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COHOMOLOGY AND COMBINATORICS OF ABELIAN ARRANGEMENTS

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Abstract

The study of hyperplane arrangements, originating in the 1960s, has seen recent advancements that renewed interest in generalizing classical results to broader contexts. This thesis aims to extend foundational results by investigating both cohomology and combinatorics in the wider framework of abelian arrangements.

We begin by presenting the cohomology ring of the complement of abelian arrangements. Using a technique that pushes forward cohomological relations from the real hyperplane case, we develop an Orlik-Solomon type presentation for noncompact abelian arrangements. This approach provides both an original result for the general case and a new proof of the Orlik-Solomon presentation for complexified hyperplane arrangements, as well as the De Concini-Procesi presentation for unimodular toric arrangements.

We then turn to combinatorial aspects, introducing definitions for inductively and divisionally free abelian arrangements based on poset structures. After proving the factorization of their characteristic polynomial, we show that inductively free arrangements include strictly supersolvable arrangements as a proper subclass, extending a well known result of Jambu and Terao. We further apply these findings to toric arrangements associated with ideals of root systems of types A, B and C, showing their inductiveness and providing a formula to compute all exponents.

Finally, we move beyond the abelian context to study elliptic arrangements from a new perspective: we focus on elliptic curves with complex multiplication, where the endomorphism ring strictly contains $\mathbb{Z}$, and this leads to significantly different behaviors. We compute the number of connected components in the intersections of any subset of the arrangement and use this result to associate an arithmetic matroid structure to an elliptic arrangement, opening new possibilities for further studies in generalized arrangement theory.

Contents

Introduction

The study of hyperplane arrangements is a classical subject whose study goes back to the sixties. It is originally motivated by the relations with braid groups [Del72], configuration spaces [CLM76], singularity theory and, from a more combinatorial point of view, by matroids.

In the last few years, new developments in matroid theory renewed the interest in the topology of arrangements and their compactifications: June Huh proved inequalities for invariants of realizable matroids by using properties of hyperplane arrangements [Huh12], that were later generalized to all matroids [AHK18]. Furthermore, hyperplane arrangements have found applications as local models in algebraic geometry, especially in moduli problems [dCHM21, MMP23].

This renewed interest has led to an increasing attention towards extending the known properties of hyperplane arrangements to toric arrangements and, more generally, to abelian arrangements. In the 2000s, De Concini and Procesi [DCP05], motivated by the application to knapsack problem, started the study of toric arrangements, presenting the cohomology ring of their complement in unimodular cases, a result later generalized by [CDD$^+$20]. D'Adderio and Moci [DM13], and Branden and Moci [BM14] further developed the theory of arithmetic matroids, offering a combinatorial counterpart to the geometry of toric arrangements. Finally, most recently, Bibby [Bib16] introduced the definition of abelian arrangements and, together with Delucchi [BD22], investigated in this context properties such as supersolvability and fiber-type.

This thesis aims to delve deeper into the study of abelian arrangements, broadening and generalizing concepts that are well known in classical cases. In the first chapter, we present the cohomology ring of the complement of abelian arrangements. We then turn to the combinatorial aspects in the second chapter, providing a new definition of inductive and divisionally free abelian arrangements and exploring their properties. In the final chapter, we shift our focus to investigate elliptic arrangements, a relatively unexplored topic in the literature. We provide a more general definition of elliptic arrangements which, in the case of complex multiplication, leads to considerably different phenomena. Let us proceed with a deeper discussion of each chapter.

In Chapter 1, we provide a presentation of the cohomology ring of the complement of an abelian arrangement. Before proceeding to this, we first introduce definitions and notations of abelian arrangements and review classical cases, which have been studied since the 1960s and of which abelian arrangements are a generalization.

A milestone is the famous result by Orlik and Solomon [OS80], who presented the cohomology ring of complex hyperplane arrangements through generators and relations. In the first section of the chapter, we present their construction of an algebra, defined as the quotient of an exterior algebra by an ideal, that they proved to be isomorphic to the cohomology and fully combinatorially determined, finding a basis of the so called *nbc*-sets.

We then focus, in section 2, on real hyperplane arrangements, highlighting the work of Gelfand and Varchenko [VG87]. They provide a presentation of the cohomology ring of the complement as the ring of functions from the chambers to the integers, with the generators represented by Heaviside functions. Although they did not find a geometric connection between the real and complex cases, they observed that one of their relation is an analogue of a relation of Orlik and Solomon for differential forms.

Chronologically following these results are those concerning subspace arrangements. Goresky and MacPherson [GM88] determined the cohomology of subspace arrangements as a module, and a rational model was provided by De Concini and Procesi [DCP95]. The multiplicative structure of the integral cohomology ring was later studied by Feichtner and Ziegler [FZ00], de Longueville and Schultz [dLS01] and Deligne, Goresky and MacPherson [DGM00]. Then, Moseley [Mos17] found an isomorphism between the algebra of Gelfand and Varchenko and the cohomology of (subspace) arrangements in $(\mathbb{R}^3)^n$.

A significant generalization involves toric arrangements. De Concini and Procesi [DCP05] provided an Orlik-Solomon type presentation for toric arrangements in the unimodular case, which is discussed in section 3 of this chapter. This result was later extended to all toric arrangements in [CD17, CDD$^+$20], employing the technique of separating covers, which we utilize in our work.

In the broader context of abelian arrangements, however, few results are currently known. Bibby [Bib16] computed the Euler characteristic for complex groups, while Liu, Tran, and Yoshinaga [LTY21] described the additive structure of the cohomology in the case of noncompact abelian groups. In the final section of the first chapter, we present a new and uniform approach to the cohomology ring of noncompact abelian arrangements. Our technique consists in the pushforward of the cohomological relations from the real hyperplane case, [VG87], to the general case of noncompact abelian arrangements. The main result of this chapter (Theorem 1.5.29) is an Orlik-Solomon type presentation of the cohomology ring of noncompact abelian arrangements, stating the following

Theorem A. *Let $\mathcal{A}$ be an arrangement in G^r, where $G = \mathbb{R}^b \times (S^1)^a$ with $b > 0$. The integer cohomology of the complement $H^*(M(\mathcal{A}); \mathbb{Z})$ is generated by classes $\eta_{W,A,B}$ (see Equation (1.15)) with relations (1.23)-(1.26).*

Through this, we obtain a new proof of the Orlik-Solomon result [OS80] for complexified hyperplane arrangements and the De Concini-Procesi presentation [DCP05] for unimodular toric arrangements, as well as a presentation for subspace arrangements. In all other cases our result is original. This is a joint work with Evienia Bazzocchi and Roberto Pagaria [BPP24].

Beside the study of the cohomology ring of the complement, arrangements have been widely investigated under many aspects and one of these is freeness. An hyperplane arrangement is said to be *free* if its module of logarithmic derivations is a free module. A remarkable theorem connecting algebra and combinatorics of arrangements, due to Terao, asserts that if an arrangement is free, then it is factorable and its combinatorial exponents coincide with the degrees of a basis for the derivation module. Based on this, Terao conjectured that freeness is a combinatorial property [OT92, Conjecture 4.138], and this conjecture remains open till now even in dimension 3.

A natural approach to Terao's conjecture is to identify classes of arrangements whose freeness is combinatorially determined. In [Ter80], Terao introduced the class of inductively free arrangements in the context of proving the famous addition-deletion theorem for free arrangements. This class consists of arrangements constructed from the empty arrangement by sequentially adding hyperplanes, subject to conditions on inductive freeness and a divisibility condition on the characteristic polynomials. Jambu and Terao [JT84] showed that this class includes the significant one of supersolvable arrangements, first defined by Stanley [Sta72]. Later, Abe [Abe16], improving the addition-deletion theorem, refined this concept by defining a proper subclass of arrangements called divisionally free arrangements. Both inductively and divisionally free arrangements are combinatorially determined and factorable, forming proper subclasses of free arrangements.

The main motivation of Chapter 2 is a pursuit of a theory for *free abelian arrangements*. Nothing is still known about how to pass from algebraic consideration of freeness of hyperplane arrangements to abelian or just toric arrangements. However, at the purely combinatorial level using only information from the posets, it is possible to define and study the combinatorial structures of abelian arrangements and geometric posets in the same way that inductive freeness and divisional freeness do for hyperplane arrangements and geometric lattices.

In the first section of Chapter 2, we provide a brief survey on supersolvability, both in hyperplane case [Sta72] and in abelian case [BD22], freeness and their related topological consequences. Right after, we define inductive and divisional arrangements in the abelian context, starting from combinatorial properties of the associated poset of layers, and prove the factorization of their characteristic polynomials.

Theorem B. *Let $\mathcal{A}$ be an abelian arrangement. If $\mathcal{A}$ is divisional, then it is factorable.*

In section 3, we generalize the classical result of Jambu and Terao [JT84] for hyperplane arrangements which states that the class of supesolvable arrangements is a proper subclass of free arrangements.

Theorem C. *Let $\mathcal{A}$ be an abelian arrangement. If $\mathcal{A}$ is strictly supersolvable, then it is inductively free.*

Finally, applying this notion to toric arrangements, we obtain the following

Theorem D. *Every toric arrangement defined by an arbitrary ideal of a root system of type A, B or C with respect to the root lattice is inductively free.*

We not only prove that all toric arrangements of this type are inductively free, but we also provide a formula to compute all exponents. The content of this chapter is a result of a joint work with Roberto Pagaria, Tan Nhat Tran and Lorenzo Vecchi [PPTV23].

The research presented in the first two chapters focuses on noncompact abelian arrangements, primarily because the compact case is considerably more difficult to handle, especially from a cohomological point of view. Indeed, the existing literature on compact abelian arrangements is limited, with results only in low-dimensional cases. For instance, Bibby, in [Bib16], constructed a spectral sequence for the cohomology of the complement of abelian arrangements when $G = (S^1)^2$, known as elliptic arrangements, but even basic invariants as Betti numbers remain unknown. Motivated by these gaps, we have chosen to explore this topic further, though with a different perspective from the traditional approaches found in the literature. This is the focus of Chapter 3, which presents the first results in this direction of an on-going project with Roberto Pagaria and Alejandro Vargas.

This chapter aims to explore elliptic arrangements in the context of elliptic curves with complex multiplication $\mathcal{E} = \mathbb{C}/\Lambda$. As mentioned, research has focused on cases where the endomorphism ring of an elliptic curve is $\mathbb{Z}$, restricting to only multiplication by integers. In the case of complex multiplication, the endomorphism ring of $\mathcal{E}$ is an order in the integers ring of an imaginary quadratic number field (see Lemma 3.1.3).

The first section of this chapter defines elliptic arrangements with complex multiplication and provides a description of $R := \mathrm{End}(\mathcal{E})$. Then, the second section addresses a fundamental question that arises when dealing with arrangements that are not hyperplane arrangements: how to compute the number of connected components in the intersection of any subset of the subvarieties of an arrangement. We compute these components, proving the theorem below using techniques involving snake lemma, short exact sequences and Smith normal forms.

Theorem E. *Let $\mathcal{A}$ be an elliptic arrangement and S be a subset of its elements. Then, the number of connected components in the intersection $\cap S$ is given by*

$$m_{\mathcal{A}}(S) = \#\,\mathrm{tor}\,\mathrm{coker}\,A_{\Lambda}[S] = \#\,\mathrm{tor}\,\mathrm{coker}\,A_R[S],$$

where A is the matrix associated with the arrangement.

Finally, in section 3, we complete our combinatoric focus by showing that it is possible to associate the structure of an arithmetic matroid to an elliptic arrangement, as it happens in the toric case. Indeed,

Theorem F. *The triple $([k], \mathrm{rk}_{\mathcal{A}}, m_{\mathcal{A}})$, encoding the information of codimension and number of layers of intersections of an elliptic arrangement, defines an arithmetic matroid.*

The hope is that this work will open the door to a deeper study of a new and more general type of arrangements.

Chapter 1

Cohomology

This chapter is dedicated to presenting the cohomology ring of the complement of an abelian arrangement. The first section aims to give a brief introduction to the definitions and notations in arrangement theory, applied in the abelian context. The following sections review the state of the art, discussing results for complex and real hyperplane arrangements and toric arrangements. In the final section, we provide an Orlik-Solomon type presentation of the cohomology of the complement in the abelian case (Theorem 1.5.29), addressing both central and non-central, unimodular and non-unimodular cases. We conclude with an application of the main theorem to the arrangements of type A_n improving a result of [CT78] on configuration spaces in $\mathbb{R}^b \times (S^1)^a$.

1.1 Introduction to abelian arrangements

The central subject, around which the entire work is built, is abelian arrangements. Let us therefore begin by providing some preliminaries, including their definition and few important related aspects.

Definition 1.1.1. Let G be an abelian connected Lie group, $G = \mathbb{R}^b \times (S^1)^a$, of real dimension $g := a + b$ canonically oriented, $e = (0, \cdots, 0, 1, \cdots, 1)$ the unit of G and let $\Lambda \cong \mathbb{Z}^r$ be a lattice with the choice of an orientation, so that G^r has a natural orientation. Every $\chi \in \Lambda$ defines a morphism $\chi : G^r \to G$. Let E be a finite set with a total order, then an abelian arrangement is a finite collection of connected subvarieties in G^r defined by a finite list of functions χ_i and elements $g_i \in G$ indexed by E, i.e.

$$\mathcal{A} := \{H_i := \chi_i^{-1}(g_i)\}_{i \in E}.$$

This definition, when applied to specific values of (a, b), corresponds to well known types of arrangements. For instance, if $(a, b) = (0, 1)$, then $G = \mathbb{R}$ and $\mathcal{A}$ is a real hyperplane arrangement, while for $(a, b) = (0, 2)$ it becomes a complex hyperplane arrangement with rational coordinates. If $(a, b) = (1, 1)$ then $G = \mathbb{C}^*$ and $\mathcal{A}$ is a toric arrangement. Finally, if $a = 0$, we obtain arrangements that are particular cases of subspace arrangements.

Abelian arrangements are the main characters of the first two chapter of this thesis. It is important to observe, however, that in both chapters we exclusively consider cases where G is *noncompact*, i.e. $b > 0$.

Definition 1.1.2. Given an arrangement $\mathcal{A} = \{H_i\}_{i \in E}$, the *complement* of the arrangement is

$$M(\mathcal{A}) = G^{\Lambda} \setminus \bigcup_{i \in E} H_i.$$

We also denote $M(\mathcal{A})$ by $M^{a,b}(\mathcal{A})$, when we need to specify what abelian group $G = \mathbb{R}^b \times (S^1)^a$ we are working with.

By examining the intersections of the arrangement's subvarieties, we introduce two further important definitions.

Definition 1.1.3. An arrangement $\mathcal{A} = \{H_i\}_{i \in E}$ is called *central* if $\cap_{i \in E} H_i \neq \emptyset$. In this case, you may assume that for each $i \in E$, $g_i = e \in G$.
An arrangement $\mathcal{A}$ is called *unimodular* if all the possible intersections of elements in $\mathcal{A}$ are either connected or empty.

This chapter gives a presentation of the cohomology of $M(\mathcal{A})$, covering all possible cases, both central and noncentral and distinguishing between unimodular and nonunimodular. Instead, the following two chapters focus exclusively on the central case.

One of the first aspects to consider when studying an arrangement is its combinatorial structure, for instance the number of the layers, the connected components of the intersections and hence the poset of layers.

Definition 1.1.4. Given an abelian arrangement $\mathcal{A} = \{H_i\}_{i \in E}$, its *poset of layers* $L(\mathcal{A})$ is the set

$$L(\mathcal{A}) := \{\text{nonempty connected components of } \cap_{i \in B} H_i \mid B \subseteq E\},$$

ordered by reverse inclusion. Define $\mathrm{rk}(\mathcal{A})$ to be the rank of $L(\mathcal{A})$, i.e. the rank of a maximal element in $L(\mathcal{A})$.

Remark 1.1.5. It is both interesting and useful to highlight the structure of the poset of layers of an arrangement. For central hyperplane arrangements, this poset is a geometric lattice. However, in more general cases, the uniqueness of the join is not satisfied due to the possibility of multiple connected components in the intersections. Nevertheless, it keeps similar properties that reflect the underlying geometry. Indeed, the poset of layers in toric or more general abelian arrangements is a geometric poset [Bib22]. We will delve deeper in this topic in Chapter 2.

Having shown that a poset structure can be associated with an arrangement, we now introduce a classical object that can be studied in relation to this structure.

Definition 1.1.6. Given an arrangement $\mathcal{A}$, the *characteristic polynomial* $\chi_{\mathcal{A}}(t)$ of $\mathcal{A}$ is defined by

$$\chi_{\mathcal{A}}(t) := \sum_{X \in L(\mathcal{A})} \mu(T, X) t^{\dim_{\mathbb{R}}(X)}.$$

Here $\mu := \mu_{L(\mathcal{A})}$ is the Möbius function of $L(\mathcal{A})$.

Remark 1.1.7. The characteristic polynomial of an arrangement $\mathcal{A}$ is closely related to the classical characteristic polynomial of $L(\mathcal{A})$. Indeed, note that $\chi_{\mathcal{A}}(t) = t^{g(\ell - \mathrm{rk}(\mathcal{A}))} \cdot \chi_{L(\mathcal{A})}(t^g)$ which has degree $g\ell$. In particular, if $\mathcal{A}$ is *essential*, i.e. $\mathrm{rk}(\mathcal{A}) = \ell$, and $g = 1$, then $\chi_{\mathcal{A}}(t) = \chi_{L(\mathcal{A})}(t)$.

When dealing with arrangements, a fundamental technical tool is the method of *deletion-contraction*, which allows induction on the number of subvarieties in an arrangement.

Definition 1.1.8. Given an abelian arrangement $\mathcal{A}$ and fix a subvariety $H \in \mathcal{A}$, we define the *deletion* of H as $\mathcal{A}' := \mathcal{A} \setminus \{H\}$ and the *restriction* to H as $\mathcal{A}'' := \{H \cap K \mid K \in \mathcal{A}'\}$. When the hyperplane H with respect to which we are applying deletion and restriction is clear from context, we denote the deletion by $\mathcal{A}_H$ and the restriction by $\mathcal{A}^H$.

Lemma 1.1.9. *Let $\mathcal{A}$ be an abelian arrangement. Let $H \in \mathcal{A}$ and $X \in L(\mathcal{A})$ be the corresponding layer. Then $L(\mathcal{A}_H) \simeq L(\mathcal{A})_{\leq X}$ and $L(\mathcal{A}^H) = L(\mathcal{A})_{\geq X}$.*

From this Lemma, we have that $L(\mathcal{A}') = L(\mathcal{A})'$ and $L(\mathcal{A}'') = L(\mathcal{A})''$. For the sake of notation we denote $L(\mathcal{A})'$ as L' and $L(\mathcal{A})''$ as L''.

The following theorem is well-known in the literature, with different proofs depending on the context and type of arrangement.

Theorem 1.1.10. *Let $\mathcal{A}$ be a nonempty abelian arrangement and $H \in \mathcal{A}$. The following deletion-restriction formula holds*

$$\chi_{\mathcal{A}}(t) = \chi_{\mathcal{A}'}(t) - \chi_{\mathcal{A}''}(t).$$

It is natural to ask whether the characteristic polynomial factors and what its roots are. Indeed, its factorization is a consequence of several properties that have been studied in the context of hyperplane arrangements since the beginning, such as supersolvability and freeness, both of which have significant implications not only combinatorially but also topologically. With the recent renewed interest in arrangements, research has focused on extending all these concepts to broader contexts. In this regard, let us define the Poincaré polynomial of an arrangement, which, as we will see later in the chapter, is closely related to the characteristic polynomial.

Definition 1.1.11. Given an arrangement $\mathcal{A}$, the associated *Poincaré polynomial* is

$$P_{\mathcal{A}}(t) := \sum_{p \geq 0} \dim H^p(M(\mathcal{A}))t^p = \sum_{p \geq 0} b_p(M(\mathcal{A}))t^p.$$

1.2 Complex hyperplane arrangements

The complement of certain hyperplanes in complex space has been an important area of study since the '60s. The first arrangement that has been considered is the braid arrangement $\mathcal{A}_\ell = \{H_{i,j} := \ker(z_i - z_j) \mid i, j \in [\ell],\ i \neq j\}$ with complement given by $M(\mathcal{A}_\ell) = \{z \in \mathbb{C}^\ell | z_i \neq z_j \text{ for } i \neq j\}$. In [Arn69], Arnol'd proved that

$$P_{\mathcal{A}_\ell}(t) = (1+t)(1+2t)\dots(1+(l-1)t).$$

He constructed a graded algebra A as the quotient of an exterior algebra by a homogeneous ideal and showed that there is an isomorphism of graded algebras $H^*(M(\mathcal{A}_\ell)) \cong A$. This provides a presentation of the cohomology ring of the pure braid space in terms of generators and relations. Brieskorn, in [Bri72], extended Arnold's work by generalizing the symmetric group and braid arrangement to a finite Coxeter group and its reflection representation. He also proved a result that became one of the most powerful tools for understanding and studying the cohomology of the arrangement complements, a result that will see further generalizations in this chapter. This significant result, known as Brieskorn's lemma, gives a sense of how the cohomology ring of an arrangement's complement can be described by local properties.

Lemma 1.2.1 (Brieskorn's Lemma, [Bri72]). *Let $\mathcal{A}$ be an arrangement. For all k the map*

$$\bigoplus_{X\in L,\ \mathrm{rk}(X)=k} H^k(M(\mathcal{A}_X);\mathbb{Z}) \to H^k(M(\mathcal{A});\mathbb{Z})$$

induced by the inclusions $M(\mathcal{A}) \hookrightarrow M(\mathcal{A}_X)$ is an isomorphism of groups.

The study of the cohomology of the complement of a hyperplane arrangement in a complex space finds his first generalization in the work of Orlik and Solomon, [OS80]. They described the cohomology, giving generators and relations, by contructing an algebra which is the quotient of an exterior algebra by an ideal. Let us go through their ideas, which will be elaborated upon later in the final section, where original results concerning a generalization of these arrangements will be presented. Here we avoid technical proofs for a matter of length.

Let $\mathcal{A} = (H_1, \dots, H_n)$, where $H_i = \ker(\chi_i : \mathbb{C}^r \to \mathbb{C})$ for each $i \in [n]$, be an hyperplane arrangement in a complex vector space V, and $M := M(\mathcal{A}) = V \setminus \bigcup_{H_i\in\mathcal{A}} H_i$ its complement. First, observe that the complement of a single hyperplane $M_i := V\setminus H_i$ is homotopy equivalent to $\mathbb{C}^*$ through the projection $\chi_i : M_i \to \mathbb{C}^*$ onto a complex line that intersects H_i transversely. A generator of $H^1(\mathbb{C}^*) \simeq \mathbb{Z}$ is represented by the form

$$\omega := \frac{1}{2\pi i}\frac{dz}{z}.$$

The pullback in cohomology of the inclusion $M \hookrightarrow M_i$ gives us an element of $H^1(M)$ of the form

$$\omega_i := \frac{1}{2\pi i}\frac{d\chi_i}{\chi_i},$$

where χ_i is the linear form associated to H_i. As we will see later on this section, these cohomology classes ω_i generate $H^*(M)$. However, to give a presentation of the cohomology we need more ingredients.

Denote by E_1 the free abelian group generated by $\{e_H\}_{H\in\mathcal{A}}$, i.e. $E_1 := \oplus_{H\in\mathcal{A}}\mathbb{Z}e_H$. We will write $e_i := e_{H_i}$. Furthermore, let E be the exterior algebra of E_1, $E = E(\mathcal{A}) = \Lambda(E_1)$. The algebra E is graded indeed, if for every $[n] \supseteq S = \{i_1, \dots, i_k\}$ we denote

$$e_S = e_{i_1} \wedge \cdots \wedge e_{i_k},$$

and the k-th graded piece E_k is the free $\mathbb{Z}$-module generated by the elements $\{e_S|\ |S|=k\}$. The derivation $\partial : E \mapsto E$ defined by

$$\partial 1 = 0,\ \partial e_H = 1 \text{ and, for } k \geq 2,\ \ \partial e_S = \sum_{j=1}^{k} (-1)^{j-1} e_{S\setminus\{i_j\}}$$

endow E with the structure of a differential graded algebra.

For any $S = \{i_1, \ldots, i_k\},\ S \subseteq [n]$, we denote $\cap S = \cap_{j\in S} H_j$. We say that S is *dependent* if $r(\cap S) \neq \emptyset$ and $= \operatorname{codim}(\cap S) < |S|$, and *independent* if $r(\cap S) = |S|$. Being dependent for S is equivalent to the corresponding linear forms defining the hyperplanes $\chi_{i_1}, \ldots, \chi_{i_k}$ to be linear dependent on $\mathbb{C}$. These dependent sets have a crucial role in the description of the relations of the cohomology, as shown by the following results.

Definition 1.2.2. Let $\mathcal{A}$ be an arrangement, the *Orlik-Solomon ideal* of $\mathcal{A}$ is the ideal $I = I(\mathcal{A})$ of E generated by all monomials e_S with $\cap S = \emptyset$ and all elements ∂e_S such that S is dependent, namely

$$I = (e_S| \cap S = \emptyset) + (\partial e_S|\ S \text{ dependent}).$$

The quotient algebra $A = A(\mathcal{A}) = E/I$ is called the Orlik-Solomon algebra of $\mathcal{A}$. Note that I is a homogeneous ideal, hence A is a graded algebra.

The set of generators in Definition 1.2.2 can be reduced in number. We call *circuits* the dependent subsets of $[n]$ minimal with respect to inclusion, then we have the following.

Lemma 1.2.3. *The Orlik-Solomon ideal $I(\mathcal{A})$ is generated by all monomial e_S with $\cap S = \emptyset$ and the elements ∂e_T for all circuits T.*

Remark 1.2.4. In [OS], the study of A was divided into two cases: when the arrangement is central, and when it is affine. The central case is more straightforward, for instance, there are no monomials e_S with $\cap S = \emptyset$, and hence the structure of I is easier to handle. In the noncentral case they applied a coning construction to centralize the arrangement. Once the necessary results were established in the coned case, they extended the conclusions to the general case via short exact sequences.

The Orlik-Solomon algebra we just constructed is crucial, as it precisely corresponds to the cohomology we wanted to investigate. Indeed, if we define a homomorphism of graded algebras

$$\phi : E/I \mapsto H^*(M)$$

sending $e_i \to \omega_i$, the following holds.

Theorem 1.2.5 (Orlik-Solomon's Theorem, [OS80])**.** *The map ϕ is an isomorphism, i.e. $A(\mathcal{A}) \cong H^*(M(\mathcal{A}))$.*

They proved this theorem by employing a third isomorphic graded module, whose basis is constructed over the so-called "non-broken circuits". To offer a brief overview and give a sense of the underlying concepts, we now introduce a few key definitions. These constructions rely on the choice of a total order on $\mathcal{A}$, and, for simplicity, we identify this order with the natural ordering on $[n]$.

Definition 1.2.6. The set $S \subset [n]$ is a *broken circuit* if there exists an index i such that $S \cup \{i\}$ is a circuit and $i > j$ for all $j \in S$. If a set $S \subseteq [n]$ does not contain broken circuits, it is referred to as an *nbc*-set, and the corresponding monomial $e_S \in E$ is called an *nbc*-monomial.
The *broken circuit module* $C = C(\mathcal{A})$ is defined as follows. Let $C_0 = \mathbb{Z}$, and for $k \geq 2$ let C_k be the free $\mathbb{Z}$-module with basis $\{e_S | S \ nbc\text{−set s.t. } |S| = k\}$. Let $C = \oplus_{k \geq 0} C_k$, it is a free graded $\mathbb{Z}$-module.

The idea of using non-broken circuit sets to construct a basis has its origins in matroid theory, introduced by Stanley in the case of supersolvable arrangements [Sta72]. This concept was later generalized to arbitrary hyperplane arrangements by Orlik and Solomon, who established the following isomorphism.

Theorem 1.2.7 ([OS80, Theorem 3.7])**.** *For every arrangement $\mathcal{A}$, the broken circuit module $C(\mathcal{A})$ and the Orlik-Solomon algebra $A(\mathcal{A})$ are isomorphic as graded $\mathbb{Z}$-modules and $\{e_S + I \in A(\mathcal{A}) | S \ nbc - set\}$ is a basis for $A(\mathcal{A})$.*

Remark 1.2.8. Notice that this result proves that $A(\mathcal{A})$ is a purely combinatorial object that only depends on the poset of layers $L(\mathcal{A})$. Hence, the cohomology ring admits a presentation that is entirely combinatorial.

Employing the non broken circuit basis, deletion and restriction arguments and the Brieskorn lemma, Orlik and Solomon managed to prove that $A(\mathcal{A}) \cong H^*(M)$.

In addition to the foundational works on the cohomology of complex arrangements, such as those by Arnold, Brieskorn, and Orlik-Solomon, several other important contributions have since emerged in the literature, each employing distinct techniques and approaches. One of the key works is by Goresky and MacPherson [GM88], where they employed stratified Morse theory to study the cohomology of real subspace arrangements. They introduced a stratification of the Euclidean space determined by the arrangement, indexed by flats, and realized the complement as one of open stratum. Moreover, Deligne, Goresky, and MacPherson explored the multiplicative structure of the cohomology, in [DGM00]. Further advances have been made by De Concini and Procesi, in [DCP95], who introduced a model, known as "wonderful model ", for the complement of the arrangement, which is an explicit, combinatorially defined sequence of blow-ups. Employing Morgan technique they were able to compute the rational cohomology of the complement. These works all focus on the more generalized case of subspace arrangements. Lastly, Yuzvinsky, in [Yuz91], simplified the De Concini-Procesi model linking the topological insights of Goresky-MacPherson and the geometric models of De Concini-Procesi in the divisorial case.

1.3 Real hyperplane arrangements

In the previous section, we discussed the first generalization of the study of the complement of arrangements, specifically focusing on the complex case. Now, we turn our attention to the case of real hyperplane arrangements. In 1987, Gelfand and Varchenko, in [VG87], presented a significant result on the cohomology of a real hyperplane arrangement. In this section, we will provide an overview of their approach and give a brief insight into their findings.

If $\mathcal{A}$ is a real hyperplane arrangement, the topology of $M(\mathcal{A})$ does not look very complicated: the complement is a collection of open polyhedral cones. Zavlasky counted the number $C(\mathcal{A})$ of regions forming this space in terms of the Mobius function of the poset of layers of the arrangement.

Theorem 1.3.1 (Zaslavsky's Theorem, [Zas75]). *Let $\mathcal{A}$ be a real hyperplane arrangement. The number of regions of $M(\mathcal{A})$, called chambers is given by*

$$C(\mathcal{A}) = \sum_{X \in L(\mathcal{A})} |\mu(0, X)| = \chi_{\mathcal{A}}(-1).$$

Indeed, with this result it is possible to give an isomorphism of algebras

$$H^*(M(\mathcal{A})) = H^0(M(\mathcal{A})) \cong \mathbb{Z}^{|C(\mathcal{A})|}$$

for any real arrangement $\mathcal{A}$ where $C(\mathcal{A})$ is the set of the connected components of the complement $M(\mathcal{A})$.

Given a real arrangement $\mathcal{A}$, Gelfand and Varchenko constructed an associated ring of functions $VG(\mathcal{A})$ that is isomorphic to the cohomology ring of the complement of $\mathcal{A}$. Namely, $VG(\mathcal{A})$ is the ring of function from the chambers of the arrangement to the integers, with pointwise addition and multiplication. They provided an alternative presentation of this ring, along with a filtration and its associated graded ring, whose Hilbert series, which is completely determined by $L(\mathcal{A})$, is the Poincaré polynomial of the complement. Let us now give some definition to understand their ideas.

Definition 1.3.2. Let $\mathcal{A} = (H_1, \dots, H_n)$ be a real arrangement, M its complement and $\chi_1, \dots, \chi_n$ the associated linear forms defining the hyperplanes. For each hyperplane consider the *Heaviside functions* on M, namely for every $i \in [n]$ let $e_i = \delta_{H_i^+}$, i.e. $e_i(v) = 1$ if $\chi_i(v) > 0$ and $e_i(v) = 0$ if $\chi_i(v) < 0$. Let $VG(\mathcal{A})$ be the ring generated by these functions, called the *Varchenko-Gelfand ring*, hence the ring whose elements are polynomials in the e_i with integer coefficients.

Let us define an increasing filtration

$$0 \subset P_0 \subset P_1 \subset \cdots \subset P_n = VG(\mathcal{A}),$$

where P_k is the subspace of functions representable by polynomials, in the elements e_i, of degree at most k. With this filtration, Varchenko and Gelfand proved an analogue version of the Brieskorn Lemma in the real case.

Lemma 1.3.3. *The natural map*

$$\bigoplus_{X \in L(\mathcal{A})\, \mathrm{rk}(X)=k} P_k(M^X)/P_{k-1}(M^X) \mapsto P_k(M)/P_{k-1}(M)$$

is an isomorphism for any $k > 0$.

To give another presentation of the Varchenko-Gelfand ring, which is isomorphic to the one just defined and offers a better understanding of its structure, we first need to introduce a few additional notions. Let us begin with the definition of a *signed circuit* $C = (c_1, \dots, c_k)$, that is, as the name suggests, a circuit with signs $s_i \in \{+, -\}$, satisfying

$$\sum_{i=1}^{k} s_i m_i \chi_{c_i} = 0,$$

for some positive integers m_i. Let us write $C = C^+ \sqcup C^-$, where C^+ is the set of indices such that $s_i = +$ and C^- the complement.
The following result is fundamental for our purpose.

Theorem 1.3.4. *The Heaviside functions e_i, $i \in [n]$, satisfy the following relations*

- *for any i, $e_i^2 - e_i = 0$;*
- *for any signed circuit $C = C^+ \sqcup C^-$,*

$$\prod_{j \in C^+} e_j \prod_{k \in C^-} (e_k - 1) - \prod_{j \in C^+} (e_j - 1) \prod_{k \in C^-} e_k = 0.$$

Let I be the ideal of $P(M)$ generated by these relations.

Having defined the ideal I, we are now able to state the main results.

Theorem 1.3.5 ([VG87, Theorem 6]). *Let $\mathcal{A}$ be a real arrangement and I defined as above, then*

$$VG(\mathcal{A}) \cong \mathbb{Z}[e_1, \dots, e_n] \big/ I.$$

Furthermore, it holds

$$VG(\mathcal{A}) \cong H^*(M(\mathcal{A})).$$

Note that the cohomology ring is entirely generated in degree 0. Furthermore Gelfand and Varchenko, analogously to the complex case, proved that $VG(\mathcal{A})$ has a $\mathbb{Z}$-basis of monomials indexed by no broken circuit sets of $\mathcal{A}$. In addition, since the number of broken circuits of $\mathcal{A}$ is equal to the number of chambers of $\mathcal{A}$, they proved the following.

Theorem 1.3.6. *Let $\mathcal{A}$ be a hyperplane arrangement, then*

$$H^*(M(\mathcal{A}); \mathbb{Z}) = H^0(M(\mathcal{A}); \mathbb{Z}) \cong \mathbb{Z}^{|nbc\text{-}sets\ of\ \mathcal{A}|} \cong \mathbb{Z}^{|C(\mathcal{A})|}.$$

To conclude and provide a complete overview on the topology of complements of real arrangements, we refer to the work in [DB23], where Dorpalen-Barry extended this study. While the results presented here and all Gelfand and Varchenko's analysis focus on central arrangements, she extends their work to cones, which are intersections of open half-spaces defined by some of the hyperplanes in $\mathcal{A}$. Cones connect central and affine arrangements while generalizing both, and they can be viewed as conditional oriented matroids. The techniques employed are inspired by Grobner basis theory.

The presentation of the cohomology of the complement of real arrangements will be revisited later in the final section of this chapter, where it will prove useful, with slight changes to the notation.

1.4 Toric arrangements

The development in the theory of arrangements has progressed by further generalizing to the study of toric arrangements, which can be seen as a periodic analogue of hyperplane arrangements. In this case, unlike the approach taken in previous sections, the techniques used to describe the cohomology of the complement of the arrangements do not involve constructing an alternative algebra to which the cohomology is isomorphic. In this section, we follow the important work of De Concini and Procesi [DCP05], which is among the first contributions to this topic.

We begin by introducing some basic definitions. Recall that the algebraic torus over $\mathbb{C}$ can be expressed as $T = \hom_{\mathbb{Z}}(\Lambda, \mathbb{C}^*)$, where Λ is its character group, $\Lambda \cong \mathbb{Z}^r$. As mentioned in the first chapter, we consider the analogue of a hyperplane to be a translate of the kernel of a character χ, specifically the hypersurface $H_{g,\chi}$ given by the equation $1 - g\chi = 0$ for some $g \in \mathbb{C}^*$. A toric arrangement is a finite collection of hypersurfaces indexed by a set with a total ordering E, namely $\mathcal{A} = \{H_i := H_{g_i,\chi_i}\}_{i \in E}$. Denote its complement by $M(\mathcal{A})$.
Let us now consider some differential forms that will play a crucial role in the description of the cohomology of $M(\mathcal{A})$. First, recall that the cohomology of the torus T is generated by the closed differential forms

$$d \log \chi, \ \chi \in \Lambda.$$

The space of these invariant 1-forms is isomorphic to $\Lambda \otimes \mathbb{R}$, which is a vector space of dimension equal to the rank of the torus. These forms not only generate the basic cohomology of the torus, but they form part of the cohomology basis for each component W of the arrangement, where W is isomorphic to a lower-dimensional torus. We write, for a first set of generators,

$$\psi_i := d \log \chi_i, \text{ for any } H_i \in \mathcal{A}.$$

The second set of generators comes from the hypersurfaces that define the toric arrangement, encoding its combinatorics, and is given by

$$\omega_i := d \log(1 - g_i \chi_i), \text{ for any } H_i \in \mathcal{A}$$

In general, these forms are not sufficient to generate the full cohomology unless the arrangement is unimodular (where certain conditions on linear independence hold). Recall that an arrangement is unimodular if all the possible intersections are either connected or empty. Nevertheless De Concini and Procesi managed to give a satisfactory description in term of explicit differential forms for $H^*(M(\mathcal{A});\mathbb{C})$.

Theorem 1.4.1 ([DCP05, Theorem 4.2]). *For each integer $i \geq 0$, we have a (non canonical) decomposition, as W runs over the components of the arrangement*

$$H^i(M(\mathcal{A});\mathbb{C}) = \bigoplus_{W \in L(\mathcal{A})} H^{i-\operatorname{codim} W}(W) \bigotimes V_W,$$

where each W is isomorphic to a torus, and its cohomology is generated by the forms $d\log\chi$ associated with characters in Λ. The space V_W, which depends on the combinatorics of the arrangement, can be identified to the top cohomology of the hyperplane arrangement defined by the differential, at a given point of W, of the functions ω_i vanishing on W.

More precisely, the following result identifies the generators for the cohomology as well as the relations among them.
Let us write ω_A for the product of all forms ω_i where $i \in A \subseteq [n]$ and similarly for ψ_A.

Proposition 1.4.2. *Let $\mathcal{A} = \{H_i\}_{i\in[n]}$ be a unimodular toric arrangement. Then $H^*(M(\mathcal{A});\mathbb{C})$ is the ring $H^*((\mathbb{C}^*)^r)[\omega_i]_{i\in[n]}$ with relations*

- $\omega_i\psi_i = 0$ *for all i;*
- *for each circuit C, let $c = \max C$ and $S = C \setminus c$,*

$$\omega_S = \sum (-1)^{|I|+|S|+\ell(I)} \omega_{\tilde{I}}\psi_{\tilde{B}},$$

where the sum is taken over all $I \subsetneq S$ with complement $B = S\setminus I$, $\ell(I)$ is the parity of the permutation reordering (I, B) and $\tilde{B} = B\setminus\{\min B\}$, $\tilde{I} = I \cup c$.

The proof of this theorem is remarkable, as it does not involve particular techniques, relying instead on proving the following basic identity for all $n \in \mathbb{N}$.

$$1 - \prod_{i=1}^{n} x_i = \sum_{I\subseteq\{1,2,\cdots,n\}} \prod_{i\in I} x_i \prod_{j\notin I}(1 - x_j).$$

In the unimodular case, the cohomology algebra is formal, meaning that it can be described entirely by its generators that are in degree one and their relations, without any need for higher degree corrections. In contrast, in the nonunimodular case, additional higher-order relations appear due to the more complicated combinatorics of the arrangement. These higher-order relations involve more dependencies among the characters in $\mathcal{A}$ and they can be understood in terms of D-module techniques.

For a detailed discussion see [CDD$^+$20], which provides a complete description of the nonunimodular case. We will not revisit their results here, as they will be presented and applied in the following section as part of our generalization.

1.5 Abelian arrangements

In the previous sections, we reviewed the state of the art in the study of the cohomology of arrangement complements, focusing on the description of generators, relations, and the main results for the cases considered so far, which mainly include complex, real, and toric arrangements. In this section, we present new and original results that build on those discussed earlier, generalizing them to the cohomology of noncompact abelian arrangements, where $G = \mathbb{R}^b \times (S^1)^a$, $b > 0$. We begin by introducing fundamental definitions, including certain classes arising from pullbacks, which we will show to be the generators of the cohomology of the complement. The discussion will then be divided into three main subsections: first, we consider the case where the arrangement is central and unimodular, followed by the case without the centrality assumption, and finally, without the unimodularity condition. In the final subsection, we will apply these computations to describe the cohomology algebra of ordered configuration spaces in $\mathbb{R}^b \times (S^1)^a$.

1.5.1 Definitions and notations

Consider the noncompact abelian group $G = \mathbb{R}^b \times (S^1)^a$, $b > 0$, canonically oriented and let $e = (0, \dots, 0, 1, \dots, 1)$ be the unit of G. We sometimes denote $d := \dim(G) - 1 = a + b - 1$. Let us start with a discussion on the cohomology of the ambient space G^r.

In order to define some classes in $H^{a+b-1}(G \backslash \{e\})$, consistent with the previous section, we consider the immersion of a small sphere $i \colon S^{a+b-1} \hookrightarrow G \backslash e$ centered in e. Let W be the top dimensional class generating $H_{a+b-1}(S^{a+b-1})$. We denote by ω the class in $H^{a+b-1}(G \setminus \{e\})$ dual to $i_*(W)$ with the orientation given by the outer normal first rule.

For $j = 1, \cdots, a$, let Y^j be the standard generators of $H_1(S^1)$ and let Y

be the top class in $H_a((S^1)^a)$. For $x \in \mathbb{R}^b \setminus \{0\}$ we consider the immersion

$$\begin{aligned} i_x\colon \quad (S^1)^a &\hookrightarrow G \setminus \{e\} = (\mathbb{R}^b \times (S^1)^a) \setminus (0, 1, \cdots, 1) \\ z &\mapsto (x, z). \end{aligned}$$

Up to homotopy, i_x only depends on the connected component of $\mathbb{R}^b \setminus \{0\}$ x belongs to. In the case $b = 1$ we choose $x < 0$, otherwise the choice of the point does not matter. Denote by ψ^j the class in $H^1(G \setminus \{e\})$ dual to $(i_x)_*(Y^j)$ and by ψ the class in $H^a(G \setminus \{e\})$ dual to $(i_x)_*(Y)$.

Let $\{\chi_i : G^r \to G\}_{i \in E}$ be a finite collection of morphisms defining the abelian arrangement $\mathcal{A} = \{H_i := \chi_i^{-1}(g_i)\}_{i \in E}$, recall definitions of Section 1.1.

For any $i \in E$, the map χ_i restricts to $(\chi_i - g_i)|_{M^{a,b}(\mathcal{A})}\colon M^{a,b}(\mathcal{A}) \to G \setminus \{e\}$. So we set

$$\begin{aligned} \omega_i &= (\chi_i - g_i|_{M^{a,b}(\mathcal{A})})^*(\omega) \in H^{a+b-1}(M^{a,b}(\mathcal{A})), \\ \psi_i^j &= (\chi_i|_{M^{a,b}_{\mathcal{A}}})^*(\psi^j) \in H^1(M^{a,b}(\mathcal{A})), \\ \psi_i &= (\chi_i|_{M^{a,b}_{\mathcal{A}}})^*(\psi) \in H^a(M^{a,b}(\mathcal{A})). \end{aligned}$$

Note that, when $b = 1$, the classes ψ_i's have degree $a = a + b - 1$, the same degree of the elements ω_i's.

1.5.2 Central and unimodular case

Consider a proper map $f\colon N \to M$ between oriented manifolds, it induces a pushforward map in cohomology f_* obtained from the pushforward in Borel-Moore homology (for a general reference [HR96]) by composing with the Poincaré duality isomorphism.

$$\begin{array}{ccc} H^k(N) & \xrightarrow{\ f_*\ } & H^{\dim M - \dim N + k}(M) \\ {\scriptstyle \mathrm{PD}_N}\big\downarrow & & \big\uparrow{\scriptstyle \mathrm{PD}_M^{-1}} \\ H^{\mathrm{BM}}_{\dim N - k}(N) & \xrightarrow[\ f_*^{\mathrm{BM}}\ ]{} & H^{\mathrm{BM}}_{\dim N - k}(M) \end{array}$$

The map f_* increases the degree by $\dim M - \dim N$ and has the following properties:

1. (Functoriality) $f_* \circ g_* = (f \circ g)_*$;
2. (Projection formula) $f_*(f^*(y) \smile x) = y \smile f_*(x)$ for any $x \in H^*(N)$ and any $y \in H^*(M)$;

3. (Naturality) For any pullback diagram

$$\begin{array}{ccc} N' & \xrightarrow{h} & N \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ M' & \xrightarrow[g]{} & M \end{array}$$

with f and f' proper maps and $\dim M - \dim N = \dim M' - \dim N'$, we have

$$g^* \circ f_* = f'_* \circ h^*;$$

4. (Embedding) If f is a closed embedding then f_* is the composition of the Thom isomorphism for the normal bundle

$$f_*\colon H^*(N) \xrightarrow{\text{Th}} H^{*+d}(T, T \setminus N) \to H^{*+d}(M, M \setminus N) \to H^{*+d}(M)$$

where we identify the normal bundle with a tubular neighborhood $N \subseteq T \subseteq M$ and $d = \dim M - \dim N$. The second map is provided by excision theorem of $M \setminus T$ and the last one is the map from the long exact sequence of the pair $(M, M \setminus N)$.

We consider the pushforward in cohomology $i_*\colon H^0(\mathbb{R}\setminus\{0\}) \to H^{a+b-1}(G\setminus\{e\})$ induced by the closed inclusion $i\colon \mathbb{R}\setminus\{0\} \to G\setminus\{e\}$, $i(x) = (x, 0, \cdots, 0, 1, \cdots, 1)$. Let w^+ and w^- be the two standard generators of $H^0(\mathbb{R} \setminus \{0\})$. Poincaré Duality maps w^+ and w^- to the classes in $H_1^{\text{BM}}(\mathbb{R} \setminus \{0\})$ represented by the infinite chains c^+ and c^-, corresponding respectively to the positive and negative semi-axes. By checking intersection numbers, we obtain $\text{PD}(\omega) = j_*c^+$ and, when $b = 1$, $\text{PD}(\psi) = j_*c^+ + j_*c^- = j_*1$. Note that, if $b > 1$, $j_*c^+ + j_*c^- = 0$ in $H_1^{\text{BM}}(G \setminus \{e\})$. Summarizing

$$i_*(w^+) = \omega, \quad i_*(w^-) = \begin{cases} \psi - \omega & \text{if } b = 1, \\ -\omega & \text{otherwise} \end{cases}, \quad i_*(1) = \begin{cases} \psi & \text{if } b = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

Remark 1.5.1. Consider a central arrangement $\mathcal{A}$, the linear relations among the elements $\chi_i \in \Lambda \simeq \mathbb{Z}^r$ define a representable matroid. For any subvariety $H_i = \ker(\chi_i) = \ker(-\chi_i)$ we choose one of the two descriptions. These choices determine an oriented matroid.

We will denote the *circuits* of the matroid by C and the *oriented circuits* by $C = C^+ \sqcup C^-$.

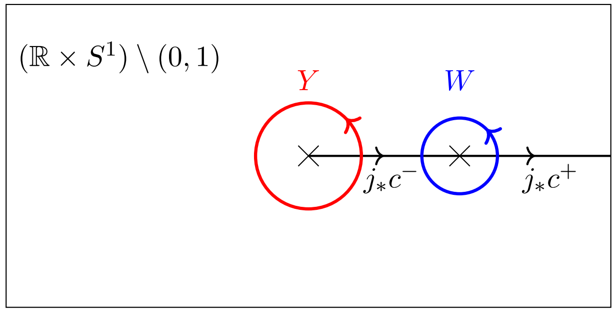


Figure 1.1: In the case $(a, b) = (1, 1)$, Y in red and W in blue are the homology classes. While, $j_*c^+, j_*c^- \in H_1^{\mathrm{BM}}((\mathbb{R} \times S^1) \setminus (0, 1))$ are the pushforward of the Borel Moore homology classes $c^+, c^- \in H_1^{\mathrm{BM}}(\mathbb{R} \setminus \{0\})$.

Now we consider $\mathcal{A}$ as a real arrangement, for any $i \in E$ let $w_i^+ = \chi_i^*(w^+)$ and $w_i^- = \chi_i^*(w^-)$ be the classes in $H^0(M^{\mathbb{R}}(\mathcal{A}))$ corresponding to the inverse image of $\mathbb{R}_+$ and $\mathbb{R}_-$ through $\chi_i \colon \mathbb{R}^r \to \mathbb{R}$. We now employ this slightly different notation, which is more advantageous for the purpose of the following results.

Theorem 1.5.2 ([VG87, Theorem 5]). *Let $\mathcal{A}$ be a central real arrangement. The ring $H^0(M^{\mathbb{R}}(\mathcal{A}))$ is generated by the classes w_i^+, w_i^- with $i \in E$ and subject to the following relations:*

- $w_i^- = 1 - w_i^+$;
- $w_i^+ w_i^- = 0$;
- *for any oriented circuit* $C = C^+ \sqcup C^-$, $\prod_{i \in C^+} w_i^+ \prod_{j \in C^-} w_j^- = 0$.

We will write, for $I \subseteq E$, $\omega_I^+ = \prod_{i \in I} \omega_i^+$ and $\omega_I^- \prod_{i \in I} \omega_i^-$. Similarly, $\omega_I = \omega_{i_1} \omega_{i_2} \dots \omega_{i_k}$ and $\psi_I = \psi_{i_1} \psi_{i_2} \dots \psi_{i_k}$ where $I = \{i_1, i_2, \dots, i_k\}$ and the product is taken in the order induced by the ordered ground-set E.

Remark 1.5.3. From the first and second relation it follows that $(w_i^{\pm})^2 = w_i^{\pm}$, which was exactly the relation obtained by Gelfand and Varchenko. We consider a central circuit C so that also the opposite $-C$ is a circuit. Then, using the first and the third relation for C, we get

$$0 = \prod_{i \in C^+} w_i^+ \prod_{j \in C^-} (1 - w_j^+) = \sum_{J \subseteq C^-} (-1)^{|J|} w_{C^+}^+ w_J^+ = \sum_{J \subseteq C^-} (-1)^{|J|} w_{C^+ \sqcup J}^+ \tag{1.2}$$

and for the opposite circuit

$$0 = \prod_{i \in C^-} w_i^+ \prod_{j \in C^+} (1 - w_j^+) = \sum_{J \subseteq C^+} (-1)^{|J|} w_J^+ w_{C^-}^+ = \sum_{J \subseteq C^+} (-1)^{|J|} w_{J \sqcup C^-}^+. \tag{1.3}$$

By summing the equations above with an opportune sign, we get the following relations

$$\sum_{J \subsetneq C^-} (-1)^{|J|} w_{C^+ \sqcup J}^+ - (-1)^{|C|} \sum_{J \subsetneq C^+} (-1)^{|J|} w_{J \sqcup C^-}^+ = 0$$

$$\sum_{\substack{K \subseteq C^- \\ K \neq \emptyset}} (-1)^{|K|} w_{C \setminus K}^+ - \sum_{\substack{K \subseteq C^+ \\ K \neq \emptyset}} (-1)^{|K|} w_{C \setminus K}^+ = 0 \tag{1.4}$$

From this identities, we will obtain a relation in $H^*(M^{a,b}(\mathcal{A}))$ by applying the map i_*.

Definition 1.5.4. Let A, B be disjoint subsets of E. We denote by $\ell(A,B)$ the sign of the permutation taking the $A \sqcup B$ as a ordered subset of E of $\mathcal{A}$ to the concatenation of A and B.

Definition 1.5.5. Let A and B be disjoint subsets of E and define

$$\eta_{A,B} := (-1)^{d\ell(A,B)} \omega_A \psi_B \in H^*(M^{a,b}(\mathcal{A})).$$

Let $B \subseteq E$ a basis of the matroid, we denote by $\operatorname{sgn}(B)$ the sign of the determinant of the matrix whose columns represent the elements χ_b (in a positive basis of Λ) with the order induced by E.

Remark 1.5.6. Let $\mathcal{B} \subset \mathcal{A}$ be a sub-arrangement and $j \colon M_{\mathcal{A}}^{a,b} \hookrightarrow M_{\mathcal{B}}^{a,b}$ be the inclusion of the complements. From the definitions follow that $j^* \eta_{C,D}^{\mathcal{B}} = \eta_{C,D}^{\mathcal{A}}$. So in the following we will not specify the ambient space.

Lemma 1.5.7. *Let $\mathcal{A}$ be a totally unimodular central arrangement of rank n. Consider the closed immersion $i \colon M^{\mathbb{R}}(\mathcal{A}) \to M^{a,b}(\mathcal{A})$ and its pushforward in cohomology $i_* \colon H^0(M^{\mathbb{R}}(\mathcal{A})) \to H^{n(a+b-1)}(M^{a,b}(\mathcal{A}))$. Let $I \subseteq E$ be an independent set and $B \subseteq E$ a basis containing I:*

- *If $b = 1$, then $i_*(w_I^+) = \operatorname{sgn}(B)^d \eta_{I, B \setminus I}$.*
- *If $b > 1$, then* $i_*(w_I^+) = \begin{cases} 0 & \textit{if } \operatorname{rk}(I) \neq n, \\ \operatorname{sgn}(I)^d \omega_I & \textit{otherwise.} \end{cases}$

Proof. Let $\mathcal{B} \subset \mathcal{A}$ be the sub-arrangement $\{H_b \mid b \in B\}$. Consider the inclusions $j^{\mathbb{R}} \colon M^{\mathbb{R}}(\mathcal{A}) \hookrightarrow M^{\mathbb{R}}(\mathcal{B})$ and $j^{a,b} \colon M^{a,b}(\mathcal{A}) \hookrightarrow M^{a,b}(\mathcal{B})$. Recall that $d = a + b - 1$ is the codimension of $\mathbb{R}$ in G. The following diagram commutes by the naturality of pushforward in cohomology (property (3))

$$\begin{array}{ccccc} H^0(M^{\mathbb{R}}(\mathcal{A})) & \xleftarrow{j^{\mathbb{R}*}} & H^0(M^{\mathbb{R}}(\mathcal{B})) & \xleftarrow[\simeq]{k^{\mathbb{R}}} & H^0(\mathbb{R} \setminus 0)^{\otimes n} \\ \downarrow{\scriptstyle i_{\mathcal{A}*}} & & \downarrow{\scriptstyle i_{\mathcal{B}*}} & & \downarrow{\scriptstyle i_*^{\otimes n}} \\ H^{nd}(M^{a,b}(\mathcal{A})) & \xleftarrow[j^{a,b*}]{} & H^{nd}(M^{a,b}(\mathcal{B})) & \xleftarrow[k^{a,b}]{\simeq} & H^d(G \setminus e)^{\otimes n} \end{array}$$

The horizontal maps in the left-hand square are induced by the inclusion of the complement of $\mathcal{A}$ in the complement of $\mathcal{B}$, in $\mathbb{R}^n$ and G^n respectively. Since $M^{\mathbb{R}}(\mathcal{B}) \cong (\mathbb{R} \setminus 0)^n$ and $M^{a,b}(\mathcal{B}) \cong (G \setminus e)^n$, the horizontal maps in the right-hand square are given by the Künneth isomorphisms. Note that the Künneth isomorphism $k^{a,b}$ depends on the sign of the basis B.

For the sake of notation we assume that I are the first elements of B, the general case differs by the sign $(-1)^{d\ell(I,B\setminus I)}$ given by the reordering of ω_i and ψ_b for $i \in I$ and $b \in B \setminus I$. We have

$$i_{\mathcal{A}*}(\omega_I^+) = i_{\mathcal{A}*} j^{\mathbb{R}*} k^{\mathbb{R}}((w^+)^{\otimes |I|} \otimes 1^{\otimes n-|I|}) = j^{a,b*} k^{a,b}((i_* w^+)^{\otimes |I|} \otimes i_* 1^{\otimes n-|I|}).$$

We use eq. (1.1) distinguishing the cases $b = 1$ and $b > 1$. If $b = 1$,

$$i_{\mathcal{A}*}(\omega_I^+) = j^{a,b*} k^{a,b}(\omega^{\otimes |I|} \otimes \psi^{\otimes n-|I|}) = j^{a,b*}(\operatorname{sgn}(B)^d \omega_I \psi_{B\setminus I}) = \operatorname{sgn}(B)^d \eta_{I,B\setminus I},$$

where in the last equality we use Remark 1.5.6. If $b > 1$, $i_*(1) = 0$ and hence $i_{\mathcal{A}*}(w_I^+) = 0$ if $\operatorname{rk}(I) \neq n$. Otherwise, I is a basis and

$$i_{\mathcal{A}*}(\omega_I^+) = j^{a,b*} k^{a,b}(\omega^{\otimes n}) = j^{a,b*}(\operatorname{sgn}(I)^d \omega_I) = \operatorname{sgn}(I)^d \omega_I.$$

This completes the proof. □

Remark 1.5.8. Note that, also in the case $b = 1$, the image of i_* does not depend on the choice of the (unimodular) basis B. In fact, if $B' \subseteq E$ is another (unimodular) basis containing I, $\operatorname{sgn}(B)^d \psi_{B\setminus I} \omega_I = \operatorname{sgn}(B')^d \psi_{B'\setminus I} \omega_I$ by using the linear relations among the classes ψ coming from the linear dependencies among the characters of B and B'.

Definition 1.5.9. A subset $A \subseteq \mathcal{A}$ is a *generalized circuit* if $\operatorname{rk}(A) = |A| - 1$. We denote by $C(A)$ the unique circuit contained in A.

Definition 1.5.10. Let $C \subseteq E$ be a circuit and let $B \supset C$ such that $B \setminus \{i\}$ is a basis for any $i \in C$. We denote by c_i the sign of the determinant of the matrix whose columns are the elements in $B \setminus \{i\}$.

Remark 1.5.11. Up to a global sign, the c_i's do not depend on the choice of B. The collection of these signs defines a chirotope (or equivalently an oriented matroid), cf [BLVS$^+$99].

Lemma 1.5.12. *Let $\mathcal{A}$ be a central arrangement and $C \subseteq E$ an unimodular oriented circuit with $C = C^+ \sqcup C^-$. The following relations holds in $H^*(M^{a,b}(\mathcal{A}))$:*

- *If $b = 1$,*
$$\sum_{\substack{K \subseteq C^- \\ K \neq \emptyset}} (-1)^{|K|} c^d_{i_K} \eta_{C \setminus K, K \setminus i_K} - \sum_{\substack{K \subseteq C^+ \\ K \neq \emptyset}} (-1)^{|K|} c^d_{i_K} \eta_{C \setminus K, K \setminus i_K} = 0, \quad (1.5)$$
where $i_K \in K$;

- *If $b > 1$ and d odd,*
$$\sum_{i \in C} (-1)^{|C_{<i}|} \omega_{C \setminus \{i\}} = 0; \quad (1.6)$$

- *If $b > 1$ and d even,*
$$\sum_{i \in C^-} \omega_{C \setminus \{i\}} - \sum_{i \in C^+} \omega_{C \setminus \{i\}} = 0. \quad (1.7)$$

Proof. Let r be the rank of the arrangement $\mathcal{A}$ and n be the rank of the circuit C. Consider the additive map given by the following composition:

$$H^0(M^{\mathbb{R}}(C)) \xrightarrow{i_*} H^{n(a+b-1)}(M^{a,b}(C)) \xrightarrow{p^*} H^{n(a+b-1)}(M^{a,b}(\mathcal{A}))$$

where p is the restriction of the projection $G^r \to G^r / \cap_{c \in C} H_c \simeq G^n$.

We consider the relation (1.4) in $H^0(M^{\mathbb{R}}(C))$ and we apply Lemma 1.5.7 to the essentialization of the arrangement C. For any independent $J \subsetneq C^+$ (resp. $J \subsetneq C^-$), the choice of $i_K \in K = C^+ \setminus J$ (resp. $i_K \in K = C^- \setminus J$) corresponds to complete $C^+ \bigsqcup J = C \setminus K$ to a basis $C \setminus i_K$ of the sublattice generated by the elements of C. By Lemma 1.5.7 and Remark 1.5.6 we have

$$p^*(i_*(\omega^+_{C \setminus K})) = p^*(c^d_{i_K} \eta_{C \setminus K, K \setminus i_K}) = c^d_{i_K} \eta_{C \setminus K, K \setminus i_K},$$

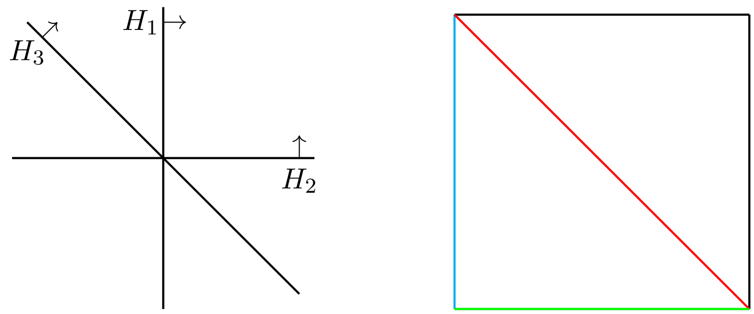


Figure 1.2: A picture of the arrangement in Example 1.5.14 for $(a, b) = (1, 0)$ on the left, and $(a, b) = (0, 1)$ on the right.

and so Equation (1.4) is mapped to

$$\sum_{\substack{K\subseteq C^-\\ K\neq\emptyset}} (-1)^{|K|} c^d_{i_K} \eta_{C\setminus K, K\setminus i_K} - \sum_{\substack{K\subseteq C^+\\ K\neq\emptyset}} (-1)^{|K|} c^d_{i_K} \eta_{C\setminus K, K\setminus i_K} = 0$$

for $b = 1$, and to

$$\sum_{i\in C^-} c^d_i \omega_{C\setminus i} - \sum_{i\in C^+} c^d_i \omega_{C\setminus i} = 0$$

for $b > 1$. Since C is unimodular, the dependence relation among the characters of the circuit is $\sum_{i\in C} (-1)^{|C_{<i}|} c_i \chi_i = 0$, hence $i \in C^+$ if and only if $c_i = (-1)^{|C_{<i}|}$. □

Remark 1.5.13. Lemma 1.5.12 holds also for a central circuit C in a non-central arrangement $\mathcal{A}$. It is sufficient to consider the map $H^*(M^{a,b}(C)) \to H^*(M^{a,b}(\mathcal{A}))$ induced by the inclusion $M^{a,b}(\mathcal{A}) \hookrightarrow M^{a,b}(C)$.

Example 1.5.14. Let us consider the central arrangement $\mathcal{A} = \{H_i\}_{i=1,2,3}$ in G^2 defined by the columns of the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

(see Figure 1.2). This is an arrangement of type A_3, hence $M(\mathcal{A})$ is the configuration space of 3 points in G (c.f. Section 1.5.5). The subspace $H^*(G^2) \subset H^*(M(\mathcal{A}))$ is generated by the classes ψ^j_1, ψ^j_2 and $\psi^j_3 = \psi^j_1 + \psi^j_2$, with $j = 1, \cdots, a$. Furthermore, we have defined the classes $\omega_i \in H^{a+b-1}(M(\mathcal{A}))$ for $i = 1, 2, 3$. The circuit $C = C^+ \sqcup C^- = \{1, 2\} \sqcup \{3\}$ gives the following relations

- if $b = 1$,
$$\omega_1\omega_2 - (-1)^d\omega_2\omega_3 - \omega_1\omega_3 - \omega_3\psi_1 = 0, \tag{1.8}$$
- if $b > 1$,
$$\omega_1\omega_2 - (-1)^d\omega_2\omega_3 - \omega_1\omega_3 = 0. \tag{1.9}$$

1.5.3 Noncentral case

Now, we deal with the case of noncentral totally unimodular arrangements. Firstly, we prove another variation of the Brieskorn lemma, a deletion-restriction short exact sequence, and an identity between Poincaré and characteristic polynomial. We will use these results in Section 1.5.4 for unimodular covers.

Recall that, given $H \in \mathcal{A}$ a subvariety of the arrangement, we denote by $\mathcal{A}' := \mathcal{A} \setminus \{H\}$ the *deletion* of H and $\mathcal{A}'' := \{H \cap K \mid K \in \mathcal{A}'\}$ the *restriction* to H. If $L(\mathcal{A})$ is the poset of layer of the arrangement $\mathcal{A}$, let us denote $L^r(\mathcal{A})$ as the set of elements in $L(\mathcal{A})$ of rank r. For any layer $p \in L(\mathcal{A})$ we consider the *local arrangement* $\mathcal{A}_p := \{H \in \mathcal{A} \mid p \subseteq H\}$ at p and let $g_p \colon M(\mathcal{A}) \to M(\mathcal{A}_p)$ be the inclusion of the complements.

Theorem 1.5.15. *Let $\mathcal{A}$ be an arrangement of rank r and $H \in \mathcal{A}$ a subvariety, then*

1. *$\iota_* \colon H^*(M(\mathcal{A}'')) \to H^{*+a+b}(M(\mathcal{A}'))$ is the zero map,*
2. *the map $\oplus_{p\in L^r(\mathcal{A})} g_p^* \colon \bigoplus_{p\in L^r(\mathcal{A})} H^*(M(\mathcal{A}_p)) \to H^*(M(\mathcal{A}))$ is surjective.*

Proof. In the central case the theorem follows from [LTY21, Theorem 7.6]. We prove the claims by induction on the cardinality of the arrangement $\mathcal{A}$. The base case is a central arrangement and the results hold. The assumption that $\mathcal{A}$ is not central implies $|\mathcal{A}_p| < |\mathcal{A}|$ for any $p \in L^r(\mathcal{A})$. Consider $p \in L(\mathcal{A}'')$ and let $\tilde{p} \in L(\mathcal{A}')$ be the unique minimal layer containing p (if H is not a coloop then $p = \tilde{p}$), the diagram

$$\begin{array}{ccc} M(\mathcal{A}'') & \xrightarrow{\iota} & M(\mathcal{A}') \\ {\scriptstyle g''_p}\downarrow & & \downarrow{\scriptstyle g'_{\tilde{p}}} \\ M(\mathcal{A}''_p) & \xrightarrow[\iota_p]{} & M(\mathcal{A}'_{\tilde{p}}) \end{array}$$

is a pullback diagram and so $\iota_* \circ g_p''^* = g_{\tilde{p}}'^* \circ \iota_{p*}$. Since the boundary map of the long exact sequence and the Thom isomorphism are functorial, the following diagram commutes

$$\begin{array}{ccccccc} H^{k-d-1}(M(\mathcal{A}''_p)) & \xrightarrow{\iota_p^*} & H^k(M(\mathcal{A}'_{\tilde{p}})) & \xrightarrow{j_p^*} & H^k(M(\mathcal{A}_p)) & \xrightarrow{\delta_p} & H^{k-d}(M(\mathcal{A}''_p)) \\ \downarrow g_p''^* & & \downarrow g_{\tilde{p}}'^* & & \downarrow g_p^* & & \downarrow g_p''^* \\ H^{k-d}(M(\mathcal{A}'')) & \xrightarrow{\iota_*} & H^k(M(\mathcal{A}')) & \xrightarrow{j^*} & H^k(M(\mathcal{A})) & \xrightarrow{\delta} & H^{k-d}(M(\mathcal{A}'')) \end{array}$$

For $p \in L(\mathcal{A}) \setminus L(\mathcal{A}'')$ we have a similar diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^k(M(\mathcal{A}'_p)) & \xrightarrow{j_p^*} & H^k(M(\mathcal{A}_p)) & \longrightarrow & 0 \\ \downarrow & & \downarrow g_p'^* & & \downarrow g_p^* & & \downarrow \\ H^{k-d}(M(\mathcal{A}'')) & \xrightarrow{\iota_*} & H^k(M(\mathcal{A}')) & \xrightarrow{j^*} & H^k(M(\mathcal{A})) & \xrightarrow{\delta} & H^{k-d}(M(\mathcal{A}'')) \end{array}$$

because $\mathcal{A}'_p = \mathcal{A}_p$. We consider the direct sum of previous sequences for all $p \in L(\mathcal{A})$. Since $|\mathcal{A}''| < |\mathcal{A}|$, the map $\oplus_{p \in L^r(\mathcal{A})} g_p''^*$ is surjective and $\iota_{p*} = 0$ by inductive step

$$\begin{array}{ccccccc} \oplus_p H^{k-d}(M(\mathcal{A}''_p)) & \xrightarrow{0} & \oplus_{\tilde{p}} H^k(M(\mathcal{A}'_{\tilde{p}})) & \xrightarrow{\oplus_p j_p^*} & \oplus_p H^k(M(\mathcal{A}_p)) & \xrightarrow{\oplus_p \delta_p} & \oplus_p H^{k-d}(M(\mathcal{A}''_p)) \\ \downarrow\!\!\downarrow \oplus_p g_p''^* & & \downarrow\!\!\downarrow \oplus_p g_{\tilde{p}}'^* & & \downarrow \oplus_p g_p^* & & \downarrow\!\!\downarrow \oplus_p g_p''^* \\ H^{k-d}(M(\mathcal{A}'')) & \xrightarrow{\iota_*} & H^k(M(\mathcal{A}')) & \xrightarrow{j^*} & H^k(M(\mathcal{A})) & \xrightarrow{\delta} & H^{k-d}(M(\mathcal{A}'')) \end{array}$$

It follows that $\iota_* = 0$. The two rows are exact because they are long exact sequences of pairs and by diagram chasing the map $\oplus_{p \in L^r(\mathcal{A})} g_p^*$ is surjective. □

The corollary below follows directly from Theorem 1.5.15.

Corollary 1.5.16. *Let $\mathcal{A}$ be an arrangement of rank r and $H \in \mathcal{A}$ a subvariety. The following sequence is exact*

$$0 \to H^*(M(\mathcal{A}')) \xrightarrow{j^*} H^*(M(\mathcal{A})) \xrightarrow{\text{res}} H^{*-(a+b-1)}(M(\mathcal{A}'')) \to 0.$$

Corollary 1.5.17. *For any abelian arrangement $\mathcal{A}$, the Poincaré polynomial only depends on the characteristic polynomial of the arrangement*

$$P_{\mathcal{A}}(t) = (-t^{a+b-1})^r \chi_{\mathcal{A}}\Big(-\frac{(1+t)^a}{t^{a+b-1}}\Big).$$

Proof. The identity follows by induction using the exact sequence in Theorem 1.5.15. □

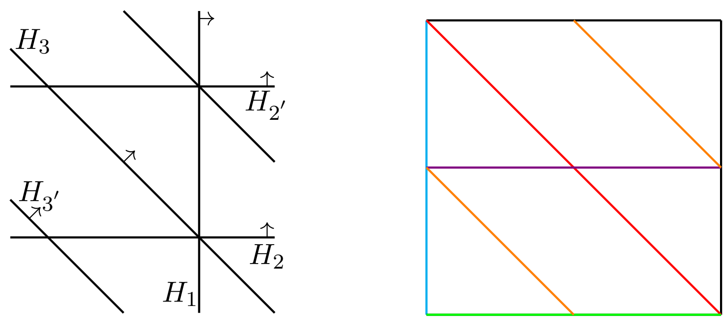


Figure 1.3: A picture of the noncentral arrangement in Example 1.5.18 when $(a, b) = (1, 0)$ on the left, and $(a, b) = (0, 1)$ on the right.

The same formula for the Poincaré polynomial is proven for central abelian arrangements in [LTY21, Theorem 7.7].

Example 1.5.18. Let us consider the arrangement obtained from the arrangement in the example 1.5.14 by adding the two hyperplanes $H_{2'} = \{\chi_2^{-1}(-e)\}$ and $H_{3'} = \{(\chi_1 + \chi_2)^{-1}(-e)\}$ as shown in Figure 1.3.
The arrangement is not central anymore, but it is still unimodular. As in the previous example, the cohomology $H^*(M(\mathcal{A}))$ is generated by the classes $\psi_1^j, \psi_2^j, \psi_3^j = \psi_1^j + \psi_2^j$, $j = 1, \cdots, a$ and $\omega_1, \omega_2, \omega_{2'}, \omega_3, \omega_{3'}$. There are two central circuits:

$$C = C^+ \sqcup C^- = \{1, 2\} \sqcup \{3\}, \quad C' = (C')^+ \sqcup (C')^- = \{1, 2'\} \sqcup \{3'\}.$$

When $b = 1$, the first circuit leads to the same relation eq. (1.8) of Example 1.5.14, while the second one to the new relation

$$\omega_1\omega_{2'} + (-1)^d \omega_{2'}\omega_{3'} - \omega_1\omega_{3'} - \omega_{3'}\psi_1 = 0. \tag{1.10}$$

The case $b > 1$ is analogous. The Poincaré polynomial of the complement is given by $P_{\mathcal{A}}(t) = (1 + t)^{2a} + 5(1 + t)^a + 6$. The four noncentral circuits provide the following relations

$$\omega_2\omega_{2'} = 0, \qquad \omega_3\omega_{3'} = 0,$$
$$\omega_1\omega_2\omega_{3'} = 0, \qquad \omega_1\omega_{2'}\omega_3 = 0,$$

see eq. (1.23) of Theorem 1.5.29.

1.5.4 Nonunimodular case

The poset of layers of a nonunimodular arrangement $\mathcal{A}$ is no longer a geometric lattice, due to multiple connected components of the intersections, but it has the structure of a geometric poset, as deeper discussed in Chapter 2. Moreover, the combinatoric data of $\mathcal{A}$ are encoded in a generalization of a matroid, basically a matroid together with a multiplicity function, known as *arithmetic matroid* and first introduced by D'Adderio and Moci in [DM13], which will be expanded in the last chapter. We deal with the nonunimodular case adapting the argument in [CDD$^+$20], by constructing an unimodular covering for an abelian arrangement $\mathcal{A}$.

Definition 1.5.19. Let Λ be a lattice and $\mathcal{A}$ an arrangement in G^Λ, define the *multiplicity*

$$m(A) = [\Lambda^A : \Lambda_A],$$

where $\Lambda_A := \langle A \rangle \subseteq \Lambda$ and Λ^A is the radical of Λ_A in Λ, i.e. $\Lambda^A = (\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda_A) \cap \Lambda$. This number $m(A)$ is the cardinality of the torsion subgroup of Λ/Λ_A and the multiplicity function of the matroid associated to $\mathcal{A}$.

Suppose $\mathcal{A} = \{\chi_0, \cdots, \chi_r\}$ consists of one generalized circuit $X = \{0, \cdots, r\} = C \sqcup F$, with $n = \operatorname{rk}(C)$. Let

$$a_i = \begin{cases} m(X) \prod_{j \in C \setminus \{i\}} m(C \setminus \{j\}) & \text{for } i \in C \\ m(X) & \text{for } i \in F \end{cases}$$

and let Λ' be the lattice in $\Lambda \otimes \mathbb{Q}$ generated by the characters $\frac{\chi_i}{a_i}$.

Lemma 1.5.20. *The lattice Λ' contains Λ and the inclusion induces a covering $\pi \colon G^{\Lambda'} \to G^\Lambda$ of degree $m(C)^a m(X)^{a(r-1)} \prod_{i \in C} m(C \setminus i)^{a(n-1)}$.*

Proof. The proof of the first statement is analogous to that of [CDD$^+$20, Lemma 6.4]. Let $i \in C$, as in [CDD$^+$20, Lemma 6.5] we compute the index of Λ in Λ'

$$\begin{aligned}[\Lambda' : \Lambda] &= \frac{[\Lambda' : \Lambda_{X \setminus i}]}{[\Lambda : \Lambda_{X \setminus i}]} = \frac{\prod_{j \in X \setminus i} a_j}{m(X \setminus i)} = \frac{m(C) m(X)^r \prod_{j \in C \setminus i} \prod_{l \in C \setminus j} m(C \setminus l)}{m(X) m(C \setminus i)} \\ &= m(C) m(X)^{r-1} \prod_{i \in C} m(C \setminus i)^{n-1}.\end{aligned}$$

It follows that the degree of the covering is $[\Lambda' : \Lambda]^a$. ∎

Let $\mathcal{A}_U$ be the arrangement in $U := G^{\Lambda'}$ given by the set of connected components of the preimages of the subvarieties in $\mathcal{A}$

$$\mathcal{A}_U = \bigcup_{H \in \mathcal{A}} \pi_0(\pi^{-1}(H)).$$

Lemma 1.5.21 ([CDD+20, Lemma 6.7])**.** *The arrangement $\mathcal{A}_U$ is unimodular.*

Let $H_i \in \mathcal{A}$, L a connected component of $\pi^{-1}(H_i)$ and $q \in L$. The subvariety L has equation $\frac{\hat{\chi}_i}{a_i} = \frac{\hat{\chi}_i}{a_i}(q)$, where $\hat{\chi} = \chi \circ \pi$. We define

$$\omega_i^U(q) = \left(\frac{\hat{\chi}_i}{a_i} - \frac{\hat{\chi}_i}{a_i}(q)\right)\Big|_{M(\mathcal{A}_U)}^{*}(\omega) \quad \in H^{a+b-1}(M(\mathcal{A}_U)), \tag{1.11}$$

$$\psi_i^{j,U} = \left(\frac{\hat{\chi}_i}{a_i}\right)\Big|_{M(\mathcal{A}_U)}^{*}(\psi^j) \quad \in H^1(M(\mathcal{A}_U)), \tag{1.12}$$

$$\psi_i^{U} = \left(\frac{\hat{\chi}_i}{a_i}\right)\Big|_{M(\mathcal{A}_U)}^{*}(\psi) \quad \in H^a(M(\mathcal{A}_U)). \tag{1.13}$$

Remark 1.5.22. For $H_i \in \mathcal{A}$, let L be a connected component of $\pi^{-1}(H_i)$ and $p, q \in L$. Then we have $\frac{\hat{\chi}_i}{a_i}(q) = \frac{\hat{\chi}_i}{a_i}(p)$ and hence $\omega_i^U(q) = \omega_i^U(p)$.

As in Section 1.5.2, for $I \subseteq E$, we write $\omega_I^U = \omega_{i_1}^U \omega_{i_2}^U \dots \omega_{i_k}^U$ and $\psi_I^U = \psi_{i_1}^U \psi_{i_2}^U \dots \psi_{i_k}^U$ where $I = \{i_1, i_2, \dots, i_k\}$ and the product is taken in the order induced by the ordered groundset E.

Let $A \subseteq E$, W a connected component of $\bigcap_{i\in A} H_i$ and $p \in W$. Since π^* is injective, we define $\omega_{W,A}$ as the unique class in $H^*(M(\mathcal{A}))$ such that

$$\pi^*(\omega_{W,A}) = \frac{1}{|L \cap \pi^{-1}(p)|} \sum_{q \in \pi^{-1}(p)} \omega_A^U(q) = \frac{1}{|L \cap \pi^{-1}(p)|} \sum_{\tilde{L} \text{ c.c. of } \pi^{-1}(W)} \omega_A^U(q_{\tilde{L}}), \tag{1.14}$$

where L is any connected component of $\pi^{-1}(W)$ and $q_{\tilde{L}} \in \tilde{L} \cap \pi^{-1}(p)$ for each connected component $\tilde{L}$ of $\pi^{-1}(W)$. For any $A, B \subseteq E$ disjoint, we denote

$$\eta_{A,B}^U(q) = (-1)^{d\ell(A,B)} \omega_A^U(q) \psi_B^U \in H^*(M(\mathcal{A}_U))$$

and

$$\eta_{W,A,B} = (-1)^{d\ell(A,B)} \omega_{W,A} \psi_B \in H^*(M(\mathcal{A})). \tag{1.15}$$

Lemma 1.5.23. *Let $A, B \subseteq E$ such that $A \cup B$ is a dependent set and let W be a connected component of $\bigcap_{i\in A} H_i$. Then, for any $j = 1, \cdots, a$,*

$$\omega_{W,A} \psi_B^j = 0. \tag{1.16}$$

Proof. By definition, it is enough to prove that, for any connected component L of $\pi^{-1}(W)$, $\omega_A^U(q_L)\psi_B^{j,U} = 0$ in $H^*(M(\mathcal{A}_U))$. This follows from $(\psi_i^{j,U})^2 = 0$, $\omega_i^U(q_L)\psi_i^{j,U} = 0$ and from the linear dependences between $\{\chi_i, i \in A \cup B\}$. □

Lemma 1.5.24. *Let X be a generalized circuit and $A, B \subseteq X$ such that $A \sqcup B$ is a maximal independent set. Let W be a connected component of $\bigcap_{i\in A} H_i$ and $p \in W$. Then*

$$\pi^*(\eta_{W,A,B}) = \frac{m(A \cup B)^a}{m(A)^a} \sum_{q\in\pi^{-1}(p)} \eta^U_{A,B}(q).$$

Proof. By definition,

$$\pi^*(\psi_B) = \big(\prod_{i\in B} a_i^a\big)\psi^U_B.$$

The computation of $|L \cap \pi^{-1}(p)|$ is analogous to that in Lemma 6.8 in [CDD+20]. The cardinality of the preimage of p is the degree of the covering. The number of connected components of $\pi^{-1}(W)$ is

$$([\Lambda(A) : \Lambda^A])^a = \left(\frac{\prod_{i\in A} a_i}{m(A)}\right)^a.$$

Therefore,

$$\begin{aligned}
|L \cap \pi^{-1}(p)| &= \frac{m(C)^a m(X)^{a(r-1)} \prod_{i\in C} m(C\setminus i)^{a(n-1)} m(A)^a}{\prod_{i\in A} a_i^a} \\
&= \frac{m(C)^a m(X)^{a(r-1)} \prod_{i\in C} m(C\setminus i)^{a(n-1)} m(A)^a}{m(X)^{a|A|} \prod_{i\in A\cap C} \prod_{j\in C\setminus i} m(C\setminus j)^a} \\
&= m(C)^a m(X)^{a(r-|A|-1)} m(A)^a \prod_{i\in C} m(C\setminus i)^{a(n-|A\cap C|-1)} \prod_{i\in A\cap C} m(C\setminus i)^a.
\end{aligned}$$

It follows that

$$\pi^*(\eta_{W,A,B}) = \frac{\prod_{i\in B} a_i^a}{|L\cap \pi^{-1}(p)|} (-1)^{d\ell(A,B)} \sum_{q\in\pi^{-1}(p)} \omega^U_A(q)\psi^U_B,$$

where

$$\begin{aligned}
\frac{\prod_{i\in B} a_i^a}{|L\cap\pi^{-1}(p)|} &= \frac{m(X)^{a|B|}\prod_{i\in C} m(C\setminus i)^{a|B\cap C|}}{\prod_{i\in B\cap C} m(C\setminus i)^a} \cdot \frac{1}{|L\cap\pi^{-1}(p)|} \\
&= \frac{m(X)^{a(|A|+|B|-r+1)} \prod_{i\in C} m(C\setminus i)^{a(|B\cap C|+|A\cap C|-n+1)}}{m(C)^a m(A)^a \prod_{i\in A\cap C} m(C\setminus i)^a \prod_{i\in B\cap C} m(C\setminus i)^a} \\
&= \frac{m(X)^a \prod_{i\in C} m(C\setminus i)^a}{m(C)^a m(A)^a \prod_{i\in (A\cup B)\cap C} m(C\setminus i)^a} \\
&= \frac{m(X)^a m((A\cup B)\cap C)^a}{m(C)^a m(A)^a} = \frac{m(A\cup B)^a}{m(A)^a}.
\end{aligned}$$

In this last computation the final equality comes from property (3) of arithmetic matroids, see [DM13]. ☐

Lemma 1.5.25. *Let $X = C \sqcup F$ be a generalized oriented circuit, Y be a connected component of $\bigcap_{i\in X} H_i$, then*

- *If $b = 1$,*

$$\sum_{\substack{K\subseteq C^-\\K\neq\emptyset}} (-1)^{|K|} c_{i_K}^d \frac{m(X\setminus K)^a}{m(X\setminus i_K)^a}\eta_{W,X\setminus K,K\setminus i_K} - \sum_{\substack{K\subseteq C^+\\K\neq\emptyset}} (-1)^{|K|} c_{i_K}^d \frac{m(X\setminus K)^a}{m(X\setminus i_K)^a}\eta_{W,X\setminus K,K\setminus i_K} = 0, \tag{1.17}$$

where $i_K \in K$ and, for each summand, W is the connected component of $\bigcap_{i\in X\setminus K} H_i$ such that $Y \subseteq W$.

- *If $b > 1$ and d odd,*

$$\sum_{i\in C} (-1)^{|C_{<i}|}\omega_{Y,X\setminus i} = 0. \tag{1.18}$$

- *If $b > 1$ and d even,*

$$\sum_{i\in C^-} \omega_{Y,X\setminus i} - \sum_{i\in C^+} \omega_{Y,X\setminus i} = 0. \tag{1.19}$$

Proof. We apply Lemma 1.5.12 to the circuit C in the abelian variety $U = G^{\Lambda'}$. Let $p \in Y$ be any point. If $b = 1$, we have

$$\sum_{\substack{K\subseteq C^-\\K\neq\emptyset}} (-1)^{|K|} c_{i_K}^d \eta^U_{C\setminus K,K\setminus i_K}(q) - \sum_{\substack{K\subseteq C^+\\K\neq\emptyset}} (-1)^{|K|} c_{i_K}^d \eta^U_{C\setminus K,K\setminus i_K}(q) = 0,$$

for all $q \in \pi^{-1}(p)$. We multiply this equation by $\omega_F^U(q)$, and we get

$$\sum_{\substack{K\subseteq C^-\\K\neq\emptyset}} (-1)^{|K|} c_{i_K}^d \eta^U_{X\setminus K,K\setminus i_K}(q) - \sum_{\substack{K\subseteq C^+\\K\neq\emptyset}} (-1)^{|K|} c_{i_K}^d \eta^U_{X\setminus K,K\setminus i_k}(q) = 0.$$

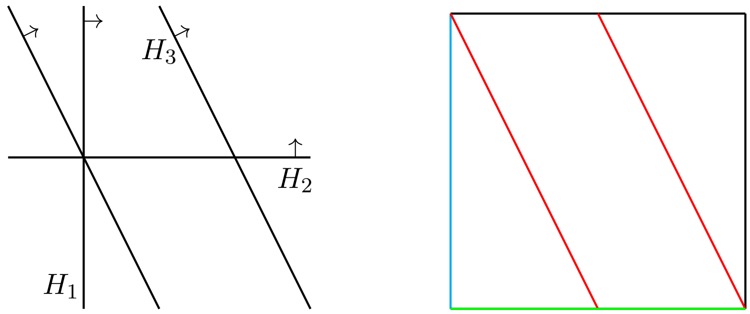


Figure 1.4: A picture of the nonunimodular arrangement in Example 1.5.26 when $(a,b)=(1,0)$ on the left, and $(a,b)=(0,1)$ on the right.

Summing over all $q \in \pi^{-1}(p)$, we obtain

$$\begin{aligned}
0 &= \sum_{q\in\pi^{-1}(p)} \sum_{\substack{K\subseteq C^-\\ K\neq\emptyset}} (-1)^{|K|} c^d_{i_K} \eta^U_{X\setminus K, K\setminus i_K}(q) - \sum_{q\in\pi^{-1}(p)} \sum_{\substack{K\subseteq C^+\\ K\neq\emptyset}} (-1)^{|K|} c^d_{i_K} \eta^U_{X\setminus K, K\setminus i_K}(q) \\
&= \sum_{\substack{K\subseteq C^-\\ K\neq\emptyset}} (-1)^{|K|} c^d_{i_K} \sum_{q\in\pi^{-1}(p)} \eta^U_{X\setminus K, K\setminus i_K}(q) - \sum_{\substack{K\subseteq C^+\\ K\neq\emptyset}} (-1)^{|K|} c^d_{i_K} \sum_{q\in\pi^{-1}(p)} \eta^U_{X\setminus K, K\setminus i_K}(q) \\
&= \sum_{\substack{K\subseteq C^-\\ K\neq\emptyset}} (-1)^{|K|} c^d_{i_K} \frac{m(X\setminus K)^a}{m(X\setminus i_K)^a} \pi^*(\eta_{W,X\setminus K, K\setminus i_K}) \\
&\quad - \sum_{\substack{K\subseteq C^+\\ K\neq\emptyset}} (-1)^{|K|} c^d_{i_K} \frac{m(X\setminus K)^a}{m(X\setminus i_K)^a} \pi^*(\eta_{W,X\setminus K, K\setminus i_K}).
\end{aligned}$$

Since π^* is injective, equation (1.17) follows.

If $b>1$, through an analogous computation, equations (1.18) and (1.19) hold.

□

Example 1.5.26. Let us consider the arrangement $\mathcal{A} = \{H_i\}_{i=1,2,3,4}$ in G^3 associated to the matrix

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

(see Figure 1.4). The generalized circuits are $C = C^+ \sqcup C^- = \{1,2\} \sqcup \{3\}$ and $X = C \sqcup F = \{1,2,3\} \sqcup \{4\}$.

Let W be the layer $H_1 \cap H_2 \cap H_3$. If $b > 1$, applying formula (1.17) to the central circuit C we obtain the relation

$$\omega_{W,12} + (-1)^d \omega_{W,23} - \omega_{W,13} + \frac{1}{2^a}\omega_3\psi_2 = 0. \tag{1.20}$$

Notice that in the case $a = 1$ the unimodular covering of $\mathcal{A}$ is the arrangement in Example 1.5.18, and the pullback of eq. (1.20) is the sum of equations (1.8) and (1.10).

Now consider the generalized circuit X, we get 2^a relations because the intersection $H_1 \cap H_2 \cap H_3 \cap H_4 = W \cap H_4$ has 2^a distinct connected components, $p_1, \cdots, p_{2^a}$ and for any $i = 1, \cdots, 2^a$

$$\omega_{p_i,124} + (-1)^d \omega_{p_i,234} - \omega_{p_i,134} - \frac{1}{4^a}\omega_{Z,34}\psi_2 = 0, \tag{1.21}$$

where Z is the layer $H_3 \cap H_4$. The Poincaré polynomial of the complement is $(t+1)^{3a} + 4(t+1)^{2a} + 7(t+1)^a + 6$.

Lemma 1.5.27. *The classes $\omega_{W,A}$ for all independent sets $A \subseteq E$ and all connected components W of $\bigcap_{i\in A} H_i$, generate $H^*(M(\mathcal{A}))$ as a $H^*(G^r)$-module.*

Proof. Fix $i \in E$, we proceed by induction using the exact sequence (of $\mathbb{Z}$-module) of Corollary 1.5.16 given by deletion and contraction with respect to H_i:

$$0 \to H^*(M(\mathcal{A}')) \xrightarrow{j^*} H^*(M(\mathcal{A})) \xrightarrow{\text{res}} H^{*-(a+b-1)}(M(\mathcal{A}'')) \to 0.$$

Let $\mathcal{K} \subseteq H^*(M(\mathcal{A}))$ be the $H^*(G^r)$-submodule generated by all $\omega_{W,A}$, $\mathcal{K}' \subseteq H^*(M(\mathcal{A}'))$ the $H^*(G^r)$-submodule generated by all $\omega'_{W,A}$ with $i \notin A$, and $\mathcal{K}'' \subseteq H^*(M(\mathcal{A}''))$ the $H^*(G^{r-1})$-submodule generated by all $\omega''_{W,A}$ with $i \in A$.

By definition of the classes $\omega_{W,A}$ we have $j_*(\omega'_{W,A}) = \omega_{W,A}$ and

$$\text{res}(\omega_{W,A}) = \begin{cases} \omega''_{W,A\setminus\{i\}} & \text{if } i \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Since the pullback map in cohomology $H^*(G^r) \twoheadrightarrow H^*(G^{r-1})$ is surjective and the residue map is a morphism of $H^*(G^r)$-modules, we have

$$\begin{array}{ccccccccc} & & \mathcal{K}' & \longrightarrow & \mathcal{K} & \twoheadrightarrow & \mathcal{K}'' & & \\ & & \wr\downarrow & & \downarrow & & \wr\downarrow & & \\ 0 & \longrightarrow & H^*(M(\mathcal{A}')) & \longrightarrow & H^*(M(\mathcal{A})) & \longrightarrow & H^*(M(\mathcal{A}'')) & \longrightarrow & 0 \end{array}$$

By diagram chasing the inductive step follows, this completes the proof. □

Definition 1.5.28. Let R be the free $H^*(G^r)$-module generated by the classes $\omega_{W,A} \in H^*(M(\mathcal{A}))$ with $A \subseteq E$ independent and W connected component of $\bigcap_{i \in A} H_i$. We endow R with a ring structure by defining the multiplication

$$\omega_{W,A}\omega_{W',A'} = \begin{cases} 0 & \text{if } A, A' \text{ are not disjoint or } A \sqcup A' \text{ dependent,} \\ (-1)^{d\ell(A,A')} \sum_{L \text{ c.c. of} W \cap W'} \omega_{L, A \sqcup A'} & \text{otherwise.} \end{cases} \tag{1.22}$$

Notice that, for any $A, B \subseteq E$, and for any connected component W of $\bigcap_{i \in A} H_i$, $\eta_{W,A,B} \in R$.

Theorem 1.5.29. *Let $\mathcal{A}$ be an arrangement in G^r, where $G = \mathbb{R}^b \times (S^1)^a$. The integer cohomology of the complement $H^*(M(\mathcal{A}); \mathbb{Z})$ is the quotient of R by the following relations:*

- *Whenever $A \sqcup B$ and $A' \sqcup B'$ are not disjoint or $A \sqcup B \sqcup A' \sqcup B'$ is dependent,*

$$\eta_{W,A,B}\eta_{W',A',B'} = 0. \tag{1.23}$$

- *For any generalized circuit $X = C \sqcup F$ with $C = C^+ \sqcup C^-$, and for any connected component Y of $\bigcap_{i \in X} H_i$,*
  - *If $b = 1$,*

$$\sum_{\substack{K \subseteq C^- \\ K \neq \emptyset}} (-1)^{|K|} c_{i_K}^d \frac{m(X \setminus K)^a}{m(X \setminus i_K)^a} \eta_{W, X \setminus K, K \setminus i_K} - \sum_{\substack{K \subseteq C^+ \\ K \neq \emptyset}} (-1)^{|K|} c_{i_K}^d \frac{m(X \setminus K)^a}{m(X \setminus i_K)^a} \eta_{W, X \setminus K, K \setminus i_K} = 0, \tag{1.24}$$

    for some $i_K \in K$ and where, for each summand, W is the connected component of $\bigcap_{i \in X \setminus K} H_i$ such that $Y \subseteq W$.
  - *If $b > 1$ and d odd,*

$$\sum_{i \in C} (-1)^{|C_{<i}|} \eta_{Y, X \setminus i, \emptyset} = 0. \tag{1.25}$$

– *If $b > 1$ and d even,*

$$\sum_{i \in C^-} \eta_{Y,X\setminus i,\emptyset} - \sum_{i \in C^+} \eta_{Y,X\setminus i,\emptyset} = 0. \tag{1.26}$$

Proof. The cohomology ring is a quotient of R by Lemma 1.5.27. Relation (1.23) holds by Lemma 1.5.23 and relations (1.24), (1.25) by Lemma 1.5.25.

Let I be the ideal given by relations (1.23)-(1.25). There is a surjective map $R/I \to H^*(M(\mathcal{A}))$. We prove that this map is an isomorphism by induction on the cardinality of $\mathcal{A}$. The base case is trivial.

Let us fix $i \in E$. Let R', R'', I' and I'' be defined analogously with respect to the deletion $\mathcal{A}'$ and the contraction $\mathcal{A}''$ respectively. By induction hypothesis $R'/I' \cong H^*(M(\mathcal{A}'))$ and $R''/I'' \cong H^*(M(\mathcal{A}''))$. Let us consider the following diagram

$$\begin{array}{ccccccccc}
 & & R'/I' & \longrightarrow & R/I & \longrightarrow & R''/I'' & & \\
 & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p''} & & \\
0 & \longrightarrow & H^*(M(\mathcal{A}')) & \xrightarrow{j^*} & H^*(M(\mathcal{A})) & \xrightarrow{\text{res}} & H^{*-(a+b-1)}(M(\mathcal{A}'')) & \longrightarrow & 0
\end{array}$$

where:

- the first horizontal map is defined by the inclusions $R' \subseteq R$ and $I' \subseteq I$;
- the second horizontal map is induced by the map $g\colon R \to R''$ given by

$$\omega_{W,A} z \mapsto \begin{cases} \omega_{W,A\setminus i} \bar{z}, & \text{if } i \in A, \\ 0 & \text{otherwise,} \end{cases}$$

  for any $z \in H^*(G^r)$, where $\bar{z}$ is the image of z through $r\colon H^*(G^r) \to H^*(H_i)$. The map is well defined since the image of I is I'';
- the second row is the deletion-contraction exact sequence of Corollary 1.5.16.

The left square of diagram commutes because $j^*(\eta'_{W,A,B}) = \eta_{W,A,B}$, as the right one, since $\text{res}(\eta_{W,A,B}) = \eta''_{W,A\setminus\{i\},B}$ if $i \in A$ and $\text{res}(\eta_{W,A,B}) = 0$ otherwise. This implies that the first horizontal map is injective and the second one surjective. We now want to prove that the central vertical map is injective.

Firstly, observe that for any relation r of type (1.24) and any generator $\eta_{W,A,B}$ the element $r\eta_{W,A,B}$ is linear combination of relations of types (1.23) and (1.24). In particular, the ideal I is generated by (1.23) and (1.24) as $\mathbb{Z}$-module. It follows that the map $g_{|I}\colon I \twoheadrightarrow I''$ is surjective.

Consider the short exact sequence $0 \to R' \to R \xrightarrow{g} R'' \to 0$ that induces the top row of the diagram. Let $z \in R$ such that $p(z) = 0 \in H^*(M(\mathcal{A}))$. By commutativity of the diagram above, $g(z) = r'' \in I''$. Since $I \twoheadrightarrow I''$, exists $r \in I$ such that $g(r) = r''$. By exactness of $0 \to R' \to R \xrightarrow{g} R'' \to 0$, we can write $z = r + z'$ for some $z' \in \mathrm{Ker}(g)$, i.e. $z' \in I' \subseteq I$. It follows that $z \in I$. $\square$

1.5.5 Configuration spaces

A straghtforward application of what we have just proved can be found in the study of the cohomology algebra of ordered configuration spaces in $\mathbb{R}^b \times (S^1)^a$.
Let X be a topological space, the configuration space of n points in X is:

$$\mathrm{Conf}_n(X) = \{(x_1, \dots, x_n) \in X^n \mid x_i \neq x_j \,\forall i \neq j\}.$$

As above, let $G = \mathbb{R}^b \times (S^1)^a$, and consider the totally unimodular and central arrangement

$$\mathcal{A}_n = \{H_{ij}\}_{\substack{i,j \in [n],\\ i<j}}, \quad H_{ij} = (\chi_i - \chi_j)^{(-1)}(e),$$

where $\chi_i\colon G^n \to G$ is the projection on the i-th component. Notice that

$$\mathrm{Conf}_n(G) = M^{a,b}(\mathcal{A}_n).$$

Cohen and Taylor [CT78] used a spectral sequence to compute the cohomology of configuration spaces in $\mathbb{R}^b \times M$, for $b \geq 1$ and M a connected manifold (see [CT78, Example 1]). More precisely, there exists a filtration F_* on $H^*(\mathrm{Conf}_n(\mathbb{R}^b \times M); \mathbb{K})$ for a field $\mathbb{K}$ and they described explicitly the associated graded

$$\mathrm{gr}_{F_*} H^*(\mathrm{Conf}_n(\mathbb{R}^b \times M); \mathbb{K})$$

as a ring. In the case $M = (S^1)^a$ and $b > 1$, Theorem 1.5.29 implies that

$$\mathrm{gr}_{F_*} H^*(\mathrm{Conf}_n(G); \mathbb{K}) \simeq H^*(\mathrm{Conf}_n(G); \mathbb{K})$$

as ring. In the case $b = 1$, we have the opposite behaviour: indeed, the two rings $\mathrm{gr}_{F_*} H^*(\mathrm{Conf}_n(G; \mathbb{K}))$ and $H^*(\mathrm{Conf}_n(G); \mathbb{K})$ are not canonically isomorphic. Finally, our result extend the coefficients from fields to integers.

Now, we write down the presentation of $H^*(\mathrm{Conf}_n(G);\mathbb{Z})$ by using Theorem 1.5.29. The ground set of the associated matroid is the set $E = \{ij | i, j \in [n], i < j\}$ with the lexicographic order. The poset of layers is $L(\mathcal{A}_n) = \Pi_{[n]}$ the poset of partitions of $[n]$. The circuits of the arrangement are of the form $\{ij, ik, jk\}$ with $i, j, k \in [n]$ and $i < j < k$.

By Theorem 1.5.29, the cohomology of the configuration space is generated as a $H^*(G^n)$-module by the classes $\omega_{ij} \in H^d(\mathrm{Conf}_n(G))$ with $ij \in E$. Relations (1.23)-(1.25) become

- $\omega_{ij}\psi_{ij} = 0$, $\forall ij \in E$;
- for $b = 1$,
$$\omega_{ij}\omega_{jk} - \omega_{ij}\omega_{ik} + \omega_{jk}\omega_{ik} - \psi_{ij}\omega_{ik} = 0; \tag{1.27}$$
- for $b > 1$,
$$\omega_{ij}\omega_{jk} - \omega_{ij}\omega_{ik} + \omega_{jk}\omega_{ik} = 0. \tag{1.28}$$

Recall that in $H^*(G^n)$, $\psi_{ik} = \psi_{ij} + \psi_{jk}$ and notice that relation (1.23) $\omega_{ij}\omega_{jk}\omega_{ik} = 0$ follows by multiplying equation (1.27) (resp. eq. (1.28)) by ω_{ik}. In the case $b = 1$, correction terms appear in our formulas. In the article [CT78] and in this one, the element ω_{ij} represent the cohomological class of H_{ij} and hence there is no canonical morphism between the two algebras $\mathrm{gr}_{F_*} H^*(\mathrm{Conf}_n(G;\mathbb{K}))$ and $H^*(\mathrm{Conf}_n(G);\mathbb{K})$.

Chapter 2

Combinatorics

This chapter explores key combinatorial properties of abelian arrangements, focusing on inductiveness and divisionality. After a brief introduction on supersolvability, freeness, and their consequences, in section 2 we extend inductiveness and divisionality to abelian arrangements, proving the factorization of their characteristic polynomials. Later we generalize a classical result of [JT84], showing that, in the abelian framework, strictly supersolvable arrangements form a proper subclass of the inductive ones (Theorem 2.3.6). Finally, in the last section, we apply these results to toric arrangements of ideals of root systems of type A, B and C, proving their inductiveness (Theorem 2.4.12) and presenting an algorithm for computing their exponents.

2.1 Supersolvability and Freeness

Supersolvability is a combinatorial property of lattices that has been introduced in the 1950's, inspired by the pioneering work of Birkhoff [Bir40] and Stanley [Sta72]. Significant progress was made by Terao [Ter86], who proved that this property, when applied to the lattice of flats associated with an arrangement, has interesting topological implications, as for instance the property of being fiber-type, first introduced by Falk and Randell in [FR85]. More recently, Bibby and Delucchi [BD22] extended the definition of supersolvability to a more general class of posets, specifically for geometric and locally geometric posets, hence including all posets arising from abelian arrangements. We will first consider the classical framework of lattices, that includes the case of central hyperplane arrangements, before going through generalizations. Let us begin with few preliminary definitions, mainly following [OT92].

Definition 2.1.1. A lattice L is called *geometric* if for all $x, y \in L$: $x \lessdot y$ if and only if there is an atom $a \in A(L)$ with $a \not\leq x$, $y = x \vee a$. Equivalently, a geometric lattice is an atomistic and semimodular finite lattice.

Definition 2.1.2. An element x in a geometric lattice L is *modular* if for all $z \leq x$ and all $y \in L$:

$$x \wedge (y \vee z) = (x \wedge y) \vee z.$$

In every lattice, examples of modular elements include the minimum, the maximum, and all atoms. The definition of supersolvability is closely tied to these modular elements.

Definition 2.1.3. Let L be a geometric lattice with rank $r(L) = \ell$. We call L a *supersolvable lattice* if it has a maximal chain of modular elements

$$\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_\ell = \hat{1}.$$

A hyperplane arrangement $\mathcal{A}$ is said to be a *supersolvable arrangement* if its lattice of flats $L(\mathcal{A})$ is supersolvable.

Supersolvability has been explored in the context of arrangements due to the significant topological properties it induces. Specifically, it translates a purely combinatorial characteristic of the lattice into the topological property of being fiber-type. The concept of fiber-type arrangements was first introduced by Terao in [Ter86], where he established their equivalence with supersolvable arrangements. Later, Falk and Randell explored fiber-type arrangements in a broader topological context in [FR85]. We now recall the formal definition of fiber-type arrangements.

Definition 2.1.4. Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^\ell$ and $M(\mathcal{A})$ its complement. $\mathcal{A}$ is said to be *strictly linearly fibered* if, after a suitable linear change of coordinates, the restriction of the projection of $M(\mathcal{A})$ to the first $(\ell - 1)$ coordinates is a fiber bundle projection whose base space B is the complement of an arrangmenet in $\mathbb{C}^{\ell-1}$, and whose fiber is the complex line $\mathbb{C}$ with finitely many points removed.

Definition 2.1.5. Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^\ell$,

1. If $\ell = 1$, then $\mathcal{A}$ is *fiber-type*;
2. For $\ell \geq 2$, $\mathcal{A}$ is *fiber-type* if it is strictly linearly fibered with base $B = M(\mathcal{B})$ and $\mathcal{B}$ is an arrangement in $\mathbb{C}^{\ell-1}$ of fiber-type.

A first significant consequence of being fiber-type is immediate to prove and it is the following.

Proposition 2.1.6. *If $\mathcal{A}$ is fiber-type, then it is $K(\pi, 1)$.*

As introduced before, for a hyperplane arrangement, the properties of being supersolvable and being fiber-type are equivalent. The first proof of this fundamental result, known as the Fibration Theorem, can be found in [Ter86].

Theorem 2.1.7 (The Fibration Theorem, [Ter86, Corollary 2.17])**.** *Let $\mathcal{A}$ be an essential central arrangement. Then $\mathcal{A}$ is fiber-type if and only if $\mathcal{A}$ is supersolvable.*

In the context of supersolvability, Bibby and Delucchi [BD22] proposed an alternative definition for posets that are not lattices, arising from a bigger class of arrangements. While the classical notion of supersolvability relies on the presence of modular elements in a lattice, this definition cannot be directly applied when dealing with non-lattice structures. Bibby and Delucchi identify a substructure of the poset that plays a similar role, adapting the concept of modularity to the case of geometric and locally geometric posets. This generalization is particularly useful for studying all abelian arrangements, as it allows similar topological results achieved in traditional lattice theory as above to be extended. Let us go through their ideas.

Definition 2.1.8. A graded, bounded below poset P is called *locally geometric* if $P_{\leq x}$ is a geometric lattice for every $x \in P$.

Note that if P is a locally geometric poset, then so are $P_{\leq x}$ and $P_{\geq x}$.

Definition 2.1.9. Let P be a poset and $B \subseteq A(P)$. The *subposet of P generated by B* is the subposet whose elements include those in B along with all possible joins of them. We denote it by $P(B)$.

If P is a locally geometric poset (or lattice) and $B \subseteq A(P)$, then the subposet $P(B)$ generated by B is also a locally geometric poset (or lattice). Now let us give some basic definitions.

Definition 2.1.10. Let P be a locally geometric poset. An *order ideal* in P is a downward-closed subset. The poset P (or an order ideal of P) is called *pure* if all maximal elements have the same rank.

An order ideal Q of P is *join-closed* if $T \subseteq Q$ implies $\bigvee T \subseteq Q$. Finally, we denote by $\max(P)$ the set of maximal elements in P.

We are ready to introduce the structure that extends the concept of modularity to the non-lattice context.

Definition 2.1.11. [BD22, Definitions 2.4.1 and 5.1.1] An *M-ideal* of a locally geometric poset P is a pure, join-closed, order ideal $Q \subseteq P$ satisfying the following two conditions:

1. $|a \vee y| \geq 1$ for any $y \in Q$ and $a \in A(P) \setminus A(Q)$,
2. for every $x \in \max(P)$, there is some $y \in \max(Q)$ such that y is a modular element in the geometric lattice $P_{\leq x}$.

An M-ideal $Q \subseteq P$ is called a *TM-ideal* if condition (1) above is replaced by the following stronger condition

1* $|a \vee y| = 1$ for any $y \in Q$ and $a \in A(P) \setminus A(Q)$.

Note that the element y in Definition 2.1.11(2) is necessarily unique since Q is join-closed. Furthermore, motivated by geometry, if we restrict to a smaller class of posets, which is defined immediately below, it is possible to give a different characterization of an M-ideal.

Definition 2.1.12. A locally geometric poset P is *geometric* if for all $x, y \in P$ with $\mathrm{rk}(x) < \mathrm{rk}(y)$ and for all subsets of the set of atoms $I \subseteq A(P)$ such that $y \in \bigvee I$ and $|I| = \mathrm{rk}(y)$, there exists $a \in I$ such that $a \not\leq x$ and $a \vee x \neq \emptyset$.

Lemma 2.1.13. *Let P be a geometric poset, and let Q be a pure, join-closed, proper order ideal of P. Then Q is an M-ideal with $\mathrm{rk}(Q) = \mathrm{rk}(P) - 1$ if and only if for any two distinct $a_1, a_2 \in A(P) \setminus A(Q)$ and every $x \in a_1 \vee a_2$ there exists $a_3 \in A(Q)$ such that $x > a_3$.*

Now, in line with the definition of supersolvability for lattices, Bibby and Delucchi provided the following definition. Instead of a chain of modular elements, here there is a chain of M-ideals. Additionally, a slightly stronger version, known as strictly supersolvability, arises when the chain is composed of TM-ideals.

Definition 2.1.14. A locally geometric poset P is *supersolvable* (resp., *strictly supersolvable)* if there is a chain, called an *M-chain* (resp., a *TM-chain)*

$$\{\hat{0}\} = Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_r = P,$$

where each Q_i is an M-ideal (resp., a TM-ideal) of Q_{i+1} with $\mathrm{rk}(Q_i) = i$.

One of the first consequences that they have proved concerns the factorization of the characteristic polynomial. Once the chain of TM-ideals is given, the exponents are explicitly determined by them.

Theorem 2.1.15 ([BD22, Theorem 5.2.1, Corollary 5.2.6])**.** *Let Q be a TM-ideal of a locally geometric poset P with $\mathrm{rk}(Q) = \mathrm{rk}(P) - 1$, and let $d = |A(P) \setminus A(Q)|$. Then*

$$\chi_P(t) = (t - d)\chi_Q(t).$$

In particular, if P is strictly supersolvable with a TM-chain $\{\hat{0}\} = Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_r = P$, and $d_i = |A(Q_i) \setminus A(Q_{i-1})|$ for each i, then

$$\chi_P(t) = \prod_{i=1}^{r}(t - d_i),$$

this means it is factorable with exponents $\exp(P) = \{d_1, \ldots, d_r\}$. If $\mathcal{A}$ is an essential abelian arrangement whose poset of layers is P, then the Poincaré polynomial of its complement is

$$\mathrm{Poin}(t) = \prod_{i=1}^{n}((1 + t)^a + d_i t^{a+b-1}).$$

Definition 2.1.16. A locally geometric poset P is *locally supersolvable* if $P_{\leq x}$ is supersolvable for every $x \in P$.

Remark 2.1.17. Denote by **SSS**, **SS** and **LSS** the class of strictly supersolvable, supersolvable and locally supersolvable posets, respectively.

$$\mathbf{SSS} \subsetneq \mathbf{SS} \subsetneq \mathbf{LSS}.$$

Moreover, if L is a geometric lattice, then $L \in \mathbf{SSS}$ if and only if $L \in \mathbf{LSS}$, i.e. all these classes coincide.

To translate these notions from the poset framework to the arrangement counterpart, the following result on the structure of the poset of layers is fundamental.

Theorem 2.1.18 ([Bib22, Corollary 13.11], [BD22, Corollary 4.4.6])**.** *Let $\mathcal{A}$ be an abelian arrangement. Then $L(\mathcal{A})$ is a geometric poset.*

Definition 2.1.19. An abelian arrangement $\mathcal{A}$ is *supersolvable* (resp., *strictly supersolvable)* if its poset of layers $L(\mathcal{A})$ is supersolvable (resp., strictly supersolvable).

This extended definition of supersolvability, applied to geometric and locally geometric posets, allows to prove topological results analogous to those of the lattice case when derived from abelian arrangements. In particular, the property of being fiber-type holds. In the abelian context, the following is a definition of the fiber-type property analogous to Definition 2.1.5 in the hyperplane case.

Definition 2.1.20. Let $\mathcal{A}$ be an arrangement in G^ℓ,

1. If $\ell = 1$, then $\mathcal{A}$ is *fiber-type*;
2. For $\ell \geq 2$, $\mathcal{A}$ is fiber-type if after a suitable change of coordinates, the restriction of the projection of $M(\mathcal{A})$ to the first $(\ell - 1)$ coordinates is a fiber bundle whose fibers are homeomorphic to G with finitely many points removed and whose base is the complement of a fiber-type arrangement in $G^{\ell-1}$.

Theorem 2.1.21 ([BD22, Theorem 3.4.3, Theorem 5.3.1])**.** *An essential arrangement $\mathcal{A}$ is fiber-type if and only if it is supersolvable.*

As a corollary we have that if the poset of layers of a linear, toric, or elliptic arrangement is supersolvable, then the arrangement complement is a $K(\pi, 1)$ space ([BD22, Corollary 3.4.4]). Moreover, if it is strictly supersolvable then the fundamental group has the structure of an iterated semidirect product of free groups ([BD22, Corollary 5.3.4]).
As the above results emphasize, the properties of supersolvability and strict supersolvability are significant not only from a combinatorial perspective but also because they lead to important topological consequences. These properties have been widely studied since the 1970s, particularly regarding their connection to freeness, which we will define and explore in the following paragraphs.

Freeness has been extensively studied since the late 1970s, with the definition initially introduced by Terao in [Ter80]. It is a purely algebraic property that leads to a variety of important topological and combinatorial implications. In this section, we focus on one key aspect, its factorability, which links freeness to other classes that will be introduced below. We will also explore how these various classes are interconnected and the relationships they establish within this framework. Let us start with the definition.

Given a $\mathbb{K}$ vector space V, let $x_1, \ldots, x_\ell$ be a basis for the dual V^*, then the symmetric algebra of V^* is $S \simeq \mathbb{K}[x_1, \ldots, x_\ell]$.

Definition 2.1.22. A $\mathbb{K}$ linear map $\theta : S \to S$ is called a *derivation* if

$$\theta(fg) = \theta(f)g + f\theta(g),$$

for all $f, g \in S$. Let $\mathrm{Der}(S)$ be the set of all derivations of S. It is a free S-module with a basis $\{\partial/\partial x_1, \ldots, \partial/\partial x_\ell\}$ consisting of the usual partial derivatives.
We say that a nonzero derivation $\theta = \sum_{i=1}^{\ell} f_i \partial/\partial x_i$ is *homogeneous of degree* p if each nonzero coefficient f_i is a homogeneous polynomial of degree p.

Definition 2.1.23. Let $\mathcal{A} = \{H_1, \ldots, H_k\}$ be a central hyperplane arrangement, and $\chi_1, \ldots, \chi_k$ the linear forms defining the hyperplanes. The *module* $D(\mathcal{A})$ *of logarithmic derivations* is defined by

$$D(\mathcal{A}) := \{\theta \in \mathrm{Der}(S) |\, \theta(\chi_i) \in \chi_i S \text{ for all } i \in [n]\}.$$

The arrangement $\mathcal{A}$ is *free* if the module $D(\mathcal{A})$ is a free S-module. Denote by $\mathbf{F}$ the class of free arrangements.

If $\mathcal{A} \in \mathbf{F}$, we may choose a basis $\{\theta_1, \ldots, \theta_\ell\}$ consisting of homogeneous derivations for $D(\mathcal{A})$ [OT92, Proposition 4.18]. Although a basis is not unique, the degrees of the derivations in a basis are uniquely determined by $\mathcal{A}$ [OT92, Proposition A.24].

Before introducing the next definition, we define the following notation: if an element e appears $d \geq 0$ times in a multiset M, we write $e^d \in M$.

Definition 2.1.24. An arrangement $\mathcal{A}$ is called *factorable* if its poset of layers $L(\mathcal{A})$ is factorable, i.e. if its characteristic polynomial has all positive integer roots. In this case we also call the roots of $\chi_{\mathcal{A}}(t)$ the *(combinatorial) exponents* of $\mathcal{A}$ and use the notation $\exp(\mathcal{A})$ to denote the multiset of exponents. Denote by $\mathbf{FR}$ the class of factorable arrangements. If $\mathcal{A} \in \mathbf{FR}$, then

$$\exp(\mathcal{A}) = \{0^{k-\mathrm{rk}(\mathcal{A})}\} \cup \{\exp(L(\mathcal{A}))\}.$$

The following theorem of Terao connects freeness and combinatorial properties of an arrangement.

Theorem 2.1.25 ([Ter81, Main Theorem])**.** *If a central hyperplane arrangement $\mathcal{A}$ is free, then it is factorable with combinatorial exponents given by the degrees of the elements in any basis for $D(\mathcal{A})$.*

Based on this, Terao conjectured that freeness is a combinatorial property [OT92, Conjecture 4.138]. Although Terao's conjecture remains open, there are certain subclasses of free arrangements that are known to be combinatorially determined. This is precisely the case for the two subclasses defined below. Before presenting the definition, recall that given a fixed hyperplane $H \in \mathcal{A}$, the deletion is defined as $\mathcal{A}' := \mathcal{A}\backslash\{H\}$, and the restriction as $\mathcal{A}'' := \{H \cap K \mid K \in \mathcal{A}'\}$.

Definition 2.1.26 ([OT92, Definition 4.53])**.** The class **IF** of *inductively free arrangements* is the smallest class of arrangements which satisfies

1. $\emptyset_\ell \in \mathbf{IF}$ for $\ell \geq 1$,

2. $\mathcal{A} \in \mathbf{IF}$ if there exists $H \in \mathcal{A}$ such that $\mathcal{A}'' \in \mathbf{IF}$, $\mathcal{A}' \in \mathbf{IF}$, and $\chi_{\mathcal{A}''}(t)$ divides $\chi_{\mathcal{A}'}(t)$.

Definition 2.1.27 ([Abe16, Theorem–Definition 4.3])**.** The class **DF** of *divisionally free arrangements* is the smallest class of arrangements which satisfies

1. $\emptyset_\ell \in \mathbf{DF}$ for $\ell \geq 1$,

2. $\mathcal{A} \in \mathbf{DF}$ if there exists $H \in \mathcal{A}$ such that $\mathcal{A}'' \in \mathbf{DF}$ and $\chi_{\mathcal{A}''}(t)$ divides $\chi_{\mathcal{A}}(t)$.

Remark 2.1.28. Supersolvability, inductive and divisional freeness and factorizability of central hyperplane arrangements all are combinatorial properties. Here is an overview of the relationships among the concepts we have defined so far

$$\mathbf{SSS} = \mathbf{SS} \subsetneq^1 \mathbf{IF} \subsetneq^2 \mathbf{DF} \subsetneq^3 \mathbf{F} \subsetneq^4 \mathbf{FR}.$$

1. The first containment is proved by Jambu and Terao [JT84, Theorem 4.2], but it is not an equality, as shown by Hultman, who found counterexamples. The arrangement associated with a root system of type D_ℓ for $\ell \geq 4$ (see Section 3.3) is inductively free, but not supersolvable (e.g., [Hul16, Theorem 6.6]);

2. The second containment follows from the deletion-restriction formula $\chi_{\mathcal{A}}(t) = \chi_{\mathcal{A}'}(t) - \chi_{\mathcal{A}''}(t)$ (e.g., [OT92, Theorem 2.56]). However, this too is not an equality: it is sufficient to consider the arrangement defined by the exceptional complex reflection group of type G_{31} which is known to be divisionally free [Abe16, Theorem 1.6] but not inductively free [HR15, Theorem 1.1];

3. The third containment, proven by Abe [Abe16, Theorem 1.1], is also not an equality. A counterexample is provided by the *intermediate arrangement* $\mathcal{A}_\ell^0(r)$ for $\ell \geq 3$, $r \geq 3$, as shown in [Abe16, Theorem 5.6];

4. Finally, the last containment is proved by Theorem 2.1.25. There are many examples of factorable but not free arrangements, for instance those described in [FR86].

All of the definitions and results introduced from the discussion of freeness onward have been presented in the classical context of hyperplane arrangements. As Bibby and Delucchi did with supersolvability, in the following section, our aim is to extend these combinatorial concepts to the case of abelian arrangements.

2.2 Inductive and divisional arrangements

In this section, we will generalize the concepts of inductiveness and divisionality to the abelian framework by first examining those properties defined on posets and then extending these ideas to the context of arrangements. From now on unless otherwise stated, we will assume that P is a locally geometric poset, and denote $A = A(P)$ the set of atoms and $r = \mathrm{rk}(P)$. Let us start with a discussion on the characteristic polynomial.

Definition 2.2.1. Fix an atom $a \in A$. Let $P' := P(A \setminus \{a\})$ be the subposet of P generated by the set of joins of the elements of $A \setminus \{a\}$ and define $P'' := P_{\geq a}$. We call (P, P', P'') the triple of posets with distinguished atom a.

Remark 2.2.2. Note that for each $a \in A$, we have $\mathrm{rk}(P) = \mathrm{rk}(P') + \epsilon(a)$, where $\epsilon(a)$ is either 0 or 1. Indeed, if $x \in \max(P)$ has $\mathrm{rk}(x) = r$ and $a \not\leq x$, then $\mathrm{rk}(P') = r$. Otherwise, setting $Q := P_{\leq x}$, we have that $a \in A(Q)$. In this case, let (Q, Q', Q'') represent the triple of posets with distinguished atom a. Since Q is a geometric lattice with $\mathrm{rk}(Q) = r$, it follows that $\mathrm{rk}(Q') \leq r \leq \mathrm{rk}(Q') + 1$, and as Q' is a subposet of P', we obtain $r \geq \mathrm{rk}(P') \geq \mathrm{rk}(Q') \geq r - 1$, as desired.

Definition 2.2.3. Let P be a poset and $a \in A$ one of its atoms. As noticed above, $\mathrm{rk}(P) = \mathrm{rk}(P') + \epsilon(a)$. If $\epsilon(a) = 1$, the atom a is called a *separator*.

For each $x \in P$, define

$$A_x := \{a \in A \mid a \leq x\}.$$

Lemma 2.2.4. *Let P be a geometric lattice. For $x, y \in P$ with $x \leq y$, let $S(x,y)$ be the set of all subsets $B \subseteq A$ such that $A_x \subseteq B$ and $\max(P(B)) = y$. Then*

$$\mu(x,y) = \sum_{B \in S(x,y)} (-1)^{|B \setminus A_x|}.$$

The proof can be found in [OT92, Lemma 2.35].
The following are useful and important results regarding the characteristic polynomial of a poset.

Lemma 2.2.5. *Let P be a locally geometric poset. Then the characteristic polynomial $\chi_P(t)$ strictly alternates in sign, i.e., if*

$$\chi_P(t) = c_r t^r + c_{r-1} t^{r-1} + \cdots + c_0,$$

then $(-1)^{r-i} c_i > 0$ for $0 \leq i \leq r$.

Proof. By definition, for each $0 \leq i \leq r$ we have

$$(-1)^{r-i} c_i = \sum_{\mathrm{rk}(x) = r-i} (-1)^{r-i} \mu(\hat{0}, x).$$

Note that the characteristic polynomial of a geometric lattice strictly alternates in sign (e.g., [Sta07, Corollary 3.5]). Thus $(-1)^{\mathrm{rk}(x)} \mu(\hat{0}, x) > 0$ since $P_{\leq x}$ is a geometric lattice for every $x \in P$. Hence $(-1)^{r-i} c_i > 0$ for each $0 \leq i \leq r$. ☐

We show below that the characteristic polynomials of locally geometric posets satisfy a deletion-restriction recurrence, which is crucial for the subsequent discussion. This formula is already proved for geometric lattices, e.g., see [Bra92, Theorem 1.2.20]. The method used in that proof can be can be naturally extended to locally geometric posets, and we include a proof here for completeness.

Theorem 2.2.6. *Let P be a locally geometric poset and fix $a \in A$. Then*

$$\chi_P(t) = t^{\epsilon(a)} \cdot \chi_{P'}(t) - \chi_{P''}(t).$$

Here $\epsilon(a) = \mathrm{rk}(P) - \mathrm{rk}(P')$ is either 0 or 1 by Remark 2.2.2.

Proof. Since $P_{\leq x}$ is a geometric lattice for every $x \in P$, by Lemma 2.2.4 we have

$$\begin{aligned}
\chi_P(t) &= \sum_{x\in P} \sum_{\substack{B\subseteq A_x \\ x = \max(P(B))}} (-1)^{|B|} t^{r-\mathrm{rk}(x)} \\
&= \sum_{x\in P} \sum_{\substack{a\notin B\subseteq A_x \\ x = \max(P(B))}} (-1)^{|B|} t^{r-\mathrm{rk}(x)} + \sum_{x\in P} \sum_{\substack{a\in B\subseteq A_x \\ x = \max(P(B))}} (-1)^{|B|} t^{r-\mathrm{rk}(x)} \\
&= \sum_{x\in P'} \sum_{\substack{B\subseteq A_x \\ x = \max(P(B))}} (-1)^{|B|} t^{\mathrm{rk}(P')+\epsilon(a)-\mathrm{rk}(x)} - \sum_{x\in P_{\geq a}} \sum_{B\in S(a,x)} (-1)^{|B\setminus A_a|} t^{r-\mathrm{rk}(x)} \\
&= t^{\epsilon(a)} \cdot \chi_{P'}(t) - \sum_{x\in P''} \mu(a,x) t^{\mathrm{rk}''(P'')-\mathrm{rk}''(x)} \\
&= t^{\epsilon(a)} \cdot \chi_{P'}(t) - \chi_{P''}(t).
\end{aligned}$$

□

Now we introduce the protagonists of this section.

Definition 2.2.7. The class **IP** of *inductive posets* is the smallest class of locally geometric posets which satisfies

1. $\{\hat{0}\} \in \mathbf{IP}$,
2. $P \in \mathbf{IP}$ if there exists an atom $a \in A$ such that $P'' \in \mathbf{IP}$, $P' \in \mathbf{IP}$, and $\chi_{P''}(t)$ divides $\chi_{P'}(t)$.

Definition 2.2.8. The class **DP** of *divisional posets* is the smallest class of locally geometric posets which satisfies

1. $\{\hat{0}\} \in \mathbf{DP}$,
2. $P \in \mathbf{DP}$ if there exists an atom $a \in A$ such that $P'' \in \mathbf{DP}$ and $\chi_{P''}(t)$ divides $\chi_P(t)$.

The following result directly follows from the definition.

Proposition 2.2.9. *Let P, Q be two isomorphic locally geometric posets. Then $P \in \mathbf{IP}$ (resp., $P \in \mathbf{DP}$) if and only if $Q \in \mathbf{IP}$ (resp., $Q \in \mathbf{DP}$).*

Proposition 2.2.10. *If a locally geometric poset P is inductive, then it is divisional. This means $\mathbf{IP} \subseteq \mathbf{DP}$.*

Proof. We argue by induction on $r = \mathrm{rk}(P) \geq 0$. The assertion clearly holds true when $r = 0$. Suppose $r > 0$. Since $P \in \mathbf{IP}$, there exists an atom $a \in A$ such that $P'' \in \mathbf{IP}$ and $\chi_{P''}(t)$ divides $\chi_{P'}(t)$. By the induction hypothesis, $P'' \in \mathbf{DP}$. Furthermore, by Theorem 2.2.6, $\chi_{P''}(t)$ divides $\chi_P(t)$. Note that $t \nmid \chi_{P''}(t)$ by Lemma 2.2.5, thus $P \in \mathbf{DP}$ as desired. □

Remark 2.2.11. We address here some remarks about the relation of our inductive and divisional posets with some known concepts in literature.

1. Brandt [Bra92, Definition 1.2.21] defined the class **IL** of *inductive lattices* to be the smallest class of geometric lattices which satisfies: (1) $\{\hat{0}\} \in \mathbf{IL}$ and (2) $P \in \mathbf{IL}$ if there exists an atom $a \in A$ such that $P'' \in \mathbf{IL}$, $P' \in \mathbf{IL}$, and $\chi_{P''}(t)$ divides $\chi_{P'}(t)$. Thus for a geometric lattice P, we have that $P \in \mathbf{IL}$ if and only if $P \in \mathbf{IP}$.

2. A central hyperplane arrangement $\mathcal{A}$ in $V = \mathbb{K}^l$ is inductively free (resp., divisionally free) in Definition 2.1.26 (resp., 2.1.27) if and only if the (geometric) intersection lattice $L(\mathcal{A})$ of $\mathcal{A}$ is inductive (resp., divisional). In particular, $\mathbf{IP} \subsetneq \mathbf{DP}$ which follows from Remark 2.1.28.

Now, we present one of the key results of this section. Recall Definition 2.1.24, which applies equally in the abelian case: an abelian arrangement $\mathcal{A}$ is called factorable, and we write $\mathcal{A} \in \mathbf{FR}$, if its poset of layers is factorable, with the multiset of (combinatorial) exponents denoted as $\exp(\mathcal{A})$.

Theorem 2.2.12. *If a locally geometric poset is divisional, then it is factorable. This means* $\mathbf{DP} \subseteq \mathbf{FR}$.

Proof. We need to show that if $P \in \mathbf{DP}$ with $r = \mathrm{rk}(P) \geq 1$, then there are positive integers $d_1, \cdots, d_r \in \mathbb{Z}_{>0}$ such that

$$\chi_P(t) = \prod_{i=1}^{r}(t - d_i).$$

We argue by induction on r. If $r = 1$ then $\chi_P(t) = t - |A|$ and the assertion clearly holds. Suppose $r > 1$. Since $P \in \mathbf{DP}$, there exists an atom $a \in A$ such that $P'' \in \mathbf{DP}$ and $\chi_{P''}(t)$ divides $\chi_P(t)$. By the induction hypothesis, there exist positive integers $d_1, \cdots, d_{r-1} \in \mathbb{Z}_{>0}$ and an integer $d_r \in \mathbb{Z}$ such that

$$\chi_{P''}(t) = \prod_{i=1}^{r-1}(t - d_i),$$
$$\chi_P(t) = (t - d_r)\chi_{P''}(t).$$

Moreover, $d_1 d_2 \cdots d_r > 0$ by Lemma 2.2.5. Thus $d_r > 0$. □

Thus the divisionality of a poset is a sufficient condition for its factorability. The following necessary and sufficient condition for a poset to be divisional is immediate from Definition 2.2.8. Note that the sum of all exponents of a divisional poset equals the number of atoms.

Theorem 2.2.13. *A locally geometric poset P of rank r is divisional if and only if there exists a chain, called a divisional chain*

$$\hat{0} = x_0 < x_1 < \cdots < x_r,$$

such that $\mathrm{rk}(x_i) = i$ *and* $\chi_{Q_i}(t)$ *divides* $\chi_{Q_{i-1}}(t)$ *where* $Q_i := P_{\geq x_i}$ *for each* $1 \leq i \leq r$. *In this case,* $\exp(P) = \{d_1, \cdots, d_r\}$ *where* $d_i := |A(Q_{i-1})| - |A(Q_i)|$.

Remark 2.2.14. The converse of Theorem 2.2.12 is not true in general. Namely, there exists a factorable poset that is not divisional. An example from hyperplane arrangements is already mentioned in Remark 2.1.28. We give here an example of a poset that is not a lattice. In [Hal17, Example 4.6], the *weighted partition poset* $P := \Pi_3^w$ of rank 3 is given with the characteristic polynomial $\chi_P(t) = (t-3)^2$ (see Figure 2.1). However, P is not divisional because $\chi_{P_{\geq x}}(t) = t - 2$ does not divide $\chi_P(t)$ for any atom x.

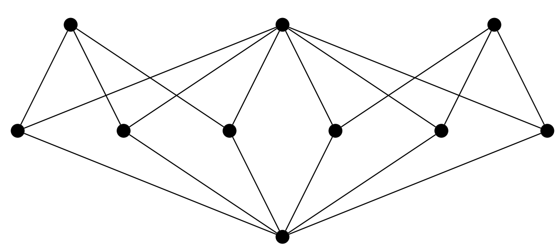


Figure 2.1: The weighted partition poset Π_3^w.

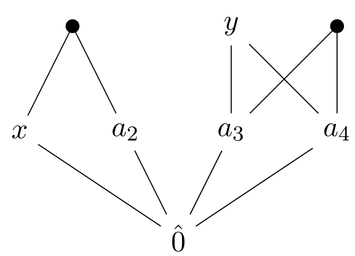
By Proposition 2.2.10, the exponents of an inductive poset are defined naturally. The following "addition" theorem for inductive posets follows readily from Definition 2.2.7 and Theorem 2.2.6.

Theorem 2.2.15. *Let P be a locally geometric poset with $A \neq \emptyset$ and let $a \in A$.*

1. *Suppose that a is not a separator of P. If* $P'' \in \mathbf{IP}$ *with* $\exp(P'') = \{d_1, \cdots, d_{\ell-1}\}$ *and* $P' \in \mathbf{IP}$ *with* $\exp(P') = \{d_1, \cdots, d_{\ell-1}, d_\ell\}$, *then* $P \in \mathbf{IP}$ *with* $\exp(P) = \{d_1, \cdots, d_{\ell-1}, d_\ell + 1\}$.

2. *Suppose that a is a separator of P. If $P'' \in \mathbf{IP}$, $P' \in \mathbf{IP}$ with* $\exp(P'') = \exp(P') = \{d_1, \cdots, d_{\ell-1}\}$, *then $P \in \mathbf{IP}$ with* $\exp(P) = \{1, d_1, \cdots, d_{\ell-1}\}$.

The process of constructing an inductive poset P from the trivial lattice (or more generally, from an inductive subposet generated by some atoms) by adding an atom one at a time with the aid of Theorem 2.2.15 is called an *induction table*. Each row of the table records the exponents of P' and P'' and the atom a added at each step. The last row displays the exponents of P. An example of this is in Figure 2.2 below, which depicts an inductive poset that is not geometric.



$\exp(P')$	a	$\exp(P'')$
$\emptyset$	x	$\emptyset$
1	a_3	$\emptyset$
2	a_4	2
1, 2	a_2	1
1, 3		

Figure 2.2: An inductive poset that is not geometric (left) and an induction table for its inductiveness (right). The elements labelled by x and y do not satisfy the requirement of Definition 2.1.12.

Having presented several combinatorial properties of posets, we now turn our attention to the implications when these posets represent the posets of layers of abelian arrangements. This will allow us to identify analogous classes of arrangements to those defined in the previous section, but in a more general context. It is important to point out and recall that in this chapter we focus exclusively on the case of central arrangements. Before proceeding, it is necessary to establish some necessary tools.

Definition 2.2.16. A property C of arrangements is called a *combinatorial property (or combinatorially determined)* if for any distinct arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ in an arbitrary vector space V having the same combinatorics, i.e., their intersection posets are isomorphic $L(\mathcal{A}_1) \simeq L(\mathcal{A}_2)$, then $\mathcal{A}_1$ has property P if and only if $\mathcal{A}_2$ has property P.

It is well known that the combinatorial structure associated with an arrangement is its poset of layers. Our initial focus will be on investigating the properties of this poset and examining whether it is possible to factorize its characteristic polynomial. First, let us restate a fundamental result on its structure.

Theorem 2.2.17 ([Bib22, Corollary 13.11], [BD22, Corollary 4.4.6]). *Let $\mathcal{A}$ be an abelian arrangement. Then $L(\mathcal{A})$ is a geometric poset.*

Definition 2.2.18. Similar to Definition 2.1.24, we call an abelian arrangement $\mathcal{A}$ *factorable* if its intersection poset $L(\mathcal{A})$ is factorable. In this case, we call the roots of $\chi_{\mathcal{A}}(t^{1/g})$ the *(combinatorial) exponents* of $\mathcal{A}$ and use the notation $\exp(\mathcal{A})$ to denote the multiset of exponents. Denote also by **FR** the class of factorable abelian arrangements.

By Remark 1.1.7, $\mathcal{A} \in \mathbf{FR}$ if and only if there are positive integers $d_1, \cdots, d_{\mathrm{rk}(\mathcal{A})} \in \mathbb{Z}_{>0}$ such that

$$\chi_{\mathcal{A}}(t) = t^{g(\ell - \mathrm{rk}(\mathcal{A}))} \cdot \prod_{i=1}^{\mathrm{rk}(\mathcal{A})} (t^g - d_i).$$

In this case,

$$\exp(\mathcal{A}) = \{0^{\ell - \mathrm{rk}(\mathcal{A})}\} \cup \exp(L(\mathcal{A})).$$

A crucial aspect to verify is whether the deletion and restriction operations associated with arrangements correspond with coherent and consistent operations in the poset of layers.

Now we are ready to introduce the protagonists of this section in the arrangement framework. First recall that, fixed $H \in \mathcal{A}$, $\mathcal{A}' := \mathcal{A} \setminus \{H\}$ is an arrangement in G^l, $\mathcal{A}'' := \mathcal{A}^H$ in $G^{\ell-1}$ and that $L(\mathcal{A}') = L(\mathcal{A})'$ and $L(\mathcal{A}'') = L(\mathcal{A})''$. We call $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ the triple of arrangements associated to H.

Definition 2.2.19. The class **IA** of *inductive (abelian) arrangements* is the smallest class of abelian arrangements which satisfies

1. $\emptyset_\ell \in \mathbf{IA}$ for $\ell \geq 1$,
2. $\mathcal{A} \in \mathbf{IA}$ if there exists $H \in \mathcal{A}$ such that $\mathcal{A}'' \in \mathbf{IA}$, $\mathcal{A}' \in \mathbf{IA}$, and $\chi_{\mathcal{A}'}(t) = (t^g - d) \cdot \chi_{\mathcal{A}''}(t)$ for some $d \in \mathbb{Z}$.

Definition 2.2.20. The class **DA** of *divisional (abelian) arrangements* is the smallest class of abelian arrangements which satisfies

1. $\emptyset_\ell \in \mathbf{DA}$ for $\ell \geq 1$,

2. $\mathcal{A} \in \mathbf{DA}$ if there exists $H \in \mathcal{A}$ such that $\mathcal{A}'' \in \mathbf{DA}$ and $\chi_{\mathcal{A}}(t) = (t^g - d) \cdot \chi_{\mathcal{A}''}(t)$ for some $d \in \mathbb{Z}$.

We now show that inductiveness and divisionality depend only on the combinatorics of arrangements. Furthermore, they precisely coincide with the definitions of inductive and divisional posets introduced at the beginning of the section.

Theorem 2.2.21. *Let $\mathcal{A}$ be an abelian arrangement. Then $\mathcal{A} \in \mathbf{IA}$ (resp., $\mathbf{DA}$) if and only if $L(\mathcal{A}) \in \mathbf{IP}$ (resp., $\mathbf{DP}$).*

Proof. We show the assertion for inductiveness by double induction on $\mathrm{rk}(\mathcal{A})$ and $|\mathcal{A}|$. The assertion for divisionality can be proved by induction on $\mathrm{rk}(\mathcal{A})$ by a similar (and easier) argument.

The assertion is clearly true when $\mathrm{rk}(\mathcal{A}) = 0$ or $|\mathcal{A}| = 0$ (i.e., $\mathcal{A} = \varnothing$). Suppose $\mathrm{rk}(\mathcal{A}) \geq 1$ and $|\mathcal{A}| \geq 1$. Suppose $\mathcal{A} \in \mathbf{IA}$. Then there exists $H \in \mathcal{A}$ such that $\mathcal{A}'' \in \mathbf{IA}$, $\mathcal{A}' \in \mathbf{IA}$, and $\chi_{\mathcal{A}'}(t) = (t^g - d) \cdot \chi_{\mathcal{A}''}(t)$ for some $d \in \mathbb{Z}$. Note that $|\mathcal{A}'| < |\mathcal{A}|$ and $\mathrm{rk}(\mathcal{A}'') < \mathrm{rk}(\mathcal{A})$. By the induction hypothesis, $L'' = L(\mathcal{A}'') \in \mathbf{IP}$ and $L' = L(\mathcal{A}') \in \mathbf{IP}$. Moreover, if $\mathrm{rk}(\mathcal{A}) = \mathrm{rk}(\mathcal{A}') + 1$, then by Remark 1.1.7,

$$t^g \cdot \chi_{L'}(t^g) = (t^g - d) \cdot \chi_{L''}(t^g).$$

Hence $\chi_{L'}(t) = \chi_{L''}(t)$ since $t \nmid \chi_{L''}(t)$. Similarly, if $\mathrm{rk}(\mathcal{A}) = \mathrm{rk}(\mathcal{A}')$, then $\chi_{L'}(t) = (t - d)\chi_{L''}(t)$. In either case, $\chi_{L''}(t)$ divides $\chi_{L'}(t)$. Thus $L(\mathcal{A}) \in \mathbf{IP}$. A similar argument shows that if $L \in \mathbf{IP}$ then $\mathcal{A} \in \mathbf{IA}$, which completes the proof. □

Corollary 2.2.22. *The property of being inductive or divisional of an abelian arrangement is a combinatorial property.*

Proof. It follows from Proposition 2.2.9 and Theorem 2.2.21 above. □

Hence, wecan state the following result.

Theorem 2.2.23. *Let $\mathcal{A}$ be an abelian arrangement. If $\mathcal{A}$ is divisional, then it is factorable.*

2.3 Strictly supersolvable implies inductive

It is particularly interesting to study how these combinatorial classes of abelian arrangements interact, and whether there are any implications among them, as it has been observed in the case of hyperplane arrangement (see Remark 2.1.28). Let us begin by examining the poset framework, proving that strictly supersolvability implies inductiveness. In order to do that, we first need to establish some additional fundamental facts regarding M-ideals.

Lemma 2.3.1. *If a poset P has an M-ideal Q with $\mathrm{rk}(Q) = \mathrm{rk}(P) - 1$, then P is necessarily pure.*

Proof. First note that $A(P) \setminus A(Q) \neq \emptyset$ since Q is join-closed. Fix an arbitrary $x \in \max(P)$. If $x \in Q$, then by Condition 2.1.11(1) for any $a \in A(P) \setminus A(Q)$ there exists $b \in a \vee x$ such that $x < b$, a contradiction. We may assume $x \in P \setminus Q$. Then by Condition 2.1.11(2), there exists $y \in \max(Q)$ such that $y < x$. Thus $\mathrm{rk}(x) > \mathrm{rk}(Q)$ and hence $\mathrm{rk}(x) = \mathrm{rk}(P)$. □

Lemma 2.3.2 ([BD22, Lemma 2.4.6])**.** *Let Q be an M-ideal of a poset P with $\mathrm{rk}(Q) = \mathrm{rk}(P) - 1$ and let $a \in P$. Then $a \in A(P) \setminus A(Q)$ if and only if $y \wedge a = \hat{0}$ for all $y \in \max(Q)$.*

Proposition 2.3.3 ([BD22, Proposition 2.4.7])**.** *Let Q be an M-ideal of a poset P with $\mathrm{rk}(Q) = \mathrm{rk}(P) - 1$. Fix $x \in P \setminus Q$ and let y be an element in $\max(P)$ such that $x \leq y$. Let y' be the unique element in $\max(Q)$ such that (y covers y' and) y' is a modular element in the geometric lattice $P_{\leq y}$ (Definition 2.1.11). Then $x' := y' \wedge x$ is the unique element in Q such that x covers x' and x' is modular in $P_{\leq x}$.*

Now we prove a new property of a TM-ideal, extending a well-known property [Sta71, Lemma 1] of a modular element in a finite geometric lattice.

Lemma 2.3.4. *If Q is a TM-ideal of a poset P with $\mathrm{rk}(Q) = \mathrm{rk}(P) - 1$, then for any $a \in A(P) \setminus A(Q)$ there is a poset isomorphism $Q \simeq P_{\geq a}$.*

Proof. Fix $a \in A(P) \setminus A(Q)$ and denote $\mathcal{R} := P_{\geq a}$. Owing to Definition 2.1.11(1*) and Proposition 2.3.3, two poset maps σ and τ below are well-defined

$$\sigma : Q \longrightarrow \mathcal{R} \text{ via } x \mapsto x \vee a, \quad \tau : \mathcal{R} \longrightarrow Q \text{ via } x \mapsto x'.$$

We show that σ is a poset isomorphism whose inverse is exactly τ. First we show that both maps are order-preserving. The assertion for σ is easy.

To show the assertion for τ note that for $x_1 \leq_{\mathcal{R}} x_2$, if $y \in \max(P)$ and $x_2 \leq_{\mathcal{R}} y$, then $\tau(x_1) = y' \wedge x_1$ and $\tau(x_2) = y' \wedge x_2$ where y' is the unique element in $\max(Q)$ such that y' is modular in $P_{\leq y}$. Thus $\tau(x_1) \leq_Q \tau(x_2)$ follows easily.

Now we show $\sigma \circ \tau = \tau \circ \sigma = \mathrm{id}$. If $x \in \mathcal{R}$, then $(\sigma \circ \tau)(x) = \sigma(x') = x' \vee a = x$ where the last equality follows from Definition 2.1.11(1*) since $x \in x' \vee a$.

Let $x \in Q$, then $(\tau \circ \sigma)(x) = \tau(x \vee a) = (x \vee a)'$. It remains to show $(x \vee a)' = x$. If x and $(x \vee a)'$ are incomparable, then $x \vee a \in (x \vee a)' \vee x$ which contradicts the join-closeness of Q. Note that $\mathrm{rk}(x \vee a) > \mathrm{rk}(x)$ hence it cannot happen that $x > (x \vee a)'$. Thus we may assume $x \leq (x \vee a)'$. Let $y \in \max(P)$ so that $x \vee a \leq y$. Let y' be the unique element in $\max(Q)$ such that y' is modular in $P_{\leq y}$. Then

$$(x \vee a)' = y' \wedge (x \vee a) = x \vee (y' \wedge a) = x \vee \hat{0} = x,$$

where the second equality follows from the modularity 2.1.2 of y' in $P_{\leq y}$ with $x \leq y'$, and the third equality follows from Lemma 2.3.2. □

The lemma above allows us to prove the following.

Lemma 2.3.5. *Let Q be a TM-ideal of a poset P with $\mathrm{rk}(Q) = \mathrm{rk}(P) - 1$. If $Q \in \mathbf{IP}$ (resp., $Q \in \mathbf{DP}$), then $P \in \mathbf{IP}$ (resp., $P \in \mathbf{DP}$) with*

$$\exp(P) = \exp(Q) \cup \{|A(P) \setminus A(Q)|\}.$$

Proof. First we show the assertion for divisionality. Fix $a \in A(P) \setminus A(Q)$. By Lemma 2.3.4, $Q \simeq P'' = P_{\geq a}$. Suppose $Q \in \mathbf{DP}$. Then $P'' \in \mathbf{DP}$ by Proposition 2.2.9. Moreover, by Theorem 2.1.15,

$$\chi_P(t) = (t - m)\chi_Q(t),$$

where $m := |A(P) \setminus A(Q)|$. Therefore, $\chi_{P''}(t)$ divides $\chi_P(t)$. Hence $P \in \mathbf{DP}$ with $\exp(P) = \exp(Q) \cup \{m\}$ as desired.

Now we show the assertion for inductiveness by adding the atoms from $A(P) \setminus A(Q)$ to $A(Q)$ in any order successively with the aid of Theorem 2.2.15. Write $A(P) \setminus A(Q) = \{a_1, \ldots, a_m\}$. Let $A_i := A(Q) \cup \{a_1, \ldots, a_i\}$ and $P_i := P(A_i)$ for each $1 \leq i \leq m$.

First note that by Lemma 2.3.1, the poset P is pure. We observe that $\mathrm{rk}(P_i) = \mathrm{rk}(P) = r$ for every $1 \leq i \leq m$. It is because $|a_i \vee y| = 1$ and $\mathrm{rk}(a_i \vee y) = r$ for any $y \in \max(Q)$ and $a_i \in A_i \setminus A(Q) \subseteq A \setminus A(Q)$.

We claim that Q is a TM-ideal of rank $r-1$ of P_i for every $1 \le i \le m$. (The case $i = m$ is obviously true.) Condition 2.1.11(1*) is clear. It suffices to show Condition 2.1.11(2). First consider $i = m-1$. Fix $x \in \max(P_{m-1}) \subseteq \max(P)$. Denote $L := P_{\le x}$ and $L_{m-1} := (P_{m-1})_{\le x}$. Therefore L and L_{m-1} are geometric lattices sharing top element x. We need to show that there is some $y \in \max(Q)$ such that y is a modular element in L_{m-1}. Since Q is a TM-ideal of P, there exists $y' \in \max(Q)$ such that y' is a modular element in L. If $x \not> a_m$ then $L = L_{m-1}$. We may take $y = y'$. If $x > a_m$ then $L_{m-1} = L(A(L) \setminus \{a_m\})$. Since $y' \not> a_m$, we must have that $y' \in L_{m-1}$ and y' is also a modular element in L_{m-1} by [JT84, Lemma 4.6]. Again take $y = y'$. Use this argument repeatedly, we may show the claim holds true for every $1 \le i \le m-1$.

Now we show that $P_i \in \mathbf{IP}$ with $\exp(P_i) = \exp(Q) \cup \{i\}$ for every $1 \le i \le m$. Note that by Lemma 2.3.4, $Q \simeq P_{\ge a}$ for any $a \in A(P) \setminus A(Q)$. It is not hard to check that $(P_1, P_1' = Q, P_1'' \simeq Q)$ is the triple of posets with distinguished atom a_1, and that a_1 is a separator of P_1. Hence $P_1 \in \mathbf{IP}$ with $\exp(P_1) = \exp(Q) \cup \{1\}$ by Theorem 2.2.15. Similarly, $(P_2, P_2' = P_1, P_2'' \simeq Q)$ is the triple with distinguished atom a_2, and that a_2 is not a separator of P_2. Hence $P_2 \in \mathbf{IP}$ with $\exp(P_2) = \exp(Q) \cup \{2\}$. Use this argument repeatedly, we may show the claim holds true for every $1 \le i \le m$. The case $i = m$ yields $P \in \mathbf{IP}$ with $\exp(P) = \exp(Q) \cup \{m\}$ as desired. □

Now, as a direct corollary of these lemmas, we have the following important result.

Theorem 2.3.6. *If a poset is strictly supersolvable, then it is inductive.*

Proof. Note that the trivial lattice is inductive. Apply Lemma 2.3.5 repeatedly to the elements in any TM-chain of a strictly supersolvable poset P. □

Example 2.3.7. The *Dowling posets* are proved to be strictly supersolvable [BD22, Example 5.1.8]. The poset of layers of the toric arrangement of an arbitrary ideal of a type C root system with respect to the *integer lattice* is also strictly supersolvable (Theorem 2.4.15). Hence these posets are inductive by Theorem 2.3.6.

Remark 2.3.8. The main result of [JT84] by Jambu-Terao mentioned in Remark 2.1.28 is a special case of our Theorem 2.3.6 when the poset is a geometric lattice. An induction table for a strictly supersolvable poset can easily be constructed using the argument in the proof of Lemma 2.3.5.

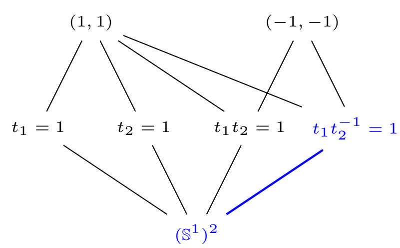
The converse of Theorem 2.3.6 is not true in general. There are many known examples of central hyperplane arrangements whose intersection lattices are inductive but not (strictly) supersolvable (see e.g., Theorem 2.4.11). In the final section, we will see in Corollary 2.4.21 and Theorem 2.4.23 new examples from toric arrangements: The poset of layers of the toric arrangement of a type B_ℓ root system for $\ell \geq 3$ is inductive, but not supersolvable. That arises from type B_2 depicted in Figure 2.3 below is inductive and supersolvable, but not strictly supersolvable.

Thus for locally geometric posets, we have proved

$$\mathbf{SSS} \subsetneq \mathbf{IP} \subsetneq \mathbf{DP} \subsetneq \mathbf{FR}.$$

Compared with the relation described in Remark 2.1.28, supersolvable posets do not form a subclass of inductive posets. An example can be found in the poset of layers of the toric arrangement of a type D_2 root system (the subposet of the poset in Figure 2.3 generated by $\{t_1t_2 = 1, t_1t_2^{-1} = 1\}$) that is supersolvable but not inductive.

The containment $\mathbf{IP} \subsetneq \mathbf{DP}$ is strict as can be seen by considering the same example of Remark 2.1.28.



$\exp(P')$	a	$\exp(P'')$
$\emptyset$	$t_1 = 1$	$\emptyset$
1	$t_2 = 1$	1
$1, 1$	$t_1t_2 = 1$	1
$1, 2$	$t_1t_2^{-1} = 1$	2
$2, 2$		

Figure 2.3: The toric arrangement of a type B_2 root system with its poset P of layers (left) and an induction table for inductiveness (right). The induction table is derived thanks to Theorem 2.2.15 which deduces that P is inductive with exponents $\exp(P) = \{2, 2\}$. In addition, P is supersolvable with the elements of a rank-1 M-ideal colored in blue. However, P is not strictly supersolvable since it has no TM-ideal of rank 1.

The containments shown for geometric posets yield direct analogous results in the context of abelian arrangements. Indeed, we can apply the same arguments and discussions within the framework of arrangements.

Remark 2.3.9. By Remark 2.3.8 and Theorem 2.2.21, we have

$$\mathbf{SSS} \subsetneq \mathbf{IA} \subseteq \mathbf{DA} \subsetneq \mathbf{FR}.$$

It is an open question to us whether or not the containment $\mathbf{IA} \subseteq \mathbf{DA}$ is strict. The example of a hyperplane arrangement that is divisionally free but not inductively free in Remark 2.1.28 is not an integral arrangement.

Note that an abelian arrangement is inductive if it can be constructed from the empty arrangement by adding an element (= a connected component of a hyperplane) one at a time with the aid of the following "addition" theorem at each addition step. It thus also makes sense to speak of an induction table for an inductive arrangement in a similar way as of inductive posets.

Theorem 2.3.10. *Let $\mathcal{A} \neq \emptyset$ be an abelian arrangement in $T \simeq G^\ell$ and let $H \in \mathcal{A}$. If $\mathcal{A}'' \in \mathbf{IA}$ with $\exp(\mathcal{A}'') = \{d_1, \dots, d_{\ell-1}\}$ and $\mathcal{A}' \in \mathbf{IA}$ with $\exp(\mathcal{A}') = \{d_1, \dots, d_{\ell-1}, d_\ell\}$, then $\mathcal{A} \in \mathbf{IA}$ with $\exp(\mathcal{A}) = \{d_1, \dots, d_{\ell-1}, d_\ell + 1\}$.*

Proof. It follows directly from Definition 2.2.19 and Theorem 1.1.10. □

We complete this discussion by describing an arrangement theoretic characterization for (strict) supersolvability. For this discussion it is helpful to provide an alternative definition of M-ideals and TM-ideals, equivalent to Definition 2.1.11.

Definition 2.3.11. Given a subarrangement $\mathcal{B}$ of an abelian arrangement $\mathcal{A}$, we say $\mathcal{B}$ is an *M-ideal* of $\mathcal{A}$ if $L(\mathcal{B})$ is a proper order ideal of $L(\mathcal{A})$, and for any two distinct $H_1, H_2 \in \mathcal{A} \setminus \mathcal{B}$ and every connected component C of the intersection $H_1 \cap H_2$ there exists $H_3 \in \mathcal{B}$ such that $C \subseteq H_3$. More strongly, an M-ideal $\mathcal{B}$ is called a *TM-ideal* of $\mathcal{A}$ if

(*) for any $X \in L(\mathcal{B})$ and $H \in \mathcal{A} \setminus \mathcal{B}$ the intersection $X \cap H$ is connected.

Theorem 2.3.12. *Let $\mathcal{A}$ be an arrangement of rank r in $T \simeq G^\ell$. Then $\mathcal{A}$ is supersolvable (resp., strictly supersolvable) (Definition 2.1.19) if and only if there is a chain, called an M-chain (resp., a TM-chain)*

$$\emptyset = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \cdots \subseteq \mathcal{A}_r = \mathcal{A},$$

such that each $\mathcal{A}_i$ is an M-ideal (resp., a TM-ideal) of $\mathcal{A}_{i+1}$.

Proof. Observe that if $\mathcal{B} \subseteq \mathcal{A}$, then $L(\mathcal{B})$ is a pure, join-closed ideal of $L(\mathcal{A})$. Note also that the poset of layers of an abelian arrangement is a geometric poset by Theorem 2.2.17. Thus by Lemma 2.1.13, if $\mathcal{B}$ is an M-ideal (resp., a TM-ideal) of $\mathcal{A}$, then $L(\mathcal{B})$ is an M-ideal (resp., a TM-ideal) of $L(\mathcal{A})$ such that, due to the assumption that the arrangement is central, $\mathrm{rk}(\mathcal{B}) = \mathrm{rk}(\mathcal{A}) - 1$. Therefore, if there exists an M-chain (resp., a TM-chain)

$$\emptyset = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \cdots \subseteq \mathcal{A}_r = \mathcal{A},$$

then $L(\mathcal{A})$ is supersolvable (resp., strictly supersolvable) with an M-chain (resp., a TM-chain)

$$\{\hat{0}\} = L(\emptyset) \subseteq L(\mathcal{A}_1) \subseteq \cdots \subseteq L(\mathcal{A}_r) = L(\mathcal{A}),$$

Conversely, if Q is an M-ideal (resp., a TM-ideal) of $L(\mathcal{A})$ with $\mathrm{rk}(Q) = \mathrm{rk}(\mathcal{A}) - 1$, then again by Lemma 2.1.13, the set $A(Q)$ of atoms is an M-ideal (resp., a TM-ideal) of $\mathcal{A}$. Thus if $L(\mathcal{A})$ is supersolvable (resp., strictly supersolvable), then any M-chain (resp., TM-chain) of $L(\mathcal{A})$ induces an M-chain (resp., a TM-chain) for $\mathcal{A}$. □

As a concluding note for this section, it is interesting to highlight an important difference between (strictly) supersolvability and inductiveness or divisionality. Note from Remark 2.1.17 that (strict) supersolvability is closed under taking localization: If $\mathcal{A} \in \mathbf{SS}$ (resp., $\mathcal{A} \in \mathbf{SSS}$), then $\mathcal{A}_X \in \mathbf{SS}$ (resp., $\mathcal{A}_X \in \mathbf{SSS}$) for every $X \in L(\mathcal{A})$. We will see that in general it is not the case for inductiveness or divisionality. More explicitly, we give an example of an inductive toric arrangement with a non-factorable localization.

First let us recall the definition of central (real) hyperplane and toric arrangements as abelian arrangements when the Lie group G is $\mathbb{R}$ and S^1, respectively. Let A be a finite set of integral vectors in $\mathbb{Z}^\ell$. Given a vector $\alpha = (a_1, \ldots, a_\ell) \in A$, we may define the hyperplane

$$H_{\alpha,\mathbb{R}} := \{x \in \mathbb{R}^\ell \mid a_1x_1 + \cdots + a_\ell x_\ell = 0\},$$

and the hypertorus

$$H_{\alpha,S^1} := \{t \in (S^1)^\ell \mid t_1^{a_1} \cdots t_\ell^{a_\ell} = 1\}.$$

The set $A \subseteq \mathbb{Z}^\ell$ defines the central hyperplane arrangement

$$\mathcal{H} := \{H_{\alpha,\mathbb{R}} \mid \alpha \in A\}.$$

and the central toric arrangement

$$\mathcal{A} := \{\text{connected components of } H_{\alpha,S^1} \mid \alpha \in A\}.$$

Alternatively, given an integral matrix $S \in \mathrm{Mat}_{\ell\times m}(\mathbb{Z})$, we may view each column as a vector in $\mathbb{Z}^\ell$ so that we may define the central hyperplane and toric arrangements from S as above.

Example 2.3.13. Let $S \in \mathrm{Mat}_{3\times 6}(\mathbb{Z})$ be an integral matrix defined as below:

$$S = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{bmatrix}. \tag{2.1}$$

Let $\mathcal{H}_S$ and $\mathcal{A}_S$ be the central hyperplane and toric arrangements defined by S, respectively. Note that by definition of localization we may write $\mathcal{H}_S = (\mathcal{A}_S)_X$ where X denotes the layer $(1,1,1) \in L(\mathcal{A}_S)$.

In fact, $\mathcal{H}_S$ is linearly isomorphic to the *essentialization* of the *cone* of the *digraphic Shi arrangement* defined by the path $3 \to 2 \to 1$ in [Ath98, Figure 3]. The characteristic polynomial of $\mathcal{H}_S$ is given by

$$\chi_{\mathcal{H}_S}(t) = (t-1)(t^2 - 5t + 7),$$

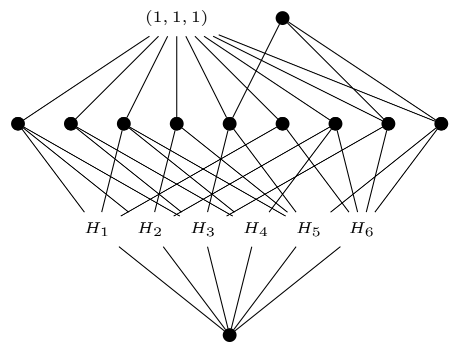
which implies that $\mathcal{H}_S$ is not divisional hence not inductive.

However, we may show that $\mathcal{A}_S$ is inductive with exponents $\{2,2,2\}$. Let H_i denote the (connected) hypertorus defined by the i-th column of the matrix S. The poset of layers of $\mathcal{A}_S$ and an induction table are given in Figure 2.4. (Observe also that $\mathcal{A}_S$ is not locally supersolvable since the localization $\mathcal{H}_S$ is not supersolvable by the preceding discussion).

It happens quite often that the hyperplane arrangement defined by a matrix is inductive, but the toric arrangement defined by the same matrix is not. Example 2.3.13 above deduces that the converse is also possible. This is a rare, perhaps counter-intuitive example that toric arrangement could be inductive, while hyperplane arrangement cannot be.

2.4 Application to toric arrangements of ideals in root systems

In this section, we will apply the previously discussed results on inductiveness and divisionality to examine toric arrangements associated to ideals of root systems. We will prove that these classes of arrangements are inductive



$\exp(\mathcal{A}')$	H	$\exp(\mathcal{A}'')$
0, 0, 0	H_1	0, 0
0, 0, 1	H_2	0, 1
0, 1, 1	H_3	0, 1
0, 1, 2	H_4	1, 2
1, 1, 2	H_6	1, 2
1, 2, 2	H_5	2, 2
2, 2, 2		

Figure 2.4: The poset of layers of the toric arrangement $\mathcal{A}_S$ defined by matrix S in (2.1) and an induction table for its inductiveness.

when defined by ideals of root systems of types A, B or C, up until now, the only cases considered. Before proceeding with this analysis, we will first provide a brief overview and basic definitions related to root systems, mainly following [Hum72]. In particular, in the context of hyperplane and toric arrangements arising from root systems, their combinatorial and arithmetic properties have been extensively studied. In this context, the arithmetic Tutte polynomials of the classical root systems have been computed in [ACH14], describing structural properties of their arithmetic matroids and enabling the computation of various combinatorial and topological invariants of associated toric arrangements.
Moreover, for types A, B, and C, the associated hyperplane arrangements are known to be supersolvable ([Hul16]), making their inductiveness a natural property to investigate. On the other hand, the case of type D, where supersolvability does not hold in general, presents a different landscape that remains to be explored.

Given any Euclidean vector space E, we will consider $E = \mathbb{R}^\ell$ in our subsequent results, a vector α defines a map $\sigma_\alpha : E \to E$ that reflects every vector over the hyperplane that is orthogonal to α. Specifically,

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha,$$

where $\beta \in E$ and $(\cdot,\cdot)$ is the standard product on E. Indeed, note that $\sigma_\alpha(\alpha) = -\alpha$ and if $(\beta,\alpha) = 0$, then $\sigma_\alpha(\beta) = \beta$, fixing all the vectors in the hyperplane.
For convenience, if we denote

$$\langle\beta,\alpha\rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)},$$

then $\langle\cdot,\cdot\rangle$ is linear in the first variable and we have $\sigma_\alpha(\beta) = \beta - \langle\beta,\alpha\rangle\alpha$.

Definition 2.4.1. A *root system* is a set of vectors $\Phi \subset E$ such that

1. Φ is finite, $\text{span}(\Phi) = E$ and $0 \notin \Phi$;
2. If $\alpha \in \Phi$, then the only scalar multiples of α in Φ are $\pm\alpha$;
3. If $\alpha \in \Phi$, then $\sigma_\alpha(\Phi) = \Phi$, hence σ_α permutes Φ;
4. If $\alpha, \beta \in \Phi$, then $\langle\beta,\alpha\rangle \in \mathbb{Z}$.

We define the *rank* of a root system $\text{rk}(\Phi)$ to be the dimension of Euclidean space it spans.

In this dissertation as in general, we focus our attention to a specific type of root systems, known as irreducible.

Definition 2.4.2. A root system Φ is *reducible* if there exixts root systems $\Phi_1, \Phi_2 \subseteq \Phi$ such that $\Phi = \Phi_1 \sqcup \Phi_2$ and $\text{span}(\Phi) = \text{span}(\Phi_1) \oplus \text{span}(\Phi_2)$. Any root system that is not reducible is called *irreducible.*

Definition 2.4.3. Given $\Delta \subseteq \Phi$, we say that Δ is a *simple system* or *base* if

1. Δ is a basis of $\text{span}(\Phi) = E$, hence $|\Delta| = \ell$ and the roots in $\Delta = \{\alpha_1, \cdots, \alpha_\ell\}$ are called *simple*;
2. Each root β can be written as $\beta = \sum_{i=1}^{\ell} k_i\alpha_i$, with integer coefficients k_i all nonnegative or all nonpositive.

Note that the expression for β in the second condition is unique, which allows to define *positive roots* as $\Phi^+ = \{\beta \in \Phi \mid k_i \geq 0,\ \forall i = 1, \cdots, \ell\}$ and similarly for *negative roots* Φ^-. This leads to a partition of the root system given by $\Phi = \Phi^+ \sqcup \Phi^-$. Since Δ is not necessarily unique, whenever any definition depends on Δ, a choice is made that must be taken into account.

For each $\Psi \subseteq \Phi^+$, let S_Ψ denote the coefficient matrix of Ψ with respect to the base Δ, i.e., $S_\Psi = [s_{ij}]$ is the $\ell \times |\Psi|$ integral matrix that satisfies

$$\Psi = \left\{ \sum_{i=1}^{\ell} s_{ij}\alpha_i \,\middle|\, 1 \leq j \leq |\Psi| \right\}.$$

The matrix S_Ψ depends only upon Φ.

Definition 2.4.4. Following the previous section, we define $\mathcal{A}_\Psi := \mathcal{A}_{S_\Psi}(\Phi)$ and $\mathcal{H}_\Psi := \mathcal{H}_{S_\Psi}(\Phi)$ as the central toric and hyperplane arrangements defined by S_Ψ respectively. We call these arrangements the arrangements with respect to the *root lattice*.

It is possible to define a partial order $\geq$ on Φ^+ such that $\beta_1 \geq \beta_2$ if and only if $\beta_1 - \beta_2 = \sum_{\alpha\in\Delta} n_\alpha \alpha$ has all nonnegative coefficients, i.e. $n_\alpha \in \mathbb{Z}_{\geq 0}$, $\forall \alpha \in \Delta$.

Definition 2.4.5. Given a root system $\Phi = \Phi^+ \sqcup \Phi^-$, a subset $I \subseteq \Phi^+$ is called an *ideal* if, for every $\beta_1, \beta_2 \in \Phi^+$, $\beta_1 \in I$ implies $\beta_2 \in I$.

A key definition that will frequently arise in this section is the following

Definition 2.4.6. The *height* of a root $\beta \in \Phi$ relative to Δ is defined as

$$\mathrm{ht}(\beta) = \sum_{\alpha\in\Delta} k_\alpha,$$

where k_α are the coefficients of β in the basis Δ, as in Definition 2.4.3.
Let I be an ideal of Φ^+ and set $M := \max\{\mathrm{ht}(\beta) \mid \beta \in I\}$. Let $t_k := |\{\beta \in I \mid \mathrm{ht}(\beta) = k\}|$ for $1 \leq k \leq M$. The sequence $(t_1, \cdots, t_k, \cdots, t_M)$ is called the *height distribution* if I and the *dual partition* $\mathrm{DP}(I)$ of the height distribution of I is defined as the multiset of nonnegative integers

$$\mathrm{DP}(I) := \{0^{\ell - t_1}, 1^{t_1 - t_2}, \cdots, M^{t_M}\}.$$

In classical contexts, standard irreducible root systems are classified into types A_ℓ, B_ℓ, C_ℓ and D_ℓ, of rank ℓ, and five other "exceptional"types

E_6, E_7, E_8, F_4 and G_2. Following [Bou68, Chapter VI, §4] (see it for a deeper analysis), we will provide a description of the root systems of types B_ℓ and C_ℓ, as these will be central to our discussion and results later in this section.

First, we need a construction of root systems of these types via a choice of basis for $\mathbb{R}^\ell$.

Let $\mathcal{E} := \{\epsilon_1, \dots, \epsilon_\ell\}$ be an orthonormal basis for V. For $\ell \geq 1$,

$$\Phi(B_\ell) = \{\pm\epsilon_i \, (1 \leq i \leq \ell), \pm(\epsilon_i \pm \epsilon_j) \, (1 \leq i < j \leq \ell)\}$$

is an irreducible root system of type B_ℓ. We may choose a positive system

$$\Phi^+(B_\ell) = \{\epsilon_i \, (1 \leq i \leq \ell), \epsilon_i \pm \epsilon_j \, (1 \leq i < j \leq \ell)\}.$$

Define $\alpha_i := \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq \ell - 1$, and $\alpha_\ell := \epsilon_\ell$. Then $\Delta(B_\ell) = \{\alpha_1, \dots, \alpha_\ell\}$ is the base associated to $\Phi^+(B_\ell)$. We may express

$$\Phi^+(B_\ell) = \Big\{\epsilon_i = \sum_{i \leq k \leq \ell} \alpha_k \, (1 \leq i \leq \ell), \epsilon_i - \epsilon_j = \sum_{i \leq k < j} \alpha_k \, (1 \leq i < j \leq \ell),$$
$$\epsilon_i + \epsilon_j = \sum_{i \leq k < j} \alpha_k + 2 \sum_{j \leq k \leq \ell} \alpha_k \, (1 \leq i < j \leq \ell)\Big\}.$$

For $\Psi \subseteq \Phi^+(B_\ell)$, write $T_\Psi = [t_{ij}]$ for the coefficient matrix of Ψ with respect to the basis $\mathcal{E}$. The matrices T_Ψ and S_Ψ are related by $T_\Psi = P(B_\ell) \cdot S_\Psi$, where $P(B_\ell)$ is an unimodular matrix of size $\ell \times \ell$ given by

$$P(B_\ell) = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & -1 & 1 \end{bmatrix}.$$

Similarly, an irreducible root system of type C_ℓ for $\ell \geq 1$ is given by

$$\begin{aligned}
\Phi(C_\ell) &= \{\pm 2\epsilon_i \, (1 \leq i \leq \ell), \pm(\epsilon_i \pm \epsilon_j) \, (1 \leq i < j \leq \ell)\}, \\
\Phi^+(C_\ell) &= \{2\epsilon_i \, (1 \leq i \leq \ell), \epsilon_i \pm \epsilon_j \, (1 \leq i < j \leq \ell)\}, \\
\Delta(C_\ell) &= \{\alpha_i = \epsilon_i - \epsilon_{i+1} \, (1 \leq i \leq \ell - 1), \, \alpha_\ell = 2\epsilon_\ell\}, \\
\Phi^+(C_\ell) &= \{2\epsilon_i = 2 \sum_{i \leq k < \ell} \alpha_k + \alpha_\ell \, (1 \leq i \leq \ell), \epsilon_i - \epsilon_j = \sum_{i \leq k < j} \alpha_k \, (1 \leq i < j \leq \ell), \\
&\quad \epsilon_i + \epsilon_j = \sum_{i \leq k < j} \alpha_k + 2 \sum_{j \leq k < \ell} \alpha_k + \alpha_\ell \, (1 \leq i < j \leq \ell)\}.
\end{aligned}$$

$$P(C_\ell) = \begin{bmatrix} 1 & & & & & \\ -1 & 1 & & & & \\ & -1 & & & & \\ & & \ddots & & & \\ & & & & 1 & \\ & & & & -1 & 2 \end{bmatrix}.$$

Example 2.4.7. Let $\Phi = B_2$ with $\Phi^+ = \{\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2, \alpha_1 + \alpha_2 = \epsilon_1, \alpha_1 + 2\alpha_2 = \epsilon_1 + \epsilon_2\}$ where $\Delta = \{\alpha_1, \alpha_2\}$ and $\mathcal{E} = \{\epsilon_1, \epsilon_2\}$. The coefficient matrices of Φ^+ w.r.t. Δ and $\mathcal{E}$ are given by

$$S_{\Phi^+} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}, \quad T_{\Phi^+} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 \end{pmatrix}.$$

Let $\Phi = C_2$. The coefficient matrix of Φ^+ w.r.t. Δ is S_{Φ^+} above with rows switched (this is not the case when $\ell \geq 3$). The coefficient matrix of Φ^+ w.r.t. $\mathcal{E} = \{\epsilon_1, \epsilon_2\}$ is given by

$$T_{\Phi^+} = \begin{pmatrix} 1 & 0 & 1 & 2 \\ -1 & 2 & 1 & 0 \end{pmatrix}.$$

Definition 2.4.8. Let $\Phi = B_\ell$ or C_ℓ. For $\Psi \subseteq \Phi^+$, denote by $\mathcal{A}_{T_\Psi}$ and $\mathcal{H}_{T_\Psi}$ the central toric and hyperplane arrangements defined by the matrix T_Ψ, respectively. We call these arrangements the arrangements with respect to the *integer lattice*.

Remark 2.4.9. Since the matrix $P(B_\ell)$ is unimodular, for every $\Psi \subseteq \Phi^+(B_\ell)$ we have an isomorphism of posets of layers: $L(\mathcal{A}_\Psi) \simeq L(\mathcal{A}_{T_\Psi})$ (see e.g., [PP21, §5]). However, $\det P(C_\ell) = 2$. In general, $L(\mathcal{A}_\Psi) \not\simeq L(\mathcal{A}_{T_\Psi})$ for $\Psi \subseteq \Phi^+(C_\ell)$ (although $L(\mathcal{H}_\Psi) \simeq L(\mathcal{H}_{T_\Psi})$).

A positive system $\Phi^+(A_{\ell-1})$ of an irreducible root system Φ of type $A_{\ell-1}$ for $\ell \geq 2$ can be defined as the ideal of $\Phi^+(B_\ell)$ (or $\Phi^+(C_\ell)$) generated by $\epsilon_1 - \epsilon_\ell = \sum_{k=1}^{\ell-1} \alpha_k$. Thus $L(\mathcal{A}_\Psi) \simeq L(\mathcal{A}_{T_\Psi})$ for every $\Psi \subseteq \Phi^+(A_{\ell-1})$.

To describe the exponents of $\mathcal{A}_I$ when Φ is B_ℓ or C_ℓ, we need information from the *signed graph* associated to I.

Definition 2.4.10. Let $\Phi = B_\ell$ or C_ℓ. For $\Psi \subseteq \Phi^+$ and $1 \leq i \leq \ell$, define the subset $E_i = E_i(\Psi) \subseteq \Psi$ by

$$E_i := E_i^+ \sqcup E_i^-, \text{ where } E_i^\pm := \{\epsilon_i \pm \epsilon_j \in \Psi \mid i < j\} \subseteq \Psi.$$

For $\alpha \in E_i$, let H_α denote the hypertorus defined by α. For example, $\alpha = \epsilon_i + \epsilon_j$ defines the hypertorus $H_\alpha = \{t_i t_j = 1\}$. We then define the subarrangement $\mathcal{B}_i = \mathcal{B}_i(\Psi) \subseteq \mathcal{A}_\Psi$ by

$$\mathcal{B}_i := \mathcal{B}_i^+ \sqcup \mathcal{B}_i^+, \text{ where } \mathcal{B}_i^\pm := \{H_\alpha \mid \alpha \in E_i^\pm\} \subseteq \mathcal{A}_\Psi.$$

Finally, define $b_i^\pm := |\mathcal{B}_i^\pm|$ and $b_i := |\mathcal{B}_i| = b_i^+ + b_i^-$.

In the language of signed graphs (e.g., following [Zas81, §5]), the elements in $E_i^+(\Psi)$ and $E_i^-(\Psi)$ correspond to the *negative* and *positive edges* of the signed graph defined by Ψ, respectively.

It is not hard to see that for each ideal I of $\Phi^+(B_\ell)$ or $\Phi^+(C_\ell)$, the elements of the dual partition $\mathrm{DP}(I)$ can be expressed in terms of $b_i(I)$'s and vice versa. However, the numbers b_i's are a bit more convenient for our subsequent discussion.

In the context of hyperplane arrangements, inductiveness and exponents of those arrangements arising from ideals of root systems have been studied since the early 2000s and the main result is the following.

Theorem 2.4.11 ([ST06, ABC$^+$16, Hul16, Rö17, CRS19])**.** *If I is an ideal of an irreducible root system Φ, then $\mathcal{H}_I$ is inductive with exponents* $\mathrm{DP}(I)$*. Moreover, $\mathcal{H}_I$ is supersolvable if Φ is A_ℓ, B_ℓ, C_ℓ, or G_2.*

In contrast to the hyperplane arrangement case, the toric arrangement $\mathcal{A}_I$ is not factorable for most cases even when $I = \Phi^+$. It is known that the characteristic polynomial of the central toric arrangement defined by an arbitrary matrix S coincides with the last constituent of the *characteristic quasi-polynomial* $\chi_S^{\mathrm{quasi}}(q)$ defined by S, that is a periodic polynomial with integral period [LTY21, Corollary 5.6]. Furthermore, an explicit computation shows that the last constituent of $\chi_{S_{\Phi^+}}^{\mathrm{quasi}}(q)$ factors with all integer roots if and only if Φ is A_ℓ, B_ℓ or C_ℓ [KTT10, Sut98]. Thus, $\mathcal{A}_{\Phi^+}$ is factorable if and only if Φ is of one of these three types.

Additionally, a stronger assertion has been proven, that is if I is an ideal of an irreducible root system of type A, B or C, then $\mathcal{A}_I$ is factorable and its combinatorial exponents can be described by the *signed graph* associated to I, see [Tra19].

The theorem we are presenting strengthens this result. We give an explicit description of the exponents of $\mathcal{A}_I$ derived from an explicit induction table, which turns out to be equivalent to the ones in [Tra19]. We also give a characterization for supersolvability of $\mathcal{A}_{\Phi^+}$ when Φ is of type B (Theorem 2.4.23).

Theorem 2.4.12. *The toric arrangement defined by an arbitrary ideal of a root system of type A, B or C with respect to the root lattice is inductive.*

Before presenting the proof of this theorem, we analyze the different cases of types A, B, and C separately, discussing the relevant results for each, in order to integrate them and conclude.

The proof of this result for the type A case in is a simple consequence of Theorem 2.4.11, which we give below.

Corollary 2.4.13. *If I is an ideal of a root system of type A, then the toric arrangement $\mathcal{A}_I$ with respect to the root lattice is strictly supersolvable (equivalently, supersolvable) hence inductive with exponents* $\mathrm{DP}(I)$*.*

Proof. It is not hard to see that for any $\Psi \subseteq \Phi^+(A_\ell)$, each layer in $L(\mathcal{A}_\Psi(A_\ell))$ is connected. Thus $L(\mathcal{A}_\Psi(A_\ell)) \simeq L(\mathcal{H}_\Psi(A_\ell))$ which is a geometric lattice. By Remark 2.1.17, its supersolvability and strict supersolvability are equivalent. Moreover, $\mathcal{A}_I$ is indeed supersolvable with exponents $\mathrm{DP}(I)$ by Theorem 2.4.11. □

Hence we are left with the computation on types B and C.

2.4.1 Type C.

We first present the results on type C as the proofs are simpler than those on type B. We begin by proving a lemma which serves as a template for some arguments later.

Lemma 2.4.14. *Let $I \subseteq \Phi^+(C_\ell)$ be an ideal such that $E_1(I) \neq \emptyset$. Define*

$$F := \begin{cases} I \setminus (E_1(I) \cup \{2\epsilon_1\}) & \text{if } 2\epsilon_1 \in I, \\ I \setminus E_1(I) & \text{otherwise.} \end{cases}$$

Then F can be regarded as an ideal of $\Phi^+(C_{\ell-1})$ and $\mathcal{A}_{T_F}$ is a TM-ideal of $\mathcal{A}_{T_I}$.

Proof. The first assertion is clear via the transformation $x_i \mapsto x_{i-1}$ for $2 \leq i \leq \ell$. Denote $\mathcal{A} := \mathcal{A}_{T_I}$ and $\mathcal{F} := \mathcal{A}_{T_F}$. There do not exist $X In L(\mathcal{F})$ and $Y In L(\mathcal{A}) \setminus L(\mathcal{F})$ such that $X \subseteq Y$ since the defining equations of any $X In L(\mathcal{F})$ do not involve t_1. Therefore, $L(\mathcal{F})$ is a proper order ideal of $L(\mathcal{A})$. Note also that the power of variable t_1 in the defining equation of any $H In \mathcal{A} \setminus \mathcal{F}$ is equal to 1. This shows Condition 2.3.11(*).

It remains to show that for any two distinct $H_1, H_2 \in \mathcal{A} \setminus \mathcal{F}$ and every connected component C of the intersection $H_1 \cap H_2$, there exists $H_3 \in \mathcal{F}$

such that $C \subseteq H_3$. We consider three main cases, the remaining cases are similar to one of these.

(a) Assume $H_1 = \{t_1t_j = 1\}$ (i.e., $\epsilon_1 + \epsilon_j \in I$) and $H_2 = \{t_1t_k^{-1} = 1\}$ for $j > 1$, $k > 1$, $j \neq k$. Then by the definition of an ideal we must have $\epsilon_j + \epsilon_k InF$ (since $\epsilon_1 + \epsilon_j > \epsilon_j + \epsilon_k$). Hence $H_3 := \{t_jt_k = 1\} \in \mathcal{F}$. Moreover, $H_1 \cap H_2$ is connected and $H_1 \cap H_2 \subseteq H_3$.

(b) Assume $H_1 = \{t_1t_j = 1\}$ and $H_2 = \{t_1t_j^{-1} = 1\}$ for $j > 1$. Then $H_3 := \{t_j = 1\} \in \mathcal{F}$ and $H_3' := \{t_j = -1\} \in \mathcal{F}$ (since $\epsilon_1 + \epsilon_j > 2\epsilon_j$). Moreover, $H_1 \cap H_2$ has two connected components; one is contained in H_3, the other is contained in H_3'.

(c) Assume $H_1 = \{t_1 = 1\}$ (i.e., $2\epsilon_1 \in I$) and $H_2 = \{t_1t_j = 1\}$ for $j > 1$. Then $H_3 := \{t_j = 1\} \in \mathcal{F}$ (since $2\epsilon_1 > 2\epsilon_j$). Moreover, $H_1 \cap H_2$ is connected and $H_1 \cap H_2 \subseteq H_3$.

This concludes that $\mathcal{F}$ is a TM-ideal of $\mathcal{A}$ as desired. □

Theorem 2.4.15. *Let $I \subseteq \Phi^+(C_\ell)$ be an ideal. Define*

$$n := \begin{cases} \min\{1 \leq i \leq \ell \mid E_i(I) \neq \emptyset\} & \text{if } I \neq \emptyset, \\ \ell + 1 & \text{otherwise,} \end{cases}$$

$$s := \begin{cases} \min\{1 \leq i \leq \ell \mid 2\epsilon_i \in I\} & \text{if there exists } 2\epsilon_i \in I \text{ for some } 1 \leq i \leq \ell, \\ \ell + 1 & \text{otherwise.} \end{cases}$$

Then the toric arrangement $\mathcal{A}_{T_I}$ with respect to the integer lattice is strictly supersolvable with exponents

$$\exp(\mathcal{A}_{T_I}) = \{0^{n-1}\} \cup \{b_i\}_{i=n}^{s-1} \cup \{2(\ell - i + 1)\}_{i=s}^{\ell}.$$

(See Definition 2.4.10 for the definition of b_i's.)

Proof. Denote $\mathcal{A} := \mathcal{A}_{T_I}$. Note that $n \leq s$ and $b_i = 0$ for $1 \leq i < n$. If $2\epsilon_i \notin I$ for all $1 \leq i \leq \ell$, then I can be regarded as an ideal of $\Phi^+(A_{\ell-1})$ by Remark 2.4.9, thus, $L(\mathcal{A}_I) \simeq L(\mathcal{A}_{T_I})$. By Corollary 2.4.13, $\mathcal{A} \in$ **SSS** with exponents $\mathrm{DP}(I) = \{b_1, \ldots, b_\ell\}$. Now we may assume $1 \leq n \leq s \leq \ell$. Then $2\epsilon_i \in I$ and $E_i(I) \neq \emptyset$ for all $s \leq i \leq \ell$. Define

$$\mathcal{A}_i := \begin{cases} \bigcup_{j=i}^{\ell} \left(\mathcal{B}_j \cup \{t_j^2 = 1\}\right) & \text{if } s \leq i \leq \ell, \\ \bigcup_{j=i}^{s-1} \mathcal{B}_j \cup \mathcal{A}_s & \text{if } n \leq i < s. \end{cases}$$

In particular, $\mathcal{A}_s$ can be identified with $\mathcal{A}_{T_{\Phi^+}}(C_{\ell-s+1})$ (via $x_i \mapsto x_{i-s+1}$ for $s \leq i \leq \ell$). Then $b_i = 2(\ell - i)$ for $s \leq i \leq \ell$.
By Theorem 2.3.12, it suffices to show that the chain

$$\emptyset \subsetneq \mathcal{A}_\ell \subsetneq \cdots \subsetneq \mathcal{A}_n = \mathcal{A}$$

is a TM-chain of $\mathcal{A}$. A similar argument as in the proof of Lemma 2.4.14 shows that $\mathcal{A}_{i+1}$ is a TM-ideal of $\mathcal{A}_i$ for each $n \leq i \leq \ell - 1$ and hence $\mathcal{A} \in$ **SSS** with the desired exponents. □

Recall the definitions of the parameters $n \leq s$ in Theorem 2.4.15.

Theorem 2.4.16. *Let $I \subseteq \Phi^+(C_\ell)$ be an ideal. Then the toric arrangement $\mathcal{A}_I$ with respect to the root lattice is inductive with exponents*

$$\exp(\mathcal{A}_I) = \{0^{n-1}\} \cup \{b_i\}_{i=n}^{s-1} \cup \{2(\ell - i)\}_{i=s}^{\ell-1} \cup \{\ell - s + 1\}.$$

Proof. Denote $\mathcal{A} := \mathcal{A}_I$.

Case 1. First, we prove the assertion for the case when $s = 1$, where $I = \Phi^+$. We show that $\mathcal{A} \in$ **IA** with the desired exponents by induction on ℓ. The case $\ell = 1$ is clear.
Suppose $\ell \geq 2$. Let $\delta := 2\epsilon_1 = 2\sum_{1 \leq k < \ell} \alpha_k + \alpha_\ell$ denote the highest root of Φ^+. Define

$$F := \Phi^+ \setminus (E_1(\Phi^+) \cup \{\delta\}), \text{ and } \mathcal{F} := \mathcal{A}_F.$$

Then $F = \Phi^+(C_{\ell-1})$ (via $x_i \mapsto x_{i-1}$). By the induction hypothesis, $\mathcal{F} \in$ **IA** with exponents

$$\exp(\mathcal{F}) = \{2(\ell - i)\}_{i=2}^{\ell-1} \cup \{\ell - 1\}.$$

Denote $\mathcal{A}' := \mathcal{A} \setminus \{H_\delta\}$. Note that $\mathcal{A}' \setminus \mathcal{F}$ consists of the hypertori defined by the roots in $E_1(\Phi^+)$. These roots are given by

$$\begin{aligned} \epsilon_1 - \epsilon_j &= \sum_{1 \leq k < j} \alpha_k \quad (1 < j \leq \ell), \\ \epsilon_1 + \epsilon_j &= \sum_{1 \leq k < j} \alpha_k + 2 \sum_{j \leq k < \ell} \alpha_k + \alpha_\ell \quad (1 < j \leq \ell). \end{aligned}$$

Again, using a similar argument as in the proof of Lemma 2.4.14, we may show that $\mathcal{F}$ is an M-ideal of $\mathcal{A}'$. Moreover, it is indeed a TM-ideal since Condition 2.3.11(*) is satisfied because the coefficient at the simple α_1 of all roots in $E_1(\Phi^+)$ is 1, while that of the roots in F is 0. Apply Lemma 2.3.5 for $L(\mathcal{F})$ and $L(\mathcal{A}')$ we have that $\mathcal{A}' \in$ **IA** with exponents

$$\exp(\mathcal{A}') = \exp(\mathcal{F}) \cup \{2(\ell - 1)\} = \{2(\ell - i)\}_{i=1}^{\ell-1} \cup \{\ell - 1\}.$$

Furthermore,it can be shown that the restriction $\mathcal{A}^{H_\delta}$ corresponds to $\mathcal{A}_{T_{\Phi^+}}(C_{\ell-1})$ by setting $t_\ell = t_1^{-2} \cdots t_{\ell-1}^{-2}$ in the equations involving t_ℓ. For instance, the equation $t_2^2 \cdots t_{\ell-1}^2 t_\ell = 1$ becomes $t_1^2 = 1$. Thus by Theorem 2.4.15, $\mathcal{A}^{H_\delta} \in \mathbf{IA}$ with exponents

$$\exp(\mathcal{A}^{H_\delta}) = \{2(\ell - i)\}_{i=1}^{\ell-1}.$$

Apply Theorem 2.3.10, we know that $\mathcal{A} \in \mathbf{IA}$ with the desired exponents

$$\exp(\mathcal{A}) = \{2(\ell - i)\}_{i=1}^{\ell-1} \cup \{\ell\}.$$

Case 2. Now we prove the assertion when $s > 1$. The set

$$\mathcal{J} := I \setminus \bigcup_{i=n}^{s-1} E_i(I)$$

can be identified with $\Phi^+(C_{\ell-s+1})$. By Case 1 above, $\mathcal{P} := \mathcal{A}_{\mathcal{J}} \in \mathbf{IA}$ with exponents

$$\exp(\mathcal{P}) = \{2(\ell - i)\}_{i=s}^{\ell-1} \cup \{\ell - s + 1\}.$$

Using a similar argument as in Case 1, we may show that the sets $E_i(I)$ for $n \le i \le s - 1$ give rise to a chain of TM-ideals for $\mathcal{A}$ starting from $\mathcal{P}$. Applying Lemma 2.3.5 repeatedly, we may conclude that $\mathcal{A} \in \mathbf{IA}$ with the desired exponents. $\square$

Example 2.4.17. Table 2.1 shows an ideal $I \subsetneq \Phi^+(C_5)$, in enclosed region, with $n = 1$, $s = 3$. By Theorem 2.4.15, $\mathcal{A}_{T_I} \in \mathbf{SSS}$ with exponents $\{4, 6, 6, 4, 2\}$ and by Theorem 2.4.16, $\mathcal{A}_I \in \mathbf{IA}$ with exponents $\{4, 6, 4, 2, 3\}$.

2.4.2 Type B.

The restriction of an ideal toric arrangement of type B is in general not an ideal toric arrangement. To solve this, we need an extension of the ideals so that the corresponding arrangements contain sufficient deletions and restrictions in order to apply Theorem 2.3.10 to guarantee the inductiveness.

Lemma 2.4.18. *Let $I \subseteq \Phi^+(B_\ell)$ be an ideal such that $E_1^+(I) \neq \emptyset$ and let $m = m(I)$ be the integer so that $\epsilon_1 + \epsilon_m$ is the highest root in $E_1^+(I)$, in particular, $2 \le m \le \ell$ and $2\ell - m = b_1$. Let $1 \le p \le \ell + 1$, define the extension $I(p)$ of I with parameter p as follows:*

$$I(p) := (I \setminus \{\epsilon_i \mid p \le i \le \ell\}) \cup \{2\epsilon_i \mid p \le i \le \ell\}.$$

If $m < p$, then $\mathcal{A}_{T_{I(p)}}$ is inductive with exponents

$$\exp(\mathcal{A}_{T_{I(p)}}) = \{2\ell - p + 1\} \cup \{b_i\}_{i=1}^{\ell-1}.$$

Height					
9	$2\epsilon_1$				
8	$\epsilon_1+\epsilon_2$				
7	$\epsilon_1+\epsilon_3$	$2\epsilon_2$			
6	$\epsilon_1+\epsilon_4$	$\epsilon_2+\epsilon_3$			
5	$\epsilon_1+\epsilon_5$	$\epsilon_2+\epsilon_4$	$2\epsilon_3$		
4	$\epsilon_1-\epsilon_5$	$\epsilon_2+\epsilon_5$	$\epsilon_3+\epsilon_4$		
3	$\epsilon_1-\epsilon_4$	$\epsilon_2-\epsilon_5$	$\epsilon_3+\epsilon_5$	$2\epsilon_4$	
2	$\epsilon_1-\epsilon_3$	$\epsilon_2-\epsilon_4$	$\epsilon_3-\epsilon_5$	$\epsilon_4+\epsilon_5$	
1	$\epsilon_1-\epsilon_2$	$\epsilon_2-\epsilon_3$	$\epsilon_3-\epsilon_4$	$\epsilon_4-\epsilon_5$	$2\epsilon_5$
	E_1	E_2	E_3	E_4	$E_5=\emptyset$

Table 2.1: An ideal I in $\Phi^+(C_5)$.

Proof. Denote $\mathcal{A} := \mathcal{A}_{T_{I(p)}}$. We may write

$$\mathcal{A} = \mathcal{A}_{T_I} \cup \{t_i = -1 \mid p \leq i \leq \ell\}.$$

We show that $\mathcal{A} \in \mathbf{IA}$ with the desired exponents by induction on ℓ. If $\ell \leq 2$, then $\mathcal{A}$ is always strictly supersolvable except when $p = 3$ and $I = I(3) = \Phi^+(B_2)$, in which case $\mathcal{A}$ is indeed inductive with exponents $\{2, 2\}$ by Figure 2.3. Now suppose $\ell \geq 3$. Since $\epsilon_1 + \epsilon_m \in I$, we must have $\epsilon_2 + \epsilon_m \in I$. Define

$$\mathcal{J} := I \setminus (E_1(I) \cup \{\epsilon_1\}).$$

Then $\mathcal{J}$ can be regarded as an ideal of $\Phi^+(B_{\ell-1})$, via $x_i \mapsto x_{i-1}$, with $m(\mathcal{J}) \leq m(I) - 1$. Also, $E_i^{\pm}(\mathcal{J}) = E_{i+1}^{\pm}(I)$ hence $b_i(\mathcal{J}) = b_{i+1}(I)$ for all $1 \leq i \leq \ell - 1$. Moreover, $I(p) \setminus (E_1(I) \cup \{\epsilon_1\})$ can be identified with the extension $\mathcal{J}(p-1)$ since $2 \leq m < p$. By the induction hypothesis, $\mathcal{P} := \mathcal{A}_{T_{\mathcal{J}(p-1)}} \in \mathbf{IA}$ with exponents

$$\exp(\mathcal{P}) = \{2\ell - p\} \cup \{b_i(I)\}_{i=2}^{\ell-1}. \tag{2.2}$$

Define

$$F := I(p) \setminus \{\epsilon_1 + \epsilon_i \mid m \leq i \leq p-1\}, \text{ and } \mathcal{F} := \mathcal{A}_{T_F}.$$

Since $2\epsilon_i \in F$ for all $p \leq i \leq \ell$, using a similar argument as in the proof of Lemma 2.4.14 we may show that $\mathcal{P}$ is a TM-ideal of $\mathcal{F}$. Applying Lemma

2.3.5 for $L(\mathcal{F})$ and $L(\mathcal{P})$ we have that $\mathcal{F} \in \mathbf{IA}$ with exponents

$$\exp(\mathcal{F}) = \exp(\mathcal{P}) \cup \{2\ell - p + 1\} = \{2\ell - p + 1, 2\ell - p\} \cup \{b_i(I)\}_{i=2}^{\ell-1}.$$

Now we show that adding the $p-m$ hypertori $t_1 t_{p-1} = 1, t_1 t_p = 1, \dots, t_1 t_m = 1$ to $\mathcal{F}$ in any order and applying Theorem 2.3.10 to each addition step, we are able to conclude that $\mathcal{A} \in \mathbf{IA}$ with the desired exponents. Since $2\ell - m = b_1$, it is sufficient to show that the restriction at each addition step is inductive with exponents $\{2\ell-p+1\} \cup \{b_i(I)\}_{i=2}^{\ell-1}$. Indeed, the restriction at each step has the form $\mathcal{P} \cup \{H_k\}$, where H_k denotes the hypertorus $t_k = -1$ for some $m \leq k \leq p-1$. Fix $m \leq k \leq p-1$. Note that, since $\epsilon_1 + \epsilon_k \in I$, $\epsilon_i + \epsilon_k \in I \subseteq \mathcal{J}(p-1)$ for all $1 < i \neq k$. Thus the restriction $(\mathcal{P} \cup \{H_k\})^{H_k}$ can be identified with the arrangement $\mathcal{A}_{T_{R(1)}}$, where $R(1)$ is the extension with parameter $p = 1$ of an ideal R of $\Phi^+(B_{\ell-2})$ (via $x_i \mapsto x_{i-1}$ $(2 \leq i < k)$ and $x_i \mapsto x_{i-2}$ $(k < i \leq \ell)$) with $b_i^{\pm}(R) = b_{i+1}^{\pm}(I) - 1$ for $1 \leq i \leq \ell - 2$. Note that the equations $b_i^{\pm}(R) = b_{i+1}^{\pm}(I) - 1$ for $k - 1 \leq i \leq \ell - 2$ follow from the fact that $\bigcup_{i=k-1}^{\ell-2} (E_i(R) \cup \{2\epsilon_i\})$ is a root system of type C. Now, using a similar argument as in the proof of Theorem 2.4.15, we know that $(\mathcal{P} \cup \{H_k\})^{H_k}$ is strictly supersolvable hence inductive with exponents

$$\exp((\mathcal{P} \cup \{H_k\})^{H_k}) = \{b_i(R) + 2\}_{i=1}^{\ell-2} = \{b_i(I)\}_{i=2}^{\ell-1}.$$

By Theorem 2.3.10 and Equation (2.2) above, we know that $\mathcal{P} \cup \{H_k\} \in \mathbf{IA}$ for every $m \leq k \leq p-1$ with the desired exponents

$$\exp(\mathcal{P} \cup \{H_k\}) = \{2\ell - p + 1\} \cup \{b_i(I)\}_{i=2}^{\ell-1}.$$

This completes the proof. □

Theorem 2.4.19. *Let $I \subseteq \Phi^+(B_\ell)$ be an ideal such that $\epsilon_k \in I$ for some $1 \leq k \leq \ell$. Define*

$$\begin{aligned} n &:= \min\{1 \leq i \leq \ell \mid E_i(I) \neq \emptyset\}, \\ a &:= \min\{n \leq i \leq \ell \mid \epsilon_i \in I \text{ and } E_i^+(I) = \emptyset\}, \\ s &:= \min\{a \leq i \leq \ell \mid E_i^+(I) \neq \emptyset\}. \end{aligned}$$

For each $s \leq i \leq \ell$, let $m(i)$ be the integer so that $\epsilon_i + \epsilon_{m(i)}$ is the highest root in $E_i(I)$, in particular, $m(j) \leq m(i)$ if $i < j$. Let $s \leq p \leq \ell + 1$, recall the definition of the extension $I(p)$ of I with parameter p in Lemma 2.4.18. Define

$$t := \min\{s \leq i \leq \ell \mid m(i) < p\}.$$

Then $\mathcal{A}_{T_{I(p)}}$ *is inductive with exponents*

$\exp(\mathcal{A}_{T_{I(p)}}) = \{0^{n-1}, 2\ell{-}p{-}t{+}2\} \cup \{b_i{+}1 \mid iIn[a, t{-}1]\} \cup \{b_i \mid iIn[n, \ell{-}1] \setminus [a, t{-}1]\ \}.$

Proof. Denote $\mathcal{A} := \mathcal{A}_{T_{I(p)}}$. The set

$$(I(p) \setminus \bigcup_{i=n}^{a-1} E_i(I)) \setminus \bigcup_{i=a}^{t-1} (E_i(I)) \cup \{\epsilon_i\})$$

can be identified with the extension $\mathcal{J}(p-t+1)$, where $\mathcal{J}$ is an ideal of $\Phi^+(B_{\ell-t+1})$ with $m(i) < p-t+1$ for all $1 \leq i \leq \ell-t+1$. By Lemma 2.4.18, $\mathcal{P} := \mathcal{A}_{T_{\mathcal{J}(p-t+1)}} \in \mathbf{IA}$ with exponents

$$\exp(\mathcal{P}) = \{2\ell - p - t + 2\} \cup \{b_i(I)\}_{i=t}^{\ell-1}.$$

Similarly as in the proof of Lemma 2.4.14, we may show that the sets $E_i(I)$ for $n \leq i \leq a-1$ and $E_i(I) \cup \{\epsilon_i\}$ for $a \leq i \leq t-1$ give rise a chain of TM-ideals for $\mathcal{A}$ starting from $\mathcal{P}$. Note that by definition $m(i) \geq p$ for all $s \leq i \leq t-1$. By applying Lemma 2.3.5 repeatedly, we may conclude that $\mathcal{A} \in \mathbf{IA}$ with the desired exponents. Indeed, the sets above contribute to $\exp(\mathcal{A})$ the exponents b_i for $n \leq i \leq a-1$ and $b_i + 1$ for $a \leq i \leq t-1$. □

Example 2.4.20. Table 2.2 shows the extension $I(4)$ of an ideal $I \subsetneq \Phi^+(B_5)$ with parameter $p = 4$. In this case, $n = a = s = 1$ and $t = 2$ with $m(t) = 3 < p$, and by Theorem 2.4.19, $\mathcal{A}_{T_{I(4)}} \in \mathbf{IA}$ with exponents $\{6, 7, 6, 4, 2\}$.

Recall from Remark 2.4.9 that $\mathcal{A}_\Psi$ and $\mathcal{A}_{T_\Psi}$ have isomorphic poset of layers for every $\Psi \subseteq \Phi^+(B_\ell)$.

Corollary 2.4.21. *If* $I \subseteq \Phi^+(B_\ell)$*, then the toric arrangement* $\mathcal{A}_I$ *with respect to the root lattice is inductive.*

Proof. If $\epsilon_i \notin I$ for all $1 \leq i \leq \ell$, then I can be regarded as an ideal of $\Phi^+(A_{\ell-1})$ and hence $\mathcal{A}_I$ is indeed strictly supersolvable hence inductive by Corollary 2.4.13. Otherwise, we know that $\mathcal{A}_{T_I}$ is inductive which follows from Theorem 2.4.19 by letting $p = \ell + 1$. □

Now, by adding together the results outlined above, the proof of Theorem 2.4.12 is completed.

Example 2.4.22. From Theorems 2.4.16, 2.4.19 and Corollary 2.4.21, we deduce that both $\mathcal{A}_{\Phi^+}(B_\ell)$ and $\mathcal{A}_{\Phi^+}(C_\ell)$ are inductive with the same multiset of exponents $\{\ell, 2, 4, \dots, 2(\ell-1)\}$. This fact is similar to the hyperplane arrangement case.

Height					
9	$\epsilon_1+\epsilon_2$				
8	$\epsilon_1+\epsilon_3$				
7	$\epsilon_1+\epsilon_4$	$\epsilon_2+\epsilon_3$			
6	$\epsilon_1+\epsilon_5$	$\epsilon_2+\epsilon_4$			
5	ϵ_1	$\epsilon_2+\epsilon_5$	$\epsilon_3+\epsilon_4$		
4	$\epsilon_1-\epsilon_5$	ϵ_2	$\epsilon_3+\epsilon_5$		
3	$\epsilon_1-\epsilon_4$	$\epsilon_2-\epsilon_5$	ϵ_3	$\epsilon_4+\epsilon_5$	
2	$\epsilon_1-\epsilon_3$	$\epsilon_2-\epsilon_4$	$\epsilon_3-\epsilon_5$	$2\epsilon_4$	
1	$\epsilon_1-\epsilon_2$	$\epsilon_2-\epsilon_3$	$\epsilon_3-\epsilon_4$	$\epsilon_4-\epsilon_5$	$2\epsilon_5$

Table 2.2: Extension of an ideal I in $\Phi^+(B_5)$ with parameter $p = 4$.

In contrast to the inductiveness, the toric arrangement of a root system of type B_ℓ is not supersolvable for most cases.

Theorem 2.4.23. *Suppose $\Phi = B_\ell$ for $\ell \geq 1$. Then $\mathcal{A}_{T_{\Phi^+}}$ is supersolvable if and only if $\ell \leq 2$.*

Proof. Let $\mathcal{A} := \mathcal{A}_{T_{\Phi^+}}$, denote $L = L(\mathcal{A})$ and $x = (-1, -1, \ldots, -1) \in L$. By Lemma 1.1.9, $L_{\leq x}$ is isomorphic to the intersection lattice $L(\mathcal{H}_{T_{\Phi^+}}(D_\ell))$ of the hyperplane arrangement of a root system of type D_ℓ.
If $\ell \geq 4$, then $L_{\leq x}$ is not supersolvable by Remark 2.1.28. Therefore, L is not locally supersolvable hence not supersolvable.
When $\ell \leq 3$, however, $L_{\leq x}$ is always supersolvable. We need a direct examination for the supersolvability of L. The assertion is clear when $\ell = 1$. The case $\ell = 2$ is shown in Figure 2.3. Now we show that L is not supersolvable, though locally supersolvable, when $\ell = 3$ by showing that L does not have an M-ideal of rank 2.
Suppose to the contrary that such an M-ideal exists and call it Q. Denote $H^+_{ij} := \{t_i t_j = 1\}$ and $H^-_{ij} := \{t_i t_j^{-1} = 1\}$. First, notice that a rank-2 element of the form $t_i = t_j = -1$ covers exactly two atoms, namely H^+_{ij} and H^-_{ij}. If these atoms are not in Q, then Lemma 2.1.13 fails, and hence at least one of them belongs to Q for every pair of indices $i \neq j \in \{1, 2, 3\}$. Moreover, we may deduce that exactly one of H^+_{ij} and H^-_{ij} belongs to Q. Otherwise, the join $H^+_{ij} \vee H^-_{ij} \vee H$ where H is either H^+_{jk} or H^-_{jk} for $k \notin \{i, j\}$ contains an element of rank 3, which contradicts the join-closedness of Q.

We consider two main cases, the remaining cases are similar to one of these:

(a) If H^+_{12}, H^+_{13}, and H^+_{23} all belong to Q, then their join consists of rank-3 elements, which leads to a contradiction;

(b) If H^+_{12}, H^+_{13}, and H^-_{23} all belong to Q, then Q has no atom of the form $t_i = 1$. Otherwise, joining it with $H^+_{12} \vee H^+_{13} \vee H^-_{23}$ would yield a rank-3 element within Q. Hence, the only rank-2 element in Q would be $H^+_{12} \vee H^+_{13} \vee H^-_{23} = \{t_2 = t_3 = t_1^{-1}\}$. However, this is not an element of $L_{\leq(1,-1,-1)}$, which contradicts Condition 2.1.11(2).

This completes the proof. □

Chapter 3

Generalized elliptic arrangements

This chapter studies elliptic arrangements in the context of elliptic curves with complex multiplication $\mathcal{E} = \mathbb{C}/\Lambda$. The first section defines these arrangements and recall the structure of the endomorphism ring of $\mathcal{E}$. In the following sections, we compute the number of connected components in intersections of subvarieties (Theorem 3.2.11) and show that this information, along with the rank, associate to an elliptic arrangement the structure of an arithmetic matroid (Theorem 3.3.3).

3.1 Definition of elliptic arrangements

The protagonists of this chapter are, as already mentioned, elliptic curves with complex multiplication $\mathcal{E} = \mathbb{C}/\Lambda$. Let us consider a morphism

$$\begin{aligned} \Phi : \quad \mathcal{E}^n &\longrightarrow \quad \mathcal{E} \\ p \quad &\mapsto \quad \langle \alpha, p \rangle = \alpha_1 p_1 + \cdots + \alpha_n p_n, \end{aligned}$$

where $\alpha = (\alpha_1, \cdots, \alpha_n) \in \operatorname{End}(\mathcal{E})^n$. The kernel of Φ defines a subvariety in $\mathcal{E}^n$, denoted by $H_\Phi := \ker(\Phi)$. Let us introduce a notation that will be consistently used throughout this chapter: we denote the endomorphism ring of the elliptic curve $\mathcal{E}$ as $R := \operatorname{End}(\mathcal{E})$.

Definition 3.1.1. Let $\mathcal{E}$ be an elliptic curve with complex multiplication. An *elliptic arrangement* is a finite collection of subvarieties

$$\mathcal{A} := \{H_i := H_{\Phi_i} = \ker(\Phi_i)\}_{i \in E},$$

where, for every $i \in [k]$, $\Phi \in \mathrm{Hom}(\mathcal{E}^n, \mathcal{E})$ and $\Phi_i(p) = \langle \alpha_i, p \rangle$, with $\alpha = (\alpha_{i1}, \cdots, \alpha_{in}) \in \mathrm{End}(\mathcal{E})^n$, and E is a finite set with a total order, $|E| = k$. Let $A = (\alpha_{ij})_{i\in[k], j\in[n]} \in \mathrm{Mat}_{k\times n}(R)$ be the matrix associated to the arrangement with coefficient in the endomorphism ring of the elliptic curve.

What distinguishes these arrangements from traditional ones lies in the structure of the endomorphism ring of the elliptic curve $\mathcal{E}$. When only integer multiplication is considered, the endomorphism ring is $\mathbb{Z}$, classifying the elliptic arrangement in the class of abelian arrangements, as defined in Chapter 1. However, new phenomena arise when the endomorphism ring strictly contains $\mathbb{Z}$. This section will hence focus on discussing and studying the ring $R := \mathrm{End}(\mathcal{E})$, with [ST92] as our primary reference.

Let $\mathcal{E}$ be a genus-1 smooth Riemann surface with $\mathrm{End}(\mathcal{E}) \ncong \mathbb{Z}$. All such $\mathcal{E}$ admit a representation of the form $\mathbb{C}/\Lambda$, with $\Lambda = \mathrm{span}_{\mathbb{Z}}\{1, \tau\}$. We recall some properties of $\mathrm{End}(\mathcal{E})$ and Λ.

Let m be a square-free positive integer, and consider $K_m = \mathbb{Q}(\sqrt{-m})$ an imaginary quadratic number field. Let

$$\omega = \begin{cases} \frac{1+\sqrt{-m}}{2} & \text{if } m \equiv 3 \mod 4, \\ \sqrt{-m} & \text{otherwise}, \end{cases} \tag{3.1}$$

and recall that $\mathcal{O}_m = \mathbb{Z}[\omega]$ is the ring of integers of K_m, i.e. the subring of algebraic numbers $\alpha \in K_m$ whose minimal polynomial f_α over $\mathbb{Z}$ is monic. In the first case of Equation (3.1) we have $f_\omega(\omega) = \omega^2 - \omega + m' = 0$ where $m = 4m' - 1$, while in the second case $f_\omega(\omega) = \omega^2 + m = 0$. There exist integers a, b, c with $\gcd(a, b, c) = 1$ such that the number $\tau = \frac{a+b\omega}{c} \in K_m$ generates a lattice $\Lambda = \mathrm{span}_{\mathbb{Z}}\{1, \tau\}$ with $\mathcal{E} \cong \mathbb{C}/\Lambda$ and the group structure of $\mathcal{E}$ coincides with the additive group structure of $\mathbb{C}/\Lambda$.
The following two results give an explicit presentation of $\mathrm{End}(\mathcal{E})$. We denote by A_α the linear map given by multiplication by α, i.e. $z \mapsto \alpha z$.

Lemma 3.1.2. *With the notation above, the ring* $\mathrm{End}(\mathcal{E})$ *is isomorphic to the ring*

$$R = \{\alpha \in \Lambda \ : \ \alpha\Lambda \subset \Lambda\}$$

via the isomorphism $R \to \mathrm{End}(\mathcal{E})$ *defined by* $\alpha \mapsto A_\alpha$.

Proof. An endomorphism of $\mathcal{E}$ is a holomorphic function $f : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$. This means that, in a neighborhood of 0, the map f is given by a convergent power series. Furthermore, f must preserve the group structure, hence

$$f(z_1 + z_2) - f(z_1) - f(z_2) \in \Lambda,$$

for all z_1, z_2 in a neighborhood of $0 \in \mathcal{E}$. Let U be such a neighborhood and let us consider a map $g : U \times U \to \Lambda$ such that $g(z_1, z_2) = f(z_1+z_2) - f(z_1) - f(z_2)$. This is a continuous function and it is constant. After substituting we obtain $g(0,0) = -f(0)$, hence for all $z \in U$

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(h)}{h} = \lim_{h\to 0} \frac{f(z) - f(0)}{h} = f'(0).$$

Thus, f is linear and since $f(0) = 0$ it exists $\alpha \in \Lambda$ such that $f(z) = \alpha z$ and clearly $\alpha\Lambda \subset \Lambda$. □

Lemma 3.1.3. *The ring R is an order of $\mathcal{O}_m$ and a basis is given by $\{1, N\tau\}$, where*

$$N = c^2/\gcd(c, c^2 \det A_\tau).$$

Proof. The first step of this proof consists in showing that R is a subring of $\mathcal{O}_m$.

Note that $\alpha\Lambda \subset \Lambda$ if and only if $\alpha \cdot 1 \in \Lambda$ and $\alpha \cdot \tau \in \Lambda$. The first condition is equivalent to $\alpha = x + y\tau$ for some $x, y \in \mathbb{Z}$, thus the second condition becomes $\alpha\tau = x\tau + y\tau^2 \in \Lambda$ and hence $y\tau^2 \in \Lambda$, i.e. $y\tau^2 = h\tau + k$ for some $h, k \in \mathbb{Z}$. If we multiply this last condition by y we obtain the monic relation $(y\tau)^2 - h'(y\tau) - yk' = 0$, hence $y\tau \in \mathcal{O}_m$ as desired.

Now, in order to compute the generators of R and describe its structure, write

$$y\tau = \frac{ay}{c} + \frac{by}{c}\omega.$$

Since 1 and ω generate $\mathcal{O}_m$ and $y\tau \in \mathcal{O}_m$, we have that $c \mid ay$ and $c \mid by$, thus $c \mid \gcd(a, b)y$. By assumption $\gcd(\gcd(a, b), c) = 1$, so we conclude that $c \mid y$, i.e. $y = cy'$ for some $y' \in \mathbb{Z}$. The last step is to verify for which y' we have $y'c\tau^2 = y'(a + b\omega) \in \Lambda$. Recall that the minimal polynomial over $\mathbb{Q}$ of τ is $f_\tau^{\mathbb{Q}} = z^2 - \operatorname{tr} A_\tau z + \det A_\tau$, where

$$\operatorname{tr} A_\tau = \begin{cases} (2a+b)/c & \text{if } m \equiv 3 \mod 4, \\ 2a/c & \text{else,} \end{cases} \tag{3.2}$$

$$\det A_\tau = \begin{cases} (a^2+ab+b^2m')/c^2 & \text{if } m \equiv 3 \mod 4, \\ (a^2+b^2m)/c^2 & \text{else.} \end{cases} \tag{3.3}$$

We write $\operatorname{tr} A_\tau = \gamma/c$ and $\det A_\tau = \delta/c^2$, so $\tau^2 = \frac{\gamma}{c}\tau - \frac{\delta}{c^2}$. With this notation, $y'c\tau^2 \in \Lambda$ is equivalent to:

$$\gamma y' \in \mathbb{Z} \quad \text{and} \quad \frac{\delta y'}{c} \in \mathbb{Z}.$$

The former is trivially true, and the latter implies y' is divisible by c/g with $g = \gcd(c, \delta)$.
We now have that a basis for R is $\{1, N\tau\}$, where $N = c^2/\gcd(c, c^2 \det A_\tau)$ as desired, and hence R is an order of $\mathcal{O}_m$. □

3.2 Description of connected components

Given an elliptic arrangement $\mathcal{A}$, we define $A = (\alpha_{ij})_{i\in[k],j\in[n]} \in \mathrm{Mat}_{k\times n}(R)$ to be the matrix associated to $\mathcal{A}$, whose i-th rows represents the i-th subvariety of the arrangement. It gives rise to maps $A_R : R^n \to R^k$, $A_\Lambda \colon \Lambda^n \to \Lambda^k$, $A_\mathbb{C} \colon \mathbb{C}^n \to \mathbb{C}^k$ and $A_\mathcal{E} \colon \mathcal{E}^n \to \mathcal{E}^k$, and for convenience if we omit the subscript then we mean A_Λ.
We are interested in the number of connected components of all intersections, i.e. in the number of layers of $\mathcal{A}_S$ for every $S \subseteq [k]$. To describe $\mathcal{A}_S$ let $A[S]$ be the submatrix of A of those rows indexed by $i \in S$. Likewise write $A_R[S]$, $A_\Lambda[S]$ and $A_\mathbb{C}[S]$ for the maps as before. Thus, $\mathcal{A}_S = \bigcap_{i\in S} H_i = \ker A_\mathcal{E}[S]$.
In order to present the main result of this section, let us first consider the following diagram, where the second and third rows describe the elliptic arrangement and the first and fourth make an exact sequence, with ∂ the map obtained via the snake lemma:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \ker A[S] & \longrightarrow & \ker A_\mathbb{C}[S] & \longrightarrow & \mathcal{A}_S & \xrightarrow{\partial} & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Lambda^n & \longrightarrow & \mathbb{C}^n & \longrightarrow & \mathcal{E}^n & \longrightarrow & 0 \\
 & & \downarrow A[S] & & \downarrow A_\mathbb{C}[S] & & \downarrow A_\mathcal{E}[S] & & \\
0 & \longrightarrow & \Lambda^S & \xrightarrow{\iota} & \mathbb{C}^S & \longrightarrow & \mathcal{E}^S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & \longrightarrow & \operatorname{coker} A[S] & \xrightarrow{\bar{\iota}} & \operatorname{coker} A_\mathbb{C}[S] & \to & \operatorname{coker} A_\mathcal{E}[S] & \longrightarrow & 0.
\end{array}
$$

Diagram 3.2.1.

From this diagram, we obtain a short exact sequence with middle term $\mathcal{A}_S$ such that, as stated in the following lemma, the sequence splits and gives us a decomposition of $\mathcal{A}_S$. This decomposition allows us to derive an initial result regarding number of layers of $\mathcal{A}_S$.

Lemma 3.2.2. *Let $\mathcal{A}$ be an arrangement in $\mathcal{E}^n$ where $\mathcal{E}$ is an elliptic curve with complex multiplication. For all $S \subset [k]$ we have the following SES:*

$$0 \to {}^{\ker A_{\mathbb{C}}[S]}\!\big/\!{}_{\ker A_{\Lambda}[S]} \to \mathcal{A}_S \to \operatorname{tor}\operatorname{coker} A_{\Lambda}[S] \to 0.$$

SES 3.2.3.

Moreover, this sequence splits as $\mathbb{Z}$-modules.

Proof. First, we show $\operatorname{Im}\partial = \ker\bar{\iota}$. Indeed, since $\operatorname{coker} A_{\mathbb{C}}[S] \cong \mathbb{C}^{s-r}$ with $s = \#S$, it has no torsion, thus $\operatorname{tor}\operatorname{coker} A[S] \subset \operatorname{Im}\partial$. The other inclusion follows from the fact that for $p \in \mathcal{E}^n$ we have $\partial(p) = \iota^{-1}(A_{\mathbb{C}}[S](x)) + A[S](\Lambda^n)$. Here x is any lift in $\mathbb{C}^n$ of p, and since $p \in \ker A_{\mathcal{E}}[S]$ we have that $A_{\mathbb{C}}[S](x)$ is in Λ^S. This implies that the coordinates of x are rational, so they can be cleared out by a positive integer. Therefore $\operatorname{Im}\partial = \operatorname{tor}\operatorname{coker} A[S]$.

From the first and fourth row of Diagram 3.2.1 and the snake lemma we readily get the SES 3.2.3. Moreover, ${}^{\ker A_{\mathbb{C}}[S]}\!\big/\!{}_{\ker A_{\Lambda}[S]}$ is a divisible abelian group, thus as a $\mathbb{Z}$-module it is injective and the sequence splits. □

Remark 3.2.4. Note that, as a $\mathbb{Z}$-module or as a R-module $\operatorname{tor}\operatorname{coker} A[S]$ is the same group.

Lemma 3.2.5. *Let $\mathcal{A}$ be an arrangement in $\mathcal{E}^n$ where $\mathcal{E}$ is an elliptic curve with complex multiplication. For all $S \subset [k]$ the number of layers in $\mathcal{A}_S$ is:*

$$m(S) = \#\textit{connected components}(\mathcal{A}_S) = \#\operatorname{tor}\operatorname{coker} A_{\Lambda}[S].$$

Proof. From Lemma 3.2.2 we get that

$$\mathcal{A}_S \cong \operatorname{tor}\operatorname{coker} A[S] \oplus {}^{\ker A_{\mathbb{C}}[S]}\!\big/\!{}_{\ker A[S]}.$$

Moreover, $\ker A_{\mathbb{C}}[S] \cong \mathbb{C}^{n-r}$ with $r = \operatorname{rk} A[S]$ and $\ker A[S] \subset \ker A_{\mathbb{C}}[S]$ is a lattice, so ${}^{\ker A_{\mathbb{C}}[S]}\!\big/\!{}_{\ker A[S]}$ is a finite product of elliptic curves, thus connected, and the result follows.

□

Remark 3.2.6. The behaviour of the SES (3.2.3) is more intrincate when regarded as R-modules. In that case, it is not possible to apply the same argument as in the proof of Lemma 3.2.2, since ${}^{\ker A_{\mathbb{C}}[S]}\!\big/\!{}_{\ker A_{\Lambda}[S]}$ is not an injective R-module and hence the sequence does not necessarily split.

While the earlier remark stands, it is still possible to achieve a stronger result. Let us start with a motivating example.

Example 3.2.7. When $n = k = 1$ we have a map of the form A_α for some $\alpha = x + yN\tau \in R$. We claim that

$$\Lambda/\alpha\Lambda \cong R/\alpha R$$

as $\mathbb{Z}$-modules. For this we write A_α as a matrix A in the basis $\{1, \tau\}$ of Λ, and as a matrix $\tilde{A}$ in the basis $\{1, N\tau\}$, and we compare their Smith normal form. As in the previous section, let γ, δ be the integers such that $\operatorname{tr} A_\tau = \gamma/c$ and $\det A_\tau = \delta/c^2$. Moreover, set $g = \gcd(c, \delta)$ and $c = gc'$ and $\delta = g\delta'$. So $N = c^2/g = g(c')^2$. We get

$$A = \begin{pmatrix} x & -y\delta' \\ yg(c')^2 & x + y\gamma c' \end{pmatrix} \qquad \tilde{A} = \begin{pmatrix} x & -y\delta' g(c')^2 \\ y & x + y\gamma c' \end{pmatrix}$$

Clearly $\det A = \det \tilde{A}$. We are done if we prove that the gcd of the entries of A and of $\tilde{A}$ coincide. By Euclidean algorithm this is equivalent to:

$$\gcd\left(x, y\gcd\left(g(c')^2, \delta', \gamma c'\right)\right) = \gcd\left(x, y\gcd\left(1, \delta' g(c')^2, \gamma c'\right)\right),$$

which is true if

$$\gcd\left(g(c')^2, \delta', \gamma c'\right) = 1 \tag{3.4}$$

Since by definition $\gcd(c', \delta') = 1$, we cancel out c' and we are left with proving that $\gcd(g, \delta', \gamma) = 1$. This is proven in Lemma 3.2.8 below.

Lemma 3.2.8. *For $\tau = (a + b\omega)/c$ with $\gcd(a, b, c) = 1$, we have that*

$$\gcd\left(\gcd(c, c^2 \det A_\tau), \frac{c^2 \det A_\tau}{\gcd(c, c^2 \det A_\tau)}, c \operatorname{tr} A_\tau\right) = \gcd(g, \delta', \gamma) = 1.$$

Proof. Let us consider the discriminant of $cf_\tau^{\mathbb{Q}}$ which, recalling eq. (3.2) and eq. (3.3), is given by

$$\gamma^2 - 4\delta = \begin{cases} -b^2 m & \text{if } m \equiv 3 \mod 4, \\ -4b^2 m & \text{else.} \end{cases} \tag{3.5}$$

Suppose there is p prime that divides $\gcd(g, \delta', \gamma)$. So $p^2 \mid \gamma^2$, and since $\delta = g\delta'$ we also have $p^2 \mid \delta$. If $m \equiv 3 \mod 4$, then $-b^2 m \equiv \gamma^2 - 4\delta \equiv 0 \mod p^2$. Otherwise $-4b^2 m \equiv \gamma^2 - 4\delta \equiv 0 \mod p^2$. In both cases m is square-free, so we get $p \mid b$, respectively $p \mid 2b$. In the first case $2a + b \equiv \gamma \equiv 0$

mod p, in the second case $p \mid 2a$, so in both cases we have that $p \mid 2\gcd(a,b)$. As $p \mid g$, and $g \mid c$, and $\gcd(a,b,c) = 1$, we are left with $p \mid 2$.

To conclude, assume that p equals 2. If $m \equiv 3 \mod 4$, by the previous reasoning $2 \mid b$ and $4 \mid \delta$ so $0 \equiv \delta \equiv a^2 + ab + b^2 m' \equiv a^2 \mod 2$. This implies that $\gcd(a,b,c) = 2$, a contradiction. If $m \not\equiv 3 \mod 4$, then $4 \mid \delta$ and so $0 \equiv \delta \equiv a^2 + b^2 m \mod 4$. If b is even, this forces a to be even and we arrive to a contradiction as before. Thus, b is odd, so $b^2 \equiv 1 \mod 4$ and we have $a^2 + m \equiv 0 \mod 4$. If a is even, then $4 \mid m$, contradicting that m is square-free. If a is odd, then $m \equiv 3 \mod 4$, contradicting that $m \not\equiv 3 \mod 4$, so we are done. □

Example 3.2.7 together with Lemma 3.2.8 prove, in the case $n = k = 1$, the isomorphism ${}^{\Lambda}\!/\!_{\alpha\Lambda} \cong {}^{R}\!/\!_{\alpha R}$ as $\mathbb{Z}$-modules, while not necessarily as R-modules. The following lemmas provide a result for arbitrary n, k.

Lemma 3.2.9. *The primitive minimal polynomial $f_\tau^{\mathbb{Z}}$ of τ over $\mathbb{Z}$ equals $N f_\tau^{\mathbb{Q}}$ where $f_\tau^{\mathbb{Q}} = \tau^2 - \operatorname{tr} A_\tau \tau + \det A_\tau$.*

Proof. We have

$$\begin{aligned} N f_\tau^{\mathbb{Q}}(z) &= N z^2 - N\frac{\gamma}{c} z + N\frac{\delta}{c^2} \\ &= g(c')^2 z^2 - c'\gamma z + \delta', \end{aligned}$$

the three coefficients of the last polynomial are coprime by Equation (3.4). □

Now, consider the basis $\{1, \tau\}$ of Λ, each entry of A_Λ expands into a 2×2 matrix, as in Example 3.2.7, to get a matrix in $\operatorname{Mat}_{\mathbb{Z}}(2k, 2n)$ representing A_Λ. Likewise, the basis $\{1, N\tau\}$ gives a matrix in $\operatorname{Mat}_{\mathbb{Z}}(2k, 2n)$ representing A_R. By reordering the bases, we can write:

$$A_\Lambda = \begin{pmatrix} X & -Y\delta' \\ YN & X + Y\gamma c' \end{pmatrix} \qquad A_R = \begin{pmatrix} X & -Y\delta' N \\ Y & X + Y\gamma c' \end{pmatrix},$$

with $X, Y \in \operatorname{Mat}_{\mathbb{Z}}(k, n)$. In the following, we claim that the cokernels of both matrices concide.

Lemma 3.2.10. *Given $A_R \colon R^n \to R^k$ and $A_\Lambda \colon \Lambda^n \to \Lambda^k$ we have that*

$$\Lambda^k / A_\Lambda(\Lambda^n) \cong R^k / A_R(R^n)$$

as additive groups.

Proof. We regard R and Λ as $\mathbb{Z}$-modules again. Write A for A_Λ and $\tilde{A}$ for A_R. Since equality of submodules is a local property, the goal is to prove that for all primes $p \in \mathbb{Z}$ the localizations $A_{(p)} \colon (\mathbb{Z}_{(p)})^{2n} \to (\mathbb{Z}_{(p)})^{2k}$ and $\tilde{A}_{(p)} \colon (\mathbb{Z}_{(p)})^{2n} \to (\mathbb{Z}_{(p)})^{2k}$ have isomorphic cokernels. First assume that p does not divide N. Let I_k be the $k \times k$ identity, and I_n the $n \times n$ identity. We have

$$A = \begin{pmatrix} X & -Y\delta' \\ YN & X + Y\gamma c' \end{pmatrix} = \begin{pmatrix} I_k & 0 \\ 0 & NI_k \end{pmatrix} \begin{pmatrix} X & -Y\delta' N \\ Y & X + Y\gamma c' \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & N^{-1}I_n \end{pmatrix} = S\tilde{A}T,$$

where S and T are suitable matrices. If p does not divide N, both S and T are invertible matrices in $\mathbb{Z}_{(p)}$, thus $\operatorname{Im} A = \operatorname{Im}(S\tilde{A}T) = S(\operatorname{Im} \tilde{A})$ shows $\operatorname{Im} A$ and $\operatorname{Im} \tilde{A}$ are isomorphic.

When $p \mid N$ we modify A and $\tilde{A}$ in the following way:

$$\begin{aligned} A' = S_a A T_a &= \begin{pmatrix} I_k & 0 \\ -aI_k & I_k \end{pmatrix} \begin{pmatrix} X & -Y\delta' \\ YN & X + Y\gamma c' \end{pmatrix} \begin{pmatrix} I_n & 0 \\ aI_n & I_n \end{pmatrix} \\ &= \begin{pmatrix} X - Ya\delta' & -Y\delta' \\ Y(N + a\gamma c' + a^2\delta') & X + Y(\gamma c' + \delta' a) \end{pmatrix} \\ \tilde{A}' = \tilde{S}_{a\delta'} \tilde{A} \tilde{T}_{a\delta'} &= \begin{pmatrix} I_k & -\delta a I_k \\ 0 & I_k \end{pmatrix} \begin{pmatrix} X & -Y\delta' N \\ Y & X + Y\gamma c' \end{pmatrix} \begin{pmatrix} I_n & \delta a I_n \\ 0 & I_n \end{pmatrix} \\ &= \begin{pmatrix} X - Ya\delta' & -Y\delta'(N + a\gamma c' + a^2\delta') \\ Y & X + Y(\gamma c' + \delta' a) \end{pmatrix} \end{aligned}$$

for some $a \in \mathbb{Z}$. Note that A' is of the form of the previous case. Write $N'(a)$ for $N + a\gamma c' + a^2\delta'$. If there is a choice of a such that $N'(a) \not\equiv 0 \mod p$, then we can conclude. Now we show that such an a always exists. Indeed, Regarding $N'(a)$ as a polynomial over $\mathbb{Z}/p\mathbb{Z}$, it is non-zero because $\gcd(\delta', \gamma c', N) = 1$ as in Example 3.2.7. Thus it has at most 2 roots, this means we are done if $p \geq 3$. If $p = 2$, then $N \equiv gc' \equiv 0 \mod 2$, and 2 divides either c' or g. If $2 \mid c'$, then $N'(a) \equiv a^2 \mod 2$ and we are done. Similarly, if $m \not\equiv 3 \mod 4$, then $2 \mid \gamma$ and again $N'(a) \equiv a^2 \mod 2$. Finally, if $2 \mid g$ and $m \equiv 3 \mod 4$, then $2 \mid \delta$, that is $a^2 + ab + b^2 \equiv 0 \mod 2$. As we argued before, the only way this can be true is if $a \equiv b \equiv 0 \mod 2$, but since $2 \mid g$ then $2 \mid c$, contradicting that $\gcd(a, b, c) = 1$. □

The previous discussion and Lemma 3.2.10 allow us to state the following theorem, which extends Lemma 3.2.5 and gives a complete characterization of the number or layers of an elliptic arrangement $\mathcal{A}$.

Theorem 3.2.11. *Let $A \in \mathrm{Mat}_{k\times n}(R)$ describe an elliptic arrangement $\mathcal{A}$ in an elliptic curve $\mathcal{E} = {}^{\mathbb{C}}\!/_{\Lambda}$ that has complex multiplication. For all $S \subseteq [k]$ we have*

$$m(S) = \#\textit{connected components}(\mathcal{A}_S) = \#\operatorname{tor}(\operatorname{coker} A_\Lambda[S]) = \#\operatorname{tor}(\operatorname{coker} A_R[S]).$$

Moreover, the modules ${}^{\Lambda^S}\!/_{A_\Lambda[S](\Lambda^n)}$ and ${}^{R^S}\!/_{A_R[S](R^n)}$ are isomorphic as $\mathbb{Z}$-modules, but not as R-modules.

3.3 Arithmetic matroid

The goal of this section is to prove that it is possible to associate the structure of an arithmetic matroid to an elliptic arrangement, similar to the approach of the toric case. Before showing the construction of it and its associated properties, we will provide a brief introduction.

Definition 3.3.1. Let E be a finite ground set. A matroid on E is given by a function $\mathrm{rk} : \mathcal{P}(E) \to \mathbb{N}$ that satisfies:

(r1) $\mathrm{rk}\,\emptyset = 0$,

(r2) $\mathrm{rk}\,X \leq \mathrm{rk}(X \cup i) \leq \mathrm{rk}\,X + 1$ for every $X \subset E$ and $i \in E$,

(r3) $\mathrm{rk}(X \cup Y) + \mathrm{rk}(X \cap Y) \leq \mathrm{rk}\,X + \mathrm{rk}\,Y$ for every $X, Y \subset E$.

These axioms represent an abstraction of the following example, known as the *realizable matroid*. Given a list of vectors $(v_e)_{e\in E}$ indexed by E in some finite dimensional vector space V over some field K, set $\mathrm{rk}\,S = \dim_K\langle v_e \mid e \in S\rangle$ for every subset $S \subset E$. The connection to arrangements becomes clear when considering a list of functionals defining a hyperplane arrangement $\mathcal{A}$ in K^n. The realizable matroid associated to them is cryptomorphic to the matroid defined via the lattice of intersections: the rank of x in a lattice is the length of a maximal chain in the downset $L(\mathcal{A})_{\leq x}$.

Going further, arithmetic matroids originally arose when considering arrangements in a torus instead of in V [DM13]. Here the functionals defining the arrangement live in a module M over $\mathbb{Z}$. We set $\mathrm{rk}\,S$ to the rank of the submodule $\langle v_e \mid e \in S\rangle_{\mathbb{Z}}$, which satisfies axioms (r1), (r2), and (r3). Moreover, consider the number $m(S)$ of connected components in the intersection $\bigcap_{i\in S} H_i$. The question of axiomatizing $m(S)$ was addressed in [DM13, BM14], which we recall now.

Denote by $[X,Y]$ the interval $\{S \subset E : X \subseteq S \subseteq Y\}$ in $(\mathcal{P}(E), \subseteq)$. We say that $[X,Y]$ is a *molecule* if we can write Y as a disjoint union $Y = X \sqcup F \sqcup T$ such that for each $S \in [X,Y]$ we have

$$\mathrm{rk}(S) = \mathrm{rk}(X) + \#X \cap F.$$

Definition 3.3.2. An *arithmetic matroid* (E, rk, m) is an underlying matroid (E, rk) plus a function $m \colon \mathcal{P}(E) \to \mathbb{N}$ such that the following algebraic axioms are satisfied:

(A1) For all $S \subset E$ and $i \in E$: if $\mathrm{rk}(S \cup i) = \mathrm{rk}(S)$, then $m(S \cup i)$ divides $m(S)$; otherwise $m(S)$ divides $m(S \cup i)$.

(A2) If $[X,Y]$ is a molecule then

$$m(X)m(Y) = m(X \cup F)m(X \cup T).$$

Moreover, the following geometric axiom must be satisfied, which intuitively expresses a count of connected components via inclusion-exclusion:

(P) If $[X,Y]$ is a molecule, $Y = X \sqcup F \sqcup T$, then the number $\rho(X,Y)$ given by

$$\rho(X,Y) = (-1)^{|T|} \sum_{S \in [X,Y]} (-1)^{|Y|-|S|} m(S)$$

is greater or equal than 0.

In the context of elliptic arrangements, it is possible to obtain similar results to those of the toric case. Indeed, as stated in the following theorem, the information of codimension and number of layers of intersections of an elliptic arrangement still defines an arithmetic matroid.

Theorem 3.3.3. *Let $\mathcal{A} = \{H_i = \ker(\Phi)_i \mid \Phi_i \in Hom(\Lambda^n, \Lambda)\}_{i \in E}$ be an elliptic arrangement. Given a subset $S \subset E$, set $\mathrm{rk}_{\mathcal{A}}(S) = \operatorname{codim} \mathcal{A}_S$ and $m_{\mathcal{A}}(S) = \#$connected components$(\mathcal{A}_S)$. The triple $([k], \mathrm{rk}_{\mathcal{A}}, m_{\mathcal{A}})$ is an arithmetic matroid.*

We prove individually each axiom. First, recall that by Theorem 3.2.11 the multiplicity $m(S) := m_{\mathcal{A}}(S) = \# \operatorname{tor}(\Lambda^S / \operatorname{Im} A_S)$. For convenience, write G_S for $\operatorname{tor}(\Lambda^S / \operatorname{Im} A_S)$. Note that if $X \subset Y$, then the natural projection $\pi \colon \Lambda^Y \to \Lambda^X$ induces a map $\bar{\pi} : G_Y \to G_X$ with the following properties.

Lemma 3.3.4. *Let $\mathcal{A} = \{H_i\}_{i\in[k]}$ be an elliptic arrangement, $S \subseteq [k]$ a set and $i \in E$ an element. Consider the map $\bar{\pi}\colon G_{S\cup i} \to G_S$ induced by projection.*

(1) If $\operatorname{rk}(S\cup i) = \operatorname{rk}(S)$, then $\bar{\pi}$ is injective.

(2) If $\operatorname{rk}(S\cup i) \neq \operatorname{rk}(S)$, then $\bar{\pi}$ is surjective.

Proof. First, notice that there exists a nonzero integer k with $ke_i \in \operatorname{Im} A_{S\cup i}$ if and only if $\operatorname{rk}(S\cup i) > \operatorname{rk}(S)$. This is because $\operatorname{rk}(S\cup i) = \operatorname{rk}(S)$ if and only if the i-th coordinate is linearly dependent on those indexed by S.

Item (1): let $\bar{v} \in G_{S\cup i}$ be nonzero, and $v \in \Lambda^{S\cup i}$ a representative. This means $mv \in \operatorname{Im} A_{S\cup i}$ for a nonzero m. If $\bar{v} \in \ker\bar{\pi}$, there exists $x \in \Lambda^n$ such that $\pi(v) = A_S(x)$. Thus, $v + \lambda e_i \in \operatorname{Im} A_{S\cup i}$ for $\lambda = A_{S\cup i}(x)_i - v_i$; also $m\lambda e_i = m(v+\lambda e_i) - mv$ is in $\operatorname{Im} A_{S\cup i}$. By the first remark $m\lambda = 0$, so $\lambda = 0$ and $\ker\bar{\pi}$ is trivial.

Item (2): Let $\bar{v} \in G_S$, and $v \in \Lambda^S$ a representative. This means $mv \in \operatorname{Im} A_S$ for a nonzero m. Thus, by a similar argument as before, there exists λ such that $(mv, 0) + \lambda e_i$ is in $A_{S\cup i}$. Moreover, by the first remark let $k \in \mathbb{Z}\setminus 0$ such that $ke_i \in \operatorname{Im} A_{S\cup i}$. Thus, $km(v,0) = k((mv,0)+\lambda e_i) - m\lambda(ke_i)$ is in $\operatorname{Im} A_{S\cup i}$, hence $\overline{(v,0)}$ is a torsion element in $G_{S\cup i}$ and also a lift of $\overline{v}$, proving that $\overline{\pi}$ is surjective. ☐

Corollary 3.3.5. *The triple $([k], \operatorname{rk}_{\mathcal{A}}, m_{\mathcal{A}})$ satisfies Axiom (A1).*

Next, if $[X, Y]$ is a molecule with $Y = X \sqcup F \sqcup T$, we can consider the maps from the previous lemma to get a commutative square.

$$
\begin{array}{ccccc}
0 & \longrightarrow & G_{X\sqcup F\sqcup T} & \longrightarrow & G_{X\sqcup F} \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & G_{X\sqcup T} & \longrightarrow & G_X \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0
\end{array}
$$

Diagram 3.3.6

This diagram can be extended and used to verify Axiom (A2), as shown in the following lemma.

Lemma 3.3.7. *In the context of Lemma 3.3.3, the triple $([k], \operatorname{rk}_{\mathcal{A}}, m_{\mathcal{A}})$ satisfies Axiom (A2). That is, if $[X, Y]$ is a molecule with $Y = X \sqcup F \sqcup T$ we have that*

$$m_{\mathcal{A}}(X) \cdot m_{\mathcal{A}}(X \sqcup F \sqcup T) = m_{\mathcal{A}}(X \sqcup F) \cdot m_{\mathcal{A}}(X \sqcup T).$$

Proof. We complete Diagram 3.3.6 and apply snake lemma to get Diagram 3.3.8:

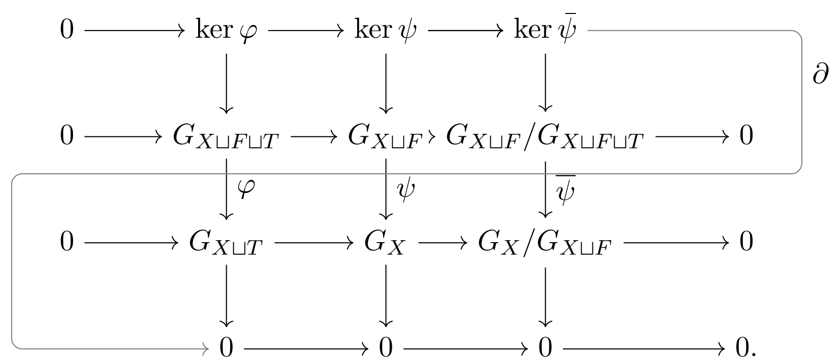


Diagram 3.3.8

The result follows from the third column if we prove that $\ker \overline{\psi}$ is trivial, that is $\ker \varphi$ and $\ker \psi$ are isomorphic. We then proceed showing that $\ker \varphi \to \ker \psi$ is surjective.
If $\bar{y}$ is in $\ker \psi$, there is a representative in $\Lambda^{X \sqcup F}$ of the form $(0, v)$, where the zeros are for the coordinates indexed by X, such that there is a nonzero m with $(0, mv) = A_{X \sqcup F}(x)$ for some $x \in \Lambda^n$. Since $\mathrm{rk}(X \sqcup T) = \mathrm{rk}(X)$, the coordinates indexed by T are dependent on those indexed by X. Thus, $(0, mv) = A_{X \sqcup F}(x)$ implies $(0, mv, 0) = A_{X \sqcup F \sqcup T}(x)$. Hence $\overline{(0, v, 0)}$ is a torsion element in $G_{X \sqcup F \sqcup T}$ and also the desired lift for $\bar{y}$. $\square$

So far we have proved the truth of Axiom (A1) and Axiom (A2). To verify Axiom (P), we will introduce a new combinatorial object: the dual matroid. First, let us start with its definition.

Definition 3.3.9. Given a triple $M = (E, \mathrm{rk}, m)$ we define the *dual rank function* $\mathrm{rk}^*(S)$ and the *dual multiplicity* $m^*(S)$ as

$$\mathrm{rk}^*(S) = \#S - (\mathrm{rk}\, E - \mathrm{rk}(E \setminus S)), \quad m^*(S) = m(E \setminus S). \tag{3.6}$$

By [DM13, Lemma 2.2], if M is an arithmetic matroid, then so is $M^* = (E, \mathrm{rk}^*, m^*)$ and we call it the *dual arithmetic matroid*. Notice that $(M^*)^* = M$. By [BM14, Section 2] Axiom (P) is equivalent to Axiom (A2) together with:

(P1) If $X \subset Y \subset E$ and $\mathrm{rk}\, X = \mathrm{rk}\, Y$, then $\rho(X, Y) \geq 0$.

(P2) If $X \subset Y \subset E$ and $\operatorname{rk}^* X = \operatorname{rk}^* Y$, then $\rho^*(X,Y) \geq 0$, where ρ^* is the analogous expression for the dual matroid.

We will first prove that our construction satisfies Axiom (P1). Following this, we will develop a dual construction to verify Axiom (P2) right after.

In this setting, in order to verify Axiom (P1), we need to prove that, for an elliptic arrangement $\mathcal{A}$,

$$\rho(X,Y) = \sum_{S \in [X,Y]} (-1)^{|S|-|X|} m(S)$$

is non-negative. We abuse notation by writing $\operatorname{CC}(S)$ for the connected components of $\mathcal{A}_S = \bigcap_{i \in S} H_i$.

Lemma 3.3.10. *Let $\mathcal{A} = \{H_i\}_{i\in[k]}$ be an elliptic arrangement. For all $X, Y \subset [k]$ with $\operatorname{rk} X = \operatorname{rk} Y$ we have*

$$\rho(X,Y) = \#\Big(\operatorname{CC}(X) \setminus \bigcup_{i \in Y \setminus X} \operatorname{CC}(X \cup i)\Big).$$

Proof. Since $\operatorname{rk} X = \operatorname{rk} Y$, we have that set-theoretically $\operatorname{CC}(S) \cap \operatorname{CC}(T) = \operatorname{CC}(S \cup T)$ for all $S, T \subset Y$. In particular, it follows $\operatorname{CC}(T) \supset \operatorname{CC}(S)$ when $S \subset T$. Furthermore, recalling that $m(S) = \#\operatorname{CC}(S)$, the result follows from a straightforward application of the inclusion-exclusion argument. □

Corollary 3.3.11. *In the context of Lemma 3.3.3, the triple $([k], \operatorname{rk}_{\mathcal{A}}, m_{\mathcal{A}})$ satisfies Axiom (P1).*

Regarding Axiom (P2), it will naturally follow from duality once we realize M^* with our elliptical arrangement construction. This is precisely the goal of the final part of this section.

Since a similar construction has been made for the toric case, for which the structure of a matroid and a dual matroid associated to an arrangement are analogous to our case, we will follow the ideas of D'Adderio and Moci of our main reference [DM13, Section 3.4] and generalize to elliptic arrangements. We will refer to this as a *weak dual* because the operation of dualization, when applied twice, does not return the original matroid.

First, let us recall arithmetic matroids realizable by toric arrangements. Fixed k elements $\mathcal{P} = \{p_1, \ldots, p_k\} \subset \mathbb{Z}^n$ (the same construction works when considering a set of points in any finitely generated abelian group), the associated arithmetic matroid $M_{\mathcal{P}}$ is defined as follows. For any $S \subset \mathcal{P}$, let G_S be the largest subgroup of $\mathbb{Z}^n$ such that the index $[G_S : \langle p \mid p \in S\rangle]$ is finite. Consider then

$$\mathrm{rk}_{\mathcal{P}} = \mathrm{rk}(S) = \mathrm{rk}\langle p \mid p \in S\rangle \text{ and } m_{\mathcal{P}}(S) = [G_S : \langle p \mid p \in S\rangle].$$

The triple $M_{\mathcal{P}} = ([k], \mathrm{rk}_{\mathcal{P}}, m_{\mathcal{P}})$ is an arithmetic matroid.

Remark 3.3.12. Note that in $\mathbb{Z}^m$, as proved by Stanley in [Sta91, Theorem 2.2], the multiplicity of $M_{\mathcal{P}}$ is the greatest common divisor of the nonzero minors of maximal rank of the matrix associated to $\mathcal{P}$, i.e. the matrix whose i-the row is p_i.

Second, recall arithmetic matroid contraction. Given an arithmetic matroid $M = (E, \mathrm{rk}, m)$, the contraction M/T by a set $T \subset E$ is an arithmetic matroid on $E \setminus T$ with rank and multiplicity given by

$$r_{M/T}(A) = \mathrm{rk}(A \cup T) - \mathrm{rk}(T) \text{ and } m_{M/T}(A) = m(A \cup T),$$

for each $A \subset E \setminus T$. It is straightforward to see that any axiom satisfied by M is also satisfied by $M/T = (E \setminus T, r_{M/T}, m_{M/T})$, which we will denote, for simplicity, $(E \setminus T, r', m')$.

Finally, let $\mathcal{Q} = \{q_1, \ldots, q_n\} \subset \mathbb{Z}^k$ be the columns of the $(k \times n)$-matrix A whose i-th row is p_i. Also, let e_i be the i-th standard vector of $\mathbb{Z}^k$ and $\mathcal{B} = \{e_1, \ldots, e_k\} \subset \mathbb{Z}^k$. Now we consider the matrid $M_{\mathcal{B}\cup\mathcal{Q}}$ associated to $\mathcal{B} \cup \mathcal{Q}$.

Lemma 3.3.13 ([DM13, Theorem 3.8]). *For k elements $\mathcal{P} = \{p_1, \ldots, p_k\} \subset \mathbb{Z}^n$ the dual $(M_{\mathcal{P}})^*$ is isomorphic to the contraction $(M_{\mathcal{B}\cup\mathcal{Q}}/\mathcal{Q}) = (\mathcal{B}, \mathrm{rk}', m')$ via the map $S \mapsto \mathcal{B}_S$ where $\mathcal{B}_S = \{e_i \mid i \in S\}$.*

Proof. Let A be the matrix associated to $\mathcal{P}$, hence the matrix whose i-th row is p_i as above. Consider $\mathcal{B}_S \subset \mathcal{B}$ for some $S \subset [k]$ with $s = |S|$ elements. We compute $\mathrm{rk}'(\mathcal{B}_S) = \mathrm{rk}_{\mathcal{B}\cup\mathcal{Q}}(\mathcal{B}_S \cup \mathcal{Q}) - \mathrm{rk}_{\mathcal{B}\cup\mathcal{Q}}\, \mathcal{Q}$ and $m'(\mathcal{B}_S) = m_{\mathcal{B}\cup\mathcal{Q}}(\mathcal{B}_S \cup \mathcal{Q})$ using the $(s+n)\times k$-matrix $(I_k[S]\ A)^\top$, where I_k is the $k \times k$ identity matrix. Both quantities remain constant if we sum any integral multiple of a row to another of $(I_k[S]\ A)^\top$. With the rows of $I_k[S]$, we clear the columns indexed by S and get $(I_k[S]\ \hat{A})^\top$. Here the j-th column of $\hat{A}$ equals that of A if $j \notin S$, and 0 otherwise. E.g. if S were $\{j\}$, we would have

$$\begin{pmatrix} 0 & \ldots & 1 & \ldots & 0 \\ a_{11} & \ldots & a_{j1} & \ldots & a_{k1} \\ & & \vdots & & \\ a_{1n} & \ldots & a_{jn} & \ldots & a_{kn} \end{pmatrix} \mapsto \begin{pmatrix} 0 & \ldots & 1 & \ldots & 0 \\ a_{11} & \ldots & 0 & \ldots & a_{k1} \\ & & \vdots & & \\ a_{1n} & \ldots & 0 & \ldots & a_{kn} \end{pmatrix}$$

Straightaway we get a rank which coincides with Equation (3.6):

$$r'(\mathcal{B}_S) = \operatorname{rk}(I_k[S]\ \hat{A})^\top - \operatorname{rk}\mathcal{Q} = s + \operatorname{rk} A[S^c] - \operatorname{rk} A^\top = \#S - (\operatorname{rk} M_{\mathcal{A}} - \operatorname{rk}([k] \setminus S)).$$

Regarding the multiplicity, as stated in Remark 3.3.12 $m(\mathcal{B}_S \cup \mathcal{Q})$ equals the greatest common divisor of all maximal minors of $(I_k[S]\ \hat{A})^\top$. Any nonzero maximal $r \times r$-minor includes all columns indexed by S. Indeed, if the j-th column missed for $j \in S$, also the row e_j would miss, otherwise the minor would be zero. But then the j-th column and the e_j row can be added to get a nonzero $(r+1) \times (r+1)$-minor, contradicting maximality. Also all rows e_j with $j \in S$ are in the maximal minor, otherwise a column would be zero. The remaining rows are from $\hat{A}$. By a cofactor expansion we reduce the calculations to $A[S^c]$, and hence $m'(\mathcal{B}_S) = m([k] \setminus S)$, as desired. □

Let us now consider an elliptic arrangement $\mathcal{A}$ in $(\mathcal{E})^n$ defined by the family of morphisms $\Phi = \{\phi_i \colon \mathcal{E}^n \mapsto \mathcal{E}\}_{i \in [k]}$, given by $\phi_i(x) = \langle a_i, x \rangle$, where $a_i \in R^n$. Let A' be the matrix associated to the arrangement $\mathcal{A}$ and, identifying with Φ the set of the elements $a_i \in R^n$, let $M_{\mathcal{A}} := M_\Phi$ be the associated arithmetic matroid.

Since $R = \langle 1, N\tau \rangle$ is a lattice in $\mathbb{C}$, let us consider $\mathbb{C}/R$, the dual elliptic curve of $\mathcal{E}$, and the elliptic arrangement in $(\mathbb{C}/R)^n$ defined by the matrix $A := (A')^\top$. Let $\Psi := \{\psi_j\}_{j \in [n]}$, $\Psi_j : (\mathbb{C}/R)^k \to (\mathbb{C}/R)$, be the family of the transposed morphisms associated to the rows of A and $\Omega := \{\varepsilon_\ell\}_{\ell \in [k]}$ the family of the standard morphisms, i.e. $\varepsilon_\ell : (\mathbb{C}/R)^k \to (\mathbb{C}/R)$ is the projections into the ℓ-th coordinate.

The union $\Omega \cup \Psi$ gives an arrangement of $n+k$ hypersurfaces in $(\mathbb{C}/R)^n$, whose associated matrix is $\begin{pmatrix} I_k \\ A \end{pmatrix}^\top \in \operatorname{Mat}_{(k+n)\times k}(R)$.

In this setting, the following lemma is essential for concluding our discussion and presenting the construction we refer to as the *weak dual*.

Lemma 3.3.14. *The dual $(M_{\mathcal{A}})^* = ([k], r^*_{\mathcal{A}}, m^*_{\mathcal{A}})$ is isomorphic to the contraction $M_{\Omega\cup\Psi}/\Psi = (\Omega, r', m')$ via the map $S \mapsto \Omega_S$ with $\Omega_S = \{\varepsilon_i \mid i \in S\}$.*

Proof. It follows from Lemma 3.2.10 and from the proof of Lemma 3.3.13. In this case, the matrix to consider is $(I_{2k}[\bar{S}]\ A_{\mathbb{Z}})^\top$ where $\bar{S} = \{i, \bar{i} \mid i \in S\}$. □

Following this lemma, we refer to $M_{\Omega\cup\Psi}$ as the *weak dual* of $M_{\mathcal{A}}$: performing dualization twice does not yield the identity, and both the dimensions and the number of hypersurfaces do not correspond to what one would expect in a dual structure. To better understand this structure, consider the following diagram,

$$
\begin{array}{ccc}
\left\{\begin{array}{c}\text{Elliptic arrangements of } k \\ \text{hypersurfaces in } (\mathbb{C}/\Lambda)^n \\ \mathcal{A}\end{array}\right\} & \xrightarrow{WD} & \left\{\begin{array}{c}\text{Elliptic arrangements of } n+k \\ \text{hypersurfaces in } (\mathbb{C}/R)^k \\ M_{\Omega\cup\Psi}\end{array}\right\} \\
\downarrow & & \downarrow q \\
\left\{\begin{array}{c}\text{Triples } ([n], \mathrm{rk}=k, m) \\ M_{\mathcal{A}}\end{array}\right\} & \xrightarrow{\quad * \quad} & \left\{\begin{array}{c}\text{Triples } ([n], \mathrm{rk}^*=k-n, m^*) \\ (M_{\mathcal{A}})^*\end{array}\right\}
\end{array}
$$

where WD denotes the weak duality just defined, $*$ represents the well-known classical duality, and q is the map that quotients by Ψ, as described in Lemma 3.3.14. This leads to the following corollary.

Corollary 3.3.15. *In the context of Lemma 3.3.3, the triple* $([k], \mathrm{rk}_{\mathcal{A}}, m_{\mathcal{A}})$ *satisfies Axiom (P2).*

Proof. By Corollary 3.3.11 the triple $M_{\mathcal{B}\cup\mathcal{Q}} = (\mathcal{B}\cup\mathcal{Q}, \mathrm{rk}_{\mathcal{B}\cup\mathcal{Q}}, m_{\mathcal{B}\cup\mathcal{Q}})$ satisfies Axiom (P1). This implies that the minor $M_{\mathcal{B}\cup\mathcal{Q}}/\mathcal{Q}$ satisfies Axiom (P1); to prove the relevant equality just consider the molecule $[X \cup \mathcal{Q}, Y \cup \mathcal{Q}]$ in $M_{\mathcal{B}\cup\mathcal{Q}}$ when given a molecule $[X, Y]$ in $M_{\mathcal{B}\cup\mathcal{Q}}/\mathcal{Q}$. Now by Lemma 3.3.13, Axiom (P1) holds for $(M_{\mathcal{A}})^*$, which as observed before proves Axiom (P2) for $M_{\mathcal{A}}$ by duality. □

By combining all the above results, it is possible to modify the diagram above by replacing *triples* with *arithmetic matroids*, and we have a proof of Theorem 3.3.3.

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