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SUPERGEOMETRY, NONCOMMUTATIVE GEOMETRY AND INVARIANT THEORY

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Abstract

This thesis covers various questions in *supergeometry*, *noncommutative geometry* and *invariant theory*. A brief summary of each of these topics is given below.

- The *fundamental theorems of invariant theory* characterize the ring of invariants $\mathbb{C}[\mathrm{M}_{r\times p}]^{\mathrm{SL}_r(\mathbb{C})}$, under a canonical action of special linear group $\mathrm{SL}_r(\mathbb{C})$ on the algebra of polynomials $\mathbb{C}[\mathrm{M}_{r\times p}]$. In this case, the generators correspond to the $r \times r$ minors and satisfy well-known *Plücker relations*. We present here a super version of this question. Following a necessary modification, we prove the first fundamental theorem for the case of *special linear supergroup* $\mathrm{SL}(r|s)$. By employing an approach taken from the classical case based on the well-known *Jacobi's complementary minor theorem*, we establish certain relations, what we refer to as *super Plücker relations*. For the case of $\mathrm{SL}(1|1)$, we are able to prove that the super Plücker relations given here completely characterize the ring of invariants. For the general case, we conjecture that the super Plücker relations proposed here fully characterize the ring of invariants.

- In a recent work by R. O. Buachalla and P. Somberg, *Lusztig's positive root vectors*, with respect to a distinguished choice of reduced decomposition of the longest element of the *Weyl group*, were shown to give a quantum tangent space for every A-series *full quantum flag manifold* $\mathcal{O}_q(\mathrm{F}_n)$. Moreover, the associated differential calculus $\Omega_q^{(0,\bullet)}(\mathrm{F}_n)$ was shown to have classical dimension, giving a direct q-deformation of the classical anti-holomorphic Dolbeault complex of F_n. Here we examine in detail the rank two case, namely the full quantum flag manifold of $\mathcal{O}_q(\mathrm{SU}_3)$. In particular, we examine the $*$-differential calculus associated to $\Omega_q^{(0,\bullet)}(\mathrm{F}_3)$ and its non-commutative complex geometry. We find that the number of *almost-complex structures* reduces from 8 to 4. Moreover, we show that each of these almost-complex structures is integrable, which is to say, each of them is a *complex structure*.

♦ A *quantization* of the complex *Minkowski space* described as the big cell inside grassmannian $\mathrm{Gr}(2,4)$, due to R. Fioresi and others, is well-known. We extend this approach to the case of $N = 2$ *Minkowski superspace*. We give the superalgebra of $N = 2$ antichiral quantum superfields realized as a subalgebra of the quantum supergroup $\mathbb{C}_q[\mathrm{SL}(4|2)]$. The multiplication law in the quantum supergroup induces a coaction on the set of antichiral superfields. We also realize the quantum deformation of the Minkowski superspace as a *quantum principal bundle*.

Contents

Dedicated to my loving family.

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Chapter 1

Introduction

> *"Quantum mechanics requires the introduction into physical theory of a vast new domain of pure mathematics - the whole domain connected with non-commutative multiplication. This, coming on top of the introduction of new geometries by the theory of relativity, indicates a trend which we may expect to continue. We may expect that in the future further big domains of pure mathematics will have to be brought in to deal with the advances in fundamental physics." [26]*

- Paul Dirac, *1939*

Quantum physics is regarded as one of the greatest accomplishments of humankind in the 20$^{\text{th}}$ century. It certainly enabled us to comprehend nature better, but it also gave rise to several extremely fascinating subjects in pure mathematics. The title of this dissertation contains the names of two of these topics, *i.e. supergeometry* and *non-commutative geometry*, as well as one very classical area of mathematics, namely, *invariant theory*. In this chapter, we briefly summarise the themes of each of these subjects, followed by an introduction to the results accomplished in this thesis.

1.1 Supergeometry

Elementary particles are classified into two classes based on their *spin quantum number*. Particles having an integer spin (e.g. *photons, gluons* etc.) are referred to as *bosons*, whereas those having a half-integer spin (e.g. *electron, protons*, etc.) are known as *fermions*. The *Pauli exclusion principle* and *spin-statistics Theorem* states that when two identical particles are exchanged in a quantum system, the total wavefunction is symmetric for bosons and anti-symmetric for fermions [75]. The *principle of supersymmetry* (SUSY) [31], which postulates a symmetry between bosons and fermions, is the foundation of *supergeometry*.

Supergeometry is a generalization of the classical geometry which allows a space to have both type of coordinates, *i.e.* the coordinates that commute and the coordinates that anti-commute, called *even* and *odd coordinates* respectively. Chapter 3 provides a brief summary of several supergeometric concepts employed in this thesis. For more details see [19, 39, 75].

1.2 Noncommutative geometry

In the *Hamiltonian* formulation of classical mechanics, the phase space of a system is parameterized by a symplectic manifold M whereas the observables are given by the families $\{f_t\} \subset C^\infty(M \times \mathbb{R})$ of smooth functions on M, [72, Chap. 2]. The dynamics is controlled via a Hamiltonian $H \in C^\infty(M)$ and the equation of motion for an observable f_t is given as:

$$\frac{d}{dt} f_t = \{f_t, H\}.$$

On the other hand, in quantum mechanics, the states of a system are defined by rays in a Hilbert space $\mathcal{H}$, the observables are defined via self-adjoint operators on $\mathcal{H}$, [72, Chap. 4]. The dynamics is controlled via a Hamiltonian H and the equation of motion for an observable F_t is given as:

$$\frac{d}{dt} F_t = \frac{\iota}{\hbar}[H, F_t].$$

Quantum physics highlights a striking difference; the non-commutativity of physical observables. In [25], Dirac proposed the famous *canonical commutation relations*:

$$[\hat{A}, \hat{B}] = \iota\hbar\widehat{[A, B]}$$

where $\hbar$ is the reduced *Plancks constant*, that is the founding observation in noncommutative geometry as we are going to see in this thesis. The transition of results from classical to quantum descriptions is often regarded as quantization. The purpose is to provide a unified framework to study both classical and quantum worlds. Mathematically, one can formalize *canonical quantization* as a correspondence:

$$Q : f \mapsto Q(f)$$

where $f \in C^\infty(M)$ and $Q(f) \in \mathcal{H}$ satisfying:

$$[Q(f), Q(g)] = \imath\hbar Q(\{f, g\}).$$

However, this approach presents difficulties. The *Groenewold No-go Theorem* [43], tells that there does not exist such a nice correspondence in general. One of the ways to move forward is known as *deformation quantization*, see [78] for details. The rough idea is to associate to a Poisson manifold M a family $\{\mathcal{A}_\hbar\}$ of non-commutative algebras parameterized by specific admissible values of Planck's constant $\hbar$ such that at $\hbar = 0$ we recover the algebra $\mathcal{A}_0 = C^\infty(M)$ of smooth functions on M.

In pure mathematics, this concept motivated the discipline of *noncommutative geometry*. The idea is to replace the commutative algebra of functions on a space with a noncommutative associative algebra, which is considered the algebra of functions on a noncommutative space. See [17, Chap. 1] for more details on the motivation of this brilliant idea.

In chapter 2, we provide a brief summary of several concepts from noncommutative geometry that are employed in this thesis. For more details see [14, 57, 58, 63].

1.3 Invariant theory

Invariant theory is a very classical area of mathematics which dates back to the 19^{th} century that provided important results, whose applications involve both, mathematics and physics, see [10] for an historical review of the theory. Many renowned mathematicians including P. Gordan, J. J. Sylvester, D. Hilbert, E. Noether made fundamental contributions to this subject. For details, see [67].

Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a linear group, and,

$$\rho : \mathbb{C}[x_1, \cdots, x_m] \times G \longrightarrow \mathbb{C}[x_1, \cdots, x_m]$$

be an action of G on the polynomial algebra $\mathbb{C}[x_1, \cdots, x_m]$. A fundamental question of invariant theory is to characterize in terms of generators and relations, the ring of invariants $\mathbb{C}[x_1, \cdots, x_m]^G$ with respect to ρ.

In chapter 4, we study this question for the case of special linear supergroup $\mathrm{SL}(r|s)$.

1.4 Outline of the main results in this thesis

In this thesis, we discuss some works that lies at the intersections of above described subjects. Most of the material is borrowed from the following:

- R. Fioresi, M. A. Lledo, J. Razzaq, *N=2 quantum chiral superfields and quantum superbundles*, J. Phys. A. : Math. Theor. 55, 384012, (2022).
- R. Fioresi, M. A. Lledo, J. Razzaq, *Quantum Chiral Superfields*, J. Phy.: Conf. Ser. 2531 012015, (2023).
- R. Fioresi, J. Razzaq, *Quantum N=2 Minkowski Superspace*, Proceedings of Science, (2023).
- Two more projects that are not published yet.

A brief summary of each of these topics is described below.

1.5 Fundamental theorems of super invariant theory

Consider the polynomial functions on the set of $r \times p$ ($r \leq p$) complex matrices $\mathrm{M}_{r\times p}$, that are invariant under the action of the complex special linear group $\mathrm{SL}_r(\mathbb{C})$:

$$\begin{array}{ccc} \mathbb{C}[\mathrm{M}_{r\times p}] \times \mathrm{SL}_r(\mathbb{C}) & \longrightarrow & \mathbb{C}[\mathrm{M}_{r\times p}] \\ (f,g) & \longrightarrow & f.g, \end{array}$$

where $(f.g)(M) := f(gM)$ and $\mathbb{C}[\mathrm{M}_{r\times p}]$ denotes the algebra of polynomials functions on the entries of $\mathrm{M}_{r\times p}$. In this case, the First and Second fundamental theorems of invariant theory characterize the ring of invariants. (See section 4.1 for precise statements.)

We want to provide a generalization of this result to supergeometry. The generalization of some questions of invariant theory to the super setting appeared early in the literature: for example, the *Young super tableaux* and techniques regarding their manipulation, appeared first in [27] and at the same time in [2], to address questions of representation theory of Lie superalgebras and the *double commutant theorem*, a fundamental result, originally proven, in the ordinary setting by *Schur* [71]. In particular, in [2], the authors prove also a version of the *straightening algorithm* for Young super tableaux, of uttermost importance in representation theory and directly linked to the Plücker relations [33]. Later on, in [44], appeared such algorithm in the super setting. Their idea, involving the so called *virtual variables method*, provides an elegant approach which encompasses both, the ordinary and the super setting at once. More recently in [79, 29] appeared generalizations to other supergroups as the orthosymplectic one and in [23] a categorical approach to super invariant theory allowed to go beyond the characteristic zero setting (see also [24]).

In the present work, however, we are interested in a related, and yet different topic. It has its significance rooted into the geometric meaning of these combinatorial questions. In the ordinary setting, fundamental theorems of invariant theory provides with a presentation of the ring of invariants via the *Plücker relations*, which give an embedding of the grassmannian manifold into a suitable projective space. In supergeometry, super grassmannians do not admit, in general, projective embeddings (see [61, Chap. 4]). However, in [73], the authors define a new graded version of super projective space, with negative grading, which allows for such an embedding. Indeed, the *Berezinians* of sub-supermatrices replace truly the notion of minors, which give the projective coordinates for the classical *Plücker embedding*. Then, they allow to proceed and give the correct supergeometric counterpart of the Plücker embedding [73]. However, due to the very nature of berezinians, which are defined only when the supermatrix is invertible, one is forced to restrict the analysis to an open set.

Once this geometric constraint is set in place, we are able to prove the super version of the first fundamental theorem (FFT). By employing an approach from the classical case based on the well-known *Jacobi's complementary minor theorem*, we establish certain relations, what we refer to as *super Plücker relations*, which are in agreement with the relations appeared in [73]. Furthermore, we are able to prove that the super Plücker relations given here completely characterize the ring of invariants for the case of $\mathrm{SL}(1|1)$. However, for the general case, we conjecture that the super Plücker relations proposed here are all the relations.

1.6 Complex structures on the full quantum flag of $\mathcal{O}_q(\mathrm{SU}_3)$

Constructing a theory of noncommutative geometry for *quantum homogeneous spaces* is an extremely important, but a very challenging question. Despite numerous significant contributions over the past three decades, this field remains largely under development. Throughout the literature, the essential example has been the celebrated *Podleś sphere* $\mathcal{O}_q(S^2)$, which serves as a fundamental test for evaluating new ideas, see [8, 28, 65].

The Podleś sphere is the simplest example of a *quantum flag manifold*, see [11]. For the last two decades, the noncommutative geometry community has tried to extend its understanding of the Podleś sphere to this general class of examples. In particular, attention has focused on those quantum flag manifolds of *irreducible* type, a special, more tractable, subfamily of the general quantum flag manifolds. Many results appeared in the literature, most notable the proof, in [54], that the irreducible quantum flag manifolds admit an essentially unique q-deformed de Rham complex, directly generalising Podleś' construction and classification of differential calculi for $\mathcal{O}_q(S^2)$. Also, the noncommutative complex and Kähler geometry of the Podleś sphere extends to the irreducible quantum flag setting, see [65, 66].

While many interesting and challenging problems remain in the irreducible setting, the time has now come to start examining the non-irreducible situation. Recently, in [15], Somberg and Buachalla constructed an anti-holomorhic Dolbeault complex $\Omega_q^{(0,\bullet)}(\mathrm{F}_n)$ for the A-series full quantum flag manifolds using *Lusztig's root vectors* and extended the Borel–Weil theorem to this setting.

In this thesis, we restrict to the simplest example of a full quantum flag manifold after the Podleś sphere, namely $\mathcal{O}_q(\mathrm{F}_3)$ the *full quantum flag manifold* of $\mathcal{O}_q(\mathrm{SU}_3)$. This offers an accessible and tractable example, making it an excellent starting point for future research in the non-irreducible setting. Just as Podleś' work advanced our understanding of the irreducible setting, $\mathcal{O}_q(\mathrm{F}_3)$ has the potential to do the same for the non-irreducible case.

We examine the maximal prolongation of the associated differential $*$-calculus, showing that it has classical dimension. Notably, unlike for the special anti-holomorphic subcalculus, σ is not of classical type. We next classify the left $\mathcal{O}_q(\mathrm{SU}_3)$-covariant almost-complex structures on $\Omega_q^\bullet(\mathrm{F}_3)$. We find that the number of almost-complex structures reduces from $2^{|\Delta^+|}$ (where Δ^+ is a choice of positive roots for $\mathfrak{sl}_3$) to $2^{|\Pi|}$ (where Π is the set of associated simple roots). This is because certain almost-complex classical decompositions fail to be bimodule decompositions in the quantum setting, due to the involved bimodule structure of the differential calculus. An almost-complex structure admits a q-deformation only if it is integrable. When it does, integrability carries over to the quantum setting, meaning that we do not have any non-integrable noncommutative almost-complex structures. We contrast this with the irreducible quantum flag manifolds that have a unique complex structure, up to identification of opposite complex structures. We conjecture that this fact generalizes to all A-series full quantum flag manifolds.

1.7 $N = 2$ Minkowski superspace and its quantization

According to the Penrose's twistor space approach [68], the complex *Minkowski space* can be realized as the big cell inside grassmannian $\mathrm{Gr}(2,4)$ which serves as the *conformal space*, see [39, Chap. 2], [61, Chap. 1, §3] for more details. A *quantization* of this Minkowski space description appeared in [34, 35, 39]. In [42], we extend this approach to the case of $N = 2$ (anti-chiral) Minkowski superspace.

It is well known that the $N = 1$ *superconformal superspace*, in its complexified version [61, 75], is the superflag $\mathrm{Fl}(2|0, 2|1, 4|1)$, on which the conformal supergroup $\mathrm{SL}(4|1)$ acts naturally. The space $\mathbb{C}^{4|1}$, underlying the defining representation of $\mathrm{SL}(4|1)$, is the space of supertwistors [68, 30].

Dealing with the complexified version has the advantage of seeing this structure, while the conditions for the real form can be imposed later on [75]. For the super grassmannians, only the extreme cases $\mathrm{Gr}(p|0, m|n)$ or $\mathrm{Gr}(p|n, m|n)$ are superprojective and are both embedded into the projective superspace $\mathbb{P}^{M|N}$ for suitable M and N, see [39]. This is different from the classical setting where all grassmannians are projective varieties. These super grassmannians $\mathrm{Gr}(p|0, m|n)$ and $\mathrm{Gr}(p|n, m|n)$ are dual to each other and corresponds to the antichiral and chiral superspaces respectively, in the physics literature. (See [39]).

The superflag $\mathrm{Fl}(2|0, 2|1, 4|1)$ can be embedded in the product,

$$\mathrm{Fl}(2|0, 2|1, 4|1) \subset \mathrm{Gr}(2|0, 4|1) \times \mathrm{Gr}(2|1, 4|1)\,,$$

and using the *super Segre embedding* [40] the superflag is embedded into the projective superspace $\mathbb{P}^{80|64}$, see [18, 39]. For $N = 2$ we can reproduce the same situation with,

$$\mathrm{Fl}(2|0, 2|2, 4|2) \subset \mathrm{Gr}(2|0, 4|2) \times \mathrm{Gr}(2|2, 4|2)\,,$$

but this superflag is too big. The scalar superfields associated to it have too many field components to be useful in the formulation of supersymmetric field theories. Still, the antichiral $\mathrm{Gr}(2|0, 4|2)$ and chiral $\mathrm{Gr}(2|2, 4|2)$ superspaces do have physical applications so it is useful to study them. They are both embedded in $\mathbb{P}^{8|8}$.

Here we will consider only the (anti-)chiral superspace. Our aim is to quantize it by substituting the supergroup $\mathrm{SL}(4|2)$ by the quantum group $\mathrm{SL}_q(4|2)$ (in the sense of Manin [59]) and trying to define appropriately the quantum super grassmannian as an homogeneous superspace. This appeared for $N = 1$ in [18, 39]. As we will see, the $N = 2$ case has its own peculiarities.

The Minkowski $N = 2$ superspace $\mathbf{M}$ emerges naturally in this context as the big cell in the super grassmannian $\mathrm{Gr}(2|0, 4|2)$, see [39, Chap. 4] for $N = 1$ case. However, as remarked above, the $N = 2$ SUSY has its own peculiarities, which make the theory richer. We view the big cell in $\mathrm{Gr}(2|0, 4|2)$ as the subsupermanifold containing certain $2|0$ subspaces and we realize it as the set S of pairs of vectors in $\mathbb{C}^{4|2}$ modulo the natural right $\mathrm{GL}(2)$-action, which accounts for basis change. Hence, we construct $\mathbf{M}$ as the quotient of S modulo the ordinary general linear group $\mathrm{GL}(2)$. The quantization of $\mathbf{M}$ is obtained, as expected, as the subring of a localization of $\mathrm{SL}_q(4|2)$, generated by the quantum coinvariants with respect to the coaction of quantum $\mathrm{GL}_q(2)$. The presentation of this quantum superring via generators and relations, makes an essential use of the commutation relations among the quantum determinants appearing in the definition of the quantum $\mathrm{Gr}(2|0, 4|2)$ and the Plücker relations. Moreover, the quantum Minkowski space, $\mathbf{M}_q$, is isomorphic to the quantum Manin superalgebra, that is, the quantum super bialgebra of matrices, as described in [59]. This fact is highly non obvious, it depends on the quite involved commutation relations of quantum determinants and it shows how this framework is natural and suitable for more exploration.

The antichiral Minkowski $N = 2$ superspace, being a quotient, appears then naturally also as a principal bundle for the action of $\mathrm{GL}(2)$. There is an extensive literature regarding the quantization of principal bundles (see [3, 11, 50, 55] and references therein). In particular the notion of *Hopf-Galois extension* [60] appears to be the right one to formulate, in the affine setting, the theory of principal bundles to obtain their quantum deformations.

We hence proceed to define Hopf-Galois extensions in the SUSY framework and prove that the chiral Minkowski $N = 2$ superspace $\mathbf{M}$ is the base for a principal bundle S for the supergroup $\mathrm{GL}(2)$, by realizing it as a trivial Hopf-Galois extension. Next, we construct a quantum deformation $\mathbf{M}_q$ of $\mathbf{M}$, by taking advantage of our previous realization and show that $\mathbf{M}_q$ is the quantum space, base for the quantum principal bundle S_q, for $\mathrm{GL}_q(2)$.

1.8 The organization of the thesis

The thesis is organized as follows.

In Chapter 2, we introduce briefly some basic and well-known material from noncommutative geometry including the notion of Hopf algebra and some interesting examples, quantum groups, quantum homogeneous spaces, Takeuchi's equivalence, differential calculus, and the notion of almost complex structures, from a noncommutative point of view.

In Chapter 3, we introduce very briefly some basic notions of supergeometry including the notion of super vector spaces, superalgebras, supermatrices, super Cramer's rule, supermanifolds, and supergroups.

In Chapter 4, firstly, we introduce the (classical) fundamental theorems of invariant theory. Then, we state the problem in supergeometry and prove the first fundamental theorem for special linear supergroup. Furthermore, we establish a super-version of a well-known determinant identity called *Jacobi's complementary minor theorem* that helps us to construct what we refer to as *super Plücker relations*. Finally, we show that the proposed super Plücker relations are all the relations for the case of $\mathrm{SL}(1|1)$, and we conjecture this statement for the general case.

In Chapter 5, we discuss the *Lusztig differential calculus* on full quantum flag manifold $\mathcal{O}_q(\mathrm{F}_3)$ and then we extend it to a $*$-calculus and study the almost complex and complex structures on $\mathcal{O}_q(\mathrm{F}_3)$ in this description.

In Chapter 6, we recall the notion of quantum Minkowski space and then extend it to the $N = 2$ quantum Minkowski superspace following a description of Minkowski superspace being the big cell inside super grassmannian $\mathrm{Gr}(2|0, 4|2)$.

Chapter 2

Noncommutative geometry

This chapter is devoted to introducing some of the notions and concepts from noncommutative geometry. The description given here is very brief, for details see the standard references [14, 57, 58, 63].

The chapter is organized as follows. In Section 2.1, we recall the notion of an *algebra* and a *coalgebra*. In Section 2.2, we introduce the notion of *Hopf algebra* and state some of their properties. In Section 2.3, we introduce an important class of Hopf algebras known as *Drinfeld–Jimbo quantized enveloping algebras*. In Section 2.5, we discuss the notion of *quantum homogeneous spaces*. In Section 2.6, we discuss a categorical equivalence called *Takeuchi's equivalence* which is fundamental to a classification of differential calculi on quantum homogeneous spaces. In Section 2.7, the notion of *first-order differential calculus* (FODC for short) in noncommutative setting is introduced. In Section 2.8, we introduce the notion of *covariant first-order differential calculus* and some characterization results are presented. In Section 2.9, the notion of *almost complex structure* and *complex structure* in noncommutative setting is discussed.

In this thesis, by a *quantum group* we mean certain Hopf algebras which are deformations of the enveloping Hopf algebras of semisimple Lie algebras or of the algebras of regular functions on algebraic groups.

2.1 Algebras and coalgebras

Symmetry is one of the greatest ideas in mathematics. In noncommutative geometry, it is captured by *Hopf algebras*. One may think of them as generalizations of ordinary groups, see [57, §1.1]. A Hopf algebra is an algebra together with some additional structures. In this thesis, by an algebra we always mean an *associative unital algebra*.

Definition 2.1.1. An *algebra* is a pair $(\mathcal{A}, .)$ where $\mathcal{A}$ is a vector space over $\mathbb{K}$ and $. : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ is the product map such that:

(i) $a.(b.c) = (a.b).c$

(ii) $a.(b + c) = a.b + a.c$

(iii) $(a + b).c = a.c + b.c$

(iv) $k(a.b) = ka.b = a.kb$

(v) there exists an element $1_{\mathcal{A}} \in \mathcal{A}$ satisfying $1_{\mathcal{A}}.a = a$.

for all $a, b, c \in \mathcal{A}$ and $k \in \mathbb{K}$. Alternatively, using the universal property of tensor products one can define an algebra as a triplet $(\mathcal{A}, \mathrm{m}, \eta)$ where $\mathrm{m} : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ and $\eta : \mathbb{K} \longrightarrow \mathcal{A}$ are linear maps, called the *multiplication map* and *unit* respectively, obeying:

$$\mathrm{m} \circ (\mathrm{id} \otimes \mathrm{m}) = \mathrm{m} \circ (\mathrm{m} \otimes \mathrm{id}), \tag{2.1}$$

$$\mathrm{m} \circ (\mathrm{id} \otimes \eta) = \mathrm{id} = \mathrm{m} \circ (\eta \otimes \mathrm{id}). \tag{2.2}$$

The properties (ii), (iii) and (iv) in the first definition above are captured by the tensor product while (i) (associativity of multiplication) and (v) are equivalent to Equations (2.1) and (2.2) respectively. Equivalently, one may write the Equations (2.1) and (2.2) by requiring the following diagrams to commute:

$$\begin{array}{ccc} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\mathrm{m} \otimes \mathrm{id}} & \mathcal{A} \otimes \mathcal{A} \\ \Big\downarrow{\scriptstyle \mathrm{id} \otimes \mathrm{m}} & & \Big\downarrow{\scriptstyle \mathrm{m}} \\ \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\quad \mathrm{m} \quad} & \mathcal{A}, \end{array} \tag{2.3}$$

$$\begin{array}{ccccc} \mathbb{K} \otimes \mathcal{A} & \xrightarrow{\eta \otimes \mathrm{id}} & \mathcal{A} \otimes \mathcal{A} & \xleftarrow[\mathrm{id} \otimes \eta]{} & \mathcal{A} \otimes \mathbb{K} \\ \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle \mathrm{m}} & & \Big\downarrow{\scriptstyle \mathrm{id}} \\ \mathbb{K} \otimes \mathcal{A} & \xrightarrow{\quad \cong \quad} & \mathcal{A} & \xleftarrow[\cong]{\quad\quad} & \mathcal{A} \otimes \mathbb{K}. \end{array} \tag{2.4}$$

Now, we introduce another object, namely the *coalgebra*, by reversing the arrows in the diagrams (2.3) and (2.4).

Definition 2.1.2. A *coalgebra* is a triple $(\mathcal{C}, \Delta, \varepsilon)$ where $\mathcal{C}$ is a vector space over $\mathbb{K}$, $\Delta : \mathcal{C} \longrightarrow \mathcal{C} \otimes \mathcal{C}$ and $\varepsilon : \mathcal{C} \longrightarrow \mathbb{K}$ are linear maps called *comultiplication* and *counit* respectively, such that the following diagrams commutes:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad\Delta\quad} & \mathcal{C} \otimes \mathcal{C} \\ \Big\downarrow{\scriptstyle \Delta} & & \Big\downarrow{\scriptstyle \mathrm{id}\otimes\Delta} \\ \mathcal{C} \otimes \mathcal{C} & \xrightarrow{\Delta\otimes\mathrm{id}} & \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C}, \end{array} \tag{2.5}$$

$$\begin{array}{ccccc} \mathbb{K} \otimes \mathcal{C} & \xleftarrow[\varepsilon\otimes\mathrm{id}]{} & \mathcal{C} \otimes \mathcal{C} & \xrightarrow{\mathrm{id}\otimes\varepsilon} & \mathcal{C} \otimes \mathbb{K} \\ {\scriptstyle \mathrm{id}}\Big\uparrow & & {\scriptstyle \Delta}\Big\uparrow & & {\scriptstyle \mathrm{id}}\Big\uparrow \\ \mathbb{K} \otimes \mathcal{C} & \xleftarrow[\cong]{} & \mathcal{C} & \xrightarrow{\cong} & \mathcal{C} \otimes \mathbb{K}. \end{array} \tag{2.6}$$

The commutativity of diagrams in (2.5) and (2.6) are referred to as *coassociativity* of comultiplication Δ and *counit condition* for ε respectively.

Let $\mathcal{C}$ and $\tilde{\mathcal{C}}$ b two coalgebras. A *coalgebra morphism* is a linear map $\varphi : \mathcal{C} \longrightarrow \tilde{\mathcal{C}}$ such that the following diagrams commute:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Delta_{\mathcal{C}}} & \mathcal{C} \otimes \mathcal{C} \\ \Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle \varphi\otimes\varphi} \\ \tilde{\mathcal{C}} & \xrightarrow{\Delta_{\tilde{\mathcal{C}}}} & \tilde{\mathcal{C}} \otimes \tilde{\mathcal{C}}, \end{array} \qquad \begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad\varphi\quad} & \tilde{\mathcal{C}} \\ & {\scriptstyle \varepsilon_{\mathcal{C}}}\searrow & \Big\downarrow{\scriptstyle \varepsilon_{\tilde{\mathcal{C}}}} \\ & & \mathbb{K}. \end{array} \tag{2.7}$$

A coalgebra $(\mathcal{C}, \Delta, \varepsilon)$ is *cocommutative* if $\tau \circ \Delta = \Delta$, where $\tau : \mathcal{C} \otimes \mathcal{C} \longrightarrow \mathcal{C} \otimes \mathcal{C}$ is the usual flip morphism sending $a \otimes b$ to $b \otimes a$.

The category of coalgebras also admits a monoidal structure. For any two coalgebras $\mathcal{C}$ and $\tilde{\mathcal{C}}$, their tensor product $\mathcal{C} \otimes \tilde{\mathcal{C}}$ is also a coalgebra together with the comultiplication and counit defined below:

$$\Delta_{\mathcal{C}\otimes\tilde{\mathcal{C}}} := (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta_{\mathcal{C}} \otimes \Delta_{\tilde{\mathcal{C}}}),$$

$$\varepsilon_{\mathcal{C}\otimes\tilde{\mathcal{C}}} := \varepsilon_{\mathcal{C}} \otimes \varepsilon_{\tilde{\mathcal{C}}}.$$

Most of the concepts related to algebras can be dualized to coalgebras in a similar fashion (i.e. reversing the arrows in their diagrammatic formulations). We discuss some of these below.

Definition 2.1.3.

- A *subcoalgebra* $\mathcal{B}$ of a coalgebra $\mathcal{A}$ is a linear subspace of $\mathcal{A}$ such that:

$$\Delta(\mathcal{B}) \subset \mathcal{B} \otimes \mathcal{B}.$$

- A *coideal* $\mathcal{I}$ of a coalgebra $\mathcal{A}$ is a linear subspace of $\mathcal{A}$ such that:

$$\Delta(\mathcal{I}) \subset \mathcal{A} \otimes \mathcal{I} + \mathcal{I} \otimes \mathcal{A}, \qquad \text{and} \qquad \varepsilon(\mathcal{I}) = \{0\}.$$

- A (left) $\mathcal{C}$-*comodule* over a coalgebra $\mathcal{C}$ is a pair (N, Δ_N) where N is a vector space and $\Delta_N : N \longrightarrow \mathcal{C} \otimes N$ is a linear map, called the *coaction*, such that the following diagrams commute:

$$\begin{array}{ccc} N & \xrightarrow{\Delta_N} & \mathcal{C} \otimes N \\ \downarrow{\scriptstyle \Delta_N} & & \downarrow{\scriptstyle \mathrm{id} \otimes \Delta_N} \\ \mathcal{C} \otimes N & \xrightarrow{\Delta \otimes \mathrm{id}} & \mathcal{C} \otimes \mathcal{C} \otimes N, \end{array} \tag{2.8}$$

$$\begin{array}{ccc} N & \xrightarrow{\Delta_N} & \mathcal{C} \otimes N \\ & \searrow^{\cong} & \downarrow{\scriptstyle \varepsilon_{\mathcal{C}} \otimes \mathrm{id}} \\ & & \mathbb{K} \otimes N. \end{array} \tag{2.9}$$

Let N and N' be two $\mathcal{C}$-comodules. A $\mathcal{C}$-*comodule morphism* $f : N \longrightarrow N'$ is a linear map such that the following diagram commute:

$$\begin{array}{ccc} N & \xrightarrow{\Delta_N} & \mathcal{C} \otimes N \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id} \otimes f} \\ N' & \xrightarrow{\Delta_{N'}} & \mathcal{C} \otimes N'. \end{array} \tag{2.10}$$

Similarly, the notion of a right-comodule can be defined.

Notation 2.1.4. Using *Sweedler's notation* greatly simplifies working with a coalgbera. For an element $x \in \mathcal{C}$, we can write $\Delta(x) \in \mathcal{C} \otimes \mathcal{C}$ as follows:

$$\Delta(x) = \sum_i x^i_{(1)} \otimes x^i_{(2)}.$$

In *Sweedler's notation*, one omit the summation sign and indices and set:

$$\Delta(x) = x_{(1)} \otimes x_{(2)}. \tag{2.11}$$

Furthermore, utilizing the coassociativity of the coproduct, one denotes:

$$x_{(1)} \otimes x_{(2)} \otimes x_{(3)} := ((\mathrm{id} \otimes \Delta) \circ \Delta)(x) = ((\Delta \otimes \mathrm{id}) \circ \Delta)(x). \tag{2.12}$$

Similarly, one sets the higher order coproducts. For more details see [58, §3.1].

2.2 Hopf algebras

A vector space having both algebra and coalgebra structure coexisting in a compatible way is called a bialgebra.

Definition 2.2.1. A *bialgebra* is a vector space $\mathcal{A}$ together with both algebra and coalgebra structures such that the comultiplication $\Delta : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$ and counit $\varepsilon : \mathcal{A} \longrightarrow \mathbb{K}$ are algebra morphisms. A *bialgebra morphism* is a linear map which is both algebra and coalgebra morphism.

Example 2.2.2. Let $q \in \mathbb{C}^\times$ be non-zero complex number and set $\nu := q - q^{-1}$. The *algebra of quantum matrices* $\mathbb{C}_q[\mathrm{M}_n]$ is defined as the quotient algebra:

$$\mathbb{C}_q[\mathrm{M}_n] := \mathbb{C}\langle u_{ij} |\ i, j = 1, \cdots, n\rangle / \mathrm{I},$$

where I is the ideal generated by following relations (often called *Manin relations*):

$$\begin{array}{rll} u_{ik}u_{jk} - q u_{jk}u_{ik}, & u_{ki}u_{kj} - q u_{kj}u_{ki}, & 1 \le i < j \le n,\ 1 \le k \le n, \\ u_{il}u_{jk} - u_{jk}u_{il}, & u_{ik}u_{jl} - u_{jl}u_{ik} - \nu u_{il}u_{jk}, & 1 \le i < j \le n,\ 1 \le k < l \le n. \end{array}$$

$\mathbb{C}_q[\mathrm{M}_n]$ turns into a bialgebra together with the coproduct and counit defined as follows:

$$\Delta(u_{ij}) := \sum_{k=1}^{n} u_{ik} \otimes u_{kj} \qquad \varepsilon(u_{ij}) := \delta_{ij}.$$

Now, we are ready to define the fundamental notion of this section.

Definition 2.2.3. A *Hopf algebra* is a bialgebra $\mathcal{A}$ together with a linear map $S : \mathcal{A} \longrightarrow \mathcal{A}$, called *antipode*, such that the following diagram commute:

$$\begin{array}{ccccc} \mathcal{A} \otimes \mathcal{A} & \xleftarrow[\Delta]{} & \mathcal{A} & \xrightarrow{\Delta} & \mathcal{A} \otimes \mathcal{A} \\ \big\downarrow{\scriptstyle \mathrm{id} \otimes S} & & \big\downarrow{\scriptstyle \eta \circ \varepsilon} & & \big\downarrow{\scriptstyle S \otimes \mathrm{id}} \\ \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\mathrm{m}} & \mathcal{A} & \xleftarrow[\mathrm{m}]{} & \mathcal{A} \otimes \mathcal{A}. \end{array} \tag{2.13}$$

In terms of the *Sweedler's notation*, the commutativity of above diagram 2.13 reads;

$$\sum a_{(1)}S(a_{(2)}) = \varepsilon(a).1_{\mathcal{A}} = \sum S(a_{(1)})a_{(2)}, \qquad \forall\ a \in \mathcal{A}.$$

Definition 2.2.4. Recall that a $*$-*vector space* over $\mathbb{C}$ is a vector space V together with a map $v \mapsto v^*$, called *involution*, such that:

$$(av_1 + bv_2)^* = \bar{a}v_1^* + \bar{b}v_2^*, \qquad (v^*)^* = v, \qquad \forall\ a, b \in \mathbb{C}, v_1, v_2 \in V.$$

A $*$-*algebra* $\mathcal{A}$ is an algebra such that the underlying vector space is a $*$-vector space and

$$(v_1v_2)^* = v_2^*v_1^*, \qquad \forall\ v_1, v_2 \in \mathcal{A}.$$

Similarly, A $*$-*coalgebra* $\mathcal{C}$ is a coalgebra such that the underlying vector space is a $*$-vector space and

$$\Delta(c^*) = c^*_{(1)} \otimes c^*_{(2)}, \qquad \forall\ c \in \mathcal{C}.$$

A *Hopf* $*$-*algebra* is a Hopf algebra such that the underlying bialgebra is both a $*$-algebra and a $*$-coalgebra.

There are many consequences of these definitions. We list some of the important ones below.

Proposition 2.2.5.

(i) For a bialgebra $\mathcal{A}$, the multiplication $\mathrm{m} : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ *and unit* $\eta : \mathbb{K} \longrightarrow \mathcal{A}$ *are coalgebra morphisms.*

(ii) For a bialgebra $\mathcal{A}$, if an antipode exists, then it is unique.

(iii) The antipode of a Hopf algebra is an algebra anti-morphism, i.e.

$$S(ab) = S(b)S(a), \qquad \forall\ a, b \in \mathcal{A}, \qquad \textit{and} \qquad S(1) = 1.$$

(iv) The antipode of a Hopf algebra is a coalgebra anti-morphism, i.e.

$$\Delta \circ S = \tau \circ (S \otimes S) \circ \Delta, \qquad \varepsilon \circ S = \varepsilon.$$

In terms of Sweedler's notation, the first identity means:

$$\sum S(a)_{(1)} \otimes S(a)_{(2)} = \sum S(a_{(2)}) \otimes S(a_{(1)}).$$

(v) If a Hopf algebra is commutative or cocommutative, then $S^2 = \mathrm{id}$.

Proof. See [57, Chap. 1, Prop. 5] and [58, Th. 3.4]. □

Below we define another key notion associated with the theory of Hopf algebras.

Definition 2.2.6. Let G and H be two Hopf algebras. A *dual pairing* is a bilinear map

$$\langle -, - \rangle : G \otimes H \longrightarrow \mathbb{C}$$

such that:

$$\langle g, hh' \rangle = \langle g_{(1)}, h \rangle \langle g_{(2)}, h' \rangle, \qquad \langle gg', h \rangle = \langle g, h_{(1)} \rangle \langle g', h_{(2)} \rangle, \tag{2.14}$$

$$\langle g, 1_H \rangle = \varepsilon(g), \qquad \langle 1_G, h \rangle = \varepsilon(h), \qquad \langle S(g), h \rangle = \langle g, S(h) \rangle, \tag{2.15}$$

for all $g, g' \in G$ and $h, h' \in H$.

In case, G and H be two Hopf $*$-algebras (as defined in Definition 2.2.4), a *dual $*$-pairing* is a pairing,

$$\langle -, - \rangle : G \otimes H \longrightarrow \mathbb{C}$$

satisfying the following identities:

$$\langle g^*, h \rangle = \overline{\langle g, S(h)^* \rangle}, \qquad \langle g, h^* \rangle = \overline{\langle S(g)^*, h \rangle} \tag{2.16}$$

for all $g \in G$ and $h \in H$, in addition to the identities in (2.14) and (2.15).

An important consequence of the above definition is described in the following proposition.

Proposition 2.2.7. *Let G and H be two dually paired Hopf algebras with the dual pairing:*

$$\langle -, - \rangle : G \otimes H \longrightarrow \mathbb{C}$$

Then G acts on H by:

$$g \triangleright h := \sum h_{(1)} \langle g, h_{(2)} \rangle. \tag{2.17}$$

Proof. See [64, Prop. 2.8]. □

We now examine a few key examples of Hopf algebras that are of interest in this thesis.

Example 2.2.8. Let G be a finite group. By $\mathcal{O}(G)$ we denote the algebra of all complex-valued functions on G with pointwise algebraic operations. Then, $\mathcal{O}(G)$ is a commutative Hopf algebra togther with the coproduct, counit and antipode defined as follows:

$$\Delta(f)(g, g') := f(gg'), \qquad \varepsilon(f) := f(e), \qquad S(f)(g) := f(g^{-1}),$$

for all $g, g' \in G$ and $f \in \mathcal{O}(G)$, and e denotes the identity element of G.

In the above example we notice that starting from a finite group one can construct a commutative Hopf algebra. In fact, this construction provides an (anti-)equivalence between the category of affine algebraic groups and commutative Hopf algebras (see [49]). The idea of Drinfeld was to *quantize* a Hopf algebra $\mathcal{O}(G)$ by deforming it to a non-commutative Hopf algebra $\mathcal{O}_q(G)$. Such a deformed Hopf algebra is known as a *quantum group*. Some quantum groups corresponding to matrix Hopf algebras are discussed below.

Example 2.2.9. Let $\mathbb{C}_q[\mathrm{M}_n]$ be the quantum matrix bialgebra as introduced in Example 2.2.2. Define the *quantum determinant* to be:

$$\det_q := \sum_{\sigma \in \mathrm{S}_n} (-q)^{l(\sigma)} u_{1\sigma(1)} \cdots u_{n\sigma(n)}, \tag{2.18}$$

where $l(\sigma)$ denotes the length of σ. It turns out that, $\det_q$ is central and group-like (*i.e.* $\Delta(\det_q) = \det_q \otimes \det_q$), see [57, Chap. 9, Prop. 7, Prop. 9].

The *quantum general linear group* $\mathbb{C}_q[\mathrm{GL}_n]$ is defined as the quotient algebra:

$$\mathbb{C}_q[\mathrm{GL}_n] := \mathbb{C}_q[\mathrm{M}_n][T]/\langle T\det_q - 1\rangle$$

where $\mathbb{C}_q[\mathrm{M}_n][T]$ denotes the algebra of polynomials in T over $\mathbb{C}_q[\mathrm{M}_n]$. The coproduct and counit are extended by defining:

$$\Delta(T) := T \otimes T, \qquad \varepsilon(T) := 1. \tag{2.19}$$

As it is clear from the definition, it is a standard practice to denote T by $\det_q^{-1}$.

In fact, the bialgebra $\mathbb{C}_q[\mathrm{GL}_n]$ can be endowed with a Hopf algebra structure by defining the antipode S as follows:

$$S(u_{ij}) := \tilde{u}_{ij}\det_q^{-1} \qquad S(\det_q^{-1}) := \det_q \tag{2.20}$$

where $\tilde{u}_{ij}$ is the ij-th entry of the matrix $\tilde{\mathbf{u}}$ (called the cofactor matrix of $\mathbf{u} = [u_{ij}]$) such that the following matrix equation holds:

$$\mathbf{u}\tilde{\mathbf{u}} = \tilde{\mathbf{u}}\mathbf{u} = \det_q \mathrm{I}_n.$$

Explicitly,

$$S(u_{ij}) = (-q)^{i-j} \sum_{\sigma \in \mathrm{S}_{n-1}} (-q)^{l(\sigma)} u_{k_1\sigma(l_1)} \cdots u_{k_{n-1}\sigma(l_{n-1})} \det_q^{-1}, \tag{2.21}$$

where $\{k_1, \cdots, k_{n-1}\} := \{1, \cdots, n\} - \{j\}$ and $\{l_1, \cdots, l_{n-1}\} := \{1, \cdots, n\} - \{i\}$ as ordered sets.

The *quantum special linear group* $\mathbb{C}_q[\mathrm{SL}_n]$ is defined as the Hopf algebra quotient:

$$\mathbb{C}_q[\mathrm{SL}_n] := \mathbb{C}_q[\mathrm{GL}_n]/\langle \det_q - 1 \rangle. \tag{2.22}$$

Furthermore, the Hopf algebras $\mathbb{C}_q[\mathrm{GL}_n]$ and $\mathbb{C}_q[\mathrm{SL}_n]$ also admits a $*$-structure (see Definition 2.2.4) given by:

$$(\det_q^{-1})^* := \det_q, \qquad (u_{ij})^* := S(u_{ji}). \tag{2.23}$$

The Hopf algebras $\mathbb{C}_q[\mathrm{GL}_n]$ and $\mathbb{C}_q[\mathrm{SL}_n]$ togther with this $*$-structure are called *quantum unitary group* $\mathbb{C}_q[\mathrm{U}_n]$ and *quantum special unitary group* $\mathbb{C}_q[\mathrm{SU}_n]$ respectively.

2.3 Drinfeld–Jimbo quantized enveloping algebras

In this section, we will introduce an important class of noncommutative Hopf algebras known as *Drinfeld–Jimbo quantized enveloping algebras*.

Firstly, we recall some basic notions from the theory of *Lie algebras* (we will always mean *complex Lie algebras*). The purpose here is just to set the notation, for details see [48, 32].

Definition 2.3.1. A Lie algebra is called *simple* if it is non-abelian and contains no non-zero proper ideals. A Lie algebra is called *semi-simple* if it is isomorphic to a direct sum of *simple Lie algebras*.

To each complex semi-simple Lie algebra L, on associate a set Δ of vectors in an *Euclidean space* E satisfying certain geometric axioms, called the *root system* of L (see [48, Chap. 9]). These root systems completely characterize semi-simple Lie algebras up to isomorphism (see [48, §14.2]).

Definition 2.3.2. The *Weyl group* W of a root system $\Delta \subset E$ is the subgroup of $\mathrm{GL}(E)$ generated by reflections through hyperplanes associated to the roots in Δ.

Definition 2.3.3. A subset $S \subset \Delta$ of root system is called a *base* if,

(i) S is a basis of E,

(ii) each root β can be written as $\beta = \sum_\alpha k_\alpha \alpha$, $\alpha \in S$, with integral coefficients k_α are all non-positive or all non-negative.

If all k_α are non-negative, we call β a *positive* root and if all k_α are non-positive, we call β a *negative* root, relative to a fixed base S. Moreover, the vectors in S are called *simple* roots.

Definition 2.3.4. Given a *root system* Δ in an Euclidean space E and a *base* $\{\alpha_1, \cdots \alpha_l\}$ of Δ, one define a square matrix $C = (a_{ij})_{1\leq i,j\leq l}$ called the *Cartan matrix* whose entries are given by:

$$a_{ij} = \langle \alpha_i, \alpha_j \rangle := \frac{2(\alpha_j, \alpha_i)}{(\alpha_j, \alpha_j)}.$$

The following important result, called *Serre's Theorem*, captures the significance of the notion of *Cartan matrix*.

Theorem 2.3.5 (Serre's Theorem)**.** *Given a root system Δ in an l-dimensional Euclidean space E, and $C = (a_{ij})_{1\leq i,j\leq l}$ its Cartan matrix, define the Lie algebra L generated by $3l$ generators $\{e_i, f_i, h_i : 1 \leq i \leq l\}$ and the following* Serre's *relations:*

$$[h_i, h_j] = 0, \qquad [e_i, f_i] = h_i, \qquad [e_i, f_j] = 0, \qquad i \neq j,$$

$$[h_i, e_j] = a_{ij} e_j, \qquad [h_i, f_j] = -a_{ij} f_j,$$

$$\mathrm{ad}(e_i)^{-a_{ij}+1}(e_j) = 0, \qquad \mathrm{ad}(e_i)^{-a_{ij}+1}(e_j) = 0, \qquad i \neq j.$$

Then, L is a finite dimensional Lie algebra with the Cartan subalgebra generated by $\{h_i : 1 \leq i \leq l\}$ and having the root system Δ.

Proof. See [48, §18.1]. ∎

Proposition 2.3.6. *Let $U(\mathfrak{g})$ be the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. Then, together with the following data:*

$$\Delta(X) := X \otimes 1 + 1 \otimes X, \qquad \varepsilon(X) = 0, \qquad S(X) = -X, \qquad \forall\ X \in U(\mathfrak{g}). \quad (2.24)$$

$U(\mathfrak{g})$ turns into a Hopf algebra.

Proof. See [57, §1.2.6]. □

Now, we are ready to define an important class of Hopf algebras which is a non-commutative version of complex semi-simple Lie algebras as characterized in the theorem 2.3.5.

Definition 2.3.7. Let $\mathfrak{g}$ be a finite dimensional, complex, semi-simple Lie algebra of rank l, and (a_{ij}) denotes its Cartan matrix, and we fix $q_i := q^{(\alpha_i,\alpha_i)/2}$.

The *Drinfeld–Jimbo quantized enveloping algebra* $U_q(\mathfrak{g})$ is the algebra generated by the elements E_i, F_i, K_i and K_i^{-1} subject to the following relations:

$$K_iE_j = q_i^{a_{ij}}E_jK_i, \qquad K_iF_j = q_i^{-a_{ij}}F_jK_i, \qquad K_iK_j = K_jK_i, \quad (2.25)$$

$$K_iK_i^{-1} = 1 = K_i^{-1}K_i, \qquad E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad (2.26)$$

where $i, j \in \{1, \cdots, l\}$, and the *quantum Serre relations*:

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \left[\binom{1-a_{ij}}{r}\right]_{q_i} E_i^{1-a_{ij}-r}E_jE_i^r = 0, \qquad i \neq j, \quad (2.27)$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \left[\binom{1-a_{ij}}{r}\right]_{q_i} F_i^{1-a_{ij}-r}F_jF_i^r = 0, \qquad i \neq j, \quad (2.28)$$

where:

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad \left[\binom{n}{r}\right]_q := \frac{[n]_q!}{[r]_q![n-r]_q!}, \qquad \forall\ n, r \in \mathbb{N}, \qquad r \leq n, \quad (2.29)$$

$$[n]_q! := [1]_q[2]_q \cdots [n]_q \qquad [0]_q! := 1 \quad (2.30)$$

The expressions $[n]_q$ are called *q-numbers.*

The following theorem deforms the Hopf algebra structure described in proposition 2.3.6.

Theorem 2.3.8. *$U_q(\mathfrak{g})$ admits a unique Hopf algebra structure with comultiplication Δ, counit ε and antipode S defined as:*

$$\Delta(E_i) := E_i \otimes K_i + 1 \otimes E_i, \qquad \Delta(F_i) := K_i^{-1} \otimes F_i + F_i \otimes 1, \tag{2.31}$$
$$\Delta(K_i) := K_i \otimes K_i, \qquad \Delta(K_i^{-1}) := K_i^{-1} \otimes K_i^{-1}, \tag{2.32}$$

$$\varepsilon(K_i) := 1, \qquad \varepsilon(E_i) := \varepsilon(F_i) = 0, \tag{2.33}$$

$$S(E_i) := -E_i K_i^{-1}, \qquad S(F_i) := -K_i F_i, \qquad S(K_i) := K_i^{-1}. \tag{2.34}$$

Proof. See [57, Chap. 6, Prop. 5]. □

The algebraic properties of $U_q(\mathfrak{g})$ are very similar to that of $U(\mathfrak{g})$, therefore, it is considered to be a fundamental object to study in noncommutative geometry. We mention one of the key results below.

Theorem 2.3.9. *Any finite-dimensional irreducible representation of a Drinfeld–Jimbo algebra is a weight representation and a representation with highest weight. Such a representation is uniquely determined by its highest weight.*

Proof. See [57, Chap. 7]. □

The example given below describes a dual pairing between $\mathcal{O}_q(\mathrm{SU}_n)$ and $U_q(\mathfrak{sl}_\mathfrak{n})$. Some of the other fundamental examples can be found in [57, §9.4].

Example 2.3.10. A dual pairing of Hopf algebras between $\mathcal{O}_q(\mathrm{SL}_n)$ and $U_q(\mathfrak{sl}_\mathfrak{n})$ is given by:

$$\langle u_{i+1,i}, E_i \rangle = 1, \qquad \langle u_{i,i+1}, F_i \rangle = 1, \tag{2.35}$$
$$\langle u_{ii}, K_j \rangle = q^{\delta_{j+1,i} - \delta_{ij}}, \qquad \langle u_{ii}, K_j^{-1} \rangle = q^{\delta_{ij} - \delta_{j+1,i}}, \tag{2.36}$$

and requiring all other pairings to be zero, where u_{ij} denotes the generators of $\mathcal{O}_q(\mathrm{SL}_n)$ and E_i, F_i, K_i are the generators of $U_q(\mathfrak{sl}_\mathfrak{n})$.

Furthermore, this pairing respects the $*$-structure as in (2.16). Therefore, it is also a $*$-pairing between $\mathcal{O}_q(\mathrm{SU}_n)$ and $U_q(\mathfrak{sl}_\mathfrak{n})$.

2.4 Lusztig's root vectors

In this section, we introduce the notion of *Lusztig's root vectors*, that is of primary importance to us in Chapter 5. We follow here the description given in [57, §6.2].

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra of rank l and (a_{ij}) denotes its Cartan matrix. The product $a_{ij}a_{ji}$ may take the values in $\{0, 1, 2, 3\}$. Set m_{ij} to be $2, 3, 4$ or 6 when the product $a_{ij}a_{ji}$ is $0, 1, 2$ or 3 respectively.

Definition 2.4.1. Let the notation be as above. The *braid group* $\mathfrak{B}_{\mathfrak{g}}$ is the group generated by $s_1, \cdots, s_l$ subject to the following relations:

$$s_i s_j s_i s_j \cdots = s_j s_i s_j s_i \cdots, \qquad i \neq j, \tag{2.37}$$

where there are m_{ij} number of s on each side.

The following theorem captures the significance of the braid group defined above.

Theorem 2.4.2. *To every generator $s_i \in \mathfrak{B}_{\mathfrak{g}}$, there corresponds an algebra automorphism T_i of $\mathrm{U}_q(\mathfrak{g})$ which acts on the generators as follows:*

$$T_i(K_j) = K_j K_i^{-a_{ij}}, \qquad T_i(E_i) = -F_i K_i, \qquad T_i(F_i) = -K_i^{-1} E_i, \tag{2.38}$$

$$T_i(E_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q_i^{-r} (E_i)^{(-a_{ij}-r)} E_j (E_i)^{(r)}, \qquad i \neq j, \tag{2.39}$$

$$T_i(F_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q_i^{r} (F_i)^{(r)} F_j (F_i)^{(-a_{ij}-r)}, \qquad i \neq j, \tag{2.40}$$

where,

$$(E_i)^{(n)} := \frac{E_i^n}{[n]_{q_i}!}, \qquad (F_i)^{(n)} := \frac{F_i^n}{[n]_{q_i}!}. \tag{2.41}$$

Moreover, the map,

$$s_i \mapsto T_i$$

determines a homomorphism of the braid group $\mathfrak{B}_{\mathfrak{g}}$ into the group of algebra automorphisms of $\mathrm{U}_q(\mathfrak{g})$.

Proof. See [57, §6.2, Th. 22]. □

Let Δ be a root system and W be its Weyl group with simple roots $S = \{\alpha_1, \cdots \alpha_l\}$ and the corresponding set $W_S = \{w_1, \cdots w_l\} \subset W$ of simple reflections.

Definition 2.4.3. Let the notation be as above. For any element w of the Weyl group W, an expression $w = w_{i_1} \cdots w_{i_n}$ with each $w_{i_k} \in W_S$ and n being the minimum such number, is called a *reduced decomposition* of w.

Moreover, we define the *length* of w to be n where $w = w_{i_1} \cdots w_{i_n}$ is a reduced decomposition of w.

It turn out that, there exist a unique element $w_0 \in W$ with highest length. We call such an element w_0 the *longest element* of the Weyl group W. Let $w_0 = w_{i_1} \cdots w_{i_n}$ be a fixed reduced decomposition of w_0. Then, the list:

$$\beta_1 := \alpha_{i_1}, \qquad \beta_k := w_{i_1} \cdots w_{i_k - 1}(\alpha_{i_k}), \tag{2.42}$$

exhausts all the positive roots of $\mathfrak{g}$. (See [57, Chap. 4, Prop. 4], also [48, §10.3]).

Definition 2.4.4. The elements,

$$E_{\beta_r} := T_{i_1} \cdots T_{i_{r-1}}(E_r) \qquad \text{and} \qquad F_{\beta_r} := T_{i_1} \cdots T_{i_{r-1}}(F_r) \tag{2.43}$$

are called the *Lusztig's root vectors* corresponding to β_r and β_{-r} respectively.

Lusztig's root vectors play a very important role in the representation theory of $\mathrm{U}_q(\mathfrak{g})$ as they helps us to prove *PBW theorem* for $\mathrm{U}_q(\mathfrak{g})$. However, we will not go into further details here, see [57, §6.2]. Below, we present the example of $\mathfrak{g} = \mathfrak{sl}_3$ that is of our interest in Chapter 5.

Example 2.4.5. Let $\mathfrak{g} = \mathfrak{sl}_3$. The longest element $w_0 = (13)$ of the Weyl group $W \cong \mathrm{S}_3$ admits exactly two reduced decompositions:

$$w_1 w_2 w_1 \qquad \text{and} \qquad w_2 w_1 w_2,$$

where we denote by $w_1 = (12)$ and $w_2 = (23)$. The corresponding list of root vectors are:

$$\begin{array}{lll} E_1, & T_1(E_2) = [E_1, E_2]_{q^{-1}}, & T_1 T_2(E_1) = E_2, \\ E_2, & T_2(E_1) = [E_2, E_1]_{q^{-1}}, & T_2 T_1(E_2) = E_1. \end{array}$$

2.5 Quantum homogeneous spaces

In this section, we will briefly describe the notion of *homogeneous spaces* in this non-commutative setting. We follow the description given in [13].

Let $(\mathcal{A}, \mathrm{m}, \Delta, \eta, \varepsilon)$ and $(\mathcal{H}, \mathrm{m}_{\mathcal{H}}, \Delta_{\mathcal{H}}, \eta_{\mathcal{H}}, \varepsilon_{\mathcal{H}})$ be Hopf algebras, and $\pi : \mathcal{A} \longrightarrow \mathcal{H}$ be a surjective Hopf algebra morphism. We view $\mathcal{A}$ as a right $\mathcal{H}$-comodule (see definition 2.1.3) algebra via coaction $\Delta_{\mathcal{A}} := (\mathrm{id} \otimes \pi) \circ \Delta : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{H}$. With all this datum, the space of coinvariants:

$$\mathcal{B} := \mathcal{A}^{\mathrm{co}(\mathcal{H})} = \{a \in \mathcal{A} : \Delta_{\mathcal{A}}(a) = a \otimes 1\}$$

is a right coideal (*i.e.* $\Delta(\mathcal{B}) \subset \mathcal{B} \otimes \mathcal{A}$) subalgebra of $\mathcal{A}$, see [74, Prop. 1].

Definition 2.5.1. We call $\mathcal{B} := \mathcal{A}^{\mathrm{co}(\mathcal{H})}$ a *quantum homogeneous space* if $\mathcal{A}$ is faithfully flat as a right $\mathcal{B}$-module, which is to say that the functor,

$$\mathcal{A} \otimes_{\mathcal{B}} - : {}_{\mathcal{B}}\mathcal{M} \longrightarrow {}_{\mathbb{C}}\mathcal{M}$$

from the category of left $\mathcal{B}$-modules to the category of complex vector spaces maps a sequence to an exact sequence if and only if the original sequence is exact.

Example 2.5.2. A Hopf algebra $\mathcal{A}$ s itself a trivial example of a quantum homogeneous space where one consider $\pi = \varepsilon : \mathcal{A} \longrightarrow \mathbb{K}$.

As a non-trivial example, we introduce the notion of *quantum projective space*. Later, in Chapter 5, we will discuss more general examples of *quantum flag manifolds*.

Example 2.5.3. Let $\pi : \mathbb{C}_q[\mathrm{SU}_n] \longrightarrow \mathbb{C}_q[\mathrm{U}_{n-1}]$ be the Hopf algebra surjection defined as:

$$\pi(u_{11}) := \det_q^{-1}, \qquad \pi(u_{1i}) = \pi(u_{i1}) = 0, \qquad \pi(u_{ij}) := u_{i-1,j-1} \tag{2.44}$$

for $i, j = 2, \cdots, n$. The *quantum projective space* $\mathbb{C}_q[\mathbb{CP}^{n-1}]$ is defined as the quantum homogeneous space $\mathbb{C}_q[\mathrm{SU}_n]^{\mathrm{co}(\mathbb{C}_q[\mathrm{U}_{n-1}])}$. For further details see [62, 11]

2.6 Takeuchi's equivalence

In this section, we briefly describe a categorical equivalence [74], that helps to classify some nice differential structures on quantum homogeneous spaces as we will see in the next section. Here, we employ Sweedler's notation as in Notation 2.1.4. We are following the description given in [11, §2.2].

Let $\mathcal{B} := \mathcal{A}^{\mathrm{co}(\mathcal{H})}$ be a quantum homogeneous space. We denote by ${}^{\mathcal{A}}_{\mathcal{B}}\mathcal{M}_{\mathcal{B}}$ the category of $\mathcal{B}$-bimodules M with a left $\mathcal{A}$-coaction Δ_L satisfying:

$$\Delta_L(bmb') = b_{(1)}m_{(-1)}b'_{(1)} \otimes b_{(2)}m_{(0)}b'_{(2)}, \qquad \forall\ m \in M,\ b, b' \in \mathcal{B}, \tag{2.45}$$

and $\mathcal{M}^{\mathcal{H}}_{\mathcal{B}}$ the category of right $\mathcal{B}$-modules N (we denote the right action by $\triangleleft$) with a right $\mathcal{H}$-coaction Δ_R satisfying:

$$\Delta_R(n \triangleleft b) = n_0 \triangleleft b_{(2)} \otimes S(\pi(b_{(1)}))n_{(1)}, \qquad \forall\ n \in N,\ b \in \mathcal{B}. \tag{2.46}$$

Furthermore, we define a pair (Φ, Ψ) of functors. The functor,

$$\Phi : {}^{\mathcal{A}}_{\mathcal{B}}\mathcal{M}_{\mathcal{B}} \longrightarrow \mathcal{M}^{\mathcal{H}}_{\mathcal{B}} \tag{2.47}$$

is defined on objects as:

$$M \mapsto \overline{M} := M/\mathcal{B}^+M,$$

while on morphism is defined as follows:

$$(\phi : M \longrightarrow M') \mapsto (\overline{\phi} : \overline{M} \longrightarrow \overline{M'}), \qquad \overline{\phi}([m]) := [\phi(m)].$$

Notice that, one can consider any object $M \in {}^{\mathcal{A}}_{\mathcal{B}}\mathcal{M}_{\mathcal{B}}$ as an object in $\mathcal{M}^{\mathcal{H}}_{\mathcal{B}}$ by neglecting the left $\mathcal{B}$-action and by projecting the left $\mathcal{A}$-coaction ${}_M\Delta$, explicitly:

$$\Delta_R(m) := m_0 \otimes S(\pi(m_{(-1)})). \tag{2.48}$$

Also, it is easy to verify using our assumption (of being a quantum homogeneous space) that $\mathcal{B}^+M$ is a subobject of M in $\mathcal{M}^{\mathcal{H}}_{\mathcal{B}}$. Moreover, the right $\mathcal{H}$-colinearity and right $\mathcal{B}$-linearity of $\overline{\phi}$ follows from the corresponding properties of ϕ. Hence, the map Φ as a functor is well-defined.

The second functor,

$$\Psi : \mathcal{M}^{\mathcal{H}}_{\mathcal{B}} \longrightarrow {}^{\mathcal{A}}_{\mathcal{B}}\mathcal{M}_{\mathcal{B}} \tag{2.49}$$

is defined on objects as:

$$N \mapsto \mathcal{A}\Box_{\mathcal{H}}N := \{\sum a^i \otimes n^i : \sum a^i_{(1)} \otimes \pi(a^i_{(2)}) \otimes n^i = \sum a^i \otimes n^i_{(-1)} \otimes n^i_{(0)}\},$$

while on morphisms is defined as follows:

$$(\psi : N \longrightarrow N') \mapsto (\mathrm{id} \otimes \psi : \mathcal{A}\Box_{\mathcal{H}}N \longrightarrow \mathcal{A}\Box_{\mathcal{H}}N').$$

The left $\mathcal{B}$-action and left $\mathcal{A}$-coaction on $\mathcal{A}\Box_{\mathcal{H}}N$ are given by:

$$b(\sum a^i \otimes n^i) := \sum ba^i \otimes n^i, \tag{2.50}$$

$$\sum a^i \otimes n^i \mapsto \sum a^i_{(1)} \otimes a^i_{(2)} \otimes n^i, \tag{2.51}$$

respectively. While, the right $\mathcal{B}$-action is given by:

$$(\sum a^i \otimes n^i)b := \sum a^i b_{(1)} \otimes (n^i \triangleleft b_{(2)}). \tag{2.52}$$

The objects $\mathcal{A}\Box_{\mathcal{H}}N$ is referred to as the *cotensor product* of $\mathcal{A}$ and N over $\mathcal{H}$. Moreover, it is easy to verify the details for Ψ to be well-defined.

Having all this setup, one has the following important theorem.

Theorem 2.6.1 (Takeuchi, [74])**.** *For a quantum homogeneous space $\mathcal{B} = \mathcal{A}^{\mathrm{co}(\mathcal{H})}$, the pair of functors (Φ, Ψ) establish an equivalence of categories. The natural transformations are given by:*

$$\begin{aligned} C : (\Phi \circ \Psi)(N) &\longrightarrow N \\ \sum \overline{a^i \otimes n^i} &\mapsto \sum \varepsilon(a^i)n^i \end{aligned} \tag{2.53}$$

and

$$\begin{aligned} U : M &\longrightarrow (\Psi \circ \Phi)(M) \\ m &\mapsto m_{(-1)} \otimes \overline{m_{(0)}} \end{aligned} \tag{2.54}$$

For more details of this description of Takeuchi equivalence, see [11].

Corollary 2.6.2. *For any Hopf algebra $\mathcal{A}$, fix $\pi := \varepsilon : \mathcal{A} \longrightarrow \mathbb{K}$. Then, $\mathcal{A}$ is a trivial homogeneous space over itself. In this case, the above equivalence reads:*

$$\begin{aligned} \Phi : {}^{\mathcal{A}}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}} &\longrightarrow \mathcal{M}_{\mathcal{A}}, & \Psi : \mathcal{M}_{\mathcal{A}} &\longrightarrow {}^{\mathcal{A}}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}} \\ M &\mapsto \overline{M} := M/\mathcal{A}^+M & N &\mapsto A \otimes N \end{aligned}$$

and

$$\begin{aligned} M &\xrightarrow{\cong} \Psi(\Phi(M)) = A \otimes \overline{M}, & \Phi(\Psi(N)) &= \overline{\mathcal{A} \otimes N} \xrightarrow{\cong} N, \\ m &\mapsto m_{(-1)} \otimes [m_{(0)}], & [a \otimes n] &\mapsto \varepsilon(a)n. \end{aligned}$$

This special case is often known as *fundamental theorem of Hopf modules*. (See [14, Lemma 2.17].)

2.7 First-order differential calculus

In this section, we see how non-commutative geometry generalizes the idea of *exterior derivative*. For a detailed discussion, we refer to [14, Chap. 1].

Definition 2.7.1. A *first-order differential calculus* (abbreviated as FODC) over an algebra $\mathcal{B}$ is a tuple (Ω^1, d), where, Ω^1 is a $\mathcal{B}$-bimodule, and,

$$\mathrm{d} : \mathcal{B} \longrightarrow \Omega^1,$$

(called the exterior derivative) is a linear map obeying:

(i) $\mathrm{d}(ab) = \mathrm{d}a.b + a.\mathrm{d}b \qquad \forall\, a, b \in \mathcal{B},$ (Leibniz Rule)

(ii) $\Omega^1 = \mathrm{span}\{a\mathrm{d}b : a, b \in \mathcal{B}\},$

(iii) $\ker \mathrm{d} = \mathbb{K}.1.$ (Connected)

A $*$-FODC over a $*$-algebra $\mathcal{B}$ means a differential calculus Ω^1 and an anti-linear map $* : \Omega^1 \longrightarrow \Omega^1$ that commutes with d and respects the bimodule structure in the sense that:

$$(a.\omega)^* = \omega^*.a^* \qquad \text{for all } a \in \mathcal{B},\ \omega \in \Omega^1. \tag{2.55}$$

A *morphism* from a FODC (Ω^1, d) on $\mathcal{B}$ to a FODC $(\tilde{\Omega}^1, \tilde{\mathrm{d}})$ on $\tilde{\mathcal{B}}$ is a pair (Φ, φ) where Φ is a bimodule morphism $\Omega^1 \longrightarrow \tilde{\Omega}^1$, φ is an algebra map $\mathcal{B} \longrightarrow \tilde{\mathcal{B}}$ such that the following diagram commute:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\varphi} & \tilde{\mathcal{B}} \\ \Big\downarrow{\scriptstyle \mathrm{d}} & & \Big\downarrow{\scriptstyle \tilde{\mathrm{d}}} \\ \Omega^1 & \xrightarrow{\Phi} & \tilde{\Omega}^1. \end{array}$$

The category $\underline{\mathrm{DC}^1}$ of all FODCs also admits a monoidal structure. Explicitly, given any objects $(\mathcal{B}, \Omega^1, \mathrm{d})$ and $(\tilde{\mathcal{B}}, \tilde{\Omega}^1, \tilde{\mathrm{d}})$ their tensor product is defined as $(\mathcal{B} \otimes \tilde{\mathcal{B}}, \Omega^1_{\mathcal{B}\otimes\tilde{\mathcal{B}}}, \mathrm{d}_{\mathcal{B}\otimes\tilde{\mathcal{B}}})$ where:

$$\Omega^1_{\mathcal{B}\otimes\tilde{\mathcal{B}}} := \Omega^1 \otimes \tilde{\mathcal{B}} \ \oplus \ \mathcal{B} \otimes \tilde{\Omega}^1,$$

and,

$$\mathrm{d}_{\mathcal{B}\otimes\tilde{\mathcal{B}}} := \mathrm{d} \otimes \mathrm{id}_{\tilde{\mathcal{B}}} \oplus \mathrm{id}_{\mathcal{B}} \otimes \tilde{\mathrm{d}}.$$

Example 2.7.2 (Universal first-order differential calculus)**.** Given any algebra $\mathcal{B}$, define:

$$\Omega^1_u := \ker(\mathrm{m}) = \{\sum a \otimes b \in \mathcal{B} \otimes \mathcal{B} : \sum ab = 0\},$$

and:

$$\mathrm{d}_u(a) := 1 \otimes a - a \otimes 1, \qquad \forall\, a \in \mathcal{B}.$$

Then, it is easy to verify that the pair $(\Omega^1_u, \mathrm{d}_u)$ is a first-order differential calculus, see [14, Prop. 1.5]. We call it the *universal first-order differential calculus*.

As the name suggests, the universal differential calculus is more than just an example. Its significance can be read through following proposition.

Proposition 2.7.3. *Any differential calculus* (Ω^1, d) *on* $\mathcal{B}$ *is isomorphic to some quotient calulus* $(\Omega^1_u/N, \mathrm{d}_N)$ *where* N *is a sub-bimodule and* $\mathrm{d}_N := \pi \circ \mathrm{d}_u$.

Proof. See [77, Th. 1.1]. □

Similarly, the notion of higher-order forms is defined in this general setting.

Definition 2.7.4. A *differential calulus* (abbreviated as DC) on an algebra $\mathcal{A}$ is a triplet $(\Omega, \wedge, \mathrm{d})$, where $\Omega = \oplus_n \Omega^n$ is a graded-algebra,

$$\wedge : \Omega \otimes \Omega \longrightarrow \Omega \qquad \text{and} \qquad \mathrm{d} : \Omega \longrightarrow \Omega$$

are linear maps called *wedge product* and *exterior derivative* respectively, such that:

(i) $\Omega^k \wedge \Omega^l \subset \Omega^{k+l}, \qquad \mathrm{d}(\Omega^k) \subset \Omega^{k+1}, \qquad \forall\, k, l \in \mathbb{N}_0,$

(ii) The wedge product $\wedge$ is unital and associative,

(iii) $\mathrm{d}^2 = 0$, and $\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^{\deg(\omega)} \omega \wedge \mathrm{d}\eta$, for all $\eta, \omega \in \Omega$, ω being homogeneous,

(iv) $\Omega^0 = \mathcal{A}, \qquad \Omega^n = \operatorname{span}\{a_0 \mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_n : a_0, \cdots a_n \in \mathcal{A}\}.$

If we drop condition (iv), the triplet $(\Omega, \wedge, \mathrm{d})$ is often called a *differential graded algebra* DGA.

Given any first-order differential calculus (Ω^1, d) on $\mathcal{A}$, one can obtain a differential calculus on $\mathcal{A}$ called *the maximal prolongation* of (Ω^1, d). The construction works as follows. Let $(\Omega^1_u/N, \mathrm{d})$ be a FODC, where Ω^1_u denotes the universal FODC as introduced in example 2.7.2. Define,

$$\Omega^\bullet(\mathcal{A}) := \oplus_{k=0}^{\infty} (\Omega^1(\mathcal{A}))^{\otimes_{\mathcal{A}} k} / \langle \mathrm{d}(N) \rangle,$$

where $\langle \mathrm{d}(N) \rangle$ denotes the subalgebra of the tensor algebra generated by $\mathrm{d}(N)$. It is easy to verify that d extends to a unique map $\Omega^\bullet(\mathcal{A}) \longrightarrow \Omega^\bullet(\mathcal{A})$ and gives us a total differential calculus on $\mathcal{A}$. For more details, see [11, §2.5, §5.1].

2.8 Covariant first-order differential calculus

If a given algebra has some additional structures, we get interested in those FODCi that are compatible with the additional structure. In this section, we will see a characterization of these nice FODCi.

Definition 2.8.1. A first-order differential calculus (Ω^1, d) on a left $\mathcal{A}$-comodule algebra $\mathcal{B}$ together with a left $\mathcal{A}$-coaction is called *left $\mathcal{A}$-covariant* if Ω^1 also admit a left $\mathcal{A}$-coaction Φ_L such that:

$$\Phi_L(a\sigma b) = \Delta_L(a)\Phi_L(\sigma)\Delta_L(b) \qquad \text{for all } a, b \in \mathcal{B},\ \sigma \in \Omega^1,$$

and the following diagram commute,

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\Delta_L} & \mathcal{A}\otimes\mathcal{B} \\ \Big\downarrow{\scriptstyle \mathrm{d}} & & \Big\downarrow{\scriptstyle \mathrm{id}\otimes\mathrm{d}} \\ \Omega^1 & \xrightarrow{\Phi_L} & \mathcal{A}\otimes\Omega^1. \end{array}$$

In case, $\mathcal{B}=\mathcal{A}$ and the left coaction is exactly the coproduct Δ, a left $\mathcal{A}$-covaraint FODC is simply called a *left-covariant first order differential calculus*.

Similarly, the notion of right $\mathcal{A}$-covariant or $\mathcal{A}$-bicovariant differential calculus can be defined, see [14, 57, 77].

In [77], Woronowicz proved that every left-covariant first order differential calculus on a Hopf algebra $\mathcal{A}$ can be obtained via a right ideal of $\mathcal{A}$ contained in $\ker\varepsilon$. More precisely, the result states the following.

Theorem 2.8.2 (Woronowicz)**.** *Let $\mathcal{A}$ be a Hopf algebra, $\mathcal{R}$ be a right ideal contained in $\ker\varepsilon$ and $N=r^{-1}(\mathcal{A}\otimes\mathcal{R})$ where $r:\mathcal{A}\otimes\mathcal{A}\longrightarrow\mathcal{A}\otimes\mathcal{A}$ defined as:*

$$r(a\otimes b)=(a\otimes 1)\Delta(b).$$

Then, N is a sub-bimodule of Ω^1_u and $(\Omega^1_u/N,\mathrm{d}_N)$ is a left-covariant first-order differential calculus. Moreover, every left-covariant first-order differential calculus can be obtained (up to isomorphism) in this way.

Proof. See [77, Th. 1.5]. □

There are also similar results for right-covariant and bicovariant first order differential calculi on a Hopf algebra $\mathcal{A}$, see [14, 77].

The Theorem 2.8.2 is special to a very particular situation of the Definition 2.8.1, namely, $\mathcal{B}=\mathcal{A}$ and the coaction is exactly the coproduct. However, thanks to the Takeuchi equivalence, one can generalize this result to the case of quantum homogeneous spaces. We report it below.

Theorem 2.8.3 (Hermisson, [51])**.** *Let $\mathcal{B}=\mathcal{A}^{co\mathcal{H}}$ be a quantum homogeneous space. Then there is a bijective correspondence,*

$$\left\{\begin{array}{l}\textit{left } \mathcal{A}\textit{-covariant first order}\\ \textit{differential calculus on } \mathcal{B}\end{array}\right\}\longleftrightarrow\left\{\begin{array}{l}\textit{sub-objects } I^{(1)}\subset\mathcal{B}^+\\ \textit{in } \mathcal{M}^{\mathcal{H}}_{\mathcal{B}}\end{array}\right\}.$$

Explicitly:

(i) The sub-object corresponding to the calculus $\Omega^1(\mathcal{B})$ *is:*

$$I^{(1)} := \Big\{ \sum_i \varepsilon(a_i) b_i^+ | \ \sum_i a_i \mathrm{d} b_i = 0 \Big\}, \tag{2.56}$$

where $b_i^+ := b_i - \varepsilon(b_i).1$

(ii) Denoting $V^1 := \mathcal{B}^+ / I^{(1)}$*, we have an isomorphism,*

$$\begin{aligned} \sigma : \Phi(\Omega^1(\mathcal{B})) &\longrightarrow V^1 \\ \overline{a \mathrm{d} b} &\mapsto \varepsilon(a) \overline{b^+}. \end{aligned} \tag{2.57}$$

(iii) Conversely, for $I^{(1)} \subset \mathcal{B}^+$ *in* $\mathcal{M}^{\mathcal{H}}_{\mathcal{B}}$*, define* $\Omega^1 := \mathcal{A} \square_{\mathcal{H}} \mathcal{B}^+ / I^{(1)}$ *with* $\mathcal{B}$*-bimodule structure and left* $\mathcal{A}$*-coaction as:*

$$b(a^i \otimes [c^i])b' := b a^i b'_{(1)} \otimes [c^i \triangleleft b'_{(2)}], \qquad {}_{\Omega^1}\Delta := \Delta \otimes \mathrm{id}, \tag{2.58}$$

and, $\mathrm{d} : \mathcal{B} \longrightarrow \Omega^1$ *defined as:*

$$\mathrm{d}(b) := b_{(1)} \otimes \pi_I((b_{(2)})^+). \tag{2.59}$$

Proof. See [13, Th. 2.10]. □

Definition 2.8.4. Let the notation be as in Theorem 2.8.3. We call $V^1 := \mathcal{B}^+ / I^{(1)}$ the *cotangent space* of the calculus $\Omega^1(\mathcal{B})$.

A further classification is given through the notion of *quantum tangent spaces*.

Let $\mathcal{A}$ be a Hopf algebra, and $W \subseteq \mathcal{A}^\circ$ a Hopf subalgebra of $\mathcal{A}^\circ$ (the restricted dual Hopf algebra of $\mathcal{A}$, see [57, §1.2.8]), such that the space of invariants:

$$\mathcal{B} := {}^W\!\mathcal{A} = \Big\{ b \in \mathcal{A} \,|\, \varepsilon(w) b = \ b_{(1)} \langle w, b_{(2)} \rangle, \ \forall \ w \in W \Big\}$$

under the natural left action of W is a quantum homogeneous A-space, and denote by B° its dual coalgebra.

Definition 2.8.5. A *quantum tangent space* for $\mathcal{B}$ is a subspace $T \subseteq \mathcal{B}^\circ$ such that $T \oplus \mathbb{C}1$ is a right coideal of $\mathcal{B}^\circ$ and $WT \subseteq T$.

For any quantum tangent space T, a right $\mathcal{B}$-ideal of $\mathcal{B}^+$ is given by

$$I^{(1)} := \big\{x \in \mathcal{B}^+ \mid X(x) = 0, \text{ for all } X \in T\big\},$$

meaning that the quotient $V^1 := \mathcal{B}^+/I^{(1)}$ is naturally an object in the category $\mathcal{M}_{\mathcal{B}}^{\pi_{\mathcal{B}}}$. We call V^1 the *cotangent space* of T.

Theorem 2.8.6. *Consider the object,*

$$\Omega^1(\mathcal{B}) := \mathcal{A}\square_{\pi_{\mathcal{B}}} V^1.$$

If $\{X_i\}_{i=1}^n$ *is a basis for* T*, and* $\{e_i\}_{i=1}^n$ *is the dual basis of* V^1*, then the map*

$$\mathrm{d} : \mathcal{B} \to \Omega^1(\mathcal{B}), \qquad\qquad a \mapsto \sum_{i=1}^n (X_i^+ \triangleright a) \otimes e_i$$

is a derivation, and the pair $(\Omega^1(\mathcal{B}), \mathrm{d})$ *is a left* $\mathcal{A}$*-covariant FODC over* $\mathcal{B}$*. This gives a bijective correspondence between isomorphism classes of finite-dimensional tangent spaces and finitely-generated left* $\mathcal{A}$*-covariant FODC.*

Proof. See [52, Prop. 4]. □

Now, we describe how the relations for the *maximal prolongation* (as described in 2.7) of a left covariant FODC can be constructed in this setting.

Definition 2.8.7. Let $I^{(1)} \subset \mathcal{B}^+$ be the ideal classifying the FODC $\Omega^1(\mathcal{B})$, and $V^1 := \mathcal{B}^+/I^{(1)}$ be the cotangent space of $\Omega^1(\mathcal{B})$. Consider the subspace:

$$I^{(2)} := \Big\{\omega(x) := [x^+_{(1)}] \otimes [x^+_{(2)}] \mid x \in I^{(1)}\Big\} \subset V^1 \otimes V^1. \tag{2.60}$$

From the tensor algebra $\mathcal{T}(V^1)$, construct the $\mathbb{Z}_{\geq 0}$-graded algebra,

$$V^\bullet := \bigoplus_{\mathbb{Z}_{\geq 0}} V^k := \mathcal{T}(V^1)/\langle I^{(2)}\rangle \tag{2.61}$$

We call $V^\bullet$ the *quantum exterior algebra* of $\Omega^1(\mathcal{B})$.

It follows from Takeuchi's equivalence, that the k-th homogeneous component of maximal prolongation $\Phi(\Omega^k(\mathcal{B}))$ is isomorphic to V^k, where Φ is the first Takeuchi's functor as in (2.47). Explicitly, an isomorphism is given by:

$$\begin{aligned} \Phi(\Omega^k(\mathcal{B})) &\longrightarrow V^k \\ \overline{b_0 \mathrm{d} b_1 \otimes \cdots \otimes \mathrm{d} b_k} &\mapsto \epsilon(b_0)\overline{b_1^+} \wedge \cdots \wedge \overline{b_k^+} \end{aligned} \tag{2.62}$$

where $\wedge$ denotes multiplication in $V^\bullet$. For more details see [13, Lemma 5.1].

2.9 Noncommutative complex structures

We end this chapter by introducing the notion of *complex structure* in this noncommutative setting. We followed closely the description given in [12, 13]. As usual in the theory of differential calculi, we find it convenient to initially work at the level of FODC and then discuss the extension to higher forms. This motivates the following general definition.

Definition 2.9.1. A *first-order almost complex structure* (sometimes abbreviated as FOACS) for a $*$-FODC $\Omega^1(\mathcal{B})$ over an algebra $\mathcal{B}$ is a direct sum decomposition of $\mathcal{B}$-bimodules,

$$\Omega^1(\mathcal{B}) \simeq \Omega^{(1,0)} \oplus \Omega^{(0,1)} \tag{2.63}$$

such that $(\Omega^{(1,0)})^* = \Omega^{(0,1)}$ or equivalently $(\Omega^{(0,1)})^* = \Omega^{(1,0)}$.

Definition 2.9.2. An *almost complex structure* for a differential $*$-calculus $\Omega^\bullet(A)$ is an $\mathbb{N}_0^2$-algebra grading $\Omega^\bullet(A) = \oplus_{(p,q)}\Omega^{(p,q)}$ such that:

(i) $\Omega^k(A) = \oplus_{p+q=k}\Omega^{(p,q)}$,

(ii) $(\Omega^{(p,q)})^* = \Omega^{(q,p)}$.

The elements of $\Omega^{(p,q)}$ are called (p,q)-forms.

Define the projections of differential operator d as follows:

$$\partial := \mathrm{proj}_{\Omega^{(p+1,q)}} \circ \mathrm{d}, \qquad \overline{\partial} := \mathrm{proj}_{\Omega^{(p,q+1)}} \circ \mathrm{d}, \tag{2.64}$$

where the maps $\mathrm{proj}_{\Omega^{(p+1,q)}}$ and $\mathrm{proj}_{\Omega^{(p,q+1)}}$ are projections from Ω^{p+q+1} to $\Omega^{(p+1,q)}$ and $\Omega^{(p,q+1)}$ respectively.

Definition 2.9.3. An almost complex structure is said to be *integrable* if $\mathrm{d} = \partial + \overline{\partial}$. Moreover, an integrable almost complex structure is called a *complex structure*.

We report below two important results followed by these definitions.

The following result gives some equivalent criterion to check whether an almost complex structure is integrable or not.

Proposition 2.9.4. *For an almost complex structure, the following are equivalent:*

(i) $\mathrm{d} = \partial + \overline{\partial}$*,*

(ii) $\partial^2 = 0$*,*

(iii) $\overline{\partial}^2 = 0$*,*

(iv) $\mathrm{d}(\Omega^{(1,0)}) \subset \Omega^{(2,0)} \oplus \Omega^{(1,1)}$*,*

(v) $\mathrm{d}(\Omega^{(0,1)}) \subset \Omega^{(1,1)} \oplus \Omega^{(0,2)}$*.*

Proof. See [12, Lemma 3.2]. □

Proposition 2.9.5. *Given a complex structure* $\Omega^\bullet(A) = \oplus_{(p,q)}\Omega^{(p,q)}$*, we have:*

(i) $\partial(a^*) = (\overline{\partial}(a))^*$ *and* $\overline{\partial}(a^*) = (\partial(a))^*$*, for all* $a \in A$*,*

(ii) ∂ *and* $\overline{\partial}$ *satisfies graded Leibniz rule.*

Proof. See [12, Prop. 3.8]. □

Chapter 3

Supergeometry

This chapter is devoted to introduce some preliminary topics in supergeometry that are used in the subsequent parts of the thesis. All of the content in this chapter is widely known, thus many technical/computational details have been omitted, for details see [19, 39, 61, 75].

In Section 3.1, we introduce some of the fundamental supergeometric objects including *super vector spaces*, *superalgebras*, *supermodules* and *supermatrices*. In Section 3.2, we introduce the notion of *berezinian* of a supermatrix which is the super-version of *determinant*. The Section 3.3 is devoted to stating and proving *super Cramer's rule*. In Section 3.4, the notions of *supermanifolds* and *superschemes* are presented briefly. In Section 3.5, we introduce the concept of *supergroup* and present the example of the *special linear supergroup* $\mathrm{SL}(r|s)$ which is of our primary interest in Chapter 4.

3.1 Super linear algebra

In this section, we introduce some fundamental notions from super linear algebra. We mainly follow the description given in [19, Chap. 1].

Linear algebra is the cornerstone of classical geometry. Thus, one must first modify linear algebra in transition from classical geometry to the supergeometry.

Definition 3.1.1. A *super vector space* V is a $\mathbb{Z}_2$-graded vector space:

$$V = V_0 \oplus V_1.$$

The elements of V_0 and V_1 are called the *even* and *odd* elements of V respectively. The *dimension* of V is denoted by $p|q$ where p and q represents the dimensions of V_0 and V_1 respectively. The *parity* of a *homogeneous* vector $v \in V$ is defined as,

$$|v| := \begin{cases} 0 & if \ v \in V_0, \\ 1 & if \ v \in V_1. \end{cases} \quad . \tag{3.1}$$

A *morphism* $\varphi : V \longrightarrow W$ from a super vector space V to another super vector space W is a parity-preserving linear transformation.

The category of super vector spaces also admits a monoidal structure.

Definition 3.1.2. For two super vector spaces V and W, their tensor product $V \otimes W$ is a super vector space, with even and odd part being:

$$(V \otimes W)_0 := (V_0 \otimes W_0) \oplus (V_1 \otimes W_1) \tag{3.2}$$

$$(V \otimes W)_1 := (V_0 \otimes W_1) \oplus (V_1 \otimes W_0). \tag{3.3}$$

This tensor product is well-behaved and the details are very similar to the classical setting. (See [19, §1.1].)

Definition 3.1.3. A *superalgebra* is a super vector space $\mathcal{A}$ together with a multiplication morphism:

$$\begin{aligned} \mathrm{m} : \mathcal{A} \otimes \mathcal{A} &\longrightarrow \mathcal{A} \\ a \otimes b &\mapsto ab \end{aligned}$$

following the usual properties (as already given in Definition 2.1.1). A superalgebra $\mathcal{A}$ is (super) *commutative* if for all homogeneous $a, b \in \mathcal{A}$:

$$ab = (-1)^{|a||b|} ba$$

where $|c|$ denotes the parity of a homogeneous element $c \in \mathcal{A}$ as defined in (3.1).

The tensor product of two superalgebras $\mathcal{A}$ and $\tilde{\mathcal{A}}$ is a superalgebra with the multiplication defined as:

$$(a \otimes b)(c \otimes d) := (-1)^{|b||c|}(ac \otimes bd).$$

We discuss below a simple but very important example. It replaces the notion of polynomial algebra in the super setting.

Example 3.1.4. Consider,

$$\mathcal{A} = \mathbb{K}[x_1, \cdots, x_p, \alpha_1, \cdots, \alpha_q] := \mathbb{K}\langle x_1, \cdots, x_p, \alpha_1, \cdots, \alpha_q\rangle/\mathrm{I}$$

where I is the ideal generated by:

$$x_i x_j - x_j x_i, \qquad x_i \alpha_k - \alpha_k x_i, \qquad \alpha_k \alpha_l + \alpha_l \alpha_k, \tag{3.4}$$

where, $1 \leq i, j \leq p$, $1 \leq k, l \leq q$. In other words, *latin and greek* generators denote even and odd generators respectively. This is a commutative superalgebra with even part being:

$$\mathcal{A}_0 = \{f_0 + \sum_{|I|\ even} f_I \alpha_I : I = (i_1 < \cdots < i_r)\},$$

where $|I| := r$, $\alpha_I := \alpha_{i_1} \cdots \alpha_{i_r}$, $f_0, f_I \in \mathbb{K}[x_1, \cdots, x_p]$ and, odd part being:

$$\mathcal{A}_1 = \{\sum_{|J|\ odd} f_J \alpha_J : J = (j_1 < \cdots < j_s)\}.$$

See [19, Example 1.1.9] for more details.

Having these algebraic structures it is natural to also define the notion of *modules* over superalgebras.

Definition 3.1.5. Let $\mathcal{A}$ be a superalgebra. A *left-module* over $\mathcal{A}$ is a super vector space M together with a morphism:

$$\begin{aligned} \mathcal{A} \otimes M &\longrightarrow M \\ a \otimes m &\mapsto am, \end{aligned}$$

obeying the following usual properties, $\forall\ a, b \in \mathcal{A}$, $m, n \in M$:

1. $a(m + n) = am + an$,
2. $(a + b)m = am + bm$,
3. $(ab)m = a(bm)$,
4. $1m = m$.

The notion of a *right-module* can be defined in a similar way. Let M and N be two $\mathcal{A}$-modules. A *module morphism* $\varphi : M \longrightarrow N$ is a morphism of super vector spaces respecting the module structure, *i.e.* obeying:

$$\varphi(am) = a\varphi(m), \qquad \forall\ a \in \mathcal{A},\ m \in M.$$

Example 3.1.6. For a superalgebra $\mathcal{A}$ over a field $\mathbb{K}$, define the left $\mathcal{A}$-module $\mathcal{A}^{p|q}$ as,

$$\mathcal{A}^{p|q} := \mathcal{A} \otimes \mathbb{K}^{p|q},$$

where $\mathbb{K}^{p|q} := (\underbrace{\mathbb{K} \oplus \cdots \oplus \mathbb{K}}_{p-times}) \oplus (\underbrace{\mathbb{K} \oplus \cdots \oplus \mathbb{K}}_{q-times})$. The module structure is given simply by:

$$\tilde{a} \otimes (a \otimes k) \mapsto \tilde{a}a \otimes k, \qquad \forall\ \tilde{a} \in \mathcal{A}, (a \otimes k) \in \mathcal{A}^{p|q}.$$

Definition 3.1.7. A module M over a superalgebra $\mathcal{A}$ is said to be *free* if for some p and q:

$$M \cong \mathcal{A}^{p|q}.$$

Alternatively, from Definition 3.1.2 it follows that, an A-module M is *free* if there exists even elements $\{e_1, ..., e_p\}$, and odd elements $\{\epsilon_1, ..., \epsilon_q\} \subset M$ such that:

$$M_0 = \mathrm{span}_{A_0}\{e_1, ..., e_p\} \oplus \mathrm{span}_{A_1}\{\epsilon_1, ..., \epsilon_q\}, \tag{3.5}$$

$$M_1 = \mathrm{span}_{A_1}\{e_1, ..., e_p\} \oplus \mathrm{span}_{A_0}\{\epsilon_1, ..., \epsilon_q\}. \tag{3.6}$$

As it turns out that when a basis is fixed, studying classical linear algebra amounts to manipulating matrices. Hence, in super linear algebra matrices play a very important role. Moreover, matrices in the super setting admits a special 2×2 block structure.

Definition 3.1.8. Let $T : A^{p|q} \longrightarrow A^{r|s}$ be a morphism of free A-modules. Fixing a basis of both $\mathcal{A}^{p|q}$ and $\mathcal{A}^{r|s}$ as above gives a 2×2 block (as it is evident from *equations* (3.5) and (3.6)) matrix T of size $(r + s) \times (p + q)$,

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}, \tag{3.7}$$

where T_1 is of size $r \times p$, T_2 is of size $r \times q$, T_3 is of size $s \times p$ and T_4 is of size $s \times q$. Moreover, the entries of T_1 and T_4 are even while the entries of T_2 and T_3 are odd. Matrices as in (3.7) are known as *even supermatrices* of order $r|s \times p|q$. The set of all even supermatrices over $\mathcal{A}$ of order $r|s \times p|q$ is denoted by $\mathrm{M}_{r|s \times p|q}(\mathcal{A})$. Moreover, the set of all invertible even supermatrices over $\mathcal{A}$ of order $p|q \times p|q$ is denoted by $\mathrm{GL}_{p|q}(\mathcal{A})$.

For our purpose there is also a need to introduce two non-standard types of matrices. These are called 'wrong supermatrices' in [9], we prefer the terminology 'fake'.

Definition 3.1.9. A *fake supermatrix of type-I* is a matrix obtained by inserting an odd vector at an even vector's position of an even supermatrix. A *fake supermatrix of type-II* is a matrix obtained by inserting an even vector at an odd vector's position of an even supermatrix.

As usual, for any even supermatrix or fake supermatrix, we define its *parity reversed matrix*, denoted by B^{Π} by swapping B_1 with B_4 and B_2 with B_3, i.e.

$$\left(\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right)^{\Pi} := \left(\begin{array}{c|c} B_4 & B_3 \\ \hline B_2 & B_1 \end{array}\right).$$

Notice that if B is a fake supermatrix of type I, then B^{Π} is a fake supermatrix of type II and viceversa.

3.2 Berezinian

In the classical setting, one of the most important concepts associated to a matrix is its *determinant*. In supergeometry, there is a generalization, known as *berezinian*, named after *F. Berezin* [6]. The motivation behind it lies in the theory of integration over supermanifolds, see [75, §3.6]. When looking at the concept of berezinian we see an important difference, because berezinian is not defined for all supermatrices. In literature, mostly, berezinian is defined only for invertible supermatrices. However, we will define it here on a larger class, though it is not possible to define it for all supermatrices.

Definition 3.2.1. Let B be an even supermatrix or a fake supermatrix of type-I:

$$B = \left(\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right). \tag{3.8}$$

Assume that B_4 is invertible. Then, the *berezinian* of B is defined as:

$$\operatorname{Ber}(B) := \det B_4^{-1} \det(B_1 - B_2 B_4^{-1} B_3).$$

It turns out that, a supermatrix B is invertible if and only if both B_1 and B_4 are invertible, see [19, Prop. 1.5.1]. In this case, another equivalent definition of the *berezinian* is:

$$\operatorname{Ber}(B) = \det B_1 \det(B_4 - B_3 B_1^{-1} B_2)^{-1}.$$

Definition 3.2.2. Let C be an even supermatrix or a fake supermatrix of type-II:

$$C = \left(\begin{array}{c|c} C_1 & C_2 \\ \hline C_3 & C_4 \end{array}\right) \tag{3.9}$$

Assume C_1 is invertible, define the *inverted berezinian* of B:

$$\mathrm{Ber}^* B := \mathrm{Ber} B^{\Pi}.$$

It is clear that for an invertible supermatrix B:

$$\mathrm{Ber}^* B = \mathrm{Ber} B^{-1}. \tag{3.10}$$

One of the key properties that berezinian shares with the classical determinant is its multiplicative property.

Theorem 3.2.3. *For any supermatrix $B' \in \mathrm{GL}_{p|q}(\mathcal{A})$ and any even supermatrix (or fake supermatric of type-I) B with B_4 invertible, we have:*

$$\mathrm{Ber}(B'B) = \mathrm{Ber}(B')\mathrm{Ber}(B). \tag{3.11}$$

Proof. See [19, Prop 1.5.4]. □

Remark 3.2.4. In [19], the multiplicative property is discussed for the case when both B and B' are invertible, as berezinian is defined only for invertible supermatrices in [19]. However, the same proof works for the statement in Theorem 3.2.3.

Notation 3.2.5. For an even supermatrix B, we denote with i the even indices and with $\hat{\imath}$ the odd ones, as we exemplify below:

$$B = \left(\begin{array}{ccc|ccc} b_{11} & \dots & b_{1p} & b_{1\hat{1}} & \dots & b_{1\hat{q}} \\ . & \dots & . & . & \dots & . \\ . & \dots & . & . & \dots & . \\ b_{r1} & \dots & b_{rp} & b_{r\hat{1}} & \dots & b_{r\hat{q}} \\ \hline b_{\hat{1}1} & \dots & b_{\hat{1}p} & b_{\hat{1}\hat{1}} & \dots & b_{\hat{1}\hat{q}} \\ . & \dots & . & . & \dots & . \\ . & \dots & . & . & \dots & . \\ b_{\hat{s}1} & \dots & b_{\hat{s}r} & b_{\hat{s}\hat{1}} & \dots & b_{\hat{s}\hat{q}} \end{array}\right).$$

3.3 Super Cramer's rule

Recall that, in linear algebra, *Cramer's rule* is a fundamental technique for solving an appropriate system of linear equations, which gives us the solution in terms of quotients of some determinants. A generalization of it in the super setting is presented below. It also reflects how beautifully the notion of berezinian is replacing determinant.

We give now *super Cramer's rule*. This is a known result, see [9]. However, for completeness, we present an original proof, obtained with different, elementary methods.

Theorem 3.3.1 (*Super Cramer's rule*)**.** *Let $M \in \mathrm{GL}_{r|s}(\mathcal{A})$ be an invertible even supermatrix and let $\mathbf{b} \in \mathcal{A}^{r|s}$ be an even vector of size $r|s$. Then, the solution to the equation,*

$$M\mathbf{x} = \mathbf{b},$$

is given by:

$$x_i = \frac{\mathrm{Ber} M_i(\mathbf{b})}{\mathrm{Ber} M}, \qquad \textit{for} \quad i = 1, ..., r,$$

$$x_{\hat{\jmath}} = \frac{\mathrm{Ber}^* M_{\hat{\jmath}}(\mathbf{b})}{\mathrm{Ber}^* M}, \qquad \textit{for} \quad \hat{\jmath} = \hat{1} \cdots \hat{s},$$

where $M_i(\mathbf{b})$ is the supermatrix obtained by replacing i-th even column of M with $\mathbf{b}$ and $M_{\hat{\jmath}}(\mathbf{b})$ is the fake supermatrix of type-II obtained by replacing $\hat{\jmath}$-th odd column of M with $\mathbf{b}$.

Proof. We need first some notation: for any even vector $\mathbf{b}$ of size $r|s$, we denote by $\mathbf{b}^{(e)}$ and $\mathbf{b}^{(o)}$ the column vectors consisting of even and odd coefficients of $\mathbf{b}$, respectively:

$$\mathbf{b} = \begin{pmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_r \\ \hline b_{\hat{1}} \\ \cdot \\ \cdot \\ \cdot \\ b_{\hat{s}} \end{pmatrix}, \quad \mathbf{b}^{(e)} = \begin{pmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_r \end{pmatrix} \quad \mathbf{b}^{(o)} = \begin{pmatrix} b_{\hat{1}} \\ \cdot \\ \cdot \\ \cdot \\ b_{\hat{s}} \end{pmatrix}.$$

We have to show that the above statement holds for all $M \in \mathrm{GL}_{r|s}(\mathcal{A})$. We will do it in three steps.

1. One can decompose any supermatrix $M \in \mathrm{GL}(r|s)$ as a product $M = M_+ M_0 M_-$, see [19, Prop. 1.5.4], where M_+, M_0 and M_- are supermatrices of the following types respectively:

$$\left(\begin{array}{c|c} I & X \\ \hline O & I \end{array}\right), \qquad \left(\begin{array}{c|c} V & O \\ \hline O & W \end{array}\right) \qquad \text{and} \qquad \left(\begin{array}{c|c} I & O \\ \hline Z & I \end{array}\right),$$

where I denotes the identity matrix and O denotes the null matrix. For $M = \left(\begin{array}{c|c} M_1 & M_2 \\ \hline M_3 & M_4 \end{array}\right)$, one gets:

$$\begin{aligned} X &= M_2 M_4^{-1}, \\ V &= M_1 - M_2 M_4^{-1} M_3, \\ W &= M_4, \\ Z &= M_4^{-1} M_3. \end{aligned}$$

2. The statement of the theorem holds for supermatrices of the form $C = AB$, where A and B are of the form M_0 and M_- respectively.

Let,

$$A = \left(\begin{array}{c|c} V & O \\ \hline O & W \end{array}\right) \qquad \text{and} \qquad B = \left(\begin{array}{c|c} I & O \\ \hline Z & I \end{array}\right),$$

Therefore,

$$C = \left(\begin{array}{c|c} V & O \\ \hline WZ & W \end{array}\right), \qquad \text{and}$$

$$C^{-1} = \left(\begin{array}{c|c} V^{-1} & O \\ \hline -ZV^{-1} & W^{-1} \end{array}\right) = \left(\begin{array}{c|c} \left[\dfrac{\det V_i(e_j)}{\det V}\right] & O \\ \hline \left[\dfrac{-\sum_{i=1}^{r} z_{\hat{k}i} \det V_i(e_j)}{\det V}\right] & \left[\dfrac{\det W_k(e_l)}{\det W}\right] \end{array}\right),$$

where e_j denotes the column vector having 1 at j-th place and 0 at all other places. Therefore,

$$C^{-1}\mathbf{b} = \begin{pmatrix} \dfrac{\sum_{j=1}^{r} \det V_1(e_j) b_j}{\det V} \\ \vdots \\ \dfrac{\sum_{j=1}^{r} \det V_r(e_j) b_j}{\det V} \\ \hline -\dfrac{\sum_{i,j=1}^{r} z_{\hat{1}i} \det V_i(e_j) b_j}{\det V} + \dfrac{\sum_{l=\hat{1}}^{\hat{s}} \det W_1(e_l) b_l}{\det W} \\ \vdots \\ -\dfrac{\sum_{i,j=1}^{r} z_{\hat{s}i} \det V_i(e_j) b_j}{\det V} + \dfrac{\sum_{l=\hat{1}}^{\hat{s}} \det W_s(e_l) b_l}{\det W} \end{pmatrix}. \tag{3.12}$$

On the other hand, using the linearity of determinant, we have:

$$\frac{\mathrm{Ber} C_i(\mathbf{b})}{\mathrm{Ber} C} = \frac{\det W^{-1} \det V_i(\mathbf{b}^{(e)})}{\det W^{-1} \det V} = \frac{\det V_i(\mathbf{b}^{(e)})}{\det V} = \frac{\sum_{j=1}^{r} \det V_i(e_j) b_j}{\det V}. \tag{3.13}$$

and

$$\begin{aligned} \frac{\mathrm{Ber}^* C_{\hat{k}}(\mathbf{b})}{\mathrm{Ber}^* C} &= \frac{\det V^{-1} \det \left(W_k(\mathbf{b}^{(o)}) - WZV^{-1} O_k(\mathbf{b}^{(e)})\right)}{\det V^{-1} \det W} = \\ & \frac{\det W_k(\mathbf{b}^{(o)}) - \det W_k(WZV^{-1}\mathbf{b}^{(e)})}{\det W} = \\ & \frac{\sum_{l=\hat{1}}^{\hat{s}} \det W_k(e_l) b_l}{\det W} - \frac{\sum_{i,j=1}^{r} z_{\hat{k}i} \det V_i(e_j) b_j}{\det V}. \end{aligned} \tag{3.14}$$

The last two equalities follow from the multi-linearity property of determinant and ordinary Cramer's rule respectively. Hence, comparing Equations (3.12), (3.13) and (3.14) one realizes that the theorem holds for supermatrices of type C.

3. The theorem holds for supermatrices of type $D = EF$ where $F = \left(\begin{array}{c|c} U & Y \\ \hline V & W \end{array}\right)$ is an even supermatrix which already satisfies the theorem and $E = \left(\begin{array}{c|c} I & X \\ \hline O & I \end{array}\right)$ with X having only one non-zero entry. Without loss of generality, we suppose that the top-left entry $x_{1\hat{1}} \neq 0$.

Therefore,

$$D = \left(\begin{array}{ccc|ccc} u_{11} + x_{1\hat{1}} v_{\hat{1}1} & \cdots & u_{1r} + x_{1\hat{1}} v_{\hat{1}r} & y_{1\hat{1}} + x_{1\hat{1}} w_{\hat{1}\hat{1}} & \cdots & y_{1\hat{s}} + x_{1\hat{1}} w_{\hat{1}\hat{s}} \\ u_{21} & \cdots & u_{2r} & y_{2\hat{1}} & \cdots & y_{2\hat{s}} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ u_{r1} & \cdots & u_{rr} & y_{r\hat{1}} & \cdots & y_{r\hat{s}} \\ \hline & V & & & W & \end{array}\right)$$

Note that,

$$\mathbf{x} = D^{-1}\mathbf{b} = F^{-1}(E^{-1}\mathbf{b}). \tag{3.15}$$

Fix,

$$\mathbf{c} = E^{-1}\mathbf{b} = \left(\begin{array}{c} b_1 - x_{1\hat{1}} b_{\hat{1}} \\ \cdot \\ \cdot \\ b_r \\ \hline b_{\hat{1}} \\ \cdot \\ \cdot \\ b_{\hat{s}} \end{array}\right).$$

Since F satisfies the theorem, equation (3.15) implies:

$$x_i = \frac{\mathrm{Ber} F_i(\mathbf{c})}{\mathrm{Ber} F} = \frac{\det\left(U_i(\mathbf{c}^{(e)}) - YW^{-1}V_i(\mathbf{b}^{(o)})\right)}{\det(U - YW^{-1}V)}, \tag{3.16}$$

and

$$x_{\hat{\jmath}} = \frac{\mathrm{Ber}^* F_{\hat{\jmath}}(\mathbf{c})}{\mathrm{Ber}^* F} = \frac{\det\left(W_{\hat{\jmath}}(\mathbf{c}^{(o)}) - VU^{-1}Y_{\hat{\jmath}}(\mathbf{c}^{(e)})\right)}{\det(W - VU^{-1}Y)}. \tag{3.17}$$

On the other hand,

$$\begin{aligned}
&\frac{\mathrm{Ber} D_i(\mathbf{b})}{\mathrm{Ber} D} = \\
&\frac{\det\left((U+XV)_i(\mathbf{b}^{(e)}) - (Y+XW)W^{-1}V_i(\mathbf{b}^{(o)})\right)}{\det(U - YW^{-1}V)} = \\
&\frac{\det\left((U+XV)_i(\mathbf{b}^{(e)}) - XV_i(\mathbf{b}^{(o)}) - YW^{-1}V_i(\mathbf{b}^{(o)})\right)}{\det(U - YW^{-1}V)} = \\
&\frac{\det\left(U_i(\mathbf{c}^{(e)}) - YW^{-1}V_i(\mathbf{b}^{(o)})\right)}{\det(U - YW^{-1}V)},
\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
&\frac{\mathrm{Ber}^* D_{\hat{\jmath}}(\mathbf{b})}{\mathrm{Ber}^* D} = \frac{\det\left(W_{\hat{\jmath}}(\mathbf{b}^{(o)}) - V(U+XV)^{-1}(Y+XW)_{\hat{\jmath}}(\mathbf{b}^{(e)})\right)}{\det\left(W - V(U+XV)^{-1}(Y+XW)\right)} = \\
&\frac{\det\left(W_{\hat{\jmath}}(\mathbf{b}^{(o)}) - V(U^{-1} - U^{-1}XVU^{-1})\left(Y_{\hat{\jmath}}(\mathbf{c}^{(e)}) + XW_{\hat{\jmath}}(\mathbf{b}^{(o)})\right)\right)}{\det\left(W - V(U^{-1} - U^{-1}XVU^{-1})(Y+XW)\right)} = \\
&\frac{\det(W_{\hat{\jmath}}(\mathbf{b}^{(o)}) - VU^{-1}Y_{\hat{\jmath}}(\mathbf{c}^{(e)}) - VU^{-1}XW_{\hat{\jmath}}(\mathbf{b}^{(o)}) - VU^{-1}XVU^{-1}Y_{\hat{\jmath}}(\mathbf{c}^{(e)}))}{\det(W - VU^{-1}Y - VU^{-1}XW + VU^{-1}XVU^{-1}Y)} = \\
&\frac{\det(I - VU^{-1}X)\det\left(W_{\hat{\jmath}}(\mathbf{c}^{(o)}) - VU^{-1}Y_{\hat{\jmath}}(\mathbf{c}^{(e)})\right)}{\det(I - VU^{-1}X)\det(W - VU^{-1}Y)} = \\
&\frac{\det\left(W_{\hat{\jmath}}(\mathbf{c}^{(o)}) - VU^{-1}Y_{\hat{\jmath}}(\mathbf{c}^{(e)})\right)}{\det(W - VU^{-1}Y)}.
\end{aligned} \tag{3.19}$$

The second and third equality follows since X contains only one non-zero entry (*i.e.* $x_{1\hat{1}}$) which is odd, therefore,

$$(U+XV)^{-1} = U^{-1} - U^{-1}XVU^{-1},$$

$$\mathbf{c}^{(e)} = \mathbf{b}^{(e)} - X\mathbf{b}^{(o)}.$$

and,

$$VU^{-1}XVU^{-1}X = \mathbf{0}.$$

By comparing Equations (3.16), (3.17), (3.18) and (3.19), one concludes that the matrices of type F satisfy the theorem.

Now, one can easily observe that every even supermatrix $M \in \mathrm{GL}(r|s)$ is of type F and this completes the proof. □

Remark 3.3.2. It is important to note that, in the above proof, we never use the fact that $\mathbf{b}$ is even. The same expressions work for the equation $M\mathbf{x} = \mathbf{b}$ if $\mathbf{b}$ is an odd vector of size $r|s$.

3.4 Supermanifolds and superschemes

In this section, we introduce the notions of *superspaces* and *supermanifolds*. One of the main ideas to properly define these super geometric objects is to modify the sheaf-theoretic definitions of their classical counterparts. However, in this picture one loses somehow the 'geometric' intuition. To restore some of the intuition, we employ the notion of *functor of points*. For details, see [19, Chap. 3]

Definition 3.4.1. A *super ringed space* S is a tuple $(|S|, \mathcal{O}_S)$, where $|S|$ is a topological space and $\mathcal{O}_S$ is a sheaf of super commutative rings, called the structure sheaf of S. A *superspace* is a super ringed space S such that the stalk at each point $x \in |S|$ is a local ring, *i.e.* it has a unique *maximal ideal*.

Let S and $\tilde{S}$ be two superspaces. A *morphism* $\varphi : S \longrightarrow \tilde{S}$ is a continuous map $|\varphi| : |S| \longrightarrow |\tilde{S}|$ together with a sheaf morphism $\varphi^* : \mathcal{O}_{\tilde{S}} \longrightarrow \varphi_*\mathcal{O}_S$ such that $\varphi_x^*(\mathfrak{m}_{\tilde{S},|\varphi|(x)})$ is contained in $\mathfrak{m}_{S,x}$, where $\mathfrak{m}_{S,x}$ is the maximal ideal in $\mathcal{O}_{S,x}$ while $\mathfrak{m}_{\tilde{S},|\varphi|(x)}$ is the maximal ideal in $\mathcal{O}_{\tilde{S},|\varphi|(x)}$ and φ_x^* is the stalk map.

One may consider ordinary manifolds (or more generally, algebraic schemes) as examples of superspaces where the sheaves of functions are sheaves of super commutative rings having trivial odd part. We discussed a non-trivial example below, which, in fact captures the idea of a *supermanifold*.

Example 3.4.2. Let M be a real differentiable manifold, $|M|$ denotes the underlying topological space and C^∞_M be the sructure sheaf of M. Define the sheaf of (commutative) superalgebras as follows:

$$\text{for each open } V \subset M, \qquad V \mapsto \mathcal{O}_M(V) := C^\infty_M(V)[\theta_1, \cdots, \theta_q]$$

where $C^\infty_M(V)[\theta_1, \cdots, \theta_q] = C^\infty_M(V) \otimes \wedge(\theta_1, \cdots, \theta_q)$. Then, $(|M|, \mathcal{O}_M)$ is a superspace. In the case, when $M = \mathbb{R}^p$ we define (abusing notation a bit):

$$\mathbb{R}^{p|q} := (\mathbb{R}^p, C^\infty_{\mathbb{R}^p}[\theta_1, \cdots, \theta_q]). \tag{3.20}$$

Definition 3.4.3. A *supermanifold* $M = (|M|, \mathcal{O}_M)$ of dimension $p|q$ is a superspace which is locally isomorphic to $\mathbb{R}^{p|q}$ (as defined in Equation 3.20). It means, given any point $x \in M$, $\exists$ a neighbourhood V of x with q odd indeterminates such that,

$$V \cong \tilde{V} \text{ open in } \mathbb{R}^p, \qquad \text{and} \qquad \mathcal{O}_M|_V \cong C^\infty_{\mathbb{R}^p}(\tilde{V})[\theta_1, \cdots, \theta_q].$$

A *morphism* $\varphi : M \longrightarrow \tilde{M}$ of supermanifolds is a morphism of underlying superspaces. We denote the category of all supermanifolds by (smflds).

Definition 3.4.4. A *superscheme* S is a superspace $(|S|, \mathcal{O}_S)$ such that $(|S|, \mathcal{O}_{S,0})$ is an ordinary scheme and $\mathcal{O}_{S,1}$ is a quasi-coherent sheaf of $\mathcal{O}_{S,0}$-modules. Similarly, *morphisms* of superschemes are morphisms of underlying superspaces. For details see [19, Chap. 2].

Let us discuss an example that is of fundamental interest to us.

Example 3.4.5. Consider the vector space $\mathrm{M}_{p|q}$ of all even supermatrices of size $p|q \times p|q$. $\mathrm{M}_{p|q}$ can be viewed as a superscheme where the underlying topological space is $\mathrm{M}_p \times \mathrm{M}_q$ together with the Zariski topology. The superalgebra of global sections of $\mathrm{M}_{p|q}$ is $\mathbb{K}[t_{ij}, \theta_{kl}]$, $1 \le i, j \le p$ or $1 \le i, j \le q$ and $1 \le k \le p$, $p+1 \le l \le p+q$ or $p+1 \le k \le p+q$, $1 \le l \le p$. Let U be the open subset of $\mathrm{M}_p \times \mathrm{M}_q$ consisting of those points $A \times B \in \mathrm{M}_p \times \mathrm{M}_q$ such that both A and B are invertible (ordinary) matrices. Then, $\mathrm{GL}_{p|q} := (U, \mathcal{O}_{\mathrm{M}_{p|q}}|_U)$ is a superscheme called *general linear superscheme*.

Now, we present the notion of *functor of points* of a superscheme.

Definition 3.4.6. Let S and T be two superschemes. A T-*point* of S is a morphism $S \longrightarrow T$. We denote by $S(T)$ the set of all T-points of S, i.e. $S(T) = \mathrm{Hom}(S, T)$. The *functor of points* of the superscheme S is the functor:

$$S : (\text{sschemes})^{\text{op}} \longrightarrow (sets), \qquad T \mapsto S(T), \qquad S(\varphi)f = f \circ \varphi.$$

3.5 Supergroups

In this section we define the notion of a supergroup. For simplicity, by a supergroup, we will always mean what in literature is called *affine algebraic supergroup*. For details see [19, Chap. 10].

Similar to the ordinary algebraic geometry, a *supergroup* is a group-valued functor. Moreover, as it turns out that the functor of points of a superscheme is completely determined by its restriction to affine superschemes (see [19, Prop. 10.1.3]), whose category is equivalent to the category of (commutative) superalgebras (see [19, Prop. 10.1.9]), therefore, one can define a supergroup as below.

Definition 3.5.1. A *supergroup* G is an (affine) superscheme whose functor of points $G : (\text{salg}) \longrightarrow (\text{sets})$ is group-valued. This is equivalent to say that G is an affine superscheme such that, $G(\mathcal{A})$ is a group for any superalgebra $\mathcal{A}$, and for any morphism φ, $G(\varphi)$ is a group homomorphism.

The notion of a *Hopf superalgebra* is defined similar to the notion of a *Hopf algebra* (as defined in Section 2.2) by replacing *algebras* with *superalgebras* and *algebra morphisms* with *superalgebra morphisms*. This gives us the following important result (already pointed out in Example 2.2.8 for the classical case).

Theorem 3.5.2. *Let G be an affine superscheme. Then, $G = \mathcal{O}_G(|G|)$ is a supergroup if and only if $\mathcal{O}(G)$ is a Hopf superalgebra. Moreover, we identify the category of affine supergroups with the category of commutative Hopf superalgebras.*

Proof. See [19, Prop. 11.1.2]. □

Finally, we are ready to introduce the fundamental examples we are interested in.

Example 3.5.3. Define a functor,

$$\begin{aligned} \mathrm{GL}_{\mathrm{m|n}} : (\text{salg}) &\longrightarrow (\text{sets}) \\ \mathcal{A} &\mapsto \mathrm{GL}_{\mathrm{m|n}}(\mathcal{A}), \end{aligned}$$

where $\mathrm{GL}_{\mathrm{m|n}}(\mathcal{A})$ denoted the set of all automorphisms of $\mathcal{A}$-*supermodule* $\mathcal{A}^{m|n}$. We call $\mathrm{GL}_{\mathrm{m|n}}$ the *general linear supergroup.*

Similarly, the functor,

$$\begin{aligned} \mathrm{SL}_{\mathrm{m|n}} : (\text{salg}) &\longrightarrow (\text{sets}) \\ \mathcal{A} &\mapsto \mathrm{SL}_{\mathrm{m|n}}(\mathcal{A}), \end{aligned}$$

where $\mathrm{SL_{m|n}}(\mathcal{A})$ denote the group of all automorphisms in $\mathrm{GL_{m|n}}(\mathcal{A})$ of $\mathcal{A}$-*supermodule* $\mathcal{A}^{m|n}$ whose Berezinian is equal to 1. We call $\mathrm{SL_{m|n}}$ the *special linear supergroup.*

The functors $\mathrm{GL_{m|n}}$ and $\mathrm{SL_{m|n}}$ are represented by the Hopf superalgebra structures on the superalgebras:

$$\mathcal{O}(\mathrm{GL_{m|n}}) = \mathbb{K}[z_{ij}, \xi_{kl}][d_1^{-1}, d_2^{-1}],$$

and,

$$\mathcal{O}(\mathrm{SL_{m|n}}) = \mathbb{K}[z_{ij}, \xi_{kl}][d_1^{-1}, d_2^{-1}]/(\mathrm{Ber} - 1),$$

respectively, where, $\mathbb{K}[z_{ij}, \xi_{kl}]$ denotes the polynomial superalgebra (as described in example 3.1.4) generated by the even variables,

$$z_{ij}, \qquad \text{for} \quad 1 \leq i, j \leq m, \quad \text{or} \qquad m+1 \leq i, j \leq m+n,$$

and by the odd variables,

$$\xi_{kl}, \qquad \begin{array}{l} \text{for} \quad 1 \leq k \leq m, \quad m+1 \leq l \leq m+n, \\ \text{or} \ \ m+1 \leq k \leq m+n, \quad 1 \leq l \leq m, \end{array}$$

and,

$$\begin{aligned} d_1 &:= \sum_{s \in \mathrm{S}_m} (-1)^{l(s)} z_{1,s(1)} \cdots z_{m,s(m)}, \\ d_2 &:= \sum_{t \in \mathrm{S}_n} (-1)^{l(t)} z_{m+1,m+t(1)} \cdots z_{m+n,m+t(n)}. \end{aligned}$$

The Hopf superalgebra structure on these superalgebras is a (super-) modification of the Hopf algebra structure on general linear group and special linear group respectively, as presented in example 2.2.9. For explicit details, see [19, Example 11.1.3].

Below we present a quantum version of the special linear supergroup that will be of interest to us in Chapter 6. This is due to *Manin* [59], see also [39].

Definition 3.5.4. The *quantum matrix superalgebra* $\mathrm{M}_q(m|n)$ is defined as:

$$\mathrm{M}_q(m|n) =_{\mathrm{def}} \mathbb{C}_q\langle z_{ij}, \xi_{kl}\rangle / \mathcal{I}_M,$$

where $\mathbb{C}_q\langle z_{ij}, \xi_{kl}\rangle$ denotes the free superalgebra over $\mathbb{C}_q = \mathbb{C}[q, q^{-1}]$ generated by the even variables,

$$z_{ij}, \qquad \text{for} \quad 1 \le i, j \le m, \quad \text{or} \qquad m+1 \le i, j \le m+n,$$

and by the odd variables,

$$\xi_{kl} \qquad \begin{array}{l} \text{for} \quad 1 \le k \le m, \quad m+1 \le l \le m+n, \\ \text{or} \;\; m+1 \le k \le m+n, \quad 1 \le l \le m, \end{array}$$

satisfying the relations $\xi_{kl}^2 = 0$, and $\mathcal{I}_M$ is an ideal that we describe below. We can visualize the generators as a matrix:

$$\begin{pmatrix} z_{m\times m} & \xi_{m\times n} \\ \xi_{n\times m} & z_{n\times n} \end{pmatrix}. \tag{3.21}$$

It is convenient sometimes to have a common notation for even and odd variables.

$$a_{ij} = \begin{cases} z_{ij} & 1 \le i, j \le m, \;\text{or} \quad m+1 \le i, j \le m+n, \\ & \\ \xi_{ij} & 1 \le i \le m, \quad m+1 \le j \le m+n, \;\text{or} \\ & m+1 \le i \le m+n, \quad 1 \le j \le m\,. \end{cases}$$

We assign a parity to the indices: $p(i) = 0$ if $1 \le i \le m$ and $p(i) = 1$ if $m+1 \le i \le m+n$. The parity of a_{ij} is $\pi(a_{ij}) = p(i) + p(j)$ mod 2. Then, the ideal $\mathcal{I}_M$ is generated by the relations [59]:

$$\begin{aligned}
& a_{ij}a_{il} = (-1)^{\pi(a_{ij})\pi(a_{il})} q^{(-1)^{p(i)+1}} a_{il}a_{ij}, && \text{for } j < l \\
& a_{ij}a_{kj} = (-1)^{\pi(a_{ij})\pi(a_{kj})} q^{(-1)^{p(j)+1}} a_{kj}a_{ij}, && \text{for } i < k \\
& a_{ij}a_{kl} = (-1)^{\pi(a_{ij})\pi(a_{kl})} a_{kl}a_{ij}, && \text{for } i < k,\ j > l \\
& && \text{or } i > k,\ j < l \\
& a_{ij}a_{kl} - (-1)^{\pi(a_{ij})\pi(a_{kl})} a_{kl}a_{ij} = (-1)^{\pi(a_{ij})\pi(a_{kl})} (q^{-1} - q) a_{kj}a_{il}, & & \\
& && \text{for } i < k, j < l
\end{aligned} \tag{3.22}$$

There is also a comultiplication,

$$\mathrm{M}_q(m|n) \xrightarrow{\Delta} \mathrm{M}_q(m|n) \otimes \mathrm{M}_q(m|n)$$

$$\Delta(a_{ij}) := \sum_k a_{ik} \otimes a_{kj},$$

and a counit,

$$\varepsilon(a_{ij}) = \delta_{ij}.$$

One can define $\mathrm{SL}_q(m|n)$ as the quotient

$$\mathrm{SL}_q(m|n) := \mathrm{M}_q(m|n)/\langle \mathrm{Ber}_q - 1 \rangle$$

where Ber_q denotes the quantum Berezinian (for the definition, see [39, Def. 5.4.6]). $\mathrm{SL}_q(m|n)$ is a Hopf superalgebra, for an explicit description of the antipode see [59] or [39, Appendix E].

Chapter 4

Fundamental theorems of super invariant theory

In this chapter, we discuss the invariant theory of special linear supergroup. In Section 4.1, we recall the classical results for special linear group very briefly. In Section 4.2, we state and prove first fundamental theorem of super invariant theory. In Section 4.3, we recall a classical determinant identity, known as *Jacobi's identity*, and show how (classical) *Plücker relations* can be reconstructed from this identity. In Section 4.4, we obtain a *super Jacobi identity*, a generalization of the Jacobi identity, to the super setting. In Section 4.5, we use a similar strategy as in Section 4.3 to construct *super Plücker relations* and to prove second fundamental theorem of super invariant theory for $\mathrm{SL}(1|1)$. In Section 4.6, we construct what we call *super Plücker relations* for $\mathrm{SL}(r|s)$ using the same technique, however, we conjecture that they are all the relations.

4.1 Fundamental theorems of invariant theory

Consider the polynomial functions on the set of $r \times p$ $(r \leq p)$ complex matrices $\mathrm{M}_{r\times p}$, and the following action of the complex special linear group $\mathrm{SL}_r(\mathbb{C})$:

$$\begin{aligned} \mathbb{C}[\mathrm{M}_{r\times p}] \times \mathrm{SL}_r(\mathbb{C}) &\longrightarrow \mathbb{C}[\mathrm{M}_{r\times p}] \\ (f, g) &\longrightarrow f.g, \end{aligned} \tag{4.1}$$

where $(f.g)(M) := f(gM)$ and $\mathbb{C}[\mathrm{M}_{r\times p}]$ denotes the algebra of polynomials functions on the entries of $\mathrm{M}_{r\times p}$.

Let $X_{i_1\ldots i_r}$ denote the $r \times r$ minor formed with the columns $(i_1, \ldots, i_r)$ of a matrix in $\mathrm{M}_{r\times p}$. The First and Second fundamental Theorems of invariant theory for $\mathrm{SL}_r(\mathbb{C})$

state the following (see [45, 32, 33] for a full account).

Theorem 4.1.1. *Let the notation be as above.*

1. **First Fundamental Theorem (FFT) of invariant theory.** *The ring of invariants* $\mathbb{C}[\mathrm{M}_{r\times p}]^{\mathrm{SL}_r(\mathbb{C})}$ *is generated by the minors* $X_{i_1\dots i_r}$ *of the* $r\times r$ *submatrices in* $\mathrm{M}_{r\times p}$.

2. **Second Fundamental Theorem (SFT) of invariant theory.** *We have a presentation of the ring of invariants via the ideal* I *of the Plücker relations:*

$$\mathbb{C}[\mathrm{M}_{r\times p}]^{\mathrm{SL}_r(\mathbb{C})} \cong \mathbb{C}[X_{i_1\dots i_r}] \,/\, \mathrm{I}, \qquad \text{with}$$

$$\mathrm{I} := \left(\sum_{k=1}^{r+1} (-1)^k X_{i_1\dots i_{r-1} j_k} X_{j_1\dots \tilde{\jmath}_k \dots j_{r+1}} \right), \tag{4.2}$$

where,

$$1 \le i_1 < \cdots < i_{r-1} \le p,$$
$$1 \le j_1 < \cdots < \tilde{\jmath}_k < \cdots < j_{r+1} \le p,$$

and $\tilde{\jmath}_k$ *means that the index is removed.*

Proof. See [33, §9.2, Prop. 2]. □

4.2 First fundamental theorem for $\mathrm{SL}(r|s)$

As we just observed that the First Fundamental Theorem of ordinary invariant theory realizes the ring of invariants with respect to this action, as the subring of $\mathbb{C}[\mathrm{M}_{r\times p}]$ generated by the determinants of the $r\times r$ minors in $\mathrm{M}_{r\times p}(\mathbb{C})$.

We now extend the action (4.1) to the super setting and compute the superalgebra of invariants. Let $\mathrm{M}_{r|s\times p|q}$ be the super vector space of supermatrices and let $\mathcal{O}$ be the superalgebra of 'functions' on $\mathrm{M}_{r|s\times p|q}$. For notational purposes, let us organize the generators of $\mathcal{O}$ (or 'coordinates') in a supermatrix, as follows:

$$A = \left(\begin{array}{ccc|ccc} x_{11} & ... & x_{1p} & \alpha_{1\hat{1}} & ... & \alpha_{1\hat{q}} \\ . & ... & . & . & ... & . \\ . & ... & . & . & ... & . \\ x_{r1} & ... & x_{rp} & \alpha_{r\hat{1}} & ... & \alpha_{r\hat{q}} \\ \hline \beta_{\hat{1}1} & ... & \beta_{\hat{1}p} & y_{\hat{1}\hat{1}} & ... & y_{\hat{1}\hat{q}} \\ . & ... & . & . & ... & . \\ . & ... & . & . & ... & . \\ \beta_{\hat{s}1} & ... & \beta_{\hat{s}p} & y_{\hat{s}\hat{1}} & ... & y_{\hat{s}\hat{q}} \end{array}\right), \tag{4.3}$$

where the hat denotes the odd indices. Then,

$$\mathcal{O} = k[x_{ij}, y_{\hat{k}\hat{l}}, \alpha_{i\hat{k}}, \beta_{\hat{l}j}],$$

where x_{ij} and $y_{\hat{k}\hat{l}}$ are even variables while $\alpha_{i\hat{k}}$ and $\beta_{\hat{l}j}$ are odd variables.

Let $\mathcal{A}$ be a commutative superalgebra. For every supermatrix in the set of $\mathcal{A}$-points of $\mathrm{M}_{r|s\times p|q}$, $M \in \mathrm{M}_{r|s\times p|q}(\mathcal{A})$ and $f \in \mathcal{O}$ one can compute an element of $\mathcal{A}$. This is denoted, in the terminology of the A-points, as $f(M)$ instead of the, perhaps more appropriate, $M(f)$.

Let $\mathrm{SL}(r|s)$ be the special linear supergroup (as introduced in Example 3.5.3). Its $\mathcal{A}$-points are the matrices in $\mathrm{GL}(r|s)(\mathcal{A})$ with Berezinian equal to 1. In terms of $\mathcal{A}$-points, one can write a left action of the special linear supergroup on the supermatrices:

$$\begin{array}{ccc} \mathrm{SL}(r|s)(\mathcal{A}) \times \mathrm{M}_{r|s\times p|q}(\mathcal{A}) & \longrightarrow & \mathrm{M}_{r|s\times p|q}(\mathcal{A}) \\ (g, M) & \longrightarrow & gM. \end{array}$$

One can also define a right action of $\mathrm{SL}(r|s)$ on $\mathcal{O}$, exactly as in the ordinary case (4.1). For $g \in \mathrm{SL}(r|s)(\mathcal{A})$ and $f \in \mathcal{O}$ one has,

$$(f.g)(M) := f(gM), \quad M \in \mathrm{M}_{r|s\times p|q}(\mathcal{A}).$$

We want to be able to define the Berezinians of all $r|s \times r|s$ minors appearing in the matrix (4.3). We then have to localize $\mathcal{O}$ at the determinants:

$$D_{i_1,..,i_r} := \det \begin{pmatrix} x_{1i_1} & ... & x_{1i_r} \\ . & ... & . \\ . & ... & . \\ x_{ri_1} & ... & x_{ri_r} \end{pmatrix}, \qquad D_{\hat{j}_1\cdots\hat{j}_s} := \det \begin{pmatrix} y_{\hat{1}\hat{j}_1} & \cdots & y_{\hat{1}\hat{j}_s} \\ . & \cdots & . \\ . & \cdots & . \\ y_{\hat{s}\hat{j}_1} & \cdots & y_{\hat{s}\hat{j}_s} \end{pmatrix},$$

in the diagonal blocks, in the expression (4.3). We can now look at the action of the supergroup $\mathrm{SL}(r|s)$ on the localization:

$$\tilde{\mathcal{O}} := \mathcal{O}[D^{-1}_{i_1\cdots i_r}, D^{-1}_{\hat{\jmath}_1\cdots\hat{\jmath}_s}]. \tag{4.4}$$

where $1 \leq i_1 \leq \cdots \leq i_r \leq p$ and $1 \leq j_1 \leq \cdots \leq j_s \leq q$. This corresponds to take the subset $\tilde{\mathrm{M}}_{r|s\times p|q}(\mathcal{A}) \subset \mathrm{M}_{r|s\times p|q}(\mathcal{A})$ where all the corresponding minors are invertible.

Remark 4.2.1. In general, localization of noncommutative rings is not always possible. One needs the extra *Ore condition*. However, in our case, the set $[D^{-1}_{i_1\cdots i_r}, D^{-1}_{\hat{\jmath}_1\cdots\hat{\jmath}_s}]$ contains even elements which are regular and central. Therefore, the Ore condition is trivially satisfied.

In this way we can formulate the super version of the First Fundamental Theorem of invariant theory.

Theorem 4.2.2. *Let $A_{i_1\cdots i_r|\hat{\jmath}_1\cdots\hat{\jmath}_s}$ be the supermatrix formed with the columns $(i_1,\dots,i_r|\hat{\jmath}_1,\dots,\hat{\jmath}_s)$ of the supermatrix A in (4.3). Then, the ring of invariants $\tilde{\mathcal{O}}^{\mathrm{SL}(r|s)}$ is generated by the superminors:*

$$X_{i_1\cdots i_r|\hat{\jmath}_1\cdots\hat{\jmath}_s} := \mathrm{Ber}A_{i_1\cdots i_r|\hat{\jmath}_1\cdots\hat{\jmath}_s} \tag{4.5}$$
$$X^*_{i_1\cdots i_r|\hat{\jmath}_1\cdots\hat{\jmath}_s} := \mathrm{Ber}^*A_{i_1\cdots i_r|\hat{\jmath}_1\cdots\hat{\jmath}_s} \tag{4.6}$$

and the 'fake' superminors,

$$X_{i_1\cdots i_{k-1}\hat{\jmath}_k i_{k+1}\cdots i_r|\hat{\jmath}_1\cdots\hat{\jmath}_s} := \mathrm{Ber}A_{i_1\cdots i_{k-1}\hat{\jmath}_k i_{k+1}\cdots i_r|\hat{\jmath}_1\cdots\hat{\jmath}_s} \tag{4.7}$$
$$X^*_{i_1\cdots i_r|\hat{\jmath}_1\cdots\hat{\jmath}_{l-1} i_l \hat{\jmath}_{l+1}\cdots\hat{\jmath}_s} := \mathrm{Ber}^*A_{i_1\cdots i_r|\hat{\jmath}_1\cdots\hat{\jmath}_{l-1} i_l \hat{\jmath}_{l+1}\cdots\hat{\jmath}_s} \tag{4.8}$$

where $1 \leq i_1 \leq \cdots \leq i_r \leq p$ and $1 \leq \hat{\jmath}_1 \leq \cdots \leq \hat{\jmath}_s \leq q$.

Proof. It is clear from the multiplicative property of Berezinian (see Theorem 3.2.3), that superminors and fake superminors are invariants.

To show that they generate the ring $\tilde{\mathcal{O}}^{\mathrm{SL}(r|s)}$, we decompose the matrix of coordinate functions (4.3) as $A = \tilde{A}_{1\cdots r|\hat{1}\cdots\hat{s}}B$, where

$$\tilde{A}_{1\cdots r|\hat{1}\cdots\hat{s}} = \left(\begin{array}{ccc|ccc} \dfrac{x_{11}}{X_{1\cdots r|\hat{1}\cdots\hat{s}}} & \cdots & x_{1r} & \alpha_{1\hat{1}} & \cdots & \alpha_{1\hat{s}} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \dfrac{x_{r1}}{X_{1\cdots r|\hat{1}\cdots\hat{s}}} & \cdots & x_{rr} & \alpha_{r\hat{1}} & \cdots & \alpha_{r\hat{s}} \\ \hline \dfrac{\beta_{\hat{1}1}}{X_{1\cdots r|\hat{1}\cdots\hat{s}}} & \cdots & \beta_{\hat{1}r} & y_{\hat{1}\hat{1}} & \cdots & y_{\hat{1}\hat{s}} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \dfrac{\beta_{\hat{s}1}}{X_{1\cdots r|\hat{1}\cdots\hat{s}}} & \cdots & \beta_{\hat{s}r} & y_{\hat{s}\hat{1}} & \cdots & y_{\hat{s}\hat{s}} \end{array}\right),$$

so we have

$$B = \tilde{A}^{-1}_{1\cdots r|\hat{1}\cdots\hat{s}}A.$$

Using super Cramer's rule we can find the entries of $B = \left(\begin{array}{c|c} U & V \\ \hline W & Z \end{array}\right)$:

$$U_{ij} = \begin{cases} X_{j\cdots r|\hat{1}\cdots\hat{s}} & \text{for } i = 1, 1 \leq j \leq p, \\ X^*_{1\cdots r|\hat{1}\cdots\hat{s}} X_{1\cdots(i-1)j(i+1)\cdots r|\hat{1}\cdots\hat{s}} & \text{for } 1 < i \leq r,\ 1 \leq j \leq p \end{cases} \tag{4.9}$$

$$W_{\hat{k}j} = X_{1\cdots r|\hat{1}\cdots\hat{s}} X^*_{1\cdots r|\hat{1}\cdots(\hat{k}-\hat{1})j(\hat{k}+\hat{1})\cdots\hat{s}} \qquad \text{for } 1 \leq k \leq s,\ 1 \leq j \leq p. \tag{4.10}$$

$$V_{i\hat{l}} = \begin{cases} X_{\hat{l}\cdots r|\hat{1}\cdots\hat{s}} & \text{for } i = 1,\ 1 \leq l \leq q, \\ X^*_{1\cdots r|\hat{1}\cdots\hat{s}} X_{1\cdots(i-1)\hat{l}(i+1)\cdots r|\hat{1}\cdots\hat{s}} & \text{for } 1 < i \leq r,\ 1 \leq l \leq q. \end{cases} \tag{4.11}$$

$$Z_{\hat{k}\hat{l}} = X_{1\cdots r|\hat{1}\cdots\hat{s}} X^*_{1\cdots r|\hat{1}\cdots(\hat{k}-\hat{1})\hat{l}(\hat{k}+\hat{1})\cdots\hat{s}} \qquad \text{for } 1 \leq k \leq s,\ 1 \leq l \leq q. \tag{4.12}$$

Let $\mathcal{A}$ be any superalgebra and consider an arbitrary element $T \in \tilde{M}_{r|s\times p|q}(\mathcal{A})$. We can write T as a product,

$$T = \tilde{T}_{1\cdots r|\hat{1}\cdots\hat{s}}R \tag{4.13}$$

where, $\tilde{T}_{1\cdots r|\hat{1}\cdots\hat{s}}$ is an element of $\mathrm{SL}(r|s)(\mathcal{A})$ and $R \in \tilde{M}_{r|s\times p|q}(\mathcal{A})$ computed in terms of superminors and fake superminors. Explicitly, each entry of R can be computed by evaluating the corresponding superminor or fake superminor as in B above.

Let $f \in \tilde{\mathcal{O}}^{\mathrm{SL}(r|s)}$ be an invariant. Then, by definition, we have:

$$f.g(R) = f(gR) = f(R), \qquad \text{for all } R \in \tilde{M}_{r|s\times p|q}(\mathcal{A}) \text{ and } g \in \mathrm{SL}(r|s)(\mathcal{A}).$$

But then, for any $T \in \tilde{M}_{r|s\times p|q}(\mathcal{A})$,

$$f(T) = f(\tilde{T}_{1\cdots r|\hat{1}\cdots\hat{s}}R) = f(R),$$

where R is as in (4.13). Therefore, f is a function of superminors and fake superminors, as we wanted to prove. □

4.3 Classical Plücker relations revisited

In this section, we revisit the classical Plücker relations. We start by recalling a determinant identity due to Jacobi, the *Jacobi complementary minor theorem* [56] or the *Jacobi identity* for short.

Theorem 4.3.1. *(The Jacobi complementary minor theorem) Let A be an invertible $n \times n$ matrix. Fix the two sets of indices:*

$$u = (n - r + 1, ..., n) \qquad \textit{and} \qquad v = (1, ..., r)$$

Let $\tilde{u}$ and $\tilde{v}$ denote the complements of u and v respectively in $(1, \dots, n)$. Then:

$$\det A \det(A^{-1})^{\tilde{v}}_{\tilde{u}} = (-1)^{r(n+1)} \det A^u_v \tag{4.14}$$

where A^u_v denotes the matrix obtained from A by deleting its rows and columns whose indices are contained in u and v, respectively.

Proof. We denote $A = \left[a_{ij}\right]_{n\times n}$ and, $A^{-1} = \left[b_{kl}\right]_{n\times n}$. Therefore,

$$(A^{-1})^{\tilde{v}}_{\tilde{u}} = \begin{pmatrix} b_{1(n-r+1)} & \cdots & b_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \ddots \\ b_{r(n-r+1)} & \cdots & b_{rn} \end{pmatrix}.$$

Notice that,

$$\det(A^{-1})_{\tilde{u}}^{\tilde{v}} = \det \begin{pmatrix} b_{1(n-r+1)} & \cdots & b_{1n} \\ . & \cdots & . \\ . & \cdots & . \\ b_{r(n-r+1)} & \cdots & b_{rn} \end{pmatrix} =$$

$$\det \begin{pmatrix} b_{1(n-r+1)} & \cdots & b_{1n} & 0 & \cdots & 0 \\ . & \cdots & . & . & \cdots & . \\ . & \cdots & . & . & \cdots & . \\ b_{r(n-r+1)} & \cdots & b_{rn} & 0 & \cdots & 0 \\ b_{(r+1)(n-r+1)} & \cdots & b_{(r+1)n} & 1 & \cdots & 0 \\ . & \cdots & . & . & \cdots & . \\ . & \cdots & . & . & \cdots & . \\ b_{n(n-r+1)} & \cdots & b_{nn} & 0 & \cdots & 1 \end{pmatrix}_{n\times n} \tag{4.15}$$

Let us denote the $n \times n$ matrix appeared in Equation (4.15) by T. Hence, using the multiplicative property of the determinant we get:

$$\det A \det(A^{-1})_{\tilde{u}}^{\tilde{v}} = \det A \det T = \det(AT) =$$

$$\det \begin{pmatrix} 0 & \cdots & 0 & a_{1,r+1} & \cdots & a_{1n} \\ . & \cdots & . & . & \cdots & . \\ . & \cdots & . & . & \cdots & . \\ 0 & \cdots & 0 & a_{(n-)(r+1)} & \cdots & a_{(n-r)n} \\ 1 & \cdots & 0 & a_{n-r+1,r+1} & \cdots & a_{(n-r+1)n} \\ . & \cdots & . & . & \cdots & . \\ . & \cdots & . & . & \cdots & . \\ 0 & \cdots & 1 & a_{n(r+1)} & \cdots & a_{nn} \end{pmatrix} =$$

$$(-1)^{r(n+1)} \det \begin{pmatrix} a_{1,r+1} & \cdots & a_{1n} \\ . & \cdots & . \\ . & \cdots & . \\ a_{n-r,r+1} & \cdots & a_{(n-r)n} \end{pmatrix} =$$

$$(-1)^{r(n+1)} \det(A_v^u).$$

as desired. □

There are several and more general versions of the Jacobi identity, (see [46, §6] and references therein). However, we show now that the Jacobi identity, as in Theorem 4.3.1, is enough to recover the ordinary Plücker relations as in Equation (4.2) of Theorem 4.1.1.

Proposition 4.3.2. *Let $X_{i_1 \cdots i_r}$ denote the minor of $M \in \mathrm{M}_{r\times p}$ $(r \leq p)$ corresponding to the submatrix formed by the columns $(i_1, \ldots, i_r)$ of M.*

$$\sum_{k=1}^{r+1} (-1)^k X_{i_1 \cdots i_{r-1} j_k} X_{j_1 \cdots \tilde{j}_k \cdots j_{r+1}} = 0$$

come directly from the Jacobi identity (4.14).

Proof. Fix an $r \times p$ matrix:

$$C = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{r1} & a_{r2} & \cdots & a_{rp} \end{pmatrix}.$$

Moreover, fix any two sets $\{i_1, ..., i_r\}$ and $\{j_1, ..., j_r\}$ of ordered indices from $\{1, ..., p\}$ and define the following $2r \times 2r$ matrix:

$$A = \begin{pmatrix} a_{1i_1} & \cdots & a_{1i_r} & a_{1j_1} & \cdots & a_{1j_r} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ a_{ri_1} & \cdots & a_{ri_r} & a_{rj_1} & \cdots & a_{rj_r} \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Clearly, $\det(A) = X_{i_1 \cdots i_r}$. Suppose that $X_{i_1 \cdots i_r}$ is invertible, and fix,

$$u = (r+1, \ldots, 2r), \qquad \text{and} \qquad v = (1, \ldots, r),$$

which implies,

$$\tilde{u} = (1, \ldots, r), \qquad \text{and} \qquad \tilde{v} = (r+1, \ldots, 2r).$$

Therefore,

$$A_v^u = \begin{pmatrix} a_{1j_1} & \cdots & a_{1j_r} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ a_{rj_1} & \cdots & a_{rj_r} \end{pmatrix},$$

which implies, $\det(A_v^u) = X_{j_1\cdots j_r}$. Putting this in Equation (4.14) one gets:

$$X_{j_1\cdots j_r} = (-1)^r X_{i_1\cdots i_r} \det(A^{-1})_{\tilde{u}}^{\tilde{v}}. \tag{4.16}$$

Moreover,

$$(A^{-1})_{\tilde{u}}^{\tilde{v}} = \left[(A^{-1})_{kl}\right]_{\substack{k=1,\dots,r \\ l=r+1,\dots,2r}}.$$

By Cramer's rule, we know that:

$$(A^{-1})_{kl} = \frac{\det A_k(e_l)}{\det(A)}.$$

Therefore, one can easily compute,

$$(A^{-1})_{\tilde{u}}^{\tilde{v}} = \begin{pmatrix} -X_{j_1 i_2\cdots i_r} X^{-1}_{i_1\cdots i_r} & -X_{j_2 i_2\cdots i_r} X^{-1}_{i_1\cdots i_r} & \cdots & -X_{j_r i_2\cdots i_r} X^{-1}_{i_1\cdots i_r} \\ -X_{i_1 j_1 i_3\cdots i_r} X^{-1}_{i_1\cdots i_r} & -X_{i_1 j_2 i_3\cdots i_r} X^{-1}_{i_1\cdots i_r} & \cdots & -X_{i_1 j_r i_3\cdots X_r} X^{-1}_{i_1\cdots X_r} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ -X_{i_1 i_2\cdots j_1 i_r} X^{-1}_{i_1\cdots i_r} & -X_{i_1 i_2\cdots j_2 i_r} X^{-1}_{i_1\cdots i_r} & \cdots & -X_{i_1 i_2\cdots j_r i_r} X^{-1}_{i_1\cdots i_r} \\ -X_{i_1 i_2\cdots i_{r-1} j_1} X^{-1}_{i_1\cdots i_r} & -X_{i_1 i_2\cdots i_{r-1} j_2} X^{-1}_{i_1\cdots i_r} & \cdots & -X_{i_1 i_2\cdots i_{r-1} j_r} X^{-1}_{i_1\cdots i_r} \end{pmatrix}.$$

The Laplace expansion of the determinant, using the first row, gives us:

$$\det(A^{-1})_{\tilde{u}}^{\tilde{v}} = (X_{i_1\cdots i_r})^{-1} \sum_{t=1}^{r} -X_{j_t i_2\cdots i_r} c_{1,t}, \tag{4.17}$$

where $c_{1,t}$ is the cofactor of the entry $\left((A^{-1})_{\tilde{u}}^{\tilde{v}}\right)_{1t}$. Therefore, substituting these in Equation (4.16) we get:

$$X_{j_1\cdots j_r} = (-1)^{r+1} \sum_{t=1}^{r} X_{j_t i_2\cdots i_r} c_{1,t}. \tag{4.18}$$

To calculate $c_{1,1}$, consider the same calculations with a matrix A' where we fix $j_1 = i_1$. This gives us:

$$(A'^{-1})^{\tilde{v}}_{\tilde{u}} = \begin{pmatrix} -1 & -X_{j_2 i_2 \cdots i_r} X^{-1}_{i_1 \cdots i_r} & \cdots & -X_{j_r i_2 .. i_r} X^{-1}_{i_1 \cdots i_r} \\ 0 & -X_{i_1 j_2 i_3 .. i_r} X^{-1}_{i_1 \cdots i_r} & \cdots & -X_{i_1 j_r i_3 .. i_r} X^{-1}_{i_1 \cdots i_r} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ 0 & -X_{i_1 i_2 \cdots j_2 i_r} X^{-1}_{i_1 \cdots i_r} & \cdots & -X_{i_1 \cdots j_r i_r} X^{-1}_{i_1 \cdots i_r} \\ 0 & -X_{i_1 i_2 \cdots i_{r-1} j_2} X^{-1}_{i_1 \cdots i_r} & \cdots & -X_{i_1 i_2 \cdots i_{r-1} j_r} X^{-1}_{i_1 \cdots i_r} \end{pmatrix},$$

and from Equation (4.14) it follows that:

$$\begin{aligned} X_{i_1 j_2 \cdots j_r} &= (-1)^r X_{i_1 \cdots i_r} \det(A'^{-1})^{\tilde{v}}_{\tilde{u}} \\ &= (-1)^{r+1} X_{i_1 \cdots i_r} \, c_{1,1}. \end{aligned}$$

Hence, we get:

$$c_{1,1} = (-1)^{r+1} X_{i_1 j_2 \cdots j_r} (X_{i_1 \cdots i_r})^{-1}.$$

Similarly, one can get:

$$c_{1,t} = (-1)^{r+1} X_{j_1 \cdots j_{t-1} i_1 j_{t+1} \cdots j_r} (X_{i_1 \cdots i_r})^{-1}.$$

By putting this in equation (4.18) we get:

$$X_{i_1 \cdots i_r} X_{j_1 \cdots j_r} = \sum_{t=1}^{r} X_{j_t i_2 \cdots i_r} X_{j_1 \cdots j_{t-1} i_1 j_{t+1} \cdots j_r}, \tag{4.19}$$

which are exactly the Plücker relations in (4.2), up to renaming some indices. □

We explain this result below with an easy example.

Example 4.3.3. Fix a matrix C of size 2×4,

$$C = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix},$$

and set $X_{ij} := \det \begin{pmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{pmatrix}$. Suppose that X_{12} is invertible. Now, consider the following matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \text{which satisfies} \qquad \det(A) = X_{12}.$$

Setting $u = (3,4)$ and $v = (1,2)$, gives us:

$$\det A_v^u = \det \begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix} = X_{34}.$$

Moreover, using Cramer's rule, one can easily compute:

$$(A^{-1})_{\tilde{u}}^{\tilde{v}} = \begin{pmatrix} (A^{-1})_{13} & (A^{-1})_{14} \\ (A^{-1})_{23} & (A^{-1})_{24} \end{pmatrix} = \begin{pmatrix} X_{23}X_{12}^{-1} & X_{24}X_{12}^{-1} \\ -X_{13}X_{12}^{-1} & -X_{14}X_{12}^{-1} \end{pmatrix}.$$

Hence, substituting these values in Jacobi's identity, we get:

$$\begin{aligned} & \det(A_v^u) = \det(A)\det(A^{-1})_{\tilde{u}}^{\tilde{v}}, \\ \Rightarrow \quad & X_{34} = X_{12}\det \begin{pmatrix} X_{23}X_{12}^{-1} & X_{24}X_{12}^{-1} \\ -X_{13}X_{12}^{-1} & -X_{14}X_{12}^{-1} \end{pmatrix}, \\ \Rightarrow \quad & X_{12}X_{34} - X_{13}X_{24} + X_{14}X_{23} = 0, \end{aligned}$$

which is exactly the Plücker relation in this case.

4.4 The super Jacobi identity

In this section we want to obtain a super Jacobi identity, a generalization of the Jacobi identity, Theorem 4.3.1, to the super setting. We first introduce some notation.

Notation. Let A be an invertible supermatrix of size $p|q \times p|q$. Let $u = (p - r + 1, \dots, p|\hat{q} - \hat{s} + \hat{1}, \dots, \hat{q})$ and $v = (1, \dots, r|\hat{1}, \dots, \hat{s})$ be two sets of indices, $r < p$, $\hat{s} < \hat{q}$. Let A_v^u denote the matrix obtained from A by deleting its rows and columns contained in u and v, respectively, *i.e.* by deleting the last r even rows, the last $\hat{s}$ odd rows, first r even columns and first $\hat{s}$ odd columns. Also, set $\tilde{u} = (1, \dots, p - r|\hat{1}, \dots, \hat{q} - \hat{s})$ and $\tilde{v} = (r + 1, \dots, p|\hat{s} + \hat{1}, \dots, \hat{q})$ as the complements of u and v, respectively.

Theorem 4.4.1. *(The super Jacobi Identity) Let the notation be as above. If A_v^u and $(A^{-1})_{\tilde{u}}^{\tilde{v}}$ are invertible, then*

$$\mathrm{Ber}A\,\mathrm{Ber}(A^{-1})_{\tilde{u}}^{\tilde{v}} = (-1)^t \mathrm{Ber}A_v^u, \tag{4.20}$$

where $t = r(p+1) + s(q+1)$.

We call (4.20) the *super Jacobi identity.*

Proof. A similar strategy to what we used in the proof of Theorem 4.3.1 will work here. Let

$$A = \left(\begin{array}{c|c} [a_{ij}] & [a_{i\hat{k}}] \\ \hline [a_{\hat{l}j}] & [a_{\hat{l}\hat{k}}] \end{array}\right) \qquad \text{and} \qquad A^{-1} = \left(\begin{array}{c|c} [b_{ij}] & [b_{i\hat{k}}] \\ \hline [b_{\hat{l}j}] & [b_{\hat{l}\hat{k}}] \end{array}\right).$$

According to our notation,

$$(A^{-1})_{\tilde{u}}^{\tilde{v}} = \left(\begin{array}{ccc|ccc} b_{1(p-r+1)} & \cdots & b_{1p} & b_{1(\hat{q}-\hat{s}+\hat{1})} & \cdots & b_{1\hat{q}} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ b_{r(p-r+1)} & \cdots & b_{rp} & b_{r(\hat{q}-\hat{s}+\hat{1})} & \cdots & b_{r\hat{q}} \\ \hline b_{\hat{1}(p-r+1)} & \cdots & b_{\hat{1}p} & b_{\hat{1}(\hat{q}-\hat{s}+\hat{1})} & \cdots & b_{\hat{1}\hat{q}} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ b_{\hat{s}(p-r+1)} & \cdots & b_{\hat{s}p} & b_{\hat{s}(\hat{q}-\hat{s}+\hat{1})} & \cdots & b_{\hat{s}\hat{q}} \end{array}\right).$$

One can see that:

$$\mathrm{Ber}(A^{-1})_{\tilde{u}}^{\tilde{v}} = \mathrm{Ber}\, T, \tag{4.21}$$

where T is the super matrix of size $p|q \times p|q$ described as follows:

$$T_j = \begin{cases} (A^{-1})_{p-r+j} & \text{for} \quad 1 \leq j \leq r, \\ e_j & \text{for} \quad r+1 \leq j \leq p, \end{cases}$$

where T_j denotes the j-th even column of T, $(A^{-1})_k$ is the k-th even column of A^{-1} and e_j denotes an even vector of size $p|q$, whose entries are all 0 except at the j-th even position, where the entry is 1.

On the other hand,

$$T_{\hat{l}} = \begin{cases} (A^{-1})_{\hat{q}-\hat{s}+\hat{l}} & \text{for} \quad \hat{1} \leq \hat{l} \leq \hat{s}, \\ e_{\hat{l}} & \text{for} \quad \hat{s}+\hat{1} \leq \hat{l} \leq \hat{q}, \end{cases}$$

where $T_{\hat{l}}$ denotes the $\hat{l}$-th odd column of T, $(A^{-1})_{\hat{k}}$ is the $\hat{k}$-th-odd column of A^{-1} and $e_{\hat{l}}$ denotes an odd vector of size $p|q$ whose entries are all 0 except at the $\hat{l}$-th odd position, where the entry is 1.

In other words, T is a supermatrix whose first r even columns coincide with the last r even columns of A^{-1}, the first $\hat{s}$ odd columns coincide with the last $\hat{s}$ odd columns of A^{-1} and the remaining even and odd columns are e_j and $e_{\hat{l}}$, respectively.

Therefore, using Equation (4.21) and the multiplicative property of berezinian, we have:

$$\text{Ber } A \,\text{Ber}(A^{-1})_{\tilde{u}}^{\tilde{v}} = \text{Ber } A \,\text{Ber } T = \text{Ber}(AT). \tag{4.22}$$

Let us denote $C = AT$, then, using above notation:

$$C_j = \begin{cases} e_{p-r+j} & \text{for} \quad 1 \leq j \leq r, \\ A_j & \text{for} \quad r+1 \leq j \leq p, \end{cases}$$

and

$$C_{\hat{l}} = \begin{cases} e_{\hat{q}-\hat{s}+\hat{l}} & \text{for} \quad \hat{1} \leq \hat{l} \leq \hat{s}, \\ A_{\hat{l}} & \text{for} \quad \hat{s}+\hat{1} \leq \hat{l} \leq \hat{q}. \end{cases}$$

Then, one can compute:

$$\text{Ber } A = \text{Ber } C = (-1)^{r(p+1)+s(q+1)} \text{Ber} A_v^u, \tag{4.23}$$

and by putting this into Equation (4.22) the desired identity is obtained. □

4.5 Second fundamental theorem for SL(1|1)

In this section we present the second fundamental theorem for SL(1|1). We want to use the super Jacobi identity as in (4.4.1) to derive the relations among the Berezinians in the superalgebra $\tilde{\mathcal{O}}$ defined in (4.4). We illustrate in detail in this section the case of $r = s = 1$, the general case will be discussed shortly, in the next section, following the same strategy. Then, we proved that this gives us all the relations among the generators for the case of SL(1|1).

Theorem 4.5.1. *(The super Plücker relations for* SL(1|1)*) Let the notation be as in (4.6), (4.7) and (4.8) for $r = s = 1$. Then:*

$$X_{i|\hat{\mu}} X^*_{i|\hat{\mu}} = 1, \tag{4.24}$$

$$X_{i|\hat{\mu}} X_{j|\hat{\nu}} = \text{Ber} \left(\begin{array}{c|c} X_{j|\hat{\mu}} & X_{\hat{\nu}|\hat{\mu}} \\ \hline X^*_{i|j} & X^*_{i|\hat{\nu}} \end{array} \right), \tag{4.25}$$

$$X_{i|\hat{\mu}} X_{\hat{\lambda}|\hat{\nu}} = \text{Ber} \left(\begin{array}{c|c} X_{\hat{\lambda}|\hat{\mu}} & X_{\hat{\nu}|\hat{\mu}} \\ \hline X^*_{i|\hat{\lambda}} & X^*_{i|\hat{\nu}} \end{array} \right), \tag{4.26}$$

$$X^*_{i|\hat{\mu}} X^*_{j|k} = \text{Ber}^* \left(\begin{array}{c|c} X_{j|\hat{\mu}} & X_{k|\hat{\mu}} \\ \hline X^*_{i|j} & X^*_{i|k} \end{array} \right), \tag{4.27}$$

where the Latin (even) indices run from 1 to p and the Greek (odd) indices run from 1 to q.

We call these the *super Plücker relations* for SL(1|1).

Proof. The relations (4.24) follow from the Equation (3.10). Now, for any indices $(i|\hat{\mu})$ and $(j|\hat{\nu})$, fix the following matrix:

$$B = \left(\begin{array}{cc|cc} x_{1i} & x_{1j} & x_{1\hat{\mu}} & x_{1\hat{\nu}} \\ 0 & 1 & 0 & 0 \\ \hline x_{\hat{1}i} & x_{\hat{1}j} & x_{\hat{1}\hat{\mu}} & x_{\hat{1}\hat{\nu}} \\ 0 & 0 & 0 & 1 \end{array}\right).$$

Clearly,

$$\text{Ber } B = X_{i|\hat{\mu}}. \tag{4.28}$$

By setting $u = (2, \hat{2})$ and $v = (1, \hat{1})$, we get:

$$\text{Ber } B_v^u = \text{Ber} \left(\begin{array}{c|c} x_{1j} & x_{1\hat{\nu}} \\ \hline x_{\hat{1}j} & x_{\hat{1}\hat{\nu}} \end{array}\right) = X_{j|\hat{\nu}}. \tag{4.29}$$

Moreover, since $\tilde{u} = (1|\hat{1})$ and $\tilde{v} = (2|\hat{2})$, using the super Cramer's rule one can compute:

$$(B^{-1})_{\tilde{u}}^{\tilde{v}} = \left(\begin{array}{c|c} (B^{-1})_{12} & (B^{-1})_{1\hat{2}} \\ \hline (B^{-1})_{\hat{1}2} & (B^{-1})_{\hat{1}\hat{2}} \end{array}\right) = \left(\begin{array}{c|c} -X_{j|\hat{\mu}} X^*_{i|\hat{\mu}} & -X_{\hat{\nu}|\hat{\mu}} X^*_{i|\hat{\mu}} \\ \hline -X^*_{i|j} X_{i|\hat{\mu}} & -X^*_{i|\hat{\nu}} X_{i|\hat{\mu}} \end{array}\right).$$

Hence,

$$\text{Ber}(B^{-1})_{\tilde{u}}^{\tilde{v}} = (X^*_{i|\hat{\mu}})^2 \text{Ber} \left(\begin{array}{c|c} X_{j|\hat{\mu}} & X_{\hat{\nu}|\hat{\mu}} \\ \hline X^*_{i|j} & X^*_{i|\hat{\nu}} \end{array}\right). \tag{4.30}$$

Substituting Equations (4.5.5), (4.29) and (4.30) in the super Jacobi identity, one arrives to the relations (4.25).

For the other relations, we would need to generalize the spirit of the proof for the super Jacobi identity to include some fake supermatrices. Consider the matrix,

$$B = \left(\begin{array}{cc|cc} x_{1i} & 0 & x_{1\hat{\mu}} & x_{1\hat{\nu}} \\ 0 & 1 & 0 & 0 \\ \hline x_{\hat{1}i} & 0 & x_{\hat{1}\hat{\mu}} & x_{\hat{1}\hat{\nu}} \\ 0 & 0 & 0 & 1 \end{array}\right), \tag{4.31}$$

with Ber $B = X_{i|\hat{\mu}}$. We construct a fake supermatrix T in the following way: the first even column is the second even column of B^{-1}, the first odd column is the second odd column of B^{-1}; at the second even column position we have $B^{-1}t$ where

$$t := \left(\begin{array}{c} x_{1\hat{\lambda}} \\ 0 \\ \hline x_{\hat{1}\hat{\lambda}} \\ 0 \end{array}\right).$$

and at the second odd column position we have $e_{\hat{2}}$. Then, clearly:

$$\mathrm{Ber}(BT) = \mathrm{Ber}\left(\begin{array}{cc|cc} 0 & x_{1\hat{\lambda}} & 0 & x_{1\hat{\nu}} \\ 1 & 0 & 0 & 0 \\ \hline 0 & x_{\hat{1}\hat{\lambda}} & 0 & x_{\hat{1}\hat{\nu}} \\ 0 & 0 & 1 & 1 \end{array}\right) = X_{\hat{\lambda}|\hat{\nu}}. \tag{4.32}$$

On the other hand, one can explicitly compute, using super Cramer's rule:

$$T = \left(\begin{array}{cc|cc} 0 & X_{\hat{\lambda}|\hat{\mu}}X^*_{i|\hat{\mu}} & -X_{\hat{\nu}|\hat{\mu}}X^*_{i|\hat{\mu}} & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & X^*_{i|\hat{\lambda}}X_{i|\hat{\mu}} & -X^*_{i|\hat{\nu}}X_{i|\hat{\mu}} & 0 \\ 0 & 0 & 1 & 1 \end{array}\right). \tag{4.33}$$

Therefore,

$$\mathrm{Ber}\ T = (X^*_{i|\hat{\mu}})^2 \mathrm{Ber}\left(\begin{array}{c|c} X_{\hat{\lambda}|\hat{\mu}} & X_{\hat{\nu}|\hat{\mu}} \\ \hline X^*_{i|\hat{\lambda}} & X^*_{i|\hat{\nu}} \end{array}\right). \tag{4.34}$$

Using the multiplicative property of the Berezinian and equation (4.32) one arrives at the relations (4.26). Similarly, the relations (4.27) follows. □

Now, we want to prove that the relations appeared in Equations (4.24), (4.25), (4.26), and (4.27) completely characterize the ring of invariants $\tilde{\mathcal{O}}^{\mathrm{SL}(1|1)}$. Let us recall the following basic fact from linear algebra.

Lemma 4.5.2. *Let $\{v_1, v_2, \cdots\}$ be an ordered (countable) basis of a vector space V. For any non-zero vector $v = \sum_i a_i v_i$, define the leading term of v as $Lt(v) := e_{i_0}$, where $a_{i_0} \neq 0$, and $a_i = 0$ for all $v_i < v_{i_0}$. Then, any collection of vectors in V with distinct leading terms is linearly independent.*

The superalgebra $\tilde{\mathcal{O}}^{\mathrm{SL}(1|1)}$ is generated by $X_{i|\hat{\mu}}$, $X_{\hat{\mu}|\hat{\nu}}$, $X^*_{j|\hat{\nu}}$, and $X^*_{i|j}$. Each element of $\tilde{\mathcal{O}}^{\mathrm{SL}(1|1)}$ is a linear combination of expressions of the following form:

$$\prod_{i|\hat{\mu},j|\hat{\lambda},\hat{\eta}|\hat{\nu},k|l} X_{i|\hat{\mu}}^{d_{i|\hat{\mu}}} X_{\hat{\eta}|\hat{\nu}}^{d_{\hat{\eta}|\hat{\nu}}} X_{j|\hat{\lambda}}^{*d_{j|\hat{\lambda}}} X_{k|l}^{*d_{k|l}}, \tag{4.35}$$

where $d_{i|\hat{\mu}}, d_{j|\hat{\lambda}} \in \{0,1,2,...\}$ and $d_{\hat{\eta}|\hat{\nu}}, d_{k|l} \in \{0,1\}$. However, super Plücker relations implies:

$$X_{i|\hat{\mu}} = X^*_{1|\hat{1}} X_{1|\hat{\mu}} X_{i|\hat{1}} - X^*_{1|\hat{1}} X_{\hat{\mu}|\hat{1}} X^2_{1|\hat{\mu}} X^*_{1|i}, \tag{4.36}$$

$$X^*_{j|\hat{\lambda}} = X_{1|\hat{1}} X^*_{j|\hat{1}} X^*_{1|\hat{\lambda}} - X_{1|\hat{1}} X^*_{1|j} X^{*2}_{j|\hat{1}} X_{\hat{\lambda}|\hat{1}}, \tag{4.37}$$

$$X_{\hat{\eta}|\hat{\nu}} = X^*_{1|\hat{1}} X_{1|\hat{\nu}} X_{\hat{\eta}|\hat{1}} - X^*_{1|\hat{1}} X_{\hat{\nu}|\hat{1}} X^2_{1|\hat{\nu}} X^*_{1|\hat{\eta}}, \tag{4.38}$$

$$X^*_{k|l} = X_{1|\hat{1}} X^*_{k|\hat{1}} X^*_{1|l} - X_{1|\hat{1}} X^*_{1|k} X^{*2}_{k|\hat{1}} X_{l|\hat{1}}. \tag{4.39}$$

Therefore, every expression in (4.35) can be written and arranged as a linear combination of the expression of the following form:

$$P = P_{1|\underline{\hat{\mu}}} P_{\underline{i}|\hat{1}} P_{\underline{\hat{\eta}}|\hat{1}} P^*_{1|\underline{\hat{\lambda}}} P^*_{\underline{j}|\hat{1}} P^*_{1|\underline{l}}, \tag{4.40}$$

where:

$$P_{1|\underline{\hat{\mu}}} := X_{1|\hat{\mu}_1}^{d_{\hat{\mu}_1}} \cdots X_{1|\hat{\mu}_\alpha}^{d_{\hat{\mu}_\alpha}}, \qquad P_{\underline{i}|\hat{1}} := X_{i_1|\hat{1}}^{d_{i_1}} \cdots X_{i_a|\hat{1}}^{d_{i_a}}, \qquad P_{\underline{\hat{\eta}}|\hat{1}} := X_{\hat{\eta}_1|\hat{1}} \cdots X_{\hat{\eta}_\beta|\hat{1}},$$

$$P^*_{1|\underline{\hat{\lambda}}} := X_{1|\hat{\lambda}_1}^{*d_{\hat{\lambda}_1}} \cdots X_{1|\hat{\lambda}_\gamma}^{*d_{\hat{\lambda}_\gamma}}, \qquad P^*_{\underline{j}|\hat{1}} := X_{j_1|\hat{1}}^{*d_{j_1}} \cdots X_{j_b|\hat{1}}^{*d_{j_b}}, \qquad P^*_{1|\underline{l}} := X^*_{1|l_1} \cdots X^*_{1|l_c},$$

and,

$$1 \leq \mu_1 < \cdots < \mu_\alpha \leq q, \qquad 1 < i_1 < \cdots < i_a \leq p,$$

$$1 < \eta_1 < \cdots < \eta_\beta \leq q, \qquad 1 \leq \lambda_1 < \cdots < \lambda_\gamma \leq q,$$

$$1 < j_1 < \cdots < j_b \leq p, \qquad 1 < l_1 < \cdots < l_c \leq p,$$

such that:

- $\mu_\epsilon \neq \lambda_\delta$ for any $\epsilon, \delta \in \{1, \cdots, q\}$,
- $i_m \neq j_n$ for any $m, n \in \{2, \cdots, p\}$.

Definition 4.5.3. We call the generators in the set

$$\{X_{1|\hat{\mu}}, X_{i|\hat{1}}, X^*_{1|\hat{\mu}}, X^*_{i|\hat{1}}, X_{\hat{\lambda}|\hat{1}}, X^*_{1|j} | 1 \leq i,j \leq p,\ 2 \leq \mu, \lambda \leq q\}$$

as *standard algebra generators* for and the expressions P in (4.40) as *standard products*.

Lemma 4.5.4. *The standard products are linearly independent and hence forms a vector space basis of* $\tilde{\mathcal{O}}^{\mathrm{SL}(1|1)}$.

Proof. Let $\mathcal{B}$ be the monomial basis of $\tilde{\mathcal{O}}$, i.e. it contains the expressions of the following form:

$$\Pi_{(i,j)} x_{ij}^{a_{ij}} (x_{ij}^{-1})^{b_{ij}} \Pi_{(k,l)} x_{kl}^{c_{kl}},$$

where:

- x_{ij} and x_{kl} denotes even and odd generators respectively,
- $a_{ij}, b_{ij} \in \{0,1,2,\cdots\}$, $c_{kl} \in \{0,1\}$,
- for a fixed (i,j) at least one of a_{ij} or b_{ij} is zero.

Now, consider the following ordering of tuples,

$$(1,1) < \cdots < (1,p) < (\hat{1},\hat{1}) < \cdots < (\hat{1},\hat{q})$$
$$< (1,1)^{-1} < \cdots < (1,p)^{-1} < (\hat{1},\hat{1})^{-1} < \cdots < (\hat{1},\hat{q})^{-1}$$
$$< (1,\hat{1}) < \cdots < (1,\hat{q}) < (\hat{1},1) < \cdots < (\hat{1},p),$$

where $(i,j)^{-1}$ is just a notation for denoting the generators x_{ij}^{-1}. This gives us a total ordering on the basis $\mathcal{B}$ of $\tilde{\mathcal{O}}$ as follows:

$$\Pi_{(i,j)} x_{ij}^{a_{ij}} (x_{ij}^{-1})^{b_{ij}} \Pi_{(k,l)} x_{kl}^{c_{kl}} < \Pi_{(i,j)} x_{ij}^{a'_{ij}} (x_{ij}^{-1})^{b'_{ij}} \Pi_{(k,l)} x_{kl}^{c'_{kl}}$$

if one of the following holds:

(i) $\exists$ an (i,j) for which $a_{ij} \neq a'_{ij}$ and (i,j) is smallest such tuple, then $a_{ij} > a'_{ij}$.

(ii) $a_{ij} = a'_{ij}$ for all (i,j), and $\exists$ and (i,j) for which $b_{ij} \neq b'_{ij}$ and (i,j) is smallest such tuple then, $b_{ij} < b'_{ij}$.

(iii) $a_{ij} = a'_{ij}$ and $b_{ij} = b'_{ij}$ for all (i,j), and (k,l) is smallest tuple for which $c_{kl} \neq c'_{kl}$, then $c_{kl} > c'_{kl}$.

With this ordering the leading terms of $P_{1|\underline{\hat{\mu}}}$, $P_{\underline{i}|\hat{1}}$, $P_{\underline{\hat{\eta}}|\hat{1}}$, $P^*_{1|\underline{\hat{\lambda}}}$, $P^*_{\underline{j}|\hat{1}}$, $P^*_{1|\underline{l}}$ turns out to be:

$$Lt(P_{1|\underline{\hat{\mu}}}) = x_{11}^{d_{\underline{\mu}}}(x^{-1}_{\hat{1}\hat{\mu}_1})^{d_{\hat{\mu}_1}}...(x^{-1}_{\hat{1}\hat{\mu}_\alpha})^{d_{\hat{\mu}_\alpha}}, \qquad Lt(P_{\underline{i}|\hat{1}}) = x_{1i_1}^{d_{i_1}}...x_{1i_a}^{d_{i_a}}(x^{-1}_{\hat{1},\hat{1}})^{d_{\underline{i}}},$$

$$Lt(P_{\underline{\hat{\eta}}|\hat{1}}) = (x^{-1}_{\hat{1}\hat{1}})^{\beta}x_{1\hat{\eta}_1}...x_{1\hat{\eta}_\beta}, \qquad Lt(P^*_{1|\underline{\hat{\lambda}}}) = x_{\hat{1}\hat{\lambda}_1}^{d_{\hat{\lambda}_1}}...x_{\hat{1}\hat{\lambda}_\gamma}^{d_{\hat{\lambda}_\gamma}}(x^{-1}_{11})^{d_{\underline{\hat{\lambda}}}},$$

$$Lt(P^*_{\underline{j}|\hat{1}}) = x_{\hat{1}\hat{1}}^{d_{\underline{j}}}(x^{-1}_{1j_1})^{d_{j_1}}...(x^{-1}_{1j_1})^{d_{j_b}}, \qquad Lt(P^*_{1|\underline{l}}) = (x^{-1}_{11})^{c}x_{\hat{1}l_1}...x_{\hat{1}l_c},$$

and the leading term $Lt(P)$ of the standard generator $P = P_{1|\underline{\hat{\mu}}}P_{\underline{i}|\hat{1}}P_{\underline{\hat{\eta}}|\hat{1}}P^*_{1|\underline{\hat{\lambda}}}P^*_{\underline{j}|\hat{1}}P^*_{1|\underline{l}}$ is the product of these leading terms. Therefore, distinct standard generators have distinct leading terms, and, by Lemma 4.5.2, they are linearly independent. □

The lemma 4.5.4 implies that the super Plücker relations are all the algebraic relations among the generators. We state it precisely in the theorem below.

Theorem 4.5.5. *(Second fundamental theorem for* SL(1|1)*) The super Plücker relations given in equations* (4.24), (4.25), (4.26) *and* (4.27) *completely characterize the ring of invariants* $\tilde{\mathcal{O}}^{\mathrm{SL}(1|1)}$.

Proof. Let R be another relation, say:

$$\sum a_I S_I = 0. \tag{4.41}$$

where $a_I \in \mathbb{C}$ are scalars, and S_I denotes some products of the generators $X_{i|\hat{\mu}}$, $X_{\hat{\mu}|\hat{\nu}}$, $X^*_{j|\hat{\nu}}$, and $X^*_{i|j}$ and I is some muti-index encoding the generators in the product S_I.

Now, according to the super Plücker relations, as explicitly written in Equations (4.36), (4.37), (4.38), and (4.39), we can turn R into an expression in the standard algebra generators (see Definition 4.5.3). But, then all the expressions in standard algebra generators can be reordered as standard products (see Definition 4.5.3), which are linearly independent (as proved in Lemma 4.5.4), therefore, all the scalars a_I are zero, which completes the proof. □

4.6 Super Plücker relations for SL(r|s)

In this section, we construct super Plücker relations for SL(r|s) using the same approach as we did in the previous section for the case when $r = s = 1$.

Theorem 4.6.1. *Let the notation be as above. The ring of invariants $\tilde{\mathcal{O}}^{\mathrm{SL}(r|s)}$ is characterized by the following relations:*

$$X_{\underline{i}|\underline{\hat{\mu}}} X^*_{\underline{i}|\underline{\hat{\mu}}} = 1 \tag{4.42}$$

$$(X_{\underline{i}|\underline{\hat{\mu}}})^{r+s-1} X_{\underline{j}|\underline{\hat{\nu}}} = \mathrm{Ber} \left(\begin{array}{c|c} X_{\underline{i}_a(j_t)|\underline{\hat{\mu}}} & X_{\underline{i}_a(\nu_\beta)|\underline{\hat{\mu}}} \\ \hline X^*_{\underline{i}|\underline{\hat{\mu}}_\alpha(j_t)} & X^*_{\underline{i}|\underline{\hat{\mu}}_\alpha(\nu_\beta)} \end{array} \right)_{\substack{a,t=1,\dots,r \\ \alpha,\beta=1,\dots,s}} \tag{4.43}$$

$$(X_{\underline{i}|\underline{\hat{\mu}}})^{r+s-1} X_{j_1\dots j_{r-1}\hat{\lambda}|\underline{\hat{\nu}}} = \mathrm{Ber} \left(\begin{array}{cc|c} X_{\underline{i}_a(j_t)|\underline{\hat{\mu}}} & X_{\underline{i}_a(\hat{\lambda})|\underline{\hat{\mu}}} & X_{\underline{i}_a(\nu_\beta)|\underline{\hat{\mu}}} \\ \hline X^*_{\underline{i}|\underline{\hat{\mu}}_\alpha(j_t)} & X^*_{\underline{i}|\underline{\hat{\mu}}_\alpha(\hat{\lambda})} & X^*_{\underline{i}|\underline{\hat{\mu}}_\alpha(\nu_\beta)} \end{array} \right)_{\substack{a=1,\dots,r \\ t=1,\dots,r-1 \\ \alpha,\beta=1,\dots,s}} \tag{4.44}$$

$$(X^*_{\underline{i}|\underline{\hat{\mu}}})^{-r-s+1} X^*_{\underline{j}|\nu_1\dots\nu_{s-1}y} = \mathrm{Ber}^* \left(\begin{array}{c|cc} X_{\underline{i}_a(j_t)|\underline{\hat{\mu}}} & X_{\underline{i}_a(\nu_\beta)|\underline{\hat{\mu}}} & X_{\underline{i}_a(y)|\underline{\hat{\mu}}} \\ X^*_{\underline{i}|\underline{\hat{\mu}}_\alpha(j_t)} & X^*_{\underline{i}|\underline{\hat{\mu}}_\alpha(\nu_\beta)} & X^*_{\underline{i}|\underline{\hat{\mu}}_\alpha(y)} \end{array} \right)_{\substack{a,t=1,\dots,r \\ \alpha=1,\dots,s \\ \beta=1,\dots,s-1}} \tag{4.45}$$

where $\underline{i}_a(j_t)|\underline{\mu}$ denotes replacing the a-th index of $\underline{i} = i_1, ..., i_r$ with j_t and similarly others. We call these the *super Plücker relations* for SL$(r|s)$.

Proof. For any ordered indices $\underline{i}|\underline{\hat{\mu}} = (i_1, ..., i_r|\hat{\mu}_1, ..., \hat{\mu}_s)$ and $\underline{j}|\underline{\hat{\nu}} = (j_1, ..., j_r|\hat{\nu}_1, ..., \hat{\nu}_s)$ taken from $(1, \dots, p|\hat{1}, \dots, \hat{q})$, fix the following matrix of size $2r|2s \times 2r|2s$:

$$A = \left(\begin{array}{cc|cc} A_1 & A_2 & A_3 & A_4 \\ O & I_r & O & O \\ \hline A_5 & A_6 & A_7 & A_8 \\ O & O & O & I_s \end{array} \right).$$

where O and I denotes the null and identity matrices respectively and

$$A_1 = \left[x_{ki_l}\right], \quad A_2 = \left[x_{kj_l}\right], \quad A_3 = \left[x_{k\hat{\mu}_n}\right], \quad A_4 = \left[x_{k\hat{\nu}_n}\right],$$
$$A_5 = \left[x_{\hat{m}i_l}\right], \quad A_6 = \left[x_{\hat{m}j_l}\right], \quad A_7 = \left[x_{\hat{m}\hat{\mu}_n}\right], \quad A_8 = \left[x_{\hat{m}\hat{\nu}_n}\right],$$

where $k, l = 1, ..., r$ and $m, n = 1, ..., s$. Then, clearly,

$$\mathrm{Ber}(A) = X_{\underline{i}|\underline{\hat{\mu}}}. \tag{4.46}$$

Moreover, setting,

$$u = (r+1, \ldots, 2r|s \hat{+} 1, \ldots, \hat{2s}), \qquad \text{and} \qquad v = (1, \ldots, r|\hat{1}, \ldots \hat{s}),$$

gives us:

$$\mathrm{Ber}(A_v^u) = \mathrm{Ber}\left(\begin{array}{c|c} A_2 & A_4 \\ \hline A_6 & A_8 \end{array}\right) = X_{\underline{j}|\underline{\hat{\nu}}}. \tag{4.47}$$

On the other hand:

$$(A^{-1})_{\tilde{u}}^{\tilde{v}} = \left(\begin{array}{c|c} (A^{-1})_{ab} & (A^{-1})_{a\hat{d}} \\ \hline (A^{-1})_{\hat{c}b} & (A^{-1})_{\hat{c}\hat{d}} \end{array}\right),$$

where $a = 1, ..., r$, $b = r+1, ..., 2r$, $c = 1, ..., s$ and $d = s+1, ..., 2s$.

These entries of A^{-1} can be computed using super Cramer's rule. For instance,

$$(A^{-1})_{ab} = \frac{\mathrm{Ber}(A_a(e_b))}{\mathrm{Ber}(A)} = -X_{\underline{i}_a(j_t)|\underline{\hat{\mu}}} X^*_{\underline{i}|\underline{\hat{\mu}}},$$

where $t = b - r$. Similarly, the other entries are obtained and we get:

$$(A^{-1})_{\tilde{u}}^{\tilde{v}} = \left(\begin{array}{c|c} -X_{\underline{i}_a(j_t)|\underline{\hat{\mu}}} X^*_{\underline{i}|\underline{\hat{\mu}}} & -X_{\underline{i}_a(\nu_\beta)|\underline{\hat{\mu}}} X^*_{\underline{i}|\underline{\hat{\mu}}} \\ \hline -X^*_{\underline{i}|\underline{\hat{\mu}}_\alpha(j_t)} X_{\underline{i}|\underline{\hat{\mu}}} & -X^*_{\underline{i}|\underline{\hat{\mu}}_\alpha(\nu_\beta)} X_{\underline{i}|\underline{\hat{\mu}}} \end{array}\right)_{\substack{a,t=1,\ldots,r \\ \alpha,\beta=1,\ldots,s}}. \tag{4.48}$$

Substituting Eqs. (4.46), (4.47) and (4.48) into the super Jacobi identity, we get relations in Eq. (4.43). Similarly, the other super Plücker relations in Eqs. (4.44) and (4.45) are obtained by modifying the strategy used in the proof of super Jacobi identity, as we explicitly presented in the previous section for the case of SL(1|1). □

We end this chapter by stating an open problem that remained unsolved in this thesis, that we plan to settle in future.

Conjecture 4.6.2. *$\tilde{\mathcal{O}}^{\mathrm{SL}(r|s)}$ is isomorphic to the superalgebra generated by $Y_{\underline{i}|\underline{\hat{\mu}}}$, $Y_{\underline{i}_a(\hat{\nu})|\underline{\hat{\mu}}}$, $Y^*_{\underline{i}|\underline{\hat{\mu}}_\alpha(j)}$ and $Y^*_{\underline{i}|\underline{\hat{\mu}}}$ subject to the super Plücker relations given in Equations (4.42), (4.43), (4.44) and (4.45).*

Chapter 5

Complex structures on the full quantum flag manifold $\mathcal{O}_q(\mathrm{F}_3)$

A theory of noncommutative geometry for quantum homogeneous spaces is an important, but a challenging question. In this chapter, we study an important example of quantum full flag manifold: $\mathcal{O}_q(\mathrm{F}_3)$, a quantization of the full flag F_3 in $\mathbb{C}^3$.

The chapter is organized as follows. Section 5.1 is devoted to introduce the notion of *quantum full flag manifold* $\mathcal{O}_q(\mathrm{F}_3)$. In Section 5.2, we introduce a *quantum tangent space* to construct a DC on $\mathcal{O}_q(\mathrm{F}_3)$. In Section 5.3, we calculate the second order exterior relations to explicitly present the quantum exterior algebra for the maximal prolongation of FODC on $\mathcal{O}_q(\mathrm{F}_3)$. In section Section 5.4, we examine covariant almost complex structures on $\mathcal{O}_q(\mathrm{F}_3)$ in this setting.

5.1 The full quantum flag manifold $\mathcal{O}_q(\mathrm{F}_3)$

In this section, we give a quick introduction to the notion of quantum flag manifolds, for more details, see [22, 20, 15, 21, 66].

Let $\mathfrak{sl}_{n+1}\mathbb{C}$ denote the *special linear Lie algebra*, *i.e.* the Lie algebra of *traceless* matrices. We can define its *Drinfeld-Jimbo quantized enveloping algebra* $\mathrm{U}_q(\mathfrak{sl}_{n+1})$ as the algebra generated by the elements E_i, F_i, K_i and K_i^{-1} where $1 \leq i \leq n$, subject to the relations presented in Definition 2.3.7. Also, recall the notion of *quantum special unitary group* $\mathcal{O}_q(\mathrm{SU}_{n+1})$ as presented in Example 2.2.9.

Moreover, as described in Example 2.3.10, there is a dual pairing of Hopf algebras between $\mathrm{U}_q(\mathfrak{sl}_{n+1})$ and $\mathcal{O}_q(\mathrm{SU}_{n+1})$. Therefore, according to the Proposition 2.2.7, this dual pairing gives a natural $\mathrm{U}_q(\mathfrak{sl}_{n+1})$-module structure on $\mathcal{O}_q(\mathrm{SU}_{n+1})$:

$$f \triangleright b := \sum b_{(1)} \langle f, b_{(2)} \rangle \qquad \text{for } f \in \mathrm{U}_q(\mathfrak{sl}_{n+1}),\ b \in \mathcal{O}_q(\mathrm{SU}_{n+1}). \tag{5.1}$$

Now, let $S \subset \Pi$ be a proper subset of simple roots Π in the root system Δ of $\mathfrak{sl}_{n+1}\mathbb{C}$ (recall the Definition 2.3.3), and consider the Hopf subalgebra $\mathrm{U}_q(\mathfrak{l}_S)$ of $\mathrm{U}_q(\mathfrak{sl}_{n+1})$ given by:

$$\mathrm{U}_q(\mathfrak{l}_S) := \langle K_i^{\pm}, E_j, F_j | i = 1, \cdots, n \, ; j \in S \rangle. \tag{5.2}$$

Having set all these notations, we define the notion of *quantum flag manifolds*, as given in [66, 21, 22], in the next definition.

Definition 5.1.1. Let the notation be as above. We define the *quantum flag manifold* $\mathcal{O}_q(\mathrm{SU}_{n+1}/L_S)$, as the subalgebra of $\mathrm{U}_q(\mathfrak{l}_S)$-invariants:

$$\begin{aligned} \mathcal{O}_q(\mathrm{SU}_{n+1}/L_S) &:= {}^{\mathrm{U}_q(\mathfrak{l}_S)}\mathcal{O}_q(\mathrm{SU}_{n+1}), \\ &= \Big\{ b \in \mathcal{O}_q(\mathrm{SU}_{n+1}) \mid w \triangleright b = \varepsilon(w) b,\ \forall\ w \in \mathrm{U}_q(\mathfrak{l}_S) \Big\} \end{aligned}$$

with respect to the natural left $\mathrm{U}_q(\mathfrak{sl}_{n+1})$-module structure on $\mathcal{O}_q(\mathrm{SU}_{n+1})$ (as in Equation (5.1)).

In a similar way, the notion of quantum flag manifolds can be defined for other series in the list of *Dynkin diagrams*. However, from now on, by a quantum flag manifold we will always mean an A-series one, as defined in Definition 5.1.1.

Definition 5.1.2. In case the complement S^c of S in the set of simple roots Π is a singleton set $\{\alpha_r\}$, the corresponding quantum flag manifold is called *the r-plane quantum grassmannian* and is denoted by $\mathcal{O}_q(\mathrm{Gr}_{r,n+1})$. In case $S = \emptyset$, *i.e.* $\mathrm{U}_q(\mathfrak{l}_S) = \mathrm{U}_q(\mathfrak{h}) := \{K_i^{\pm} | i = 1, ..., n\}$, the corresponding quantum flag manifold is called *the full quantum flag manifold* and is denoted by $\mathcal{O}_q(\mathrm{F}_{n+1})$. (See [15]).

These are examples of quantum homogeneous spaces, as described in Definition 2.5.1, see [22, §2.5] and [20, §5.3-5.4], and they provide an interesting deformation of their classical counterparts.

The main object of study in this chapter is the *full quantum flag manifold* $\mathcal{O}_q(\mathrm{F}_3)$, *i.e.* the space of invariants in $\mathcal{O}_q(\mathrm{SU}_3)$ with respect to the restriction of the action of $U_q(\mathfrak{sl}_3)$ to $U_q(\mathfrak{h}) := \{K_i^{\pm}|i = 1,2\}$,

$$\mathcal{O}_q(\mathrm{F}_3) := {}^{U_q(\mathfrak{h})}\mathcal{O}_q(\mathrm{SU}_3).$$

The proposition given below gives us a set of generators for $\mathcal{O}_q(\mathrm{F}_3)$ as a subalgebra of $\mathcal{O}_q(\mathrm{SU}_3)$.

Proposition 5.1.3. *$\mathcal{O}_q(\mathrm{F}_3)$ as a subalgebra of $\mathcal{O}_q(\mathrm{SU}_3)$ is generated by the following elements:*

$$z_{ij}^{\alpha_1} := u_{i1}u_{j1}^* = u_{i1}S(u_{1j}), \qquad z_{ij}^{\alpha_2} := u_{i3}u_{j3}^* = u_{i3}S(u_{3j}), \qquad \text{for } i,j = 1,2,3. \quad (5.3)$$

where u_{kl} are the generators of $\mathcal{O}_q(\mathrm{SU}_3)$ as in Example 2.2.9.

Proof. See [53, Prop. 3.2] for a general proof. □

The subalgebra generated by the elements $z_{ij}^{\alpha_1}$ is the *quantum projective plane* as introduced in Example 2.5.3. Similarly, an isomorphic copy of the quantum projective plane is generated by the elements $z_{ij}^{\alpha_2}$, and together they generate $\mathcal{O}_q(\mathrm{F}_3)$ as an algebra.

5.2 A quantum tangent space for $\mathcal{O}_q(\mathrm{F}_3)$

In this section, we construct a quantum tangent space for $\mathcal{O}_q(\mathrm{F}_3)$, by generalizing a celebrated construction due to Heckenberger and Kolb for the case of quantum grassmannians $\mathcal{O}_q(\mathrm{Gr}_{r,n+1})$ in [54].

The *quantum grassmannian* $\mathcal{O}_q(\mathrm{Gr}_{r,n+1})$ (as in Definition 5.1.2) admits a unique covariant differential calculus $\Omega_q(\mathrm{Gr}_{r,n+1})$ that provides a q-deformation of the classical de Rham complex of grassmannian manifolds, such that the dimension of each homogeneous component coincides with the classical one. (See [54]). Moreover, this differential calculus exhibits a *Kähler structure*. (See [66]). In fact, a more general class known as the *irreducible quantum flag manifolds* is covered by the results published in [54]. Nevertheless, these are precisely the quantum grassmannians for the A-series, which is the sole series we are discussing about in this thesis.

We refer to the calculus $\Omega_q(\mathrm{Gr}_{r,n+1})$ appeared in [54] as the *Heckenberger–Kolb* calculus (abbreviated as *HK* calculus) on quantum grassmannian $\mathcal{O}_q(\mathrm{Gr}_{r,n+1})$. The first-order homogeneous component $\Omega^1_q(\mathrm{Gr}_{r,n+1})$ of the *HK* calculus $\Omega_q(\mathrm{Gr}_{r,n+1})$ can be written as:

$$\Omega^1_q(\mathrm{Gr}_{(n+1,r)}) = \Omega^{(1,0)}_q(\mathrm{Gr}_{r,n+1}) \oplus \Omega^{(0,1)}_q(\mathrm{Gr}_{r,n+1}).$$

We call $\Omega^{(1,0)}_q(\mathrm{Gr}_{r,n+1})$ and $\Omega^{(0,1)}_q(\mathrm{Gr}_{r,n+1})$ the *holomorphic* and *anti-holomorphic* Heckenberger–Kolb FODCi respectively, as they classically correspond to the holomorphic and anti-holomorphic calculi on the complex grassmannain manifold $\mathrm{Gr}_{r,n+1}$, [66].

According to the Theorem 2.8.6, in this setting, there is a 1 : 1 correspondence between finite-dimensional first-order differential calculi and quantum tangent spaces. The quantum tangent spaces corresponding to $\Omega^{(1,0)}_q(\mathrm{Gr}_{r,n+1})$ and $\Omega^{(0,1)}_q(\mathrm{Gr}_{r,n+1})$ are defined by:

$$T^{(1,0)} := \mathrm{U}_q(\mathfrak{l}_S)F_r, \qquad T^{(0,1)} := \mathrm{U}_q(\mathfrak{l}_S)E_r. \tag{5.4}$$

We call $T^{(1,0)}$ and $T^{(0,1)}$ as the *holomorphic* and *anti-holomorphic HK* quantum tangent spaces for $\mathcal{O}_q(\mathrm{Gr}_{r,n+1})$ respectively. (See [15, §4.4]).

Proposition 5.2.1. *Let the notation be as above. Then, we can express the holomorphic and anti-holomorphic quantum tangent spaces for the quantum grassmannian* $\mathcal{O}_q(\mathrm{Gr}_{(n+1,r)})$ *as:*

$$T^{(1,0)} = \mathrm{span}_{\mathbb{C}}\{F_\beta \mid \beta \in \overline{\Delta^+_S}\}, \tag{5.5}$$

$$T^{(0,1)} = \mathrm{span}_{\mathbb{C}}\{E_\beta \mid \beta \in \overline{\Delta^+_S}\}, \tag{5.6}$$

where E_β and F_β are as defined in [15].

Proof. See [54, Prop. 3.3]. □

Now, to define a FODC on $\mathcal{O}_q(\mathrm{F}_3)$, we define below a quantum tangent space T generalizing this construction of Heckenberger and Kolb. The idea is to make use of *Lusztig's root vectors* (see Section 2.4). In fact, this is a general theory given in [15], that for a particular choice of reduced decomposition of the longest element of the *Weyl group*, the space spanned by the *Lusztig's root vectors* is a quantum tangent space for $\mathcal{O}_q(\mathrm{SU}_{n+1})$, whose restriction to the case of $\mathcal{O}_q(\mathrm{Gr}_{r,n+1})$ gives the *anti-holomorphic HK* quantum tangent space. See [15] for further details.

Recall, from the Example 2.4.5, for the case of $\mathfrak{sl}_3\mathbb{C}$, and the choice $w_0 = w_2w_1w_2$ (this is the choice fixed in [15]) of reduced decomposition of the longest element w_0 of the Weyl group $W \cong \mathrm{S}_3$, the list of root vectors is given by:

$$E_{\alpha_1} := E_1, \qquad E_{\alpha_2} := E_2, \qquad \text{and} \qquad E_{\alpha_1+\alpha_2} := [E_2, E_1]_{q^{-1}}. \tag{5.7}$$

where E_1 and E_2 are as in the definition of *Drinfeld-Jimbo quantized enveloping algebra* $\mathrm{U}_q(\mathfrak{sl}_3)$.

Proposition 5.2.2. *Let the notation be as above. Then,*

$$T^{(0,1)} := \mathrm{span}_{\mathbf{C}}\Big\{E_{\alpha_1},\, E_{\alpha_2}, E_{\alpha_1+\alpha_2}\Big\} \tag{5.8}$$

is a quantum tangent space.

Proof. See [15, Corollary 3.4]. □

To study the complex geometry of $\mathcal{O}_q(\mathrm{F}_3)$, we need to generalize the notion of *HK holomorphic quantum tangent space* as appeared in Proposition 5.2.1. For this purpose, we define:

$$T^{(1,0)} := (T^{(0,1)})^*, \tag{5.9}$$

where $*$ is the $*$-structure on $U_q(\mathfrak{sl}_3)$ (see [57]). We see it is spanned by the elements:

$$F_{\alpha_1} := E^*_{\alpha_1} = K_1F_1, \quad F_{\alpha_2} := E^*_{\alpha_2} = K_2F_2, \quad F_{\alpha_1+\alpha_2} := E^*_{\alpha_1+\alpha_2} = q^{-1}K_1K_2[F_1, F_2]_{q^{-1}}.$$

Moreover, by a direct calculation, we can conclude the following coproduct formula,

$$\Delta(F_{\alpha_1+\alpha_2}) = F_{\alpha_1+\alpha_2} \otimes K_1K_2 + \nu F_{\alpha_1} \otimes F_{\alpha_2}K_1 + 1 \otimes F_{\alpha_1+\alpha_2}. \tag{5.10}$$

Definition 5.2.3. We define the $*$-extension T of $T^{(0,1)}$ as:

$$T := T^{(1,0)} \oplus T^{(0,1)}. \tag{5.11}$$

where $T^{(1,0)}$ and $T^{(0,1)}$ are as defined in Equations (5.9) and (5.8) respectively. It is clear from Equation (5.10) and Proposition 5.2.2 that T is a quantum tangent space.

Notation 5.2.4. It follows now from the Theorem 2.8.6, there is a FODC on $\mathcal{O}_q(\mathrm{SU}_3)$ and on $\mathcal{O}_q(\mathrm{F}_3)$ associated to T (as defined in Definition 5.2.3), we denote those by $\Omega^1_q(\mathrm{SU}_3)$ and $\Omega^1_q(\mathrm{F}_3)$ respectively, and their cotangent spaces by Λ^1 and V^1 respectively (see Section 2.8), and the basis of Λ^1 dual to the defining basis of T by,

$$\text{basis of } \Lambda^1 = \Big\{e_\gamma,\, f_\gamma \,|\, \gamma \in \Delta^+\Big\}, \tag{5.12}$$

where $\Delta^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$ is the set of positive roots.

The following lemma gives explicit representatives for the cosets of the dual basis. These representatives will be used in the calculations in Section 5.3.

Remember the notation from Section 2.8, $\Lambda^1 := \mathcal{O}_q(\mathrm{SU}_3)^+/I^{(1)}$ where $I^{(1)}$ is the ideal classifying the FODC corresponding to T.

Lemma 5.2.5. *Let the notation be as in* (5.12)*. It holds that:*

$$\begin{aligned} e_{\alpha_1} &= [u_{21}], & e_{\alpha_2} &= [u_{32}], & e_{\alpha_1+\alpha_2} &= [u_{31}], \\ f_{\alpha_1} &= [qu_{12}], & f_{\alpha_2} &= [qu_{23}], & f_{\alpha_1+\alpha_2} &= [q^2u_{13}]. \end{aligned}$$

where u_{ij} denotes the generators of $\mathcal{O}_q(\mathrm{SU}_3)$ as in Example 2.2.9.

Proof. These are direct calculations. For example, the calculation:

$$\langle F_{\alpha_1}, u_{12}\rangle = \langle K_1F_1, u_{12}\rangle = \langle K_1, u_{11}\rangle\langle F_1, u_{12}\rangle = q^{-1},$$

implies that qu_{12} is a representative for the coset f_{α_1}. Here, $\langle -, -\rangle$ denotes the dual pairing between $\mathcal{O}_q(\mathrm{SU}_3)$ and $\mathrm{U}_q(\mathfrak{sl}_3)$ as described in Example 2.3.10. □

Notation 5.2.6. Let u_{ij} denotes the the generators of $\mathcal{O}_q(\mathrm{SU}_3)$ as in Example 2.2.9 and recall that $\nu := q - q^{-1}$. Let $(-, -)$ denotes the standard inner product on $\mathbb{R}^2$ (the space spanned by the root vectors for the case of $\mathfrak{sl}_3(\mathbb{C})$). Moreover, let us fix the set of simple roots:

$$\alpha_1 := \alpha_{12} := \varepsilon_1 - \varepsilon_2 \qquad \text{and} \qquad \alpha_2 := \alpha_{23} := \varepsilon_2 - \varepsilon_3,$$

where ε_k denotes the standard basis of $\mathbb{R}^2$. Also, the positive non-simple root is $\alpha_1+\alpha_2 = \varepsilon_1 - \varepsilon_3$.

The following proposition determines the right $\mathcal{O}_q(\mathrm{SU}_3)$-module structure of Λ^1 (the cotangent space of $\mathcal{O}_q(\mathrm{SU}_3)$ as in Notation 5.2.4).

Proposition 5.2.7. *Let us adopt the notation presented in* 5.2.6*. The right $\mathcal{O}_q(\mathrm{SU}_3)$-module structure of Λ^1 is determined by:*

$$\begin{aligned} e_\gamma u_{kk} &= q^{-(\gamma,\varepsilon_k)}e_\gamma, & f_\gamma u_{kk} &= q^{-(\gamma,\varepsilon_k)}f_\gamma, \\ e_{\alpha_1}u_{32} &= \nu e_{\alpha_1+\alpha_2}, & f_{\alpha_1}u_{23} &= q^{-1}\nu f_{\alpha_1+\alpha_2}, \end{aligned}$$

with all other actions by the generators u_{ij} being zero.

Proof. The proof is a direct check. For example, for computing $e_{\alpha_1}u_{ij} = [u_{21}u_{ij}]$, we need to find the dual pairing (as introduced in Example 2.3.10) of $u_{21}u_{ij}$ with all the basis elements in T given in Definition 5.2.3.

$$\begin{aligned}
\langle u_{21}u_{ij}, E_{\alpha_1}\rangle &= \langle u_{21}, E_{\alpha_1}\rangle\langle u_{ij}, K_1\rangle + \langle u_{21}, 1\rangle\langle u_{ij}, E_{\alpha_1}\rangle \qquad \because\ \langle gg', h\rangle = \langle g, h_{(1)}\rangle\langle g', h_{(2)}\rangle \\
&= \langle u_{ij}, K_1\rangle \qquad \because\ \langle u_{21}, E_{\alpha_1}\rangle = 1 \text{ and } \langle u_{21}, 1\rangle = 0 \\
&= \begin{cases} q^{-1} & \text{for } i = j = 1, \\ q & \text{for } i = j = 2, \\ 1 & \text{for } i = j = 3, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Similarly, one can compute:

$$\langle u_{21}u_{ij}, E_{\alpha_2}\rangle = 0 = \langle u_{21}u_{ij}, F_{\alpha_1}\rangle = \langle u_{21}u_{ij}, F_{\alpha_2}\rangle = \langle u_{21}u_{ij}, F_{\alpha_3}\rangle$$

and,

$$\langle u_{21}u_{ij}, E_{\alpha_3}\rangle = \begin{cases} \nu & \text{for } (i,j) = (3,2), \\ 0 & \text{otherwise.} \end{cases}$$

These calculations establish the action of u_{ij} on e_{α_1}. Similarly, the action on other elements is computed and we finally arrive at the relations given in the statement of the proposition. □

Proposition 5.2.8. *Let us adopt the notation presented in* 5.2.4. *Then, we have an isomorphism in the category of* $U_q(\mathfrak{h})$*-modules given by:*

$$V^1 \to \Lambda^1, \qquad\qquad [b] \mapsto [b].$$

Remark 5.2.9. We slightly abuse notation here denoting the cosets in both spaces with the same symbol.

Proof. The fact that this is an injective module map follows from [65, Theorem 2.1]. (See also the discussion in [15, §4.2].) Surjectivity follows from the fact that each basis element of the tangent space T pairs non-trivially with an element of $\mathcal{O}_q(\mathrm{F}_3)$. Explicitly, the pairings

$$\begin{array}{llll}
\langle E_{\alpha_1}, z_{21}^{\alpha_1}\rangle, & \langle E_{\alpha_2}, z_{32}^{\alpha_2}\rangle, & \langle E_{\alpha_1+\alpha_2}, z_{31}^{\alpha_1}\rangle, & \langle E_{\alpha_1+\alpha_2}, z_{31}^{\alpha_2}\rangle, \\
\langle F_{\alpha_1}, z_{12}^{\alpha_1}\rangle, & \langle F_{\alpha_2}, z_{23}^{\alpha_2}\rangle, & \langle F_{\alpha_1+\alpha_2}, z_{13}^{\alpha_1}\rangle, & \langle F_{\alpha_1+\alpha_2}, z_{13}^{\alpha_2}\rangle.
\end{array}$$

are all non-zero scalars, where z_{ij} denotes the generators of $\mathcal{O}_q(\mathrm{F}_3)$ as appeared in Proposition 5.1.3. □

5.3 The higher forms

We observed in Section 2.7, we can prolong each FODC to a differential calculus called the *maximal prolongation* of FODC. In this section, we use the results of [13, §5] to calculate the degree two relations for the maximal prolongation of $\Omega^1_q(\mathrm{F}_3)$ the FODC on $\mathcal{O}_q(\mathrm{F}_3)$ corresponding to the quantum tangent space T introduced in the Definition 5.2.3.

Theorem 5.3.1. *Let us fix Notation 5.2.4 and Notation 5.2.6. Let $V^\bullet$ denote the quantum exterior algebra (as defined in Definition 2.8.7) for the maximal prolongation of $\Omega^1(\mathrm{F}_3)$. Then, a full set of relations for $V^\bullet$ is given by following three sets of identities:*

1)

$$e_\gamma \wedge e_\beta = -q^{(\beta,\gamma)} e_\beta \wedge e_\gamma, \quad f_\gamma \wedge f_\beta = -q^{-(\beta,\gamma)} f_\beta \wedge f_\gamma, \quad \text{for all } \beta \leq \gamma \in \Delta^+,$$

2)

$$e_\gamma \wedge f_\beta = -q^{(\beta,\gamma)} f_\beta \wedge e_\gamma, \qquad \text{for all } \beta \neq \gamma \in \Delta^+, \text{ or for } \beta = \gamma = \alpha_1 + \alpha_2,$$

3)

$$\begin{aligned} e_{\alpha_1} \wedge f_{\alpha_1} &= -q^2 f_{\alpha_1} \wedge e_{\alpha_1} - \nu f_{\alpha_1+\alpha_2} \wedge e_{\alpha_1+\alpha_2}, \\ e_{\alpha_2} \wedge f_{\alpha_2} &= -q^2 f_{\alpha_2} \wedge e_{\alpha_2} + \nu f_{\alpha_1+\alpha_2} \wedge e_{\alpha_1+\alpha_2}, \end{aligned}$$

where an order $\leq$ on the set of positive roots $\Delta^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$ is fixed as follows:

$$\alpha_2 \leq \alpha_1 + \alpha_2 \leq \alpha_1.$$

Proof. Using the description of the right $\mathcal{O}_q(\mathrm{SU}_3)$-module structure of Λ^1 (the cotangent space of $\mathcal{O}_q(\mathrm{SU}_3)$ as in Notation 5.2.4) given above in Proposition 5.2.7, one can observe the following set of identities, analogous to those in Lemma 5.2.5:

$$\begin{aligned} [S(u_{21})] &= -q e_{\alpha_1}, & [S(u_{32})] &= -q e_{\alpha_2}, & [S(u_{31})] &= -q e_{\alpha_1+\alpha_2}, \\ [S(u_{12})] &= -q^{-2} f_{\alpha_1}, & [S(u_{23})] &= -q^{-2} f_{\alpha_2}, & [S(u_{13})] &= -q^{-5} f_{\alpha_1+\alpha_2}. \end{aligned}$$

Moreover, we have the second set of identities, analogous to those in Proposition 5.2.7:

$$e_\gamma S(u_{kk}) = q^{(\gamma,\varepsilon_k)} e_\gamma, \qquad f_\gamma S(u_{kk}) = q^{(\gamma,\varepsilon_k)} f_\gamma, \tag{5.13}$$

$$e_{\alpha_1} S(u_{32}) = -\nu e_{\alpha_1+\alpha_2}, \qquad f_{\alpha_1} S(u_{23}) = -q^{-3} \nu f_{\alpha_1+\alpha_2}, \tag{5.14}$$

with all other actions by the antipoded generators $S(u_{ij})$ being zero.

From these identities, we can now see that the following set is the dual basis of V^1 (the cotangent space of $\mathcal{O}_q(\mathrm{F}_3)$ as in Notation 5.2.4):

$$\begin{aligned} e_{\alpha_1} &= q^{-1}[z_{21}^{\alpha_1}], & e_{\alpha_2} &= -q^{-1}[z_{32}^{\alpha_2}], & e_{\alpha_1+\alpha_2} &= q^{-1}[z_{31}^{\alpha_1}] = -q^{-1}[z_{31}^{\alpha_2}] \\ f_{\alpha_1} &= -q^2[z_{12}^{\alpha_1}], & f_{\alpha_2} &= q^2[z_{23}^{\alpha_2}], & f_{\alpha_1+\alpha_2} &= -q^5[z_{13}^{\alpha_1}] = q^3[z_{13}^{\alpha_2}], \end{aligned}$$

where z_{ij} denotes the generators of $\mathcal{O}_q(\mathrm{F}_3)$ as appeared in Proposition 5.1.3.

We next introduce a set of generators for the ideal I of the tangent space (recall from section 2.8 that $V^1 = \mathcal{O}_q(\mathrm{F}_3)^+/I^{(1)}$) from the description of the right $\mathcal{O}_q(\mathrm{SU}_3)$-module structure of Λ^1 given above. We divide the set of generators according to their polynomial degree. To do so we find it convenient to introduce the subset of $\mathbb{Z}^3_{>0}$,

$$B := \Big\{(1,2,1),(1,1,2),(1,3,1),(1,1,3),(2,3,2),(2,2,3),(2,3,1),(2,1,3)\Big\}.$$

Consider now the degree one polynomials,

$$G_1 := \Big\{z_{ab}^{\alpha_i} \,|\, (i,a,b) \notin B\Big\} \cup \Big\{z_{31}^{\alpha_1} + z_{31}^{\alpha_2},\, q^2 z_{13}^{\alpha_1} + z_{13}^{\alpha_2}\Big\}.$$

Next consider the quadratic polynomials,

$$\begin{aligned} G_2 :=& \Big\{z_{kl}^{\alpha_i}(z_{ab}^{\alpha_p})^+ \,|\, (i,k,l) \in B\backslash\{(1,2,1),\,(2,3,2)\},\, p=1,2,\, a,b=1,2,3\Big\}, \\ G_3 :=& \Big\{z_{kl}^{\alpha_i}(z_{ab}^{\alpha_p})^+ \,|\, (i,k,l) \in B,\, (p,a,b) \neq (2,3,2),(2,2,3)\Big\}, \\ G_4 :=& \Big\{z_{21}^{\alpha_1} z_{32}^{\alpha_2} - \nu z_{31}^{\alpha_2},\, z_{12}^{\alpha_1} z_{23}^{\alpha_2} - \nu z_{13}^{\alpha_1}\Big\}. \end{aligned}$$

Collecting these elements together gives us our proposed set of generators

$$G := G_1 \cup G_2 \cup G_3 \cup G_4.$$

Indeed, since it is clear that

$$\dim\big(\mathcal{O}_q(\mathrm{F}_3)^+/\langle G\rangle\big) \leq 6,$$

where $\langle G\rangle$ is the right ideal of $\mathcal{O}_q(\mathrm{F}_3)^+$ generated by the elements of G, we see that G gives a full set of generators.

Calculating the action of the map ω, see [13, §5], on these generators is now some tedious calculations. (See [11, Proposition 5.8] for the case of quantum projective space.) As an example, here we take the generator $z^{\alpha_1}_{22}$, and note that,

$$\begin{aligned}\omega(z^{\alpha_1}_{22}) &= \sum_{a,b}[u_{2a}S(u_{b2})] \otimes [z^{\alpha_1}_{ab}] \\ &= [u_{22}S(u_{12})] \otimes [z^{\alpha_1}_{21}] + [u_{23}S(u_{12})] \otimes [z^{\alpha_1}_{31}] + \\ &\quad [u_{21}S(u_{22})] \otimes [z^{\alpha_1}_{12}] + [u_{21}S(u_{32})] \otimes [z^{\alpha_1}_{13}]. \end{aligned} \tag{5.15}$$

Using identities in (5.13) and (5.14), we computed:

$$[u_{22}S(u_{12})] = -q^{-2}f_{\alpha_1}, \qquad [u_{23}S(u_{12})] = 0,$$

$$[u_{21}S(u_{22})] = q^{-1}e_{\alpha_1}, \qquad [u_{21}S(u_{32})] = -\nu e_{\alpha_1+\alpha_2}.$$

Substituting these values in Equation (5.15) we get:

$$w(z^{\alpha_1}_{22}) = -q^{-1}f_{\alpha_1} \otimes e_{\alpha_1} - q^{-3}e_{\alpha_1} \otimes f_{\alpha_1} + q^{-5}\nu e_{\alpha_1+\alpha_2} \otimes f_{\alpha_1+\alpha_2}.$$

Therefore, the relation $w(z^{\alpha_1}_{22}) = 0$ gives us:

$$e_{\alpha_1} \wedge f_{\alpha_1} = -q^2 f_{\alpha_1} \wedge e_{\alpha_1} + q^{-2}\nu e_{\alpha_1+\alpha_2} \wedge f_{\alpha_1+\alpha_2}.$$

Continuing as such gives us the claimed set of relations. Finally, we observe the description of the right $\mathcal{O}_q(\mathrm{SU}_3)$-module structure of Λ^1 given in Proposition 5.2.7 implies that the relations form a right $\mathcal{O}_q(\mathrm{F}_3)$-submodule of $V^1 \otimes V^1$. Thus, they give a full set of relations. □

Corollary 5.3.2. *For $k = 1, \ldots, 6 = |\Delta|$, a basis for V^k (the homogeneous component of degree k of $V^\bullet$) is given by:*

$$\Big\{e_{\gamma_1} \wedge \cdots \wedge e_{\gamma_a} \wedge f_{\gamma_1} \wedge \cdots \wedge f_{\gamma_b} \,|\, \gamma_1 < \cdots < \gamma_k \in \Delta^+\Big\}.$$

In particular, it holds that

$$\dim\Big(V^k\Big) = \binom{|\Delta|}{k}, \qquad \textit{and} \qquad \dim\Big(V^\bullet\Big) = 2^{|\Delta|}.$$

Proof. It is clear from the set of relations given in Theorem 5.3.1 that the proposed basis is a spanning set. To prove that its elements are linearly independent, let $\langle V^1 \rangle$

be the free monoid generated by elements of V^1 and let $S_{\Delta^+} := (W_{\Delta^+}, f_{\Delta^+})$ be the reduction system in the free algebra $\mathbb{C}\langle V^1\rangle$ corresponding to the set of relations 5.3, namely,

$$\begin{aligned}
&(e_\gamma \otimes e_\beta,\ -q^{(\beta,\gamma)} e_\beta \otimes e_\gamma), \quad (f_\gamma \otimes f_\beta, -q^{-(\beta,\gamma)} f_\beta \otimes f_\gamma), \quad \text{for all } \beta \leq \gamma \in \Delta^+,\\
&(e_\gamma \otimes f_\beta,\ -q^{(\beta,\gamma)} f_\beta \otimes e_\gamma) \text{ for all } \beta \neq \gamma \in \Delta^+, \text{ or for } \beta = \gamma = \alpha_1 + \alpha_2,\\
&(e_{\alpha_1} \otimes f_{\alpha_1},\ -q^2 f_{\alpha_1} \otimes e_{\alpha_1} - \nu f_{\alpha_1+\alpha_2} \otimes e_{\alpha_1+\alpha_2},)\\
&(e_{\alpha_2} \otimes f_{\alpha_2},\ -q^2 f_{\alpha_2} \otimes e_{\alpha_2} + \nu f_{\alpha_1+\alpha_2} \otimes e_{\alpha_1+\alpha_2}).
\end{aligned}$$

Let $\ll$ denote the total ordering such that for every $\beta, \gamma \in \Delta^+$,

$$f_\beta \ll e_\gamma$$

and,

$$\beta \leq \gamma \in \Delta^+ \Rightarrow e_\beta \ll e_\gamma, \quad f_\gamma \ll f_\beta.$$

Then S_{Δ^+} is a reduction system compatible with the ordering $\ll$ and it is easy to verify that it has no ambiguities, hence from Bergmann's diamond lemma, see [5], the set of algebra relations 5.3 is linearly independent and the spanning set given above is a basis. □

5.4 Almost-complex structures on $\mathcal{O}_q(\mathrm{F}_3)$

In this section we examine covariant complex and almost complex structures for the differential calculus: $\Omega_q^\bullet(\mathrm{F}_3)$, the maximal prolongation of the $\Omega_q^1(\mathrm{F}_3)$ established in the previous section. We follow closely the description given in [13, 12]. We observe that the number of almost-complex structures decreases from 8 (the classical case, which is 2 to the number of positive roots of $\mathfrak{sl}_3\mathbb{C}$) to 4 (which is 2 to the number of simple roots of $\mathfrak{sl}_3$). Furthermore, we demonstrate that all of these almost-complex structures are integrable, which is to say, they are complex structures.

Firstly, we briefly recall the covariant almost complex structures for the classical flag manifold F_3. We do so to highlight the novel non-classical behaviour occurring for the quantum case.

By [1], a choice of complex structure on the full flag manifold F_3 corresponds to a choice of a base for Δ the root system of $\mathfrak{sl}_3\mathbb{C}$. The Weyl group S_3 of $\mathfrak{sl}_3\mathbb{C}$ acts transitively on the set of bases for Δ, and hence on the set of covariant almost-complex structures.

Now we classify the covariant complex structures on the differential calculus $\Omega^\bullet_q(\mathrm{F}_3)$. We find that two of the classical almost structures fail to extend to the quantum setting. In particular, one of the bases of the root system of $\mathfrak{sl}_3\mathbb{C}$ fails to have a corresponding FODCi in the quantum setting.

As usual in the theory of differential calculi, we initially work at the level of FODC and then discuss the extension to higher forms. (See Section 2.9).

Theorem 5.4.1. *The first-order differential calulus $\Omega^1_q(\mathrm{F}_3)$ admits, up to identification of opposite structures, two covariant first-order almost complex structures. Explicitly, one decomposition of V^1 is given by:*

$$V^{(1,0)} = \mathrm{span}_{\mathbb{C}}\Big\{e_{\alpha_1},\, e_{\alpha_2},\, e_{\alpha_1+\alpha_2}\Big\}, \qquad V^{(0,1)} := \mathrm{span}_{\mathbb{C}}\Big\{f_{\alpha_1},\, f_{\alpha_2},\, f_{\alpha_1+\alpha_2}\Big\},$$

and the other is given by:

$$V^{(1,0)} = \mathrm{span}_{\mathbb{C}}\Big\{f_{\alpha_1},\, e_{\alpha_2},\, e_{\alpha_1+\alpha_2}\Big\}, \qquad V^{(0,1)} := \mathrm{span}_{\mathbb{C}}\Big\{e_{\alpha_1},\, f_{\alpha_2},\, f_{\alpha_1+\alpha_2}\Big\},$$

Proof. Consider a general left $\mathcal{O}_q(\mathrm{SU}_3)$-covariant first-order almost complex structure on $\Omega^1_q(\mathrm{F}_3)$, and denote by

$$V^1 := V^{(1,0)} \oplus V^{(0,1)},$$

the corresponding decomposition of the cotangent space V^1 into two left $\mathcal{O}(\mathfrak{h})$-comodule right $\mathcal{O}_q(\mathrm{F}_3)$-modules. Since the basis elements all have mutually distinct weights, we see that each basis element is contained in either $V^{(1,0)}$ or $V^{(0,1)}$. The right $\mathcal{O}_q(\mathrm{F}_3)$-module requirement, together with Lemma 5.2.5, implies that if e_{α_1} is contained in $V^{(1,0)}$, then $e_{\alpha_1+\alpha_2}$ is also contained in $V^{(1,0)}$, and analogously, if f_{α_1} is contained in $V^{(0,1)}$, then $f_{\alpha_+\alpha_2}$ is contained in $V^{(0,1)}$. In other words, any complex structure is determined by knowing whether the basis elements e_α, f_α, for $\alpha \in \{\alpha_1, \alpha_2\}$ (the set of simple roots), are contained in $V^{(1,0)}$ or $V^{(0,1)}$.

We now note that any such $\mathcal{O}_q(\mathrm{F}_3)$-decomposition of V^1 will necessarily be a decomposition of right $\mathcal{O}_q(\mathrm{F}_3)$-modules. Considering V^1 as a subspace of Λ^1, the cotangent space of the FODC $\Omega^1_q(\mathrm{SU}_3)$, and recalling that $e^*_\gamma = f_\gamma$, for all $\gamma \in \Delta^+$, we now see that the only possible decompositions are those two decompositions given in the statement of the theorem. □

Given a first-order almost complex structure on a FODC, there is at most one extension to an almost complex structure on its maximal prolongation, or indeed any quotient thereof (see [13, Prop. 6.1] for details). The following proposition tells that both our FOCASs extend.

Corollary 5.4.2. *Both FOACSs on $\Omega^1_q(\mathrm{F}_3)$ extend to a factorisable almost complex structure on $\Omega^\bullet_q(\mathrm{F}_3)$.*

Proof. The fact that both first-order structures extend to covariant almost-complex structures, follows directly from the explicit form of the relations given in section 5.3 and [13, Theorem 6.4]. Moreover, factorisability follows from the explicit form of the relations in [13, Corollarly 6.8]. □

5.5 Integrable almost-complex structures

In this section we observe that both the covariant almost-complex structures on $\Omega^\bullet_q(\mathrm{F}_3)$ (as shown in Theorem 5.4.1) are integrable. In other words, both the covariant almost-complex structures on $\Omega^\bullet_q(\mathrm{F}_3)$ are *complex structures*.

An almost-complex structure $\Omega^{(\bullet,\bullet)}$ on a differential calculus $\Omega^\bullet$ is integrable if and only if the maximal prolongation of the FODC $\Omega^{(0,1)}$ is isomorphic to the subalgebra $\Omega^{(0,\bullet)}$. (See [13, Lemma 7.2]). Using this reformulation of integrability, we now prove the following final result.

Proposition 5.5.1. *Both covariant almost-complex structures of the differential calculus $\Omega^\bullet_q(\mathrm{F}_3)$ are integrable.*

Proof. We will treat the case of the almost-complex structure:

$$V^{(0,1)} = \{e_{\alpha_1},\, e_{\alpha_2},\, e_{\alpha_1+\alpha_2}\},$$

the other case being entirely analogous. We need to calculate the dimension of the maximal prolongation of the associated FODC $\Omega^{(0,1)}$. We note that, $\Omega^{(0,1)}_q(\mathrm{SU}_3)$ is a framing calculus for $\Omega^{(0,1)}_q(\mathrm{F}_3)$, see [15, Lemma 5.4], allowing us to calculate the degree two relations of the maximal prolongation of $\Omega^{(0,1)}_q(\mathrm{F}_3)$.

We see that the ideal $I' \subseteq \mathcal{O}_q(\mathrm{F}_3)^+$ corresponding to the $\Omega^{(0,1)}_q(\mathrm{F}_3)$ contains the elements:

$$I \cup \{z^{\alpha_1}_{12},\, z^{\alpha_2}_{23},\, z^{\alpha_1}_{13}\}.$$

where I is the ideal appeared in Theorem 5.3.1. Moreover, since the quotient of $\mathcal{O}_q(\mathrm{F}_3)^+$ by I' is three dimensional, we see that this is in fact the whole ideal.

Operating on the elements of I by ω (as defined in Definition 2.8.7) we clearly reproduce the degree-$(0,2)$ elements from those given in Theorem 5.3.1. For the element $z_{23}^{\alpha_2}$, we see that,

$$\omega(z_{23}^{\alpha_2}) = \omega(u_{23}S(u_{33})) = \sum_{a=1}^{3}[(u_{23}S(u_{b3})] \otimes [S(u_{3b})^+] + \sum_{a=1}^{3}[S(u_{b3})^+] \otimes [(u_{23}^+S(u_{3b})]$$
$$+ \sum_{a=1}^{3}[(u_{2a}^+S(u_{b3})] \otimes [u_{a3}^+S(u_{3b})].$$

Since each of the elements:

$$u_{23},\ u_{13},\ u_{22}^+,\ u_{23}$$

pair trivially with each element of $T^{(0,1)}$, we now see that $\omega(z_{23}^{\alpha_2}) = 0$. Analogous calculations establish that:

$$\omega(z_{23}^{\alpha_2}) = \omega(z_{13}^{\alpha_1}) = 0.$$

Thus, we see that the maximal prolongation of $\Omega_q^{(0,1)}(\mathrm{F}_3)$ is isomorphic to the subalgebra $\Omega_q^{(0,\bullet)}(\mathrm{F}_3)$, and so, the almost-complex structure is integrable. $\square$

Chapter 6

N=2 Minkowski superspace and its quantization

In this final chapter, we discuss some work that is relevant from a physics perspective. According to the current understanding of quantum physics, the space-time manifold in low-energy depiction collapses at extremely small scales. On the other hand, since the geometry of space-time is fundamental to the Einstein theory of general relativity, it is reasonable to assume that, in order to fully comprehend both small and cosmological scales, some quantum notion must be introduced. Noncommutative geometry is a suitable framework to better comprehend this issue.

As a mathematical model, here we give the superalgebra of $N = 2$ antichiral quantum superfields realized as a subalgebra of the quantum supergroup $\mathbb{C}_q[\mathrm{SL}(4|2)]$. The multiplication law in the quantum supergroup induces a coaction on the set of antichiral superfields. We also realize the quantum deformation of the Minkowski superspace as a quantum principal bundle. Most of the work in this chapter is taken from [42], [39], [38] and [41].

The chapter is organized as follows. In Section 6.1, we briefly recall the notion of complex Minkowski space realized as the big-cell inside $\mathrm{Gr}(2, 4)$, which plays the role of the conformal space. In Section 6.2, a quantization of complex Minkowski space is introduced via Manin relations as the big cell for the quantum grassmannian $\mathrm{Gr}_q(2, 4)$, together with a coaction of quantum Lorentz group. In Section 6.3, we introduce a Lorentz-covariant FODC on quantum Minkowski space. In Section 6.4, we present the super Plücker embedding for super grassmannian $\mathrm{Gr}(2|0, 4|2)$. In Section 6.5, a quantum version of $\mathrm{Gr}(2|0, 4|2)$ is obtained via super Manin relations. In Section 6.6, we discuss the notion of $N = 2$ Minkowski superspace. In Section 6.7, the notion of $N = 2$ quantum Minkowski superspace is presented and we also realize it as a trivial quantum principal superbundle.

6.1 Complex Minkowski space

The special theory of relativity suggests that spacetime is a 4-dimensional manifold whose symmetries are encoded by *Poincaré group*. On the other hand, *Maxwell's equations of electromagnetism* possess some more general symmetries called the *conformal transformations*, whose collection is the *conformal group* $\mathrm{SO}(2,4)$. Thus, we expect that the fundamental framework would be conformal, subject to conformal invariance. This is Penrose's perspective on Minkowski space in his twistor space approach [68]. In this section, we will briefly recall it. For more details see [39, Chap. 2], [61, Chap. 1].

Recall that, the complex grassmannian $\mathrm{Gr}(2,4)$ is the space of 2-dimensional planes in $\mathbb{C}^4$. It admits a natural transitive action of $\mathrm{SL}(4,\mathbb{C})$ given by:

$$\begin{aligned} \mathrm{SL}(4,\mathbb{C}) \times \mathrm{Gr}(2,4) &\longrightarrow \mathrm{SL}(4,\mathbb{C}) \\ g.\ \mathrm{span}\{a,b\} &\mapsto \mathrm{span}\{ga, gb\}. \end{aligned}$$

On the other hand, $\mathrm{SL}(4,\mathbb{C})$ is the complexification of the spin group of the connected component of the identity $\mathrm{SU}(2,2)$ for the conformal group $\mathrm{SO}(2,4)$. The idea is to realize complex Minkowski space as a dense open subset of $\mathrm{Gr}(2,4)$. The grassmannian $\mathrm{Gr}(2,4)$ is embedded into $\mathbb{P}(\bigwedge^2 \mathbb{C}^4)$ via Plücker embedding,

$$\begin{aligned} \mathrm{Gr}(2,4) &\longrightarrow \mathbb{P}(\bigwedge^2 \mathbb{C}^4) \cong \mathbb{P}^5, \\ \mathrm{span}\{a,b\} &\mapsto [y_{12}, y_{13}, y_{14}, y_{23}, y_{24}, y_{34}], \end{aligned} \tag{6.1}$$

where $y_{ij} := a_i b_j - a_j b_i$. Moreover, it can be realized as a projective variety being the zero set of the following Plücker relation:

$$y_{12}y_{34} - y_{13}y_{24} + y_{14}y_{23} = 0. \tag{6.2}$$

Therefore, the algebra of functions on $\mathbb{C}[\mathrm{Gr}(2,4)]$ is:

$$\mathbb{C}[\mathrm{Gr}(2,4)] \cong \mathbb{C}[y_{12}, y_{13}, y_{14}, y_{23}, y_{24}, y_{34}]/\mathrm{I}\,, \tag{6.3}$$

where I is the ideal generated by $y_{12}y_{34} - y_{13}y_{24} + y_{14}y_{23}$.

Define the *big cell* U to be the following Zariski open set:

$$U = \{P \in \mathrm{Gr}(2,4) : y_{12} \neq 0\} \subset \mathrm{Gr}(2,4)$$

It is easy to see that $U \cong \mathbb{C}^4$.

Theorem 6.1.1. *The subgroup of* $\mathrm{SL}(4,\mathbb{C})$ *that leaves big cell invariant is:*

$$F_0^c := \left\{ \begin{bmatrix} L & O \\ NL & R \end{bmatrix} \mid \textit{with } L \textit{ and } R \textit{ being invertible } 2 \times 2 \textit{ matrices.} \right\}.$$

Moreover, the induced action on U *is given by:*

$$A \mapsto N + RAL^{-1}. \tag{6.4}$$

Proof. See [39], Prop. 2.7.1. □

The subgroup F_0^c is called the *complex Poincaré group times dilations*. Moreover, the structure of F_0^c is

$$F_0^c \cong (\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \times \mathbb{C}^\times) \ltimes \mathbb{M}_2(\mathbb{C}), \tag{6.5}$$

that remind us the (real) Poincaré group. Define the *complex Poincaré group* to be,

$$P_0^c := \left\{ \begin{bmatrix} L & O \\ NL & R \end{bmatrix} \mid \text{with } \det(L).\det(R) = 1. \right\}.$$

Definition 6.1.2. The subset $U \subset \mathrm{Gr}(2,4)$, together with this action of complex Poincare group P_0^c is called the *complex Minkowski space* M.

So, with this definition we are able to embed complex Minkowski space M into its compactification $\mathrm{Gr}(2,4)$ such that the action of the complex Poincarè group times dilations is coming as the action of the subgroup F_0^c of $\mathrm{SL}(4,\mathbb{C})$ leaving big cell invariant. See [39, 75] for further details.

6.2 Quantum Minkowski space

In this section, we will briefly introduce a quantization of complex Minkowski space using the quantum group approach. This work is well-known and details can be found in [39, Chap. 4], [41], [34] and [35].

Note that, we can realize $\mathbb{C}[\mathrm{Gr}(2,4)]$ (as introduced in (6.3)) as a subalgebra of $\mathbb{C}[\mathrm{SL}(4,\mathbb{C})]$,

$$\begin{aligned} &\mathbb{C}[\mathrm{Gr}(2,4)] \longrightarrow \mathbb{C}[\mathrm{SL}(4,\mathbb{C})] \\ &y_{ij} \mapsto d_{ij} = x_{i1}x_{j2} - x_{j1}x_{i2}. \end{aligned} \tag{6.6}$$

Therefore, we can simply quantize this notion of complex conformal space by introducing the Manin relations. To be consistent with the references [39, 42], in this chapter, we replace q with q^{-1} and vice versa in the definition of quantum matrix bialgebra and quantum genral linear group, furthermore, we take q as a parameter, see [39, Chap. 5].

Definition 6.2.1. Define the *quantum conformal space* $\mathbb{C}_q[\mathrm{Gr}(2,4)]$ as the subalgebra of $\mathbb{C}_q[\mathrm{SL}(4,\mathbb{C})]$ generated by:

$$D_{ij} := x_{i1}x_{j2} - q^{-1}x_{j1}x_{i2}, \qquad 1 \le i < j \le 4. \tag{6.7}$$

The following theorem completely characterize $\mathbb{C}_q[\mathrm{Gr}(2,4)]$ in terms of generators and relations.

Theorem 6.2.2. *We have the following identification:*

$$\mathbb{C}_q[\mathrm{Gr}(2,4)] \cong \mathbb{C}_q\langle\lambda_{ij}\rangle/\mathrm{I}, \tag{6.8}$$

where I *is the ideal generated by the following relations. Let* $<$ *be the lexicographic order,*

$$\lambda_{ij}\lambda_{kl} = q^{-1}\lambda_{kl}\lambda_{ij}, \qquad (i,j) < (k,l) \qquad i,j,k,l \text{ not all distinct}, \tag{6.9}$$

$$\lambda_{12}\lambda_{34} = q^{-2}\lambda_{34}\lambda_{12}, \qquad \lambda_{14}\lambda_{23} = \lambda_{23}\lambda_{14}, \tag{6.10}$$

$$\lambda_{13}\lambda_{24} = q^{-2}\lambda_{24}\lambda_{13} - (q^{-1} - q)\lambda_{12}\lambda_{34}, \tag{6.11}$$

$$\lambda_{12}\lambda_{34} - q^{-1}\lambda_{13}\lambda_{24} + q^{-2}\lambda_{14}\lambda_{23} = 0. \qquad \text{(quantum Plücker relation)} \tag{6.12}$$

Proof. See [39, Prop. 5.2.4]. □

We notice in Section 6.1 that $\mathrm{Gr}(2,4)$ admits a natural action of $\mathrm{SL}(4,\mathbb{C})$. Similarly, in this dual quantum picture, $\mathbb{C}_q[\mathrm{Gr}(2,4)]$ admits a coaction of $\mathbb{C}_q[\mathrm{SL}(4,\mathbb{C})]$ by restricting the product.

Theorem 6.2.3. $\mathbb{C}_q[\mathrm{Gr}(2,4)]$ *is a* $\mathbb{C}_q[\mathrm{SL}(4,\mathbb{C})]$*-comodule, where the coaction is induced simply by restricting the coproduct.*

Proof. See [39], Prop. 5.2.5. □

Definition 6.2.4. Define the *quantum Minkowski space* $\mathbb{C}_q[\mathrm{M}]$ as the projective localization of $\mathbb{C}_q[\mathrm{Gr}(2,4)]$ at D_{12}. In other words, it is generated by:

$$m_{11} := D_{23}D_{12}^{-1}, \qquad m_{12} := D_{13}D_{12}^{-1}, \qquad m_{21} := D_{24}D_{12}^{-1} \qquad \text{and} \qquad m_{22} := D_{14}D_{12}^{-1}.$$

It is important to note that, D_{12}^{-1} q-commutes (*i.e.* up to multiplication by a power of q) with the generators of $\mathbb{C}_q[\mathrm{Gr}(2,4)]$, therefore, this localization is well-defined.

Theorem 6.2.5. *The quantum Minkowski space* $\mathbb{C}_q[\mathrm{M}]$ *as a bialgebra is isomorhic to the bialgebra* $\mathbb{C}_q[\mathrm{M}_2]$ *of quantum* 2×2 *matrices. The isomorphism is given by:*

$$\begin{aligned} \mathbb{C}_q[\mathrm{M}] &\longrightarrow \mathbb{C}_q[\mathrm{M}_2] \\ m_{11} &\mapsto u_{12} \\ m_{12} &\mapsto u_{11} \\ m_{21} &\mapsto u_{22} \\ m_{22} &\mapsto u_{21}. \end{aligned}$$

Proof. See [39, Prop. 5.2.13]. □

Moreover, the *quantum complex Lorentz group* is defined as $\mathbb{C}_q[\mathrm{SL}_2] \oplus \mathbb{C}_q[\mathrm{SL}_2]$, and the quantum Minkowski space $\mathbb{C}_q[\mathrm{M}]$ admits a coaction of this as follows:

$$\phi : \mathbb{C}_q[\mathrm{M}] \longrightarrow (\mathbb{C}_q[\mathrm{SL}_2] \oplus \mathbb{C}_q[\mathrm{SL}_2]) \otimes \mathbb{C}_q[\mathrm{M}]$$

$$m_{ij} \mapsto \sum_{s,r} y_{is} S(x_{rj}) \otimes m_{sr} \tag{6.13}$$

where y_{is} and x_{rj} denotes the generators of first and second copy of $\mathbb{C}_q[\mathrm{SL}_2]$ in quantum Lorentz group respectively. See [39, Chap. 5] and [41] for further details.

6.3 A Lorentz-covariant differential structure on $\mathbb{C}_q[\mathrm{M}]$

In this section, we explicitly described a Lorentz-covariant first-order differential calculus on quantum Minkowski space $\mathbb{C}_q[\mathrm{M}]$ as presented in the previous section.

Let us identify:

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{6.14}$$

In [63], given an R-matrix deformation of a commutative algebra, there is a recipe to construct a quantum first-order differential calculus. Namely we write, in short:

$$M_1 \mathbf{d} M_2 = R_{21} \mathbf{d} M_2 M_1 R. \tag{6.15}$$

For notational details, see [63, Chap. 10].

This leads to a differential calculus $\Omega(\mathrm{M})$ on $\mathbb{C}_q[\mathrm{M}]$ whose bimodule structure is described as:

$$\begin{aligned} a.\mathbf{d}a &= q^2 \mathbf{d}a.a \\ b.\mathbf{d}a &= q\mathbf{d}a.b \\ c.\mathbf{d}a &= q\mathbf{d}a.c + (q - q^{-1})a.\mathbf{d}c \\ d.\mathbf{d}a &= \mathbf{d}a.d + (q - q^{-1})\mathbf{d}c.b \end{aligned}$$

$$\begin{aligned} a.\mathbf{d}b &= q\mathbf{d}b.a + (q - q^{-1})b.\mathbf{d}a \\ b.\mathbf{d}b &= q^2 \mathbf{d}b.b \\ c.\mathbf{d}b &= \mathbf{d}b.c + (q - q^{-1})(\mathbf{d}d.a + d.\mathbf{d}a) \\ d.\mathbf{d}b &= q\mathbf{d}b.d + (q - q^{-1})b.\mathbf{d}d \end{aligned}$$

$$\begin{aligned} a.\mathbf{d}c &= q\mathbf{d}c.a \\ b.\mathbf{d}c &= \mathbf{d}c.b \\ c.\mathbf{d}c &= q^2 \mathbf{d}c.c \\ d.\mathbf{d}c &= q\mathbf{d}c.d \end{aligned}$$

$$
\begin{aligned}
a.\mathbf{d}d &= \mathbf{d}d.a + (q - q^{-1})\mathbf{d}c.b \\
b.\mathbf{d}d &= q\mathbf{d}d.b \\
c.\mathbf{d}d &= q\mathbf{d}d.c + (q - q^{-1})d.\mathbf{d}c \\
d.\mathbf{d}d &= q^2\mathbf{d}d.d
\end{aligned} \tag{6.16}
$$

Theorem 6.3.1. *The first-order differential calculus $\Omega(\mathrm{M})$ defined on $\mathbb{C}_q[\mathrm{M}]$ is left-covariant under the quantum Lorentz coaction ϕ as in (6.13).*

Proof. Define:

$$
\begin{gathered}
\Phi : \Omega(\mathrm{M}) \longrightarrow (\mathbb{C}_q[\mathrm{SL}_2] \oplus \mathbb{C}_q[\mathrm{SL}_2]) \otimes \Omega(\mathrm{M}) \\
(\mathbf{d}m_{ij}) \mapsto (\mathrm{id} \otimes \mathbf{d})\phi(m_{ij})
\end{gathered} \tag{6.17}
$$

and,

$$
\Phi(x\rho y) = \phi(x)\Phi(\rho)\phi(y) \qquad x, y \in \mathbb{C}_q[\mathrm{M}], \rho \in \Omega(\mathrm{M}). \tag{6.18}
$$

Clearly, by definition:

$$
(\mathrm{id} \otimes \mathbf{d}) \circ \phi = \mathbf{d} \circ \Phi.
$$

To prove that $\Omega(\mathrm{M})$ is left-covariant with respect to ϕ, we only need to verify if the map Φ is well-defined (*i.e.* it respects the bimodule structure), and it comprises of various calculation checks. For example,

$$
\begin{aligned}
\Phi(a.\mathbf{d}a) &= \phi(a)(\mathrm{id} \otimes \mathbf{d})\phi(a) \\
&= (y_{11}x_{22} \otimes a - q^{-1}y_{11}x_{21} \otimes b + y_{12}x_{22} \otimes c - q^{-1}y_{12}x_{21} \otimes d) \\
&\quad (y_{11}x_{22} \otimes \mathbf{d}a - q^{-1}y_{11}x_{21} \otimes \mathbf{d}b + y_{12}x_{22} \otimes \mathbf{d}c - q^{-1}y_{12}x_{21} \otimes \mathbf{d}d) \\
&= A^2 \otimes a.\mathbf{d}a + AB \otimes a.\mathbf{d}b + AC \otimes a.\mathbf{d}c + AD \otimes a.\mathbf{d}d \\
&\quad + BA \otimes b.\mathbf{d}a + B^2 \otimes b.\mathbf{d}b + BC \otimes b.\mathbf{d}c + BD \otimes b.\mathbf{d}d \\
&\quad + CA \otimes c.\mathbf{d}a + CB \otimes c.\mathbf{d}b + C^2 \otimes c.\mathbf{d}c + CD \otimes c.\mathbf{d}d \\
&\quad + DA \otimes d.\mathbf{d}a + DB \otimes d.\mathbf{d}b + DC \otimes d.\mathbf{d}c + D^2 \otimes d.\mathbf{d}d
\end{aligned} \tag{6.19}
$$

where we fix:

$$
A := y_{11}x_{22} \qquad B := -q^{-1}y_{11}x_{21} \qquad C := y_{12}x_{22} \qquad D := -q^{-1}y_{12}x_{21}.
$$

Now, using the relations in (6.16) we can write:

$$\begin{aligned}\Phi(a.\mathbf{d}a) \;=\; & q^2A^2\mathbf{d}a.a + AB\otimes(q\mathbf{d}b.a+(q^2-1)da.b) + qAC\otimes\mathbf{d}c.a \\ & +AD\otimes(\mathbf{d}d.a+(q-q^{-1})\mathbf{d}c.b) + qBA\otimes\mathbf{d}a.b + q^2B^2\otimes\mathbf{d}b.b \\ & BC\otimes\mathbf{d}c.b + qBD\otimes\mathbf{d}d.b + CA\otimes(q\mathbf{d}a.c+(q^2-1)\mathbf{d}c.a) \\ & CB\otimes(\mathbf{d}b.c+(q-q^{-1})\mathbf{d}d.a+(q-q^{-1})\mathbf{d}a.d+(q-q^{-1})^2\mathbf{d}c.b) \\ & q^2C^2\otimes\mathbf{d}c.c + CD\otimes(q\mathbf{d}d.c+(q^2-1)\mathbf{d}c.d) + DA\otimes(\mathbf{d}a.d+(q-q^{-1})\mathbf{d}c.b) \\ & +DB\otimes(q\mathbf{d}b.d+(q^2-1)\mathbf{d}d.b) + qDC\otimes\mathbf{d}c.d + q^2\otimes\mathbf{d}d.d.\end{aligned}$$

Rearranging the R.H.S of above equation gives us:

$$\Phi(a.\mathbf{d}a) = q^2\Phi(\mathbf{d}a.a).$$

Therefore, Φ respects the first relation in (6.16). Similarly, the other checks are made and it completes the proof. □

6.4 Super Plücker embedding of $\mathrm{Gr}(2|0,4|2)$

Now, our aim is to generalize the construction of quantum Minkowski space as presented above to $N=2$ *Minkowski superspace*.

In this section, we are going to give an embedding of $\mathrm{Gr}(2|0,4|2)$ in the projective superspace $\mathbb{P}^{8|8}$, generalizing the classical Plücker embedding, see [33, Chap. 9]. Let $E = \bigwedge^2 \mathbb{C}^{4|2}$ and $\{e_1,\dots,e_4,\epsilon_5,\epsilon_6\}$ be a homogeneous basis for $\mathbb{C}^{4|2}$ (for notational details, see [39, §1.4, §4.8]), we then have a basis for E as:

$$\begin{array}{ll} e_i\wedge e_j,\quad 1\le i<j\le 4,\quad \epsilon_5\wedge\epsilon_5,\quad \epsilon_6\wedge\epsilon_6,\quad \epsilon_5\wedge\epsilon_6, & \text{(even)},\\ e_k\wedge\epsilon_5,\quad e_k\wedge\epsilon_6\quad 1\le k\le 4, & \text{(odd)}.\end{array}$$

So $E\simeq\mathbb{C}^{9|8}$ and $\mathbb{P}(E)\simeq\mathbb{P}^{8|8}$. An element of E is given as:

$$Q = q + \lambda_5\wedge\epsilon_5 + \lambda_6\wedge\epsilon_6 + a_{55}\epsilon_5\wedge\epsilon_5 + a_{66}\epsilon_6\wedge\epsilon_6 + a_{56}\epsilon_5\wedge\epsilon_6\,,$$

with,

$$q = q_{ij}e_i\wedge e_j,\quad \lambda_m=\lambda_{im}e_i,\qquad i,j=1,\dots,4,\quad m=5,6\,.$$

Let us denote $\mathrm{Gr} = Gr(2|0,4|2)$ and consider the super Plücker map,

$$\begin{aligned} \mathfrak{P} : \mathrm{Gr} &\longrightarrow \mathbb{P}^{8|8} \\ \mathrm{span}\{a,b\} &\longrightarrow [a \wedge b]\,, \end{aligned} \tag{6.20}$$

Any element $Q \in E$ is decomposable (and hence belongs to the image of $\mathfrak{P}$) if:

$$Q = a \wedge b,\ \text{for some},\ a = r + \xi_5\epsilon_5 + \xi_6\epsilon_6,\ \ b = s + \eta_5\epsilon_5 + \eta_6\epsilon_6 \in \mathbb{C}^{4|2}$$

with $r = r_i e_i$, $s = s_i e_i$. Therefore, one obtains the following equalities:

$$\begin{aligned} &q = r \wedge s, \\ &\lambda_5 = \xi_5 s - \eta_5 r, \qquad \lambda_6 = \xi_6 s - \eta_6 r, \\ &a_{55} = \xi_5\eta_5, \qquad a_{66} = \xi_6\eta_6, \qquad a_{56} = \xi_5\eta_6 + \xi_6\eta_5. \end{aligned} \tag{6.21}$$

Now taking all the (possible) wedge products and products among the expressions in (6.21) gives us,

$$\begin{aligned} &q \wedge q = 0, \\ &q \wedge \lambda_5 = 0, \qquad q \wedge \lambda_6 = 0, \\ &\lambda_5 \wedge \lambda_5 = -2a_{55}q, \qquad \lambda_6 \wedge \lambda_6 = -2a_{66}q, \qquad \lambda_5 \wedge \lambda_6 = -a_{56}q, \\ &\lambda_5 a_{55} = 0, \qquad \lambda_6 a_{66} = 0, \\ &\lambda_5 a_{66} = -\lambda_6 a_{56}, \qquad \lambda_6 a_{55} = -\lambda_5 a_{56}, \\ &a_{55}^2 = 0, \qquad a_{66}^2 = 0, \qquad a_{56}a_{56} = -2a_{55}a_{66}, \\ &a_{55}a_{56} = 0, \qquad a_{66}a_{56} = 0\,. \end{aligned} \tag{6.22}$$

Relations (6.22) are called the *super Plücker relations*. We can write them in coordinates in the following way (always $1 \leq i < j < k \leq 4$ and $5 \leq n \leq 6$):

$$\begin{aligned} &q_{12}q_{34} - q_{13}q_{24} + q_{14}q_{23} = 0, \qquad \text{(Plücker relation)} \\ &q_{ij}\lambda_{kn} - q_{ik}\lambda_{jn} + q_{jk}\lambda_{in} = 0, \\ &\lambda_{i5}\lambda_{j6} + \lambda_{i6}\lambda_{j5} = a_{56}q_{ij}, \qquad \lambda_{in}\lambda_{jn} = a_{nn}q_{ij}, \\ &\lambda_{in}a_{nn} = 0, \qquad \lambda_{i5}a_{66} = -\lambda_{i6}a_{56}, \qquad \lambda_{i6}a_{55} = -\lambda_{i5}a_{56}, \\ &a_{nn}^2 = 0, \qquad a_{55}a_{56} = 0, \qquad a_{66}a_{56} = 0, \\ &a_{56}a_{56} = -2a_{55}a_{66}\,. \end{aligned} \tag{6.23}$$

We will denote as $\mathcal{I}_P$ the ideal generated by them in the affine superspace $\mathbb{A}^{9|8}$ (with generators $q_{ij}, a_{nm}, \lambda_{kn}$). They are homogeneous quadratic equations, so they are defined in the projective space $\mathbb{P}^{8|8}$.

Now, we realize $\mathbb{C}[\mathrm{Gr}]$ as a subalgebra of $\mathbb{C}[\mathrm{SL}(4|2)]$. Let us display the generators of this algebra in matrix form:

$$\left(\begin{array}{cccc|cc} g_{11} & g_{12} & g_{13} & g_{14} & \gamma_{15} & \gamma_{16} \\ g_{21} & g_{22} & g_{23} & g_{24} & \gamma_{25} & \gamma_{26} \\ g_{31} & g_{32} & g_{33} & g_{34} & \gamma_{35} & \gamma_{36} \\ g_{41} & g_{42} & g_{43} & g_{44} & \gamma_{45} & \gamma_{46} \\ \hline \gamma_{51} & \gamma_{52} & \gamma_{53} & \gamma_{54} & g_{55} & g_{56} \\ \gamma_{61} & \gamma_{62} & \gamma_{63} & \gamma_{64} & g_{65} & g_{66} \end{array}\right), \tag{6.24}$$

then,

$$\mathbb{C}[\mathrm{SL}(4|2)] = \mathbb{C}[g_{ij}, g_{mn}, \gamma_{im}, \gamma_{nj}]/(\mathrm{Ber} - 1)\,,$$

where Ber is the Berezinian of the matrix and $1 \leq i, j \leq 4$ and $5 \leq m, n \leq 6$.

Proposition 6.4.1. *The superring* $\mathbb{C}[\mathrm{Gr}]$ *is generated as a subring of* $\mathbb{C}[\mathrm{SL}(4|2)]$ *by the elements:*

$$y_{ij} = g_{i1}g_{j2} - g_{i2}g_{j1}, \qquad \eta_{kn} = g_{k1}\gamma_{n2} - g_{k2}\gamma_{n1},$$
$$x_{55} = \gamma_{51}\gamma_{52}, \qquad x_{66} = \gamma_{61}\gamma_{62}, \qquad x_{56} = \gamma_{51}\gamma_{62} + \gamma_{61}\gamma_{52},$$

with the homomorphism,

$$\begin{array}{ccc} \mathbb{C}[\mathrm{Gr}] & \longrightarrow & \mathbb{C}[\mathrm{SL}(4|2)] \\ q_{ij}, \lambda_{kn} & \longrightarrow & y_{ij}, \eta_{kn}, \\ a_{55}, a_{66}, a_{56} & \longrightarrow & x_{55}, x_{66}, x_{56}. \end{array}$$

Proof. The proof uses an argument similar to the one used to obtain (6.21). Instead of taking the vectors a and b we have to take the first two columns of the matrix (6.24). □

Now, we have the following.

Proposition 6.4.2. *The superring associated to the image of* Gr *under the super Plücker embedding is*

$$\mathbb{C}[\mathrm{Gr}] \cong \mathbb{C}[q_{ij}, a_{nm}, \lambda_{kn}]/\mathcal{I}_P\,,$$

that is, the relations in $\mathcal{I}_P$ *are all the relations satisfied by the generators* $q_{ij}, a_{nm}, \lambda_{kn}$.

Proof. We give here a brief sketch of the argument, which is essentially a modification of straightening algorithm in the classical case, see [70, 33].

A generic monomial in $\mathbb{C}[\mathrm{Gr}]$ is of the form:

$$(\Pi_{(i,j)}q_{ij}^{c_{ij}})\lambda_{k_1 5}^{c_{k_1 5}}\cdots\lambda_{k_u 5}^{c_{k_u 5}}\lambda_{l_1 6}^{c_{l_1 6}}\cdots\lambda_{l_v 6}^{c_{l_v 6}}a_{55}^{c_{55}}a_{56}^{c_{56}}a_{66}^{c_{66}} \tag{6.25}$$

where (i,j) are ordered lexicographically, $k_1<\cdots<k_u$ and $l_1<\cdots<l_v$, $c_{ij}\in\mathbb{N}$, $c_{k_c 5}, c_{l_d 6}, c_{55}, c_{66}\in\{0,1\}$, and $c_{56}\in\{0,1,2\}$.

Using the relation $\lambda_{in}\lambda_{jn}=a_{nn}q_{ij}$ in (6.23), we can convert a monomial as in (6.25) to the form:

$$(\Pi_{(i,j)}q_{ij}^{c_{ij}})\lambda_{k5}^{c_{k5}}\lambda_{l6}^{c_{l6}}a_{55}^{c_{55}}a_{56}^{c_{56}}a_{66}^{c_{66}}. \tag{6.26}$$

Similarly, using the last seven relations in (6.23), we can assume following for each monomial in (6.26):

(i) either c_{k5} or c_{55} is zero.

(ii) either c_{l6} or c_{66} is zero.

(iii) either c_{l6} or c_{56} is zero.

(iv) either c_{l6} or c_{55} is zero.

(v) either c_{55} or c_{56} is zero.

(vi) either c_{66} or c_{56} is zero.

(vii) $c_{56}\in\{0,1\}$

Furthermore, similar to the classical case of $\mathrm{Gr}(2,n)$, using the first three relations in (6.23), we can write each such monomial (6.26) as a linear combination of monomials of the form as in (6.26) satisfying (i)-(vii), such that the Young's tableaux corresponding to the $(\Pi_{(i,j)}q_{ij}^{c_{ij}})\lambda_{k5}^{c_{k5}}\lambda_{l6}^{c_{l6}}$ is standard, i.e. it is strictly increasing along each row and weakly increasing down each column (see [70] for details). We call such monomials as *standard.*

Now, if we consider $\mathbb{C}[\mathrm{Gr}]$ as a subalgebra of $\mathbb{C}[\mathrm{SL}(4|2)]$ (as in Proposition 6.4.1), and consider the lexicographic ordering on the generators, due to straightening algorithm on $(\Pi_{(i,j)}q_{ij}^{c_{ij}})\lambda_{k5}^{c_{k5}}\lambda_{l6}^{c_{l6}}$ and (i)-(vii), it turns out that the leading terms of distinct standard monomials are distinct, and according to the Lemma 4.5.2 they form a basis of $\mathbb{C}[\mathrm{Gr}]$. Therefore, similar to the argument in Theorem 4.5.5, it proves that the relations in (6.23) are all the relations. □

6.5 The quantum grassmannian Gr_q

We can now define the quantum Grassmannian Gr_q mimicking Proposition 6.4.1.

Definition 6.5.1. The quantum super Grassmannian $\mathrm{Gr}_q := \mathrm{Gr}_q(2|0,4|2)$ is the subalgebra of $\mathbb{C}_q[\mathrm{SL}(4|2)]$ (as defined in Definition 3.5.4) generated by the elements:

$$\begin{aligned} D_{ij} &:= a_{i1}a_{j2} - q^{-1}a_{i2}a_{j1}, & D_{in} &:= a_{i1}a_{n2} - q^{-1}a_{i2}a_{n1}, \\ D_{55} &:= a_{51}a_{52}, & D_{66} &:= a_{61}a_{62}, \\ D_{56} &= a_{51}a_{62} - q^{-1}a_{52}a_{61}, \end{aligned}$$

with $1 \le i < j \le 4$ and $n = 5, 6$.

We want to give a presentation in terms of generators and relations, as in Proposition 6.4.2 for the classical case. Note that, first of all, we have to compute the commutation rules among the D's.

Let $1 \le i < j < l \le 4$, then:

$$\begin{aligned} D_{ij}D_{il} &= (a_{i1}a_{j2} - q^{-1}a_{i2}a_{j1})(a_{i1}a_{l2} - q^{-1}a_{i2}a_{l1}) \\ &= a_{i1}a_{j2}a_{i1}a_{l2} - q^{-1}a_{i1}a_{j2}a_{i2}a_{l1} - q^{-1}a_{i2}a_{j1}a_{i1}a_{l2} + q^{-2}a_{i2}a_{j1}a_{i2}a_{l1} \\ &= a_{i1}(a_{i1}a_{j2} - (q^{-1} - q)a_{i2}a_{j1})a_{l2} - q^{-1}a_{i2}a_{i1}a_{j2}a_{l1} \\ &\quad -qa_{i1}a_{i2}a_{j1}a_{l2} + q^{-3}a_{i2}a_{l1}a_{i2}a_{j1} \\ &= q^{-1}a_{i1}a_{l2}a_{i1}a_{j2} + (q^{-2} - 1)a_{i1}a_{i2}a_{l1}a_{j2} - (q^{-1} - q)a_{i1}a_{i2}a_{j1}a_{l2} \\ &\quad -q^{-2}a_{i2}a_{l1}a_{i1}a_{j2} - qa_{i1}a_{i2}(a_{l2}a_{j1} + (q^{-1} - q)a_{j2}a_{l1}) + q^{-3}a_{i2}a_{l1}a_{i2}a_{j1} \\ &= q^{-1}a_{i1}a_{l2}a_{i1}a_{j2} - q^{-2}a_{i2}a_{l1}a_{i1}a_{j2} + q^{-3}a_{i2}a_{l1}a_{i2}a_{j1} \\ &\quad +(q^{-2} - 1)a_{i1}a_{i2}a_{l1}a_{j2} - (q^{-1} - q)a_{i1}a_{i2}a_{j1}a_{l2} - a_{i1}a_{l2}a_{i2}a_{j1} - (1 - q^2)a_{i1}a_{i2}a_{j2}a_{l1} \\ &= q^{-1}a_{i1}a_{l2}a_{i1}a_{j2} - q^{-2}a_{i2}a_{l1}a_{i1}a_{j2} + q^{-3}a_{i2}a_{l1}a_{i2}a_{j1} \\ &\quad +(q^{-1} - q)^2a_{i1}a_{i2}a_{l1}a_{j2} - (q^{-1} - q)a_{i1}a_{i2}a_{j1}a_{l2} - a_{i1}a_{l2}a_{i2}a_{j1} \\ &= q^{-1}a_{i1}a_{l2}a_{i1}a_{j2} - q^{-2}a_{i2}a_{l1}a_{i1}a_{j2} + q^{-3}a_{i2}a_{l1}a_{i2}a_{j1} \\ &\quad +(q^{-1} - q)^2a_{i1}a_{i2}a_{l1}a_{j2} - (q^{-1} - q)a_{i1}a_{i2}\{a_{l2}a_{j1} + (q^{-1} - q)a_{j2}a_{l1}\} - a_{i1}a_{l2}a_{i2}a_{j1} \\ &= q^{-1}a_{i1}a_{l2}a_{i1}a_{j2} - q^{-2}a_{i2}a_{l1}a_{i1}a_{j2} + q^{-3}a_{i2}a_{l1}a_{i2}a_{j1} \\ &\quad -q^{-2}a_{i1}a_{l2}a_{i2}a_{j1} \\ &= q^{-1}D_{il}D_{ij}. \end{aligned}$$

Similarly, after case by case calculations, we arrive at:

- Let $1 \leq i, j, k, l \leq 6$ be not all distinct, and D_{ij}, D_{kl} not both odd. Then
$$D_{ij} D_{kl} = q^{-1} D_{kl} D_{ij}, \qquad (i,j) < (k,l),\ i < j, k < l\,, \tag{6.27}$$
where the ordering '<' of pairs is the lexicographical ordering.

- Let $1 \leq i, j, k, l \leq 6$ be all distinct, and D_{ij}, D_{kl} not both odd and $D_{ij}, D_{kl} \neq D_{56}$. Then
$$\begin{aligned}
&D_{ij} D_{kl} = q^{-2} D_{kl} D_{ij}, && 1 \leq i < j < k < l \leq 6,\\
&D_{ij} D_{kl} = q^{-2} D_{kl} D_{ij} - (q^{-1} - q) D_{ik} D_{jl}, && 1 \leq i < k < j < l \leq 6,\\
&D_{ij} D_{kl} = D_{kl} D_{ij}. && 1 \leq i < k < l < j \leq 6,
\end{aligned} \tag{6.28}$$

- Let $1 \leq i < j \leq 4$, $5 \leq n \leq m \leq 6$. Then
$$\begin{aligned}
&D_{in} D_{jn} = -q^{-1} D_{jn} D_{in} - (q^{-1} - q) D_{ij} D_{nn} = -q D_{jn} D_{in},\\
&D_{ij} D_{nm} = q^{-2} D_{nm} D_{ij},\\
&D_{i5} D_{j6} = -q^{-2} D_{j6} D_{i5} - (q^{-1} - q) D_{ij} D_{56},\\
&D_{i6} D_{j5} = -D_{j5} D_{i6},\\
&D_{i5} D_{i6} = -q^{-1} D_{i6} D_{i5},\\
&D_{i5} D_{i6} = -q^{-1} D_{i6} D_{i5},\\
&D_{55} D_{66} = q^{-2} D_{66} D_{55}\,,\\
&D_{55} D_{56} = 0\,.
\end{aligned} \tag{6.29}$$

The Plücker relations are modified. One has for $1 \leq i < j < k \leq 4$ and $n = 5, 6$:

$$\begin{aligned}
&D_{12} D_{34} - q^{-1} D_{13} D_{24} + q^{-2} D_{14} D_{23} = 0,\\
&D_{ij} D_{kn} - q^{-1} D_{ik} D_{jn} + q^{-2} D_{jk} D_{in} = 0,\\
&D_{i5} D_{j6} + q^{-1} D_{i6} D_{j5} = q D_{ij} D_{56},\\
&D_{in} D_{jn} = q D_{ij} D_{nn},\\
&D_{in} D_{nn} = 0,\\
&D_{i5} D_{66} = -q^{-1} D_{i6} D_{56},\\
&D_{i6} D_{55} = -q^{2} D_{i5} D_{56}\\
&D_{nn}^2 = 0,\\
&D_{55} D_{56} = 0,\\
&D_{66} D_{56} = 0\\
&D_{56} D_{56} = (q^{-1} - 3q) D_{55} D_{66}\,.
\end{aligned} \tag{6.30}$$

The first relation in (6.29) has been simplified with the use of the fourth relation in (6.30). For the case of $\mathrm{Gr}(2|0,4|1)$, these relations are given in [39, Chap. 5].

To finish the interpretation of the quantum super grassmannian as $\mathrm{SL}_q(4|2)$-comodule, we have to see how the coaction restricts to Gr_q. Let us denote with Δ both, the comultiplication and its restriction to Gr_q in order not to burden the notation. The meaning should be clear from the context.

Proposition 6.5.2. *The restriction of the comultiplication in* $\mathrm{SL}_q(4|2)$,

$$\begin{array}{ccc} \mathrm{SL}_q(4|2) & \xrightarrow{\Delta} & \mathrm{SL}_q(4|2)\otimes \mathrm{SL}_q(4|2) \\ a_{ij} & \longrightarrow & \Delta(a_{ij}) = \sum_{k=1}^{6} a_{ik}\otimes a_{kj}, \end{array}$$

to the subalgebra Gr_q *is of the form,*

$$\mathrm{Gr}_q \xrightarrow{\Delta} \mathrm{SL}_q(4|2)\otimes \mathrm{Gr}_q\,.$$

Proof. The coaction property is guaranteed by the associativity of the coproduct, so we only have to check that,

$$\Delta(D_{ij}), \Delta(D_{im}), \Delta(D_{mn}) \in \mathrm{SL}_q(4|2)\otimes \mathrm{Gr}_q\,.$$

Let us denote as $D_{ij}^{kl} = a_{ik}a_{jl} - q^{-1}a_{il}a_{jk}$, so in the previous notation $D_{ij} = D_{ij}^{12}$. After some calculations one can prove

1. Let us call P the condition $1 \le k, l \le 6$ and at least one of the two indices is less that 5. For $1 \le i < j \le 4$:

$$\begin{aligned}\Delta(D_{ij}) = &\sum_{P\cap(k<l)} D_{ij}^{kl}\otimes D_{kl}^{12} - (a_{i5}a_{j6} + q^{-1}a_{i6}a_{j5})\otimes D_{56} \\ &- (1+q^{-2})\sum_{5\le k\le 6} a_{ik}a_{jk}\otimes D_{kk}\,.\end{aligned}$$

2. For $1 \le i \le 4$ and $5 \le m \le 6$:

$$\begin{aligned}\Delta(D_{im}) = &\sum_{\substack{k<5\\ k<l}} a_{ik}a_{ml}\otimes D_{kl} - q^{-1}\sum_{\substack{k<5\\ l<k}} a_{ik}a_{ml}\otimes D_{lk} \\ &+ (a_{i5}a_{m6} + q^{-1}a_{i6}a_{m5})\otimes D_{56} \\ &+ (1+q^{-2})\sum_{5\le k\le 6} a_{ik}a_{mk}\otimes D_{kk} + q^{-1}\sum_{\substack{k\ge 5\\ l<5}} a_{ik}a_{ml}\otimes D_{lk}\,.\end{aligned}$$

3. For $5 \leq m, n \leq 6$:

$$\Delta(D_{56}) = \sum_{\substack{k<5 \\ k<l}} a_{5k}a_{6l} \otimes D_{kl} - q^{-1} \sum_{\substack{k<5 \\ l<k}} a_{5k}a_{6l} \otimes D_{lk}$$
$$+ (a_{55}a_{66} + q^{-1}a_{56}a_{65}) \otimes D_{56}$$
$$+ (1+q^{-2}) \sum_{5\leq k\leq 6} a_{5k}a_{6k} \otimes D_{kk} + q^{-1} \sum_{\substack{k\geq 5 \\ l<5}} a_{5k}a_{6l} \otimes D_{lk}\,,$$

and

$$\Delta(D_{nn}) = \sum_{1\leq k<l\leq 6} a_{nk}a_{nl} \otimes D_{kl} + \sum_{5\leq k\leq 6} a_{nk}^2 \otimes D_{kk}\,.$$

This proves our statement. □

6.6 $N = 2$ Minkowski superspace

In this section we introduce the notion of $N = 2$ Minkowski superspace realizing it as the big cell inside $\mathrm{Gr}(2|0, 4|2)$. Consider the set of $4 \times 2\,|\,2 \times 2$ supermatrices with complex entries,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{41} \\ \alpha_{51} & \alpha_{52} \\ \alpha_{61} & \alpha_{62} \end{pmatrix}. \tag{6.31}$$

This can be seen as the affine superspace $\mathbb{A}^{8|4}$ described by the coordinate superalgebra $\mathbb{C}[a_{ij}, \alpha_{kl}]$ with $i = 1, \ldots, 4,\ j, l = 1, 2,\ k = 5, 6$. As in the ordinary setting, we can view elements in $\mathbb{A}^{8|4}$ as $2|0$ subspaces of $\mathbb{C}^{4|2}$:

$$W = \mathrm{span}\{a_1, a_2\} \subset \mathbb{C}^{4|2}.$$

In this way, W may also be viewed as an element in $\mathrm{Gr}(2|0, 4|2)$.

In the superspace $\mathbb{A}^{8|4}$ consider the open subset S consisting of matrices such that the minor formed with a_{ij}, $i, j = 1, 2$ is invertible. This open subset S is described by its coordinate superalgebra:

$$\mathbb{C}[S] = \mathbb{C}[a_{ij}, \alpha_{kl}][T]/((a_{11}a_{22} - a_{12}a_{21})T - 1).$$

We have a right action of $\mathrm{GL}_2(\mathbb{C})$ on S corresponding to the change of basis of such subspaces:

$$\mathrm{span}\{a_1, a_2\}, g \mapsto \mathrm{span}\{a_1 \cdot g, a_2 \cdot g\}, \qquad g \in \mathrm{GL}_2(\mathbb{C})\,.$$

Definition 6.6.1. Define the $N = 2$ *Minkowski superspace* $\mathbf{M}$ as the quotient of S by the right $\mathrm{GL}_2(\mathbb{C})$-action.

Proposition 6.6.2. *Let the notation be as above. Then,* $\mathbf{M}$ *(the* $N = 2$ *Minkowski superspace) is an affine superspace of dimension* $4|4$*.*

Proof. We can write:

$$\mathbf{M} = \left\{ (a_1, a_2),\ a_1, a_2 \in \mathbb{C}^{4|2} \mid \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \neq 0 \right\} \Big/ \mathrm{GL}_2(\mathbb{C}). \tag{6.32}$$

In the quotient $\mathbf{M}$ we can choose a (unique) representative (u, v) for (a_1, a_2) of the form:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ u_1 \\ u_2 \\ \nu_3 \\ \nu_4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ v_1 \\ v_2 \\ \eta_3 \\ \eta_4 \end{pmatrix} \right\}, \tag{6.33}$$

so $\mathbf{M}$ is $\mathbb{C}^{4|4}$. □

We notice that $\mathbf{M}$ is naturally identified with the dense open set of the Grassmannian Gr in the Plücker embedding, determined by the invertibility of the coordinate q_{12} in $\mathbb{P}^{8|8}$.

We now would like to retrieve a set of global coordinates for $\mathbf{M}$ starting from the global coordinates a_{ij} for S. Let $\mathbb{C}[\mathrm{GL}_2] = \mathbb{C}[g_{ij}][T]/((g_{11}g_{22} - g_{12}g_{21})T - 1)$ be the coordinate algebra for the algebraic group $\mathrm{GL}_2(\mathbb{C})$. Let us write heuristically the equation relating the generators of $\mathbb{C}[S]$, $\mathbb{C}[\mathrm{GL}_2]$ and the polynomial superalgebra $\mathbb{C}[\mathbf{M}] := \mathbb{C}[u_{ij}, \nu_{kl}]$,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \\ \alpha_{51} & \alpha_{52} \\ \alpha_{61} & \alpha_{62} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ u_{31} & u_{32} \\ u_{41} & u_{42} \\ \nu_{51} & \nu_{52} \\ \nu_{61} & \nu_{62} \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} . \tag{6.34}$$

We obtain immediately:

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} .$$

and then with a short calculation,

$$u_{i1} = -d_{2i} d_{12}^{-1}, \qquad u_{i2} = d_{1i} d_{12}^{-1},$$
$$\nu_{k1} = -d_{2k} d_{12}^{-1}, \qquad \nu_{k2} = d_{1k} d_{12}^{-1},$$

for $i = 3, 4$ and $k = 5, 6$, where:

$$d_{rs} := a_{r1} a_{s2} - a_{r2} a_{s1}, \qquad r < s \, .$$

Proposition 6.6.3. *Let the notation be as above.*

1. *The complex supermanifold S is diffeomorphic to the supermanifold $\mathbb{C}^{4|4} \times \mathrm{GL}_2(\mathbb{C})$:*

   $$S \xrightarrow{\ \psi\ } \mathbb{C}^{4|4} \times \mathrm{GL}_2(\mathbb{C}) \, ,$$

   with

   $$\begin{aligned} &\psi^*(g_{ij}) = a_{ij}, \\ &\psi^*(u_{i1}) = -d_{2i} d_{12}^{-1}, \qquad && \psi^*(u_{i2}) = d_{1i} d_{12}^{-1}, \\ &\psi^*(\nu_{k1}) = -d_{2k} d_{12}^{-1}, \qquad && \psi^*(\nu_{k2}) = d_{1k} d_{12}^{-1}. \end{aligned}$$

2. *The diffeomorphism ψ is $\mathrm{GL}_2(\mathbb{C})$-equivariant with respect to the right $\mathrm{GL}_2(\mathbb{C})$ action, hence $S/\mathrm{GL}_2(\mathbb{C}) \cong \mathbb{C}^{4|4}$.*

Proof. We notice that ψ is invertible, ψ^{-1} is given by:

$$(\psi^{-1})^*(a_{ij}) = g_{ij},$$

and the rest follows from equation (6.34). The right equivariance of ψ is a simple calculation, taking into account that the determinants d_{ij} transform as $d_{ij} \det g'$, were $g' \in \mathrm{GL}_2(\mathbb{C})$. □

6.7 Quantum $N = 2$ Minkowski superspace

In this section we introduce the notion of *quantum $N = 2$ Minkowski superspace*. We also want to reinterpret our construction in the framework of quantum principal bundles, as in [3, 4] and references therein. We recall here briefly, the key definitions in order to put in the correct framework.

Definition 6.7.1. Let (H, Δ, ϵ, S) be a Hopf superalgebra and A be an H-comodule superalgebra with coaction $\delta : A \longrightarrow A \otimes H$. Let

$$B := A^{\mathrm{coinv}(H)} := \{a \in A \,|\, \delta(a) = a \otimes 1\} \; . \tag{6.35}$$

The extension A of the superalgebra B is called *H-Hopf-Galois* (or simply Hopf-Galois) if the map:

$$\chi : A \otimes_B A \longrightarrow A \otimes H, \qquad \chi = (m_A \otimes \mathrm{id})(\mathrm{id} \otimes_B \delta)$$

called the canonical map, is bijective (m_A denotes the multiplication in A).

The extension $B = A^{\mathrm{coinv}(H)} \subset A$ is called *H-principal comodule superalgebra* if it is Hopf-Galois and A is H-equivariantly projective as a left B-supermodule, i.e., there exists a left B-supermodule and right H-comodule morphism $s : A \to B \otimes A$ that is a section of the (restricted) product $m : B \otimes A \to A$.

We now follow [3, §2], in giving the definition of quantum principal bundle, though it differs slightly from the one given in the literature, which also requires the existence of a differential structure (see e.g. [14, Chap. 5]).

Definition 6.7.2. We define *quantum principal bundle* a pair (A, B), where A is an H-Hopf Galois extension and A is H-equivariantly projective as a left B-supermodule, that is, A is an H-principal comodule superalgebra.

There is a special case of Hopf-Galois extensions, corresponding to a globally trivial principal bundle. In this case the technical conditions of Definition 6.7.2 are automatically satisfied. We shall focus on this case leaving aside the general one.

Definition 6.7.3. Let H be a Hopf superalgebra and A an H-comodule superalgebra. The algebra extension $A^{\mathrm{coinv}(H)} \subset A$ is called a *cleft extension* if there is a right H-comodule map $j : H \to A$, called *cleaving map*, that is convolution invertible, i.e. there exists a map $h : H \to A$ such that the convolution product $j \star h$ satisfies,

$$j \star h := m_A \circ (j \otimes h) \circ \Delta(f) = \epsilon(f) \cdot 1$$

or, in Sweedler's notation,

$$\sum j(f_{(1)})h(f_{(2)}) = \epsilon(f) \cdot 1$$

for all $f \in H$. The map h is the convolution inverse of j.

An extension $A^{\mathrm{coinv}(H)} \subset A$ is called a *trivial extension* if there exists such map. Notice that when j is an algebra map, its convolution inverse is just $h = j \circ S^{-1}$.

We now examine the example of $N = 2$ Minkowski superspace.

Lemma 6.7.4. *The coordinate superalgebra* $\mathbb{C}[\mathbf{M}] := \mathbb{C}[u_{ij}, \nu_{kl}]$ *is isomorphic to the subalgebra of coinvariants:*

$$\mathbb{C}[S]^{\mathrm{coinv}\,\mathbb{C}[\mathrm{GL}_2]} := \{g \in \mathbb{C}[S] \mid \delta(g) = g \otimes 1\},$$

in $\mathbb{C}[S]$ *with respect to the* $\mathbb{C}[\mathrm{GL}_2]$ *right coaction* δ*:*

$$\begin{array}{ccc} \mathbb{C}[S] & \stackrel{\delta}{\longrightarrow} & \mathbb{C}[S] \otimes \mathbb{C}[\mathrm{GL}_2] \\ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \\ \alpha_{51} & \alpha_{52} \\ \alpha_{61} & \alpha_{62} \end{pmatrix} & \longrightarrow & \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \\ \alpha_{51} & \alpha_{52} \\ \alpha_{61} & \alpha_{62} \end{pmatrix} \otimes \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}. \end{array}$$

Proof. In our calculations we computed expressions (given above in Proposition 6.6.3) for the coordinates on $\mathbf{M}$. We claim that these are coinvariants, so we need to show $\delta(c) = c \otimes 1$ for any $c \in \{u_{ij}, \nu_{kl}\}$. A little calculation gives us:

$$\delta(d_{rs}) = d_{rs} \otimes (g_{11}g_{22} - g_{12}g_{21}) \;\Rightarrow\; \delta(d_{rs}d_{12}^{-1}) = d_{rs}d_{12}^{-1} \otimes 1$$

which proves our claim.

On the other hand, the space of functions on S that are invariant under the right translation of the group can be identified with the space of functions on the quotient $S/\mathrm{GL}_2(\mathbb{C})$. Since we have global coordinates in $\mathbf{M}$, any other invariant will be a function of these coordinates. In the Hopf algebra language, this means that $\{u_{ij}, \nu_{kl}\}$ are the only independent coinvariants. □

Proposition 6.7.5. *Let the notation be as above. The natural projection* $p : S \longrightarrow S/\mathrm{GL}_2(\mathbb{C})$ *is a trivial principal bundle on* $\mathbf{M}$*.*

Proof. We will show that $\mathbb{C}[S]$ is a trivial $\mathbb{C}[\mathrm{GL}_2]$-Hopf Galois extension of $\mathbb{C}[\mathbf{M}]$. In Lemma 6.7.4, we proved that $\mathbb{C}[\mathbf{M}] \cong \mathbb{C}[S]^{\mathrm{coinv}\,\mathbb{C}[\mathrm{GL}_2]}$, so if we give an algebra cleaving map we are done.

We define:

$$\begin{array}{ccc} \mathbb{C}[\mathrm{GL}_2] & \xrightarrow{j} & \mathbb{C}[S] \\ g_{ij} & \longrightarrow & a_{ij}\,. \end{array}$$

It is easy to check (below in Theorem 6.7.8, for the quantum case, the explicit details are given) that j is convolution invertible with convolution inverse:

$$h = j \circ S\,.$$

Moreover, the calculation below shows that j is a $\mathbb{C}[\mathrm{GL}_2(\mathbb{C})]$-comodule map,

$$(\delta \circ j)(g_{ij}) = \delta(a_{ij}) = \sum a_{ik} \otimes g_{kj}.$$

$$((j \otimes \mathrm{id}) \circ \Delta)(g_{ij}) = (j \otimes \mathrm{id})(\sum g_{ik} \otimes g_{kj}) = \sum a_{ik} \otimes g_{kj}.$$

$$\Rightarrow \quad \delta \circ j = (j \otimes \mathrm{id}) \circ \Delta.$$

This proves the result. □

We now go to the quantum setting, where we lose the geometric interpretation and we retain only the algebraic point of view. Hence a quantum principal super bundle over an affine base is just understood as a Hopf-Galois extension with the properties mentioned in Definition 6.7.2.

We want to study the quantization of the example studied above. Let $\mathbb{C}_q[S]$ be the quantization of the superalgebra $\mathbb{C}[S]$ obtained by taking the super Manin relations among the entries still denoted as a_{ij} and α_{kl}, with $i, j = 1, \dots, 4$ and $k, l = 5, 6$.

Definition 6.7.6. The $N = 2$ *quantum chiral Mikowski superspace* $\mathbb{C}_q[\mathbf{M}]$ is defined as the subalgebra of $\mathbb{C}_q[S][D_{12}^{-1}]$ generated by the elements:

$$\tilde{u}_{i1} = -q^{-1} D_{2i} D_{12}^{-1}, \qquad \tilde{u}_{i2} = D_{1i} D_{12}^{-1},$$
$$\tilde{\nu}_{k1} = -q^{-1} D_{2k} D_{12}^{-1}, \qquad \tilde{\nu}_{k2} = D_{1k} D_{12}^{-1}\,,$$

for $i = 3, 4$, and $k = 5, 6$, in $\mathbb{C}_q[\mathrm{Gr}]$, where:

$$D_{rs} := a_{r1} a_{s2} - q^{-1} a_{r2} a_{s1}, \qquad r < s.$$

Notice that, D_{12} q-commutes (commutes up to multiplication by a power of q) with the generators of $\mathbb{C}_q[S]$, therefore, the localization $\mathbb{C}_q[S][D_{12}^{-1}]$ is well-defined.

Similar to Theorem 6.2.5, we have the following.

Proposition 6.7.7. *The quantum chiral Minkowski superspace* $\mathbb{C}_q[M]$ *is isomorphic to the quantum superalgebra of matrices* $\mathrm{M}_q(2|2)$*.*

Proof. We define the map $\beta : \mathbb{C}_q[M] \longrightarrow \mathrm{M}_q(2|2)$ by giving it on the generators as follows:

$$\beta(\tilde{u}_{ij}) := z_{rs}, \qquad \text{where} \;\; r = i-2, \;\; \text{and} \;\; s = \begin{cases} 1 \;\; \text{if} \;\; j = 2, \\ 2 \;\; \text{if} \;\; j = 1, \end{cases}$$

$$\beta(\tilde{\nu}_{kl}) := \xi_{mn}, \qquad \text{where} \;\; m = k-2, \;\; \text{and} \;\; n = \begin{cases} 1 \;\; \text{if} \;\; l = 2, \\ 2 \;\; \text{if} \;\; l = 1\,. \end{cases}$$

It is clearly bijective. Using our previous computations for commutation relations among D_{rs}'s we get the commutation relations among $\tilde{u}_{ij}$'s and $\tilde{\nu}_{kl}$'s and comparing with the commutation relations for $\mathrm{M}_q(2|2)$ we observe that it is an isomorphism. □

We now present the final result for this section.

Theorem 6.7.8. *The quantum superalgebra* $\mathbb{C}_q[S]$ *is a trivial quantum principal super bundle on the quantum chiral Minkowski superspace.*

Proof. We need to show that $\mathbb{C}_q[S]$ is a trivial Hopf-Galois extension of $\mathbb{C}_q[M]$. We will proceed similarly to the classical case. It is easy to see that the quantum version of Lemma 6.7.4 also holds. It is enough to check that:

$$\delta_q(D_{rs}) = D_{rs} \otimes (g_{11}g_{22} - q^{-1}g_{12}g_{21})\,.$$

Therefore, we need to give an algebra cleaving map $j_q : \mathbb{C}_q[\mathrm{GL}_2(\mathbb{C})] \longrightarrow \mathbb{C}_q[S]$.

Define:

$$j_q(g_{ij}) := a_{ij}, \qquad h_q = j_q \circ S_q.$$

Therefore,

$$h_q : \mathbb{C}_q[\mathrm{GL}_2(\mathbb{C})] \longrightarrow \mathbb{C}_q[S]$$
$$h_q(g_{11}) := D^{-1}a_{22}, \qquad h_q(g_{12}) := -qD^{-1}a_{12},$$
$$h_q(g_{21}) := -q^{-1}D^{-1}a_{21}, \qquad h_q(g_{22}) := D^{-1}a_{11},$$

where $D := a_{11}a_{22} - q^{-1}a_{12}a_{21}$. One can observe that:

$$j_q \star h_q = \varepsilon.1 = h_q \star j_q,$$

where $\star$ denotes the convolution product, i.e j_q is convolution invertible. Moreover, a similar calculation to the one given in Proposition 6.7.5 shows that j_q is a $\mathbb{C}_q[\mathrm{GL}_2]$-comodule map, i.e. $\delta_q \circ j_q = (j_q \otimes \mathrm{id}) \circ \Delta$. Therefore, j_q is an algebra cleaving map and $\mathbb{C}_q[M] \subset \mathbb{C}_q[S]$ is a trivial extension. Hence, the theorem is proven. $\square$

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