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(Non-)perturbative techniques in de Sitter space

Presentata da

Polina Petriakova

Coordinatore Dottorato
Prof. Alessandro Gabrielli

Supervisore
Prof. Alexandr Kamenchtchik

Co-supervisore
Prof. Roberto Casadio

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Valutatori

Prof. Vincent Vennin
Prof. Richard P. Woodard

Abstract

In this thesis, we develop computational techniques for calculating correlation functions of a spectator scalar field in de Sitter space within the long-wavelength approximation. Our main objectives are threefold: to obtain perturbative quantum field theory results, to compare them with the Starobinsky-Yokoyama stochastic approach, and to extract non-perturbative information solely from perturbative data. To initiate this program, we introduce a method for incorporating mass into the truncated Yang-Feldman equation. Unlike the standard theory of a massive scalar field based on the de Sitter-invariant vacuum, we develop a framework that does not require de Sitter invariance. Our framework yields a smooth massless limit for the infrared part of correlation functions, thereby bridging the gap between massless and massive perturbative results in the literature. Remarkably, the two-point correlation function of a free massive scalar field matches that of the Ornstein-Uhlenbeck stochastic process, which has a clear physical interpretation. By iteratively employing the massive Yang-Feldman-type equation, we compute the two-point correlation function to three-loop order and the four-point correlation function to one-loop order. Our approach is equivalent to constructing a perturbative series in the mass for the massless scalar field, summing leading logarithms, and organizing the perturbative massive series in terms of the self-interaction coupling constant λ. We also establish the correspondence between integral structures arising from the massive Yang-Feldman-type equation and the Schwinger-Keldysh ("in-in") formalism. Our perturbative results at late times agree with both diagrammatic calculations and the Starobinsky-Yokoyama stochastic approach, which describes the near-equilibrium regime. Within the stochastic framework, we derive non-perturbative results for 2n-point correlation functions using recurrence relations in the massive case with a quartic self-interaction. We further propose an alternative method for computing perturbative series for equal- and multi-time correlation functions. Rather than employing the massive Yang-Feldman-type equation or diagrammatic techniques, we derive a simple first-order differential equation from the Fokker-Planck or forward Kolmogorov equation. Its formal solution connects various correlation functions at different orders in the self-interaction coupling constant λ. Our method unifies the Starobinsky-Yokoyama stochastic approach with quantum field theory results, extending this consistency beyond the equal-time and stationary cases. Finally, we construct a renormalization group-inspired autonomous equation for the long-wavelength part of the expectation value $\left\langle \phi^2(t,\vec{x}) \right\rangle$. Integrating its approximate form, we obtain an expression that is non-analytic in the self-interaction coupling constant λ. This solution reproduces the correct perturbative series up to two-loop order. In the late-time limit, it almost coincides with the result obtained within the Starobinsky-Yokoyama stochastic approach across the entire interval $0 \leq \frac{\pi^2 m^4}{3\lambda H^4} < \infty$.

Declaration

This dissertation is based on the results presented in the following published articles:

- **IR finite correlation functions in de Sitter space, a smooth massless limit, and an autonomous equation**
  Alexander Kamenshchik and Polina Petriakova, *Journal of High Energy Physics* 04 (2025) **127**, DOI: 10.1007/JHEP04(2025)127; arXiv: [2410.16226][hep-th];
- **From the Fokker-Planck equation to perturbative QFT's results in de Sitter space**
  Alexander Kamenshchik and Polina Petriakova, *Journal of High Energy Physics* 08 (2025) **063**, DOI: 10.1007/JHEP08(2025)063; arXiv: [2504.20646][hep-th].

During my PhD programme (01/11/2022 – 30/10/2025), I have also been involved in research projects outside the scope of this thesis. These led to the following publications:

- **Newman-Janis algorithm's application to regular black hole models**
  Alexander Kamenshchik and Polina Petriakova, *Physical Review D* **107**, 124020 (2023), DOI: 10.1103/PhysRevD.107.124020; arXiv: [2305.04697][gr-qc];
- **Bianchi cosmologies, magnetic fields, and singularities**
  Roberto Casadio, Alexander Kamenshchik, Panagiotis Mavrogiannis, and Polina Petriakova, *Physical Review D* **108**, 084059 (2023), DOI: 10.1103/PhysRevD.108.084059; arXiv: [2307.12830][gr-qc];
- **Regular Friedmann universes and matter transformations**
  Alexander Kamenshchik and Polina Petriakova, *Universe* **10**, (3), 137 (2024); DOI: 10.3390/universe10030137; arXiv: [2403.08400][gr-qc].

Additional publications:

- **Self-tuning inflation**
  Polina Petriakova and Sergey Rubin, *The European Physical Journal C* **82**, 1048 (2022), DOI: 10.1140/epjc/s10052-022-10983-w; arXiv: [2204.04647][gr-qc];
- **Flexible extra dimensions**
  Polina Petriakova, Arkady Popov, and Sergey Rubin, *The European Physical Journal C* **83**, 371 (2023), DOI: 10.1140/epjc/s10052-023-11542-7; arXiv: [2303.04785][gr-qc].

Contents

1 Introduction

Friedmann's relativistic model of a non-stationary, isotropic, and spatially homogeneous universe, derived from the Einstein field equations, laid the theoretical foundation for the concept of the expanding universe [1, 2]. This framework was linked to observations via the empirical velocity–distance relation, first proposed by Lemaître [3] and later confirmed experimentally by Hubble [4]. These discoveries gave rise to fundamental questions of modern cosmology: what is the origin of our universe, and how has it evolved to its present state?

Although classical gravity and the hot expanding universe model based on it have been highly successful in explaining many cosmological phenomena [5, 6], they fall short in addressing key foundational issues. For instance, the model assumes, rather than derives, the initial high-temperature, homogeneous, and isotropic state of the universe. The inflationary paradigm [7, 8, 9, 10, 11] was introduced to address this challenge. It postulates a period of rapid expansion in the early universe, which accounts for the large-scale homogeneity and flatness observed nowadays and provides, through quantum fluctuations, a mechanism for generating the primordial density perturbations that seeded the formation of cosmic structures [12, 13, 14, 15, 16, 17]. Observational evidence for the accelerated phases of the universe's expansion came a few decades ago [18, 19, 20, 21].

The solution to the vacuum Einstein field equations with a positive cosmological constant, known as de Sitter space [22], serves as an important model for studying the accelerated expansion of the universe. It provides insights into both the inflationary epoch of the early universe and the present phase of accelerated expansion. In addition to its relevance to observational cosmology, de Sitter space is a maximally symmetric spacetime, with an isometry group of the same dimension as the Poincaré group in Minkowski space. Its high degree of symmetry provides mathematically tractable yet physically rich framework for exploring quantum field theory in curved spacetime, revealing non-trivial quantum effects distinct from those in Minkowski space [23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33].

Given this, quantum field theory in curved de Sitter spacetime is of great interest from both theoretical and phenomenological perspectives. Its relevance to the cosmology of the very early universe has become especially important due to the explosive growth of observational data. Even before the inflationary paradigm was proposed, the study of quantum field theory in de Sitter space was pioneered by Chernikov and Tagirov [34, 35], who examined the free massive scalar field. Nowadays, scalar fields are thought to play a pivotal role during the inflationary (quasi-)de Sitter phase of the universe's evolution and to seed, through their quantum fluctuations, the formation of observable large-scale structures. Even if scalar fields do not drive inflation, they can still play a significant role in various physical processes. In particular, light fields exhibit strong semi-classical fluctuations on super-horizon scales, which lead to secular growth behavior and non-perturbative infrared effects [36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47]. This issue presents both mathematical and physical challenges. Understanding infrared effects is crucial because the late-time behavior of quantum fields during inflation can influence post-inflationary physics and observable cosmological parameters; see, e.g., the review by Hu [48]. The study of light scalar fields, with masses much smaller than the curvature scale, $m < H$, is also of particular interest due to its relevance to inflationary cosmology and the absence of a flat-space counterpart.

Applying quantum field theory techniques to inflationary or fixed de Sitter backgrounds is considerably more challenging than in flat spacetime, due to the nontrivial form of the mode functions. While various well-established tools exist for flat spacetime, methods for treating cosmological backgrounds are still under development. In addition, in curved spacetime, the choice of the vacuum state becomes a non-trivial problem: quantum field theory does not provide a unique prescription to define the vacuum state; see, e.g, the classic book by Birrell and Davies [49] and references therein. Since de Sitter space is maximally symmetric, one might attempt to define a vacuum state that remains invariant under the full de Sitter isometry group. Meanwhile, for a massive scalar field, there exists a one-parameter family of de Sitter-invariant vacuum states, whereas for a massless field no such vacuum exists [40, 50, 51]. Consequently, there is no regular massless limit connecting perturbative results for massive and massless scalar field theories in de Sitter space.

In this thesis, we propose an alternative framework that resolves this issue. In contrast to the standard theory of a massive scalar field based on the de Sitter-invariant vacuum, we develop a vacuum-independent approach that does not require de Sitter invariance but results in a smooth massless limit of the infrared part of the correlation functions. Even in Minkowski space, examining states that break Poincaré invariance can be insightful, since such states carry information about the dynamical properties of the theory. Furthermore, the most symmetric vacuum does not necessarily correspond to the physically realized one. We explore a rather particular theory of a minimally coupled massive scalar field

$$\mathcal{L}_m = \sqrt{-g}\left(\frac{1}{2}\,\phi^{,\mu}\phi_{,\mu} - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\,\phi^4\right) \tag{1.1}$$

living in the flat de Sitter background expressed in planar coordinates:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - a^2(t)\,d\vec{x}^2, \qquad a(t) = \mathrm{e}^{Ht}. \tag{1.2}$$

We consider only the long-wavelength modes: the simplest from a technical standpoint and potentially the most relevant for upcoming observational tests.

Our framework not only bridges the massive-massless perturbative results but also provides a unified picture of the Starobinsky-Yokoyama stochastic approach [52, 53] and quantum field theory methods, laying the groundwork for non-perturbative resummation techniques in de Sitter space.

Starobinsky's stochastic approach originated in [52] and was applied to inflation in various early works [54, 55, 56, 57, 58, 59, 60, 61, 62] and more recent ones [63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74]. Within the stochastic approach, one decomposes the quantum field operator $\phi\,(t,\vec{x})$ in the Heisenberg representation into the super-Hubble or long-wavelength, $k < He^{Ht}$, and the sub-Hubble or short-wavelength, $k > He^{Ht}$, parts using a window function, which one chooses to be the dynamical Heaviside step function. The long-wavelength part $\phi^{\text{l-w}}(t,\vec{x})$ satisfies the equation of motion $\square\,\phi = -V'_{\phi}(\phi)$, and for the slowly varying field, it takes the local Langevin-like form [52], where the noise term represents short-wavelength modes that continually shift into the long-wavelength ones. Therefore, the long-wavelength part of the quantum scalar field $\phi^{\text{l-w}}(t,\vec{x})$ can be treated as the classical stochastic field $\varphi(t,\vec{x})$ with a probability distribution function $\mathcal{P}\left[\varphi(t,\vec{x})\right]$ that satisfies the Fokker-Planck or Einstein-Smoluchowski equation, associated with the corresponding local Langevin-like equation [53]. The first non-perturbative development for computing the vacuum expectation value of the coarse-grained, long-wavelength massless minimally coupled scalar field with a quartic self-interaction in de Sitter space was proposed in the seminal paper by Starobinsky and Yokoyama [53].

We begin with the computation of two-point and four-point correlation functions of a massive minimally coupled scalar field (1.1) within the long-wavelength approximation in Chapter 2. For this purpose, we employ the Yang-Feldman-type equation.

The Yang-Feldman formalism [75] recursively defines the interacting field as a formal power series in the coupling constant, expressed in terms of the free field; see also the works of Källén [76, 77]. Within this formalism, one solves perturbatively the field equation for the quantum field in the Heisenberg representation. By iterating the Yang-Feldman equation to the desired order in the coupling constant, one constructs the formal perturbative series for the Heisenberg operator and computes the vacuum expectation value of products of field operators. In the literature, that integral equation is widely exploited often without being explicitly referred to as the Yang–Feldman equation. The expectation value of the stress-energy tensor up to first order in the coupling constant for interacting scalar fields in de Sitter space was first computed by Ford [78] using this form of the equation. Subsequently, in the seminal works [44, 45] Tsamis and Woodard have shown that the Yang-Feldman formalism is rather convenient to catch the leading infrared logarithms in cosmic time t, namely, $\ln\left(a(t)\right) = Ht$, for the massless minimally coupled scalar field with a quartic self-interaction in de Sitter space. They have also established the perturbative agreement between the Starobinsky and Yokoyama stochastic approach [52, 53] and the quantum field theory results in de Sitter space. This remarkable discovery launched the program to examine the relation between the stochastic approach and quantum field theory [79, 80, 81, 82, 83, 84, 85, 86] and some modern techniques [87, 88, 89, 90, 91]. Furthermore, it stimulated efforts to generalize the resummation techniques to more complicated models [92, 93, 94, 95, 96, 97]; see also the present-day review [98].

In Chapter 2, we follow this established program and propose a method to include the mass into the truncated Yang-Feldman equation for a massless minimally coupled scalar field derived by Tsamis and Woodard [44, 45]. In addition, we define the massive scalar field in terms of the massless field. To define it, we invert the integral Yang-Feldman equation considering only the massive term in the potential. The resulting relation allows us to compute the vacuum expectation value of the free massive scalar field relying only on the known vacuum expectation value of the free massless one. Our treatment could be considered a theory of a massive scalar field with the vacuum “inherited” from a massless field. This is not problematic, as the most symmetric vacuum is not always the physical one. Remarkably, the obtained free massive two-point correlation function coincides with that of the well-known Ornstein–Uhlenbeck stochastic process [99]. This process is the unique Gaussian and Markov stochastic process with a stationary state [100, 101, 102, 103, 104]. By its virtue, the obtained two-point correlation function exhibits a smooth massless limit and tends over time to the equilibrium de Sitter-invariant state. This attractor feature was also discovered from other considerations in [53, 105, 106]. We employed the Yang-Feldman-type equation to compute the two-point correlation function up to order λ^3 and the four-point correlation function up to order λ^2. Our resulting series are equivalent to the resummation of leading infrared logarithms in the perturbative expansion in m^2/H^2 at each order of the self-interaction coupling constant λ. Within this scheme, the infrared divergences in cosmic time t of the massless self-interacting scalar field are fully resummed in the massive series.

In Chapter 3, we establish the correspondence between the integral structures arising in the iterative Yang-Feldman-type equation and diagrams within the Schwinger-Keldysh, or “in-in”, or “closed-time-path”, formalism [107, 108] and compare our findings with the stochastic diagrammatic approach [81]. All types of diagrams have the same topological structure, but in the stochastic case, one encounters a single vertex and three propagators, while within the Schwinger-Keldysh formalism, there are two types of vertices and four types of propagators; see [109, 110, 111, 112, 113, 114, 115, 116] for an introduction to this formalism and some applications. The Schwinger-Keldysh, or "in-in", formalism [107, 108], is one of the most efficient techniques for computing correlation functions in cosmology. In contrast to the scattering computations, where the system evolves from $t = -\infty$ ("in" state) to $+\infty$ ("out" state) and both states are specified, in cosmology we are interested in evolving the quantum system up to some finite moment of time. One specifies only the initial state, and both the "in" state, $|\text{in}\rangle$, and its Hermitian conjugate, $\langle\text{in}|$, evolve from $t = -\infty$ to a certain time, and the field operators are sandwiched between unitary time-evolution operators. To spot what kind of correspondence might exist, in Chapter 3 we rely on results for the two-point correlation function of a massive scalar field calculated at the one-loop and two-loop orders via the Schwinger-Keldysh diagrammatic technique in [117] by Kamenshchik, Starobinsky, and Vardanyan, and in [118] by Gautier and Serreau in the p-representation [119]. In the massless case at equal times, we compare our outcomes with [83, 84, 120, 121, 122, 123]. For the massive case, Garbrecht, Rigopoulos, and Zhu introduced [81] a functional representation of equations within Starobinsky’s stochastic approach. In the proposed technique, the expectation value of an operator is given through distribution averages. In the late-time limit, they have computed the topologically distinct diagrams for the two-point correlation function at equal times, using both the stochastic approach and the "in-in" formalism; see also [82]. At late times and the late-time limit at equal times, our results are in agreement with those obtained from diagrammatic techniques. We proceed with our reasoning, supported by the two-point correlation function results and existing literature, to compute the four-point correlation function up to the one-loop order in Chapter 3. For the two-point correlation function at the three-loop level, we present the diagrams and their contributions in Appendix B.

We provide the basics of Starobinsky's stochastic approach [52] in Chapter 4. Within this approach, the stationary solution of the corresponding Fokker-Planck equation and the related method to calculate the equal-time correlation functions at equilibrium were derived by Starobinsky and Yokoyama in [53] for a massless minimally coupled scalar field with a quartic self-interaction. We explore the Starobinsky and Yokoyama stochastic approach for the model of a massive scalar field with a quartic self-interaction (1.1). With the use of recursion relations involving the modified Bessel functions of the second kind, we derive the non-perturbative late-time stochastic expectation value $\langle \varphi^{2n} \rangle^{\text{late-time}}_{\text{non-pert}}$. When expanded in the small self-interaction coupling constant λ regime, the obtained series are in agreement with results obtained from the massive Yang-Feldman-type equation in Chapter 2 at the late-time limit. To make an additional contact between approaches, we follow the method proposed by Tsamis and Woodard in their seminal work [45]. We extract a first-order differential equation for the stochastic expectation values from the Fokker-Planck equation and the corresponding series in the free massive case and in the massless case with a quartic self-interaction. One readily observes the leading infrared logarithm structure in cosmic time t in the obtained series.

In Chapter 5, we develop an alternative approach to compute the perturbative series for the equal-time and multi-time correlation functions within the leading logarithm approximation [124]. Instead of processing the massive Yang-Feldman-type equation or dealing with the diagrammatic techniques, one derives a simple first-order differential equation from the Fokker-Planck or forward Kolmogorov equation. The formal integral solution to that equation relates various correlation functions at different self-interaction coupling constant λ orders. One solves it iteratively. The starting point is the non-interacting case, and the exact solutions are well known in that context. Further, we related these correlation functions to ones at different orders in λ by the integral relation. In the case of equal-time correlation functions, that relation provides the results order by order in λ via straightforward integration. In the multi-time case, we have the formal solution to the corresponding first-order differential equation as an indefinite integral. To obtain the multi-time correlation function in the free massive case, we extensively exploit the Markov property, formulated in terms of the conditional (or transition) probability density. The local Langevin-like equation underlying the Fokker–Planck equation in Starobinsky's stochastic approach [52] is a first-order differential equation. Therefore, for each noise realization, one determines uniquely the stochastic field if its value at the initial time is given. Moreover, since the fluctuating or noise term is delta-correlated [52], its values at different times are statistically independent. Thus, the solution to the local Langevin-like equation has the Markov property: the noise-term values at previous times, $t < t_0$, do not influence the conditional probabilities for $t > t_0$ times. After integration of the obtained first-order differential equation for the correlation functions, one has the antiderivative and the unknown function, which one defines by matching it to the known result for the equal-time correlation function or to the previous "iteration" step by step. We provide the detailed expressions to illustrate the proposed strategy. The obtained results are in agreement with the corresponding perturbative series from quantum field theory for the two-point and four-point correlation functions. This treatment provides an additional confirmation that the Starobinsky-Yokoyama stochastic approach precisely reproduces quantum field theory results in de Sitter space for scalar field potential models at all orders within the leading logarithm approximation, extending its consistency beyond the equal-time and stationary cases.

Contrary to the basic assumption, the perturbation theory might miserably fail due to the growth of higher-order terms in both classical and quantum areas of physics. The presence of these secular growth terms could stimulate the development of some analytical methods to improve perturbative expansion. Some kind of quantum field theory renormalization group technique [125] can be employed in quite different contexts of physics (both classical and quantum) and mathematics [126, 127, 128, 129, 130, 131, 132]; see the corresponding developments in de Sitter space [133, 134] and in stochastic inflation [135]. For instance, Chen, Goldenfeld, and Oono have shown that one can improve the naive perturbative terms that exhibit secular growth arising at the iterative solution of some complicated differential equation [136]. The authors of [136] developed the dynamical renormalization group method by considering differential equations that involve a small parameter and whose zeroth order solutions are bounded functions, while the first iteration reveals the presence of secularly growing terms. In contrast to their development, we have only some pieces of that perturbative information for the long-wavelength part of the correlation function and do not have a general differential equation. Following the spirit of [125, 128, 130, 136], Kamenshchik and Vardanyan constructed the renormalization group inspired autonomous first-order differential equation for the massless minimally coupled scalar field with a quartic self-interaction in de Sitter space [122]. Within the proposed techniques to treat the series of secularly growing terms, the long-wavelength part of the vacuum expectation values $\left\langle \phi^2_{m=0}(t,\vec{x}) \right\rangle$ and $\left\langle \phi^4_{m=0}(t,\vec{x}) \right\rangle$ in the late-time limit are finite and closely agree with the Starobinsky-Yokoyama stochastic approach results. The massive expectation values do not contain such secularly growing terms, however, it would be interesting to obtain the non-perturbative result. The autonomous equation technique was also applied in [137, 138], while a slightly different but conceptually similar resummation technique was implemented in [123, 139, 140]. In Chapter 6, we construct an autonomous equation for the long-wavelength part of the massive vacuum expectation value $\left\langle \phi^2(t,\vec{x}) \right\rangle$ relying on the obtained perturbative series in the previous chapters. Before that, we consider the so-called Hartree-Fock, or Gaussian, approximation. We have found the non-perturbative solution within this approximation and revealed what it resums. With the help of our extracted diagram contributions for the two-point function at the one-, two-, and three-loop levels from Chapter 3 and Appendix B, we establish that the Hartree-Fock, or Gaussian, approximation resums only the so-called "Cactus"-type diagrams. Further, we construct the autonomous first-order differential equation, relying on the obtained perturbative series from the previous chapters. In order to capture the absentee "Sunset" contribution at the two-loop level, we linearize the obtained autonomous equation and integrate it. The full non-perturbative result in the self-interaction coupling constant λ is the sum of that obtained from the Hartree-Fock approximation and the solution of the linearized autonomous equation. It reproduces the correct perturbative series for $\left\langle \phi^2(t,\vec{x}) \right\rangle$ up to the two-loop level. In the late-time limit, it almost coincides with the result obtained within the Starobinsky-Yokoyama stochastic approach over the entire interval of the dimensionless parameter $0 \leq \frac{\pi^2 m^4}{3\lambda H^4} < \infty$.

We recap all findings in detail in Chapter 7 and list potential extensions of the applied techniques in Chapter 8.

2 Yang-Feldman-type equation

In this chapter, we derive the truncated Yang-Feldman-type equation within the long-wavelength approximation for a minimally coupled massive scalar field (1.1) on a flat de Sitter background (1.2). Using this equation, we then compute the two-point and four-point correlation functions.

2.1 Truncated Yang-Feldman-type equation for a massive scalar field in de Sitter space

The Yang-Feldman formalism [75, 76, 77] recursively defines the interacting field as a formal power series in the coupling constant, expressed through the free field in the Heisenberg representation. Following this approach, we start from the equation of motion for a massless scalar field

$$\Box\phi(t,\vec{x}) = -V'_{\phi}(\phi) \tag{2.1}$$

and recast it as the integral equation:

$$\phi(t,\vec{x}) = \phi_0(t,\vec{x}) - \int d^4x'\sqrt{-g(x')}\,G_{\rm R}(t,\vec{x};t',\vec{x}')\,V'_{\phi}\big(\phi(t',\vec{x}')\big), \tag{2.2}$$

where $\phi_0(t,\vec{x})$ is the free field satisfying the homogeneous equation, $\Box\phi_0(t,\vec{x}) = 0$, and the retarded Green's function satisfies

$$\Box\, G_{\rm R}(t,\vec{x};t',\vec{x}') = \frac{\delta\,(t-t')\,\delta\,(\vec{x}-\vec{x}')}{\sqrt{-g(x')}} \tag{2.3}$$

with the retarded boundary condition $G_{\rm R}(t,\vec{x};t',\vec{x}') = 0$ for $t \leq t'$.

The solution to equation (2.3) can be expressed as

$$G_{\rm R}(t,\vec{x};t',\vec{x}') = i\Theta(t-t')\,\langle\big[\phi_0(t,\vec{x}),\phi_0(t',\vec{x}')\big]\rangle. \tag{2.4}$$

The scalar field appearing in the commutator can be expanded in terms of canonically normalized creation and annihilation operators as follows:

$$\phi_0(t,\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}}\left(u_k(t)\,\mathrm{e}^{i\vec{k}\vec{x}}\,\hat{a}_{\vec{k}} + u^*_k(t)\,\mathrm{e}^{-i\vec{k}\vec{x}}\,\hat{a}^\dagger_{\vec{k}}\right), \tag{2.5}$$

where the modes $u_k(t)$ on a flat de Sitter background (1.2) are solutions to the linear equation

$$\ddot{u}_k + 3H\dot{u}_k + k^2 e^{-2Ht} u_k = 0, \tag{2.6}$$

and they must be normalized through the Wronskian

$$W\left[u_k(t), u_k^*(t)\right] = \dot{u}_k u_k^* - u_k \dot{u}_k^* = -i e^{-3Ht} \tag{2.7}$$

as a consequence of the canonical commutation relations

$$\left[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}\right] = \left[\hat{a}^\dagger_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}\right] = 0, \qquad \left[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}\right] = \delta(\vec{k} - \vec{k}'). \tag{2.8}$$

The expression for the retarded Green's function (2.4) can be written straightforwardly as

$$G_{\mathrm{R}}(t, \vec{x}; t', \vec{x}') = i\Theta(t - t') \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\,(\vec{x}-\vec{x}')} \Big(u_k(t) u_k^*(t') - u_k^*(t)\, u_k(t')\Big). \tag{2.9}$$

We are interested in the contribution of the super-Hubble or long-wavelength (l-w) modes, whose wave numbers are small, $H < k \ll H e^{Ht}$. Thus, in equation (2.6), one can neglect the last term proportional to k^2, leading to the simple general solution:

$$u_k^{\text{l-w}}(t) = c_1 + c_2\, e^{-3Ht}. \tag{2.10}$$

By employing the Wronskian (2.7) with (2.10), one obtains $c_1^* c_2 - c_1 c_2^* = i/3H$. Therefore, from (2.9) we obtain[1]

$$G_{\mathrm{R}}^{\text{l-w}}(t, \vec{x}; t', \vec{x}') = \frac{\Theta(t - t')}{3H}\Big(e^{-3Ht'} - e^{-3Ht}\Big)\, \delta(\vec{x} - \vec{x}'). \tag{2.13}$$

1 Let us also argue on a bit more detailed derivation. One can keep the full solution to (2.6), which is

$$u_k(t) = c_1\, e^{-\frac{3}{2}Ht}\, J_{-3/2}\bigg(\frac{k}{H}\, e^{-Ht}\bigg) + c_2\, e^{-\frac{3}{2}Ht}\, Y_{-3/2}\bigg(\frac{k}{H}\, e^{-Ht}\bigg). \tag{2.11}$$

Here, $J_{-3/2}(z)$ and $Y_{-3/2}(z)$ are the Bessel functions of the first and second kinds. From the Wronskian (2.7)

$$W[u_k(t), u_k{}^*(t)] \overset{(2.7)}{=} k\, e^{-4Ht}(c_1 c_2^* - c_2 c_1^*)\left(J_{-1/2}\bigg(\frac{k}{H}\, e^{-Ht}\bigg) Y_{-3/2}\bigg(\frac{k}{H}\, e^{-Ht}\bigg) - J_{-3/2}\bigg(\frac{k}{H}\, e^{-Ht}\bigg) Y_{-1/2}\bigg(\frac{k}{H}\, e^{-Ht}\bigg)\right),$$

in the long-wavelength approximation, $k \ll H\, e^{Ht}$, where the entering Bessel functions are [141]

$$\begin{aligned}
J_{-1/2}\bigg(\frac{k}{H}\, e^{-Ht}\bigg) &\sim -\frac{\sqrt{2H}}{\sqrt{\pi k}}\, e^{\frac{Ht}{2}} - \frac{\sqrt{k^3}}{\sqrt{2\pi H^3}}\, e^{-\frac{3Ht}{2}} + \ldots\,; \qquad J_{-3/2}\bigg(\frac{k}{H}\, e^{-Ht}\bigg) \sim -\frac{\sqrt{2H^3}}{\sqrt{\pi k^3}}\, e^{\frac{3Ht}{2}} + \ldots\,; \\
Y_{-1/2}\bigg(\frac{k}{H}\, e^{-Ht}\bigg) &\sim \frac{\sqrt{2k}}{\sqrt{\pi H}}\, e^{-\frac{Ht}{2}} - \frac{\sqrt{k^5}}{3\sqrt{2\pi H^5}}\, e^{-\frac{5Ht}{2}} + \ldots\,; \qquad Y_{-3/2}\bigg(\frac{k}{H}\, e^{-Ht}\bigg) \sim -\frac{\sqrt{2k^3}}{3\sqrt{\pi H^3}}\, e^{-\frac{3Ht}{2}} + \ldots,
\end{aligned} \tag{2.12}$$

we have $c_1 c_2^* - c_1^* c_2 = -i\pi/2H$, and $u_k(t) u_k^*(t') - u_k^*(t)\, u_k(t') = \frac{2}{3\pi}(c_1 c_2^* - c_1^* c_2)\Big(e^{-3Ht'} - e^{-3Ht}\Big)$ in (2.9) becomes precisely one that was obtained in a bit more "cheating" way, leading to (2.13).

Substituting this expression into (2.2), one observes the multiplication by the measure $\sqrt{-g(x')}$ under the integral. When combined with the measure, the first exponent in the brackets in (2.13) always prevails, whereas the second is "switched on" only at the upper limit of integration in (2.2) and is truncated to isolate the leading logarithmic order and prevent the appearance of sub-leading contributions; see the seminal works [44, 45]. Therefore, at the leading logarithmic order in cosmic time t, i.e., $\ln\big(a(t)\big) = Ht$, equation (2.2) reduces to the truncated form:

$$\phi(t,\vec{x}) = \phi_0(t,\vec{x}) - \frac{1}{3H}\int\limits_0^t dt'\, V'_\phi\big(\phi(t',\vec{x})\big). \tag{2.14}$$

Note that we did not use any particular choice of the vacuum. However, one can arrive at the retarded Green's function (2.13) owing to the explicit form of the basis functions of the chosen vacuum in the Fock space, the so-called Bunch-Davies vacuum [142]. Tsamis and Woodard derived the truncated form (2.14) for the massless, minimally coupled scalar field with a quartic self-interaction in [44, 45]. To do this, they used the explicit form of the Bunch-Davies basis functions and expanded them up to the third order, taking into account only the contribution of the long-wavelength modes, $H < k < He^{Ht}$.

One can now iteratively use equation (2.14) to construct the formal perturbative series for the Heisenberg operator and compute the vacuum expectation value of $\phi(t,\vec{x})$ solely from the known vacuum expectation value of the free massless field $\phi_0(t,\vec{x})$.

Through (2.14), we define the free massive field $\tilde{\phi}(t,\vec{x})$ in terms of the free massless field $\phi_0(t,\vec{x})$ as follows:

$$\tilde{\phi}(t,\vec{x}) = \phi_0(t,\vec{x}) - \frac{m^2}{3H}\,e^{-\frac{m^2 t}{3H}}\int\limits_0^t dt'\, e^{\frac{m^2 t'}{3H}}\,\phi_0(t',\vec{x}); \tag{2.15}$$

see appendix A, namely (A.3). The relation introduced above, together with the massive Yang-Feldman-type equation discussed below, is important for our proposed treatment. We employ it to calculate the vacuum expectation value of the free massive scalar field, relying only on the infrared part of the known vacuum expectation value of the free massless field.

Furthermore, we derive the analog of equation (2.14) in the massive case:

$$\phi(t,\vec{x}) = \tilde{\phi}(t,\vec{x}) - \frac{\lambda}{3H}\,e^{-\frac{m^2 t}{3H}}\int\limits_0^t dt'\, e^{\frac{m^2 t'}{3H}}\,\phi^3(t',\vec{x}); \tag{2.16}$$

see (A.5) and the nearby reasoning. Note that equations (2.14) and (2.16) have the same structure with two differences: the free massless field $\phi_0(t,\vec{x})$ is replaced by the free massive field $\tilde{\phi}(t,\vec{x})$, and the retarded Green's function for the massive field acquires an additional exponential factor.

The massive Yang-Feldman-type equation (2.16) can be iteratively expanded to the first few terms of the formal perturbative expansion for the Heisenberg operator as

$$\begin{aligned}\phi(t,\vec{x}) = \tilde{\phi}(t,\vec{x}) &- \frac{\lambda}{3H}\mathrm{e}^{-\frac{m^2 t}{3H}}\int_0^t dt'\,\mathrm{e}^{\frac{m^2 t'}{3H}}\,\tilde{\phi}^3(t',\vec{x}) \\ &+ \frac{\lambda^2}{3H^2}\mathrm{e}^{-\frac{m^2 t}{3H}}\int_0^t dt'\,\tilde{\phi}^2(t',\vec{x})\int_0^{t'} dt''\mathrm{e}^{\frac{m^2 t''}{3H}}\,\tilde{\phi}^3(t'',\vec{x}) \\ &- \frac{\lambda^3}{3H^3}\,\mathrm{e}^{-\frac{m^2 t}{3H}}\Bigg(\int_0^t dt'\,\tilde{\phi}^2(t',\vec{x})\int_0^{t'} dt''\,\tilde{\phi}^2(t'',\vec{x})\int_0^{t''} dt'''\mathrm{e}^{\frac{m^2 t'''}{3H}}\,\tilde{\phi}^3(t''',\vec{x}) \\ &\qquad + \frac{1}{3}\int_0^t dt'\,\mathrm{e}^{-\frac{m^2 t'}{3H}}\,\tilde{\phi}(t',\vec{x})\left(\int_0^{t'} dt''\,\mathrm{e}^{\frac{m^2 t''}{3H}}\,\tilde{\phi}^3(t'',\vec{x})\right)^2\Bigg) + O(\lambda^4)\end{aligned} \tag{2.17}$$

to calculate the series for the vacuum expectation value of the massive field $\phi(t,\vec{x})$ in terms of the known vacuum expectation value of the free massless field $\phi_0(t,\vec{x})$ with the help of (2.15).

Our proposed approach is equivalent to the following: we construct the perturbative series in m^2 for the massless scalar field, then sum the leading logarithms (thanks to the massive theory being Gaussian and to the relation (2.15)), and finally, organize the perturbative series for the massive field in terms of the self-interaction coupling constant λ.

2.2 Two-point correlation function

We begin with the two-point correlation function of the free massive scalar field $\tilde{\phi}(t,\vec{x})$ for coincident spatial points but different times.

Using the relation derived in (2.15), we express the two-point correlation function of the free massive scalar field $\tilde{\phi}\,(t,\vec{x})$ as follows:

$$\begin{aligned}\left\langle\tilde{\phi}(t_1,\vec{x})\,\tilde{\phi}(t_2,\vec{x})\right\rangle := \left\langle\tilde{\phi}(t_1)\tilde{\phi}(t_2)\right\rangle = \left\langle\phi_0(t_1)\phi_0(t_2)\right\rangle &- \frac{m^2}{3H}\,\mathrm{e}^{-\frac{m^2 t_1}{3H}}\int_0^{t_1} dt'\mathrm{e}^{\frac{m^2 t'}{3H}}\left\langle\phi_0(t')\phi_0(t_2)\right\rangle \\ &- \frac{m^2}{3H}\,\mathrm{e}^{-\frac{m^2 t_2}{3H}}\int_0^{t_2} dt'\mathrm{e}^{\frac{m^2 t'}{3H}}\left\langle\phi_0(t_1)\phi_0(t')\right\rangle \\ &+ \frac{m^4}{9H^2}\,\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2)}\int_0^{t_1} dt'\mathrm{e}^{\frac{m^2 t'}{3H}}\int_0^{t_2} dt''\mathrm{e}^{\frac{m^2 t''}{3H}}\left\langle\phi_0(t')\phi_0(t'')\right\rangle.\end{aligned} \tag{2.18}$$

Hereafter, we omit the argument $\vec{x}$, as it is the same in all contributions. To compute (2.18), we substitute the known result for the infrared part of the free massless field [83]:

$$\left\langle \phi_0(t_1,\vec{x})\,\phi_0(t_2,\vec{x})\right\rangle := \left\langle \phi_0(t_1)\phi_0(t_2)\right\rangle = \frac{H^3}{4\pi^2}\cdot\begin{cases} t_2\,, & t_2\le t_1\,;\\ t_1\,, & t_2\ge t_1\,.\end{cases} \tag{2.19}$$

In the equal-time case, this simplifies to the well-known result [37, 38, 40]: $\left\langle \phi_0^2(t)\right\rangle = \dfrac{H^3 t}{4\pi^2}$.

By inserting this expression for the infrared part of the free massless field (2.19) into equation (2.18) and performing the time-ordered integration, we obtain:

$$\begin{aligned}
\left\langle \tilde{\phi}(t_1)\tilde{\phi}(t_2)\right\rangle\Big|_{t_2\le t_1} &= \frac{H^3}{4\pi^2}\Bigg(t_2 - \frac{m^2}{3H}\,\mathrm{e}^{-\frac{m^2 t_1}{3H}}\left(t_2\int_{t_2}^{t_1} dt'\,\mathrm{e}^{\frac{m^2 t'}{3H}} + \int_0^{t_2} dt'\,\mathrm{e}^{\frac{m^2 t'}{3H}}\,t'\right) \\
&\quad - \frac{m^2}{3H}\,\mathrm{e}^{-\frac{m^2 t_2}{3H}}\int_0^{t_2} dt'\,\mathrm{e}^{\frac{m^2 t'}{3H}}\,t' + \frac{m^4}{9H^2}\,\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2)}\Bigg(\int_0^{t_2} dt'\,\mathrm{e}^{\frac{m^2 t'}{3H}}\,t'\int_{t'}^{t_2} dt''\,\mathrm{e}^{\frac{m^2 t''}{3H}} \\
&\quad + \int_{t_2}^{t_1} dt'\,\mathrm{e}^{\frac{m^2 t'}{3H}}\int_0^{t_2} dt''\,\mathrm{e}^{\frac{m^2 t''}{3H}}\,t'' + \int_0^{t_2} dt'\,\mathrm{e}^{\frac{m^2 t'}{3H}}\int_0^{t'} dt''\,\mathrm{e}^{\frac{m^2 t''}{3H}}\,t''\Bigg)\Bigg) \\
&= \frac{3H^4}{8\pi^2 m^2}\left(\mathrm{e}^{-\frac{m^2}{3H}(t_1-t_2)} - \mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2)}\right);
\end{aligned} \tag{2.20}$$

and, analogously, for the reversed time ordering:

$$\begin{aligned}
\left\langle \tilde{\phi}(t_1)\tilde{\phi}(t_2)\right\rangle\Big|_{t_2\ge t_1} &= \frac{H^3}{4\pi^2}\Bigg(t_1 - \frac{m^2}{3H}\,\mathrm{e}^{-\frac{m^2 t_1}{3H}}\int_0^{t_1} dt'\,\mathrm{e}^{\frac{m^2 t'}{3H}}\,t' \\
&\quad - \frac{m^2}{3H}\,\mathrm{e}^{-\frac{m^2 t_2}{3H}}\left(\int_0^{t_1} dt'\,\mathrm{e}^{\frac{m^2 t'}{3H}}\,t' + t_1\int_{t_1}^{t_2} dt'\,\mathrm{e}^{\frac{m^2 t'}{3H}}\right) + \frac{m^4}{9H^2}\,\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2)}\Bigg(\int_0^{t_1} dt'\,\mathrm{e}^{\frac{m^2 t'}{3H}}\,t'\int_{t'}^{t_1} dt''\,\mathrm{e}^{\frac{m^2 t''}{3H}} \\
&\quad + \int_0^{t_1} dt'\,\mathrm{e}^{\frac{m^2 t'}{3H}}\,t'\int_{t_1}^{t_2} dt''\,\mathrm{e}^{\frac{m^2 t''}{3H}} + \int_0^{t_1} dt'\,\mathrm{e}^{\frac{m^2 t'}{3H}}\int_0^{t'} dt''\,\mathrm{e}^{\frac{m^2 t''}{3H}}\,t''\Bigg)\Bigg) \\
&= \frac{3H^4}{8\pi^2 m^2}\left(\mathrm{e}^{\frac{m^2}{3H}(t_1-t_2)} - \mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2)}\right).
\end{aligned} \tag{2.21}$$

In the general case, it suffices to add the modulus to the first term, resulting in:

$$\left\langle \tilde{\phi}(t_1)\tilde{\phi}(t_2)\right\rangle = \frac{3H^4}{8\pi^2 m^2}\left(\mathrm{e}^{-\frac{m^2}{3H}|t_1-t_2|} - \mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2)}\right). \tag{2.22}$$

Hereafter, we apply the same time-ordering "tricks" to compute higher-order corrections to the two-point and four-point correlation functions. The obtained correlation function (2.22) serves as the primary input for computing perturbative series by iteratively applying the Yang-Feldman-type equation (2.17). This result coincides precisely with the correlation function of the Ornstein-Uhlenbeck process [99], which is the unique Gaussian and Markov stochastic process with a stationary state [101, 102, 103]. Its key feature is the drift tendency towards the average value, with the mean-reversion rate given by $m^2/3H$. In our framework, correlation functions at each perturbative order have a smooth massless limit, matching the expressions for massless and massive scalar fields found in the literature, obtained within diagrammatic quantum field theory and stochastic methods, as will be discussed in Chapters 3 and 4. Our treatment can be considered a theory of a massive scalar field, where the vacuum is "inherited" from the massless one. The transition between massive and massless field perturbative results is smooth, even though de Sitter invariance is broken. However, by virtue of the correlation function (2.22), this tends over time to the equilibrium state, which depends only on the time difference and turns out to be de Sitter-invariant.

Building on (2.22), one can organize the perturbative loop series within the massive Yang-Feldman-type equation (2.16) or its explicit iterated form (2.17):

$$\begin{aligned}\big\langle\phi(t_1)\phi(t_2)\big\rangle &= \big\langle\tilde{\phi}(t_1)\tilde{\phi}(t_2)\big\rangle + \big\langle\phi(t_1)\phi(t_2)\big\rangle_{\lambda} + \big\langle\phi(t_1)\phi(t_2)\big\rangle_{\lambda^2}\\ &\quad + \big\langle\phi(t_1)\phi(t_2)\big\rangle_{\lambda^3} + O\big(\lambda^4\big).\end{aligned} \tag{2.23}$$

Once again, the basic idea of the Yang-Feldman formalism is to recursively define the interacting field as a formal perturbative series in the self-interaction coupling constant λ for the field's operator expressed in the Heisenberg representation.

At linear order in λ, the Yang-Feldman-type equation (2.17) yields

$$\begin{aligned}\big\langle\phi(t_1)\phi(t_2)\big\rangle_{\lambda} &= -\frac{\lambda}{3H}\left(\mathrm{e}^{-\frac{m^2 t_1}{3H}}\int_0^{t_1} dt'\mathrm{e}^{\frac{m^2 t'}{3H}}\big\langle\tilde{\phi}(t_2)\tilde{\phi}^3(t')\big\rangle + \mathrm{e}^{-\frac{m^2 t_2}{3H}}\int_0^{t_2} dt'\mathrm{e}^{\frac{m^2 t'}{3H}}\big\langle\tilde{\phi}(t_1)\tilde{\phi}^3(t')\big\rangle\right)\\ &= -\frac{\lambda}{H}\mathrm{e}^{-\frac{m^2 t_1}{3H}}\int_0^{t_1} dt'\mathrm{e}^{\frac{m^2 t'}{3H}}\big\langle\tilde{\phi}(t')\tilde{\phi}(t_2)\big\rangle\big\langle\tilde{\phi}^2(t')\big\rangle \;+\; \big(t_1\leftrightarrow t_2\big)\end{aligned} \tag{2.24}$$

with time intervals $t' \le t_2 \le t_1$ and $t_2 \le t' \le t_1$, and the one-loop contribution to (2.23) is

$$\begin{aligned}\big\langle\phi(t_1)\phi(t_2)\big\rangle_{\lambda} = -\frac{27\lambda H^8}{128\,\pi^4 m^6}\Bigg(& 2\,\mathrm{e}^{-\frac{m^2}{3H}|t_1-t_2|} + \frac{4m^2}{3H}\Big(|t_1-t_2| - (t_1+t_2)\Big)\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2)}\\ & - \mathrm{e}^{\frac{m^2}{3H}\left(|t_1-t_2|-2(t_1+t_2)\right)} + \frac{2m^2}{3H}|t_1-t_2|\left(\mathrm{e}^{-\frac{m^2}{3H}|t_1-t_2|} - \mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2)}\right)\\ & - \mathrm{e}^{-\frac{m^2}{3H}\left(|t_1-t_2|+2(t_1+t_2)\right)} + \mathrm{e}^{-\frac{m^2}{3H}\left(2|t_1-t_2|+(t_1+t_2)\right)} - \mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2)}\Bigg).\end{aligned} \tag{2.25}$$

Correspondingly, at the two-loop level or λ^2, we have

$$\left\langle\phi(t_1)\phi(t_2)\right\rangle_{\lambda^2} = \frac{\lambda^2}{3H^2}e^{-\frac{m^2t_1}{3H}}\int_0^{t_1}dt'\int_0^{t'}dt''e^{\frac{m^2t''}{3H}}\left\langle\tilde{\phi}(t_2)\tilde{\phi}^2(t')\tilde{\phi}^3(t'')\right\rangle \tag{2.26}$$
$$+\frac{\lambda^2}{3H^2}e^{-\frac{m^2t_2}{3H}}\int_0^{t_2}dt'\int_0^{t'}dt''e^{\frac{m^2t''}{3H}}\left\langle\tilde{\phi}(t_1)\tilde{\phi}^2(t')\tilde{\phi}^3(t'')\right\rangle$$
$$+\frac{\lambda^2}{9H^2}e^{-\frac{m^2}{3H}(t_1+t_2)}\int_0^{t_1}dt'e^{\frac{m^2t'}{3H}}\int_0^{t_2}dt''e^{\frac{m^2t''}{3H}}\left\langle\tilde{\phi}^3(t')\tilde{\phi}^3(t'')\right\rangle$$
$$=\frac{\lambda^2}{H^2}e^{-\frac{m^2t_1}{3H}}\int_0^{t_1}dt'\int_0^{t'}dt''e^{\frac{m^2t''}{3H}}\Bigg(2\left\langle\tilde{\phi}(t')\tilde{\phi}(t_2)\right\rangle\left\langle\tilde{\phi}(t')\tilde{\phi}(t'')\right\rangle\left\langle\tilde{\phi}^2(t'')\right\rangle \tag{2.27}$$
$$+2\left\langle\tilde{\phi}(t'')\tilde{\phi}(t_2)\right\rangle\left(\left\langle\tilde{\phi}(t')\tilde{\phi}(t'')\right\rangle\right)^2+\left\langle\tilde{\phi}(t'')\tilde{\phi}(t_2)\right\rangle\left\langle\tilde{\phi}^2(t')\right\rangle\left\langle\tilde{\phi}^2(t'')\right\rangle\Bigg)$$
$$+\frac{\lambda^2}{H^2}e^{-\frac{m^2t_2}{3H}}\int_0^{t_2}dt'\int_0^{t'}dt''e^{\frac{m^2t''}{3H}}\Bigg(2\left\langle\tilde{\phi}(t_1)\tilde{\phi}(t')\right\rangle\left\langle\tilde{\phi}(t')\tilde{\phi}(t'')\right\rangle\left\langle\tilde{\phi}^2(t'')\right\rangle$$
$$+2\left\langle\tilde{\phi}(t_1)\tilde{\phi}(t'')\right\rangle\left(\left\langle\tilde{\phi}(t')\tilde{\phi}(t'')\right\rangle\right)^2+\left\langle\tilde{\phi}(t_1)\tilde{\phi}(t'')\right\rangle\left\langle\tilde{\phi}^2(t')\right\rangle\left\langle\tilde{\phi}^2(t'')\right\rangle\Bigg)$$
$$+\frac{\lambda^2}{H^2}e^{-\frac{m^2}{3H}(t_1+t_2)}\int_0^{t_1}dt'e^{\frac{m^2t'}{3H}}\int_0^{t_2}dt''e^{\frac{m^2t''}{3H}}\left(\frac{2}{3}\left(\left\langle\tilde{\phi}(t')\tilde{\phi}(t'')\right\rangle\right)^3+\left\langle\tilde{\phi}^2(t')\right\rangle\left\langle\tilde{\phi}(t')\tilde{\phi}(t'')\right\rangle\left\langle\tilde{\phi}^2(t'')\right\rangle\right).$$

Here, time integration intervals for the first double integral in (2.27) are

$$t_2\le t''\le t'\le t_1\,,\quad t''\le t_2\le t'\le t_1\,,\quad \text{and}\quad t''\le t'\le t_2\le t_1\,; \tag{2.28}$$

for the second, $t''\le t'\le t_2\le t_1$, and for the third,

$$t''\le t'\le t_2\le t_1\,,\quad t''\le t_2\le t'\le t_1\,,\quad \text{and}\quad t'\le t''\le t_2\le t_1\,. \tag{2.29}$$

These integrals result in the following expression:

$$\left\langle\phi(t_1)\phi(t_2)\right\rangle_{\lambda^2} = \frac{81\,\lambda^2H^{12}}{2048\,\pi^6m^{10}}\Bigg(\left(30+\frac{12m^2}{H}|t_1-t_2|+\frac{2m^4}{3H^2}|t_1-t_2|^2\right)e^{-\frac{m^2}{3H}|t_1-t_2|} \tag{2.30}$$
$$+2e^{-\frac{m^2}{H}|t_1-t_2|}-5\,e^{-\frac{m^2}{H}(t_1+t_2)}+\left(36+\frac{2m^2}{H}\left(9|t_1-t_2|-14(t_1+t_2)\right)\right.$$

$$
\begin{aligned}
&- \frac{2m^4}{3H^2}\Big(|t_1 - t_2| - 2(t_1 + t_2)\Big)^2\Big)\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2)} + \frac{15}{2}\mathrm{e}^{-\frac{m^2}{3H}\left(3|t_1-t_2|+2(t_1+t_2)\right)} \\
&+ \left(48 + \frac{2m^2}{H}\Big(7|t_1 - t_2| + 2(t_1 + t_2)\Big)\right)\mathrm{e}^{-\frac{m^2}{3H}\left(2|t_1-t_2|+(t_1+t_2)\right)} - \frac{15}{2}\mathrm{e}^{-\frac{m^2}{3H}\left(2|t_1-t_2|+3(t_1+t_2)\right)} \\
&- \left(45 + \frac{2m^2}{H}\Big(|t_1 - t_2| + 8(t_1 + t_2)\Big)\right)\mathrm{e}^{-\frac{m^2}{3H}\left(|t_1-t_2|+2(t_1+t_2)\right)} - \frac{15}{2}\mathrm{e}^{\frac{m^2}{3H}\left(2|t_1-t_2|-3(t_1+t_2)\right)} \\
&- \left(\frac{117}{2} - \frac{2m^2}{H}\Big(7|t_1 - t_2| - 8(t_1 + t_2)\Big)\right)\mathrm{e}^{\frac{m^2}{3H}\left(|t_1-t_2|-2(t_1+t_2)\right)}\Bigg).
\end{aligned}
$$

Finally, at three-loop order, we obtain the following expression:

$$
\begin{aligned}
\langle\phi(t_1)\phi(t_2)\rangle_{\lambda^3} = &-\frac{\lambda^3}{H^3}\mathrm{e}^{-\frac{m^2 t_2}{3H}}\int_0^{t_2} dt' \int_0^{t'} dt'' \int_0^{t''} dt''' \mathrm{e}^{\frac{m^2 t'''}{3H}}\Bigg(\langle\tilde\phi(t_1)\tilde\phi(t''')\rangle\langle\tilde\phi^2(t')\rangle \\
&\times \langle\tilde\phi^2(t'')\rangle\langle\tilde\phi^2(t''')\rangle + 4\,\langle\tilde\phi(t_1)\tilde\phi(t')\rangle\langle\tilde\phi(t')\tilde\phi(t'')\rangle\langle\tilde\phi(t'')\tilde\phi(t''')\rangle\langle\tilde\phi^2(t''')\rangle \\
&+ 2\,\langle\tilde\phi(t_1)\tilde\phi(t')\rangle\langle\tilde\phi(t')\tilde\phi(t''')\rangle\langle\tilde\phi^2(t'')\rangle\langle\tilde\phi^2(t''')\rangle \\
&+ 4\,\langle\tilde\phi(t_1)\tilde\phi(t')\rangle\langle\tilde\phi(t')\tilde\phi(t''')\rangle\Big(\langle\tilde\phi(t'')\tilde\phi(t''')\rangle\Big)^2 \\
&+ 2\,\langle\tilde\phi(t_1)\tilde\phi(t'')\rangle\langle\tilde\phi^2(t')\rangle\langle\tilde\phi(t'')\tilde\phi(t''')\rangle\langle\tilde\phi^2(t''')\rangle \\
&+ 4\,\langle\tilde\phi(t_1)\tilde\phi(t'')\rangle\langle\tilde\phi(t')\tilde\phi(t'')\rangle\langle\tilde\phi(t')\tilde\phi(t''')\rangle\langle\tilde\phi^2(t''')\rangle \\
&+ 4\,\langle\tilde\phi(t_1)\tilde\phi(t'')\rangle\Big(\langle\tilde\phi(t')\tilde\phi(t''')\rangle\Big)^2\langle\tilde\phi(t'')\tilde\phi(t''')\rangle \\
&+ 2\,\langle\tilde\phi(t_1)\tilde\phi(t''')\rangle\Big(\langle\tilde\phi(t'')\tilde\phi(t''')\rangle\Big)^2\langle\tilde\phi^2(t')\rangle \\
&+ 2\,\langle\tilde\phi(t_1)\tilde\phi(t''')\rangle\Big(\langle\tilde\phi(t')\tilde\phi(t'')\rangle\Big)^2\langle\tilde\phi^2(t''')\rangle \\
&+ 8\,\langle\tilde\phi(t_1)\tilde\phi(t''')\rangle\langle\tilde\phi(t')\tilde\phi(t'')\rangle\langle\tilde\phi(t')\tilde\phi(t''')\rangle\langle\tilde\phi(t'')\tilde\phi(t''')\rangle \\
&+ 2\,\langle\tilde\phi(t_1)\tilde\phi(t''')\rangle\Big(\langle\tilde\phi(t')\tilde\phi(t''')\rangle\Big)^2\langle\tilde\phi^2(t'')\rangle\Bigg) \\
&- \frac{\lambda^3}{H^3}\mathrm{e}^{-\frac{m^2 t_2}{3H}}\int_0^{t_2} dt'\mathrm{e}^{\frac{-m^2 t'}{3H}} \int_0^{t'} dt''\mathrm{e}^{\frac{m^2 t''}{3H}} \int_0^{t'} dt'''\mathrm{e}^{\frac{m^2 t'''}{3H}}\Bigg(\langle\tilde\phi(t_1)\tilde\phi(t')\rangle\langle\tilde\phi^2(t'')\rangle\times \\
&\times\langle\tilde\phi(t'')\tilde\phi(t''')\rangle\langle\tilde\phi^2(t''')\rangle + \frac{2}{3}\langle\tilde\phi(t_1)\tilde\phi(t')\rangle\Big(\langle\tilde\phi(t'')\tilde\phi(t''')\rangle\Big)^3 \\
&+ 2\langle\tilde\phi(t_1)\tilde\phi(t'')\rangle\langle\tilde\phi(t')\tilde\phi(t'')\rangle\langle\tilde\phi(t'')\tilde\phi(t''')\rangle\langle\tilde\phi^2(t''')\rangle \\
&+ \langle\tilde\phi(t_1)\tilde\phi(t'')\rangle\langle\tilde\phi(t')\tilde\phi(t''')\rangle\langle\tilde\phi^2(t'')\rangle\langle\tilde\phi^2(t''')\rangle \\
&+ 2\langle\tilde\phi(t_1)\tilde\phi(t'')\rangle\langle\tilde\phi(t')\tilde\phi(t''')\rangle\Big(\langle\tilde\phi(t'')\tilde\phi(t''')\rangle\Big)^2 \\
&+ \langle\tilde\phi(t_1)\tilde\phi(t''')\rangle\langle\tilde\phi(t')\tilde\phi(t'')\rangle\langle\tilde\phi^2(t'')\rangle\langle\tilde\phi^2(t''')\rangle
\end{aligned}
\tag{2.31}
$$

$$
\begin{aligned}
&+ 2\langle\tilde{\phi}(t_1)\tilde{\phi}(t''')\rangle\langle\tilde{\phi}(t')\tilde{\phi}(t'')\rangle\Big(\langle\tilde{\phi}(t'')\tilde{\phi}(t''')\rangle\Big)^2 \\
&+ 2\langle\tilde{\phi}(t_1)\tilde{\phi}(t''')\rangle\langle\tilde{\phi}(t')\tilde{\phi}(t''')\rangle\langle\tilde{\phi}^2(t'')\rangle\langle\tilde{\phi}(t'')\tilde{\phi}(t''')\rangle\Big) \\
&- \frac{\lambda^3}{H^3}\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2)}\int_0^{t_1} dt'\,\mathrm{e}^{\frac{m^2 t'}{3H}}\int_0^{t_2} dt''\int_0^{t''} dt'''\,\mathrm{e}^{\frac{m^2 t'''}{3H}}\Big(2\langle\tilde{\phi}^2(t')\rangle\langle\tilde{\phi}(t')\tilde{\phi}(t'')\rangle \\
&\times\langle\tilde{\phi}(t'')\tilde{\phi}(t''')\rangle\langle\tilde{\phi}^2(t''')\rangle + 2\Big(\langle\tilde{\phi}(t')\tilde{\phi}(t'')\rangle\Big)^2\langle\tilde{\phi}(t')\tilde{\phi}(t''')\rangle\langle\tilde{\phi}^2(t''')\rangle \\
&+ 4\langle\tilde{\phi}(t')\tilde{\phi}(t'')\rangle\Big(\langle\tilde{\phi}(t')\tilde{\phi}(t''')\rangle\Big)^2\langle\tilde{\phi}(t'')\tilde{\phi}(t''')\rangle + \langle\tilde{\phi}^2(t')\rangle\langle\tilde{\phi}(t')\tilde{\phi}(t''')\rangle\langle\tilde{\phi}^2(t'')\rangle\langle\tilde{\phi}^2(t''')\rangle \\
&+ 2\langle\tilde{\phi}^2(t')\rangle\langle\tilde{\phi}(t')\tilde{\phi}(t''')\rangle\Big(\langle\tilde{\phi}(t'')\tilde{\phi}(t''')\rangle\Big)^2 + \frac{2}{3}\Big(\langle\tilde{\phi}(t')\tilde{\phi}(t''')\rangle\Big)^3\langle\tilde{\phi}^2(t'')\rangle\Big) \\
&+ \big(t_1 \leftrightarrow t_2\big).
\end{aligned}
$$

Here, the time intervals are somewhat more involved:

$$
\begin{aligned}
Ia:&\quad t''' \le t'' \le t' \le t_2 \le t_1; & Ib:&\quad t_2 \le t''' \le t'' \le t' \le t_1; \\
&& & \quad t''' \le t_2 \le t'' \le t' \le t_1; \\
&& & \quad t''' \le t'' \le t_2 \le t' \le t_1; \\
&& & \quad t''' \le t'' \le t' \le t_2 \le t_1; \\
IIa:&\quad t''' \le t'' \le t' \le t_2 \le t_1; & IIb:&\quad t''' \le t'' \le t' \le t_2 \le t_1; \\
&\quad t'' \le t''' \le t' \le t_2 \le t_1; & & \quad t''' \le t'' \le t_2 \le t' \le t_1; \\
&& & \quad t''' \le t_2 \le t'' \le t' \le t_1; \\
&& & \quad t_2 \le t''' \le t'' \le t' \le t_1; \\
&& & \quad t'' \le t''' \le t' \le t_2 \le t_1; \\
&& & \quad t'' \le t''' \le t_2 \le t' \le t_1; \\
&& & \quad t'' \le t_2 \le t''' \le t' \le t_1; \\
&& & \quad t_2 \le t'' \le t''' \le t' \le t_1; \\
IIIa:&\quad t''' \le t'' \le t_2 \le t' \le t_1; & IIIb:&\quad t''' \le t'' \le t' \le t_2 \le t_1; \\
&\quad t''' \le t'' \le t' \le t_2 \le t_1; & & \quad t''' \le t' \le t'' \le t_2 \le t_1; \\
&\quad t''' \le t' \le t'' \le t_2 \le t_1; & & \quad t' \le t''' \le t'' \le t_2 \le t_1; \\
&\quad t' \le t''' \le t'' \le t_2 \le t_1; & & \quad t''' \le t' \le t_2 \le t'' \le t_1; \\
&& & \quad t' \le t''' \le t_2 \le t'' \le t_1; \\
&& & \quad t' \le t_2 \le t''' \le t'' \le t_1.
\end{aligned}
\tag{2.32}
$$

The notation I, II, and III refer to the first, second, and third triple integrals, respectively. The index a corresponds to (2.31) and b to the case involving the permutation $(t_1 \leftrightarrow t_2)$. After performing the integration of (2.31) over the time intervals given in (2.32), one has

$$
\begin{aligned}
\big\langle\phi(t_1)\phi(t_2)\big\rangle_{\lambda^3} = &-\frac{729\lambda^3H^{16}}{4096\pi^8 m^{14}}\Bigg(\left(26+\frac{85m^2}{6H}|t_1-t_2|+\frac{5m^4}{6H^2}|t_1-t_2|^2+\frac{m^6}{54H^3}|t_1-t_2|^3\right)e^{-\frac{m^2}{3H}|t_1-t_2|} \\
&+\left(7+\frac{3m^2}{2H}|t_1-t_2|\right)e^{-\frac{m^2}{H}|t_1-t_2|}+\frac{5}{4}e^{-\frac{m^2}{3H}\left(4|t_1-t_2|+(t_1+t_2)\right)}+\Bigg(\frac{223}{2}+\frac{11m^2}{H}\Big(4|t_1-t_2|+(t_1+t_2)\Big) \\
&+\frac{m^4}{12H^2}\Big(7|t_1-t_2|+2(t_1+t_2)\Big)^2\Bigg)e^{-\frac{m^2}{3H}\left(2|t_1-t_2|+(t_1+t_2)\right)}+\Bigg(\frac{2135}{12}+\frac{m^2}{3H}\Big(96|t_1-t_2|-119(t_1+t_2)\Big) \\
&-\frac{m^4}{12H^2}\Big(17|t_1-t_2|-26(t_1+t_2)\Big)\Big(|t_1-t_2|-2(t_1+t_2)\Big)+\frac{m^6}{54H^3}\Big(|t_1-t_2|-2(t_1+t_2)\Big)^3\Bigg)\times \\
&\times e^{-\frac{m^2}{3H}(t_1+t_2)}+\left(\frac{345}{8}+\frac{5m^2}{8H}\Big(17|t_1-t_2|+8(t_1+t_2)\Big)\right)e^{-\frac{m^2}{3H}\left(3|t_1-t_2|+2(t_1+t_2)\right)} \\
&-\left(\frac{339}{4}+\frac{m^2}{4H}\Big(3|t_1-t_2|+208(t_1+t_2)\Big)+\frac{m^4}{12H^2}\Big(|t_1-t_2|+8(t_1+t_2)\Big)^2\right)e^{-\frac{m^2}{3H}\left(|t_1-t_2|+2(t_1+t_2)\right)} \\
&-\left(\frac{1203}{8}-\frac{m^2}{8H}\Big(441|t_1-t_2|-488(t_1+t_2)\Big)+\frac{m^4}{12H^2}\Big(7|t_1-t_2|-8(t_1+t_2)\Big)^2\right)e^{\frac{m^2}{3H}\left(|t_1-t_2|-2(t_1+t_2)\right)} \\
&-\left(\frac{505}{16}-\frac{15m^2}{4H}\Big(|t_1-t_2|-2(t_1+t_2)\Big)\right)e^{-\frac{m^2}{H}(t_1+t_2)}+\frac{175}{48}e^{-\frac{m^2}{3H}\left(4|t_1-t_2|+3(t_1+t_2)\right)} \\
&-\left(\frac{675}{16}+\frac{5m^2}{8H}\Big(7|t_1-t_2|+18(t_1+t_2)\Big)\right)e^{-\frac{m^2}{3H}\left(2|t_1-t_2|+3(t_1+t_2)\right)}-\frac{35}{16}e^{\frac{m^2}{3H}\left(|t_1-t_2|-4(t_1+t_2)\right)} \\
&-\left(\frac{2395}{48}-\frac{5m^2}{8H}\Big(17|t_1-t_2|-18(t_1+t_2)\Big)\right)e^{\frac{m^2}{3H}\left(2|t_1-t_2|-3(t_1+t_2)\right)}-\frac{35}{16}e^{-\frac{m^2}{3H}\left(|t_1-t_2|+4(t_1+t_2)\right)} \\
&-\frac{175}{48}e^{-\frac{m^2}{3H}\left(3|t_1-t_2|+4(t_1+t_2)\right)}-\frac{175}{48}e^{\frac{m^2}{3H}\left(3|t_1-t_2|-4(t_1+t_2)\right)}\Bigg). \qquad (2.33)
\end{aligned}
$$

At late times, using the perturbative results (2.25)–(2.33) for the series (2.23), we obtain the corresponding expression:

$$
\begin{aligned}
\big\langle\phi(t_1)\phi(t_2)\big\rangle \xrightarrow[\text{times}]{\text{late}} \frac{3H^4}{8\pi^2m^2}\Bigg(& e^{-\frac{m^2}{3H}|t_1-t_2|}-\frac{9\lambda H^4}{8\pi^2m^4}\left(1+\frac{m^2}{3H}|t_1-t_2|\right)e^{-\frac{m^2}{3H}|t_1-t_2|} \\
&+\frac{27\lambda^2H^8}{128\,\pi^4m^8}\left(15+\frac{6m^2}{H}|t_1-t_2|+\frac{m^4}{3H^2}|t_1-t_2|^2+e^{-\frac{2m^2}{3H}|t_1-t_2|}\right)e^{-\frac{m^2}{3H}|t_1-t_2|} \\
&-\frac{243\lambda^3H^{12}}{512\,\pi^6m^{12}}\Bigg(26+\frac{85m^2}{6H}|t_1-t_2|+\frac{5m^4}{6H^2}|t_1-t_2|^2+\frac{m^6}{54H^3}|t_1-t_2|^3 \\
&\qquad+\left(7+\frac{3m^2}{2H}|t_1-t_2|\right)e^{-\frac{2m^2}{3H}|t_1-t_2|}\Bigg)e^{-\frac{m^2}{3H}|t_1-t_2|}\Bigg)+O(\lambda^4); \qquad (2.34)
\end{aligned}
$$

while the massless limit of the series (2.23) with $t_1 \geq t_2$ is the following:

$$\left\langle \phi(t_1)\phi(t_2) \right\rangle \xrightarrow{m\to 0} \frac{H^3 t_2}{4\pi^2} - \frac{\lambda H^5}{96\,\pi^4}\left(3\,t_1^2\,t_2 + t_2^3\right) + \frac{\lambda^2 H^7}{1536\,\pi^6}\left(11\,t_1^4 t_2 + 2\,t_1^2 t_2^3 + \frac{31}{5}\,t_2^5\right) \tag{2.35}$$
$$- \frac{\lambda^3 H^9}{184320\pi^8}\left(471 t_1^6 t_2 + 55 t_1^4 t_2^3 + 93 t_1^2 t_2^5 + 160 t_1 t_2^6 + \frac{1331}{7}\,t_2^7\right) + O\left(\lambda^4\right) := \left\langle \phi_{m=0}(t_1)\phi_{m=0}(t_2) \right\rangle.$$

Our late-time series (2.34) is in agreement up to $O\left(\lambda^3\right)$ with those obtained from quantum field theory's diagrammatic techniques [117, 118]; see the next Chapter 3.

At equal times, i.e., $t_1 = t_2 := t$, expressions (2.25)–(2.33) for series (2.23) lead to

$$\left\langle \phi^2(t) \right\rangle = \frac{3H^4}{8\pi^2 m^2}\left(1 - e^{-\frac{2m^2 t}{3H}}\right) - \frac{27\lambda H^8}{64\,\pi^4 m^6}\left(1 - \frac{4m^2 t}{3H} e^{-\frac{2m^2 t}{3H}} - e^{-\frac{4m^2 t}{3H}}\right) \tag{2.36}$$
$$+ \frac{81\lambda^2 H^{12}}{64\,\pi^6 m^{10}}\left(1 + \left(\frac{21}{8} - \frac{3m^2 t}{2H} - \frac{m^4 t^2}{3H^2}\right) e^{-\frac{2m^2 t}{3H}} - \left(3 + \frac{2m^2 t}{H}\right) e^{-\frac{4m^2 t}{3H}} - \frac{5}{8} e^{-\frac{2m^2 t}{H}}\right)$$
$$- \frac{2187\lambda^3 H^{16}}{4096\,\pi^8 m^{14}}\left(11 + \left(\frac{872}{9} - \frac{172 m^2 t}{9H} - \frac{16 m^4 t^2}{3H^2} - \frac{32 m^6 t^3}{81 H^3}\right) e^{-\frac{2m^2 t}{3H}}\right.$$
$$\left. - \left(64 + \frac{72 m^2 t}{H} + \frac{128 m^4 t^2}{9H^2}\right) e^{-\frac{4m^2 t}{3H}} - \left(40 + \frac{20 m^2 t}{H}\right) e^{-\frac{2m^2 t}{H}} - \frac{35}{9} e^{-\frac{8m^2 t}{3H}}\right) + O(\lambda^4),$$

and at the late-time limit this series (as well as all massive series from above) is

$$\left\langle \phi^2(t) \right\rangle \xrightarrow{t\to\infty} \frac{3H^4}{8\pi^2 m^2} - \frac{27\lambda H^8}{64\,\pi^4 m^6} + \frac{81\lambda^2 H^{12}}{64\,\pi^6 m^{10}} - \frac{24057\lambda^3 H^{16}}{4096\,\pi^8 m^{14}} + O\left(\lambda^4\right). \tag{2.37}$$

The obtained equilibrium result matches those from [81, 82, 117, 118] up to $O(\lambda^3)$, and agrees with the Starobinsky-Yokoyama stochastic approach results in the perturbative regime. We will discuss this in detail in Chapters 3, 4, and 5.

While in the smooth massless limit, expression (2.36) leads to

$$\left\langle \phi^2(t) \right\rangle \xrightarrow{m\to 0} \frac{H^3 t}{4\,\pi^2} - \frac{\lambda H^5 t^3}{24\,\pi^4} + \frac{\lambda^2 H^7 t^5}{80\,\pi^6} - \frac{53\,\lambda^3 H^9 t^7}{10080\,\pi^8} + O\left(\lambda^4\right) := \left\langle \phi_{m=0}^2(t) \right\rangle. \tag{2.38}$$

This expression reproduces the well-known secular growth behavior. To a given order, it agrees with results of several studies [45, 83, 84, 120, 121, 122]. It is important to note that this perturbative series (2.36) not only reduces to (2.38) in the smooth massless limit, but also approximates this behavior in the regime $t \ll H/m^2$, as observed at the tree level in [54, 143]. Specifically, it recovers the Wiener process (2.19), discussed in [37, 38, 40].

The resulting series (2.36) is equivalent to the resummation of the leading infrared logarithms in the perturbative expansion in m^2/H^2, performed at each order of the self-interaction coupling constant λ. A nice way to see it is to expand the obtained expression in the m^2/H^2 regime:

$$\begin{aligned}\left\langle\phi^2(t)\right\rangle = \frac{H^3 t}{4\pi^2}\Bigg(&\left(1-\frac{m^2}{3H^2}(Ht)+\frac{2m^4}{27H^4}(Ht)^2-\frac{m^6}{81H^6}(Ht)^3+\dots\right)\\ &-\frac{\lambda}{6\pi^2}(Ht)^2\left(1-\frac{2m^2}{3H^2}(Ht)+\frac{11m^4}{45H^4}(Ht)^2-\frac{26m^6}{405H^6}(Ht)^3+\dots\right)\\ &+\frac{\lambda^2}{20\pi^4}(Ht)^4\left(1-\frac{26m^2}{27H^2}(Ht)+\frac{40m^4}{81H^4}(Ht)^2-\frac{101m^6}{567H^6}(Ht)^3+\dots\right)\\ &-\frac{53\,\lambda^3}{2520\,\pi^6}(Ht)^6\left(1-\frac{200m^2}{159H^2}(Ht)+\frac{3562m^4}{4293H^2}(Ht)^2-\frac{24514m^6}{64395H^6}(Ht)^3+\dots\right)\Bigg)+O(\lambda^4).\end{aligned} \tag{2.39}$$

One identifies the leading infrared logarithm structure in cosmic time t, $\ln\left(a(t)\right)=Ht$, and establishes in the $m^2=0$ case the agreement with the massless series (2.38) and with the results of [45, 83, 84, 120, 121, 122]. Within our framework, infrared divergences in cosmic time t of the massless scalar field (2.38) are fully resummed in the massive series (2.36).

2.3 Four-point correlation function

In the case of the four-point correlation function, where the spacetime points have different times but coinciding spatial arguments, we compute the perturbative series in the self-interaction coupling constant λ in the same manner. Suppose the time ordering is:

$$t_1 \geq t_2 \geq t_3 \geq t_4. \tag{2.40}$$

Then, one has the series for the four-point correlation function

$$\begin{aligned}\left\langle\phi(t_1)\phi(t_2)\phi(t_3)\phi(t_4)\right\rangle &= \left\langle\phi(t_1)\phi(t_2)\phi(t_3)\phi(t_4)\right\rangle_0+\left\langle\phi(t_1)\phi(t_2)\phi(t_3)\phi(t_4)\right\rangle_\lambda\\ &+\left\langle\phi(t_1)\phi(t_2)\phi(t_3)\phi(t_4)\right\rangle_{\lambda^2}+O(\lambda^3).\end{aligned} \tag{2.41}$$

At the λ^0 order, we already have the answer:

$$\begin{aligned}\left\langle\phi(t_1)\phi(t_2)\phi(t_3)\phi(t_4)\right\rangle_0 &= \left\langle\tilde\phi(t_1)\tilde\phi(t_2)\right\rangle\left\langle\tilde\phi(t_3)\tilde\phi(t_4)\right\rangle\\ &+\left\langle\tilde\phi(t_1)\tilde\phi(t_3)\right\rangle\left\langle\tilde\phi(t_2)\tilde\phi(t_4)\right\rangle+\left\langle\tilde\phi(t_1)\tilde\phi(t_4)\right\rangle\left\langle\tilde\phi(t_2)\tilde\phi(t_3)\right\rangle.\end{aligned} \tag{2.42}$$

This expression corresponds to the product of two-point correlation functions from (2.22) arranged in an appropriate manner and results in

$$\begin{aligned}\left\langle\phi(t_1)\phi(t_2)\phi(t_3)\phi(t_4)\right\rangle_0 = \frac{9H^8}{64\,\pi^4 m^4}\,\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2+t_3+t_4)}\Bigg(&\mathrm{e}^{\frac{2m^2}{3H}(t_2+t_4)}\\ &+2\,\mathrm{e}^{\frac{2m^2}{3H}(t_3+t_4)}-\mathrm{e}^{\frac{2m^2 t_2}{3H}}-2\,\mathrm{e}^{\frac{2m^2 t_3}{3H}}-3\,\mathrm{e}^{\frac{2m^2 t_4}{3H}}+3\Bigg)\xrightarrow{m\to 0}\end{aligned} \tag{2.43}$$

$$\xrightarrow{m\to 0} \frac{H^6}{16\,\pi^4}\Big(t_2 t_4 + 2\,t_3 t_4\Big) := \big\langle \phi_{m=0}(t_1)\phi_{m=0}(t_2)\phi_{m=0}(t_3)\phi_{m=0}(t_4)\big\rangle_0. \tag{2.44}$$

At the λ–linear order, we have a partial result, as the correlation function splits into connected and disconnected diagram contributions. The complete expression, including the appropriate permutations, is:

$$\begin{aligned}\big\langle\phi(t_1)\phi(t_2)\phi(t_3)\phi(t_4)\big\rangle_\lambda &= -\frac{\lambda}{H}\mathrm{e}^{-\frac{m^2 t_1}{3H}}\int_0^{t_1} dt'\,\mathrm{e}^{\frac{m^2 t'}{3H}}\Big(\big\langle\tilde\phi^2(t')\big\rangle\big\langle\tilde\phi(t')\tilde\phi(t_2)\big\rangle\big\langle\tilde\phi(t_3)\tilde\phi(t_4)\big\rangle\\
&+\big\langle\tilde\phi^2(t')\big\rangle\big\langle\tilde\phi(t')\tilde\phi(t_3)\big\rangle\big\langle\tilde\phi(t_2)\tilde\phi(t_4)\big\rangle+\big\langle\tilde\phi^2(t')\big\rangle\big\langle\tilde\phi(t')\tilde\phi(t_4)\big\rangle\big\langle\tilde\phi(t_2)\tilde\phi(t_3)\big\rangle\\
&+\underbrace{2\big\langle\tilde\phi(t')\tilde\phi(t_2)\big\rangle\big\langle\tilde\phi(t')\tilde\phi(t_3)\big\rangle\big\langle\tilde\phi(t')\tilde\phi(t_4)\big\rangle}_{\text{connected}}\Big) + (t_1\leftrightarrow t_2)+(t_1\leftrightarrow t_3)+(t_1\leftrightarrow t_4),\end{aligned} \tag{2.45}$$

where the time intervals for the corresponding permutations above, with the chosen time ordering given in (2.40), are

$$\begin{aligned}
&a. && t'\le t_4\le t_3\le t_2\le t_1; && b. && t'\le t_4\le t_3\le t_2\le t_1;\\
& && t_4\le t'\le t_3\le t_2\le t_1; && && t_4\le t'\le t_3\le t_2\le t_1;\\
& && t_4\le t_3\le t'\le t_2\le t_1; && && t_4\le t_3\le t'\le t_2\le t_1;\\
& && t_4\le t_3\le t_2\le t'\le t_1; && && \\
&c. && t'\le t_4\le t_3\le t_2\le t_1; && d. && t'\le t_4\le t_3\le t_2\le t_1.\\
& && t_4\le t'\le t_3\le t_2\le t_1; && &&
\end{aligned} \tag{2.46}$$

One can directly calculate the complete correlation function via (2.45) with (2.46), as we have done. However, the contribution from the disconnected diagrams can be obtained from the previously computed results for the two-point correlation function at different λ orders; see (2.51) below. According to (2.45) with (2.46), the contribution from the connected diagrams is

$$\begin{aligned}\big\langle\phi(t_1)\phi(t_2)\phi(t_3)\phi(t_4)\big\rangle_\lambda^{\text{con}} &= -\frac{81\,\lambda H^{12}}{512\,\pi^6 m^8}\,\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2+t_3+t_4)}\Bigg(\Big(4+\frac{4m^2}{3H}(t_2-t_3)\Big)\,\mathrm{e}^{\frac{2m^2}{3H}(t_3+t_4)}\\
&-\mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_2-t_3-t_4)}-\mathrm{e}^{\frac{4m^2 t_4}{3H}}+\mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_2-t_3)}+\mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_2-t_4)}+\mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_3-t_4)}\\
&+\mathrm{e}^{-\frac{2m^2}{3H}(t_2-t_3-t_4)}-\Big(4+\frac{4m^2}{3H}(t_2-t_3)\Big)\,\mathrm{e}^{\frac{2m^2 t_3}{3H}}-\Big(12+\frac{4m^2}{3H}(t_2+2t_3-3t_4)\Big)\,\mathrm{e}^{\frac{2m^2 t_4}{3H}}\\
&+12+\frac{4m^2}{3H}(t_2+2t_3+3t_4)-\mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_2)}-\mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_3)}-\mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_4)}-\mathrm{e}^{-\frac{2m^2}{3H}(t_2-t_3)}\\
&-\mathrm{e}^{-\frac{2m^2}{3H}(t_2-t_4)}-\mathrm{e}^{-\frac{2m^2}{3H}(t_3-t_4)}+\mathrm{e}^{-\frac{2m^2 t_1}{3H}}+\mathrm{e}^{-\frac{2m^2 t_2}{3H}}+\mathrm{e}^{-\frac{2m^2 t_3}{3H}}+\mathrm{e}^{-\frac{2m^2 t_4}{3H}}\Bigg)\xrightarrow{m\to 0}\end{aligned} \tag{2.47}$$

$$\xrightarrow{m\to 0} -\frac{\lambda H^8}{32\pi^6} t_1 t_2 t_3 t_4 := \left\langle \phi_{m=0}(t_1)\phi_{m=0}(t_2)\phi_{m=0}(t_3)\phi_{m=0}(t_4)\right\rangle_\lambda^{\text{con}}. \tag{2.48}$$

At the coinciding times, one has

$$\left\langle \phi^4(t)\right\rangle_\lambda^{\text{con}} = -\frac{81\,\lambda H^{12}}{256\,\pi^6 m^8}\left(1-6\mathrm{e}^{-\frac{2m^2t}{3H}}+\left(3+\frac{4m^2t}{H}\right)\mathrm{e}^{-\frac{4m^2t}{3H}}+2\mathrm{e}^{-\frac{2m^2t}{H}}\right) \xrightarrow{m\to 0} \tag{2.49}$$

$$\xrightarrow{m\to 0} -\frac{\lambda H^8 t^4}{32\,\pi^6} := \left\langle \phi_{m=0}^4(t)\right\rangle_\lambda^{\text{con}}. \tag{2.50}$$

The obtained massless limit (2.50) agrees with the corresponding results from [122, 144]. For the disconnected contribution at order λ, the result is the sum of the products of the zeroth and the linear λ orders of the already known two-point correlation functions

$$\begin{aligned}\left\langle \phi(t_1)\phi(t_2)\phi(t_3)\phi(t_4)\right\rangle_\lambda^{\text{discon}} &= \left\langle\tilde\phi(t_1)\tilde\phi(t_2)\right\rangle\left\langle\phi(t_3)\phi(t_4)\right\rangle_\lambda + \left\langle\tilde\phi(t_1)\tilde\phi(t_3)\right\rangle\left\langle\phi(t_2)\phi(t_4)\right\rangle_\lambda \\ &+ \left\langle\tilde\phi(t_1)\tilde\phi(t_4)\right\rangle\left\langle\phi(t_2)\phi(t_3)\right\rangle_\lambda + \left\langle\tilde\phi(t_2)\tilde\phi(t_3)\right\rangle\left\langle\phi(t_1)\phi(t_4)\right\rangle_\lambda \\ &+ \left\langle\tilde\phi(t_2)\tilde\phi(t_4)\right\rangle\left\langle\phi(t_1)\phi(t_3)\right\rangle_\lambda + \left\langle\tilde\phi(t_3)\tilde\phi(t_4)\right\rangle\left\langle\phi(t_1)\phi(t_2)\right\rangle_\lambda,\end{aligned} \tag{2.51}$$

or, once again, one computes (2.45) with (2.46). The final result for the disconnected contribution is

$$\begin{aligned}\left\langle \phi(t_1)\phi(t_2)\phi(t_3)\phi(t_4)\right\rangle_\lambda^{\text{discon}} &= -\frac{81\,\lambda H^{12}}{1024\,\pi^6 m^8}\left(\left(4+\frac{2m^2}{3H}(t_1-t_2+t_3-t_4)\right)\mathrm{e}^{\frac{2m^2}{3H}(t_2+t_4)}\right. \\ &+\left(8+\frac{4m^2}{3H}(t_1+t_2-t_3-t_4)\right)\mathrm{e}^{\frac{2m^2}{3H}(t_3+t_4)}+\mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_2-t_4)}+2\mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_3-t_4)} \\ &+2\,\mathrm{e}^{-\frac{2m^2}{3H}(t_2-t_3-t_4)}+\mathrm{e}^{\frac{2m^2}{3H}(t_2-t_3+t_4)}-\left(3+\frac{2m^2}{3H}(t_1-t_2+t_3+3t_4)\right)\mathrm{e}^{\frac{2m^2t_2}{3H}} \\ &-\left(6+\frac{4m^2}{3H}(t_1+t_2-t_3+3t_4)\right)\mathrm{e}^{\frac{2m^2t_3}{3H}}-\left(9+\frac{2m^2}{3H}(3t_1+5t_2+7t_3-3t_4)\right)\mathrm{e}^{\frac{2m^2t_4}{3H}} \\ &-\mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_2)}-2\,\mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_3)}-3\,\mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_4)}-2\,\mathrm{e}^{-\frac{2m^2}{3H}(t_2-t_3)}-3\,\mathrm{e}^{-\frac{2m^2}{3H}(t_2-t_4)} \\ &-3\,\mathrm{e}^{-\frac{2m^2}{3H}(t_3-t_4)}-\mathrm{e}^{\frac{2m^2}{3H}(t_2-t_3)}-\mathrm{e}^{\frac{2m^2}{3H}(t_2-t_4)}-2\,\mathrm{e}^{\frac{2m^2}{3H}(t_3-t_4)}+6+\frac{2m^2}{3H}(3t_1+5t_2+7t_3+9t_4) \\ &\left.+3\mathrm{e}^{-\frac{2m^2t_1}{3H}}+3\mathrm{e}^{-\frac{2m^2t_2}{3H}}+3\,\mathrm{e}^{-\frac{2m^2t_3}{3H}}+3\,\mathrm{e}^{-\frac{2m^2t_4}{3H}}\right)\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2+t_3+t_4)} \xrightarrow{m\to 0}\end{aligned} \tag{2.52}$$

$$\begin{aligned}\xrightarrow{m\to 0} -\frac{\lambda H^8}{384\,\pi^6}&\left(3\,t_1^2t_2t_4+6\,t_1^2t_3t_4+t_2t_4^3+2\,t_3^3t_4+t_2^3t_4+6\,t_2^2t_3t_4\right. \\ &\left.+3\,t_2t_3^2t_4+2\,t_3t_4^3\right) := \left\langle \phi_{m=0}(t_1)\phi_{m=0}(t_2)\phi_{m=0}(t_3)\phi_{m=0}(t_4)\right\rangle_\lambda^{\text{discon}};\end{aligned} \tag{2.53}$$

and, once again, at the coinciding times, one has

$$\left\langle\phi^4(t)\right\rangle_\lambda^{\text{discon}} = -\frac{243\,\lambda H^{12}}{256\,\pi^6 m^8}\left(1-\left(1+\frac{4m^2t}{3H}\right)\mathrm{e}^{-\frac{2m^2t}{3H}}-\left(1-\frac{4m^2t}{3H}\right)\mathrm{e}^{-\frac{4m^2t}{3H}}+\mathrm{e}^{-\frac{2m^2t}{H}}\right) \tag{2.54}$$

$$\xrightarrow{m\to 0} -\frac{\lambda H^8 t^4}{16\pi^6} := \left\langle\phi^4_{m=0}(t)\right\rangle_\lambda^{\text{discon}}. \tag{2.55}$$

We provide the final answer, representing the sum of these contributions (2.47) and (2.52):

$$\begin{aligned}\left\langle\phi(t_1)\phi(t_2)\phi(t_3)\phi(t_4)\right\rangle_\lambda &= -\frac{81\lambda H^{12}}{1024\,\pi^6 m^8}\Bigg(\left(4+\frac{2m^2}{3H}(t_1-t_2+t_3-t_4)\right)\mathrm{e}^{\frac{2m^2}{3H}(t_2+t_4)} \\ &+\left(16+\frac{4m^2}{3H}(t_1+3t_2-3t_3-t_4)\right)\mathrm{e}^{\frac{2m^2}{3H}(t_3+t_4)}-2\,\mathrm{e}^{\frac{4m^2t_4}{3H}}-2\,\mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_2-t_3-t_4)} \\ &-\left(3+\frac{2m^2}{3H}(t_1-t_2+t_3+3t_4)\right)\mathrm{e}^{\frac{2m^2t_2}{3H}}-\left(14+\frac{4m^2}{3H}(t_1+3t_2-3t_3+3t_4)\right)\mathrm{e}^{\frac{2m^2t_3}{3H}} \\ &+\mathrm{e}^{\frac{2m^2}{3H}(t_2-t_3+t_4)}-\left(33+\frac{2m^2}{H}(t_1+3t_2+5t_3-5t_4)\right)\mathrm{e}^{\frac{2m^2t_4}{3H}}+2\,\mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_2-t_3)} \\ &+3\,\mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_2-t_4)}+4\,\mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_3-t_4)}+4\,\mathrm{e}^{-\frac{2m^2}{3H}(t_2-t_3-t_4)}-\mathrm{e}^{\frac{2m^2}{3H}(t_2-t_3)} \\ &-\mathrm{e}^{\frac{2m^2}{3H}(t_2-t_4)}-2\,\mathrm{e}^{\frac{2m^2}{3H}(t_3-t_4)}+30+\frac{2m^2}{H}(t_1+3t_2+5t_3+7t_4)-3\,\mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_2)} \\ &-4\,\mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_3)}-5\,\mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_4)}-4\,\mathrm{e}^{-\frac{2m^2}{3H}(t_2-t_3)}-5\,\mathrm{e}^{-\frac{2m^2}{3H}(t_2-t_4)}-5\,\mathrm{e}^{-\frac{2m^2}{3H}(t_3-t_4)} \\ &+5\,\mathrm{e}^{-\frac{2m^2t_1}{3H}}+5\,\mathrm{e}^{-\frac{2m^2t_2}{3H}}+5\,\mathrm{e}^{-\frac{2m^2t_3}{3H}}+5\,\mathrm{e}^{-\frac{2m^2t_4}{3H}}\Bigg)\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2+t_3+t_4)} \xrightarrow{m\to 0}\end{aligned} \tag{2.56}$$

$$\begin{aligned}\xrightarrow{m\to 0} -\frac{\lambda H^8}{384\,\pi^6}&\Big(3\,t_1^2\,t_2t_4+6\,t_1^2\,t_3t_4+12\,t_1t_2t_3t_4+t_2^3\,t_4+6\,t_2^2\,t_3t_4+3\,t_2t_3^2\,t_4 \\ &+t_2t_4^3+2\,t_3^3\,t_4+2\,t_3t_4^3\Big) := \left\langle\phi_{m=0}(t_1)\phi_{m=0}(t_2)\phi_{m=0}(t_3)\phi_{m=0}(t_4)\right\rangle_\lambda.\end{aligned} \tag{2.57}$$

Finally, regarding the correction at order λ^2, we repeat the same reasoning. The complete expression for the correlation function at the λ^2 level with the necessary permutations is

$$\begin{aligned}\left\langle\phi(t_1)\phi(t_2)\phi(t_3)\phi(t_4)\right\rangle_{\lambda^2} &= \frac{2\lambda^2}{H^2}\mathrm{e}^{-\frac{m^2t_1}{3H}}\int_0^{t_1}dt'\int_0^{t'}dt''\mathrm{e}^{\frac{m^2t''}{3H}}\Big(\left(\left\langle\tilde\phi(t')\tilde\phi(t'')\right\rangle\right)^2 \\ &\times\left\langle\tilde\phi(t'')\tilde\phi(t_2)\right\rangle\left\langle\tilde\phi(t_3)\tilde\phi(t_4)\right\rangle+\left(\left\langle\tilde\phi(t')\tilde\phi(t'')\right\rangle\right)^2\left\langle\tilde\phi(t'')\tilde\phi(t_3)\right\rangle\left\langle\tilde\phi(t_2)\tilde\phi(t_4)\right\rangle \\ &+\left(\left\langle\tilde\phi(t')\tilde\phi(t'')\right\rangle\right)^2\left\langle\tilde\phi(t'')\tilde\phi(t_4)\right\rangle\left\langle\tilde\phi(t_2)\tilde\phi(t_3)\right\rangle \\ &+\left\langle\tilde\phi(t')\tilde\phi(t'')\right\rangle\left\langle\tilde\phi(t')\tilde\phi(t_2)\right\rangle\left\langle\tilde\phi^2(t'')\right\rangle\left\langle\tilde\phi(t_3)\tilde\phi(t_4)\right\rangle\end{aligned} \tag{2.58}$$

$$
\begin{aligned}
&+ \underbrace{2\left\langle\tilde{\phi}(t')\tilde{\phi}(t'')\right\rangle\left\langle\tilde{\phi}(t')\tilde{\phi}(t_2)\right\rangle\left\langle\tilde{\phi}(t'')\tilde{\phi}(t_3)\right\rangle\left\langle\tilde{\phi}(t'')\tilde{\phi}(t_4)\right\rangle}_{\text{connected}} \\
&+ \left\langle\tilde{\phi}(t')\tilde{\phi}(t'')\right\rangle\left\langle\tilde{\phi}(t')\tilde{\phi}(t_3)\right\rangle\left\langle\tilde{\phi}^2(t'')\right\rangle\left\langle\tilde{\phi}(t_2)\tilde{\phi}(t_4)\right\rangle \\
&+ \underbrace{2\left\langle\tilde{\phi}(t')\tilde{\phi}(t'')\right\rangle\left\langle\tilde{\phi}(t')\tilde{\phi}(t_3)\right\rangle\left\langle\tilde{\phi}(t'')\tilde{\phi}(t_2)\right\rangle\left\langle\tilde{\phi}(t'')\tilde{\phi}(t_4)\right\rangle}_{\text{connected}} \\
&+ \left\langle\tilde{\phi}(t')\tilde{\phi}(t'')\right\rangle\left\langle\tilde{\phi}(t')\tilde{\phi}(t_4)\right\rangle\left\langle\tilde{\phi}^2(t'')\right\rangle\left\langle\tilde{\phi}(t_2)\tilde{\phi}(t_3)\right\rangle \\
&+ \underbrace{2\left\langle\tilde{\phi}(t')\tilde{\phi}(t'')\right\rangle\left\langle\tilde{\phi}(t')\tilde{\phi}(t_4)\right\rangle\left\langle\tilde{\phi}(t'')\tilde{\phi}(t_2)\right\rangle\left\langle\tilde{\phi}(t'')\tilde{\phi}(t_3)\right\rangle}_{\text{connected}} \\
&+ \underbrace{\left\langle\tilde{\phi}(t')\tilde{\phi}(t_2)\right\rangle\left\langle\tilde{\phi}(t')\tilde{\phi}(t_3)\right\rangle\left\langle\tilde{\phi}^2(t'')\right\rangle\left\langle\tilde{\phi}(t'')\tilde{\phi}(t_4)\right\rangle}_{\text{connected}} \\
&+ \underbrace{\left\langle\tilde{\phi}(t')\tilde{\phi}(t_2)\right\rangle\left\langle\tilde{\phi}(t')\tilde{\phi}(t_4)\right\rangle\left\langle\tilde{\phi}^2(t'')\right\rangle\left\langle\tilde{\phi}(t'')\tilde{\phi}(t_3)\right\rangle}_{\text{connected}} \\
&+ \underbrace{\left\langle\tilde{\phi}(t')\tilde{\phi}(t_3)\right\rangle\left\langle\tilde{\phi}(t')\tilde{\phi}(t_4)\right\rangle\left\langle\tilde{\phi}^2(t'')\right\rangle\left\langle\tilde{\phi}(t'')\tilde{\phi}(t_2)\right\rangle}_{\text{connected}} \\
&+ \frac{1}{2}\left\langle\tilde{\phi}^2(t')\right\rangle\left\langle\tilde{\phi}^2(t'')\right\rangle\left\langle\tilde{\phi}(t'')\tilde{\phi}(t_2)\right\rangle\left\langle\tilde{\phi}(t_3)\tilde{\phi}(t_4)\right\rangle \\
&+ \frac{1}{2}\left\langle\tilde{\phi}^2(t')\right\rangle\left\langle\tilde{\phi}^2(t'')\right\rangle\left\langle\tilde{\phi}(t'')\tilde{\phi}(t_3)\right\rangle\left\langle\tilde{\phi}(t_2)\tilde{\phi}(t_4)\right\rangle \\
&+ \frac{1}{2}\left\langle\tilde{\phi}^2(t')\right\rangle\left\langle\tilde{\phi}^2(t'')\right\rangle\left\langle\tilde{\phi}(t'')\tilde{\phi}(t_4)\right\rangle\left\langle\tilde{\phi}(t_2)\tilde{\phi}(t_3)\right\rangle \\
&+ \underbrace{\left\langle\tilde{\phi}^2(t')\right\rangle\left\langle\tilde{\phi}(t'')\tilde{\phi}(t_2)\right\rangle\left\langle\tilde{\phi}(t'')\tilde{\phi}(t_3)\right\rangle\left\langle\tilde{\phi}(t'')\tilde{\phi}(t_4)\right\rangle}_{\text{connected}} \\
&\quad + \ (t_1 \leftrightarrow t_2) \ + \ (t_1 \leftrightarrow t_3) \ + \ (t_1 \leftrightarrow t_4)
\end{aligned}
$$

$$
\begin{aligned}
+ \frac{\lambda^2}{H^2} e^{-\frac{m^2}{3H}(t_1+t_2)} \int_0^{t_1} dt' \, e^{\frac{m^2 t'}{3H}} \int_0^{t_2} dt'' e^{\frac{m^2 t''}{3H}} \Bigg(&\left\langle\tilde{\phi}^2(t')\right\rangle\left\langle\tilde{\phi}(t')\tilde{\phi}(t'')\right\rangle\left\langle\tilde{\phi}^2(t'')\right\rangle\left\langle\tilde{\phi}(t_3)\tilde{\phi}(t_4)\right\rangle \\
&+ \frac{2}{3}\left(\left\langle\tilde{\phi}(t')\tilde{\phi}(t'')\right\rangle\right)^3\left\langle\tilde{\phi}(t_3)\tilde{\phi}(t_4)\right\rangle \\
&+ \underbrace{2\left\langle\tilde{\phi}(t')\tilde{\phi}(t_3)\right\rangle\left\langle\tilde{\phi}(t')\tilde{\phi}(t_4)\right\rangle\left\langle\tilde{\phi}(t')\tilde{\phi}(t'')\right\rangle\left\langle\tilde{\phi}^2(t'')\right\rangle}_{\text{connected}} \\
&+ \left\langle\tilde{\phi}^2(t')\right\rangle\left\langle\tilde{\phi}(t')\tilde{\phi}(t_3)\right\rangle\left\langle\tilde{\phi}(t'')\tilde{\phi}(t_4)\right\rangle\left\langle\tilde{\phi}^2(t'')\right\rangle \\
&+ \underbrace{2\left\langle\tilde{\phi}(t')\tilde{\phi}(t_3)\right\rangle\left(\left\langle\tilde{\phi}(t')\tilde{\phi}(t'')\right\rangle\right)^2\left\langle\tilde{\phi}(t'')\tilde{\phi}(t_4)\right\rangle}_{\text{connected}} \\
&+ \left\langle\tilde{\phi}^2(t')\right\rangle\left\langle\tilde{\phi}(t')\tilde{\phi}(t_4)\right\rangle\left\langle\tilde{\phi}^2(t'')\right\rangle\left\langle\tilde{\phi}(t'')\tilde{\phi}(t_3)\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+\underbrace{2\left\langle\tilde{\phi}(t')\tilde{\phi}(t_4)\right\rangle\Big(\left\langle\tilde{\phi}(t')\tilde{\phi}(t'')\right\rangle\Big)^2\left\langle\tilde{\phi}(t'')\tilde{\phi}(t_3)\right\rangle}_{\text{connected}}\\
&+\underbrace{2\left\langle\tilde{\phi}^2(t')\right\rangle\left\langle\tilde{\phi}(t')\tilde{\phi}(t'')\right\rangle\left\langle\tilde{\phi}(t'')\tilde{\phi}(t_3)\right\rangle\left\langle\tilde{\phi}(t'')\tilde{\phi}(t_4)\right\rangle}_{\text{connected}}\Bigg)\\
+&\ \big(t_2\leftrightarrow t_3\big)\ +\ \big(t_2\leftrightarrow t_4\big)+\ \big(t_1\leftrightarrow t_2\,;\,t_2\leftrightarrow t_3\,;\,t_3\leftrightarrow t_1\big)\\
+&\ \big(t_1\leftrightarrow t_2\,;\,t_2\leftrightarrow t_4\,;\,t_3\leftrightarrow t_1\big)\ +\ \big(t_1\leftrightarrow t_3\,;t_2\leftrightarrow t_4\big).
\end{aligned}
$$

The corresponding time intervals are:

$$
\begin{array}{llll}
Ia. & t''\le t'\le t_4\le t_3\le t_2\le t_1; & Ib. & t''\le t'\le t_4\le t_3\le t_2\le t_1;\\
 & t''\le t_4\le t'\le t_3\le t_2\le t_1; & & t''\le t_4\le t'\le t_3\le t_2\le t_1;\\
 & t''\le t_4\le t_3\le t'\le t_2\le t_1; & & t''\le t_4\le t_3\le t'\le t_2\le t_1;\\
 & t''\le t_4\le t_3\le t_2\le t'\le t_1; & & t_4\le t''\le t_3\le t'\le t_2\le t_1;\\
 & t_4\le t''\le t_3\le t_2\le t'\le t_1; & & t_4\le t_3\le t''\le t'\le t_2\le t_1;\\
 & t_4\le t_3\le t''\le t_2\le t'\le t_1; & & t_4\le t''\le t'\le t_3\le t_2\le t_1;\\
 & t_4\le t_3\le t_2\le t''\le t'\le t_1; & & \\
 & t_4\le t''\le t_3\le t'\le t_2\le t_1; & & \\
 & t_4\le t_3\le t''\le t'\le t_2\le t_1; & & \\
 & t_4\le t''\le t'\le t_3\le t_2\le t_1; & & \\
Ic. & t''\le t'\le t_4\le t_3\le t_2\le t_1; & Id. & t''\le t'\le t_4\le t_3\le t_2\le t_1;\\
 & t''\le t_4\le t'\le t_3\le t_2\le t_1; & & \\
 & t_4\le t''\le t'\le t_3\le t_2\le t_1; & & \\
IIa. & t''\le t'\le t_4\le t_3\le t_2\le t_1; & IIb. & t''\le t'\le t_4\le t_3\le t_2\le t_1;\\
 & t''\le t_4\le t'\le t_3\le t_2\le t_1; & & t''\le t_4\le t'\le t_3\le t_2\le t_1;\\
 & t''\le t_4\le t_3\le t'\le t_2\le t_1; & & t''\le t_4\le t_3\le t'\le t_2\le t_1;\\
 & t''\le t_4\le t_3\le t_2\le t'\le t_1; & & t''\le t_4\le t_3\le t_2\le t'\le t_1;\\
 & t_4\le t''\le t_3\le t_2\le t'\le t_1; & & t_4\le t''\le t_3\le t_2\le t'\le t_1;\\
 & t_4\le t''\le t_3\le t'\le t_2\le t_1; & & t_4\le t''\le t_3\le t'\le t_2\le t_1;\\
 & t_4\le t''\le t'\le t_3\le t_2\le t_1; & & t_4\le t''\le t'\le t_3\le t_2\le t_1;\\
 & t_4\le t_3\le t''\le t_2\le t'\le t_1; & & t_4\le t'\le t''\le t_3\le t_2\le t_1;\\
 & t_4\le t_3\le t''\le t'\le t_2\le t_1; & & t'\le t_4\le t''\le t_3\le t_2\le t_1;\\
 & t_4\le t_3\le t'\le t''\le t_2\le t_1; & & t'\le t''\le t_4\le t_3\le t_2\le t_1;\\
 & t_4\le t'\le t_3\le t''\le t_2\le t_1; & & \\
 & t_4\le t'\le t''\le t_3\le t_2\le t_1; & & \\
 & t'\le t_4\le t''\le t_3\le t_2\le t_1; & &
\end{array}
\tag{2.59}
$$

$$
\begin{aligned}
& t' \le t'' \le t_4 \le t_3 \le t_2 \le t_1; \\
& t' \le t_4 \le t_3 \le t'' \le t_2 \le t_1;
\end{aligned}
$$

$$
\begin{aligned}
IIc. \quad & t'' \le t' \le t_4 \le t_3 \le t_2 \le t_1; \\
& t'' \le t_4 \le t' \le t_3 \le t_2 \le t_1; \\
& t'' \le t_4 \le t_3 \le t' \le t_2 \le t_1; \\
& t'' \le t_4 \le t_3 \le t_2 \le t' \le t_1; \\
& t' \le t'' \le t_4 \le t_3 \le t_2 \le t_1;
\end{aligned}
$$

$$
\begin{aligned}
IId. \quad & t'' \le t' \le t_4 \le t_3 \le t_2 \le t_1; \\
& t'' \le t_4 \le t' \le t_3 \le t_2 \le t_1; \\
& t'' \le t_4 \le t_3 \le t' \le t_2 \le t_1; \\
& t_4 \le t'' \le t_3 \le t' \le t_2 \le t_1; \\
& t_4 \le t'' \le t' \le t_3 \le t_2 \le t_1; \\
& t_4 \le t' \le t'' \le t_3 \le t_2 \le t_1; \\
& t' \le t_4 \le t'' \le t_3 \le t_2 \le t_1; \\
& t' \le t'' \le t_4 \le t_3 \le t_2 \le t_1;
\end{aligned}
$$

$$
\begin{aligned}
IIe. \quad & t'' \le t' \le t_4 \le t_3 \le t_2 \le t_1; \\
& t'' \le t_4 \le t' \le t_3 \le t_2 \le t_1; \\
& t'' \le t_4 \le t_3 \le t' \le t_2 \le t_1; \\
& t' \le t'' \le t_4 \le t_3 \le t_2 \le t_1;
\end{aligned}
$$

$$
\begin{aligned}
IIg. \quad & t'' \le t' \le t_4 \le t_3 \le t_2 \le t_1; \\
& t'' \le t_4 \le t' \le t_3 \le t_2 \le t_1; \\
& t' \le t'' \le t_4 \le t_3 \le t_2 \le t_1.
\end{aligned}
$$

Here, we refer to the first and the second double integral by I and II in our expression (2.58), and the letters link the time intervals with the corresponding permutations listed above.

From (2.58), by splitting into different parts with respect to (2.59) and computing all the integrals appropriately, the contribution from the connected part takes the following form:

$$
\begin{aligned}
\big\langle \phi(t_1)\phi(t_2)\phi(t_3)\phi(t_4)\big\rangle^{\mathrm{con}}_{\lambda^2} = {} & \frac{729\,\lambda^2 H^{16}}{4096\,\pi^8 m^{12}}\, \mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2+t_3+t_4)} \Bigg(\mathrm{e}^{\frac{m^2}{3H}(t_2+t_3+2\,t_4)} \qquad (2.60) \\
& + \left(\frac{67}{2} + \frac{4m^2}{3H}\big(t_1+9t_2-9t_3-t_4\big) + \frac{4m^4}{9H^2}\big(t_2-t_3\big)\big(t_1+2t_2-2t_3-t_4\big)\right) \mathrm{e}^{\frac{2m^2}{3H}(t_3+t_4)} \\
& - \left(1+\frac{2m^2}{3H}\big(t_3-t_4\big)\right) \mathrm{e}^{\frac{m^2}{3H}(t_2-t_3+4\,t_4)} - \left(\frac{21}{2} + \frac{m^2}{3H}\big(t_1+3t_2+3t_3-7t_4\big)\right) \mathrm{e}^{\frac{4m^2 t_4}{3H}} \\
& + \frac{1}{2}\mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_3-2\,t_4)} - \left(\frac{23}{2} + \frac{m^2}{3H}\big(9t_1-5t_2-3t_3-t_4\big)\right) \mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_2-t_3-t_4)} + \frac{1}{2}\mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_2-2\,t_4)} \\
& + \frac{1}{2}\mathrm{e}^{-\frac{2m^2}{3H}(t_1-2\,t_3-t_4)} + \frac{1}{2}\mathrm{e}^{-\frac{2m^2}{3H}(t_2-2\,t_3-t_4)} + \frac{1}{2}\mathrm{e}^{-\frac{2m^2}{3H}(t_2-t_3-2\,t_4)} - \mathrm{e}^{\frac{m^2}{3H}(t_2+t_3)} \\
& + \left(2-\frac{4m^4}{9H^2}\big(t_3-t_4\big)^2\right) \mathrm{e}^{\frac{m^2}{3H}(t_2-t_3+2\,t_4)} - \mathrm{e}^{\frac{m^2}{3H}(t_2-3\,t_3+4\,t_4)} - \left(\frac{63}{2} + \frac{2m^2}{3H}\big(2t_1+17t_2-17t_3+6t_4\big)\right. \\
& \left. + \frac{4m^4}{9H^2}\big(t_2-t_3\big)\big(t_1+2t_2-2t_3+3t_4\big)\right) \mathrm{e}^{\frac{2m^2 t_3}{3H}} - \left(146 + \frac{2m^2}{3H}\big(6t_1+29t_2+64t_3-75t_4\big)\right. \\
& \left. + \frac{4m^4}{9H^2}\Big(\big(t_2+2t_3-3t_4\big)t_1 + \big(2t_2+9t_3-10t_4\big)t_2 + \big(6t_3-15t_4\big)t_3 + 8t_4^2\Big)\right) \mathrm{e}^{\frac{2m^2 t_4}{3H}} \\
& - \frac{1}{2}\mathrm{e}^{-\frac{2m^2}{3H}(t_1-2\,t_3)} + \left(11 + \frac{m^2}{3H}\big(9t_1-5t_2-3t_3+3t_4\big)\right) \mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_2-t_3)} + \left(15 + \frac{m^2}{3H}\big(9t_1-5t_2\right.
\end{aligned}
$$

$$+ 7t_3 - 7t_4)\Big) e^{-\frac{2m^2}{3H}(t_1-t_2-t_4)} + \Big(21 + \frac{m^2}{3H}(9t_1+13t_2-11t_3-7t_4)\Big) e^{-\frac{2m^2}{3H}(t_1-t_3-t_4)}$$

$$-\frac{3}{2} e^{-\frac{2m^2}{3H}(t_1-2t_4)} - \frac{5}{2} e^{-\frac{2m^2}{3H}(2t_1-t_2-t_3-t_4)} + \Big(\frac{51}{2}+\frac{m^2}{3H}(t_1+21t_2-11t_3-7t_4)\Big)e^{-\frac{2m^2}{3H}(t_2-t_3-t_4)}$$

$$-\frac{1}{2} e^{-\frac{2m^2}{3H}(t_2-2t_3)} - \frac{3}{2} e^{-\frac{2m^2}{3H}(t_2-2t_4)} - \frac{1}{2} e^{-\frac{2m^2}{3H}(t_3-2t_4)} - \frac{4m^2}{3H}(t_3-t_4) e^{\frac{m^2}{3H}(t_2-3t_3+2t_4)}$$

$$+ \frac{4m^2}{3H}(t_3-t_4)\Big(1+\frac{m^2}{3H}(t_3-t_4)\Big) e^{\frac{m^2}{3H}(t_2-t_3)} - \Big(2+\frac{2m^2}{3H}(t_2-t_3)\Big)e^{\frac{2m^2}{3H}(t_3-t_4)} + \frac{223}{2}$$

$$+ \frac{4m^2}{3H}\Big(3t_1+14t_2+30t_3+55t_4\Big) + \frac{4m^4}{9H^2}\Big((t_2+2t_3+3t_4)t_1 + (2t_2+9t_3+12t_4)t_2$$

$$+ (6t_3+23t_4)t_3 + 14t_4^4\Big) - \Big(\frac{29}{2}+\frac{m^2}{3H}(9t_1-5t_2+7t_3+9t_4)\Big)e^{-\frac{2m^2}{3H}(t_1-t_2)} + 2\,e^{-\frac{2m^2}{3H}(t_1+t_2-t_3-t_4)}$$

$$-\Big(\frac{41}{2}+\frac{m^2}{3H}(9t_1+13t_2-11t_3+9t_4)\Big)e^{-\frac{2m^2}{3H}(t_1-t_3)} + e^{-\frac{2m^2}{3H}(t_1-t_2+t_3-t_4)}$$

$$-\Big(\frac{57}{2}+\frac{m^2}{3H}(9t_1+13t_2+17t_3-19t_4)\Big)e^{-\frac{2m^2}{3H}(t_1-t_4)} + \frac{1}{2} e^{-\frac{2m^2}{3H}(t_1-t_2-t_3+t_4)} + \frac{5}{2} e^{-\frac{2m^2}{3H}(2t_1-t_2-t_3)}$$

$$+ \frac{5}{2} e^{-\frac{2m^2}{3H}(2t_1-t_2-t_4)} + \frac{5}{2} e^{-\frac{2m^2}{3H}(2t_1-t_3-t_4)} + \frac{5}{2} e^{-\frac{2m^2}{3H}(2t_2-t_3-t_4)} + \Big(2+\frac{4m^2}{3H}(t_3-t_4)\Big)e^{\frac{m^2}{3H}(t_2-3t_3)}$$

$$-\Big(25+\frac{m^2}{3H}(t_1+21t_2-11t_3+9t_4)\Big)e^{-\frac{2m^2}{3H}(t_2-t_3)} - \Big(33+\frac{m^2}{3H}(t_1+21t_2+17t_3-19t_4)\Big)e^{-\frac{2m^2}{3H}(t_2-t_4)}$$

$$-\Big(\frac{75}{2}+\frac{m^2}{3H}(t_1+5t_2+29t_3-15t_4)\Big)e^{-\frac{2m^2}{3H}(t_3-t_4)} - \Big(\frac{3}{2}+\frac{2m^2}{3H}(t_3-t_4)\Big)e^{\frac{m^2}{3H}(t_2-t_3-2t_4)}$$

$$-\frac{1}{2} e^{\frac{m^2}{3H}(t_2-5t_3+2t_4)} - \frac{5}{2} e^{-\frac{2m^2}{3H}(2t_1-t_2)} - \frac{5}{2} e^{-\frac{2m^2}{3H}(2t_1-t_3)} - \frac{5}{2} e^{-\frac{2m^2}{3H}(2t_1-t_4)} - 2\,e^{-\frac{2m^2}{3H}(t_1+t_2-t_3)}$$

$$-2\,e^{-\frac{2m^2}{3H}(t_1+t_2-t_4)} - 2\,e^{-\frac{2m^2}{3H}(t_1+t_3-t_4)} - e^{-\frac{2m^2}{3H}(t_1-t_2+t_3)} - e^{-\frac{2m^2}{3H}(t_1-t_2+t_4)} - e^{-\frac{2m^2}{3H}(t_1-t_3+t_4)}$$

$$+\Big(28+\frac{m^2}{3H}(9t_1+13t_2+17t_3+21t_4)\Big)e^{-\frac{2m^2t_1}{3H}} - 2\,e^{-\frac{2m^2}{3H}(t_2+t_3-t_4)} - e^{-\frac{2m^2}{3H}(t_2-t_3+t_4)}$$

$$-\frac{5}{2} e^{-\frac{2m^2}{3H}(2t_2-t_3)} - \frac{5}{2} e^{-\frac{2m^2}{3H}(2t_2-t_4)} + \Big(\frac{65}{2}+\frac{m^2}{3H}(t_1+21t_2+17t_3+21t_4)\Big)e^{-\frac{2m^2t_2}{3H}}$$

$$-2\,e^{-\frac{2m^2}{3H}(2t_3-t_4)} + \Big(35+\frac{m^2}{3H}(t_1+5t_2+29t_3+25t_4)\Big)e^{-\frac{2m^2t_3}{3H}} + \frac{1}{2} e^{\frac{m^2}{3H}(t_2-t_3-4t_4)}$$

$$+\Big(43+\frac{m^2}{3H}(t_1+5t_2+11t_3+43t_4)\Big)e^{-\frac{2m^2t_4}{3H}} + \frac{1}{2} e^{\frac{m^2}{3H}(t_2-5t_3)} - e^{\frac{m^2}{3H}(t_2-3t_3-2t_4)}$$

$$+2\,e^{-\frac{2m^2}{3H}(t_1+t_2)} + 2\,e^{-\frac{2m^2}{3H}(t_1+t_3)} + 2\,e^{-\frac{2m^2}{3H}(t_1+t_4)} + 2\,e^{-\frac{2m^2}{3H}(t_2+t_3)} + 2\,e^{-\frac{2m^2}{3H}(t_2+t_4)} + 3\,e^{-\frac{2m^2}{3H}(t_3+t_4)}$$

$$+\frac{5}{2}e^{-\frac{4m^2t_1}{3H}} + \frac{5}{2}e^{-\frac{4m^2t_2}{3H}} + 2e^{-\frac{4m^2t_3}{3H}} + 2e^{-\frac{4m^2t_4}{3H}}\Big) \xrightarrow{m\to 0}$$

$$\xrightarrow{m\to 0} \frac{\lambda^2 H^{10}}{768\,\pi^8}\Big(11\, t_1^3 t_2 t_3 t_4 + t_1 t_2^3 t_3 t_4 + t_1 t_2 t_3^3 t_4 + t_1 t_2 t_3 t_4^3 \tag{2.61}$$
$$+ 5\, t_2^4 t_3 t_4 + 3\, t_3^5 t_4 + 2\, t_4^6\Big) := \big\langle \phi_{m=0}(t_1)\phi_{m=0}(t_2)\phi_{m=0}(t_3)\phi_{m=0}(t_4)\big\rangle^{\text{con}}_{\lambda^2},$$

and the equal time result is

$$\big\langle \phi^4(t)\big\rangle^{\text{con}}_{\lambda^2} = \frac{5103\lambda^2 H^{16}}{2048\,\pi^8 m^{12}}\left(1 - \left(8 + \frac{8m^2 t}{7H}\right)\mathrm{e}^{-\frac{2m^2 t}{3H}} - \left(\frac{18}{7} - \frac{48m^2 t}{7H} - \frac{16m^4 t^2}{7H^2}\right)\mathrm{e}^{-\frac{4m^2 t}{3H}}\right. \tag{2.62}$$
$$\left. + \left(8 + \frac{40m^2 t}{7H}\right)\mathrm{e}^{-\frac{2m^2 t}{H}} + \frac{11}{7}\,\mathrm{e}^{-\frac{8m^2 t}{3H}}\right) \xrightarrow{m\to 0} \frac{\lambda^2 H^{10}\, t^6}{32\,\pi^8} := \big\langle \phi^4_{m=0}(t)\big\rangle^{\text{con}}_{\lambda^2}. \tag{2.63}$$

As before, the contribution of the disconnected diagrams can be calculated directly via (2.58) and playing the game with (2.59) or can be obtained by combining the zeroth-order, linear-λ, and quadratic-λ^2 contributions from the previously presented two-point correlation function results as follows:

$$\big\langle \phi(t_1)\phi(t_2)\phi(t_3)\phi(t_4)\big\rangle^{\text{discon}}_{\lambda^2} = \big\langle \tilde\phi(t_1)\tilde\phi(t_2)\big\rangle\big\langle \phi(t_3)\phi(t_4)\big\rangle_{\lambda^2} \tag{2.64}$$
$$+ \big\langle \tilde\phi(t_1)\tilde\phi(t_3)\big\rangle\big\langle \phi(t_2)\phi(t_4)\big\rangle_{\lambda^2} + \big\langle \tilde\phi(t_1)\tilde\phi(t_4)\big\rangle\big\langle \phi(t_2)\phi(t_3)\big\rangle_{\lambda^2} + \big\langle \tilde\phi(t_2)\tilde\phi(t_3)\big\rangle\big\langle \phi(t_1)\phi(t_4)\big\rangle_{\lambda^2}$$
$$+ \big\langle \tilde\phi(t_2)\tilde\phi(t_4)\big\rangle\big\langle \phi(t_1)\phi(t_3)\big\rangle_{\lambda^2} + \big\langle \tilde\phi(t_3)\tilde\phi(t_4)\big\rangle\big\langle \phi(t_1)\phi(t_2)\big\rangle_{\lambda^2} + \big\langle \phi(t_1)\phi(t_2)\big\rangle_{\lambda}\big\langle \phi(t_3)\phi(t_4)\big\rangle_{\lambda}$$
$$+ \big\langle \phi(t_1)\phi(t_3)\big\rangle_{\lambda}\big\langle \phi(t_2)\phi(t_4)\big\rangle_{\lambda} + \big\langle \phi(t_1)\phi(t_4)\big\rangle_{\lambda}\big\langle \phi(t_2)\phi(t_3)\big\rangle_{\lambda}.$$

We computed it straightforwardly with (2.58) and (2.59). The resulting explicit expression is

$$\big\langle \phi(t_1)\phi(t_2)\phi(t_3)\phi(t_4)\big\rangle^{\text{discon}}_{\lambda^2} = \frac{243\,\lambda^2 H^{16}}{8192\,\pi^8 m^{12}}\,\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2+t_3+t_4)}\left(\mathrm{e}^{-\frac{2m^2}{3H}(t_1-2\,t_2-t_4)}\right. \tag{2.65}$$
$$+ \left(6 + \frac{m^2}{3H}(t_1 - t_2 + t_3 - t_4)\right)\left(6 + \frac{m^2}{H}(t_1 - t_2 + t_3 - t_4)\right)\mathrm{e}^{\frac{2m^2}{3H}(t_2+t_4)} + \mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_3-2\,t_4)}$$
$$+ \mathrm{e}^{-\frac{2m^2}{3H}(t_1-2\,t_3-t_4)} + \mathrm{e}^{\frac{2m^2}{3H}(t_2-t_3+2\,t_4)} + \mathrm{e}^{-\frac{2m^2}{3H}(t_2-t_3-2\,t_4)} + \mathrm{e}^{-\frac{2m^2}{3H}(t_2-2\,t_3-t_4)}$$
$$+ \left(4 + \frac{2m^2}{3H}(t_1 + t_2 - t_3 - t_4)\right)\left(18 + \frac{m^2}{H}(t_1 + t_2 - t_3 - t_4)\right)\mathrm{e}^{\frac{2m^2}{3H}(t_3+t_4)}$$
$$+ \left(27 + \frac{m^2}{H}(9t_1 - 5t_2 + t_3 - t_4)\right)\mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_2-t_4)} - \mathrm{e}^{-\frac{2m^2}{3H}(t_1-2\,t_3)} - \mathrm{e}^{-\frac{2m^2}{3H}(t_1-2\,t_4)}$$
$$+ \left(54 + \frac{2m^2}{H}(9t_1 + t_2 - 3t_3 - 3t_4)\right)\mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_3-t_4)} - \mathrm{e}^{-\frac{2m^2}{3H}(t_2-2\,t_3)} - \mathrm{e}^{-\frac{2m^2}{3H}(t_2-2\,t_4)}$$
$$+ \left(27 + \frac{m^2}{H}(t_1 - t_2 + 9t_3 - 5t_4)\right)\mathrm{e}^{\frac{2m^2}{3H}(t_2-t_3+t_4)} + \left(54 + \frac{2m^2}{H}(t_1 + 9t_2 - 3t_3 - 3t_4)\right)\times$$
$$\times\, \mathrm{e}^{-\frac{2m^2}{3H}(t_2-t_3-t_4)} - \left(\frac{m^2}{H}(7t_1 - 7t_2 + 7t_3 + 29t_4) + \frac{m^4}{3H^2}(t_1 - t_2 + t_3 + 3t_4)^2\right)\mathrm{e}^{\frac{2m^2 t_2}{3H}}$$

$$- e^{-\frac{2m^2}{3H}(t_3-2t_4)} - \left(\frac{2m^2}{H}(7t_1+7t_2-7t_3+29t_4) + \frac{2m^4}{3H^2}(t_1+t_2-t_3+3t_4)^2\right) e^{\frac{2m^2 t_3}{3H}}$$

$$-\Bigg(\frac{m^2}{H}(21t_1+43t_2+65t_3-21t_4) + \frac{m^4}{3H^2}\Big((3t_1+10t_2+14t_3-6t_4)t_1 + (11t_2+18t_3$$

$$-10t_4)t_2 + (19t_3-14t_4)t_3+3t_4^2\Big)\Bigg) e^{\frac{2m^2t_4}{3H}} + \frac{15}{4}e^{-\frac{2m^2}{3H}(2t_1-t_2-t_4)} + \frac{15}{2}e^{-\frac{2m^2}{3H}(2t_1-t_3-t_4)}$$

$$-\left(\frac{51}{2}+\frac{m^2}{H}(9t_1-5t_2+t_3+3t_4)\right)e^{-\frac{2m^2}{3H}(t_1-t_2)} - \left(51+\frac{2m^2}{H}(9t_1+t_2-3t_3+5t_4)\right)\times$$

$$\times e^{-\frac{2m^2}{3H}(t_1-t_3)} - \left(\frac{153}{2}+\frac{m^2}{H}(27t_1+9t_2+11t_3-7t_4)\right)e^{-\frac{2m^2}{3H}(t_1-t_4)} + \frac{3}{2}e^{-\frac{2m^2}{3H}(t_1-t_2+t_3-t_4)}$$

$$+\frac{15}{2}e^{-\frac{2m^2}{3H}(2t_2-t_3-t_4)} - \left(\frac{51}{2}+\frac{m^2}{H}(t_1-t_2+9t_3+7t_4)\right)e^{\frac{2m^2}{3H}(t_2-t_3)}$$

$$-\left(\frac{129}{4}+\frac{m^2}{H}(t_1-t_2+t_3+15t_4)\right)e^{\frac{2m^2}{3H}(t_2-t_4)} - \left(51+\frac{2m^2}{H}(t_1+9t_2-3t_3+5t_4)\right)e^{-\frac{2m^2}{3H}(t_2-t_3)}$$

$$+3e^{-\frac{2m^2}{3H}(t_1+t_2-t_3-t_4)} - \left(\frac{333}{4}+\frac{m^2}{H}(3t_1+33t_2+11t_3-7t_4)\right)e^{-\frac{2m^2}{3H}(t_2-t_4)} - e^{-\frac{2m^2}{3H}(t_1-2t_2)}$$

$$-\left(90+\frac{m^2}{H}(3t_1+5t_2+39t_3-7t_4)\right)e^{-\frac{2m^2}{3H}(t_3-t_4)} - \left(\frac{129}{2}+\frac{2m^2}{H}(t_1+t_2-t_3+15t_4)\right)\times$$

$$\times e^{\frac{2m^2}{3H}(t_3-t_4)} - \frac{207}{2}+\frac{2m^2}{H}(9t_1+19t_2+29t_3+39t_4)+\frac{m^4}{3H^2}\Big((3t_1+10t_2+14t_3+18t_4)t_1$$

$$+(11t_2+18t_3+30t_4)t_2+(19t_3+42t_4)t_3+27t_4^2\Big) + \frac{15}{4}e^{\frac{2m^2}{3H}(t_2-2t_3+t_4)} - \frac{15}{4}e^{\frac{2m^2}{3H}(t_2-2t_3)}$$

$$-\frac{15}{4}e^{\frac{2m^2}{3H}(t_2-2t_4)} + \left(72+\frac{m^2}{H}(27t_1+9t_2+11t_3+13t_4)\right)e^{-\frac{2m^2t_1}{3H}} - \frac{5}{2}e^{\frac{2m^2}{3H}(t_2-t_3-t_4)}$$

$$-\frac{15}{2}e^{\frac{2m^2}{3H}(t_3-2t_4)} - \frac{15}{4}e^{-\frac{2m^2}{3H}(2t_1-t_2)} - \frac{15}{2}e^{-\frac{2m^2}{3H}(2t_1-t_3)} - \frac{45}{4}e^{-\frac{2m^2}{3H}(2t_1-t_4)} - 3e^{-\frac{2m^2}{3H}(t_1+t_2-t_3)}$$

$$-\frac{11}{2}e^{-\frac{2m^2}{3H}(t_1+t_2-t_4)} - \frac{3}{2}e^{-\frac{2m^2}{3H}(t_1-t_2+t_3)} - \frac{3}{2}e^{-\frac{2m^2}{3H}(t_1-t_2+t_4)} - \frac{11}{2}e^{-\frac{2m^2}{3H}(t_1+t_3-t_4)}$$

$$-4e^{-\frac{2m^2}{3H}(t_1-t_3+t_4)} - \frac{15}{2}e^{-\frac{2m^2}{3H}(2t_2-t_3)} + \left(\frac{315}{4}+\frac{m^2}{H}(3t_1+33t_2+11t_3+13t_4)\right)e^{-\frac{2m^2t_2}{3H}}$$

$$-\frac{45}{4}e^{-\frac{2m^2}{3H}(2t_2-t_4)} - \frac{11}{2}e^{-\frac{2m^2}{3H}(t_2+t_3-t_4)} - 4e^{-\frac{2m^2}{3H}(t_2-t_3+t_4)} - \frac{45}{4}e^{-\frac{2m^2}{3H}(2t_3-t_4)}$$

$$+\left(\frac{171}{2}+\frac{m^2}{H}(3t_1+5t_2+39t_3+13t_4)\right)e^{-\frac{2m^2t_3}{3H}} + \frac{11}{2}e^{-\frac{2m^2}{3H}(t_1+t_2)} + \frac{11}{2}e^{-\frac{2m^2}{3H}(t_1+t_3)}$$

$$+\left(\frac{369}{4}+\frac{m^2}{H}(3t_1+5t_2+7t_3+45t_4)\right)e^{-\frac{2m^2t_4}{3H}} + \frac{11}{2}e^{-\frac{2m^2}{3H}(t_1+t_4)} + \frac{11}{2}e^{-\frac{2m^2}{3H}(t_2+t_3)}$$

$$+\frac{11}{2}e^{-\frac{2m^2}{3H}(t_2+t_4)} + \frac{11}{2}e^{-\frac{2m^2}{3H}(t_3+t_4)} + \frac{45}{4}e^{-\frac{4m^2t_1}{3H}} + \frac{45}{4}e^{-\frac{4m^2t_2}{3H}} + \frac{45}{4}e^{-\frac{4m^2t_3}{3H}} + \frac{45}{4}e^{-\frac{4m^2t_4}{3H}}\Bigg)$$

$$
\begin{aligned}
&\xrightarrow{m\to 0} \frac{\lambda^2 H^{10}}{6144\,\pi^8}\Big(11\, t_1^4\, t_2 t_4 + 22\, t_1^4 t_3 t_4 + 2\, t_1^2\, t_2^3\, t_4 + 12\, t_1^2\, t_2^2\, t_3 t_4 + 6\, t_1^2\, t_2 t_3^2\, t_4 \\
&\quad + 2\, t_1^2\, t_2 t_4^3 + 4\, t_1^2\, t_3^3\, t_4 + 4\, t_1^2\, t_3 t_4^3 + \frac{31}{5}\, t_2^5\, t_4 + 22\, t_2^4\, t_3 t_4 + 2\, t_2^3\, t_3^2 t_4 \\
&\quad + \frac{2}{3}\, t_2^3\, t_4^3 + 4\, t_2^2\, t_3^3\, t_4 + 4\, t_2^2\, t_3 t_4^3 + 11\, t_2 t_3^4\, t_4 + 2\, t_2 t_3^2\, t_4^3 + \frac{31}{5}\, t_2 t_4^5 + \frac{62}{5}\, t_3^5\, t_4 \\
&\quad + \frac{4}{3}\, t_3^3\, t_4^3 + \frac{62}{5}\, t_3 t_4^5\Big) := \big\langle \phi_{m=0}(t_1)\phi_{m=0}(t_2)\phi_{m=0}(t_3)\phi_{m=0}(t_4)\big\rangle^{\text{discon}}_{\lambda^2},
\end{aligned}
\tag{2.66}
$$

and at equal times it becomes

$$
\begin{aligned}
\big\langle \phi^4(t)\big\rangle^{\text{discon}}_{\lambda^2} &= \frac{729\,\lambda^2 H^{16}}{4096\,\pi^8 m^{12}}\Bigg(19 + \left(26 - \frac{32m^2 t}{H} - \frac{16m^4 t^2}{3H^2}\right)\mathrm{e}^{-\frac{2m^2 t}{3H}} \\
&\quad - \left(96 + \frac{8m^2 t}{H} - \frac{32m^4 t^2}{3H^2}\right)\mathrm{e}^{-\frac{4m^2 t}{3H}} + \left(38 + \frac{40m^2 t}{H}\right)\mathrm{e}^{-\frac{2m^2 t}{H}} + 13\,\mathrm{e}^{-\frac{8m^2 t}{3H}}\Bigg)
\end{aligned}
\tag{2.67}
$$

$$
\xrightarrow{m\to 0} \frac{23\,\lambda^2 H^{10}\, t^6}{960\,\pi^8} := \big\langle \phi^4_{m=0}(t)\big\rangle^{\text{discon}}_{\lambda^2}.
\tag{2.68}
$$

The full four-point correlation function at order λ^2, as the sum of (2.60) and (2.65), is

$$
\begin{aligned}
&\big\langle \phi(t_1)\phi(t_2)\phi(t_3)\phi(t_4)\big\rangle_{\lambda^2} = \frac{729\,\lambda^2 H^{16}}{4096\,\pi^8 m^{12}}\, \mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2+t_3+t_4)}\Bigg(\frac{1}{6}\,\mathrm{e}^{\frac{2m^2}{3H}(t_2-t_3+2t_4)} \\
&+ \mathrm{e}^{\frac{m^2}{3H}(t_2+t_3+2t_4)} + \left(6 + \frac{m^2}{3H}(t_1 - t_2 + t_3 - t_4)\right)\left(1 + \frac{m^2}{6H}(t_1 - t_2 + t_3 - t_4)\right)\mathrm{e}^{\frac{2m^2}{3H}(t_2+t_4)} \\
&- \left(1 + \frac{2m^2}{3H}(t_3 - t_4)\right)\mathrm{e}^{\frac{m^2}{3H}(t_2-t_3+4\,t_4)} + \Bigg(\frac{91}{2} + \frac{4m^2}{3H}(3t_1 + 11t_2 - 11t_3 - 3t_4) \\
&+ \frac{m^4}{9H^2}(t_1 + 3t_2 - 3t_3 - t_4)^2\Bigg)\mathrm{e}^{\frac{2m^2}{3H}(t_3+t_4)} - \left(\frac{21}{2} + \frac{m^2}{3H}(t_1 + 3t_2 + 3t_3 - 7t_4)\right)\mathrm{e}^{\frac{4m^2 t_4}{3H}} \\
&+ \frac{1}{6}\,\mathrm{e}^{-\frac{2m^2}{3H}(t_1-2\,t_2-t_4)} - \left(\frac{23}{2} + \frac{m^2}{3H}(9t_1 - 5t_2 - 3t_3 - t_4)\right)\mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_2-t_3-t_4)} \\
&+ \frac{1}{2}\,\mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_2-2\,t_4)} + \frac{2}{3}\,\mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_3-2\,t_4)} + \frac{2}{3}\,\mathrm{e}^{-\frac{2m^2}{3H}(t_1-2\,t_3-t_4)} + \frac{2}{3}\,\mathrm{e}^{-\frac{2m^2}{3H}(t_2-t_3-2\,t_4)} \\
&+ \frac{2}{3}\,\mathrm{e}^{-\frac{2m^2}{3H}(t_2-2\,t_3-t_4)} - \left(\frac{m^2}{6H}(7t_1 - 7t_2 + 7t_3 + 29t_4) + \frac{m^4}{18H^2}(t_1 - t_2 + t_3 + 3t_4)^2\right)\mathrm{e}^{\frac{2m^2 t_2}{3H}} \\
&- \mathrm{e}^{\frac{m^2}{3H}(t_2+t_3)} + \left(\frac{9}{2} + \frac{m^2}{6H}(t_1 - t_2 + 9t_3 - 5t_4)\right)\mathrm{e}^{\frac{2m^2}{3H}(t_2-t_3+t_4)} - \mathrm{e}^{\frac{m^2}{3H}(t_2-3\,t_3+4\,t_4)} \\
&+ \left(2 - \frac{4m^4}{9H^2}(t_3 - t_4)^2\right)\mathrm{e}^{\frac{m^2}{3H}(t_2-t_3+2\,t_4)} - \left(\frac{63}{2} + \frac{m^2}{3H}(11t_1 + 41t_2 - 41t_3 + 41t_4)\right.
\end{aligned}
\tag{2.69}
$$

$$
\begin{aligned}
&+ \frac{m^4}{9H^2}\big(t_1 + 3t_2 - 3t_3 + 3t_4\big)^2\Big) \mathrm{e}^{\frac{2m^2 t_3}{3H}} - \Big(146 + \frac{m^2}{2H}\big(15t_1 + 53t_2 + 107t_3 - 107t_4\big) \\
&+ \frac{m^4}{18H^2}\Big(3\big(t_1 + 3t_2\big)^2 + \big(30t_1 + 90t_2 + 67t_3 - 134t_4\big)\,t_3 - \big(30t_1 + 90t_2 - 67t_4\big)\,t_4\Big)\Big) \mathrm{e}^{\frac{2m^2 t_4}{3H}} \\
&- \frac{5}{2}\,\mathrm{e}^{-\frac{2m^2}{3H}(2t_1-t_2-t_3-t_4)} + \Big(11 + \frac{m^2}{3H}\big(9t_1 - 5t_2 - 3t_3 + 3t_4\big)\Big) \mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_2-t_3)} \\
&- \frac{1}{6}\,\mathrm{e}^{-\frac{2m^2}{3H}(t_1-2t_2)} + \Big(\frac{39}{2} + \frac{m^2}{2H}\big(9t_1 - 5t_2 + 5t_3 - 5t_4\big)\Big) \mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_2-t_4)} \\
&+ \Big(30 + \frac{2m^2}{3H}\big(9t_1 + 7t_2 - 7t_3 - 5t_4\big)\Big) \mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_3-t_4)} - \frac{2}{3}\,\mathrm{e}^{-\frac{2m^2}{3H}(t_1-2t_3)} - \frac{5}{3}\,\mathrm{e}^{-\frac{2m^2}{3H}(t_1-2t_4)} \\
&+ \Big(\frac{69}{2} + \frac{2m^2}{3H}\big(t_1 + 15t_2 - 7t_3 - 5t_4\big)\Big) \mathrm{e}^{-\frac{2m^2}{3H}(t_2-t_3-t_4)} - \frac{2}{3}\,\mathrm{e}^{-\frac{2m^2}{3H}(t_2-2t_3)} - \frac{5}{3}\,\mathrm{e}^{-\frac{2m^2}{3H}(t_2-2t_4)} \\
&- \frac{2}{3}\,\mathrm{e}^{-\frac{2m^2}{3H}(t_3-2t_4)} - \Big(\frac{17}{4} + \frac{m^2}{6H}\big(t_1 - t_2 + 9t_3 + 7t_4\big)\Big) \mathrm{e}^{\frac{2m^2}{3H}(t_2-t_3)} + \frac{5}{8}\,\mathrm{e}^{\frac{2m^2}{3H}(t_2-2t_3+t_4)} \\
&- \frac{4m^2}{3H}\big(t_3 - t_4\big)\,\mathrm{e}^{\frac{m^2}{3H}(t_2-3t_3+2t_4)} - \Big(\frac{43}{8} + \frac{m^2}{6H}\big(t_1 - t_2 + t_3 + 15t_4\big)\Big) \mathrm{e}^{\frac{2m^2}{3H}(t_2-t_4)} \\
&- \Big(\frac{51}{4} + \frac{m^2}{3H}\big(t_1 + 3t_2 - 3t_3 + 15t_4\big)\Big) \mathrm{e}^{\frac{2m^2}{3H}(t_3-t_4)} + \frac{4m^2}{3H}\big(t_3 - t_4\big)\Big(1 + \frac{m^2}{3H}\big(t_3 - t_4\big)\Big) \mathrm{e}^{\frac{m^2}{3H}(t_2-t_3)} \\
&+ \frac{377}{4} + \frac{m^2}{3H}\big(21t_1 + 75t_2 + 149t_3 + 259t_4\big) + \frac{m^4}{18H^2}\Big(\big(3t_1 + 18t_2 + 30t_3 + 42t_4\big)t_1 \\
&+ \big(27t_2 + 90t_3 + 126t_4\big)t_2 + \big(67t_3 + 226t_4\big)t_3 + 139t_4^2\Big) + \frac{5}{2}\,\mathrm{e}^{-\frac{2m^2}{3H}(2t_1-t_2-t_3)} \\
&+ \frac{25}{8}\,\mathrm{e}^{-\frac{2m^2}{3H}(2t_1-t_2-t_4)} + \frac{15}{4}\,\mathrm{e}^{-\frac{2m^2}{3H}(2t_1-t_3-t_4)} + \frac{5}{2}\,\mathrm{e}^{-\frac{2m^2}{3H}(t_1+t_2-t_3-t_4)} + \frac{5}{4}\,\mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_2+t_3-t_4)} \\
&- \Big(\frac{75}{4} + \frac{m^2}{2H}\big(9t_1 - 5t_2 + 5t_3 + 7t_4\big)\Big) \mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_2)} + \frac{1}{2}\,\mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_2-t_3+t_4)} \\
&- \Big(29 + \frac{2m^2}{3H}\big(9t_1 + 7t_2 - 7t_3 + 7t_4\big)\Big) \mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_3)} + \frac{15}{4}\,\mathrm{e}^{-\frac{2m^2}{3H}(2t_2-t_3-t_4)} \\
&- \Big(\frac{165}{4} + \frac{5m^2}{6H}\big(9t_1 + 7t_2 + 9t_3 - 9t_4\big)\Big) \mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_4)} - \Big(\frac{67}{2} + \frac{2m^2}{3H}\big(t_1 + 15t_2 - 7t_3 + 7t_4\big)\Big)\times \\
&\times \mathrm{e}^{-\frac{2m^2}{3H}(t_2-t_3)} - \Big(\frac{375}{8} + \frac{5m^2}{6H}\big(t_1 + 15t_2 + 9t_3 - 9t_4\big)\Big) \mathrm{e}^{-\frac{2m^2}{3H}(t_2-t_4)} \\
&- \Big(\frac{105}{2} + \frac{m^2}{6H}\big(5t_1 + 15t_2 + 97t_3 - 37t_4\big)\Big) \mathrm{e}^{-\frac{2m^2}{3H}(t_3-t_4)} - \frac{5}{8}\,\mathrm{e}^{\frac{2m^2}{3H}(t_2-2t_3)} - \frac{1}{2}\,\mathrm{e}^{\frac{m^2}{3H}(t_2-5t_3+2t_4)} \\
&+ \Big(2 + \frac{4m^2}{3H}\big(t_3 - t_4\big)\Big) \mathrm{e}^{\frac{m^2}{3H}(t_2-3t_3)} - \frac{5}{12}\,\mathrm{e}^{\frac{2m^2}{3H}(t_2-t_3-t_4)} - \Big(\frac{3}{2} + \frac{2m^2}{3H}\big(t_3 - t_4\big)\Big) \mathrm{e}^{\frac{m^2}{3H}(t_2-t_3-2t_4)}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{5}{8}\,e^{\frac{2m^2}{3H}(t_2-2t_4)}-\frac{5}{4}\,e^{\frac{2m^2}{3H}(t_3-2t_4)}+\left(40+\frac{5m^2}{6H}(9t_1+7t_2+9t_3+11t_4)\right)e^{-\frac{2m^2t_1}{3H}}\\
&-\frac{25}{8}\,e^{-\frac{2m^2}{3H}(2t_1-t_2)}-\frac{15}{4}\,e^{-\frac{2m^2}{3H}(2t_1-t_3)}-\frac{35}{8}\,e^{-\frac{2m^2}{3H}(2t_1-t_4)}-\frac{5}{2}\,e^{-\frac{2m^2}{3H}(t_1+t_2-t_3)}\\
&-\frac{35}{12}\,e^{-\frac{2m^2}{3H}(t_1+t_2-t_4)}-\frac{35}{12}\,e^{-\frac{2m^2}{3H}(t_1+t_3-t_4)}-\frac{5}{4}\,e^{-\frac{2m^2}{3H}(t_1-t_2+t_3)}-\frac{5}{4}\,e^{-\frac{2m^2}{3H}(t_1-t_2+t_4)}\\
&-\frac{5}{3}\,e^{-\frac{2m^2}{3H}(t_1-t_3+t_4)}-\frac{15}{4}\,e^{-\frac{2m^2}{3H}(2t_2-t_3)}+\left(\frac{365}{8}+\frac{5m^2}{6H}(t_1+15t_2+9t_3+11t_4)\right)e^{-\frac{2m^2t_2}{3H}}\\
&-\frac{35}{8}\,e^{-\frac{2m^2}{3H}(2t_2-t_4)}+\left(\frac{197}{4}+\frac{m^2}{6H}(5t_1+15t_2+97t_3+63t_4)\right)e^{-\frac{2m^2t_3}{3H}}\\
&-\frac{35}{12}\,e^{-\frac{2m^2}{3H}(t_2+t_3-t_4)}-\frac{5}{3}\,e^{-\frac{2m^2}{3H}(t_2-t_3+t_4)}+\left(\frac{467}{8}+\frac{m^2}{6H}(5t_1+15t_2+29t_3+131t_4)\right)\times\\
&\times e^{-\frac{2m^2t_4}{3H}}-\frac{31}{8}\,e^{-\frac{2m^2}{3H}(2t_3-t_4)}+\frac{1}{2}\,e^{\frac{m^2}{3H}(t_2-5t_3)}-e^{\frac{m^2}{3H}(t_2-3t_3-2t_4)}+\frac{1}{2}\,e^{\frac{m^2}{3H}(t_2-t_3-4t_4)}\\
&+\frac{35}{12}\,e^{-\frac{2m^2}{3H}(t_1+t_2)}+\frac{35}{12}\,e^{-\frac{2m^2}{3H}(t_1+t_3)}+\frac{35}{12}\,e^{-\frac{2m^2}{3H}(t_1+t_4)}+\frac{35}{12}\,e^{-\frac{2m^2}{3H}(t_2+t_3)}+\frac{35}{12}\,e^{-\frac{2m^2}{3H}(t_2+t_4)}\\
&+\frac{47}{12}\,e^{-\frac{2m^2}{3H}(t_3+t_4)}+\frac{35}{8}\,e^{-\frac{4m^2t_1}{3H}}+\frac{35}{8}\,e^{-\frac{4m^2t_2}{3H}}+\frac{31}{8}\,e^{-\frac{4m^2t_3}{3H}}+\frac{31}{8}\,e^{-\frac{4m^2t_4}{3H}}\Bigg)\xrightarrow{m\to0}
\end{aligned}
$$

$$
\begin{aligned}
\xrightarrow{m\to0}\;&\frac{\lambda^2H^{10}}{6144\,\pi^8}\Big(11\,t_1^4t_2t_4+22\,t_1^4t_3t_4+88\,t_1^3t_2t_3t_4+2\,t_1^2t_2^3t_4+12\,t_1^2t_2^2t_3t_4 \qquad (2.70)\\
&+6\,t_1^2t_2t_3^2t_4+2\,t_1^2t_2t_4^3+4\,t_1^2t_3^3t_4+4\,t_1^2t_3t_4^3+8\,t_1t_2^3t_3t_4+8\,t_1t_2t_3^3t_4+8\,t_1t_2t_3t_4^3+\frac{31}{5}\,t_2^5t_4\\
&+62\,t_2^4t_3t_4+2\,t_2^3t_3^2t_4+\frac{2}{3}\,t_2^3t_4^3+4\,t_2^2t_3^3t_4+4\,t_2^2t_3t_4^3+11\,t_2t_3^4t_4+2\,t_2t_3^2t_4^3+\frac{31}{5}\,t_2t_4^5\\
&+\frac{182}{5}\,t_3^5t_4+\frac{4}{3}\,t_3^3t_4^3+\frac{62}{5}\,t_3t_4^5+16\,t_4^6\Big):=\big\langle\phi_{m=0}(t_1)\phi_{m=0}(t_2)\phi_{m=0}(t_3)\phi_{m=0}(t_4)\big\rangle_{\lambda^2}.
\end{aligned}
$$

At equal times, this series with the obtained orders (2.43), (2.56), and (2.69) appears to be

$$
\begin{aligned}
\big\langle\phi^4(t)\big\rangle=\;&\frac{27H^8}{64\,\pi^4m^4}\left(1-e^{-\frac{2m^2t}{3H}}\right)^2-\frac{81\lambda H^{12}}{64\,\pi^6m^8}\Bigg(1-\left(\frac{9}{4}+\frac{m^2t}{H}\right)e^{-\frac{2m^2t}{3H}} \qquad (2.71)\\
&+\frac{2m^2t}{H}e^{-\frac{4m^2t}{3H}}+\frac{5}{4}e^{-\frac{2m^2t}{H}}\Bigg)+\frac{729\lambda^2H^{16}}{4096\,\pi^8m^{12}}\Bigg(33\;-\left(86+\frac{48m^2t}{H}+\frac{16m^4t^2}{3H^2}\right)e^{-\frac{2m^2t}{3H}}\\
&-\left(132-\frac{88m^2t}{H}-\frac{128m^4t^2}{3H^2}\right)e^{-\frac{4m^2t}{3H}}+\left(150+\frac{120m^2t}{H}\right)e^{-\frac{2m^2t}{H}}+35\,e^{-\frac{8m^2t}{3H}}\Bigg)+O(\lambda^3);\\
\xrightarrow{m\to0}\;&\frac{3H^6t^2}{16\,\pi^4}-\frac{3\lambda H^8t^4}{32\,\pi^6}+\frac{53\,\lambda^2H^{10}\,t^6}{960\,\pi^8}+O(\lambda^3):=\big\langle\phi^4_{m=0}(t)\big\rangle; \qquad (2.72)\\
\xrightarrow{t\to\infty}\;&\frac{27H^8}{64\,\pi^4m^4}-\frac{81\,\lambda H^{12}}{64\,\pi^6m^8}+\frac{24057\,\lambda^2H^{16}}{4096\,\pi^8m^{12}}+O(\lambda^3). \qquad (2.73)
\end{aligned}
$$

All the terms of the computed perturbative massive series are finite, have a smooth massless limit at each λ order (2.72), and approach the equilibrium value at the late-time limit (2.73). Similar to the expectation value $\langle\phi^2(t)\rangle$, the infrared divergences in cosmic time t of the massless self-interacting scalar field (2.72) are fully resummed in the massive series (2.71). This corresponds to the resummation of leading infrared logarithms in the perturbative expansion in m^2/H^2 at each order of the self-interaction coupling constant λ. Expanding the resulting expression (2.71) in the m^2/H^2 regime

$$\begin{aligned}\langle\phi^4(t)\rangle = \frac{3H^6t^2}{16\,\pi^4}\Bigg(&\left(1-\frac{2m^2}{3H^2}(Ht)+\frac{7m^4}{27H^4}(Ht)^2-\frac{2m^6}{27H^6}(Ht)^3+\ldots\right)\\ &-\frac{\lambda}{2\pi^2}(Ht)^2\left(1-\frac{46m^2}{45H^2}(Ht)+\frac{76m^4}{135H^4}(Ht)^2-\frac{1864m^6}{8505H^6}(Ht)^3+\ldots\right)\\ &+\frac{53\,\lambda^2}{180\,\pi^4}(Ht)^4\left(1-\frac{498m^2}{371H^2}(Ht)+\frac{1054m^4}{1113H^4}(Ht)^2-\frac{1552m^6}{3339H^6}(Ht)^3+\ldots\right)\Bigg)+O(\lambda^3),\end{aligned} \tag{2.74}$$

we can trace the leading infrared logarithm structure in cosmic time t and establish the correspondence to series (2.72) in the $m^2=0$ limit. Additionally, we note that $\langle\phi^4_{m=0}(t)\rangle_\lambda$ in (2.72) is in agreement with the results from [122, 123].

To conclude, in this chapter we have computed the two-point correlation function of the free massive scalar field (2.22), relying solely on the known two-point correlation function of the free massless field (2.19) with the use of the introduced relation (2.15) derived within the truncated Yang-Feldman equation (2.14) for a massless scalar field. We have further computed the quantum corrections and organized a perturbative series in the self-interaction coupling constant λ for the two-point and four-point correlation functions through the massive Yang-Feldman-type equation (2.16); see also (2.17).

3 On the correspondence between the Yang-Feldman-type equation and diagrammatic techniques

Based on the previous chapter, computing the vacuum expectation value or correlation function within the massive Yang-Feldman-type equation (2.16) involves integrating the retarded Green's function (i.e., exponential factors) together with products of two-point correlation functions of the free massive field with some combinatorial coefficients.

In this chapter, we intend to explore the correspondence between integral structures within the massive Yang-Feldman-type equation (2.16) and the Feynman or Schwinger-Keldysh [107, 108] and stochastic [81, 82] diagrammatic techniques. All types of diagrams have the same topological structure, but in the stochastic case, one encounters a single vertex and three propagators, while in the Schwinger–Keldysh formalism, there are two types of vertices and four types of propagators. To spot what kind of correspondence may exist, we shall rely on the outputs for the two-point correlation function of a massive scalar field calculated at the one-loop and two-loop levels via the Schwinger-Keldysh diagrammatic technique in [117] and [118] in the p-representation [119], for a massless one at equal times taken from [83, 84, 120, 121, 122, 123], and the stochastic ones at the equilibrium for the massive case from [81, 82]; see the original development [145].

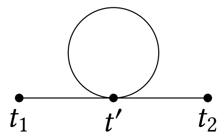


Figure 3.1: One-loop diagram for two-point correlation function.

At linear order in λ, via the Yang-Feldman-type equation (2.17), one obtains the following integral structures for (2.24)

$$\mathcal{I}^{\lambda}_{1,\mathrm{a}} = -\frac{\lambda}{H} \mathrm{e}^{-\frac{m^2 t_1}{3H}} \int_0^{t_1} dt' \mathrm{e}^{\frac{m^2 t'}{3H}} \left\langle \tilde{\phi}(t')\tilde{\phi}(t_2)\right\rangle \left\langle \tilde{\phi}^2(t')\right\rangle; \tag{3.1}$$

$$\mathcal{I}^{\lambda}_{1,\mathrm{b}} = \mathcal{I}^{\lambda}_{1,\mathrm{a}} \text{ with } \left(t_1 \leftrightarrow t_2\right).$$

The corresponding one-loop diagram is shown in Figure 3.1 and has the same topological structure for both the Schwinger-Keldysh and Feynman diagrams; see below.

All integral structures for the two-point correlation function at the two-loop level in the Yang-Feldman-type equation, see (2.27), are the following:

$$\mathcal{I}^{\lambda^2}_{1,\mathrm{a}} = \frac{2\lambda^2}{H^2} \mathrm{e}^{-\frac{m^2 t_1}{3H}} \int_0^{t_1} dt' \int_0^{t'} dt'' \mathrm{e}^{\frac{m^2 t''}{3H}} \left\langle \tilde{\phi}(t')\tilde{\phi}(t_2)\right\rangle \left\langle \tilde{\phi}(t')\tilde{\phi}(t'')\right\rangle \left\langle \tilde{\phi}^2(t'')\right\rangle; \tag{3.2}$$

$$\mathcal{I}^{\lambda^2}_{1,\mathrm{b}} = \mathcal{I}^{\lambda^2}_{1,\mathrm{a}} \text{ with } \left(t_1 \leftrightarrow t_2\right);$$

$$\mathcal{I}^{\lambda^2}_{2,\mathrm{a}} = \frac{2\lambda^2}{H^2} \mathrm{e}^{-\frac{m^2 t_1}{3H}} \int_0^{t_1} dt' \int_0^{t'} dt'' \mathrm{e}^{\frac{m^2 t''}{3H}} \left\langle \tilde{\phi}(t'')\tilde{\phi}(t_2)\right\rangle \left(\left\langle \tilde{\phi}(t')\tilde{\phi}(t'')\right\rangle\right)^2; \tag{3.3}$$

$$\mathcal{I}_{2,\text{b}}^{\lambda^2} = \mathcal{I}_{2,\text{a}}^{\lambda^2} \text{ with } (t_1 \leftrightarrow t_2);$$

$$\mathcal{I}_{3,\text{a}}^{\lambda^2} = \frac{\lambda^2}{H^2} e^{-\frac{m^2 t_1}{3H}} \int_0^{t_1} dt' \int_0^{t'} dt'' e^{\frac{m^2 t''}{3H}} \langle \tilde{\phi}(t'')\tilde{\phi}(t_2)\rangle \langle \tilde{\phi}^2(t')\rangle \langle \tilde{\phi}^2(t'')\rangle ; \tag{3.4}$$

$$\mathcal{I}_{3,\text{b}}^{\lambda^2} = \mathcal{I}_{3,\text{a}}^{\lambda^2} \text{ with } (t_1 \leftrightarrow t_2);$$

$$\mathcal{I}_{4}^{\lambda^2} = \frac{2\lambda^2}{3H^2} e^{-\frac{m^2}{3H}(t_1+t_2)} \int_0^{t_1} dt' e^{\frac{m^2 t'}{3H}} \int_0^{t_2} dt'' e^{\frac{m^2 t''}{3H}} \left(\langle \tilde{\phi}(t')\tilde{\phi}(t'')\rangle\right)^3 ; \tag{3.5}$$

$$\mathcal{I}_{5}^{\lambda^2} = \frac{\lambda^2}{H^2} e^{-\frac{m^2}{3H}(t_1+t_2)} \int_0^{t_1} dt' e^{\frac{m^2 t'}{3H}} \int_0^{t_2} dt'' e^{\frac{m^2 t''}{3H}} \langle \tilde{\phi}^2(t')\rangle \langle \tilde{\phi}(t')\tilde{\phi}(t'')\rangle \langle \tilde{\phi}^2(t'')\rangle . \tag{3.6}$$

Alongside, all possible two-loop level diagram structures are presented in Figure 3.2 below.

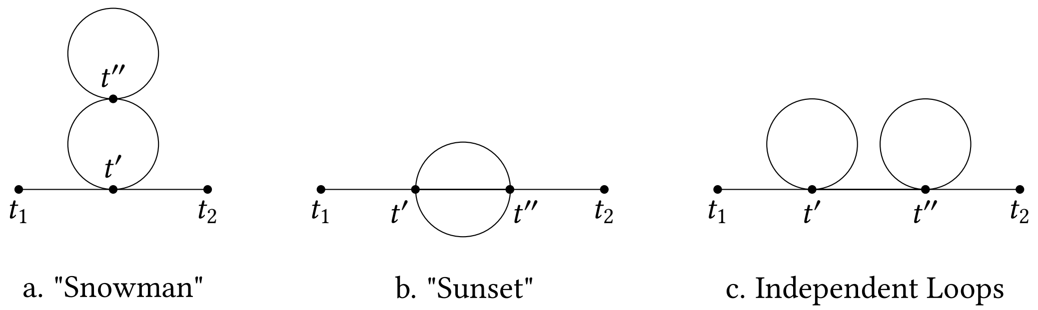


Figure 3.2: All possible two-loop-level diagram structures for the two-point correlation function.

Our correspondence hypothesis is based on the following assignment assumption:

In order to obtain a given diagram topology, the points might be connected either by an explicit correlation function in structures (3.1) and (3.2)–(3.6) or by the limits of the integration variables.

Regarding the one-loop contribution, one notices that t' is integrated up to t_1 (or t_2), while under the integrals (3.1), we have the two-point correlation function, which connects t' and t_1 (or t_2), and another correlation function, which corresponds to the loop attached at t'. Therefore, the two integral structures (3.1) correspond to the topology in Figure 3.1 and yield the result already obtained in (2.25). At late times, the obtained result (2.34) is in agreement with those from [117, 118] within the Schwinger-Keldysh diagrammatic technique [107, 108]. In the massless limit and at equal times, see (2.35) and (2.38), it matches those obtained using quantum field theory methods [83, 84, 120, 121].

At the two-loop level, let us look at the first structure (3.2). We observe that t'' is integrated from 0 to t', while t' is integrated from 0 to t_1 (or t_2). To reflect the fact that t''

is integrated up to t', we shall connect the points t' and t'' and add the line connecting the point t' with the external points t_1 (or t_2), corresponding to the absent free massive field $\tilde{\phi}(t_1)$ (or $\tilde{\phi}(t_2)$). One has another external point t_2 (or t_1), where the field $\tilde{\phi}(t_2)$ (or $\tilde{\phi}(t_1)$) is present and connected to the field at the point t' through the correlation function under the integral. There is also the two-point correlation function $\langle \tilde{\phi}(t')\tilde{\phi}(t'') \rangle$ that connects t' and t'' representing an additional line in the diagram. Finally, we have a closed loop attached to the point t''. As a result, we obtain the topology of the so-called "Snowman" diagram; see Figure 3.2a. By computing the contribution of these structures, one has

$$\begin{aligned}\langle \phi(t_1)\phi(t_2) \rangle^{\text{Snowman}}_{\lambda^2} &= \mathcal{I}^{\lambda^2}_{1,\text{a}} + \mathcal{I}^{\lambda^2}_{1,\text{b}} = \frac{243\,\lambda^2 H^{12}}{1024\,\pi^6 m^{10}} \Bigg(\left(2 + \frac{2m^2}{3H}|t_1 - t_2|\right) \mathrm{e}^{-\frac{m^2}{3H}|t_1-t_2|} \\ &- \left(1 - \frac{2m^2}{3H}\Big(|t_1-t_2| - 2(t_1+t_2)\Big) + \frac{2m^4}{9H^2}\Big(|t_1-t_2| - (t_1+t_2)\Big)^2\right) \mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2)} \\ &+ \left(2 + \frac{2m^2}{3H}\Big(|t_1-t_2| + (t_1+t_2)\Big)\right) \mathrm{e}^{-\frac{m^2}{3H}\left(2|t_1-t_2|+(t_1+t_2)\right)} + \frac{1}{2}\,\mathrm{e}^{-\frac{m^2}{3H}\left(3|t_1-t_2|+2(t_1+t_2)\right)} \\ &- \left(2 + \frac{2m^2}{3H}\Big(|t_1-t_2| + (t_1+t_2)\Big)\right) \mathrm{e}^{-\frac{m^2}{3H}\left(|t_1-t_2|+2(t_1+t_2)\right)} - \frac{1}{2}\,\mathrm{e}^{-\frac{m^2}{3H}\left(2|t_1-t_2|+3(t_1+t_2)\right)} \\ &- \left(\frac{1}{2} - \frac{2m^2}{3H}\Big(|t_1-t_2| - (t_1+t_2)\Big)\right) \mathrm{e}^{\frac{m^2}{3H}\left(|t_1-t_2|-2(t_1+t_2)\right)} - \frac{1}{2}\,\mathrm{e}^{\frac{m^2}{3H}\left(2|t_1-t_2|-3(t_1+t_2)\right)} \Bigg)\end{aligned} \tag{3.7}$$

$$\xrightarrow[\text{times}]{\text{late}} \frac{243\lambda^2 H^{12}}{512\,\pi^6 m^{10}} \left(1 + \frac{m^2}{3H}|t_1 - t_2|\right) \mathrm{e}^{-\frac{m^2}{3H}|t_1-t_2|}; \tag{3.8}$$

$$\xrightarrow[t_1 \geq t_2]{m\to 0} \frac{\lambda^2 H^7}{1920\,\pi^6}\left(5\,t_1^4\,t_2 + 3\,t_2^5\right) := \langle \phi_{m=0}(t_1)\phi_{m=0}(t_2) \rangle^{\text{Snowman}}_{\lambda^2}; \tag{3.9}$$

$$\begin{aligned}\xrightarrow[\text{times}]{\text{equal}} \frac{243\lambda^2 H^{12}}{512\,\pi^6 m^{10}} \Bigg(1 &+ \left(\frac{1}{2} - \frac{2m^2 t}{3H} - \frac{4m^4 t^2}{9H^2}\right) \mathrm{e}^{-\frac{2m^2 t}{3H}} \\ &- \left(1 + \frac{4m^2 t}{3H}\right) \mathrm{e}^{-\frac{4m^2 t}{3H}} - \frac{1}{2}\,\mathrm{e}^{-\frac{2m^2 t}{H}}\Bigg) := \langle \phi^2(t) \rangle^{\text{Snowman}}_{\lambda^2}.\end{aligned} \tag{3.10}$$

In the case of the second structure (3.3), we have a pair of two-point correlation functions connecting the points t' and t'', as well as a third line, which connects these two points because t'' is integrated up to t'. The integration limit for t' is t_1 (or t_2), and the internal two-point correlation function also connects t'' with external points t_2 (or t_1). As a result, one has the so-called "Sunset" diagram; see Figure 3.2b. Moreover, structure (3.5) has the same topology: the integration limits for t' and t'' are t_1 and t_2, respectively, so the integration effectively replaces the external fields $\tilde{\phi}(t_1)$ (or $\tilde{\phi}(t_2)$), which are absent in this structure. From the points t' and t'', one can draw two lines connecting them to the external points t_1 and t_2 corresponding to the absent fields $\tilde{\phi}(t_1)$ and $\tilde{\phi}(t_2)$, and these two internal points, t' and t'', are connected by three identical correlation functions. Therefore, their sum leads to

$$
\begin{aligned}
\left\langle \phi(t_1)\phi(t_2)\right\rangle^{\text{Sunset}}_{\lambda^2} &= \mathcal{I}^{\lambda^2}_{2,\text{a}} + \mathcal{I}^{\lambda^2}_{2,\text{b}} + \mathcal{I}^{\lambda^2}_{4} = \frac{243\,\lambda^2 H^{12}}{1024\,\pi^6 m^{10}} \Bigg(\left(1+\frac{2m^2}{3H}|t_1-t_2|\right) \mathrm{e}^{-\frac{m^2}{3H}|t_1-t_2|} \qquad (3.11)\\
&+ \frac{1}{3}\,\mathrm{e}^{-\frac{m^2}{H}|t_1-t_2|} + \left(7+\frac{2m^2}{3H}\Big(3|t_1-t_2| - 4(t_1+t_2)\Big)\right) \mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2)}\\
&+ \left(5+\frac{4m^2}{3H}|t_1-t_2|\right) \mathrm{e}^{-\frac{m^2}{3H}\left(2|t_1-t_2|+(t_1+t_2)\right)} - \left(5+\frac{4m^2}{3H}(t_1+t_2)\right) \mathrm{e}^{-\frac{m^2}{3H}\left(|t_1-t_2|+2(t_1+t_2)\right)}\\
&- \frac{1}{3}\,\mathrm{e}^{-\frac{m^2}{H}(t_1+t_2)} + \frac{1}{2}\,\mathrm{e}^{-\frac{m^2}{3H}\left(3|t_1-t_2|+2(t_1+t_2)\right)} - \frac{1}{2}\,\mathrm{e}^{-\frac{m^2}{3H}\left(2|t_1-t_2|+3(t_1+t_2)\right)}\\
&- \left(\frac{15}{2} - \frac{4m^2}{3H}\Big(|t_1-t_2| - (t_1+t_2)\Big)\right) \mathrm{e}^{\frac{m^2}{3H}\left(|t_1-t_2|-2(t_1+t_2)\right)} - \frac{1}{2}\,\mathrm{e}^{\frac{m^2}{3H}\left(2|t_1-t_2|-3(t_1+t_2)\right)} \Bigg)\\
&\xrightarrow[\text{times}]{\text{late}} \frac{243\lambda^2 H^{12}}{1024\,\pi^6 m^{10}} \left(\left(1+\frac{2m^2}{3H}|t_1-t_2|\right) \mathrm{e}^{-\frac{m^2}{3H}|t_1-t_2|} + \frac{1}{3}\,\mathrm{e}^{-\frac{m^2}{H}|t_1-t_2|} \right); \qquad (3.12)\\
&\xrightarrow[t_1\geq t_2]{m\to 0} \frac{\lambda^2 H^7}{1920\,\pi^6}\left(5\,t_1^4\,t_2 + 3\,t_2^5\right) := \left\langle \phi_{m=0}(t_1)\phi_{m=0}(t_2)\right\rangle^{\text{Sunset}}_{\lambda^2}; \qquad (3.13)\\
&\xrightarrow[\text{times}]{\text{equal}} \frac{81\lambda^2 H^{12}}{256\pi^6 m^{10}} \left(1 + \left(9 - \frac{4m^2 t}{H}\right)\mathrm{e}^{-\frac{2m^2 t}{3H}} - \left(9+\frac{4m^2 t}{H}\right)\mathrm{e}^{-\frac{4m^2 t}{3H}} - \mathrm{e}^{-\frac{2m^2 t}{H}}\right) := \left\langle \phi^2(t)\right\rangle^{\text{Sunset}}_{\lambda^2}. \qquad (3.14)
\end{aligned}
$$

In the last case of the third structure (3.4), we have closed loops attached to both internal points. There is also a correlation function connecting these two points, and there are, as in the case of the first structure, two lines connecting the moments t' and t'' with the external fields $\tilde{\phi}(t_1)$ and $\tilde{\phi}(t_2)$. In the fifth structure (3.6), we also have two closed loops attached to both internal points t' and t'', and, hence, we again obtain two independent loops, or the so-called "Double Seagull" diagram; see Figure 3.2c. Performing the computation, we get for this contribution

$$
\begin{aligned}
\left\langle \phi(t_1)\phi(t_2)\right\rangle^{\text{Ind. Loops}}_{\lambda^2} &= \mathcal{I}^{\lambda^2}_{3,\text{a}} + \mathcal{I}^{\lambda^2}_{3,\text{b}} + \mathcal{I}^{\lambda^2}_{5} = \frac{243\,\lambda^2 H^{12}}{1024\,\pi^6 m^{10}} \Bigg(\Bigg(2+\frac{2m^2}{3H}|t_1-t_2| \qquad (3.15)\\
&+ \frac{m^4}{9H^2}|t_1-t_2|^2\Bigg) \mathrm{e}^{-\frac{m^2}{3H}|t_1-t_2|} + \left(\frac{m^2}{3H}\Big(|t_1-t_2| - 2(t_1+t_2)\Big) + \frac{m^4}{9H^2}\Big(|t_1-t_2|^2\right.\\
&\left.- 2(t_1+t_2)^2\Big)\right) \mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2)} + \frac{1}{4}\,\mathrm{e}^{-\frac{m^2}{3H}\left(3|t_1-t_2|+2(t_1+t_2)\right)} + \left(1+\frac{m^2}{3H}|t_1-t_2|\right) \mathrm{e}^{-\frac{m^2}{3H}\left(2|t_1-t_2|+(t_1+t_2)\right)}\\
&- \left(\frac{1}{2} - \frac{m^2}{3H}\Big(|t_1-t_2| - 2(t_1+t_2)\Big)\right) \mathrm{e}^{-\frac{m^2}{3H}\left(|t_1-t_2|+2(t_1+t_2)\right)} - \frac{1}{4}\,\mathrm{e}^{-\frac{m^2}{3H}\left(2|t_1-t_2|+3(t_1+t_2)\right)}\\
&- \left(\frac{7}{4} - \frac{m^2}{3H}\Big(|t_1-t_2| - 2(t_1+t_2)\Big)\right) \mathrm{e}^{\frac{m^2}{3H}\left(|t_1-t_2|-2(t_1+t_2)\right)} - \frac{1}{4}\mathrm{e}^{\frac{m^2}{3H}\left(2|t_1-t_2|-3(t_1+t_2)\right)} - \frac{1}{2}\mathrm{e}^{-\frac{m^2}{H}(t_1+t_2)} \Bigg)
\end{aligned}
$$

$$\xrightarrow[\text{times}]{\text{late}} \frac{243\lambda^2 H^{12}}{1024\,\pi^6 m^{10}}\left(2+\frac{2m^2}{3H}|t_1-t_2|+\frac{m^4}{9H^2}|t_1-t_2|^2\right)\mathrm{e}^{-\frac{m^2}{3H}|t_1-t_2|}; \tag{3.16}$$

$$\xrightarrow[t_1\geq t_2]{m\to 0} \frac{\lambda^2 H^7}{64\,\pi^6}\left(\frac{1}{8}\,t_1^4\,t_2+\frac{1}{12}\,t_1^2\,t_2^3+\frac{7}{120}\,t_2^5\right):=\left\langle\phi_{m=0}(t_1)\phi_{m=0}(t_2)\right\rangle^{\text{Ind. Loops}}_{\lambda^2}; \tag{3.17}$$

$$\xrightarrow[\text{times}]{\text{equal}} \frac{243\lambda^2 H^{12}}{512\,\pi^6 m^{10}}\left(1+\left(\frac{1}{2}-\frac{2m^2 t}{3H}-\frac{4m^4t^2}{9H^2}\right)\mathrm{e}^{-\frac{2m^2t}{3H}}\right. \tag{3.18}$$

$$\left.-\left(1+\frac{4m^2t}{3H}\right)\mathrm{e}^{-\frac{4m^2t}{3H}}-\frac{1}{2}\,\mathrm{e}^{-\frac{2m^2t}{H}}\right):=\left\langle\phi^2(t)\right\rangle^{\text{Ind. Loops}}_{\lambda^2}.$$

Our results at late times, given by (3.8), (3.12), and (3.16), exactly match those computed in [117] and [118] in the p-representation [119] within the Schwinger-Keldysh, "in-in", or "closed-time-path" diagrammatic technique [107, 108]. The full expressions for the one-loop (2.25) and two-loop contributions, (3.7), (3.11), and (3.15) differ from the findings of [117, 118]. This is due to the initial choice of the so-called Bunch-Davies, de Sitter-invariant vacuum state [142] for a massive scalar field used there, as well as the specific momentum cut-off, $\underline{0} < k < He^{Ht}$. However, by virtue of our "building block" (2.22), the obtained results tend to the equilibrium, de Sitter-invariant state at late times, as seen in (2.34), and turn out to be in agreement with [117, 118]. In addition to the agreement in the final results, the structural form of the expressions is also consistent. We will demonstrate it a bit later.

At the three-loop level, we present the results for each type of diagrams in Appendix B. All types of diagrams are listed in Figure B.1. We follow the same assignments, as presented below Figure 3.2, to sort out all the integral structures in (2.31). The full expressions for each diagram type are (B.1), (B.5), (B.9), (B.13), (B.17), (B.21), (B.25), and (B.29), and their sum, obviously, is (2.33). At late times, they appear to be de Sitter-invariant; see (B.2), (B.6), (B.10), (B.14), (B.18), (B.22), (B.26), and (B.30). In the massless limit, they are equal to (B.3), (B.7), (B.11), (B.15), (B.19), (B.23), (B.27), and (B.31).

In the four-point correlation function case, we follow the same assignments in between. There is also another relation between the notion of a loop and λ order. The connected contributions from the λ−linear order in (2.45):

$$\left\langle\phi(t_1)\phi(t_2)\phi(t_3)\phi(t_4)\right\rangle^{\text{con}}_{\lambda}=-\frac{2\lambda}{H}\mathrm{e}^{-\frac{m^2t_1}{3H}}\int_0^{t_1}dt'\mathrm{e}^{\frac{m^2t'}{3H}}\left\langle\tilde\phi(t')\tilde\phi(t_2)\right\rangle\left\langle\tilde\phi(t')\tilde\phi(t_3)\right\rangle\left\langle\tilde\phi(t')\tilde\phi(t_4)\right\rangle$$

$$+\left(t_1\leftrightarrow t_2\right)+\left(t_1\leftrightarrow t_3\right)+\left(t_1\leftrightarrow t_4\right). \tag{3.19}$$

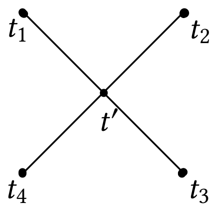


Figure 3.3: Vertex diagram at tree level contributing to the four-point correlation function.

They belong to the single vertex diagram presented in Figure 3.3. The result of that contribution is (2.47) and in the massless limit appears to be (2.48). In the equal-time case, our obtained expression reduces to (2.49), and its massless limit is in agreement with the corresponding result from [122, 144]; see (2.50). The connected part at order λ^2 in (2.58) corresponds to the one-loop contribution. It is equal to the sum of two types of diagrams; see Figure 3.4 below. The full one-loop answer is (2.60), and its massless limit is (2.61). In the late-time limit, one extracts the equilibrium values from (2.60) or (2.62).

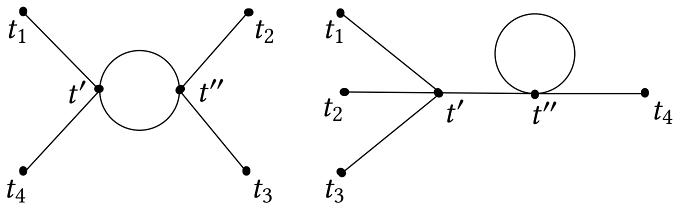


Figure 3.4: One-loop diagrams contributing to the four-point correlation function.

3.1 "In-in" formalism

The Schwinger-Keldysh, or "in-in" diagrammatic technique [107, 108], is one of the most efficient formalisms for computing correlation functions in cosmology. In contrast to the typical scattering computations, where the system evolves from $t = -\infty$ ("in" state) to $t = +\infty$ ("out" state) and both states are specified, in cosmology we are interested in evolving the quantum system up to some finite moment of time. One specifies only the initial state, and both the "in" state, $|\text{in}\rangle$, and its Hermitian conjugate, $\langle\text{in}|$, evolve from $t = -\infty$ to a certain time moment, and the field operators are sandwiched between unitary time-evolution operators. Therefore, within the Schwinger-Keldysh or "in-in" formalism, the two-point correlation function is

$$\Big\langle \phi(t_1,\vec{x}_1)\,\phi(t_2,\vec{x}_2)\Big\rangle = \Big\langle \text{in}\,\Big|\, {U_I}^\dagger(t_1,-\infty)\,\phi_I(t_1,\vec{x}_1)\,U_I(t_1,-\infty)\times \\ {U_I}^\dagger(t_2,-\infty)\,\phi_I\,(t_2,\vec{x}_2)U_I(t_2,-\infty)\Big|\,\text{in}\Big\rangle. \tag{3.20}$$

Here, the subscript I denotes the interaction representation, $\phi_I(t,\vec{x})$ is the field in the interaction picture, and its time evolution is governed by the free massive theory Hamiltonian $\mathcal{H}_0$, and $|\text{in}\rangle$ is the vacuum state of the free massive theory. The unitary time-evolution operator $U_I(t,-\infty)$ in (3.20) for the model under consideration (1.1) is defined as follows:

$$U_I(t,-\infty) = \mathcal{T}\exp\Bigg(-i\int_{-\infty}^{t} dt'\,\underbrace{e^{3Ht'}\int d^3\vec{x}'\,\frac{\lambda}{4}\,\phi_I^4(t',\vec{x}')}_{:=\mathcal{H}_I\left(\phi_I(t')\right)}\Bigg), \tag{3.21}$$

${U_I}^\dagger(t,-\infty)$ in (3.20) is its Hermitian conjugate, and $U_I{U_I}^\dagger = {U_I}^\dagger U_I = \mathbb{1}$. We denote by $\mathcal{T}$ in $U_I(t,-\infty)$ and $\bar{\mathcal{T}}$ in ${U_I}^\dagger(t,-\infty)$ the time- and anti-time-ordering respectively.

To provide some details for comparison with our findings in this chapter, we refer to the results of [117]. We represent operators as $U_I(t_1, -\infty) = U_I{}^{\dagger}(+\infty, t_1)\, U_I(+\infty, -\infty)$ and $U_I{}^{\dagger}(t_2, -\infty) = U_I{}^{\dagger}(+\infty, -\infty)\, U_I(+\infty, t_2)$ and substitute their product in (3.20), i.e., $U_I(t_1, -\infty)\, U_I^{\dagger}(t_2, -\infty) = U_I^{\dagger}(+\infty, t_1)\, U_I(+\infty, t_2)$, resulting in

$$\begin{aligned}\big\langle \phi(t_1, \vec{x}_1)\, \phi(t_2, \vec{x}_2) \big\rangle &= \big\langle \mathrm{in} \big| U_I{}^{\dagger}(t_1, -\infty)\, \phi_I(t_1, \vec{x}_1)\, U_I{}^{\dagger}(+\infty, t_1) \times \\ &\qquad U_I(+\infty, t_2)\, \phi_I\,(t_2, \vec{x}_2)\, U_I(t_2, -\infty) \big| \mathrm{in} \big\rangle \qquad (3.22) \\ &= \big\langle \mathrm{in} \big| \bar{\mathcal{T}} \Big(\phi_I(t_1, \vec{x}_1)\, \mathrm{e}^{\, i \int_{-\infty}^{+\infty} dt'\, \mathcal{H}_I(t')} \Big)\, \mathcal{T} \Big(\phi_I(t_2, \vec{x}_2)\, \mathrm{e}^{-i \int_{-\infty}^{+\infty} dt'\, \mathcal{H}_I(t')} \Big) \big| \mathrm{in} \big\rangle .\end{aligned}$$

One starts in (3.22) with some initial state at $t_0 = -\infty$ in the last line of (3.22), evolves forward through the time moment t_2, continues to $+\infty$, then backward to t_1 and to $-\infty$. That allows one to describe the non-equilibrium dynamics, as well as to express (3.22) as the time-ordered products. Thus, one introduces the following notations for the forward-evolving field $\phi_I^+(t, \vec{x})$ and for the backward-evolving field $\phi_I^-(t, \vec{x})$, and the two-point correlation function (3.22) takes the form

$$\big\langle \phi(t_1, \vec{x}_1) \phi(t_2, \vec{x}_2) \big\rangle = \big\langle \mathrm{in} \big| \mathcal{T} \Big(\phi_I^-(t_1, \vec{x}_1) \phi_I^+(t_2, \vec{x}_2)\, \mathrm{e}^{-i \int_{-\infty}^{+\infty} dt' \left(\mathcal{H}_I(\phi_I^+(t')) - \mathcal{H}_I(\phi_I^-(t')) \right)} \Big) \big| \mathrm{in} \big\rangle . \qquad (3.23)$$

All time-ordered products as a result of the Wick theorem fall into four possible combinations of field contractions, i.e., four propagator types[2]:

$$\big\langle \mathrm{in} \big| \mathcal{T} \big(\phi_I^+(t_1, \vec{x}_1) \phi_I^-(t_2, \vec{x}_2) \big) \big| \mathrm{in} \big\rangle = \big\langle \mathrm{in} \big|\, \phi_I(t_2, \vec{x}_2)\, \phi_I(t_1, \vec{x}_1) \big| \mathrm{in} \big\rangle := G^{<}\big(t_1, \vec{x}_1; t_2, \vec{x}_2\big); \qquad (3.24)$$

$$\big\langle \mathrm{in} \big| \mathcal{T} \big(\phi_I^-(t_1, \vec{x}_1) \phi_I^+(t_2, \vec{x}_2) \big) \big| \mathrm{in} \big\rangle = \big\langle \mathrm{in} \big|\, \phi_I(t_1, \vec{x}_1)\, \phi_I(t_2, \vec{x}_2) \big| \mathrm{in} \big\rangle := G^{>}\big(t_1, \vec{x}_1; t_2, \vec{x}_2\big); \qquad (3.25)$$

$$\begin{aligned}\big\langle \mathrm{in} \big| \mathcal{T} \big(\phi_I^+(t_1, \vec{x}_1)\, \phi_I^+(t_2, \vec{x}_2) \big) \big| \mathrm{in} \big\rangle &= \Theta(t_1 - t_2) \big\langle \mathrm{in} \big|\, \phi_I(t_1, \vec{x}_1)\, \phi_I(t_2, \vec{x}_2) \big| \mathrm{in} \big\rangle \qquad (3.26) \\ &\quad + \Theta(t_2 - t_1) \big\langle \mathrm{in} \big|\, \phi_I(t_2, \vec{x}_2)\, \phi_I(t_1, \vec{x}_1) \big| \mathrm{in} \big\rangle \\ &:= \Theta(t_1 - t_2) G^{>}\big(t_1, \vec{x}_1; t_2, \vec{x}_2\big) + \Theta(t_2 - t_1) G^{<}\big(t_1, \vec{x}_1; t_2, \vec{x}_2\big);\end{aligned}$$

$$\begin{aligned}\big\langle \mathrm{in} \big| \mathcal{T} \big(\phi_I^-(t_1, \vec{x}_1) \phi_I^-(t_2, \vec{x}_2) \big) \big| \mathrm{in} \big\rangle &= \Theta(t_2 - t_1) \big\langle \mathrm{in} \big|\, \phi_I(t_1, \vec{x}_1)\, \phi_I(t_2, \vec{x}_2) \big| \mathrm{in} \big\rangle \qquad (3.27) \\ &\quad + \Theta(t_1 - t_2) \big\langle \mathrm{in} \big|\, \phi_I(t_2, \vec{x}_2)\, \phi_I(t_1, \vec{x}_1) \big| \mathrm{in} \big\rangle \\ &:= \Theta(t_2 - t_1) G^{>}\big(t_1, \vec{x}_1; t_2, \vec{x}_2\big) + \Theta(t_1 - t_2) G^{<}\big(t_1, \vec{x}_1; t_2, \vec{x}_2\big).\end{aligned}$$

Here, $G^{\gtrless}\big(t_1, \vec{x}_1; t_2, \vec{x}_2\big)$ are the free Wightman functions associated with the two-point functions of the massive field evaluated in the initial vacuum state:

$$G^{\gtrless}\big(t_1, \vec{x}_1; t_2, \vec{x}_2\big) = \int \frac{d^3\vec{k}}{(2\pi)^3}\, \mathrm{e}^{i\vec{k}(\vec{x}_1 - \vec{x}_2)}\, G_k^{\gtrless}\big(t_1, t_2\big), \qquad (3.28)$$

and their momentum representation here is expressed through the mode functions as

2 Note that four propagator types are not independent: (3.24)+(3.25)=(3.26)+(3.27).

follows:

$$G_k^>(t_1,t_2) = u_k(t_1)u_k^*(t_2) \quad \text{and} \quad G_k^<(t_1,t_2) = u_k^*(t_1)u_k(t_2). \tag{3.29}$$

To perform the computation using the Schwinger-Keldysh, or "in-in", formalism, one expands the operators in (3.23) as a series in λ and sequentially carries out field contractions for the expectation values of the field operators at specific time moments for the given initial state.

At the λ–linear order, the correction to the two-point correlation function is

$$\left\langle \phi(t_1,\vec{x}_1)\,\phi(t_2,\vec{x}_2)\right\rangle_\lambda = -\frac{i\lambda}{4}\int_{-\infty}^{+\infty} dt' e^{3Ht'}\int d^3\vec{x}'\,\langle \text{in}|\,\mathcal{T}\Big(\phi_I^-(t_1,\vec{x}_1)\,\phi_I^+(t_2,\vec{x}_2)\times \\ \times\left(\left(\phi_I^+(t',\vec{x}')\right)^4 - \left(\phi_I^-(t',\vec{x}')\right)^4\right)\Big)\,|\text{in}\rangle. \tag{3.30}$$

Thus, in addition to four types of propagators (3.24)–(3.27), we have two types of vertices with two different signs:

$$\pm\frac{i\lambda}{4}\int dt'\,e^{3Ht'}\int d^3\vec{x}'. \tag{3.31}$$

With appropriate field contractions and with the help of (3.24)+(3.25)=(3.26)+(3.27), we get for the connected diagram with the coinciding spatial points $\vec{x}_1 = \vec{x}_2 := \vec{x}$:

$$\left\langle \phi(t_1,\vec{x})\phi(t_2,\vec{x})\right\rangle_\lambda := \left\langle \phi(t_1)\phi(t_2)\right\rangle_\lambda = -3i\lambda\int_{-\infty}^{+\infty} dt' e^{3Ht'}\int d^3\vec{x}'\times \tag{3.32}$$

$$\times\Big(\langle \text{in}|\,\mathcal{T}\big(\phi_I^-(t_1,\vec{x})\phi_I^+(t',\vec{x}')\big)\mathcal{T}\big(\phi_I^+(t_2,\vec{x})\phi_I^+(t',\vec{x}')\big)\,\mathcal{T}\big(\phi_I^+(t',\vec{x}')\phi_I^+(t',\vec{x}')\big)|\text{in}\rangle \\ -\langle \text{in}|\,\mathcal{T}\big(\phi_I^-(t_1,\vec{x})\phi_I^-(t',\vec{x}')\big)\mathcal{T}\big(\phi_I^+(t_2,\vec{x})\phi_I^-(t',\vec{x}')\big)\mathcal{T}\big(\phi_I^-(t',\vec{x}')\phi_I^-(t',\vec{x}')\big)|\,\text{in}\rangle\Big)$$

$$= -3\lambda\int_{-\infty}^{+\infty} dt' e^{3Ht'}\int d^3\vec{x}'\,\Big(G_R(t_1,\vec{x};t',\vec{x}')\,G^<(t_2,\vec{x};t',\vec{x}') \\ + G_R(t_2,\vec{x};t',\vec{x}')\,G^>(t_1,\vec{x};t',\vec{x}')\Big)\,G^>(t',\vec{x}';t',\vec{x}'). \tag{3.33}$$

The disconnected contributions canceled each other. Here, the retarded Green's function is defined as $G_R(t_1,\vec{x}_1;t',\vec{x}') = i\Theta(t_1-t')\langle \text{in}|[\phi_I(t_1,\vec{x}_1),\phi_I(t',\vec{x}')]|\text{in}\rangle$, and it is the difference of (3.26)–(3.24) or (3.25)–(3.27). By substituting the long-wavelength part of the retarded Green's function (2.13) and truncating and dressing it with the mass, one gets

$$\left\langle \phi(t_1,\vec{x})\phi(t_2,\vec{x})\right\rangle_\lambda := \left\langle \phi(t_1)\phi(t_2)\right\rangle_\lambda = -\frac{\lambda}{H}\,e^{-\frac{m^2 t_1}{3H}}\int_0^{t_1} dt' e^{\frac{m^2 t'}{3H}}\,G^<(t_2,\vec{x};t',\vec{x})G^>(t',\vec{x};t',\vec{x})$$

$$- \frac{\lambda}{H}\, e^{-\frac{m^2 t_2}{3H}} \int\limits_0^{t_2} dt' e^{\frac{m^2 t'}{3H}} G^{>}(t_1,\vec{x};t',\vec{x}) G^{>}(t',\vec{x};t',\vec{x}). \tag{3.34}$$

Now, it is clear that the result is precisely the sum of two integral structures (2.24) or (3.1).

In [117], the authors have performed all calculations with the choice of the Bunch-Davies vacuum state [142]. They replaced the cosmic time t with the so-called conformal time, which is related as $\eta = -e^{-Ht}/H$. In the analogous expression to (3.33), the authors of [117] performed the integration by expanding Wightman's functions at small physical momenta. At the λ-linear order, the obtained result is

$$\left\langle \phi(\eta_1,\vec{x})\,\phi(\eta_2,\vec{x})\right\rangle_\lambda = -\frac{27\,\lambda H^8}{64\,\pi^4 m^6}\left(\frac{\eta_1}{\eta_2}\right)^{\frac{m^2}{3H}}\left(1-\frac{m^2}{3H}\ln\frac{\eta_1}{\eta_2}\right). \tag{3.35}$$

By replacing back the cosmic time instead of the conformal one, we get precisely (2.25) at late times; see also (2.34). The same reasoning works for each two-loop diagram as well. Our late-time contributions (3.8), (3.12), and (3.16) agree with the results of [117].

In the four-point correlation function case, the corresponding structures will appear in the Schwinger-Keldysh formalism through appropriate contractions of time-ordered products.

3.2 Stochastic diagrammatic technique

A further comparison can be made with the results of [81] obtained within the Starobinsky stochastic approach by Garbrecht, Rigopoulos, and Zhu. This formalism will be discussed in detail in the next chapter. Perhaps one can overcome this shortcoming with a less-than-sequential reading.

The authors of [81] introduced a functional representation of equations within the Starobinsky stochastic approach; see (4.2) and (4.5) below. In the proposed routine, the expectation value of an operator is given by means of distribution averages. At the late-time limit, they have computed the topologically distinct diagrams for the two-point correlation function at equal times, employing both the stochastic approach and the "in-in" formalism. Contributions for each type of diagram in the present chapter and series at the equilibrium (2.37) up to λ^2 match those derived in [81].

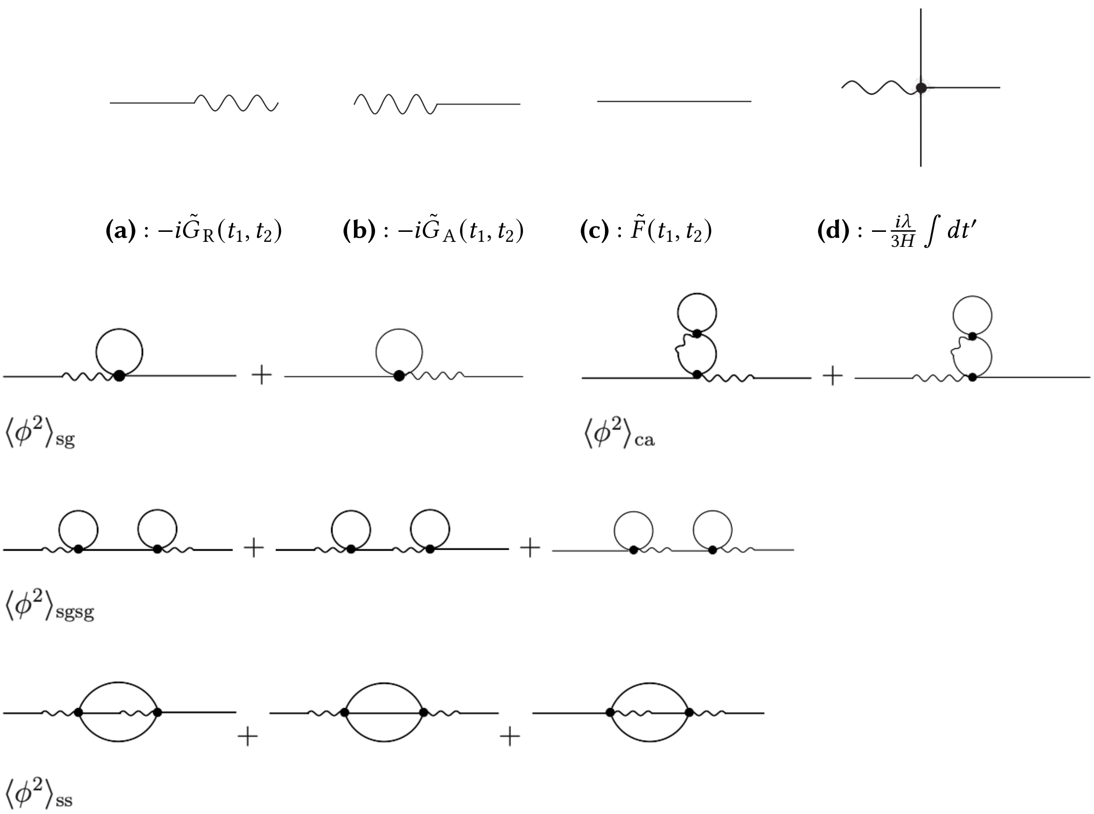
Types of integrals that come up within the Yang-Feldman-type equation (3.2)– (3.6) match the diagrammatic rules in [81, 82]; see Figure 3.5. The integration represents the vertex with one wiggle and three solid lines, and the two-point correlation functions and the retarded/advanced Green's functions (which appear as exponential factors involving the time difference) represent the propagators. Note that diagrammatic elements, shown in Figure 3.5(a)–(d), are identical to the "in-in" formalism, expressed in the so-called Keldysh basis, but with the three-wiggle-line vertex omitted. The retarded and advanced Green's functions for the operator $\partial_t + \frac{m^2}{3H}$ have the following form [81]:

$$\tilde{G}_{\mathrm{R}}(t_1, t_2) = \tilde{G}_{\mathrm{A}}(t_2, t_1) = \mathrm{e}^{-\frac{m^2}{3H}(t_1 - t_2)} \Theta(t_1 - t_2), \tag{3.36}$$

and the two-point correlation function is [81]

$$\tilde{F}(t_1, t_2) := \frac{H^3}{4\pi^2} \int_0^\infty dt' \tilde{G}_{\mathrm{R}}(t_1, t') \tilde{G}_{\mathrm{A}}(t', t_2) = \frac{3H^4}{8\pi^2 m^2} \left(\mathrm{e}^{-\frac{m^2}{3H}(t_1 - t_2)} - \mathrm{e}^{-\frac{m^2}{3H}(t_1 + t_2)} \right). \tag{3.37}$$

The retarded Green's functions (3.36) match the additional exponential factor that appeared in the massive Yang-Feldman-type equation (2.16) in the previous chapter, and the two-point correlation function (3.37) precisely matches (2.22). The number of different diagram types in [81] is the same as the number of integral structures within the massive Yang-Feldman-type equation (2.16). One establishes the one-to-one correspondence between the diagrams in [81] and our integral structures (3.1) and (3.2)–(3.6); see Figure 3.5(e) below.



(a) : $-i\tilde{G}_{\mathrm{R}}(t_1, t_2)$ **(b)** : $-i\tilde{G}_{\mathrm{A}}(t_1, t_2)$ **(c)** : $\tilde{F}(t_1, t_2)$ **(d)** : $-\frac{i\lambda}{3H}\int dt'$

(e) : The stochastic diagrams contributing to $\langle \phi(t_1)\phi(t_2) \rangle$ up to λ^2 order from [81]. Labels refer to the diagram topology: Seagull, Cactus, Double Seagull, and Sunset; see (3.38) for the link with our notations.

Figure 3.5: The stochastic diagrammatic elements and the corresponding diagrams from [81].

At the λ–linear order, we have two types of diagrams: the wiggle line corresponds to an integration, and exponents in front of the integral structure (3.1) correspond to the retarded Green's function; see $\langle\phi^2\rangle_{\rm sg}$ in Figure 3.5(e) with assignments (3.36) and (3.37). The same reasoning applies to the λ^2 or two-loop order: after a straightforward calculation, one can find the correspondence between them. The authors of [81] assign the following factors: $-16\lambda\pi^2/H^4$ and $3H^4/(8\pi^2 m^2)$ for each vertex and propagator, respectively, and divide by an appropriate symmetry factor for the diagram. The final equilibrium contributions for each type of diagram, according to [81], appear to be equal to

$$\begin{aligned}
\left\langle\phi^2(t)\right\rangle_{\lambda}^{\text{late-time}} &:= \left\langle\phi^2\right\rangle_{\rm sg} = -\frac{27\,\lambda H^8}{64\,\pi^4 m^6}\,;\\
\left\langle\phi^2(t)\right\rangle_{\text{late-time}}^{\text{Snowman}} &:= \left\langle\phi^2\right\rangle_{\rm ca} \;=\; \frac{243\,\lambda^2 H^{12}}{512\,\pi^6 m^{10}}\,;\\
\left\langle\phi^2(t)\right\rangle_{\text{late-time}}^{\text{Sunset}} &:= \left\langle\phi^2\right\rangle_{\rm ss} = \frac{81\,\lambda^2 H^{12}}{256\,\pi^6 m^{10}}\,;\\
\left\langle\phi^2(t)\right\rangle_{\text{late-time}}^{\text{Ind. Loops}} &:= \left\langle\phi^2\right\rangle_{\rm sgsg} = \frac{243\,\lambda^2 H^{12}}{512\,\pi^6 m^{10}}\,.
\end{aligned} \tag{3.38}$$

They are precisely in agreement with our limiting values of (2.34) or (2.37) and the sum of (3.8), (3.12), and (3.16) at equal times from above; see also the late-time limit of (2.25), (3.10), (3.14), and (3.18).

Hence, instead of developing the diagrammatic technique, one can discern the diagram's topology from the integral structures arising in the iterated Yang-Feldman-type equation. The processing of the Yang-Feldman-type equation (2.16) is much more economical while providing an extension to the non-equilibrium regime in the coordinate representation. At the same time, in the limiting cases, the results are fully in agreement with those obtained from diagrammatic techniques.

4 Comparison with the Starobinsky-Yokoyama stochastic approach

Since our "building block", the obtained massive correlation function (2.22), coincides with the Ornstein-Uhlenbeck mean-reverting stochastic process, results of the previous Chapter 2 must be in agreement in the late-time limit with those from the Starobinsky-Yokoyama stochastic approach [52, 53], which operates with a near-equilibrium state. In this chapter, we present the basics of this approach and deduce the recurrent expression for arbitrary 2n-point correlation functions (4.26) with (4.25) in the massive interacting case. In addition, we establish the perturbative agreement with our outcomes from Chapter 2, following the approach proposed in [45] using the Fokker-Planck equation.

4.1 Starobinsky-Yokoyama stochastic approach

The Starobinsky-Yokoyama stochastic approach [52, 53] was the first non-perturbative development for the vacuum expectation value of fluctuations of the coarse-grained, long-wavelength scalar field in de Sitter space. Such an approach identifies the long-wavelength part of the quantum scalar field with the classical stochastic field with a probability distribution function that satisfies the Fokker-Planck equation. The resummed non-perturbative quantities are free of secular growth, which occurs for the two-point correlation function of the massless scalar field already at the tree level [37, 38]; see (2.19) in our Chapter 2.

According to Starobinsky's stochastic approach, one decomposes the operator of the quantum scalar field $\phi(t,\vec{x})$ in the Heisenberg representation into the super-Hubble or long-wavelength, $k < He^{Ht}$, and sub-Hubble or short-wavelength, $k > He^{Ht}$, parts using a window function, which we choose to be the dynamical Heaviside step function, as follows:

$$\begin{aligned}\phi(t,\vec{x}) &= \phi^{\text{s-w}}(t,\vec{x}) + \phi^{\text{l-w}}(t,\vec{x}) \\ &:= \int \frac{d^3\vec{k}}{(2\pi)^{3/2}}\, \Theta\Big(|\vec{k}| - H\,\mathrm{e}^{Ht}\Big)\Big(u_k(t)\,\mathrm{e}^{i\vec{k}\vec{x}}\,\hat{a}_{\vec{k}} + u_k^*(t)\,\mathrm{e}^{-i\vec{k}\vec{x}}\,\hat{a}^\dagger_{\vec{k}}\Big) \\ &\quad + \int \frac{d^3\vec{k}}{(2\pi)^{3/2}}\, \Theta\Big(H\,\mathrm{e}^{Ht} - |\vec{k}|\Big)\Big(u_k(t)\,\mathrm{e}^{i\vec{k}\vec{x}}\,\hat{a}_{\vec{k}} + u_k^*(t)\,\mathrm{e}^{-i\vec{k}\vec{x}}\,\hat{a}^\dagger_{\vec{k}}\Big).\end{aligned} \tag{4.1}$$

In de Sitter space the splitting into the 'long' and 'short' modes does not possess an absolute physical meaning; rather, it changes over time.

The coarse-grained long-wavelength part satisfies the equation of motion $\Box\phi = -V'_\phi(\phi)$, which for the slowly varying $\phi^{\text{l-w}}(t,\vec{x})$ takes the local Langevin-like form:

$$\dot{\phi}^{\text{l-w}}(t,\vec{x}) = -\frac{1}{3H}\,V'_\phi\big(\phi^{\text{l-w}}(t,\vec{x})\big) + \eta(t,\vec{x}), \tag{4.2}$$

where the noise term $\eta(t,\vec{x})$ represents short-wavelength modes that continually shift into the long-wavelength ones. From (4.1) it is given by

$$\eta(t,\vec{x}) = H^2 e^{Ht} \int \frac{d^3\vec{k}}{(2\pi)^{3/2}}\, \delta\left(|\vec{k}| - H\,e^{Ht}\right) \left(u_k(t)\, e^{i\vec{k}\vec{x}}\, \hat{a}_{\vec{k}} + u_k^*(t)\, e^{-i\vec{k}\vec{x}}\, \hat{a}^\dagger_{\vec{k}}\right) \tag{4.3}$$

with the white noise properties [52, 53]:

$$\left\langle \eta(t,\vec{x}) \right\rangle = 0 \qquad \text{and} \qquad \left\langle \eta(t_1,\vec{x}_1)\, \eta(t_2,\vec{x}_2) \right\rangle = \frac{H^3}{4\pi^2}\, \frac{\sin\left(H\,e^{Ht}\,|\vec{x}_1 - \vec{x}_2|\right)}{H\,e^{Ht}\,|\vec{x}_1 - \vec{x}_2|}\, \delta(t_1 - t_2). \tag{4.4}$$

Clearly, at the coinciding spatial points, $\vec{x}_1 = \vec{x}_2 := \vec{x}$, the expression above takes the form:

$$\left\langle \eta(t_1,\vec{x})\, \eta(t_2,\vec{x}) \right\rangle := \left\langle \eta(t_1)\, \eta(t_2) \right\rangle = \frac{H^3}{4\pi^2}\, \delta(t_1 - t_2). \tag{4.5}$$

To obtain correlation function (4.4), one employs the canonical commutation relations (2.8), converts the integrand to polar coordinates in momentum space, and performs the integral.

The local Langevin-like equation (4.2) is a first-order differential equation. For each noise realisation $\eta(t)$, one determines uniquely $\phi^{\text{l-w}}(t)$ if the value at the initial time $\phi^{\text{l-w}}(t_0)$ is given. Moreover, since the fluctuating or noise term is delta-correlated, its values at different times are statistically independent. Thus, the solution to (4.2) has the Markov property: the noise-term values at previous times, $t < t_0$, do not influence the conditional probabilities at times $t > t_0$. We will extensively exploit Markov's property in Chapter 5. Consequently, the long-wavelength part of the quantum scalar field $\phi^{\text{l-w}}(t,\vec{x})$ can be treated as a classical stochastic field $\varphi(t,\vec{x})$, which commutes with itself $\left[\varphi(t,\vec{x}), \varphi(t',\vec{x}')\right] = 0$, with a probability density $\mathcal{P}\left[\varphi(t,\vec{x})\right]$ that satisfies the Fokker-Planck equation [52, 53]. The short-wavelength part retains the quantum character, while a classical noise assumed to be white represents its effect. The long-wavelength $\phi^{\text{l-w}}(t,\vec{x})$ part's vacuum expectation value is equal to the expectation value of the classical stochastic field $\varphi(t,\vec{x})$ at each order in perturbation theory in the leading logarithm approximation in cosmic time t.

To find the corresponding Fokker-Planck or Einstein-Smoluchowski equation for the local Langevin-like equation (4.2), one derives[3] the coefficients $\alpha^{(\mathrm{k}=1,2)}(\varphi,t)$ that enter the Kramers–Moyal expansion [147, 148] of the master equation [102, 103]:

$$\begin{gathered}
\frac{\partial}{\partial t}\mathcal{P}\left[\varphi\,(t\,,\vec{x})\right] = \sum_{\mathrm{k}=1}^{\infty} \frac{(-1)^{\mathrm{k}}}{\mathrm{k}!}\, \frac{\partial^{\mathrm{k}}}{\partial\varphi^{\mathrm{k}}} \left(\alpha^{(\mathrm{k})}(\varphi,t)\, \mathcal{P}\left[\varphi\,(t\,,\vec{x})\right]\right), \\
\alpha^{(\mathrm{k}=1,2)}(\varphi,t) = \lim_{\Delta t\to 0} \frac{\left\langle \left(\varphi(t+\Delta t) - \varphi(t)\right)^{\mathrm{k}} \right\rangle}{\Delta t}, \qquad \alpha^{(\mathrm{k}\geq 3)}(\varphi,t) = 0.
\end{gathered} \tag{4.6}$$

3 There are limits to the validity of that approach. For the problem under consideration, it does work since the transition probability enjoys the specific form, and the terms higher than second do indeed vanish in the Kramers-Moyal expansion; see the critical remarks in [102, 146].

By replacing $\phi^{\text{l-w}}(t,\vec{x})$ with $\varphi(t,\vec{x})$ and omitting the $\vec{x}$ argument in (4.2), we rewrite it in the integral equation form:

$$\varphi(t+\Delta t)-\varphi(t)=\int_t^{t+\Delta t}dt'\left(-\frac{1}{3H}V'_\varphi[\,\varphi\,(t'),t']\right)+\int_t^{t+\Delta t}dt'\eta(t'). \tag{4.7}$$

Further, we expand and iterate the above integrand in the r.h.s. (4.7) as follows:

$$\int_t^{t+\Delta t}dt'\left(-\frac{1}{3H}V'_\varphi[\varphi(t),t']-\frac{1}{3H}V''_{\varphi\varphi}[\varphi(t),t']\left(\varphi(t')-\varphi(t)\right)+\ldots\right)+\int_t^{t+\Delta t}dt'\eta(t')= \tag{4.8}$$
$$-\frac{1}{3H}\int_t^{t+\Delta t}dt'\left(V'_\varphi[\varphi(t),t']+V''_{\varphi\varphi}[\varphi(t),t']\int_t^{t'}dt''\left(-\frac{1}{3H}V'_\varphi[\varphi(t),t'']+\ldots\right)\right)+\int_t^{t+\Delta t}dt'\eta(t').$$

By averaging (4.8) and taking the limit (4.6), one gets

$$\alpha^{(1)}(\varphi,t)=-\frac{1}{3H}V'_\varphi[\,\varphi\,(t),t\,]; \tag{4.9}$$
$$\alpha^{(2)}(\varphi,t)=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_t^{t+\Delta t}dt'\int_t^{t+\Delta t}dt''\langle\eta(t')\,\eta(t'')\rangle\overset{(4.5)}{=}\frac{H^3}{4\pi^2}; \tag{4.10}$$

Thus, the corresponding Fokker-Planck or Einstein-Smoluchowski equation for (4.2) is

$$\frac{\partial}{\partial t}\mathcal{P}[\varphi\,(t\,,\vec{x})]=\frac{1}{3H}\frac{\partial}{\partial\varphi}\left(V'_\varphi(\varphi\,(t\,,\vec{x}))\,\mathcal{P}[\varphi\,(t\,,\vec{x})]\right)+\frac{H^3}{8\pi^2}\frac{\partial^2}{\partial\varphi^2}\mathcal{P}[\varphi\,(t\,,\vec{x})]. \tag{4.11}$$

Any solution to the Fokker-Planck equation (4.11) tends to the stationary one at late times [53]:

$$\mathcal{P}[\varphi(t,\vec{x})]\xrightarrow[\text{times}]{\text{late}}\mathcal{P}_{\text{st}}[\varphi]=\frac{1}{\mathcal{N}}\,\mathrm{e}^{-\frac{8\pi^2}{3H^4}V(\varphi)}, \tag{4.12}$$

where $\mathcal{N}$ is defined to ensure the normalization condition, namely, $\int_{-\infty}^{+\infty}d\varphi\,\mathcal{P}_{\text{st}}[\varphi]=1$. In the stochastic language, a de Sitter invariant state for a free massless minimally coupled scalar field does not exist, as we mentioned in the introduction, due to the nonexistence of any static solution to (4.11) with the absent potential in the so-called drift term.

In the original work by Starobinsky and Yokoyama [53], for the massless case with a quartic self-interaction, the stationary solution is

$$\mathcal{P}_{\text{st}}[\varphi]=\frac{\left(32\,\lambda\pi^2\right)^{1/4}}{3^{1/4}H\,\Gamma\!\left(\frac{1}{4}\right)}\,\mathrm{e}^{-\frac{2\pi^2\lambda}{3H^4}\varphi^4} \tag{4.13}$$

and the corresponding expectation values are

$$\left\langle\varphi^{2n}\right\rangle^{\text{late-time}}_{\text{non-pert}} = \frac{\left(32\,\lambda\pi^2\right)^{1/4}}{3^{1/4}H\,\Gamma\left(\frac{1}{4}\right)}\int_{-\infty}^{+\infty} d\varphi\,\varphi^{2n}\,e^{-\frac{2\pi^2\lambda}{3H^4}\varphi^4} = \frac{\Gamma\left(\frac{n}{2}+\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\left(\frac{3H^4}{2\pi^2\lambda}\right)^{n/2}. \tag{4.14}$$

One also computes expectation values of the stochastic variable in the massive case by expanding in the small self-interaction coupling constant λ regime using (4.12), as follows:

$$\left\langle\varphi^{2}\right\rangle^{\text{late-time}}_{\text{perturb}} = \frac{\int_{-\infty}^{+\infty} d\varphi\varphi^2\,\mathcal{P}_{\text{st}}[\varphi]}{\int_{-\infty}^{+\infty} d\varphi\mathcal{P}_{\text{st}}[\varphi]} \approx \frac{3H^4}{8\pi^2m^2} - \frac{27\lambda H^8}{64\pi^4m^6} + \frac{81\lambda^2H^{12}}{64\pi^6m^{10}} - \frac{24057\lambda^3H^{16}}{4096\,\pi^8m^{14}} + \ldots \tag{4.15}$$

$$\left\langle\varphi^{4}\right\rangle^{\text{late-time}}_{\text{perturb}} = \frac{\int_{-\infty}^{+\infty} d\varphi\,\varphi^4\,\mathcal{P}_{\text{st}}[\varphi]}{\int_{-\infty}^{+\infty} d\varphi\,\mathcal{P}_{\text{st}}[\varphi]} \approx \frac{27H^8}{64\,\pi^4m^4} - \frac{81\lambda H^{12}}{64\,\pi^6m^8} + \frac{24057\lambda^2H^{16}}{4096\,\pi^8m^{12}} + \ldots \tag{4.16}$$

which exactly coincide with our equal-time series at the late-time limit, (2.37) and (2.73) correspondingly, from the previous Chapter 2.

4.2 Recurrence relations for the massive case with quartic self-interaction

Let us derive the recurrence expressions for any 2n-point correlation function within the Starobinsky-Yokoyama stochastic approach for our massive and self-interacting theory (1.1).

The explicit form of the normalization constant was found from its definition [65, 117]:

$$\mathcal{N} = \int_{-\infty}^{+\infty} d\varphi\,e^{-\frac{8\pi^2}{3H^4}\left(\frac{m^2\varphi^2}{2}+\frac{\lambda\varphi^4}{2}\right)} = \frac{m}{\sqrt{2\lambda}}\exp\left(\frac{\pi^2m^4}{3\lambda H^4}\right)\mathcal{K}_{1/4}\left(\frac{\pi^2m^4}{3\lambda H^4}\right), \tag{4.17}$$

here $\mathcal{K}_{1/4}\left(\frac{\pi^2m^4}{3\lambda H^4}\right)$ is the modified Bessel function of the second kind.

Therefore, the expectation value $\left\langle\varphi^{2}\right\rangle^{\text{late-time}}_{\text{non-pert}}$ is

$$\left\langle\varphi^{2}\right\rangle^{\text{late-time}}_{\text{non-pert}} = \frac{1}{\mathcal{N}}\int_{-\infty}^{+\infty} d\varphi\,\varphi^2\,e^{-\frac{8\pi^2}{3H^4}V(\varphi)} = \frac{m^2}{2\lambda}\left(\frac{\mathcal{K}_{3/4}(z)}{\mathcal{K}_{1/4}(z)} - 1\right), \quad \text{where} \quad z := \frac{\pi^2m^4}{3\lambda H^4}. \tag{4.18}$$

That result can be extracted by an appropriate differentiation of (4.17) with respect to m^2

as follows:

$$\left\langle\varphi^2\right\rangle^{\text{late-time}}_{\text{non-pert}} = -\frac{3H^4}{4\pi^2}\frac{1}{\mathcal{N}}\frac{d\mathcal{N}}{dm^2} \qquad \Rightarrow \qquad \left\langle\varphi^2\right\rangle^{\text{late-time}}_{\text{non-pert}} = \frac{m^2}{2\lambda}\left(\frac{\mathcal{K}_{3/4}(z)}{\mathcal{K}_{1/4}(z)} - 1\right), \tag{4.19}$$

where we have used the recursion relations [141]:

$$\frac{d}{dz}\mathcal{K}_\nu(z) = -\frac{\nu}{z}\mathcal{K}_\nu(z) - \mathcal{K}_{\nu-1}(z) \qquad \text{and} \qquad \mathcal{K}_{-\nu}(z) = \mathcal{K}_\nu(z). \tag{4.20}$$

In the massless limit, the obtained expression (4.18) or (4.19) descends to (4.14) for n = 1 from the original work by Starobinsky and Yokoyama [53]. Let us also point out that the two-point correlation function (4.18) and (4.19) resembles the exact zero-mode propagator in Euclidean de Sitter space [149].

Onward, one can proceed and get the expression anew with the help of (4.20) for

$$\left\langle\varphi^4\right\rangle^{\text{late-time}}_{\text{non-pert}} = \left(-\frac{3H^4}{4\pi^2}\right)^2\frac{1}{\mathcal{N}}\frac{d^2\mathcal{N}}{d(m^2)^2} \qquad \Rightarrow \qquad \left\langle\varphi^4\right\rangle^{\text{late-time}}_{\text{non-pert}} = \frac{3H^4}{8\pi^2\lambda}+\frac{m^4}{2\lambda^2}\left(1-\frac{K_{3/4}(z)}{K_{1/4}(z)}\right). \tag{4.21}$$

This result again reduces to (4.14) for n = 2 in the massless limit.

Thanks to (4.20), the general structure of any 2n'th expectation value will always have the following form:

$$\frac{d^{\mathrm{n}}\mathcal{N}}{d(m^2)^{\mathrm{n}}} = \alpha_{\mathrm{n}}\,\mathrm{e}^z\,K_{1/4}(z) + \beta_{\mathrm{n}}\,\mathrm{e}^z\,K_{3/4}(z), \tag{4.22}$$

where from the definition (4.17), from (4.19), and (4.21) we have

$$\alpha_0 = \frac{m}{\sqrt{2\lambda}} \quad \text{and} \quad \beta_0 = 0; \qquad \alpha_1 = -\beta_1 = \frac{\sqrt{2}\,\pi^2 m^3}{3\,\lambda^{3/2}H^4}; \tag{4.23}$$

$$\alpha_2 = \frac{\sqrt{2}\,\pi^2 m}{3\lambda^{3/2}H^4} + \frac{4\sqrt{2}\,\pi^4 m^5}{9\lambda^{5/2}H^8}, \quad \text{and} \quad \beta_2 = -\frac{4\sqrt{2}\,\pi^4 m^5}{9\lambda^{5/2}H^8}. \tag{4.24}$$

Consequently, we observe that the defined coefficients satisfy the recurrence relations

$$\begin{aligned}\alpha_{\mathrm{n}+1} &= \frac{d\alpha_{\mathrm{n}}}{dm^2} - \frac{\alpha_{\mathrm{n}}}{2m^2} + \frac{2\pi^2 m^2}{3\lambda H^4}(\alpha_{\mathrm{n}} - \beta_{\mathrm{n}}),\\ \beta_{\mathrm{n}+1} &= \frac{d\beta_{\mathrm{n}}}{dm^2} - \frac{3\beta_{\mathrm{n}}}{2m^2} - \frac{2\pi^2 m^2}{3\lambda H^4}(\alpha_{\mathrm{n}} - \beta_{\mathrm{n}})\end{aligned} \tag{4.25}$$

and result in

$$\left\langle\varphi^{2\mathrm{n}}\right\rangle^{\text{late-time}}_{\text{non-pert}} = \left(-\frac{3H^4}{4\pi^2}\right)^{\mathrm{n}}\frac{1}{\mathcal{N}}\frac{d^{\mathrm{n}}\mathcal{N}}{d(m^2)^{\mathrm{n}}} = \frac{\sqrt{2\lambda}}{m}\left(-\frac{3H^4}{4\pi^2}\right)^{\mathrm{n}}\left(\alpha_{\mathrm{n}} + \beta_{\mathrm{n}}\frac{K_{3/4}(z)}{K_{1/4}(z)}\right). \tag{4.26}$$

Naturally, in the large-z expansion, $z \gg 1$, the obtained expressions (4.19) and (4.21) become

$$\left\langle\varphi^2\right\rangle_{\text{non-pert}}^{\text{late-time}} = \frac{m^2}{2\lambda}\left(\frac{K_{3/4}(z)}{K_{1/4}(z)} - 1\right) \longrightarrow \frac{m^2}{8\lambda z} - \frac{3m^2}{64\lambda z^2} + \frac{3m^2}{64\lambda z^3} - \frac{297m^2}{4096\lambda z^4} + \dots ; \tag{4.27}$$

$$\left\langle\varphi^4\right\rangle_{\text{non-pert}}^{\text{late-time}} = \frac{3H^4}{8\pi^2\lambda} + \frac{m^4}{2\lambda^2}\left(1 - \frac{K_{3/4}(z)}{K_{1/4}(z)}\right) \longrightarrow \frac{3H^4}{8\pi^2\lambda} - \frac{m^4}{8\lambda^2 z} + \frac{3m^4}{64\lambda^2 z^2} - \frac{3m^4}{64\lambda^2 z^3} + \frac{297m^4}{4096\lambda^2 z^4} + \dots ; \tag{4.28}$$

$$\left\langle\varphi^6\right\rangle_{\text{non-pert}}^{\text{late-time}} = \frac{m^6}{2\lambda^3}\left(\left(1 + \frac{9\lambda H^4}{8\pi^2 m^4}\right)\frac{K_{3/4}(z)}{K_{1/4}(z)} - 1 - \frac{15\lambda H^4}{8\pi^2 m^4}\right) \longrightarrow -\frac{3m^2H^4}{8\pi^2\lambda^2} + \frac{m^6}{8\lambda^3 z} - \frac{3m^6}{64\lambda^3 z^2} + \frac{3m^6}{64\lambda^3 z^3} - \frac{297m^6}{4096\,\lambda^3 z^4} + \frac{9m^2H^4}{64\pi^2\lambda^2 z} - \frac{27m^2H^4}{512\pi^2\lambda^2 z^2} + \frac{27m^2H^4}{512\pi^2\lambda^2 z^3} + \dots ; \tag{4.29}$$

and using the value of z defined in (4.18), they reproduce our series (2.37), (2.73), (4.15), and (4.16). We present (4.29) here for the upcoming comparison with the next Chapter 5. To obtain expression (4.29), we have defined α_3 and β_3 through (4.25) with the known (4.24).

4.3 One more perturbative agreement with quantum field theory results

To make an additional comparison of our perturbative results with the stochastic approach, we follow the seminal work by Tsamis and Woodard [45], in which the expectation value of a massless, minimally coupled scalar field with quartic self-interaction was derived using the Fokker–Planck equation.

The corresponding Fokker-Planck equation for our model under consideration is (4.11) with the same potential as one studies within the quantum field theory (1.1). We multiply both sides of (4.30) by $\varphi^{2\text{n}}$ and perform the integral $\int_{-\infty}^{+\infty} d\varphi$ weighted by the probability density $\mathcal{P}\left[\varphi(t,\vec{x})\right]$, and after integration by parts, this yields

$$\frac{\partial}{\partial t}\left\langle\varphi^{2\text{n}}(t,\vec{x})\right\rangle = -\frac{2\text{n}m^2}{3H}\left\langle\varphi^{2\text{n}}(t,\vec{x})\right\rangle - \frac{2\text{n}\lambda}{3H}\left\langle\varphi^{2\text{n}+2}(t,\vec{x})\right\rangle + \text{n}\left(2\text{n}-1\right)\frac{H^3}{4\pi^2}\left\langle\varphi^{2\text{n}-2}(t,\vec{x})\right\rangle. \tag{4.30}$$

Hereafter, we omit the argument $\vec{x}$ since it is the same in all our expressions. Let us also point out that this system of equations must be valid for any value of n, while in the literature one can find the equation obtained after multiplying by φ^2 only [69].

In the free massive case, $\lambda = 0$, and after some redefinitions, namely,

$$\alpha := \frac{Ht}{4\pi^2}, \qquad \bar{m}^2 := \frac{8\pi^2 m^2}{3H^2}, \quad \text{and} \quad \bar{\varphi} := \frac{\varphi}{H}, \tag{4.31}$$

one obtains a more convenient form of (4.30) with $\lambda = 0$ to deal with:

$$\frac{\partial}{\partial\alpha}\left\langle \bar{\varphi}^{2\mathrm{n}}\right\rangle_0 = -\mathrm{n}\,\bar{m}^2\left\langle \bar{\varphi}^{2\mathrm{n}}\right\rangle_0 + \mathrm{n}\,(2\mathrm{n}-1)\left\langle \bar{\varphi}^{2\mathrm{n}-2}\right\rangle_0. \tag{4.32}$$

The solution to equation (4.32) above can be found in the following way:

$$\left\langle \bar{\varphi}^{2\mathrm{n}}\right\rangle_0 = (2\mathrm{n}-1)!!\,\alpha^{\mathrm{n}}\,F_{\mathrm{n}}(\alpha\bar{m}^2), \qquad F_{\mathrm{n}}(z) = 1 + \beta^1_{\mathrm{n}} z + \beta^2_{\mathrm{n}} z^2 + \beta^3_{\mathrm{n}} z^3 + \beta^4_{\mathrm{n}} z^4 + \ldots, \tag{4.33}$$

where $F_{\mathrm{n}}(z)$ obeys

$$\mathrm{n}\,F_{\mathrm{n}}(z) + z\,F'_{\mathrm{n}}(z) = -\,\mathrm{n}\,z\,F_{\mathrm{n}}(z) + \mathrm{n}\,F_{\mathrm{n}-1}(z). \tag{4.34}$$

The recurrence relations on the unknown coefficients $\beta^{\mathrm{i}}_{\mathrm{n}}$ are

$$(\mathrm{n}+\mathrm{i})\beta^{\mathrm{i}}_{\mathrm{n}} = -\mathrm{n}\,\beta^{\mathrm{i}-1}_{\mathrm{n}} + \mathrm{n}\,\beta^{\mathrm{i}}_{\mathrm{n}-1} \quad \text{and} \quad \mathrm{i} = \overline{1,\infty}. \tag{4.35}$$

Therefore, up to $O(z^6)$ in (4.33), one can obtain the perturbative expansion:

$$\begin{aligned}\left\langle \varphi^{2\mathrm{n}}(t)\right\rangle_0 = (2\mathrm{n}-1)!!\left(\frac{H^3 t}{4\pi^2}\right)^{\mathrm{n}}\Bigg(1 &- \frac{\mathrm{n}m^2}{3H^2}(Ht) + \frac{\mathrm{n}(3\mathrm{n}+1)}{54}\frac{m^4}{H^4}(Ht)^2 - \frac{\mathrm{n}^2(\mathrm{n}+1)}{162}\frac{m^6}{H^6}(Ht)^3 \\ &+ \frac{\mathrm{n}}{29160}\left(15\mathrm{n}^3 + 30\mathrm{n}^2 + 5\mathrm{n} - 2\right)\frac{m^8}{H^8}(Ht)^4 \\ &- \frac{\mathrm{n}^2(\mathrm{n}+1)}{87480}\left(3\mathrm{n}^2 + 7\mathrm{n} - 2\right)\frac{m^{10}}{H^{10}}(Ht)^5 + \ldots\Bigg) + O(m^{12}).\end{aligned} \tag{4.36}$$

And for specific values n = 1 and n = 2, we have

$$\left\langle \varphi^2(t)\right\rangle_0 = \frac{H^3 t}{4\pi^2} - \frac{m^2H^2t^2}{12\,\pi^2} + \frac{m^4Ht^3}{54\,\pi^2} - \frac{m^6t^4}{324\,\pi^2} + \frac{m^8t^5}{2430\,H\,\pi^2} - \frac{m^{10}t^6}{21870\,H^2\,\pi^2} + \ldots; \tag{4.37}$$

$$\left\langle \varphi^4(t)\right\rangle_0 = \frac{3\,H^6t^2}{16\,\pi^4} - \frac{m^2H^5t^3}{8\pi^4} + \frac{7m^4H^4t^4}{144\,\pi^4} - \frac{m^6H^3t^5}{72\,\pi^4} + \frac{31m^8H^2t^6}{9720\,\pi^4} - \frac{m^{10}Ht^7}{1620\,\pi^4} + \ldots. \tag{4.38}$$

The obtained series are precisely the same as those obtained by expanding $\langle\varphi^2(t)\rangle_0$ and $\langle\varphi^4(t)\rangle_0$ from expressions (2.36) and (2.71) in the small m^2/H^2 regime; see the first lines of the corresponding expressions from the previous chapter, namely (2.39) and (2.74). Being so, our free massive correlation functions are exactly the resummed expressions of these series. One readily notices the leading infrared logarithm structure in cosmic time t in (4.36).

In the massless case involving a quartic self-interaction, we apply analogous redefinitions outlined in (4.31) and get

$$\frac{\partial}{\partial\alpha}\left\langle \bar{\varphi}^{2\mathrm{n}}_{m=0}\right\rangle = -\mathrm{n}\,\bar{\lambda}\left\langle \bar{\varphi}^{2\mathrm{n}+2}_{m=0}\right\rangle + \mathrm{n}\,(2\mathrm{n}-1)\left\langle \bar{\varphi}^{2\mathrm{n}-2}_{m=0}\right\rangle, \qquad \bar{\lambda} := \frac{8\pi^2}{3}\lambda. \tag{4.39}$$

The solution can be again found in the form of (4.33):

$$\left\langle \bar{\varphi}^{2\mathrm{n}}_{m=0} \right\rangle = (2\mathrm{n}-1)!!\,\alpha^{\mathrm{n}}\,G_{\mathrm{n}}\left(\alpha^2\bar{\lambda}\right), \qquad G_{\mathrm{n}}(z) = 1 + \beta^1_{\mathrm{n}} z + \beta^2_{\mathrm{n}} z^2 + \beta^3_{\mathrm{n}} z^3 + \ldots, \tag{4.40}$$

and $G_{\mathrm{n}}(z)$ obeys

$$\mathrm{n}\,G_{\mathrm{n}}(z) + 2z\,G'_{\mathrm{n}}(z) = -\mathrm{n}(2\mathrm{n}+1)\,z\,G_{\mathrm{n}+1}(z) + \mathrm{n}\,G_{\mathrm{n}-1}(z). \tag{4.41}$$

Here, the recurrence relations for coefficients are

$$(\mathrm{n}+2\mathrm{i})\beta^{\mathrm{i}}_{\mathrm{n}} = -\mathrm{n}\,(2\mathrm{n}+1)\beta^{\mathrm{i}-1}_{\mathrm{n}+1} + \mathrm{n}\,\beta^{\mathrm{i}}_{\mathrm{n}-1}, \quad \text{and} \quad \mathrm{i} = \overline{1,\infty}. \tag{4.42}$$

Therefore, up to $O(z^4)$ in (4.40) one can obtain the perturbative expansion:

$$\begin{aligned}\left\langle \varphi^{2\mathrm{n}}_{m=0}(t) \right\rangle = (2\mathrm{n}-1)!!\left(\frac{H^3 t}{4\pi^2}\right)^{\mathrm{n}}\Bigg(1 &- \frac{\lambda\,\mathrm{n}\,(\mathrm{n}+1)}{12\pi^2}\,(Ht)^2 \\ &+ \frac{\lambda^2\,\mathrm{n}}{10080\,\pi^4}\left(35\mathrm{n}^3 + 170\mathrm{n}^2 + 225\mathrm{n} + 74\right)(Ht)^4 \\ &- \frac{\lambda^3\mathrm{n}(\mathrm{n}+1)(\mathrm{n}+2)}{362880\,\pi^6}\left(35\mathrm{n}^3 + 300\mathrm{n}^2 + 685\mathrm{n} + 252\right)(Ht)^6 + \ldots\Bigg).\end{aligned} \tag{4.43}$$

Here, we extended this expression up to the next λ^3 order through the same reasoning as in the original work by Tsamis and Woodard [45] up to $O(\lambda^3)$. One can easily compare these outcomes for $\mathrm{n}=1$ and $\mathrm{n}=2$ with (2.38), (2.39), (2.72), and (2.74).

In the next chapter, inspired by [45], we develop an alternative approach to perturbative computations of expectation values in a more economical way, directly using the Fokker-Planck equation, and extend it to the case of multi-time correlation functions. Our treatment is based on the non-stationary solution for the probability distribution function, including the conditional (or transition) one, in the case of the linear drift. Through the Fokker-Planck or forward Kolmogorov equation, we gain the integral relation to various correlation functions at different self-interaction coupling constant λ orders. With an appropriate integration, it matches the corresponding perturbative quantum field theory results from our previous Chapter 2.

4.4 Why are approaches in agreement?

The key point is that the equations of motion for the long-wavelength part of the quantum scalar field at leading logarithmic order and for the classical stochastic field coincide. One catches sight of that agreement by taking the time derivative of both sides of the truncated Yang-Feldman equation (2.14) and comparing with (4.2).

At first glance, additional exponents in the modified Yang-Feldman equation (2.16) do not preserve this correspondence with the local Langevin-like equation (4.2). Nevertheless, let us substitute the relation between the massive and the massless fields (2.15) into (2.16) and take the time derivative of both its left-hand and right-hand sides:

$$\dot{\phi}(t,\vec{x}) = \dot{\phi}_0(t,\vec{x}) - \frac{m^2}{3H}\phi_0(t,\vec{x}) + \underbrace{\frac{m^4}{9H^2}\mathrm{e}^{-\frac{m^2 t}{3H}}\int_0^t dt'\,\mathrm{e}^{\frac{m^2 t'}{3H}}\phi_0(t',\vec{x})}_{=\frac{m^2}{3H}\left(\phi_0(t,\vec{x})-\tilde{\phi}(t,\vec{x})\right)} \tag{4.44}$$

$$- \frac{\lambda}{3H}\phi^3(t,\vec{x}) + \frac{m^2\lambda}{9H^2}\mathrm{e}^{-\frac{m^2 t}{3H}}\int_0^t dt'\,\mathrm{e}^{\frac{m^2 t'}{3H}}\phi^3(t',\vec{x})$$

$$= \dot{\phi}_0(t,\vec{x}) - \frac{m^2}{3H}\left(\underbrace{\tilde{\phi}(t,\vec{x}) - \frac{\lambda}{3H}\mathrm{e}^{-\frac{m^2 t}{3H}}\int_0^t dt'\,\mathrm{e}^{\frac{m^2 t'}{3H}}\phi^3(t',\vec{x})}_{=\phi(t,\vec{x})}\right) - \frac{\lambda}{3H}\phi^3(t,\vec{x}).$$

Therefore, we arrive at the following form:

$$\dot{\phi}(t,\vec{x}) = -\frac{1}{3H}V'_\phi\left(\phi(t,\vec{x})\right) + \dot{\phi}_0(t,\vec{x}), \tag{4.45}$$

which is precisely the original local Langevin-like form derived by Starobinsky [52]. The proposed way to "hang up" the mass in (2.16) using (2.15) does not go beyond the leading logarithm approximation.

5 From the Fokker-Planck equation to perturbative QFT's results

The present chapter provides an alternative approach for computing the perturbative series for equal- and multi-time correlation functions in the leading logarithm approximation beyond the equal-time and stationary cases [124]. Instead of iterating the massive Yang-Feldman-type equation or dealing with diagrammatic techniques, one derives a simple first-order differential equation through the Fokker-Planck or forward Kolmogorov equation. The formal solution to that equation relates various correlation functions at different orders in the self-interaction coupling constant λ.

5.1 Equal-time correlation functions

We shall start with the case of equal-time correlation functions. The corresponding Fokker-Planck equation for the model under consideration is (4.11) with the same potential as studied in quantum field theory; in our case, this potential is given by (1.1). As in the previous Chapter 4, one multiplies both sides by $\varphi^{2\mathrm{n}}$ and integrates over $\int_{-\infty}^{+\infty} d\varphi$ weighted by the probability distribution function $\mathcal{P}\left[\varphi(t,\vec{x})\right]$, and after integration by parts, this yields (4.30). This differential equation can be easily rewritten as

$$\mathrm{e}^{-\frac{2\mathrm{n}m^2t}{3H}}\frac{\partial}{\partial t}\left(\mathrm{e}^{\frac{2\mathrm{n}m^2t}{3H}}\left\langle\varphi^{2\mathrm{n}}(t)\right\rangle\right) = -\frac{2\mathrm{n}\lambda}{3H}\left\langle\varphi^{2\mathrm{n}+2}(t)\right\rangle + \mathrm{n}\left(2\mathrm{n}-1\right)\frac{H^3}{4\pi^2}\left\langle\varphi^{2\mathrm{n}-2}(t)\right\rangle, \tag{5.1}$$

which immediately leads to the integral relation for equal-time correlation functions

$$\left\langle\varphi^{2\mathrm{n}}(t)\right\rangle = \mathrm{e}^{-\frac{2\mathrm{n}m^2t}{3H}}\int_0^t dt'\,\mathrm{e}^{\frac{2\mathrm{n}m^2t'}{3H}}\left(-\frac{2\mathrm{n}\lambda}{3H}\left\langle\varphi^{2\mathrm{n}+2}(t')\right\rangle + \mathrm{n}\left(2\mathrm{n}-1\right)\frac{H^3}{4\pi^2}\left\langle\varphi^{2\mathrm{n}-2}(t')\right\rangle\right). \tag{5.2}$$

The obtained relation connects various correlation functions at different self-interaction coupling constant λ orders.

We begin by defining equal-time correlation functions in the free massive case without self-interaction, $\lambda = 0$. In this case, one can find an exact non-stationary solution to the Fokker-Planck equation (4.11) for the probability distribution function $\mathcal{P}_m\left[\varphi(t,\vec{x})\right]$. By taking the ansatz as

$$\mathcal{P}_m\left[\varphi(t,\vec{x})\right] = \frac{\mathrm{e}^{-\varphi^2/\zeta(t)}}{\sqrt{\pi\zeta(t)}} \tag{5.3}$$

and substituting it into (4.11), one gets a simple equation, whose solution is given by

$$\dot{\zeta} = -\frac{2m^2}{3H}\zeta(t) + \frac{H^3}{2\pi^2} \qquad \Rightarrow \qquad \zeta(t) = \frac{3H^4}{4\pi^2m^2}\left(1-\mathrm{e}^{-\frac{2m^2t}{3H}}\right). \tag{5.4}$$

Using the obtained solution, we can calculate any 2n-point correlation functions as

$$\left\langle \varphi^{2n}(t) \right\rangle = \frac{(2n-1)!!}{2^n}\, \zeta^n(t) := \left\langle \varphi^{2n}(t) \right\rangle_0 . \tag{5.5}$$

One sets on foot the iterative series $\left\langle \varphi^{2n}(t) \right\rangle = \sum_{k=0}^{\infty} \lambda^k \left\langle \varphi^{2n}(t) \right\rangle_k$ and gains from (5.2):

$$\left\langle \varphi^{2n}(t) \right\rangle_{k+1} = e^{-\frac{2nm^2 t}{3H}} \int_0^t dt'\, e^{\frac{2nm^2 t'}{3H}} \left(-\frac{2n\lambda}{3H} \left\langle \varphi^{2n+2}(t') \right\rangle_k + n\,(2n-1)\, \frac{H^3}{4\pi^2} \left\langle \varphi^{2n-2}(t') \right\rangle_{k+1} \right). \tag{5.6}$$

We rely on this interconnection in order to derive various correlation functions and to compare them with those found in Chapter 2 within the massive Yang-Feldman-type equation.

Let us start with the case of the $\left\langle \varphi^2(t) \right\rangle$ correlation function. According to (5.6), one obtains at linear order in λ

$$\left\langle \varphi^2(t) \right\rangle_\lambda = -\frac{2\lambda}{3H} e^{-\frac{2m^2 t}{3H}} \int_0^t dt' e^{\frac{2m^2 t'}{3H}} \left\langle \varphi^4(t') \right\rangle_0 = -\frac{27\,\lambda H^8}{64\pi^4 m^6} \left(1 - \frac{4m^2 t}{3H} e^{-\frac{2m^2 t}{3H}} - e^{-\frac{4m^2 t}{3H}} \right). \tag{5.7}$$

Hereafter, all correlation functions with zero subscript are related to the free massive case and are given in (5.5). Besides, $\left\langle \varphi^0(t) \right\rangle = 1$ and does not contribute to the λ–linear order. Integrating $\left\langle \varphi^0(t) \right\rangle = 1$, $\left\langle \varphi^2(t) \right\rangle$, etc. in (5.2) in the free massive case, one obtains exactly (5.5). This result for $\left\langle \varphi^2(t) \right\rangle_\lambda$ precisely coincides with our previous result (2.36).

Within the same equation (5.6), we are able to compute the four-point correlation function at linear order in λ, $\left\langle \varphi^4(t) \right\rangle_\lambda$: one should, obviously, integrate the six-point correlation function at zeroth order, $\left\langle \varphi^6(t) \right\rangle_0$, from (5.5) and the two-point correlation function at λ–linear order, $\left\langle \varphi^2(t) \right\rangle_\lambda$, which is already known from the previous step. Next, we find the two-point correlation function at λ^2 order, $\left\langle \varphi^2(t) \right\rangle_{\lambda^2}$, again, through (5.6). In that fashion, for the four-point correlation function at order λ^2, $\left\langle \varphi^4(t) \right\rangle_{\lambda^2}$, we compute the λ–linear order for $\left\langle \varphi^6(t) \right\rangle_\lambda$, and, finally, with the appropriate integration, we obtain $\left\langle \varphi^2(t) \right\rangle_{\lambda^3}$. Below we list the final results of that trivial integration performance:

$$\left\langle \varphi^2(t) \right\rangle = \frac{3H^4}{8\,\pi^2 m^2} \left(1 - e^{-\frac{2m^2 t}{3H}} \right) - \frac{27\,\lambda H^8}{64\,\pi^4 m^6} \left(1 - \frac{4m^2 t}{3H} e^{-\frac{2m^2 t}{3H}} - e^{-\frac{4m^2 t}{3H}} \right) + \frac{81\lambda^2 H^{12}}{64\,\pi^6 m^{10}} \left(1 + \left(\frac{21}{8} - \frac{3m^2 t}{2H} - \frac{m^4 t^2}{3H^2} \right) e^{-\frac{2m^2 t}{3H}} - \left(3 + \frac{2m^2 t}{H} \right) e^{-\frac{4m^2 t}{3H}} - \frac{5}{8} e^{-\frac{6m^2 t}{3H}} \right) \tag{5.8}$$

$$- \frac{2187\lambda^3 H^{16}}{4096\,\pi^8 m^{14}}\Bigg(11 + \left(\frac{872}{9} - \frac{172\,m^2 t}{9H} - \frac{16 m^4 t^2}{3H^2} - \frac{32 m^6 t^3}{81 H^3}\right) \mathrm{e}^{-\frac{2m^2 t}{3H}}$$

$$- \left(64 + \frac{72\,m^2 t}{H} + \frac{128\,m^4 t^2}{9H^2}\right)\mathrm{e}^{-\frac{4m^2 t}{3H}} - \left(40 + \frac{20\,m^2 t}{H}\right)\mathrm{e}^{-\frac{6m^2 t}{3H}} - \frac{35}{9}\,\mathrm{e}^{-\frac{8m^2 t}{3H}}\Bigg) + O(\lambda^4);$$

$$\left\langle \varphi^4(t)\right\rangle = \frac{27 H^8}{64\,\pi^4 m^4}\left(1 - \mathrm{e}^{-\frac{2m^2 t}{3H}}\right)^2 - \frac{81\lambda H^{12}}{64\,\pi^6 m^8}\Bigg(1 - \left(\frac{9}{4} + \frac{m^2 t}{H}\right)\mathrm{e}^{-\frac{2m^2 t}{3H}} \tag{5.9}$$

$$+ \frac{2m^2 t}{H}\mathrm{e}^{-\frac{4m^2 t}{3H}} + \frac{5}{4}\mathrm{e}^{-\frac{2m^2 t}{H}}\Bigg) + \frac{729\lambda^2 H^{16}}{4096\,\pi^8 m^{12}}\Bigg(33 - \left(86 + \frac{48 m^2 t}{H} + \frac{16 m^4 t^2}{3H^2}\right)\mathrm{e}^{-\frac{2m^2 t}{3H}}$$

$$- \left(132 - \frac{88 m^2 t}{H} - \frac{128 m^4 t^2}{3H^2}\right)\mathrm{e}^{-\frac{4m^2 t}{3H}} + \left(150 + \frac{120 m^2 t}{H}\right)\mathrm{e}^{-\frac{2m^2 t}{H}} + 35\,\mathrm{e}^{-\frac{8m^2 t}{3H}}\Bigg) + O(\lambda^3);$$

$$\left\langle \varphi^6(t)\right\rangle = \frac{405 H^{12}}{512\,\pi^6 m^6}\left(1 - \mathrm{e}^{-\frac{2m^2 t}{3H}}\right)^3 - \frac{18225\,\lambda H^{16}}{4096\,\pi^8 m^{10}}\Bigg(1 - \left(4 + \frac{4m^2 t}{5H}\right)\mathrm{e}^{-\frac{2m^2 t}{3H}} \tag{5.10}$$

$$+ \left(\frac{18}{5} + \frac{16 m^2 t}{5H}\right)\mathrm{e}^{-\frac{4m^2 t}{3H}} + \left(\frac{4}{5} - \frac{12 m^2 t}{5H}\right)\mathrm{e}^{-\frac{6m^2 t}{3H}} - \frac{7}{5}\,\mathrm{e}^{-\frac{8m^2 t}{3H}}\Bigg) + O(\lambda^2).$$

All of them, namely, contributions to $\langle\varphi^2(t)\rangle$ at orders λ^0, λ, λ^2, and λ^3 and to $\langle\varphi^4(t)\rangle$ at orders λ^0, λ, and λ^2 coincide with our outcomes from the previous Chapter 2; see (2.36) and (2.71) respectively.

Once again, our results (5.8)–(5.10) are equivalent to the resummation of leading infrared logarithms in the perturbative series in m^2 at each order in the self-interaction coupling constant λ. That is precisely what we revealed in the previous Chapter 2; see (2.39), (2.74), and nearby reasoning. Within this scheme, the massless infrared divergences of the self-interacting scalar field are fully resummed. Our massive-field perturbative series in λ with $m^2 \ll H^2$ and $\lambda \ll m^4/H^4$ converge and, in the late-time limit, approach

$$\left\langle \varphi^2(t)\right\rangle \xrightarrow[\text{times}]{\text{late}} \frac{3H^4}{8\,\pi^2 m^2} - \frac{27\,\lambda H^8}{64\,\pi^4 m^6} + \frac{81\lambda^2 H^{12}}{64\,\pi^6 m^{10}} - \frac{24057\lambda^3 H^{16}}{4096\,\pi^8 m^{14}} = \left\langle\varphi^2\right\rangle^{\text{late-time}}_{\text{perturb}}; \tag{5.11}$$

$$\left\langle \varphi^4(t)\right\rangle \xrightarrow[\text{times}]{\text{late}} \frac{27H^8}{64\,\pi^4 m^4} - \frac{81\lambda H^{12}}{64\,\pi^6 m^8} + \frac{24057\lambda^2 H^{16}}{4096\,\pi^8 m^{12}} + O(\lambda^3) = \left\langle\varphi^4\right\rangle^{\text{late-time}}_{\text{perturb}}; \tag{5.12}$$

$$\left\langle \varphi^6(t)\right\rangle \xrightarrow[\text{times}]{\text{late}} \frac{405 H^{12}}{512\,\pi^6 m^6} - \frac{18225\,\lambda H^{16}}{4096\,\pi^8 m^{10}} + O(\lambda^2) = \left\langle\varphi^6\right\rangle^{\text{late-time}}_{\text{perturb}}. \tag{5.13}$$

The obtained series for $\langle\varphi^2(t)\rangle$ at equilibrium (5.11) is in agreement with those obtained from diagrammatic techniques in [81, 82, 117] up to $O(\lambda^3)$. Outcomes (5.11)–(5.13) match the results from the previous Chapter 4; see (4.15), (4.16), and (4.27), (4.28), and (4.29), gained through recurrence relations (4.25), when inserted into (4.26). As expected, (5.11)–(5.13) are in agreement with (2.37) and (2.73) from Chapter 2.

Regarding the massless case, our approach is also clearly efficient. In this case, the ansatz has the same form (5.3), and the equation for the unknown function is (5.4) without the massive term. The solution to the corresponding equation is $\zeta_{m=0}(t) = H^3 t/2\pi^2$ and coincides with the smooth massless limit of the solution (5.4), as one expects. The 2n'th correlation function in the case without self-interaction is defined by a relation identical-to (5.5)

$$\left\langle \varphi^{2\mathrm{n}}_{m=0}(t) \right\rangle_0 = (2\mathrm{n}-1)!! \left(\frac{H^3 t}{4\pi^2}\right)^{\mathrm{n}}, \tag{5.14}$$

and the resulting formal solution to the corresponding first-order differential equation (4.30) in the massless case takes the form (5.6) without the exponential factors:

$$\left\langle \varphi^{2\mathrm{n}}_{m=0}(t) \right\rangle_{\mathrm{k}+1} = \int\limits_0^t dt' \left(-\frac{2\mathrm{n}\lambda}{3H} \left\langle \varphi^{2\mathrm{n}+2}_{m=0}(t') \right\rangle_{\mathrm{k}} + \mathrm{n}\left(2\mathrm{n}-1\right) \frac{H^3}{4\pi^2} \left\langle \varphi^{2\mathrm{n}-2}_{m=0}(t') \right\rangle_{\mathrm{k}+1} \right). \tag{5.15}$$

Let us also note that in our previous Chapter 2 in the massive case, the Yang-Feldman-type equation (or the retarded Green's function) acquires additional exponents in a manner analogous to the massless case; see also [81, 82, 150].

Using (5.14) and (5.15) and following the proposed routine, one calculates

$$\left\langle \varphi^{2}_{m=0}(t) \right\rangle = \frac{H^3 t}{4\pi^2} - \frac{\lambda H^5 t^3}{24\,\pi^4} + \frac{\lambda^2 H^7 t^5}{80\,\pi^6} - \frac{53\lambda^3 H^9 t^7}{10080\,\pi^8} + O\left(\lambda^4\right); \tag{5.16}$$

$$\left\langle \varphi^{4}_{m=0}(t) \right\rangle = \frac{3H^6 t^2}{16\,\pi^4} - \frac{3\lambda H^8 t^4}{32\,\pi^6} + \frac{53\,\lambda^2 H^{10}\,t^6}{960\,\pi^8} + O\left(\lambda^3\right); \tag{5.17}$$

$$\left\langle \varphi^{6}_{m=0}(t) \right\rangle = \frac{15 H^9 t^3}{64\,\pi^6} - \frac{15\lambda H^{11} t^5}{64\,\pi^8} + O\left(\lambda^2\right), \tag{5.18}$$

which are precisely the massless limits of (5.8), (5.9), and (5.10) and coincide with (2.38), (2.72), and (4.43). Outcomes also coincide with the corresponding results up to some (which are present in the references onward) perturbative orders using quantum field theory methods [83, 84, 120, 121, 122]. One recovers of the leading infrared logarithm structure in cosmic time t in the massless limit.

5.2 Multi-time correlation functions

Now we are in a position to generalize our approach to the multi-time correlation functions. The idea is the same: to derive the first-order differential equation from the Fokker-Planck equation and the integral relation for the multi-time correlation functions through it.

In that case, the conditional (or transition) probability $\mathcal{P}\left(\varphi, t \,\middle|\, \varphi_0, t_0\right)$ must also obey the Fokker-Planck equation or the forward Kolmogorov equation for $t \geq t_0$:

$$\frac{\partial}{\partial t}\mathcal{P}\left(\varphi, t \,\middle|\, \varphi_0, t_0\right) = \frac{1}{3H}\frac{\partial}{\partial\varphi}\left(\left(m^2\varphi + \lambda\varphi^3\right)\mathcal{P}\left(\varphi, t \,\middle|\, \varphi_0, t_0\right)\right) + \frac{H^3}{8\pi^2}\frac{\partial^2}{\partial\varphi^2}\mathcal{P}\left(\varphi, t \,\middle|\, \varphi_0, t_0\right) \tag{5.19}$$

and have the initial value $\mathcal{P}(\varphi, t_0 \,|\, \varphi_0, t_0) = \delta(\varphi - \varphi_0)$.

For the massive case without self-interaction, namely, $\lambda = 0$, the exact non-stationary solution to (5.19) for the conditional (or transition) probability is known:

$$\mathcal{P}_m(\varphi, t| \,\varphi_0, t_0) = \frac{2\sqrt{\pi}m}{\sqrt{3}\,H^2}\left(1 - \mathrm{e}^{-\frac{2m^2}{3H}(t-t_0)}\right)^{-1/2} \exp\left(-\frac{4\pi^2 m^2}{3H^4}\frac{\left(\varphi - \varphi_0\,\mathrm{e}^{-\frac{m^2}{3H}(t-t_0)}\right)^2}{\left(1 - \mathrm{e}^{-\frac{2m^2}{3H}(t-t_0)}\right)}\right). \quad (5.20)$$

That is nothing other than the Gaussian distribution, which can be obtained by the method of characteristics; see, e.g., the classic book by Risken [101]. Thus, one can obtain any multi-time correlation function through appropriate integration with (5.20) in the free massive case.

In the following computations, we will exploit the Markov assumption or the Markov property [101, 102, 103], formulated in terms of the conditional (or transition) probability density, which is

$$\mathcal{P}(\,\varphi_1, t_1\,;\,\varphi_2, t_2\,;\,\ldots \,|\, \sigma_1, \tau_1;\,\sigma_2, \tau_2\,;\,\ldots\,) \stackrel{\text{def}}{=} \frac{\mathcal{P}(\,\varphi_1, t_1\,;\,\varphi_2, t_2\,;\,\ldots\,;\,\sigma_1, \tau_1\,;\,\sigma_2, \tau_2\,;\,\ldots\,)}{\mathcal{P}(\,\sigma_1, \tau_1\,;\,\sigma_2, \tau_2\,;\,\ldots\,)}\,. \quad (5.21)$$

The r.h.s. of (5.21) consists solely of joint probability densities (and we assume that they exist and describe the system under consideration); σ_i are the past values of our time-dependent random variable at times τ_i, while φ_i are the future values at times t_i.

Under the Markov assumption, the conditional (or transition) probability is defined entirely by the known value of the most recent one:

$$\mathcal{P}(\,\varphi_1, t_1\,;\,\varphi_2, t_2\,;\,\ldots \,|\, \sigma_1, \tau_1\,;\,\sigma_2, \tau_2\,;\,\ldots\,) = \mathcal{P}(\,\varphi_1, t_1\,;\,\varphi_2, t_2\,;\,\ldots \,|\, \sigma_1, \tau_1\,) \quad (5.22)$$

for the time ordering $t_1 \geq t_2 \geq \ldots \geq \tau_1 \geq \tau_2 \geq \ldots$. Using the conditional (or transition) probability density definition (5.21) and the Markov assumption (5.22), one gets

$$\begin{aligned}\mathcal{P}(\,\varphi_1, t_1\,;\,\varphi_2, t_2 \,|\, \sigma_1, \tau_1\,) &\stackrel{(5.21)}{=} \mathcal{P}(\,\varphi_1, t_1 \,|\, \varphi_2, t_2\,; \sigma_1, \tau_1\,)\mathcal{P}(\,\varphi_2, t_2 \,|\, \sigma_1, \tau_1\,) \\ &\stackrel{(5.22)}{=} \mathcal{P}(\,\varphi_1, t_1 \,|\, \varphi_2, t_2\,)\mathcal{P}(\,\varphi_2, t_2 \,|\, \sigma_1, \tau_1\,),\end{aligned} \quad (5.23)$$

and in the generalized case:

$$\begin{aligned}\mathcal{P}(\,\varphi_1, t_1\,;\,\varphi_2, t_2\,;\,\varphi_3, t_3\,;\,\ldots\,;\varphi_\mathrm{n}\,, t_\mathrm{n} \,|\, \sigma_1, \tau_1\,) &\stackrel{(5.21)}{=} \mathcal{P}(\,\varphi_1, t_1 \,|\, \varphi_2, t_2\,;\,\ldots\,;\varphi_\mathrm{n}\,, t_\mathrm{n}\,;\,\sigma_1, \tau_1\,)\times \quad (5.24)\\ &\times \mathcal{P}(\,\varphi_2, t_2 \,|\, \varphi_3, t_3\,;\,\ldots\,;\varphi_\mathrm{n}\,, t_\mathrm{n}\,;\,\sigma_1, \tau_1\,)\,\mathcal{P}(\,\varphi_3, t_3 \,|\, \varphi_4, t_4\,;\,\ldots\,;\varphi_\mathrm{n}\,, t_\mathrm{n}\,;\,\sigma_1, \tau_1\,) \times \ldots \times \\ &\times\, \mathcal{P}(\,\varphi_{\mathrm{n}-1}\,, t_{\mathrm{n}-1} \,|\, \varphi_\mathrm{n}\,, t_\mathrm{n}\,;\,\sigma_1, \tau_1\,)\,\mathcal{P}(\,\varphi_\mathrm{n}\,, t_\mathrm{n} \,|\, \sigma_1, \tau_1\,) \\ &\stackrel{(5.22)}{=} \mathcal{P}(\,\varphi_1, t_1 \,|\, \varphi_2, t_2\,)\mathcal{P}(\,\varphi_2, t_2 \,|\, \varphi_3, t_3\,)\mathcal{P}(\,\varphi_3, t_3 \,|\, \varphi_4, t_4\,)\ldots\mathcal{P}(\,\varphi_n, t_n \,|\, \sigma_1, \tau_1\,). \quad (5.25)\end{aligned}$$

By definition and using the Markov property (5.23), the two-time correlation function with $t_1 \geq t_2 \geq t_0$ is

$$\Big\langle \varphi(t_1)\, \varphi(t_2) \Big| \big[\varphi_0, t_0\big] \Big\rangle_0 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\varphi_1\, d\varphi_2\, \varphi_1\, \varphi_2\, \mathcal{P}_m\big(\varphi_1, t_1;\, \varphi_2, t_2 \,\big|\, \varphi_0, t_0\big)$$
$$\stackrel{(5.22)}{=} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\varphi_1\, d\varphi_2\, \varphi_1\, \varphi_2\, \mathcal{P}_m\big(\varphi_1, t_1 \,\big|\, \varphi_2, t_2\big)\, \mathcal{P}_m\big(\varphi_2, t_2 \,\big|\, \varphi_0, t_0\big). \quad (5.26)$$

Without making any assumptions about the initial values of φ_0 and t_0, using (5.26) together with (5.20), one has

$$\Big\langle \varphi(t_1)\, \varphi(t_2) \Big| \big[\varphi_0, t_0\big] \Big\rangle_0 = \frac{3H^4}{8\,\pi^2 m^2}\left(\mathrm{e}^{-\frac{m^2}{3H}(t_1-t_2)} - \mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2-2t_0)}\right) + \frac{\varphi_0^2\left(\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2-2t_0)} - \mathrm{e}^{-\frac{m^2}{3H}(t_1+3t_2-4t_0)}\right)}{\left(1-\mathrm{e}^{-\frac{2m^2}{3H}(t_2-t_0)}\right)}. \quad (5.27)$$

We choose the initial value $\varphi_0 = 0$ at time $t_0 = 0$ so that $\big\langle \varphi(t_0)\varphi(t_0) \big| \big[\varphi_0, t_0\big] \big\rangle\big|_{t_0=0} = 0$. The chosen values $\varphi_0 = t_0 = 0$ lead to the following form of the conditional (or transition) probability $\mathcal{P}_m\big(\varphi_2, t_2 \big| \varphi_0 = 0, t_0 = 0\big)$, which is given by (5.3) together with (5.4):

$$\mathcal{P}_m\big(\varphi_2, t_2\big) = \frac{2\sqrt{\pi}\, m}{\sqrt{3}\, H^2}\left(1-\mathrm{e}^{-\frac{2m^2 t_2}{3H}}\right)^{-1/2} \exp\left(-\frac{4\pi^2 m^2 \varphi_2^2}{3H^4\left(1-\mathrm{e}^{-\frac{2m^2 t_2}{3H}}\right)}\right). \quad (5.28)$$

Remarkably, the transition probability $\mathcal{P}_m\big(\varphi_2, t_2 \big|\, \varphi_0, t_0\big)$ at the limit $t_0 \to -\infty$ goes to

$$\mathcal{P}_m\big(\varphi_2, t_2 \big|\, \varphi_0, t_0\big) \xrightarrow{t_0 \to -\infty} \mathcal{P}_{\mathrm{st}}\big(\varphi_2\big) := \frac{2\sqrt{\pi}\, m}{\sqrt{3}\, H^2} \exp\left(-\frac{4\pi^2 m^2 \varphi_2^2}{3H^4}\right) \quad (5.29)$$

the stationary distribution, and the two-time correlation function (5.27):

$$\Big\langle \varphi(t_1)\, \varphi(t_2) \Big| \big[\varphi_0, t_0\big] \Big\rangle_0 \Big|_{t_1 \geq t_2} \xrightarrow{t_0 \to -\infty} \frac{3H^4}{8\,\pi^2 m^2}\, \mathrm{e}^{-\frac{m^2}{3H}(t_1-t_2)} \quad (5.30)$$

depends only on the time difference. That is the notable feature of the stationary process, to which belongs the Fokker-Planck or forward Kolmogorov equation (5.19) with the linear drift term, i.e., in the free massive case with $\lambda = 0$. By inserting the chosen initial values $\varphi_0 = t_0 = 0$ into (5.27) or, equivalently, after straightforward integration of (5.26) with (5.20) and (5.28), one gets

$$\left\langle \varphi(t_1)\,\varphi(t_2)\Big|[0,0]\right\rangle_0 := \left\langle \varphi(t_1)\,\varphi(t_2)\right\rangle_0 = \frac{3H^4}{8\,\pi^2 m^2}\left(\mathrm{e}^{-\frac{m^2}{3H}(t_1-t_2)} - \mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2)}\right). \tag{5.31}$$

This is the well-known Ornstein-Uhlenbeck stochastic process [99, 101, 103]: the unique Gaussian and Markov process that possesses a stationary state. The obtained two-point correlation function (5.31) precisely coincides with that of the free massive scalar field (2.22).

Note that at equal times (5.31) matches (5.5) above, which can be obtained using (5.28) by straightforward integration:

$$\left\langle \varphi^2(t)\Big|[0,0]\right\rangle_0 := \left\langle \varphi^2(t)\right\rangle_0 = \int_{-\infty}^{+\infty} d\varphi\;\varphi^2\,\mathcal{P}_m(\varphi,t) = \frac{3H^4}{8\,\pi^2 m^2}\left(1-\mathrm{e}^{-\frac{2m^2 t}{3H}}\right). \tag{5.32}$$

Indeed, (5.28) is exactly our ansatz (5.3) with (5.4). Let us point out that this two-point function not only reduces to that of the Wiener process [37, 38] in the smooth massless limit but also approximates this behavior in the regime $t \ll H/m^2$, as observed in [54, 143].

To get the iterative λ series for the multi-time correlation functions, one employs (5.19) with t replaced by t_1, φ by φ_1, t_0 by t_2, and φ_0 by φ_2. For the two-time correlation function, we multiply l.h.s. and r.h.s. of (5.19) by φ_1, φ_2, and $\mathcal{P}(\varphi_2,t_2)$, integrate by parts over φ_1, φ_2, and obtain the corresponding first-order differential equation:

$$\frac{\partial}{\partial t_1}\left\langle \varphi(t_1)\,\varphi(t_2)\right\rangle = -\frac{m^2}{3H}\left\langle \varphi(t_1)\,\varphi(t_2)\right\rangle - \frac{\lambda}{3H}\left\langle \varphi^3(t_1)\,\varphi(t_2)\right\rangle. \tag{5.33}$$

Its formal solution can be written as

$$\left\langle \varphi(t_1)\,\varphi(t_2)\right\rangle_{\mathrm{k+1}} = -\frac{\lambda}{3H}\,\mathrm{e}^{-\frac{m^2 t_1}{3H}}\int dt_1\,\mathrm{e}^{\frac{m^2 t_1}{3H}}\left\langle \varphi^3(t_1)\,\varphi(t_2)\right\rangle_{\mathrm{k}}. \tag{5.34}$$

In contrast to the equal-time correlation functions, we have here the indefinite integral. After integration, one has the antiderivative and the unknown function, which is determined by matching it to the known result for the equal-time correlation function.

In the simplest example, namely at λ–linear order in (5.34), we will illustrate the proposed technique in detail. We extract the integrand[4] using the definition of the Markov

4 Here, one can recognize the discrete Furutsu-Novikov's theorem [151, 152] for a multivariate Gaussian distribution with zero mean $\langle\varphi_{\mathrm{i}}\rangle = 0$. According to that theorem, averages of $\langle\varphi_{\mathrm{i}}\, f(\vec{\varphi})\rangle$ can be obtained as

$$\begin{aligned}\left\langle\varphi_{\mathrm{i}}\, f(\vec{\varphi})\right\rangle = \sum_{\mathrm{n}}\left\langle\varphi_{\mathrm{i}}\varphi_{\mathrm{n}}\right\rangle\left\langle\frac{\partial f(\vec{\varphi})}{\partial\varphi_{\mathrm{n}}}\right\rangle\Bigg|_{f(\vec{\varphi}):=\varphi_{\mathrm{k}}\,\varphi_{\mathrm{l}}\,\varphi_{\mathrm{m}}} &= \sum_{\mathrm{n}}\left\langle\varphi_{\mathrm{i}}\varphi_{\mathrm{n}}\right\rangle\left\langle\delta_{\mathrm{kn}}\varphi_{\mathrm{l}}\varphi_{\mathrm{n}} + \delta_{\mathrm{ln}}\varphi_{\mathrm{k}}\varphi_{\mathrm{m}} + \delta_{\mathrm{mn}}\varphi_{\mathrm{k}}\varphi_{\mathrm{l}}\right\rangle \\ &= \left\langle\varphi_{\mathrm{i}}\varphi_{\mathrm{k}}\right\rangle\left\langle\varphi_{\mathrm{l}}\varphi_{\mathrm{m}}\right\rangle + \left\langle\varphi_{\mathrm{i}}\varphi_{\mathrm{l}}\right\rangle\left\langle\varphi_{\mathrm{k}}\varphi_{\mathrm{m}}\right\rangle + \left\langle\varphi_{\mathrm{i}}\varphi_{\mathrm{m}}\right\rangle\left\langle\varphi_{\mathrm{k}}\varphi_{\mathrm{l}}\right\rangle.\end{aligned} \tag{5.35}$$

For our multi-time functions, it takes Wick's form, (5.37), (5.41), etc. For the theorem's proof, see, e.g., [153].

property (5.23) and the initial conditions $\varphi_0 = t_0 = 0$ as above, obtaining

$$\begin{aligned}\Big\langle \varphi^3(t_1)\,\varphi(t_2)\Big|[0,0]\Big\rangle_0 &:= \Big\langle \varphi^3(t_1)\,\varphi(t_2)\Big\rangle_0 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\varphi_1\, d\varphi_2\, \varphi_1^3\, \varphi_2\, \mathcal{P}_m\big(\varphi_1, t_1;\, \varphi_2, t_2\,\big|\,\varphi_0, t_0\big)\\
&\overset{(5.23)}{=} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\varphi_1\, d\varphi_2\;\varphi_1^3\, \varphi_2\; \mathcal{P}_m\big(\varphi_1, t_1\,\big|\,\varphi_2, t_2\big)\,\mathcal{P}_m\big(\varphi_2, t_2\big) \qquad (5.36)\\
&= \frac{27H^8}{64\,\pi^4 m^4}\left(1 - \mathrm{e}^{-\frac{2m^2 t_1}{3H}}\right)\left(\mathrm{e}^{-\frac{m^2}{3H}(t_1-t_2)} - \mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2)}\right). \qquad (5.37)\end{aligned}$$

Therefore, one calculates indefinite integral (5.34) with the obtained integrand (5.37):

$$\Big\langle \varphi(t_1)\,\varphi(t_2)\Big\rangle_\lambda = -\frac{27\lambda H^8}{64\,\pi^4 m^6}\left(\left(\mathrm{e}^{\frac{m^2 t_2}{3H}} - \mathrm{e}^{-\frac{m^2 t_2}{3H}}\right)\left(\frac{m^2 t_1}{3H} + \frac{1}{2}\,\mathrm{e}^{-\frac{2m^2 t_1}{3H}}\right) + \mathrm{C}\,(t_2)\right)\mathrm{e}^{-\frac{m^2 t_1}{3H}}. \qquad (5.38)$$

Here, $\mathrm{C}\,(t_2)$ is an unknown function depending only on t_2. To define it, one can rely on the known answer for $\langle \varphi^2(t)\rangle_\lambda$ from (5.7). We set $t_1 = t_2$ in this expression (5.38) and equate it to (5.7), after substituting $t = t_2$, resulting in

$$\begin{aligned}\Big\langle \varphi(t_1)\,\varphi(t_2)\Big\rangle_\lambda =& -\frac{27\lambda H^8}{128\,\pi^4 m^6}\left(\left(2 + \frac{2m^2}{3H}(t_1 - t_2)\right)\mathrm{e}^{-\frac{m^2}{3H}(t_1-t_2)} + \mathrm{e}^{-\frac{m^2}{3H}(3\,t_1-t_2)}\right. \qquad (5.39)\\
&\left. - \left(1 + \frac{2m^2}{3H}(t_1 + 3t_2)\right)\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2)} - \mathrm{e}^{-\frac{m^2}{3H}(3\,t_1+t_2)} - \mathrm{e}^{-\frac{m^2}{3H}(t_1+3\,t_2)}\right).\end{aligned}$$

That expression matches one from (2.25) with the expanded moduli.

One can proceed in this way to calculate the four-point correlation function. The idea is the same: we relate different correlation functions at different orders in λ through the Fokker-Planck or forward Kolmogorov equation (5.19). We intend to compute the four-point correlation function at linear order in λ, i.e., $\langle \varphi(t_1)\,\varphi(t_2)\,\varphi(t_3)\,\varphi(t_4)\rangle_\lambda$. Not surprising to find that one needs to have at hand the six-point correlation function at zeroth order and the two-point correlation function at λ–linear order.

At zeroth order in λ, one calculates the integrand using the definition and the same initial conditions $\varphi_0 = t_0 = 0$, together with the Markov property (5.25), $t_1 \geq t_2 \geq t_3 \geq t_4 \geq t_0 = 0$:

$$\begin{aligned}\Big\langle \varphi^3(t_1)\varphi(t_2)\varphi(t_3)\varphi(t_4)\Big\rangle_0 &= \int_{-\infty}^{+\infty}..\int_{-\infty}^{+\infty}\left(\prod_{\mathrm{i}=1}^{4} d\varphi_\mathrm{i}\right)\varphi_1^3\,\varphi_2\,\varphi_3\,\varphi_4\mathcal{P}_m\big(\varphi_1, t_1\,;\varphi_2, t_2\,;\varphi_3, t_3\,;\varphi_4, t_4\,\big|\varphi_0, t_0\big)\\
&\overset{(5.25)}{=} \int_{-\infty}^{+\infty}..\int_{-\infty}^{+\infty}\left(\prod_{\mathrm{i}=1}^{4} d\varphi_\mathrm{i}\right)\varphi_1^3\,\varphi_2\,\varphi_3\,\varphi_4\left(\prod_{\mathrm{i}=1}^{3}\mathcal{P}_m\big(\varphi_\mathrm{i}, t_\mathrm{i}\,\big|\varphi_{\mathrm{i}+1}, t_{\mathrm{i}+1}\big)\right)\mathcal{P}_m\big(\varphi_4, t_4\big) \qquad (5.40)\end{aligned}$$

$$= \frac{81H^{12}}{512\,\pi^6 m^6}\, e^{-\frac{m^2 t_1}{3H}} \left(1 - e^{-\frac{2m^2 t_4}{3H}}\right)\left(e^{-\frac{2m^2 t_1}{3H}}\left(5\, e^{-\frac{m^2}{3H}(t_2+t_3-t_4)} - 4\, e^{-\frac{m^2}{3H}(t_2-t_3-t_4)}\right.\right. \tag{5.41}$$
$$\left.\left. - 3\, e^{\frac{m^2}{3H}(t_2-t_3+t_4)} + 2\, e^{\frac{m^2}{3H}(t_2+t_3+t_4)}\right) - 3\, e^{-\frac{m^2}{3H}(t_2+t_3-t_4)} + 2\, e^{-\frac{m^2}{3H}(t_2-t_3-t_4)} + e^{\frac{m^2}{3H}(t_2-t_3+t_4)}\right).$$

By setting $t_1 = t_2 = t_3 = t_4 := t$ in (5.41), one comes back to (5.5) for n = 3, as expected. One can also identify the expected form of (5.35) or the Wick form including additional permutations.

We shall start with the case where the last time differs from the others, which are equal, $\langle \varphi^3(t_1)\varphi(t_4)\rangle_\lambda$, and after iterating through $\langle \varphi^2(t_1)\varphi(t_3)\,\varphi(t_4)\rangle_\lambda$, we will have at hand the final result for $\langle \varphi(t_1)\,\varphi(t_2)\,\varphi(t_3)\,\varphi(t_4)\rangle_\lambda$.

By multiplying both l.h.s. and r.h.s. of (5.19) (with t replaced by t_1, φ by φ_1, t_0 by t_2, and φ_0 by φ_2) by φ_1^3, φ_2, and $\mathcal{P}(\varphi_2, t_2)$, integrating by parts over φ_1 and φ_2, and setting $t_2 = t_4$, one arrives at the first-order differential equation, whose formal solution takes the form:

$$\left\langle \varphi^3(t_1)\varphi(t_4)\right\rangle_{k+1} = e^{-\frac{m^2 t_1}{H}} \int dt_1 e^{\frac{m^2 t_1}{H}} \left(-\frac{\lambda}{H}\left\langle \varphi^5(t_1)\varphi(t_4)\right\rangle_k + \frac{3H^3}{4\pi^2}\left\langle \varphi(t_1)\varphi(t_4)\right\rangle_{k+1}\right). \tag{5.42}$$

For the computation at λ−linear order, we already know the integrand from (5.39) and (5.41). By taking indefinite integral (5.42), one gets an expression with the unknown function $C(t_4)$ and defines it as in the previous case: by substituting $t_1 = t_4$ in the indefinite integral's expression and equating it to $\langle \varphi^4(t_4)\rangle_\lambda$ from (5.9). The result is the following:

$$\left\langle \varphi^3(t_1)\varphi(t_4)\right\rangle_\lambda = -\frac{81\,\lambda H^{12}}{1024\,\pi^6 m^8}\left(\left(18 + \frac{2m^2}{H}(t_1 - t_4)\right)e^{-\frac{m^2}{3H}(t_1-t_4)} - 2e^{-\frac{3m^2}{3H}(t_1-t_4)}\right. \tag{5.43}$$
$$- \left(15 + \frac{2m^2}{H}(t_1 + 3\,t_4)\right)e^{-\frac{m^2}{3H}(t_1+t_4)} - \left(21 + \frac{2m^2}{H}(9\,t_1 - 5\,t_4)\right)e^{-\frac{m^2}{3H}(3t_1-t_4)} + 5\,e^{-\frac{3m^2}{3H}(t_1+t_4)}$$
$$\left. + \left(18 + \frac{2m^2}{H}(9\,t_1 + 7\,t_4)\right)e^{-\frac{m^2}{3H}(3t_1+t_4)} - 3\,e^{-\frac{m^2}{3H}(t_1+3t_4)} - 15\,e^{-\frac{m^2}{3H}(5t_1-t_4)} + 15\,e^{-\frac{m^2}{3H}(5t_1+t_4)}\right).$$

Following the same spirit, one finds the corresponding equation for $\langle \varphi^2(t_1)\varphi(t_3)\,\varphi(t_4)\rangle_\lambda$. We multiply both l.h.s. and r.h.s. of (5.19) (with t replaced by t_1, φ by φ_1, t_0 by t_3, and φ_0 by φ_3) by φ_1^2, φ_3, φ_4, $\mathcal{P}(\varphi_3, t_3 \,|\, \varphi_4, t_4)$, and $\mathcal{P}(\varphi_4, t_4)$ and integrate by parts over φ_1, φ_3 and φ_4. The formal solution in this case becomes

$$\left\langle \varphi^2(t_1)\,\varphi(t_3)\,\varphi(t_4)\right\rangle_{k+1} = e^{-\frac{2m^2 t_1}{3H}} \int dt_1\, e^{\frac{2m^2 t_1}{3H}} \left(-\frac{2\lambda}{3H}\left\langle \varphi^4(t_1)\,\varphi(t_3)\,\varphi(t_4)\right\rangle_k \right. \tag{5.44}$$
$$\left. + \frac{H^3}{4\pi^2}\left\langle \varphi(t_3)\,\varphi(t_4)\right\rangle_{k+1}\right).$$

For the λ–linear order, after straightforward integration, one obtains the indefinite integral and the unknown function of two arguments, $C(t_3, t_4)$, which we define by matching it to the previous "iteration" $\langle\varphi^3(t_1)\varphi(t_4)\rangle_\lambda$; see (5.43).

As the final step, one gets the analogous equation through (5.19), and its formal solution can be written in the same manner as above:

$$\Big\langle\varphi(t_1)\,\varphi(t_2)\,\varphi(t_3)\,\varphi(t_4)\Big\rangle_{\mathrm{k+1}} = -\frac{\lambda}{3H}\,\mathrm{e}^{-\frac{m^2 t_1}{3H}}\int dt_1\,\mathrm{e}^{\frac{m^2 t_1}{3H}}\Big\langle\varphi^3(t_1)\varphi(t_2)\varphi(t_3)\varphi(t_4)\Big\rangle_{\mathrm{k}}. \tag{5.45}$$

For the λ–linear order, that integral is equal to some expression with the unknown function of three arguments $C\,(t_2, t_3, t_4)$. We define it by setting $t_2 = t_1$ and matching it to $\langle\varphi^2(t_1)\,\varphi(t_3)\,\varphi(t_4)\rangle_\lambda$ from the previous "iteration" result. Therefore, one gets

$$\begin{aligned}
\Big\langle\varphi(t_1)\,\varphi(t_2)\,\varphi(t_3)\,\varphi(t_4)\Big\rangle_\lambda = -\frac{81\lambda H^{12}}{1024\,\pi^6 m^8}\Bigg(&\left(4+\frac{2m^2}{3H}(t_1-t_2+t_3-t_4)\right)\mathrm{e}^{-\frac{m^2}{3H}(t_1-t_2+t_3-t_4)}\\
&+\left(16+\frac{4m^2}{3H}(t_1+3\,t_2-3\,t_3-t_4)\right)\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2-t_3-t_4)}-2\,\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2+t_3-3\,t_4)}\\
&-2\,\mathrm{e}^{-\frac{m^2}{3H}(3t_1-t_2-t_3-t_4)}-\left(3+\frac{2m^2}{3H}(t_1-t_2+t_3+3\,t_4)\right)\mathrm{e}^{-\frac{m^2}{3H}(t_1-t_2+t_3+t_4)}\\
&-\left(14+\frac{4m^2}{3H}(t_1+3\,t_2-3\,t_3+3\,t_4)\right)\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2-t_3+t_4)}+\mathrm{e}^{-\frac{m^2}{3H}(t_1-t_2+3t_3-t_4)}\\
&-\left(33+\frac{2m^2}{H}(t_1+3\,t_2+5\,t_3-5\,t_4)\right)\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2+t_3-t_4)}+2\,\mathrm{e}^{-\frac{m^2}{3H}(3t_1-t_2-t_3+t_4)}\\
&+3\,\mathrm{e}^{-\frac{m^2}{3H}(3t_1-t_2+t_3-t_4)}+4\,\mathrm{e}^{-\frac{m^2}{3H}(3t_1+t_2-t_3-t_4)}+4\,\mathrm{e}^{-\frac{m^2}{3H}(t_1+3t_2-t_3-t_4)}-\mathrm{e}^{-\frac{m^2}{3H}(t_1-t_2+3t_3+t_4)}\\
&-\mathrm{e}^{-\frac{m^2}{3H}(t_1-t_2+t_3+3t_4)}+\left(30+\frac{2m^2}{H}(t_1+3\,t_2+5\,t_3+7\,t_4)\right)\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2+t_3+t_4)}\\
&-2\,\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2-t_3+3t_4)}-3\,\mathrm{e}^{-\frac{m^2}{3H}(3t_1-t_2+t_3+t_4)}-4\,\mathrm{e}^{-\frac{m^2}{3H}(3t_1+t_2-t_3+t_4)}-5\,\mathrm{e}^{-\frac{m^2}{3H}(3t_1+t_2+t_3-t_4)}\\
&-4\,\mathrm{e}^{-\frac{m^2}{3H}(t_1+3t_2-t_3+t_4)}-5\,\mathrm{e}^{-\frac{m^2}{3H}(t_1+3t_2+t_3-t_4)}-5\,\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2+3t_3-t_4)}+5\,\mathrm{e}^{-\frac{m^2}{3H}(3t_1+t_2+t_3+t_4)}\\
&+5\,\mathrm{e}^{-\frac{m^2}{3H}(t_1+3t_2+t_3+t_4)}+5\,\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2+3t_3+t_4)}+5\,\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2+t_3+3t_4)}\Bigg).
\end{aligned} \tag{5.46}$$

The last comparison will be made for the two-point correlation function at order λ^2. Our equation is (5.34) and $\langle\varphi^3(t_1)\,\varphi(t_2)\rangle_\lambda$ is taken from (5.43). Following our proposed routine, one takes the indefinite integral and finds the unknown function $C(t_2)$ by matching this expression to $\langle\varphi^2(t_2)\rangle_{\lambda^2}$ from (5.8). The final answer after all is

$$\Big\langle\varphi(t_1)\varphi(t_2)\Big\rangle_{\lambda^2} = \frac{81\,\lambda^2 H^{12}}{2048\pi^6 m^{10}}\Bigg(\left(30+\frac{12m^2}{H}(t_1-t_2)+\frac{2m^4}{3H^2}(t_1-t_2)^2\right)\mathrm{e}^{-\frac{m^2}{3H}(t_1-t_2)} \tag{5.47}$$

$$+ 2\,\mathrm{e}^{-\frac{m^2}{H}(t_1-t_2)} + \left(48 + \frac{2m^2}{H}\,(9t_1 - 5t_2)\right)\mathrm{e}^{-\frac{m^2}{3H}(3t_1-t_2)}$$
$$+ \left(36 - \frac{2m^2}{H}\,(5t_1 + 23t_2) - \frac{2m^4}{3H^2}\,(t_1 + 3t_2)^2\right)\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2)}$$
$$- \left(45 + \frac{2m^2}{H}\,(9t_1 + 7t_2)\right)\mathrm{e}^{-\frac{m^2}{3H}(3t_1+t_2)} - \left(\frac{117}{2} + \frac{2m^2}{H}\,(t_1 + 15t_2)\right)\mathrm{e}^{-\frac{m^2}{3H}(t_1+3t_2)}$$
$$\left. + \frac{15}{2}\,\mathrm{e}^{-\frac{m^2}{3H}(5t_1-t_2)} - \frac{15}{2}\,\mathrm{e}^{-\frac{m^2}{3H}(5t_1+t_2)} - 5\,\mathrm{e}^{-\frac{m^2}{3H}(3t_1+3t_2)} - \frac{15}{2}\,\mathrm{e}^{-\frac{m^2}{3H}(t_1+5t_2)}\right).$$

For the de Sitter-invariant two-time correlation functions, one can obtain perturbative series using our approach. We take the late-time behavior of $\langle\varphi^3(t_1)\,\varphi(t_2)\rangle_0$ from (5.37) and $\langle\varphi^3(t_1)\,\varphi(t_2)\rangle_\lambda$ from (5.43), which depends solely on time difference, integrate using (5.34), and match the result to equal-time results at late times, which are just constants:

$$\left\langle\varphi^3(t_1)\,\varphi(t_2)\right\rangle_0 \xrightarrow[\text{times}]{\text{late}} \frac{27\,H^8}{64\,\pi^4 m^4}\,\mathrm{e}^{-\frac{m^2}{3H}(t_1-t_2)} \quad\Rightarrow \tag{5.48}$$
$$\left\langle\varphi(t_1)\,\varphi(t_2)\right\rangle_\lambda \overset{(5.34)}{=} -\frac{9\,\lambda H^7}{64\pi^4 m^4}\Big(t_1 + \mathrm{C}(t_2)\Big)\mathrm{e}^{-\frac{m^2}{3H}(t_1-t_2)}\bigg|_{t_1=t_2} = \left\langle\varphi^2\right\rangle_\lambda^{\text{late-time}} \overset{(5.11)}{=} -\frac{27\,\lambda H^8}{64\pi^4 m^6};$$

$$\left\langle\varphi^3(t_1)\varphi(t_2)\right\rangle_\lambda \xrightarrow[\text{times}]{\text{late}} -\frac{81\lambda H^{12}}{512\,\pi^6 m^8}\left(\left(9 + \frac{m^2}{H}(t_1 - t_2)\right)\mathrm{e}^{-\frac{m^2}{3H}(t_1-t_2)} - \mathrm{e}^{-\frac{3m^2}{3H}(t_1-t_2)}\right) \quad\Rightarrow$$
$$\left\langle\varphi(t_1)\,\varphi(t_2)\right\rangle_{\lambda^2} \overset{(5.34)}{=} \frac{27\,\lambda^2 H^{11}}{512\,\pi^6 m^8}\left(\left(9\,t_1 + \frac{m^2}{2H}\,t_1^2 - \frac{m^2}{H}\,t_1 t_2\right)\mathrm{e}^{-\frac{m^2}{3H}(t_1-t_2)} \right. \tag{5.49}$$
$$\left.\left. + \frac{3H}{2m^2}\,\mathrm{e}^{-\frac{m^2}{H}|t_1-t_2|} + \mathrm{C}(t_2)\mathrm{e}^{-\frac{m^2 t_1}{3H}}\right)\right|_{t_1=t_2} = \left\langle\varphi^2\right\rangle_{\lambda^2}^{\text{late-time}} \overset{(5.11)}{=} \frac{81\,\lambda^2 H^{12}}{64\,\pi^6 m^{10}}.$$

Therefore, the final de Sitter-invariant series is the following:

$$\left\langle\varphi(t_1)\,\varphi(t_2)\right\rangle \xrightarrow[\text{times}]{\text{late}} \frac{3H^4}{8\pi^2 m^2}\,\mathrm{e}^{-\frac{m^2}{3H}(t_1-t_2)} - \frac{27\,\lambda H^8}{64\pi^4 m^6}\left(1 + \frac{m^2}{3H}(t_1 - t_2)\right)\mathrm{e}^{-\frac{m^2}{3H}(t_1-t_2)} \tag{5.50}$$
$$+ \frac{81\,\lambda^2 H^{12}}{1024\,\pi^6 m^{10}}\left(15 + \frac{6m^2}{H}(t_1 - t_2) + \frac{m^4}{3H^2}(t_1 - t_2)^2 + \mathrm{e}^{-\frac{2m^2}{3H}(t_1-t_2)}\right)\mathrm{e}^{-\frac{m^2}{3H}(t_1-t_2)} + O\big(\lambda^3\big).$$

All the obtained expressions in this part of the chapter coincide with those from our previous Chapter 2; see (2.25), (2.30), and (2.56). Our series are convergent and well defined. The two-point correlation function at late times approaches the de Sitter-invariant result (5.50) and is in agreement with (2.34) and results from the Schwinger–Keldysh formalism [117, 118]. Remarkably, our proposed technique in this chapter takes several trivial lines, in contrast to alternative approaches.

We conclude this section with the massless case. The exact solution to (5.19) for the conditional (or transition) probability is also known:

$$\mathcal{P}_{m=0}\big(\varphi, t \,\big|\, \varphi_0, t_0\big) = \frac{\sqrt{2\pi}}{\sqrt{H^3(t-t_0)}} \exp\left(-\frac{2\pi^2\,(\varphi - \varphi_0)^2}{H^3(t-t_0)}\right). \tag{5.51}$$

One finds the analogue of (5.28) and the correlation function by straightforward integration, as in (5.26), (5.36), (5.41), etc. For arbitrary initial values φ_0 and t_0, through (5.26) we have

$$\left\langle \varphi_{m=0}(t_1)\, \varphi_{m=0}(t_2) \Big| \big[\varphi_0, t_0\big] \right\rangle_0 \Big|_{t_1 \geq t_2} = \frac{H^3}{4\pi^2}\big(t_2 - t_0\big) + \varphi_0^2, \tag{5.52}$$

being the smooth massless limit of (5.27). To ensure $\langle \varphi_{m=0}(t_0)\, \varphi_{m=0}(t_0) \big| \big[\varphi_0, t_0\big] \rangle\big|_{t_0=0} = 0$, we set $\varphi_0 = 0$ at the initial time $t_0 = 0$ as before.

Following our approach, one defines the necessary first-order differential equation and, from its formal solution (without exponents), takes the indefinite integral. The result of integration should match equal-time results (or those from the previous "iteration") to fix the unknown function. We present the final expressions with the chosen initial conditions $\varphi_0 = t_0 = 0$ for the massless case:

$$\begin{aligned} \left\langle \varphi_{m=0}(t_1)\, \varphi_{m=0}(t_2) \right\rangle &= \frac{H^3 t_2}{4\pi^2} - \frac{\lambda H^5}{96\,\pi^4}\left(3\,t_1^2 t_2 + t_2^3\right) \\ &+ \frac{\lambda^2 H^7}{1536\,\pi^6}\left(11\,t_1^4 t_2 + 2\,t_1^2 t_2^3 + \frac{31}{5}\,t_2^5\right) + O(\lambda^3); \end{aligned} \tag{5.53}$$

$$\left\langle \varphi^3_{m=0}(t_1)\, \varphi_{m=0}(t_2)\, \varphi_{m=0}(t_3)\, \varphi_{m=0}(t_4) \right\rangle_0 = \frac{3H^9}{64\,\pi^6}\left(t_1 t_2 t_4 + 2\,t_1 t_3 t_4 + 2\,t_2 t_3 t_4\right); \tag{5.54}$$

$$\begin{aligned} \left\langle \varphi_{m=0}(t_1)\, \varphi_{m=0}(t_2)\, \varphi_{m=0}(t_3)\, \varphi_{m=0}(t_4) \right\rangle_\lambda &= -\frac{\lambda H^8}{384\,\pi^6}\Big(3\,t_1^2\, t_2 t_4 + 6\,t_1^2\, t_3 t_4 \\ &+ 12\,t_1 t_2 t_3 t_4 + t_2^3\, t_4 + 6\,t_2^2\, t_3 t_4 + 3\,t_2 t_3^2\, t_4 + t_2 t_4^3 + 2\,t_3^3\, t_4 + 2 t_3 t_4^3\Big). \end{aligned} \tag{5.55}$$

Indeed, all results listed above correspond to the massless limit of (5.31), (5.39), (5.41), (5.46), and (5.47), and two-point and four-point correlation functions match those obtained in Chapter 2; see (2.35) and (2.57). At equal times, those expressions reduce to (5.16)–(5.18).

Several studies have proposed modified forms of the Fokker–Planck equation to include the sub-leading corrections; see, e.g., [88, 154, 155, 156]. However, from a mathematical perspective, the question of the uniqueness of the solution to the Fokker–Planck or forward Kolmogorov equation remains non-trivial. Even for its standard form, with constant diffusion and polynomial drift terms, the uniqueness was only recently proved [157, 158]. For the corrected form of the Fokker–Planck or forward Kolmogorov equation, it remains unclear how to consistently impose a well-defined probabilistic interpretation and compute the corresponding moments.

The last comment we would like to make is about the free massive case, $\lambda = 0$. In this case, our equations (5.33) and the preceding equations leading to (5.42), (5.44), and (5.45) resemble the "formulae of differentiation" from the original work by Shapiro and Loginov [159]:

$$\frac{d}{dt_1}\Big\langle\, \alpha(t_1)\, \alpha(t_2) \ldots \alpha(t_n)\Big\rangle = -\nu \Big\langle\, \alpha(t_1)\, \alpha(t_2) \ldots \alpha(t_n)\Big\rangle \tag{5.56}$$

if $t_1 > t_2 > \ldots > t_n$ and $\alpha(t)$ is the Ornstein-Uhlenbeck stochastic process with

$$\big\langle \alpha(t)\big\rangle = 0 \qquad \text{and} \qquad \big\langle \alpha(t_1)\, \alpha(t_2)\big\rangle = \sigma^2\, \mathrm{e}^{-\nu|t_1-t_2|}; \tag{5.57}$$

see also [102, 160]. The full form of this "formulae of differentiation" is [159]:

$$\frac{d}{dt_1}\Big\langle\, \alpha(t_1)\, \Phi\,(t_1, [\alpha]\,)\Big\rangle = \Big\langle\, \alpha(t_1)\, \frac{d}{dt_1} \Phi(t_1, [\alpha]\,)\Big\rangle - \nu \Big\langle\, \alpha(t_1)\, \Phi(t_1, [\alpha]\,)\Big\rangle, \tag{5.58}$$

where $\Phi(t_1, [\alpha]\,)$ is a functional depending on t_1 and all values of $\alpha(t_i)$ at the moments t_i prior to t_1. Note that they did not operate with the Fokker-Planck equation. Shapiro and Loginov [159] have also deduced its generalized version, which is

$$\begin{aligned}\frac{d}{dt_1}\Big\langle \alpha^{\mathrm{k}}(t_1)\, \Phi(t_1, [\alpha]\,)\Big\rangle =& \Big\langle \alpha^{\mathrm{k}}(t_1)\, \frac{d}{dt_1}\Phi(t_1, [\alpha]\,)\Big\rangle - \mathrm{k}\,\nu \Big\langle \alpha(t_1)\, \Phi(t_1, [\alpha]\,)\Big\rangle \\ &+ \mathrm{k}(\mathrm{k}-1)\,\nu\, \sigma^2 \Big\langle \alpha^{\mathrm{k}-2}(t_1)\, \Phi(t_1, [\alpha]\,)\Big\rangle.\end{aligned} \tag{5.59}$$

This expression coincides with our (5.33) and the preceding equations leading to (5.42), (5.44), and (5.45) in the free massive case after restoring the parameters ν and σ^2 as given in (5.57) and (5.30).

6 Non-perturbative treatment: an autonomous equation

In this Chapter, we develop the non-perturbative framework for the long-wavelength part of the expectation value $\langle\phi^2(t,\vec{x})\rangle$. Our aim is to extract non-perturbative results, relying only on some pieces of the perturbative information. As discussed in the introduction, it is possible to improve the naive perturbative secular terms that arise in the iterative solution of some complicated differential equations; see [136] by Chen, Goldenfeld, and Oono. Following the spirit of [125, 128, 130, 136], Kamenshchik and Vardanyan constructed the autonomous first-order differential equation for a massless minimally coupled scalar field in de Sitter space [122]. Within the proposed techniques to treat the series of secular terms, they obtained finite late-time vacuum expectation values for $\langle\phi^2_{m=0}(t,\vec{x})\rangle$ and $\langle\phi^4_{m=0}(t,\vec{x})\rangle$. Although the massive case does not exhibit secular growth, it remains of interest to obtain the non-perturbative result. In this chapter, we therefore construct an autonomous equation for the massive vacuum expectation value $\langle\phi^2(t,\vec{x})\rangle$ based on the perturbative series derived in the preceding chapters. By integrating an approximate version of this equation, we derive a non-analytic expression in the self-interaction coupling constant λ that reproduces the correct perturbative expansion up to two-loop order. We start with the comparison of our outcomes with the Hartree-Fock, or Gaussian, approximation, derive the corresponding non-perturbative solution, clarify which terms it resums, and then refine it to include the two-loop correction.

6.1 Hartree-Fock approximation

We shall start with the computation of $\langle\phi^2(t,\vec{x})\rangle$ within the Hartree-Fock, or Gaussian, approximation, which is known to be exact for the free massive case. To compare our outcomes for $\langle\phi^2(t,\vec{x})\rangle := \langle\phi^2(t)\rangle$ from the previous chapters with those obtained in the Hartree-Fock approximation, one considers the equation of motion, $\Box\,\phi(t) = -V'_{\phi}(\phi)$, for (1.1). Multiplying both sides of that equation by $\phi(t)$, integrating the left-hand side by parts, taking expectation values of the field operators, and using the Hartree-Fock, or Gaussian, approximation, namely $\langle\phi^4(t)\rangle_{\rm HF} = 3\,\langle\phi^2(t)\rangle^2_{\rm HF}$, one arrives at

$$\frac{1}{2}\Box\,\langle\phi^2\rangle_{\rm HF} - \langle\phi^{,\mu}\phi_{,\mu}\rangle_{\rm HF} = -m^2\langle\phi^2\rangle_{\rm HF} - 3\lambda\langle\phi^2\rangle^2_{\rm HF}. \tag{6.1}$$

In the case of a massless and non-interacting scalar field, r.h.s. of (6.1) vanishes, the dominant contribution to the left-hand side comes from its long-wavelength part, i.e., $3H\partial_t$, and $\Box\,\langle\phi^2_{m=0}(t)\rangle = 3H^4/4\pi^2$. We dare to assume that for small mass and small self-interaction coupling constant, this approximation remains valid

$$\frac{d}{dt}\left\langle\phi^2(t)\right\rangle_{\rm HF}=\frac{H^3}{4\pi^2}-\frac{2m^2}{3H}\left\langle\phi^2(t)\right\rangle_{\rm HF}-\frac{2\lambda}{H}\left\langle\phi^2(t)\right\rangle_{\rm HF}^2. \tag{6.2}$$

The solution to the equation above is[5]

$$\left\langle\phi^2(t)\right\rangle_{\rm HF}=\frac{\frac{3H^4}{4\pi^2m^2}\left(1-\exp\left(-\frac{2m^2t}{3H}\sqrt{1+\frac{9\lambda H^4}{2\pi^2m^4}}\right)\right)}{1+\sqrt{1+\frac{9\lambda H^4}{2\pi^2m^4}}-\left(1-\sqrt{1+\frac{9\lambda H^4}{2\pi^2m^4}}\right)\exp\left(-\frac{2m^2t}{3H}\sqrt{1+\frac{9\lambda H^4}{2\pi^2m^4}}\right)}. \tag{6.5}$$

Expanding the obtained solution along a small self-interaction coupling constant λ, one gets the following series:

$$\begin{aligned}\left\langle\phi^2(t)\right\rangle_{\rm HF}\longrightarrow\ &\frac{3H^4}{8\pi^2m^2}\left(1-e^{-\frac{2m^2t}{3H}}\right)-\frac{27\lambda H^8}{64\pi^4m^6}\left(1-\frac{4m^2t}{3H}e^{-\frac{2m^2t}{3H}}-e^{-\frac{4m^2t}{3H}}\right)\\ &+\frac{243\lambda^2H^{12}}{512\,\pi^6m^{10}}\left(2+\left(1-\frac{4m^2t}{3H}-\frac{8m^4t^2}{9H^2}\right)e^{-\frac{2m^2t}{3H}}-\left(2+\frac{8m^2t}{3H}\right)e^{-\frac{4m^2t}{3H}}-e^{-\frac{2m^2t}{H}}\right)\\ &-\frac{2187\,\lambda^3H^{16}}{4096\,\pi^8m^{14}}\left(5+\left(4-\frac{4m^2t}{3H}-\frac{16m^4t^2}{9H^2}-\frac{32m^6t^3}{81H^3}\right)e^{-\frac{2m^2t}{3H}}\right.\\ &\left.-\left(4+\frac{8m^2t}{H}+\frac{32m^4t^2}{9H^2}\right)e^{-\frac{4m^2t}{3H}}-\left(4+\frac{4m^2t}{H}\right)e^{-\frac{2m^2t}{H}}-e^{-\frac{8m^2t}{3H}}\right)+O\left(\lambda^4\right).\end{aligned} \tag{6.6}$$

Now, we are in a position to compare the retrieved series with our previous outcomes, e.g., (2.36) in Chapter 2. One can compare and notice that at tree and one-loop levels, the Hartree-Fock, or Gaussian, approximation gives correct results, but fails at two-loop and three-loop orders.

5 By denoting $\left\langle\phi^2(t)\right\rangle_{\rm HF}:=f(t)$, equation (6.2) reads

$$\frac{df}{dt}=a-b\,f-c\,f^2\,,\qquad\text{where}\quad a:=\frac{H^3}{4\pi^2}\,,\quad b:=\frac{2m^2}{3H}\,,\quad\text{and}\quad c:=\frac{2\lambda}{H}. \tag{6.3}$$

Then, the solution can be defined through

$$\frac{df}{(f-f_1)(f-f_2)}=-c\,dt\qquad\Rightarrow\qquad f(t)=\frac{f_1\,f_2\left(1-e^{-c\left(f_1-f_2\right)t}\right)}{f_2-f_1\,e^{-c\left(f_1-f_2\right)t}}\,, \tag{6.4}$$

where f_1 and f_2, obviously, are the roots of r.h.s. (6.3). Substituting into this expression (6.4) the assigned values (6.3), one gets precisely the solution (6.5).

Through our outcomes for each of the two-loop diagrams in the previous Chapter 3, one notices that the Hartree-Fock approximation only resums the "Snowman" (3.10) and the "Double Seagull" (3.18) diagrams, leaving aside the "Sunset" (3.14) one. These are known as "Cactus" diagrams. At the three-loop level, the story is the same. The Hartree-Fock, or Gaussian, approximation resums the following contributions: from the "Triple Snowman", the "Mouse", the "Snowman + Independent Loop", and the "Three Independent Loops" diagrams on Figures B.1a, b, c, and d; see also the corresponding Appendix with (B.4), (B.8), (B.12), and (B.16). Therefore, the other four diagrams, namely, the ones on Figures B.1e, f, g, and h, are not captured by the Hartree-Fock approximation.

6.2 Autonomous equation

Having encountered the missing contribution from the "Sunset" diagram, in this section we go beyond the Hartree-Fock approximation (but not so far). Here, we construct through the known perturbative series for $\langle\phi^2(t)\rangle$ a simple first-order differential equation, which we refer to as the autonomous equation. The solution of such an equation is non-analytic in the self-interaction coupling constant λ while providing the correct perturbative series up to $O(\lambda^3)$, including the "Sunset" diagram. We also compare that result at the late-time limit with the non-perturbative one within the Starobinsky and Yokoyama stochastic approach.

The idea is to obtain an equation such that its expanded solution reproduces the correct perturbative series up to the two-loop level. We will rely on our outcomes from Chapter 2, i.e., series (2.36). At the tree level from (2.36) one has

$$\left\langle\phi^2(t)\right\rangle_0 = \frac{3H^4}{8\pi^2 m^2}\left(1 - e^{-\frac{2m^2 t}{3H}}\right) := f(t). \tag{6.7}$$

This tree-level expression above is a solution to the following autonomous equation:

$$\frac{d}{dt}\big(f(t)\big) = \frac{H^3}{4\pi^2} - \frac{2m^2}{3H} f(t). \tag{6.8}$$

One can take the time derivative of both sides of (6.7) and express the resulting exponential factor in terms of $f(t)$ to verify this.

Hereafter, we repeat this prescription in order to find an autonomous equation at the one-loop level. We replace the function in (6.8) with $f(t) := \langle\phi^2(t)\rangle$, taken up to linear order in the self-interaction coupling constant λ from (2.36). In this case, comparing the left- and right-hand sides of (6.8) shows that a correction term $\delta f_1(t)$ must be added

$$\frac{d}{dt}\big(f(t)\big) = \frac{H^3}{4\pi^2} - \frac{2m^2}{3H} f(t) + \delta f_1(t), \tag{6.9}$$

where

$$\delta f_1(t) := -\frac{9\lambda H^7}{32\,\pi^4 m^4}\left(1 - e^{-\frac{2m^2 t}{3H}}\right)^2 = -\frac{2\lambda}{H}\big(f(t)\big)^2. \tag{6.10}$$

As expected, this equation is precisely (6.2) in the Hartree-Fock approximation above, and the solution to such an autonomous equation is (6.5).

Proceeding with the same routine, we are also able to find the correction term for the two-loop or λ^2 level, which takes the following form:

$$\frac{d}{dt}\big(f(t)\big) = \frac{H^3}{4\pi^2} - \frac{2m^2}{3H}f(t) - \frac{2\lambda}{H}\big(f(t)\big)^2 - \frac{27\lambda^2 H^{11}}{16\,\pi^6 m^8}\Bigg(\frac{2\pi^2 m^2}{H^4}f(t) - \frac{8\pi^4 m^4}{H^8}\big(f(t)\big)^2 + \frac{128\,\pi^6 m^6}{27H^{12}}\big(f(t)\big)^3 + \frac{3}{4}\Big(1-\frac{8\pi^2 m^2}{3H^4}f(t)\Big)^2\ln\Big(1-\frac{8\pi^2 m^2}{3H^4}f(t)\Big)\Bigg). \tag{6.11}$$

This autonomous equation above in the massless limit reduces to its analogue from the preceding work by Kamenshchik and Vardanyan [122]:

$$(6.11) \xrightarrow{m\to 0} \frac{d}{dt}\big(f(t)\big) = \frac{H^3}{4\pi^2} - \frac{2\lambda}{H}\big(f(t)\big)^2 + \frac{16\,\pi^2\lambda^2}{3H^5}\big(f(t)\big)^4. \tag{6.12}$$

For our purposes, namely, to capture the contribution of the "Sunset" diagram, it is sufficient to represent a solution to the obtained equation (6.11) in the compact form:

$$f(t) = \big\langle\phi^2(t)\big\rangle_{\mathrm{HF}} + \delta f_2(t) := \frac{3H^4}{4\pi^2 m^2}\mathcal{F}_z(t) + \delta f_2(t), \tag{6.13}$$

where we have introduced

$$\mathcal{F}_z(t) := \frac{1-\mathrm{e}^{-\frac{2m^2}{3H}Zt}}{1+Z-\big(1-Z\big)\mathrm{e}^{-\frac{2m^2}{3H}Zt}} \quad \text{with} \quad Z := \sqrt{1+\frac{3}{2z}}\ ; \quad z \overset{(4.18)}{:=} \frac{\pi^2 m^4}{3\lambda H^4}. \tag{6.14}$$

Suppose that the correcting λ^2-term in our autonomous equation (6.11) can be considered small. Hence, its linearized version appears to be

$$\frac{d}{dt}\Big(\delta f_2(t)\Big) = -\left(\frac{2m^2}{3H} + \frac{3\lambda H^3}{\pi^2 m^2}\mathcal{F}_z(t)\right)\delta f_2(t) \underbrace{- \frac{81\lambda^2 H^{11}}{64\,\pi^6\,m^8}\left(2\,\mathcal{F}_z(t) - 6\,\mathcal{F}_z^{\,2}(t) + \frac{8}{3}\mathcal{F}_z^{\,3}(t) + \Big(1-2\mathcal{F}_z(t)\Big)^2\ln\Big(1-2\mathcal{F}_z(t)\Big)\right)}_{:=\mathcal{W}(t)}. \tag{6.15}$$

Hereafter, we denote the correcting term with the help of the new notation as $\mathcal{W}(t)$. Therefore, the solution to our linearized equation (6.15) can be found in the following form:

$$\delta f_2(t) = \exp\left(-\int_0^t dt'\left(\frac{2m^2}{3H} + \frac{3\lambda H^3}{\pi^2 m^2}\mathcal{F}_z(t')\right)\right)\times \tag{6.16}$$

$$\times \int_0^t dt''\, \mathcal{W}(t'') \exp\left(\int_0^{t''} dt''' \left(\frac{2m^2}{3H} + \frac{3\lambda H^3}{\pi^2 m^2} \mathcal{F}_z(t''')\right)\right).$$

Here, the first integral is

$$\int_0^t dt' \left(\frac{2m^2}{3H} + \frac{2m^2}{3H}(Z^2-1)\,\mathcal{F}_z(t')\right) = \frac{2m^2 Zt}{3H} + 2\ln\left(1 + Z + (Z-1)\mathrm{e}^{-\frac{2m^2}{3H}Zt}\right), \quad (6.17)$$

where we have used (6.14) to rewrite the integrand. Therefore, the first factor in (6.16) is

$$\exp\left(-\int_0^t dt' \left(\frac{2m^2}{3H} + \frac{2m^2}{3H}(Z^2-1)\,\mathcal{F}_z(t')\right)\right) = \left((Z+1)\,\mathrm{e}^{\frac{m^2}{3H}Zt} + (Z-1)\,\mathrm{e}^{-\frac{m^2}{3H}Zt}\right)^{-2}. \quad (6.18)$$

By straightforwardly integrating the second factor in (6.16) using the result from (6.17), the resulting expression is

$$\begin{aligned}
\delta f_2(t) = \frac{243\,\lambda^2 H^{12}}{32\,\pi^6 m^{10}} &\left((Z+1) + (Z-1)\,\mathrm{e}^{-\frac{2m^2}{3H}Zt}\right)^{-2} \Bigg(-\frac{3Z^2-3Z-2}{6Z(Z+1)} \qquad (6.19)\\
&- \frac{(Z-1)^2}{4Z}\ln\left(\frac{(Z-1)+(Z+1)\,\mathrm{e}^{-\frac{2m^2}{3H}Zt}}{(Z+1)+(Z-1)\,\mathrm{e}^{-\frac{2m^2}{3H}Zt}}\right) + \left(\frac{(Z^2-1)}{2Z}\ln\left(-\frac{(Z+1)}{(Z-1)}\mathrm{e}^{-\frac{2m^2}{3H}Zt}\right)\right.\\
&\times \ln\left(\frac{(Z-1)+(Z+1)\,\mathrm{e}^{-\frac{2m^2}{3H}Zt}}{(Z+1)+(Z-1)\,\mathrm{e}^{-\frac{2m^2}{3H}Zt}}\right) + \frac{(Z^2-1)}{2Z}\,\mathrm{Li}_2\left(-\frac{4\,Z\,\mathrm{e}^{-\frac{2m^2}{3H}Zt}}{\left((Z^2-1)+(Z-1)^2\,\mathrm{e}^{-\frac{2m^2}{3H}Zt}\right)}\right)\\
&- \frac{(Z^2-1)}{2Z}\,\mathrm{Li}_2\left(-\frac{4\,Z}{\left((Z+1)^2+(Z^2-1)\,\mathrm{e}^{-\frac{2m^2}{3H}Zt}\right)}\right) + \frac{16\,Z^2}{3(Z^2-1)^2}\ln\left((Z+1)\,\mathrm{e}^{\frac{2m^2}{3H}Zt}\right.\\
&\left.+ (Z-1)\right) + 4\left(\frac{Z^2+1}{(Z^2-1)}\ln 2 - \frac{m^2 t}{3H}\right) + \frac{(Z+1)}{(Z-1)}\ln\left(-\frac{Z}{(Z+1)+(Z-1)\,\mathrm{e}^{-\frac{2m^2}{3H}Zt}}\right)\\
&\left.+ \frac{(Z-1)}{(Z+1)}\ln\left(-\frac{Z\,\mathrm{e}^{-\frac{2m^2}{3H}Zt}}{(Z+1)+(Z-1)\,\mathrm{e}^{-\frac{2m^2}{3H}Zt}}\right)\right)\mathrm{e}^{-\frac{2m^2}{3H}Zt}\\
&+ \left(\frac{(Z+1)^2}{4Z}\ln\left(\frac{(Z-1)+(Z+1)\,\mathrm{e}^{-\frac{2m^2}{3H}Zt}}{(Z+1)+(Z-1)\,\mathrm{e}^{-\frac{2m^2}{3H}Zt}}\right) - \frac{3Z^2+3Z+2}{6Z(Z-1)}\right)\mathrm{e}^{-\frac{4m^2}{3H}Zt}\Bigg).
\end{aligned}$$

Here $\mathrm{Li}_2(z)$ is the dilogarithm function: $\mathrm{Li}_2(z) = \int\limits_1^z \frac{\ln u}{1-u}\, du$.

Immediately, the limiting value as $t \longrightarrow \infty$ from the first two terms of the obtained (6.19) is:

$$\delta f_2(t) \xrightarrow{t\to\infty} \frac{243\,\lambda^2 H^{12}}{32\,\pi^6 m^{10}} \left(-\frac{3Z^2 - 3Z - 2}{6Z(Z+1)^3} - \frac{(Z-1)^2}{4Z(Z+1)^2} \ln\left(\frac{Z-1}{Z+1}\right)\right). \tag{6.20}$$

That precisely matches the "Sunset" diagram contribution, since, when expanded, it yields

$$\delta f_2(t) \xrightarrow{t\to\infty} (6.20) \approx \frac{81\lambda^2 H^{12}}{256\,\pi^6 m^{10}} + O(\lambda^3), \tag{6.21}$$

which was precisely absent in the Hartree-Fock approximation; see Section 3, namely, (3.14).

The full non-analytic in λ result for the vacuum expectation value $\langle \phi^2(t)\rangle$ within the autonomous equation at the late-time limit is the sum of late-time outcomes (6.5) and (6.20):

$$\langle \phi^2(t)\rangle^{\mathrm{aut}} = \langle \phi^2(t)\rangle_{\mathrm{HF}} + \delta f_2(t) \xrightarrow{t\to\infty} \frac{H^2}{\pi\sqrt{\lambda}} \Bigg(\frac{\sqrt{12z+18} - \sqrt{12z}}{12} \tag{6.22}$$

$$+ \frac{\left(3\sqrt{4z^2+6z} - 2z - 9\right)\left(\sqrt{12z+18} - \sqrt{12z}\right)^3}{1728\, z\, \sqrt{4z^2+6z}} + \frac{3\left(\sqrt{2z+3} - \sqrt{2z}\right)^4}{64 z^2 \sqrt{12z+18}} \ln\left(\frac{\sqrt{2z+3} + \sqrt{2z}}{\sqrt{2z+3} - \sqrt{2z}}\right)\Bigg).$$

Expanding in the small self-interaction coupling constant λ, this expression naturally leads to (2.37) up to $O(\lambda^3)$. Let us also point out that in the smooth massless limit it recovers the late-time non-perturbative result from the preceding work by Kamenshchik and Vardanyan [122]:

$$\langle \phi^2\rangle^{\mathrm{aut}}_{t\to\infty} = (6.22) \xrightarrow{m\to 0} \frac{7\,H^2}{12\,\pi\sqrt{2\lambda}} := \langle \phi^2_{m=0}\rangle^{\mathrm{aut}}_{t\to\infty}. \tag{6.23}$$

One can straightaway compare it with (4.14) from the seminal work by Starobinsky and Yokoyama [53] to establish the accuracy of the autonomous equation:

$$\langle \phi^2_{m=0}\rangle^{\mathrm{aut}}_{t\to\infty} \approx 0.1313\, \frac{H^2}{\sqrt{\lambda}} \qquad \text{vs.} \qquad \langle \varphi^2\rangle^{\text{late-time}}_{\text{non-pert}} \approx 0.1318\, \frac{H^2}{\sqrt{\lambda}}. \tag{6.24}$$

Besides, our massive late-time non-perturbative result (6.22) almost coincides with the Starobinsky-Yokoyama stochastic approach. Using our derived expressions (4.18) and (6.22), one has

$$\frac{\langle \phi^2\rangle^{\mathrm{aut}}_{t\to\infty}}{\langle \varphi^2\rangle^{\text{late-time}}_{\text{non-pert}}} \overset{z\to 0}{\approx} \frac{7\,\Gamma\left(\frac14\right)}{12\sqrt{3}\,\Gamma\left(\frac34\right)} - \frac{7\sqrt{z}}{12\sqrt{6}} \left(4 + \frac{92}{35\sqrt{3}}\frac{\Gamma\left(\frac14\right)}{\Gamma\left(\frac34\right)} - \frac{\Gamma^2\left(\frac14\right)}{\Gamma^2\left(\frac34\right)}\right) + O(z)$$

$$\xrightarrow{z\to 0} \quad \frac{7\,\Gamma\left(\frac{1}{4}\right)}{12\sqrt{3}\,\Gamma\left(\frac{3}{4}\right)} \approx 0.9964\,; \tag{6.25}$$

$$\frac{\left\langle \phi^2 \right\rangle^{\text{aut}}_{t\to\infty}}{\left\langle \varphi^2 \right\rangle^{\text{late-time}}_{\text{non-pert}}} \quad \overset{z\to\infty}{\approx} \quad 1 + \frac{9}{256\,z^3} - \frac{81}{1024\,z^4}\left(\frac{107}{18} + \ln\frac{3}{8z}\right) + O\left(z^{-5}\right) \xrightarrow{z\to\infty} \quad 1. \tag{6.26}$$

Numerical estimates show that this relation increases to a maximum of approximately 1.0055 at $z \approx 0.145$ before decreasing to its asymptotic value.

Therefore, the amplitude of the deviation of the autonomous late-time result $\left\langle \phi^2 \right\rangle^{\text{aut}}_{t\to\infty}$, see (6.22), from the one within the Starobinsky-Yokoyama stochastic approach (4.18) does not exceed 0.6 % in the whole interval of a new dimensionless parameter z, which is

$$0 \le \frac{\pi^2 m^4}{3\lambda H^4} < \infty. \tag{6.27}$$

7 Conclusion

In the present thesis, we have developed techniques for computing correlation functions of a spectator scalar field in de Sitter space within the long-wavelength approximation. The following conclusion provides a detailed recap of the main results from each chapter.

We have started with the computation of two-point and four-point correlation functions within the long-wavelength approximation and rather particular theory (1.1) in Chapter 2. To perform it, we have employed the massive Yang-Feldman-type equation (2.16). By iterating (2.16) up to the necessary self-interaction coupling constant λ order, see (2.17), one builds up the formal perturbative series for the Heisenberg operator and computes the vacuum expectation value of products of field operators. In the case of a massless scalar field with a quartic self-interaction, the Yang-Feldman equation (2.14) appeared to be rather convenient to catch the leading logarithms in cosmic time t; see the seminal work by Tsamis and Woodard [44, 45]. We proposed a way to "hang up" the mass into that equation (2.14): the free massless field is replaced by the free massive field and the retarded Green's function acquires the additional exponential factor; see again (2.16). To define the massive scalar field in terms of the massless field, (2.15), we have inverted the integral equation (2.14) considering only the massive term in the potential. The obtained relation (2.15) allows us to compute the two-point correlation function of the free massive scalar field relying only on the known two-point correlation function of the free massless scalar field (2.19). Our treatment could be considered a theory of a massive scalar field with the vacuum "inherited" from a massless one. Remarkably, our "building block", i.e., free massive two-point correlation function (2.22) coincides with that of the well-known Ornstein–Uhlenbeck stochastic process [99]. By virtue of this, the obtained correlation function exhibits a smooth massless limit and tends over time to the equilibrium de Sitter-invariant state. We employed (2.16) to compute the two-point correlation function up to order λ^3 and the four-point correlation function up to order λ^2. Our resulting series are equivalent to the resummation of leading infrared logarithms in the perturbative series in m^2/H^2 at each order of the self-interaction coupling constant λ. Within this scheme, the infrared divergences in cosmic time t of the massless self-interacting scalar field are fully resummed in the massive series.

In the next Chapter 3, we have established the correspondence between the integral structures arising in the iterative Yang-Feldman-type equation and the diagrams within the Schwinger-Keldysh, or "in-in", formalism and the stochastic approach. All types of diagrams have the same topological structure, but in the stochastic case, one encounters the single vertex and three propagators, while in the Schwinger-Keldysh formalism, there are two types of vertices and four types of propagators. To spot what kind of correspondence may exist, we relied on the outputs for the two-point correlation function of a massive scalar field calculated at the one-loop and two-loop orders via the Schwinger-Keldysh diagrammatic technique in [117] and [118] in the p-representation [119], for the massless equal-time case from [83, 84, 120, 121, 122, 123], and the equilibrium stochastic results for the massive case [81]. Our obtained full expressions for the one-loop (2.25) and the two-loop, (3.7), (3.11), and (3.15), for each diagram's contributions to the two-point correlation function differ from findings in [117, 118]. That is due to the initial choice of the so-called Bunch-Davies, de Sitter-invariant vacuum state [142] for a massive scalar field made there. Nonetheless, by virtue of our "building block" (2.22), the obtained results tend to an equilibrium, de Sitter-invariant state at late times, see (2.34), and turn out to be in agreement with [117, 118] as well as with [81] at the late-time limit. Apart from the revealed results coincidence, one finds the same structure in the used expressions; see (3.20)–(3.35). By the same reasoning, we present the contributions of each diagram at the three-loop level in Appendix B. We have also commented on the four-point correlation function case; see (3.19) and nearby reasoning. We conclude that instead of using the diagrammatic techniques, one can discern the diagram's topology from integral structures arising in the iterated Yang-Feldman-type equation. The processing of the massive Yang-Feldman-type equation (2.16) is much more economical while providing the extension to the non-equilibrium regime in the coordinate representation.

Further, we presented in Chapter 4 the basics of the Starobinsky and Yokoyama stochastic approach [52, 53]. One decomposes the quantum field operator $\phi\,(t,\vec{x})$ in the Heisenberg representation into the super-Hubble or long-wavelength, $k < He^{Ht}$, and the sub-Hubble or short-wavelength, $k > He^{Ht}$, parts using a window function, which one chooses to be the dynamical Heaviside step function. The long-wavelength part $\phi^{\text{l-w}}(t,\vec{x})$ satisfies the equation of motion $\Box\,\phi = -V_{\phi}'(\phi)$, and for a slowly varying $\phi^{\text{l-w}}(t,\vec{x})$, it takes the local Langevin-like form [52], where the noise term represents short-wavelength modes that continually shift into the long-wavelength ones. Therefore, the long-wavelength part of the quantum scalar field $\phi^{\text{l-w}}(t,\vec{x})$ can be treated as the classical stochastic field $\varphi(t,\vec{x})$ with a probability distribution function $\mathcal{P}\left[\varphi(t,\vec{x})\right]$ that satisfies the Fokker-Planck or Einstein-Smoluchowski equation, associated with the corresponding local Langevin-like equation. We explored the Starobinsky and Yokoyama stochastic approach for the model of the massive scalar field with a quartic self-interaction. Through the recursion relations involving the modified Bessel functions of the second kind (4.20), we have derived the non-perturbative late-time stochastic expectation value $\left\langle\varphi^{2\text{n}}\right\rangle^{\text{late-time}}_{\text{non-pert}}$, see (4.26) with (4.25). Being expanded in the small self-interaction coupling constant regime, the resulting series agree at late times with those obtained from the massive Yang–Feldman-type equation in Chapter 2. To make an additional contact between approaches, we followed the method proposed by Tsamis and Woodard [45]. We extracted the first-order differential equation for the stochastic expectation values through the Fokker-Planck equation, see (4.30), and the corresponding series in the free massive case (4.36) and in the massless case with quartic self-interaction (4.43). One can readily notice the leading infrared logarithm structure in cosmic time t in the obtained series. As well as to verify that our free massive correlation functions are exactly the resummed expressions of series (4.36); see (2.39) and (2.74) at $\lambda = 0$.

In Chapter 5, we developed an alternative approach to compute the perturbative series for equal- and multi-time correlation functions in the leading logarithm approximation. Once again, instead of processing the Yang-Feldman-type equation or dealing with the diagrammatic techniques, one derives a simple first-order differential equation through the Fokker-Planck or forward Kolmogorov equation. Its formal integral solution relates various correlation functions at different orders in the self-interaction coupling constant. One solves it iteratively. The starting point is the non-interacting case, and the exact solutions are well known in that context. Further, we related these correlation functions to ones at different orders in λ by the integral relation. In the case of equal-time correlation functions, that relation provides the results order by order in λ via the straightforward integration (5.6). In the multi-time case, we have the formal solution to the corresponding first-order differential equation as an indefinite integral. To obtain the multi-time correlation function in the free massive case, we extensively exploited the Markov property, formulated in terms of the conditional (or transition) probability density (5.21); see (5.24) and (5.25). After integration, one has the antiderivative and the unknown function, which one defines by matching it to the known result for the equal-time correlation function or to the previous "iteration" step by step; see Section 5.2. The obtained results are in agreement with the corresponding quantum field theory perturbative series for two-point and four-point correlation functions. This treatment, going beyond the equal-time and stationary cases, provides further confirmation that Starobinsky and Yokoyama stochastic approach precisely reproduces quantum field theory results in de Sitter space for models with a scalar field potential within the so-called leading logarithm approximation.

Finally, in Chapter 6, our goal was to extract the non-perturbative result relying only on some pieces of the perturbative information, which we computed in the previous chapters. First of all, we compared our outcomes with the Hartree-Fock, or Gaussian, approximation. To do so, we have found the non-perturbative solution within this approximation and revealed what it resums. Using our extracted diagram contributions for the two-point function at the one-, two- and three-loop orders from Chapter 3 and Appendix B, we have established that the Hartree-Fock approximation resums only the so-called "Cactus"-type diagrams; see details in (6.6). Further, we have constructed the renormalization group-inspired autonomous first-order differential equation for the long-wavelength vacuum expectation value $\langle\phi^2(t,\vec{x})\rangle$ of the massive scalar field, see (6.11), relying on the obtained perturbative series (2.36). In order to capture the absentee "Sunset" contribution at the two-loop level, we linearized the obtained autonomous equation (6.15) and integrated it; see (6.19). The full non-perturbative result in the self-interaction coupling constant λ is the sum of that obtained from the Hartree-Fock approximation and the solution of the linearized autonomous equation. It reproduces the correct perturbative series for $\left\langle\phi^2(t,\vec{x})\right\rangle$ up to the two-loop level. In the late-time limit, it almost coincides with the result obtained within the Starobinsky-Yokoyama stochastic approach over the entire interval of the dimensionless parameter $0 \leq \frac{\pi^2 m^4}{3\lambda H^4} < \infty$; see (6.25), (6.26), and for the massless case (6.24).

8 Outlook

We would like to list potential extensions of the applied techniques and results.

Beyond de Sitter space

One can extend our approach to computations in a flat Friedmann background with a power-law scale factor [58, 161]:

$$ds^2 = dt^2 - a^2(t)\, d\vec{x}^2, \qquad a(t) \propto t^p \quad \text{and} \quad p = \frac{2}{3(1+\omega)} > 1. \tag{8.1}$$

Here, ω is the parameter relating pressure and energy density in the equation of state. Such an expansion corresponds to a negative pressure and hence inflation for $p > 2/3$, since in this case $\omega < -1/3$.

Following reasoning analogous to (2.6)–(2.10), the mode equation in this background is

$$\ddot{u}_k(t) + \frac{3p}{t}\, \dot{u}_k(t) + \frac{k^2}{t^{2p}}\, u_k(t) = 0, \tag{8.2}$$

and the Wronskian condition in that case has the following form: $W[u_k(t), u_k^*(t)] = -i\, t^{-3p}$. For the long-wavelength (l-w) modes, $k < p\, t^{p-1}$, one neglects the last term $\sim k^2$, solves (8.2), yielding $u_k^{\text{l-w}}(t) = c_1 + c_2\, t^{-3p+1}$, and from the Wronskian one finds $c_1^* c_2 - c_1 c_2^* = i/(3p-1)$.

Therefore, for the retarded Green's function (2.9) one can obtain

$$G_{\mathrm{R}}^{\text{l-w}}(t, \vec{x}\,; t', \vec{x}\,') = \frac{\Theta(t-t')}{3p-1}\Big((t')^{-3p+1} - t^{-3p+1}\Big)\, \delta(\vec{x} - \vec{x}\,'). \tag{8.3}$$

The truncated Yang-Feldman equation for the massless scalar field with a quartic self-interaction in that setting is

$$\phi(t, \vec{x}) = \phi_0(t, \vec{x}) - \frac{\lambda}{3p-1} \int_0^t dt'\ t'\, \phi^3(t', \vec{x}), \tag{8.4}$$

and the expression of the free massive scalar field through the free massless one is

$$\tilde{\phi}(t, \vec{x}) = \phi_0(t, \vec{x}) - \frac{m^2}{3p-1}\, \mathrm{e}^{-\frac{m^2 t^2}{6p-2}} \int_0^t dt'\ \mathrm{e}^{\frac{m^2 t'^2}{6p-2}}\, t'\, \phi_0(t', \vec{x}); \tag{8.5}$$

see Appendix A as a reference. Thus, in the power-law background, the Yang–Feldman-type equation (2.16) takes the form

$$\phi(t,\vec{x}) = \tilde{\phi}(t,\vec{x}) - \frac{\lambda}{3p-1}\, \mathrm{e}^{-\frac{m^2t^2}{6p-2}} \int_0^t dt'\; \mathrm{e}^{\frac{m^2t'^2}{6p-2}}\, t'\, \phi^3(t',\vec{x}). \tag{8.6}$$

What is needed is the infrared part of the correlation function of the massless scalar field, see, e.g., [162, 163, 164], and everything is almost prepared for the calculations similar to those presented in Chapter 2.

Another way to derive the truncated Yang-Feldman equation for the power-law background is to solve the corresponding equation for the long-wavelength massive zeroth modes

$$\ddot{u}_k^{\text{l-w}}(t) + \frac{3p}{t}\,\dot{u}_k^{\text{l-w}}(t) + m^2 u_k^{\text{l-w}}(t) = 0, \tag{8.7}$$

whose solution is

$$u_k^{\text{l-w}}(t) = c_1\, t^{-\nu}\, J_\nu(mt) + c_2\, t^{-\nu}\, Y_\nu(mt), \qquad \nu := \frac{3p}{2} - \frac{1}{2}. \tag{8.8}$$

Here $J_\nu(mt)$ and $Y_\nu(mt)$ are the Bessel functions of the first and second kinds respectively. Analogously, using the Wronskian condition $W[u_k(t), u_k^*(t)] = -i\,t^{-3p}$, we have

$$\left(c_1^* c_2 - c_1 c_2^*\right)\Big(J_\nu(mt)\,Y_{\nu+1}(mt) - J_{\nu+1}(mt)\,Y_\nu(mt)\Big) = -\frac{2}{\pi m t}\left(c_1^* c_2 - c_1 c_2^*\right) = \frac{i}{mt}. \tag{8.9}$$

Therefore, $c_1^* c_2 - c_1 c_2^* = -\frac{i\pi}{2}$ and the retarded Green's function (2.9) becomes

$$G_R^{\text{l-w}}(t,\vec{x};t',\vec{x}') = -\frac{\pi\Theta(t-t')}{2\,(t\,t')^{-\nu}}\Big(J_\nu(mt)\,Y_\nu(mt') - J_\nu(mt')\,Y_\nu(mt)\Big)\,\delta(\vec{x}-\vec{x}'). \tag{8.10}$$

Since one cannot treat the mass as a perturbation over time intervals such that $mt \gg 1$ [111], one can consider the region $0 < mt \ll \sqrt{\nu+1}$. In that case, the Bessel functions have the asymptotic form [141]:

$$J_\nu(mt) \;\longrightarrow\; \frac{1}{\Gamma(1+\nu)}\left(\frac{mt}{2}\right)^\nu, \qquad Y_\nu(mt) \;\longrightarrow\; -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{mt}\right)^\nu, \tag{8.11}$$

and in that regime, expression (8.10) above takes the following form

$$G_R^{\text{l-w}}(t,\vec{x};t',\vec{x}') = \frac{\Gamma(\nu)}{2\Gamma(\nu+1)}\,\frac{\Theta(t-t')}{(t\,t')^{-\nu}}\left(\left(\frac{t}{t'}\right)^\nu - \left(\frac{t'}{t}\right)^\nu\right)\delta(\vec{x}-\vec{x}'), \tag{8.12}$$

which is precisely the previous result (8.3) with the assigned ν as in (8.8). That leads to the Yang-Feldman equation (2.2) with the inserted expression for the retarded Green's function (8.12). By "truncating" the second term under the integral, one obtains (8.4). Further, one can proceed to (8.6) with (8.5) and perform the computation of vacuum expectation values.

It would certainly be interesting to attempt a non-perturbative treatment in the spirit of the stochastic formalism in a power-law background (and even more so in more general backgrounds) and to establish what it resums (if so) in the corresponding quantum field theory within a certain approximation.

Beyond white noise assumption

Apart from Starobinsky's brilliant intuition, there is no fundamental reason to split the quantum scalar field into the 'short' and 'long' parts by the chosen Heaviside step function; see again (4.1). The form of that window function was crucial for the short-wavelength contribution to induce white noise. The white-noise assumption is a great simplification and does not exist (no more than the delta function exists as a function) but serves as the model for any rapidly fluctuating force. Some studies initiate this research direction by exploring the non-white noise properties from different perspectives [165, 166, 167]. In addition, as we pointed out in the main part of the present thesis, the Starobinsky-Yokoyama stochastic approach [52, 53] does its job in the small mass regime, namely, $m \ll H$; see again [111]. However, the heavy massive field case, $m \sim H$ could possibly induce non-white noise effects. It would also certainly be interesting to approach that possible non-white property and explore it within a modified Fokker–Planck or master equation framework. The non-Markovian features are much more fruitful from the physical point of view, but they involve more complicated techniques to compute something there.

Do other choices of the window function correspond to any quantum field theory results within a certain approximation?

Does the mass control the transition between Markovian and non-Markovian regimes?

A A trick to "hang up" the mass to the Yang-Feldman equation

In this Appendix, we will show how to derive the massive Yang-Feldman-type equation starting from its massless counterpart. Let us start with the following equation

$$\phi(t,\vec{x}) = \phi_0(t,\vec{x}) + \int_0^t dt'\, F\big(\phi(t',\vec{x})\big), \tag{A.1}$$

here $F\big(\phi(t',\vec{x})\big) = \alpha\phi(t',\vec{x}) + W\big(\phi(t',\vec{x})\big)$ is a general function of the scalar field and $\phi_0(t',\vec{x})$ is a given fucntion. Introducing the new scalar field $\tilde{\phi}(t,\vec{x})$, which satisfies the equation

$$\tilde{\phi}(t,\vec{x}) = \phi_0(t,\vec{x}) + \alpha\int_0^t dt'\, \tilde{\phi}(t',\vec{x}), \tag{A.2}$$

and taking the time derivative, one finds exactly a solution to an ordinary inhomogeneous first-order linear differential equation:

$$\tilde{\phi}(t,\vec{x}) = \phi_0(t,\vec{x}) + \alpha e^{\alpha t}\int_0^t dt'\, e^{-\alpha t'}\phi_0(t',\vec{x}). \tag{A.3}$$

One can also express the known function $\phi_0(t,\vec{x})$ in terms of the new free massive one $\tilde{\phi}(t,\vec{x})$ from (A.2) and substitute this expression into (A.1), resulting in

$$\phi(t,\vec{x}) = \tilde{\phi}(t,\vec{x}) + \alpha\int_0^t dt'\, \Big(\phi(t',\vec{x}) - \tilde{\phi}(t',\vec{x})\Big) + \int_0^t dt'\, W\big(\phi(t',\vec{x})\big). \tag{A.4}$$

Hereafter, we introduce a new equation

$$\phi(t,\vec{x}) = \tilde{\phi}(t,\vec{x}) + e^{\alpha t}\int_0^t dt'\, e^{-\alpha t'} W\big(\phi(t',\vec{x})\big), \tag{A.5}$$

which is equivalent to (A.4). To establish this, we take $\phi(t,\vec{x}) - \tilde{\phi}(t,\vec{x})$ from (A.5), substitute it into (A.4), and after changing the order of integration, one finds that (A.5) reduces to (A.4):

$$\alpha\int_0^t dt'\, e^{\alpha t'}\int_0^{t'} dt''\, e^{-\alpha t''} W\big(\phi(t'',\vec{x})\big) = \alpha\int_0^t dt''\, e^{-\alpha t''} W\big(\phi(t'',\vec{x})\big)\int_{t''}^t dt'\, e^{\alpha t'}. \tag{A.6}$$

B Two-point function at three loops: diagrammatic contributions

In the present Appendix, we provide the diagram contributions at the three-loop level and the corresponding expressions, as well as their results at late and equal times and in the smooth massless limit.

We display all types of diagrams in Figure B.1 and list the following contributions to those diagrams below. We follow the same assignments as in Chapter 3:

In order to obtain a given diagram topology, the points might be connected either by an explicit correlation function in structure (2.31) *or by the limits of the integration variables.*

One defines each diagram topology through the general expression (2.31) and computes the corresponding integrals with all time-ordering tricks, see (2.32).

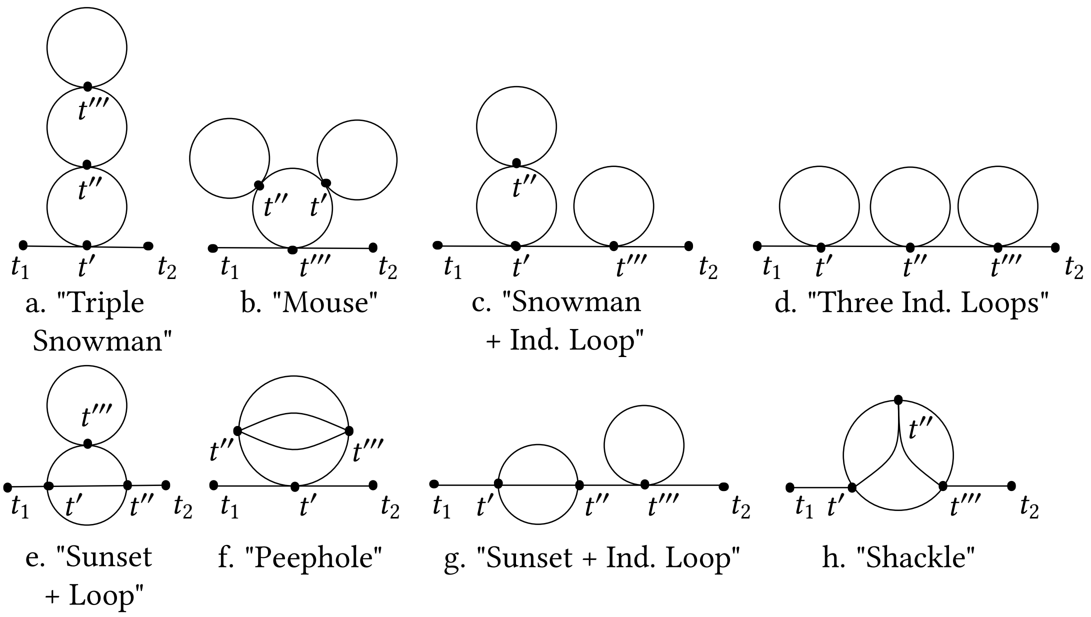


Figure B.1: All the possible three-loop-level diagrams for the two-point correlation function.

By doing so, one obtains the contribution of the first diagram type in Figure B.1, "Triple Snowman", which is

$$\left\langle\phi(t_1)\phi(t_2)\right\rangle_{\lambda^3}^{\text{Triple Snowman}} = -\frac{2187\,\lambda^3 H^{16}}{4096\,\pi^8 m^{14}}\Bigg(\left(1+\frac{m^2}{3H}|t_1-t_2|\right)\mathrm{e}^{-\frac{m^2}{3H}|t_1-t_2|} \tag{B.1}$$

$$-\left(\frac{11}{6}-\frac{m^2}{3H}\Big(|t_1-t_2|-2(t_1+t_2)\Big)+\frac{m^4}{9H^2}\Big(|t_1-t_2|-(t_1+t_2)\Big)^2-\frac{m^6}{81H^3}\Big(|t_1-t_2|\right.$$

$$
- (t_1 + t_2)\Big)^3\Big) e^{-\frac{m^2}{3H}(t_1+t_2)} + \Big(\frac{5}{4} + \frac{m^2}{2H}\Big(|t_1 - t_2| + (t_1 + t_2)\Big) + \frac{m^4}{18H^2}\Big(|t_1 - t_2| + (t_1 + t_2)\Big)^2\Big)
$$
$$
\times \Big(e^{-\frac{m^2}{3H}(2|t_1-t_2|+(t_1+t_2))} - e^{-\frac{m^2}{3H}(|t_1-t_2|+2(t_1+t_2))}\Big) + \Big(\frac{5}{4} - \frac{m^4}{18H^2}\Big(|t_1 - t_2| - (t_1 + t_2)\Big)^2\Big)\times
$$
$$
\times\, e^{\frac{m^2}{3H}(|t_1-t_2|-2(t_1+t_2))} + \Big(\frac{1}{2} + \frac{m^2}{6H}\Big(|t_1 - t_2| + (t_1 + t_2)\Big)\Big)e^{-\frac{m^2}{3H}(3|t_1-t_2|+2(t_1+t_2))} - \Big(\frac{1}{2} + \frac{m^2}{6H}\Big(|t_1
$$
$$
- t_2| + (t_1 + t_2)\Big)\Big)e^{-\frac{m^2}{3H}(2|t_1-t_2|+3(t_1+t_2))} - \Big(\frac{1}{3} - \frac{m^2}{6H}\Big(|t_1 - t_2| - (t_1 + t_2)\Big)\Big)e^{\frac{m^2}{3H}(2|t_1-t_2|-3(t_1+t_2))}
$$
$$
+ \frac{1}{12}\, e^{-\frac{m^2}{3H}(4|t_1-t_2|+3(t_1+t_2))} - \frac{1}{12}\, e^{-\frac{m^2}{3H}(3|t_1-t_2|+4(t_1+t_2))} - \frac{1}{12}\, e^{\frac{m^2}{3H}(3|t_1-t_2|-4(t_1+t_2))}\Big)
$$
$$
\xrightarrow[\text{times}]{\text{late}} -\frac{2187\,\lambda^3 H^{16}}{4096\,\pi^8 m^{14}}\Big(1 + \frac{m^2}{3H}|t_1 - t_2|\Big)\, e^{-\frac{m^2}{3H}|t_1-t_2|}; \tag{B.2}
$$
$$
\xrightarrow[t_1 \geq t_2]{m\to 0} -\frac{\lambda^3 H^9}{5760\,\pi^8}\Big(t_1^6\, t_2 + \frac{5}{7}\, t_2^7\Big) := \big\langle \phi_{m=0}(t_1)\phi_{m=0}(t_2)\big\rangle^{\text{Triple Snowman}}_{\lambda^3}; \tag{B.3}
$$
$$
\xrightarrow[\text{times}]{\text{equal}} -\frac{2187\,\lambda^3 H^{16}}{4096\,\pi^8 m^{14}}\Big(1 - \Big(\frac{7}{12} + \frac{m^2 t}{3H} + \frac{2m^4 t^2}{9H^2} + \frac{8m^6 t^3}{81H^3}\Big)e^{-\frac{2m^2 t}{3H}} \tag{B.4}
$$
$$
+ \Big(\frac{1}{2} - \frac{2m^2 t}{3H} - \frac{4m^4 t^2}{9H^2}\Big)e^{-\frac{4m^2 t}{3H}} - \Big(\frac{3}{4} + \frac{2m^2 t}{3H}\Big)e^{-\frac{6m^2 t}{3H}} - \frac{1}{6}\, e^{-\frac{8m^2 t}{3H}}\Big).
$$

The result for the "Mouse" diagram type in Figure B.1b appears to be:

$$
\big\langle \phi(t_1)\phi(t_2)\big\rangle^{\text{Mouse}}_{\lambda^3} = -\frac{2187\,\lambda^3 H^{16}}{32768\,\pi^8 m^{14}}\Big(\Big(8 + \frac{8m^2}{3H}|t_1 - t_2|\Big)\, e^{-\frac{m^2}{3H}|t_1-t_2|} - \Big(\frac{44}{3} - \frac{8m^2}{3H}\Big(|t_1 - t_2|
$$
$$
- 2(t_1 + t_2)\Big) + \frac{8m^4}{9H^2}\Big(|t_1 - t_2| - (t_1 + t_2)\Big)^2 - \frac{8m^6}{81H^3}\Big(|t_1 - t_2| - (t_1 + t_2)\Big)^3\Big)\, e^{-\frac{m^2}{3H}(t_1+t_2)}
$$
$$
+ \Big(10 + \frac{4m^2}{H}\Big(|t_1 - t_2| + (t_1 + t_2)\Big) + \frac{4m^4}{9H^2}\Big(|t_1 - t_2| + (t_1 + t_2)\Big)^2\Big)\Big(e^{-\frac{m^2}{3H}(2|t_1-t_2|+(t_1+t_2))}
$$
$$
- e^{-\frac{m^2}{3H}(|t_1-t_2|+2(t_1+t_2))}\Big) - \Big(\frac{8}{3} - \frac{4m^2}{3H}\Big(|t_1 - t_2| - (t_1 + t_2)\Big)\Big)\, e^{\frac{m^2}{3H}(2|t_1-t_2|-3(t_1+t_2))} \tag{B.5}
$$
$$
+ \Big(10 - \frac{4m^4}{9H^2}\Big(|t_1 - t_2| - (t_1 + t_2)\Big)^2\Big)\, e^{\frac{m^2}{3H}(|t_1-t_2|-2(t_1+t_2))} + \Big(4 + \frac{4m^2}{3H}\Big(|t_1 - t_2| + (t_1 + t_2)\Big)\Big)
$$
$$
\times\Big(e^{-\frac{m^2}{3H}(3|t_1-t_2|+2(t_1+t_2))} - e^{-\frac{m^2}{3H}(2|t_1-t_2|+3(t_1+t_2))}\Big) + \frac{2}{3}\, e^{-\frac{m^2}{3H}(4|t_1-t_2|+3(t_1+t_2))}
$$
$$
- \frac{2}{3}\, e^{-\frac{m^2}{3H}(3|t_1-t_2|+4(t_1+t_2))} - \frac{2}{3}\, e^{\frac{m^2}{3H}(3|t_1-t_2|-4(t_1+t_2))}\Big)
$$

$$\xrightarrow[\text{times}]{\text{late}} -\frac{2187\,\lambda^3 H^{16}}{32768\,\pi^8 m^{14}}\left(8+\frac{8m^2}{3H}|t_1-t_2|\right)\mathrm{e}^{-\frac{m^2}{3H}|t_1-t_2|}\Bigg); \tag{B.6}$$

$$\xrightarrow[t_1\geq t_2]{m\to 0} -\frac{\lambda^3 H^9}{5760\,\pi^8}\left(t_1^6\,t_2+\frac{5}{7}\,t_2^7\right) := \left\langle\phi_{m=0}(t_1)\phi_{m=0}(t_2)\right\rangle_{\lambda^3}^{\text{Mouse}}; \tag{B.7}$$

$$\xrightarrow[\text{times}]{\text{equal}} -\frac{2187\,\lambda^3 H^{16}}{4096\,\pi^8 m^{14}}\Bigg(1-\left(\frac{7}{12}+\frac{m^2 t}{3H}+\frac{2m^4t^2}{9H^2}+\frac{8m^6t^3}{81H^3}\right)\mathrm{e}^{-\frac{2m^2t}{3H}} \tag{B.8}$$
$$+\left(\frac{1}{2}-\frac{2m^2t}{3H}-\frac{4m^4t^2}{9H^2}\right)\mathrm{e}^{-\frac{4m^2t}{3H}}-\left(\frac{3}{4}+\frac{2m^2t}{3H}\right)\mathrm{e}^{-\frac{6m^2t}{3H}}-\frac{1}{6}\,\mathrm{e}^{-\frac{8m^2t}{3H}}\Bigg).$$

For the next one, Figure B.1c, the result is the following:

$$\left\langle\phi(t_1)\phi(t_2)\right\rangle_{\lambda^3}^{\text{Snowman+Ind.Loop}} = -\frac{729\,\lambda^3 H^{16}}{16384\,\pi^8 m^{14}}\Bigg(\left(24+\frac{8m^2}{H}|t_1-t_2|+\frac{4m^4}{3H^2}|t_1-t_2|^2\right)\mathrm{e}^{-\frac{m^2}{3H}|t_1-t_2|}$$
$$+\Bigg(51+\frac{2m^2}{H}\Big(|t_1-t_2|-3(t_1+t_2)\Big)-\frac{m^4}{3H^2}\bigg(\Big(|t_1-t_2|+(t_1+t_2)\Big)^2-6\Big(|t_1-t_2|^2-(t_1+t_2)^2\Big)$$
$$+3\Big(|t_1-t_2|-(t_1+t_2)\Big)^2\bigg)-\frac{4m^6}{27H^3}\Big(2|t_1-t_2|+(t_1+t_2)\Big)\Big(|t_1-t_2|-(t_1+t_2)\Big)^2\Bigg)\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2)}$$
$$+\left(18+\frac{2m^2}{H}\Big(5|t_1-t_2|+2(t_1+t_2)\Big)+\frac{4m^4}{3H^2}|t_1-t_2|\Big(|t_1-t_2|+(t_1+t_2)\Big)\right)\mathrm{e}^{-\frac{m^2}{3H}\left(2|t_1-t_2|+(t_1+t_2)\right)}$$
$$-\left(9-\frac{2m^2}{H}\Big(2|t_1-t_2|-7(t_1+t_2)\Big)-\frac{2m^4}{3H^2}\Big(|t_1-t_2|^2-5(t_1+t_2)^2\Big)\right)\mathrm{e}^{-\frac{m^2}{3H}\left(|t_1-t_2|+2(t_1+t_2)\right)}$$
$$-\Bigg(66-\frac{3m^2}{H}\Big(7|t_1-t_2|-8(t_1+t_2)\Big)+\frac{22m^4}{33H^2}\Big(3|t_1-t_2|-5(t_1+t_2)\Big)\Big(|t_1-t_2| \tag{B.9}$$
$$-(t_1+t_2)\Big)\Bigg)\mathrm{e}^{\frac{m^2}{3H}\left(|t_1-t_2|-2(t_1+t_2)\right)}+\left(9+\frac{m^2}{H}\Big(3|t_1-t_2|+2(t_1+t_2)\Big)\right)\mathrm{e}^{-\frac{m^2}{3H}\left(3|t_1-t_2|+2(t_1+t_2)\right)}$$
$$-\left(\frac{15}{2}+\frac{4m^2}{H}(t_1+t_2)\right)\mathrm{e}^{-\frac{3m^2}{3H}(t_1+t_2)}-\left(\frac{15}{2}+\frac{m^2}{H}\Big(|t_1-t_2|+4(t_1+t_2)\Big)\right)\mathrm{e}^{-\frac{m^2}{3H}\left(2|t_1-t_2|+3(t_1+t_2)\right)}$$
$$-\left(\frac{15}{2}-\frac{m^2}{H}\Big(3|t_1-t_2|-4(t_1+t_2)\Big)\right)\mathrm{e}^{\frac{m^2}{3H}\left(2|t_1-t_2|-3(t_1+t_2)\right)}+\frac{3}{2}\,\mathrm{e}^{-\frac{m^2}{3H}\left(4|t_1-t_2|+3(t_1+t_2)\right)}$$
$$-\frac{3}{2}\left(\mathrm{e}^{-\frac{m^2}{3H}\left(3|t_1-t_2|+4(t_1+t_2)\right)}+\mathrm{e}^{-\frac{m^2}{3H}\left(|t_1-t_2|+4(t_1+t_2)\right)}+\mathrm{e}^{\frac{m^2}{3H}\left(|t_1-t_2|-4(t_1+t_2)\right)}+\mathrm{e}^{\frac{m^2}{3H}\left(3|t_1-t_2|-4(t_1+t_2)\right)}\right)\Bigg)$$

$$\xrightarrow[\text{times}]{\text{late}} -\frac{729\,\lambda^3 H^{16}}{16384\,\pi^8 m^{14}}\left(24+\frac{8m^2}{H}|t_1-t_2|+\frac{4m^4}{3H^2}|t_1-t_2|^2\right)\mathrm{e}^{-\frac{m^2}{3H}|t_1-t_2|}; \tag{B.10}$$

$$\xrightarrow[t_1\geq t_2]{m\to 0} -\frac{\lambda^3 H^9}{3072\,\pi^8}\left(t_1^6 t_2+\frac{1}{3}t_1^4t_2^3+\frac{3}{5}t_1^2t_2^5+\frac{53}{105}t_2^7\right) := \left\langle\phi_{m=0}(t_1)\phi_{m=0}(t_2)\right\rangle_{\lambda^3}^{\text{Snowman + Ind.Loop}} \tag{B.11}$$

$$\xrightarrow[\text{times}]{\text{equal}} -\frac{2187\,\lambda^3 H^{16}}{2048\,\pi^8 m^{14}}\Bigg(1+\left(\frac{23}{8}-\frac{m^2 t}{6H}-\frac{5m^4t^2}{9H^2}-\frac{4m^6t^3}{81H^3}\right)\mathrm{e}^{-\frac{2m^2t}{3H}} \\ -\left(\frac{11}{4}+\frac{3m^2t}{H}+\frac{10m^4t^2}{9H^2}\right)\mathrm{e}^{-\frac{4m^2t}{3H}}-\left(\frac{7}{8}+\frac{m^2t}{H}\right)\mathrm{e}^{-\frac{6m^2t}{3H}}-\frac{1}{4}\,\mathrm{e}^{-\frac{8m^2t}{3H}}\Bigg). \tag{B.12}$$

The "Three Independent Loops" contribution is

$$\left\langle\phi(t_1)\phi(t_2)\right\rangle_{\lambda^3}^{\text{Three Ind. Loops}} = -\frac{729\,\lambda^3 H^{16}}{16384\,\pi^8 m^{14}}\Bigg(\bigg(12+\frac{4m^2}{H}|t_1-t_2|+\frac{2m^4}{3H^2}|t_1-t_2|^2 \tag{B.13}$$

$$+\frac{2m^6}{27H^3}|t_1-t_2|^3\bigg)\mathrm{e}^{-\frac{m^2}{3H}|t_1-t_2|}-\bigg(13-\frac{2m^2}{H}\Big(|t_1-t_2|-(t_1+t_2)\Big)-\frac{m^4}{3H^2}\Big(|t_1-t_2|^2$$

$$-2\big(t_1+t_2\big)^2\Big)-\frac{2m^6}{27H^3}\Big(|t_1-t_2|^3-2\big(t_1+t_2\big)^3\Big)\bigg)\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2)}+\bigg(6+\frac{2m^2}{H}|t_1-t_2|$$

$$+\frac{m^4}{3H^2}\,|t_1-t_2|^2\bigg)\,\mathrm{e}^{-\frac{m^2}{3H}\left(2|t_1-t_2|+(t_1+t_2)\right)}+\bigg(\frac{m^2}{H}\Big(|t_1-t_2|-2\big(t_1+t_2\big)\Big)+\frac{m^4}{3H^2}\Big(|t_1-t_2|^2$$

$$-2\big(t_1+t_2\big)^2\Big)\bigg)\,\mathrm{e}^{-\frac{m^2}{3H}\left(|t_1-t_2|+2(t_1+t_2)\right)}+\bigg(\frac{9}{2}-\frac{m^2}{2H}\Big(3|t_1-t_2|+4\big(t_1+t_2\big)\Big)+\frac{m^4}{3H^2}\Big(|t_1-t_2|^2$$

$$-2\big(t_1+t_2\big)^2\Big)\bigg)\,\mathrm{e}^{\frac{m^2}{3H}\left(|t_1-t_2|-2(t_1+t_2)\right)}+\bigg(\frac{3}{2}+\frac{m^2}{2H}|t_1-t_2|\bigg)\,\mathrm{e}^{-\frac{m^2}{3H}\left(3|t_1-t_2|+2(t_1+t_2)\right)}-\bigg(\frac{21}{4}-\frac{m^2}{H}$$

$$\times\Big(|t_1-t_2|-2\big(t_1+t_2\big)\Big)\bigg)\,\mathrm{e}^{-\frac{3m^2}{3H}(t_1+t_2)}-\bigg(\frac{3}{4}-\frac{m^2}{2H}\Big(|t_1-t_2|-2\big(t_1+t_2\big)\Big)\bigg)\,\mathrm{e}^{-\frac{m^2}{3H}\left(2|t_1-t_2|+3(t_1+t_2)\right)}$$

$$-\bigg(\frac{13}{4}-\frac{m^2}{2H}\Big(|t_1-t_2|-2\big(t_1+t_2\big)\Big)\bigg)\,\mathrm{e}^{\frac{m^2}{3H}\left(2|t_1-t_2|-3(t_1+t_2)\right)}+\frac{1}{4}\,\mathrm{e}^{-\frac{m^2}{3H}\left(4|t_1-t_2|+3(t_1+t_2)\right)}$$

$$-\frac{1}{4}\bigg(\mathrm{e}^{-\frac{m^2}{3H}\left(3|t_1-t_2|+4(t_1+t_2)\right)}+3\mathrm{e}^{-\frac{m^2}{3H}\left(|t_1-t_2|+4(t_1+t_2)\right)}+3\mathrm{e}^{\frac{m^2}{3H}\left(|t_1-t_2|-4(t_1+t_2)\right)}+\mathrm{e}^{\frac{m^2}{3H}\left(3|t_1-t_2|-4(t_1+t_2)\right)}\bigg)\Bigg)$$

$$\xrightarrow[\text{times}]{\text{late}} -\frac{729\,\lambda^3 H^{16}}{16384\,\pi^8 m^{14}}\left(12+\frac{4m^2}{H}|t_1-t_2|+\frac{2m^4}{3H^2}|t_1-t_2|^2+\frac{2m^6}{27H^3}|t_1-t_2|^3\right)\mathrm{e}^{-\frac{m^2}{3H}|t_1-t_2|}; \tag{B.14}$$

$$\xrightarrow[t_1\geq t_2]{m\to 0} -\frac{\lambda^3 H^9}{12288\pi^8}\left(t_1^6 t_2+t_1^4 t_2^3+\frac{7}{5}t_1^2 t_2^5+\frac{9}{35}t_2^7\right) := \left\langle\phi_{m=0}(t_1)\phi_{m=0}(t_2)\right\rangle_{\lambda^3}^{\text{Three Ind. Loops}}; \tag{B.15}$$

$$\xrightarrow[\text{times}]{\text{equal}} -\frac{2187\,\lambda^3 H^{16}}{4096\,\pi^8 m^{14}}\Bigg(1-\left(\frac{7}{12}+\frac{m^2t}{3H}+\frac{2m^4t^2}{9H^2}+\frac{8m^6t^3}{81H^3}\right)\mathrm{e}^{-\frac{2m^2t}{3H}} \\ +\left(\frac{1}{2}-\frac{2m^2t}{3H}-\frac{4m^4t^2}{9H^2}\right)\mathrm{e}^{-\frac{4m^2t}{3H}}-\left(\frac{3}{4}+\frac{2m^2t}{3H}\right)\mathrm{e}^{-\frac{6m^2t}{3H}}-\frac{1}{6}\,\mathrm{e}^{-\frac{8m^2t}{3H}}\Bigg). \tag{B.16}$$

The final expression for the "Sunset with a loop" diagram, Figure B.1e, is

$$
\begin{aligned}
\left\langle\phi(t_1)\phi(t_2)\right\rangle_{\lambda^3}^{\text{Sunset+Loop}} &= -\frac{2187\,\lambda^3 H^{16}}{4096\,\pi^8 m^{14}}\Bigg(\left(1+\frac{7m^2}{6H}|t_1-t_2|\right)\mathrm{e}^{-\frac{m^2}{3H}|t_1-t_2|} \qquad \text{(B.17)}\\
&+\left(1+\frac{m^2}{6H}|t_1-t_2|\right)\mathrm{e}^{-\frac{3m^2}{3H}|t_1-t_2|}+\left(\frac{449}{24}+\frac{m^2}{6H}\Big(17|t_1-t_2|-24(t_1+t_2)\Big)-\frac{m^4}{3H^2}\Big(|t_1-t_2|\\
&-(t_1+t_2)\Big)^2\right)\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2)}+\left(9+\frac{5m^2}{3H}\Big(2|t_1-t_2|+(t_1+t_2)\Big)+\frac{m^4}{9H^2}|t_1-t_2|\Big(3|t_1-t_2|\\
&+4(t_1+t_2)\Big)\right)\mathrm{e}^{-\frac{m^2}{3H}\left(2|t_1-t_2|+(t_1+t_2)\right)}-\left(\frac{37}{4}+\frac{m^2}{6H}\Big(7|t_1-t_2|+23(t_1+t_2)\Big)+\frac{m^4}{18H^2}\Big(3|t_1-t_2|^2\\
&+2|t_1-t_2|(t_1+t_2)+9(t_1+t_2)^2\Big)\right)\mathrm{e}^{-\frac{m^2}{3H}\left(|t_1-t_2|+2(t_1+t_2)\right)}-\left(\frac{129}{8}-\frac{11m^2}{2H}\Big(|t_1-t_2|-(t_1+t_2)\Big)\right.\\
&\left.+\frac{m^4}{2H^2}\Big(|t_1-t_2|-(t_1+t_2)\Big)^2\right)\mathrm{e}^{\frac{m^2}{3H}\left(|t_1-t_2|-2(t_1+t_2)\right)}+\left(\frac{23}{8}+\frac{m^2}{6H}\Big(5|t_1-t_2|+2(t_1+t_2)\Big)\right)\\
&\times\mathrm{e}^{-\frac{m^2}{3H}\left(3|t_1-t_2|+2(t_1+t_2)\right)}-\left(\frac{5}{8}-\frac{m^2}{6H}\Big(|t_1-t_2|-2(t_1+t_2)\Big)\right)\mathrm{e}^{-\frac{3m^2}{3H}(t_1+t_2)}\\
&+\frac{1}{8}\,\mathrm{e}^{-\frac{m^2}{3H}\left(4|t_1-t_2|+(t_1+t_2)\right)}-\left(\frac{23}{8}+\frac{m^2}{6H}\Big(2|t_1-t_2|+5(t_1+t_2)\Big)\right)\mathrm{e}^{-\frac{m^2}{3H}\left(2|t_1-t_2|+3(t_1+t_2)\right)}\\
&-\left(\frac{79}{24}-\frac{5m^2}{6H}\Big(|t_1-t_2|-(t_1+t_2)\Big)\right)\mathrm{e}^{\frac{m^2}{3H}\left(2|t_1-t_2|-3(t_1+t_2)\right)}+\frac{7}{24}\,\mathrm{e}^{-\frac{m^2}{3H}\left(4|t_1-t_2|+3(t_1+t_2)\right)}\\
&-\frac{7}{24}\,\mathrm{e}^{-\frac{m^2}{3H}\left(3|t_1-t_2|+4(t_1+t_2)\right)}-\frac{1}{8}\mathrm{e}^{-\frac{m^2}{3H}\left(|t_1-t_2|+4(t_1+t_2)\right)}-\frac{1}{8}\,\mathrm{e}^{\frac{m^2}{3H}\left(|t_1-t_2|-4(t_1+t_2)\right)}\\
&-\frac{7}{24}\,\mathrm{e}^{\frac{m^2}{3H}\left(3|t_1-t_2|-4(t_1+t_2)\right)}\Bigg)\xrightarrow[\text{times}]{\text{late}}-\frac{2187\,\lambda^3 H^{16}}{4096\,\pi^8 m^{14}}\Bigg(\left(1+\frac{7m^2}{6H}|t_1-t_2|\right)\mathrm{e}^{-\frac{m^2}{3H}|t_1-t_2|} \qquad \text{(B.18)}\\
&\qquad+\left(1+\frac{m^2}{6H}|t_1-t_2|\right)\mathrm{e}^{-\frac{3m^2}{3H}|t_1-t_2|}\Bigg);\\
&\xrightarrow[t_1\geq t_2]{m\to 0}-\frac{\lambda^3 H^9}{11520\,\pi^8}\left(7\,t_1^6 t_2+3\,t_1 t_2^6+\frac{20}{7}\,t_2^7\right):=\left\langle\phi_{m=0}(t_1)\phi_{m=0}(t_2)\right\rangle_{\lambda^3}^{\text{Sunset+Loop}}; \qquad \text{(B.19)}\\
&\xrightarrow[\text{times}]{\text{equal}}-\frac{2187\,\lambda^3 H^{16}}{4096\,\pi^8 m^{14}}\Bigg(2+\left(\frac{167}{6}-\frac{14m^2 t}{3H}-\frac{4m^4t^2}{3H^2}\right)\mathrm{e}^{-\frac{2m^2t}{3H}} \qquad \text{(B.20)}\\
&\qquad-\left(\frac{45}{2}+\frac{18m^2t}{H}+\frac{4m^4t^2}{H^2}\right)\mathrm{e}^{-\frac{4m^2t}{3H}}-\left(\frac{13}{2}+\frac{4m^2t}{H}\right)\mathrm{e}^{-\frac{6m^2t}{3H}}-\frac{5}{6}\,\mathrm{e}^{-\frac{8m^2t}{3H}}\Bigg).
\end{aligned}
$$

For the "Peephole" diagram we have

$$\big\langle\phi(t_1)\phi(t_2)\big\rangle_{\lambda^3}^{\text{Peephole}} = -\frac{729\,\lambda^3H^{16}}{2048\,\pi^8m^{14}}\Bigg(\bigg(1+\frac{m^2}{3H}|t_1-t_2|\bigg)\,\mathrm{e}^{-\frac{m^2}{3H}|t_1-t_2|} \tag{B.21}$$
$$-\bigg(\frac{35}{3}+\frac{m^2}{3H}\Big(6|t_1-t_2|-5\big(t_1+t_2\big)\Big)+\frac{m^4}{3H^2}\Big(|t_1-t_2|-\big(t_1+t_2\big)\Big)^2\bigg)\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2)}$$
$$-\bigg(\frac{3}{2}-\frac{m^2}{H}\Big(|t_1-t_2|+\big(t_1+t_2\big)\Big)\bigg)\Big(\mathrm{e}^{-\frac{m^2}{3H}\left(2|t_1-t_2|+(t_1+t_2)\right)}-\mathrm{e}^{-\frac{m^2}{3H}\left(|t_1-t_2|+2(t_1+t_2)\right)}\Big)$$
$$+\bigg(\frac{27}{2}-\frac{m^2}{2H}\Big(|t_1-t_2|-\big(t_1+t_2\big)\Big)\bigg)\,\mathrm{e}^{\frac{m^2}{3H}\left(|t_1-t_2|-2(t_1+t_2)\right)}+\bigg(3+\frac{m^2}{2H}\Big(|t_1-t_2|+\big(t_1+t_2\big)\Big)\bigg)$$
$$\times\Big(\mathrm{e}^{-\frac{m^2}{3H}\left(3|t_1-t_2|+2(t_1+t_2)\right)}-\mathrm{e}^{-\frac{m^2}{3H}\left(2|t_1-t_2|+3(t_1+t_2)\right)}\Big)-\bigg(\frac{8}{3}-\frac{m^2}{2H}\Big(|t_1-t_2|-\big(t_1+t_2\big)\Big)\bigg)$$
$$\times\,\mathrm{e}^{\frac{m^2}{3H}\left(2|t_1-t_2|-3(t_1+t_2)\right)}+\frac{1}{6}\,\mathrm{e}^{-\frac{m^2}{3H}\left(4|t_1-t_2|+3(t_1+t_2)\right)}-\frac{1}{6}\,\mathrm{e}^{-\frac{m^2}{3H}\left(3|t_1-t_2|+4(t_1+t_2)\right)}$$
$$-\frac{1}{6}\,\mathrm{e}^{\frac{m^2}{3H}\left(3|t_1-t_2|-4(t_1+t_2)\right)}\Bigg)\xrightarrow[\text{times}]{\text{late}}-\frac{729\,\lambda^3H^{16}}{2048\,\pi^8m^{14}}\bigg(1+\frac{m^2}{3H}|t_1-t_2|\bigg)\,\mathrm{e}^{-\frac{m^2}{3H}|t_1-t_2|}; \tag{B.22}$$
$$\xrightarrow[t_1\geq t_2]{m\to 0}-\frac{\lambda^3H^9}{5760\,\pi^8}\bigg(t_1^6t_2+\frac{5}{7}\,t_2^7\bigg):=\big\langle\phi_{m=0}(t_1)\phi_{m=0}(t_2)\big\rangle_{\lambda^3}^{\text{Peephole}}; \tag{B.23}$$
$$\xrightarrow[\text{times}]{\text{equal}}-\frac{729\,\lambda^3H^{16}}{2048\,\pi^8m^{14}}\Bigg(1-\bigg(\frac{79}{6}-\frac{16m^2t}{3H}+\frac{4m^4t^2}{3H^2}\bigg)\,\mathrm{e}^{-\frac{2m^2t}{3H}}+18\,\mathrm{e}^{-\frac{4m^2t}{3H}} \tag{B.24}$$
$$-\bigg(\frac{11}{2}+\frac{2m^2t}{H}\bigg)\,\mathrm{e}^{-\frac{6m^2t}{3H}}-\frac{1}{3}\,\mathrm{e}^{-\frac{8m^2t}{3H}}\Bigg).$$

For Figure B.1g, i.e., the "Sunset and Independent Loop" contribution, we obtain

$$\big\langle\phi(t_1)\phi(t_2)\big\rangle_{\lambda^3}^{\text{Sunset+Ind. Loop}} = -\frac{6561\,\lambda^3H^{16}}{8192\,\pi^8m^{14}}\Bigg(\bigg(1+\frac{2m^2}{9H}|t_1-t_2|+\frac{2m^4}{27H^2}|t_1-t_2|^2\bigg)\,\mathrm{e}^{-\frac{m^2}{3H}|t_1-t_2|}$$
$$-\frac{1}{9}\,\mathrm{e}^{-\frac{m^2}{H}|t_1-t_2|}-\bigg(\frac{17}{36}-\frac{m^2}{9H}\Big(|t_1-t_2|+6\big(t_1+t_2\big)\Big)-\frac{2m^4}{27H^2}\Big(2|t_1-t_2|^2-3\big(t_1+t_2\big)^2\Big)\bigg)\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2)}$$
$$+\bigg(\frac{7}{9}+\frac{4m^2}{9H}|t_1-t_2|+\frac{2m^4}{27H^2}\,|t_1-t_2|^2\bigg)\,\mathrm{e}^{-\frac{m^2}{3H}\left(2|t_1-t_2|+(t_1+t_2)\right)}+\frac{1}{36}\,\mathrm{e}^{-\frac{m^2}{3H}\left(4|t_1-t_2|+(t_1+t_2)\right)}$$
$$+\bigg(\frac{11}{6}+\frac{m^2}{9H}\Big(7|t_1-t_2|-9\big(t_1+t_2\big)\Big)+\frac{m^4}{27H^2}\Big(3|t_1-t_2|^2-2|t_1-t_2|\big(t_1+t_2\big)-3\big(t_1+t_2\big)^2\Big)\bigg)$$
$$\times\,\mathrm{e}^{-\frac{m^2}{3H}\left(|t_1-t_2|+2(t_1+t_2)\right)}+\bigg(\frac{7}{12}+\frac{m^2}{18H}\Big(|t_1-t_2|-18\big(t_1+t_2\big)\Big)+\frac{m^4}{27H^2}\Big(|t_1-t_2|-\big(t_1+t_2\big)\Big)$$
$$\times\Big(|t_1-t_2|+3\big(t_1+t_2\big)\Big)\bigg)\,\mathrm{e}^{\frac{m^2}{3H}\left(|t_1-t_2|-2(t_1+t_2)\right)}+\bigg(\frac{7}{12}+\frac{m^2}{6H}|t_1-t_2|\bigg)\,\mathrm{e}^{-\frac{m^2}{3H}\left(3|t_1-t_2|+2(t_1+t_2)\right)}$$

$$
- \left(\frac{43}{18} - \frac{2m^2}{9H}\left(|t_1 - t_2| - 2(t_1 + t_2)\right)\right) \mathrm{e}^{-\frac{3m^2}{3H}(t_1+t_2)} - \left(\frac{1}{2} - \frac{m^2}{18H}\left(|t_1 - t_2| - 4(t_1 + t_2)\right)\right)
\times \mathrm{e}^{-\frac{m^2}{3H}\left(2|t_1-t_2|+3(t_1+t_2)\right)} - \left(\frac{19}{18} - \frac{m^2}{18H}\left(3|t_1 - t_2| - 4(t_1 + t_2)\right)\right) \mathrm{e}^{\frac{m^2}{3H}\left(2|t_1-t_2|-3(t_1+t_2)\right)}
+ \frac{1}{18} \mathrm{e}^{-\frac{m^2}{3H}\left(4|t_1-t_2|+3(t_1+t_2)\right)} - \frac{1}{18} \mathrm{e}^{-\frac{m^2}{3H}\left(3|t_1-t_2|+4(t_1+t_2)\right)} - \frac{1}{9} \mathrm{e}^{-\frac{m^2}{3H}\left(|t_1-t_2|+4(t_1+t_2)\right)}
- \frac{1}{9} \mathrm{e}^{\frac{m^2}{3H}\left(|t_1-t_2|-4(t_1+t_2)\right)} - \frac{1}{18} \mathrm{e}^{\frac{m^2}{3H}\left(3|t_1-t_2|-4(t_1+t_2)\right)}\Bigg) \tag{B.25}
$$

$$
\xrightarrow[\text{times}]{\text{late}} -\frac{6561\lambda^3 H^{16}}{8192\,\pi^8 m^{14}}\left(\left(1 + \frac{2m^2}{9H}|t_1 - t_2| + \frac{2m^4}{27H^2}|t_1 - t_2|^2\right)\mathrm{e}^{-\frac{m^2}{3H}|t_1-t_2|} - \frac{1}{9}\mathrm{e}^{-\frac{m^2}{H}|t_1-t_2|}\right) \tag{B.26}
$$

$$
\xrightarrow[t_1 \geq t_2]{m\to 0} -\frac{\lambda^3 H^9}{46080\,\pi^8}\left(11t_1^6 t_2 + 5t_1^4 t_2^3 + 9t_1^2 t_2^5 + 4t_1 t_2^6 + \frac{13}{7}t_2^7\right) := \left\langle \phi_{m=0}(t_1)\phi_{m=0}(t_2)\right\rangle_{\lambda^3}^{\text{Sunset+Ind.Loop}} \tag{B.27}
$$

$$
\xrightarrow[\text{times}]{\text{equal}} -\frac{2187\,\lambda^3 H^{16}}{4096\,\pi^8 m^{14}}\Bigg(\frac{4}{3} + \left(\frac{1}{2} + \frac{2m^2 t}{H} - \frac{4m^4 t^2}{3H^2}\right)\mathrm{e}^{-\frac{2m^2 t}{3H}} \tag{B.28}
+ \left(\frac{9}{2} - \frac{6m^2 t}{H} - \frac{4m^4 t^2}{3H^2}\right)\mathrm{e}^{-\frac{4m^2 t}{3H}} - \left(\frac{35}{6} + \frac{8m^2 t}{3H}\right)\mathrm{e}^{-\frac{6m^2 t}{3H}} - \frac{1}{2}\,\mathrm{e}^{-\frac{8m^2 t}{3H}}\Bigg).
$$

And finally, the last contribution of the "Shackle" diagram is

$$
\left\langle \phi(t_1)\phi(t_2)\right\rangle_{\lambda^3}^{\text{Shackle}} = -\frac{2187\,\lambda^3 H^{16}}{8192\,\pi^8 m^{14}}\Bigg(\left(1 + \frac{8m^2}{3H}|t_1 - t_2|\right)\mathrm{e}^{-\frac{m^2}{3H}|t_1-t_2|} \tag{B.29}
+ \left(3 + \frac{2m^2}{3H}|t_1 - t_2|\right)\mathrm{e}^{-\frac{m^2}{H}|t_1-t_2|} + \left(\frac{595}{6} + \frac{8m^2}{3H}\left(6|t_1 - t_2| - 7(t_1 + t_2)\right)\right)\mathrm{e}^{-\frac{m^2}{3H}(t_1+t_2)}
+ \left(47 + \frac{16m^2}{H}|t_1 - t_2| + \frac{4m^4}{3H^2}|t_1 - t_2|^2\right)\mathrm{e}^{-\frac{m^2}{3H}\left(2|t_1-t_2|+(t_1+t_2)\right)} + \frac{1}{2}\mathrm{e}^{-\frac{m^2}{3H}\left(4|t_1-t_2|+(t_1+t_2)\right)}
- \left(39 - \frac{2m^2}{H}\left(|t_1 - t_2| - 9(t_1 + t_2)\right) + \frac{4m^4}{3H^2}(t_1 + t_2)^2\right)\mathrm{e}^{-\frac{m^2}{3H}\left(|t_1-t_2|+2(t_1+t_2)\right)} - \Bigg(\frac{165}{2}
- \frac{23m^2}{H}\left(|t_1 - t_2| - (t_1 + t_2)\right) + \frac{4m^4}{3H^2}\left(|t_1 - t_2| - (t_1 + t_2)\right)^2\Bigg)\mathrm{e}^{\frac{m^2}{3H}\left(|t_1-t_2|-2(t_1+t_2)\right)} + \Bigg(\frac{27}{2}
+ \frac{m^2}{H}\left(3|t_1 - t_2| + (t_1 + t_2)\right)\Bigg)\mathrm{e}^{-\frac{m^2}{3H}\left(3|t_1-t_2|+2(t_1+t_2)\right)} - \left(\frac{27}{2} + \frac{m^2}{H}\left(|t_1 - t_2| + 3(t_1 + t_2)\right)\right)
\times \mathrm{e}^{-\frac{m^2}{3H}\left(2|t_1-t_2|+3(t_1+t_2)\right)} - \left(\frac{21}{2} - \frac{2m^2}{3H}\left(2|t_1 - t_2| - 3(t_1 + t_2)\right)\right)\mathrm{e}^{-\frac{m^2}{H}(t_1+t_2)} - \Bigg(\frac{101}{6} - \frac{3m^2}{H}\times
\left(|t_1 - t_2| - (t_1 + t_2)\right)\Bigg)\mathrm{e}^{\frac{m^2}{3H}\left(2|t_1-t_2|-3(t_1+t_2)\right)} + \frac{5}{6}\mathrm{e}^{-\frac{m^2}{3H}\left(4|t_1-t_2|+3(t_1+t_2)\right)} - \frac{5}{6}\mathrm{e}^{-\frac{m^2}{3H}\left(3|t_1-t_2|+4(t_1+t_2)\right)}
$$

$$
-\frac{1}{2}\,\mathrm{e}^{-\frac{m^2}{3H}\left(|t_1-t_2|+4(t_1+t_2)\right)}-\frac{1}{2}\,\mathrm{e}^{\frac{m^2}{3H}\left(|t_1-t_2|-4(t_1+t_2)\right)}-\frac{5}{6}\,\mathrm{e}^{\frac{m^2}{3H}\left(3|t_1-t_2|-4(t_1+t_2)\right)}\Bigg)
$$

$$
\xrightarrow[\text{times}]{\text{late}}-\frac{2187\,\lambda^3H^{16}}{8192\,\pi^8m^{14}}\left(\left(1+\frac{8m^2}{3H}|t_1-t_2|\right)\mathrm{e}^{-\frac{m^2}{3H}|t_1-t_2|}+\left(3+\frac{2m^2}{3H}|t_1-t_2|\right)\mathrm{e}^{-\frac{m^2}{H}|t_1-t_2|}\right) \tag{B.30}
$$

$$
\xrightarrow[t_1\geq t_2]{m\to 0}-\frac{\lambda^3H^9}{1280\,\pi^8}\left(t_1^6t_2+\frac{2}{3}\,t_1t_2^6+\frac{5}{21}t_2^7\right):=\left\langle\phi_{m=0}(t_1)\phi_{m=0}(t_2)\right\rangle_{\lambda^3}^{\text{Shackle}}; \tag{B.31}
$$

$$
\xrightarrow[\text{times}]{\text{equal}}-\frac{2187\,\lambda^3H^{16}}{4096\,\pi^8m^{14}}\Bigg(2+\left(\frac{220}{3}-\frac{56m^2t}{3H}\right)\mathrm{e}^{-\frac{2m^2t}{3H}}-\left(54+\frac{40m^2t}{H}+\frac{16m^4t^2}{3H^2}\right)\mathrm{e}^{-\frac{4m^2t}{3H}} \tag{B.32}
$$

$$
-\left(20+\frac{8m^2t}{H}\right)\mathrm{e}^{-\frac{6m^2t}{3H}}-\frac{4}{3}\,\mathrm{e}^{-\frac{8m^2t}{3H}}\Bigg).
$$

When summed, the obtained results lead to (2.33); at late times, to (2.34); in the smooth massless limit, to (2.35); and at equal times, to (2.36) and (5.8).

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