_{11} -graph (consisting only of s_{11}). The height of s_{13} is 2, that of s_{12} is 1, and that of s_{11} is 0.

To verify whether s_{13} is per incuriam we have to check whether s_{12} is per incuriam. To check whether s_{12} is per incuriam, we have to check whether s_{11} is per incuriam. s_{11} is not per incuriam, because it did not go against any potentially binding precedent. Hence s_{12} is per incuriam. And so s_{13} is not per incuriam.

Example 3.14. To better illustrate the complexity of the definition of per incuriam, we propose two slight variants of the previous example. Suppose we introduce another state, $\tilde{s}_{11}$, which is decided before s_{12} , is relevant to s_{12} , and is decided by a court binding for s_{12} (that is, either c_0 or c_1). Therefore, $\tilde{s}_{11}$ is potentially binding for s_{12} . Assume moreover $\tilde{s}_{11}$ and s_{12} are decided in the same direction; hence, s_{12} was decided in accordance with $\tilde{s}_{11}$.

Finally, $\tilde{s}_{11}$ and s_{11} are not related; therefore, neither s_{11} nor $\tilde{s}_{11}$ went against a binding precedent and thus are not per incuriam.

FIGURE 3.9: s_{13} -Graph including $\tilde{s}_{11}$

We distinguish two possible situations:

- $\tilde{s}_{11}$ is decided by c_0 , the same court as s_{11} ($\tilde{s}_{11} \cong s_{11}$), and is decided after s_{11} ($s_{11} <_T \tilde{s}_{11}$). In this case, s_{12} is not per incuriam because it correctly followed the decision of $\tilde{s}_{11}$, which was decided after s_{11} . Consequently, s_{13} is per incuriam.
- $\tilde{s}_{11}$ is decided by c_1 , that is, by a lower court than s_{11} ($\tilde{s}_{11} \prec s_{11}$). In this case, s_{12} is per incuriam because it wrongly disregarded s_{11} , which was decided by a higher court than $\tilde{s}_{11}$. Consequently, s_{13} is not per incuriam.

As shown in previous examples, we have provided a method for checking whether a case is *per incuriam* by recursively exploring the graph ‘moving forward’. That is, given a state, to determine whether it is *per incuriam* we must check whether its successors in the graph are *per incuriam*. When moving forward through the graph—from one state to its successors—the height of the states decreases. Thus, to understand whether a case is *per incuriam*, we must examine the behavior of states of lower height in the graph. Building on these observations, we can now provide two characterizations of *per incuriam*

First characterization *per incuriam* The first characterization of *per incuriam* is given as follows:

- I) We iteratively define unary relations I^n , with $n \in \mathbb{N}$;
- II) We prove that a state is *per incuriam* if and only if it verifies I^h for h equal to its height.

The relations I^n referred to in point I) are defined as follows:

- **Step 1.** A state s satisfies I^1 iff
 - there exists a state s' such that s is *against* s' , and
 - for every state s'' such that s is *according to* s'' , either
 - s'' is decided *before* s' by the *same* court as s' , or
 - s'' is decided by a *lower* court than s' .
- **Step $n + 1$.** A state s satisfies I^{n+1} iff
 - there exists a state s' such that (s' does *not* satisfy I^n or s is decided by a *lower* court than s'), and (every state $\tilde{s}$ overruling s' before s satisfies I^n); and
 - for every state s'' such that s is *according to* s'' , at least one of the following holds:
 - s'' satisfies I^n and s is not lower in the hierarchy than s'' ;
 - there exists $\tilde{s}$ an overruling state for s'' that was decided *before* s and that does not satisfy I^n ;
 - s'' is decided *before* s' by the *same* court as s' ; or
 - s'' is decided by a *lower* court than s' .

Definition 3.43 (I^n Sem.). Given $s \in S$, we define $I^n(s)$ (where $n \in \mathbb{N}$) recursively as follows:

$$\begin{aligned}
 I^1(s) &\text{ iff } \exists s' \text{ s.t. } \textit{Against}(s, s') \text{ and} \\
 &\quad \forall s'' \text{ s.t. } \textit{According}(s, s'') : ((s'' <_T s' \text{ and } s' \cong s'') \text{ or } s'' \prec s'). \\
 I^{n+1}(s) &\text{ iff } \exists s' \in S \text{ s.t. } [\textit{Against}(s, s') \text{ and } (\text{not-}I^n(s') \text{ or } s \prec s'), \text{ and} \\
 &\quad (\forall \tilde{s} \in \textit{Overrule}_T(s', s) : I^n(\tilde{s}))] \text{ and} \\
 &\quad \forall s'' \in S \text{ s.t. } \textit{According}(s, s'') : [I^n(s'') \text{ and } s \not\prec s''] \text{ or} \\
 &\quad (\exists \tilde{s} \in \textit{Overrule}_T(s'', s) : \text{not-}I^n(\tilde{s})) \text{ or} \\
 &\quad (s'' \prec s' \text{ or } (s'' <_T s' \text{ and } s' \cong s''))].
 \end{aligned}$$

We now provide the intuition underlying the previous definition. Observe that the definition of the I^n relations mirrors the definition of *per incuriam* introduced earlier (see Definition 3.42). Recall that our ultimate goal is to establish the equivalence stated in point II), namely, to prove the following.

Proposition 3.44 (First characterization *per incuriam*). *Let s s.t. $f(s) \neq ?$ and let $h = \text{height}(s)$. So:*

$$\textit{Incuriam}(s) \Leftrightarrow I^h(s).$$

The proof proceeds by induction. Here, we outline the base case, focusing on the implication $I^h(s) \Rightarrow \textit{Incuriam}(s)$ for $h = \text{height}(s) = 1$. Assume $I^1(s)$. From the definition, we can deduce:

- there exists a state s' such that $\textit{Against}(s, s')$, and
- for all s'' such that $\textit{According}(s, s'')$, we have: $(s'' <_T s' \text{ and } s' \cong s'') \text{ or } s'' \prec s'$

Is this enough to conclude that $\textit{Incuriam}(s)$? Yes—provided that *not* $\textit{Incuriam}(s')$ and s' has not been overruled. We argue that this is indeed the case.

Since $\text{height}(s) = 1$, it follows that $\text{height}(s') = 0$. Indeed, $\text{height}(s) = 1$ means that each successor node of s —namely, each potentially binding or potentially overruling state for s —in turn has no successor nodes, and thus is of height 0. Since $\textit{Against}(s, s')$, we know that $s' \in \beta_s$, so s' is a successor of s in the graph. Hence, s' has height 0. As s' has height 0, it has no successors—meaning it admits neither binding precedents

nor potential overruling decisions.

Therefore, we can conclude that $\text{not-Incuriam}(s')$ holds and that s' is not overruled.

Now, to formally prove point II) above, we will need first an intermediate result: we show now that if s satisfies a I^k , for a $k \geq h = \text{height}(s)$, then s satisfies all I^j , for all $j \geq h$. Roughly speaking, if s satisfies *one* I^k for *one* index k at least equal to the height of s , then s satisfies *all* I^j of index j at least equal to the height of s .

Lemma 3.45. *Let $h = \text{height}(s)$*

$$\exists k \geq h \text{ s.t. } I^k(s) \Rightarrow \forall j \geq h : I^j(s). \quad (3.1)$$

Thanks to Lemma 3.45, we prove Proposition 3.44.

Second Characterization *per incuriam* To provide a second characterization of *per incuriam* we proceed as follows:

- By Lemma 3.45, we know that we can prove that if a state s verifies an I^k at height k equal at least to the height of the state, then s verifies I^j for every j equal to at least the height of the state (Lemma 3.45).
- We know that the height of each state is less than the cardinality of Names_d (Prop 3.41).
- From two previous points, we obtain that: a state s is *per incuriam* iff there exists a k lower than the cardinality of Names_d s.t. s verifies I^j for every j greater or equal to k .

Proposition 3.46 (Second Characterization *per incuriam*). *Let s such that $f(s) \neq ?$. Then:*

$$\text{Incuriam}(s) \Leftrightarrow \exists k \leq |\text{Names}_d| \text{ s.t. } \forall j \text{ with } k \leq j \leq |\text{Names}_d| : I^j(s).$$

Per Incuriam within the language We now use our modal language to represent the notion of *per incuriam*. We do so by translating the second semantic characterization of *per incuriam* into the language. We begin by representing the I^n relations within the language. Notice that the I^n relations apply to decided states. We have already noted

that decided states are named within the finite set Names_d of decided names. Accordingly, we can define the following useful formulas to express I^n relations:

$$\begin{aligned} \hat{F}\varphi &=_{\text{def}} \bigvee_{n \in \text{Names}_d} (n \wedge \hat{F}_n \varphi); \text{Against}\varphi =_{\text{def}} \bigvee_{n, m \in \text{Names}_d} (n \wedge \text{Against}_{n, m} \varphi); \text{POverruling}\varphi =_{\text{def}} \\ &\bigvee_{n, m \in \text{Names}_d} (n \wedge \text{POverruling}_{n, m} \varphi); \text{According}^\forall \varphi =_{\text{def}} \bigvee_{n \in \text{Names}_d} (n \wedge \bigwedge_{m \in \text{Names}_d} \neg \text{According}_{n, m} \neg \varphi); \\ \text{POverruling}^\forall \varphi &=_{\text{def}} \bigvee_{n \in \text{Names}_d} (n \wedge \bigwedge_{m \in \text{Names}_d} \neg \text{POverruling}_{n, m} \neg \varphi).^{23} \end{aligned}$$

The interpretation of these formulas is provided by the following proposition.

Proposition 3.47.

- 1) $(C, s) \models \hat{F}\varphi$ iff $f(s) \neq ?$ and there is s' s.t. $s <_T s'$ and $(C, s') \models \varphi$;
- 2) $(C, s) \models \text{Against}\varphi$ iff $f(s) \neq ?$ and there is s' s.t. $\text{Against}(s, s')$ and $(C, s') \models \varphi$;
- 3) $(C, s) \models \text{POverruling}\varphi$ iff $f(s) \neq ?$ and there is s' s.t. $s' \in \omega_s$ and $(C, s') \models \varphi$;
- 4) $(C, s) \models \text{According}^\forall \varphi$ iff $f(s) \neq ?$ and for all s' s.t. $\text{According}(s, s')$: $(C, s') \models \varphi$;
- 5) $(C, s) \models \text{POverruling}^\forall \varphi$ iff $f(s) \neq ?$ and for all $s' \in \omega_s$: $(C, s') \models \varphi$.

We can now recursively define formulas ι^n , i.e. the a syntactic representation of I^n in the language.

Definition 3.48 (ι^n Lang.). We define formula ι^n , recursively on $n \in \mathbb{N}$:

$$\begin{aligned} \iota^1 &=_{\text{def}} \bigvee_{m \in \text{Names}_d} \left(\text{Against}(m) \wedge \text{According}^\forall ((\hat{F}m \wedge \text{SameCourt}(m)) \vee \text{Higher}(m)) \right), \\ \iota^{n+1} &=_{\text{def}} \bigvee_{n, m \in \text{Names}_d} \left(n \wedge \text{Against}(m \wedge (\neg \iota^n \vee \text{Lower}(n))) \wedge \text{POverruling}^\forall (\hat{F}n \rightarrow \iota^n) \wedge \right. \\ &\quad \left. \text{According}^\forall ((\iota^n \wedge \neg \text{Lower}(m)) \vee \text{POverruling}(\hat{F}n \wedge \neg \iota^n) \vee ((\hat{F}m \wedge \text{SameCourt}(m)) \vee \text{Higher}(m))) \right). \end{aligned}$$

We prove that ι^n , is indeed syntactic representation of I^n in the language.

Proposition 3.49 (I^n , ι^n Repr.). For all $n \in \mathbb{N}$, $(C, s) \models \iota^n \Leftrightarrow I^n(s)$.

We can finally define *per incuriam* in the language.

Definition 3.50 (Per Incuriam Lang.). The notion that *current state is per incuriam* noted *Incuriam* is defined as $\bigvee_{k \leq |\text{Names}_d|} \bigwedge_{k \leq j \leq |\text{Names}_d|} \iota^j$.

²³The fact that Names_d is finite is crucial to ensure that the formulas are well-defined. If Names_d were not finite, the formulas would contain infinite conjunctions or disjunctions, which cannot be expressed within the logic.

By Propositions 3.46 and 3.49, we can establish the equivalence between the semantic and syntactic definitions of *per incuriam*.

Proposition 3.51 (Per Incuriam Repr.). *Let $s \in S$. $(C, s) \models \text{Incuriam} \Leftrightarrow \text{Incuriam}(s)$.*

3.3.2.3 Overruled States/Cases

We have modeled the notion of potential overruling (Definitions 3.30 and 3.31). This notion describes cases that may potentially overrule another state, unless they are themselves decided *per incuriam*. We have also modeled the notion of *per incuriam*, as defined in Definitions 3.42 and 3.50. With both notions in place, we can now model *overruling*—not merely potential, but effective—which constitutes another exception to the binding force of precedent. More precisely, an overruled case is one for which there exists a potentially overruling case that is not *per incuriam*.

Recall that in Definition 3.30 we introduced the set of potentially overruling cases that were decided before another case's decision: $\text{Overrule}_T(s, s'')$ denotes the set of potentially overruling cases for s that were decided prior to s'' . Using this notion, we can formally represent what it means for a case to be overruled (by cases not decided *per incuriam*) *before another case is decided*.

Definition 3.52 ((Before) Overruled Sem.). *s has been overruled, noted $\text{Overruled}(s)$, iff there is $s' \in \omega_s$ such that $\text{not-Incuriam}(s')$.*

s has been overruled before s'' decision, noted $\text{BeforeOverruled}_T(s, s'')$, iff there is $s' \in \text{Overrule}_T(s, s'')$ such that $\text{not-Incuriam}(s')$.

Remark 3.53. *We highlight here an important aspect. In defining the notion of per incuriam, we argued that a decision rendered per incuriam by a higher court remains binding on a lower court. This assumption accords with established judicial practice, particularly in England and Wales, where vertical authority is understood to prevail notwithstanding demonstrable error in the higher court's reasoning or decision. By contrast, according to Definition 3.52, a decision reached per incuriam cannot overrule any precedent; in particular, a per incuriam judgment of a higher court cannot overrule a decision of a lower court. In this sense, the binding authority of courts and the overruling power of courts behave differently with respect to per incuriam and vertical authority. We adopt Definition 3.52 of overruling, as we have found no evidence in*

legal practice that a *per incuriam* decision of a higher court has ever been treated as overruling a decision of a lower court. We return to this point in Section 3.5.

The language-level equivalent of *overruling* and *overruling before* another decision are given by following definitions.

Definition 3.54 ((Before) Overruled Lang.). Then notion that *current state has been overruled*, denoted as *Overruled*, is defined as $\text{POverruling}(\neg \text{Incuriam})$. The notion that *current state has been overruled, before the decision for the n -named state*, denoted as BeforeOverruled_n , is defined as $\text{POverruling}(\hat{\mathbf{F}}n \wedge \neg \text{Incuriam})$.²⁴

Proposition 3.55 (Overruled Repr.). $(C, s) \models \text{Overruled}$ iff $\text{Overruled}(s)$.

Proposition 3.56 (Before Overruled Repr.). $(C, s) \models \text{BeforeOverruled}_n$ iff $\text{BeforeOverruled}_T(s, s'')$ and $(C, s'') \models n$.

Example 3.15. By Example 3.11, we know that s_{11} is potentially overruling for s_9 . Also, by Example 3.14, s_{11} is not *per incuriam* and s_{11} has been decided before s^* . Hence, $\text{BeforeOverruled}(s_9, s^*)$. Within the language, $(C, s_9) \models \text{BeforeOverruled}_{n^*}$.

Remark 3.57. We highlight a further difference in our treatment of overruling and *per incuriam*, beyond the one noted in Remark 3.53. When introducing the notion of *per incuriam*, we argued that if a case appears to be decided *per incuriam* against another case that is itself *per incuriam*, then the former is not *per incuriam* and may therefore be reinstated as a binding precedent. By contrast, this does not hold for overruling: we do not adopt the approach according to which a case overruled by another case, which is itself subsequently overruled, may be reinstated. Although such an approach may, in principle, be legitimate, we do not endorse it. The underlying reason is conceptual: whereas a *per incuriam* case is, in some sense, a mistaken decision that should not have been rendered, an overruled case is better understood as an updated decision within an evolving body of case law; when a court overrules a prior decision, it provides a fresh legal treatment of the matter at hand.

We illustrate what we mean by means of an example in which the relevance relation is Horty's a fortiori relevance relation that we defined in Definition 3.6. Assume that

²⁴Note that the interpretation of $\hat{\mathbf{F}}n$ is provided in Proposition 3.48.

factors $\pi_1, \dots, \pi_3$ favour the plaintiff and factors $\delta_1, \dots, \delta_3$ favour the defendant. Consider the following decided and pending cases within the jurisdiction of England and Wales:

- s^* with factual situation $\{\pi_3, \pi_1, \delta_1\}$, to be decided ($f(s^*) = ?$), by the County Court c_4 ;
- s_1 with factual situation $\{\delta_1, \delta_2, \pi_1\}$, decided for the plaintiff ($f(s_1) = 1$), by the Hight Court c_2 ;
- s_2 with factual situation $\{\delta_1, \pi_1, \pi_2\}$, decided for the defendant ($f(s_2) = 0$), by the Court of Appeal c_1 ;
- s_3 with factual situation $\{\delta_1, \delta_3, \pi_2\}$, decided for the plaintiff ($f(s_3) = 1$) by the UK Supreme Court c_0 .

The temporal relation is given by $s_3 <_T s_2 <_T s_1 <_T s^*$. According to the a fortiori relevance relation, we obtain: $s_1 \mathcal{R}_F s^*, s_2 \mathcal{R}_F s_1, s_3 \mathcal{R}_F s_2$ (but not $s_3 \mathcal{R}_F s^*$). Considering the judicial hierarchy, we observe that s_2 is potentially binding for s_1 , yet s_1 was decided in the opposite direction. Similarly, s_3 is potentially binding for s_2 , but s_2 was decided in the opposite direction. A formal inspection of the s_1 -graph therefore shows that s_2 is per incuriam, whereas s_1 is not. Consequently, s_1 may be reinstated as a binding precedent for s^* . Let us examine the same situation from the perspective of the preference relations among factors that can be extracted from the factual situation of the cases (Definition 1.1).²⁵ Case s_3 establishes $\{\delta_1, \delta_3\} < \{\pi_2\}$, i.e., π_2 alone outweighs δ_1 and δ_3 , and hence, a fortiori, $\{\delta_1\} < \{\pi_2, \pi_1\}$. By contrast, s_2 establishes $\{\pi_1, \pi_2\} < \{\delta_1\}$, which violates the balance of factors determined by s_3 . Accordingly, the decision in s_2 should have favored the plaintiff; thus, s_2 was decided per incuriam. In this context, the judge deciding s_1 may have recognized that s_2 was decided per incuriam and was therefore not constrained by a binding precedent, so that it could legitimately decide either way.

We now reverse the scenario. Consider:

²⁵We emphasize that, strictly speaking, the preference relation in Definition 1.1 is introduced for cases in the reason and result models (and not for our cases, which also include names and courts). Nevertheless, the notion of preference within the result model can be readily adapted to our framework by focusing solely on the factual situation of each case.

- again s^* with with factual situation $\{\pi_3, \pi_1, \delta_1\}$, to be decided ($f(s^*) = ?$), by court c_4
- s_4 with factual situation $\{\delta_1, \delta_2, \pi_1\}$, decided for the plaintiff ($f(s_4) = 1$) by court c_2 ;
- s_5 with factual situation $\{\delta_1, \pi_1, \pi_2\}$, decided for the defendant ($f(s_5) = 0$) by court c_1 ;
- s_6 with factual situation $\{\delta_1, \delta_3, \pi_2\}$, decided for the plaintiff ($f(s_6) = 1$) by court c_0 .

The temporal relation is given by $s_4 <_T s_5 <_T s_6 <_T s^*$. According to the a fortiori relevance relation, we obtain: $s_4 \mathcal{R}_{FS} s^*, s_4 \mathcal{R}_{FS} s_5, s_5 \mathcal{R}_{FS} s_6$ (but not $s_6 \mathcal{R}_{FS} s^*$).

From the perspective of the judicial hierarchy, s_5 is potentially overruling with respect to s_4 , and s_6 is potentially overruling with respect to s_5 . None of these cases is per incuriam; hence, s_4 was indeed overruled by s_5 , and s_5 was in turn overruled by s_6 .²⁶ However, s_4 , despite an “overruling of an overruling”, is not reinstated as binding for s^* . Also, s_6 is not binding for s because it is not relevant to it. At first sight, this appears to conflict with the behaviour observed in the per incuriam situation. Let us therefore analyse the preference ordering.

Case s_4 establishes $\{\delta_1, \delta_2\} < \{\pi_1\}$, and thus, a fortiori, $\{\delta_1\} < \{\pi_1\}$. Case s_5 establishes $\{\pi_1, \pi_2\} < \{\delta_1\}$, and hence, a fortiori, $\{\pi_1\} < \{\delta_1, \delta_2\}$, thereby rewriting the preference ordering among the relevant factors established in s_4 . Case s_5 also implies a fortiori $\{\pi_2\} < \{\delta_1, \delta_3\}$. However, case s_6 overruled s_5 , establishing $\{\delta_1, \delta_3\} < \{\pi_2\}$. Does this restore the original ordering $\{\delta_1, \delta_2\} < \{\pi_1\}$ established by s_4 ? It does not.

²⁶In the book *The Logic of Precedent* [52], published while the core material of this thesis was being finalized, Horty discusses the notion of overruling also in relation to the reason model, offering some examples (in this footnote we refer in particular to Section 4.2). The temporal dimension of cases, the hierarchical relationships among them, and the interaction between overruling and *per incuriam* are not taken into consideration. It seems to us that, in the case of simple case bases (e.g., containing a single precedent and involving a single case to be decided), and setting aside temporal and hierarchical considerations, overruling in the result model (which, we recall, can be understood as a specific instance of the reason model) works similarly to the account we propose. A more detailed comparison is left for future work.

For this reason, we adopt a cautious approach and assume that the overruling of s_5 over s_4 remains valid. And s_4 is not reinstated as binding for s^* .²⁷

3.3.2.4 Binding precedents *without exception*

Finally, we can formalize the notion of *binding precedents without exceptions* for a state s : these are the potentially binding precedents that (a) were not overruled *prior to the decision of s* , and (b) if decided *per incuriam*, were issued by a court higher than s . To clarify condition (a), potentially binding precedents for s that were overruled *only after*—or *simultaneously with*—the decision of s were still binding at the time s was decided. Condition (b), in turn, reflects the principle that *per incuriam* precedents issued by a higher court remain binding on lower courts.

Definition 3.58 (Binding without except. Sem.). s' is a binding precedent (without exceptions) for s iff $s' \in \beta_s$ and $(\text{Incuriam}(s) \Rightarrow s \prec s')$ and not $\text{BeforeOverruled}(s', s)$. $\bar{\beta}_s$ is the set of binding precedents without exception for s .

Definition 3.59 (Binding without except. Lang.). The notion that *for the n -named state a binding state without exceptions is m -named and a ϕ -state*, denoted $\text{Binding}_n \phi$, is defined as $\bigvee_{m \in \text{Names}_d} \text{PBinding}_{n,m} \left(\phi \wedge \neg \text{BeforeOverruled}_n \wedge (\text{Incuriam} \rightarrow \text{Lower}(n)) \right)$.

As shown in the proposition below, our modal language can be used to represent the concept of binding precedents without exceptions.

Proposition 3.60 (Binding without except. Repr.). $(C, s) \models \text{Binding}_n \phi$ iff $(C, s) \models n$ and there is $s' \in \bar{\beta}_s$ s.t. $(C, s') \models \phi$.

Example 3.16. By Example 3.10, $s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{16}$ are potentially binding for s^* . We know that:

- s_9 has been overruled by s_{11} before a decision for s^* (see Example 3.15).
- s_{12} is *per incuriam* (see Example 3.14).
- s_{10}, s_{11}, s_{13} and s_{16} are not *per incuriam* neither overruled.

²⁷Actually, it may even be argued that s_5 is only *partially* overruled: case s_6 revises the ordering $\{\pi_1, \pi_2\} < \{\delta_1\}$ by asserting $\{\delta_1, \delta_3\} < \{\pi_2\}$, while it may still be the case that $\{\pi_1\} < \{\delta_1\}$, as partially maintained by s_5 . However, a detailed investigation of the notion of partial overruling is left for future work.

So, the binding precedents without exception for s^* are s_{10} , s_{11} , s_{13} and s_{16} . Hence $\overline{\beta}_{s^*} = \{s_{10}, s_{11}, s_{13}, s_{16}\}$. Elements in $\overline{\beta}_{s^*}$ are decided both as 0 and as 1. Within the language, $(C_{ex}, s^*) \models \text{Binding}_{n*}(t(1)) \wedge \text{Binding}_{n*}(t(0))$. We still have a conflict of binding precedents. We need a principle to solve the conflict.

3.4 Resolving Precedents Conflict for New Cases

In the previous section, we showed how it is possible to identify binding precedents without exceptions for a given case. However, as illustrated in Example 3.16, a new case under consideration may be subject to two or more binding precedents decided in opposite directions and without exceptions. In such situations, we face a conflict of binding precedents, and the decision-making process for the new case cannot be determined unambiguously.

In this section we want to address the following research question:

Q3) *How can conflicts among binding precedents be resolved in a legally grounded way?*

To address this question, we propose two principles for resolving such conflicts, grounded in the temporal and hierarchical dimensions of cases. These principles are, in fact, widely applied in legal practice [66].

The first principle is the *Hierarchical Principle*: in the event of conflicting precedents, the cases decided by the higher court should be followed. This principle is founded on the hierarchical structure of judicial authority.

We will see, however, that there are situations in which the Hierarchical Principle alone is insufficient—specifically, when there exist multiple binding precedents established by the higher court. In such cases, a second principle may be useful, one that refines the Hierarchical Principle: a *Temporal Hierarchical Principle*. Under this principle, in the event of conflicting binding precedents, priority is given to the most recent decision of the higher court. The reason is that political, economic, or social changes may influence how a court interprets precedent [66]. This principle, for example, is applied

by courts in the United States [66, 22]. We will see that this principle is sufficient to resolve conflicts whenever the precedents at issue are not simultaneous.

In this section, we formally define decision-making processes for a new case s^* ($f(s^*) = ?$), based on the two principles described, and we examine under what conditions such processes yields an unambiguous outcome.

Definition 3.61 (Decision making process/function). Let $s^* \in S$. A *decision making process/function for s^** , is any function $f^* : \{s^*\} \rightarrow 2^{\{0,1\}}$ s.t. $f^*(s^*) = V \in 2^{\{0,1\}}$.

So, a decision making process for s^* is any function that to s^* assigns a subset V of $\{0, 1\}$ (possibly empty). On the basis of the cardinality of V we determine whether the decision process is unambiguous.

Definition 3.62. Let $f^* : \{s^*\} \rightarrow 2^{\{0,1\}}$ be a decision making process. We say that:

- 1) *no decision can be made for s^** iff $f^*(s^*) = \emptyset$;
- 2) *f^* is unambiguous for s^** iff $|f^*(s^*)| = 1$;
- 3) *f^* is ambiguous for s^** iff $|f^*(s^*)| > 1$.

Thus, the decision making function f^* is ambiguous for s^* iff $f^*(s^*) = \{0, 1\}$, i.e. when s^* can be decided both as 0 and as 1 (thus, we have a conflict situation).

A first approach to set a decision making process could rely on binding precedents without exceptions. Let s^* be as in Example 3.16 and suppose to define a decision making function associating to s^* the values (0 or 1) assumed by the set of binding precedents without exception for s^* (precedents in $\bar{\beta}_{s^*}$), i.e. s.t. $f^*(s^*) = \{f(s') \mid s' \in \bar{\beta}_{s^*}\}$. So, since $s_{10}, s_{11}, s_{13}, s_{16} \in \bar{\beta}_{s^*}$, $f^*(s^*) = \{0, 1\}$. We obtain an ambiguous decision making process for s^* .

We now formalise two decision making processes. The first is based on the *Hierarchical Principle*, the second is based on the *Temporal Hierarchical Principle*.

Hierarchical Principle We formalise a decision making process based on the *Hierarchical Principle*. First, we define the *best hierarchical binding precedents*— those from higher courts.

Definition 3.63 (Best-Hier. Bind. Sem.). The set of best hierarchical binding precedents for $s \in S$ is $Best_H(\bar{\beta}_s) = \{s' \in \bar{\beta}_s \mid \forall s'' \in \bar{\beta}_s: s' \not\prec s''\}$.

Recall that the hierarchical relation satisfies *tree-likeness*, which requires that for any two distinct courts that are superior to the same court, one must be hierarchically higher than the other. Recall also that, by condition SD, a binding case for the case at hand is decided either by the same court or by a higher court (see Proposition 3.25). For this reason, the set of best hierarchical binding precedents without exceptions for a given case consists of cases decided by the same court. Namely, we have the following result.

Proposition 3.64. For all $s', s'' \in Best_H(\bar{\beta}_s)$, $s' \cong s''$.

The best binding precedents can be expressed by our modal formulas, as shown below.

Definition 3.65 (Best-Hier. Bind. Lang.). That for the n -named state, there is a best hierarchical binding state without exception in which φ holds, denoted as $Best_H Binding_n \varphi$, is defined as

$$\bigvee_{m \in Names_d} \left(Binding_n(m \wedge \varphi) \wedge \neg Binding_n(Lower(m)) \right).$$

Proposition 3.66 (Best-Hier. Bind. Repr.). $(C, s) \models Best_H Binding_n \varphi$ iff $(C, s) \models n$ and $\exists s' \in Best_H(\bar{\beta}_s)$ s.t. $(C, s') \models \varphi$.

We finally define a decision making process based on the *Hierarchical Principle*.

Definition 3.67 (Hier. Principle). Let s^* s.t. $f(s^*) = ?$. The decision making function $f_H^* : \{s^*\} \rightarrow 2^{\{0,1\}}$, based on the *Hierarchical Principle*, is defined as $f_H^*(s^*) = \{f(s') \mid s' \in Best_H(\bar{\beta}_{s^*})\}$.

The decision process f_H^* assigns to s^* the decisions among the best temporal hierarchical binding precedents in $Best_H(\bar{\beta}_{s^*})$. Note that there may be two such precedents with conflicting outcomes — two binding precedents decided differently (as 0 and as 1), by the higher court. In such a case, the decision process may yield both outcomes, $f_H^*(s^*) = \{0, 1\}$.

Conversely, if all the best binding precedents agree on a single outcome $o \in \{0, 1\}$, we

say that, *on the basis of the Hierarchical Principle, the decision is forced to o* , namely $f_H^*(s^*) = \{o\}$.

Both scenarios can be expressed using our language. The formula $\text{Best}_H\text{Binding}_{n*}(\mathbf{t}(o)) \wedge \text{Best}_H\text{Binding}_{n*}(\mathbf{t}(\bar{o}))$ holds for s^* in the first scenario. While, in the second scenario $\text{Best}_H\text{Binding}_{n*}(\mathbf{t}(o)) \wedge \neg \text{Best}_H\text{Binding}_{n*}(\mathbf{t}(\bar{o}))$ holds for s^* .

Definition 3.68 (Hier. Forced Lang.). The notion that *for the n -named state the decision is forced to be $o \in \{0, 1\}$ on the basis of the Hierarchical Principle*, denoted $\text{HForced}_n(o)$, is defined as $\text{Best}_H\text{Binding}_n(\mathbf{t}(o)) \wedge \neg \text{Best}_H\text{Binding}_n(\mathbf{t}(\bar{o}))$.

Proposition 3.69 (Hier. Forced Repr.). $(C, s^*) \models \text{HForced}_n(o)$ iff $(C, s^*) \models n$ and $f_H^*(s^*) = \{o\}$.

Example 3.17. Recall that $s_{10}, s_{11}, s_{13}, s_{16}$ are binding without exceptions for s^* . Among these, s_{11} and s_{16} are decided by a higher court (c_0) than s_{10} and s_{13} (decided by c_1). Hence, $\text{Best}_H(\bar{\beta}_{s^*}) = \{s_{11}, s_{16}\}$. Thus, we can disregard the two conflicting precedents s_{10} and s_{13} . However, s_{11} and s_{16} are decided in two different directions (respectively as 0 and 1). Therefore, the decision-making process for s^* , based on the Hierarchical Principle, remains ambiguous: $f_H^*(s^*) = \{f(s_{11}), f(s_{16})\} = \{0, 1\}$.

Within the language, $(C_{ex}, s^*) \models \text{Best}_H\text{Binding}_{n*}(\mathbf{t}(0)) \wedge \text{Best}_H\text{Binding}_{n*}(\mathbf{t}(1))$.

The Hierarchical Principle is not enough to solve the conflict for s^* . We will consider the temporal dimension of s_{11} and s_{16} .

Temporal Hierarchical Principle We formalise now the Temporal Hierarchical Principle. First, we define the best temporal hierarchical binding precedents — those more recent and from higher court.

Definition 3.70 (Best Temp-Hier Bind. Sem.). The set of best temporal hierarchical binding precedents for $s \in S$ is $\text{Best}_{TH}(\bar{\beta}_s) = \{s' \in \text{Best}_H(\bar{\beta}_s) \mid \forall s'' \in \text{Best}_H(\bar{\beta}_s): s' \not\prec_T s''\}$.

Definition 3.71 (Best Temp-Hier Bind. Lang.). That for the n -named state, there is a best temporal hierarchical binding state without exception in which φ holds, denoted as $\text{Best}_{TH}\text{Binding}_n\varphi$, is defined as

$$\bigvee_{m \in \text{Names}_d} \left(\text{Best}_{\text{H}}\text{Binding}_n(m \wedge \varphi) \wedge \left(\left(\bigvee_{h \in \text{Names}_d} \text{Best}_{\text{H}}\text{Binding}_n(h) \right) \rightarrow @_h \neg \hat{\text{P}}m \right) \right).$$

The best temporal hierarchical binding precedents thus can be expressed by our modal formulas, as shown above, capturing those that satisfy *Temporal Hierarchical Principle*.

Proposition 3.72 (Best Temp-Hier Bind. Repr.). $(C, s) \models \text{Best}_{\text{TH}}\text{Binding}_n \varphi$ iff $(C, s) \models n$ and $\exists s' \in \text{Best}_{\text{TH}}(\bar{\beta}_s)$ s.t. $(C, s') \models \varphi$.

We finally define a decision making process based on the *Temporal Hierarchical Principle*.

Definition 3.73 (Temp. Hier. Principle). Let s^* s.t. $f(s^*) = ?$. The decision making function $f^* : \{s^*\} \rightarrow 2^{\{0,1\}}$, based on the *Temporal Hierarchical Principle*, is defined as $f_{\text{TH}}^*(s) = \{f(s') \mid s' \in \text{Best}_{\text{TH}}(\bar{\beta}_{s^*})\}$.

The decision process f_{TH}^* assigns to s^* the decisions among the best temporal hierarchical binding precedents in $\text{Best}_{\text{TH}}(\bar{\beta}_{s^*})$. Note that there may be two such precedents with conflicting outcomes — two binding precedents decided differently (as 0 and as 1), by higher court and considered to apply “simultaneously”.²⁸ In such a case, the decision process may yield both outcomes, $f_{\text{TH}}^*(s^*) = \{0, 1\}$.

Conversely, if all the best binding precedents agree on a single outcome $o \in \{0, 1\}$, we say that, *on the basis of the Temporal Hierarchical Principle, the decision is forced to o*, namely $f_{\text{TH}}^*(s^*) = \{o\}$.

Both scenarios can be expressed using our language. The formula $\text{Best}_{\text{TH}}\text{Binding}_{n*}(\text{t}(o)) \wedge \text{Best}_{\text{TH}}\text{Binding}_{n*}(\text{t}(\bar{o}))$ holds for s^* in the first scenario. While, in the second scenario $\text{Best}_{\text{TH}}\text{Binding}_{n*}(\text{t}(o)) \wedge \neg \text{Best}_{\text{TH}}\text{Binding}_{n*}(\text{t}(\bar{o}))$ holds for s^* .

Definition 3.74 (Temp-Hier. Forced Lang.). The notion that *for the n-named state the decision is forced to be $o \in \{0, 1\}$* , denoted $\text{THForced}_n(o)$, is defined as $\text{Best}_{\text{TH}}\text{Binding}_n(\text{t}(o)) \wedge \neg \text{Best}_{\text{TH}}\text{Binding}_n(\text{t}(\bar{o}))$.

²⁸When we say that the same court may adjudicate multiple cases simultaneously, we mean that different judicial panels of that court may consider separate cases in parallel. For instance, in US and UK courts of appeal or supreme courts, although there is a fixed number of judges in office, not all judges decide every case; rather, cases are typically decided by panels (i.e., subsets of judges).

Proposition 3.75 (Temp-Hier. Forced Repr.). $(C, s^*) \models \text{THForced}_n(o)$ iff $(C, s^*) \models n$ and $f_{TH}^*(s) = \{o\}$.

Example 3.18. We can resolve the conflict in Example 3.16 by applying the Temporal Hierarchical principle, according to which s^* is forced to be decided as 1. Indeed, recall from Example 3.17, that $\text{Best}_H(\bar{\beta}_{s^*}) = \{s_{11}, s_{16}\}$. We know that $s_{11} <_T s_{16}$. Hence, the set of best temporal hierarchical binding precedents is $\text{Best}_{TH}(\bar{\beta}_{s^*}) = \{s_{16}\}$. The decision-making process is therefore unambiguous: $f_{TH}^*(s^*) = \{f(s_{16})\} = \{1\}$. Within the language, we have $(C, s^*) \models \text{Best}_{TH}\text{Binding}_n(t(1)) \wedge \neg \text{Best}_{TH}\text{Binding}_n(t(0))$ and therefore, $(C, s^*) \models \text{THForced}_n(1)$.

Notice that we were able to resolve the conflict because s_{11} and s_{16} are not simultaneous. However, recall that in our framework we allow for the possibility of simultaneous cases. Indeed, suppose instead that $s_{11} =_T s_{16}$. In that case, the Temporal Hierarchical Principle would not have been sufficient to resolve the conflict.

As hinted in previous example, f_{TH}^* is an unambiguous decision making for s^* when $\leq_T$ is an order, namely also antisymmetric. Indeed, from antisymmetry it follows that cases cannot be simultaneous and there is a unique latest case decided by the higher court.

Proposition 3.76. Suppose $\bar{\beta}_{s^*} \neq \emptyset$. If $\leq_T$ is a total order (at least on $\text{Best}_H(\bar{\beta}_{s^*})$), then f_{TH}^* is unambiguous decision making for s^* .

3.5 More on Relevance Relation and Per Incuriam

In our work so far, we have imposed no conditions on the relevance relation. In what follows, we introduce some possible properties that could be applied to the relevance relation and examine whether they yield interesting consequences for the notion of *per incuriam*.

Commonly studied properties of binary relations include reflexivity, transitivity, and symmetry. In what follows, we focus on transitivity and, in particular, on the property of *left-Euclideanity*.

Definition 3.77 (Left Euclidean). A relevance relation $\mathcal{R}$ is *left Euclidean* iff $(s' \mathcal{R} s \text{ and } s'' \mathcal{R} s) \Rightarrow (s' \mathcal{R} s'')$.

Notice that, from the previous definition, it follows that if the relevance relation is left Euclidean, then whenever $(s' \mathcal{R} s \text{ and } s'' \mathcal{R} s)$, we have both $(s' \mathcal{R} s'' \text{ and } s'' \mathcal{R} s')$.²⁹ In other words, for a left Euclidean relevance relation, whenever two states are both related to a third one, they must also be related to each other.

Why are we interested in left Euclideanity?

Suppose that, on the classifier model, we have a left Euclidean relevance relation.

Suppose also that, for a given case s , we retrieve two best hierarchical binding cases without exceptions, s' and s'' , decided one strictly before the other and in opposite directions—namely, $s', s'' \in \text{Best}_H(\bar{\beta}s)$, $s' <_T s''$, and $f(s') \neq f(s'')$.

We can verify that s'' went against s' . Indeed we have the following.

- Since s' and s'' are binding without exceptions for s , they are relevant for s . And so, by left Euclideanity, they are relevant to each other. Therefore, since s' is decided strictly before s'' , s' is a precedent for s'' .
- s' and s'' are *best hierarchical* precedents for s , then by Proposition 3.64 we know that they are decided by the same court, let us say by court c' .
- c' is self-bound. Indeed if the court were not self-bound, then c' had the power to overrule itself (see Definition 3.27). Then s'' would have overruled s' , contrary to the fact that s' is a binding precedent without exceptions for s .
- But then s' was a precedent for s'' , decided by the same self-bound court c' , and thus s' was potentially binding for s'' . Hence, s'' went against s' .

Does this mean that s'' is *per incuriam*, or at least that one between s' and s'' is *per incuriam*? Recall that this would be possible, because in the set of binding precedents without exceptions for case s , we may have *per incuriam* cases decided by a higher court than s . The following proposition shows that the answer to this question is positive when, in the underlying classifier model, the temporal relation is a total order and the relevance relation is not only left-Euclidean but also transitive.

²⁹Indeed, since $(s' \mathcal{R} s \text{ and } s'' \mathcal{R} s)$, by definition we obtain $s' \mathcal{R} s''$. Moreover, by commutativity, $(s' \mathcal{R} s \text{ and } s'' \mathcal{R} s)$ is equivalent to $(s'' \mathcal{R} s \text{ and } s' \mathcal{R} s)$, and thus, by definition, $s'' \mathcal{R} s'$.

Proposition 3.78. *Let $C = (S, f, Jur, <_T, \mathcal{R})$, a temporal jurisdictional classifier, with $<_T$ a total order³⁰ and $\mathcal{R}$ transitive and left Euclidean. Let $s \in S$. If $\exists s', s'' \in \text{Best}_H(\bar{\beta}_s)$, s.t. $f(s') \neq f(s'')$, then*

- I) *Incuriam(s') or Incuriam(s'') and*
- II) *$s \prec s'$ and $s \prec s''$.*

The best hierarchically binding precedents for a case s are decided either by the same court as s or by a higher court (as a consequence of Proposition 3.25). The previous proposition shows that, if the relevance relation is left Euclidean and transitive, and the temporal relation is a total order, we cannot obtain two best hierarchically binding and conflicting precedents for s that were decided by the same court as s . In other words, if the best hierarchically binding precedents are issued by the same court as s , all the best hierarchically binding precedents will point in the same direction.

However, it may happen that we retrieve two best hierarchically binding and conflicting precedents for s that were decided by a court higher than s . In such a situation, at least one of them must be *per incuriam*. This implies that, when selecting from the set of best hierarchically binding precedents the best *temporal* hierarchical precedent (which always exists, since $<_T$ is a total temporal order), it *could* correspond to a *per incuriam* state. Although this may seem counterintuitive, it is in fact a consequence of the strong approach to vertical bindingness adopted—particularly in the jurisdiction of England and Wales—where even a *per incuriam* decision of a higher court cannot be disregarded.

Connection to Harty a fortiori relevance relation. Recall that at the end of Section 3.2.1, we defined a relevance relation derived from the *a fortiori* reasoning introduced by Harty in the context of the result model (see Definition 3.6).

We recall here Example 3.19, which was itself constructed on the basis of Example 3.1.

Example 3.19. *Let us recall that we developed Example 3.19, on the basis of Example 3.1, with which we opened this chapter. More precisely, in Example 3.1, we showed that within an inconsistent Harty case base, it is not possible to infer new decisions*

³⁰ $<_T$ is a reflexive, transitive, antisymmetric relation and such that for all $s, s' \in S$, $s <_T s'$ or $s' <_T s$.

in the presence of conflicting precedents. We also noted that, even following the approach proposed by Canavotto [25], we were unable to determine whether a decision was forced: according to Canavotto's proposal, both an outcome for 1 and an outcome for 0 were admissible. We therefore concluded in Example 3.1 that we lacked sufficient information to disambiguate the situation.

In Example 3.19, we then expanded Example 3.1 by introducing, for illustrative purposes, possible temporal and hierarchical dimensions for the cases under consideration. Our aim in that example was to provide to the readers a more concrete illustration of the relevance relation. Based on the additional information and the discussion so far, we can now examine which decision should be inferred for the new case introduced in Example 3.1.

We begin by recalling that $\tilde{s}_6$, $\tilde{s}_7$, and $\tilde{s}_8$ are assumed to be decided under the jurisdiction of England and Wales (Example 3.6). $\tilde{s}_6 = \{\pi_1, \delta_1, \delta_2, c_1, \tilde{n}_6\}$ and $f(\tilde{s}_6) = 1$; $\tilde{s}_7 = \{\pi_1, \pi_2, \delta_1, c_1, \tilde{n}_7\}$ and $f(\tilde{s}_7) = 0$; $\tilde{s}_8 = \{\pi_1, \delta_1, c_2, \tilde{n}_8\}$ and $f(\tilde{s}_8) = ?$.

The temporal relation is given by $\tilde{s}_6 <_T \tilde{s}_7 <_T \tilde{s}_8$.

Recall that it can be verified that $\tilde{s}_6 \mathcal{R}_F \tilde{s}_8$, $\tilde{s}_7 \mathcal{R}_F \tilde{s}_8$, $\tilde{s}_6 \mathcal{R}_F \tilde{s}_7$, $\tilde{s}_6 \mathcal{R}_F \tilde{s}_6$.

$\tilde{s}_6$ is not per incuriam, since it did not go against any decision.

$\tilde{s}_7$ is decided later than the relevant $\tilde{s}_6$ and by the same court c_1 which self bound. So $\tilde{s}_7$ went against $\tilde{s}_6$ and is not supported by any other decision: $\tilde{s}_7$ is per incuriam.

However, both $\tilde{s}_6$ and $\tilde{s}_7$ were decided by a higher court than $\tilde{s}_8$. So, both $\tilde{s}_6$ and $\tilde{s}_7$ are binding precedents without exception for $\tilde{s}_8$.

We know $\tilde{s}_6 <_T \tilde{s}_7$: using the temporal hierarchical principle to solve conflicts we know that $\tilde{s}_8$ should be decided as $\tilde{s}_7$.

We have thus resolved the conflict in Example 3.1. Naturally, this resolution depends heavily on the temporal and hierarchical information we assumed for the cases under consideration—different temporal or hierarchical configurations could have led to a different outcome. For instance, if $\tilde{s}_6$ and $\tilde{s}_7$ had been decided simultaneously ($\tilde{s}_6 =_T \tilde{s}_7$), the conflict could not have been resolved.

The previous example might suggest that left-Euclideanity holds for the a fortiori relevance relation $\mathcal{R}_F$ —indeed, $\tilde{s}_6$ and $\tilde{s}_7$ are both relevant to the same case $\tilde{s}_8$, and are

also relevant to each other. However, left-euclideanity does not generally hold for $\mathcal{R}_F$. Consider the following simple example. We have three states in a classifier model: s_1 , s_2 , and s_3 . State s_1 is such that $s_1 \cap \text{Facts} = \{\pi_1, \delta_1\}$ and $f(s_1) = 1$; state s_2 is such that $s_2 \cap \text{Facts} = \{\pi_1, \delta_2\}$ and $f(s_2) = 1$; and state s_3 is such that $s_3 \cap \text{Facts} = \{\pi_1\}$ and $f(s_3) = 1$. In this setting, s_1 and s_2 are both relevant to s_3 , but they are not relevant to each other.

However, for the a fortiori relevance relation $\mathcal{R}_F$, a particular form of left-euclideanity does hold: if two states decided *in opposite directions* are both relevant to the same state, then they are also relevant to each other.

Definition 3.79 (opposite-decisions left Euclidean). A relevance relation is *opposite-decisions left Euclidean* iff

$$(s\mathcal{R}s'' \text{ and } s'\mathcal{R}_F s'') \Rightarrow (s\mathcal{R}_F s' \text{ and } s'\mathcal{R}_F s)$$

for s, s' such that $f(s) = o \in \{0, 1\}$ and $f(s') = \bar{o}$,

Proposition 3.80. $\mathcal{R}_F$ is opposite-decisions left Euclidean.

The a fortiori relevance relation is also not transitive. Consider the following simple example. We have three states in a classifier model: s_1 , s_2 , and s_3 . State s_1 is such that $s_1 \cap \text{Facts} = \{\pi_1, \delta_1\}$ and $f(s_1) = 1$; s_2 is such that $s_2 \cap \text{Facts} = \{\pi_1, \pi_2, \delta_1\}$ and $f(s_2) = 0$; s_3 is such that $s_3 \cap \text{Facts} = \{\delta_1\}$ with $f(s_3) = 0$. Here, s_1 is relevant to s_2 , and s_2 is relevant to s_3 , but s_1 is not relevant to s_3 .

However, the a fortiori relevance relation satisfies a particular form of transitivity: if s is relevant to s' and s' is relevant to s'' , and if s and s' are decided in the same direction, then s is also relevant to s'' .

Definition 3.81 (same-decisions transitive). A relevance relation $\mathcal{R}$ is *same-decisions transitive* iff

$$(s\mathcal{R}s' \text{ and } s'\mathcal{R}_F s'') \Rightarrow (s\mathcal{R}_F s'')$$

for s, s' such that $f(s) = f(s') = o \in \{0, 1\}$.

Proposition 3.82. $\mathcal{R}_F$ is same-decisions transitive.

However, opposite-direction left-Euclideanity and same-direction transitivity are too weak to establish an analogue of Proposition 3.78. Consider following example.

Example 3.20. *We consider the a fortiori relevance relation $\mathcal{R}_F$. The set of factors is $\text{Facts} = \text{Facts}^1 \uplus \text{Facts}^0$. Where $\text{Facts}^1 = \{\pi_1, \dots, \pi_6\}$ and $\text{Facts}^0 = \{\delta_1, \dots, \delta_6\}$. We consider the following states decided in jurisdiction of England and Wales (in Example 3.4):*

s with factual situation $\{\pi_1, \pi_6, \delta_1, \delta_2, \delta_5\}$, to be decided $f(s) = ?$;

s' with factual situation $\{\pi_1, \pi_6, \delta_1, \delta_2, \delta_5, \delta_6\}$, and outcome $f(s') = 1$;

s'' with factual situation $\{\pi_1, \pi_5, \pi_6, \delta_1, \delta_2, \delta_5\}$, and outcome $f(s'') = 0$;

s₃ with factual situation $\{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6, \delta_1, \delta_2\}$, and outcome $f(s_3) = 0$;

s₄ with factual situation $\{\pi_1, \pi_2, \delta_1, \delta_2, \delta_3\}$ and outcome $f(s_4) = 0$;

s₅ with factual situation $\{\pi_3, \pi_4, \delta_1, \delta_2, \delta_4\}$ and outcome $f(s_5) = 1$.

The temporal relation is given by $s' <_T s_4 <_T s_3 <_T s'' <_T s_5 <_T s$.

The hierarchical relation is given by $s' \cong s'' \cong s$ and $s', s'', s \prec s_3, s_3 \prec s_4$ and $s_3 \prec s_5$. In particular s', s'', s are decided by the High Court (c_2) that is self bound. s_3 is decided by the Court of Appeal (c_1), which is self bound and binding for the High Court, s_4 and s_5 are decided by the UK supreme court (c_0), which is binding for the High Court and the Court of Appeal.

The relevance relation among the states is represented in following Figure 3.10 (e.g., s' is relevant to s).

We retrieve that s' and s'' are the best hierarchical states for s . And they are relevant one for the other. Also, s' is potentially binding for s'' (s' is decided before s'' , by the same self-bound court). However, s' and s'' are both not-per incuriam. Indeed:

- *Although s'' went against s' , s'' is not-per incuriam, because s'' was decided according to s_3 , decided by a higher court and later than s' (but before s'').*

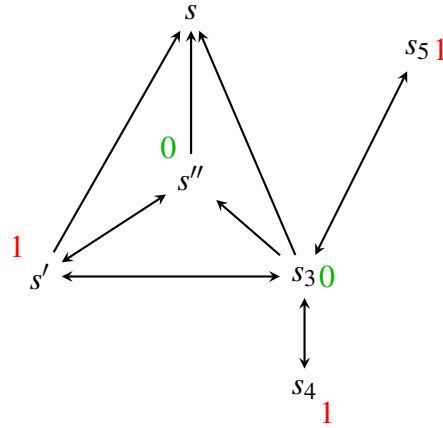


FIGURE 3.10: Relevance Relation among states in Example 3.20

- However, s' was not overruled by s_3 , indeed s_3 is per incuriam, since it went against the higher binding precedent s_4 .
- Also, s_3 is relevant to s . And s_3 is decided by a court with binding power over the court in s and higher with respect to the court of s', s'' . However s_3 is not among the best binding precedents without exceptions for s . Why? because s_3 was overruled before the decision of s , by s_5 that is relevant only to s_3 .

Leaving all technical details aside, we highlight that, in this example, the fact that s' and s'' , although relevant to each other, are not per incuriam is a consequence of two assumptions we made :

- Per incuriam precedents decided by a higher court are binding on a lower court (as is the case for s_3 with respect to s'');
- The same does not apply to overruling: a per incuriam case decided by a higher court cannot overrule a case decided by a lower court (as is the case for s_3 with respect to s').

Recall that we have already highlighted this difference in the behavior of the binding force and the overruling power of courts with respect to per incuriam and vertical authority in Remark 3.53.

The previous example seems to suggest that, if we adapt the definitions of overruling or *per incuriam* in light of the observations just made, we could obtain results analogous to Proposition 3.78—this time with the $\mathcal{R}$ relation being same-decisions transitive and opposite-decisions left-Euclidean (as in Horty's *a fortiori* relevance relation). In what follows, we focus solely on an alternative definition of *per incuriam*.

In particular, Example 3.20 suggests exploring a variant of *per incuriam*, according to which a *per incuriam* case may be disregarded by any court, including lower courts.

This requires first adapting Definition 3.42, which, by contrast, presupposed that *per incuriam* cases decided by higher courts cannot be disregarded by lower courts. We now introduce the following alternative definition.

Definition 3.83 (*Per Incuriam* alternative version). (Incuriam) Let $G_s = (V_s, E_s)$ be a s -graph. The state s was decided *per incuriam*, denoted as $Incuriam(s)$, iff

- $\exists s' \in S$ s.t. $(s, s') \in E_s$, $Against(s, s')$, $\text{not-Incuriam}(s')$, and $\forall \tilde{s} \in Overrule_T(s', s) : Incuriam(\tilde{s})$.
- $\forall s'' \in S$ if $(s, s'') \in E_s$ and $According(s, s'')$ then either $[Incuriam(s'')]$, or $[\exists \tilde{s} \in Overrule_T(s'', s) : \text{not Incuriam}(\tilde{s})]$, or $[(s'' <_T s' \text{ and } s' \cong s'') \text{ or } s'' < s']$.

A choice in favor of this version of the notion of *per incuriam* can be justified not only by the observations in Example 3.20, but also on legal grounds. Indeed, in some jurisdictions the approach to the notion of *per incuriam* differs from that in England and Wales. For example, in India a lower court may, in certain circumstances, disregard a decision of a higher court if it was rendered *per incuriam*.³¹

Consequently, Definition 3.58 of binding precedents (without exceptions) should also be adapted. Recall that we originally defined binding precedents (without exceptions) for a given case s as those that (a) were not overruled prior to the decision of s , and (b) if decided *per incuriam*, were issued by a court higher than s . In the revised version, binding precedents (without exceptions) for s are simply those that (a) were not overruled prior to the decision of s , and (b) are not *per incuriam*.

³¹"The beauty of Indian Judicial System allows the lower court to invoke the doctrine of *per incuriam* if a decision has been rendered due to oversight". Kanta Rani Anr v. State of Haryana, CrI. Rev. No. 2295 of 2015, decided on 28 February 2025 (Punjab Haryana HC).

Definition 3.84 (Binding without exception alternative version). s' is a binding precedent (without exceptions) for s iff $s' \in \beta_s$ and $\text{not-Incuriam}(s)$ and $\text{not-BeforeOverruled}(s', s)$. $\bar{\beta}_s$ is the set of binding precedents without exceptions for s .

If we leave all the other definitions unchanged but adopt the alternative definitions of *per incuriam* and of binding precedents without exception stated above, then—under a $\mathcal{R}$ relation that is same-direction transitive and opposite-direction left-Euclidean—we obtain a strengthened version of Proposition 3.78. In particular, *it is no longer possible to retrieve two best hierarchically binding precedents for the same state that are decided in opposite directions*. This accords with the intuition that Proposition 3.78 reflects the strong approach to vertical bindingness adopted—particularly in the jurisdiction of England and Wales—where a *per incuriam* decision of a higher court cannot be disregarded.

Proposition 3.85. Let $C = (S, f, \text{Jur}, <_T, \mathcal{R})$, a temporal jurisdictional classifier, with $<_T$ a total order, and $\mathcal{R}$ a same-decisions transitive and opposite-decisions left Euclidean relation. Let $s \in S$. Then $\nexists s', s'' \in \text{Best}_H(\bar{\beta}_s)$, s.t. $f(s') \neq f(s'')$.

3.6 Discussion

In this chapter, we developed models for legal CBR that are not grounded on the strong assumptions A2 and A3. In particular, our approach rests on two main intuitions.

First, the notion of precedent depends not only on factual relations between cases, but also on the jurisdiction in which they are decided and on their temporal dimension. Second, in legal practice, conflicts among binding precedents may occur, and when such conflicts arise, both the hierarchical and temporal dimensions of cases can contribute to their resolution.

Building on these points, we enriched both classifier models and the BCL framework with representations of court organization and temporal relations between cases. This allowed us to model the notions of precedent, binding precedent, and binding precedents subject to exceptions, such as precedents decided *per incuriam* or those overruled. Finally, we introduced a Hierarchical Principle and a Temporal Hierarchical

Principle for handling conflicts of binding precedent, and examined the conditions under which these principles yield an unambiguous decision process.

Some aspects deserve further discussion.

Firstly, what about the other strong assumption A1? It can be either adopted or rejected depending on the chosen notion of relevance. For instance, we have shown that in our model it is possible to represent a relevance relation that mirrors Horty's *a fortiori reasoning* in the result model, which is based on assumption A1. Other notions of relevance, however, may not rely on A1.

We have assumed the relevance relation to be a “binary” notion: a case is either relevant to another or not. In future work, we aim to refine this notion so that some cases may be considered more relevant than others.

We have also implicitly assumed that if a case is relevant, it cannot be distinguished. Broadly speaking, a precedent may be distinguished from the case at hand when a significant difference can be identified that justifies its non-application. The notion of distinguishing has been widely studied in the literature on legal CBR; see, among others, [9, 4, 52, 81, 108]. In the result model, according to *a fortiori* reasoning, if one case compels the decision in another, then the two cases cannot be distinguished.

Within our framework, by assuming that a relevant case cannot be distinguished, we have taken relevant precedents issued by a binding court to be automatically binding. For future work, it would be interesting to incorporate a mechanism of distinguishing and to explore its possible interactions with jurisdictional structure.

We highlight here a further aspect concerning distinguishing. If a previously decided case can be distinguished from the case at hand, this means—as we have just stated—that, within our model, the earlier case is not relevant to the present one. In this sense, in our model, when the current case distinguishes a prior case, it cannot simultaneously overrule it, since the earlier case was not relevant. Indeed, within our framework, the existence of a relevance relation among cases is a necessary condition for speaking of overruling. This is consistent with the conception of overruling that we adopt: when a case overrules another, it eliminates the (binding) force of that precedent for subsequent cases. By contrast, when a court distinguishes a precedent, it merely states that the earlier case may not be applied to the case under consideration, without

necessarily invalidating what was established in that case. When keeping overruling and distinguishing conceptually separate, we adhere to the account developed within the reason model in Horty's *The Logic of Precedent* [52].³² More precisely, Horty presents two different perspectives on the relationship between overruling and distinguishing, each reflecting distinct strands of thought within the philosophy of law. Some scholars question whether a genuine difference between the two exists. Schauer observes about a rule that is subject to distinguishing "that 'the rule itself furnishes no constraint' much like a rule that can simply be eliminated"[52].³³ Moreover, Alexander and Sherwin maintain that "When a judge makes an exception to a rule to accommodate a particular case, the judge is effectively eliminating the precedent rule and announcing a new rule in its place".³⁴ Horty shows that, within the standard model—where rules derived from cases are strict and non-defeasible—the distinction between overruling and distinguishing largely disappears. At the same time, Horty demonstrates that this is not the case within the reason model. From this perspective, overruling and distinguishing are indeed distinct operations: overruling is "a *destructive* operation, removing existing information from the overall priority ordering in addition to adding new information" whereas distinguishing is "entirely constructive, merely adding new information to the priority ordering, not removing any existing information" [52].³⁵

Moreover, we have treated also the notion of binding as "binary": a court either issues binding decisions or it does not. Yet, as suggested in [7], binding relations between institutional agents may be context-dependent. In particular, whether one court is binding on another can depend on procedural aspects or matters of jurisdictional competence.

³²See in particular Section 4.2 of [52].

³³[52] p. 104, which in turn cites Schauer, Frederick. *Playing by the Rules: A Philosophical Examination of Rule-Based Decision-Making in Law and Life*. Oxford University Press, 1991.

³⁴Alexander, Larry and Sherwin, Emily. *Demystifying Legal Reasoning*. Cambridge University Press, 2008. Cited in [52], p.104.

³⁵Let us clarify what might appear to be a discrepancy in our earlier claims. We previously stated that, according to a *fortiori reasoning* (and thus within the result model), if a case forces a particular decision it cannot be distinguished. Yet we have just noted that Horty discusses distinguishing from the perspective of the reason model. As we have emphasized several times, the reason model can be viewed as a particular instance of the result model. However, no contradiction arises. Our focus is on cases that force a decision under the precedential constraint (more precisely, under a *fortiori reasoning*), namely cases that impose a (factual) constraint on the case at hand. For Horty, distinguishing may occur when a case, although it could follow a rule established by a precedent, is not compelled by that precedent to be decided for the same side in order to preserve consistency within the case base. From another perspective, when a case is forced to be decided for a side according to the precedent constraint, it cannot be distinguished.

For instance, we assumed that the UK Supreme Court is not formally self-bound. In practice, however, the situation is more nuanced. The Court comprises twelve judges, though not all sit on every case. Most appeals are heard by panels of five judges, while matters of exceptional constitutional or legal significance—especially those involving a potential departure from precedent—are heard by larger panels of seven, nine, or, in rare cases, eleven judges. Thus, although the Supreme Court formally retains the authority to overrule its own precedents, it has developed a doctrine of *de facto* self-binding. Under this doctrine, only larger panels may decide to depart from earlier rulings. In other words, a five-judge panel may be regarded as self-bound by the Court's previous decisions, whereas a larger panel is not bound and may overrule them.

We will return to this and other examples of context-dependent binding in the next chapter, where we propose a conditional—rather than binary—conception of binding relations.

In this chapter, we have relied extensively on the connection between the binding relation, the overruling relation, and the hierarchical relation among courts.

First, drawing on the principles of vertical and horizontal *stare decisis*, expressed by condition SD, we established that a higher court is always binding on a lower court, that a court may be self-bound, and that a lower court is never binding on a higher one. We also defined the overruling power between courts in terms of both the binding and hierarchical relations: a court may overrule another if it is higher in the hierarchy, and it may overrule itself if it is not self-bound.

However, as we will see in the next chapter, in some legal systems—such as that of the United States—a court that is not hierarchically related may nonetheless exercise binding or overruling authority over another court. This will lead us to abandon the assumption of condition SD.

Furthermore, we will refine the conceptual framework for overruling: we will treat the overruling power between courts as a primitive relation, specified by the model and conceptually distinct from the binding power between courts. This refinement reflects the fact that the binding and overruling powers of courts are two different kinds of authority: while binding expresses a constraint, overruling establishes the possibility of removing a constraint.

As we have noted, a decision *per incuriam* is, literally, a decision taken “through lack of care”. The usual interpretation of this notion is that it refers to a decision made in

ignorance of a binding authority. The crucial question, however, is what *ignorance* should be taken to mean in this context.

From our perspective, the term can have at least two meanings. First, ignorance may mean *lack of knowledge*: a precedent was overlooked simply because it was not considered at all—presumably, it was not cited. Second, ignorance may mean *deliberate disregard*: the precedent was known and even cited, but the decision nonetheless chose to ignore it.

In our framework, precedents cannot be explicitly cited within a case, and thus these two nuances cannot be distinguished. Incorporating the possibility of citation within cases, and thus exploring the different interpretations of *per incuriam*, is left for future work.

The focus of our analysis thus far has been on binding precedents—that is, we have been primarily concerned with identifying the conditions under which a decision is binding. Nevertheless, as we briefly mentioned in the introduction to Section 3.3, there also exist non-binding but *persuasive* precedents, namely, decisions that a court is not obliged to follow but may nevertheless draw upon in its reasoning [66]. The examination of persuasive precedents is left for future research. Of particular interest will be assessing whether the notion of (supporting) precedent developed in this chapter is sufficient to capture the concept of persuasive precedent, or whether additional theoretical elements must be considered.

Finally, to model precedents, we adopted a hybrid logic approach to define the intersection between the past and relevance modalities.

We employed a very simple form of hybrid logic. In particular, following Gargov and Goranko [43], we avoided introducing satisfaction operators such as $@_n \phi$ [18] as primitives of the language. Instead, we treated names as propositional variables and derived $@_n \phi$ directly from names by means of the universal modality.

Having names for states offers an advantage over the BCL framework for legal CBR presented in the previous chapters. In BCL, each state corresponded to a case described solely in terms of factual situations, and each state was mapped by the classification function to a single outcome. This excluded the possibility of having two cases with the same factual situation but different decisions. By contrast, introducing names for cases/states allows us to represent distinct cases—with different names but identical factual situations—that are decided differently.

The remainder of this section discusses existing research situated in relation to our framework.

Previous work in legal CBR has examined both the precedential constraint and the inference of new decisions from inconsistent case bases. Among these contributions is the previously mentioned work by Canavotto [25].

In earlier research, Peters et al. [79], within the framework of *A Fortiori Case-Based Argumentation* (AF-CBA), investigate how to identify the most suitable precedents to cite, based on a form of a fortiori reasoning. In this context, they observe that “CB consistency is not always a safe assumption to make” [79]. To address this, they introduce the notion of *authoritativeness*, a numerical measure expressing the extent to which the rest of the case base supports citing a given case for a specific outcome. They propose several metrics for evaluating *authoritativeness*, all of which rely on counting the number of cases supporting one decision or the other. Both approaches in [25] and [79] rest on the assumption that all cases initially carry the same weight, although in [79] some may emerge as more authoritative under a frequentist perspective; in this sense, authority amounts to observed consensus.

Morello and Ciabattoni [70, 71] introduce probabilistic extensions of Horty’s result model to manage inconsistencies. In their framework, some cases are more reliable than others, and Bayesian inference is used to assess how confidently each instance of the constraint should be applied, based on empirical evidence from the case base. This approach also explicitly accounts for epistemic uncertainty.

Our model differs from the approaches discussed so far. We do not assume that all cases have the same importance or weight; the weight of each case is not learned from empirical evidence or observed consensus in the case base. Rather, it is determined by elements known a priori, namely their temporal and jurisdictional dimensions.

Two prior works have explored the role of time and hierarchy in legal CBR from a semantic perspective: the work by Broughton [22] and the work by Wyner and Bench-Capon [105].

In [22] two different models of precedential constraint are combined, starting from the assumption that the vertical precedent constraint is much tighter than the horizontal one; in this sense, even if not explicitly stated, the relevance relation between cases

depends on which constraint applies. We don't make such a distinction. More specifically, we do not impose any specific notion of relevance between cases, in this sense our work moves on a more abstract level. For us, the vertical constraint is stronger only in the approach to *per incuriam*, when we argue that a *per incuriam* precedent cannot be ignored by a lower court. A further distinction pertains to how violations of the vertical constraint are handled: in [22], if a case in the case base violated the vertical constraint, then a decision for a new case cannot be forced (as, in Horty's model [50], the precedential constraint cannot force a new case if the starting case base is not consistent); for us a case decided ignoring a vertical precedent is *per incuriam* and can simply be discarded. Also, differently from [22], we do not assume that horizontal constraint always applies, this depends on the binding relation associated to the classifier model. Finally, in [22] the temporal relation between the cases is a total order, for us is a total preorder; indeed, the temporal and hierarchical principle, also discussed to some extent in [22], is not always sufficient for us to solve conflicts.

The work in [105] extends abstract argumentation frameworks to model judicial reasoning. It distinguishes between two layers: object-level arguments and meta-level arguments. In a legal context, the object-level arguments are those within legal claims, while the meta-level arguments pertains the interaction between legal claims. The focus is particularly on meta-level arguments and the attacks between them, starting from the intuition that whether an argument succeeds depends on the judicial context—for example on court hierarchy, procedural admissibility, and standards of proof. Within this framework, both overruling and *lex posterior* (the equivalent of a Temporal Principle) are briefly outlined but not developed further, while the notion of *per incuriam* is not modelled at all. A detailed exploration of the interaction between the argumentation-based approach in [105] and our work is left for future research.

More generally, an interesting direction for future work would be to explore the notions proposed in this chapter within the framework of abstract argumentation [13], treating cases themselves as arguments. In particular, the concepts of *per incuriam* and overruling—as modeled through *s*-graphs—seem especially well suited to representation via attack relations among cases, defined according to whether they were decided in opposition to, or in accordance with, one another.

We have already mentioned the discussion of the relationship between overruling and

distinguishing in Horty's *The Logic of Precedent* [52]. In the book, Horty also examines the notion of overruling from a formal perspective.³⁶ We do not address the technical details here. Roughly speaking, by means of suitably enriched default theories, Horty introduces a softer, or weaker, notion of constraint. Reasoning with such soft constraints is essential for modeling overruling. In particular, Horty is concerned with how to account for the issue of *special justification*: a court with overruling authority “could overrule an earlier decision as long as it is able to identify some special property or reason [...] for overruling that decision, above and beyond the property of being, merely, wrong”[52]. The notion of special justification is examined in considerable detail in Horty's book, whereas what counts as a court's overruling authority is not formally specified. Our approach to overruling differs from that of [52]. We briefly return to the notion of special justification in Section 4.8, but we do not address it formally in this thesis. Instead, our primary aim is to provide a formal account of overruling by taking into consideration the hierarchical structure of the judiciary, courts' authority to overrule, the temporal dimension of cases, and the interaction between overruling and *per incuriam*.

³⁶We refer here in particular to Section 6.2 of [52].

4

Conditional Judicial Authority

In the previous chapter, we defined the *binding power* between courts as a Boolean relation: a court is either bound by another court or it is not. We further characterized this relation in terms of the *hierarchical structure* of the courts within a jurisdiction. Drawing on the principles of *vertical* and *horizontal stare decisis*, we established that a higher court is always binding on a lower court, that a court may be bound by its own prior decisions, and that a lower court is never binding on a higher one. Finally, we defined the *overruling power* relation between courts, deriving it from both the hierarchical relation among distinct courts and the binding relation when the same court is involved.

In this chapter, we revisit some of these aspects by addressing the following questions.

Firstly, we want to clarify:

Q4) *What is the relation between binding and overruling power?*

In this regard, we first distinguish between two separate relations: a *binding relation* and an *overruling relation*, since, as already noted in the previous chapter, they represent distinct notions. Both binding and overruling express forms of judicial authority between courts, but they involve powers exercised in different ways.

The binding relation expresses a constraining power: when one court c is said to be binding on another court c' , it means that c imposes a constraint on c' 's subsequent decisions.

The notion of *overruling power* between courts carries a different meaning. When a court c overrules a decision of another court c' , it rules upon the decision of c' , thereby modifying the line of law. But what does it mean for the line of law to be altered? It means that, as already stated in the previous chapter, an overruling decision may create a new constraint, but above all—and most importantly for our purposes—the overruled decision of c' loses its *force* with respect to subsequent decisions. Depending on how we interpret the notion of the force of precedent, we can identify several nuances of overruling. In particular, drawing on [66], we can distinguish at least two main interpretations of overruling:

- a) *Overruling in the strict sense*: a "judicial act of overturning a *binding* precedent or a line of precedent".¹
- b) *Overruling in a broader sense*: a "judicial act of rejecting a precedent (or line of precedent) having only [*persuasive*] *force* or other supporting influence".²

Recall that a precedent has *binding force* over the case at hand when the latter is required to follow it, and *persuasive force* when it does not formally bind the court in the present case, yet the court may choose to follow it if it finds its reasoning convincing — for instance, to further support its decision [66].

Recall also that the main purpose for introducing the notion of overruling in our framework was to represent exceptions to the binding force of precedent — that is, to determine which cases are effectively binding and which are not. Hence, the meaning most relevant for our purposes is that of overruling in the strict sense.

In the strict sense, while the binding relation imposes a constraint, overruling, by contrast, does not impose one but rather removes it. More precisely, if a court c overrules a court c' , this means that c may depart from a relevant decision of c' , thereby rendering c' 's decision no longer binding, typically on other courts. In common law systems, for instance, when a higher court c overrules a lower court c' , it does not lift a constraint on itself but eliminates the constraint for the courts subordinate to c' (and possibly creates

¹This is the sense of overruling also adopted by Eisenberg [39].

²Actually, in [66], both *overruling in the strict sense* and *overruling in the broader sense* are defined as *explicit* judicial acts. This is connected to the distinction between *explicit* and *silent* (or *implicit*) overruling introduced in the previous chapter. Recall that the former occurs when a court expressly states that it is overruling another decision, whereas the latter occurs when the overruling is not explicitly acknowledged. Recall also that, in our framework, we cannot distinguish between explicit and implicit overruling, since we have no means of determining which cases are explicitly cited within a given decision. Our primary concern here, rather, is to differentiate the type of *force of precedent* that overruling may remove.

a new constraint for the courts subordinate to c').

However, although our primary focus is on overruling as the act of removing the binding force of a precedent, in the following sections we will show that if we take the notion of *strict overruling* literally, we must model the overruling power between courts in a much more restrictive way than we did in the previous chapter.

Q5) *Are binding and overruling relations necessarily Boolean?*

As we will see, the answer is no. We will not assume that binding and overruling relations are Boolean, that a court either is or is not binding on another, or that a court either does or does not have overruling power over another. Rather, as suggested by Araszkiewicz and Koszowy [7], a relation between two agents within an institution may be context-dependent. There may be specific conditions under which, in deciding the cases at hand, a court may be binding on or may overrule another court. Moreover, it may even happen that, under certain conditions, one court is binding on or can overrule another, while under different conditions the binding/overruling relation between the two same courts is inverted. Where do these conditions come from? In this chapter, we identify two main sources. On the one hand, they may be *procedural*, arising from the composition of the court (for example, when the court sits in panels of judges) or from the jurisdictional role the court plays in a proceeding (such as acting as first instance court or as a court of appeal). On the other hand, the conditions may derive from *subject-matter competence*, depending on the type of case under adjudication. In the United States, for instance, the distribution of authority between federal and state courts depends on whether the case concerns federal or state law.

Q6) *Should binding and overruling relations between courts always be defined solely through the hierarchical relation among courts?*

Again, the answer is no. The *distribution of authority* in a legal system becomes far more sophisticated once contextual elements are taken into account. Both overruling and binding relations will no longer be simply defined exclusively in terms of the hierarchical relation between courts. As we will see with reference to the US court system, it may occur that a lower court is binding on a higher court, or that two courts not hierarchically related are nonetheless connected by binding or overruling relations.

The considerations outlined so far lead us to refine the framework of temporal and jurisdictional classifier models. In particular, we will enrich these models with *conditional binding* and *overruling relations* between courts. In defining the new framework, we adopt a deliberately “neutral” approach: that is, we do not impose in advance any specific constraints governing the relations of binding and overruling, their interaction with each other, or their connection to the hierarchical structure of courts. Instead, we propose a set of principles—some motivated by theoretical considerations, others derived from observed examples—that may or may not be adopted, depending on the chosen approach.

Roadmap to Chapter 4

- **Section *Two Examples: England and Wales and United States Jurisdictions* (4.1)** introduces two illustrative examples — the jurisdictions of England and Wales, and a simplified version of the US court system. These examples demonstrate that binding and overruling relation between courts are not necessarily Boolean notions but may instead be context-dependent, that is, conditional in nature. The model of the England and Wales jurisdiction reflects a less strict approach to overruling, while the US jurisdiction follows a definition of overruling in the strict sense.
- **Section *Conditional Jurisdictions* (4.2)** revises the notion of jurisdiction introduced in the previous chapter (see Definition 3.2), incorporating conditional binding and conditional overruling relations.
- **Section *Conditional Temporal and Jurisdictional Classifier Models and Logic* (4.3)** extends the framework of temporal and jurisdictional classifiers and the logical system TJCL by taking into account conditional jurisdictions.
- **Section *Admissible Conditions* (4.4)** introduces the notion of admissible conditions, namely, conditions that can be used to define the binding and overruling relations.
- **Section *Potentially Binding Precedents and Potential Overruling* (4.5)** builds on the newly defined conditional binding and conditional overruling relations to revise the notions of potentially binding precedents and potentially overruling

states introduced in the previous chapter within the unconditional framework. As we will see, these revised definitions presuppose a form of monotonicity within the states.

- **Section *Principles for Binding and Overruling* (4.6)** marks a turning point: up to this point, no specific constraints have been imposed on the binding or overruling relations, nor on their possible interactions. This section introduces principles that may be reasonably adopted to govern and structure these relations, providing a more systematic foundation for their interplay.
- **Section *Monotonicity or Non-Monotonicity?* (4.7)** presents an alternative formulation of the notion of potential binding, in which the monotonicity assumed in Section 4.5 is relaxed or abandoned, opening the way to non-monotonic interpretations.
- **Section *Discussion* (4.8)** concludes the chapter by reviewing relevant literature, summarizing the implications of the proposed framework, and outlining possible directions for future research.

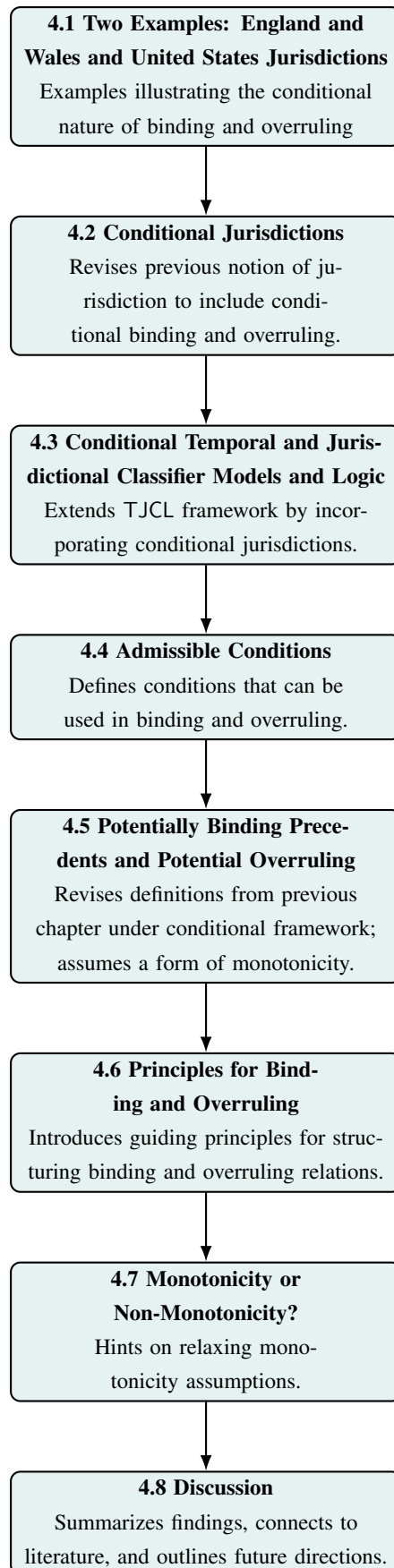


FIGURE 4.1: Roadmap to Chapter 4

4.1 Two Examples: England and Wales and United States Jurisdictions

In this section, we illustrate two examples of jurisdictions in which the binding and overruling powers of courts depend on the context. We consider the jurisdictions of England and Wales, and a simplified version of the United States system. In the first example, we assume a more flexible approach to the interpretation of *strict overruling*, whereas in the second example, we adopt a more literal interpretation of it.

Example 4.1 (England and Wales (EW)). *We will now reconsider and refine the jurisdiction of England and Wales we described in previous chapter. Recall that the hierarchy of courts is illustrated in figure below.*

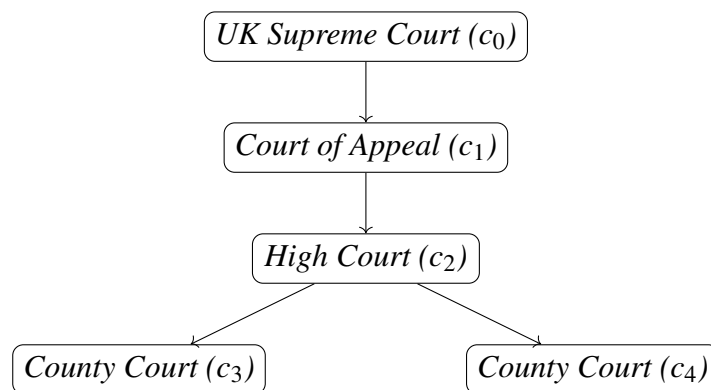


Figure 3.2. *The structure of the civil courts of England and Wales*

Vertical stare decisis continues to apply, so that higher courts are always binding on lower courts. Also, higher courts can always overrule lower courts. A court cannot overrule, and is not binding on, a higher court or a court that is not hierarchically related. Instead, whether some courts are self-bound or can do self-overruling may depend on specific conditions.

UK Supreme Court (c_0): *In the binary framework for binding relations we considered in the previous chapter, we assumed that the Supreme Court (c_0) is not strictly self-bound. This assumption can, however, be refined: the Supreme Court is self-bound, but*

only under specific conditions. In 1966, the House of Lords—the institutional predecessor of the UK Supreme Court—issued the Practice Statement [1], which granted it the discretion to depart from its prior decisions “when it appears right to do so”[65, 28].³ This jurisdictional flexibility was inherited by the UK Supreme Court upon its establishment in 2009, when it replaced the House of Lords as the highest appellate court.

In practice, the Supreme Court developed procedures governing when and how it may depart from precedents. The Court comprises twelve Judges, though not all sit on every case. Most appeals are heard by panels of five Judges. However, matters of exceptional constitutional or legal significance—particularly those involving the possible departure/overruling of established precedent—are typically heard by larger panels of seven, nine, or, in rare cases, eleven Judges.

Accordingly, while the Supreme Court formally retains the authority to overrule its own precedents, it has adopted a doctrine of *de facto* self-binding. This doctrine manifests in the requirement that only larger panels consider whether to depart from previous decisions.

In summary, under this interpretation, a **standard panel** of five-judges is subject to *binding power*: a standard panel may be regarded as *always* self-bound by previous decisions of the Court. While a **larger panel** has *always overruling power* on previous decisions of the Court.

Court of Appeal (c_1): *the Court of Appeal is self bound. And so, it cannot overrule itself.*

High Court (c_2): *The High Court hears a wide variety of cases and may be constituted in different ways depending on the function it performs. In civil matters, it may sit as an appellate court (for example, when hearing appeals from the County Courts) or as a court of first instance.*

There is interpretative debate as to whether the High Court is bound by its own decisions, depending on the jurisdiction in which it is acting [45]. According to some authorities, decisions of the High Court are binding in future cases if rendered while

³For the purposes of this discussion, we do not engage with the nuanced interpretations of “when it appears right to do so”.

the Court was sitting in its appellate capacity.⁴ By contrast, decisions made while the Court is sitting in first-instance jurisdiction are not binding on the Court in later cases. Other formulations put the matter differently, stating that “when the High Court is sitting as an appellate court, it will be bound by its own decisions”[2]. But this leaves open an important question: in its appellate capacity, is the Court bound only by previous appellate decisions, or also by those it handed down while sitting as a court of first instance?

Recent authorities appear to suggest parity between all forms of High Court precedent, regardless of whether the Court was sitting in an appellate or first-instance capacity.⁵ Just for the purposes of illustration, however, we will proceed on the assumption that the High Court, when sitting in its **appellate** capacity, is binding on all subsequent decisions. Likewise, in its **appellate** capacity, it may overrule any previous decision taken in **first-instance** capacity.

County Courts (c_3, c_4): County courts are not self bound. They can overrule themselves.

The possible conditions on courts for England and Wales jurisdiction we described so far are summarized in Table 4.1.

Court	Conditions	Type of conditions
Supreme Court (c_0)	{StandardPanel, LargerPanel}	Procedural: Composition of court
Court of Appeal (c_1)	— ⁶	—
High Court (c_2)	{FirstInstance, Appellate}	Procedural: Court jurisdictional role
County Court (c_3, c_4)	—	—

TABLE 4.1: EW conditions on courts

We began this section by stating that, in this example, we would have adopted a less strict interpretation of the meaning of overruling. Let us now clarify what we meant by that. In this example, we have assumed, as a general principle, that a higher court can always overrule a lower court and that, if a court is not self-bound, it can also overrule itself. This approach is similar to the one adopted for the notion of overruling

⁴As indicated in [45], see *Rogers v Essex County Council* [1985] 2 All ER 39, 43. See also *Watson* (n 25), 846–848; and *R v Greater Manchester Coroner ex p Tal* [1985] QB 67, 78–81.

⁵As indicated in [45], see *Coral Reef v Silverbound Enterprises* [2016] EWHC 3844 (Ch), [2018] 4 WLR 104.

⁶“—” means absence of conditions

power introduced in the previous chapter (see Definition 3.27).

On the other hand, at the beginning of this chapter we stated that our main interest in overruling concerns its strict interpretation. According to the definition, strict overruling is a “judicial act of overturning a binding precedent or a line of precedents” [66]. When we say that the Supreme Court can overrule the Court of Appeal because it is a lower court, this is consistent with the notion of strict overruling, as we are asserting that the Supreme Court may eliminate the binding force that a decision of the Court of Appeal exerts either on itself or on lower courts.

Likewise, when we say that the Supreme Court can overrule itself when it sits in a larger panel, this is also consistent with the notion of strict overruling, as it means that the court removes not a constraint upon itself, but one upon the lower courts.

However, let us now consider the County Courts, which are at the bottom of the hierarchy. These courts are neither binding on themselves nor on any other court. Therefore, when we say that a higher court can overrule the County Courts, or that a County Court can overrule itself, we move beyond the strict sense of overruling, since in these cases there is no binding constraint to remove.

This situation can be interpreted in two ways. The first is that the discrepancy is merely a side effect of the County Courts being at the lowest level of the hierarchy—if there were a court below them, we would still be adhering to the strict interpretation of overruling. The second interpretation is that, although the County Courts are not binding, they may nonetheless possess persuasive force with respect to future decisions. In that case, what we are modelling is a form of overruling understood in a broader sense.

In the previous example, we illustrated how the UK Supreme Court and the High Court are subject to forms of binding and conditional overruling that depend on procedural elements, such as the composition of the panel or the jurisdictional role of the courts. In what follows, we introduce an example—drawing inspiration from the structure of the United States judiciary—that also incorporates conditions for binding and overruling related to questions of *subject-matter competence*.

Moreover, unlike in the previous example, we choose to adopt a literal interpretation of the notion of *strict overruling*. Specifically, we will follow a stronger approach: if a court overrules another, *it removes a binding constraint*.

Example 4.2 (United States). *The United States operates a dual court system in which federal and state courts function within their respective jurisdictions and are governed*

by distinct sets of rules. This institutional design reflects the federalist principle of distributing authority between the national government and the individual states.

Federal courts are responsible for interpreting federal law, adjudicating constitutional issues, and resolving specific disputes—such as those between citizens of different states. State courts, established under each state’s own laws, handle the vast majority of legal matters, including civil, criminal, family, and probate cases. Although independent, the two systems occasionally intersect: federal courts may apply state law, and state courts must observe federal constitutional principles [58].

In what follows, we focus on the court system of Illinois, examining both its state and federal courts, with particular emphasis on civil cases. The hierarchy is shown in Figure 4.2 and described below.

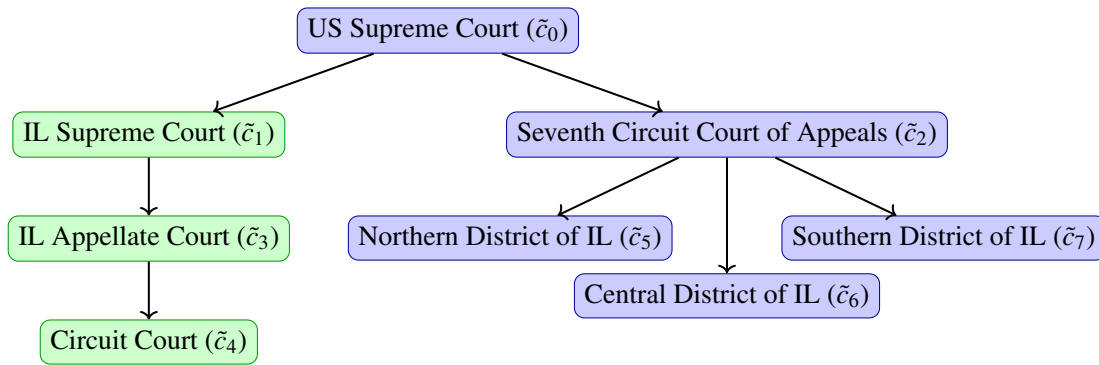


FIGURE 4.2: Structure of the US federal (violet) and Illinois (IL) state (green) courts

Illinois state courts. Illinois state courts handle most civil disputes. These cases typically begin in the Circuit Courts, which serve as trial courts of general jurisdiction across all counties. Illinois has 25 judicial circuits; for simplicity, we assume there is only one ($\tilde{c}_4$). Above the Circuit Courts is the Illinois Appellate Court ($\tilde{c}_3$) and above it lies the Illinois Supreme Court ($\tilde{c}_1$) [54].

Illinois federal courts. The federal trial courts in Illinois are the US District Courts, divided into three districts: the Northern District ($\tilde{c}_5$), the Central District ($\tilde{c}_6$), and the Southern District ($\tilde{c}_7$). Each district court functions as a trial-level federal court. Above the District Courts there is the Seventh Circuit Court of Appeals ($\tilde{c}_2$). At the top of the federal hierarchy stands the US Supreme Court ($\tilde{c}_0$).

*As the highest court in the country, the US Supreme Court ranks above the Illinois Supreme Court.*⁷

We now consider the relations of binding and overruling among the courts.

Binding and overruling within state courts.

- No self-binding. *We found no conclusive evidence that Illinois Circuit, Appellate, or Supreme Courts treat their own prior decisions as strictly binding.*
- Vertical binding. *Higher courts are **always** binding on lower courts in matters of both **federal** and **state law**.*
- Strict overruling. *For illustrative purposes, in this example we adhere literally to the strict notion of overruling—namely, that a court overrules only when it removes a binding constraint. So, the analysis of overruling, becomes more complex:*
 - *Strict vertical overruling. The general principle holds that higher courts may overrule lower courts. However:*
 - * *Higher state courts can overrule lower state courts— on matters of both **federal** and **state law** — except with respect to the Circuit Court.*
 - * *The Circuit Court ($\tilde{c}_4$), being the lowest state court and not self-bound, creates no binding precedent for other courts. Consequently, when the Illinois Supreme Court or the Illinois Appellate Court decides a case differently from a prior Circuit Court decision, this does not constitute strict overruling, since no binding constraint is removed.*
 - *Strict horizontal overruling. Since state courts are not self-bound, they may overrule their own prior decisions—both in matters of **federal law** and in matters of **state law**. The exception, once again, is the Circuit Court: since its decisions are not binding on any court (including itself), departing from a prior Circuit Court decision does not constitute strict self-overruling.*

⁷According to some interpretations, the federal and state court systems may be viewed as hierarchically unrelated: the US Supreme Court is not superior to the Illinois State Supreme Court. Thus, there would be two distinct branches of courts. This poses no problem for us, since—as we will see—we do not assume that jurisdictional hierarchies (see Definition 4.2) have a root.

Binding and overruling within federal courts.

- Vertical Binding. *Higher federal courts are binding on lower federal courts only on matters of **federal law**.*
- Strict vertical overruling. *With the strict notion of overruling:*
 - *A higher federal court may overrule a lower federal court on matters of **federal law**-except with respect to District Courts.*
 - *District Courts $\tilde{c}_5$, $\tilde{c}_6$, and $\tilde{c}_7$ do not create binding precedent for other courts; thus, when the Seventh Circuit or the US Supreme Court departs from a district court decision, this is not strict overruling (mirroring the discussion for Circuit Court above).*
 - *We do not treat higher federal courts as strictly overruling lower federal courts on matters of **state law**. Indeed, federal courts decisions on state law are not binding on any other court; therefore, a departure from federal courts decision on state law removes no binding constraint. Hence, according to our strong interpretation, higher federal courts cannot overrule lower federal courts on matters of **state law**.*
- Self-binding and strict horizontal overruling.
 - *District Courts and the US Supreme Court are not assumed to be self-bound.⁸*
 - *According to our strict interpretation of overruling:*
 - * *The US Supreme Court can overrule its own prior decisions on matters of **federal law**. However, it cannot strictly overrule its own prior decisions on matters of **state law**, since US court decisions on state law are not binding on other courts.*

⁸The Supreme Court generally follows stare decisis, adhering to prior rulings absent a “special justification” or, at minimum, “strong grounds” to overturn precedent. See, e.g., *Janus v. AFSCME*, 585 US (2018). The Court, however, characterizes stare decisis as a policy guiding judicial discretion, not an absolute rule [55]. See, for example, *Citizens United v. FEC*, 558 US 310, 378 (2010) (Roberts, C.J., concurring) (noting that none of the Justices regard stare decisis as “absolute”); and *Helvering v. Hallock*, 309 US 106, 119 (1940) (“[Stare decisis] is a principle of policy rather than a mechanical formula of adherence to the latest decision . . .”). For these reasons, here we assume the Court is not self-bound.

* *All District Courts are non-binding on any other court; therefore, District Courts cannot overrule themselves in the strict sense.*

- The Seventh Circuit Court ($\tilde{c}_2$) internal practice. *In the US, each federal court of appeals has multiple active judges (e.g., 11 in the Seventh Circuit). In most federal circuits, three-judge panels cannot self-overrule prior decisions of the same court; self-overruling typically requires an en banc hearing of all active judges [22].⁹ The Seventh Circuit, however, follows a distinctive practice under “Rule 40(e)”: a three-judge panel may overrule a prior Seventh Circuit Court’s decision if it circulates the proposed opinion to all active judges and a majority declines to hear the case en banc [56]. In practical terms, if a panel’s proposed opinion is circulated and no majority of active judges objects, the panel is effectively authorized to overrule prior Seventh Circuit Court’s precedents; if a majority objects, the case must be reheard en banc and the panel is not authorized to overrule. Decisions of the Seventh Circuit on matters of state law are not binding on any court. Therefore, in the strict sense of overruling, we treat this authorization mechanism as relevant only to matters of federal law.*

Summarizing, on *matters of federal law*, if **not authorized** by a majority of active judges, a **partial** three-judge panel is strictly bound by the Seventh Circuit’s prior decisions; if **authorized** by a majority of active judges, the **partial** panel may overrule a precedent without convening the full court. In all events, an **en banc** court retains authority to overrule the Seventh Circuit’s prior decisions *on matters of federal law*.

Relations between federal and state courts. *Binding and overruling relations also arise between the two systems, because federal courts sometimes apply state law and state courts may decide questions of federal law.*

- When a federal court applies **state law**, the Illinois State Supreme court’s interpretation is binding [77, 3].¹⁰

⁹*En banc*—French for “on the bench”—refers to the procedure by which all active judges of a court hear and decide a case [94].

¹⁰See *Erie Railroad Co. v. Tompkins*, 304 US 64 (1938).

- Under the strict approach to overruling, we do not assume that the Illinois State Supreme can overrule a federal court on matters of **state law**, *because federal-courts' determinations of state law are not binding on other courts*.
- All state courts may decide questions of **federal law**, but they are bound only by the US Supreme Court, *which is the final authority on federal law [77]*.
- The US Supreme Court may overrule a state courts - except the Circuit Court ($\tilde{c}_4$)- on matters of **federal law** [3]. *This formulation aligns with the fact that state courts, except the Circuit Court, bind lower state courts—even when addressing federal questions*.

Important remark. Notice that in the US system—unlike in the EW system, at least in our formalization—binding and overruling relations may exist:

- between courts that are not vertically related (*e.g., the Illinois Supreme Court binds federal courts other than the US Supreme Court, with respect to matters of state law*); and
- between a court and a higher court. *For instance, the Illinois Supreme Court is binding for the higher US Supreme Court—on matters of that state's law.*

We propose two summary tables. Table 4.2 summarizes the binding and overruling relations introduced so far, while Table 4.3 summarizes the possible conditions concerning the courts.

Courts	Domain	Binding Relation	Strict Overruling Relation
State Courts (Illinois)			
IL Supreme Court ($\tilde{c}_1$) $\rightarrow$ IL Appellate Court ($\tilde{c}_3$)	State/Federal law	Binding	Can overrule
IL Supreme Court ($\tilde{c}_1$), IL Appellate Court ($\tilde{c}_3$) $\rightarrow$ Circuit Court ($\tilde{c}_4$)	State/Federal law	Binding	Not overruling (no constraint to remove)
Any state court except Circuit Court $\rightarrow$ itself	State/Federal Law	Not binding on itself	Can overrule
Circuit Court ($\tilde{c}_4$) $\rightarrow$ itself	State/Federal law	Not binding	Not overruling (no constraint to remove)
Federal Courts			
US Supreme Court ($\tilde{c}_0$) $\rightarrow$ Seventh Circuit CoA ($\tilde{c}_2$)	Federal law	Binding	Can overrule
US Supreme Court ($\tilde{c}_0$), Seventh Circuit CoA ($\tilde{c}_2$) $\rightarrow$ District Courts ($\tilde{c}_5$ – $\tilde{c}_7$)	Federal law	Binding	Not overruling (no constraint to remove)
US Supreme Court ($\tilde{c}_0$) $\rightarrow$ Seventh Circuit CoA ($\tilde{c}_2$)	State law	Not binding	Not overruling (no constraint to remove)
US Supreme Court ($\tilde{c}_0$), Seventh Circuit CoA ($\tilde{c}_2$) $\rightarrow$ District Courts ($\tilde{c}_5$ – $\tilde{c}_7$)	State law	Not binding	Not overruling (no constraint to remove)
US Supreme Court ($\tilde{c}_0$) $\rightarrow$ itself	Federal law	Not self-bound	Can overrule own decisions
US Supreme Court ($\tilde{c}_0$) $\rightarrow$ itself	State law	Not binding	Not overruling (no constraint to remove)
Seventh Circuit CoA ($\tilde{c}_2$) $\rightarrow$ itself	Federal law	Bound unless authorized by majority or en banc	Can overrule if authorized or en banc
Seventh Circuit CoA ($\tilde{c}_2$) $\rightarrow$ itself	State law	Not binding	Not overruling (no constraint to remove)
District Courts ($\tilde{c}_5$ – $\tilde{c}_7$) $\rightarrow$ itself	Federal/State law	Not binding	Not overruling (no constraint to remove)
Relations Between State and Federal Courts			
IL Supreme Court ($\tilde{c}_1$) $\rightarrow$ All federal courts	State law	Binding	Not overruling (no constraint to remove)
US Supreme Court ($\tilde{c}_0$) $\rightarrow$ State courts (except Circuit Court)	Federal law	Binding	Can overrule state courts (on federal law)
US Supreme Court ($\tilde{c}_0$) $\rightarrow$ Circuit Court ($\tilde{c}_4$)	Federal law	Binding	Not overruling (no constraint to remove)

TABLE 4.2: Binding and strict overruling relations among US federal and Illinois state courts.

Remark 4.1. Our focus on subject-matter conditions in binding and overruling relations may recall Hart's notion of the domain of competence of a power. More precisely,

Court	Conditions	Type of conditions
US Supreme Court ($\tilde{c}_0$)	{FederalLaw, StateLaw}	Subject-matter competence
IL Supreme Court ($\tilde{c}_1$)	{FederalLaw, StateLaw}	Subject-matter competence
7th Circuit Court of Appeal ($\tilde{c}_2$)	{FederalLaw, StateLaw}	Subject-matter competence
7th Circuit Court of Appeal ($\tilde{c}_2$)	{Partial, Authorized, Unauthorized, EnBanc}	Procedural: Composition of court
IL Court of Appeal ($\tilde{c}_3$)	{FederalLaw, StateLaw}	Subject-matter competence
Circuit Court ($\tilde{c}_4$)	{FederalLaw, StateLaw}	Subject-matter competence
Northern District of Illinois ($\tilde{c}_5$)	{FederalLaw, StateLaw}	Subject-matter competence
Central District of Illinois ($\tilde{c}_6$)	{FederalLaw, StateLaw}	Subject-matter competence
Southern District of Illinois ($\tilde{c}_7$)	{FederalLaw, StateLaw}	Subject-matter competence

TABLE 4.3: Conditions for US federal and Illinois (IL) state courts.

in *The Concept of Law*¹¹, Hart introduces the idea of rules that confer legal powers of a public or official nature. “In the case of a court, some rules specify the subject-matter and content of the judge’s jurisdiction or, as we say, give him ‘power to try’ certain types of case.” A specific example Hart discusses is the section of the County Courts Act 1959, which confers jurisdiction on county courts to try actions for the recovery of land: “[...] where the net annual value for rating of the land in question does not exceed one hundred pounds.”

Our notion of subject-matter competence is related to this discussion, but in a different way. For us, subject-matter competence (e.g., federal law, state law) does not primarily refer to a condition determining whether a court has jurisdiction to decide a case. Rather, it serves to establish whether a court has the power to bind or overrule another court while deciding a case. In this sense, subject-matter competence defines jurisdictional rules concerning the power of binding and overruling.¹²

4.2 Conditional Jurisdictions

We aim now to enrich the classifier framework introduced in the previous chapter to be able to model situations like those in the proposed examples.

To achieve this, we must first expand the notion of jurisdiction (Definition 3.2). Specifically, for each court within a jurisdiction, we need to define a set of conditions that the court may verify (recall, for example, StandardPanel i.e. "sitting on a partial" or

¹¹[47], pp. 28–29

¹²The author thanks Giovanni Sileno for drawing attention to this comparison.

LargerPanel i.e. "sitting in a larger panel" constitute conditions for the UK Supreme Court).

We also introduce a relation specifying when two sets of conditions are *incompatible*, that is, when they cannot be satisfied simultaneously by a court in deciding a case. Consider, for instance, StandardPanel and LargerPanel. These conditions are *incompatible* in the sense that, if a court deciding a case sits in a larger panel, then it does not sit in a standard panel, and vice versa.

We will also require that the binding relation, $\mathcal{B}$, is no longer binary but “conditional”: for each pair of courts (c_i, c_j) , $\mathcal{B}$ specifies the conditions under which the first court, c_i , is binding on the second court, c_j . Finally, unlike what was done previously, we will introduce an overruling relation, also conditional, like the binding relation. This will allow us to study, in a more nuanced manner than we did before, the possible interactions between binding and overruling.

Definition 4.2. (Conditional Jurisdiction) A triple $(\text{Courts}, \{\text{Cond}_c\}_{c \in \text{Courts}}, \text{INC}, \mathcal{H}, \mathcal{B}, \mathcal{O})$ is called a *conditional jurisdiction*, denoted as *Jur*, when:

- Courts is a finite set of courts,
- $\{\text{Cond}_c\}_{c \in \text{Courts}}$ is the collection of finite of set Cond_c of conditions on courts $c \in \text{Courts}$. $\text{Cond} = \bigcup_{c \in \text{Courts}} \text{Cond}_c$ is the set of all conditions.
- $\text{INC} \subseteq 2^{\text{Cond}} \times 2^{\text{Cond}}$ is a irreflexive and symmetric relation.¹³
- $\mathcal{H} \subseteq \text{Courts} \times \text{Courts}$, is a transitive, irreflexive and tree like relation.¹⁴
- $\mathcal{B} \subseteq 2^{\text{Cond}} \times 2^{\text{Cond}} \times \text{Courts} \times \text{Courts}$ is such that $\forall c_i, c_j \in \text{Courts}, \mathcal{B}(c_i, c_j \mid X_i, X_j) \Rightarrow X_i \subseteq \text{Cond}_{c_i}$ and $X_j \subseteq \text{Cond}_{c_j}$.
- $\mathcal{O} \subseteq 2^{\text{Cond}} \times 2^{\text{Cond}} \times \text{Courts} \times \text{Courts}$ is such that $\forall c_i, c_j \in \text{Courts}, \mathcal{O}(c_i, c_j \mid X_i, X_j) \Rightarrow X_i \subseteq \text{Cond}_{c_i}$ and $X_j \subseteq \text{Cond}_{c_j}$

For $X_i, X_j \subseteq \text{Cond}$, where Cond denotes conditions on courts, $\text{INC}(X_i, X_j)$ is read as “ X_i and X_j are incompatible conditions”. As in previous chapter, for $c_i, c_j \in \text{Courts}$,

¹³Irreflexive: not- $\text{INC}(X_i, X_i)$. Symmetric: $(\text{INC}(X_i, X_j) \Rightarrow \text{INC}(X_j, X_i))$.

¹⁴See Definition 3.2

$\mathcal{H}(c_i, c_j)$ is read as “court c_i is hierarchically higher than c_j ”. For $X_i, X_j \subseteq \text{Cond}$, $c_i, c_j \in \text{Courts}$, $\mathcal{B}(c_i, c_j \mid X_i, X_j)$ is read as “ c_i , when acting under conditions X_i , issues binding decisions for court c_j , when the latter acts under conditions X_j ”. For $X_i, X_j \subseteq \text{Cond}$, $c_i, c_j \in \text{Courts}$, $\mathcal{O}(c_i, c_j \mid X_i, X_j)$ is read as “ c_i , when acting under conditions X_i , can overrule court c_j , when the latter acts under conditions X_j ”.¹⁵

For the incompatibility relation, we require irreflexivity: a set of conditions is not incompatible with itself. We also require symmetry, namely that incompatibility is non-directional; for example, there is no difference between $\text{INC}(\{\text{StandardPanel}\}, \{\text{LargerPanel}\})$ and $\text{INC}(\{\text{LargerPanel}\}, \{\text{StandardPanel}\})$.

Two sets of conditions X_i and X_j that are not in the incompatibility relation (i.e., $\text{not-INC}(X_i, X_j)$) are said to be *compatible*.

Notice that for the conditional binding relation $\mathcal{B}(c_i, c_j \mid X_i, X_j)$, we require that the conditions X_i and X_j pertain to the respective courts c_i and c_j — that is, $\mathcal{B}(c_i, c_j \mid X_i, X_j) \Rightarrow X_i \subseteq \text{Cond}_{c_i}$ and $X_j \subseteq \text{Cond}_{c_j}$. Same applies to conditional overruling.

Note also that in the previous definition, the set of conditions may be empty for either court, i.e., $X_i = \emptyset$ or $X_j = \emptyset$. An *empty set of conditions means "always"*. For instance, $\mathcal{B}(c_i, c_j \mid \emptyset, X_j)$ indicates that c_i is always binding on c_j , unconditionally, provided that c_j meets the conditions X_j . In other words, c_i is always binding on c_j whenever c_j satisfies X_j . We will return on this aspect later.

Example 4.3 (EW Conditional Jurisdiction). *We now model the England and Wales court system within the new framework, taking into account the considerations discussed in Example 4.1.*

Courts and Hierarchy. *The courts and the hierarchical relation among them are described in Figure 3.2.*

Conditions on Courts. *The possible, non-empty conditions are those concerning the Supreme Court (c_0) and the High Court (c_2). Specifically, for c_0 , in Table 4.1, we identified $\text{Cond}_{c_0} = \{\text{StandardPanel}, \text{LargerPanel}\}$. With respect to c_2 , from Table 4.1, we know that $\text{Cond}_{c_2} = \{\text{FirstInstance}, \text{Appellate}\}$.*

¹⁵ $\mathcal{O}(c_i, c_j \mid X_i, X_j)$ is the conditional counterpart of the overruling power in Definition 3.27.

Incompatibility. As we already noted, it seems reasonable to require that $\{\text{StandardPanel}\}$ and $\{\text{LargerPanel}\}$ are incompatible: indeed, if a court sits in a larger panel, it cannot at the same time sit in a standard panel, and vice versa. Namely, we require $\text{INC}(\{\text{StandardPanel}\}, \{\text{LargerPanel}\})$.

It is also reasonable to require that $\{\text{FirstInstance}\}$ and $\{\text{Appellate}\}$ are incompatible: if a court acts in a first-instance capacity, it cannot act in an appellate capacity, and vice versa. So we could require $\text{INC}(\{\text{Appellate}\}, \{\text{FirstInstance}\})$.

All the incompatibility relations we have introduced are symmetric (e.g., we have both $\text{INC}(\{\text{StandardPanel}\}, \{\text{LargerPanel}\})$ and $\text{INC}(\{\text{LargerPanel}\}, \{\text{StandardPanel}\})$).

Binding and overruling relations. From Example 4.1 we know that vertical stare decisis holds unconditionally; a higher court is always binding for lower courts. So, for example, the Supreme Court is always binding for the Court of Appeal; namely, $\mathcal{B}(c_0, c_1 \mid \emptyset, \emptyset)$.

Also a higher court can always overrule a lower court. So, for example, the Supreme Court can always overrule the Court of Appeal; namely, $\mathcal{O}(c_0, c_1 \mid \emptyset, \emptyset)$.

We now consider self-binding and self-overruling.

- **Supreme Court** (c_0). We noted that a standard five-judge panel may be regarded as self-bound by previous decisions of the Court, while a larger panel can overrule previous decisions of the Court. The binding relation can then be modeled as $\mathcal{B}(c_0, c_0 \mid \emptyset, \{\text{StandardPanel}\})$, meaning that when c_0 sits as a standard panel, it is always self-bound by its own previous decisions. Similarly, the overruling relation can be modeled as $\mathcal{O}(c_0, c_0 \mid \{\text{LargerPanel}\}, \emptyset)$, meaning that when c_0 sits as a larger panel, it can always overrule its own previous decisions.
- **Court of Appeal** (c_1). The court of appeal is self-bound, and so it cannot overrule its own previous decisions. We can take the set of conditions on this court as empty ($\text{Cond}_{c_1} = \emptyset$). The self-binding relation is given by $\mathcal{B}(c_1, c_1 \mid \emptyset, \emptyset)$. We have no self-overruling relation.
- **High Court** (c_2). We know that the High Court may sit as an appellate court or as a court of first instance. Just for the purposes of illustration, we assumed that when the High Court sits in its appellate capacity it is binding on any subsequent decision of the High Court. Hence $\mathcal{B}(c_2, c_2 \mid \{\text{Appellate}\}, \emptyset)$. Also the High Court in its appellate capacity can overrule any subsequent decision of the High

Court in first instance capacity. Pertaining to the overruling relation we have $\mathcal{O}(c_2, c_2 \mid \{\text{Appellate}\}, \{\text{FirstInstance}\})$.

- **County Courts** (c_3, c_4). *We stated that County Courts are not self bound and they can overrule themselves. So, $\mathcal{O}(c_3, c_3 \mid \emptyset, \emptyset)$, $\mathcal{O}(c_4, c_4 \mid \emptyset, \emptyset)$.*

Tables 4.4 and 4.5 present the binding and overruling relations for England and Wales discussed thus far.

$c_i \backslash c_j$	c_0	c_1	c_2	c_3	c_4
c_0	$\mathcal{B}(c_0, c_0 \mid \emptyset, \{\text{StandardPanel}\})$	$\mathcal{B}(c_0, c_1 \mid \emptyset, \emptyset)$	$\mathcal{B}(c_0, c_2 \mid \emptyset, \emptyset)$	$\mathcal{B}(c_0, c_3 \mid \emptyset, \emptyset)$	$\mathcal{B}(c_0, c_4 \mid \emptyset, \emptyset)$
c_1	— ¹⁶	$\mathcal{B}(c_1, c_1 \mid \emptyset, \emptyset)$	$\mathcal{B}(c_1, c_2 \mid \emptyset, \emptyset)$	$\mathcal{B}(c_1, c_3 \mid \emptyset, \emptyset)$	$\mathcal{B}(c_1, c_4 \mid \emptyset, \emptyset)$
c_2	—	—	$\mathcal{B}(c_2, c_2 \mid \{\text{Appellate}\}, \emptyset)$	$\mathcal{B}(c_2, c_3 \mid \emptyset, \emptyset)$	$\mathcal{B}(c_2, c_4 \mid \emptyset, \emptyset)$
c_3	—	—	—	—	—
c_4	—	—	—	—	—

TABLE 4.4: Binding relations $\mathcal{B}(c_i, c_j \mid X_i, X_j)$ in England and Wales

$c_i \backslash c_j$	c_0	c_1	c_2	c_3	c_4
c_0	$\mathcal{O}(c_0, c_0 \mid \{\text{LargerPanel}\}, \emptyset)$	$\mathcal{O}(c_0, c_1 \mid \emptyset, \emptyset)$	$\mathcal{O}(c_0, c_2 \mid \emptyset, \emptyset)$	$\mathcal{O}(c_0, c_3 \mid \emptyset, \emptyset) (\star)$ ¹⁷	$\mathcal{O}(c_0, c_4 \mid \emptyset, \emptyset) (\star)$
c_1	— ¹⁸	—	$\mathcal{O}(c_1, c_2 \mid \emptyset, \emptyset)$	$\mathcal{O}(c_1, c_3 \mid \emptyset, \emptyset) (\star)$	$\mathcal{O}(c_1, c_4 \mid \emptyset, \emptyset) (\star)$
c_2	—	—	$\mathcal{O}(c_2, c_2 \mid \{\text{Appellate}\}, \{\text{FirstInstance}\})$	$\mathcal{O}(c_2, c_3 \mid \emptyset, \emptyset) (\star)$	$\mathcal{O}(c_2, c_4 \mid \emptyset, \emptyset) (\star)$
c_3	—	—	—	$\mathcal{O}(c_3, c_3 \mid \emptyset, \emptyset) (\star)$	—
c_4	—	—	—	—	$\mathcal{O}(c_4, c_4 \mid \emptyset, \emptyset) (\star)$

TABLE 4.5: Overruling relations $\mathcal{O}(c_i, c_j \mid X_i, X_j)$ in England and Wales

Example 4.4 (US Conditional Jurisdiction). *We now model the system of US courts described in Example 4.2.*

Courts and Hierarchy. *The courts and the hierarchical relation between them are described in Figure 4.2.*

Conditions on courts. *We refer to Table 4.3. US Supreme Court, Illinois Supreme Court, the Circuit Courts and the District Court have the same set of conditions. Namely $\text{Cond}_{\tilde{c}_i} = \{\text{StateLaw}, \text{FederalLaw}\}$, for $i \in \{0, 1, 3, 4, 5, 6, 7\}$.*

Pertaining the Seventh Circuit Court of Appeal $\tilde{c}_2$, the set of conditions is $\text{Cond}_{\tilde{c}_2} = \{\text{StateLaw}, \text{FederalLaw}, \text{EnBanc}, \text{Partial}, \text{Authorized}, \text{Unauthorized}\}$.

¹⁶—" indicates that no binding relation exists between the courts.

¹⁷The meaning of $(\star)$ will be revealed in section. 4.6.1

¹⁸—" means there is no overruling relation between the courts.

Incompatibility. It is reasonable to assume that $\{\text{EnBanc}\}$ and $\{\text{Partial}\}$ are incompatible: if a court sits en banc, it cannot simultaneously sit in a partial panel, and vice versa; that is $\text{INC}(\{\text{EnBanc}\}, \{\text{Partial}\})$.

Also, $\{\text{Authorized}\}$ and $\{\text{Unauthorized}\}$ can be regarded as incompatible: if a court is authorized by the majority of active judges, it is not unauthorized by the majority of active judges, and vice versa; that is, $\text{INC}(\{\text{Unauthorized}\}, \{\text{Authorized}\})$.

The main problem concerns $\{\text{FederalLaw}\}$ and $\{\text{StateLaw}\}$. In general, these two conditions are not incompatible, since a case may involve both state-law and federal-law questions. However, in this work we consider situations in which a single issue with a single legal concern is examined (see Section 1.4). Under this assumption, it is reasonable to suppose that such an issue pertains either to state law or to federal law, but not to both. Therefore, $\{\text{FederalLaw}\}$ and $\{\text{StateLaw}\}$ may be considered incompatible, i.e. $\text{INC}(\{\text{FederalLaw}\}, \{\text{StateLaw}\})$.

All the incompatibility relations we have introduced are symmetric.

Finally, as a general principle we require that if $\text{INC}(X, Y)$, then $\text{INC}(X', Y')$ for all $X \subseteq X'$ and $Y \subseteq Y'$. For example consider court $\tilde{c}_2$. $\{\text{FederalLaw}, \text{Partial}, \text{Unauthorized}\}$ and $\{\text{FederalLaw}, \text{Partial}, \text{Authorized}\}$ are possible sets of conditions on $\tilde{c}_2$. They are incompatible, because $\{\text{Partial}\}$ and $\{\text{Unauthorized}\}$ are incompatible. The same holds for $\{\text{FederalLaw}, \text{EnBanc}\}$ and $\{\text{FederalLaw}, \text{Partial}, \text{Unauthorized}\}$.

Binding and overruling relations.

- **US Supreme Court ($\tilde{c}_0$).**
 - $\tilde{c}_0$ it is not self bound and it can always overrule itself on matter of federal law: so $\mathcal{O}(\tilde{c}_0, \tilde{c}_0 \mid \{\text{FederalLaw}\}, \{\text{FederalLaw}\})$.
 - On matter of federal law, $\tilde{c}_0$ is binding for the Illinois Supreme Court ($\tilde{c}_1$), Illinois Court of Appeal ($\tilde{c}_3$), and the Circuit Court ($\tilde{c}_4$). So, for $i \in \{1, 3, 4\}$ we have: $\mathcal{B}(\tilde{c}_0, \tilde{c}_i \mid \{\text{FederalLaw}\}, \{\text{FederalLaw}\})$.
 - On matter of federal law, $\tilde{c}_0$ can overrule the Illinois Supreme Court ($\tilde{c}_1$), Illinois Court of Appeal ($\tilde{c}_3$). So, for $i \in \{1, 3\}$ we have $\mathcal{O}(\tilde{c}_0, \tilde{c}_i \mid \{\text{FederalLaw}\})$
 - On matter of federal law, $\tilde{c}_0$ is binding for all federal courts, namely the Seventh Circuit Court of Appeals ($\tilde{c}_2$) and Northern District of Illinois ($\tilde{c}_5$), the Central District of Illinois ($\tilde{c}_6$), and the Southern District of Illinois ($\tilde{c}_7$). So, for $i \in \{2, 5, 6, 7\}$ we have: $\mathcal{B}(\tilde{c}_0, \tilde{c}_i \mid \{\text{FederalLaw}\}, \{\text{FederalLaw}\})$.

- *On matter of federal law, $\tilde{c}_0$ can overrule the Seventh Circuit Court of Appeals ($\tilde{c}_2$). So $\mathcal{O}(\tilde{c}_0, \tilde{c}_2 \mid \{\text{FederalLaw}\}, \{\text{FederalLaw}\})$.*
- **Illinois Supreme Court ($\tilde{c}_1$).** *Besides the binding and overruling relation with the US supreme court already described we have:*
 - $\tilde{c}_1$ *it is not self bound and it can always overrule itself. So $\mathcal{O}(\tilde{c}_1, \tilde{c}_1 \mid \emptyset, \emptyset)$.*
 - *on matter of state law, $\tilde{c}_1$ is binding for the US Supreme Court ($\tilde{c}_0$), the Seventh Circuit Court of Appeal ($\tilde{c}_2$), and the three District courts ($\tilde{c}_5, \tilde{c}_6, \tilde{c}_7$). So, for $i \in \{0, 2, 5, 6, 7\}$ we have: $\mathcal{B}(\tilde{c}_1, \tilde{c}_i \mid \{\text{StateLaw}\}, \{\text{StateLaw}\})$.*
 - $\tilde{c}_1$ *is always binding for all state courts, namely the Illinois Court of Appeal ($\tilde{c}_3$), and the Circuit Court ($\tilde{c}_4$). So, for $i \in \{3, 4\}$, $\mathcal{B}(\tilde{c}_1, \tilde{c}_i \mid \emptyset, \emptyset)$.*
 - $\tilde{c}_1$ *can always overrule the Illinois Court of Appeal ($\tilde{c}_3$). So, $\mathcal{O}(\tilde{c}_1, \tilde{c}_3 \mid \emptyset, \emptyset)$.*
- **Seven Circuit Court of Appeal ($\tilde{c}_2$).** *Besides the relation with the US supreme Court, the Illinois supreme Court, we know that:*
 - *on federal law, the Seven Circuit Court of Appeal is binding for District Courts. So, for $i \in \{5, 6, 7\}$ we have: $\mathcal{B}(\tilde{c}_2, \tilde{c}_i \mid \{\text{FederalLaw}\}, \{\text{FederalLaw}\})$.*
 - *Also, it is self bound and can do self overruling under specific conditions. Indeed we know that:*
 - *on federal law, if not authorized by a majority of judges, a partial three-judge panel is strictly bound by its own prior decisions:*
 $\mathcal{B}(\tilde{c}_2, \tilde{c}_2 \mid \{\text{FederalLaw}\}, \{\text{FederalLaw}, \text{Partial}, \text{Unauthorized}\})$;
 - *on federal law, if authorized by a majority, the partial panel may overrule a precedent without convening the full court:*
 $\mathcal{O}(\tilde{c}_2, \tilde{c}_2 \mid \{\text{FederalLaw}, \text{Partial}, \text{Authorized}\}, \{\text{FederalLaw}\})$;
 - *In any case, on federal law, an en banc panel retains the authority to overrule previous decisions of the Court:*
 $\mathcal{O}(\tilde{c}_2, \tilde{c}_2 \mid \{\text{EnBanc}, \text{FederalLaw}\}, \{\text{FederalLaw}\})$.
- **Illinois Court of Appeal ($\tilde{c}_3$).** *Besides the binding and overruling relation with the US Supreme Court and the Illinois Supreme Court already described, we have that Illinois Court of Appeal can overrule itself ($\mathcal{O}(\tilde{c}_3, \tilde{c}_3 \mid \emptyset, \emptyset)$). Also it is binding for the Circuit Court ($\tilde{c}_4$). So, $\mathcal{B}(\tilde{c}_3, \tilde{c}_4 \mid \emptyset, \emptyset)$.*

Tables 4.6 and 4.7 present the binding and overruling relations for the United States discussed thus far.

$\tilde{c}_i \backslash \tilde{c}_j$	$\tilde{c}_0$	$\tilde{c}_1$	$\tilde{c}_2$	$\tilde{c}_3$	$\tilde{c}_4$	$\tilde{c}_5$	$\tilde{c}_6$	$\tilde{c}_7$
$\tilde{c}_0$	—	$\mathcal{B}(\tilde{c}_0, \tilde{c}_1 \mid \{\text{FederalLaw}\}, \{\text{FederalLaw}\})$	$\mathcal{B}(\tilde{c}_0, \tilde{c}_2 \mid \{\text{FederalLaw}\}, \{\text{FederalLaw}\})$	$\mathcal{B}(\tilde{c}_0, \tilde{c}_3 \mid \{\text{FederalLaw}\}, \{\text{FederalLaw}\})$	$\mathcal{B}(\tilde{c}_0, \tilde{c}_4 \mid \{\text{FederalLaw}\}, \{\text{FederalLaw}\})$	$\mathcal{B}(\tilde{c}_0, \tilde{c}_5 \mid \{\text{FederalLaw}\}, \{\text{FederalLaw}\})$	$\mathcal{B}(\tilde{c}_0, \tilde{c}_6 \mid \{\text{FederalLaw}\}, \{\text{FederalLaw}\})$	$\mathcal{B}(\tilde{c}_0, \tilde{c}_7 \mid \{\text{FederalLaw}\}, \{\text{FederalLaw}\})$
$\tilde{c}_1$	$\mathcal{B}(\tilde{c}_1, \tilde{c}_0 \mid \{\text{StateLaw}\}, \{\text{StateLaw}\})$	—	$\mathcal{B}(\tilde{c}_1, \tilde{c}_2 \mid \{\text{StateLaw}\}, \{\text{StateLaw}\})$	$\mathcal{B}(\tilde{c}_1, \tilde{c}_3 \mid \mathbf{0}, \mathbf{0})$	$\mathcal{B}(\tilde{c}_1, \tilde{c}_4 \mid \mathbf{0}, \mathbf{0})$	$\mathcal{B}(\tilde{c}_1, \tilde{c}_5 \mid \{\text{StateLaw}\}, \{\text{StateLaw}\})$	$\mathcal{B}(\tilde{c}_1, \tilde{c}_6 \mid \{\text{StateLaw}\}, \{\text{StateLaw}\})$	$\mathcal{B}(\tilde{c}_1, \tilde{c}_7 \mid \{\text{StateLaw}\}, \{\text{StateLaw}\})$
$\tilde{c}_2$	—	—	$\mathcal{B}(\tilde{c}_2, \tilde{c}_2 \mid \{\text{FederalLaw}\}, \{\text{Partial}, \text{Unauthorized}, \text{FederalLaw}\})$	—	—	$\mathcal{B}(\tilde{c}_2, \tilde{c}_5 \mid \{\text{FederalLaw}\}, \{\text{FederalLaw}\})$	$\mathcal{B}(\tilde{c}_2, \tilde{c}_6 \mid \{\text{FederalLaw}\}, \{\text{FederalLaw}\})$	$\mathcal{B}(\tilde{c}_2, \tilde{c}_7 \mid \{\text{FederalLaw}\}, \{\text{FederalLaw}\})$
$\tilde{c}_3$	—	—	—	—	$\mathcal{B}(\tilde{c}_3, \tilde{c}_4 \mid \mathbf{0}, \mathbf{0})$	—	—	—
$\tilde{c}_4$	—	—	—	—	—	—	—	—
$\tilde{c}_5$	—	—	—	—	—	—	—	—
$\tilde{c}_6$	—	—	—	—	—	—	—	—
$\tilde{c}_7$	—	—	—	—	—	—	—	—

TABLE 4.6: Binding relations $\mathcal{B}(\tilde{c}_i, \tilde{c}_j \mid X_i, X_j)$ among US courts

$\tilde{c}_i \backslash \tilde{c}_j$	$\tilde{c}_0$	$\tilde{c}_1$	$\tilde{c}_2$	$\tilde{c}_3$	$\tilde{c}_4$	$\tilde{c}_5$	$\tilde{c}_6$	$\tilde{c}_7$
$\tilde{c}_0$	$\mathcal{O}(\tilde{c}_0, \tilde{c}_0 \mid \{\text{FederalLaw}\}, \{\text{FederalLaw}\})$	$\mathcal{O}(\tilde{c}_0, \tilde{c}_1 \mid \{\text{FederalLaw}\}, \{\text{FederalLaw}\})$	$\mathcal{O}(\tilde{c}_0, \tilde{c}_2 \mid \{\text{FederalLaw}\}, \{\text{FederalLaw}\})$	$\mathcal{O}(\tilde{c}_0, \tilde{c}_3 \mid \{\text{FederalLaw}\}, \{\text{FederalLaw}\})$	—	—	—	—
$\tilde{c}_1$	—	$\mathcal{O}(\tilde{c}_1, \tilde{c}_1 \mid \mathbf{0}, \mathbf{0})$	—	$\mathcal{O}(\tilde{c}_1, \tilde{c}_3 \mid \mathbf{0}, \mathbf{0})$	—	—	—	—
$\tilde{c}_2$	—	—	$\mathcal{O}(\tilde{c}_2, \tilde{c}_2 \mid \{\text{FederalLaw}, \text{Partial}, \text{Authorized}\}, \{\text{FederalLaw}\})$	—	—	—	—	—
$\tilde{c}_3$	—	—	$\mathcal{O}(\tilde{c}_2, \tilde{c}_2 \mid \{\text{EnBanc}, \text{FederalLaw}\}, \{\text{FederalLaw}\})$	—	$\mathcal{O}(\tilde{c}_3, \tilde{c}_3 \mid \mathbf{0}, \mathbf{0})$	—	—	—
$\tilde{c}_4$	—	—	—	—	—	—	—	—
$\tilde{c}_5$	—	—	—	—	—	—	—	—
$\tilde{c}_6$	—	—	—	—	—	—	—	—
$\tilde{c}_7$	—	—	—	—	—	—	—	—

TABLE 4.7: Overruling relations $\mathcal{O}(\tilde{c}_i, \tilde{c}_j \mid X_i, X_j)$ among US courts

4.3 Conditional Temporal and Jurisdictional Classifier Models and Logic

In this section, we adapt the framework of Temporal and Jurisdictional Classifier Models and the logic TJCL to incorporate conditions on binding and overruling relations and operators.

First, we introduce the modified temporal and jurisdictional classifier models.

As in previous chapter there is a set of possible input values of the classifiers, Atm_0 . In Atm_0 there are: input variables describing the facts of the case, belonging to the countable set $Facts$; the courts that can make the decision, belonging to the finite set $Courts$; the finite set of conditions $Cond$, which in turn is the union of all the finite set of conditions on court – $Cond = \bigcup_{c \in Courts} Cond_c$; the name of each case belonging to the countable set $Names$. Pertaining the $Names$ we assume that there is a finite subset of names $Names_d \subseteq Names$ for decided cases. The sets $Facts$, $Courts$, and $Names$ and $Cond$ are non empty and pairwise disjoint to form Atm_0 . In other words, $Atm_0 = Facts \uplus Names \uplus Courts \uplus Cond$.¹⁹

The classifier outputs have values in $Val = \{1, 0, ?\}$ where elements stand for *plaintiff wins*, *defendant wins* and *absent decision* respectively. For $o \in \{0, 1\}$, the “opposite” $\bar{o}$ is noted for the value $1 - o$.²⁰

Definition 4.3. A conditional temporal jurisdictional classifier model (TJCM) is a tuple $C = (S, f, Jur, \leq_T, \mathcal{R})$ which satisfies these conditions:

- $S \subseteq 2^{Atm_0}$ is a non-empty set s.t. $\forall s \in S \exists ! c \in Courts : c \in s$;
- $f : S \longrightarrow Val$ is called a decision (or classification) function;
- $|S_d|$ is finite, with $S_d = \{s \mid f(s) \neq ?\}$ be the set of decided states;
- Jur is a *conditional jurisdiction*;
- $INC(X_i, X_j) \Rightarrow \forall s \in S : (X_i \not\subseteq s \vee X_j \not\subseteq s)$;
- $\leq_T$ is a total preorder on S ;²¹
- $\mathcal{R} \subseteq S \times S$ is a relation of relevant cases;
- (*name existence*) $\forall s \exists n \in Names : n \in s$;

¹⁹ $\uplus$ denotes disjoint union.

²⁰The opposite is defined only for 0, 1, see Footnote 13.

²¹ $\leq_T$ is a transitive and reflexive relation, with $\forall s, s' \in S$, either $s \leq_T s'$ or $s' \leq_T s$.

- (name for decided states) $\forall s \in S \forall n \in \text{Names} \cap s : s \in S_d \Leftrightarrow n \in \text{Names}_d$;
- (no shared name) $\forall s, s' \in S : (s \neq s' \Rightarrow s \cap s' \cap \text{Names} = \emptyset)$.

The only additional condition with respect to the temporal and jurisdictional classifier models (see Definition 3.3) concerns the incompatibility relation. This condition captures the intuition underlying the notion of incompatibility. Namely, we require that if two sets of conditions X_i and X_j are incompatible, then they cannot be satisfied simultaneously in any state/case. As a consequence, the empty set is incompatible with respect to a $X \subseteq \text{Cond}$, only if conditions X are never satisfied by any state of the model. More precisely, $\text{INC}(X, \emptyset)$ or $\text{INC}(\emptyset, X)$ if and only if for all $s \in S$, $X \not\subseteq s$.

The class of temporal and jurisdictional classifier models is **TJCM** – *Cond*.

We introduce now the language to deal with conditions.

Definition 4.4 (Language). The modal language $\mathcal{L}^{\text{cond}}(\text{Atm})$ is defined as:

$$\varphi ::= p \mid \mathbf{t}(o) \mid \text{INC}(X_i, X_j) \mid \mathbf{H}(c, c) \mid \mathbf{B}(c, c \mid X_i, X_j) \mid \mathbf{O}(c, c \mid X_i, X_j) \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid [\leq_T]\varphi \mid \mathbf{R}^\forall\varphi$$

where $p \in \text{Atm}_0$, $\mathbf{t}(o) \in \text{Dec}$, $c \in \text{Courts}$ and $X_i \subseteq \text{Cond}$.

$\text{INC}(X_i, X_j)$ is read as “conditions X_i, X_j are incompatible”. $\mathbf{H}(c_i, c_j)$ is read as “court c_i is hierarchically higher than the court c_j ”. $\mathbf{B}(c_i, c_j \mid X_i, X_j)$ is read as “court c_i , under conditions X_i issues binding decisions for the court c_j , under conditions X_j ”. $\mathbf{O}(c_i, c_j \mid X_i, X_j)$ is read as “court c_i , under conditions X_i can overrule court c_j , under conditions X_j ”. All the other operators are to be read as in the previous chapter (see Definition 3.7).

Definition 4.5 (Satisfaction relation). Let $C = (S, f, \text{Jur}, \leq_T, \mathcal{R})$ be a temporal jurisdictional classifier model. Let $s \in S$. Then:

$$\begin{aligned} (C, s) &\models p \iff p \in s, \\ (C, s) &\models \mathbf{t}(o) \iff f(s) = o, \\ (C, s) &\models \text{INC}(X_i, X_j) \iff \text{INC}(X_i, X_j) \\ (C, s) &\models \mathbf{H}(c_i, c_j) \iff \mathcal{H}(c_i, c_j) \\ (C, s) &\models \mathbf{B}(c_i, c_j \mid X_i, X_j) \iff \mathcal{B}(c_i, c_j \mid X_i, X_j), \\ (C, s) &\models \mathbf{O}(c_i, c_j \mid X_i, X_j) \iff \mathcal{O}(c_i, c_j \mid X_i, X_j) \end{aligned}$$

$$\begin{aligned}
(C, s) &\models \neg\varphi \iff (C, s) \not\models \varphi \\
(C, s) &\models \varphi \wedge \psi \iff (C, s) \models \varphi \text{ and } (C, s) \models \psi, \\
(C, s) &\models \Box\varphi \iff \forall s' \in S : (C, s') \models \varphi, \\
(C, s) &\models [\leq_T]\varphi \iff \forall s' \in \leq_T[s] : (C, s') \models \varphi, \\
(C, s) &\models R^\forall\varphi \iff \forall s' \in \mathcal{R}(s) : (C, s') \models \varphi.
\end{aligned}$$

Notice that the satisfiability of the incompatibility, hierarchical, binding, and overruling operators is independent of the states of the model; rather, it depends solely on the hierarchical, binding, and overruling relations defined within the model.

Definition 4.6 (Axiomatics of TJCL-Cond). We define TJCL-Cond as the extension of propositional logic obtained by taking the axioms and rules of TJCL (see Definition 3.14), replacing **GlobBind** with **GlobBind1**, and adding **GlobOv**, **GlobME**, **INC**, **irrINC** and **symmINC**.

$$\begin{aligned}
B(c_i, c_j \mid X_i, X_j) &\rightarrow \Box B(c_i, c_j \mid X_i, X_j) && \textbf{(GlobBind1)} \\
O(c_i, c_j \mid X_i, X_j) &\rightarrow \Box O(c_i, c_j \mid X_i, X_j) && \textbf{(GlobOv)} \\
INC(X_i, X_j) &\rightarrow \Box INC(X_i, X_j \mid X_i, X_j) && \textbf{(GlobME)} \\
INC(X_i, X_j) &\rightarrow \Box \left(\bigvee_{k \in X_i} \neg k \vee \bigvee_{l \in X_j} \neg l \right) && \textbf{(INC)} \\
\neg INC(X_i, X_i) &&& \textbf{(irrINC)} \\
INC(X_i, X_j) &\rightarrow INC(X_j, X_i) && \textbf{(symmINC)}
\end{aligned}$$

GlobBind1, **GlobOv** and **GlobME** capture the global nature of the binding, overruling, and incompatibility relations: whenever they hold, they hold across all states/cases. **INC** captures the semantic condition on the model, requiring that two incompatible sets of conditions cannot be satisfied in the same state. **irrINC** and **symmINC** reflect respectively the irreflexivity and symmetry of incompatibility relation.

Proposition 4.7. *TJCL-Cond is sound and complete with respect to TJCM – Cond.*

4.4 Admissible Conditions

Notice that in the models of UK and US jurisdictions all set of conditions X_i, X_j that appear in the binding and overruling relations ($\mathcal{B}(c_i, c_j \mid X_i, X_j)$ or ($\mathcal{O}(c_i, c_j \mid X_i, X_j)$) are "admissible", in the sense that all the elements within X_i (resp. X_j) are not incompatible. For example, there is no relation in the EW jurisdiction stating that $\mathcal{B}(c_0, c_0 \mid \{\text{StandardPanel}, \text{LargerPanel}\}, \{\text{StandardPanel}\})$: this seems reasonable since the elements in $\{\text{StandardPanel}, \text{LargerPanel}\}$ are incompatible according to Example 4.3. On the basis of this we define the notion of admissible set of conditions.

Definition 4.8 (Admissible Conditions Sem.). $X \subseteq \text{Cond}$ is *admissible*, noted $\text{Adm}(X)$ iff for all $Y, Z \subseteq X, Y \neq \emptyset$ and $Z \neq \emptyset$: $\text{not-INC}(Y, Z)$

Notice that the empty set is always admissible; that is, $\text{Adm}(\emptyset)$ always holds, since every its subset is trivially empty.

From now on we will assume that:

$$\begin{aligned} \mathcal{B}(c_i, c_j \mid X_i, X_j) &\Rightarrow \text{Adm}(X_i) \text{ and } \text{Adm}(X_j) \\ \mathcal{O}(c_i, c_j \mid X_i, X_j) &\Rightarrow \text{Adm}(X_i) \text{ and } \text{Adm}(X_j) \end{aligned} \quad (\mathbf{Adm})$$

We define the language-level counterparts of admissibility and condition **Adm**.

Definition 4.9 (Admissible Conditions Lang.). The notion that $X \subseteq \text{Cond}$ is *admissible*, noted $\text{Adm}(X)$, is defined as $\bigwedge_{\substack{Y \neq \emptyset, Z \neq \emptyset \\ Y, Z \subseteq X}} \neg \text{ME}(Y, Z)$

$$\begin{aligned} \mathbf{B}(c_i, c_j \mid X_i, X_j) &\rightarrow \text{Adm}(X_i) \wedge \text{Adm}(X_j) \\ \mathbf{O}(c_i, c_j \mid X_i, X_j) &\rightarrow \text{Adm}(X_i) \wedge \text{Adm}(X_j) \end{aligned} \quad (\mathbf{LAdm})$$

4.5 Potentially Binding Precedents and Potential Overruling

We now aim to incorporate into this new conditional framework all the notions introduced in the previous chapter—namely, *precedent*, *potentially binding precedent*, *potentially overruling cases*, *per incuriam cases*, and *overruled cases*. The only concepts that require adaptation are those of *potentially binding precedents* and *potentially overruling cases*, as these now take into account the conditional binding and overruling relations. All other definitions remain unchanged.

Throughout this section, given states s and s' , we denote by $c \in s \cap \text{Courts}$ and $c' \in s' \cap \text{Courts}$ their respective courts.

For ease of reference, we restate here the definitions of *precedent* and *supporting cases* introduced in the previous chapter.

Definition 3.16 (Precedent Sem). Let $s, s' \in S$ and $o \in \{0, 1\}$. s' is a (*supporting*) *precedent for s in the direction of o* , noted $\Pi(s', s, o)$, iff $f(s') = o$, $s' \in \mathcal{R}(s)$, and $s' <_T s$.

Definition 3.19 (Precedent Lang.). The notion that *for the n -named state a precedent supporting state is m -named and a φ -state*, denoted as $\text{Supporting}_{n,m}\varphi$, is defined as $(\hat{P} \cap \mathcal{R}^\exists)_{n,m}(\neg t(?) \wedge \varphi)$.

We must, however, modify the notions of potentially binding precedent and potentially overruling, taking into account the fact that the binding relation/operator and the overruling relation/operator are conditional.

Definition 4.10 (Pot. bind. Sem.). Let $s, s' \in S$ and $o \in \{0, 1\}$. s' is *potentially a binding precedent for s for a decision as o* , denoted as $\beta(s', s, o)$, iff

- $\Pi(s', s, o)$ and
- $\exists X_1 \subseteq s' \exists X_2 \subseteq s$ such that $\mathcal{B}(c', c \mid X_1, X_2)$.

We simply write $\beta(s', s)$ iff there is $o \in \{0, 1\}$ s.t. $\beta(s', s, o)$. We define $\beta_s = \{s' \mid \beta(s', s)\}$ the *set of potentially binding precedents for s* .

Put simply, a state s' is said to be potentially binding for another state s if there exists a conditional binding relation between the courts deciding them, $\mathcal{B}(c', c \mid X_1, X_2)$, and if the states satisfy the conditions imposed on their respective courts: $X_1 \subseteq s'$ and $X_2 \subseteq s$.

Note, first, that $\mathcal{B}(c, c' \mid \emptyset, \emptyset)$ does not always hold; therefore, the conditions X_1, X_2 may be non-empty and play an essential role in determining whether s' is binding for s . Recall also that $X_1 \subseteq \text{Cond}_{c'}$ and $X_2 \subseteq \text{Cond}_c$. Hence, if X_1, X_2 are non-empty, the fact that s' is potentially binding for s implies that $\text{Cond}_{c'} \cap s' \neq \emptyset$ and $\text{Cond}_c \cap s \neq \emptyset$.

When introducing the conditional binding relation in the context of the definition of conditional jurisdiction (see Definition 4.2), we stated that $\mathcal{B}(c', c \mid X_1, X_2)$ is read as: “ c' , when acting under conditions X_i , issues binding decisions for court c , when the latter acts under conditions X_j ”. According to the previous definition of *potentially binding*, we can refine the interpretation of the binding relation $\mathcal{B}(c', c \mid X_1, X_2)$ as follows: “ c' , when acting under *at least* conditions X_1 , issues binding decisions for c , when the latter acts under *at least* conditions X_2 .”

Indeed, the definition of potential binding implies a form of monotonicity *within the states*. Namely, when s' is potentially binding for s , and this holds due to the relation $\mathcal{B}(c', c \mid X_1, X_2)$, we have that $X_1 \subseteq s'$ and $X_2 \subseteq s$. However, s' may contain *additional conditions* beyond those in X_1 and still remain binding; it only needs to include at least X_1 . Similarly, s may contain *additional conditions* beyond those in X_2 and still be bound by s' .

We must be careful, however: this kind of monotonicity within the states is *formally* distinct from the monotonicity of the binding relation itself. The latter would require that $\mathcal{B}(c', c \mid X_1, X_2)$ implies $\mathcal{B}(c', c \mid Y, X_2)$ for $X_1 \subseteq Y$, or analogously $\mathcal{B}(c', c \mid X_1, Z)$ for $X_2 \subseteq Z$. We will return to this point in Section 4.6.6.

Example 4.5. Consider a simple example within the US jurisdiction. Suppose we have a very simple classifier model, with three states $S = \{s, s', s''\}$. Suppose s' and s'' were decided by court $\tilde{c}_2$, and s is to be decided by court $\tilde{c}_5$. We also know that s, s' and s'' pertain to federal law, hence $\{\text{FederalLaw}\} \subseteq s, s', s''$. Moreover, we know that s' was decided by a court sitting en banc, namely $\{\text{EnBanc}\} \subseteq s'$, while s'' was decided by a court sitting in a partial panel, $\{\text{Partial}\} \subseteq s''$. Suppose we also retrieve that s' and s'' serve as precedents for s , namely $\Pi(s', s, o)$ and $\Pi(s'', s, o)$. We know, from Table 4.6, that, under federal law, $\tilde{c}_2$ is binding on $\tilde{c}_5$: $\mathcal{B}(\tilde{c}_2, \tilde{c}_5 \mid \{\text{FederalLaw}\}, \{\text{FederalLaw}\})$.

Hence, both s' and s'' are potentially binding for s . The fact that s' was decided en banc and s'' by a partial panel plays no role in establishing the potentially binding relation.

We can express the new definition of *potential binding* at the language level.

Definition 4.11 (Pot. bind. Lang.). The notion that for the n -named state a potentially binding state is a m -named φ -state, noted as $\text{PBinding}_{n,m}\varphi$, is defined as

$$\bigvee_{c_i, c_j \in \text{Courts}} \left(\text{B}(c_i, c_j \mid X_i, X_j) \wedge \bigwedge_{k \in X_j} k \wedge c_j \wedge \text{Supporting}_{n,m}(c_i \wedge \bigwedge_{l \in X_i} l \wedge \varphi) \right).$$

Proposition 4.12 (Pot. bind. Repr.). Let $s \in S$, $o \in \{0, 1\}$. $(C, s) \models \text{PBinding}_{m,n}(\text{t}(o) \wedge \varphi)$ iff $(C, s) \models n$ and exists $s' \in S$, s.t. $\beta(s', s, o)$ and $(C, s') \models m \wedge \varphi$.

Similarly to what has been done for *potential binding* between states, we can express *potential overruling* between states.

Definition 4.13 (Pot.Overrul. Sem.). The case s' has potentially overruling power on s , denoted as $O(s', s)$, iff

- $\Pi(s, s', o)$, $f(s') = \bar{o}$, and
- $\exists X_1 \subseteq s' \exists X_2 \subseteq s$ s.t. $\mathcal{O}(c', c \mid X_1, X_2)$.

We write $\omega_s = \{s' \mid O(s', s)\}$ for the set of potentially overruling states for s .

The considerations we have just outlined for potential binding apply equally to potential overruling.

We can express the new definition of *potential overruling* at the language level.

Definition 4.14 (Pot.Overrul. Lang.). The notion that for the n -named state a potentially overruling state is m -named and a φ -state, denoted as $\text{POverruling}_{n,m}\varphi$, is defined as

$$\bigvee_{c_i, c_j} \left(n \wedge c_i \wedge \text{O}(c_j, c_i \mid X_j, X_i) \wedge \bigwedge_{k \in X_i} k \wedge \diamond(\varphi \wedge \bigwedge_{k \in X_j} k \wedge c_j \wedge \text{t}(\bar{o}) \wedge \text{Supporting}_{m,n}\text{t}(o)) \right)$$

Proposition 4.15 (Pot.Overrul. Repr.). Let $s \in S$, $o \in \{0, 1\}$. $(C, s) \models \text{POverruling}_{n,m}(\text{t}(o) \wedge \varphi)$ iff $(C, s) \models n$ and exists $s' \in S$, s.t. $O(s', s)$ and $(C, s') \models m \wedge \varphi$.

4.6 Principles for Binding and Overruling

So far, we have not imposed any constraints on the binding relation, the overruling relation, or on their possible interactions. This choice was deliberate, in order to keep our model potentially applicable across different jurisdictions. In this section, we introduce a set of candidate principles that may reasonably be adopted. We emphasize that our focus is on the notions of binding power and overruling power in a strictly legal and judicial sense.

We propose a collection of principles describing properties concerning binding and overruling authority. We do not claim to provide an exhaustive account of all the properties these relations may exhibit in law. Rather, we introduce these properties for two main reasons: (a) they emerge from the analysis of the examples drawn from the jurisdictions of England and Wales and of the United States, and capture aspects that may reasonably be expected in similar settings; (b) some of these properties are less intuitive, yet they will prove useful in establishing results presented in Section 4.6.9, where we address how to guarantee, even in the context of conditional jurisdiction, the acyclicity of the s -graphs, on which our notion of *per incuriam* is computed (see Definition 3.38).

The following tables provide a summary of the proposed principles, indicating the sections in which they are introduced, their informal interpretation in natural language, illustrative examples drawn from the jurisdictions of England and Wales and the United States (where applicable), and whether they are relevant to Section 4.6.9.

TABLE 4.8: **Interpretation of strict overruling**

Section	Principle	Intuition	Example	Used Sec. 4.6.9
4.6.1	StOv (Strict Overruling)	Formalizes a literal interpretation of strict overruling: <i>strict overruling</i> occurs when a court removes a binding precedent.	The UK Supreme Court can strictly overrule the High Court, since decisions of the Court of Appeal are binding on itself and on lower courts. By contrast, the UK Supreme Court cannot strictly overrule a County Court, because County Court decisions are not binding.	NO

TABLE 4.9: **Relation between conditions**

Section	Principle	Intuition	Example	Used Sec. 4.6.9
4.6.2	BRelCond (Shared macro-category for conditions in binding relations)	Some conditions on the binding court and some conditions on the bounded court belong to the same macro-category (either subject-matter competence or procedural).	The Seventh Circuit Court, when deciding on federal law, is binding on its subsequent decision on federal law taken when sitting in a partially unauthorised panel. There exists a subset of the binding and bound conditions (federal law) that belongs to the same macro-category (subject-matter competence).	NO

4.6.2	ORelCond (Shared macro-category for conditions in overruling relations)	Some conditions on the overruling court and some conditions on the overruled court belong to the same macro-category (either subject-matter competence or procedural).	The Seventh Circuit Court, when sitting on a partially authorised panel and deciding on federal law, can overrule its own previous decision on federal law. There exists a subset of the overruling and overruled conditions (federal law) that belongs to the same macro-category (subject-matter competence).	NO
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TABLE 4.10: Directionality of powers

Section	Principle	Intuition	Example	Used Sec. 4.6.9
4.6.3	AS1 (Asymmetry of Binding)	If a court is binding on another under certain conditions, the reverse cannot hold <i>under the same conditions</i> .	The US Supreme Court is binding on the Illinois Supreme Court regarding federal law, but the Illinois Supreme Court is not binding on the US Supreme Court on federal law.	YES
4.6.3	AS2 (Asymmetry of Overruling)	If a court can overrule another under certain conditions, the reverse cannot hold <i>under the same conditions</i> .	The US Supreme Court may overrule the Illinois Supreme Court on federal law, but the Illinois Supreme Court cannot overrule the US Supreme Court on federal law.	YES

4.6.3	AS3 (Binding excludes symmetric Overruling)	If a court is binding on another under certain conditions, the latter cannot overrule the former under <i>under the same conditions</i> .	The US Supreme Court is binding on the Illinois Supreme Court regarding federal law, but the Illinois Supreme Court cannot overrule the US Supreme Court on federal law.	YES
4.6.3	AS4 (Overruling excludes symmetric Binding)	If a court can overrule another under certain conditions, the latter cannot be binding on the former under <i>under the same conditions</i> .	The US Supreme Court can overrule the Illinois Supreme Court regarding federal law, but the Illinois Supreme Court is not binding on the US Supreme Court on federal law.	YES
4.6.3	CB1 (Strong Asymmetry of Binding)	If a court is binding on another under certain conditions, the reverse cannot hold <i>under any other conditions</i> .	The US Supreme Court is binding on the Seventh Circuit in matters of federal law. By contrast, the Seventh Circuit is never binding on the US Supreme Court under any conditions.	YES
4.6.3	CO1 (Strong Asymmetry of Overruling)	If a court can overrule another under certain conditions, the reverse cannot hold <i>under any other conditions</i> .	The US Supreme Court can overrule the Seventh Circuit in matters of federal law, but the Seventh Circuit can never overrule the US Supreme Court under any other conditions.	YES

4.6.3	CBO1 (Strong Asymmetry of Binding and Overruling)	If a court is binding (resp. overruling) on another under certain conditions, it cannot be overruled (resp. bound) by that court <i>under any other conditions</i> .	The US Supreme Court is binding on (and can overrule) the Seventh Circuit in matters of federal law, whereas the Seventh Circuit can never overrule (and is never binding on) the US Supreme Court.	YES
4.6.3	CB2 (Compatible-Condition Asymmetry of Binding)	If a court is binding on another under certain conditions, the reverse cannot hold <i>under compatible conditions</i> .	The US Supreme Court is binding on Illinois Supreme State Court in matters of federal law. Illinois Supreme State Court is binding on the US Supreme Court in matters of state law. But federal-law and state-law conditions are not compatible (since single-issue cases typically involve either federal law or state law, but not both).	YES
4.6.3	CO2 (Compatible-Condition Asymmetry of Overruling)	If a court can overrule another under certain conditions, the reverse cannot hold <i>under compatible conditions</i> .	No example in US and EW jurisdiction.	YES

4.6.3	CBO2,CBO3 (Compatible-Condition Asymmetry of Binding and Overruling)	If a court is binding (resp. overruling) on another under certain conditions, it cannot be overruled (resp. bound) by that court <i>under compatible conditions</i> . We will see that CBO3 is a stronger version of CBO2 .	The US Supreme Court can overrule the Illinois Supreme State Court in matters of federal law. Illinois Supreme State Court is binding on the US Supreme Court in matters of state law. But federal-law and state-law conditions are not compatible	YES
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TABLE 4.11: **Binding, overruling and judicial hierarchy**

Section	Principle	Intuition	Example	Used Sec. 4.6.9
4.6.4	BSD (Binding Strong Hierarchical Dependence)	A higher court is always binding on a lower court; any binding court must be either higher or the same court.	This holds in England and Wales, but not in the US (the Illinois Supreme Court is binding on the US Supreme Court in matters of state law, even though it is a lower court).	NO
4.6.4	wBSD (Binding Hierarchical Dependence)	A binding court must be either higher or the same court (a higher court is not always necessarily binding).	This holds in England and Wales, but not in the US (the Illinois Supreme Court is binding on the US Supreme Court in matters of state law, even though it is a lower court).	NO
4.6.4	OSD (Overruling Strong Hierarchical Dependence)	A higher court may always overrule a lower court; any overruling court must be either higher or the same court.	This holds in England and Wales, and in the US	NO

Section	Principle	Intuition	Example	Used Sec. 4.6.9
4.6.4	wOSD (Overruling Hierarchical Dependence)	An overruling court must be either higher or the same court (a higher court has not always overruling power).	This holds in the US and in England and Wales.	NO
4.6.4	HBO (Overruling Hierarchical Dependence)	If we have two concatenated binding or overruling relations (c_2 is binding/overruling on c_1 and c_3 is binding/overruling on c_2), then either A) a form of cyclicity holds ($c_3 = c_1$), or B) there exists a hierarchical relationship between the courts c_1 and c_2 and between the courts c_2 and c_3	A) The US Supreme Court is binding on the Illinois Supreme Court (on matters of federal law). The Illinois Supreme Court is binding on the US Supreme Court (on matters of state law). In this case, the initially binding court and the ultimately bound court coincide. B) The US Supreme Court is binding on the Illinois Supreme Court (on federal law). The Illinois Supreme Court is binding on the Circuit Court (on federal law and state law). US Supreme Court is higher than Circuit Court.	YES

TABLE 4.12: Symmetry between binding and overruling

Section	Principle	Intuition	Example	Used Sec. 4.6.9
4.6.5	S1 (Symmetry between binding and overruling across courts)	If a court is binding on another under certain conditions, then it can overrule it under the same conditions (and vice versa).	The US Supreme Court is binding on and may overrule the Illinois Supreme Court in federal matters.	YES
4.6.5	S2 (Symmetry between binding and overruling within same court)	A court is binding on itself under some non-empty and pairwise non-identical set of conditions if and only if it can overrule itself under those same conditions.	No example in EW and US.	YES

TABLE 4.13: Monotonicity of binding and overruling relations

Section	Principle	Intuition	Example	Used Sec. 4.6.9
4.6.6	MonB1	Monotonicity for being bound: if a court is bound under certain conditions, it remains bound under stronger compatible conditions.	The Seventh Circuit is bound by the US Supreme Court on matters of federal law. When sitting en banc, it remains bound by the US Supreme Court on matters of federal law.	YES
4.6.6	MonB2	Monotonicity for binding: if a court is binding under certain conditions, it remains binding under stronger compatible conditions.	The Seventh Circuit is binding on District Courts on matters of federal law. When sitting en banc, it remains binding on District Courts on matters of federal law.	YES

4.6.6	MonO1	Monotonicity for being overruled: if a court is overruled under certain conditions, it remains overruled also under stronger compatible conditions.	The Seventh Circuit is overruled by the US Supreme Court on matters of federal law. When sitting en banc, it remains overruled by the US Supreme Court on matters of federal law.	YES
4.6.6	MonO2	Monotonicity for overruling: if a court can overrule under certain conditions, it can overrule under stronger compatible conditions.	The Seventh Circuit may overrule District Courts on matters of federal law. When sitting en banc, it may still overrule District Courts on matters of federal law.	YES
4.6.7	E-MonB1 (Monotonicity for being always bound)	When a court is bound under the empty set of conditions, it is bound unconditionally; that is, it is bound under all sets of admissible conditions for that court.	The High Court of England and Wales is always bound by its own previous decisions issued in its appellate capacity. Accordingly, whether sitting as a court of appeal or as a court of first instance, it remains bound by its prior decisions that were issued when acting in its appellate capacity.	YES

4.6.7	E-MonB2 (Monotonicity for being always binding)	When a court is binding under the empty set of conditions, it is binding unconditionally; that is, it is binding under all sets of admissible conditions for that court.	The UK Supreme Court is always binding on its subsequent decisions to be taken when sitting in a standard panel. Accordingly, decisions issued when the Court sits in either a standard or a larger panel are binding on its subsequent decisions taken when sitting in a standard panel.	YES
4.6.7	E-MonO1 (Monotonicity for being always overruled)	When a court is overruled under the empty set of conditions, it is overruled unconditionally; that is, it is overruled under all sets of admissible conditions for that court.	The UK Supreme Court is always overruled by its subsequent decision taken when sitting in larger panel. Consequently, when sitting in a larger panel or a standard panel, it is overruled by its subsequent decision taken when sitting in larger panel.	YES
4.6.7	E-MonO2 (Monotonicity for being always overruling)	When a court overrules under the empty set of admissible conditions, the overruling is unconditional; that is, it holds under all sets of admissible conditions for the court.	The UK Supreme Court always has the power to overrule the Court of Appeal. Accordingly, whether sitting in a standard panel or in a larger panel, the UK Supreme Court may overrule the Court of Appeal.	YES

TABLE 4.14: **Transitivity of binding, overruling, and their interaction**

Section	Principle	Intuition	Example	Used Sec. 4.6.9
4.6.8	TB (Transitivity of Binding)	If c_i is binding on c_j , and c_j is binding on c_k , then c_i is binding on c_k .	The US Supreme Court binds the Federal Court of Appeals, and the Federal Court of Appeals court binds a District Court. The US Supreme Court binds a District Court.	YES
4.6.8	TO (Transitivity of Overruling)	If c_i can overrule c_j , and c_j can overrule c_k , then c_i can overrule c_k .	The US Supreme Court overrules the Federal Court of Appeals, and the Federal Court of Appeals overrules a District Court. The US Supreme Court overrules a District Court.	YES
4.6.8	TBO (Mixed Transitivity)	If c_i is binding (resp. overruling) on c_j , and c_j can overrule (resp. is binding on) c_k , then c_i can overrule (resp. is binding on) c_k .	The US Supreme Court overrules (resp. is binding on) the Federal Court of Appeals, and the Federal Court of Appeals is binding (resp. overrules) for District Court. The US Supreme Court overrules (resp. is binding on) a District Court.	YES

TABLE 4.15: Acyclicity of the s -graph

Section	Principle	Intuition	Example	Used Sec. 4.6.9
4.6.9	L1 (Acyclicity of s -Graphs)	We require a property that guarantees the acyclicity of the s -graph.	–	YES

4.6.1 Overruling in the "Strict Sense"

At the beginning of this chapter, we observed that the notion of *overruling* is not univocal and may take different forms. Following [66], we distinguished two. In the *strict sense*, overruling is an act of overturning a *binding* precedent or line of precedent. In the *broader sense*, overruling refers to the rejection of a precedent that possesses only *persuasive force* or other supporting influence.

We also recalled that in the previous chapter the notion of *overruling* was introduced to identify *exceptions* to precedent binding—that is, to determine whether certain precedents, potentially binding in the case at hand (because issued by courts with binding authority), should instead be regarded as non-binding because they have been *overruled*. Our primary interest, therefore, lies in *overruling in the strict sense*.

Intuitively, if we interpret the definition of *overruling in the strict sense* literally, overruling occurs when a court c_i removes a binding *constraint* that a court c_j , in an earlier decision, has imposed on other courts (or on itself). For example, in common-law systems, when a higher court “overrules” a lower court, it is not lifting a constraint from itself but rather eliminating the constraint that previously bound the lower courts.

This interpretation suggests a possible principle to capture the notion of *strict overruling*: it makes sense to speak of *strict overruling* by c_i over c_j only if the decisions of c_j actually bind some court c_k . Otherwise, c_i would not be removing any binding constraint. Put differently, for c_i to have the power to overrule c_j , it must be the case that c_j binds at least one court c_k —possibly even itself. This idea is captured by the following principle:

$$\mathcal{O}(c_i, c_j \mid X_i, X_j) \Rightarrow \exists c_k \in \text{Courts such that } \mathcal{B}(c_j, c_k \mid X_j, X_k), \text{ for some } X_k \subseteq \text{Cond}_{c_k} \quad (\text{StOv})$$

We now consider the jurisdictions modeled earlier, starting with *England and Wales* as described in Example 4.1 and modeled in Example 4.3. This jurisdiction does *not* satisfy the principle at issue, and this depends on the overruling relations we marked with the symbol $(\star)$ in Table 4.5. For instance, we required that $\mathcal{O}(c_i, c_4 \mid \emptyset, \emptyset)$ (for every $i \in \{0, 1, 2, 4\}$), in line with the general approach according to which a court

may always overrule a lower court and, if it is not self-bound, may also engage in *self-overruling*. However, the *County Court* c_4 does not exert any binding force over other courts or over itself. It follows that the principle **StOv** fails.

As already noted in Example 4.1, this situation can be interpreted in two ways. The first is that the discrepancy is merely a side effect of the County Court c_4 being at the lowest level of the hierarchy—if there were a court below it, we would still be adhering to the strict interpretation of overruling. The second interpretation is that, although the County Court c_4 is not binding, it may nonetheless possess *persuasive force* with respect to future decisions. In this case, asserting, for example, that $\mathcal{O}(c_1, c_4 \mid \emptyset, \emptyset)$ can be read as the idea that a decision of c_1 eliminates (or at least systematically weakens) even the line of *persuasive* precedents produced by c_4 .

By contrast, the US jurisdiction described in Example 4.2 and modeled in Example 4.4 does satisfy Principle **StOv**. This is a direct consequence of our decision to model that jurisdiction in a way that reflects, quite literally, the strict definition of overruling. In doing so, we deliberately adopted restrictive assumptions. For instance, we assumed that the Illinois State Supreme Court cannot overrule federal courts on questions of state law, since federal-court decisions on such matters are never binding—although they may be persuasive for all courts. For this reason, if one wishes to adopt a broader conception of overruling in the US context, additional overruling relations should be introduced to capture this persuasive dimension as well.

4.6.2 Relation Between Conditions

In the definition of binding $\mathcal{B}(c_i, c_j \mid X_i, X_j)$ and overruling $\mathcal{O}(c_i, c_j \mid X_i, X_j)$ (see Definition 4.2), we did not impose any relationship between the set of conditions X_i and X_j .²² However, by examining the relations extracted for the courts of England and Wales and those of the United States, we observe that X_i and X_j within the same relation sometimes concern the same macro-category, for instance *procedural competence* or *subject-matter competence*. For instance, for the binding relation between the US Supreme Court ($\tilde{c}_0$) and the Illinois Supreme Court ($\tilde{c}_1$), we obtain $\mathcal{B}(\tilde{c}_0, \tilde{c}_1 \mid \{\text{FederalLaw}\}, \{\text{FederalLaw}\})$, where the binding and bounded conditions coincide

²²The author thanks Giovanni Sileno for suggesting investigating the relation between X_i and X_j .

and both trivially relate to issues of subject-matter competence. Also, for the self-overruling relation of the High Court of England and Wales (c_2), we have $\mathcal{O}(c_2, c_2 \mid \{\text{Appellate}\}, \{\text{FirstInstance}\})$, where $\{\text{Appellate}\}$ and $\{\text{FirstInstance}\}$ are both conditions related to procedural matters. We might therefore be tempted to require that the conditions involved in the same binding or overruling relation be related to each other, in the sense that they belong to the same category of conditions.

However, this is not the case for the self-overruling conditions of the Seventh Circuit Court ($\tilde{c}_2$). For example, we have $\mathcal{O}(\tilde{c}_2, \tilde{c}_2 \mid \{\text{FederalLaw}, \text{Partial}, \text{Authorized}\}, \{\text{FederalLaw}\})$, where the conditions simultaneously concern both procedural issues and questions of subject-matter competence. Nevertheless, there exists a subset of the binding and bound conditions (FederalLaw) that belongs to the same macro-category (subject-matter competence).

Motivated by this last example, a weaker requirement can be formulated: for X_i and X_j appearing in the same overruling or binding relation, at least a subset of X_i and a subset of X_j should belong to the same macro-category. To properly formalise this, we should formally represent the macro-category. Namely, we should assume that there exist sets $\text{Cat}_1, \dots, \text{Cat}_n$ such that $\text{Cond} = \text{Cat}_1 \cup \dots \cup \text{Cat}_n$. For instance, in our examples we may have $\text{Cat}_1 = \text{Procedural}$ and $\text{Cat}_2 = \text{SubjectMatter}$. More specifically, for the UK jurisdiction, $\text{Procedural} = \{\text{StandardPanel}, \text{LargerPanel}, \text{FirstInstance}, \text{Appellate}\}$ and $\text{SubjectMatter} = \emptyset$. For the US jurisdiction, $\text{SubjectMatter} = \{\text{FederalLaw}, \text{StateLaw}\}$ and $\text{Procedural} = \{\text{Partial}, \text{EnBanc}, \text{Authorized}, \text{Unauthorized}\}$.

We can then formalise the following two principles:

$$\mathcal{B}(c_i, c_j \mid X_i, X_j) \Rightarrow \exists \tilde{X}_i \subseteq X_i, \tilde{X}_j \subseteq X_j, \text{Cat}_k \subseteq \text{Cond} \text{ such that } \tilde{X}_i, \tilde{X}_j \subseteq \text{Cat}_k$$

(BRelCond)

$$\mathcal{O}(c_i, c_j \mid X_i, X_j) \Rightarrow \exists \tilde{X}_i \subseteq X_i, \tilde{X}_j \subseteq X_j, \text{Cat}_k \subseteq \text{Cond} \text{ such that } \tilde{X}_i, \tilde{X}_j \subseteq \text{Cat}_k$$

(ORelCond)

Note that in both binding and overruling relations the set of conditions may be empty, which, as already stated, means *always*. Consider, for example, the self-binding relation of the UK Supreme Court: $\mathcal{B}(c_0, c_0 \mid \emptyset, \{\text{StandardPanel}\})$. This relation means

that the court is always binding on itself in subsequent decisions taken as a standard panel. Principle **BRelCond** also captures this situation, since trivially $\emptyset \subseteq \text{Procedural}$.

4.6.3 Directionality and Mutual Exclusivity of Powers

As stated in the introduction of this chapter, both *binding* and *overruling* are forms of judicial authority between courts, but they are exercised in different ways. The *binding* relation expresses a constraining power: when one court is binding on another, the former constrains the latter's subsequent decisions. By contrast, *overruling* does not impose a constraint; rather—at least in the strict sense described in the previous section—it *removes* (or defeats) an existing binding constraint generated by a prior decision.

It may be argued that the very notion of power fundamentally presupposes a form of directionality among the agents involved. Moreover, when different forms of power produce complementary effects, they are not expected to operate simultaneously. We propose here some overarching criteria based on these considerations. We note at the outset, however, that such criteria may be accepted or rejected depending on the approach taken.

I) Directionality of Binding Authority. We know that a court may possess binding authority over another court, or even over itself, under specific conditions. This authority is inherently *directional*: if a court c_i can exercise binding power over another court c_j under a given set of circumstances, the reverse is generally not possible. In other words, court c_j cannot exercise binding authority over c_i under the same or compatible conditions as those in which the original binding occurred. This criteria ensures that the exercise of judicial power remains structured and prevents contradictory or reciprocal applications of authority under similar circumstances.

II) Directionality of Overruling Authority. We know that a court may possess overruling authority over another court, or even over itself, under specific conditions. This authority is inherently *directional*: if a court c_i can exercise overruling power over another court c_j under a given set of circumstances, the reverse

is generally not possible. In other words, court c_j cannot exercise overruling authority over c_i under the same or compatible conditions as those in which the original overruling occurred.

III) Mutual Exclusivity of Binding and Overruling Authority. When a court is said to be binding on another, the latter is obligated to follow the decisions of the former. Conversely, when a court has the authority to overrule another, it may not only disregard that court's decisions but also remove the force that such decisions exert on other courts. Accordingly, if under certain conditions a court is binding on another, it is expected that the latter cannot overrule the former under the same or compatible conditions; otherwise, the binding relationship would not, in fact, hold. Similarly, if a court can overrule another under specific conditions, it is expected that the latter cannot be binding on the former under the same or compatible conditions, as this would preclude the exercise of overruling authority.

Criterion **II)** warrants a closer examination. Binding power, as a constraining force, reasonably excludes the possibility that two courts could bind one another under same or compatible conditions. Overruling, by contrast, functions as the removal of a constraint (or force), and it is less clear whether the same expectation of directionality should apply. We will return to this point later.

Below, we propose a series of principles corresponding to the formalization of **I)**, **II)**, and **III)** at various levels.

Asymmetry under same conditions

Let us suppose that a court c_i , under conditions X_i , is binding on c_j , under conditions X_j , i.e., $\mathcal{B}(c_i, c_j \mid X_i, X_j)$. In accordance with the criterion set out in **I)**, the binding relation is expected to exhibit directionality: specifically, c_j , under conditions X_j , would not be binding on c_i under conditions X_i . Formally, this is expressed as $not - \mathcal{B}(c_j, c_i \mid X_j, X_i)$.

$$\text{For } X_i \neq X_j \text{ or } c_i \neq c_j : \mathcal{B}(c_i, c_j \mid X_i, X_j) \Rightarrow not - \mathcal{B}(c_j, c_i \mid X_j, X_i).^{23} \quad (\text{AS1})$$

We note that principle **AS1** establishes a form of directionality in the binding relation on two levels. First, **AS1** imposes directionality or asymmetry between different courts. If a court is binding for another different court, under specific conditions, the binding relation cannot be reverted under the same conditions; that is, for $c_i \neq c_j$, $\mathcal{B}(c_i, c_j \mid X_i, X_j) \Rightarrow \text{not-}\mathcal{B}(c_j, c_i \mid X_j, X_i)$. Second, **AS1** establishes directionality or asymmetry between different conditions for the same court. If a court is self binding under specific conditions the binding relation cannot be reverted under the same conditions. That is, for $X_i \neq X_j$, $\mathcal{B}(c_i, c_i \mid X_i, X_j) \Rightarrow \text{not-}\mathcal{B}(c_i, c_i \mid X_j, X_i)$.

Suppose now a court c_i , under conditions X_i , has overruling power over c_j , acting under conditions X_j , namely $\mathcal{O}(c_i, c_j \mid X_i, X_j)$. If we adhere to criterion in **II**), we would expect a form of directionality of overruling relation: namely we would expect that c_j , under conditions X_j , cannot overrule c_i under conditions, X_i . Namely, we expect $\text{not} - \mathcal{O}(c_j, c_i \mid X_j, X_i)$

$$\text{For } X_i \neq X_j \text{ or } c_i \neq c_j : \mathcal{O}(c_i, c_j \mid X_i, X_j) \Rightarrow \text{not} - \mathcal{O}(c_j, c_i \mid X_j, X_i).^{24} \quad (\text{AS2})$$

As for **AS1**, principle **AS2** establishes a form of directionality in the overruling relation on two levels, between courts and between set of conditions.

Suppose that c_i , under conditions X_i , is binding on c_j , under conditions X_j , i.e., $\mathcal{B}(c_i, c_j \mid X_i, X_j)$. If we adopt the criterion in **III**), then c_j cannot overrule c_i under the same conditions, i.e., $\text{not} - \mathcal{O}(c_j, c_i \mid X_j, X_i)$. Otherwise, c_i would not actually be binding on c_j . Similarly, if c_i , under conditions X_i , can overrule c_j , under conditions X_j , i.e., $\mathcal{O}(c_i, c_j \mid X_i, X_j)$, then c_j cannot bind c_i under the same conditions, i.e., $\text{not} - \mathcal{B}(c_j, c_i \mid X_j, X_i)$.

$$\mathcal{B}(c_i, c_j \mid X_i, X_j) \Rightarrow \text{not} - \mathcal{O}(c_j, c_i \mid X_j, X_i).^{25} \quad (\text{AS3})$$

$$\mathcal{O}(c_i, c_j \mid X_i, X_j) \Rightarrow \text{not} - \mathcal{B}(c_j, c_i \mid X_j, X_i).^{26} \quad (\text{AS4})$$

²³We require $X_i \neq X_j$ or $c_i \neq c_j$ to not impose $\mathcal{B}(c_i, c_i \mid X_i, X_i) \Rightarrow \text{not} - \mathcal{B}(c_i, c_i \mid X_i, X_i)$

²⁴We require $X_i \neq X_j$ or $c_i \neq c_j$ to not impose $\mathcal{O}(c_i, c_i \mid X_i, X_i) \Rightarrow \text{not} - \mathcal{O}(c_i, c_i \mid X_i, X_i)$

²⁵Notice that, differently from **AS1** and **AS2**, we did not require that For $X_i \neq X_j$ or $c_i \neq c_j$. Indeed, for example, $\mathcal{B}(c_i, c_i \mid X_i, X_i) \Rightarrow \text{not} - \mathcal{O}_{X_i, X_i}(c_i, c_i)$, does not lead to any contradiction and is in line with criterion **III**).

²⁶Analogous to Footnote 25.

As in **AS1** and **AS2**, principles **AS3** and **AS4** establish a form of directionality in the interaction between the binding and overruling relations, operating on two levels: between courts and between sets of conditions.

We denote **AS** the conjunction of all AS-principles.²⁷

AS1 & AS2 & AS3 & AS4 (**AS**)

Figure 4.3 represents the **AS** principles.

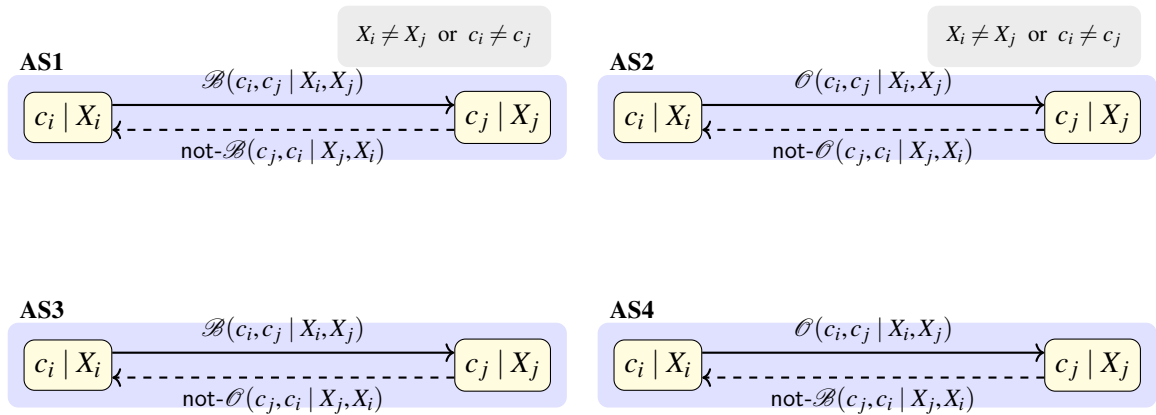


FIGURE 4.3: **AS**-principles

Example 4.6. *The England and Wales jurisdiction in Example 4.3 satisfies **AS**. The US jurisdiction in Example 4.4 satisfies **AS**.*

Strong Asymmetry and Asymmetry under compatible conditions

Suppose we have both $\mathcal{B}(c_i, c_j | X_i, X_j)$ and $\mathcal{B}(c_j, c_i | \tilde{X}_j, \tilde{X}_i)$.

Suppose we have cases s_1, s_2, s_3 and s_4 such that $s_1 <_T s_2$ and $s_3 <_T s_4$. Let s_1 be decided by c_i and s_2 by c_j , while s_3 and s_4 are to be decided by c_j and c_i , respectively. Assume further that s_1 is a precedent for s_2 and s_3 is a precedent for s_4 . Court c_i , in deciding s_1 and s_4 , satisfies both conditions X_i and $\tilde{X}_i$, while court c_j , in deciding s_2 and s_3 , satisfies both conditions X_j and $\tilde{X}_j$.

²⁷Henceforth, we use & to denote the conjunction of semantic conditions.

This would mean that court c_j , in deciding s_2 , is bound by c_i (because $\mathcal{B}(c_i, c_j \mid X_i, X_j)$), while court c_i , in deciding s_4 , is bound by c_j (because of $\mathcal{B}(c_j, c_i \mid \tilde{X}_j, \tilde{X}_i)$). Yet c_i , in deciding s_1 and s_3 , satisfies the same conditions $X_i, \tilde{X}_i$, and c_j , in deciding s_2 and s_4 , also satisfies the same conditions $X_j, \tilde{X}_j$. This contradicts criterion **I**) of the directionality of binding power that we highlighted earlier. The situation discussed above is illustrated in Figure 4.4.



FIGURE 4.4: $\mathcal{B}(c_i, c_j \mid X_i, X_j)$ and $\mathcal{B}(c_j, c_i \mid \tilde{X}_j, \tilde{X}_i)$

A similar undesirable situation may be obtained if we replace the binding relation with the overruling relation. Namely, we would obtain that c_j in deciding s_2 can overrule c_i , while court c_i in deciding s_4 can overrule c_j , *under the same conditions*. This would go against criterion **II**).

Finally, suppose also that we know $\mathcal{B}(c_i, c_j \mid X_i, X_j)$ and $\mathcal{O}(c_j, c_i \mid \tilde{X}_j, \tilde{X}_i)$. This would mean that, *under the same conditions*, c_j in deciding s_2 is bound by c_i , having at the same time the possibility to overrule c_i . This would go against criterion **III**).

We note that in the previous situations we started by considering binding/overruling relations between the states s_1, s_2, s_3, s_4 . However, in what follows we move directly to the root of the issue and focus instead on the possible relations between the courts themselves. In particular, to manage previous situations we propose principles to address how to handle:

- 1) $\mathcal{B}(c_i, c_j \mid X_i, X_j)$ and $\mathcal{B}(c_j, c_i \mid \tilde{X}_j, \tilde{X}_i)$;
- 2) $\mathcal{O}(c_i, c_j \mid X_i, X_j)$ and $\mathcal{O}(c_j, c_i \mid \tilde{X}_j, \tilde{X}_i)$;
- 3) $\mathcal{B}(c_i, c_j \mid X_i, X_j)$ and $\mathcal{O}(c_j, c_i \mid \tilde{X}_j, \tilde{X}_i)$.

Strong Asymmetry

A very first and strong principle that solves problem 1) is to require that if a court c_i is binding on another court c_j , under certain conditions, then court c_j is never binding on court c_i , *under any conditions*.

$$\text{For } X_i \neq \tilde{X}_j \text{ or } X_j \neq \tilde{X}_i \text{ or } c_i \neq c_j : \mathcal{B}(c_i, c_j \mid X_i, X_j) \Rightarrow \text{not} - \mathcal{B}(c_j, c_i \mid \tilde{X}_j, \tilde{X}_i).^{28} \quad (\text{CB1})$$

An analogous principle may be applied to the overruling relation to address situation 2).

$$\text{For } X_i \neq \tilde{X}_j \text{ or } X_j \neq \tilde{X}_i \text{ or } c_i \neq c_j : \mathcal{O}(c_i, c_j \mid X_i, X_j) \Rightarrow \text{not} - \mathcal{O}(c_j, c_i \mid \tilde{X}_j, \tilde{X}_i).^{29} \quad (\text{CO1})$$

An analogous principle could be applied to the mutual interaction between the binding and overruling relations in order to address situation 3).

$$(\mathcal{O}(c_i, c_j \mid X_i, X_j) \Rightarrow \text{not} - \mathcal{B}(c_j, c_i \mid \tilde{X}_j, \tilde{X}_i)) \text{ and} \\ (\mathcal{B}(c_i, c_j \mid X_i, X_j) \Rightarrow \text{not} - \mathcal{O}(c_j, c_i \mid \tilde{X}_j, \tilde{X}_i)). \quad (\text{CBO1})$$

We denote **C1** the conjunction of **CB1**, **CO1** and **CBO1**.

$$\text{CB1 \& CO1 \& CBO1} \quad (\text{C1})$$

Figure 4.5 represents the **C1** principles.

Notice that **C1**-principles imply the **AS**-principles. This reflects the intuition that the **C1**-principles are stronger than the **AS**-principles.

Proposition 4.16. $\text{C1} \Rightarrow \text{AS}$

The principles in **C1**, while resolving problems 1), 2), and 3), are quite strong. In fact, they are not satisfied by either of the two jurisdictions we have modeled.

²⁸We require $X_i \neq \tilde{X}_j$ or $X_j \neq \tilde{X}_i$ or $c_i \neq c_j$ to not impose $\mathcal{B}(c_i, c_i \mid X_i, X_i) \Rightarrow \text{not} - \mathcal{B}(c_i, c_i \mid X_i, X_i)$.

²⁹We require $X_i \neq \tilde{X}_j$ or $X_j \neq \tilde{X}_i$ or $c_i \neq c_j$ to not impose $\mathcal{O}(c_i, c_i \mid X_i, X_i) \Rightarrow \text{not} - \mathcal{O}(c_i, c_i \mid X_i, X_i)$.

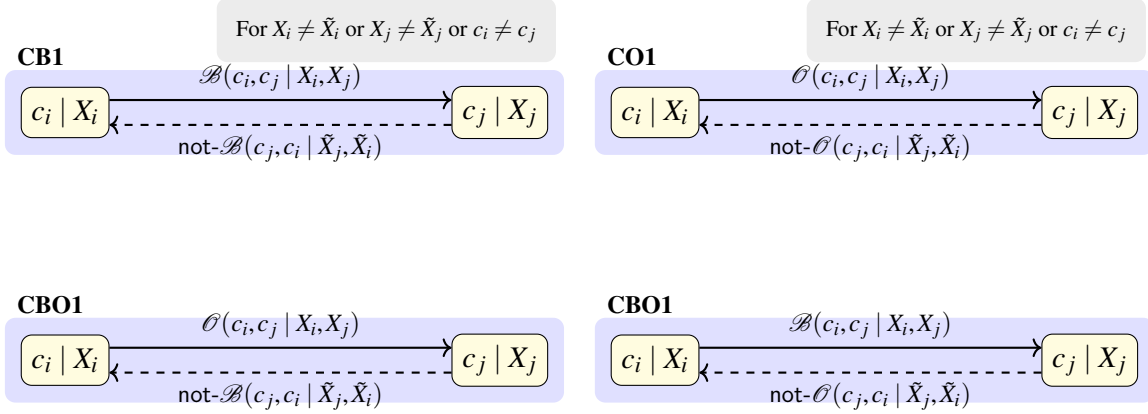


FIGURE 4.5: C1-principles

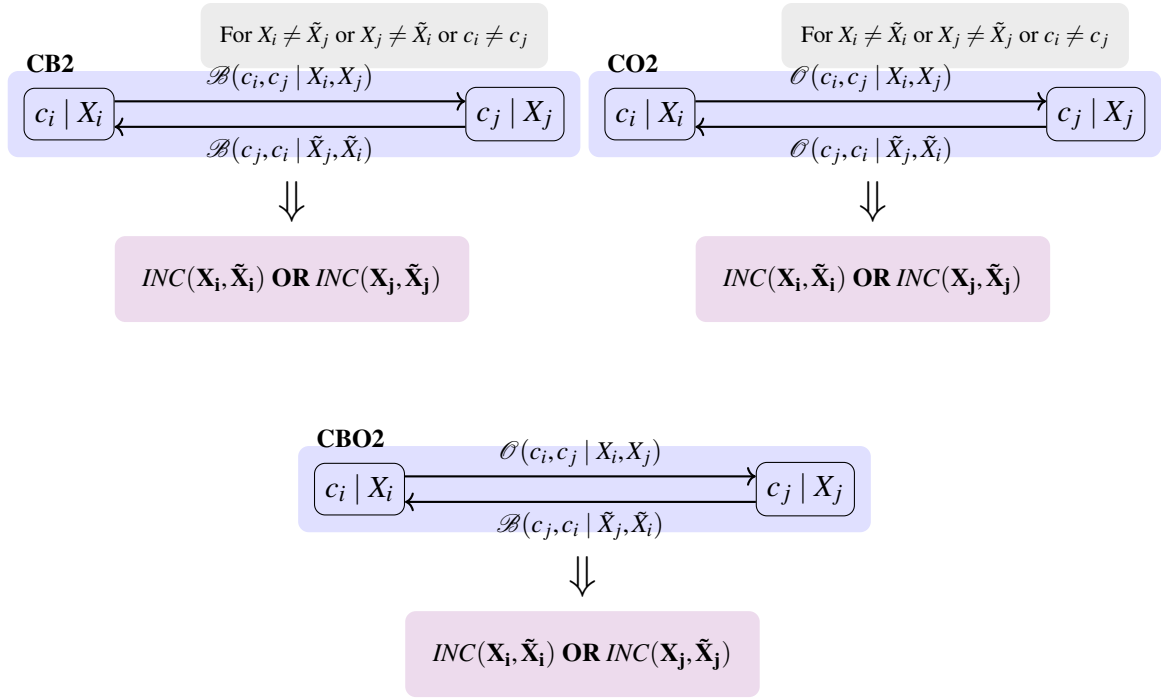
Example 4.7. Principles **C1** are not satisfied by the EW system. Indeed, court c_0 violates **CBO1**, with respect to self binding and self overruling (we have both $\mathcal{B}(c_0, c_0 | \emptyset, \{\text{StandardPanel}\})$ and $\mathcal{O}(c_0, c_0 | \{\text{LargerPanel}\}, \emptyset)$).

Principles **C1** are not satisfied by the US system. Indeed, for example, court $\tilde{c}_0$ and $\tilde{c}_1$ violate **CB1** (we have both $\mathcal{B}(\tilde{c}_0, \tilde{c}_1 | \{\text{FederalLaw}\}, \{\text{FederalLaw}\})$ and $\mathcal{B}(\tilde{c}_1, \tilde{c}_0 | \{\text{StateLaw}\}, \{\text{StateLaw}\})$).

Asymmetry under compatible conditions

Other possible principles to solve problems 1), 2) and 3) may rely on the notion of *incompatible conditions* (Definition 4.2). Recall that incompatible conditions are conditions that are never satisfied (i.e. contained) simultaneously in two states (Definition 4.3).

For instance, incompatibility may provide a solution to handle situations in 1). Indeed we could require that if $\mathcal{B}(c_i, c_j | X_i, X_j)$ and $\mathcal{B}(c_j, c_i | \tilde{X}_j, \tilde{X}_i)$, then $X_i, \tilde{X}_i$ are incompatible ($INC(X_i, \tilde{X}_i)$) or $X_j, \tilde{X}_j$ are incompatible ($INC(X_j, \tilde{X}_j)$). In other words, we require that no asymmetry arises under compatible conditions. In this way, if s_i and s_j are assessed respectively by courts c_i and c_j , then either s_i does not satisfy both X_i and $\tilde{X}_i$, or s_j does not satisfy both X_j and $\tilde{X}_j$.

FIGURE 4.6: **C2**-principles

Similar principles may apply to overruling (thus to handle situation 2)) and to interaction of overruling and binding (thus to handle situation 3)).

For $X_i \neq \tilde{X}_j$ or $X_j \neq \tilde{X}_i$ or $c_i \neq c_j$:

$$\mathcal{B}(c_i, c_j \mid X_i, X_j) \ \& \ \mathcal{B}(c_j, c_i \mid \tilde{X}_j, \tilde{X}_i) \Rightarrow INC(X_i, \tilde{X}_i) \text{ or } INC(X_j, \tilde{X}_j).^{30} \quad (\mathbf{CB2})$$

For $X_i \neq \tilde{X}_j$ or $X_j \neq \tilde{X}_i$ or $c_i \neq c_j$:

$$\mathcal{O}(c_i, c_j \mid X_i, X_j) \ \& \ \mathcal{O}(c_j, c_i \mid \tilde{X}_j, \tilde{X}_i) \Rightarrow INC(X_i, \tilde{X}_i) \text{ or } INC(X_j, \tilde{X}_j).^{31} \quad (\mathbf{CO2})$$

$$\mathcal{O}(c_i, c_j \mid X_i, X_j) \ \& \ \mathcal{B}(c_j, c_i \mid \tilde{X}_j, \tilde{X}_i) \Rightarrow INC(X_i, \tilde{X}_i) \text{ or } INC(X_j, \tilde{X}_j). \quad (\mathbf{CBO2})$$

We denote **C2** the conjunction of **CB2**, **CO2** and **CBO2**.

$$\mathbf{CB2} \ \& \ \mathbf{CO2} \ \& \ \mathbf{CBO2} \quad (\mathbf{C2})$$

Figure 4.6 represents the **C2** principles.

³⁰We require $X_i \neq \tilde{X}_j$ or $X_j \neq \tilde{X}_i$ or $c_i \neq c_j$ to highlight that $\mathcal{B}(c_i, c_j \mid X_i, X_j) \neq \mathcal{B}(c_j, c_i \mid \tilde{X}_j, \tilde{X}_i)$.

³¹We require $X_i \neq \tilde{X}_j$ or $X_j \neq \tilde{X}_i$ or $c_i \neq c_j$ to highlight that $\mathcal{O}(c_i, c_j \mid X_i, X_j) \neq \mathcal{O}(c_j, c_i \mid \tilde{X}_j, \tilde{X}_i)$.

C2 is less demanding than **C1**, and it may in fact be satisfied in one of our jurisdictions: that of England and Wales.

Example 4.8. *Principle C2 is satisfied by England and Wales jurisdiction. More precisely, the only possibilities of retrieving conflictual unions are given by a) $\mathcal{B}(c_0, c_0 \mid \emptyset, \text{StandardPanel})$, $\mathcal{O}(c_0, c_0 \mid \{\text{LargerPanel}\}, \emptyset)$ or b) $\mathcal{B}(c_2, c_2 \mid \{\text{Appellate}\}, \emptyset)$, $\mathcal{O}(c_2, c_2 \mid \{\text{Appellate}\}, \{\text{FirstInstance}\})$. Recall that, in Example 4.3, we stated that $\{\text{LargerPanel}\}$ and $\{\text{StandardPanel}\}$ are considered incompatible. Also we stated that $\{\text{FirstInstance}\}$ and $\{\text{Appellate}\}$ are considered incompatible.*

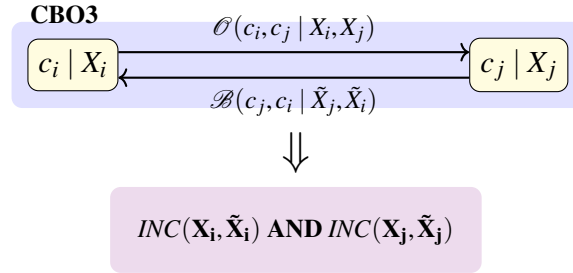
US jurisdiction does not satisfy C2, since it not satisfy CO2. Indeed, we know that $\mathcal{O}(\tilde{c}_2, \tilde{c}_2 \mid \{\text{FederalLaw}, \text{Partial}, \text{Authorized}\}, \{\text{FederalLaw}\})$ and $\mathcal{O}(\tilde{c}_2, \tilde{c}_2 \mid \{\text{FederalLaw}, \text{EnBanc}\}, \{\text{FederalLaw}\})$. In Example 4.4, we did not state that $\{\text{FederalLaw}, \text{Partial}, \text{Authorized}\}$ and $\{\text{FederalLaw}\}$ are incompatible, nor did we state that $\{\text{FederalLaw}, \text{EnBanc}\}$ and $\{\text{FederalLaw}\}$ are incompatible.

The previous example for US jurisdiction, allows us to reconsider criterion **II**) on the directionality of overruling. Indeed, $\mathcal{O}(\tilde{c}_2, \tilde{c}_2 \mid \{\text{FederalLaw}, \text{Partial}, \text{Authorized}\}, \{\text{FederalLaw}\})$ means that when court $\tilde{c}_2$ decides on federal matter, acting as a partial and authorized panel, it can overrule its own prior decisions on federal matter—even those made when it was sitting en banc.

Similarly, $\mathcal{O}(\tilde{c}_2, \tilde{c}_2 \mid \{\text{FederalLaw}, \text{EnBanc}\}, \{\text{FederalLaw}\})$ means that when court $\tilde{c}_2$ decides on federal law, acting *en banc*, it can overrule its own prior decisions on federal matter, including decisions taken when it acted as a partial and authorized panel.

One aspect should be stressed here: we are dealing with cases of self-overruling, that is, situations where court $\tilde{c}_2$ overrules itself. Consequently, any notion of directionality in this overruling relation concerns the conditions rather than the courts.

With this in mind, the example illustrates how criterion **II**) of directionality in overruling can be questioned—at least in cases of self-overruling. Extending this to the more general framework, it raises the question of whether insisting on directionality in overruling relations between two courts might be too strong an assumption. If one court has the authority to overturn the binding precedents of another, does it necessarily follow that the second court cannot overturn the precedents of the first? We believe that the answer to this question is not univocal and depends on the chosen approach.

FIGURE 4.7: **CBO3**-principle

Returning to the previous example, the US system fails to satisfy **C2** because of **CO2**. It satisfies instead a stronger version of **CBO2**, which will be useful later in Section 4.6.9. Indeed, it could be verified that if $\mathcal{O}(c_i, c_j | X_i, X_j) \& \mathcal{B}(c_j, c_i | \tilde{X}_j, \tilde{X}_i)$, then $X_i, \tilde{X}_i$ are incompatible *and* $X_j, \tilde{X}_j$ are incompatible—whereas **CO2** requires only that either $X_i, \tilde{X}_i$ be incompatible *or* $X_j, \tilde{X}_j$, be incompatible.

$$\mathcal{O}(c_i, c_j | X_i, X_j) \& \mathcal{B}(c_j, c_i | \tilde{X}_j, \tilde{X}_i) \Rightarrow INC(X_i, \tilde{X}_i) \text{ and } INC(X_j, \tilde{X}_j). \quad (\mathbf{CBO3})$$

It is straightforward to verify that **CBO3** implies **CBO2**. Figure 4.7 represents **CBO3**.

Example 4.9. *To verify that the US system satisfies **CBO3**, we need to examine the conditions involved in the binding and overruling relations between $\tilde{c}_1$ and $\tilde{c}_0$. These conditions are $\{\text{StateLaw}\}$ and $\{\text{FederalLaw}\}$, which are incompatible (see Example 4.4). We must also examine the conditions governing the self-binding and self-overruling relations of $\tilde{c}_2$. In particular, we need to determine whether $\{\text{FederalLaw}, \text{Partial}, \text{Unauthorized}\}$ and $\{\text{FederalLaw}, \text{Partial}, \text{Authorized}\}$ are incompatible, and whether $\{\text{FederalLaw}, \text{EnBanc}\}$ and $\{\text{FederalLaw}, \text{Partial}, \text{Unauthorized}\}$ are incompatible. These mutual-exclusivity claims follow from Example 4.4.*

We now provide some results about the connection of **C2**-principles and of **CBO3** with the other asymmetry principles stated earlier.

First, notice that **C2**-principles imply the **AS**-principles. In this sense, **C2**-principles are a generalisation of **AS**-principles. Also, **CBO3** implies **AS3** as well as **AS4**.

Proposition 4.17. $C2 \Rightarrow AS$, $CBO3 \Rightarrow (AS3 \& AS4)$.

Also, notice that **C1**-principles imply the **C2**-principles. This reflects the intuition that **C1**-principles are indeed stronger than **C2**-principles. Also, **CBO1** implies **CBO3**.

Proposition 4.18. $C1 \Rightarrow C2$, $CBO1 \Rightarrow CBO3$.

By the two previous propositions and Proposition 4.16, we may conclude that the **C1**-principles are the strongest asymmetry principles, while the **C2**-principles are weaker than the **C1**-principles but stronger than the **AS**-principles.

Table 4.16 groups the principles introduced in this section according to the criterion they express.

Criterion I)	Criterion II)	Criterion III)
AS1	AS1	AS3, AS4
CB1	CO1	CBO1
CB2	CO2	CBO2
		CBO3

TABLE 4.16: Summary of Principles in Section 4.6.3 by criterion.

4.6.4 Binding, overruling and hierarchy

From the previous chapter, we already know that in England and Wales jurisdiction a connection can be drawn between binding authority and hierarchical relations. In such a system: 1) vertical stare decisis applies: decisions of higher courts bind lower courts; 2) lower courts have no binding authority over higher courts, and courts that are not hierarchically related do not issue binding decisions upon one another; 3) horizontal stare decisis may apply, meaning that some courts may be bound by their own prior decisions. Accordingly, we can formulate the following principle, corresponding to SD in the previous chapter.

$$\begin{aligned} \mathcal{H}(c_1, c_2) &\Rightarrow \mathcal{B}(c_1, c_2 \mid \emptyset, \emptyset), \text{ and} \\ \mathcal{B}(c_1, c_2 \mid X_1, X_2) &\Rightarrow \mathcal{H}(c_1, c_2) \text{ or } c_1 = c_2. \end{aligned} \quad (\text{BSD})$$

A weaker version of the previous principle can be adopted, stating that if one court is binding on another, it must either be a higher court or the same court, without requiring that a higher court is always binding on a lower one.

$$\mathcal{B}(c_1, c_2 \mid X_1, X_2) \Rightarrow \mathcal{H}(c_1, c_2) \text{ or } c_1 = c_2. \quad (\mathbf{wBSD})$$

The same two principles could be required for overruling.

$$\begin{aligned} \mathcal{H}(c_1, c_2) &\Rightarrow \mathcal{O}(c_1, c_2 \mid \emptyset, \emptyset), \text{ and} \\ \mathcal{O}(c_1, c_2 \mid X_1, X_2) &\Rightarrow \mathcal{H}(c_1, c_2) \text{ or } c_1 = c_2. \end{aligned} \quad (\mathbf{OSD})$$

$$\mathcal{O}(c_1, c_2 \mid X_1, X_2) \Rightarrow \mathcal{H}(c_1, c_2) \text{ or } c_1 = c_2. \quad (\mathbf{wOSD})$$

It can be verified that **wBSD** (resp. **wOSD**) indeed represents the weak form of **BSD** (resp. **OSD**).

Proposition 4.19. $\mathbf{BSD} \Rightarrow \mathbf{wBSD}$. $\mathbf{OSD} \Rightarrow \mathbf{wOSD}$.

Figures 4.8 and 4.9 represent **BSD**, **wBSD**, **OSD** and **wOSD**.

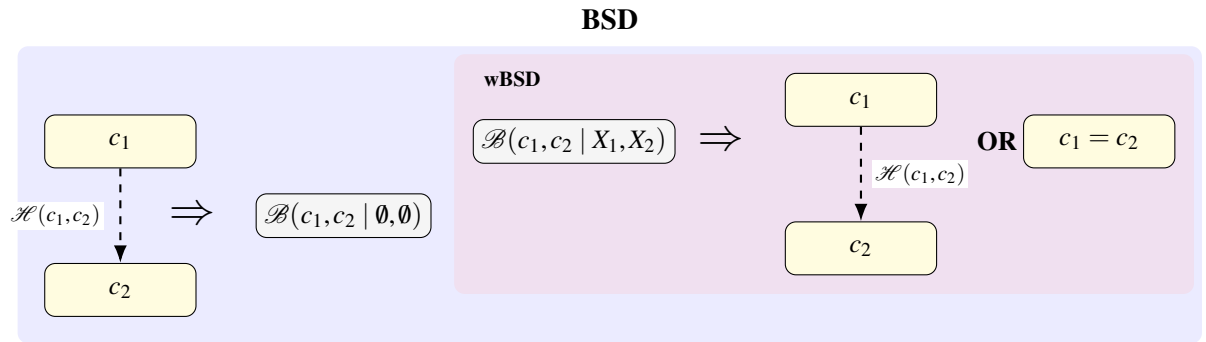
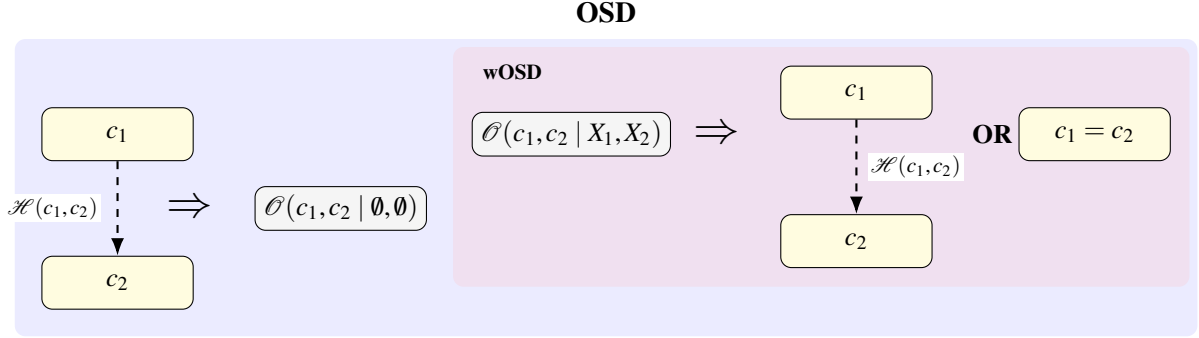


FIGURE 4.8: **BSD** and **wBSD** representation

The England and Wales jurisdiction satisfies all previous principles.

In contrast, the US court system satisfies the principles of *stare decisis* with respect to overruling authority (**wOSD** and **OSD**), but it fails to satisfy the principles of *stare*

FIGURE 4.9: **OSD** and **wOSD** representation

decisis with respect to binding authority (**wBSD** and **BSD**).³²

It fails to satisfy **wBSD** (and thus **BSD**), for instance, because of the relationship between the US Supreme Court ($\tilde{c}_0$) and the Illinois Supreme Court ($\tilde{c}_1$) with respect to state law ($\mathcal{B}(\tilde{c}_1, \tilde{c}_0 \mid \{\text{StateLaw}\}, \{\text{StateLaw}\})$). Specifically, under state law, the Illinois Supreme Court is binding on the US Supreme Court, even though the latter is higher in the overall judicial hierarchy.

Nevertheless, the US system satisfies another principle that may prove useful in Section 4.6.9.

$$Rel_1(c_2, c_1 \mid X_1, X_2) \ \& \ Rel_2(c_3, c_2 \mid X_3, \tilde{X}_2) \Rightarrow (c_1 = c_3) \text{ or } (\mathcal{H}(c_3, c_2) \text{ and } \mathcal{H}(c_2, c_1))$$

(HBO)

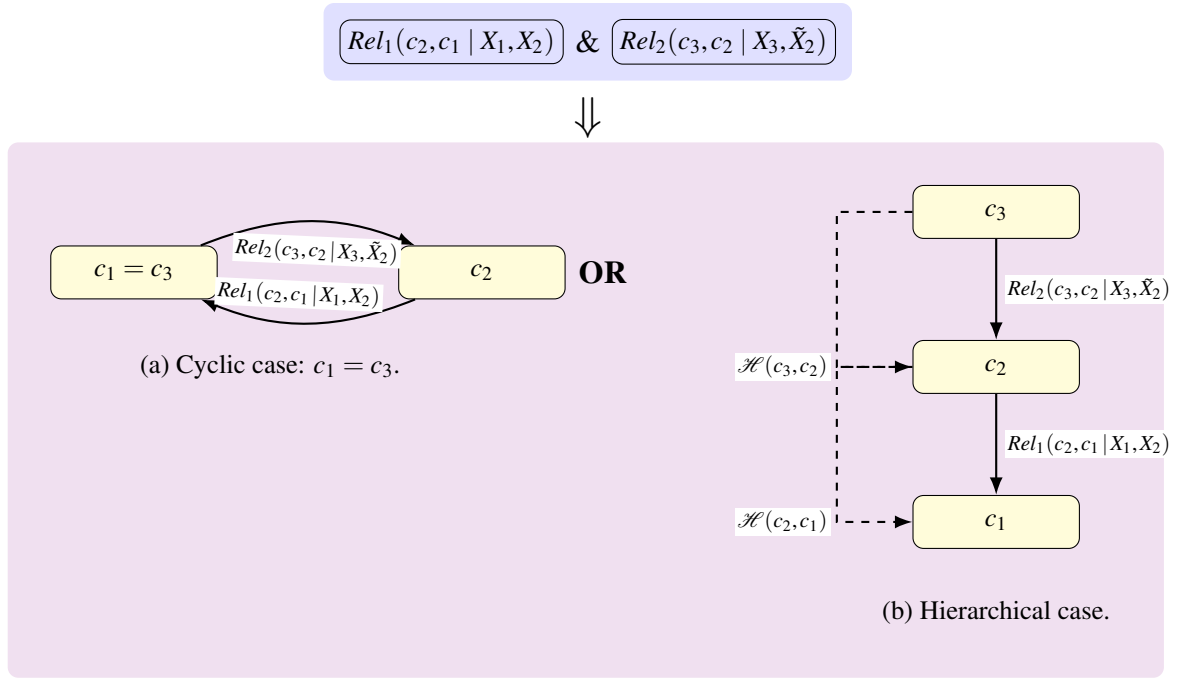
where $Rel_i = \mathcal{B}$ or $Rel_i = \mathcal{O}$ for $i \in \{1, 2\}$.

The previous principle requires that if we have two concatenated binding or overruling relations, namely $Rel_1(c_2, c_1 \mid X_1, X_2)$ and $Rel_2(c_3, c_2 \mid X_3, \tilde{X}_2)$, then either a form of symmetry holds ($c_3 = c_1$), or there exists a hierarchical relationship between the courts c_1 and c_2 ($\mathcal{H}(c_2, c_1)$) and between the courts c_2 and c_3 ($\mathcal{H}(c_3, c_2)$).³³

In other words, as illustrated in Figure 4.10, when two binding and/or overruling relations are concatenated, we can identify either a cyclic structure among the courts involved or a hierarchical chain connecting them.

³²The fact that US jurisdiction satisfies (**wOSD**) and (**OSD**) follows from the literal approach to strict overruling that we adopted. If, for instance, we assumed that the Illinois Supreme Court could overrule federal courts on matters of state law, then (**wOSD**) and (**OSD**) would no longer hold.

³³Notice that from $\mathcal{H}(c_2, c_1)$ and $\mathcal{H}(c_3, c_2)$, together with the transitivity of $\mathcal{H}$, it follows that $\mathcal{H}(c_3, c_1)$.

FIGURE 4.10: **HBO** representation

It can be verified that the weak *stare decisis* principles for binding and overruling (**wBSD**, **wOSD**) together imply principle **HBO**.

Proposition 4.20. $wBSD \ \& \ wOSD \Rightarrow HBO$.

Example 4.10. *US jurisdiction satisfies HBO. We refer the reader to Tables 4.6 and 4.7. EW jurisdiction satisfies HBO, since it satisfies wBSD and wOSD.*

4.6.5 Symmetry between Binding and Overruling

We have introduced in Section 4.6.3 some possible connections between binding and overruling on the basis of criteria **I**), **II**) and **III**). More specifically, some of the principles introduced in in Section 4.6.3 established different forms of exclusivity between binding authority and overruling authority. In this section, we instead propose principles that capture a form of symmetry between these two powers. To better explain, let us consider the England and Wales jurisdiction. In such a system, one court is binding on a different court only if it is higher in the hierarchy, under all circumstances. Likewise, a court can overrule another different court only if it is higher in the hierarchy,

under all circumstances. This would seem to mean that if a court has binding power over another (under certain conditions), then it also has overruling power over it (under certain conditions), and viceversa.

In this situation, there appears to be a one-to-one correspondence between binding and overruling. This intuition is behind the following principle.

$$\text{For } c_i \neq c_j : \mathcal{B}(c_i, c_j \mid X_i, X_j) \Leftrightarrow \mathcal{O}(c_i, c_j \mid X_i, X_j). \quad (\text{S1})$$

The principle that there is a one-to-one correspondence between binding and overruling powers, could be also extended to the same court.

$$\mathcal{B}(c, c \mid X_i, X_j) \Leftrightarrow \mathcal{O}(c, c \mid X_i, X_j). \quad (\text{S2}^*)$$

But, previous principle could be problematic. Indeed

- Suppose that $\mathcal{B}(c, c \mid X_i, X_i) \Leftrightarrow \mathcal{O}(c, c \mid X_i, X_i)$. This would conflict with criterion **III**), and in particular with **AS3**, as it would imply that c , acting under condition X_i , is both self-bound and capable of overruling itself.
- Suppose that $\mathcal{B}(c, c \mid X_i, \emptyset) \Leftrightarrow \mathcal{O}(c, c \mid \emptyset, X_i)$. This situation would conflict as well with criterion **III**). Indeed, recall that the $\emptyset$ -condition denotes an “always” condition. It would therefore imply that court c , when acting under condition X_i , is both self-bound and capable of overruling itself, which contradicts criterion **III**).

For these reasons a revised version of **S2*** could be proposed.

$$\text{For } X_i \neq X_j \text{ and } X_i \neq \emptyset \text{ and } X_j \neq \emptyset : \mathcal{B}(c, c \mid X_i, X_j) \Leftrightarrow \mathcal{O}(c, c \mid X_i, X_j). \quad (\text{S2})$$

We denote **S** the conjunction of **S1** and **S2**.

$$\text{S1 \& S2} \quad (\text{S})$$

Figure 4.11 represents **S**-principles.

Example 4.11. *The EW jurisdiction satisfies **S1** but not **S2**. It does not satisfy **S2**, since $\mathcal{O}(c_2, c_2 \mid \{\text{Appellate}\}, \{\text{FirstInstance}\})$ but not $\mathcal{B}(c_2, c_2 \mid \{\text{Appellate}\}, \{\text{FirstInstance}\})$.*

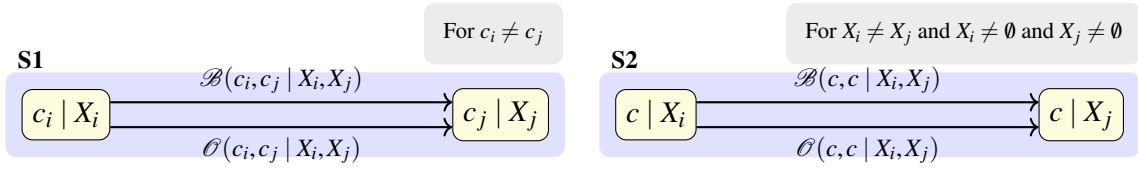


FIGURE 4.11: S-principles

The US jurisdiction does not satisfy **S1** and **S2**. It does not satisfy **S1**, since, for example, we have $\mathcal{B}(\tilde{c}_1, \tilde{c}_0 \mid \{\text{StateLaw}\}, \{\text{StateLaw}\})$ but not $\mathcal{O}(\tilde{c}_1, \tilde{c}_0 \mid \{\text{StateLaw}\}, \{\text{StateLaw}\})$ (this is a consequence of the strict approach to overruling). It also does not satisfy **S2**, since $\mathcal{B}(\tilde{c}_2, \tilde{c}_2 \mid \{\text{FederalLaw}\}, \{\text{Partial, Unauthorized, FederalLaw}\})$ but not $\mathcal{O}(\tilde{c}_2, \tilde{c}_2 \mid \{\text{FederalLaw}\}, \{\text{Partial, Unauthorized, FederalLaw}\})$.

So far, for many principles we have presented three versions: one concerning the binding relation, one concerning the overruling relation, and one capturing the interaction between the two. We have now introduced the **S** principles, which establish a one-to-one correspondence between binding and overruling relations. This naturally raises the question: given the **S** principles, can the version of a given principle for the overruling relation be derived directly from its counterpart for the binding relation (and vice versa)? It depends—among the principles introduced so far, this holds only for those that pertain to the hierarchical structure underlying binding and overruling.³⁴ Specifically, we obtain the following results.

Proposition 4.21. $S1 \& BSD(\text{resp. } OSD) \Rightarrow OSD(\text{resp. } wBSD)$

Proposition 4.22. $S1 \& wBSD(\text{resp. } wOSD) \Rightarrow wOSD(\text{resp. } wBSD)$

4.6.6 Monotonicity

In the discussion of potential binding (Definition 4.10), we observed that a form of monotonicity holds *within states*. In particular, we stated that $\mathcal{B}(c_i, c_j \mid X_i, X_j)$ is to be read as: “court c_i , acting *at least* under conditions X_i , binds court c_j , acting *at least* under conditions X_j .” Under this interpretation, we can formalise principles that make

³⁴In Section 4.6.8, we will use **S1** to establish a correspondence between the binding and overruling counterparts of the *transitivity principles*.

this monotonicity explicit directly at the level of the binding relation. A first principle requires that, for any set of conditions $X_k \subseteq \text{Cond}_{c_j}$ that is the same as or stronger than X_j (more precisely, $X_j \subseteq X_k$), we also have $\mathcal{B}(c_i, c_j \mid X_i, X_k)$. In line with **Adm**, we require that the set of conditions $X_k \subseteq \text{Cond}_{c_j}$ be admissible. Analogous principles apply to the first component X_i of the binding relation, as well as to both components of the overruling relation.

$$\mathcal{B}(c_i, c_j \mid X_i, X_j) \Rightarrow \forall X_k \subseteq \text{Cond}_{c_j} \text{ s.t. } \text{Adm}(X_k) \text{ and } X_j \subseteq X_k : \mathcal{B}(c_i, c_j \mid X_i, X_k). \quad (\text{MonB1})$$

$$\mathcal{B}(c_i, c_j \mid X_i, X_j) \Rightarrow \forall X_k \subseteq \text{Cond}_{c_i} \text{ s.t. } \text{Adm}(X_k) \text{ and } X_i \subseteq X_k : \mathcal{B}(c_i, c_j \mid X_k, X_j). \quad (\text{MonB2})$$

$$\mathcal{O}(c_i, c_j \mid X_i, X_j) \Rightarrow \forall X_k \subseteq \text{Cond}_{c_j} \text{ s.t. } \text{Adm}(X_k) \text{ and } X_j \subseteq X_k : \mathcal{O}(c_i, c_j \mid X_i, X_k). \quad (\text{MonO1})$$

$$\mathcal{O}(c_i, c_j \mid X_i, X_j) \Rightarrow \forall X_k \subseteq \text{Cond}_{c_i} \text{ s.t. } \text{Adm}(X_k) \text{ and } X_i \subseteq X_k : \mathcal{O}(c_i, c_j \mid X_k, X_j). \quad (\text{MonO2})$$

We denote **Mon** the conjunction of all monotonicity principles.

$$\text{MonB1} \ \& \ \text{MonB2} \ \& \ \text{MonO1} \ \& \ \text{MonO2} \quad (\text{Mon})$$

The jurisdictions of England and Wales and the US do not satisfy condition **Mon**. We refer the reader to Example 4.12 in the next section.

4.6.7 The Role of Empty Sets of Conditions and Monotonicity

Recall that, put simply, an empty set of conditions means “always.” In other words, an empty set of conditions in a binding relation indicates that the court in question is always binding on—or bound by—the other court with which it stands in relation. More precisely, $\mathcal{B}(c_i, c_j \mid X_i, \emptyset)$, means that court c_j is always bound—regardless of the conditions it satisfies—by court c_i , when c_i acts under the conditions X_i . In this sense, we expect that under any admissible conditions $X_j \subseteq \text{Cond}_{c_j}$, we should have $\mathcal{B}(c_i, c_j \mid X_i, X_j)$. This is nothing other than a particular instance of the monotonicity principle discussed in the previous section. In other words, we are dealing with monotonicity *with respect to the empty set*—we can speak of monotonicity with respect to the empty

set because, trivially, $\emptyset \subseteq X_j$ for all X_j . The reasoning developed so far can be adapted also to $\mathcal{B}(c_i, c_j \mid \emptyset, X_j)$ and $\mathcal{O}(c_i, c_j \mid X_i, \emptyset)$, $\mathcal{O}(c_i, c_j \mid \emptyset, X_j)$.

These instances of monotonicity with respect to the empty set can be stated as follows:

$$\mathcal{B}(c_i, c_j \mid X_i, \emptyset) \Rightarrow \forall X_j \subseteq \text{Cond}_{c_j} \text{ s.t. } \text{Adm}(X_i) : \mathcal{B}(c_i, c_j \mid X_i, X_j). \quad (\mathbf{E-MonB1})$$

$$\mathcal{B}(c_i, c_j \mid \emptyset, X_j) \Rightarrow \forall X_i \subseteq \text{Cond}_{c_i} \text{ s.t. } \text{Adm}(X_i) : \mathcal{B}(c_i, c_j \mid X_i, X_j). \quad (\mathbf{E-MonB2})$$

$$\mathcal{O}(c_i, c_j \mid X_i, \emptyset) \Rightarrow \forall X_j \subseteq \text{Cond}_{c_j} \text{ s.t. } \text{Adm}(X_j) : \mathcal{O}(c_i, c_j \mid X_i, X_j). \quad (\mathbf{E-MonO1})$$

$$\mathcal{O}(c_i, c_j \mid \emptyset, X_j) \Rightarrow \forall X_i \subseteq \text{Cond}_{c_i} \text{ s.t. } \text{Adm}(X_i) : \mathcal{O}(c_i, c_j \mid X_i, X_j). \quad (\mathbf{E-MonO2})$$

We denote **E-Mon** the conjunction of all empty-set monotonicity principles.

$$\mathbf{E-MonB1} \ \& \ \mathbf{E-MonB2} \ \& \ \mathbf{E-MonO1} \ \& \ \mathbf{E-MonO2} \quad (\mathbf{E-Mon})$$

Obviously the general monotonicity principles **Mon** imply empty-set monotonicity principles **E-Mon**.

Proposition 4.23. *Mon $\Rightarrow$ E-Mon.*

Example 4.12. *The jurisdiction of England and Wales in Example 4.3 does not satisfy condition **E-Mon** (and hence does not satisfy **Mon** either). Indeed, as shown in Table 4.4, we know, for instance, that $\mathcal{B}(c_0, c_0 \mid \emptyset, \text{StandardPanel})$. Hence, to obtain monotonicity, we would need to require both $\mathcal{B}(c_0, c_0 \mid \{\text{StandardPanel}\}, \{\text{StandardPanel}\})$ and $\mathcal{B}(c_0, c_0 \mid \{\text{LargerPanel}\}, \{\text{StandardPanel}\})$. The overruling and binding relation closed under monotonicity are reported in Tables 4.17 and 4.18.*

*Similarly, the US jurisdiction in Example 4.4 also fails to satisfy **E-Mon** (and hence does not satisfy **Mon** either). For instance, to obtain monotonicity with respect to the empty set we should require $\mathcal{B}(\tilde{c}_1, \tilde{c}_3 \mid \{\text{FederalLaw}\}, \{\text{FederalLaw}\})$ and $\mathcal{B}(\tilde{c}_1, \tilde{c}_3 \mid \{\text{StateLaw}\}, \{\text{StateLaw}\})$. For space reasons, we do not report overruling and binding relation for US jurisdiction closed under monotonicity in tables.*

We would like to point out that the monotonicity conditions—both those concerning the empty set and, and even more so, the more general principles—may require the specification of a large number of binding and overruling relations when the set of conditions applicable to a court is extensive. It may therefore be tempting to avoid

specifying them. However, in the absence of closure under monotonicity, undesirable consequences may arise. For readers interested in further details, we refer to the Section D.1 in Appendix D, where we discuss the issue from the perspective of empty set monotonicity.

We also note that, in the case of the EW jurisdiction, closure under monotonicity and closure under empty-set monotonicity coincide. Why is this the case?

Looking at Tables 4.4 and 4.5, we observe that the only courts involved in binding and overruling relations with a non-empty set of conditions are courts c_0 and c_2 . Consider, for example, court c_0 . Recall that $\text{Cond}_{c_0} = \{\text{StandardPanel}, \text{LargerPanel}\}$. We know that $\mathcal{B}(c_0, c_0 \mid \emptyset, \{\text{StandardPanel}\})$. In Example 4.3, we assumed that $\{\text{StandardPanel}\}$ and $\{\text{LargerPanel}\}$ are incompatible. We also note that we can “expand” a binding or overruling relation according to monotonicity (**Mon**) only when considering *admissible* sets—namely, sets that do not contain incompatible subsets.

So, if we consider the binding relation $\mathcal{B}(c_0, c_0 \mid \emptyset, \{\text{StandardPanel}\})$, we cannot derive from monotonicity principle **MonB1** that $\mathcal{B}(c_0, c_0 \mid \emptyset, \{\text{StandardPanel}, \text{LargerPanel}\})$, since StandardPanel and LargerPanel are incompatible. The only admissible way to expand $\mathcal{B}(c_0, c_0 \mid \emptyset, \{\text{StandardPanel}\})$ is by expanding the first (empty) component, obtaining instead $\mathcal{B}(c_0, c_0 \mid \{\text{StandardPanel}\}, \{\text{StandardPanel}\})$ and $\mathcal{B}(c_0, c_0 \mid \{\text{LargerPanel}\}, \{\text{StandardPanel}\})$. These two relations can be derived directly by applying empty-set monotonicity. Same reasoning applies for conditions in c_2 .

So, for the EW jurisdiction, closure under monotonicity and closure under empty-set monotonicity coincide. This is because, for each court, the possible non-empty subsets of conditions associated with the court are incompatible. Formally, in EW, the following holds true.

$$\forall c_i \in \text{Courts}, \forall X, Y \subseteq \text{Cond}_{c_i} : (X \neq Y \& X \neq \emptyset \& Y \neq \emptyset) \Rightarrow \text{INC}(X, Y) \quad (\text{ME1})$$

Specifically, we can show that **ME1** and **E-Mon**, taken together, imply **Mon**.

Proposition 4.24. *ME1 & E-Mon $\Rightarrow$ Mon.*

$c_i \backslash c_j$	c_0	c_1	c_2	c_3	c_4
c_0	$\mathcal{B}(c_0, c_0 \mid \emptyset, \{\text{StandardPanel}\})$ $\mathcal{B}(c_0, c_0 \mid \{\text{StandardPanel}\}, \{\text{StandardPanel}\})$ $\mathcal{B}(c_0, c_0 \mid \{\text{LargerPanel}\}, \{\text{StandardPanel}\})$	$\mathcal{B}(c_0, c_1 \mid \emptyset, \emptyset)$ $\mathcal{B}(c_0, c_1 \mid \{\text{StandardPanel}\}, \emptyset)$ $\mathcal{B}(c_0, c_1 \mid \{\text{LargerPanel}\}, \emptyset)$	$\mathcal{B}(c_0, c_2 \mid \emptyset, \emptyset)$ $\mathcal{B}(c_0, c_2 \mid \{\text{StandardPanel}\}, \emptyset)$ $\mathcal{B}(c_0, c_2 \mid \{\text{LargerPanel}\}, \emptyset)$ $\mathcal{B}(c_0, c_2 \mid \emptyset, \{\text{FirstInstance}\})$ $\mathcal{B}(c_0, c_2 \mid \emptyset, \{\text{Appellate}\})$ $\mathcal{B}(c_0, c_2 \mid \{\text{StandardPanel}\}, \{\text{FirstInstance}\})$ $\mathcal{B}(c_0, c_2 \mid \{\text{StandardPanel}\}, \{\text{Appellate}\})$ $\mathcal{B}(c_0, c_2 \mid \{\text{LargerPanel}\}, \{\text{FirstInstance}\})$ $\mathcal{B}(c_0, c_2 \mid \{\text{LargerPanel}\}, \{\text{Appellate}\})$	$\mathcal{B}(c_0, c_3 \mid \emptyset, \emptyset)$ $\mathcal{B}(c_0, c_3 \mid \{\text{StandardPanel}\}, \emptyset)$ $\mathcal{B}(c_0, c_3 \mid \{\text{LargerPanel}\}, \emptyset)$	$\mathcal{B}(c_0, c_4 \mid \emptyset, \emptyset)$ $\mathcal{B}(c_0, c_4 \mid \{\text{StandardPanel}\}, \emptyset)$ $\mathcal{B}(c_0, c_4 \mid \{\text{LargerPanel}\}, \emptyset)$
c_1	—	$\mathcal{B}(c_1, c_1 \mid \emptyset, \emptyset)$	$\mathcal{B}(c_1, c_2 \mid \emptyset, \emptyset)$ $\mathcal{B}(c_1, c_2 \mid \emptyset, \{\text{Appellate}\})$ $\mathcal{B}(c_1, c_2 \mid \emptyset, \{\text{FirstInstance}\})$	$\mathcal{B}(c_1, c_3 \mid \emptyset, \emptyset)$	$\mathcal{B}(c_1, c_4 \mid \emptyset, \emptyset)$
c_2	—	—	$\mathcal{B}(c_2, c_2 \mid \{\text{Appellate}\}, \emptyset)$ $\mathcal{B}(c_2, c_2 \mid \{\text{Appellate}\}, \{\text{Appellate}\})$ $\mathcal{B}(c_2, c_2 \mid \{\text{Appellate}\}, \{\text{FirstInstance}\})$	$\mathcal{B}(c_2, c_3 \mid \emptyset, \emptyset)$ $\mathcal{B}(c_2, c_3 \mid \{\text{Appellate}\}, \emptyset)$ $\mathcal{B}(c_2, c_3 \mid \{\text{FirstInstance}\}, \emptyset)$	$\mathcal{B}(c_2, c_4 \mid \emptyset, \emptyset)$ $\mathcal{B}(c_2, c_4 \mid \{\text{Appellate}\}, \emptyset)$ $\mathcal{B}(c_2, c_4 \mid \{\text{FirstInstance}\}, \emptyset)$
c_3	—	—	—	—	—
c_4	—	—	—	—	—

TABLE 4.17: Binding relations $\mathcal{B}(c_i, c_j \mid X_i, X_j)$ in England and Wales, closed under empty-set monotonicity

$c_i \backslash c_j$	c_0	c_1	c_2	c_3	c_4
c_0	$\mathcal{O}(c_0, c_0 \mid \{\text{LargerPanel}\}, \emptyset)$ $\mathcal{O}(c_0, c_0 \mid \{\text{LargerPanel}\}, \{\text{StandardPanel}\})$ $\mathcal{O}(c_0, c_0 \mid \{\text{LargerPanel}\}, \{\text{LargerPanel}\})$	$\mathcal{O}(c_0, c_1 \mid \emptyset, \emptyset)$ $\mathcal{O}(c_0, c_1 \mid \{\text{StandardPanel}\}, \emptyset)$ $\mathcal{O}(c_0, c_1 \mid \{\text{LargerPanel}\}, \emptyset)$	$\mathcal{O}(c_0, c_2 \mid \emptyset, \emptyset)$ $\mathcal{O}(c_0, c_2 \mid \{\text{StandardPanel}\}, \emptyset)$ $\mathcal{O}(c_0, c_2 \mid \{\text{LargerPanel}\}, \emptyset)$ $\mathcal{O}(c_0, c_2 \mid \emptyset, \{\text{FirstInstance}\})$ $\mathcal{O}(c_0, c_2 \mid \emptyset, \{\text{Appellate}\})$ $\mathcal{O}(c_0, c_2 \mid \{\text{StandardPanel}\}, \{\text{FirstInstance}\})$ $\mathcal{O}(c_0, c_2 \mid \{\text{StandardPanel}\}, \{\text{Appellate}\})$ $\mathcal{O}(c_0, c_2 \mid \{\text{LargerPanel}\}, \{\text{FirstInstance}\})$ $\mathcal{O}(c_0, c_2 \mid \{\text{LargerPanel}\}, \{\text{Appellate}\})$	$\mathcal{O}(c_0, c_3 \mid \emptyset, \emptyset)$ $\mathcal{O}(c_0, c_3 \mid \{\text{StandardPanel}\}, \emptyset)$ $\mathcal{O}(c_0, c_3 \mid \{\text{LargerPanel}\}, \emptyset)$	$\mathcal{O}(c_0, c_4 \mid \emptyset, \emptyset)$ $\mathcal{O}(c_0, c_4 \mid \{\text{StandardPanel}\}, \emptyset)$ $\mathcal{O}(c_0, c_4 \mid \{\text{LargerPanel}\}, \emptyset)$
c_1	—	—	$\mathcal{O}(c_1, c_2 \mid \emptyset, \emptyset)$ $\mathcal{O}(c_1, c_2 \mid \emptyset, \{\text{Appellate}\})$ $\mathcal{O}(c_1, c_2 \mid \emptyset, \{\text{FirstInstance}\})$	$\mathcal{O}(c_1, c_3 \mid \emptyset, \emptyset)$	$\mathcal{O}(c_1, c_4 \mid \emptyset, \emptyset)$
c_2	—	—	$\mathcal{O}(c_2, c_2 \mid \{\text{Appellate}\}, \{\text{FirstInstance}\})$	$\mathcal{O}(c_2, c_3 \mid \emptyset, \emptyset)$ $\mathcal{O}(c_2, c_3 \mid \{\text{Appellate}\}, \emptyset)$ $\mathcal{O}(c_2, c_3 \mid \{\text{FirstInstance}\}, \emptyset)$	$\mathcal{O}(c_2, c_4 \mid \emptyset, \emptyset)$ $\mathcal{O}(c_2, c_4 \mid \{\text{Appellate}\}, \emptyset)$ $\mathcal{O}(c_2, c_4 \mid \{\text{FirstInstance}\}, \emptyset)$
c_3	—	—	—	—	—
c_4	—	—	—	—	—

TABLE 4.18: Overruling relations $\mathcal{O}(c_i, c_j \mid X_i, X_j)$ in England and Wales, closed under empty-set monotonicity

4.6.8 Transitivity

Another principle that can be proposed is closure under transitivity. Specifically, if a court c_2 , under conditions X_2 , is binding over court c_1 , and a court c_3 is binding over c_2 *under the same conditions* X_2 , then it could be required that c_3 is binding over c_1 with respect to the established conditions. The same principle may apply to the overruling relation.

$$\mathcal{B}(c_2, c_1 \mid X_2, X_1) \text{ and } \mathcal{B}(c_3, c_2 \mid X_3, X_2) \Rightarrow \mathcal{B}(c_3, c_1 \mid X_3, X_1). \quad (\mathbf{TB})$$

$$\mathcal{O}(c_2, c_1 \mid X_2, X_1) \text{ and } \mathcal{O}(c_3, c_2 \mid X_3, X_2) \Rightarrow \mathcal{O}(c_3, c_1 \mid X_3, X_1). \quad (\mathbf{TO})$$

The transitivity principle for overruling (resp. binding) can be retrieved from the transitivity principle for binding (resp. overruling), together with **S1** and **AS**.

Proposition 4.25. ***S1** & **AS** & **TB** (resp. **TO**) $\Rightarrow$ **TO** (resp. **TB**).*

The question now is whether an equivalent of transitivity holds for the combination of binding and overruling relations. In other words, whether the following holds:

$$\begin{aligned} &\mathcal{B}(c_2, c_1 \mid X_2, X_1) \text{ and } \mathcal{O}(c_3, c_2 \mid X_3, X_2) \Rightarrow \mathcal{O}(c_3, c_1 \mid X_3, X_1); \\ &\mathcal{O}(c_2, c_1 \mid X_2, X_1) \text{ and } \mathcal{B}(c_3, c_2 \mid X_3, X_2) \Rightarrow \mathcal{B}(c_3, c_1 \mid X_3, X_1). \end{aligned} \quad (\mathbf{TBO})$$

Previous principle may seem rather arbitrary and lack any clear underlying intuition. However, **TBO** can be derived from **TB** and **TO**, by imposing the **S**-principles, which establish a correspondence between binding and overruling, together with **E-Mon**-principles and the **AS**-principles.

Proposition 4.26. ***S** & **E-Mon** & **AS** & **TB** (resp. **TO**) $\Rightarrow$ **TBO***

We denote **T** the conjunction of all transitivity principles.

$$\mathbf{TB} \& \mathbf{TO} \& \mathbf{TBO} \quad (\mathbf{T})$$

Example 4.13. *EW does not satisfy T , because it does not satisfy TBO . Indeed, we have $\mathcal{B}(c_0, c_0 \mid \emptyset, \{\text{StandardPanel}\})$ and $\mathcal{O}(c_0, c_0 \mid \{\text{LargerPanel}\}, \emptyset)$. But we don't have $\mathcal{O}(c_0, c_0 \mid \{\text{LargerPanel}\}, \{\text{StandardPanel}\})$. However, this latter relation does hold in the EW jurisdiction when closure under monotonicity is assumed (see Table 4.18).*

US does not satisfy T , because it does not satisfy TB . Indeed, we know $\mathcal{B}(\tilde{c}_0, \tilde{c}_1 \mid \{\text{FederalLaw}\}, \{\text{FederalLaw}\})$ and $\mathcal{B}(\tilde{c}_1, \tilde{c}_0 \mid \{\text{StateLaw}\}, \{\text{StateLaw}\})$. But it does not hold $\mathcal{B}(\tilde{c}_0, \tilde{c}_1 \mid \{\text{FederalLaw}\}, \{\text{StateLaw}\})$ nor $\mathcal{B}(\tilde{c}_1, \tilde{c}_0 \mid \{\text{StateLaw}\}, \{\text{FederalLaw}\})$. Similarly, it can be verified that it does not satisfy TO .

4.6.9 Per Incuriam and Acyclic s-Graphs

Recall from previous chapter that to determine whether a state $s \in S$ is *per incuriam* we can refer to the s -graph. We restate the definition here for convenience.

Definition 3.38 (s -Graph). A structure $G_s = (V_s, E_s) = \bigcup_{n \geq 0} G_n$ is called *the s -graph*, when $f(s) \neq ?$ and every G_n satisfies:

- $G_0 = (V_0, E_0)$, where $V_0 = \{s\}$ and $E_0 = \emptyset$;
- $G_{n+1} = (V_{n+1}, E_{n+1})$, where
 - $V_{n+1} = V_n \cup \{s' \mid s' \in \beta_{s''} \text{ or } s' \in \omega_{s''}, \text{ with } s'' \in V_n\}$ and
 - $E_{n+1} = E_n \cup \{(s'', s') \mid s' \in \beta_{s''} \text{ or } s' \in \omega_{s''}, \text{ with } s'' \in V_n\}$.

To compute whether a state is *per incuriam*, we need to ensure that the s -graph is acyclic. In the previous chapter, where we considered binary binding and overruling relations, acyclicity was ensured through the stare decisis conditions (SD) and the incompatibility of the binding and overruling relations on the same court (see Definition 3.27). Here, we propose a general principle to impose acyclicity on every s -graph, when we deal with conditional binding and overruling relations. To formulate this principle, we begin with the following considerations.

Consider the s -graph, $G_s = (V_s, E_s)$.

- 1) If s_2 is the successor of s_1 in the s -graph, i.e. $(s_1, s_2) \in E_s$, then either $s_2 \in \omega_{s_1}$ or $s_2 \in \beta_{s_1}$. But then this means $Rel(c_2, c_1 \mid X_2, \tilde{X}_1)$ with $c_1 \in s_1$, $c_2 \in s_2$, $X_2 \subseteq s_2$, $\tilde{X}_1 \subseteq s_1$ and $Rel = \mathcal{B}$ or $Rel = \mathcal{O}$.
- 2) If in a s -graph there is a cycle, this means that there is a sequence of states $s_1, s_2, \dots, s_n, s_{n+1}$ s.t. $(s_i, s_{i+1}) \in E_s$ and $s_1 = s_{n+1}$.
- 3) By two previous points, this means that we have $Rel_1(c_2, c_1 \mid X_2, \tilde{X}_1)$, $Rel_2(c_3, c_2 \mid X_3, \tilde{X}_2)$, ..., $Rel_{n-1}(c_n, c_{n-1} \mid X_n, \tilde{X}_{n-1})$ and $Rel_n(c_1, c_n \mid X_1, \tilde{X}_n)$, where for all $i \in \{1, \dots, n\}$, $X_i, \tilde{X}_i \subseteq s_i$ and $Rel_i = \mathcal{B}$ or $Rel_i = \mathcal{O}$.
- 4) Notice that if $Rel_1 = Rel_2 = \dots = Rel_n = \mathcal{O}$, then this means that $s_2 \in \omega_{s_1}$, $s_3 \in \omega_{s_2}, \dots, s_1 \in \omega_{s_n}$. Namely we know that s_2 potentially overrules s_1 , s_3 potentially overrules s_2 and so on. This implies a strict temporal relation among the cases: $s_1 <_T s_2 <_T \dots <_T s_n <_T s_1$. By the transitivity of the temporal relation, we would then have $s_1 <_T s_1$ — a contradiction, since $<_T$ is irreflexive.
- 5) Notice that if $Rel_1 = Rel_2 = \dots = Rel_n = \mathcal{B}$ then this means $s_2 \in \beta_{s_1}$ and $s_3 \in \beta_{s_2}, \dots, s_1 \in \beta_{s_n}$. Namely we know that s_2 potentially binds s_1 , s_3 potentially binds s_2 and so on. This implies a strict temporal relation among the cases: $s_1 <_T s_n <_T \dots <_T s_3 <_T s_2 <_T s_1$. By the transitivity of the temporal relation, we would then have $s_1 <_T s_1$ — a contradiction, since $<_T$ is irreflexive.

Keeping in mind the above observations, we want to ensure that cycles do not exist in the s -graph. To this end, we require that any chain of relations $Rel_1(c_2, c_1 \mid X_2, \tilde{X}_1)$, $Rel_2(c_3, c_2 \mid X_3, \tilde{X}_2) \dots Rel_{n-1}(c_n, c_{n-1} \mid X_n, \tilde{X}_{n-1})$ and $Rel_n(c_1, c_n \mid X_1, \tilde{X}_n)$, does not actually give rise to a cycle $s_1, s_2, \dots, s_n, s_{n+1}$ with $s_1 = s_{n+1}$.

This can be ensured by requiring that:

- A) either $X_i, \tilde{X}_i$ are incompatible, for some i . Then, the chain is inapplicable to any sequence $s_1, s_2, \dots, s_n, s_{n+1}$, with $s_1 = s_{n+1}$, since it is impossible to have $X_i, \tilde{X}_i \subseteq s_i$ for all i . Hence, point 3) in the previous list does not hold.
- B) or $Rel_1 = Rel_2 = \dots = Rel_n$. Then, the chain is inapplicable to any sequence $s_1, s_2, \dots, s_n, s_{n+1}$, with $s_1 = s_{n+1}$, because of irreflexivity of time order (see points 4) and 5) in previous list).

On the basis of A) and B), the following principle can be formalised to ensure the acyclicity of any s -graph.

$$\begin{aligned}
 & Rel_1(c_2, c_1 \mid X_2, X_1) \& Rel_2(c_3, c_2 \mid X_3, \tilde{X}_2) \& \dots \& Rel_{n-1}(c_n, c_{n-1} \mid X_n, \tilde{X}_{n-1}) \& Rel_n(c_1, c_n \mid X_1, \tilde{X}_n) \\
 & \Rightarrow [\exists i \in \{1, \dots, n\} \text{ s.t. } INC(X_i, \tilde{X}_i)] \text{ or } [\forall j \in \{1, \dots, n-1\}, Rel_j = Rel_{j+1}], \\
 & \text{where } c_i \in \text{Courts}, X_i, \tilde{X}_i \subseteq \text{Cond}, Rel_i = \mathcal{B} \text{ or } Rel_i = \mathcal{O}, \text{ for } i \in \{1, \dots, n\}. \quad (\mathbf{L1})
 \end{aligned}$$

It can indeed be verified that **L1** ensures the acyclicity of the s -graph G_s .

Proposition 4.27. *Assume **L1**. Then G_s is acyclic, for every $s \in S$.*

One interesting question is now whether **L1** can be retrieved combining principles already defined in previous sections. The answer is yes. We provide the two following results.

Firstly, **L1** can be derived by combining **CBO3**, which was formalised for asymmetry under compatible conditions, and **HBO**, which concerns the relationship between overruling/binding and the hierarchical structure of courts. In broad terms, **HBO** is applied to verify **L1** in the case of chains of relations involving different courts, while **CBO3** is applied to verify **L1** in the case of chains involving the same court.

Proposition 4.28. ***HBO** & **CBO3** $\Rightarrow$ **L1***

Example 4.14. *We know that US and EW jurisdiction satisfy **HBO**, **CBO3**. Therefore, they also satisfy **L1**.*

L1 can also be retrieved by combining the principles of monotonicity, transitivity, admissibility and asymmetry under compatible conditions.

Proposition 4.29. ***Mon** & **T** & **Adm** & **C2** $\Rightarrow$ **L1***

By Propositions 4.16, 4.18, 4.17, 4.24 and 4.26, we obtain several corollaries. We list some of them below.

Proposition 4.30. ***Mon** & **T** & **Adm** & **C1** $\Rightarrow$ **L1***

Proposition 4.31. ***E-Mon** & **ME1** & **T** & **Adm** & **C2** $\Rightarrow$ **L1***

Proposition 4.32. *E-Mon & ME1 & S & C2 & TB (rep. TO) & Adm $\Rightarrow$ LI*

Proposition 4.33. *E-Mon & ME1 & S & C1 & TB (rep. TO) & Adm $\Rightarrow$ LI*

Example 4.15. *The EW jurisdiction closed under monotonicity satisfies T, C2, Adm. So it verifies LI.*

4.6.10 Language-Level Representation of Principles

All the principles introduced so far can be expressed within the language. For instance, the representation principle **AS1**, at the language level is given below:

$$B(c_i, c_j \mid X_i, X_j) \rightarrow \neg B(c_j, c_i \mid X_j, X_i), \text{ for } X_i \neq X_j \text{ or } c_i \neq c_j \quad (\mathbf{LA1})$$

The Appendix D reports in Section D.2 the language-level representations of all the principles proposed so far.

4.7 Monotonicity or Non-Monotonicity?

As noted in connection with the notion of *potential binding*, we assumed that this concept presupposes a certain form of monotonicity, at least within the states. Namely, when s' is potentially binding for s , and this holds because of the relation $\mathcal{B}(c', c \mid X_1, X_2)$, we have that $X_1 \subseteq s'$ and $X_2 \subseteq s$. However, s' may contain *additional conditions* beyond those in X_1 and still be binding; s' only needs to include at least X_1 .

We also stated that, from this perspective, $\mathcal{B}(c', c \mid X_1, X_2)$ can be read as: “ c' , when acting under *at least* conditions X_1 , issues binding decisions for c , when the latter acts under *at least* conditions X_2 .”

However, it might legitimately be argued that the monotonicity presupposed in Definition 4.34 is too strong. That is, while $\mathcal{B}(c', c \mid X_1, X_2)$ indicates that c' , under conditions X_1 , binds c , which operates under conditions X_2 , it might also be the case that, if c acts

under stronger conditions or c' acts under weaker conditions, the direction of the binding relation may be reversed.

By *stronger conditions* for c , we mean that, for a superset $\tilde{X}_2 \supset X_2$, c may constrain c' acting under X_1 —that is, $\mathcal{B}(c, c' \mid \tilde{X}_2, X_1)$.

By *weaker conditions* for c' , we mean that for a subset $\tilde{X}_1 \subset X_1$, c' might instead be bounded by c acting under X_2 —i.e. $\mathcal{B}(c, c' \mid X_2, \tilde{X}_1)$.

In other words, under an alternative approach, both $\mathcal{B}(c', c \mid X_1, X_2)$ and $\mathcal{B}(c, c' \mid \tilde{X}_2, \tilde{X}_1)$ may hold when $\tilde{X}_1 \subset X_1$ and $X_2 \subset \tilde{X}_2$. In such cases, the binding relation would not be monotonic—at least with respect to the second component of the conditions in the binding relation.

It is worth noting, however, that while this line of reasoning is plausible, our examination of the jurisdictions of England and Wales and of the United States has revealed no evidence of such situations. Binding relations appear to exhibit a monotonic structure, and we have identified no counterexamples suggesting otherwise. More specifically, once a binding or overruling relation is established under a given set of conditions, extending those conditions does not appear to invalidate it.

In any case, if this alternative approach is adopted, the definition of *potential binding* should be revised accordingly. We therefore propose one possible variant below.

Definition 4.34 (Pot. bind. 2). Let $s, s' \in S$ and $o \in \{0, 1\}$. We say that s' is *potentially a binding precedent* for s for a decision with outcome o , denoted as $\beta(s', s, o)$, if and only if:

- $\Pi(s', s, o)$; and
- there exist $X_1 \subseteq s'$ and $X_2 \subseteq s$ such that $\mathcal{B}(c', c \mid X_1, X_2)$; and
- there do **not** exist $\tilde{X}_1 \subseteq s'$ or $\tilde{X}_2 \subseteq s$ such that $\mathcal{B}(c, c' \mid \tilde{X}_2, \tilde{X}_1)$ holds with $\tilde{X}_1 \subseteq X_1$ or $X_2 \subseteq \tilde{X}_2$.

In a similar manner, and for analogous reasons, the definition of potential overruling could likewise be revised accordingly.

We should note, however, that adopting new definitions of potential binding and potential overruling would require revising some of the principles introduced earlier. For instance, conditions **CB1**, **CO1**, **CB2**, and **CO2** would need to be reconsidered. To provide an example, condition **CB2** requires that, if we have both $\mathcal{B}(c_i, c_j \mid X_i, X_j)$

and $\mathcal{B}(c_j, c_i \mid \tilde{X}_j, \tilde{X}_i)$, then X_i and $\tilde{X}_i$ are incompatible, or X_j and $\tilde{X}_j$ are incompatible. This requirement could prevent the possibility of having both $\mathcal{B}(c, c' \mid \tilde{X}_2, \tilde{X}_1)$ and $\mathcal{B}(c, c' \mid X_2, X_1)$ at the same time, when $\tilde{X}_1 \subset X_1$ or $X_2 \subset \tilde{X}_2$, contradicting the scenario discussed above. Therefore, we should require incompatibility only for those pairs $X_i, \tilde{X}_i$ or $X_j, \tilde{X}_j$ such that $\tilde{X}_i \not\subseteq X_i$ or $\tilde{X}_j \not\subseteq X_j$.

For $\tilde{X}_i \not\subseteq X_i$ or $\tilde{X}_j \not\subseteq X_j$:

$$\mathcal{B}(c_i, c_j \mid X_i, X_j) \ \& \ \mathcal{B}(c_j, c_i \mid \tilde{X}_j, \tilde{X}_i) \Rightarrow INC(X_i, \tilde{X}_i) \text{ or } INC(X_j, \tilde{X}_j). \quad (\mathbf{VCB3})$$

4.8 Discussion

In this chapter, we have refined the notions of the *binding* and *overruling* relations introduced in the previous chapter, by modeling their *conditional* counterparts. We have also discussed some principles that could be adopted for these relations.

Some aspects deserve further reflection and discussion.

The conditions examined in connection to binding and overruling relations are of a procedural character or pertain to subject-matter competence. However, there are additional, more nuanced conditions that may also be identified. As noted in Footnote 8, the US Supreme Court generally adheres to the doctrine of *stare decisis*, following its prior rulings unless there is a “special justification” or, at a minimum, “strong grounds” to overturn precedent. Similarly, the UK Supreme Court, under the Practice Statement, may depart from its previous decisions “when it appears right to do so”[1]. Both of these formulations identify conditions governing overruling and binding that are grounded in judicial discretion and entail a corresponding duty of justification—that is, judges must articulate the reasons to depart from precedent. Further investigation of this issue is left for future research. In particular, it would be interesting to explore a possible interaction between our framework and the notion of overruling introduced by Horty in [52], which is grounded centrally in the concept of special justification, as briefly discussed in Section 3.6.

Our analysis has made extensive use of the notion of incompatibility, particularly in addressing potential overlaps that may arise between two binding relations, two overruling relations, or between a binding and an overruling relation. For some conditions, their incompatibility is straightforward—for example, sitting *en banc* versus sitting in partial panel: if a court sits *en banc*, it does not sit as a partial panel, and vice versa. For other conditions, the incompatibility is less obvious. We treated as incompatible the conditions of deciding under state law versus federal law. We did so on the premise that each case involves a single issue with a single concern, and that any given issue consists exclusively of either state-law questions or federal-law questions.

Although this assumption may seem reasonable, we believe it warrants further examination. To begin with, federal courts may adjudicate cases that include state-law questions when federal jurisdiction is properly invoked; conversely, state courts may adjudicate cases involving federal questions when state-law issues arise. Thus, in certain cases, federal and state law questions may coexist. Otherwise, the federal and state court systems would be entirely separate: purely federal cases would be addressed exclusively by federal courts, while purely state cases would be handled by state courts. We therefore identify two questions for future detailed investigation.

- 1) Is it possible that, within a single issue (and concern), both federal and state-law questions are addressed simultaneously? If so, how should such situations be handled or modeled?
- 2) How should we treat cases that present multiple issues (or concerns), where different issues fall under different competences—one state and another federal?

In discussing the notions of binding and overruling—particularly strict overruling—we have referred to different forms of judicial power. Binding represents a form of power that imposes a constraint, whereas overruling constitutes a form of power that removes a binding constraint. At this point, it would be interesting to explore the relationship between our framework and that proposed by Hohfeld [49]—and formally elaborated, for instance, by Markovich [67] and Pascucci and Sileno [92]. In Hohfeld’s system, power is defined as the capacity to modify another’s legal position—for example, to create, extinguish, or alter rights and duties. From this perspective, binding may be understood as the exercise of a power that generates a constraint, that is, one that transforms a given configuration of conditions or decisions into a bound state. Overruling, by contrast, expresses a higher-order directive: it does not directly affect the content of

the decision, but rather its binding force itself, by removing or altering the relation of obligatoriness that binding had created. A systematic analysis of the analogy between our framework and the Hohfeldian system is left for future work.

As already mentioned several times, the notion of *binding* establishes a constraint. Moreover, the binding operator is *directed*, since it defines a constraint exercised by one court over another, and it is *conditional*, as it specifies the conditions under which such a constraint applies. The notion of constraint echoes, at least partially, the notion of obligation. In deontic logic, both operators for *conditional obligation* [46, 96] (already mentioned in Chapter 2) and operators for *directed obligation* [48, 41, 67] have been introduced. For both operators different approaches were proposed. We do not enter here into technical details; our aim is rather to provide an intuitive understanding that the notion of *directed and conditional binding* introduced here differs from that of directed or conditional obligation.

Directed obligation operators are typically expressed in the form $O_{a \rightarrow b}(\varphi)$ which reads as “agent a is obliged to bring about φ towards b ”. The binding operator $B(c_i, c_j \mid X_i, X_j)$, however, does not state that c_j has an obligation *towards* c_i , but rather that c_j has the obligation to *follow the decision* of c_i .

The *dyadic obligation* operator, expressed as $O(\psi \mid \varphi)$, establishes that the obligation ψ is conditional upon φ . According to the most common semantics, this means that in the best worlds where φ holds, ψ also holds. Whether the obligation holds therefore depends on the set of worlds of the model. In our framework, by contrast, the constraint between two courts does not depend on the states of the classifier model—it is assumed as given. What we do, rather, is determine for which states such a constraint applies. The binding operator requires that two conditions—each possibly pertaining to a distinct court (and thus, potentially, to two distinct states)—be satisfied in order to assert that the obligation of one court to follow the decision of another is in force.

Conclusions

Thesis Overview

The thesis develops logical models for legal case-based reasoning, relaxing some of the strong assumptions underlying existing approaches in the field. We recall that the assumptions challenged in this work are the following:

- A1) Legal cases can be represented as sets of *case features* (e.g. *factors*), each corresponding to a legally relevant factual pattern that favors one of the two parties (e.g., the plaintiff or the defendant). The set of *factors* and their *directionality*—that is, which party they favor — are known *a priori*.
- A2) Factual similarity (or comparison) *alone* is taken to determine whether a precedent case exerts a constraining effect on the decision in a given case (e.g., by comparing the factors present in each case).
- A3) The case base is constructed so that no conflict of precedents may occur — i.e., there are no situations in which, for the same case to be decided, two precedents force opposite outcomes (e.g., one favoring the plaintiff and another the defendant).

Chapter 2 focuses specifically on CBR models that abandon assumptions A1, and A3 while retaining A2. *Chapter 3*, by contrast, examines the consequences of abandoning both A2 and A3, independently of whether A1 is adopted. *Chapter 4* further refines some aspects introduced in Chapter 3, with a particular focus on conditional judicial authority.

Chapter 2 addresses the following question: *Is it possible to identify, ex post, which features of the cases in the case base effectively behave as factors, and what their directionality is?* To answer this, the chapter focuses on the logical identification of factors among the features present in a legal case base, adopting the BCL framework. The proposed approach enables a *pro tanto* analysis of features by observing their behavior within the case base—specifically, by verifying whether they are discriminating (i.e., supporting) for one of the parties involved. Each feature is classified as a *factor* (consistently supporting one outcome), *irrelevant* (having no discriminative power), or *ambiguous* (supporting opposite outcomes in different contexts). Unlike traditional models that discard inconsistencies, this approach explicitly accounts for ambiguous features, providing an inference mechanism capable of reasoning through conflicting precedents. This mechanism builds on the intuition that some cases are more significant than others, extending BCL to incorporate a *preorder* between states to represent different levels of importance.

Chapter 3 addresses the following questions: *Which precedents are binding for the case at hand, considering both the factual relevance between the cases and the jurisdictional relations among the courts that decided them?* and *How can conflicts among binding precedents be resolved in a legally grounded way?* The chapter develops a hybrid logic extending BCL, enriched with state names and new modal operators that capture relevance, temporal relations among cases and institutional hierarchies among courts. Through this formalization, it models the notion of legal precedents, distinguishing between binding precedents and those subject to exceptions, such as *per incuriam* or overruled decisions. To address conflicts among binding precedents without exceptions, both hierarchical and temporal–hierarchical principles are proposed.

Chapter 4 refines the notions of *binding* and *overruling* relations between courts, moving beyond the binary perspective adopted in the previous chapter. It builds on the intuition that these relations are often *context-dependent*, shaped by procedural aspects (such as court composition or jurisdictional role) or by subject-matter competence (for instance, federal versus state competence). Drawing on examples from the jurisdictions of England and Wales and of the United States, the chapter illustrates how these relations vary across institutional settings. To capture the different approaches to binding and overruling, it introduces a set of possible principles governing these relations, their mutual interaction, and their connection to the hierarchical structure of courts.

Future works

At the end of each chapter, we proposed possible directions for future research. Here, we summarize some broader research areas that emerge across chapters and appear promising for further investigation.

Multi-step legal reasoning

In this thesis, we focused on cases involving a single issue and a single concern within each issue. On this basis, we assumed that the decision-making process for each new case can be modeled as a single-step procedure that, on the basis of binding precedents, leads directly from the facts of the case to its outcome. Within the enriched frameworks of TJCL, each case is described primarily in terms of the facts that characterize it and the court that adjudicated it. In TJCL-Cond, we additionally specify whether, for each case, certain conditions related to subject-matter competence or procedural aspects are satisfied.

However, in legal practice, cases are not defined solely by their facts but also by their *ratio decidendi*—the reason that underlies the court’s decision. The *ratio decidendi* is generally binding on lower courts and subsequent judgments [61]. In the *reason model* [50], the *ratio decidendi* is represented as a rule directly linking specific facts of a case to its conclusion. Yet, as argued by Branting [20] and Canavotto [27], judicial reasoning typically proceeds through a series of inferences connecting subsets of facts to intermediate conclusions supported by those facts. According to Branting, these intermediate rules—referred to as *precedent constituents*—should themselves constrain future cases.

Different approaches in legal CBR have been proposed to capture how portions of precedents influence later decisions. In systems such as CATO [4] and IBP [23], the basic factors of a case are organized into issues. In CATO, however, an additional layer of *intermediate* or *abstract* factors – representing, to some extent, intermediate concerns– exists between the issues and the base-level factors. According to Bench-Capon and Atkinson [16, 15], only decisions concerning issues (and not intermediate

factors) can constrain future cases, whereas Canavotto et al. [27, 26] contend that decisions involving intermediate factors and intermediate concerns may also impose constraints on subsequent judgments.

In a recent work [34], we proposed a semantic extension of classifier models to address the problems of *ratio decidendi* and precedent constituents outlined above. This work developed rule-based classifier models that integrate rules into the existing classifier framework, enriching classifiers so that their states are defined not only in terms of facts but also in terms of sets of rules. As a starting point, we built classifiers modeled on the CBR framework proposed in [27]. The latter extends the reason model to incorporate a hierarchy of factors, so that the final decision in a case depends on the outcomes of intermediate concerns and the rules applied in reaching those outcomes.

For future work, we aim to develop a formal logic extending TJCL and TJCL-Cond, capable of capturing the semantics of the rule-based classifier models and demonstrating the applicability of our framework to multi-step—namely, multi-concern and multi-issue—forms of legal reasoning.

Refining Conditional Authority: Judicial Discretion and Multi-Level Conditions

The introduction of conditional binding and overruling represents a significant advance toward a more realistic account of judicial reasoning. Future research should continue to investigate context-dependent authority, with particular attention to discretionary conditions—for example, the “special justification” standard adopted by the US Supreme Court for self-overruling, or the “when it appears right to do so” clause of the UK Practice Statement, which similarly governs self-overruling for UK Supreme Court. In particular, will we start by considering the account already developed by Horty [52] in this regard.

Another promising avenue concerns the interaction between conditionality and multi-step reasoning. In the US jurisdiction, for instance, cases may involve both federal and state competences. Also, as outlined above, judicial decisions can exhibit multi-step reasoning that encompasses multiple issues or concerns. Hence, for example, within a single case, one issue or concern may pertain to state law while another pertains to federal law, and the overall outcome of the case depends on the determinations made

with respect to each issue or concern. Extending the notions of conditional binding and overruling to multi-level conditionality—where distinct issues trigger different binding or overruling conditions—would substantially enhance the scope and applicability of the proposed framework.

Logic, Power, and Normativity: Foundational Reflections

Some conceptual connections invite deeper exploration. One promising avenue lies in relating the binding operator and bounded decisions to deontic logic.

On the one hand, as observed in the discussion of Chapter 4, the binding operator is directional and conditional in ways that depart from classical accounts of directed [48, 41, 67] or dyadic obligation operators [46, 96].

On the other hand, in Chapters 2 and 3, when defining the conditions under which a decision is forced or bounded by previous cases, we selected the best precedents. This approach parallels that adopted in dyadic deontic logic, according to which a conditional obligation of φ given ψ holds if the best worlds satisfying ψ also satisfy φ .

Exploring the intersections among the binding operator, bounded decisions, and established forms of deontic logic may yield a more comprehensive and refined theory of jurisdictional obligation.

Another line of inquiry involves examining how *binding* and *overruling* correspond to Hohfeldian forms of power [49]. As previously suggested, binding can be understood as a first-order directive that generates a constraint, whereas overruling constitutes a second-order directive that modifies or removes it. A systematic comparison between this framework and Hohfeld's theory of legal relations could shed new light on the structure of judicial authority.

Argumentation-Based Approaches

In the Introduction, we argued that modal logic provides a suitable framework for our purposes, as it allows us to highlight the relations between states (and between courts) and, through this relational perspective, to formalize key elements of our work, such as the notions of precedent and binding precedent.

Conversely, within the field of legal case-based reasoning, a formal approach that is not logic-based but argumentation-based has been widely adopted ([97] provides a brief overview of argumentation in case-based reasoning; see also, among many others, [17, 104, 83, 82, 101, 81]). Argumentation-based approaches have the advantage of emphasizing the dialectical and interactive aspects characteristic of legal (defeasible) reasoning with precedents. For example, among many other contributions, they enable a formalization of the interactions between the parties within a single judicial proceeding, for instance by allowing the identification of the parties' winning strategies based on citable precedents, achieved through emphasizing and downplaying relevant distinctions.

The body of work and research lines on argumentation-based approaches to case-based reasoning is vast, and we do not aim to provide an exhaustive overview here. To the best of our knowledge, the work most closely related to this thesis is that of Wyner and Bench-Capon [105], who extend abstract argumentation frameworks to model judicial reasoning by distinguishing between two layers: object-level arguments and meta-level arguments. In a legal context, the object-level arguments are those within legal claims, while the meta-level arguments pertain to the interaction between legal claims. Their analysis focuses especially on meta-level arguments and the attacks among them, based on the intuition that whether an argument succeeds depends on the judicial context—for example, on court hierarchy, procedural admissibility, and standards of proof.

Hence, a further strand of future research concerns the integration of our framework with argumentation-based models of legal reasoning, with particular reference to the framework proposed in [105]. The underlying intuition is that cases themselves can be viewed as arguments; this would allow notions such as binding, overruled, or *per incuriam* precedents to be modeled in terms of attacks or defeats among arguments.

Applicability to Computationally Oriented Models.

We note that the languages of the Temporal and Jurisdictional Classifier Logic and of the Conditional Temporal Classifier Logic comprise two distinct modules: (i) a notion of relevance together with a classification function, which identify the content of the

constraint—that is, how a new case should be decided and on the basis of which elements; and (ii) the set of courts, along with the hierarchical, binding, and overruling relations among them, which instead identify the normative source, the effective applicability of the constraint (i.e., whether and to what extent a decision is binding), and the priority structure among sources. In future work, it would be interesting to separate these two components—content and source of the constraint—so as to make the framework applicable beyond the strictly legal domain and more suitable for computationally oriented approaches. For example in automated processes in organizational and computational settings, the distribution of rules is a recurrent problem that concerns how decisions and constraints are spread across multiple layers—for example, policies, governance and coordination mechanisms, and local service behavior [91, 75]. Investigating whether our proposed framework may prove useful for addressing these issues is left for future work.³⁵

³⁵The author thanks Giovanni Sileno for suggesting this direction for future research.

A

Appendix Chapter1

Proposition 1.8. *Let CB a consistent case base, and s a fact situation to be decided. If $\exists k' \in CB$ s.t. $\text{forces}(k', s, o)$ then $\nexists k'' \in CB$ s.t. $\text{forces}(k'', s, \bar{o})$.*

Proof. Assume by contradiction that $\text{forces}(k', s, o)$ and $\text{forces}(k'', s, \bar{o})$. Assume $k' = (s', X', o)$ and $k'' = (s'', X'', \bar{o})$. By $\text{forces}(k', s, o)$, we know $X' \subseteq s \cap \text{Atm}_0^o$ and $s \cap \text{Atm}_0^{\bar{o}} \subseteq s' \cap \text{Atm}_0^{\bar{o}}$. Hence, $s \cap \text{Atm}_0^{\bar{o}} <_{k'} s \cap \text{Atm}_0^o$. By $\text{forces}(k'', s, \bar{o})$, we know $X'' \subseteq s \cap \text{Atm}_0^{\bar{o}}$ and $s \cap \text{Atm}_0^o \subseteq s'' \cap \text{Atm}_0^o$. Hence, $s \cap \text{Atm}_0^o <_{k''} s \cap \text{Atm}_0^{\bar{o}}$. By $s \cap \text{Atm}_0^{\bar{o}} <_{k'} s \cap \text{Atm}_0^o$ and $s \cap \text{Atm}_0^o <_{k''} s \cap \text{Atm}_0^{\bar{o}}$, contradiction because of consistency. $\square$

B

Appendix Chapter 2

Proposition 2.6.

$$\begin{aligned}
& \models_{\mathbf{CM}^{ic}} \bigwedge_{p \in \text{Atm}_0} \left(\text{Factor}(p, 1) \vee \text{Factor}(p, 0) \vee \text{Irrelevant}(p) \vee \text{Amb}(p) \right). \\
& \models_{\mathbf{CM}^{ic}} \bigwedge_{p \in \text{Atm}_0} \left(\left(\bigvee_{o \in \{0,1\}} \text{Factor}(p, o) \right) \rightarrow \neg \text{Irrelevant}(p) \wedge \neg \text{Amb}(p) \right). \\
& \models_{\mathbf{CM}^{ic}} \bigwedge_{p \in \text{Atm}_0} \left(\text{Irrelevant}(p) \rightarrow \neg \text{Factor}(p, 0) \wedge \neg \text{Factor}(p, 1) \wedge \neg \text{Amb}(p) \right). \\
& \models_{\mathbf{CM}^{ic}} \bigwedge_{p \in \text{Atm}_0} \left(\text{Amb}(p) \rightarrow \neg \text{Factor}(p, 0) \wedge \neg \text{Factor}(p, 1) \wedge \neg \text{Irrelevant}(p) \right).
\end{aligned}$$

Proof. We prove the first validity. Let (C, s) a pointed model, $C \in \mathbf{CM}^{ic}$. Let p s.t., for all $o \in \{0, 1\}$, $(C, s) \models \neg \text{Factor}(p, o)$. Then, $(C, s) \models (\Diamond \text{SCfXp}(p, o) \wedge \Diamond \text{SCfXp}(p, \bar{o})) \vee (\neg \Diamond \text{SCfXp}(p, o) \wedge \neg \Diamond \text{SCfXp}(p, \bar{o}))$. That is $(C, s) \models \text{Amb}(p) \vee \text{Irrelevant}(p)$. $\square$

Proposition 2.12. *Let CB be a consistent Horty case base and C its genuine classifier. Let Atm_0 be the set of features. Then a dichotomy $\text{Atm}_0 = \text{Plt}' \uplus \text{Dfd}'$ is admissible, if and only if for any state s ,*

1. $(C, s) \models \neg \text{Amb}(p)$ for any feature $p \in \text{Atm}_0$
2. if $(C, s) \models \text{SetFactor}(X, 1) \wedge \text{SetFactor}(Y, 0)$ for some $X, Y \subseteq \text{Atm}_0$, then $X \subseteq \text{Plt}', Y \subseteq \text{Dfd}'$.

Proof. We claim that if $p \in Plt$ then $(C, s) \models \neg \Diamond SCfXp(p, 0)$; and if $p \in Dfd$ then $(C, s) \models \neg \Diamond SCfXp(p, 1)$. And all the statements in the proposition follow as corollaries. To prove the claim, suppose towards a contradiction that $p \in Plt$ but $\exists s' \in 2^{Atm_0}$ s.t. $(C, s') \models t(1) \wedge \neg p \Rightarrow t(0)$. This violates the a fortiori reasoning: since C is genuine to CB , $f(s') = 1$ because some precedent $(s_1, X_1, 1) \in CB$ s.t. $X_1 \subseteq s' \cap Plt$ and $s_1 \cap Dfd \supseteq s' \cap Dfd$. Then a fortiori, $f(s' \setminus \{p\}) = 1$ for the same reason. But this contradicts $(C, s') \models t(1) \wedge \neg p \Rightarrow t(0)$. $\square$

Proposition 2.14. *Let $C = (S, f)$ be a classifier model. Then C is the genuine classifier model for some consistent Horty case base with an admissible dichotomy $Atm_0 = Plt \uplus Dfd$, if and only if $C \in \mathbf{CM}^{ci}$, C is pro tanto monotone, C does not have ambiguous features, $f(\emptyset) = ?$ and $\forall p \in Atm_0^o, f(\{p\}) \neq \bar{o}$.*

To outline the proof of previous proposition, we will need following Definitions B.1, B.2 and B.3, taken from [64].

Definition B.1. Let λ denote a conjunction of finitely many literals, where a literal is an atom p (positive literal) or its negation $\neg p$ (negative literal). We write $lit(\lambda), lit^+(\lambda), lit^-(\lambda)$ to mean all literals, all positive literals and all negative literals in λ respectively. Moreover let $Atm(\lambda)$ denote the atoms occurring in λ .

Definition B.2 (Implicant (Imp) and prime implicant (PImp)). We write $Imp(\lambda, c)$ to mean that λ is an *implicant* for c and define it as $Imp(\lambda, c) =_{def} \Box(\lambda \rightarrow t(c))$. We write $PImp(\lambda, c)$ to mean that λ is a *prime implicant* for c and define it as

$$PImp(\lambda, c) =_{def} \Box \left(\lambda \rightarrow (t(c) \wedge \bigwedge_{p \in Atm(\lambda)} \langle Atm(\lambda) \setminus \{p\} \rangle \neg t(c)) \right).$$

Definition B.3 (Abductive explanation (AXp)). We write $AXp(\lambda, o)$ to mean that λ *abductively explains* the decision o and define it as $AXp(\lambda, o) =_{def} \lambda \wedge PImp(\lambda, o)$.

Proof of Proposition 2.16 (Sketch.) Let C be the genuine CM for an Horty case base. We show that it satisfies all constraints. Obviously $C \in \mathbf{CM}^{ic}$, since S need equal 2^{Atm_0} ; any precedent shall not have an empty fact situation by definition; also for any $p \in Plt$, it shall never be a reason of a defendant case and similarly for a Dfd -factor. Now assume towards a contradiction that C does not satisfy CMon. Then suppose w.l.o.g., there are two sets $X, Y \subseteq Atm_0$ such that $SetFactor(X, 1) \wedge SetFactor(Y, 0)$ holds in C .

Moreover, there is a state $s_1 = X_1 \cup Y_1 \cup Z$, with $X_1 \subseteq X, Y_1 \subseteq Y, Z \subseteq \text{Atm}_0 \setminus (X \cup Y)$. However, there is another state $s_2 = X_2 \cup Y_2 \cup Z$ s.t. $X_1 \subseteq X_2 \subseteq X, Y_2 \subseteq Y_1$ but $f(s_2) = 0$.

Now recall C is the genuine CM for some case base. We know there must be $(s_3, X_3, 1), (s_4, Y_4, 0) \in CB$ s.t. $X_3 \subseteq s_1 \cap \text{Plt}, s_3 \cap \text{Dfd} \supseteq s_1 \cap \text{Dfd}$, and $Y_4 \subseteq s_2 \cap \text{Dfd}, s_4 \cap \text{Plt} \supseteq s_2 \cap \text{Plt}$, which allow the genuine classifier has $f(s_1) = 1$ and $f(s_2) = 0$ via a fortiori reasoning. Now we consider the state $s_5 = X_3 \cup Y_4 \cup Z$. The a fortiori reasoning forces s_5 to be both 1 and 0, contradicting that CB is consistent.

For the other direction let us build a CB from a C satisfying all constraints, while taking care of the dichotomy of Atm_0 and the consistency of CB . Taking advantage of Proposition 1 in [64], we simply do that for any state s and term λ , if $(C, s) \models \text{AXp}(\lambda, 1)$ with $\text{lit}^+(\lambda) = X$ and $\text{lit}^-(\lambda) = Y$, then put $(s, X, 1)$ in CB , put members of X in Plt and members of Y in Dfd . We do the similar for $\text{AXp}(\lambda, 0)$. Eventually we obtain CB and let Plt what it is, and simply $\text{Dfd} = \text{Atm}_0 \setminus \text{Plt}$ to have $\text{Plt} \uplus \text{Dfd} = \text{Atm}_0$.

For the consistency of CB , suppose not towards a contradiction, then there are two precedents $(s_1, X, 1)$ and $(s_0, Y, 0)$ s.t. $X \subseteq s_0 \cap \text{Plt} \subseteq \text{Plt}$ and $Y \subseteq s_1 \cap \text{Dfd} \subseteq \text{Dfd}$. We show the supposition impossible by using AXp and PImp . In light of the construction, we have that $X \in \text{lit}^+(\lambda_1), Y \in \text{lit}^+(\lambda_0)$ with $(C, s_i) \models \text{AXp}(\lambda_i, i)$ with $i \in \{1, 0\}$. By definition of AXp , we know also $(C, s_i) \models \text{PImp}(\lambda_i, i)$ with $i \in \{1, 0\}$. Consider the term $\text{cn}_{X \cup Y, (s_0 \cap \text{Plt}) \cup (s_1 \cap \text{Dfd})}$. Since it propositionally entails both λ_1 (the negative part of λ_1 is always “a subset of” $s_0 \cap \text{Dfd}$) and λ_2 (in the same way), by definition of PImp any state verifying it shall have outputs both 0 and 1, contradicting that f is a function.

□

Proposition 2.21. $(C, s) \models [\text{Best}(\psi)]\phi \Leftrightarrow \forall s' \in S : ((C, s') \models \psi \text{ and } \nexists s'' \in S \text{ s.t. } s' < s'' \text{ and } (C, s'') \models \psi) \Rightarrow (C, s') \models \phi$.

Proof. $\Rightarrow$) Assume $(C, s) \models [\text{Best}(\psi)]\phi$. Let $s' \in S$ s.t. $(C, s') \models \psi$ and $\nexists s'' \in S$ s.t. $s' < s'', (C, s'') \models \psi$. We want to verify that $(C, s') \models \phi$. We know $(C, s') \models \psi$. So by hypothesis $(C, s') \models \langle \leq \rangle (\psi \wedge [\leq] (\psi \rightarrow \phi))$. Let s''' s.t. $s' \leq s'''$ and $(C, s''') \models \psi \wedge [\leq] (\psi \rightarrow \phi)$. Since there is no s'' s.t. $s' < s''$ and $(C, s'') \models \psi$, this means that $s' \leq s'''$ and $s''' \leq s'$. But then, since $s''' \leq s'$, $(C, s') \models \psi \rightarrow \phi$. So, $(C, s') \models \phi$.

$\Leftarrow$) Let s s.t. $(C, s) \models \psi$. Let $S_\psi = \{\tilde{s} \mid (C, \tilde{s}) \models \psi\}$. Let $Max(S_\psi) = \{s \in S_\psi \mid s'' \leq s, \forall s'' \in S_\psi\}$ (it is not empty since $\leq$ is total on finite sets and it is non-empty since S_ψ is not empty—indeed $(C, s) \models \psi$). Let $s' \in Max(S_\psi)$. So $s \leq s'$ and $(C, s') \models \psi$. Let s''' , s.t. $s' \leq s'''$ and $(C, s''') \models \psi$. Then $s''' \in Max(S_\psi)$. So, $\nexists s_1 \in S$ s.t. $s''' < s_1$ and $(C, s_1) \models \psi$. By hypothesis $(C, s''') \models \phi$. So the thesis. $\square$

Proposition 2.25. *For all $s \in S$, $k \geq 0$, $o \in \{0, 1\}$, $(C_{or}, s) \models Cl^{j \leq k}(o)$ iff $(C_{or}^k, s) \models t(o)$.*

Proof. (sketch) Let $k = 0$. Then $(C_{or}, s) \models Cl^0(o)$, $c \in \{0, 1\}$ iff $(C, s) \models t(o)$ iff $(C_{or}^0, s) \models t(o)$.

Suppose that for all $k \geq 1$, $(C_{or}, s) \models Cl^{j \leq k}(o)$, iff $(C_{or}^k, s) \models t(o)$.

Suppose now that $(C_{or}, s) \models Cl^{j \leq k+1}(o)$. So, $(C_{or}, s) \models \neg Cl^{j \leq k}(\bar{o}) \wedge \text{SetFactor}^k(F_1^k, 1) \wedge \text{SetFactor}^k(F_0^k, 0) \wedge \bigwedge_{X_1 \subseteq F_1^k, X_0 \subseteq F_0^k, Z \not\subseteq F_1 \cup F_0} \text{cn}_{X_1 \cup X_0 \cup Z, \text{Atm}_0} \wedge [\text{Best}(\text{PtForces}^k(X_1, X_0, Z))] Cl^{j \leq k}(o)$.

By induction hypothesis, we can verify that $(C_{or}^k, s) \models \neg t(\bar{o}) \wedge \text{SetFactor}^0(F_1^0, 1) \wedge \text{SetFactor}^0(F_0^0, 0) \wedge \bigwedge_{X_1 \subseteq F_1^0, X_0 \subseteq F_0^0, Z \not\subseteq F_1 \cup F_0} \text{cn}_{X_1 \cup X_0 \cup Z, \text{Atm}_0} \wedge [\text{Best}(\text{PtForces}^0(X_1, X_0, Z))] t(o)$. Hence we have $(C_{or}^k, s) \models Cl^1(o)$. So $(C_{or}^{k+1}, s) \models t(o)$.

To prove that $(C_{or}^{k+1}, s) \models t(o)$ implies $(C_{or}, s) \models Cl^{k+1}(o)$, proceed in “reverse order”.

$\square$

B.1 Completeness

This entire section is dedicated to proving completeness; specifically, we aim to establish the following result.

Proposition 2.19 (Completeness). *OBCL is sound and complete relative to the class CM^{or} .*

To prove previous proposition, we need some intermediate definitions and results. First of all we will recall some well-known definitions.

Definition B.4. [OBCL-consistency] A set of formulas Γ is OBCL-consistent if $\Gamma \not\vdash_{\text{TJCL}} \perp$, and OBCL-inconsistent otherwise.

Definition B.5 (OBCL-MCS). A set of formulas Γ is maximal OBCL-consistent if Γ is OBCL-consistent, and any set of formulas properly containing Γ is OBCL-inconsistent. If Γ is a maximal OBCL-consistent set of formulas then we say it is a OBCL-MCS.

The following proposition specifies some usual properties of MCSs.

Proposition B.6. *Let Γ be a MCS and let $\phi, \psi \in \mathcal{L}$. Then:*

- *if $\phi, \phi \rightarrow \psi \in \Gamma$ then $\psi \in \Gamma$*
- *$\phi \in \Gamma$ or $\neg\phi \in \Gamma$;*
- *$\phi \in \Gamma$ and $\psi \in \Gamma$ iff $\phi \wedge \psi \in \Gamma$*

We recall Lindebaum's Lemma, whereby any set of formulas can be extended to MCS.

Lemma B.7 (Lindenbaum). *If Γ is a OBCL-consistent set of formulas then there is there is a Γ^+ OBCL-MCS s.t. $\Gamma \subseteq \Gamma^+$.*

Proof. standard see [18] pag. 197 □

In proving strong completeness, we will refer the following well-known result (see [18], pag 194).

Proposition B.8. *A logic Λ is strongly complete with respect to a class of structures $\mathcal{S}$ iff every Λ -consistent set of formulas is satisfiable on some $G \in \mathcal{S}$.*

To prove strong completeness, we mimic the standard construction of the canonical model. But, we actually construct a ordered classifier model for each set OBCL-consistent. Namely, we fix Θ , OBCL-consistent. We build, on the basis of MCS, a ordered classifier model, in which Θ is satisfiable. Then, the result follows from B.8. In constructing the classifier model for Θ , from the MCSs, we shall ensure that:

- A) the universality of $\Box$ is captured in the model;
- B) the state of the model are subsets of 2^{Atm_0}

The proof proceeds as follows:

- We fix a OBCL-consistent set θ . Using the Lindenbaum Lemma, we know that we can extend θ to a maximal consistent set, θ^+ .
- We fix Ψ^{Θ^+} the set of all $\Box$ and $\neg\Box$ formulas in the extension θ^+ . We prove Ψ^{Θ^+} is consistent (Prop. B.9).
- So, by Lindenbaum Lemma, we can define E , the set of all MCSs that extend Ψ^{Θ^+} (Def. B.10). By construction, all elements of E contain the same $\Box$ formulas (Proposition B.11). This will allow us to guarantee point A).
- Recall that we need to construct a classifier model, whose states are subsets of 2^{Atm_0} . So, we partition the MCSs in E into equivalence classes, placing in the same equivalence class all the MCSs that share exactly the same atoms in Atm_0 . We can then define the sets that will be later be the states of our model: these are given by the intersection of each equivalence class with Atm_0 (Def. B.12).
- We define a relation between the equivalence classes (Def. B.13). We prove that the relation is a preorder (Prop B.14). We also prove a characterization for the preorder (Proposition B.15). This characterization allow us to prove that the preorder is a total relation (Prop B.16), and will later be also useful to prove the Existential lemma (Lemma B.19).
- We finally construct the classification model (Def B.17 and Prop B.18): each state is defined from an equivalence class; the function assigns to each state the unique decision atom in the MCSs in the equivalence class (unicity and existence of the decision atom is guaranteed by axioms **AtMostValue**, **AtLeastValue**); the preorder between states is derived from the relations between the equivalence classes.
- Once the model is constructed, we prove the Existence and Truth Lemmas (Lemmas B.19 and B.20). Thus, we obtain completeness (Prop 2.19).

So, we fix a OBCL-consistent set θ . We consider a MCS Θ^+ that extends Θ (it exists for the Lindenbaum Lemma). We define $\Psi^{\Theta^+} = \{\Box\varphi \in \Theta^+\} \cup \{\neg\Box\varphi \in \Theta^+\}$.

Proposition B.9. Ψ^{Θ^+} is consistent.

Proof. $\Psi^{\Theta^+} \subseteq \Theta^+$. We know that Θ^+ is consistent. Hence the thesis. $\square$

In constructing the model, we will therefore only consider a OBCL-MCS extending Ψ^{Θ^+} . More precisely, we define the following.

Definition B.10. Let $E =_{def} \{\Gamma : \Gamma \text{ is TJCL-MCS extending } \Psi^{\Theta^+}\}$.

Recall Θ^+ is a maximal consistent set. So, by construction, all elements of E contain the same $\Box$ formulas. This will guarantee A). Namely we have the following.

Proposition B.11. $\forall \Gamma \in E: \Box \varphi \in \Gamma \Leftrightarrow \Box \varphi \in \Theta^+$

To guarantee B), we will partition the MCSs according to the atoms in Atm_0 they share. On the basis of this partition we will then define the states of the classification model (these states will be the $s_{[\Gamma]}$ introduced following definition). More precisely we have the following.

Definition B.12. For all $\Gamma, \Gamma' \in E$, let $\Gamma \equiv_{Atm_0} \Gamma'$, iff $\Gamma \cap Atm_0 = \Gamma' \cap Atm_0$. $\equiv_{Atm_0}$ is an equivalence relation on E . For all $\Gamma \in E$, $[\Gamma] = \{\Gamma' \mid \Gamma \equiv_{Atm_0} \Gamma'\}$ is the equivalence class of Γ wrt to $\equiv_{Atm_0}$. $E / \equiv$ is the quotient set of E wrt to $\equiv_{Atm_0}$. For notational conciseness, we define $s_{[\Gamma]} =_{def} \Gamma' \cap Atm_0$, for any $\Gamma' \in [\Gamma]$.

Before defining the canonical classifier model, we need some additional notions and results for the construction of the preorder. More precisely, as we mentioned before, the states of the classifier model will be the $s_{[\Gamma]}$ defined from the equivalence classes $[\Gamma]$. We will now define a preorder between the equivalence classes. From these relations we will then induce a preorder between the states $s_{[\Gamma]}$.

Definition B.13. Let, $[\Gamma], [\Delta]$ (hence, $\Gamma, \Delta \in E$):

- $[\Gamma] \preceq [\Delta]$ iff $\forall \Gamma' \in [\Gamma], \exists \Delta' \in [\Delta]: \forall \varphi \in \mathcal{L} ([\leq] \varphi \in \Gamma' \Rightarrow \varphi \in \Delta')$

Proposition B.14. $\preceq$ is a pre-order on $E / \equiv$.

Proof. • Reflexivity: trivially for all $\Gamma' \in \llbracket \Gamma \rrbracket$, if $[\leq]\varphi \in \Gamma'$ then $\varphi \in \Gamma'$ (ax. **T**_[≤T]).

Hence $\llbracket \Gamma \rrbracket \preceq \llbracket \Gamma \rrbracket$

- Transitivity : suppose $\llbracket \Gamma \rrbracket \preceq \llbracket \Delta \rrbracket$ and $\llbracket \Delta \rrbracket \preceq \llbracket Y \rrbracket$. Let $\Gamma' \in \llbracket \Gamma \rrbracket$. Suppose $[\leq]\varphi \in \Gamma'$. Then $[\leq][\leq]\varphi \in \Gamma'$ (ax. **4**_[≤T]). Then $\exists \Delta' \in \llbracket \Delta \rrbracket$ s.t. $[\leq]\varphi \in \Delta'$ by the assumption $\llbracket \Gamma \rrbracket \preceq \llbracket \Delta \rrbracket$. Hence, since $\llbracket \Delta \rrbracket \preceq \llbracket Y \rrbracket$, there is $Y' \in \llbracket Y \rrbracket$ s.t. $\varphi \in Y'$. So, $\llbracket \Gamma \rrbracket \preceq \llbracket Y \rrbracket$.

□

We prove a characterization of the preorder (Proposition B.15). This characterization allow us to prove that preorder is a total relation (Prop B.16), and will later be also useful to prove the Existential lemma (Lemma B.19).

Proposition B.15. *Let $\Gamma, \Delta \in E$. Then:*

$$i) \llbracket \Gamma \rrbracket \preceq \llbracket \Delta \rrbracket \Leftrightarrow \exists \Gamma' \in \llbracket \Gamma \rrbracket, \exists \Delta' \in \llbracket \Delta \rrbracket : \forall \varphi \in \mathcal{L} ([\leq]\varphi \in \Gamma' \Rightarrow \varphi \in \Delta')$$

Proof. We prove i).

$\Rightarrow$) trivial.

$\Leftarrow$) Suppose Γ' and Δ' satisfy the right-hand side of the equivalence. Let $\Gamma'' \in \llbracket \Gamma \rrbracket$. So, $\text{cn}_{s[\Gamma], \text{Atm}_0} \in \Gamma' \cap \Gamma''$. Suppose $[\leq]\varphi \in \Gamma''$. Then, by axiom **FunctT**, we have $\Box(\text{cn}_{s[\Gamma], \text{Atm}_0} \rightarrow [\leq]\varphi) \in \Gamma''$. By proposition B.11, $\Box(\text{cn}_{s[\Gamma], \text{Atm}_0} \rightarrow [\leq]\varphi) \in \Gamma'$. So $\text{cn}_{s[\Gamma], \text{Atm}_0} \rightarrow [\leq]\varphi \in \Gamma'$ (ax **T**_□). By **MP**, $[\leq]\varphi \in \Gamma'$. So, by hypothesis, $\varphi \in \Delta'$. Hence the thesis.

□

Proposition B.16. $\preceq$ is total on $E / \equiv$.

Proof. We prove the thesis using Proposition B.15. Let $\llbracket \Gamma \rrbracket$ and $\llbracket \Delta \rrbracket$ in $E / \equiv$. We prove that $\llbracket \Gamma \rrbracket \preceq \llbracket \Delta \rrbracket$ or $\llbracket \Delta \rrbracket \preceq \llbracket \Gamma \rrbracket$. Let $\Gamma' \in \llbracket \Gamma \rrbracket$, then $\text{cn}_{s[\Gamma], \text{Atm}_0} \in \Gamma'$. Let $\Delta' \in \llbracket \Delta \rrbracket$, then $\text{cn}_{s[\Delta], \text{Atm}_0} \in \Delta'$. By Axiom **Total**_[≤T], $\Box((\text{cn}_{s[\Gamma], \text{Atm}_0} \wedge [\leq_T]\varphi) \rightarrow \Box(\text{cn}_{s[\Delta], \text{Atm}_0} \rightarrow \varphi)) \vee \Box((\text{cn}_{s[\Delta], \text{Atm}_0} \wedge [\leq_T]\varphi) \rightarrow \Box(\text{cn}_{s[\Gamma], \text{Atm}_0} \rightarrow \varphi)) \in \Gamma'$.

- Suppose $\Box \left((cn_{s_{[\Gamma]}.Atm_0} \wedge [\leq_T] \varphi) \rightarrow \Box (cn_{s_{[\Delta]}.Atm_0} \rightarrow \varphi) \right) \in \Gamma'$. Then $(cn_{s_{[\Gamma]}.Atm_0} \wedge [\leq_T] \varphi) \rightarrow \Box (cn_{s_{[\Delta]}.Atm_0} \rightarrow \varphi) \in \Gamma'$. Let $[\leq_T] \varphi \in \Gamma'$. Since $cn_{s_{[\Gamma]}.Atm_0} \in \Gamma'$, then $\Box (cn_{s_{[\Delta]}.Atm_0} \rightarrow \varphi) \in \Gamma'$. So, by Prop. B.11, $\Box (cn_{s_{[\Delta]}.Atm_0} \rightarrow \varphi) \in \Delta'$. Then, by $\mathbf{T}_\Box$, $cn_{s_{[\Delta]}.Atm_0} \rightarrow \varphi \in \Delta'$. Since $cn_{s_{[\Delta]}.Atm_0} \in \Delta'$, then $\varphi \in \Delta'$. Hence $[\Gamma] \preceq [\Delta]$.
- Suppose $\Box \left((cn_{s_{[\Delta]}.Atm_0} \wedge [\leq_T] \varphi) \rightarrow \Box (cn_{s_{[\Gamma]}.Atm_0} \rightarrow \varphi) \right) \in \Gamma'$. Then $\Box \left((cn_{s_{[\Delta]}.Atm_0} \wedge [\leq_T] \varphi) \rightarrow \Box (cn_{s_{[\Gamma]}.Atm_0} \rightarrow \varphi) \right) \in \Delta'$. $(cn_{s_{[\Delta]}.Atm_0} \wedge [\leq_T] \varphi) \rightarrow \Box (cn_{s_{[\Gamma]}.Atm_0} \rightarrow \varphi) \in \Delta'$. Let $[\leq_T] \varphi \in \Delta'$. Since $cn_{s_{[\Delta]}.Atm_0} \in \Delta'$, then $\Box (cn_{s_{[\Gamma]}.Atm_0} \rightarrow \varphi) \in \Delta'$. So, by Prop. B.11 $\Box (cn_{s_{[\Gamma]}.Atm_0} \rightarrow \varphi) \in \Gamma'$. Then, by $\mathbf{T}_\Box$, $cn_{s_{[\Gamma]}.Atm_0} \rightarrow \varphi \in \Gamma'$. Since $cn_{s_{[\Gamma]}.Atm_0} \in \Delta'$, then $\varphi \in \Gamma'$. Hence $[\Delta] \preceq [\Gamma]$.

□

Finally, we can define the classifier model associated to Θ and Θ^+ .

Definition B.17 (Model associated to Θ and Θ^+). The ordered classifier model associated to Θ and Θ^+ , is the tuple $\mathcal{C} = (S^c, f^c, \leq^c, \mathcal{R}^c, Jur^c)$, where

- $S^c = \{s_{[\Gamma]} \mid \Gamma \in E\} \subseteq 2^{Atm_0}$
- $f^c : S^c \rightarrow Val$, $f(s_{[\Gamma]}) = o$ iff for all $\Gamma' \in [\Gamma]$, $t(o) \in \Gamma'$.
- $s_{[\Gamma]} \leq^c s_{[\Delta]}$ iff $[\Gamma] \preceq [\Delta]$

Proposition B.18. $\mathcal{C}$ is indeed a ordered classifier model.

Proof. • By definition, for all $\Gamma \in E$, $s_{[\Gamma]} \subseteq 2^{Atm_0}$.

- f^c is a well defined function. Indeed, let $\Gamma', \Gamma'' \in s_{[\Gamma]}$. Then $cn_{s_{[\Gamma]}.Atm_0} \in \Gamma' \cap \Gamma''$. By axiom **AtLeastValue**, we know that there are $t(o) \in \Gamma'$ and $t(o') \in \Gamma''$. By axiom **Funct**, we have that $\Box (cn_{s_{[\Gamma]}.Atm_0} \rightarrow t(o)) \in \Gamma'$. Hence, by Proposition B.11, we have that $\Box (cn_{s_{[\Gamma]}.Atm_0} \rightarrow t(o)) \in \Gamma''$. Hence, $cn_{s_{[\Gamma]}.Atm_0} \rightarrow t(o) \in \Gamma''$. By **MP**, $t(o) \in \Gamma''$. So, by ax. **AtMostValue**, it must be $t(o') = t(o)$.
- $\leq^c$ is a total preorder. It follows directly from the fact that $\preceq$ is a preorder (Prop. B.14) and it is total (Prop. B.16).

□

We can finally prove the existential and truth lemma, and so completeness.

Proposition B.19 (Existence Lemma). *Let $\Gamma \in E$. Then*

- i) *If $\neg \Box \varphi \in \Gamma$, then there is $\Delta \in E$ s.t $\neg \varphi \in \Delta$*
- ii) *If $\neg[\leq] \varphi \in \Gamma$ then there is $\Delta \in E$ s.t $s_{[\Gamma]} \leq^c s_{[\Delta]}$ and $\neg \varphi \in \Delta$*

Proof. i) Let $\Delta^- = \{\psi \mid \Box \psi \in \Theta^+\} \cup \{\neg \psi \mid \neg \Box \psi \in \Theta^+\}$. Then $\neg \varphi \in \Delta^-$.

- Δ^- is consistent. Indeed, suppose it is not. Then, there are $\Box \psi_1, \dots, \Box \psi_n \in \Theta^+$ and $\neg \Box \tau_1, \dots, \neg \Box \tau_m \in \Theta^+$ s.t $\vdash \neg(\bigwedge_i \psi_i \wedge \bigwedge_j \neg \tau_j)$. Namely, $\vdash \neg \bigwedge_i \psi_i \vee \bigvee_j \tau_j$. Then, $\vdash \bigwedge_i \psi_i \rightarrow \bigvee_j \tau_j$. From **Nec** $_{\Box}$, and **K** $_{\Box}$, we have $\vdash \bigwedge_i \Box \psi_i \rightarrow \bigvee_j \Box \tau_j$. Since $\bigwedge_i \Box \psi_i \in \Gamma$, then, by **MP**, $\bigvee_j \Box \tau_j \in \Gamma$. Contradiction, by Prop. B.11.
- We can extend Δ^- to a OBCL-MCS, Δ . Then, $\neg \varphi \in \Delta$, by construction of Δ^- .
- $\Delta \in E$. Namely, $\Box \psi \in \Delta$ iff $\Box \psi \in \Theta^+$. Indeed, if $\Box \psi \in \Theta^+$ then $\Box \Box \psi \in \Theta^+$, so $\Box \psi \in \Delta$. Viceversa, suppose by contraposition that $\neg \Box \psi \in \Theta^+$. So, $\neg \psi \in \Delta$. Hence $\neg \Box \psi \in \Delta$ (by ax. **T** $_{\Box}$).

ii) Let $\Delta^- = \{\neg \varphi\} \cup \{\psi \mid \Box \psi \in \Theta^+\} \cup \{\psi \mid [\leq] \psi \in \Gamma\} \cup \{\neg \Box \psi \in \Theta^+\}$.

- Δ^- is consistent. Suppose by contradiction it is not. Namely, there are $\Box \psi_1, \dots, \Box \psi_n \in \Theta^+$, $[\leq] \xi_1, \dots, [\leq] \xi_m \in \Gamma$, $\neg \Box \tau_1, \dots, \neg \Box \tau_l \in \Theta^+$, s.t $\vdash \neg(\bigwedge_i \psi_i \wedge \bigwedge_j \xi_j \wedge \bigwedge_k \neg \tau_k \wedge \neg \varphi)$. So, $\vdash \neg(\bigwedge_i \psi_i \wedge \bigwedge_j \xi_j) \vee \neg(\bigwedge_k \neg \tau_k \wedge \neg \varphi)$. Then, $\vdash \neg(\bigwedge_i \psi_i \wedge \bigwedge_j \xi_j) \vee (\bigvee_k \Box \tau_k \vee \varphi)$. Hence, $\vdash (\bigwedge_i \psi_i \wedge \bigwedge_j \xi_j) \rightarrow (\bigvee_k \Box \tau_k \vee \varphi)$. By **Nec** $_{[\leq]}$ and **K** $_{[\leq]}$, $\vdash [\leq](\bigwedge_i \psi_i \wedge \bigwedge_j \xi_j) \rightarrow [\leq](\bigvee_k \Box \tau_k \vee \varphi)$. Hence, $\vdash (\bigwedge_i [\leq] \psi_i \wedge \bigwedge_j [\leq] \xi_j) \rightarrow \bigvee_k [\leq] \Box \tau_k \vee [\leq] \varphi$. Since $\bigwedge_i \Box \psi_i \in \Gamma$, then $\bigwedge_i [\leq] \psi_i \in \Gamma$ (by, **MIX** $_{\Box, [\leq]}$). Moreover, $\bigwedge_j [\leq] \xi_j \in \Gamma$. So, by **MP**, we have $\bigvee_k [\leq] \Box \tau_k \vee [\leq] \varphi \in \Gamma$. By hypothesis, $[\leq] \varphi \notin \Gamma$. Moreover since $\neg \Box \tau_k \in \Gamma$, we also know (**T** $_{[\leq]}$), that $\neg[\leq] \Box \tau_k \in \Gamma$. Hence contradiction. So Γ^- is consistent.
- Then, by Lindebaum Lemma we can extend Δ^- , to a MCS Δ . Δ is in E by construction. Namely, $\Box \psi \in \Delta$ iff $\Box \psi \in \Theta^+$. Indeed, if $\Box \psi \in \Theta^+$ then

$\Box\Box\psi \in \Theta^+$, so $\Box\psi \in \Delta$. Viceversa, suppose that $\neg\Box\psi \in \Theta^+$, then by definition of Δ^- , $\neg\Box\psi \in \Delta$.

- Moreover, for all ψ , we know that if $[\leq]\psi \in \Gamma$ then $\psi \in \Delta$. Hence, by proposition B.15, $[\Gamma] \preceq [\Delta]$. So, $s_{[\Gamma]} \leq^c s_{[\Delta]}$.

□

Lemma B.20 (Truth Lemma). *Let $\mathcal{C}$ the ordered classifier model associated to Θ and Θ^+ . Then for all $\Gamma \in E$, for all $\varphi \in \mathcal{L}$*

$$(\mathcal{C}, s_{[\Gamma]}) \models \varphi \Leftrightarrow \varphi \in \Gamma$$

Proof. By induction on φ . Base case

- $(\mathcal{C}, s_{[\Gamma]}) \models p \Leftrightarrow p \in s_{[\Gamma]} \Leftrightarrow \forall \Gamma' \in [\Gamma], p \in \Gamma' \cap \text{Atm}_0 \Leftrightarrow p \in \Gamma$.
- $(\mathcal{C}, s_{[\Gamma]}) \models \mathbf{t}(o) \Leftrightarrow f^c(s_{[\Gamma]}) = o \Leftrightarrow \forall \Gamma' \in [\Gamma], \mathbf{t}(o) \in \Gamma' \cap \text{Atm}_0 \Leftrightarrow \mathbf{t}(o) \in \Gamma$.

Inductive Hypothesis: Assume the thesis holds for φ and ψ .

- $(\mathcal{C}, s_{[\Gamma]}) \models \neg\varphi \Leftrightarrow (\mathcal{C}, s_{[\Gamma]}) \not\models \varphi \Leftrightarrow (\text{inductive hyp.}) \varphi \notin \Gamma$
- $(\mathcal{C}, s_{[\Gamma]}) \models \varphi \wedge \psi \Leftrightarrow (\mathcal{C}, s_{[\Gamma]}) \models \varphi \text{ and } (\mathcal{C}, s_{[\Gamma]}) \models \psi \Leftrightarrow (\text{inductive hyp.}) \varphi \in \Gamma \text{ and } \psi \in \Gamma \Leftrightarrow \varphi \wedge \psi \in \Gamma$.
- Suppose $\neg\Box\varphi \in \Gamma$. Then, by Existence Lemma, there is $\Delta \in E$ s.t. $\neg\varphi \in \Delta$. Hence (ind. hypoth) $(\mathcal{C}, s_{[\Delta]}) \not\models \varphi$. So, $(\mathcal{C}, s_{[\Gamma]}) \not\models \Box\varphi$. Viceversa, suppose that $\Box\varphi \in \Gamma$. Then, by definition of E (see prop. B.11), for all $\Delta \in E$, $\Box\varphi \in \Delta$. Hence (from ax. $\mathbf{T}_\Box$) for all $\Delta \in E$, $\varphi \in \Delta$. Hence, by ind. hypoth, for all $\Delta \in E$, $(\mathcal{C}, s_{[\Delta]}) \models \varphi$. So $(\mathcal{C}, s_{[\Gamma]}) \models \Box\varphi$.
- Suppose $\neg[\leq]\varphi \in \Gamma$. Then, by Existence Lemma, there is $\Delta \in E$ s.t. $\neg\varphi \in \Delta$ and $s_{[\Gamma]} \leq^c s_{[\Delta]}$. Hence (ind. hypoth) $(\mathcal{C}, s_{[\Delta]}) \not\models \varphi$. So $(\mathcal{C}, s_{[\Gamma]}) \not\models [\leq]\varphi$. Viceversa, suppose that $[\leq]\varphi \in \Gamma$. Let $\Delta \in E$, s.t. $s_{[\Gamma]} \leq^c s_{[\Delta]}$. Hence, by definition of $\leq^c$, there is $\Delta' \in [\Delta]$ s.t. $\varphi \in \Delta'$. Hence, by inductive hypothesis $(\mathcal{C}, s_{[\Delta']}) \models \varphi$. Since $\Delta' \in [\Delta]$, then $s_{[\Delta']} = s_{[\Delta]}$. Hence, $(\mathcal{C}, s_{[\Delta]}) \models \varphi$. So, $(\mathcal{C}, s_{[\Gamma]}) \models [\leq]\varphi$.

□

Finally, we can prove completeness.

Proposition 2.19 (Completeness). *OBCL is sound and complete relative to the class $\mathbf{CM}^{\text{or}}$.*

Proof. Let Θ be a OBCL- consistent set of formulas. Let Θ^+ a MCS s.t. $\Theta \subseteq \Theta^+$. Define $\mathcal{C}$ the ordered classifier model associated to Θ and Θ^+ . Notice that $\Theta^+ \in E$. Then by Truth lemma, $(\mathcal{C}, s_{\llbracket \Theta^+ \rrbracket}) \models \Theta$. Hence, by Proposition B.8, the thesis. □

We conclude by noting that we have chosen this line of proof because it will serve as a useful guideline for the completeness proofs presented in the next chapter.

C

Appendix Chapter 3

Proposition 3.5 (States Hierarchy Properties). *Given $s, s', s'' \in S$:*

- (irreflexive) $s \not\prec s$
- (transitive) *if $s \prec s'$ and $s' \prec s''$, then $s \prec s''$*
- (tree-like) *if $s \prec s'$ and $s \prec s''$, then $s' \prec s''$ or $s'' \prec s'$ or $s'' \cong s'$.*

Proof. We prove only the tree-like property.

If $s \prec s'$ and $s \prec s''$, then $c' \mathcal{H} c$ and $c'' \mathcal{H} c$, where $c \in s \cap \text{Courts}$, $c' \in s' \cap \text{Courts}$, $c'' \in s'' \cap \text{Courts}$. $\mathcal{H}$ is tree like. So $c'' \mathcal{H} c'$ or $c' \mathcal{H} c''$ or $c' = c''$. Then, $s'' \prec s'$ or $s' \prec s''$ or $s'' \cong s'$. $\square$

Proposition 3.10 (Future Repr.). $(C, s) \models \hat{F}_n \varphi$ iff $(C, s) \models n$ and $\exists s' \in S$ s.t. $s <_T s'$ and $(C, s') \models \varphi$.

Proof. $\Rightarrow$) Suppose $(C, s) \models \hat{F}_n \varphi$. We want to prove that $(C, s) \models n$ and there exists s' s.t. $s <_T s'$ and $(C, s') \models \varphi$.

- 1) By hypothesis, $(C, s) \models n$.
- 2) By hypothesis, $(C, s) \models \langle \leq_T \rangle (\neg \langle \leq_T \rangle n \wedge \varphi)$. Hence there is s' s.t. $s \leq_T s'$ and $(C, s') \models \neg \langle \leq_T \rangle n \wedge \varphi$.
- 3) By previous point, $(C, s') \models \varphi$.
- 4) We want to prove that $s <_T s'$.
 - Since $(C, s') \models \neg \langle \leq_T \rangle n$, it is the case that, for all $s'' \in S$, if $s' \leq_T s''$ then $(C, s'') \models \neg n$.

- We know, by 1), $(C, s) \models n$. So, by previous point, $s' \not\leq_T s$.
- Hence $s <_T s'$.

5) The thesis can be inferred from 1), 3) and 4).

$\Leftarrow$) Suppose $(C, s) \models n$ and exists s' s.t. $s <_T s'$ (#1) and $(C, s') \models \varphi$. We have to prove that $(C, s) \models n \wedge \langle \leq_T \rangle (\neg \langle \leq_T \rangle n \wedge \varphi)$.

6) By hypothesis we know that $(C, s) \models n$.

7) We will prove that $(C, s') \models \neg \langle \leq_T \rangle n \wedge \varphi$.

- By hypothesis, we know that $(C, s') \models \varphi$.
- Suppose by contradiction that $(C, s') \models \langle \leq_T \rangle n$. Hence, there is $s'' \in S$ s.t. $s' \leq_T s''$ (#2) and $(C, s'') \models n$. But $s'' \cap s \cap \text{Names} \neq \emptyset$. Hence, by definition of classifier models, it follows that $s = s''$. By (#2), $s' \leq_T s$, contradicting (#1).

8) Since $s <_T s'$, it is $s \leq_T s'$. By (7), $(C, s) \models \langle \leq_T \rangle (\neg \langle \leq_T \rangle n \wedge \varphi)$.

9) By previous and point 6) we have the thesis

□

Proposition 3.11 (Past Repr.). $(C, s) \models \hat{\mathbb{P}}_{n,m} \varphi$ iff $(C, s) \models n$ and $\exists s' \in S$ s.t. $(C, s') \models m \wedge \varphi$ and $s' <_T s$.

Proof. $\Rightarrow$) Suppose $(C, s) \models \hat{\mathbb{P}}_{n,m} \varphi$. We want to prove that $(C, s) \models n$ and there exists $s' \in S$ s.t. $(C, s') \models m$, $s' <_T s$, and $(C, s') \models \varphi$.

1) By hypothesis, $(C, s) \models n$.

2) By hypothesis there is $s' \in S$, s.t. $(C, s') \models \varphi \wedge \hat{\mathbb{F}}_m n$.

3) By $(C, s') \models \hat{\mathbb{F}}_m n$, Prop. 3.10 and point 2),

- $(C, s') \models m$ and
- there is s'' s.t. $s' <_T s''$ and $(C, s'') \models n$.

4) $s = s''$. Indeed, by 1) and 3), it follows $s \cap s'' \cap \text{Names} \neq \emptyset$. So, by definition of classifier models (Def. 3.3), $s = s''$.

5) By 3) and 4), $s' <_T s$

6) By 1), 3) and 5), the thesis.

$\Leftarrow$) Suppose $(C, s) \models n$ and there exists s' s.t. $(C, s') \models m$, $s' <_T s$ and $(C, s') \models \varphi$. We want to prove that $(C, s) \models n \wedge \diamond (\varphi \wedge \hat{\mathbb{F}}_m n)$.

7) $(C, s) \models n$, by hypothesis.

8) $(C, s') \models \varphi$ by hypothesis.

9) By Prop. 3.10 and by hypothesis, we can prove that $(C, s') \models \hat{F}_m n$.

- $(C, s') \models m$ and
- there is a $s \in S$ s.t. $s' <_T s$ and $(C, s) \models n$.

10) The thesis follows from 7), 8), and 9).

□

Proposition 3.18 (Past-Relevance Intersection Repr.). $(C, s) \models (\hat{P} \cap R^\exists)_{n,m} \varphi$ iff $(C, s) \models n$ and there is $s' \in S$ s.t. $(C, s') \models m$, $s' <_T s$, $s' \in \mathcal{R}(s)$ and $(C, s') \models \varphi$.

Proof. $\Rightarrow$) Suppose $(C, s) \models (\hat{P} \cap R^\exists)_{n,m} \varphi$. Then, we have $(C, s) \models \hat{P}_{n,m} \varphi \wedge R^\exists(\varphi \wedge m)$. So, by Prop. 3.11 and Def. 4.5,

- $(C, s) \models n$ and there is $s' \in S$ s.t. $s' <_T s$ and $(C, s') \models m \wedge \varphi$, as well as
- there is $s'' \in S$ s.t. $s'' \in \mathcal{R}(s)$ and $(C, s'') \models m \wedge \varphi$.

But then $s' = s''$ (since they have the same name m). Hence, the thesis.

$\Leftarrow$) Suppose $(C, s) \models n$. Let $s' \in S$ s.t. $(C, s') \models m$, $s' <_T s$, and $s' \in \mathcal{R}(s)$ and $(C, s') \models \varphi$. Then, by Prop. 3.11 and Def. 4.5, $(C, s') \models \hat{P}_{n,m} \varphi \wedge R^\exists(m \wedge \varphi)$. Hence $(C, s) \models (\hat{P} \cap R^\exists)_{n,m} \varphi$

□

Proposition 3.20 (Precedent Repr.). $(C, s) \models \text{Supporting}_{n,m} \varphi$ iff $(C, s) \models n$ and exists $s' \in S$ s.t. $\Pi(s', s, f(s'))$ and $(C, s') \models m \wedge \varphi$.

Proof. Let $s \in S$ and $o \in \{0, 1\}$. Notice that $\text{Supporting}_{n,m} \varphi$ is $(\hat{P} \cap R^\exists)_{n,m}(\neg t(?) \wedge \varphi)$.

$\Rightarrow$) Suppose $(C, s) \models \text{Supporting}_{n,m}(\neg t(?) \wedge \varphi)$. Then, by Prop. 3.18 we can verify that $(C, s) \models n$ and there is $s' \in S$ s.t. $s' <_T s$, $s' \in \mathcal{R}(s)$ and $(C, s') \models m \wedge \neg t(?) \wedge \varphi$. Hence, $f(s') \neq ?$ and so $f(s') = o \in \{0, 1\}$. From $f(s') = o \in \{0, 1\}$, $s' <_T s$, and $s' \in \mathcal{R}(s)$, by Def. 3.19, we know that $\Pi(s', s, o)$. The thesis holds.

$\Leftarrow$) Suppose $(C, s) \models n$ and there is a $s' \in S$ s.t. $\Pi(s', s, o)$ and $(C, s') \models m \wedge \varphi$. From $\Pi(s', s, o)$ and Def. 3.19, it follows:

- $f(s') \neq ?$, $s' \in \mathcal{R}(s)$, and $s' <_T s$.

So, $(C, s') \models \neg t(?)$. Additionally, from $s' <_T s$, $s' \in \mathcal{R}(s)$, $(C, s) \models n$, $(C, s') \models m \wedge \varphi$, and Prop. 3.18, it is $(C, s) \models (\hat{P} \cap R^\exists)_{n,m}(\neg t(?) \wedge \varphi)$. $\square$

Proposition 3.23 (Pot. bind. Repr.). $(C, s) \models \text{PBinding}_{m,n}\varphi$ iff $(C, s) \models n$ and exists $s' \in S$ s.t. $\beta(s', s, f(s'))$ and $(C, s') \models m \wedge \varphi$.

Proof. $\Rightarrow$) Suppose $(C, s) \models \text{PBinding}_{n,m}\varphi$. Let $c_i, c_j \in \text{Courts}$ s.t. $(C, s) \models B(c_i, c_j) \wedge c_j \wedge \text{Supporting}_{n,m}(c_i \wedge \varphi)$.

- 1) $c_i \mathcal{B} c_j$, since $(C, s) \models B(c_i, c_j)$
- 2) $\Pi(s', s, f(s'))$ (#1) and $(C, s') \models c_i \wedge \varphi \wedge m$ (#2). Indeed, by $(C, s) \models \text{Supporting}_{n,m}(c_i \wedge \varphi)$ and Prop. 3.20, it follows that $(C, s) \models n$ and there is $s' \in S$ s.t. $\Pi(s', s, f(s'))$ and $(C, s') \models c_i \wedge \varphi \wedge m$.
- 3) $\beta(s', s, f(s'))$, by 1) and (#1)
- 4) By 1) and (#2), the thesis.

$\Leftarrow$) Suppose $(C, s) \models n$ and exists $s' \in S$ s.t. $\mathcal{B}(s', s, f(s'))$ (#3) and $(C, s') \models m \wedge \varphi$ (#4).

- 5) By Def. 3.21, from $\mathcal{B}(s', s, f(s'))$ it follows $\Pi(s', s, f(s'))$ (#5), $f(s') = o$ (#6), $c_i \mathcal{B} c_j$ (#7), where $c_i \in s'$ (#8), $c_j \in s$ (#9).
- 6) By (#5), (#6), (#4), (#8) and Prop. 3.20, $(C, s) \models \text{Supporting}_{n,m}(c_i \wedge \varphi)$
- 7) By (#7) and $c_j \in s$, we know $(C, s) \models B(c_i, c_j) \wedge c_j$.
- 8) By 6 and 7), the thesis.

$\square$

Proposition 3.24. Let $o \in \{0, 1\}$. $(C, s) \models \text{PBinding}_{m,n}(t(o) \wedge \varphi)$ iff $(C, s) \models n$ and exists $s' \in S$, s.t. $\beta(s', s, o)$ and $(C, s') \models m \wedge \varphi$.

Proof. $\Leftarrow$) Suppose $(C, s) \models \text{PBinding}_{n,m}(t(o) \wedge \varphi)$. Then by Prop. 3.23, exists $s' \in S$ s.t. $\mathcal{B}(s', s, f(s'))$ and $(C, s') \models t(o) \wedge \varphi \wedge m$. Since $(C, s') \models t(o)$, then $f(s') = o$. So, the thesis.

$\Rightarrow$) Suppose $(C, s) \models n$ and exists $s' \in S$, s.t. $\beta(s', s, o)$ and $(C, s') \models m \wedge \varphi$. Since $\beta(s', s, o)$, then $f(s') = o$. So $(C, s') \models o$. Then by 3.23, $(C, s) \models \text{PBinding}_{n,m}(t(o) \wedge \varphi)$.

$\square$

Proposition 3.25. *If $s' \in \beta_s$ then $s \cong s'$ or $s \prec s'$.*

Proof. Suppose $s' \in \beta_s$. So, there is $o \in \{1, 0\}$ such that $\beta(s', s, o)$, which means $\Pi(s', s, o), c' \mathcal{B} c$. According to (SD), from $c' \mathcal{B} c$, it follows that $(c, c') \in \mathcal{H} \cup \mathcal{I}$.

- If $c' \mathcal{H} c$, obviously we have $s \prec s'$ by Def. 3.4;
- If $c = c'$, by Def. 3.4 again, we have $s \cong s'$.

□

Proposition 3.26. *If $\Pi(s', s, f(s'))$ and $s \prec s'$, then $s' \in \beta_s$.*

Proof. Suppose $\Pi(s', s, f(s'))$ (#1) and $s \prec s'$ (#2). Let $c \in s \cap \text{Courts}$ and $c' \in s' \cap \text{Courts}$. By (#2), $c' \mathcal{H} c$. By (#2) and SD, $c' \mathcal{B} c$ (#3). Hence, by (#1), (#3) and Def. 3.22 the thesis.

□

Proposition 3.29 (Overruling Power Repr.). *Let $c \in s \cap \text{Courts}$. So, $(C, s) \models \text{PwOver}(c_i)$ iff $O(c, c_i)$.*

Proof. Let $c \in \text{Courts} \cap s$.

$$\begin{aligned}
 & (C, s) \models \text{PwOver}(c_i) \\
 \Leftrightarrow & (C, s) \models \bigvee_{c \neq c_i} (c \wedge \text{H}(c, c_i)) \vee (c_i \wedge \neg \text{B}(c_i, c_i)) \\
 \Leftrightarrow & (C, s) \models \text{H}(c, c_i) \text{ or } (C, s) \models c_i \wedge \neg \text{B}(c_i, c_i) \\
 \Leftrightarrow & c \mathcal{H} c_i \text{ or } (\text{not } - c_i \mathcal{B} c_i \text{ and } c = c_i) \\
 \Leftrightarrow & O(c, c_i).
 \end{aligned}$$

□

Proposition 3.32 (Pot. Over. Repr.). *$(C, s) \models \text{POverruling}_{n,m} \phi$ iff $(C, s) \models n$ and exists $s' \in S$ s.t. $O(s', s)$ and $(C, s') \models m \wedge \phi$.*

Proof. $\Rightarrow$) Suppose $(C, s) \models \text{POverruling}_{n,m} \phi$. Let $o \in \{0, 1\}$, $c \in \text{Courts}$ s.t. $(C, s) \models n \wedge c$.

- By hypothesis there is $s' \in S$ s.t. $(C, s') \models \varphi \wedge t(\bar{o}) \wedge \text{PwOver}(c) \wedge \text{Supporting}_{m,n}(t(o))$.
- So $f(s') = \bar{o}$ (#1) and $(C, s') \models \varphi$ (#2).
- Let $c' \in s' \cap \text{Courts}$ (it exists by def of classifier model).
- Since $(C, s') \models \text{PwOver}(c)$, by Prop. 3.29, we know $O(c', c)$ (#3).
- We know $(C, s') \models \text{Supporting}_{m,n}(t(o))$. Recall $n \in s$. By Prop. 3.20, we retrieve $(C, s') \models m$ (#4) and $\Pi(s, s', o)$ (#5).
- By Def. 3.30, (#1), (#3), (#5), we have $O(s', s)$.
- By the previous point, and (#2) and (#4), we have the thesis.

$\Leftarrow$) Assume $(C, s) \models n$ (#6), $O(s', s)$, and $(C, s') \models m \wedge \varphi$ (#7).

- Since $O(s', s)$, by Def. 3.30, there is $o \in \{0, 1\}$ s.t. $f(s') = \bar{o}$, $\Pi(s, s', o)$, and $O(c', c)$, where $c' \in s'$ (#8).
- By (#8) and Def. 3.16, we know $f(s) = o$. Let $c \in s \cap \text{Courts}$. Then $(C, s) \models t(o) \wedge c$ (#9).
- By (#8) and Prop. 3.29, we obtain that $(C, s') \models \text{PwOver}(c)$ (#10).
- By (#7), (#8), and (#10), it follows $(C, s') \models \varphi \wedge t(\bar{o}) \wedge \text{PwOver}(c)$.
- By (#6), (#7), (#8) and Prop. 3.20 we have $(C, s') \models \text{Supporting}_{m,n}(t(o))$.
- By two previous points and (#6) and $(C, s) \models c$ (by (#9)), we have the thesis.

□

Proposition 3.33. *If $s' \in \omega_s$ then $s \prec s'$ or $s \cong s'$.*

Proof. Let $s' \in \omega_s$. So $O(s', s)$. Hence, $\Pi(s, s', o)$, $f(s') = \bar{o}$, and $O(c', c)$. By $O(c', c)$, either $c' \mathcal{H} c$ or $c' = c$. By Def. 3.4, we then have $s' \prec s$ or $s \cong s'$.

□

Proposition 3.34. *The following statements hold:*

- 1) *Let $c \in s$ and $c' \in s'$. $s' \in \beta_s \Rightarrow \text{not } O(c, c')$.*
- 2) *$s' \in \beta_s \Rightarrow s \notin \omega_{s'}$.*
- 3) *Let $c \in s$ and $c' \in s'$. $s' \in \omega_s \Rightarrow \text{not } c \mathcal{B} c'$.*
- 4) *$s' \in \omega_s \Rightarrow s \notin \beta_{s'}$.*

Proof. We prove 1). 2) is a direct consequence of 1). The proof for 3) and 4) is similar. $s' \in \beta_s$. Then, by Def. 3.21, there is a $o \in \{0, 1\}$ such that $\beta(s', s, o)$. We want to show it is not the case that $O(c, c')$. Suppose by contradiction that $O(c, c')$

- From $\beta(s', s, o)$ and Prop. 3.25, we know $c\mathcal{B}c$ (#1) and so (#2): $c'\mathcal{H}c$ or $c' = c$
- By $O(c, c')$, and Def. 3.27 we know (#3): either $c'\mathcal{H}c'$ or $(c' = c$ and $\text{not } -c\mathcal{B}c)$
- By (#1), (#2), (#3) contradiction.

□

Proposition 3.37 (Against/According Repr.). *The following statements hold:*

- I) $(C, s) \models \text{Against}_{n,m}\varphi$ iff $(C, s) \models n$ and $\exists s'$ s.t. $\text{Against}(s, s')$ and $(C, s') \models m \wedge \varphi$;
- II) $(C, s) \models \text{According}_{n,m}\varphi$ iff $(C, s) \models n$ and $\exists s'$ s.t. $\text{According}(s, s')$ and $(C, s') \models m \wedge \varphi$.

Proof. We prove 1). The proof for 2) is similar.

$\Rightarrow$) Assume $(C, s) \models \text{Against}_{n,m}\varphi$.

- 1) By hypothesis, $(C, s) \models t(o)$ with $o \in \{0, 1\}$. So $f(s) = o$ (#1).
- 2) By hypothesis, $(C, s) \models \text{PBinding}_{n,m}(\varphi \wedge t(\bar{o}))$.
- 3) By Prop. 3.24, $(C, s) \models n$ and $\exists s' \in S$ s.t. $\beta(s', s, \bar{o})$ (#2) and $(C, s') \models m \wedge \varphi$.
- 4) By (#1) and (#2), we have the thesis.

$\Leftarrow$) Let $(C, s) \models n$ and $\exists s'$ s.t. $\text{Against}(s, s')$ and $(C, s') \models m \wedge \varphi$ (#3).

- 1) By $\text{Against}(s, s')$, it implies that $f(s) = o \in \{0, 1\}$ (#4) and $\beta(s', s, \bar{o})$ (#5)
- 2) By (#3), (#5) and Prop. 3.24, $(C, s) \models \text{PBinding}_{n,m}(\varphi \wedge t(\bar{o}))$
- 3) By (#4) and point 2), we have the thesis.

□

We state here Propositions C.1, C.2 and C.3, which will be useful for proof of Proposition 3.39.

Proposition C.1. *For $s \in S$, $s \notin \beta_s$ and $s \notin \omega_s$.*

Proof. Suppose by contradiction that $s \in \beta_s$. Then, by def 3.21, we have $\Pi(s, s, f(s))$. So, by Def. 3.16, $s <_T s$. Contradiction. If $s \in \omega_s$, then by Def. 3.30, we have $\Pi(s, s, f(s))$. So by Def. 3.16 $s <_T s$. Contradiction. □

Proposition C.2. *Given $s \in S$,*

- I) *if $s' \in \beta_s \cup \omega_s$, then $s' \in S_d$.*
- II) *If exists $s' \in S$ s.t. $\text{Against}(s, s')$, then $s \in s' \in S_d$ and $s' \in S_d$.*

III) If exists $s' \in S$ s.t. According (s, s') , then $s \in s' \in S_d$ and $s' \in S_d$.

Proof. I) By Def. 3.21, 3.30, if $s' \in \beta_s \cup \omega_s$, then $f(s') \neq ?$. Hence $s' \in S_d$.

II) and III) follow from I) and Def. 3.35. □

Proposition C.3. $V_s \subseteq S_d$

Proof. By Def. 3.38, we know $s \in V_s$ and $f(s) \neq ?$, hence $s \in S_d$. For $s' \in V_s$, with $s' \neq s$, we have $s' \in \beta_s$ or $s' \in \omega_s$. Then by Prop. C.2, $s' \in S_d$. □

Proposition 3.39. G_s is a finite directed acyclic graph.

sketch. G_s is directed by definition. First, we will prove that G_s is acyclic.

- Suppose by contradiction that there is directed cycle, namely a sequence of states $s_1, s_2, \dots, s_n$ s.t. $(s_i, s_{i+1}) \in E$ and $s_1 = s_n$ (#1).
- Notice that if $(s_i, s_{i+1}) \in E$ then $s_i \prec s_{i+1}$ or $s_i \cong s_{i+1}$ (#2). Indeed, since $(s_i, s_{i+1}) \in E$, then $s_{i+1} \in \omega_{s_i}$ or $s_{i+1} \in \beta_{s_i}$ (#3), then, by Prop. 3.25 and Prop. 3.33, it follows that $s_i \prec s_{i+1}$ or $s_i \cong s_{i+1}$.
- Suppose $s_i \prec s_{i+1}$, for some $i \in \{1, \dots, n-1\}$.
 - Then by (#2) and (#1), we obtain $s_{n-1} \prec s_1$ (#4).
 - By the transitivity of $\prec$ and $\cong$, we also know that either $s_1 \prec s_{n-1}$ or $s_1 \cong s_{n-1}$. However this leads to a contradiction, because of (#4).
- Suppose instead $s_i \cong s_{i+1}$ for all $i \in \{1, \dots, n-1\}$. Then, there is $c \in s_i \cap \text{Courts}$, for all $i \in \{1, \dots, n\}$ (#5). Since $(s_{n-1}, s_1) \in E$, we have two possibilities $s_1 \in \omega_{s_{n-1}}$ or $s_1 \in \beta_{s_{n-1}}$.
 - If $s_1 \in \omega_{s_{n-1}}$, then by Prop. 3.34, *not* $-c\mathcal{B}c$ (#6). Since $(s_1, s_2) \in E$, it must be $s_2 \in \omega_{s_1}$ —indeed, we know (#3) and it cannot be $s_2 \in \beta_{s_1}$, because $c \in s_1, c \in s_2$ (by (#5)) and (#6). For the same reason it must be $s_3 \in \omega_{s_2}$. We can reiterate the same reasoning until $s_{n-2} \in \omega_{s_{n-1}}$. We know then that $s_1 \in \omega_{s_{n-1}}, s_2 \in \omega_{s_1}, \dots, s_3 \in \omega_{s_2}, s_{n-2} \in \omega_{s_{n-1}}$. By definition 3.30 it follows that $s_{n-1} <_T s_1, s_1 <_T s_2 <_T \dots <_T s_{n-2} <_T s_{n-1}$. Namely, we have both $s_{n-1} <_T s_1$ and $s_1 <_T s_{n-1}$. Contradiction ($<_T$ is antisymmetric).

- If $s_1 \in \beta_{s_{n-1}}$, then by Prop. 3.34, $not - O(c, c)$ (#7). Since $(s_1, s_2) \in E$, it must be $s_2 \in \beta_{s_1}$ – indeed, we know (#3) and it cannot be $s_2 \in \omega_{s_1}$, because $c \in s_1$, $c \in s_2$ (by (#5)) and (#7). We can reiterate the same for all states. Then, we know that $s_1 \in \beta_{s_{n-1}}$, $s_{n-1} \in \beta_{s_{n-2}}$, ..., $s_2 \in \beta_{s_1}$. Hence, by definition 3.21, we have $s_1 <_T s_{n-1}$, and $s_{n-1} <_T s_{n-2} <_T \dots <_T s_2 <_T s_1$. Namely, we have both $s_1 <_T s_{n-1}$ and $s_{n-1} <_T s_1$. Contradiction ($<_T$ is antisymmetric).

Hence G_s is acyclic.

We prove G_s is finite: 1) the set of nodes V_s is finite, since, by Prop. C.3, $V_s \subseteq S_d$, and S_d is finite; 2) the set of edges E_s is finite, by Prop. C.1, and acyclicity. $\square$

Proposition 3.41. *Let $s \in S$. $height(s) \leq |Names_d|$.*

Proof. (sketch)

- Note that $height(s) \leq |S_d|$. Indeed the graph G_s contains only states in S_d and G_s is a finite, directed acyclic graph.
- By definition of classifier model each decided state in S_d has a name in $Names_d$. And two states have no name in common. Hence, there is an injection from S_d to $Names_d$. So $|S_d| \leq |Names_d|$.
- So, $height(s) \leq |Names_d|$.

$\square$

Following Propositions C.4, C.5, C.6, C.7, C.8, C.9 are useful to proof Propositions 3.45 and 3.44.

Proposition C.4. *Given $G_s = (V_s, E_s)$, for any $s' \in V_s$, it is $G_{s'} \subseteq G_s$.*

Proof. Follow directly from definition 3.38. $\square$

By previous proposition we obtain that the height of a decided state is the same in all the graph which the state belongs to.

Proposition C.5. *$height(s') = height_{G_s}(s')$, for any $s' \in V_s$.*

Proof. Let $G_{s'} = (V_{s'}, E_{s'})$ a s' -graph. Let P' a longest path connecting s' to a sink node in $G_{s'}$. Let P a longest path connecting s' to a sink node in G_s . We want to prove that $|P'| = |P|$ ($|\cdot|$ counts the number of edges in the paths)

- By Prop. C.4, $|P'| \leq |P|$.
- We prove that $|P'| \leq |P|$. P is of the form $P = s', s_1, \dots, s_n$, where s_n is a sink node and $(s', s_1) \in E_s$ and $(s_i, s_{i+1}) \in E_s$ for $1 \leq i \leq n$. Since $(s', s_1) \in E_s$ then $s_1 \in \omega_{s'}$ or $s_1 \in \beta_{s'}$. But then, $s_1 \in V_{s'}$ and $(s', s_1) \in E_{s'}$. Similarly, we obtain that $s_i \in V_{s'}$ and $(s_i, s_{i+1}) \in E_{s'}$ for all $1 \leq i \leq n$. Hence P is a path connecting s' to a sink node in $G_{s'}$. But then, since P' a longest path connecting s' to a sink node in $G_{s'}$, we have $|P'| \leq |P|$.

□

If a state is of height 0 then it does not admit potentially binding nor potentially overruling states.

Proposition C.6. $height(s) = 0 \Rightarrow (\omega_s = \emptyset \ \& \ \beta_s = \emptyset)$

Proof. Since $height(s) = 0$, then s is a sink node, i.e. there is no outgoing edge from s . By definition of s -graph this means that $\omega_s = \emptyset$ and $\beta_s = \emptyset$. □

The height of each state, different than s and belonging to the s -graph, is lower than the height of s .

Proposition C.7. Let $G_s = (V_s, E_s)$, a s -graph. Let $s' \in V_s$, with $s \neq s'$. Then $height(s') < height(s)$.

Proof. By Prop. C.5, $height(s') = height_{G_s}(s')$. Suppose by contradiction $height(s') \geq height(s)$. Hence $height_{G_s}(s') \geq height(s)$. Contradiction (since s is the origin of the G_s graph). □

In particular we will need following result.

Lemma C.8. Let s s.t. $f(s) \neq ?$ and $h = height(s)$.

- 1) Given s' s.t. Against(s, s') and $\tilde{s} \in Overrule_T(s', s)$, then $height(s') \leq h - 1$ and $height(\tilde{s}) \leq h - 2$.

II) Given s'' s.t. $\text{According}(s, s'')$ and $\tilde{s} \in \text{Overrule}_T(s'', s)$, then $\text{height}(s'') \leq h - 1$ and $\text{height}(\tilde{s}) \leq h - 2$.

Proof. We prove I). The proof for II) is similar. Let s' s.t. $\text{Against}(s, s')$ and $\tilde{s} \in \text{Overrule}_T(s', s)$. Let $G_s = (V_s, E_s)$, the s -graph.

- Since $\text{Against}(s, s')$, then, by Def. 3.35, $s' \in \beta_s$. Hence $s \neq s'$, and $s' \in V_s$. Then by Prop. C.7 $\text{height}(s') < h$. So $\text{height}(s') \leq h - 1$
- Let $\tilde{s} \in \text{Overrule}_T(s', s)$. Then, by Def 3.30, $\tilde{s}$ in $\omega_{s'}$. This means that $\tilde{s} \neq s'$, $\tilde{s} \in V_{s'}$. Then by Prop. C.7 $\text{height}(\tilde{s}) < \text{height}(s') \leq h - 1$. Hence $\text{height}(\tilde{s}) \leq h - 2$.

□

Lemma C.9. If $\text{height}(s) = 1$ and $s' \in \omega_s \cup \beta_s$, then

- $\forall j$, not- $I^j(s')$;
- not- $\text{Incuriam}(s')$.
- $\text{Overrule}_T(s', s) = \emptyset$

Proof. Let $G_s = (V_s, E_s)$. Assume $\text{height}(s) = 1$ (#1). Let $s' \in \omega_s \cup \beta_s$.

- $\text{height}(s') = 0$ (#3). Indeed since, $s' \in \omega_s \cup \beta_s$, then $s' \in V_s$. Then by (#1) and Prop. C.7, $\text{height}(s') = 0$.
- By (#3) and Prop. C.6, $\nexists \tilde{s} \in \beta_{s'}$. Hence $\nexists \tilde{s}$ s.t $\text{Against}(s', \tilde{s})$. Then, respectively by Def. 3.43 and 3.42, trivially $(\forall j : \text{not-}I^j(s'))$ and not- $\text{Incuriam}(s')$. So, a) and b) hold.
- By (#3) and Prop. C.6, $\omega_{s'} = \emptyset$. Hence, by Def. 3.30, $\text{Overrule}_T(s', s) = \emptyset$. Hence c) holds.

□

Lemma 3.45. Let $h = \text{height}(s)$

$$\exists k \geq h \text{ s.t. } I^k(s) \Rightarrow \forall j \geq h : I^j(s). \quad (3.1)$$

Proof. We prove the thesis by induction on $\text{height}(s)$

Base cases. Assume $h = \text{height}(s) = 1$ (#1). Assume $\exists k \geq 1$ s.t. $I^k(s)$. Let $j \geq 1$, we prove that $I^j(s)$.

- Since $I^k(s)$, then by Def. 3.43, we know there is $s' \in S$, s.t. $\text{Against}(s, s')$ (#2). Hence, by Def. 3.36, $s' \in \beta_s$ (#3).
- If $j > 1$ then $\text{not} - I^{j-1}(s')$ (#4) and $\text{OVERRULE}_T(s', s) = \emptyset$ (#5). (#4) follows from (#1), (#3), and Prop. C.9 a). (#5) follows from (#1), (#3), and Prop. C.9 c).
- Suppose there is $s'' \in \text{According}(s, s'')$. Since, by hypothesis, $I^k(s)$ then $(I^{k-1}(s''))$ and $s \not\prec s''$ or $(\exists \tilde{s} \in \text{OVERRULE}_T(s'', s) : \text{not} - I^{k-1}(\tilde{s}))$ or $(s'' \prec s' \text{ or } (s'' <_T s' \text{ and } s' \cong s''))$ (#6).
- Since $s'' \in \text{According}(s, s'')$, then, by Def. 3.36, $s'' \in \beta_s$ (#7). By (#1), (#7) and Prop. C.9 a) and c), we know $\text{not} - I^{k-1}(s'')$ (#8) and $\text{OVERRULE}_T(s'', s) = \emptyset$ (#9).
- By (#6), (#8) and (#9), we have for all $s'' \in \text{According}(s, s'')$: $[(s'' \prec s' \text{ or } (s'' <_T s' \text{ and } s' \cong s''))]$ (#10).
- Now we distinguish two cases:
 - $j = 1$. By (#2) and (#10) we obtain $I^1(s)$
 - $j > 1$. By (#2), (#4), (#5), and (#10) we obtain $I^j(s)$.

Inductive step. Assume the thesis holds true for $\text{height}(s) = h > 1$. Namely assume

$$\exists k \geq h \text{ s.t. } I^k(s) \Rightarrow \forall j \geq h : I^j(s) \quad (\text{C.1})$$

Notice that this is equivalent to

$$\exists k \geq h \text{ s.t. } \text{not} - I^k(s) \Rightarrow \forall j \geq h : \text{not} - I^j(s) \quad (\text{C.2})$$

We prove the thesis holds for $\text{height}(s) = h + 1$ (#a). Namely we want to prove that

$$\exists k \geq h + 1 \text{ s.t. } I^k(s) \Rightarrow \forall j \geq h + 1 : I^j(s) \quad (\text{C.3})$$

So let $k \geq h + 1$ s.t. $I^k(s)$ and fix $j \geq h + 1$.

- $k - 1 \geq h$ (#11). Indeed we know that $k \geq h + 1$.
- $j - 1 \geq h$ (#12). Indeed we know that $j \geq h + 1$.
- Since $I^k(s)$ then (#13): $\exists s' \in S$ s.t. $[\text{Against}(s, s'), (\text{not} - I^{k-1}(s') \text{ or } s \prec s'), \text{ and } (\forall \tilde{s} \in \text{OVERRULE}_T(s', s) : I^{k-1}(\tilde{s}))]$ and $\forall s'' \in S$ s.t. $\text{According}(s, s'') : [(I^{k-1}(s'')) \text{ and } s \not\prec s''] \text{ or } (\exists \tilde{s} \in \text{OVERRULE}_T(s'', s) : \text{not} - I^{k-1}(\tilde{s})) \text{ or } (s'' \prec s' \text{ or } (s'' <_T s' \text{ and } s' \cong s''))]$

- By #a) and Lemma C.8, we know that $height(s') \leq h$ (#14). Also, by Lemma C.8, if $\tilde{s} \in Overrule_T(s', s)$, then $height(\tilde{s}) \leq h - 1$ (#15).
- By (#11) and (#14), $k - 1 \geq height(s')$ (#16). By (#12) and (#14), $j - 1 \geq height(s')$ (#17).
- By (#11) and (#15), $k - 1 \geq height(\tilde{s})$ (#18). By (#12) and (#15), $j - 1 \geq height(\tilde{s})$ (#19).
- Suppose $\text{not-}I^{k-1}(s')$. Since we have (#14), we can apply inductive hypothesis (C.2): by, $\text{not-}I^{k-1}(s')$, (#16), (#17) and (C.2), we obtain $\text{not-}I^{j-1}(s')$ (#20).
- Suppose $\tilde{s} \in Overrule_T(s', s)$, then by (#13) $I^{k-1}(\tilde{s})$. Since we have (#15), we can apply inductive hypothesis (C.1). By $I^{k-1}(\tilde{s})$, (#18), (#19) and (C.1), we obtain $I^{j-1}(\tilde{s})$ (#21).
- Let s'' s.t. $According(s, s'')$. If $I^{k-1}(s'')$, as in previous points, using Prop. C.8, we obtain $I^{j-1}(s'')$ (#22).
- Suppose $\exists \tilde{s} \in Overrule_T(s'', s)$ and $\text{not-}I^{k-1}(\tilde{s})$. As in previous points, using Prop. C.8, we obtain $\text{not-}I^{j-1}(\tilde{s})$ (#23).
- By (#13), (#21), (#22), (#23) : $\exists s' \in S$ s.t. $[Against(s, s'), (\text{not-}I^{j-1}(s') \text{ or } s \prec s'), \text{ and } (\forall \tilde{s} \in Overrule_T(s', s) : I^{j-1}(\tilde{s}))]$ and $\forall s'' \in S$ s.t. $According(s, s'') : [(I^{j-1}(s'') \text{ and } s \not\prec s'') \text{ or } (\exists \tilde{s} \in Overrule_T(s'', s) : \text{not-}I^{j-1}(\tilde{s})) \text{ or } (s'' \prec s' \text{ or } (s'' <_T s' \text{ and } s' \cong s''))]$
- Hence $I^j(s)$

□

Proposition 3.44 (First characterization *per incuriam*). *Let s s.t. $f(s) \neq ?$ and let $h = height(s)$. So:*

$$Incuriam(s) \Leftrightarrow I^h(s).$$

Proof. We prove by induction on $h = height(s)$.

Base cases. Assume $h = 1$ (#1). Assume $Incuriam(s)$. We prove $I^1(s)$.

- 1) By hypothesis $\exists s' \in S$ s.t. $(s, s') \in E_s$ and $Against(s, s')$ (#2) and $(Incuriam(s') \Rightarrow s \prec s')$ and $(\forall \tilde{s} \in Overrule_T(s', s) : Incuriam(\tilde{s}))$
- 2) By hypothesis, $\forall s'' \in S$ if $(s, s'') \in E_s$ and $According(s, s'')$ then either $(Incuriam(s'') \text{ and } s \not\prec s'')$ or $(\exists \tilde{s} \in Overrule_T(s'', s) \text{ s.t. } Incuriam(\tilde{s}))$ or $((s'' <_T s' \text{ and } s' \cong s'') \text{ or } s'' \prec s')$.
- 3) Let s'' s.t. $According(s, s'')$. Also, by Lemma C.8 II), and (#1), $height(s'') = 0$. So, by Prop. C.6, $\omega_{s''} = \emptyset$. Hence $Overrule_T(s'', s) = \emptyset$ (#3). Also, we know that

$s'' \in \beta_s$, so by Lemma C.9.b) and $height(s'') = 0$, we retrieve not $Incuriam(s'')$ (#4).

- 4) By (#2), 2), (#3) and (#4), $\exists s' \in S$, s.t. $Against(s, s')$ and $\forall s''$ s.t. $According(s, s'')$: $[(s'' <_T s' \text{ and } s' \cong s'') \text{ or } s'' \prec s']$.
- 5) So $I^1(s)$.

Assume $I^1(s)$. We prove $Incuriam(s)$.

- 6) By hypothesis, $\exists s'$ s.t. $Against(s, s')$ and $\forall s''$, if $According(s, s'')$, then $(s'' <_T s' \text{ and } s' \cong s'') \text{ or } s'' \prec s'$.
- 7) Notice that for any s' s.t. $Against(s, s')$, we have $s' \in \beta_s$. So $(s, s') \in E_s$ (#5).
- 8) Let s' s.t. $Against(s, s')$. Then, by Lemma C.8 I), and (#1), $height(s') = 0$. So, by Prop. C.6, $\omega_{s'} = \emptyset$. Hence, by Def. 3.30, $Overrule_T(s', s) = \emptyset$. Also by Lemma C.9 b), we obtain that not- $Incuriam(s')$. So we trivially have $(Incuriam(s') \Rightarrow s \prec s')$ and $(\exists \tilde{s} \in Overrule_T(s', s) \Rightarrow Incuriam(\tilde{s}))$ (#6).
- 9) By 6), (#5) and (#6), we obtain that $\exists s' \in S$ s.t. $(s, s') \in E_s$ and $Against(s, s')$ and $(Incuriam(s') \Rightarrow s \prec s')$ and $(\forall \tilde{s} \in Overrule_T(s', s) : Incuriam(\tilde{s}))$ and $\forall s''$ if $(s, s'') \in E_s$ and $According(s, s'')$ then $(s'' <_T s' \text{ and } s' \cong s'') \text{ or } s'' \prec s'$.
- 10) Hence $Incuriam(s)$.

Inductive steps. We assume the thesis hold for all $k \leq h$. We prove $h + 1$. Assume $height(s) = h + 1$

Assume $Incuriam(s)$, we prove $I^{h+1}(s)$

- 11) By hypothesis, $\exists s'$ s.t. $(s, s') \in E_s$ and $Against(s, s')$ and $(Incuriam(s') \Rightarrow s \prec s')$ and $(\forall \tilde{s} \in Overrule_T(s', s) : Incuriam(\tilde{s}))$
- 12) By hypothesis, $\forall s''$ if $(s, s'') \in E_s$ and $According(s, s'')$ then $(Incuriam(s'') \text{ and } s \not\prec s' \text{ or } (\exists \tilde{s} \in Overrule_T(s'', s) \text{ s.t. } Incuriam(\tilde{s})) \text{ or } ((s'' <_T s' \text{ and } s' \cong s'') \text{ or } s'' \prec s'))$. Notice that for all s'' s.t. $According(s, s'')$, then, by Def. 3.35 and 3.38, $(s, s'') \in E_s$ (#c).
- 13) Let s' s.t. $Against(s, s')$. By Lemma C.8.I), $h' = height(s') \leq h < h + 1 = height(s)$. By inductive hypothesis if not- $Incuriam(s')$ then not- $I^{h'}(s')$. By Lemma 3.45, since $height(s') = h' \leq h$ and not- $I^{h'}(s')$, we obtain not- $I^h(s')$ (#7).
- 14) Let $\tilde{s} \in Overrule_T(s', s)$. By Lemma C.8.I), $\tilde{h} = height(\tilde{s}) < h$. By 11), $Incuriam(\tilde{s})$. By inductive hypothesis $I^{\tilde{h}}(\tilde{s})$. By Lemma 3.45, since $height(\tilde{s}) = \tilde{h} < h$ and $I^{\tilde{h}}(\tilde{s})$, we obtain $I^h(\tilde{s})$ (#8).

- 16) By inductive hypothesis and the same reasoning patterns as in two previous point if $According(s, s'')$ and $Incuriam(s'')$ then $I^h(s'')$ (#9); if $\tilde{s} \in Overrule_T(s'', s)$ and not $Incuriam(\tilde{s})$ then not- $I^h(\tilde{s})$ (#10).
- 17) Summarizing, by 11), 12), (#c), (#7), (#8), (#9), (#10), $\exists s' \in S$ s.t. $Against(s, s') :$ $[(\text{not-}I^h(s') \text{ or } s \prec s'), \text{ and } (\forall \tilde{s} \in Overrule_T(s', s) : I^h(\tilde{s}))]$ and $\forall s'' \in S$ s.t. $According(s, s'') :$ $[(I^h(s'') \text{ and } s \not\prec s'') \text{ or } (\exists \tilde{s} \in Overrule_T(s'', s) : \text{not-}I^h(\tilde{s})) \text{ or } (s'' \prec s' \text{ or } (s'' <_T s' \text{ and } s' \cong s'')))]$.
- 18) hence $I^{h+1}(s)$.

Assume $I^{h+1}(s)$. We will prove $Incuriam(s)$.

- 19) For $s' \in S$ if $Against(s, s')$, then, by Def. 3.35, it is $(s, s') \in E$.
- 21) By hypothesis $\exists s' \in S$ s.t. $[Against(s, s') \text{ and } (\text{not-}I^h(s') \text{ or } s \prec s') \forall \tilde{s} \in Overrule_T(s', s) : I^h(\tilde{s})]$ and $\forall s'' \in S$ s.t. $According(s, s'') : [(I^h(s'') \text{ and } s \not\prec s'') \text{ or } (\exists \tilde{s} \in Overrule_T(s'', s) \text{ and not-}I^h(\tilde{s})) \text{ or } (s'' \prec s' \text{ or } (s'' <_T s' \text{ and } s' \cong s''))]$.
- 22) Suppose for $s' \in S$ s.t. $Against(s, s')$ it is not $I^h(s')$ (#11). By Lemma C.8.I), $h' = height(s') \leq h$ (#12). By (#11), (#12), Lemma 3.45, we obtain that not- $I^{h'}(s')$. By inductive hypothesis, not- $Incuriam(s')$ (#13).
- 23) Let $\tilde{s} \in Overrule_T(s', s)$. Then it follows $I^h(\tilde{s})$ (#14). By Lemma C.8.I), $\tilde{h} = height(\tilde{s}) < h$ (#15). By (#14), (#15) and Lemma 3.45, we obtain $I^{\tilde{h}}(\tilde{s})$. Hence, by inductive hypothesis $Incuriam(\tilde{s})$ (#16).
- 24) Same reasoning pattern apply to $s'' \in S$ s.t. $According(s, s'')$ and $I^h(s'')$: we retrieve that $Incuriam(s')$ (#17). Also for $\tilde{s} \in Overrule_T(s'', s)$ and not- $I^h(\tilde{s})$: we retrieve that not- $Incuriam(\tilde{s})$ (#18).
- 25) Summarising, by 19), 21), (#13), (#16), (#17), (#18),
- $\exists s' \in S$ s.t. $(s, s') \in E_s$, $Against(s, s')$, $(Incuriam(s') \Rightarrow s \prec s')$, and $\forall \tilde{s} \in Overrule_T(s', s) : Incuriam(\tilde{s})$.
 - $\forall s'' \in S$ if $(s, s'') \in E_s$ and $According(s, s'')$ then either $[Incuriam(s'') \text{ and } s \not\prec s'']$, or $[\exists \tilde{s} \in Overrule_T(s'', s) : Incuriam(\tilde{s})]$, or $[(s'' <_T s' \text{ and } s' \cong s'') \text{ or } s'' \prec s']$.
- 26) Hence $Incuriam(s)$.

□

Proposition 3.46 (Second Characterization *per incuriam*). *Let s such that $f(s) \neq ?$. Then:*

$$Incuriam(s) \Leftrightarrow \exists k \leq |\text{Names}_d| \text{ s.t. } \forall j \text{ with } k \leq j \leq |\text{Names}_d| : I^j(s).$$

Proof. (sketch)

$\Rightarrow$) Suppose $Incuriam(s)$, then by Prop. 3.44, $I^h(s)$, for $h = \text{height}(s)$. By Prop. 3.45, we know that $I^j(s)$, for all $j \geq h$. By Prop 3.41, we know $h = \text{height}(s) \leq |\text{Names}_d|$. Hence, trivially, $I^j(s)$, for all j s.t. $h \leq j \leq |\text{Names}_d|$. Hence the thesis.

$\Leftarrow$) Suppose $\exists k \leq |\text{Names}_d|$ s.t. $\forall j$ with $k \leq j \leq |\text{Names}_d| : I^j(s)$. We prove $I^h(s)$. Indeed if $k \leq h$, then automatically we have $I^h(s)$, by hypothesis. If $k > h$, then by Lemma 3.45, we have $I^h(s)$. From $I^h(s)$ and Prop. 3.44, we have the thesis. $\square$

Proposition 3.47.

- 1) $(C, s) \models \hat{F}\varphi$ iff $f(s) \neq ?$ and there is s' s.t. $s <_T s'$ and $(C, s') \models \varphi$;
- 2) $(C, s) \models \text{Against}\varphi$ iff $f(s) \neq ?$ and there is s' s.t. $\text{Against}(s, s')$ and $(C, s') \models \varphi$;
- 3) $(C, s) \models \text{POverruling}\varphi$ iff $f(s) \neq ?$ and there is s' s.t. $s' \in \omega_s$ and $(C, s') \models \varphi$;
- 4) $(C, s) \models \text{According}^\forall \varphi$ iff $f(s) \neq ?$ and for all s' s.t. $\text{According}(s, s') : (C, s') \models \varphi$;
- 5) $(C, s) \models \text{POverruling}^\forall \varphi$ iff $f(s) \neq ?$ and for all $s' \in \omega_s : (C, s') \models \varphi$.

Proof. We prove 1), 2) and 4). The other proofs are similar.

- 1) $\Rightarrow$) Suppose $(C, s) \models \hat{F}\varphi$. By hypothesis, there is $n \in \text{Names}_d$ s.t. $n \in s$. So, by Def. 3.3, $s \in S_d$. Hence, there is $o \in \{0, 1\}$, s.t. $f(s) = o$. Hence $f(s) \neq ?$. We know $(C, s) \models \hat{F}_n\varphi$. Hence by Prop. 3.10, there is s' s.t. $s <_T s'$ and $(C, s') \models \varphi$.
 $\Leftarrow$) Suppose $f(s) \neq ?$ and there is s' s.t. $s <_T s'$ and $(C, s') \models \varphi$; Since $f(s) \neq ?$, there is $o \in \{0, 1\}$ s.t. $(C, s) \models t(o)$ (#1). Since $f(s) \neq ?$, then $s \in S_d$. So, by Def. 3.3, there is $n \in \text{Names}_d$ s.t. $n \in s$ (#2). Since there is s' s.t. $s <_T s'$ and $(C, s') \models \varphi$, then, by Prop. 3.10, $(C, s) \models \hat{F}_n\varphi$ (#3). By (#1), (#2) and (#3), $(C, s) \models \hat{F}\varphi$.
- 2) $\Rightarrow$) Suppose $(C, s) \models \text{Against}\varphi$. Since $(C, s) \models \text{Against}_{n,m}\varphi$, by Prop. 3.37, there is s' s.t. $\text{Against}(s, s')$ (#4) and $(C, s') \models \varphi$ (#5). By (#4) and Prop. C.2 $f(s) \neq ?$ (#6). By (#4), (#5) and (#6), the thesis.

- $\Leftarrow$) Suppose $f(s) \neq ?$ and there is s' s.t. $\text{Against}(s, s')$ and $(C, s') \models \varphi$ (#7). Since $f(s) \neq ?$, then $s \in S_d$. So, there is $n \in \text{Names}_d$ s.t. $n \in s$ (#8). Since s' is s.t. $\text{Against}(s, s')$, the by Prop. C.2, $s' \in S_d$. So, there is $m \in \text{Names}_d$ s.t. $m \in s'$ (#9), By Prop. 3.37, $(C, s) \models \text{Against}_{n,m} \varphi$ (#10). By (#8), (#9) and (#10), $(C, s) \models \text{Against} \varphi$.
- 4) $\Rightarrow$) Suppose $(C, s) \models \text{According}^\forall \varphi$. By hypothesis, there is $n \in \text{Names}_d$ s.t. $n \in s$. So, by Def. 3.3, $s \in S_d$. Hence, there is $o \in \{0, 1\}$, s.t. $f(s) = o$. Hence $f(s) \neq ?$ (#12). Let s' s.t. $\text{According}(s, s')$. Suppose by contradiction that $(C, s') \models \neg \varphi$ (#13). By $\text{According}(s, s')$ and Prop.C.2, $s' \in S_d$. So, there is $m \in \text{Names}_d$ s.t. $m \in s'$ (#7). Hence by (#11), (#12), (#13), and by Prop. 3.37, $(C, s) \models \text{According}_{n,m} \neg \varphi$. Contradiction, since, by hypothesis $(C, s) \models \neg \text{According}_{n,m} \neg \varphi$
- $\Leftarrow$) Suppose $f(s) \neq ?$ and for all s' s.t. $\text{According}(s, s')$: $(C, s') \models \varphi$. Since $f(s) \neq ?$, there is $o \in \{0, 1\}$ s.t. $(C, s) \models t(o)$ (#14). Since $f(s) \neq ?$, then $s \in S_d$. So there is $n \in \text{Names}_d$ s.t. $n \in s$ (#15). Suppose by contradiction that $(C, s) \models \bigvee_m \text{According}_{n,m} \neg \varphi$. Then, by Prop. 3.37, there is s' s.t.: a) $m \in s'$ and b) $\text{According}(s, s')$ and c) $(C, s') \models \neg \varphi$. Contradiction (because of the hypothesis). So, $(C, s) \models \neg \bigvee_m \text{According}_{n,m} \neg \varphi$ (#16). By (#14), (#15) and (#16), the thesis.

□

Proposition 3.49 (I^n, ι^n Repr.). For all $n \in \mathbb{N}$, $(C, s) \models \iota^n \Leftrightarrow I^n(s)$.

Proof. We first verify that:

- A) $[(C, s) \models \bigvee_{n \in \text{Names}_d} (n \wedge \text{Against}(\text{POverruling}^\forall(\hat{F}n \rightarrow \varphi)))] \Leftrightarrow [\exists n \in s \cap \text{Names}_d \text{ and } \exists s' \text{ s.t. } \text{Against}(s, s') \text{ and } \forall \tilde{s} \in \text{Overrule}_T(s', s): (C, \tilde{s}) \models \varphi]$.
- $\Rightarrow$) Since $(C, s) \models \bigvee_{n \in \text{Names}_d} n$, then $\exists n \in s \cap \text{Names}_d$ (#1). By hypothesis and by Prop. 3.47 exists s' s.t. $\text{Against}(s, s')$ (#2) and $(C, s') \models \text{POverruling}^\forall(\hat{F}n \rightarrow \varphi)$. By prop 3.47, for all $\tilde{s} \in \omega_s$: $(C, \tilde{s}) \models \hat{F}n \rightarrow \varphi$ (#3). Hence $(C, \tilde{s}) \models \hat{F}n \rightarrow \varphi$. Then by Prop. 3.47: if there is $\tilde{s}$ s.t. $\tilde{s} \models n$ and $\tilde{s} <_T \tilde{s}$ then $(C, \tilde{s}) \models \varphi$. Recall n is the name of s , so (#4): if $\tilde{s} <_T s$ then $(C, \tilde{s}) \models \varphi$. By (#2) and (#3), and Def 3.30: $\forall \tilde{s} \in \text{Overrule}_T(s', s): (C, \tilde{s}) \models \varphi$ (#5). By (#1), (#2), (#3) and (#4), the thesis.
- $\Leftarrow$) Suppose $\exists n \in s \cap \text{Names}_d$ (#6) and $\exists s' \text{ s.t. } \text{Against}(s, s')$ (#7) and $\forall \tilde{s} \in \text{Overrule}_T(s', s): (C, \tilde{s}) \models \varphi$ (#8). By (#6), $(C, s) \models \bigvee_{n \in \text{Names}_d} n$ (#9). Let

$\tilde{s} \in \omega_{s'}$. By (#8), if $\tilde{s} <_T s$ then $(C, \tilde{s}) \models \varphi$ (#10). By (#6), (#10) and Prop. 3.47, $(C, s') \models \text{POverruling}^\forall(\hat{F}n \rightarrow \varphi)$ (#11). By (#7), (#11) and Prop. 3.47, $(C, s) \models n \wedge \text{Against}(\text{POverruling}^\forall(\hat{F}n \rightarrow \varphi))$ (#12). By (#12) and (#9), the thesis.

- B) $[(C, s) \models \bigvee_{n \in \text{Names}_d} (n \wedge \text{According}^\forall(\text{POverruling}(\hat{F}n \wedge \varphi)))] \Leftrightarrow [\exists n \in s \cap \text{Names}_d \text{ and } \forall s'' \text{ s.t. } \text{According}(s, s''): \exists \tilde{s} \in \text{Overrule}_T(s'', s) \text{ s.t. } (C, \tilde{s}) \models \varphi]$. The proof is similar as in A)

We prove the thesis by induction on n .

Base cases. We prove the thesis for $n = 1$. $\Rightarrow$) Assume $(C, s) \models \iota^i$. We prove $I^1(s)$.

- By Hypothesis $(C, s) \models \bigvee_{m \in \text{Names}_d} (\text{Against}(m) \wedge \text{According}^\forall((\hat{F}m \wedge \text{SameCourt}(m)) \vee \text{Higher}(m)))$. By Prop. 3.47,
 - there is s' s.t. $(C, s') \models m$ and $\text{Against}(s, s')$ (#13) and
 - for all s'' s.t. $\text{According}(s'', s)$: $(C, s'') \models (\hat{F}m \wedge \text{SameCourt}(m)) \vee \text{Higher}(m)$.

Since m is the name of s' , by Prop. 3.47 and 3.13, it follows that for all s'' s.t. $\text{According}(s'', s)$: $[(s'' <_T s' \text{ and } s' \cong s'') \text{ or } s'' \prec s']$ (#14).

- By (#13) and (#14), $I^1(s)$

$\Leftarrow$) Assume $I^1(s)$. We prove that $(C, s) \models \iota^1$.

- By hypothesis $\exists s' \text{ s.t. } \text{Against}(s, s') \text{ and } \forall s'' \text{ s.t. } \text{According}(s, s''): [(s'' <_T s' \text{ and } s' \cong s'') \text{ or } s'' \prec s']$ (#15). Since $\text{Against}(s, s')$, then by Prop. C.2, $s' \in S_d$. So, there is $m \in s' \cap \text{Names}_d$ (#16). By (#15), (#16), Prop. 3.47 and 3.13, for all s'' s.t. $\text{According}(s'', s)$ we have $(C, s'') \models (\hat{F}m \wedge \text{SameCourt}(m)) \vee \text{Higher}(m)$ (#17). By (#15), (#17), and Prop. 3.47, $(C, s) \models \bigvee_{m \in \text{Names}_d} \text{Against}(m) \wedge \text{According}^\forall((\hat{F}m \wedge \text{SameCourt}(m)) \vee \text{Higher}(m))$.
- Hence, $(C, s) \models \iota^1$.

Inductive steps. Assume the thesis hold for n , we prove it for $n + 1$.

$$(C, s) \models \iota^{n+1} \Leftrightarrow (C, s) \models \bigvee_{n, m \in \text{Names}_d} \left(n \wedge (\text{Against}(m \wedge (\neg \iota^n \vee \text{Lower}(n))) \wedge \text{POverruling}^\forall(\hat{F}n \rightarrow \iota^n)) \wedge \text{According}^\forall \left((\iota^n \wedge \neg \text{Lower}(m)) \vee \text{POverruling}(\hat{F}n \wedge \iota^n) \vee ((\hat{F}m \wedge \text{SameCourt}(m)) \vee \text{Higher}(m)) \right) \right) \xLeftrightarrow{*}$$

$\exists s' \in S$ s.t. $[\text{Against}(s, s'), (\text{not-}I^n(s') \text{ or } s \prec s'), \text{ and } (\forall \tilde{s} \in \text{Overrule}_T(s', s) : I^n(\tilde{s}))]$ and $\forall s'' \in S$ s.t. $[\text{According}(s, s'') \Rightarrow (I^n(s'') \text{ and } s \not\prec s'') \text{ or } (\exists \tilde{s} \in \text{Overrule}_T(s'', s) : \text{not-}I^n(\tilde{s})) \text{ or } (s'' \prec s' \text{ or } (s'' <_T s' \text{ and } s' \cong s''))]$.

(*: by Prop. 3.47, (A), (B), Prop.3.13 and inductive hypothesis)

□

Proposition 3.56 (Before Overruled Repr.). $(C, s) \models \text{BeforeOverruled}_n$ iff $\text{BeforeOverruled}_T(s, s'')$ and $(C, s'') \models n$.

Proof. $\Rightarrow$) Assume $(C, s) \models \text{BeforeOverruled}_n$. Then $(C, s) \models \text{POverruling}(\hat{F}n \wedge \neg \text{Incuriam})$. So, by Prop. 3.47, there is s' s.t. $s' \in \omega_s$ and $(C, s') \models \hat{F}n \wedge \neg \text{Incuriam}$. So, by Prop. 3.51, $\text{not-Incuriam}(s')$. Also $(C, s') \models \hat{F}n$. Hence, by prop 3.47, exists $s'' \in S$ s.t. $(C, s'') \models n$ and $s' <_T s''$. So $s' \in \text{Overrule}_T(s, s'')$. So $\text{BeforeOverruled}_T(s, s'')$.

$\Leftarrow$) Assume $\text{BeforeOverruled}_T(s, s'')$ and $(C, s'') \models n''$. So, by Def 3.52, $\exists s' \in \text{Overruled}_T(s, s'')$ s.t. $\text{not-Incuriam}(s')$. Then, by Prop. 3.51, $(C, s') \models \neg \text{Incuriam}$. Also, since $s' \in \text{Overruled}_T(s, s'')$, then $s' <_T s''$. Hence, $(C, s') \models \hat{F}n$. Since $s' \in \text{Overruled}_T(s, s'')$, then $s' \in \omega_s$. So, by Prop. 3.47, $(C, s) \models \text{POverruling}(\hat{F}n \wedge \neg \text{Incuriam})$. □

Proposition 3.60 (Binding without except. Repr.). $(C, s) \models \text{Binding}_n \varphi$ iff $(C, s) \models n$ and there is $s' \in \bar{\beta}_s$ s.t. $(C, s') \models \varphi$.

Proof. By Prop. 3.24, Prop 3.56, 3.51, Prop 3.13. A more detailed proof is given in the following. $\Rightarrow$) Assume $(C, s) \models \text{Binding}_n$. Since $\text{PBinding}_{n, m}(\varphi \wedge \neg \text{BeforeOverruled}_n \wedge (\text{Incuriam} \rightarrow \text{Lower}(n)))$, then by Prop. 3.24, $(C, s) \models n$ and exists $s' \in \beta_s$, s.t. $(C, s') \models m \wedge \varphi \wedge \neg \text{BeforeOverruled}_n \wedge \text{Incuriam} \rightarrow \text{Lower}(n)$. By $(C, s') \models \neg \text{BeforeOverruled}(n)$, an proposition 3.56, $\text{not-}(\text{BeforeOverruled}_T(s', s''') \text{ and } (C, s''') \models n)$. Since $(C, s) \models n$. Then $\text{not-} \text{BeforeOverruled}_T(s', s)$. By $(C, s') \models (\text{Incuriam} \rightarrow \text{Lower}(n))$ and Prop. 3.51, Prop. 3.13, $(\text{Incuriam}(s') \Rightarrow s \prec s')$. So the thesis.

$\Leftarrow$) Assume $(C, s) \models n$ and there is $s' \in \overline{\beta}_s$ s.t. $(C, s') \models \varphi$. So $s' \in \beta_s$ and $(\text{Incuriam}(s') \Rightarrow s \prec s')$ and $\text{not-BeforeOverruled}(s', s)$ and $(C, s') \models \varphi$. By, $(\text{Incuriam}(s') \Rightarrow s \prec s')$ and Prop. 3.51, Prop. 3.13, $(C, s') \models (\text{Incuriam} \rightarrow \text{Lower}(n))$. By $\text{not-BeforeOverruled}(s', s)$ and Prop. 3.56, $(C, s') \models \neg \text{BeforeOverruled}_n$. By Prop. 3.24, the thesis.

□

Proposition 3.64. *For all $s', s'' \in \text{Best}_H(\overline{\beta}_s)$, $s' \cong s''$.*

Proof. Suppose $s', s'' \in \text{Best}_H(\overline{\beta}_s)$. Since $s', s'' \in \text{Best}_H(\overline{\beta}_s)$, then $s', s'' \in \beta_s$. Then, by 3.25, $(s' \cong s \text{ or } s \prec s')$ and $(s'' \cong s \text{ or } s \prec s'')$.

- If $s'' \cong s$ and $s' \cong s$, then $s'' \cong s'$. So the thesis
- If $s'' \cong s$ and $s \prec s'$, then $s'' \prec s'$. Contradiction, because $s'' \in \text{Best}_H(\overline{\beta}_s)$
- If $s' \cong s$ and $s \prec s''$, then $s' \prec s''$. Contradiction, because $s' \in \text{Best}_H(\overline{\beta}_s)$
- If $s \prec s'$ and $s \prec s''$, then by prop 3.5, $s'' \prec s'$ or $s' \prec s''$ or $s'' \cong s'$. If $s'' \prec s'$ or $s' \prec s''$, contradiction as in previous points. Hence, $s'' \cong s'$.

□

Proposition 3.66 (Best-Hier. Bind. Repr.). $(C, s) \models \text{Best}_H \text{Binding}_n \varphi$ iff $(C, s) \models n$ and $\exists s' \in \text{Best}_H(\overline{\beta}_s)$ s.t. $(C, s') \models \varphi$.

(Sketch). $\Rightarrow$) Suppose $(C, s) \models \text{Best}_H \text{Binding}_n \varphi$, then, by Prop. 3.60, $(C, s) \models n$ and there are $m \in \text{Names}_d$ and $s' \in \overline{\beta}_s$ (#1) s.t. $(C, s') \models \varphi$ (#2) and $m \in s'$ (#3). Also, by Prop. 3.60, $\nexists s'' \in \overline{\beta}_s$ s.t. $(C, s'') \models \text{Lower}(m)$. From the latter and (#3), we retrieve that $\nexists s'' \in \overline{\beta}_s$ s.t. $s' \prec s''$. So $\forall s'' \in \overline{\beta}_s$, $s' \not\prec s''$. So $s' \in \text{Best}_{TH}(\overline{\beta}_s)$ (#4). By (#2) and (#4) the thesis.

$\Leftarrow$) Suppose $(C, s) \models n$ and $\exists s' \in \text{Best}_H(\overline{\beta}_s)$ (#5) s.t. $(C, s') \models \varphi$ (#6). Let $m \in \text{Names}_d$, s.t. $(C, s') \models m$ (#7) (exist such $m \in \text{Names}_d$, since s' is a decided state).

- $(C, s) \models \text{Binding}_n(m \wedge \varphi)$. Indeed, by (#5), $s' \in (\overline{\beta}_s)$. So, by 3.60 and (#6), $(C, s) \models \text{Binding}_n(m \wedge \varphi)$

- $(C, s) \models \neg \text{Binding}_n(\text{Lower}(m))$. Indeed, by (#5), $\forall s'' \in \bar{\beta}_s, s' \not\prec s''$. Hence $\nexists s'' \in \bar{\beta}_s$, s.t. $s' \prec s''$. So, by (#7), $\nexists s'' \in \bar{\beta}_s$ s.t. $(C, s'') \models \text{Lower}(m)$. By 3.60, $(C, s) \models \neg \text{Binding}_n(\text{Lower}(m))$.

□

Proposition 3.69 (Hier. Forced Repr.). $(C, s^*) \models \text{HForced}_n(o)$ iff $(C, s^*) \models n$ and $f_H^*(s^*) = \{o\}$.

Proof. $\Rightarrow$) Suppose $(C, s^*) \models \text{HForced}_n(o)$. We have to prove that $f_H^*(s) = \{o\}$, namely that forall $s' \in \text{Best}_H(\bar{\beta}_s)$, $f(s') = o$.

1. We know $(C, s^*) \models \text{Best}_H \text{Binding}_n(t(o))$. Then, by Prop. 3.66, there is $s' \in \text{Best}_H(\bar{\beta}_s)$, s.t. $f(s') = o$. Also, $(C, s) \models n$
2. We know $(C, s) \models \neg \text{Best}_H \text{Binding}_n(t(o))$. Then, by Prop. 3.66, there is no $s'' \in \text{Best}_H(\bar{\beta}_s)$, s.t. $f(s'') = \bar{o}$.
3. By 1. and 2. the thesis

$\Leftarrow$) Suppose $f_H^*(s^*) = \{o\}$ and $(C, s) \models n$.

4. Then, there is $s' \in \text{Best}_H(\bar{\beta}_s)$, s.t. $f(s') = o$. By Prop. 3.66, $(C, s) \models \text{Best}_H \text{Binding}_n(t(o))$
5. There is not $s'' \in \text{Best}_H(\bar{\beta}_s)$, s.t. $f(s'') = o$. By Prop. 3.66, $(C, s) \models \neg \text{Best}_H \text{Binding}_n(t(\bar{o}))$
6. By 1. and 2. the thesis.

□

Proposition 3.72 (Best Temp-Hier Bind. Repr.). $(C, s) \models \text{Best}_{TH} \text{Binding}_n \varphi$ iff $(C, s) \models n$ and $\exists s' \in \text{Best}_{TH}(\bar{\beta}_s)$ s.t. $(C, s') \models \varphi$.

(Sketch). $\Rightarrow$) Suppose $(C, s) \models \text{Best}_{TH} \text{Binding}_n \varphi$, then, by Prop. 3.66, $(C, s) \models n$ and there are $m \in \text{Names}_d$ and $s' \in \text{Best}_H(\bar{\beta}_s)$ (#1) s.t. $(C, s') \models \varphi$ (#2) and $m \in s'$ (#3).

- $\forall s'' \in \text{Best}_H(\bar{\beta}_s): s' \not\prec_T s''$ (#4). Indeed, suppose $s'' \in \text{Best}_H(\bar{\beta}_s)$. Then, there is $h \in \text{Names}_d$ s.t. $(C, s'') \models h$ (since s'' is a decided state). Then, by Prop. 3.66, $(C, s') \models \bigvee_{h \in \text{Names}_d} \text{Best}_H \text{Binding}_n(h)$. So, by hypothesis, $(C, s) \models @_h \neg \hat{P}m$. Since $(C, s'') \models h$, then $(C, s'') \models \neg \hat{P}m$, so, by (#3), $s' \not\prec_T s''$.

By (#1), (#2) and (#4), the thesis.

$\Leftarrow$) Suppose $(C, s) \models n$ and $\exists s' \in \text{Best}_{TH}(\bar{\beta}_s)$ (#5) s.t. $(C, s') \models \varphi$ (#6). Let $m \in \text{Names}_d$, s.t. $(C, s') \models m$ (#7) (exist such $m \in \text{Names}_d$, since s' is a decided state).

- $(C, s) \models \text{Best}_H \text{Binding}_n(m \wedge \varphi)$. Indeed, by (#5), $s' \in \text{Best}_H(\bar{\beta}_s)$. So, by Prop. 3.66 and (#6), $(C, s) \models \text{Best}_H \text{Binding}_n(m \wedge \varphi)$
- $(C, s) \models (\bigvee_{h \in \text{Names}_d} \text{Best}_H \text{Binding}_n(h)) \rightarrow @_h \neg \hat{P}m$. Indeed, by (#5), $\forall s'' \in \text{Best}_H(\bar{\beta}_s)$, $s' \not\prec_T s''$ (#8). So, suppose $(C, s) \models \bigvee_{h \in \text{Names}_d} \text{Best}_H \text{Binding}_n(h)$. Then, by Prop. 3.66, there is $s'' \in \text{Best}_H(\bar{\beta}_s)$ s.t. $(C, s'') \models h$ (#9). By (#7) and (#8), $(C, s'') \models \neg \hat{P}m$. By the latter and (#9) $(C, s) \models @_h \neg \hat{P}m$. So $(C, s) \models (\bigvee_{h \in \text{Names}_d} \text{Best}_H \text{Binding}_n(h)) \rightarrow @_h \neg \hat{P}m$.

□

Proposition 3.75 (Temp-Hier. Forced Repr.). $(C, s^*) \models \text{THForced}_n(o)$ iff $(C, s^*) \models n$ and $f_{TH}^*(s) = \{o\}$.

Proof. $\Rightarrow$) Suppose $(C, s^*) \models \text{THForced}_n(o)$. We have to prove that $f_{TH}^*(s) = \{o\} = \{o\}$, namely that forall $s' \in \text{Best}_{TH}(\bar{\beta}_s)$, $f(s') = o$.

1. We know $(C, s^*) \models \text{Best}_{TH} \text{Binding}_n(\text{t}(o))$. Then, by Prop. 3.72, there is $s' \in \text{Best}_{TH}(\bar{\beta}_s)$, s.t. $f(s') = o$. Also $(C, s) \models n$
2. We know $(C, s) \models \neg \text{Best}_{TH} \text{Binding}_n(\text{t}(o))$. Then, by Prop. 3.72, there is no $s'' \in \text{Best}_{TH}(\bar{\beta}_s)$, s.t. $f(s'') = \bar{o}$.
3. By 1. and 2. the thesis

$\Leftarrow$) Suppose $f_{TH}^*(s) = \{o\}$ and $(C, s) \models n$.

4. Then, there is $s' \in \text{Best}_{TH}(\bar{\beta}_s)$, s.t. $f(s') = o$. By Prop. 3.72, $(C, s) \models \text{Best}_{TH} \text{Binding}_n(\text{t}(o))$
5. There is not $s'' \in \text{Best}_{TH}(\bar{\beta}_s)$, s.t. $f(s'') = \bar{o}$. By Prop 3.72, $(C, s) \models \neg \text{Best}_{TH} \text{Binding}_n(\text{t}(\bar{o}))$
6. By 1. and 2. the thesis.

□

Proposition 3.76. Suppose $\bar{\beta}_{s^*} \neq \emptyset$. If $\leq_T$ is a total order (at least on $\text{Best}_H(\bar{\beta}_{s^*})$), then f_{TH}^* is unambiguous decision making for s^* .

Proof. Suppose $\leq_T$ is an order on $\text{Best}_{TH}(\bar{\beta}_{s^*})$. Suppose $\bar{\beta}_{s^*} \neq \emptyset$. Suppose by contradiction that f_{TH}^* is not unambiguous decision making for s^* . Then we can have two possibilities:

- $f_{TH}^*(s^*) = \emptyset$; but since $\bar{\beta}_{s^*} \neq \emptyset$ then $\text{Best}_{TH}(\bar{\beta}_{s^*}) \neq \emptyset$. Namely, there is $s \in \text{Best}_{TH}(\bar{\beta}_{s^*})$ and so $f(s) \in f_{TH}^*(s^*)$. Contradiction.

- $|f_{TH}^*(s^*)| > 1$; then, there are $s, s' \in Best_{TH}(\bar{\beta}_{s^*})$, s.t. $f(s) \neq f(s')$ (and so $s \neq s'$). Notice that since $s, s' \in Best_{TH}(\bar{\beta}_{s^*})$, then $s, s' \in Best_H(\beta_{s^*})$. Since $s \in Best_{TH}(\bar{\beta}_{s^*})$, it must be $s \leq_T s'$. Since $s \in Best_{TH}(\bar{\beta}_{s^*})$ it must be $s' \leq_T s$. Contradiction since $\leq_T$ is an order on $Best_H(\bar{\beta}_{s^*})$, so antisymmetric, and $s \neq s'$.

□

Proposition 3.78. *Let $C = (S, f, Jur, <_T, \mathcal{R})$, a temporal jurisdictional classifier, with $<_T$ a total order¹ and $\mathcal{R}$ transitive and left Euclidean. Let $s \in S$. If $\exists s', s'' \in Best_H(\bar{\beta}_s)$, s.t. $f(s') \neq f(s'')$, then*

- I) *Incuriam(s') or Incuriam(s'') and*
- II) *$s \prec s'$ and $s \prec s''$.*

Proof. We prove I).

- 1) Assume by contradiction that not *Incuriam(s')* and not *Incuriam(s'')*.
- 2) Without loss of generality assume $s' <_T s''$, by that $<_T$ is a total order.
- 3) $s' \in \mathcal{R}(s'')$. Indeed $s' \in \mathcal{R}(s)$, $s'' \in \mathcal{R}(s)$. Then, since $\mathcal{R}$ is left euclidean, $s' \in \mathcal{R}(s'')$.
- 3a) $s \preceq s', s''$. Indeed $s', s'' \in Best_H(\bar{\beta}_s)$.
- 4) not *BeforeOverruled_T(s', s)*, since $s' \in \bar{\beta}_s$.
- 4a) not *BeforeOverruled_T(s'', s)*, since $s'' \in \bar{\beta}_s$.
- 5) $s' \cong s''$ and so there is $\tilde{c} = s' \cap Courts = s'' \cap Courts$. It follows from $s', s'' \in Best_H(\beta_s)$ and by proposition 3.64.
- 6) $s' <_T s'' <_T s$. Indeed, since $s', s'' \in Best_H(\bar{\beta}_s)$, then $s', s'' \in \beta_s$. So $s', s'' <_T s$. By point 2) $s' <_T s'' <_T s$.
- 7) $\Pi(s', s'', f(s'))$, this follows from $s' \in \mathcal{R}(s'')$ (by point 3) and $s' <_T s''$ (by point 2).
- 9) $\mathcal{B}(\tilde{c}, \tilde{c})$. Indeed suppose not- $\mathcal{B}(\tilde{c}, \tilde{c})$. Then, by Def. 3.27, $O(\tilde{c}, \tilde{c})$. So, by point 5) and 7) and $f(s') \neq f(s'')$, $O(s'', s')$. But then, since $s'' <_T s$ and not-*Incuriam(s'')*, *BeforeOverruled_T(s', s)*. Contradiction, because of point 4).
- 10) $\beta(s', s'', f(s'))$. By point 5), 7) and point 9).
- 11) *Against(s', s'')*. By 10) and $f(s') \neq f(s'')$ (by hypothesis).
- 12) $(s'', s') \in E_{s''}$, since $\beta(s', s'', f(s'))$.
- 13) $s' \in \bar{\beta}_{s''}$. Indeed $\beta(s', s'', f(s'))$, and not *Incuriam(s')* (by 1)) and not *BeforeOverruled_T(s', s'')* (since not *BeforeOverruled_T(s', s)* by point 4), and $s'' <_T s$. Hence, $s' \in \bar{\beta}_{s''}$.

¹ $<_T$ is a reflexive, transitive, antisymmetric relation and such that for all $s, s' \in S$, $s <_T s'$ or $s' <_T s$.

14) We know that $(s'', s') \in E_{s''}$ and $\text{Against}(s', s'')$. By 1) not $\text{Incuriam}(s'')$. Then, by Definition 3.42:

A) $\nexists s' \in S$ s.t. $(s'', s') \in E_{s''}$, $\text{Against}(s'', s')$, $(\text{Incuriam}(s') \Rightarrow s \prec s')$, and $\forall \tilde{s} \in \text{Overrule}_T(s', s'') : \text{Incuriam}(\tilde{s})$. Or

B) $\exists s_1 \in S$ s.t. $(s, s'') \in E_{s''}$ and $\text{According}(s'', s_1)$ and not- $[\text{Incuriam}(s_1)$ and $s'' \not\prec s_1]$, and not- $[\exists \tilde{s} \in \text{Overrule}_T(s_1, s'') : \text{not } \text{Incuriam}(\tilde{s})]$, and not- $[(s_1 <_T s' \text{ and } s' \cong s_1) \text{ or } s_1 \prec s']$.

15) A is not the case. Indeed we know that $\text{Against}(s, s'')$, not- $\text{Incuriam}(s')$. Also we know not- $\text{BeforeOverruled}_T(s', s)$. So there there is no $\tilde{s} \in \text{Overruled}_T(s', s'')$ s.t. not $\text{Incuriam}(\tilde{s})$.

16) So B). Since $<_T$ is a total order and $\mathcal{H}$ is tree like this means that

B') $\exists s_1 \in S$ s.t. $(s'', s_1) \in E_{s''}$ and $\text{According}(s'', s_1)$ and $[\text{not-}\text{Incuriam}(s_1)$ or $s'' \prec s_1]$, and $[\forall \tilde{s} \in \text{Overrule}_T(s_1, s'') : \text{Incuriam}(\tilde{s})]$, and $[(s' <_T s_1 \text{ and } s' \preceq s_1) \text{ or } s' \prec s_1]$

17) $f(s_1) = f(s'')$, since $\text{According}(s'', s_1)$.

18) $\beta(s_1, s'', f(s_1))$, since $\text{According}(s'', s_1)$

18a) $s_1 <_T s$. By 18), $s_1 <_T s''$ and by hypothesis $s'' <_T s$. Hence, $s_1 <_T s$.

19) $s_1 \in \mathcal{R}(s'')$, by 18).

20) $s'', s' \preceq s_1$. By 18) and Prop. 3.25, $s'' \preceq s_1$. Also $s'' \cong s'$. Hence $s' \preceq s_1$.

20a) $s \preceq s_1$. Indeed $s'' \preceq s_1$ (by 20a)) and $s \preceq s''$ (by 3a).

21) $s_1 \in \mathcal{R}(s)$. Indeed, we know that $s'' \in \mathcal{R}(s)$ (by 3)) and $s_1 \in \mathcal{R}(s'')$ (by 19)).

Then, by transitivity $s_1 \in \mathcal{R}(s)$.

22) $s_1 \in \mathcal{R}(s')$ and $s' \in \mathcal{R}(s_1)$. Indeed, by 19), $s_1 \in \mathcal{R}(s'')$. Also $s' \in \mathcal{R}(s'')$ (by 3)).

By left euclideanity, $s_1 \in \mathcal{R}(s')$ and $s' \in \mathcal{R}(s_1)$.

23) $f(s_1) \neq f(s')$. Indeed, since $\text{According}(s'', s_1)$, then $f(s_1) = f(s'')$; and we know that $f(s'') \neq f(s')$ (by hypothesis).

24) $\mathcal{B}(s_1, s, f(s_1))$. Indeed we know $s_1 \in \mathcal{R}(s)$ (by 21)), $s_1 <_T s$ (by 18a), and $s \preceq s_1$ (by 20a)).

25) We prove it cannot be $s', s'' \prec s_1$. Indeed suppose by contradiction that $s', s'' \prec s_1$.

- the only possibility is that $\text{BeforeOverruled}(s_1, s)$, namely s_1 has been overruled before s decision (it can be *per incuriam*, but is not a problem since $s \prec s_1$). Indeed we know $\mathcal{B}(s_1, s, f(s_1))$ (by 24)). Since we assumed $s', s'' \prec s_1$, and by

point 3a), we know $s \prec s_1$. But then, since s', s'' are the best hierarchical states for s , and $s', s'' \prec s_1$ this means that the only possibility is that s_1 is subject to exception, otherwise s', s'' would not have been best hierarchical states for s . In particular s_1 the only possibility is that s_1 has been overruled before s decision (it can be *per incuriam*, it is not a problem since $s \prec s_1$).

- So there is $\tilde{s} \in \text{Overrule}_T(s_1, s)$ and not-*Incuriam*($\tilde{s}$).
- $f(\tilde{s}) = f(s')$ and $f(\tilde{s}) \neq f(s'')$. Indeed, since $\tilde{s} \in \text{Overrule}_T(s_1, s)$, then $f(\tilde{s}) \neq f(s_1)$. So, $f(\tilde{s}) = f(s')$ and $f(\tilde{s}) \neq f(s'')$, by 17) and 23)
- $\tilde{s} <_T s$. This follows from definition of $\tilde{s} \in \text{Overrule}_T(s_1, s)$.
- $s'' <_T \tilde{s}$. Indeed, by B'), we have $[\forall \tilde{s} \in \text{Overrule}_T(s_1, s'') : \text{Incuriam}(\tilde{s})]$. But we also know that $\tilde{s} \in \text{Overrule}_T(s_1, s)$ and not-*Incuriam*($\tilde{s}$). So it must be $\tilde{s} <_T s$.
- $s'' \prec \tilde{s}$. Indeed $\tilde{s} \in \text{Overrule}_T(s_1, s)$, so $s_1 \preceq \tilde{s}$. And we assumed $s'' \prec s_1$.
- $s'' \in \mathcal{R}(\tilde{s})$. Indeed $\tilde{s} \in \text{Overrule}_T(s_1, s)$, so $s_1 \in \mathcal{R}(\tilde{s})$. Also we know $s' \in \mathcal{R}(s_1)$ (by 22)). Hence, by transitivity $s' \in \mathcal{R}(\tilde{s})$. We know also $s'' \in \mathcal{R}(s')$. So, by transitivity, $s'' \in \mathcal{R}(\tilde{s})$.
- But then *BeforeOverruled*(s'', s). Indeed by $s'' \in \mathcal{R}(\tilde{s})$, and $s'' \prec \tilde{s}$, $\Pi(s'', \tilde{s}, f(\tilde{s}))$. So, since $f(\tilde{s}) \neq f(s'')$ and $s'' <_T \tilde{s} <_T s$, then $\tilde{s} \in \text{Overruled}(s'', s)$. Since not-*Incuriam*($\tilde{s}$), *BeforeOverruled*(s'', s).
- contradiction because of *BeforeOverruled*(s'', s) and 4a).

26) By 20) and 25), $s', s'' \cong s_1$.

27) By B') and $s', s'' \cong s_1$, this means not-*Incuriam*(s_1).

28 By B') and $s', s'' \cong s_1$ this means $s_1 <_T s'$.

29) since $s_1 <_T s'$, and $s_1 \in \mathcal{R}(s')$ (by 22), then $\Pi(s_1, s', f(s_1))$.

- $s_1 \in \beta_{s'}(\Pi(s_1, s', f(s_1)))$ and $s' \cong s_1$ and the court $\tilde{c}$ in s' and s_1 is self-bound, by 9)).
- We know $f(s_1) \neq f(s')$. So *Against*(s', s_1).
- But by 1), not – *Incuriam*(s'). So, by definition of *per incuriam*, there is s_2 s.t. s_2 s.t. *According*(s', s_2). Doing similar reasoning done so far, it can be verified that not-*Incuriam*(s_2) and not-*BeforeOverruled*(s_2, s) and $s_2 <_T s_1$ – from the latter, it follows s_2 is different from all already found states. Also, $f(s'') \neq f(s_2)$, s_2 is potentially binding for s'' , so *Against*(s'', s_2). Then, there must be s_3 s.t. *According*(s'', s_3), different from all already found states ...

30) Applying reasoning similar to the one done so far, we iteratively find other cases $s_3, s_4, s_5 \dots$. It can be verified that these cases are different from already found cases.

We know that the set of states is finite. So we will reach a contradiction.

II) is a direct consequence of I) and of the fact that $s', s'' \in \bar{\beta}_s$.

□

Proposition 3.80. $\mathcal{R}_F$ is opposite-decisions left Euclidean.

Proof. Let $s, s', s'' \in S$, s.t. $f(s) = o$, $f(s') = \bar{o}$. Suppose $s \mathcal{R}_F s''$ and $s' \mathcal{R}_F s''$. Then we have both

$$\begin{aligned} s \cap \text{Facts}^o &\subseteq s'' \cap \text{Facts}^o \text{ and } s'' \cap \text{Facts}^{\bar{o}} \subseteq s \cap \text{Facts}^{\bar{o}} \\ s' \cap \text{Facts}^{\bar{o}} &\subseteq s'' \cap \text{Facts}^{\bar{o}} \text{ and } s'' \cap \text{Facts}^o \subseteq s' \cap \text{Facts}^o \end{aligned}$$

Hence,

$$s' \cap \text{Facts}^{\bar{o}} \subseteq s \cap \text{Facts}^{\bar{o}} \text{ and } s \cap \text{Facts}^o \subseteq s' \cap \text{Facts}^o$$

So $s \mathcal{R}_F s'$ and $s' \mathcal{R}_F s$.

□

Proposition 3.82. $\mathcal{R}_F$ is same-decisions transitive.

Proof. Let $s, s', s'' \in S$, s.t. $f(s) = f(s') = o$. Suppose $s \mathcal{R}_F s'$ and $s' \mathcal{R}_F s''$. Then we have both

$$\begin{aligned} s \cap \text{Facts}^o &\subseteq s' \cap \text{Facts}^o \text{ and } s' \cap \text{Facts}^{\bar{o}} \subseteq s \cap \text{Facts}^{\bar{o}} \\ s' \cap \text{Facts}^o &\subseteq s'' \cap \text{Facts}^o \text{ and } s'' \cap \text{Facts}^{\bar{o}} \subseteq s' \cap \text{Facts}^{\bar{o}} \end{aligned}$$

Hence,

$$s \cap \text{Facts}^o \subseteq s'' \cap \text{Facts}^o \text{ and } s'' \cap \text{Facts}^{\bar{o}} \subseteq s \cap \text{Facts}^{\bar{o}}$$

So $s \mathcal{R}_F s''$.

□

Proposition 3.85. Let $C = (S, f, \text{Jur}, <_T, \mathcal{R})$, a temporal jurisdictional classifier, with $<_T$ a total order, and $\mathcal{R}$ a same-decisions transitive and opposite-decisions left Euclidean relation. Let $s \in S$. Then $\nexists s', s'' \in \text{Best}_H(\bar{\beta}_s)$, s.t. $f(s') \neq f(s'')$.

Proof. Suppose by contradiction that $\exists s', s'' \in \text{Best}_H(\bar{\beta}_s)$, s.t. $f(s') \neq f(s'')$.

- 1) By definition, not $\text{Incuriam}(s')$ and not $\text{Incuriam}(s'')$.
- 2) Without loss of generality assume $s' <_T s''$, by that $<_T$ is a total order.
- 3) $s' \in \mathcal{R}(s'')$. Indeed $s' \in \mathcal{R}(s)$, $s'' \in \mathcal{R}(s)$. Then, since $\mathcal{R}$ is opposite-decisions euclidean, $s' \in \mathcal{R}(s'')$.
- 3a) since $s', s'' \in \text{Best}_H(\bar{\beta}_s)$, then $s \preceq s', s''$
- 4) not $\text{BeforeOverruled}_T(s', s)$, since $s' \in \bar{\beta}_s$.
- 5) since $s', s'' \in \text{Best}_H(\beta_s)$, by proposition 3.64, $s' \cong s''$, namely, there is $\tilde{c} = s' \cap \text{Courts} = s'' \cap \text{Courts}$.
- 6) Since, $s', s'' \in \text{Best}_H(\bar{\beta}_s)$, then $s', s'' \in \beta_s$. So $s', s'' <_T s$. By point 2) $s' <_T s'' <_T s$.
- 7) $\Pi(s', s'', f(s'))$, this follows from $s' \in \mathcal{R}(s'')$ (by point 3) and $s' <_T s''$ (point 2)).
- 9) $\mathcal{B}(\tilde{c}, \tilde{c})$. Indeed suppose not- $\mathcal{B}(\tilde{c}, \tilde{c})$. Then, by Def. 3.27, $O(\tilde{c}, \tilde{c})$. So by point 5) and 7) and $f(s') \neq f(s'')$, $O(s'', s')$. But then since $s'' <_T s$, and not- $\text{Incuriam}(s'')$, $\text{BeforeOverruled}_T(s', s)$. Contradiction, because of point 4).
- 10) By point 5), 7) and point 9), $\beta(s', s'', f(s'))$
 - since $\beta(s', s'', f(s'))$, $(s'', s') \in E_s$. Since $f(s') \neq f(s'')$, $\text{Against}(s', s'')$.
- 11) $s' \in \bar{\beta}_{s''}$. Indeed $\beta(s', s'', f(s'))$. By 1) not $\text{Incuriam}(s')$. Also not $\text{BeforeOverruled}_T(s', s'')$ (since not $\text{BeforeOverruled}_T(s', s)$ by point 4), and $s'' <_T s$. Hence, $s' \in \bar{\beta}_{s''}$.
- 12) We know that $(s'', s') \in E_s$ and $\text{Against}(s', s'')$. By 1) not $\text{Incuriam}(s'')$, then, by definition 3.42
 - A) $\nexists s' \in S$ s.t. $(s'', s') \in E_s$, $\text{Against}(s'', s')$, (not- $\text{Incuriam}(s')$), and $\forall \tilde{s} \in \text{Overrule}_T(s', s'') : \text{Incuriam}(\tilde{s})$. Or
 - B) $\exists s_1 \in S$ s.t. $(s'', s_1) \in E_s$ and $\text{According}(s'', s_1)$ and [not- $\text{Incuriam}(s_1)$], and not- $[\exists \tilde{s} \in \text{Overrule}_T(s_1, s'') : \text{not } \text{Incuriam}(\tilde{s})]$, and not- $[(s_1 <_T s' \text{ and } s' \cong s_1) \text{ or } s_1 \prec s']$.
- 13) A) is not the case. Indeed we know that $\text{Against}(s, s'')$, not- $\text{Incuriam}(s')$. Also we know, by 4), that not- $\text{BeforeOverruled}_T(s', s)$. So there there is no $\tilde{s} \in \text{Overruled}_T(s', s'')$ s.t. not $\text{Incuriam}(\tilde{s})$.
- 14) So B). Since $<_T$ is a total order and $\mathcal{H}$ is tree like this means that
 - B') $\exists s_1 \in S$ s.t. $(s'', s_1) \in E_s$ and $\text{According}(s'', s_1)$ and [not- $\text{Incuriam}(s_1)$], and $[\forall \tilde{s} \in \text{Overrule}_T(s_1, s'') : \text{Incuriam}(\tilde{s})]$, and $[(s' <_T s_1 \text{ and } s' \preceq s_1) \text{ or } s' \prec s_1]$
- 15) $f(s_1) = f(s'')$, since $\text{According}(s'', s_1)$.
- 16) $s_1 \in \mathcal{R}(s'')$, since $\text{According}(s'', s_1)$.

- 17) $s'', s' \preceq s_1$. Indeed since $According(s'', s_1)$, then $s'' \preceq s_1$. Also $s'' \cong s'$. Hence $s' \preceq s_1$.
- 18) $f(s_1) \neq f(s')$. Indeed by 15), $f(s_1) = f(s'')$. And we know by hypothesis that $f(s'') \neq f(s')$.
- 19) $s_1 \in \mathcal{R}(s')$ and $s' \in \mathcal{R}(s_1)$. Indeed, since $(s'', s_1) \in E_s$ then $s_1 \in \mathcal{R}(s'')$. Also $s' \in \mathcal{R}(s'')$. By opposite-direction euclideanity and 18), $s_1 \in \mathcal{R}(s')$ and $s' \in \mathcal{R}(s_1)$.
- 20) We distinguish two possibilities $s_1 \leq_T s'$ or $s' \leq_T s_1$.
- 21) Suppose $s_1 \leq_T s'$
- We know $s' \preceq s_1$. Then s' was bound by s_1 . So $s_1 \in \beta_{s'}(\Pi(s_1, s', f(s_1)))$ and s_1 is decided either by the same binding court or by a higher court than s' - in the latter case SD applies).
 - We know $f(s_1) \neq f(s')$. So $Against(s', s_1)$.
 - But by 1), *not* $-Incuriam(s')$. So, by definition of *per incuriam* be s_2 s.t. s_2 s.t. $According(s', s_2)$. In particular, similar reasoning done in point 11) applies. Notice that s_2 is different from all already found states.
- 22) Suppose $s' <_T s_1$.
- 23 If $s' \prec s_1$, then s_1 overruled s' , before a decision for s . Contradiction
- If $s' \cong s_1$, then s_1 was bound by s' . But if $s' \cong s_1$, then by B') *not-Incuriam*(s_1). So, by definition of *per incuriam* there must be s_3 s.t. $According(s_1, s_3)$. In particular, similar reasoning done in point 11) applies. Notice that s_3 is different from all already found states.
- 23) Applying reasoning similar to the one done so far, we iteratively find other cases s.t. $s_3, s_4, s_5, \dots$. It can be verified that these cases are different from already found cases. We know that the set of states is finite. So we will reach a contradiction.

□

C.1 Completeness

This entire section is dedicated to proving completeness; specifically, we want to establish the following results.

Proposition C.10 (Completeness TJCL). *TJCL is complete with respect to TJCM.*

Proposition C.11 (Completeness TJCL⁺). *TJCL⁺ is complete with respect to $\overline{\text{TJCM}}$.*

Firstly, we prove Proposition C.10. To this end, we need some intermediate definitions results. First of all we will recall some well-known definitions.

Definition C.12. [TJCL-consistency] A set of formulas Γ is TJCL- consistent if $\Gamma \not\vdash_{\text{TJCL}} \perp$, and TJCL-inconsistent otherwise.

Definition C.13 (TJCL-MCS). A set of formulas Γ is maximal TJCL-consistent if Γ is TJCL-consistent, and any set of formulas properly containing Γ is TJCL-inconsistent. If Γ is a maximal TJCL-consistent set of formulas then we say it is a TJCL-MCS.

The following proposition specifies some usual properties of MCSs.

Proposition C.14. *Let Γ be a MCS and let $\phi, \psi \in \mathcal{L}$. Then:*

- *if $\phi, \phi \rightarrow \psi \in \Gamma$ then $\psi \in \Gamma$*
- *$\phi \in \Gamma$ or $\neg\phi \in \Gamma$;*
- *$\phi \in \Gamma$ and $\psi \in \Gamma$ iff $\phi \wedge \psi \in \Gamma$*

We will focus on MCSs that are named, that is that contain a name.

Definition C.15. We say that a MCS, Γ is named iff there is $n \in \Gamma \cap \text{Names}$.

We propose a variant of Lindebaum's Lemma, whereby any set of formulas can be extended to a named MCS .

Lemma C.16 (Lindenbaum for names). *If Γ is a TJCL-consistent set of formulas then there is a named Γ^+ TJCL-MCS s.t. $\Gamma \subseteq \Gamma^+$.*

Proof. Let Γ a TJCL-consistent set of formulas.

- We prove that it can be extended to a named Γ^+ , namely s.t. there is $n \in \Gamma^+ \cap \text{Names}$. We distinguish two cases:

- there is $n \in \Gamma \cap \text{Names}$. Then, we simply apply Lindenbaum Lemma (standard see [18] pag. 197)
- there is not $n \in \Gamma \cap \text{Names}$. Then we enumerate elements in Names . Consider $\Gamma_n = \Gamma \cup \{n\}$, where n is the first element in the enumeration. Then Γ_n is consistent. Suppose it is not, then there must be θ in Γ , s.t. $\vdash n \rightarrow \neg\theta$. Since n , does not occur in Γ , from rule **NAME** it follows that $\vdash \neg\theta$. But this contradicts consistency of Γ . So Γ_n is consistent. Then we simply apply Lindabaum lemma to Γ_n .

□

In proving strong completeness, we will refer the following well-known result (see [18], pag 194).

Proposition C.17. *A logic Λ is strongly complete with respect to a class of structures $\mathcal{S}$ iff every Λ -consistent set of formulas is satisfiable on some $G \in \mathcal{S}$.*

To prove strong completeness, we mimic the standard construction of the canonical model. But, we actually construct a *temporal jurisdictional classifier model* for each set TJCL-consistent. Namely, we fix Θ , TJCL-consistent. We build, on the basis of named MCS, a *temporal jurisdictional classifier model*, in which Θ is satisfiable. Then, the result follows from C.17. In constructing the classifier model for Θ , from the MCSs, we shall ensure that:

- A) the universality of $\Box$ is captured in the model;
- B) the worlds of the model are subsets of 2^{Atm_0}
- C) a hierarchical structure can be constructed, i.e. hierarchical and binding relations globally satisfied in the model can be defined;
- D) each state of the model has a name

Actually, we will see that C) is a direct consequence of A) (and axioms **GlobHier**, **GlobBind**).

The proof proceeds as follows:

- We fix a TJCL-consistent set θ . Using our variant of the Lindenbaum Lemma, we know that we can extend θ to a named maximal consistent set, θ^+ .
- We fix Ψ^{Θ^+} the set of all $\Box$ and $\neg\Box$ formulas in the extension θ^+ . We prove Ψ^{Θ^+} is consistent Prop. C.18.
- So, by our variant of Lindenbaum Lemma, we can define E , the set of all named MCSs that extend Ψ^{Θ^+} (Def. C.19). By construction, all elements of E contain the same $\Box$ formulas (Proposition C.20) and the same formulas of the form $H(c_i, c_j), B(c_i, c_j)$ (Proposition C.21). This will allow us to guarantee points A) and C).
- Recall that we need to construct a classifier model, whose states are subsets of 2^{Atm_0} . So we partition the MCSs in E into equivalence classes, placing in the same equivalence class all MCSs that share exactly the same atoms in Atm_0 . We can then define the sets that will be later be the states of our model: these are given by the intersection of each equivalence class with Atm_0 (Def. C.22).
- We prove that all elements in an equivalence class contain the same and unique name (Prop C.23).
- We define a temporal and a relevance relation between the equivalence classes (Def. C.24). We prove that the temporal relation is transitive and reflexive (Prop C.25). We also prove a characterization of the temporal and relevance relations (Proposition C.26). This characterization allow us to prove that the temporal relation is a total relation (Prop C.27), and will later be also useful to prove the Existential lemma (Lemma C.32).
- We finally construct the classification model (Def C.28 and Prop C.29): each state is defined from an equivalence class; the function assigns to each state the unique decision atom in the MCSs in the equivalence class (unicity and existence of the decision atom is guaranteed by axioms **AtMostValue**, **AtLeastValue**); the the temporal and relevance relations between states are derived from the relations between the equivalence classes. The jurisdiction is defined relying on Prop. C.21.
- Once the model is constructed, we prove the existence and truth lemmas (Lemmas C.32 and C.33). Thus, we obtain completeness (Prop C.35).

So, we fix a TJCL-consistent set θ . We consider a named MCS Θ^+ that extends Θ (it exists for the Lindenbaum Lemma). We define $\Psi^{\Theta^+} = \{\Box\varphi \in \Theta^+\} \cup \{\neg\Box\varphi \in \Theta^+\}$.

Proposition C.18. Ψ^{Θ^+} is consistent.

Proof. $\Psi^{\Theta^+} \subseteq \Theta^+$ We know that Θ^+ is consistent. Hence the thesis. $\square$

In constructing the model, we will therefore only consider the TJCL-MCS extending Ψ^{Θ^+} . More precisely, we define the following.

Definition C.19. Let $E =_{def} \{\Gamma : \Gamma \text{ is named TJCL-MCS extending } \Psi^{\Theta^+}\}$.

Remind that Θ^+ is a maximal consistent set. Therefore, by construction, all elements of E contain the same $\Box$ formulas, this will guarantee A). Namely we have the following.

Proposition C.20. $\forall \Gamma \in E: \Box\varphi \in \Gamma \Leftrightarrow \Box\varphi \in \Theta^+$

As we will see, C) will follow directly from the following proposition.

Proposition C.21. Let $\Gamma \in E$. Hence

- i) $H(c_i, c_j) \in \Gamma \text{ iff } H(c_i, c_j) \in \Theta^+$
- ii) $B(c_i, c_j) \in \Gamma \text{ iff } B(c_i, c_j) \in \Theta^+$

Proof. We prove i). The proof for ii) is similar.

- Suppose $H(c_i, c_j) \in \Gamma$, then by axiom **GlobHier**, $\Box H(c_i, c_j) \in \Gamma$. Hence by definition of E (more precisely, prop. C.20), $\Box H(c_i, c_j) \in \Theta^+$. Hence, $H(c_i, c_j) \in \Theta^+$ (by ax. **T_□**).
- We prove the other direction by contraposition. Suppose $H(c_i, c_j) \notin \Theta^+$. Then $\neg H(c_i, c_j) \in \Theta^+$. Hence, $\neg\Box H(c_i, c_j) \in \Theta^+$ (from ax. **T_□**). So, by prop. C.20, $\neg\Box H(c_i, c_j) \in \Theta^+$. By axiom **GlobHier**, we can verify that $\neg H(c_i, c_j) \in \Theta^+$.

$\square$

To guarantee B), we will partition the MCSs according to the atoms in Atm_0 they share. On the basis of this partition we will then define the states of the classification model (these states will be the $s_{[\Gamma]}$ introduced following definition). More precisely we have the following.

Definition C.22. For all $\Gamma, \Gamma' \in E$, let $\Gamma \equiv_{Atm_0} \Gamma'$, iff $\Gamma \cap Atm_0 = \Gamma' \cap Atm_0$. $\equiv_{Atm_0}$ is an equivalence relation on E . For all $\Gamma \in E$, $[\Gamma] = \{\Gamma' \mid \Gamma \equiv_{Atm_0} \Gamma'\}$ is the equivalence class of Γ wrt to $\equiv_{Atm_0}$. $E / \equiv$ is the quotient set of E wrt to $\equiv_{Atm_0}$. For notational conciseness, we define $s_{[\Gamma]} =_{def} \Gamma' \cap Atm_0$, for any $\Gamma' \in [\Gamma]$.

We can prove that the same name is contained in every MCS of the same equivalence class. This will guarantee D).

Proposition C.23. Let $\Gamma \in E$ then $\exists n \in \Gamma \cap Names$, for all $\Gamma' \in [\Gamma]$.

Proof. • $\exists n \in \Gamma \cap Names$. This follows directly from Lindebaum Lemma.

- Let $n \in \Gamma \cap Names$. Let $\Gamma' \in [\Gamma]$. Then $\Gamma \cap Atm_0 = \Gamma' \cap Atm_0$. So, since $Names \subseteq Atm_0$, $n \in \Gamma'$.

□

Before defining the classification model, we need some additional notions and results for the construction of the temporal and relevance relations. More precisely, as we mentioned before, the states of the classifier model will be the $s_{[\Gamma]}$ defined from the equivalence classes $[\Gamma]$.

We will now define a preorder and a relevance relation between the equivalence classes. From these relations we will then induce a preorder and a notion of relevance between the states $s_{[\Gamma]}$.

Definition C.24. Let, $[\Gamma], [\Delta]$ (hence, $\Gamma, \Delta \in E$):

- $[\Gamma] \preceq [\Delta]$ iff $\forall \Gamma' \in [\Gamma], \exists \Delta' \in [\Delta]: \forall \varphi \in \mathcal{L} ([\leq_T] \varphi \in \Gamma' \Rightarrow \varphi \in \Delta')$
- $[\Gamma] R [\Delta]$ iff $\forall \Gamma' \in [\Gamma], \exists \Delta' \in [\Delta]: \forall \varphi \in \mathcal{L} (R^\forall \varphi \in \Gamma' \Rightarrow \varphi \in \Delta')$.

Proposition C.25. $\preceq$ is a pre-order on $E / \equiv$.

Proof. • Reflexivity: trivially for all $\Gamma' \in \llbracket \Gamma \rrbracket$, if $[\leq_T]\varphi \in \Gamma'$ then $\varphi \in \Gamma'$ (ax. $\mathbf{T}_{[\leq_T]}$). Hence $\llbracket \Gamma \rrbracket \preceq \llbracket \Gamma \rrbracket$

• Transitivity : suppose $\llbracket \Gamma \rrbracket \preceq \llbracket \Delta \rrbracket$ and $\llbracket \Delta \rrbracket \preceq \llbracket Y \rrbracket$. Let $\Gamma' \in \llbracket \Gamma \rrbracket$. Suppose $[\leq_T]\varphi \in \Gamma'$. Then $[\leq_T][\leq_T]\varphi \in \Gamma'$ (ax. $\mathbf{4}_{[\leq_T]}$). Then $\exists \Delta' \in \llbracket \Delta \rrbracket$ s.t. $[\leq_T]\varphi \in \Delta'$ by the assumption $\llbracket \Gamma \rrbracket \preceq \llbracket \Delta \rrbracket$. Hence, since $\llbracket \Delta \rrbracket \preceq \llbracket Y \rrbracket$, there is $Y' \in \llbracket Y \rrbracket$ s.t. $\varphi \in Y'$. So, $\llbracket \Gamma \rrbracket \preceq \llbracket Y \rrbracket$.

□

We prove a characterization of the temporal and relevance relations (Proposition C.26). This characterization allow us to prove that the temporal relation is a total relation (Prop C.27), and will later be also useful to prove the Existential lemma (Lemma C.32).

Proposition C.26. *Let $\Gamma, \Delta \in E$. Then:*

- i) $\llbracket \Gamma \rrbracket \preceq \llbracket \Delta \rrbracket \Leftrightarrow \exists \Gamma' \in \llbracket \Gamma \rrbracket, \exists \Delta' \in \llbracket \Delta \rrbracket: \forall \varphi \in \mathcal{L} ([\leq_T]\varphi \in \Gamma' \Rightarrow \varphi \in \Delta')$
- ii) $\llbracket \Gamma \rrbracket R \llbracket \Delta \rrbracket \Leftrightarrow \exists \Gamma' \in \llbracket \Gamma \rrbracket, \exists \Delta' \in \llbracket \Delta \rrbracket: \forall \varphi \in \mathcal{L} (R^\forall \varphi \in \Gamma' \Rightarrow \varphi \in \Delta')$

Proof. We prove i). The proof for ii) is similar.

$\Rightarrow$) trivial.

$\Leftarrow$) Suppose Γ' and Δ' satisfy the right-hand side of the equivalence. Let $\Gamma'' \in \llbracket \Gamma \rrbracket$. So, by proposition C.23 there is $n \in \Gamma' \cap \Gamma''$. Suppose $[\leq_T]\varphi \in \Gamma''$. Then, by axiom **nam1**, we have $@_n[\leq_T]\varphi \in \Gamma''$. Hence $\Box(n \rightarrow [\leq_T]\varphi) \in \Gamma''$. By proposition C.20, $\Box(n \rightarrow [\leq_T]\varphi) \in \Gamma'$. Hence $n \rightarrow [\leq_T]\varphi \in \Gamma'$ (ax $\mathbf{T}_\Box$). By **MP**, $[\leq_T]\varphi \in \Gamma'$. So, by hypothesis, $\varphi \in \Delta'$. Hence the thesis.

□

Proposition C.27. $\preceq$ is total on $E / \equiv$.

Proof. We prove the thesis using Prop.C.26. Let $\llbracket \Gamma \rrbracket$ and $\llbracket \Delta \rrbracket$ in $E / \equiv$. We prove that $\llbracket \Gamma \rrbracket \preceq \llbracket \Delta \rrbracket$ or $\llbracket \Delta \rrbracket \preceq \llbracket \Gamma \rrbracket$. Let $\Gamma' \in \llbracket \Gamma \rrbracket$; by proposition C.23, exists $n \in \Gamma'$. Let $\Delta' \in \llbracket \Delta \rrbracket$; by proposition C.23, exists $m \in \Delta'$. By Axiom **Total** $_{[\leq_T]}$, $@_n([\leq_T]\varphi \rightarrow @_m\varphi) \vee @_m([\leq_T]\varphi \rightarrow @_n\varphi)$.

- Suppose $@_n([\leq_T]\varphi \rightarrow @_m\varphi)$. We want to prove that if $[\leq_T]\varphi \in \Gamma'$ then $\varphi \in \Delta'$.

If so, by proposition C.26, $\llbracket \Gamma \rrbracket \preceq \llbracket \Delta \rrbracket$. So:

Suppose $[\leq_T]\varphi \in \Gamma'$. We now $n \in \Gamma$. Then, applying twice **MP**, $@_m\varphi \in \Gamma'$.

Hence $\Box(m \rightarrow \varphi) \in \Gamma'$. As consequence of Proposition C.20, $\Box(m \rightarrow \varphi) \in \Delta'$. So, $m \rightarrow \varphi \in \Delta'$. Since $m \in \Delta'$, then $\varphi \in \Delta'$.

Hence, $\llbracket \Gamma \rrbracket \preceq \llbracket \Delta \rrbracket$

- If $@_m([\leq_T]\varphi \rightarrow @_n\varphi)$, in a similar way we obtain that $\llbracket \Delta \rrbracket \preceq \llbracket \Gamma \rrbracket$

□

Finally, we can define the classifier model associated to Θ and Θ^+ .

Definition C.28 (Model associated to Θ and Θ^+). The temporal jurisdictional classifier model associated to Θ and Θ^+ , is the tuple $\mathcal{C} = (S^c, f^c, \leq^c, \mathcal{R}^c, Jur^c)$, where

- $S^c = \{s_{\llbracket \Gamma \rrbracket} \mid \Gamma \in E\} \subseteq 2^{Atm_0}$
- $f^c : S^c \rightarrow Val$, $f(s_{\llbracket \Gamma \rrbracket}) = o$ iff for all $\Gamma' \in \llbracket \Gamma \rrbracket$, $t(o) \in \Gamma'$.
- $s_{\llbracket \Gamma \rrbracket} \leq^c s_{\llbracket \Delta \rrbracket}$ iff $\llbracket \Gamma \rrbracket \preceq \llbracket \Delta \rrbracket$
- $s_{\llbracket \Gamma \rrbracket} \mathcal{R}^c s_{\llbracket \Delta \rrbracket}$ iff $\llbracket \Gamma \rrbracket R \llbracket \Delta \rrbracket$
- $Jur^c = (Courts, \mathcal{H}^c, \mathcal{B}^c)$ is defined as follows:
 - $c_i \mathcal{H}^c c_j$ iff $H(c_i, c_j) \in \Theta^+$
 - $c_i \mathcal{B}^c c_j$ iff $B(c_i, c_j) \in \Theta^+$

Proposition C.29. $\mathcal{C}$ is indeed a temporal jurisdictional classifier model.

Proof. • By definition, for all $\Gamma \in E$, $s_{\llbracket \Gamma \rrbracket} \subseteq 2^{Atm_0}$. Moreover, $\forall s_{\llbracket \Gamma \rrbracket} \in S^c$, $\exists! c \in Courts \cap s_{\llbracket \Gamma \rrbracket}$. This follows directly from Axioms **AtLeastCourt**, **AtMostCourt**

- f^c is a well defined function. Indeed, let $\Gamma', \Gamma'' \in s[\Gamma]$. By axiom **AtLeast-Value**, we know that there are $t(o) \in \Gamma'$ and $t(o') \in \Gamma''$. By Proposition C.23 and axiom **nam1**, we obtain that there is $n \in \text{Names}$ s.t. $@_n t(o) \in \Gamma'$. Namely $\Box(n \rightarrow t(o)) \in \Gamma'$. Hence, by Proposition C.20, we have that $\Box(n \rightarrow t(o)) \in \Gamma''$. Hence, $n \rightarrow t(o) \in \Gamma''$. By **MP**, and Proposition C.23, $t(o) \in \Gamma''$. So, by ax. **At-MostValue**, it must be $t(o') = t(o)$.
- $\leq^c$ is a total preorder. It follows directly from the fact that $\preceq$ is a preorder (Prop. C.25) and it is total (Prop. C.27)
- The set of decided states $S_d^c = \{s[\Gamma] \mid f^c(s[\Gamma]) \neq ?\}$ is finite. This follows from axiom **nam2** and the fact that Names_d is finite.
- Jur^c is a jurisdiction. We prove that $\mathcal{H}^c$ satisfied condition required in Definition 3.3:
 - $\mathcal{H}^c$ is irreflexive. Let $c_i \in \text{Courts}$, then not $c_i \mathcal{H}^c c_i$, since $\neg H(c_i, c_i) \in \Theta^+$ (from ax. **IrrHierarchy**).
 - $\mathcal{H}^c$ is transitive. Suppose $c_i \mathcal{H}^c c_j$ and $c_j \mathcal{H}^c c_k$. Then, $H(c_i, c_j) \in \Theta^+$ and $H(c_j, c_k) \in \Theta^+$. So, from **TrHierarchy**, $H(c_i, c_k) \in \Theta^+$. Hence, $c_i \mathcal{H}^c c_k$.
 - $\mathcal{H}^c$ is tree-like. Let $c_i, c_j, c_k \in \text{Courts}$, $c_j \neq c_k$ s.t. $\mathcal{H}^c(c_j, c_k)$ and $\mathcal{H}^c(c_j, c_i)$. Then $H(c_j, c_i) \in \Theta^+$ and $H(c_k, c_i) \in \Theta^+$. Then, from axiom **TreeLike-Hier**, $H(c_j, c_k) \in \Theta^+$ or $H(c_k, c_j) \in \Theta^+$. So, $\mathcal{H}^c(c_j, c_k)$ or $\mathcal{H}^c(c_k, c_j)$.
- We have to check the constraint on names:
 - $\forall s[\Gamma] \in S^c, \exists n \in \text{Names}: n \in s[\Gamma]$. This follows from proposition C.23.
 - $\forall s[\Gamma] \in S^c \forall n \in \text{Names} \cap s[\Gamma] : s[\Gamma] \in S_d^c \Leftrightarrow n \in \text{Names}_d$. This follows from axiom **nam2**.
 - Let $s[\Gamma], s[\Delta]$ s.t. $s[\Gamma] \neq s[\Delta]$. We prove that $s[\Gamma] \cap s[\Delta] \cap \text{Names} = \emptyset$:
 - * Since $s[\Gamma] \neq s[\Delta]$, then $[\Gamma] \neq [\Delta]$. Namely $\Gamma \cap \text{Atm}_0 \neq \Delta \cap \text{Atm}_0$. Without loss of generality we can assume that there is $p \in \Gamma \cap \text{Atm}_0$ s.t. $p \notin \Delta \cap \text{Atm}_0$ (@1).
 - * Assume by contradiction that exists $n \in s[\Gamma] \cap s[\Delta] \cap \text{Names}$. This means that $n \in \Gamma \cap \Delta$.

- * We know, by (@1) and previous point, that $p \wedge n \in \Gamma$. Hence, it can be proven that, $\Diamond(p \wedge n) \in \Gamma$. By axiom **nam1**, and **MP**, $@_n p \in \Gamma$. Hence, $\Box(n \rightarrow p) \in \Gamma$. By Prop C.20, we know that $\Box(n \rightarrow p) \in \Delta$; so, by **T** $_{\Box}$, $n \rightarrow p \in \Delta$. By **MP** $p \in \Delta$. Contradiction.

□

Once the model is constructed, we prove existential Lemma and truth lemma. To prove the existential Lemma for $R^{\forall}$, we need the followig two results.

Proposition C.30. *Let Γ a MCS. If $\exists \varphi \in \mathcal{L}$ s.t. $R^{\exists} \varphi \in \Gamma$ then $\nexists \psi \in \mathcal{L}$ s.t. $R^{\forall}(\psi) \wedge R^{\forall}(\neg \psi) \in \Gamma$*

Proof. Suppose, $R^{\exists} \varphi \in \Gamma$. Then $\neg R^{\forall} \neg \varphi \in \Gamma$. Suppose by contradiction that there is $\psi \in \mathcal{L}$ s.t. $R^{\forall}(\psi) \wedge R^{\forall}(\neg \psi) \in \Gamma$. We know by propositional logic, that $\vdash \psi \wedge \neg \psi \rightarrow \neg \varphi$. Then by **Nec** $_{R^{\forall}}$, **K** $_{R^{\forall}}$, and **MP**, $R^{\forall} \neg \varphi \in \Gamma$. Contradiction. □

Proposition C.31. $\vdash R^{\exists} \Box \varphi \rightarrow \Box \varphi$

Proof. First of all $\vdash \Diamond \Box \varphi \rightarrow \Box \varphi$. Indeed

- I. $\vdash \Box \varphi \rightarrow \Box \Box \varphi$.
- II. From I., $\vdash \Diamond \Box \varphi \rightarrow \Diamond \Box \Box \varphi$
- III. From **B** $_{\Box}$, $\vdash \Diamond \neg \varphi \rightarrow \Box \Diamond \Diamond \neg \varphi$
- IV. By contraposition and III. $\vdash \Diamond \Box \Box \varphi \rightarrow \Box \varphi$
- V. From II. and IV. $\vdash \Diamond \Box \varphi \rightarrow \Box \varphi$

We prove the thesis

2. $\vdash R^{\exists} \Box \varphi \leftrightarrow \neg R^{\forall} \neg \Box \varphi$
3. From, **MIX** $_{\Box, R^{\forall}}$, $\vdash \neg R^{\forall} \neg \Box \varphi \rightarrow \neg \Box \neg \Box \varphi$

4. $\vdash \neg \Box \neg \Box \varphi \rightarrow \Diamond \Box \varphi$
5. From 2,3,4, 1, $R^{\exists} \Box \varphi \rightarrow \Box \varphi$

□

We can finally prove the existential and truth lemma, and so completeness.

Proposition C.32 (Existence Lemma). *Let $\Gamma \in E$. Then*

- i) *If $\neg \Box \varphi \in \Gamma$, then there is $\Delta \in E$ s.t $\neg \varphi \in \Delta$*
- ii) *If $\neg [\leq_T] \varphi \in \Gamma$ then there is $\Delta \in E$ s.t $s_{[\Gamma]} \leq^c s_{[\Delta]}$ and $\neg \varphi \in \Delta$*
- iii) *If $\neg R^{\forall} \varphi \in \Gamma$ then there is $\Delta \in E$ s.t $s_{[\Gamma]} R^c s_{[\Delta]}$ and $\neg \varphi \in \Delta$*

Proof. i) Let $\Delta^- = \{\psi \mid \Box \psi \in \Theta^+\} \cup \{\neg \psi \mid \neg \Box \psi \in \Theta^+\}$. Then $\neg \varphi \in \Delta^-$.

- Δ^- is consistent. Indeed, suppose it is not. Then, there are $\Box \psi_1, \dots, \Box \psi_n \in \Theta^+$ and $\neg \Box \tau_1, \dots, \neg \Box \tau_m \in \Theta^+$ s.t $\vdash \neg (\bigwedge_i \psi_i \wedge \bigwedge_j \neg \tau_j)$. Namely, $\vdash \neg \bigwedge_i \psi_i \vee \bigvee_j \tau_j$. Then, $\vdash \bigwedge_i \psi_i \rightarrow \bigvee_j \tau_j$. From **Nec** $_{\Box}$, and **K** $_{\Box}$, we have $\vdash \bigwedge_i \Box \psi_i \rightarrow \bigvee_j \Box \tau_j$. Since $\bigwedge_i \Box \psi_i \in \Gamma$, then, by **MP**, $\bigvee_j \Box \tau_j \in \Gamma$. Contradiction, by Prop. C.20.
- We can extend Δ^- to a TJCL-MCS Δ . Then, $\neg \varphi \in \Delta$, by construction of Δ^- .
- $\Delta \in E$. Namely, $\Box \psi \in \Delta$ iff $\Box \psi \in \Theta^+$. Indeed, if $\Box \psi \in \Theta^+$ then $\Box \Box \psi \in \Theta^+$, so $\Box \psi \in \Delta$. Viceversa, suppose by contraposition that $\neg \Box \psi \in \Theta^+$. So, $\neg \psi \in \Delta$. Hence $\neg \Box \psi \in \Delta$ (by ax. **T** $_{\Box}$).

ii) $\Delta^- = \{\neg \varphi\} \cup \{\psi \mid \Box \psi \in \Theta^+\} \cup \{\psi \mid [\leq_T] \psi \in \Gamma\} \cup \{\neg \Box \psi \in \Theta^+\}$.

- Δ^- is consistent. Suppose by contradiction it is not. Namely, there are $\Box \psi_1, \dots, \Box \psi_n \in \Theta^+$, $[\leq_T] \xi_1, \dots, [\leq_T] \xi_m \in \Gamma$, $\neg \Box \tau_1, \dots, \neg \Box \tau_l \in \Theta^+$, s.t $\vdash \neg (\bigwedge_i \psi_i \wedge \bigwedge_j \xi_j \wedge \bigwedge_k \neg \tau_k \wedge \neg \varphi)$. So, $\vdash \neg (\bigwedge_i \psi_i \wedge \bigwedge_j \xi_j) \vee \neg (\bigwedge_k \neg \tau_k \wedge \neg \varphi)$. Then, $\vdash \neg (\bigwedge_i \psi_i \wedge \bigwedge_j \xi_j) \vee (\bigvee_k \Box \tau_k \vee \varphi)$. Hence, $\vdash (\bigwedge_i \psi_i \wedge \bigwedge_j \xi_j) \rightarrow (\bigvee_k \Box \tau_k \vee \varphi)$. By **Nec** $_{[\leq_T]}$ and **K** $_{[\leq_T]}$, $\vdash [\leq_T] (\bigwedge_i \psi_i \wedge \bigwedge_j \xi_j) \rightarrow [\leq_T] (\bigvee_k \Box \tau_k \vee \varphi)$. Hence, $\vdash (\bigwedge_i [\leq_T] \psi_i \wedge \bigwedge_j [\leq_T] \xi_j) \rightarrow \bigvee_k [\leq_T] \Box \tau_k \vee [\leq_T] \varphi$

Since $\bigwedge_i \Box \psi_i \in \Gamma$, then $\bigwedge_i [\leq_T] \psi_i \in \Gamma$ (by, **MIX** _{$\Box, [\leq_T]$}). Moreover, $\bigwedge_j [\leq_T] \xi_j \in \Gamma$. So, by **MP**, we have $\bigvee_k [\leq_T] \Box \tau_k \vee [\leq_T] \varphi \in \Gamma$. By hypothesis, $[\leq_T] \varphi \notin \Gamma$. Moreover since $\neg \Box \tau_k \in \Gamma$, we also know (**T** _{$[\leq_T]$}), that $\neg [\leq_T] \Box \tau_k \in \Gamma$. Hence contradiction. So Γ^- is consistent.

- Then, by Lindebaum Lemma we can extend Δ^- , to a MCS Δ . Δ is in E by construction. Namely, $\Box \psi \in \Delta$ iff $\Box \psi \in \Theta^+$. Indeed, if $\Box \psi \in \Theta^+$ then $\Box \Box \psi \in \Theta^+$, so $\Box \psi \in \Delta$. Viceversa, suppose that $\neg \Box \psi \in \Theta^+$, then by definition of Δ^- , $\neg \Box \psi \in \Delta$.
- Moreover, for all ψ , we know that if $[\leq_T] \psi \in \Gamma$ then $\psi \in \Delta$. Hence, by proposition C.26, $\llbracket \Gamma \rrbracket \preceq \llbracket \Delta \rrbracket$. So, $s_{\llbracket \Gamma \rrbracket} \leq^c s_{\llbracket \Delta \rrbracket}$.

iii) $\Delta^- = \{\neg \varphi\} \cup \{\psi \mid \Box \psi \in \Theta^+\} \cup \{\psi \mid R^\forall \psi \in \Gamma\} \cup \{\neg \Box \psi \in \Theta^+\}$.

- Δ^- is consistent. Suppose by contradiction it is not. Namely, there are $\Box \psi_1, \dots, \Box \psi_n \in \Theta^+$, $R^\forall \xi_1, \dots, R^\forall \xi_m \in \Gamma$, $\neg \Box \tau_1, \dots, \neg \Box \tau_l \in \Theta^+$, s.t. $\vdash \neg (\bigwedge_i \psi_i \wedge \bigwedge_j \xi_j \wedge \bigwedge_k \neg \Box \tau_k \wedge \neg \varphi)$. So, $\vdash \neg (\bigwedge_i \psi_i \wedge \bigwedge_j \xi_j) \vee \neg (\bigwedge_k \neg \Box \tau_k \wedge \neg \varphi)$. Then, $\vdash \neg (\bigwedge_i \psi_i \wedge \bigwedge_j \xi_j) \vee (\bigvee_k \Box \tau_k \vee \varphi)$. Hence, $\vdash (\bigwedge_i \psi_i \wedge \bigwedge_j \xi_j) \rightarrow (\bigvee_k \Box \tau_k \vee \varphi)$. By **Nec** _{$R^\forall$} and **K** _{$[\leq_T]$} , $\vdash R^\forall (\bigwedge_i \psi_i \wedge \bigwedge_j \xi_j) \rightarrow R^\forall (\bigvee_k \Box \tau_k \vee \varphi)$. Hence, $\vdash \bigwedge_i R^\forall \psi_i \wedge \bigwedge_j R^\forall \xi_j \rightarrow \bigvee_k R^\forall \Box \tau_k \vee R^\forall \varphi$. Since $\bigwedge_i \Box \psi_i \in \Gamma$, then $\bigwedge_i R^\forall \psi_i \in \Gamma$ (**MIX** _{$\Box, R^\forall$}). Moreover, $\bigwedge_j R^\forall \xi_j \in \Gamma$. So, by **MP**, we have $\bigvee_k R^\forall \Box \tau_k \vee R^\forall \varphi \in \Gamma$. By hypothesis, $R^\forall \varphi \notin \Gamma$. We can verify (see (+) below) that for all k , $\neg R^\forall \Box \tau_k \notin \Gamma$. Hence contradiction. Γ^- is consistent.

(+) Suppose by contradiction that $R^\forall \Box \tau_k \in \Gamma$. Then,

- * By Proposition C.31, since $\neg \Box \tau_k \in \Gamma$ then $\neg R^\exists \Box \tau_k \in \Gamma$. Namely $R^\forall \neg \Box \tau_k \in \Gamma$. Namely, $R^\forall \Box \tau_k \wedge R^\forall \neg \Box \tau_k \in \Gamma$.
- * By hypothesis $\neg R^\forall \varphi \in \Gamma$. Namely, $R^\exists \neg \varphi \in \Gamma$. Contradiction, because of Proposition C.30.

- By Lindebaum Lemma we can extend Δ^- , to a MCS Δ . Δ is in E by construction. Namely, $\Box \psi \in \Delta$ iff $\Box \psi \in \Theta^+$. Indeed, if $\Box \psi \in \Theta^+$ then $\Box \Box \psi \in \Theta^+$, so $\Box \psi \in \Delta$. Viceversa, suppose that $\neg \Box \psi \in \Theta^+$, then by definition of Δ^- , $\neg \Box \psi \in \Delta$.
- Moreover, for all ψ , we know that $R^\forall \psi \in \Gamma$ then $\psi \in \Delta$. Hence by proposition C.26, $\llbracket \Gamma \rrbracket R \llbracket \Delta \rrbracket$. So, $s_{\llbracket \Gamma \rrbracket} \mathcal{R}^c s_{\llbracket \Delta \rrbracket}$.

□

Lemma C.33 (Truth Lemma). *Let $\mathcal{C}$ the temporal jurisdictional classifier model associated to Θ and Θ^+ . Then for all $\Gamma \in E$, for all $\varphi \in \mathcal{L}$*

$$(\mathcal{C}, s_{\llbracket \Gamma \rrbracket}) \models \varphi \Leftrightarrow \varphi \in \Gamma$$

Proof. By induction on φ . Base case

- $(\mathcal{C}, s_{\llbracket \Gamma \rrbracket}) \models p \Leftrightarrow p \in s_{\llbracket \Gamma \rrbracket} \Leftrightarrow \forall \Gamma' \in \llbracket \Gamma \rrbracket, p \in \Gamma' \cap \text{Atm}_0 \Leftrightarrow p \in \Gamma$.
- $(\mathcal{C}, s_{\llbracket \Gamma \rrbracket}) \models \mathbf{t}(o) \Leftrightarrow f^c(s_{\llbracket \Gamma \rrbracket}) = o \Leftrightarrow \forall \Gamma' \in \llbracket \Gamma \rrbracket, \mathbf{t}(o) \in \Gamma' \cap \text{Atm}_0 \Leftrightarrow \mathbf{t}(o) \in \Gamma$.
- $(\mathcal{C}, s_{\llbracket \Gamma \rrbracket}) \models \mathbf{H}(c_i, c_j) \Leftrightarrow c_i \mathcal{H}^c c_j \Leftrightarrow \mathbf{H}(c_i, c_j) \in \Theta^+ \Leftrightarrow \mathbf{H}(c_i, c_j) \in \Gamma$ (by prop. C.21).
- $(\mathcal{C}, s_{\llbracket \Gamma \rrbracket}) \models \mathbf{B}(c_i, c_j) \Leftrightarrow c_i \mathcal{B}^c c_j \Leftrightarrow \mathbf{B}(c_i, c_j) \in \Theta^+ \Leftrightarrow \mathbf{B}(c_i, c_j) \in \Gamma$ (by prop. C.21).

Inductive Hypothesis: Assume the thesis holds for φ and ψ .

- $(\mathcal{C}, s_{\llbracket \Gamma \rrbracket}) \models \neg \varphi \Leftrightarrow (\mathcal{C}, s_{\llbracket \Gamma \rrbracket}) \not\models \varphi \Leftrightarrow (\text{inductive hyp.}) \varphi \notin \Gamma$
- $(\mathcal{C}, s_{\llbracket \Gamma \rrbracket}) \models \varphi \wedge \psi \Leftrightarrow (\mathcal{C}, s_{\llbracket \Gamma \rrbracket}) \models \varphi \text{ and } (\mathcal{C}, s_{\llbracket \Gamma \rrbracket}) \models \psi \Leftrightarrow (\text{inductive hyp.}) \varphi \in \Gamma \text{ and } \psi \in \Gamma \Leftrightarrow \varphi \wedge \psi \in \Gamma$
- Suppose $\neg \Box \varphi \in \Gamma$. Then By Existence Lemma, there is $\Delta \in E$ s.t $\neg \varphi \in \Delta$. Hence (ind. hypoth) $(\mathcal{C}, s_{\llbracket \Delta \rrbracket}) \not\models \varphi$. So, $(\mathcal{C}, s_{\llbracket \Gamma \rrbracket}) \not\models \Box \varphi$. Viceversa suppose that $\Box \varphi \in \Gamma$. Then, by definition of E (see prop. C.20), for all $\Delta \in E$, $\Box \varphi \in \Delta$. Hence (from ax. $\mathbf{T}_\Box$) for all $\Delta \in E$, $\varphi \in \Delta$. Hence, by ind. hypoth, for all $\Delta \in E$, $(\mathcal{C}, s_{\llbracket \Delta \rrbracket}) \models \varphi$. So $(\mathcal{C}, s_{\llbracket \Gamma \rrbracket}) \models \Box \varphi$
- Suppose $\neg [\leq_T] \varphi \in \Gamma$. Then By Existence Lemma, there is $\Delta \in E$ s.t $\neg \varphi \in \Delta$ and $s_{\llbracket \Gamma \rrbracket} \leq^c s_{\llbracket \Delta \rrbracket}$. Hence (ind. hypoth) $(\mathcal{C}, s_{\llbracket \Delta \rrbracket}) \not\models \varphi$. So $(\mathcal{C}, s_{\llbracket \Gamma \rrbracket}) \not\models [\leq_T] \varphi$. Viceversa, suppose that $[\leq_T] \varphi \in \Gamma$. Let $\Delta \in E$, s.t $s_{\llbracket \Gamma \rrbracket} \leq^c s_{\llbracket \Delta \rrbracket}$. Hence, by definition of $\leq^c$, there is $\Delta' \in \llbracket \Delta \rrbracket$ s.t. $\varphi \in \Delta'$. Hence, by inductive hypothesis $(\mathcal{C}, s_{\llbracket \Delta' \rrbracket}) \models \varphi$. Since $\Delta' \in \llbracket \Delta \rrbracket$, then $s_{\llbracket \Delta' \rrbracket} = s_{\llbracket \Delta \rrbracket}$. Hence, $(\mathcal{C}, s_{\llbracket \Delta \rrbracket}) \models \varphi$. So, $(\mathcal{C}, s_{\llbracket \Gamma \rrbracket}) \models [\leq_T] \varphi$.

- Suppose $\neg R^\forall \varphi \in \Gamma$. Then By Existence Lemma, there is $\Delta \in E$ s.t $\neg \varphi \in \Delta$ and $s_{[\Gamma]} \mathcal{R}^c s_{[\Delta]}$. Hence (ind. hypoth) $(\mathcal{C}, s_{[\Delta]}) \not\models \varphi$. So $(\mathcal{C}, s_{[\Gamma]}) \not\models R^\forall \varphi$. Viceversa, suppose that $R^\forall \varphi \in \Gamma$. Let $\Delta \in E$, s.t $s_{[\Gamma]} \mathcal{R}^c s_{[\Delta]}$. Hence, by definition of $\mathcal{R}^c$, there is $\Delta' \in [\Delta]$ s.t. $\varphi \in \Delta'$. Hence, by inductive hypothesis $(C, s_{[\Delta']}) \models \varphi$. Since $\Delta' \in [\Delta]$, then $s_{[\Delta']} = s_{[\Delta]}$. Hence, $(C, s_{[\Delta]}) \models \varphi$. So, $(\mathcal{C}, s_{[\Gamma]}) \models R^\forall \varphi$.

□

Finally, we can prove completeness.

Proposition C.34 (Completeness TJCL). *TJCL is complete with respect to TJCM.*

Proof. Let Θ be a L - consistent set of formulas. Let Θ^+ a named MCS s.t. $\Theta \subseteq \Theta^+$. Define $\mathcal{C}$ the temporal jurisdictional classifier model associated to Θ and Θ^+ . Notice that $\Theta^+ \in E$. Then by Truth lemma, $(\mathcal{C}, s_{[\Theta^+]}) \models \Theta$. Hence, from Proposition C.17, the thesis. □

The proof of the completeness of $TJCL^+$ with respect to $\overline{TJCM}$ is almost the same as that of Proposition C.10.

Proposition C.35 (Completeness $TJCL^+$). *$TJCL^+$ is complete with respect to $\overline{TJCM}$.*

sketch. The proof is almost the same as that of Proposition C.10. We just need to prove that the canonical model defined in a similar way as in C.28, satisfies SD. This is a consequence of Axiom **SL**.

$\mathcal{H}^c \subseteq \mathcal{B}^c$. Indeed, let c_i, c_j s.t. $c_i \mathcal{H}^c c_j$. As in Proposition C.29 we can prove that $\mathcal{H}$ is irreflexive so $c_i \neq c_j$. We know that $H(c_i, c_j) \in \Theta^+$. Then, by axiom **SL** $B(c_i, c_j) \in \Theta^+$. So, $\mathcal{B}(c_i, c_j)$.

$\mathcal{B}^c \subseteq \mathcal{H}^c \cup \mathcal{I}$. Indeed, suppose by contradiction that $\mathcal{B} \not\subseteq \mathcal{H} \cup \mathcal{I}$. Then there are let c_i, c_j , $c_i \neq c_j$ s.t. $c_i \mathcal{B}^c c_j$ and not- $c_i \mathcal{H}^c c_j$. Since $c_i \mathcal{B}^c c_j$, then $B(c_i, c_j) \in \Theta^+$ (#1). Since not- $c_i \mathcal{H}^c c_j$, then $H(c_i, c_j) \notin \Theta^+$. So, $\neg H(c_i, c_j) \in \Theta^+$. by Axiom **SL**, $\neg B(c_i, c_j) \in \Theta^+$. $B(c_i, c_j) \notin \Theta^+$. Contradiction (because of (#1)).

□

D

Appendix Chapter 4

D.1 Again on the Emptyset

In section 4.6.7 we proposed principles for empty-set monotonicity. We started by the intuition that in the binding and overruling relation $\emptyset$ means "always". We also outlined that the empty-set monotonicity principle might require specifying numerous binding and overruling relations if the number of conditions applicable to a court is large. For this reason, it may be reasonable to not impose them. However some undesirable consequences may follow without empty-set monotonicity. We propose here some of them.

Consider for example $\mathcal{B}(c_i, c_j \mid \emptyset, X_j)$. The relation means that court c_i is always binding — regardless of the conditions c_i satisfies — on court c_j when the latter acts under conditions X_j . If we adhere to criterion **I** , then we should require that c_j , under the conditions X_j is never binding on c_i — regardless of the conditions c_i satisfies. Namely we expect that for all X_i , $\text{not-}\mathcal{B}(c_j, c_i \mid X_j, X_i)$.

Similarly, if we adhere to criterion **III**) we expect that c_j , under the conditions X_j can never overrule c_i — regardless of the conditions c_i satisfies — otherwise c_i would not be binding for c_j . Namely we expect that for all X_i , $\text{not-}\mathcal{O}(c_j, c_i \mid X_j, X_i)$.

The same reasoning applies to $\mathcal{O}(c_i, c_j \mid \emptyset, X_j)$, and to $\mathcal{O}(c_i, c_j \mid X_i, \emptyset)$, $\mathcal{B}(c_i, c_j \mid X_i, \emptyset)$.

Notice that the constraints we have just discussed may resemble the **AS**-principles. However, the **AS**-principles alone are not sufficient to guarantee these constraints; additional principles are required.

$$[c_i \neq c_j \text{ and } Rel_1(c_i, c_j \mid \emptyset, X_j)] \Rightarrow \text{not-}Rel_2(c_j, c_i \mid X_j, X_i),$$

$$\text{for } Rel_1, Rel_2 \in \{\mathcal{B}, \mathcal{O}\}.$$
¹ (E1)

$$[c_i \neq c_j \text{ and } Rel_1(c_i, c_j \mid X_i, \emptyset)] \Rightarrow \text{not-}Rel_2(c_j, c_i \mid X_j, X_i),$$

$$\text{for } Rel_1, Rel_2 \in \{\mathcal{B}, \mathcal{O}\}.$$
² (E2)

$$Rel_1(c_i, c_i \mid \emptyset, X_j) \Rightarrow \text{not-}Rel_2(c_i, c_i \mid X_i, X_j),$$

$$\text{for } Rel_1, Rel_2 \in \{\mathcal{B}, \mathcal{O}\} \text{ and } Rel_1 \neq Rel_2.$$
 (E3)

$$Rel_1(c_i, c_i \mid X_j, \emptyset) \Rightarrow \text{not-}Rel_2(c_i, c_i \mid X_i, X_j),$$

$$\text{for } Rel_1, Rel_2 \in \{\mathcal{B}, \mathcal{O}\} \text{ and } Rel_1 \neq Rel_2.$$
 (E4)

Similarly, consider $\mathcal{B}(c_i, c_j \mid \emptyset, \emptyset)$. This relation indicates that court c_i is always binding—regardless of the conditions c_i satisfies—on court c_j , regardless of the conditions c_j satisfies. If we adhere to criterion **I**, then we should require that c_j is never binding on c_i —regardless of the conditions c_i and c_j satisfy. In other words, we expect that for all X_i, X_j , $\text{not-}\mathcal{B}(c_j, c_i \mid X_j, X_i)$. The same reasoning applies to $\mathcal{O}(c_i, c_j \mid \emptyset, \emptyset)$.

$$[c_i \neq c_j \text{ and } Rel_1(c_i, c_j \mid \emptyset, \emptyset)] \Rightarrow \text{not-}Rel_2(c_j, c_i \mid X_j, X_i),$$

$$\text{for } Rel_1, Rel_2 \in \{\mathcal{B}, \mathcal{O}\}.$$
³ (E5)

$$Rel_1(c_i, c_i \mid \emptyset, \emptyset) \Rightarrow \text{not-}Rel_2(c_i, c_i \mid X_i, X_j),$$

$$\text{for } Rel_1, Rel_2 \in \{\mathcal{B}, \mathcal{O}\} \text{ and } Rel_1 \neq Rel_2.$$
 (E6)

Finally, suppose to have $\mathcal{B}(c_i, c_j \mid \emptyset, X_j)$ and $\mathcal{B}(c_j, c_i \mid \emptyset, X_i)$.

The first relation, $\mathcal{B}(c_i, c_j \mid \emptyset, X_j)$, states that c_i is always binding on c_j , when c_j acts under conditions X_j . Then we expect that if c_i satisfies condition X_i , then it is still

¹We require $c_i \neq c_j$ to not impose for example $\mathcal{B}(c_i, c_i \mid \emptyset, X_j)$ and $\text{not-}\mathcal{B}(c_i, c_i \mid X_i, X_j)$. Indeed, this would be not in line with the interpretation of $\mathcal{B}(c_i, c_i \mid \emptyset, X_j)$, according to which c_i under any set of condition (and so also under X_i) is binding for c_i under set of condition X_j . In any case it could be legitimate to impose **E3** which specifies analogous of **E1** for $\mathcal{B}(c_i, c_o \mid \emptyset, X_j)$.

²We require $c_i \neq c_j$ for similar reason reported in footnote 1.

³We require $c_i \neq c_j$ to not impose for example $\mathcal{B}(c_i, c_i \mid \emptyset, \emptyset)$ and $\text{not-}\mathcal{B}(c_i, c_i \mid X_i, X_i)$. In any case it could be legitimate to impose **E6** which specifies analogous of **E4** for $\mathcal{B}(c_i, c_i \mid \emptyset, \emptyset)$.

binding on c_j acting under conditions X_j .

The second relation, $\mathcal{B}(c_j, c_i \mid \emptyset, X_i)$, states that c_j is always binding on c_i , when c_i acts under conditions X_i . Then, we expect that if c_j satisfies condition X_j , then it is still binding on c_i acting under conditions X_i .

These two consequences, however, are in conflict according to criterion **I** .

For similar reasons, having both $\mathcal{O}(c_i, c_j \mid \emptyset, X_j)$ and $\mathcal{O}(c_j, c_i \mid \emptyset, X_i)$ would conflict with criterion **II**). Likewise, having both $\mathcal{O}(c_i, c_j \mid \emptyset, X_j)$ and $\mathcal{B}(c_j, c_i \mid \emptyset, X_i)$ would conflict with criterion **III**).

$$[(X_i \neq X_j \text{ or } c_i \neq c_j) \text{ and } Rel_1(c_i, c_j \mid \emptyset, X_j)] \Rightarrow \text{not-}Rel_2(c_j, c_i \mid \emptyset, X_i),$$

for $Rel_1, Rel_2 \in \{\mathcal{B}, \mathcal{O}\}$.⁴ (E7)

$$[(X_i \neq X_j \text{ or } c_i \neq c_j) \text{ and } Rel_1(c_i, c_j \mid X_i, \emptyset)] \Rightarrow \text{not-}Rel_2(c_j, c_i \mid X_j, \emptyset),$$

for $Rel_1, Rel_2 \in \{\mathcal{B}, \mathcal{O}\}$. (E8)

$$Rel_1(c_i, c_i \mid \emptyset, X_j) \Rightarrow \text{not-}Rel_2(c_i, c_i \mid \emptyset, X_j),$$

for $Rel_1, Rel_2 \in \{\mathcal{B}, \mathcal{O}\}$ and $Rel_1 \neq Rel_2$. (E9)

$$Rel_1(c_i, c_i \mid X_i, \emptyset) \Rightarrow \text{not-}Rel_2(c_i, c_i \mid X_i, \emptyset),$$

for $Rel_1, Rel_2 \in \{\mathcal{B}, \mathcal{O}\}$ and $Rel_1 \neq Rel_2$. (E10)

We denote **E** the conjunction of all E-principles.

$$\mathbf{E1 \& E2 \& E4 \& E3 \& E4 \& E7 \& E8 \& E9 \& E10} \quad (\mathbf{E})$$

Example D.1. *Both jurisdiction of England and Wales and the US jurisdictions satisfies condition **E**.*

We introduced the **E**-principles in relation to empty-set monotonicity principles. And we also noticed that **E**-principles may resemble **AS**-principles. Is there a connection? It can be can verified that the empty-set monotonicity principles **E-Mon** and **AS**-principles, together with the admissibility principle **Adm**, allows for retrieving all the **E**-principles.

Proposition D.1. ***E-Mon** & **AS** & **Adm** $\Rightarrow$ **E***

⁴We require $X_i \neq \tilde{X}_i$ or $c_i \neq c_j$ to not impose for example $\mathcal{B}(c_i, c_i \mid \emptyset, X_i)$ and $\text{not-}\mathcal{B}(c_i, c_i \mid \emptyset, X_i)$.

D.2 Language-level Principles

We present the language-level representations of the principles introduced in Section 4.6. For each principle, the corresponding language-level equivalent bears the same name preceded by the letter L; for example, the equivalent of principle **StOv** is denoted as **LStOv**.

$$\mathcal{O}(c_i, c_j \mid X_i, X_j) \rightarrow \bigvee_{c_k \in \text{Courts}, X_k \subseteq \text{Cond}_{c_k}} \mathcal{B}(c_j, c_k \mid X_j, X_k) \quad (\text{LStOv})$$

$$\mathcal{B}(c_i, c_j \mid X_i, X_j) \rightarrow \neg \mathcal{B}(c_j, c_i \mid X_j, X_i), \text{ for } X_i \neq X_j \text{ or } c_i \neq c_j \quad (\text{LAS1})$$

$$\mathcal{O}(c_i, c_j \mid X_i, X_j) \rightarrow \neg \mathcal{O}(c_j, c_i \mid X_j, X_i), \text{ for } X_i \neq X_j \text{ or } c_i \neq c_j \quad (\text{LAS2})$$

$$\mathcal{B}(c_i, c_j \mid X_i, X_j) \rightarrow \neg \mathcal{O}(c_j, c_i \mid X_j, X_i) \quad (\text{LAS3})$$

$$\mathcal{O}(c_i, c_j \mid X_i, X_j) \rightarrow \neg \mathcal{B}(c_j, c_i \mid X_j, X_i) \quad (\text{LAS4})$$

$$\mathcal{B}(c_i, c_j \mid X_i, X_j) \rightarrow \neg \mathcal{B}(c_j, c_i \mid \tilde{X}_j, \tilde{X}_i), \text{ for } X_i \neq \tilde{X}_i \text{ or } X_j \neq \tilde{X}_j \text{ or } c_i \neq c_j \quad (\text{LCB2})$$

$$\mathcal{O}(c_i, c_j \mid X_i, X_j) \rightarrow \neg \mathcal{O}(c_j, c_i \mid \tilde{X}_j, \tilde{X}_i), \text{ for } X_i \neq \tilde{X}_i \text{ or } X_j \neq \tilde{X}_j \text{ or } c_i \neq c_j \quad (\text{LCO2})$$

$$(\mathcal{O}(c_i, c_j \mid X_i, X_j) \rightarrow \neg \mathcal{B}(c_j, c_i \mid \tilde{X}_j, \tilde{X}_i)) \wedge (\mathcal{B}(c_i, c_j \mid X_i, X_j) \rightarrow \neg \mathcal{O}(c_j, c_i \mid \tilde{X}_j, \tilde{X}_i)) \quad (\text{LCBO2})$$

$$\mathcal{B}(c_i, c_j \mid X_i, X_j) \wedge \mathcal{B}(c_j, c_i \mid \tilde{X}_j, \tilde{X}_i) \rightarrow \text{ME}(X_i, \tilde{X}_i) \vee \text{ME}(X_j, \tilde{X}_j), \text{ for } X_i \neq \tilde{X}_i \text{ or } X_j \neq \tilde{X}_j \text{ or } c_i \neq c_j \quad (\text{LCB3})$$

$$\mathcal{B}(c_i, c_j \mid X_i, X_j) \wedge \mathcal{B}(c_j, c_i \mid \tilde{X}_j, \tilde{X}_i) \rightarrow \text{ME}(X_i, \tilde{X}_i) \vee \text{ME}(X_j, \tilde{X}_j), \text{ for } X_i \neq \tilde{X}_i \text{ or } X_j \neq \tilde{X}_j \text{ or } c_i \neq c_j : \quad (\text{LCB3})$$

$$\mathcal{O}(c_i, c_j \mid X_i, X_j) \wedge \mathcal{O}(c_j, c_i \mid \tilde{X}_j, \tilde{X}_i) \rightarrow \text{ME}(X_i, \tilde{X}_i) \vee \text{ME}(X_j, \tilde{X}_j), \text{ for } X_i \neq \tilde{X}_i \text{ or } X_j \neq \tilde{X}_j \text{ or } c_i \neq c_j : \quad (\text{LCO3})$$

$$\mathcal{O}(c_i, c_j \mid X_i, X_j) \wedge \mathcal{B}(c_j, c_i \mid \tilde{X}_j, \tilde{X}_i) \rightarrow \text{ME}(X_i, \tilde{X}_i) \vee \text{ME}(X_j, \tilde{X}_j) \quad (\text{LCBO3})$$

$$\mathcal{O}(c_i, c_j \mid X_i, X_j) \wedge \mathcal{B}(c_j, c_i \mid \tilde{X}_j, \tilde{X}_i) \rightarrow \text{ME}(X_i, \tilde{X}_i) \wedge \text{ME}(X_j, \tilde{X}_j) \quad (\text{LCBO4})$$

$$\bigwedge_{c_i \neq c_j} \left((\mathcal{H}(c_i, c_j) \rightarrow \mathcal{B}(c_i, c_j \mid \emptyset, \emptyset)) \wedge (\neg \mathcal{H}(c_i, c_j) \rightarrow \neg \bigvee_{X_i, X_j} \mathcal{B}(c_i, c_j \mid X_i, X_j)) \right) \quad (\text{LsBSD})$$

$$\bigwedge_{c_i \neq c_j} \left((\mathsf{H}(c_i, c_j) \rightarrow \mathsf{O}(c_i, c_j \mid \emptyset, \emptyset)) \wedge (\neg \mathsf{H}(c_i, c_j) \rightarrow \neg \bigvee_{X_i, X_j} \mathsf{O}(c_i, c_j \mid X_i, X_j)) \right) \quad (\mathbf{LsOSD})$$

$$\bigwedge_{c_i \neq c_j} \mathsf{B}(c_i, c_j \mid X_i, X_j) \rightarrow \mathsf{H}(c_i, c_j) \quad (\mathbf{LBSD})$$

$$\bigwedge_{c_i \neq c_j} \mathsf{O}(c_i, c_j \mid X_i, X_j) \rightarrow \mathsf{H}(c_i, c_j) \quad (\mathbf{LOSD})$$

$$\bigwedge_{c_1 \neq c_3} (\mathsf{Op}_1(c_2, c_1 \mid X_1, X_2) \wedge \mathsf{Op}_2(c_3, c_2 \mid X_3, \tilde{X}_2) \rightarrow \mathsf{H}(c_3, c_1)), \text{ where } \mathsf{Op}_i = \mathsf{B} \text{ or } \mathsf{Op}_i = \mathsf{O} \text{ for } i \in \{1, 2\} \quad (\mathbf{LHBO})$$

$$\mathsf{B}(c_i, c_j \mid X_i, X_j) \leftrightarrow \mathsf{O}(c_i, c_j \mid X_i, X_j), \text{ for } c_i \neq c_j \quad (\mathbf{LS1})$$

$$\mathsf{B}(c, c \mid X_i, X_j) \leftrightarrow \mathsf{O}(c, c \mid X_i, X_j), \text{ for } X_i \neq X_j, X_i \neq \emptyset, X_j \neq \emptyset \quad (\mathbf{LS2})$$

$$\mathsf{B}(c_i, c_j \mid X_i, \emptyset) \rightarrow \bigwedge_{X_j \subseteq \text{Cond}_{c_j}, \text{Adm}(X_j)} \mathsf{B}(c_i, c_j \mid X_i, X_j) \quad (\mathbf{LE-MonB1})$$

$$\mathsf{B}(c_i, c_j \mid \emptyset, X_j) \rightarrow \bigwedge_{X_i \subseteq \text{Cond}_{c_i}, \text{Adm}(X_i)} \mathsf{B}(c_i, c_j \mid X_i, X_j) \quad (\mathbf{LE-MonB2})$$

$$\mathsf{O}(c_i, c_j \mid X_i, \emptyset) \rightarrow \bigwedge_{X_j \subseteq \text{Cond}_{c_j}, \text{Adm}(X_j)} \mathsf{O}(c_i, c_j \mid X_i, X_j) \quad (\mathbf{LE-MonO1})$$

$$\mathsf{O}(c_i, c_j \mid \emptyset, X_j) \rightarrow \bigwedge_{X_i \subseteq \text{Cond}_{c_i}, \text{Adm}(X_i)} \mathsf{O}(c_i, c_j \mid X_i, X_j) \quad (\mathbf{LE-MonO2})$$

$$\mathsf{B}(c_i, c_j \mid X_i, X_j) \rightarrow \bigwedge_{X_j \subseteq X_k \subseteq \text{Cond}_{c_j}, \text{Adm}(X_k)} \mathsf{B}(c_i, c_j \mid X_i, X_k) \quad (\mathbf{L-MonB1})$$

$$\mathsf{B}(c_i, c_j \mid X_i, X_j) \rightarrow \bigwedge_{X_i \subseteq X_k \subseteq \text{Cond}_{c_i}, \text{Adm}(X_k)} \mathsf{B}(c_i, c_j \mid X_k, X_j) \quad (\mathbf{L-MonB2})$$

$$\mathsf{O}(c_i, c_j \mid X_i, X_j) \rightarrow \bigwedge_{X_j \subseteq X_k \subseteq \text{Cond}_{c_j}, \text{Adm}(X_k)} \mathsf{O}(c_i, c_j \mid X_k, X_j) \quad (\mathbf{L-MonO1})$$

$$\mathsf{O}(c_i, c_j \mid X_i, X_j) \rightarrow \bigwedge_{X_i \subseteq X_k \subseteq \text{Cond}_{c_i}, \text{Adm}(X_k)} \mathsf{O}(c_i, c_j \mid X_k, X_j) \quad (\mathbf{L-MonO2})$$

$$\bigwedge_{c_i \in \text{Courts}} \left(\bigwedge_{X, Y \subseteq \text{Cond}_{c_i}, X \neq Y, X \neq \emptyset, Y \neq \emptyset} \text{ME}(X, Y) \right) \quad (\mathbf{LME1})$$

$$B(c_2, c_1 \mid X_2, X_1) \wedge B(c_3, c_2 \mid X_3, X_2) \rightarrow B(c_3, c_1 \mid X_3, X_1) \quad (\text{LTB})$$

$$O(c_2, c_1 \mid X_2, X_1) \wedge O(c_3, c_2 \mid X_3, X_2) \rightarrow O(c_3, c_1 \mid X_3, X_1) \quad (\text{LTO})$$

$$(B(c_2, c_1 \mid X_2, X_1) \wedge O(c_3, c_2 \mid X_3, X_2) \rightarrow O(c_3, c_1 \mid X_3, X_1)) \wedge \\ (O(c_2, c_1 \mid X_2, X_1) \wedge B(c_3, c_2 \mid X_3, X_2) \rightarrow B(c_3, c_1 \mid X_3, X_1)) \quad (\text{LTBO})$$

$$Op_1(c_2, c_1 \mid X_2, X_1) \wedge Op_2(c_3, c_2 \mid X_3, \tilde{X}_2) \wedge \dots \wedge Op_n(c_1, c_n \mid X_1, \tilde{X}_n) \rightarrow \\ \bigvee_{i \in \{1, \dots, n\}} ME(X_i, \tilde{X}_i) \vee \bigwedge_{j \in \{1, \dots, n-1\}} Op_j = Op_{j+1}$$

$$\text{where } c_i \in \text{Courts}, X_i, \tilde{X}_i \subseteq \text{Cond}, Op_i = B \text{ or } Op_i = O \text{ for } i \in \{1, \dots, n\} \quad (\text{LL1})$$

$$Op_1(c_i, c_j \mid \emptyset, X_j) \rightarrow \neg Op_2(c_j, c_i \mid X_j, X_i), \text{ for } c_i \neq c_j \text{ and } Op_1, Op_2 \in \{B, O\} \quad (\text{LE1})$$

$$Op_1(c_i, c_j \mid X_i, \emptyset) \rightarrow \neg Op_2(c_j, c_i \mid X_j, X_i), \text{ for } c_i \neq c_j \text{ and } Op_1, Op_2 \in \{B, O\} \quad (\text{LE2})$$

$$Op_1(c_i, c_i \mid \emptyset, X_j) \rightarrow \neg Op_2(c_i, c_i \mid X_i, X_j), \text{ for } Op_1, Op_2 \in \{B, O\} \text{ and } Op_1 \neq Op_2 \quad (\text{LE3})$$

$$Op_1(c_i, c_i \mid X_j, \emptyset) \rightarrow \neg Op_2(c_i, c_i \mid X_i, X_j), \text{ for } Op_1, Op_2 \in \{B, O\} \text{ and } Op_1 \neq Op_2 \quad (\text{LE4})$$

$$Op_1(c_i, c_j \mid \emptyset, \emptyset) \rightarrow \neg Op_2(c_j, c_i \mid X_j, X_i), \text{ for } c_i \neq c_j \text{ and } Op_1, Op_2 \in \{B, O\} \quad (\text{LE5})$$

$$Op_1(c_i, c_i \mid \emptyset, \emptyset) \rightarrow \neg Op_2(c_i, c_i \mid X_i, X_j), \text{ for } Op_1, Op_2 \in \{B, O\} \text{ and } Op_1 \neq Op_2 \quad (\text{LE6})$$

$$Op_1(c_i, c_j \mid \emptyset, X_j) \rightarrow \neg Op_2(c_j, c_i \mid \emptyset, X_i), \text{ for } (X_i \neq X_j \text{ or } c_i \neq c_j) \text{ and } Op_1, Op_2 \in \{B, O\} \quad (\text{LE7})$$

$$Op_1(c_i, c_j \mid X_i, \emptyset) \rightarrow \neg Op_2(c_j, c_i \mid X_j, \emptyset), \text{ for } (X_i \neq X_j \text{ or } c_i \neq c_j) \text{ and } Op_1, Op_2 \in \{B, O\} \quad (\text{LE8})$$

$$Op_1(c_i, c_i \mid \emptyset, X_j) \rightarrow \neg Op_2(c_i, c_i \mid \emptyset, X_j), \text{ for } Op_1, Op_2 \in \{B, O\} \text{ and } Op_1 \neq Op_2 \quad (\text{LE9})$$

$$Op_1(c_i, c_i \mid X_i, \emptyset) \rightarrow \neg Op_2(c_i, c_i \mid X_i, \emptyset), \text{ for } Op_1, Op_2 \in \{B, O\} \text{ and } Op_1 \neq Op_2 \quad (\text{LE10})$$

D.3 Completeness

Proposition 4.7. *TJCL-Cond is sound and complete with respect to TJCM – Cond.*

The proof of completeness essentially follows that given for TJCL (Proposition C.10 in Section C.1). In particular, Proposition C.21 and Definition C.28 and Proposition C.29, must be revised to take into account the conditions and the overruling operator.

Proposition D.2 and Definition D.3 and Proposition D.4 below are respectively the equivalent of Proposition C.21, Definition C.28 and Proposition C.29.

Proposition D.2. *Let $\Gamma \in E$. Hence*

- i) $H(c_i, c_j) \in \Gamma$ iff $H(c_i, c_j) \in \Theta^+$*
- ii) $B(c_i, c_j \mid X_i, X_j) \in \Gamma$ iff $B(c_i, c_j \mid X_i, X_j) \in \Theta^+$*
- iii) $O(c_i, c_j \mid X_i, X_j) \in \Gamma$ iff $O(c_i, c_j \mid X_i, X_j) \in \Theta^+$*
- iv) $INC(X_i, X_j) \in \Gamma$ iff $INC(X_i, X_j) \in \Theta^+$*

Proof. The proof of $H(c_i, c_j)$ is as in C.21 . We prove ii). The proof for iii) and iv) is similar.

- Suppose $B(c_i, c_j \mid X_i, X_j) \in \Gamma$, then by axiom **GlobBind1**, $\Box B(c_i, c_j \mid X_i, X_j) \in \Gamma$. Hence by definition of E (more precisely, prop. C.20), $\Box B(c_i, c_j \mid X_i, X_j) \in \Theta^+$. Hence, $B(c_i, c_j \mid X_i, X_j) \in \Theta^+$ (by ax. **T $\Box$**).
- We prove the other direction by contraposition. Suppose $B(c_i, c_j \mid X_i, X_j) \notin \Theta^+$. Then $\neg B(c_i, c_j \mid X_i, X_j) \in \Theta^+$. Hence, $\neg \Box B(c_i, c_j \mid X_i, X_j) \in \Theta^+$ (from ax. **T $\Box$**). So, by prop. C.20, $\neg \Box B(c_i, c_j \mid X_i, X_j) \in \Theta^+$. By axiom **GlobBind1**, we can verify that $\neg B(c_i, c_j \mid X_i, X_j) \in \Theta^+$.

□

Definition D.3 (Model associated to Θ and Θ^+). The temporal jurisdictional classifier model associated to Θ and Θ^+ , is the tuple $\mathcal{C} = (S^c, f^c, Jur^c, \leq^c, \mathcal{R}^c,)$, where

- $S^c = \{s_{\llbracket \Gamma \rrbracket} \mid \Gamma \in E\} \subseteq 2^{Atm_0}$
- $f^c : S^c \rightarrow Val$, $f(s_{\llbracket \Gamma \rrbracket}) = o$ iff for all $\Gamma' \in \llbracket \Gamma \rrbracket$, $t(o) \in \Gamma'$.
- $s_{\llbracket \Gamma \rrbracket} \leq^c s_{\llbracket \Delta \rrbracket}$ iff $\llbracket \Gamma \rrbracket \preceq \llbracket \Delta \rrbracket$
- $s_{\llbracket \Gamma \rrbracket} \mathcal{R}^c s_{\llbracket \Delta \rrbracket}$ iff $\llbracket \Gamma \rrbracket R \llbracket \Delta \rrbracket$
- $Jur^c = (Courts, \{\text{Cond}_c\}_{c \in \text{Courts}}, INC^c, \mathcal{H}^c, \mathcal{B}^c, \mathcal{O}^c)$ is defined as follows:
 - $INC^c(X_i, X_j)$ iff $INC(X_i, X_j) \in \Theta^+$
 - $c_i \mathcal{H}^c c_j$ iff $H(c_i, c_j) \in \Theta^+$
 - for $c_i \in \text{Courts}$, we define

$$\text{Cond}_{c_i} = \bigcup (\{X_i \mid B(c_i, c_j \mid X_i, X_j) \in \Theta^+\} \cup \{X_i \mid B(c_j, c_i \mid X_j, X_i) \in \Theta^+\})$$

$$\cup \bigcup (\{X_i \mid 0(c_i, c_j \mid X_i, X_j) \in \Theta^+\} \cup \{X_i \mid 0(c_j, c_i \mid X_j, X_i) \in \Theta^+\})$$
 - $\mathcal{B}^c(c_i, c_j \mid X_i, X_j)$ iff $B(c_i, c_j \mid X_i, X_j) \in \Theta^+$
 - $\mathcal{O}^c(c_i, c_j \mid X_i, X_j)$ iff $0(c_i, c_j \mid X_i, X_j) \in \Theta^+$

Proposition D.4. $\mathcal{C}$ is indeed a temporal jurisdictional classifier model.

Proof. We prove that Jur^c is a conditional jurisdiction. The rest of the proof is similar to C.29. In particular we prove that:

- $INC^c(X_i, X_j)$ is irreflexive and symmetric. It follows from Axioms **irrINC** and **symmINC** respectively.
- if $\mathcal{B}^c(c_i, c_j \mid X_i, X_j)$ then $X_i \subseteq \text{Cond}_{c_i}$ and $X_j \subseteq \text{Cond}_{c_j}$. Indeed suppose $\mathcal{B}^c(c_i, c_j \mid X_i, X_j)$. Then by definition $B(c_i, c_j \mid X_i, X_j) \in \Theta^+$. Since we know from the definition that $\text{Cond}_{c_i} \supseteq \bigcup (\{X_i \mid B(c_i, c_j \mid X_i, X_j) \in \Theta^+\} \cup \{X_i \mid B(c_j, c_i \mid X_j, X_i) \in \Theta^+\})$, this means that $X_i \subseteq \text{Cond}_{c_i}$. Similarly $X_j \subseteq \text{Cond}_{c_j}$.

We also have to prove that: if $INC^c(X_i, X_j)$ then for all $s_{\llbracket \Gamma \rrbracket} \in S^c$: $X_i \not\subseteq s_{\llbracket \Gamma \rrbracket}$ or $X_j \not\subseteq s_{\llbracket \Gamma \rrbracket}$. This follows from Def. D.3, Prop. D.2 and Axiom **INC**. $\square$

D.4 Proofs of Chapter 4

Proposition 4.12 (Pot. bind. Repr.). *Let $s \in S$, $o \in \{0, 1\}$. $(C, s) \models \text{PBinding}_{m,n}(\text{t}(o) \wedge \varphi)$ iff $(C, s) \models n$ and exists $s' \in S$, s.t. $\beta(s', s, o)$ and $(C, s') \models m \wedge \varphi$.*

Proof. $\Rightarrow$) Suppose $(C, s) \models \text{POverruling}_{n,m} \varphi$. Let $o \in \{0, 1\}$, $c \in \text{Courts}$ s.t. $(C, s) \models n \wedge c$.

- By hypothesis $(C, s) \models n \wedge c_i \wedge \emptyset(c_j, c_i \mid X_j, X_i) \wedge \bigwedge_{k \in X_i} k$. So $(C, s) \models n$ (#1), $X_i \subseteq s$ (#2), $c_i \in s$ (#3), $n \in s$ and $\mathcal{O}(c_j, c_i \mid X_j, X_i)$ (#4)
- By hypothesis there is $s' \in S$ s.t. $(C, s') \models (\varphi \wedge \bigwedge_{k \in X_j} k \wedge c_j \wedge \text{t}(\bar{o}) \wedge \text{Supporting}_{m,n} \text{t}(o))$
- So $f(s') = \bar{o}$ (#5), $(C, s') \models \varphi$ (#6), $X_j \subseteq s'$ (#7).
- We know $(C, s') \models \text{Supporting}_{m,n}(\text{t}(o))$. Recall $n \in s$. By Prop. 3.20, we retrieve $(C, s') \models m$ (#8) and $\Pi(s, s', o)$ (#9).
- By Def. 4.14, (#2), (#3), (#4), (#5), (#7) we have $\omega(s', s, o)$.
- By the previous point and (#1), (#6), (#8) we have the thesis.

$\Leftarrow$) Assume $(C, s) \models n$ (#9), $\omega(s', s, o)$, and $(C, s') \models m \wedge \varphi$ (#10).

- Since $\omega(s', s, o)$, by Def. 4.14, there is $o \in \{0, 1\}$ s.t. $f(s') = \bar{o}$, $\Pi(s, s', o)$ (#11), $\mathcal{O}(c_j, c_i \mid X_j, X_i)$ (#12), where $c_j \in s'$ (#13), $c_i \in s$ (#14), $X_i \subseteq s$ (#16) and $X_j \subseteq s'$ (#17).
- By (#16), $(C, s) \models \bigwedge_{k \in X_i} k$ (#18). By (#17), $(C, s') \models \bigwedge_{l \in X_j} l$ (#19)
- By (#12), $(C, s) \models \emptyset(c_j, c_i \mid X_j, X_i)$ (#20)
- By (#13), $(C, s') \models c_j$ (#21). By (#14), $(C, s) \models c_i$ (#22)
- By Prop. 3.20, (#9), (#10), (#11), (#20), (#21), (#17), (#18), (#19), the thesis.

□

Proposition 4.15 (Pot. Overrul. Repr.). *Let $s \in S$, $o \in \{0, 1\}$. $(C, s) \models \text{POverruling}_{n,m}(\text{t}(o) \wedge \varphi)$ iff $(C, s) \models n$ and exists $s' \in S$, s.t. $O(s', s)$ and $(C, s') \models m \wedge \varphi$.*

Proof. $\Rightarrow$) Suppose $(C, s) \models \text{POverruling}_{n,m} \varphi$. Let $o \in \{0, 1\}$, $c \in \text{Courts}$ s.t. $(C, s) \models n \wedge c$.

- By hypothesis $(C, s) \models n \wedge c_i \wedge \emptyset(c_j, c_i \mid X_j, X_i) \wedge \bigwedge_{k \in X_i} k$. So $(C, s) \models n$ (#1), $X_i \subseteq s$ (#2), $c_i \in s$ (#3), $n \in s$ and $\mathcal{O}(c_j, c_i \mid X_j, X_i)$ (#4)
- By hypothesis there is $s' \in S$ s.t. $(C, s') \models (\varphi \wedge \bigwedge_{k \in X_j} k \wedge c_j \wedge \text{t}(\bar{o}) \wedge \text{Supporting}_{m,n} \text{t}(o))$

- So $f(s') = \bar{o}$ (#5), $(C, s') \models \varphi$ (#6), $X_j \subseteq s'$ (#7).
- We know $(C, s') \models \text{Supporting}_{m,n}(t(o))$. Recall $n \in s$. By Prop. 3.20, we retrieve $(C, s') \models m$ (#8) and $\Pi(s, s', o)$ (#9).
- By Def. 4.14, (#2), (#3), (#4), (#5), (#7) we have $\omega(s', s, o)$.
- By the previous point and (#1), (#6), (#8) we have the thesis.

$\Leftarrow$) Assume $(C, s) \models n$ (#9), $\omega(s', s, o)$, and $(C, s') \models m \wedge \varphi$ (#10).

- Since $\omega(s', s, o)$, by Def. 4.14, there is $o \in \{0, 1\}$ s.t. $f(s') = \bar{o}$, $\Pi(s, s', o)$ (#11), $\mathcal{O}(c_j, c_i \mid X_j, X_i)$ (#12), where $c_j \in s'$ (#13), $c_i \in s$ (#14), $X_i \subseteq s$ (#16) and $X_j \subseteq s'$ (#17).
- By (#16), $(C, s) \models \bigwedge_{k \in X_i} k$ (#18). By (#17), $(C, s') \models \bigwedge_{l \in X_j} l$ (#19)
- By (#12), $(C, s) \models 0(c_j, c_i \mid X_j, X_i)$ (#20)
- By (#13), $(C, s') \models c_j$ (#21). By (#14), $(C, s) \models c_i$ (#22)
- By Prop. 3.20, (#9), (#10), (#11), (#20), (#21), (#17), (#18), (#19), the thesis.

□

Proposition 4.16. $CI \Rightarrow AS$

Proof. We only prove **CB1** $\Rightarrow$ **AS1**. Assume **CB1**. Let $X_i \neq X_j$ or $c_i \neq c_j$. Let $\mathcal{B}(c_i, c_j \mid X_i, X_j)$. Let $\tilde{X}_i = X_i$ and $\tilde{X}_j = X_j$. Since $X_i \neq X_j$ or $c_i \neq c_j$, then $(X_i \neq \tilde{X}_j$ and $X_j \neq \tilde{X}_i)$ or $c_i \neq c_j$. Then by **CB1** not- $\mathcal{B}(c_j, c_i \mid \tilde{X}_j, \tilde{X}_i)$. So, not- $\mathcal{B}(c_j, c_i \mid X_j, X_i)$. □

Proposition 4.17. $C2 \Rightarrow AS$, $CBO3 \Rightarrow (AS3 \ \& \ AS4)$.

Proof. We only prove **CB2** $\Rightarrow$ **AS1**. Assume **CB2**. Let $X_i \neq X_j$ or $c_i \neq c_j$. Let $\mathcal{B}(c_i, c_j \mid X_i, X_j)$. We prove not- $\mathcal{B}(c_j, c_i \mid X_j, X_i)$. Assume by contradiction $\mathcal{B}(c_j, c_i \mid X_j, X_i)$. Then by **CB2**, since $\mathcal{B}(c_i, c_j \mid X_i, X_j)$ and $\mathcal{B}(c_j, c_i \mid X_j, X_i)$, this means have that X_i and X_j are incompatible or X_j and X_j are incompatible. Contradiction, since the incompatibility relation is irreflexive. □

Proposition 4.18. $CI \Rightarrow C2$, $CBO1 \Rightarrow CBO3$.

Proof. We only prove **CB1** $\Rightarrow$ **CB2**. Assume **CB1**. Let $X_i \neq \tilde{X}_j$ or $X_j \neq \tilde{X}_i$ or $c_i \neq c_j$. Then, by **CB1** it cannot be $\mathcal{B}(c_i, c_j \mid X_i, X_j)$ and $\mathcal{B}(c_j, c_i \mid \tilde{X}_j, \tilde{X}_i)$. So **CB2** is trivially satisfied. □

Proposition 4.20. $wBSD \ \& \ wOSD \Rightarrow HBO$.

Proof. Assume $Rel_1(c_1, c_2 \mid X_1, X_2)$ & $Rel_2(c_3, c_2 \mid X_3, \tilde{X}_2)$, where $Rel_i = \mathcal{O}$ for $i \in \{1, 2\}$. By $Rel_1(c_1, c_2 \mid X_1, X_2)$ and **wBSD** & **wOSD**, we know $\mathcal{H}(c_2, c_1)$ or $c_2 = c_1$. By $Rel_2(c_3, c_2 \mid X_3, \tilde{X}_2)$ and **wBSD** & **wOSD**, we know $\mathcal{H}(c_3, c_2)$ or $c_3 = c_2$. Hence, by transitivity of equality and $\mathcal{H}$, we retrieve $c_3 = c_1$ or $(\mathcal{H}(c_3, c_2)$ and $\mathcal{H}(c_2, c_1))$. $\square$

Proposition 4.21. $S1 \& BSD(resp. OSD) \Rightarrow OSD(resp. wBSD)$

Proof. We prove that **S1** & **BSD** $\Rightarrow$ **OSD**. The proof that **S1** & **OSD** $\Rightarrow$ **BSD** is similar. Assume **S1** and **BSD**.

We prove that if $\mathcal{H}(c_1, c_2)$, then $\mathcal{O}(c_1, c_2 \mid \emptyset, \emptyset)$.

- Suppose $\mathcal{H}(c_1, c_2)$.
- Then, by irreflexivity of hierarchy relation $c_1 \neq c_2$.
- Also, by **BSD**, $\mathcal{B}(c_1, c_2 \mid \emptyset, \emptyset)$.
- By $c_1 \neq c_2$, **S1** and $\mathcal{B}(c_1, c_2 \mid \emptyset, \emptyset)$, $\mathcal{O}(c_1, c_2 \mid \emptyset, \emptyset)$.

We prove that if $\mathcal{O}(c_1, c_2 \mid \emptyset)$ then $c_1 = c_2$ or $\mathcal{H}(c_1, c_2)$

- Suppose $\mathcal{O}(c_1, c_2 \mid \emptyset)$.
- if $c_1 = c_2$, then the thesis.
- If $c_1 \neq c_2$ then by **S1** and $\mathcal{O}(c_1, c_2 \mid \emptyset, \emptyset)$, $\mathcal{B}(c_1, c_2 \mid \emptyset, \emptyset)$. Then by **BSD**, $\mathcal{H}(c_1, c_2)$ (since $c_1 \neq c_2$)

$\square$

Proposition 4.24. $ME1 \& E-Mon \Rightarrow Mon$.

Proof. We prove that **ME1** & **E-Mon** implies **MonB1**. The other proofs are similar.

So assume $\mathcal{B}(c_i, c_j \mid X_i, X_j)$. We want to prove that $\forall X_k \subseteq \text{Cond}_{c_j}$ s.t. $\text{Adm}(X_k)$ and $X_j \subset X_k : \mathcal{B}(c_i, c_j \mid X_i, X_k)$.

Let $X_k \subseteq \text{Cond}_{c_j}$ s.t. $\text{Adm}(X_k)$. If $X_j = X_k$ trivially the thesis holds. Suppose $X_j \subset X_k$.

- Since $X_j \subset X_k$, $X_j \neq X_k$.
- Since $X_j \subset X_k$, $X_k \neq \emptyset$
- Since $X_j \subseteq X_k$, by $\text{Adm}(X_k)$ and Definition 4.8, this means that either $X_j = \emptyset$ or $X_k = \emptyset$; or $X_j, X_k \neq \emptyset$ and not- $\emptyset$ X_j, X_k .

- We know it cannot be $X_k = \emptyset$.
- Suppose $X_j, X_k \neq \emptyset$ and not- $_j, X_k$). Contradiction because of **ME1**
- So suppose $X_j = \emptyset$. Then we can apply **E-MonB1** to $\mathcal{B}(c_i, c_j \mid X_i, \emptyset)$, obtaining $\mathcal{B}(c_i, c_j \mid X_i, X_k)$.

□

Proposition 4.25. $S1 \ \& \ AS \ \& \ TB \ (resp. \ TO) \Rightarrow TO \ (resp. \ TB)$.

Proof. We prove that from **S1** and **TB** we obtain **TO**. Suppose $\mathcal{O}(c_2, c_1 \mid X_2, X_1)$ (#1) and $\mathcal{O}(c_3, c_2 \mid X_3, X_2)$ (#2).

- $c_3 \neq c_1$ or $X_3 \neq X_1$ (#3). Indeed, if $c_3 = c_1$ and $X_3 = X_1$ then $\mathcal{O}(c_2, c_1 \mid X_2, X_1)$ and $\mathcal{O}(c_1, c_2 \mid X_1, X_2)$, against **AS2**.
- Assume $(c_2 \neq c_1 \text{ or } X_2 \neq X_1)$ and $(c_3 \neq c_2 \text{ or } X_3 \neq X_2)$:
 - By **S1** and $\mathcal{O}(c_2, c_1 \mid X_2, X_1)$, we have $\mathcal{B}(c_2, c_1 \mid X_2, X_1)$. By **S1** and $\mathcal{O}(c_3, c_2 \mid X_3, X_2)$ (#4), we have $\mathcal{B}(c_3, c_2 \mid X_3, X_2)$ (#5).
 - So, by **TB**, (#4) and (#5), $\mathcal{B}(c_3, c_1 \mid X_3, X_1)$.
 - Since (#3), by $\mathcal{B}(c_3, c_1 \mid X_3, X_1)$ and **S1**, then $\mathcal{O}(c_3, c_1 \mid X_3, X_1)$. So the thesis
- We have to verify the thesis for $(c_2 = c_1 \text{ and } X_2 = X_1)$ or $(c_3 = c_2 \text{ and } X_3 = X_2)$
- Assume $c_2 = c_1$ and $X_2 = X_1$. By (#1), $\mathcal{O}(c_1, c_1 \mid X_1, X_1)$. By (#2), $\mathcal{O}(c_3, c_1 \mid X_3, X_1)$. Hence trivially $\mathcal{O}(c_3, c_1 \mid X_3, X_1)$. So the thesis
- Assume $c_3 = c_2$ and $X_3 = X_2$. By (#1), $\mathcal{O}(c_2, c_1 \mid X_2, X_1)$. By (#2), $\mathcal{O}(c_2, c_2 \mid X_2, X_2)$. Hence trivially $\mathcal{O}(c_3, c_1 \mid X_3, X_1)$, since $c_3 = c_2$ and $X_3 = X_2$. So, the thesis.

□

Proposition 4.26. $S \ \& \ E\text{-Mon} \ \& \ AS \ \& \ TB \ (resp. \ TO) \Rightarrow TBO$

Proof. Assume **S** & **E-Mon** & **AS** & **TB**. By proposition 4.25, from **S1** & **AS** & **TB**, we obtain **TO**.

We prove that $\mathcal{B}(c_2, c_1 \mid X_2, X_1)$ and $\mathcal{O}(c_3, c_2 \mid X_3, X_2) \Rightarrow \mathcal{O}(c_3, c_1 \mid X_3, X_1)$. The proof for other implication in **TBO** is similar.

Assume $\mathcal{B}(c_2, c_1 \mid X_2, X_1)$ (#1) and $\mathcal{O}(c_3, c_2 \mid X_3, X_2)$ (#2).

- Suppose $c_2 \neq c_1$. Then by (#1) and **S1**, we obtain $\mathcal{O}(c_2, c_1 \mid X_2, X_1)$. From $\mathcal{O}(c_2, c_1 \mid X_2, X_1)$, (#2) and **TO**, $\mathcal{O}(c_3, c_2 \mid X_3, X_2)$. So the thesis.
- Suppose $c_2 = c_1$. So from (#1) and (#2), we have $\mathcal{B}(c_1, c_1 \mid X_2, X_1)$ (#3) and $\mathcal{O}(c_3, c_1 \mid X_3, X_2)$ (#4).
 - If $c_3 \neq c_2$. Then, from (#4) and **S1**, we obtain $\mathcal{B}(c_3, c_1 \mid X_3, X_2)$. So, by $\mathcal{B}(c_3, c_1 \mid X_3, X_2)$, (#3) and **TB**, we obtain $\mathcal{B}(c_3, c_1 \mid X_3, X_1)$ (#4). Since $c_3 \neq c_2$ and $c_1 = c_2$, then $c_3 \neq c_1$. Then, by **S1**, (#5), it follows $\mathcal{O}(c_3, c_1 \mid X_3, X_1)$. Hence, the thesis.
 - So we assume $c_3 = c_2 = c_1$. So, from (#4), we have $\mathcal{O}(c_1, c_1 \mid X_3, X_2)$ (#6).
 - If $X_2 \neq X_1$ and $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$, then by **S2** and (#3), we obtain $\mathcal{O}(c_1, c_1 \mid X_2, X_1)$. So, from the latter, (#6) and **TO**, the thesis.
 - If $X_2 = X_1$ then we know, from (#3), $\mathcal{B}(c_1, c_1 \mid X_1, X_1)$ and, from (#6), $\mathcal{O}(c_1, c_1 \mid X_3, X_1)$. Then trivially the thesis.
 - If $X_1 = \emptyset$, then, from (#3), $\mathcal{B}(c_1, c_1 \mid X_2, \emptyset)$ (#7). Consider $\mathcal{O}(c_1, c_1 \mid X_3, X_2)$. By definition 4.2, $X_3 \subseteq \text{Cond}_{c_1}$. Then, from (#7) and $X_3 \subseteq \text{Cond}_{c_1}$, by **E-Mon**, $\mathcal{B}(c_2, c_1 \mid X_2, X_3)$. From the latter contradiction, because of (#6), **AS3/AS4**.
 - If $X_2 = \emptyset$, then, from (#3), $\mathcal{B}(c_1, c_1 \mid \emptyset, X_1)$ and, from (#6), $\mathcal{O}(c_1, c_1 \mid X_3, \emptyset)$. Then by **E-Mon**, we obtain both $\mathcal{B}(c_1, c_1 \mid X_3, X_1)$ and $\mathcal{O}(c_1, c_1 \mid X_3, X_1)$. Contradiction because of **AS3/AS4**.

□

Proposition 4.27. Assume **L1**. Then G_s is acyclic, for every $s \in S$.

Proof. We prove that G_s is acyclic.

- 1) Suppose by contradiction that there is a cycle, namely a sequence of states $s_1, s_2, \dots, s_n, s_{n+1}$ s.t. $(s_i, s_{i+1}) \in E$ and $s_1 = s_{n+1}$.
- 2) Let $c_i \in s_i \cap \text{Courts}$.
- 3) $(s_1, s_2) \in E$. Then there are $\tilde{X}_1, X_2 \subseteq \text{Cond}$ s.t. $\tilde{X}_1 \subseteq s_1$, $X_2 \subseteq s_2$ and $\text{Rel}_1(c_2, c_1 \mid X_2, \tilde{X}_1)$, with $\text{Rel}_1 = \mathcal{B}$ or $\text{Rel}_1 = \mathcal{O}$.
- 4) $(s_2, s_3) \in E$. Then there are $X_3, \tilde{X}_2 \subseteq \text{Cond}$ s.t. $X_3 \subseteq s_3$, $\tilde{X}_2 \subseteq s_2$ and $\text{Rel}_2(c_2, c_1 \mid X_3, \tilde{X}_2)$, with $\text{Rel}_2 = \mathcal{B}$ or $\text{Rel}_2 = \mathcal{O}$.
- 5) Notice that $X_2 \subseteq s_2$ and $\tilde{X}_2 \subseteq s_2$. So X_2 and $\tilde{X}_2$ are not incompatible.

- 6) By reiterating the same reasoning for all $(s_i, s_{i+1}) \in E$, we obtain that, for $i \in \{1, \dots, n\}$, there are $X_i, \tilde{X}_i \subseteq \text{Cond}$ such that $\text{Rel}_1(c_2, c_1 \mid X_2, \tilde{X}_1)$ and $\text{Rel}_2(c_3, c_2 \mid X_3, \tilde{X}_2) \dots \text{Rel}_{n-1}(c_n, c_{n-1} \mid X_n, \tilde{X}_{n-1})$ and $\text{Rel}_n(c_1, c_n \mid X_1, \tilde{X}_n)$.
- 7) For all $i \in \{1, \dots, n\}$, $X_i, \tilde{X}_i \subseteq s_i$. So $X_i, \tilde{X}_i$ are not incompatible.
- 8) It cannot be $s_2 \in \omega_{s_1}, s_3 \in \omega_{s_2} \dots s_1 \in \omega_{s_n}$ otherwise we would have $s_1 <_T s_2 <_T s_n <_T s_1$ and so a contradiction. Hence it cannot be $\text{Rel}_1 = \text{Rel}_2 = \dots = \text{Rel}_n = \mathcal{O}$
- 9) It cannot be $s_2 \in \beta_{s_1}$ and $s_3 \in \beta_{s_2} \dots s_1 \in \beta_{s_n}$ otherwise we would have $s_1 <_T s_n <_T s_3 <_T s_2 <_T s_1$ and so a contradiction. So, it is not $\text{Rel}_1 = \text{Rel}_2 = \dots = \text{Rel}_n = \mathcal{B}$
- 10) By 8) and 9), $\exists j \in \{1, \dots, n-1\}$ s.t. $\text{Rel}_j \neq \text{Rel}_{j+1}$.
- 8) By 6), 7) and 10) contradiction because of **L1**.

□

Proposition 4.28. *HBO & CBO3 $\Rightarrow$ LI*

Proof. • Let $c_1, \dots, c_n \in \text{Courts}$; and $X_1, \tilde{X}_1, \dots, X_n \subseteq \text{Cond}$.

- Assume $\text{Rel}_1(c_2, c_1 \mid X_2, \tilde{X}_1)$ and $\text{Rel}_2(c_3, c_2 \mid X_3, \tilde{X}_2) \dots \text{Rel}_{n-1}(c_n, c_{n-1} \mid X_n, \tilde{X}_{n-1}), \text{Rel}_n(c_{n+1}, c_n \mid X_1, \tilde{X}_n)$, where $c_{n+1} = c_1$ (#1), $\text{Rel}_i = \mathcal{B}$ or $\text{Rel}_i = \mathcal{O}$ for all $i \in \{1, \dots, n\}$
- Assume $\text{Rel}_j \neq \text{Rel}_{j+1}$ for some $j \in \{1, \dots, n-1\}$ We want to prove that $X_i, \tilde{X}_i$ are incompatible for some $i \in \{1, \dots, n\}$.
- We know $\text{Rel}_i(c_{i+1}, c_i \mid X_{i+1}, \tilde{X}_i) \& \text{Rel}_{i+1}(c_{i+2}, c_{i+1} \mid X_{i+2}, \tilde{X}_{i+1})$ for $i \in \{1, \dots, n-1\}$.
By **HBO**, we have that either $(\mathcal{H}(c_{i+1}, c_i)$ and $\mathcal{H}(c_{i+2}, c_{i+1}))$ or $c_{i+2} = c_i$
- Suppose $\mathcal{H}(c_{i+1}, c_i)$ and $\mathcal{H}(c_{i+2}, c_{i+1})$, for some $i \in \{1, \dots, n-1\}$.
 - By transitivity of $\mathcal{H}$ we obtain $\mathcal{H}(c_{n+1}, c_1)$ However this leads to a contradiction, because of (#1) and irreflexivity of $\mathcal{H}$.
- Suppose for all $i \in \{1, \dots, n\}$, $c_{i+2} = c_i$.
 - We know $\text{Rel}_j \neq \text{Rel}_{j+1}$ for some $j \in \{1, \dots, n-1\}$.
 - Without loss of generality assume $\mathcal{B}(c_{j+1}, c_j \mid X_{j+1}, \tilde{X}_j) \& \mathcal{O}(c_{j+2}, c_{j+1} \mid X_{j+2}, \tilde{X}_{j+1})$.
We know $c_j = c_{j+2}$. Hence, $\mathcal{B}(c_{j+1}, c_j \mid X_{j+1}, \tilde{X}_j) \& \mathcal{O}(c_j, c_{j+1} \mid X_{j+2}, \tilde{X}_{j+1})$.
By **CBO3**, this means that $\tilde{X}_{j+1}$ and X_{j+1} are incompatible. Hence, the thesis.

□

Proposition 4.29. *Mon & T & Adm & C2 $\Rightarrow$ LI*

sketch. • Let $c_1, \dots, c_n \in \text{Courts}$; and $X_1, \tilde{X}_1, \dots, X_n \subseteq \text{Cond}$.

- Assume $\text{Rel}_1(c_2, c_1 \mid X_2, \tilde{X}_1)$ and $\text{Rel}_2(c_3, c_2 \mid X_3, \tilde{X}_2) \dots \text{Rel}_{n-1}(c_n, c_{n-1} \mid X_n, \tilde{X}_{n-1}), \text{Rel}_n(c_{n+1}, c_n \mid X_1, \tilde{X}_n)$, with $c_{n+1} = c_1$ (#1), $\text{Rel}_i = \mathcal{B}$ or $\text{Rel}_i = \mathcal{O}$ for all $i \in \{1, \dots, n\}$
- Assume by contradiction $\text{Rel}_j \neq \text{Rel}_{j+1}$ for some $j \in \{1, \dots, n-1\}$ and for all $i \in \{1, \dots, n\}$ $X_i, \tilde{X}_i$ are not incompatible.
- Consider for example $\text{Rel}_1(c_2, c_1 \mid X_2, \tilde{X}_1)$ and $\text{Rel}_2(c_3, c_2 \mid X_3, \tilde{X}_2)$. We assumed that X_2 and $\tilde{X}_2$ are not incompatible. So we can use **Mon** and obtain $\text{Rel}_1(c_2, c_1 \mid X_2 \cup \tilde{X}_2, \tilde{X}_1)$ as we all as $\text{Rel}_2(c_3, c_2 \mid X_3, \tilde{X}_2 \cup X_2)$. So we can apply transitivity, obtaining $\text{Rel}_2(c_3, c_1 \mid X_3, \tilde{X}_1)$.
- We know that there is j s.t. $\text{Rel}_j \neq \text{Rel}_{j+1}$. Let l the greatest index s.t. $\text{Rel}_l \neq \text{Rel}_{l+1}$.
- We split the chain in two chains $\text{Rel}_1(c_2, c_1 \mid X_2, \tilde{X}_1) \dots \text{Rel}_l(c_{l+1}, c_l \mid X_{l+1}, \tilde{X}_l)$ and $\text{Rel}_{l+1}(c_{l+2}, c_{l+1} \mid X_{l+2}, \tilde{X}_{l+1}) \dots \text{Rel}_n(c_1, c_n \mid X_1, \tilde{X}_n)$ (recall $c_{n+1} = c_1$)
- We can apply transitivity to the two chains obtaining $\text{Rel}_l(c_{l+1}, c_1 \mid X_{l+1}, \tilde{X}_1)$ and $\text{Rel}_n(c_1, c_{l+1} \mid X_1, \tilde{X}_{l+1})$.
- Since l the greatest index s.t. $\text{Rel}_l \neq \text{Rel}_{l+1}$, then $\text{Rel}_l \neq \text{Rel}_n$.
- Hence, by **C2**, we have X_{l+1} and $\tilde{X}_{l+1}$ are incompatible or X_1 and $\tilde{X}_1$ are incompatible. Contradiction.

□

Proposition D.1. $E\text{-Mon} \& \text{AS} \& \text{Adm} \Rightarrow E$

Proof. Assume **E-Mon** & **AS** & **Adm**

- We verify **E1**. The proof for **E2** is similar. Assume $c_i \neq c_j$ and $\text{Rel}_1, \text{Rel}_2 \in \{\mathcal{B}, \mathcal{O}\}$
 - Assume $\text{Rel}_1(c_i, c_j \mid \emptyset, X_j)$
 - If $X_i \not\subseteq \text{Cond}_{c_i}$ then, by Definition 4.2, not $\text{Rel}_2(c_j, c_i \mid X_i, X_j)$.
 - If not- $\text{Adm}(X_i)$ then by **Adm** not $\text{Rel}_2(c_j, c_i \mid X_i, X_j)$
 - If $X_i \subseteq \text{Cond}_{c_i}$ and $\text{Adm}(X_i)$, then by **E-Mon**, we have $\text{Rel}_1(c_i, c_j \mid X_i, X_j)$.
 - Since $c_i \neq c_j$, then by **AS**, we have not- $\text{Rel}_2(c_j, c_i \mid X_j, X_i)$
- We prove **E3**. The proof for **E4** is similar. Let $\text{Rel}_1, \text{Rel}_2 \in \{\mathcal{B}, \mathcal{O}\}$, with $\text{Rel}_1 \neq \text{Rel}_2$
 - Assume $\text{Rel}_1(c_i, c_i \mid \emptyset, X_j)$.
 - Since $\text{Rel}_1(c_i, c_i \mid \emptyset, X_j)$, then, by Definition 4.2 and **Adm**, $X_j \subseteq \text{Cond}_{c_j}$, $\text{Adm}(X_j)$

- By **E-Mon**, we have $Rel_1(c_i, c_i \mid X_j, X_j)$. By **AS3** or **AS4**, $\text{not-}Rel_2(c_i, c_i \mid X_j, X_j)$.
- We prove **E5**. Assume $c_i \neq c_j$ and $Rel_1, Rel_2 \in \{\mathcal{B}, \mathcal{O}\}$
 - Assume $Rel_1(c_i, c_j \mid \emptyset, \emptyset)$
 - If $X_i \not\subseteq \text{Cond}_{c_i}$ or $\text{not } X_j \not\subseteq \text{Cond}_{c_j}$ then by Definition 4.2, $\text{not-}Rel_2(c_j, c_i \mid X_j, X_i)$.
 - If $\text{not-Adm}(X_i)$ or $\text{not-Adm}(X_j)$ then, by **Adm**, $\text{not-}Rel_2(c_j, c_i \mid X_j, X_i)$
 - Let $X_i \subseteq \text{Cond}_{c_i}$, $\text{Adm}(X_i)$, $X_j \subseteq \text{Cond}_{c_j}$, $\text{Adm}(X_j)$
 - By **E-Mon** and $Rel_1(c_i, c_j \mid \emptyset, \emptyset)$, then $Rel_1(c_i, c_j \mid \emptyset, X_j)$. So, by **E-Mon** $Rel_1(c_i, c_j \mid X_i, X_j)$. By **AS** $\text{not-}Rel_2(c_j, c_i \mid X_i, X_i)$.
- We prove **E6**. Let $Rel_1, Rel_2 \in \{\mathcal{B}, \mathcal{O}\}$, with $Rel_1 \neq Rel_2$
 - Assume $Rel_1(c_i, c_i \mid \emptyset, \emptyset)$.
 - If $X_i \not\subseteq \text{Cond}_{c_i}$ or $\text{not } X_j \not\subseteq \text{Cond}_{c_j}$ then, by Definition 4.2, $\text{not-}Rel_2(c_i, c_i \mid X_j, X_i)$.
 - If $\text{not-Adm}(X_i)$ or $\text{not-Adm}(X_j)$ then, by **Adm**, $\text{not-}Rel_2(c_i, c_i \mid X_j, X_i)$
 - Let $X_i \subseteq \text{Cond}_{c_i}$, $\text{Adm}(X_i)$, $X_j \subseteq \text{Cond}_{c_j}$, $\text{Adm}(X_j)$
 - By **E-Mon** $Rel_1(c_i, c_i \mid \emptyset, X_i)$. From the latter, by **E-Mon** $Rel_1(c_i, c_i \mid X_j, X_i)$. Then, by **AS3** or **AS4** $\text{not-}Rel_1(c_j, c_i \mid X_i, X_i)$.
- We prove **E7**. The proof for **E8** is similar. Assume $c_i \neq c_j$ or $X_i \neq X_j$ Assume also, $Rel_1, Rel_2 \in \{\mathcal{B}, \mathcal{O}\}$
 - assume $Rel_1(c_i, c_j \mid \emptyset, X_j)$. Then, by Definition 4.2 and **Adm**, $X_j \subseteq \text{Cond}_{c_j}$ and $\text{Adm}(X_j)$. Assume by contradiction $Rel_2(c_j, c_i \mid \emptyset, X_i)$.
 - Then, by Definition 4.2 and **Adm**, $X_i \subseteq \text{Cond}_{c_i}$ and $\text{Adm}(X_i)$
 - By $Rel_1(c_i, c_j \mid \emptyset, X_j)$ and **E-Mon**, we have $Rel_2(c_i, c_j \mid X_i, X_j)$.
 - By $c_i \neq c_j$ or $X_i \neq X_j$ and **AS**, $\text{not-}Rel_2(c_j, c_i \mid X_j, X_i)$.
 - By $Rel_2(c_j, c_i \mid \emptyset, X_i)$ and **E-Mon**, we have $Rel_2(c_j, c_i \mid X_j, X_i)$. Contradiction because of $\text{not-}Rel_2(c_j, c_i \mid X_j, X_i)$.
- We prove **E9**. The proof for **E10** is similar. $Rel_1, Rel_2 \in \{\mathcal{B}, \mathcal{O}\}$ and $Rel_1 \neq Rel_2$
 - Assume $Rel_1(c_i, c_i \mid \emptyset, X_j)$. Then, by Definition 4.2 and **Adm**, $X_j \subseteq \text{Cond}_{c_j}$ and $\text{Adm}(X_j)$. Assume by contradiction $Rel_2(c_i, c_i \mid \emptyset, X_i)$. Then, by Definition 4.2 and **Adm**, $X_i \subseteq \text{Cond}_{c_i}$ and $\text{Adm}(X_i)$
 - By $Rel_1(c_i, c_i \mid \emptyset, X_j)$ and **E-Mon**, we have $Rel_1(c_i, c_i \mid X_i, X_j)$. By **AS3** or **AS4** $\text{not-}Rel_2(c_j, c_i \mid X_j, X_i)$.

- By $Rel_2(c_j, c_i \mid \emptyset, X_i)$ and **E-Mon**, we have $Rel_2(c_j, c_i \mid X_j, X_i)$. Contradiction because of $\text{not-}Rel_2(c_j, c_i \mid X_j, X_i)$.

□

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