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EARLY AND LATE TIME COSMOLOGY OF TYPE IIB FLUX COMPACTIFICATIONS

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To my family, and my friends. To my past and future self.

"Жизнь прекрасна. Пусть же будущие поколения очистят ее от всякого зла, угнетения и насилия и наслаждаются ею во всем ее блеске."

- Лев Тро́цкий

La vita è bella. Possano le generazioni future liberarla da ogni male, oppressione e violenza, e goderla in tutto il suo splendore.

– Lev Trotskij

Abstract

We explore various aspects of type IIB flux compactifications, focusing on the stabilisation of moduli and the implications for cosmology. We present a novel method to obtain type IIB flux vacua with flat directions at tree level, implemented in both toroidal and Calabi-Yau compactifications. We also estimate the leading higher derivative corrections to $\mathcal{N} = 1$ supergravity and their impact on moduli stabilisation and inflation models. Additionally, we develop realizations of Early Dark Energy (EDE) within type IIB string theory, identifying promising candidates for resolving the Hubble Tension. Finally, we analyse the joint distribution of the gravitino mass and the cosmological constant in KKLT and LVS models, highlighting the favourable conditions for lower scales of supersymmetry breaking.

Declaration

This manuscript is based on the results presented in the following works:

- M. Cicoli, M. Licheri, P. Piantadosi, F. Quevedo, P. Shukla. "Higher derivative corrections to string inflation". JHEP **02,** 115. arXiv: 2309.11697 [hep-th] (2024)
- M. Cicoli, M. Licheri, R. Mahanta, E. McDonough, F. G. Pedro, M. Scalisi. "Early Dark Energy in Type IIB String Theory". JHEP **06,** 052. arXiv: 2303.03414 [hep-th] (2023)
- M. Cicoli, M. Licheri, A. Maharana, K. Singh, K. Sinha. "Joint statistics of cosmological constant and SUSY breaking in flux vacua with nilpotent Goldstino". JHEP **01,** 013. arXiv: 2211.07695 [hep-th] (2023)
- M. Cicoli, M. Licheri, R. Mahanta, A. Maharana. "Flux vacua with approximate flat directions". JHEP **10,** 086. arXiv: 2209.02720 [hep-th] (2022)
- R. Allahverdi, M. Cicoli, M. Licheri, K. Sinha. "Joint statistics of cosmological constant and SUSY breaking in flux vacua with generic uplift". (in preparation)

Contents

Introduction

Potentials for moduli fields play a central role in string phenomenology. The simplest way to generate these potentials is to consider solutions with background fluxes, see e.g. [6–10]. In the type IIB setting the effect of fluxes is to stabilize the complex structure moduli and the axio-dilaton [10]. This is encoded in the Gukov-Vafa-Witten (GVW) superpotential [8]:

$$W = \int_X G_3 \wedge \Omega \,, \tag{1}$$

where $G_3 = F_3 - \phi H_3$ is the complexified 3-form flux, ϕ is the axio-dilaton[1] and Ω the holomorphic 3-form of the (orientifolded) Calabi-Yau (CY) X on which the theory is compactified. The 3-form fluxes thread 3-cycles of the CY with their integrals over the cycles satisfying Dirac quantization conditions. Depending on the choice of fluxes, minima of the associated potential can be isolated or have flat directions. Once the axio-dilaton and the complex structure moduli are integrated out, the effect of fluxes is captured by a constant superpotential:

$$W_0 \equiv \langle \int_X G_3 \wedge \Omega \rangle \,. \tag{2}$$

The value of W_0 is a key input for phenomenology.[2] Again, this is determined by the choice of flux quanta. Various recent studies have shown an interesting interplay between the existence of (approximate) flat directions and a low value of W_0 [12–16]. Ref. [17] argued that these perturbative flat directions are associated to pseudo-Goldstone bosons of a 2-parameter family of scale invariance of the classical 10-dimensional theory. One of these 2 symmetries is the scale transformation included in $SL(2,\mathbb{R})$, while the other transforms the metric. Both of them are spontaneously broken by the fact that both the metric and the dilaton acquire

[1] In Sec. 1.2.2 and in Chap. 2 we do not follow the standard convention to denote the axio-dilaton as τ to avoid confusion with the period matrix τ^{ij} of the toroidal case. In other sections it is defined as $S = 1/g_s + C_0$.

[2] For a general discussion on W_0 in the context of moduli stabilization and its role in phenomenology, see [11] and references therein.

a vacuum expectation value. When combined with axionic shift symmetries, this is reflected in the 4-dimensional effective theory in the fact that both the axio-dilaton and the overall Kähler modulus are flat directions at classical level. However, in the 10-dimensional theory also G_3 transforms with a non-zero weight. When compactifying, 3-form fluxes take quantized background values, and so act as explicit breaking parameters which lift the axio-dilaton. In [17] the explicit breaking parameter was identified in W_0 (promoted to a spurion), arguing that $W_0 = 0$ implies the existence of a flat direction, in agreement with the findings of [12–16].

Notice that the condition to have a flat axio-dilaton is that $W_0 = 0$ after the complex structure moduli have been integrated out. In fact, in this case the 4-dimensional action does not see any explicit scale breaking parameter since the flux quanta do not contribute to the scalar potential. On the other hand, $W_0 = 0$ can clearly be compatible with a stable axio-dilaton at classical level when W_0 has an appropriate dependence on it after complex structure moduli stabilization, even if the generic case would be characterized by $W_0 \neq 0$.

Let us stress that these flat directions are only approximate since they are expected to be lifted by a combination of non-perturbative and perturbative effects. Nevertheless they can have a wide variety of interesting phenomenological applications. The first is in the context of Kähler moduli stabilization. A low value of W_0 is an essential ingredient for the KKLT scenario [18] for moduli stabilization. A method to construct vacua with low W_0 has been put forward in [12].[3] This is in the large complex structure limit of the underlying CY compactification. Flux quanta are so chosen that they yield a GVW superpotential which, when computed using the perturbative part of the prepotential, is a degree-2 homogeneous polynomial. The homogeneity property and the request of a vanishing flux superpotential for non-zero values of the moduli guarantees the presence of a flat direction. This flat direction is lifted when non-perturbative corrections to the prepotential are incorporated. Hence W_0 acquires an exponentially small value (at weak string coupling). Working with a CY orientifold obtained by considering a degree-18 hypersurface in $\mathbb{CP}_{[1,1,1,6,9]}$, [12] presented an explicit choice of flux quanta corresponding to $W_0 \sim 10^{-8}$. Using the same method, an example with W_0 as low as 10^{-95} was constructed in [13, 14]. Further studies of this setup have been carried out in [15, 16, 23–26].

Low values of W_0 might also be important in the context of LVS models

[3] For earlier work on obtaining low values of W_0 see [19, 20], while for challenges in implementing moduli stabilization and obtaining dS vacua in this setting see [21, 22].

[27]. Recently explicit LVS realizations of the Standard Model have been carried out by considering D3-branes at an orientifolded dP_5 singularity [28]. Here the cancellation of all D7-charges and Freed-Witten anomalies forces the presence of a hidden D7 sector with non-zero gauge fluxes which induce a T-brane background suitable for de Sitter (dS) uplifting [29]. The T-brane contribution can give a leading order Minkowski vacuum if the value of the W_0 is exponentially small in the string coupling, i.e. it is precisely of the form described above. A dS minimum with soft terms above the TeV scale requires W_0 as small as 10^{-13}. Let us point out that, contrary to KKLT, in LVS an exponentially small value of W_0 is just a model-dependent condition since [30] presented a chiral global D3-brane model at an orientifolded dP_0 singularity which can allow for dS moduli stabilization with T-brane uplifting for $W_0 \sim \mathcal{O}(1)$.

Approximate flat directions are naturally interesting also in the context of cosmology. In fact, the idea of focusing on degree-2 homogeneous superpotentials so as to obtain flat direction(s) was first used in [31] to enhance the inflaton field range. More recently, flat directions in the type IIB flux superpotential have been used to construct models of sequestered inflation [32]. Interestingly, the predictions of the models carry signatures of the moduli space geometry. Moreover, leading order flat directions can be promising candidates to realize quintessence models in order to avoid any destabilization problem due to the inflationary energy contribution and to reproduce the correct tiny value of the cosmological constant scale [33, 34].

Regarding supersymmetry breaking, type IIB models are characterized by a no-scale relation which implies that generically the main contribution to supersymmetry breaking comes from the Kähler moduli sector. In fact, typically at semi-classical level the complex structure moduli and the axio-dilaton are fixed by setting their F-terms to zero with $W_0 \neq 0$ which induces instead non-zero F-terms for the Kähler moduli (that are still flat at this level of approximation). However in scenarios with $W_0 = 0$ and a flat axio-dilaton, all F-terms are zero at leading order and the effective field theory after integrating out the complex structure moduli has to include both the axio-dilaton and the Kähler moduli [35]. Therefore the F-term of the axio-dilaton can also play an important role in supersymmetry breaking, especially in sequestered models with D3-branes where gaugino masses are controlled by the F-term of ϕ [36, 37].

Moreover, flux vacua with a leading order axionic flat direction and $W_0 \neq 0$ have been shown in [38] to be very promising to obtain a dS uplifting contribution from non-zero F-terms of the complex structure moduli, so providing an explicit realization of the idea proposed in [39, 40] without however the assumption of continuous 3-form fluxes. More precisely, at perturbative level all F-terms of the complex structure moduli are zero with $W_0 \neq 0$ and a flat axion. Instanton corrections to the superpotential lift the axion and shift all the remaining moduli, so that the corresponding F-terms become non-zero and can act as a dS uplifting source by an appropriate tuning of background fluxes [38].

From a more theoretical point of view, developments under the name 'the tadpole problem' [41–45] seem to suggest that flat directions of the GVW superpotential might be a generic feature when the number of complex structure moduli is large. Thus classifying and studying the precise nature of flat directions, together with finding mechanisms for lifting them, are needed to develop a comprehensive understanding of the string landscape.

Understanding supersymmetric effective field theories (EFT) from string compactifications is key in order to determine most of the relevant physical implications of these frameworks. These EFTs are only known approximately, and corrections to leading order effects play an important role for the most pressing questions such as moduli stabilization and inflation from string theory.

These effects correspond to non-perturbative contributions to the superpotential W, and perturbative and non-perturbative corrections to the Kähler potential K, both in the α' and string-loop expansions. These corrections to K and W modify the standard F-term part of the scalar potential which comes from the square of the auxiliary fields at order F^2. However, there are also higher derivative F^4 corrections to the scalar potential. In the type IIB case, they have an explicit linear dependence on the two-cycle volume moduli t^i, $i = 1, ..., h^{1,1}$, and the overall volume $\mathcal{V}$ of the Calabi-Yau (CY) threefold X [46–48]:

$$V_{F^4} = \frac{\gamma}{\mathcal{V}^4} \sum_{i=1}^{h^{1,1}} \Pi_i t^i \,, \tag{3}$$

where γ is a computable constant (independent of Kähler moduli) and $\Pi_i = \int_X c_2 \wedge \hat{D}_i$ with c_2 the CY second Chern-class and $\hat{D}_i$ a basis of harmonic (1,1)-forms dual to the divisors D_i. In terms of this basis, the Kähler form J can be written as $J = t^i \hat{D}_i$.

The relevance of the corrections (3) is manifest especially for determining the structure of the scalar potential since, due to the no-scale property, the leading order, tree-level, contribution vanishes, and therefore a combination of subleading corrections has to be considered. However, these higher-derivative corrections are naturally subdominant compared with the leading order α'^3 correction at order F^2 that scales with the volume as $V_{\alpha'^3} \simeq |W_0|^2/\mathcal{V}^3$. In this sense they should not substantially modify moduli stabilization mechanisms such as KKLT and the Large Volume Scenario (LVS). However, they can play a crucial role for:

1. Lifting flat directions which are not stabilized at leading LVS order [47, 48];

2. Modifying slow-roll conditions needed for inflationary scenarios where the leading order effects leave an almost flat direction for the inflaton field.

In Chap. 3 we will concentrate on the second item, and determine under which topological conditions these higher derivative corrections vanish. For cases where they are instead non-zero, we will numerically estimate the largest value of their prefactor γ which does not ruin the flatness of the inflationary potential of different inflation models derived in the LVS framework such as blow-up inflation [30, 49, 50], fibre inflation [51–58] and poly-instanton inflation [59–63].

Using the Kreuzer-Skarke database of four-dimensional reflexive polytopes [64] and their triangulated CY database [65], we present scanning results for a set of divisor topologies corresponding to CY threefolds with $1 \leq h^{1,1} \leq 5$. These divisor topologies are relevant for various phenomenological purposes in LVS models. For inflationary model building, this includes, for example: (i) the (diagonal) del Pezzo divisors needed for generating non-perturbative superpotential corrections useful for blow-up inflation, (ii) the K3-fibration structure relevant for fibre inflation, and (iii) the so-called 'Wilson' divisors which are relevant for realizing poly-instanton inflation. In addition, we present a class of divisors which have vanishing Π.

In Chap. 4 we address the Hubble tension problem, and one candidate solution. The Hubble constant H_0, as inferred from Planck 2018 Cosmic Microwave Background (CMB) data [66], is in 5σ disagreement with the SH0ES cosmic distance ladder measurement [67]. This 'Hubble tension' has spurred on an intense experimental effort and the development of new ways to measure H_0 (see [68, 69] for reviews). The tension persists between varied early and late universe probes at the level of 4-6σ [69]. A commensurate effort has been made on the theory side, aimed at developing an alternative cosmological model to bring these measurements into agreement. Amongst the theory approaches, the modification of early universe physics holds particular promise (see [70]) by satisfying, first and foremost, the tight constraints that the CMB places on any new cosmological physics. A detailed review is provided in Sec. 4.2 (see also the review section of [71]).

Early Dark Energy (EDE) [72, 73] is an example of new physics in the early universe that resolves the Hubble tension by bringing the CMB inference in agreement with SH0ES, while leaving the former nearly indistinguishable from ΛCDM. The model proposed in [72] utilizes a scalar field with potential energy $V(\varphi) = V_0 \left[1 - \cos\left(\varphi/f\right)\right]^3$, featuring an exponent that distinguishes it from the conventional potential of an axion-like particle. This potential is motivated by data: it provides a significantly better fit to the data than a monomial $V \sim \varphi^{2n}$ [74] or a cosine with a different exponent [72]. The vast majority of work on EDE (see e.g. [71, 75, 76]) has therefore focused on this form of the potential, though alternative EDE-like models abound [77–91]. This work has elucidated challenges to the model from data, in particular, tension with large scale structure (see e.g. [75, 76]), that has motivated extensions of the EDE model, see [89, 92–94], to include an additional ultralight axion dark matter component [93, 94]. Relatively little input has come from the formal theory community, with exception of Refs. [71] and [95].

Phenomenologically attractive string compactifications can be obtained by turning on background fluxes [6–10]. In the type IIB setting, the effect of fluxes is to stabilize the complex structure moduli and axio-dilaton [10]. Furthermore, in type IIB there are various scenarios for the stabilization of the Kähler moduli [18, 27, 29, 40, 48, 96–102]. This has made type IIB flux compactifications the setting for various detailed phenomenological explorations.

The introduction of fluxes also leads to the possibility of a large multitude of solutions. Apart from construction of detailed models, string phenomenology involves developing an understanding of the broad properties of vacua. The later has motivated the statistical approach to string phenomenology [20, 103–107] (see also [108–133]). The distribution of the scale of supersymmetry breaking in the space of string vacua is of course of much interest (see [109, 123–128] for early work in this direction).

This manuscript is structured as follows:

Chap. 1: In Sec. 1.1 we review the Type IIB compactification setup. After that, in Sec. 1.2 we delve into the details of stabilization of complex structure moduli, presenting the tadpole condition and toroidal orientifolds. In Sec. 1.3 we review the different constructions to stabilize Kähler moduli and to subsequently uplift the potential to Minkowski/dS. Lastly, in Sec 1.5 we provide a brief overview of axions in string theory.

Chap. 2: In Sec. 2.1 we review the significance of flat directions in flux vacua solutions, present the outline of our strategy and the main results of the study. Particularly, we describe the choices of flux quanta that induce relations between the flux superpotential and its derivatives. Sec. 2.2 focuses on the toroidal case, presenting solutions with $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetry and arbitrarily weak coupling. We classify these solutions based on duality equivalences. Sec. 2.3 focuses instead on orientifolded CYs in the large complex structure limit. We give a detailed treatment of the $\mathbb{CP}_{[1,1,1,6,9]}[18]$ example and a preliminary analysis of the CY studied in [134] which features effectively 3 complex structure moduli.

Chap. 3: In Sec. 3.1 we presents a brief review of LVS moduli stabilization and the role of divisor topologies in LVS phenomenology. Subsequently we present a classification of the divisor topologies relevant for taming higher derivative F^4 corrections in Sec. 3.2. Sec. 3.3.1 discusses instead potential candidate CYs for realizing global embeddings of blow-up inflation and the effect of F^4 corrections on these models. The analysis of higher derivative corrections to LVS inflation models is continued in Sec. 3.3.6 which is devoted to fibre inflation, and in Sec. 3.3.11 which focuses on poly-instanton inflation.

Chap. 4: In Sec. 4.1 we present the problem of the Hubble tension while in Sec. 4.2 we introduce the concept of Early Dark Energy (EDE) and its relevance to this problem. Subsequently, we discuss the theoretical framework, including the identification of the EDE field as a C_4 or C_2 axion and the stabilization of closed string moduli within the KKLT construction in Sec. 4.3 or in the LVS in Sec. 4.4, presenting the construction of EDE models, focusing on achieving a natural hierarchy between the EDE energy scale and other fields.

Chap. 5: In Sec. 5.1 we present the joint distribution of the gravitino mass and the cosmological constant in the context of anti D3-brane uplift, with separate analyses for KKLT and LVS models, focusing on the KKLT scenario, providing both analytical estimates and numerical results and subsequently, addressing the LVS scenario, detailing the statistical properties and implications for supersymmetry breaking scales. In Sec. 5.2 we repeat a similar analysis employing the complex structure uplift approach.

Chap. 6: Finally, we summarize our study and discuss both phenomenological aspects of our key findings as well as new possible line of research and further pursuit of the concepts studied.

Notation: Throughout this work we will set the reduced Planck mass $M_P = 1$ unless explicitly stated.

Chapter I

Type IIB Flux Compactification and Moduli stabilization

Type IIB string theory is one of the five superstring theories that have been extensively studied in recent years. A notable feature of the effective field theory (EFT) at low energies is the presence of several moduli—scalar fields whose potential is vanishing. If this is the case, they could mediate a previously unseen fifth force. Therefore, it is essential to construct mechanisms that generate a potential, thereby giving mass to these fields. In this chapter, we will analyse such mechanisms and, more broadly, models that achieve complete stabilization, ultimately uplifting the potential to a de Sitter (dS) space. Finally, we will examine in detail the role of axions in the EFT and their associated potentials.

1.1 Type IIB Compactification

Let us review some of the key ingredients needed for the compactification of the moduli. We will follow the work done in [135]. The $D = 10$, $\mathcal{N} = 2$ bosonic effective action of type IIB string theory reads, in the string frame

$$S_{10}^{(s)} = \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x \sqrt{-\hat{g}} \Big[e^{-2\hat{\phi}} \left(R + 4(\partial\hat{\phi})^2 - \frac{1}{2}|\hat{H}_3|^2 \right) - \frac{1}{2}|\hat{F}_1|^2 - \frac{1}{2}|\hat{\tilde{F}}_3|^2 - \frac{1}{2}|\hat{\tilde{F}}_5|^2 \Big] , \tag{1.1}$$

where the hat denotes 10D fields, and the field content is given by

- the metric $\hat{g}_{MN}$ and its determinant $\hat{g} = \det\{\hat{g}_{MN}\}$
- dilaton $\hat{\phi}$
- Kalb-Ramond 2-form $\hat{B}_2$ and its field strength $\hat{H}_3 = \mathrm{d}\hat{B}_2$

- Ramond-Ramond p-forms ($p = 0, 2, 4$) $\hat{C}_p$ and their field strength $\hat{F}_{p+1} = \mathrm{d}\hat{C}_p$

and the definitions

$$\hat{\tilde{F}}_3 = \hat{F}_3 - \hat{C}_0 \hat{H}_3 \tag{1.2}$$

$$\hat{\tilde{F}}_5 = \hat{F}_5 + \frac{1}{2}\hat{H}_3 \,, \wedge \hat{C}_2 + \frac{1}{2}\hat{B}_2 \wedge \hat{F}_3 \,, \tag{1.3}$$

and $2\kappa_{10}^2 = (2\pi)^7 (\alpha')^4$. It is possible then to go to the Einstein frame by introducing the axio-dilaton

$$\hat{S} = e^{-\hat{\phi}} + \mathrm{i}\hat{C}_0 = \frac{1}{g_s} + \mathrm{i}\hat{C}_0 \,, \tag{1.4}$$

and the 3-form

$$\hat{G}_3 = \hat{F}_3 - \mathrm{i}\bar{\hat{S}}\hat{H}_3 \,, \tag{1.5}$$

and sending the metric to

$$\hat{g}_{MN} \to \hat{g}_{MN} e^{\hat{\phi}/2} \,, \tag{1.6}$$

leading to

$$S_{10}^{(E)} = \frac{1}{2\kappa_{10}^2} \int \left[R * 1 + \frac{\mathrm{d}\hat{S} \wedge * \mathrm{d}\hat{S}}{2\,(\mathrm{Re}\,\hat{S})^2} - \frac{\hat{G}_3 \wedge * \bar{\hat{G}}_3}{2\,\mathrm{Re}\,\hat{S}} - \frac{1}{4}\hat{\tilde{F}}_5 \wedge * \hat{\tilde{F}}_5 + \frac{\hat{C}_4 \wedge \hat{G}_3 \wedge \bar{\hat{G}}_3}{4\mathrm{i}\,\mathrm{Re}\,\hat{S}} \right] \,, \tag{1.7}$$

to which one must always provide the self-duality constraint of the 5-form imposed on the solution

$$\hat{\tilde{F}}_5 = * \hat{\tilde{F}}_5 \,. \tag{1.8}$$

One can now proceed with the general prescription of compactification on a Calabi-Yau manifold, while the simpler toroidal case will be analysed in a subsequent section. This choice ensures that we obtain a $D = 4$, $\mathcal{N} = 2$ SUSY EFT after the compactification. Assuming that the space can be factorized as $\mathcal{M}^{10} = \mathcal{M}^{1,3} \times \mathcal{X}_6$, the space-time interval takes a diagonal form

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + g_{i\bar{\jmath}}\,\mathrm{d}y^i\,\mathrm{d}\bar{y}^{\bar{\jmath}} \,, \tag{1.9}$$

where we have denoted by x the coordinates of the Minkowski space $\mathcal{M}^{1,3}$ and by y the coordinates of the Calabi Yau $\mathcal{X}_6$. Thus, introducing the Kähler form

$$J = \mathrm{i} g_{i\bar{\jmath}}\,\mathrm{d}y^i\,\mathrm{d}\bar{y}^{\bar{\jmath}} \,, \tag{1.10}$$

we obtain $h^{1,1}$ scalars v^A via the expansion

$$J = t^a(x)\omega_a \,, \qquad a = 1, ..., h^{1,1} \,, \tag{1.11}$$

with the basis $\omega_a \in H^{1,1}(\mathcal{X}_6)$. Instead, deforming the complex structure we obtain the complex structure moduli z^K as

$$\delta g_{i\bar{j}} = \frac{\mathrm{i}}{||\Omega||^2}\bar{z}^K(\bar{\chi}_I)_{i\bar{\imath}\bar{\jmath}}\Omega^{\bar{\imath}\bar{\jmath}}{}_j\,, \quad I = 1, ..., h^{1,2} \tag{1.12}$$

with Ω the unique holomorphic $(3,0)$-form, the basis $\bar{\chi}_I \in H^{2,1}(\mathcal{X}_6)$ and the definition $||\Omega||^2 = \frac{1}{3!}\Omega_{IJK}\bar{\Omega}^{IJK}$. With this knowledge we can expand in harmonics the fields of the theory obtaining

$$\hat{B}_2 = B_2(x) + b^a(x)\omega_a\,, \qquad \hat{C}_2 = C_2(x) + c^a(x)\omega_a\,, \tag{1.13}$$

$$\hat{C}_4 = D_2^a(x)\wedge\omega_a + V^I(x)\wedge\alpha_I - U_J(x)\wedge\beta^J - \theta_a(x)\tilde{\omega}^a\,, \qquad I,J = 0, ..., h^{1,2}\,, \tag{1.14}$$

with $\tilde{\omega}^a$ being the dual basis to ω_a and the symplectic basis $(\alpha_I, \beta^J) \in H^3(\mathcal{X}_6)$ satisfying

$$\int \alpha_I \wedge \beta^J = \delta_I^J\,, \quad \int \alpha_I \wedge \alpha_J = \int \beta^I \wedge \beta^J = 0\,. \tag{1.15}$$

Moreover, by imposing the self-duality of $\tilde{F}_5$ one can drop half of the d.o.f. of C_4, for example choosing to keep only the scalars θ_a and the vector field V^I. Now, adding the metric $g_{\mu\nu}$ and the scalar fields S and C_0 we can arrange the field content into $\mathcal{N}=2$, $D=4$ SUSY supermultiplets as shown in Tab.1.1. This leads to the action, setting units to one for

multiplet	#	fields
gravity	1	$(g_{\mu\nu}, V^0)$
vector	$h^{1,2}$	(V^I, z^I)
hyper	$h^{1,1}$	$(t^a, b^a, c^a, \theta_a)$
double-tensor	1	(B_2, C_2, ϕ, C_0)

Table 1.1. $\mathcal{N}=2$ supermultiplets for Type IIB string theory compactified on a Calabi-Yau manifold.

simplicity

$$S_4^{(\mathcal{N}=2)} = \int \Bigg[\frac{1}{2}R * 1 + \frac{1}{2}\mathrm{Re}\, M_{IJ}\, F^I \wedge F^J + \frac{1}{4}\mathrm{Im}\, M_{IJ} F^I \wedge *F^J - K_{I\bar{J}}\,\mathrm{d}z^I \wedge *\,\mathrm{d}\bar{z}^{\bar{J}} - h_{A\bar{B}}\,\mathrm{d}q^A \wedge *\,\mathrm{d}\bar{q}^{\bar{q}} \Bigg]\,, \tag{1.16}$$

where we have introduced the following quantities:

- Period matrix

$$M_{IJ} = \bar{\mathcal{F}}_{IJ} + 2\mathrm{i}\frac{(\mathrm{Im}\,\mathcal{F})_{IK}X^K(\mathrm{Im}\,\mathcal{F})_{JM}X^M}{X^M(\mathrm{Im}\,\mathcal{F})_{MN}X^N}\,, \tag{1.17}$$

with the periods

$$X^A = \int_{X_6} \Omega\wedge\beta^A\,, \qquad \mathcal{F}_A = \int_{X_6}\Omega\wedge\alpha_A\,, \tag{1.18}$$

and $\mathcal{F}_{AB} = \partial_{X^B}\mathcal{F}_A$, with the choice of coordinates $X^A = (1, z^A)$.

- Metric of Kähler deformations

$$K_{a\bar{b}} = \frac{\partial^2 K_{\mathrm{cs}}}{\partial z^a \partial \bar{z}^{\bar{b}}}\,, \tag{1.19}$$

with the Kähler potential given by

$$K_{\mathrm{cs}} = -\ln\left[-\mathrm{i}\int_{\mathcal{X}}\Omega\wedge\Omega\right] = -\ln\left[-\mathrm{i}\left(\bar{X}^I\mathcal{F}_I - X^I\bar{\mathcal{F}}_I\right)\right]\,, \tag{1.20}$$

where we have employed the expansion of the holomorphic 3-form Ω.

- The quaternionic metric h_{AB} whose coordinates q^A encompasses all $h^{1,1}+1$ fields of the hypermultiplets, having dualized the tensor-multiplet fields B_2 and C_2 to scalars.

1.2 Complex Structure Moduli stabilization

We now want to generate a potential for the moduli in order to give them a mass. First of all we have derived only the $\mathcal{N}=2$ SUSY theory in 4D where the superpotential is identically zero. The next step would be orientifolding and thus obtain the $\mathcal{N}=1$ SUSY theory which we will now analyse in detail.

The general prescription is to introduce the worldsheet parity operator Ω_p and an internal symmetry σ which leaves invariant J but transforms non trivially the holomorphic 3-form Ω [136-138]. We can construct two types of symmetry operator, namely

$$\text{O3/O7 planes:}\qquad \mathcal{O}_1 = (-1)^{F_L}\Omega_p\sigma^*\,, \qquad \sigma^*\Omega = -\Omega\,, \tag{1.21}$$

$$\text{O5/O9 planes:}\qquad \mathcal{O}_2 = \Omega_p\sigma^*\,, \qquad \sigma^*\Omega = \Omega\,. \tag{1.22}$$

We will focus on the first type of orientifolds. Moreover, the action of the operators is here summarized

$$\begin{aligned}
\sigma^* \quad &: (\hat\phi, \hat g, \hat B_2, \hat C_0, \hat C_2, \hat C_4) \mapsto (+\hat\phi, +\hat g, -\hat B_2, +\hat C_0, -\hat C_2, +\hat C_4)\,, && (1.23)\\
(-1)^{F_L} \quad &: (\hat\phi, \hat g, \hat B_2, \hat C_0, \hat C_2, \hat C_4) \mapsto (+\hat\phi, +\hat g, +\hat B_2, -\hat C_0, -\hat C_2, -\hat C_4)\,, && (1.24)\\
\Omega_p \quad &: (\hat\phi, \hat g, \hat B_2, \hat C_0, \hat C_2, \hat C_4) \mapsto (+\hat\phi, +\hat g, -\hat B_2, -\hat C_0, +\hat C_2, -\hat C_4)\,. && (1.25)
\end{aligned}$$

Thus, under the action of σ^* we split the cohomology groups as

$$H^{p,q} = H^{p,q}_+ \oplus H^{p,q}_- \,, \tag{1.26}$$

and the expansions of the fields is now given by

$$\begin{aligned}
&\hat B_2 = b^{a_-}(x)\omega_{a_-}\,, \qquad \hat C_2 = c^{a_-}(x)\omega_{a_-}\,, \qquad a_- = 1,...,h^{1,1}_-\,, && (1.27)\\
&\hat C_4 = D_2^{a_+}(x)\wedge\omega_{a_+} + V^{I_*}(x)\wedge\alpha_{I_+} - U_{J_+}(x)\wedge\beta^{J_+} - \theta_{a_+}(x)\tilde\omega^{a_+}\\
&\qquad\qquad\qquad\qquad a_+ = 1,...,h^{1,1}_+\,, \quad I_+, J_+ = 1,...,h^{1,2}_+\,,
\end{aligned}$$

and we can arrange now the field content into $\mathcal{N} = 1$, $D = 4$ SUSY supermultiplets as shown in in Tab.1.2. The action obtained is then written

multiplet	#	fields
gravity	1	$g_{\mu\nu}$
vector	$h^{1,2}_+$	V^{I_+}
chiral	$h^{1,2}_-$	z^{I_-}
	1	(ϕ, C_0)
	$h^{1,1}_-$	(b^{a_-}, c^{a_-})
chiral/linear	$h^{1,1}_+$	(t^{a_+}, θ_{a_+})

Table 1.2. $\mathcal{N} = 1$ supermultiplets for Type IIB string theory compactified on a Calabi-Yau manifold with O3/O7 orientifolds.

as $\mathcal{N} = 1$, $D = 4$ SUGRA EFT

$$\begin{aligned}
S_4^{(\mathcal{N}=1)} = -\int \Big[&\frac{1}{2}R * 1 + K_{i\bar\jmath}\mathcal{D}\mu^i \wedge *\mathcal{D}\bar\mu^{\bar\jmath} && (1.28)\\
&+ \frac{1}{2}\mathrm{Re}\, f_{kl}\, F^k \wedge *F^l + \frac{1}{2}\mathrm{Im}\, f_{kl}\, F^k \wedge F^l + V * 1\,, \Big]\,,
\end{aligned}$$

where we denote collectively by μ^i all the complex scalars, $F^k = \mathrm{d}V^k$, $\mathcal{D}_i = \partial_i + K_i$, where

$$K_i = \partial_i K\,, \qquad K_{i\bar\jmath} = \partial_i\partial_{\bar\jmath}K\,, \qquad K^{i\bar\jmath}K_{\bar\jmath k} = \delta^i_k\,, \tag{1.29}$$

and defined the correct chiral coordinates

$$S = e^{-\phi} + \mathrm{i}C_0 \equiv s + \mathrm{i}C_0\,, \tag{1.30}$$
$$G^{a_-} = \bar{S}b^{a_-} + \mathrm{i}c^{a_-}\,, \tag{1.31}$$
$$\tau_{a_+} = \frac{1}{2}k_{a_+b_+c_+}t^{b_+}t^{c_+}\,, \tag{1.32}$$
$$T_{a_+} = \tau_{a_+} + \mathrm{i}\theta_{a_+} - \frac{1}{2}\mathcal{K}_{a_+a_-b_-}\frac{G^{a_-}(G+\bar{G})^{b_-}}{S+\bar{S}}\,, \tag{1.33}$$

where

$$\mathcal{K}_{a_\pm b_\pm c_\pm} = \int_{\mathcal{X}} \omega_{a_\pm}\wedge\omega_{b_\pm}\wedge\omega_{c_\pm}\,, \tag{1.34}$$
$$\mathcal{K}_{a_\pm b_\pm} = k_{a_\pm b_\pm a_+}t^{a_+}\,, \tag{1.35}$$
$$\mathcal{K}_{a_+} = k_{a_+b_+c_+}t^{b_+}t^{c_+}\,, \tag{1.36}$$
$$\mathcal{K}_{a_+} = k_{a_+b_+c_+}t^{b_+}t^{c_+}\,, \tag{1.37}$$
$$\mathcal{K} = k_{a_+b_+c_+}t^{a_+}t^{b_+}t^{c_+} = 6\,\mathsf{Vol}\,(\mathcal{X}_6) \equiv 6\mathcal{V}\,, \tag{1.38}$$
$$\mathcal{K}_{a_+b_+a_-} = \mathcal{K}_{a_-b_-c_-} = \mathcal{K}_{a_+a_-} = \mathcal{K}_{a_-} = 0\,, \tag{1.39}$$

we have the Kähler potential

$$K = K_{\mathsf{cs}}(z,\bar{z}) + K_S(S,\bar{S}) + K_{\mathsf{k}}(S,\bar{S},T,\bar{T},G,\bar{G})\,, \tag{1.40}$$
$$K_{\mathsf{cs}} = -\ln\left[-\mathrm{i}\int\Omega(z)\wedge\bar{\Omega}(\bar{z})\right]\,, \tag{1.41}$$
$$K_S(S,\bar{S}) = -\ln\left(S+\bar{S}\right)\,, \tag{1.42}$$
$$K_{\mathsf{k}} = -2\ln\mathcal{V}\,, \tag{1.43}$$

noting that the K_{k} satisfies the no-scale condition

$$K_i K^{i\bar{j}} K_{\bar{j}} = 4\,, \tag{1.44}$$

and the Gukov-Vafa-Witten superpotential [8]

$$W_{\mathsf{GVW}}(S,z) = \int_{\mathcal{X}} \Omega\wedge G_3\,, \tag{1.45}$$

leading to the potential V

$$V = e^K\left(\mathcal{D}_iWK^{i\bar{j}}\overline{\mathcal{D}_jW} - 3|W|^2\right) + \frac{1}{2}\left(\mathsf{Re}\,f^{-1}\right)^{kl}D_kD_l\,, \tag{1.46}$$

and lastly the gauge kinetic function given by

$$f_{kl} = -\frac{\mathrm{i}}{2}\bar{M}_{kl}\bigg|_{z=0=\bar{z}} = -\frac{\mathrm{i}}{2}\mathcal{F}_{kl}\bigg|_{z=0=\bar{z}}\,. \tag{1.47}$$

At this stage it is immediate to see that the potential is generated only for the complex structure moduli and for the axio-dilaton, leaving the Kähler moduli massless. However, the superpotential W is identically zero except in the case where one allows for background fluxes for G_3.

The procedure to stabilize the complex structure moduli is here summarized [134, 135]. First of all, we construct the period vector Π from the expansion of $\Omega = X^A\alpha_A - \mathcal{F}_A\beta^A$

$$\Pi = \begin{pmatrix} \mathcal{F}_A \\ X^A \end{pmatrix} , \qquad A = 0, ..., h^{1,2} , \tag{1.48}$$

with $\mathcal{F}_A = \partial_{X^A}\mathcal{F}$. The prepotential $\mathcal{F}$ is a homogeneous degree two function in the projective coordinates X^A, thus we can introduce the function F as

$$F(U^i) = (X^0)^{-2}\mathcal{F} , \tag{1.49}$$

with the introduction of the flat coordinates

$$U^i = \frac{X^i}{X^0} , \qquad i = 1, ..., h^{1.2} , \tag{1.50}$$

giving

$$\Pi = \begin{pmatrix} \mathcal{F}_0 \\ \mathcal{F}_i \\ X^0 \\ X^i \end{pmatrix} = \begin{pmatrix} 2F - U^iF_i \\ F_i \\ 1 \\ U^i \end{pmatrix} \tag{1.51}$$

where we have again chosen the normalization of Ω by setting $X^0 = 1$. We now expand F_3 and H_3 in the symplectic basis

$$F_3 = a^A\alpha_A - b_A\beta^A , \tag{1.52}$$

$$H_3 = c^A\alpha_A - d_A\beta^A , \tag{1.53}$$

$$\Longrightarrow G_3 = (a^A - \mathrm{i}\bar{S}c^A)\alpha_A - (b_A - \mathrm{i}\bar{S}d_A)\beta^A , \tag{1.54}$$

from which one can introduce the vectors

$$f = \begin{pmatrix} b^A \\ a^A \end{pmatrix} , \quad h = \begin{pmatrix} d^A \\ c^A \end{pmatrix} . \tag{1.55}$$

It is crucial to note that these quantities follow the Dirac quantization and are thus integer valued

$$\frac{1}{(2\pi)^2\alpha'}\int_\gamma F_3 \in \mathbb{Z} , \qquad \frac{1}{(2\pi)^2\alpha'}\int_\gamma H_3 \in \mathbb{Z} . \tag{1.56}$$

Hence, we obtain the superpotential

$$W = (f - \mathrm{i}\bar{S}h)^t \cdot \Sigma \cdot \Pi\,, \tag{1.57}$$

and the Kähler potential

$$K_{\text{cs}} = -\ln\left(-\mathrm{i}\Pi^\dagger \cdot \Sigma \cdot \Pi\right)\,. \tag{1.58}$$

From the point of view of a $\mathcal{N} = 1$, $D = 4$ SUGRA EFT the potential generated from the superpotential (1.45), ignoring D-terms, is [139]

$$V_F = \frac{3\mathrm{i}}{2g_s \int \Omega \wedge \bar{\Omega}} \left[\int G_3 \wedge \bar{\Omega} \int \bar{G}_3 \wedge \Omega + K^{i\bar{\jmath}} \int G_3 \wedge \chi_i \int \bar{G}_3 \wedge \bar{\chi}_{\bar{\jmath}}\right]\,, \tag{1.59}$$

where χ_i is the basis of $H^{1,2}$. Moreover, denoting collectively the axiodilaton and the complex structure moduli by $\zeta_A = \{S, z^a\}$ and the Kähler moduli by Ξ_B we have the following properties

$$W = W(\zeta_A)\,, \qquad \frac{\partial W}{\partial \Xi_B} = 0\,, \qquad K_B K^{B\overline{B}} \overline{K}_{\overline{B}} = 3\,, \tag{1.60}$$

leading to the scalar potential

$$V_F = e^K \left[W_A K^{A\overline{A}} \overline{W}_{\overline{A}} + 2\text{Re}\left(\overline{W} W_A K^{A\overline{B}} \overline{K}_{\overline{B}}\right)\right] \geq 0\,. \tag{1.61}$$

Finally, in order to ensure that the solution is supersymmetric, i.e. the vanishing of W and of K, one must impose [140]

$$D_{\bar{S}} W = -\frac{1}{S + \overline{S}} \int \overline{G}_3 \wedge \Omega + -K_{ab} b^a b^b W = 0\,, \tag{1.62}$$

$$D_{z^k} W = \int G_3 \wedge \chi_k = 0\,, \tag{1.63}$$

$$D_{T_a} W = -2W \frac{t^a}{\mathcal{K}} = 0\,, \tag{1.64}$$

$$D_{G^a} W = 2\mathrm{i} K_{ab} b^b W = 0\,, \tag{1.65}$$

which leads to

$$\int G_3 \wedge \Omega = \int \overline{G}_3 \wedge \Omega = \int G_3 \wedge \chi_k = 0\,, \tag{1.66}$$

and, decomposing G_3 into terms of $H^{p,q}$

$$G_3 = G_3^{3,0} + G_3^{2,1} + G_3^{1,2} + G_3^{0,3}\,, \tag{1.67}$$

we find that the SUSY conditions are satisfied if

$$G_3^{3,0} = G_3^{0,3} = G_3^{1,2} = 0\,, \tag{1.68}$$

implying that G_3 is a $(2,1)$-form which satisfies the ISD condition and is primitive, i.e.

$$G_3 \wedge J = 0\,. \tag{1.69}$$

1.2.1 D3-tadpole cancellation

It is now necessary to check the consistency of the setup by ensuring the cancellation of the RR tadpole. In fact, adding 3-form background fluxes, one modifies the Bianchi identity of the RR 5-form as

$$\mathrm{d}\hat{\tilde{F}}_5 = \hat{H}_3 \wedge \hat{F}_3 + 2\kappa_{10}^2 \mu_3 \rho_3^{\text{local}} \,, \tag{1.70}$$

with contributions to the density ρ_3^{local} coming from O3-planes and D3-branes, and

$$\mu_p = \frac{(\alpha')^{-\frac{p+1}{2}}}{(2\pi)^p} \,. \tag{1.71}$$

In general, a Dp-brane that fills the directions X^i with $i = 0, \ldots, p$ and is transverse to the directions x^j with $j = p+1, \ldots, 9$, is charged under the $(p+1)$-form $\hat{C}_{p+1}$ and satisfies the following action

$$S = -\frac{1}{2} \int_{\mathcal{M}} \hat{F}_{p+2} \wedge *\hat{F}_{p+2} + \mu_p^{\text{tot}} \int_{\mathcal{W}_p} \hat{C}_{p+1} \,, \tag{1.72}$$

with $\mathcal{W}_p$ its worldvolume. Hence, introducing the current for the brane as

$$J_p = \delta^{9-p}(x^j)\, \mathrm{d}x^{p+1} \wedge \ldots \wedge \mathrm{d}x^9 \,, \tag{1.73}$$

we obtain the generalized Gauss law for an extended charged object with charge density ρ_p, with eom

$$\mathrm{d} * \hat{F}_{p+2} = (-1)^p \mu_p^{\text{tot}} J_p = (-1)^p \rho_p \,. \tag{1.74}$$

Thus, taking a compact and unbounded space for the transverse directions, integrating the left-hand side will yield a vanishing result, which imposes that the total charge must also be zero.

Let us recall what objects are charged under the $\hat{C}_{p+1}$ RR-form:

- Dp-branes: following from the Chern-Simons action

$$S_{CS} = \mu_p \int_{\mathcal{W}_p} \hat{C}_{p+1} + \ldots \,, \tag{1.75}$$

  we find the charge μ_p.

- $\overline{\mathrm{D}p}$-branes: opposite charge of Dp-branes, i.e. $-\mu_p$.

- Op-planes [141]: these objects are non-dynamical, and their charge is computed by invoking certain string dualities, resulting in $-2^{p-4}\mu_p$. In this calculation, we are considering branes in the upstairs geometry, where branes and their reflections under Z_2 are treated as two distinct objects with the same charge.

Now coming back to our specific case with D3-branes and O3-planes, the total 10D action is given by

$$S_{10}^{\text{flux}} = S_{10}^{E} + Q_3 \int_{M^{1,3}} \left(\frac{1}{2}\hat{C}_4 - *_4 1\right) , \tag{1.76}$$

with

$$Q_3 = \mu_3 \left(N_{\text{D3}} - \frac{1}{2} N_{\text{O3}}\right) \tag{1.77}$$

and the $1/2$ factor is introduced due to the self-duality of $\hat{\tilde{F}}_5$. Thus, taking once again the Chern-Simons action for type IIB string theory

$$S_{CS} = \frac{1}{2\kappa_{10}^2} \int \frac{\hat{C}_4 \wedge \hat{G}_3 \wedge \bar{\hat{G}}_3}{4\text{i}\,\text{Re}\,\hat{S}} = \frac{1}{2\kappa_{10}} \int \frac{1}{2} \hat{C}_4 \wedge \hat{F}_3 \wedge \hat{H}_3 \,, \tag{1.78}$$

we obtain that the tadpole cancellation condition can be expressed as

$$N_{\text{D3}} - \frac{1}{2} N_{\text{O3}} + N_{\text{flux}} = 0 \,, \tag{1.79}$$

with

$$N_{\text{flux}} = \frac{1}{2\kappa_{10}\mu_3} \int_{\mathcal{X}} \hat{H}_3 \wedge \hat{F}_3 = \frac{1}{(\alpha')^2 (2\pi)^4} \int_{\mathcal{X}} \hat{H}_3 \wedge \hat{F}_3 \,. \tag{1.80}$$

Moreover, the 3-form term generates a potential in the $D = 4$ EFT given by

$$V_G = \frac{1}{2\kappa_{10}^2} \int_{X_6} \frac{\hat{G}_3 \wedge *\overline{\hat{G}}_3}{4\text{Re}\,\hat{S}} \,, \tag{1.81}$$

and we can furthermore split $\hat{G}_3$ into imaginary self-dual (ISD) and imaginary anti-self-dual (IASD) components, where the dual operation must be intended restricted to the compact 6D space [142]

$$\hat{G}_3 = \hat{G}_3^{\text{(ISD)}} + \hat{G}_3^{\text{(IASD)}} \,, \qquad \begin{cases} \hat{G}_3^{\text{(ISD)}} = \text{i}\hat{G}_3^{\text{(ISD)}} \\ \hat{G}_3^{\text{(IASD)}} = -\text{i}\hat{G}_3^{\text{(IASD)}} \end{cases} \,, \tag{1.82}$$

leading to

$$V_G = \frac{1}{\kappa_{10}^2} \int_{X_6} \frac{2\hat{G}_3^{\text{(ISD)}} \wedge *\hat{G}_3^{\text{(ISD)}} + \text{i}\hat{G}_3 \wedge \overline{\hat{G}}_3}{4\text{Re}\,\hat{S}} \tag{1.83}$$

$$= \frac{1}{\kappa_{10}^2} \int_{X_6} \frac{2\hat{G}_3^{\text{(IASD)}} \wedge *\hat{G}_3^{\text{(IASD)}} - \text{i}\hat{G}_3 \wedge \overline{\hat{G}}_3}{4\text{Re}\,\hat{S}} \,, \tag{1.84}$$

where the last term is a topological invariant given by

$$\frac{1}{\kappa_{10}^2} \int_{X_6} \frac{\hat{G}_3 \wedge \overline{\hat{G}}_3}{4\text{i}\,\text{Re}\,\hat{S}} = \mu_3 N_{\text{flux}} \,, \tag{1.85}$$

allowing us to rewrite

$$V_G = V^{\mathsf{ISD}} - \mu_3 N_{\mathsf{flux}} = V^{\mathsf{IASD}} + \mu_3 N_{\mathsf{flux}} \,. \tag{1.86}$$

Thus, the total potential also receives a contribution from the second term in (1.76), giving

$$V = V_G + Q_3 = V_G + \mu_3 N_{\mathsf{flux}} = V_G^{\mathsf{ISD}} = V_G^{\mathsf{IASD}} + 2\mu_3 N_{\mathsf{flux}} \,, \tag{1.87}$$

meaning that the potential is identically zero if $\hat{G}_3$ is IASD while it is non-negative if it is ISD. Moreover, the contribution from N_{flux} can be shown to be positive in the ISD case, since

$$*\hat{G}_3 = \mathrm{i}\left(\hat{F}_3 + \hat{C}_0\hat{H}_3 - \frac{\mathrm{i}}{g_s}\hat{H}_3\right) \implies *\left(\frac{\hat{H}_3}{g_s}\right) = -\hat{F}_3 + \hat{C}_0\hat{H}_3 \,, \tag{1.88}$$

thus leading to

$$(2\pi)^4(\alpha')^2 N_{\mathsf{flux}} = \int_{X_6} \hat{H}_3 \wedge \hat{F}_3 = -\frac{1}{g_s}\int_{X_6} \hat{H}_3 \wedge *\hat{H}_3 = \frac{1}{g_s}\int_{X_6} |\hat{H}_3|^2 * 1 > 0 \,. \tag{1.89}$$

1.2.2 Toroidal Orientifolds

In this section we review some basic ingredients of type IIB compactifications on the $T^6/\mathbb{Z}_2$ orientifold with non-trivial 3-form fluxes turned on. Notice that in this section and in Chap. 2, in order to follow the conventions of [142] in our treatment, we will make use of the following redefinition of the axio-dilaton field

$$S \longrightarrow \phi = \mathrm{i}\bar{S} \,. \tag{1.90}$$

The type IIB supergravity action in Einstein frame is:

$$\begin{aligned} S_{\,\mathsf{IIB}} = &\frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{-g}\left(R - \frac{\partial_{\mathcal{M}}\phi\partial^{\mathcal{M}}\phi}{2\,(\mathsf{Im}\,\phi)^2} - \frac{G_3\cdot\bar{G}_3}{2\cdot 3!\,\mathsf{Im}\,\phi} - \frac{\tilde{F}_5^2}{4\cdot 5!}\right) \\ &+\frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\,\frac{C_4\wedge G_3\wedge\bar{G}_3}{4\,\mathsf{Im}\,\phi} + S_{\,\mathsf{local}} \,, \end{aligned} \tag{1.91}$$

where:

$$\begin{aligned} &\phi = C_0 + \mathrm{i}/g_s \,, \qquad F_3 = \mathrm{d}C_2 \,, \qquad H_3 = \mathrm{d}B_2 \,, \\ &G_3 = F_3 - \phi H_3 \,, \qquad \tilde{F}_5 = F_5 - \frac{1}{2}C_2\wedge H_3 + \frac{1}{2}F_3\wedge B_2 \,, \qquad *\tilde{F}_5 = \tilde{F}_5 \,. \end{aligned} \tag{1.92}$$

Upon compactifying on the $T^6/\mathbb{Z}_2$ orientifold with spacetime-filling D3-branes, the D3 tadpole condition is (setting $2\pi\sqrt{\alpha'}=1$):

$$\frac{1}{2}N_{\text{flux}} + N_{D3} - 16 = 0\,, \qquad N_{\text{flux}} \equiv \int_{T^6} H_3 \wedge F_3\,, \tag{1.93}$$

where N_{D3} is the number of D3-branes. The flux contribution can be shown to be positive semi-definite. The negative contribution arises from the 2^6 O3-planes. Clearly this condition implies $0 < N_{\text{flux}} \leq 32$.[1]

The geometry of the torus will be parametrised as follows. The 6 real periodic coordinates on T^6 are denoted as x^i, y^i, $i=1,2,3$ with $x^i \sim x^i+1$, $y^i \sim y^i+1$. The holomorphic 1-forms are taken to be $\mathrm{d}z^i = \mathrm{d}x^i + \tau^{ij}\mathrm{d}y^j$, where τ^{ij} is the period matrix. The choice of orientation is:

$$\int \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}y^3 = 1\,. \tag{1.94}$$

We will make use of the following orthonormal basis $\{\alpha_0, \alpha_{ij}, \beta^{ij}, \beta^0\}$ for $H^3(T^6,\mathbb{Z})$:

$$\begin{aligned} &\alpha_0 = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\,, \qquad \alpha_{ij} = \frac{1}{2}\epsilon_{ilm}\mathrm{d}x^l \wedge \mathrm{d}x^m \wedge \mathrm{d}y^j\,, \\ &\beta^{ij} = -\frac{1}{2}\epsilon_{jlm}\mathrm{d}y^l \wedge \mathrm{d}y^m \wedge \mathrm{d}x^i\,, \qquad \beta^0 = \mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}y^3\,, \quad i,j=1,2,3\,, \end{aligned} \tag{1.95}$$

with:

$$\int \alpha_I \wedge \beta^J = \delta_I^J\,. \tag{1.96}$$

Finally, the holomorphic 3-form is taken to be $\Omega = \mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3$. The NSNS and RR fluxes can be expanded in terms of the orthonormal basis as:

$$\begin{aligned} F_3 &= a^0\alpha_0 + a^{ij}\alpha_{ij} + b_{ij}\beta^{ij} + b_0\beta^0\,, \\ H_3 &= c^0\alpha_0 + c^{ij}\alpha_{ij} + d_{ij}\beta^{ij} + d_0\beta^0\,, \end{aligned} \tag{1.97}$$

where the Dirac quantitation condition requires $(a^0, a^{ij}, b_{ij}, b_0, c^0, c^{ij}, d_{ij}, d_0)$ to be integers. We will restrict them to be even integers so as to avoid the need for any discrete flux on the orientifold planes.[2] The flux contribution to the D3 tadpole (1.93) takes the form:

$$N_{\text{flux}} = (c^0 b_0 - a^0 d_0) + (c^{ij}b_{ij} - a^{ij}d_{ij})\,, \tag{1.98}$$

[1] We do not consider the $N_{\text{flux}}=0$ case since, due to the imaginary self-duality condition on the fluxes, it corresponds to either $g_s \to \infty$ or trivial flux quanta.

[2] See [142, 143] for a discussion on this point.

while the GVW superpotential (1.45) becomes:

$$W = (a^0 - \phi c^0)\det\tau - (a^{ij} - \phi c^{ij})(\text{cof }\tau)_{ij} - (b_{ij} - \phi d_{ij})\,\tau^{ij} - (b_0 - \phi d_0)\,. \quad (1.99)$$

Supersymmetry is preserved when the F-flatness conditions of this superpotential are satisfied, together with $W = 0$ (which can be thought of as the F-flatness condition for the Kähler moduli) and the requirement of primitivity of G_3 (i.e. the existence of a Kähler form such that $J \wedge G_3 = 0$). We will use the method described in Chap. 2 to obtain supersymmetric solutions with at least one flat direction. The F-flatness and $W = 0$ conditions are equivalent to:

$$\begin{aligned} f^{(1)} &\equiv a^0 \det\tau - a^{ij}(\text{cof }\tau)_{ij} - b_{ij}\tau^{ij} - b_0 = 0\,, \quad (1.100)\\ f^{(2)} &\equiv c^0 \det\tau - c^{ij}(\text{cof }\tau)_{ij} - d_{ij}\tau^{ij} - d_0 = 0\,,\\ f^{(3)}_{kl} &\equiv (a^0 - \phi c^0)(\text{cof }\tau)_{kl} - (a^{ij} - \phi c^{ij})\epsilon_{kim}\epsilon_{ljn}\tau^{mn} - (b_{ij} - \phi d_{ij})\delta^i_k\delta^j_l = 0\,. \end{aligned}$$

The primitivity of G_3 will be imposed as a final condition. We will see that a suitable choice of the Kähler form satisfying the primitivity condition can be found for all cases.

1.3 Kähler Moduli stabilization

Until now, we have discussed only the stabilization of the axio-dilaton and of complex structure moduli, let us now shift our focus onto the Kähler moduli instead. We first review the low-energy effective field theory of the KKLT [18] and LVS [27, 99] approaches to moduli stabilization.

We will hereby assume that the axion-dilaton and complex structure moduli are stabilized at a higher scale, and that all quantities, implicitly depending on these fields, such as the flux-induced superpotential W_0 or the prefactor of instanton corrections, can be regarded as constant. The F-term scalar potential is then calculated employing the supergravity formula (1.46) without D-terms

$$V = e^K\left(\mathcal{D}_I W K^{I\bar{J}}\overline{\mathcal{D}_J W} - 3|W|^2\right)\,. \quad (1.101)$$

Moreover, the gravitino mass is defined as

$$m_{3/2} = e^{K/2}|W|\,. \quad (1.102)$$

1.3.1 KKLT

Let us first focus on KKLT [18]. We consider a simple model consisting in only one Kähler modulus $T = \tau + i\theta$.

The Kähler potential in KKLT may be written as (with $\mathcal{V} = (2\tau)^{3/2}$)

$$K = -3\ln\left(T + \bar{T}\right)\,, \tag{1.103}$$

while the superpotential reads

$$W = W_0 + A\, e^{-\mathfrak{a}T}\,. \tag{1.104}$$

Writing $W_0 = |W_0|\, e^{i\phi_{W_0}}$ and $A = |A|\, e^{i\phi_A}$, without loss of generality, we set $\phi_{W_0} = \pi$ and $\phi_A = 0$, so that $W_0 = -|W_0|$ and $A = |A| \in \mathbb{R}^+$. Using (1.101) one can compute the uplifted scalar potential

$$V_{\text{KKLT}} = \frac{\mathfrak{a}^2 A^2 e^{-2\mathfrak{a}\tau}}{6\tau} + \frac{\mathfrak{a}A^2 e^{-2\mathfrak{a}\tau}}{2\tau^2} - \frac{\mathfrak{a}A|W_0|e^{-\mathfrak{a}\tau}}{2\tau^2}\cos(\mathfrak{a}\theta)\,. \tag{1.105}$$

The minimum for the axion lies at the origin: $\theta = 0$. Note that a different choice of the phase of W_0 would give a different location of the axion minimum. For example, choosing $\phi_{W_0} = 0$ would imply $\theta = \pi/\mathfrak{a}$. As we will see in Sec. 4.3 and 4.4, the choice that leads to $\theta = 0$ is however important for the derivation of the EDE potential.

With this minimization condition we get

$$V_{\text{KKLT}} = \frac{\mathfrak{a}^2 A^2 e^{-2\mathfrak{a}\tau}}{6\tau} + \frac{\mathfrak{a}A^2 e^{-2\mathfrak{a}\tau}}{2\tau^2} - \frac{\mathfrak{a}A|W_0|e^{-\mathfrak{a}\tau}}{2\tau^2}\,. \tag{1.106}$$

This scalar potential admits an AdS minimum with $|W_0|$ given by

$$|W_0| = \frac{2}{3}A\,\mathfrak{a}\tau\, e^{-\mathfrak{a}\tau}\left(1 + \frac{5}{2\mathfrak{a}\tau}\right)\,, \tag{1.107}$$

where the gravitino mass in Planck units scales as

$$m_{3/2} = \frac{A\,\mathfrak{a}}{3\sqrt{2\tau}}\, e^{-\mathfrak{a}\tau}. \tag{1.108}$$

1.3.2 Large Volume Scenario

We now turn to the LVS [27, 99]. We will divide this discussion in two, each with a different choice of the underlying structure of the Calabi-Yau (CY) manifold. First, we assume a so called 'Swiss cheese' CY, where, given the field content $T_b = \tau_b + i\theta_b$ and $T_s = \tau_s + i\theta_s$ (representing 'big' and 'small' 4-cycle volume moduli, respectively), the total volume takes the form

$$\mathcal{V} = \tau_b^{3/2} - \tau_s^{3/2}\,. \tag{1.109}$$

We have the following Kähler potential

$$K = -2\ln\left(\mathcal{V} + \frac{\hat{\xi}}{2}\right)\,, \tag{1.110}$$

where $\mathcal{O}(\alpha'^3)$ corrections are proportional to $\hat{\xi} \equiv \xi\, g_s^{-3/2}$ with $\xi = -\frac{\zeta(3)\chi}{2(2\pi)^3}$ where χ is the CY Euler number. Furthermore, in order to generate a potential for the fields, we take a superpotential with the contribution coming from non-perturbative corrections as

$$W = W_0 + +A_s\, e^{-\mathfrak{a}_s T_s} + A_b\, e^{-\mathfrak{a}_b T_b}\,. \tag{1.111}$$

Computing the scalar potential we find 2 contributions: a leading one responsible for stabilizing the volume, τ_s and the axion θ_s in a AdS vacuum, and a subleading one stabilizing the axion θ_b. The scalar potential is then given by

$$V = V_{\mathsf{LVS}}\left(\mathcal{V}, \tau_s, \theta_s\right) + V_b\left(\theta_b\right)\,, \tag{1.112}$$

where, in detail, we have the LVS potential (at leading order in the $\mathcal{V} \gg 1$ and $\mathfrak{a}_s\tau_s \gg 1$ expansions, and setting again $W_0 = -|W_0|$ and $A_s = |A_s|$)

$$V_{\mathsf{LVS}} = \frac{8\mathfrak{a}_s^2 A_s^2 e^{-2\mathfrak{a}_s\tau_s}\sqrt{\tau_s}}{3\mathcal{V}} - \frac{4\mathfrak{a}_s A_s \tau_s |W_0|\, e^{-\mathfrak{a}_s\tau_s}}{\mathcal{V}^2}\cos\left[\mathfrak{a}_s\theta_s\right] + \frac{3|W_0|^2\hat{\xi}}{4\mathcal{V}^3}\,, \tag{1.113}$$

where one can immediately see that the axion gets stabilized at $\theta_s = 0$, and the potential for the axion θ_b

$$V_b = -\frac{4\mathfrak{a}_b A_b |W_0|\, e^{-\mathfrak{a}_b\mathcal{V}^{2/3}}}{\mathcal{V}^{4/3}}\cos\left(\mathfrak{a}_b\theta_b\right)\,, \tag{1.114}$$

which fixes the axion θ_b at $\theta_b = 0$. The leading order LVS potential, at $\theta_s = 0$, therefore reads

$$V_{\mathsf{LVS}} = \frac{8\mathfrak{a}_s^2 A_s^2 e^{-2\mathfrak{a}_s\tau_s}\sqrt{\tau_s}}{3\mathcal{V}} - \frac{4\mathfrak{a}_s A_s \tau_s |W_0|\, e^{-\mathfrak{a}_s\tau_s}}{\mathcal{V}^2} + \frac{3|W_0|^2\hat{\xi}}{4\mathcal{V}^3}\,. \tag{1.115}$$

We will now concern ourselves with the potential (1.115) after the stabilization of the axions. To find the minimum of the theory we first solve for τ_s by deriving V, finding

$$e^{-\mathfrak{a}_s\tau_s} = \frac{3|W_0|\sqrt{\tau_s}}{\mathfrak{a}_s A_s \mathcal{V}}\,\frac{\mathfrak{a}_s\tau_s - 1}{4\mathfrak{a}_s\tau_s - 1} \simeq \frac{3|W_0|\sqrt{\tau_s}}{4\mathfrak{a}_s A_s\mathcal{V}}\,. \tag{1.116}$$

Moreover, in order to simplify the equations, we perform the following change of variables

$$\psi \equiv \mathfrak{a}_s\tau_s\,, \qquad \mathcal{V} = \beta\sqrt{\frac{\psi}{\mathfrak{a}_s}}\,e^{\psi}\left(\frac{1-1/\psi}{1-1/(4\psi)}\right)\,, \qquad \beta = \frac{3|W_0|}{4\mathfrak{a}_s A_s}\,, \tag{1.117}$$

which implies $\psi \simeq \ln \mathcal{V}$. Thus, at leading order in $\psi \gg 1$, the scalar potential looks like

$$V(\psi) = -\frac{3\,|W_0|^2}{2\beta^3}\, e^{-3\psi} \left[1 - \left(\frac{\mathfrak{a}_s}{\psi}\right)^{3/2} \frac{\hat{\xi}}{2}\right] \left(1 + \frac{9}{4\psi}\right) , \tag{1.118}$$

where we kept the first correction in the $\psi \gg 1$ expansion. Solving for the minimum and requiring it to be Minkowski (i.e. $\partial V = V = 0$) we find the minima conditions to be

$$\frac{\hat{\xi}}{2} = \left(\frac{\psi}{\mathfrak{a}_s}\right)^{3/2} - \frac{9}{10\mathfrak{a}_s}\left(\frac{\psi}{\mathfrak{a}_s}\right)^{1/2} \simeq \left(\frac{\psi}{\mathfrak{a}_s}\right)^{3/2} \equiv \tau_s^{3/2} , \tag{1.119}$$

where again we included only the first correction for $\psi \gg 1$. The value of the volume at this global minimum is

$$\mathcal{V} \simeq \frac{3|W_0|\sqrt{\tau_s}}{4\mathfrak{a}_s A_s}\, e^{\mathfrak{a}_s \tau_s} \simeq |W_0|\, e^{\frac{\mathfrak{a}_s}{g_s}\left(\frac{\xi}{2}\right)^{2/3}} . \tag{1.120}$$

Moreover, the gravitino mass turns out to be

$$m_{3/2} = \frac{|W_0|}{\mathcal{V}} \simeq e^{-\frac{\mathfrak{a}_s}{g_s}\left(\frac{\xi}{2}\right)^{2/3}} . \tag{1.121}$$

We now turn to a Calabi-Yau with a K3 or T^4 fibration over a $\mathbb{P}^1$ base and a diagonal del Pezzo divisor [144, 145]. In this case the volume is written in terms of the 3 Kähler moduli as

$$\mathcal{V} = \sqrt{\tau_1}\tau_2 - \tau_s^{3/2} . \tag{1.122}$$

This class of manifolds was used in the context of Fibre Inflation [51], where the inflaton is the direction $u = \tau_1/\tau_2$ orthogonal to the overall volume mode. The Kähler potential assumes the form[3]

$$K = -2\ln\left(\mathcal{V} + \frac{\hat{\xi}}{2}\right) . \tag{1.123}$$

Furthermore, the superpotential takes the form

$$W = W_0 + A_s\, e^{-\mathfrak{a}_s T_s} + A_1\, e^{-\mathfrak{a}_1 T_1} + A_2\, e^{-\mathfrak{a}_2 T_2} . \tag{1.124}$$

After stabilizing the axion at $\theta_s = 0$, one finds again the potential given in (1.140), following the same minimization conditions as in the Swiss

[3]The actual moduli dependence of the uplifting contribution in fibred CY cases might be more complicated since it might involve both the overall volume and the fibre modulus. In this case, perturbative corrections to K should be used to fix the fibre modulus in terms of the overall volume to obtain an uplifting term of the standard form.

cheese case. Given that in this case, in the $\tau_s \to 0$ limit, the volume is determined by 2 fields, rather than the 1 of the simpler Swiss cheese geometry, one direction in the (τ_1, τ_2)-plane is left flat by LVS stabilization. This flat direction can be lifted by string loops [51, 52, 54, 146], higher derivative [48] or non-perturbative corrections [62, 147] to the action and can play a role either in early [47, 48, 51–54, 59] or late time cosmology [148]. Finally, the potential of the two bulk axions θ_1 and θ_2 is generated at an even subleading order by the T_1- and T_2-dependent non-perturbative contributions to the superpotential.

1.4 Uplifting

In all previously discussed stabilization mechanisms, we typically achieve an Anti-de Sitter (AdS) minimum. Attaining a de Sitter (dS) vacuum, however, requires additional contributions for uplift, as outlined in [149]. In this section, we detail two such methods: anti-brane uplift and complex structure F-terms. It's worth noting other uplift schemes, which we summarize below:

- A non-zero Fayet-Iliopoulos term via T-brane configurations with gauge fluxes in a hidden D7-brane sector [28–30, 134, 150, 151].
- D-terms from magnetized branes, introducing gauge field fluxes on a D7-brane [152].
- Competition between α' corrections to the Kähler potential and non-perturbative effects [98, 153–155].
- Non-perturbative effects from E(-1)-instantons or D3-branes at singularities, affecting the dilaton [100].
- Perturbative loop corrections with logarithmic or power-law behaviour from intersecting D7-branes [97, 101, 156].
- Single-step stabilization of all moduli using tree-level potentials with non-perturbative corrections [157].
- Positive 4D EFT contributions from negative curvature in extra dimensions [158, 159].

1.4.1 Anti D3-branes

Spontaneous supersymmetry breaking in (effective) supergravity theories leads to the super-Higgs effect – the gravitino eats the goldstino to become massive. If this phenomenon takes place at energies which are low compared with the Planck mass, the goldstino couplings can be described by making use of a (constrained) independent superfield. Supersymmetry is then non-linearly realized as in the Volkov-Akulov formalism. There are several approaches to describe the low energy dynamics of the goldstino in terms of spurion or constrained superfields (see for instance [160] and references therein). Our focus will be on the approach where the goldstino is described in terms of a chiral superfield X that is constrained to be nilpotent, i.e. $X^2 = 0$. This has the ingredients necessary to describe supersymmetry breaking induced by the presence of an anti-D3 brane at the tip of a warped throat in flux compactifications [161–163].

The effective field theory involving a nilpotent chiral superfield X can be described in terms of a Kähler potential K, a superpotential W and a gauge kinetic function f whose general forms are:

$$K = K_0 + K_1 X + \bar{K}_1 \bar{X} + K_2 X\bar{X}, \qquad W = \rho X + \tilde{W}, \qquad f = f_0 + f_1 X, \tag{1.125}$$

where K_0, K_1, K_2, ρ, $\tilde{W}$, f_0, f_1 are functions of other low energy fields. Higher powers of X are absent in K and W since $X^2 = 0$. The nilpotency condition implies a constraint on the components of X. Expanding X in superspace

$$X = X_0(y) + \sqrt{2}\psi(y)\theta + F(y)\theta\bar{\theta}\,, \tag{1.126}$$

(where as usual, $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$), $X^2 = 0$ implies

$$X_0 = \frac{\psi\psi}{2F}\,. \tag{1.127}$$

Thus, unless the fermion ψ condenses in the vacuum, the vacuum expectation value of X_0 (the scalar component of X) vanishes.

For an anti-D3 brane at the bottom of a warped throat of a type IIB flux compactification, the description in terms of X is very convenient. It allows to treat its effects in terms of supergravity: its contribution to the scalar potential, the gravitino mass and the couplings to moduli and matter fields are easily obtained. It was shown in [163] that nilpotent superfield(s) capture all degrees of freedom of an anti-D3 brane when it is placed on top of an orientifold plane. The presence of the fluxes and the orientifold projection leave the massless goldstino as a low energy propagating degree of freedom, in keeping with the use of a nilpotent superfield X to describe the system. The simplest example is that of an O3-plane and an anti-D3 brane at the bottom of a warped throat. In this case, there is no modulus associated with the position of the anti-D3 brane (in contrast to the case of D3-branes in the bulk). This corresponds to the fact that the scalar component of X is not a propagating field. Furthermore, there is no contribution from X_0 to the scalar potential and it can be consistently set to zero while looking for vacuum solutions.

Next, let us turn to the form of the Kähler potential and superpotential in (1.125). In the case of a single Kähler modulus T (with $T = \mathcal{V}^{2/3} + i\psi \equiv \tau + i\psi$ is a complex field obtained by pairing the volume modulus and its axionic partner), the functions $\tilde{W}$ and ρ have no dependence on T at the perturbative level, as a result of holomorphy and the Peccei-Quinn shift symmetry $T \mapsto T + ic$. The zeroth order term in the Kähler potential of (1.125), $K_0 = -2\ln\mathcal{V}$ is invariant (up to a Kähler transformation) under the full modular transformation $T \to (aT - ib)/(icT + d)$ (which is a generalization of the shift symmetry). If X transforms suitably, i.e. as a modular form of weight κ, the quadratic coefficient takes the form $K_2 = \beta\tau^{-\kappa}$ (where β is a constant). Moreover, if the term linear in X is absent, the only contribution of X to the F-term potential is a positive definite term:

$$V_{\text{up}} = e^K K^{-1}_{X\bar{X}} \left\| \frac{\partial W}{\partial X} \right\|^2 = \frac{|\rho|^2}{\tau^{3-\kappa}} \,. \tag{1.128}$$

This precisely coincides with form of the contribution of an anti-D3 brane at the tip of a warped throat with hierarchy $|\rho|^2/\beta$ as computed from direct dimensional reduction (see equation (B.8)), if the modular weight is $\kappa = 1$[4]. Since the anti-D3 brane is localized at a particular point in the compactification manifold, direct couplings to gauge fields located at distant D3 or D7-branes are difficult. This implies the absence of terms linear in X in the gauge kinetic functions. In summary, at leading order in the α' expansion the effective field theory is specified by:

$$K = -2\ln\mathcal{V} + \beta\frac{X\bar{X}}{\tau}, \qquad W = \tilde{W} + \rho X, \qquad f = f_0 \,, \tag{1.129}$$

where c, ρ, $\tilde{W}$, f_0 are constants and $|\rho|^2/\beta$ is identified with the hierarchy y as defined in (B.5), i.e. $y = |\rho|^2/\beta$. Furthermore, the superpotential only receives non-perturbative corrections. Nilpotency of X implies that the Kähler potential in (1.129) can be written as:

$$K = -3\ln\left(\tau - \frac{\beta}{3}X\bar{X}\right) \,. \tag{1.130}$$

[4] If $\kappa = 0$, one has a $(T+\bar{T})^{-3}$ dependence which corresponds to an anti-D3-brane in an unwarped region. The magnitude of the term is of order the string scale $V_{\text{up}} \sim M_s^4$, which if included in the low energy theory, would lead to a runaway potential.

Note that in the regime where the effective field theory is valid, i.e. when the anti-D3 brane is at the tip of a warped throat, the X superfield couples to T in the Kähler potential in the same way a superfield ϕ describing the D3-brane matter fields,[5] i.e. [135, 139]

$$K_{\text{D3}} = -3\ln\left(\tau - \frac{\alpha}{3}\phi\bar{\phi}\right) \sim -2\ln\mathcal{V} + \alpha\frac{\phi\bar{\phi}}{\tau} + \cdots\,, \tag{1.131}$$

where the dots indicate terms which are higher order in the $1/\tau$ expansion. It is then natural to conjecture [164] that the only effect of X in the Kähler potential is to shift the Kähler coordinate T in the same way as the field ϕ does. This was called the log hypothesis, as it leads one to write the $X\bar{X}$ term inside the log as in (1.130).

When both D3-branes and anti-D3-brane are present, generically the Kähler potential can be written as

$$K = -2\ln\mathcal{V} + \alpha\frac{\phi\bar{\phi}}{\tau^{\mu}} + \beta\frac{X\bar{X}}{\tau^{\kappa}} + \gamma\frac{X\bar{X}\,\phi\bar{\phi}}{\tau^{\zeta}} + \cdots\,, \tag{1.132}$$

with modular weights $\mu = \kappa = 1$ as discussed above. Moreover, if ϕ and X have modular weights μ and κ respectively, the modular weight for the $X\bar{X}\,\phi\bar{\phi}$ term should be $\zeta = \mu + \kappa$. In this case, $\zeta = 1 + 1 = 2$. This agrees with the log hypothesis, i.e. a Kähler potential of the form

$$K_{\text{no-scale}} = -3\ln\left(\tau - \frac{\alpha}{3}\phi\bar{\phi} - \frac{\beta}{3}X\bar{X}\right)\,. \tag{1.133}$$

In fact, expanding this in powers of $1/\tau$, one obtains (1.132) with the condition that $\gamma = \frac{\alpha\beta}{3}$. The Kähler potential in (1.133) is of the standard no-scale form [17, 165].

We conclude this subsection with comments relevant for the computation of soft masses. In the KKLT scenario, the low energy effective theory is usually written in terms of the fields with masses of order or below the gravitino mass. These include massless chiral fields (arising as excitations of open strings) and the Kähler moduli. Supersymmetry is broken at the minimum of the scalar potential. Both the F-term of X and the F-term of T are different from zero (with $F^T \ll F^X$). Thus, the goldstino is a combination of the fermion in X and the fermion in T. In LVS, even in the absence of the anti-brane, the overall volume modulus T_b breaks supersymmetry ($F^{T_b} \neq 0$). Inclusion of the nilpotent superfield in the effective action allows one to consider the breaking of supersymmetry induced by

[5] Here and in the following we will take a simplified model where we write down only 1 of the 3 complex superfields describing the D3-brane positions. Adding the other 2 would only complicate the expressions without altering our results.

fluxes and supersymmetry breaking by the anti-brane on equal footing. Again, the goldstino is a combination of the fermion components of X and of the moduli. Although the dominant component is usually the one from the T_b-field, for sequestered models the X component is relevant and its contribution to the soft terms must be accounted for.

Let us now summarize the effect of the addition of this uplift to the two stabilization mechanism explored.

KKLT: The Kähler potential in (1.103) reads

$$K = -3\ln\left(T+\bar{T}\right) + 3\,\frac{X\bar{X}}{T+\bar{T}}\,, \tag{1.134}$$

while the superpotential (1.104) becomes

$$W = W_0 + MX + A\,e^{-\mathfrak{a}T}\,. \tag{1.135}$$

Using again (1.101) the uplifted scalar potential is

$$V_{\mathsf{KKLT}} = \frac{\mathfrak{a}^2 A^2 e^{-2\mathfrak{a}\tau}}{6\tau} + \frac{\mathfrak{a}A^2 e^{-2\mathfrak{a}\tau}}{2\tau^2} - \frac{\mathfrak{a}A|W_0|e^{-\mathfrak{a}\tau}}{2\tau^2} + \frac{M^2}{12\tau^2}\,. \tag{1.136}$$

where the nilpotency condition has been imposed and θ has been minimized to its value $\theta = 0$. This scalar potential admits a Minkowski minimum with spontaneously broken supersymmetry for M and $|W_0|$ given by

$$\begin{aligned} |W_0| &= \frac{2}{3}A\,\mathfrak{a}\tau\,e^{-\mathfrak{a}\tau}\left(1+\frac{5}{2\mathfrak{a}\tau}\right)\,, \\ M &= \sqrt{2\mathfrak{a}}A\,e^{-\mathfrak{a}\tau}\sqrt{\mathfrak{a}\tau+2}\,. \end{aligned} \tag{1.137}$$

The minimum may be further lifted to a small but non-zero cosmological constant via a small shift in M. We will use (1.137) when discussing parameter values in KKLT.

LVS: Now, adding a nilpotent superfield X as in the KKLT case, the Kähler potential (1.110)

$$K = -2\ln\left(\mathcal{V}+\frac{\hat{\xi}}{2}\right) + \frac{\bar{X}X}{\mathcal{V}^{2/3}}\,, \tag{1.138}$$

while the superpotential (1.111) becomes

$$W = W_0 + MX + A_s\,e^{-\mathfrak{a}_s T_s} + A_b\,e^{-\mathfrak{a}_b T_b}\,. \tag{1.139}$$

The LVS scalar potential is then given by

$$V_{\text{LVS}} = \frac{8\mathfrak{a}_s^2 A_s^2 e^{-2\mathfrak{a}_s\tau_s}\sqrt{\tau_s}}{3\mathcal{V}} - \frac{4\mathfrak{a}_s A_s \tau_s |W_0|\, e^{-\mathfrak{a}_s\tau_s}}{\mathcal{V}^2} + \frac{3|W_0|^2\hat{\xi}}{4\mathcal{V}^3} + \frac{M^2}{\mathcal{V}^{4/3}}\,. \qquad (1.140)$$

Making again the change of variables described in (1.117), the scalar potential (1.118) becomes

$$V(\psi) = -\frac{3\,|W_0|^2}{2\beta^3}\,e^{-3\psi}\left[1-\left(\frac{\mathfrak{a}_s}{\psi}\right)^{3/2}\frac{\hat{\xi}}{2}\right]\left(1+\frac{9}{4\psi}\right)$$
$$+\frac{M^2}{\beta^{4/3}}\left(\frac{\mathfrak{a}_s}{\psi}\right)^{2/3}e^{-4\psi/3}\left(1+\frac{1}{\psi}\right)\,. \qquad (1.141)$$

Solving for the minimum and requiring it to be Minkowski (i.e. $\partial V = V = 0$) we find the minima conditions to be

$$\frac{\hat{\xi}}{2} = \left(\frac{\psi}{\mathfrak{a}_s}\right)^{3/2} - \frac{9}{10\mathfrak{a}_s}\left(\frac{\psi}{\mathfrak{a}_s}\right)^{1/2} \simeq \left(\frac{\psi}{\mathfrak{a}_s}\right)^{3/2} \equiv \tau_s^{3/2} \qquad (1.142)$$

$$M^2 = \frac{27}{20}\frac{|W_0|^2}{\mathfrak{a}_s^{2/3}\beta^{5/3}\psi^{1/3}}\,e^{-5\psi/3} \simeq \frac{27}{20}\frac{|W_0|^2}{\mathfrak{a}_s}\frac{\sqrt{\tau_s}}{5/3}\,, \qquad (1.143)$$

where again we included only the first correction for $\psi \gg 1$. Note that (1.142) and (1.143), when substituted in (1.141), would give a leading order cancellation and an AdS vacuum for $M = 0$. Lastly, let us mention that a dS minimum can be achieved by allowing small shifts of M.

Lastly, the case for a fibrated Calabi-Yau is completely analogous and will not be described here.

1.4.2 Complex structure uplift

We will now turn to the second method of uplifting of our interest, which involves employing small complex structure F-terms [40]. This approach begins from one of the usual Kähler stabilization schemes, KKLT or LVS, but introduces a small flux SUSY breaking term F_a at the level of the axio-dilaton/complex structure moduli. This introduction, implies an additional positive term in the potential given by

$$V_{\text{up}} = \frac{e^{K_{\text{cs}}}F_a\overline{F}^{\overline{a}}}{\mathcal{V}^2} \equiv \frac{|F|^2}{\mathcal{V}^2}\,. \qquad (1.144)$$

Specifically, in the KKLT scenario, there is a competition between the uplift term and the leading potential

$$V_{\text{KKLT}} \sim \frac{|W_0|^2}{\mathcal{V}^2} \implies |F|^2 \sim |W_0|^2 \qquad (1.145)$$

$$\implies |F|^2 \sim \mathcal{O}\big(|W_0|^2\big) = \mathcal{O}\Big(e^{-2\mathfrak{a}\mathcal{V}^{2/3}}\Big) \ll 1\,. \qquad (1.146)$$

Instead, in LVS, due to the smaller volume scaling of the additional term, considering an F-term such that $|F|^2 \sim |W|^2$ would inevitably lead to a runaway solution. Parametrizing the F-term as

$$F_a = \epsilon W f_a \,, \qquad \epsilon \ll 1 \,, \tag{1.147}$$

with f_a an unit flux vector, we get that

$$\frac{|F|^2}{\mathcal{V}^2} = \frac{\epsilon^2 |W|^2}{\mathcal{V}^2} \sim \mathcal{O}\big(\mathcal{V}^{-3}\big) \implies \epsilon \sim \mathcal{O}\Big(\mathcal{V}^{-1/2}\Big) \ll 1 \,, \tag{1.148}$$

and this can be achieved by suitably choosing the fluxes in such a way to achieve the right hierarchy. It is crucial to note that, from this analysis, the key point in both schemes is maintaining a large volume so that the F-term is small enough to avoid a runaway solution.

KKLT: In the case of the KKLT framework, the scalar potential becomes

$$V = \frac{\mathfrak{a}^2 A^2 e^{-2\mathfrak{a}\tau}}{6\tau} + \frac{\mathfrak{a} A^2 e^{-2\mathfrak{a}\tau}}{2\tau^2} - \frac{\mathfrak{a} A |W_0| e^{-\mathfrak{a}\tau}}{2\tau^2} + \frac{|F|^2}{\tau^3} \,, \tag{1.149}$$

where θ has been minimized to its value $\theta = 0$. This scalar potential admits a Minkowski minimum with spontaneously broken supersymmetry for $|F|^2$ and $|W_0|$ given by

$$\begin{aligned} |W_0| &= \frac{2}{3} A e^{-\mathfrak{a}\tau} \left(2 + \mathfrak{a}\tau - \frac{1}{\mathfrak{a}\tau}\right) \,, \\ |F|^2 &= \frac{1}{6} A^2 \mathfrak{a}^2 \tau^2 e^{-2\mathfrak{a}\tau} \frac{\mathfrak{a}\tau + 2}{\mathfrak{a}\tau - 1} \,. \end{aligned} \tag{1.150}$$

Again, the minimum may be further lifted to a small but non-zero cosmological constant via a small shift in $|F|^2$.

LVS: The LVS scalar potential is correct as

$$V = \frac{8\mathfrak{a}_s^2 A_s^2 e^{-2\mathfrak{a}_s\tau_s}\sqrt{\tau_s}}{3\mathcal{V}} - \frac{4\mathfrak{a}_s A_s \tau_s |W_0|\, e^{-\mathfrak{a}_s\tau_s}}{\mathcal{V}^2} + \frac{3|W_0|^2 \hat{\xi}}{4\mathcal{V}^3} + \frac{|F|^2}{\mathcal{V}^2} \,. \tag{1.151}$$

Making again the change of variables described in (1.117), the scalar potential (1.151) becomes

$$V(\psi) = -\frac{3\,|W_0|^2}{2\beta^3}\, e^{-3\psi} \left[1 - \left(\frac{\mathfrak{a}_s}{\psi}\right)^{3/2} \frac{\hat{\xi}}{2}\right] \left(1 + \frac{9}{4\psi}\right) + \frac{\mathfrak{a}_s |F|^2}{\beta^2 \psi} e^{-2\psi} \left(1 + \frac{3}{2\psi}\right) \,. \tag{1.152}$$

Solving for the minimum and requiring it to be Minkowski (i.e. $\partial V = V = 0$) we find the minima conditions to be

$$\frac{\hat{\xi}}{2} = \left(\frac{\psi}{\mathfrak{a}_s}\right)^{3/2} - \frac{3}{2\mathfrak{a}_s}\left(\frac{\psi}{\mathfrak{a}_s}\right)^{1/2} \simeq \left(\frac{\psi}{\mathfrak{a}_s}\right)^{3/2} \equiv \tau_s^{3/2} \tag{1.153}$$

$$|F|^2 = \frac{9|W_0|^2}{4\mathfrak{a}_s\beta}e^{-\psi} = 3A_s W_0\, e^{-\mathfrak{a}_s\tau_s}\,, \tag{1.154}$$

where again we included only the first correction for $\psi \gg 1$. Note that (1.153) and (1.154), when substituted in (1.152), would give a leading order cancellation and an AdS vacuum for $|F|^2 = 0$. Lastly, let us mention that a dS minimum can be achieved by allowing small shifts of $|F|^2$.

The case for a fibrated Calabi-Yau is again completely analogous and will not be described here.

1.5 Odd Axions in Type IIB String Theory

For a detailed treatment of axions in string theory, we refer the reader to [166-169]. Here we provide a brief overview, covering the necessary background for the detailed discussion to follow.

1.5.1 Axions in string theory

In addition to the fundamental axion C_0, axions emerge in the 4-dimensional low-energy effective theory of type IIB string compactifications from dimensional reduction of p-form gauge fields. The shift-symmetry which earns these fields the name 'axion' corresponds to gauge invariance of the higher dimensional theory, and the small axion mass, like a standard field theory axion, is generated by non-perturbative effects, such as gaugino condensation and instantons.

We define the 4-dimensional axion fields as,

$$b^a = \int_{\Sigma_a} B_2\,, \qquad c^a = \int_{\Sigma_a} C_2\,, \qquad \theta^\alpha = \int_{D_\alpha} C_4\,, \tag{1.155}$$

where B_2, C_2, and C_4, correspond to the Kalb-Ramond 2-form and the Ramond-Ramond 2- and 4-form fields, respectively, and Σ_a and D_α denote respectively a basis of 2-cycles and 4-cycles of the underlying CY threefold $\mathcal{X}$ with $a = 1, .., h^{1,1}_-(\mathcal{X})$ and $\alpha = 1, .., h^{1,1}_+(\mathcal{X})$. Here $h^{1,1}_\pm(\mathcal{X})$ are the so-called Hodge numbers which count the number of holomorphic $(1,1)$-forms of $\mathcal{X}$ which are even or odd under the orientifold involution with $h^{1,1} = h^{1,1}_+ + h^{1,1}_-$.

The C_4 axions are inextricably linked to stabilization of the volume moduli in both the KKLT and LVS approaches, as can be appreciated from (1.105) for KKLT and (1.113) and (1.114) for LVS. In KKLT, stabilization of the volume τ necessitates stabilization of the C_4 axion θ, and similarly in LVS stabilization of the small cycle volumes τ_s necessitates stabilization of θ_s. The θ_b axion, partner to the LVS large cycle volume τ_b, is ostensibly decoupled from stabilization of τ_b which is fixed by perturbative effects.

The B_2 axions are also linked to the stabilization of the volume moduli due to the mixing between b and τ fields in the Kähler potential which breaks the shift symmetry of the B_2 axions at the perturbative level. Consequently, any effect that stabilizes the Kähler moduli generates also a potential for the b-fields [170]. The shift symmetry is also broken by D-branes through the DBI action, which manifests in the 4-dimensional theory as a D-term [171, 172]. These effects generically stabilize b at a high energy scale, allowing b to be neglected in analyses of cosmology, as in [173].

The situation is instead different for the C_2 axions which are not a priori linked to the stabilization of the volume moduli, and therefore provide an opportunity for cosmological model building. In fact, the shift symmetry of the C_2 axions is unbroken by many of the standard ingredients of flux compactifications, but can be broken upon the inclusion of different effects we study in this work, such as fluxed ED1-instantons or gaugino condensation on D7-branes in the presence of worldvolume fluxes.

Finally, the C_0 axion is stabilized by the superpotential (1.45), induced by background 3-form fluxes, which generates a mass for C_0 that is comparable to that of the dilaton, and thus stabilization of the dilaton precludes C_0 from playing a cosmological role (with some exceptions [174]).

Let us point out that axions can also be eaten up by anomalous $U(1)$s in the process of anomaly cancellation. In this case they would become as heavy as the string scale, and so would disappear from the low energy theory. Investigations in this direction [175] have shown that C_4 and C_2 axions are eaten up by anomalous $U(1)$s only in the presence of D3-branes at singularities. Throughout Chap. 4 we will always focus on branes wrapping cycles in the geometric regime, where therefore C_4 and C_2 axions are guaranteed to survive in the 4-dimensional theory (in this case the modes eaten up correspond to open string axions).

1.5.2 Odd axions in effective field theory

Let us now focus on the description of odd axions in type IIB string theory compactified on an orientifolded Calabi-Yau manifold $\mathcal{X}$ [170], expanding the discussion done in section 1.2. Let us denote the basis of 2- and 4-forms as

$$\hat{D}_\alpha \in H^{1,1}_+(\mathcal{X}), \qquad \tilde{D}^\alpha \in H^{2,2}_+(\mathcal{X}), \qquad \alpha = 1, ..., h^{1,1}_+(\mathcal{X}), \quad (1.156)$$
$$\hat{D}_a \in H^{1,1}_-(\mathcal{X}), \qquad \tilde{D}^a \in H^{2,2}_-(\mathcal{X}), \qquad a = 1, ..., h^{1,1}_-(\mathcal{X}) \quad (1.157)$$

which lead to the normalization and intersection numbers

$$\int_{\mathcal{X}} \hat{D}_\alpha \wedge \hat{D}_\beta \wedge \hat{D}_\gamma = k_{\alpha\beta\gamma}\,, \qquad \int_{\mathcal{X}} \hat{D}_\alpha \wedge \hat{D}_a \wedge \hat{D}_b = k_{\alpha ab}\,, \quad (1.158)$$
$$\int_{\mathcal{X}} \hat{D}_\alpha \wedge \tilde{D}^\beta = \delta^\alpha_\beta\,, \qquad \int_{\mathcal{X}} \hat{D}_a \wedge \tilde{D}^b = \delta^a_b\,. \quad (1.159)$$

Furthermore, the Kähler form, C_4, C_2 and B_2 can be expanded as [135]

$$J = t^\alpha \hat{D}_\alpha\,, \qquad B_2 = b^a \hat{D}_a \qquad C_2 = c^a \hat{D}_a, \quad (1.160)$$
$$C_4 = D_2^\alpha \wedge \hat{D}_\alpha + V^K \wedge \alpha_K - V_K \wedge \beta^K - \theta_\alpha \tilde{D}^\alpha\,, \quad (1.161)$$

where $(\alpha_K, \beta^K) \in H^3_+(\mathcal{X})$ is a basis of symplectic forms such that $\int_X \alpha_K \wedge \beta^J = \delta^J_K$. These combine to give the chiral coordinates of the $\mathcal{N} = 1$ supergravity effective theory that read

$$G^a = \bar{S}b^a + c^a = \frac{b^a}{g_s} + (c^a - C_0 b^a)\,, \qquad \tau_\alpha = \frac{1}{2}\, k_{\alpha\beta\gamma}\, t^\beta t^\gamma\,, \quad (1.162)$$
$$T_\alpha = \tau_\alpha + \theta_\alpha - \frac{1}{4}\, g_s k_{\alpha ab} G^a (G + \bar{G})^b\,. \quad (1.163)$$

The CY volume is an implicit function of the T_α and G^a fields

$$\mathcal{V} = \frac{1}{6}\, k_{\alpha\beta\gamma} t^\alpha t^\beta t^\gamma\,, \quad (1.164)$$

which determines the tree-level Kähler potential

$$K = -2 \ln \mathcal{V}\,. \quad (1.165)$$

We now restrict ourselves to the simple case with $h^{1,1} = 2$ and $h^{1,1}_+ = h^{1,1}_- = 1$, where the orientifold image of the divisor D_1 is D_2. It is therefore possible to define an orientifold-even 4-cycle $D_+ \equiv D_1 \cup D_2$ and and orientifold-odd 4-cycle $D_- \equiv D_1 \cup (-D_2)$. Hence the Kähler form and the Kähler modulus take the form

$$J = t\, \hat{D}_+\,, \qquad T = \tau + i\theta - \frac{1}{4} g_s k G (G + \bar{G})\,, \quad (1.166)$$

where $k_{+--} \equiv k$, $\tau_+ \equiv \tau$ and $\hat{D}_+$ is the 2-form Poincaré dual to D_+. Defining $k_{+++} \equiv \tilde{k}$, the CY volume takes therefore the form

$$\mathcal{V} = \frac{1}{3}\sqrt{\frac{2}{\tilde{k}}}\,\tau^{3/2}\,, \qquad 2\tau = T + \bar{T} - \gamma\left(G + \bar{G}\right)^2 = 2\,\text{Re}\,(T) - \frac{4\gamma}{g_s^2}\,b^2\,, \quad (1.167)$$

where we have introduced

$$\gamma \equiv -\frac{1}{4} g_s k\,. \qquad (1.168)$$

Substituting (1.167) in (1.165), we can obtain the Kähler potential which becomes

$$K = -3\ln\left(\text{Re}\,(T) - \frac{2\gamma}{g_s^2}\,b^2\right). \qquad (1.169)$$

Note the explicit dependence of the Kähler potential on the B_2 axion, implying that this field does not enjoy any perturbative shift symmetry. Moreover, the sign of γ is required to be positive ($\gamma > 0 \Leftrightarrow k < 0$) in order to avoid the C_2 axion being a ghost. This can be seen by calculating the kinetic term for the orientifold odd axion

$$\mathcal{L}_{\text{kin}} = K_{G\bar{G}}\,\partial_\mu G\,\partial^\mu \bar{G} \supset K_{G\bar{G}}\,\partial_\mu c\,\partial^\mu c = \frac{3\gamma}{\tau}(\partial c)^2\,, \qquad (1.170)$$

where $K_{G\bar{G}} = \partial_G \partial_{\bar{G}} K$ and we have set $b = 0$. One can then define the canonically normalized field as

$$\varphi = \sqrt{\frac{6\gamma}{\tau}}\,c\,. \qquad (1.171)$$

1.5.3 Odd axions and non-perturbative effects

Type IIB compactifications feature different non-perturbative effects that can generate corrections to both the superpotential and the Kähler potential. We discuss now how these effects can break the perturbative shift symmetry of C_4 and C_2 axions.

ED3-instantons

A typical source of non-perturbative corrections to the superpotential is 4-cycles wrapped by Euclidean D3-brane (ED3) instantons [176]. The simplest configurations are fluxless but ED3-instantons can also support non-zero 2-form fluxes as studied in [177, 178]. In order to obtain a non-zero contribution to W the ED3 should wrap an orientifold-even rigid cycle, which in our simple case can only be D_+ which we assume to be a smooth and connected divisor (see Fig. 1.1). Moreover, in the case of a rank-1 instanton, a non-zero W is compatible only with a purely odd 2-form flux $F_2 = \hat{\mathfrak{f}}_- \hat{D}_-$, while rank-2 instantons can contribute to W also for even fluxes [179].

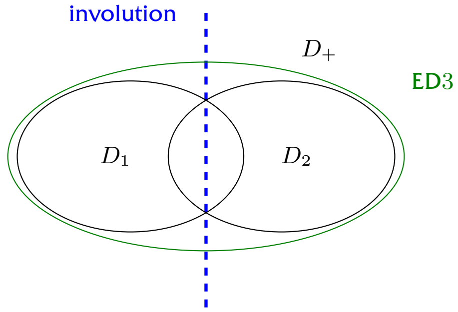

Figure 1.1. ED3-instanton wrapping the smooth orientifold-even divisor D_+.

Restricting just to odd fluxes, the resulting contribution to the superpotential is [177, 178]

$$W_{\text{ED3}} = \sum_{\substack{n\in\mathbb{N}\\ \hat{\mathfrak{f}}_-\in\mathbb{Z}}} A_{n,\hat{\mathfrak{f}}_-}\, e^{-2\pi n\left(T + k\hat{\mathfrak{f}}_- G + \frac{1}{2}k\hat{\mathfrak{f}}_-^2 \bar{S}\right)} . \tag{1.172}$$

This expression shows that in general ED3-instantons generate a scalar potential for C_4 axions, while they lift C_2 axions only in the presence of fluxes.

Interestingly, in the presence of D7-branes some of the terms of this series can be absent due to gauge invariance. To see this more precisely, let us consider a stack of N D7-branes wrapped either on D_+, as shown in Fig. 1.2, or on D_1 and its orientifold image D_2, as shown in Fig. 1.3.

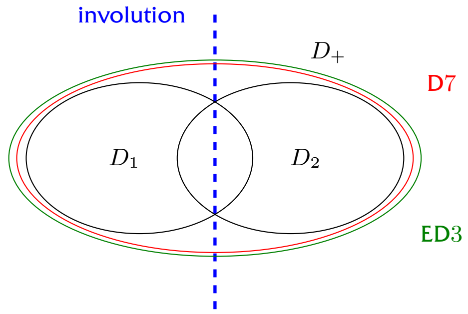

Figure 1.2. ED3-instanton and a stack of D7-branes wrapping the smooth orientifold-even divisor D_+.

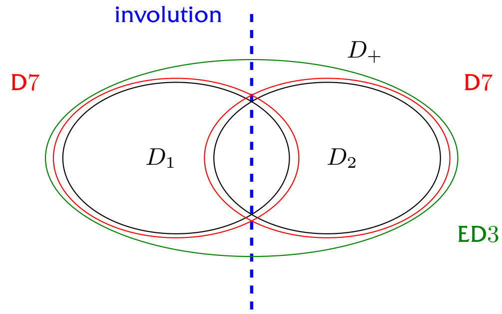

Figure 1.3. ED3-instanton wrapping the smooth orientifold-even divisor D_+ and a stack of D7-branes wrapping D_1 and its orientifold image D_2.

The T- and G-fields can get charged under the diagonal $U(1)$ of the stack of N D7-branes. G develops a non-zero charge q_G due to a geometric Stückelberg mechanism only when the D7s wrap D_1, while the $U(1)$ charge of T, q_T, is due to the world-volume flux on the D7-stack $\hat{F}_2 = \mathfrak{f}_+ \hat{D}_+ + \mathfrak{f}_- \hat{D}_-$:[6]

$$\text{D7 on } D_+ : \quad q_G = 0\,, \qquad q_T = -2N\,\tilde{k}\,\mathfrak{f}_+\,, \tag{1.173}$$

$$\text{D7 on } D_1 : \quad q_G = N\,, \qquad q_T = -N\left(\tilde{k}\,\mathfrak{f}_+ + k\,\mathfrak{f}_-\right). \tag{1.174}$$

Consequently the ED3-instanton acquires a $U(1)$ charge given by [177]

$$\text{D7 on } D_+ : \quad q = 2\,n\,N\,\tilde{k}\,\mathfrak{f}_+\,, \tag{1.175}$$

$$\text{D7 on } D_1 : \quad q = n\,N\left[k\left(\mathfrak{f}_- - \hat{\mathfrak{f}}_-\right) + \tilde{k}\,\mathfrak{f}_+\right], \tag{1.176}$$

which in general induce non-zero charges for all terms in the ED3-instanton series (1.172). In order to obtain a gauge invariant contribution to the superpotential, each of these terms has therefore to be multiplied by an operator of the form $\mathcal{O} \sim \Pi_i \Phi_i$ involving a product of open string modes whose $U(1)$-charge cancels the one of the instanton. However, if these are visible sector fields, they have to acquire a vanishing vacuum expectation value in order not to break the Standard Model gauge symmetry at high scales. Hence, if the Standard Model lives on the D7-stack under consideration, the only possibility to have a non-zero ED3-contribution to

[6] More precisely, the even flux on the ED3 and the D7-stack contains also a contribution from the B_2-field which can be either 0 or $1/2$, so that $\mathcal{F}_2 = F_2 - B_2$ for both of them. For non-spin cycles, Freed-Witten anomaly cancellation forces a half integer contribution to the even F_2 flux which can be cancelled by choosing $b_+ = 1/2$ so that the total even flux for the ED3 is zero. With this choice of B_2-field, the gauge flux on the D7s is then just given by $\hat{F}_2$ with integer quanta when the D7s wrap D_+, while a half integer contribution should be added to the even flux $\hat{F}_2$ when the D7s wrap D_1. In this case, we shall however omit this contribution and consider it implicitly included in $\mathfrak{f}_+$.

W is by choosing the flux quanta such that $q = 0$ without the need of any field-dependent prefactor. However this is never possible when the D7s wrap D_+ since $\tilde{k}\,\mathfrak{f}_+$ is necessarily non-zero given that it is proportional to the number of chiral states on the D7-stack. This kills any possible ED3 contribution to W, which is a manifestation of the known tension between chirality and moduli stabilization [180]. On the other hand, when the D7s wrap D_1, the $U(1)$-charge (1.176) can be vanishing for an appropriate value of $\hat{\mathfrak{f}}_-$. In fact, the only non-zero contribution in the ED3-expansion (1.172) is the one corresponding to $\mathfrak{f}_-$ such that

$$\hat{\mathfrak{f}}_- = \mathfrak{f}_- + \frac{\tilde{k}}{k}\,\mathfrak{f}_+ \equiv \mathfrak{f}\,. \tag{1.177}$$

Thus the ED3-series (1.172) would reduce to:

$$\text{SM D7 on } D_+ : \quad W_{\text{ED3}} = 0\,, \tag{1.178}$$

$$\text{SM D7 on } D_1 : \quad W_{\text{ED3}} = \sum_{n\in\mathbb{N}} A_{n,\mathfrak{f}}\, e^{-2\pi n\left(T + k\mathfrak{f}G + \frac{1}{2}k\mathfrak{f}^2\bar{S}\right)} \tag{1.179}$$

On the other hand, if the D7-stack wrapping D_+ or D_1 is a hidden sector, open string fields can acquire non-zero vacuum expectation values, and so all terms in (1.172) can in principle survive.

Gaugino condensation on D7-branes

The superpotential can in general also receive a non-zero contribution from gaugino condensation in the gauge theory living on a stack of N D7-branes. We shall consider the case where N D7-branes wrap D_1 and N D7-branes wrap its orientifold image D_2, as shown in Fig. 1.4 (similar considerations apply for the case when the D7-branes wrap D_+). In this case the world-volume theory is an $SU(N)$ gauge theory and we shall allow for a general gauge flux of the form $F_2 = \mathfrak{f}_+\hat{D}_+ + \mathfrak{f}_-\hat{D}_-$.

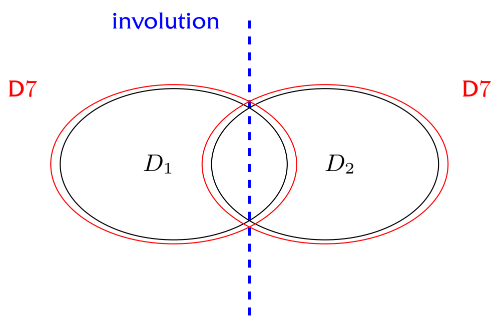

Figure 1.4. A stack of D7-branes wrapped around D_1 and its orientifold image D_2.

In this case the induced W is given in terms of the gauge kinetic function f_{D7} as

$$W_{\text{D7}} = A\, e^{-\frac{2\pi}{N} f_{\text{D7}}}\,, \tag{1.180}$$

where f_{D7} reads

$$\text{D7 on } D_+ : \quad f_{\text{D7}} = T + k\mathfrak{f}_- G + \frac{1}{2}\left(k\mathfrak{f}_-^2 + \tilde{k}\mathfrak{f}_+^2\right)\bar{S}\,, \tag{1.181}$$

$$\text{D7 on } D_1 : \quad f_{\text{D7}} = T + k\left(\mathfrak{f}_+ + \mathfrak{f}_-\right)G + \frac{1}{2}\left(k\mathfrak{f}_-^2 + \tilde{k}\mathfrak{f}_+^2 + 2k\mathfrak{f}_+\mathfrak{f}_-\right)\bar{S}\,. \tag{1.182}$$

Note that W_{D7} would have a $U(1)$-charge of the form

$$\text{D7 on } D_+ : \quad q = 2\,N\,\tilde{k}\,\mathfrak{f}_+\,, \tag{1.183}$$

$$\text{D7 on } D_1 : \quad q = N\left(\tilde{k} - k\right)\mathfrak{f}_+\,, \tag{1.184}$$

which could vanish if $\mathfrak{f}_+ = 0$ (or also for $\tilde{k} = k$ when gaugino condensation is on D_1). For $\mathfrak{f}_+ \neq 0$, the $U(1)$-charge is non-zero, and so gaugino condensation can generate an Affleck-Dine-Seiberg non-zero contribution to the superpotential [181] only in the presence of a prefactor $\mathcal{O}$ which depends on chiral matter fields with appropriate $U(1)$-charges to make W_{D7} gauge invariant. Clearly these fields need also to develop non-zero vacuum expectation values, which is not necessarily a problem if the $SU(N)$ theory undergoing gaugino condensation belongs to a hidden sector. Let us however point out that the generation of a non-zero W_{D7} should be studied carefully since this situation is more complicated than the simplest one with no gauge fluxes where the world-volume theory is a pure $SU(N)$ gauge theory that is known to undergo gaugino condensation.

Comparing (1.180) with (1.179) one immediately sees that, for $N > 1$, ED3 contributions are subleading in respect to the one from gaugino condensation on D7-branes, and thus can be safely ignored when W_{D7} is generated.

An intriguing possibility, which is clearly harder to realize explicitly, is when branes in the same stack are differently magnetized. In this case the original $SU(N)$ theory factorizes into $SU(N_1) \times SU(N_2) \times ... \times SU(N_p)$ with $N_1 + N_2 + ...N_p = N$ allowing, in principle, for multiple gaugino condensation contributions to W where each of them takes the same form as (1.180):

$$W_{\text{D7}} = \sum_{i=1}^{p} A_i\, e^{-\frac{2\pi}{N} f_{\text{D7},i}}\,. \tag{1.185}$$

Alternatively, multiple non-perturbative corrections to W due to gaugino condensation could arise from different stacks of D7-branes wrapped around 4-cycles which are distinct representatives of the same homology class [173].

ED1-instantons and gaugino condensation on D5-branes

Another potential source of non-perturbative corrections to the effective action are ED1-instantons and gaugino condensation on D5-branes wrapping internal 2-cycles. Due to holomorphy, these effects are expected to correct the Kähler potential but not the superpotential [170, 182, 183]. This can be seen as follows. Due to the general arguments presented in [176], any non-perturbative correction to the superpotential should go to zero either in the large volume limit or for vanishing string coupling. Hence ED1/D5 non-perturbative corrections to W should depend on the volume of the wrapped 2-cycle t as $W_{\text{ED1/D5}} \sim e^{-(t+G)}$, so that $W_{\text{ED1/D5}} \to 0$ for $t \to \infty$. However, as can be seen from (1.163), t is not a correct chiral coordinate for the type IIB supergravity effective theory. The correct chiral superfield is instead $T \sim t^2$. Thus, the putative superpotential $W_{\text{ED1/D5}} \sim e^{-(\sqrt{T}+G)}$ would not be a holomorphic function, and so it is expected to vanish. On the other hand, note that a non-zero non-perturbative W could arise from gaugino condensation on D5-branes wrapping vanishing 2-cycles [168], though this in turn introduces new subtleties, in particular, control of the effective field theory around the singularity, and that axions can be 'eaten' up by anomalous $U(1)$'s at singularities [134, 150, 151].

We shall therefore ignore potential ED1/D5 corrections to W but we will consider the possibility of non-perturbative corrections to K since this quantity is not protected by holomorphy. These corrections have not been computed explicitly in type IIB (see however [184] for a derivation of in type I toroidal orbifolds). However the authors of [182, 183] estimated the scaling of the leading ED1/D5 corrections to the Kähler potential. Here, we generalize their results proposing an educated guess for the series of non-perturbative corrections to K from ED1/D5-branes wrapped on an internal 2-cycle t with non-zero odd gauge flux. Making an analogy with the ED3 case (1.172), we propose

$$K_{\text{ED1}} = -3\ln\left(\text{Re}\,(T) - \frac{2\gamma}{g_s^2}\,b^2 + \ldots + \sum_{\substack{n\in\mathbb{N}\\ \hat{\mathfrak{f}}_-\in\mathbb{Z}}} A_{n,\hat{\mathfrak{f}}_-}\, e^{-2\pi n\left(\frac{t}{\sqrt{g_s}}+k\hat{\mathfrak{f}}_- G\right)}\right), \tag{1.186}$$

where

$$t = \sqrt{\frac{2}{\tilde{k}}\left(\text{Re}\,(T) - \frac{2\gamma}{g_s^2}\,b^2\right)}\,, \tag{1.187}$$

and the dots denote perturbative corrections in α' and g_s, as well as non-perturbative worldsheet α' corrections [178], which do not depend on the

C_2 axion due to its shift symmetry. In (1.186) we have absorbed in the prefactor $A_{n,\hat{\mathfrak{f}}_-}$ a potential dilaton-dependent factor $e^{-2\pi n \frac{1}{2} k \hat{\mathfrak{f}}_-^2 \bar{S}}$.

Making an analogy with the D7 case (1.185), we can also propose a similar form for the corrections to the Kähler potential for the case of multiple gaugino condensates on a stack of D5-branes (absorbing again in the prefactors potential $\bar{S}$-dependent exponents)

$$K_{\text{D5}} = -3\ln\left(\text{Re}\,(T) - \frac{2\gamma}{g_s^2}\,b^2 + ... + \sum_{i=1}^{p} A_i\, e^{-\frac{2\pi}{N_i}\left(\frac{t}{\sqrt{g_s}} + k\mathfrak{f}_i G\right)}\right). \qquad (1.188)$$

ED(-1)-instantons and gaugino condensation on D3-branes

Other possible non-perturbative effects in type IIB can be generated by ED(-1)-instantons or gaugino condensation on D3-branes at singularities. We shall however not consider these corrections to the superpotential since they depends just on the dilaton (and the blow-up mode resolving the local singularity T_{loc}) [100], $W_{\text{ED(-1)/D3}} \sim e^{-(S+T_{\text{loc}})}$. Moreover, in this case, C_4 and C_2 axions tend to be removed from the low energy theory since they get eaten up by anomalous $U(1)$s localized at the singularity [175].

1.5.4 Odd axions and D-terms

As we have explained above, the shift symmetry of C_2 axions can be broken only in the presence of non-zero 2-form fluxes. However these world-volume fluxes generate also moduli-dependent Fayet-Iliopoulos (FI) terms for the diagonal $U(1)$ of D7-branes. We need therefore to analyse these FI-terms carefully.

Fayet-Iliopoulos terms

For a stack of D7-branes wrapping the divisor D_{D7} with world-volume flux $\mathcal{F}_2 = F_2 - B_2$ the FI-term takes the form [171, 172]

$$\xi_{\text{FI}} = \frac{1}{4\pi\mathcal{V}} \int_{D_{\text{D7}}} J \wedge \mathcal{F}_2\,. \qquad (1.189)$$

The Kähler form can be expanded as $J = t^\alpha \hat{D}_\alpha$, while the gauge flux $\mathcal{F}$ can be decomposed as (without including potential half integer contributions for the even B_2 field)

$$\mathcal{F}_2 = F_2 - B_2 = \mathfrak{f}^\alpha \hat{D}_\alpha + (\mathfrak{f}^a - b^a)\,\hat{D}_a\,. \qquad (1.190)$$

Focusing for concreteness on the simple case with $h^{1,1} = 2$ and $h^{1,1}_+ = h^{1,1}_- = 1$, the exact expression of the FI-term (1.189) depends on the nature of the divisor $D_{\rm D7}$ wrapped by the D7-branes. If $D_{\rm D7} = D_+$, as in Fig. 1.2, we have

$$\xi_{\rm FI} = \frac{1}{4\pi\mathcal{V}} \int_{D_+} J \wedge \mathcal{F}_2 = \left(\frac{3\tilde{k}}{4\pi}\right) \frac{\mathfrak{f}_+}{\tau} \,, \tag{1.191}$$

which does not depend on the B_2 axion since $k_{++-} = 0$. Note that for and ED3-instanton (1.191) is identically zero since the flux can only be purely odd, i.e. $\mathfrak{f}_+ = 0$, to have a non-zero contribution to W.

On the other hand, if $D_{\rm D7} = D_1$, as in Fig. 1.3, the FI-term (1.189) takes the form

$$\begin{aligned} \xi_{\rm FI} &= \frac{1}{4\pi\mathcal{V}} \int_{D_1} J \wedge \mathcal{F}_2 = \frac{1}{8\pi\mathcal{V}} \int_{D_+} J \wedge \mathcal{F}_2 + \frac{1}{8\pi\mathcal{V}} \int_{D_-} J \wedge \mathcal{F}_2 \\ &= \frac{3}{8\pi\tau} \left[\tilde{k}\mathfrak{f}_+ + k\left(\mathfrak{f}_- - b\right)\right] , \end{aligned} \tag{1.192}$$

which introduces an explicit dependence on the B_2 axion, even for $F_2 = 0$. In what follows we shall assume, without loss of generality, $F_2 = \mathfrak{f}\hat{D}_1 - \mathfrak{f}\hat{D}_2 = \mathfrak{f}\hat{D}_-$, which implies $\mathfrak{f}_- = \mathfrak{f}$ and $\mathfrak{f}_+ = 0$. This choice guarantees that ED3/D7 non-perturbative corrections to W maintain their dependence on the G-field but simplifies the expression of the FI-term (1.192) to

$$\xi_{\rm FI} = \left(\frac{3k}{8\pi}\right) \frac{(\mathfrak{f} - b)}{\tau} \,. \tag{1.193}$$

The total D-term potential includes also open string modes χ_i with $U(1)$ charges q_i and looks like

$$V_D = \frac{g^2}{2} \left(\sum_i q_i |\chi_i|^2 - \xi_{\rm FI}\right)^2 \quad \text{with} \quad g^2 = \frac{4\pi}{\text{Re}(T)} \,. \tag{1.194}$$

The D-term potential scales as $V_D \sim \xi_{\rm FI}^2/\tau \sim \mathcal{V}^{-2}$, and so it is leading with respect to the F-term potential due to the no-scale cancellation. Hence, the minimum is located at

$$q\,|\chi|^2 \simeq \xi_{\rm FI} \,, \tag{1.195}$$

where for simplicity we have focused just on a single charged matter field whose charge q has an opposite sign with respect to the FI-term. All charged matter fields whose charge has the same sign as $\xi_{\rm FI}$ are instead fixed to zero.

As can be seen from (1.193), $\xi_{\rm FI}$ is in general a function of two fields: τ and b. Hence, the relation (1.195) fixes one direction, among $|\phi|$, τ and b, in terms of the other two which, at this level of approximation, remain still flat. They are lifted by subdominant F-term contributions. Before studying the stabilization of these two directions, let us point out that the axionic partner of the saxionic direction fixed by the D-terms is eaten up by the anomalous $U(1)$ via the Stückelberg mechanism. This axion is in general a linear combination of the axionic phase ζ of $\phi = |\phi|\, e^{i\zeta}$, the C_4 axion θ, and the C_2 axion c. The resulting $U(1)$ mass is proportional to the decay constants of these axions [185]

$$M^2_{U(1)} \sim g^2 \left(f_\zeta^2 + f_\theta^2 + f_c^2\right), \tag{1.196}$$

where

$$f_\zeta^2 \simeq \xi_{\mathsf{FI}} \sim \frac{(\mathfrak{f}-b)}{\tau}\,, \qquad f_\theta^2 \simeq K_{T\bar{T}} \sim \frac{1}{\tau^2}\,, \qquad f_c^2 \simeq K_{G\bar{G}} \sim \frac{g_s}{\tau}\,. \tag{1.197}$$

In the presence of a hierarchy among these decay constants, the combination of axions eaten up by the anomalous $U(1)$ is mostly given by the axion with the largest decay constant. We will see that F-term stabilization gives two branches for b:

- $b = 0$: in this case the Abelian gauge boson becomes massive by eating up the open string axion ζ since

$$f_\zeta^2 \sim \frac{\mathfrak{f}}{\tau} \gg f_c^2 \sim \frac{g_s}{\tau} \gg f_\theta^2 \sim \frac{1}{\tau^2} \qquad \text{for} \quad \tau^{-1} \lesssim \mathcal{O}(0.01) \ll g_s \sim \mathcal{O}(0.1) \ll \mathfrak{f} \sim \mathcal{O}(1)\,. \tag{1.198}$$

- $b = \mathfrak{f}$: in this case the Abelian gauge boson becomes massive by eating up the C_2 axion c since

$$f_c^2 \sim \frac{g_s}{\tau} \gg f_\theta^2 \sim \frac{1}{\tau^2} \gg f_\zeta^2 \sim 0 \quad \text{for} \quad \tau^{-1} \lesssim \mathcal{O}(0.01) \ll g_s \sim \mathcal{O}(0.1)\,. \tag{1.199}$$

B_2 axion stabilization

The two directions left flat by D-term stabilization are lifted by F-term contributions. The matter fields receive F-term contributions from soft supersymmetry breaking terms of order the gravitino mass

$$V_F(|\chi|) = \mathcal{C}\, m_{3/2}^2\, |\chi|^2 + \ldots = \frac{|W_0|^2}{\mathcal{V}^2}\, |\chi|^2 + \ldots\,, \tag{1.200}$$

where $\mathcal{C}$ is an $\mathcal{O}(1)$ coefficient and the dots denote potential contributions with higher powers of $|\chi|$. On the other hand, the F-term potential for the Kähler moduli τ and b is given in KKLT by (1.106) and in LVS by (1.140) after including the dependence on b through the mixing in the Kähler coordinates, as can be seen from (1.167). Using the general formalism developed in [170], in the KKLT case we obtain (after fixing the C_4 axion)

$$V_{\text{KKLT}}(\tau, b) \simeq \frac{\mathfrak{a}^2 A^2 e^{-2\mathfrak{a}(\tau+\tilde{\gamma}\, b^2)}}{6\tau} - \frac{\mathfrak{a} A |W_0| e^{-\mathfrak{a}(\tau+\tilde{\gamma}\, b^2)}}{2\tau^2} + \frac{M^2}{12\tau^2}\,, \tag{1.201}$$

where we have defined $\tilde{\gamma} \equiv 2\gamma/g_s^2$ and we have included only the leading terms for $\mathfrak{a}\tau \gg 1$ and $\tau \gg \tilde{\gamma} b^2$.

In LVS we will consider only the case where the G-modulus mixes with the big modulus T_b, i.e. $k_{s--} = 0$ while $k_{b--} \neq 0$, since the huge $e^{-\mathfrak{a}_b \tau_b} \lll 1$ suppression is crucial to reproduce the correct EDE scale. Hence the total LVS potential including the B_2 axion becomes

$$V_{\text{LVS}}(\mathcal{V}, \tau_s, \theta_b, b) = V_{\text{LVS}}(\mathcal{V}, \tau_s) - \frac{4\mathfrak{a}_b A_b |W_0|\, e^{-\mathfrak{a}_b(\tau_b+\tilde{\gamma}\, b^2)}}{\mathcal{V}^{4/3}} \cos\left(\mathfrak{a}_b \theta_b\right)\,, \tag{1.202}$$

where $V_{\text{LVS}}(\mathcal{V}, \tau_s)$ is given by (1.140) with the axion θ_s fixed at zero, and we have included again only the leading terms for $\tau_b \gg \tilde{\gamma} b^2$.

Let us analyse the KKLT and LVS cases separately for D7-branes wrapping either D_+ of D_1.

- **KKLT with D7s on D_+ or D_1:** D-term fixing gives

$$q\,|\chi|^2 \sim \frac{(\mathfrak{f} - d\, b)}{\tau}\,, \tag{1.203}$$

  where $d = 0$ and $\mathfrak{f} = \mathfrak{f}_+$ when $D_{\text{D7}} = D_+$, while $d = 1$ when $D_{\text{D7}} = D_1$. The relation (1.203) fixes $|\chi|$ in terms of τ and b (or just τ for $D_{\text{D7}} = D_+$). The remaining flat directions are fixed by $V_{\text{KKLT}}(\tau, b)$ since this potential dominates over $V_F(|\chi|)$. In fact, substituting (1.203) in (1.200) we obtain

$$V_F(|\chi|) \sim (\mathfrak{f} - d\, b)\, \frac{|W_0|^2}{\tau^4} \ll V_{\text{KKLT}} \sim \frac{|W_0|^2}{\tau^3} \quad \text{for} \quad \tau \gg 1\,. \tag{1.204}$$

  It is then straightforward to realize that V_{KKLT} fixes $b = 0$ and τ as in KKLT, implying that the axion eaten up by the anomalous $U(1)$ is ζ.

- **LVS with D7s on D_+:** D-term moduli stabilization sets

$$q\,|\chi|^2 \sim \frac{\mathfrak{f}_+}{\mathcal{V}^{2/3}}\,, \tag{1.205}$$

which fixes $|\chi|$ in terms of $\mathcal{V}$. Substituting this result in (1.200) we obtain

$$V_F(|\chi|) \sim \mathfrak{f}_+ \frac{|W_0|^2}{\mathcal{V}^{8/3}}\,, \tag{1.206}$$

that represents the standard expression for T-brane dS uplifting [29]. Hence, in this case the UV consistency of the underlying model forces the presence of a precise dS uplifting source, in addition to the potential existence of anti D3-branes. The Kähler moduli and the B_2 axion are then stabilized by (1.206) together with the LVS potential (1.202) which fix $b = 0$ and the T-moduli as in standard LVS construction. This implies that the axion eaten up by the anomalous $U(1)$ is again the open string mode ζ.

- **LVS with D7s on D_1:** D-term fixing implies

$$q\,|\chi|^2 \sim \frac{(\mathfrak{f} - b)}{\mathcal{V}^{2/3}}\,, \tag{1.207}$$

which fixes b in terms of $|\chi|$ and $\mathcal{V}$ that have to be considered as two independent variables. Given that the LVS potential (1.202) does not depend on $|\chi|$, the charged matter field has to be stabilized by its F-term potential (1.200). Two different situations can arise:

1. If $\mathcal{C} > 0$, the minimum for $|\chi|$ lies at $|\chi| = 0$. Substituting this result back in (1.207) we find $b = \mathfrak{f}$ which implies that the C_2 axion is removed from the low energy effective theory since it is eaten up by the anomalous $U(1)$. The location of the $\mathcal{V}$ minimum remains instead the same as in the LVS case without odd moduli since it is still determined by the leading order potential V_{LVS} in (1.202).
2. If $\mathcal{C} < 0$, the matter field is tachyonic and can develop a non-zero vacuum expectation value. For example, if the F-term potential for $|\chi|$ features an additional cubic contribution of the form (setting $|W_0| \sim \mathcal{O}(1)$):

$$V_F(|\chi|) = -\frac{1}{\mathcal{V}^2}\,|\chi|^2 + \frac{1}{\mathcal{V}^\alpha}\,|\chi|^3\,, \tag{1.208}$$

with $\alpha > 0$, the minimum of the charged matter field is located at $|\chi| \sim \mathcal{V}^{\alpha-2}$. Substituting this relation back in (1.207) we obtain

$$(\mathfrak{f} - b) \sim \mathcal{V}^{2\left(\alpha - \frac{5}{3}\right)}\,. \tag{1.209}$$

For $\alpha < 5/3$, $(\mathfrak{f} - b)$ is $\mathcal{V}$-suppressed, and so the solution is given again, at first approximation, by $b \simeq f$, implying that the C_2 axion is eaten up. On the other hand, for $\alpha \geq 5/3$, $(\mathfrak{f} - b)$ becomes larger than unity. Consequently, as can be seen from (1.197), the axion eaten up by the anomalous $U(1)$ becomes the open string mode ζ since its decay constant becomes larger than the one of the C_2 axion. As in the previous case, the T-moduli are still fixed by the standard LVS potential.

The best case scenario is therefore the one where the B_2 axion is fixed at zero, so that the C_2 axion can survive in the low energy effective theory. Let us finally stress that, at this level of approximation, the B_2 axion becomes massive, while the C_2 axion is still massless. If the potential for the C_2 axion is generated by subleading non-perturbative effects, the resulting moduli mass spectrum would feature a hierarchical structure with the lightest mode given by the C_2 axion.

Chapter II

Flux Vacua with Perturbative Flat Directions

In this chapter we will perform appropriate choices of flux quanta that induce relations between the flux superpotential and its derivatives. This method is implemented in toroidal and Calabi-Yau compactifications in the large complex structure limit. Explicit solutions are obtained and classified on the basis of duality equivalences. Let us recall that in this chapter we will use the notation for the axio-dilaton $\phi = \mathrm{i}\bar{S}$.

2.1 Outline of the strategy and main results

The superpotential in type IIB compactifications is given by the sum of the GVW superpotential (1.45) and non-perturbative corrections. We will work with the GVW term (the non-perturbative terms are small corrections in the large radius limit) and search for supersymmetric minima with flat directions. At this level the conditions for supersymmetry are $D_\phi W = \partial_\phi W + W\partial_\phi K = 0$ and $D_{U_\alpha} W = \partial_{U_\alpha} W + W\partial_{U_\alpha} K = 0$ where U_α $(\alpha = 1, \ldots, h_-^{2,1})$ are the complex structure moduli [10].[1] Given that (1.45) does not depend on the Kähler moduli T_i $(i = 1, \ldots, h_+^{1,1})$, the F-flatness conditions for these modes is $W = 0$ since $D_{T_i} W = W\partial_{T_i} K$ with $\partial_{T_i} K \neq 0$ for finite field values. Thus supersymmetry at classical level requires $W = \partial_\phi W = \partial_{U_\alpha} W = 0$. Notice that flat directions can clearly exist also for $W \neq 0$ where supersymmetry is definitely broken by the Kähler moduli (and potentially by the axio-dilaton and the complex structure moduli as well). Despite being interesting for phenomenological and cosmological applications, these solutions would typically be characterized

[1] For toroidal compactifications primitivity of the fluxes has to be imposed as an additional requirement. This is due to the presence of holomorphic 1-forms on tori [10]. In our study of a toroidal case we impose this condition at the very end, after having obtained solutions to the F-flatness conditions.

by large values of W_0 which are incompatible with the KKLT scenario (and some LVS models with T-brane uplifting).

Our analysis will be for toroidal orientifolds and CY compactifications in the large complex structure limit where the superpotential is a polynomial (after dropping exponentially small terms in the large complex structure limit). Thus the F-flatness conditions $W = \partial_\phi W = \partial_{U_\alpha} W = 0$ are $n + 1 = h^{2,1}_- + 2$ polynomial equations in n complex variables which in general do not have a solution since the system is overdetermined. In addition, we are interested in solutions with $p \geq 1$ flat directions which can exist if the number of linearly independent equation is reduced from $(n + 1)$ to $(n - p)$ by an appropriate flux choice. Thus the first step is to understand which choice of flux quanta can yield solutions with flat directions. At present the answer to this in full generality is unknown, and so we have to resort to a well motivated ansatz.

Before discussing our ansatz, let us recapitulate the basic idea in [12, 31]. Flux quanta were chosen so that W was a degree-2 homogeneous polynomial. For such superpotentials:

$$2W = \phi\, \partial_\phi W + U_\alpha\, \partial_{U_\alpha} W\,, \tag{2.1}$$

holds as a functional relation (i.e. on all points on the moduli space). This implies that the $W = 0$ equation is automatically satisfied once the derivatives of W vanish. Furthermore the scaling behaviour of W implies that, if $(\hat{\phi}, \hat{U}_\alpha)$ is a solution, $\phi = \lambda\hat{\phi}$, $U_\alpha = \lambda\hat{U}_\alpha$ remains a solution, signalling the existence of a flat direction parametrised by λ.[2] This implies that, on top of (2.1), $\partial_\phi W$ can be expressed as a linear combination of the derivatives of W with respect to the complex structure moduli. This can be easily seen in the $h^{1,2}_- = 1$ case where, setting $c \equiv W_{\phi\phi}W_{UU} - W_{\phi U}W_{U\phi}$, one has:

$$\begin{cases} W_{\phi\phi}W_U = W_{U\phi}W_\phi + c\,U \\ 2W_{\phi\phi}W = W_\phi^2 + c\,U^2 \end{cases} \quad \xrightarrow{c\,=\,0} \quad \begin{cases} W_U = \left(\frac{W_{U\phi}}{W_{\phi\phi}}\right) W_\phi \\ 2W_{\phi\phi}W = W_\phi^2 \end{cases}, \tag{2.2}$$

showing that the flux choice $c = 0$ (or $W_{\phi\phi}W_{UU} = W_{\phi U}W_{U\phi}$) guarantees that $W = 0$ for $U \neq 0$ and the fact that $W = \partial_U W = 0$ is an automatic consequence of $\partial_\phi W = 0$, signalling the presence of a flat direction.

The lesson to take from the above is that superpotentials where there are functional relations between W and its derivatives, such that the vanishing of some implies the vanishing of other(s), are particularly suited

[2] Unless $\hat{\phi} = 0$ and $\hat{U}_\alpha = 0\ \forall\alpha$ which is however a situation that we do not consider since it would lead to a breakdown of the effective field theory.

for obtaining solutions with flat directions. In this chapter we will focus on the more general case where W is not necessarily a homogeneous function but its derivatives are linearly dependent:

$$\lambda_\phi \, \partial_\phi W + \lambda_\alpha \, \partial_{U_\alpha} W = 0 \,, \tag{2.3}$$

where λ_ϕ and λ_α are constants with no moduli dependence. Our strategy is as follows:

1. Given a toroidal orientifold or an orientifolded CY in the large complex structure limit, we compute the superpotential in full generality as a function of the flux vectors and moduli.

2. We impose that a condition of the form (2.3) holds as a functional relation, and determine the constraints that this sets on the fluxes. At this stage λ_ϕ and λ_α are to be thought of as parameters in the ansatz for the fluxes. Thus the constrained fluxes are allowed to depend on them. This in general reduces the number of independent equations from $n + 1$ to n.

3. Taking the fluxes obtained in the previous step, we impose the F-flatness conditions and the requirement to have at least 1 flat direction. Unlike the case of a degree-2 homogeneous superpotential, a flat direction is not guaranteed if just a condition of the form (2.3) holds. When possible, the existence of a flat direction is obtained by an appropriate choice of λ_ϕ and λ_α which reduces further the number of independent equation from n to $n - p$ with $p \geq 1$. Thus the requirement of a flat direction can further constrain the fluxes.[3]

4. The end result of step 3 are solutions to the F-flatness conditions with at least 1 flat direction and flux vectors parametrised by λ_ϕ and λ_α. Of these we isolate the subset of flux vectors that satisfy the integrality and the D3 tadpole condition. We also impose physical restrictions such as the positivity of $\text{Im}\,\phi$ which sets the string coupling ($\text{Im}\,\phi = g_s^{-1}$).

5. For toroidal examples we finally impose also the primitivity of G_3 to have a supersymmetric solution.

[3] In effect, we adjust fluxes to ensure the following. We have n independent equations in n variables: $f^l(U_k) = 0, \; U_k = \phi, U^a$, after step 2. For cases with flat directions, $\det \left(\partial_k f^l\right)$ vanishes at the solution.

A few comments are in order. Implementing the procedure working with the general form of the linear relation is rather cumbersome. It is easier to work case by case with the linear relations being classified by which of the λ_ϕ and λ_α are non-vanishing. We have included the checks for the solutions being physical in step 4 of the procedure. In practice, it is easier to check for these conditions at every stage and discard any candidate solution as soon as it becomes clear that it is unphysical.

Let us highlight our key results. An explicit implementation of the algorithm has been carried out for the $T^6/\mathbb{Z}_2$ orientifold [142, 143], an orientifold of the CY obtained by considering a degree-18 hypersurface in $\mathbb{CP}_{[1,1,1,6,9]}$ (first studied in the context of mirror symmetry in [186] and also the example studied in [12]), and an orientifold of the CY discussed in [134].

- For the $T^6/\mathbb{Z}_2$ orientifold, we find solutions with 1 and 2 flat directions (and no more). The solutions fall into various families (classified according to the nature of the linear relation that holds). In all solutions the residual moduli space contains regions in which the string coupling is arbitrarily small.

- For the $T^6/\mathbb{Z}_2$ orientifold, there are solutions which preserve $N = 2$ supersymmetry in 4 dimensions. Being novel solutions with extended supersymmetry, they are interesting in their own right.

- For the $\mathbb{CP}_{[1,1,1,6,9]}[18]$ case, we find essentially 1 family of fluxes which lead to solutions with 1 flat direction corresponding to the axio-dilaton. One can ensure that the moduli take on values in the large complex structure limit (as is required for the consistency of our analysis) when the string coupling is taken to arbitrarily small values.

- We find 68 distinct solutions in the $\mathbb{CP}_{[1,1,1,6,9]}[18]$ case, 15 of which are entirely novel since the superpotential is a non-homogeneous polynomial. The remaining 53 solutions can instead be mapped by duality to the case when the superpotential is a degree-2 homogeneous polynomial. However only 2 out of these 53 solutions lie at weak string coupling and in a regime where the large complex structure limit is definitely under control, reproducing the old vacua already found in [12, 23, 26].

- For the $\mathbb{CP}_{[1,1,1,6,9]}[18]$ case, we find also solutions with 1 axionic flat direction and $W \neq 0$ which represent promising starting points for an explicit CY realization of winding dS uplift [38]. In this case, W is still a polynomial of degree 2 but not a homogeneous function.

- For the CY studied in [134], which features effectively 1 complex structure modulus more than the $\mathbb{CP}_{[1,1,1,6,9]}[18]$ example, we present a preliminary analysis where we find solutions with 2 flat directions. Again, W is a polynomial of degree 2 that is always non-homogeneous when $W \neq 0$, while it can become a homogeneous function for some flux quanta only when $W = 0$ at the minimum (as in the $\mathbb{CP}_{[1,1,1,6,9]}[18]$ case, there are $W = 0$ cases where W cannot be made non-homogeneous by duality).

2.2 Solutions in Toroidal Orientifolds

In this section we will study classical supersymmetric solutions with flat directions that can arise in the $T^6/\mathbb{Z}_2$ orientifold. This is the setting where some of the first explicit computations of the flux potential in type IIB were carried out [142, 143]. Flux vacua in the toroidal setting have been studied in much detail (see e.g. [140, 187-195] for related studies).

We will consider the class in which the flux vectors are diagonal, i.e.:

$$
\begin{aligned}
a^{ij} &= \mathrm{diag}\{a_1, a_2, a_3\}, \;\; b_{ij} = \mathrm{diag}\{b_1, b_2, b_3\}, \\
c^{ij} &= \mathrm{diag}\{c_1, c_2, c_3\}, \;\; d_{ij} = \mathrm{diag}\{d_1, d_2, d_3\}\,,
\end{aligned}
\tag{2.4}
$$

which lead to:

$$
N_{\text{flux}} = (b_0 c_0 - a_0 d_0) + (b_1 c_1 - a_1 d_1) + (b_2 c_2 - a_2 d_2) + (b_3 c_3 - a_3 d_3)\,. \tag{2.5}
$$

Given that the structure of (1.100) implies a diagonal form of the period matrix, we take:

$$
\tau^{ij} = \mathrm{diag}\{\tau_1, \tau_2, \tau_3\}\,. \tag{2.6}
$$

Note that this corresponds to a $T^2 \times T^2 \times T^2$ factorization of the T^6 with τ_α $(\alpha = 1, 2, 3)$ as the complex structure moduli of the 3 2-tori. For notational convenience we introduce:

$$
(U_1, U_2, U_3, U_4) \equiv (\tau_1, \tau_2, \tau_3, \phi)\,. \tag{2.7}
$$

With this, (1.99) takes the form:

$$
\begin{aligned}
W = {}& (a^0 - U_4 c^0) U_1 U_2 U_3 \\
& - (a_1 - U_4 c_1) U_2 U_3 - (a_2 - U_4 c_2) U_1 U_3 - (a_3 - U_4 c_3) U_1 U_2 \\
& - (b_1 - U_4 d_1) U_1 - (b_2 - U_4 d_2) U_2 - (b_3 - U_4 d_3) U_3 - (b_0 - U_4 d_0)\,,
\end{aligned}
\tag{2.8}
$$

and the system of equations (1.100) reduces to:

$$a^0 U_1 U_2 U_3 - (a_1 U_2 U_3 + a_2 U_1 U_3 + a_3 U_1 U_2) - (b_1 U_1 + b_2 U_2 + b_3 U_3) - b_0 = 0\,, \tag{2.9}$$

$$c^0 U_1 U_2 U_3 - (c_1 U_2 U_3 + c_2 U_1 U_3 + c_3 U_1 U_2) - (d_1 U_1 + d_2 U_2 + d_3 U_3) - d_0 = 0\,, \tag{2.10}$$

$$(a^0 - U_4 c^0) U_2 U_3 - ((a_2 U_3 + a_3 U_2) - U_4(c_2 U_3 + c_3 U_2)) - (b_1 - U_4 d_1) = 0\,, \tag{2.11}$$

$$(a^0 - U_4 c^0) U_1 U_3 - ((a_1 U_3 + a_3 U_1) - U_4(c_1 U_3 + c_3 U_1)) - (b_2 - U_4 d_2) = 0\,, \tag{2.12}$$

$$(a^0 - U_4 c^0) U_1 U_2 - ((a_1 U_2 + a_2 U_1) - U_4(c_1 U_2 + c_2 U_1)) - (b_3 - U_4 d_3) = 0\,. \tag{2.13}$$

In the next sections we present different families of solutions to these F-flatness and $W = 0$ conditions. We start with an example without any flat direction and we then provide our classification of the solutions with flat directions.[4] Representative examples of flux vectors satisfying the integrality condition are provided for all the families that arise in the classification. We first present solutions with 1 flat direction and then solutions with 2 flat directions (our ansatz does not lead to any solutions with higher number of flat directions). As mentioned earlier, the solutions will be classified according to the nature of the linear relation that the derivatives of the superpotential satisfy. This leads to 3 different cases (all compatible with $N_{\text{flux}} \neq 0$) for which (2.9)-(2.13) admit complex solutions, i.e. Im $U_a \neq 0\ \forall a$, with 1 or 2 flat directions:

1. Linear relation among all derivatives: $\lambda_1\, \partial_1 W + \lambda_2\, \partial_2 W + \lambda_3 \partial_3 W + \partial_4 W = 0$ with $\lambda_\alpha \neq 0\ \forall \alpha = 1,2,3$ which can allow for solutions with $W = 0$ and either 1 or 2 flat directions;

2. Linear relation among the derivatives of W with respect to the axio-dilaton and 1 complex structure modulus: $\lambda_\alpha\, \partial_\alpha W + \partial_4 W = 0$ (no sum over α) with $\alpha = 1,2,3$ which can feature solutions with $W = 0$ and 2 flat directions;

3. Linear relation among the derivatives of W with respect to 2 different complex structure moduli: $\partial_\alpha W = \lambda_\beta\, \partial_\beta W$ (no sum over β) with $\alpha \neq \beta$ and $\alpha, \beta = 1,2,3$ which can give solutions with $W = 0$ and 2 flat directions.

[4] Let us point out that this is not a full classification of the solutions since we obtain only those which satisfy the linear dependence ansatz (2.3).

2.2.1 No Flat Directions

In this section we review a solution presented in [142] which has $W = 0$ but no linearity relation among the superpotential and its derivatives. Hence it does not feature any flat direction since it can be shown that the solution is not part of a continuous family. In this case the fluxes are taken to be proportional to identity:

$$(a^{ij}, b_{ij}, c^{ij}, d_{ij}) = (a, b, c, d)\, \delta_{ij}\,, \tag{2.14}$$
$$a_0 = b_0 = c_0 = -c = -d = 2\,, \quad a = b = 0\,, \quad d_0 = -4\,,$$

and an explicit solution to (1.100) is given by:

$$\tau^{ij} = \tau\, \delta^{ij}\,, \qquad \tau = \phi = e^{\frac{2\pi}{3}}\,. \tag{2.15}$$

For a set of fluxes to find whether a given solution is isolated or part of a continuous family, we will use linearized perturbation theory.[5] For this let us write abstractly the system of equations (1.100) as:

$$f^{(I)}(U_a) = 0\,, \tag{2.16}$$

where I runs over the 11 equations and U_a runs over the 10 variables (τ^{ij}, ϕ). Then if the solution $\hat{U}_a$ is part of a continuous family, the following linear system for δU_a must have a solution:

$$\partial_{U_a} f^{(I)}\Big|_{\hat{U}_a} \delta U_a = 0\,, \tag{2.17}$$

i.e. the rank of the matrix $\partial_{U_a} f^{(I)}(\hat{U}_a)$ should be less than 10. Now, the matrix elements are given by:[6]

$$\partial_{\tau^{ij}} f^{(1)} = \frac{1}{2} a^0 \epsilon_{ikl}\epsilon_{jmn}\tau^{km}\tau^{ln} - a^{km}\epsilon_{ikl}\epsilon_{jmn}\tau^{ln} - b_{ij}\,, \tag{2.18}$$
$$\partial_{\tau^{ij}} f^{(2)} = \frac{1}{2} c^0 \epsilon_{ikl}\epsilon_{jmn}\tau^{km}\tau^{ln} - c^{km}\epsilon_{ikl}\epsilon_{jmn}\tau^{ln} - d_{ij}\,, \tag{2.19}$$
$$\partial_{\tau^{ij}} f^{(3)}_{kl} = (a^0 - \phi c^0)\epsilon_{ikm}\epsilon_{jln}\tau^{mn} - (a^{mn} - \phi c^{mn})\epsilon_{ikm}\epsilon_{ljn}\,, \tag{2.20}$$
$$\partial_{\phi} f^{(1)} = \partial_{\phi} f^{(2)} = 0\,, \tag{2.21}$$
$$\partial_{\phi} f^{(3)}_{kl} = -c^0 (\mathrm{cof}\ \tau)_{kl} + c^{ij}\epsilon_{ikm}\epsilon_{jln}\tau^{mn} + d_{kl}\,. \tag{2.22}$$

[5] This technique is not limited to diagonal fluxes. For a generic choice of fluxes, even if a given solution has diagonal τ^{ij}, that may be a part of a continuous family with non-zero off-diagonal terms. As a result, in general we must deal with a 11×10 matrix, as shown below. However only $\tau^{ij} = \tau\delta^{ij}$ can satisfy (1.100) for fluxes proportional to the identity. As a result, it is possible to work with a matrix with lower dimensions.

[6] Here we use $\partial_{\tau^{ij}} \det \tau = \frac{1}{2}\epsilon_{ikl}\epsilon_{jmn}\tau^{km}\tau^{ln}$, $\partial_{\tau^{ij}}(\mathrm{cof}\ \tau)_{ab} = \epsilon_{ial}\epsilon_{jbn}\tau^{ln}$, and repeated indices are summed.

For fluxes proportional to the identity, these matrix elements evaluated at $(\tau\delta^{ij}, \phi)$ become:

$$\partial_{\tau^{ij}} f^{(1)} = (a^0\tau^2 - 2a\tau - b)\delta_{ij}\,, \tag{2.23}$$

$$\partial_{\tau^{ij}} f^{(2)} = (c^0\tau^2 - 2c\tau - d)\delta_{ij}\,, \tag{2.24}$$

$$\partial_{\tau^{ij}} f^{(3)}_{kl} = [(a^0 - \phi c^0)\tau - (a - \phi c)](\delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk})\,, \tag{2.25}$$

$$\partial_\phi f^{(1)} = \partial_\phi f^{(2)} = 0\,, \tag{2.26}$$

$$\partial_\phi f^{(3)}_{kl} = -(c^0\tau^2 - 2c\tau - d)\delta_{kl}\,. \tag{2.27}$$

The above matrix has rank 10 at (2.15), implying that it is a solution with no flat directions.

2.2.2 1 Flat Direction

Solutions with 1 flat direction are all in 1 family. The linear relation satisfied in this family is:

$$\lambda_1\,\partial_1 W + \lambda_2\,\partial_2 W + \lambda_3\,\partial_3 W + \partial_4 W = 0\,, \tag{2.28}$$

with $\lambda_\alpha \neq 0\ \forall\alpha = 1,2,3$. The flux quanta (introduced in (2.4)) take the form:

$$\begin{aligned}
\{a_0, a_1, a_2, a_3\} &= \{0, \frac{d_3}{\lambda_2} + \frac{d_2}{\lambda_3}, -\frac{d_2\lambda_2}{\lambda_1\lambda_3}, -\frac{d_3\lambda_3}{\lambda_1\lambda_2}\}\,,\\
\{b_0, b_1, b_2, b_3\} &= \{b_0, \frac{d_0 - b_2\lambda_2 - b_3\lambda_3}{\lambda_1}, b_2, b_3\}\,,\\
\{c_0, c_1, c_2, c_3\} &= \{0, 0, 0, 0\}\,,\\
\{d_0, d_1, d_2, d_3\} &= \{d_0, -\frac{d_2\lambda_2 + d_3\lambda_3}{\lambda_1}, d_2, d_3\}\,,
\end{aligned} \tag{2.29}$$

with the condition $d_2, d_3, d_2\lambda_2 + d_3\lambda_3 \neq 0$. With this choice of fluxes N_{flux} becomes:

$$N_{\text{flux}} = \frac{2}{\lambda_1\lambda_2\lambda_3}\left(\lambda_2^2 d_2^2 + \lambda_2\lambda_3 d_2 d_3 + \lambda_3^2 d_3^2\right)\,, \tag{2.30}$$

and the GVW superpotential reduces to:

$$\begin{aligned}
W = &\frac{1}{\lambda_1}\left[(b_2\lambda_2 + b_3\lambda_3 - d_0)U_1 - \lambda_1(b_2U_2 + b_3U_3 - d_0U_4 + b_0)\right]\\
&+ \frac{d_2}{\lambda_1\lambda_3}(U_3 - \lambda_3U_4)(\lambda_2U_1 - \beta U_2) + \frac{d_3}{\lambda_1\lambda_2}(U_2 - \lambda_2U_4)(\lambda_3U_1 - \lambda_1U_3)\,.
\end{aligned} \tag{2.31}$$

Demanding that the derivatives of the superpotential vanish implies that the 3 complex structure moduli U_α, $\alpha = 1,2,3$, are related to the axio-dilaton U_4 as follows:

$$
\begin{aligned}
U_1 &= -\frac{\lambda_1(b_3 d_2 + b_2 d_3)}{2d_2 d_3} + \frac{\lambda_1 d_0(\lambda_2 d_2 + \lambda_3 d_3)}{2\lambda_2\lambda_3 d_2 d_3} + \lambda_1 U_4 \,, \\
U_2 &= -\frac{\lambda_2 b_3}{d_3} + \frac{\lambda_2^2(b_3 d_2 - b_2 d_3)}{2d_3(\lambda_2 d_2 + \lambda_3 d_3)} + \frac{\lambda_2 d_0}{2\lambda_3 d_3} + \lambda_2 U_4 \,, \\
U_3 &= -\frac{\lambda_3 b_2}{2d_2} - \frac{\lambda_3(\lambda_2 b_2 + \lambda_3 b_3)}{2(\lambda_2 d_2 + \lambda_3 d_3)} + \frac{\lambda_3 d_0}{2\lambda_2 d_2} + \lambda_3 U_4 \,.
\end{aligned}
\tag{2.32}
$$

The $W = 0$ condition instead implies:

$$
\begin{aligned}
&(\lambda_2 d_2 + \lambda_3 d_3)\left[4b_0 d_2 d_3 - 2\lambda_2\lambda_3 d_0(b_2 d_3 + b_3 d_2) + d_0^2(\lambda_2 d_2 + \lambda_3 d_3)\right] \\
&+ \lambda_2\lambda_3(b_3 d_2 - b_2 d_3)^2 = 0 \,.
\end{aligned}
\tag{2.33}
$$

Note that this can be thought of as a relation between the parameters λ_2 and λ_3. Hence the flux quanta are essentially parametrised by 2 parameters and some integers. We could have presented the flux vectors as functions of 2 parameters from the very beginning. In this case the $W = 0$ condition would have been automatically satisfied. We did not do so to avoid cluttering the notation.

In summary, the solutions are obtained by choosing the even integers b_0, b_2, b_3, d_0, d_2, d_3 and the parameters λ_α $\alpha = 1,2,3$ such that all flux quanta in (2.29) are even, the $W = 0$ condition (2.33) is met and the D3 tadpole condition $N_{\text{flux}} \leq 32$ (with N_{flux} given in (2.30)) is satisfied. Furthermore, physical consistency conditions such as $\text{Im}\,(U_4) > 0$ must be satisfied. It is easy to find explicit examples. For instance:

$$
\lambda_1 = \lambda_2 = \lambda_3 = 1 \,, \qquad b_2 = b_3 = 0 \,, \qquad d_2 = d_3 = 2 \,, \tag{2.34}
$$

and:

$$
b_0 = -4p^2 \,, \qquad d_0 = 4p \,, \quad p \in \mathbb{Z} \,, \tag{2.35}
$$

yields a family of solutions parametrised by $p \in \mathbb{Z}$. The corresponding flux quanta are:

$$
\begin{aligned}
\{a_0, a_1, a_2, a_3\} &= \{0, 4, -2, -2\} \,, \qquad \{b_0, b_1, b_2, b_3\} = \{-4p^2, 4p, 0, 0\} \,, \\
\{c_0, c_1, c_2, c_3\} &= \{0, 0, 0, 0\} \,, \qquad \{d_0, d_1, d_2, d_3\} = \{4p, -4, 2, 2\} \,.
\end{aligned}
\tag{2.36}
$$

It follows that $N_{\text{flux}} = 24$ and the superpotential can be written as:

$$
W = 2\left(2p^2 + 2p(U_4 - U_1) + U_1(U_2 + U_3 - 2U_4) + U_4(U_2 + U_3) - 2U_2 U_3\right) , \tag{2.37}
$$

which satisfies:

$$W \propto (\partial_2 W - \partial_3 W)^2 + 4(\partial_2 W + \partial_3 W)\partial_4 W + 4(\partial_4 W)^2 \,. \tag{2.38}$$

Due to above relation, solving $\partial_a W = 0$, $a = 1, \ldots, 4$ automatically sets $W = 0$, although W does not have any scaling property when $p \neq 0$.[7] For $p = 0$, W is a degree-2 homogeneous function. Let us mention that we are unable to find even integer fluxes (2.29) subject to (2.33) and $0 < N_{\text{flux}} \leq 32$, for which N_{flux} is other than 24. At the F-flatness locus the moduli take the values:

$$(U_1, U_2, U_3, U_4) = (U_4 + 2p, U_4 + p, U_4 + p, U_4)\,. \tag{2.39}$$

This might seem as giving an infinite number of solutions. However, one needs to check if the solutions are physically distinct or related by duality transformations. We give a summary of the relevant duality transformations in App. A.1. Applying these we find that the infinite class actually corresponds to just 1 distinct solution with representative the case $p = 0$.

2.2.3 2 Flat Directions

In this section we discuss solutions with 2 flat directions. Classified according to the nature of the linear relations satisfied by the derivatives of the superpotential, these solutions fall into 3 families.

Family $\mathcal{A}$: For this family the linear relation involves the derivatives of W with respect to all moduli and looks like:

$$\lambda_1\, \partial_1 W + \lambda_2\, \partial_2 W + \lambda_3\, \partial_3 W + \partial_4 W = 0\,, \qquad \lambda_1, \lambda_2, \lambda_3 \neq 0\,. \tag{2.40}$$

With this, the allowed flux quanta fall into 3 subfamilies. We will refer to them as $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$.

Subfamily $\mathcal{A}_1$: Here the flux quanta are characterized by $d_3 \neq 0$ and take the form:

$$\{a_0, a_1, a_2, a_3\} = \{0, \frac{d_3}{\lambda_2}, 0, -\frac{d_3\lambda_3}{\lambda_1\lambda_2}\}\,, \quad \{b_0, b_1, b_2, b_3\} = \{\frac{b_3 d_0}{d_3}, -\frac{b_3\lambda_3}{\lambda_1}, \frac{d_0}{\lambda_2}, b_3\}\,,$$
$$\{c_0, c_1, c_2, c_3\} = \{0, 0, 0, 0\}\,, \quad \{d_0, d_1, d_2, d_3\} = \{d_0, -\frac{d_3\lambda_3}{\lambda_1}, 0, d_3\}\,. \tag{2.41}$$

[7] By scaling property of a function $g(U_1 \ldots, U_n)$, we mean that there exists a set of numbers $\lambda_1, \ldots, \lambda_n$ not all zeros, such that

$$g(\lambda^{w_1} U_1, \ldots, \lambda^{w_n} U_n) = \lambda^{w(w_1,\ldots,w_n)} g(U_1, \ldots, U_n), \quad w(w_1, \ldots, w_n) \neq 0\,.$$

With this choice N_{flux} and W become:

$$N_{\text{flux}} = \frac{2d_3^2\lambda_3}{\lambda_1\lambda_2}\,, \qquad W = \left(d_3U_4 - \frac{d_3}{\lambda_2}U_2 - b_3\right)\left(U_3 - \frac{\lambda_3}{\lambda_1}U_1 + \frac{d_0}{d_3}\right). \quad (2.42)$$

The superpotential and its derivatives vanish when the moduli take the values:

$$(U_1, U_2, U_3, U_4) = \left(\frac{\lambda_1}{\lambda_3}\left(U_3 + \frac{d_0}{d_3}\right), \lambda_2\left(U_4 - \frac{b_3}{d_3}\right), U_3, U_4\right). \quad (2.43)$$

Note that the residual moduli space is 2-dimensional and parametrised by U_3 and U_4. Let us present an explicit solution. For $\lambda_\alpha = 1$, $\forall\alpha = 1, 2, 3$ and $d_3 = 2$, $N_{\text{flux}} = 8$ and the fluxes in (2.41) become:

$$\{a_0, a_1, a_2, a_3\} = \{0, 2, 0, -2\}\,, \quad \{b_0, b_1, b_2, b_3\} = \{\frac{b_3d_0}{2}, -b_3, d_0, b_3\}\,,$$
$$\{c_0, c_1, c_2, c_3\} = \{0, 0, 0, 0\}\,, \quad \{d_0, d_1, d_2, d_3\} = \{d_0, -2, 0, 2\}\,. \quad (2.44)$$

Clearly $b_3 = 2p$ and $d_0 = 2q$ with $p, q \in \mathbb{Z}$ retain all fluxes even. With these choices we get a quadratic superpotential:

$$W = -2\,(U_2 - U_4 + p)\,(U_3 - U_1 + q)\;, \quad (2.45)$$

and the solution to $W = \partial_a W = 0$ $\forall a = 1, \ldots, 4$ is given by:

$$(U_1, U_2, U_3, U_4) = (U_3 + q, U_4 - p, U_3, U_4)\,. \quad (2.46)$$

All the solutions in this class, parametrised by a pair of integers (p, q), are shown to be dual to 1 physically distinct solution with representative $p = q = 0$ in App. A.1.

Subfamily $\mathcal{A}_2$: In this case $d_3 \neq 0$ again but the fluxes take the form:

$$\{a_0, a_1, a_2, a_3\} = \{0, 0, \frac{d_3}{\lambda_1}, -\frac{d_3\lambda_3}{\lambda_1\lambda_2}\}\,, \quad \{b_0, b_1, b_2, b_3\} = \{\frac{b_3d_0}{d_3}, \frac{d_0}{\lambda_1}, -\frac{b_3\lambda_3}{\lambda_2}, b_3\}\,,$$
$$\{c_0, c_1, c_2, c_3\} = \{0, 0, 0, 0\}\,, \quad \{d_0, d_1, d_2, d_3\} = \{d_0, 0, -\frac{d_3\lambda_3}{\lambda_2}, d_3\}\,. \quad (2.47)$$

With this choice N_{flux} and W become:

$$N_{\text{flux}} = \frac{2d_3^2\lambda_3}{\lambda_1\lambda_2}\,, \qquad W = \left(d_3U_4 - \frac{d_3}{\lambda_1}U_1 - b_3\right)\left(U_3 - \frac{\lambda_3}{\lambda_2}U_2 + \frac{d_0}{d_3}\right). \quad (2.48)$$

The superpotential and its derivatives vanish when the moduli take the values:

$$(U_1, U_2, U_3, U_4) = \left(\lambda_1\left(U_4 - \frac{b_3}{d_3}\right), \frac{\lambda_2}{\lambda_3}\left(U_3 + \frac{d_0}{d_3}\right), U_3, U_4\right). \quad (2.49)$$

Note that the residual moduli space is 2-dimensional and parametrised by U_3 and U_4. Let us present an explicit example. For $\lambda_\alpha = 1$, $\forall \alpha = 1,2,3$ and $d_3 = 4$, $N_{\mathsf{flux}} = 32$ and the fluxes in (2.47) become:

$$\{a_0, a_1, a_2, a_3\} = \{0, 0, 4, -4\}\,, \quad \{b_0, b_1, b_2, b_3\} = \{\frac{b_3 d_0}{4}, d_0, -b_3, b_3\}\,,$$
$$\{c_0, c_1, c_2, c_3\} = \{0, 0, 0, 0\}\,, \quad \{d_0, d_1, d_2, d_3\} = \{d_0, 0, -4, 4\}\,. \tag{2.50}$$

Clearly, $b_3 = 2p$ and $d_0 = 4q$ with $p, q \in \mathbb{Z}$ retain all fluxes even. With these choices we get a quadratic superpotential:

$$W = -4\left(U_1 - U_4 + \frac{p}{2}\right)\left(U_3 - U_2 + q\right)\,, \tag{2.51}$$

and the solution to $W = \partial_a W = 0\ \forall a = 1, \ldots, 4$ is given by:

$$(U_1, U_2, U_3, U_4) = \left(U_4 - \frac{p}{2}, U_3 + q, U_3, U_4\right)\,. \tag{2.52}$$

All the solutions in this class, parametrised by a pair of integers (p, q), are shown to be dual to 2 physically distinct solutions with representatives $p = q = 0$ and $p = 1$, $q = 0$ in App. A.1.

Subfamily $\mathcal{A}_3$: Here $d_2 \neq 0$ and the fluxes look like:

$$\{a_0, a_1, a_2, a_3\} = \{0, \frac{d_2}{\lambda_3}, -\frac{d_2\lambda_2}{\lambda_1\lambda_3}, 0\}\,, \quad \{b_0, b_1, b_2, b_3\} = \{\frac{b_2 d_0}{d_2}, -\frac{b_2\lambda_2}{\lambda_1}, b_2, \frac{d_0}{\lambda_3}\}\,,$$
$$\{c_0, c_1, c_2, c_3\} = \{0, 0, 0, 0\}\,, \quad \{d_0, d_1, d_2, d_3\} = \{d_0, -\frac{d_2\lambda_2}{\lambda_1}, d_2, 0\}\,, \tag{2.53}$$

With this choice N_{flux} and the superpotential become:

$$N_{\mathsf{flux}} = \frac{2 d_2^2 \lambda_2}{\lambda_1\lambda_3}\,, \qquad W = \left(d_2 U_4 - \frac{d_2}{\lambda_3} U_3 - b_2\right)\left(U_2 - \frac{\lambda_2}{\lambda_1} U_1 + \frac{d_0}{d_2}\right)\,. \tag{2.54}$$

W and its derivatives vanish if the moduli take the values:

$$(U_1, U_2, U_3, U_4) = \left(\frac{\lambda_1}{\lambda_2}\left(U_2 + \frac{d_0}{d_2}\right), U_2, \lambda_3\left(U_4 - \frac{b_2}{d_2}\right), U_4\right)\,. \tag{2.55}$$

Note that the residual moduli space is 2-dimensional and parametrised by U_2 and U_4. Let us present an explicit example. For $\lambda_1 = \lambda_3 = 1$ and $\lambda_2 = d_2 = 2$, $N_{\mathsf{flux}} = 16$ and the fluxes in (2.53) become:

$$\{a_0, a_1, a_2, a_3\} = \{0, 2, -4, 0\}\,, \quad \{b_0, b_1, b_2, b_3\} = \{\frac{1}{2} b_2 d_0, -2b_2, b_2, d_0\}\,,$$
$$\{c_0, c_1, c_2, c_3\} = \{0, 0, 0, 0\}\,, \quad \{d_0, d_1, d_2, d_3\} = \{d_0, -4, 2, 0\}\,. \tag{2.56}$$

Clearly $b_2 = 2p$ and $d_0 = 2q$ with $p, q \in \mathbb{Z}$ retain all fluxes even. With these choices we get a quadratic superpotential:

$$W = -2\,(U_3 - U_4 + p)\,(U_2 - 2U_1 + q)\,, \tag{2.57}$$

and the solution to $W = \partial_a W = 0\ \forall a = 1, \ldots, 4$ is given by:

$$(U_1, U_2, U_3, U_4) = \left(\frac{U_2 + q}{2}, U_2, U_4 - p, U_4\right). \tag{2.58}$$

All the solutions in this class, parametrised by a pair of integers (p, q), are shown to be dual to 2 physically distinct solutions with representatives $p = q = 0$ and $p = 0$, $q = 1$ in App. A.1.

Family $\mathcal{B}$: For this family the linear relation involves derivatives of W with respect to the dilaton and 1 complex structure modulus and reads:

$$\lambda_3\, \partial_3 W + \partial_4 W = 0\,, \qquad \lambda_3 \neq 0\,. \tag{2.59}$$

Similar solutions exist for linear relations of the form $\lambda_\alpha \partial_\alpha W + \partial_4 W = 0$ with $\lambda_\alpha \neq 0$ for $\alpha = 1, 2$, and so we do not list them separately. The allowed flux quanta fall into 2 subfamilies which we call $\mathcal{B}_1$ and $\mathcal{B}_2$.

Subfamily $\mathcal{B}_1$: Here $d_2 \neq 0$, $c_3 d_0 \neq d_1 d_2$ and the fluxes take the form:

$$\{a_0, a_1, a_2, a_3\} = \{-\frac{c_3}{\lambda_3}, \frac{d_2}{\lambda_3}, \frac{d_1}{\lambda_3}, \frac{b_2 c_3}{d_2}\}\,, \quad \{b_0, b_1, b_2, b_3\} = \{\frac{b_2 d_0}{d_2}, \frac{b_2 d_1}{d_2}, b_2, \frac{d_0}{\lambda_3}\}\,,$$
$$\{c_0, c_1, c_2, c_3\} = \{0, 0, 0, c_3\}\,, \qquad \{d_0, d_1, d_2, d_3\} = \{d_0, d_1, d_2, 0\}\,. \tag{2.60}$$

With this choice N_{flux} and the superpotential become:

$$N_{\text{flux}} = \frac{2}{\lambda_3}\,(c_3 d_0 - d_1 d_2)\,, \qquad W = \left(U_4 - \frac{U_3}{\lambda_3} - \frac{b_2}{d_2}\right)(U_2(c_3 U_1 + d_2) + d_1 U_1 + d_0)\,. \tag{2.61}$$

W and its derivatives vanish at:

$$(U_1, U_2, U_3, U_4) = \left(-\frac{d_2 U_2 + d_0}{c_3 U_2 + d_1}, U_2, \lambda_3\left(U_4 - \frac{b_2}{d_2}\right), U_4\right). \tag{2.62}$$

Note that the residual moduli space is 2-dimensional and parametrised by U_2 and U_4. Let us present an explicit solution. For $\lambda_3 = 1$, $c_3 = 6$, $d_0 = d_2 = 2$ and $d_1 = 0$, $N_{\text{flux}} = 24$ and the fluxes in (2.60) become:

$$\{a_0, a_1, a_2, a_3\} = \{-6, 2, 0, 3b_2\}\,, \quad \{b_0, b_1, b_2, b_3\} = \{b_2, 0, b_2, 2\}\,,$$
$$\{c_0, c_1, c_2, c_3\} = \{0, 0, 0, 6\}\,, \quad \{d_0, d_1, d_2, d_3\} = \{2, 0, 2, 0\}\,. \tag{2.63}$$

Clearly $b_2 = 2p$ with $p \in \mathbb{Z}$ retains all fluxes even. With these choices we get a cubic superpotential:

$$W = -2\,(3U_1U_2 + U_2 + 1)\,(U_3 - U_4 + p)\,, \tag{2.64}$$

and the solution to $W = \partial_a W = 0\ \forall a = 1, \ldots, 4$ is given by:

$$(U_1, U_2, U_3, U_4) = \left(-\frac{U_2+1}{3U_2}, U_2, U_4 - p, U_4\right). \tag{2.65}$$

All the solutions in this class, parametrised by an integer p, are shown to be dual to 1 physically distinct solution with representative $p = 0$ in App. A.1.

Subfamily $\mathcal{B}_2$: In this case $c_3 \neq 0$, $c_3d_0 \neq d_1d_2$ and the fluxes read:

$$\{a_0, a_1, a_2, a_3\} = \{-\frac{c_3}{\lambda_3}, \frac{d_2}{\lambda_3}, \frac{d_1}{\lambda_3}, a_3\}\,, \quad \{b_0, b_1, b_2, b_3\} = \{\frac{a_3d_0}{c_3}, \frac{a_3d_1}{c_3}, \frac{a_3d_2}{c_3}, \frac{d_0}{\lambda_3}\}\,,$$
$$\{c_0, c_1, c_2, c_3\} = \{0, 0, 0, c_3\}\,, \quad \{d_0, d_1, d_2, d_3\} = \{d_0, d_1, d_2, 0\}\,. \tag{2.66}$$

This choice induce a flux contribution to the D3 tadpole and a superpotential of the form:

$$N_{\text{flux}} = \frac{2}{\lambda_3}\,(c_3d_0 - d_1d_2)\ , \tag{2.67}$$
$$W = \left(U_4 - \frac{U_3}{\lambda_3} - \frac{a_3}{c_3}\right)(U_2(c_3U_1 + d_2) + d_1U_1 + d_0)\ .$$

The superpotential and its derivatives vanish if the moduli take the values:

$$(U_1, U_2, U_3, U_4) = \left(-\frac{d_2U_2 + d_0}{c_3U_2 + d_1}, U_2, \lambda_3\left(U_4 - \frac{a_3}{c_3}\right), U_4\right). \tag{2.68}$$

Note that the residual moduli space is 2-dimensional and parametrised by U_2 and U_4. Let us present an explicit example. For $\lambda_3 = 1$, $c_3 = d_1 = d_2 = 2$ and $d_0 = 4$, $N_{\text{flux}} = 8$ and the fluxes in (2.66) take the form:

$$\{a_0, a_1, a_2, a_3\} = \{-2, 2, 2, a_3\}\,, \quad \{b_0, b_1, b_2, b_3\} = \{2a_3, a_3, a_3, 4\}\,,$$
$$\{c_0, c_1, c_2, c_3\} = \{0, 0, 0, 2\}\,, \quad \{d_0, d_1, d_2, d_3\} = \{4, 2, 2, 0\}\,. \tag{2.69}$$

Clearly $a_3 = 2p$ with $p \in \mathbb{Z}$ retains all fluxes even. With these choices the superpotential is cubic:

$$W = -2\,(U_1U_2 + U_1 + U_2 + 2)\,(U_3 - U_4 + p)\,, \tag{2.70}$$

and the solution to $W = \partial_a W = 0\ \forall a = 1, \ldots, 4$ is given by:

$$(U_1, U_2, U_3, U_4) = \left(-\frac{U_2+2}{U_2+1}, U_2, U_4 - p, U_4\right). \tag{2.71}$$

All the solutions in this class, parametrised by an integer p, are shown to be dual to 1 physically distinct solution with representative $p = 0$ in App. A.1.

Family $\mathcal{C}$: For this family the linear relation involves the derivatives of W with respect to 2 complex structure moduli and takes the form:

$$\partial_1 W = \lambda_2 \partial_2 W \,, \qquad \lambda_2 \neq 0 \,. \tag{2.72}$$

Similar solutions exist with linear relations of the form $\partial_\alpha W = \lambda_3\, \partial_3 W$ with $\alpha = 1, 2$ and $\lambda_3 \neq 0$, and so we do not list them separately. With a relation of the form (2.72) the allowed flux quanta fall into 3 subfamilies. We will refer to them as $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$.

Subfamily $\mathcal{C}_1$: Here $b_2 c_2 \neq a_2 d_2$ and the flux quanta look like:

$$\begin{aligned} \{a_0, a_1, a_2, a_3\} &= \{0, \frac{a_2}{\lambda_2}, a_2, 0\} \,, \quad \{b_0, b_1, b_2, b_3\} = \{0, b_2\lambda_2, b_2, 0\} \,, \\ \{c_0, c_1, c_2, c_3\} &= \{0, \frac{c_2}{\lambda_2}, c_2, 0\} \,, \quad \{d_0, d_1, d_2, d_3\} = \{0, d_2\lambda_2, d_2, 0\} \,. \end{aligned} \tag{2.73}$$

With this choice we have:

$$\begin{aligned} N_{\text{flux}} &= 2\,(b_2 c_2 - a_2 d_2) \,, \\ W &= \left(U_1 + \frac{U_2}{\lambda_2}\right)\left(U_3(c_2 U_4 - a_2) + \lambda_2 d_2 U_4 - b_2 \lambda_2\right) \,. \end{aligned} \tag{2.74}$$

The superpotential and its derivatives vanish at:

$$(U_1, U_2, U_3, U_4) = \left(-\frac{U_2}{\lambda_2}, U_2, -\lambda_2 \frac{d_2 U_4 - b_2}{c_2 U_4 - a_2}, U_4\right) . \tag{2.75}$$

Note that the residual moduli space is 2-dimensional and parametrised by U_2 and U_4. Let us present an explicit example. For $a_2 = d_2 = 0$, $b_2 = 4$ and $c_2 = 2$, $N_{\text{flux}} = 16$ and the fluxes in (2.73) become:

$$\begin{aligned} \{a_0, a_1, a_2, a_3\} &= \{0, 0, 0, 0\} \,, \quad \{b_0, b_1, b_2, b_3\} = \{0, 4\lambda_2, 4, 0\} \,, \\ \{c_0, c_1, c_2, c_3\} &= \{0, \frac{2}{\lambda_2}, 2, 0\} \,, \quad \{d_0, d_1, d_2, d_3\} = \{0, 0, 0, 0\} \,. \end{aligned} \tag{2.76}$$

Clearly $\lambda_2 = \pm 1, \pm\frac{1}{2}$ retain all fluxes even. With these choices we get a cubic superpotential:

$$W = 2\left(U_1 + \frac{U_2}{\lambda_2}\right)\left(U_3 U_4 - 2\lambda_2\right) , \tag{2.77}$$

and the solution to $W = \partial_a W = 0 \ \forall a = 1, \dots, 4$ is given by:

$$(U_1, U_2, U_3, U_4) = \left(-\frac{U_2}{\lambda_2}, U_2, \frac{2\lambda_2}{U_4}, U_4\right). \tag{2.78}$$

Subfamily $\mathcal{C}_2$: In this case $b_2, c_2, d_3 \neq 0$ and the fluxes look like:

$$\{a_0, a_1, a_2, a_3\} = \{0, \frac{b_3 c_2}{d_3 \lambda_2}, \frac{b_3 c_2}{d_3}, 0\}, \quad \{b_0, b_1, b_2, b_3\} = \{\frac{b_2 d_3 \lambda_2}{c_2}, b_2\lambda_2, b_2, b_3\},$$
$$\{c_0, c_1, c_2, c_3\} = \{0, \frac{c_2}{\lambda_2}, c_2, 0\}, \quad \{d_0, d_1, d_2, d_3\} = \{0, 0, 0, d_3\}. \tag{2.79}$$

This choices induces:

$$N_{\text{flux}} = 2b_2c_2\,, \qquad W = \left(\left(U_1 + \frac{U_2}{\lambda_2}\right) + \frac{d_3}{c_2}\right)\left(c_2 U_3 \left(U_4 - \frac{b_3}{d_3}\right) - b_2\lambda_2\right). \tag{2.80}$$

The superpotential and its derivatives vanish if the moduli take the values:

$$(U_1, U_2, U_3, U_4) = \left(-\frac{U_2}{\lambda_2} - \frac{d_3}{c_2}, U_2, \frac{b_2 d_3 \lambda_2}{c_2 d_3 U_4 - b_3 c_2}, U_4\right). \tag{2.81}$$

Note that the residual moduli space is 2-dimensional and parametrised by U_2 and U_4. Let us present an explicit solution. For $b_2 = c_2 = 4$, $N_{\text{flux}} = 32$ and the fluxes in (2.79) take the form:

$$\{a_0, a_1, a_2, a_3\} = \{0, \frac{4b_3}{\lambda_2 d_3}, \frac{4b_3}{d_3}, 0\}, \quad \{b_0, b_1, b_2, b_3\} = \{\lambda_2 d_3, 4\lambda_2, 4, b_3\},$$
$$\{c_0, c_1, c_2, c_3\} = \{0, \frac{4}{\lambda_2}, 4, 0\}, \quad \{d_0, d_1, d_2, d_3\} = \{0, 0, 0, d_3\}. \tag{2.82}$$

Clearly $\lambda_2 = 1$, $d_3 = 2p$, $b_3 = qd_3$ with $p, q \in \mathbb{Z}$ retain all fluxes even. With these choices we get a cubic superpotential:

$$W = 4\left(U_3 U_4 - qU_3 - 1\right)\left(U_1 + U_2 + \frac{p}{2}\right), \tag{2.83}$$

and the solution to $W = \partial_a W = 0 \ \forall a = 1, \dots, 4$ is given by:

$$(U_1, U_2, U_3, U_4) = \left(-U_2 - \frac{p}{2}, U_2, \frac{1}{U_4 - q}, U_4\right). \tag{2.84}$$

All the solutions in this class, parametrised by a pair of integers (p, q), are shown to be dual to 2 physically distinct solutions with representatives $p = q = 0$ and $p = 1$, $q = 0$ in App. A.1.

Subfamily $\mathcal{C}_3$: In this case $d_0, d_2 \neq 0$, $b_2 d_3 \neq b_3 d_2$ and the fluxes take the form:

$$\begin{aligned} \{a_0,a_1,a_2,a_3\} &= \{0, \frac{b_3 d_2}{d_0}, \frac{b_3 d_2 \lambda_2}{d_0}, 0\}\,, \quad \{b_0,b_1,b_2,b_3\} = \{\frac{b_2 d_0}{d_2}, b_2\lambda_2, b_2, b_3\}\,, \\ \{c_0,c_1,c_2,c_3\} &= \{0, \frac{d_2 d_3}{d_0}, \frac{d_2 d_3 \lambda_2}{d_0}, 0\}\,, \quad \{d_0,d_1,d_2,d_3\} = \{d_0, d_2\lambda_2, d_2, d_3\}\,. \end{aligned} \tag{2.85}$$

The expressions for N_{flux} and W become:

$$\begin{aligned} N_{\text{flux}} &= \frac{2 d_2 \lambda_2}{d_0}\left(b_2 d_3 - b_3 d_2\right)\,, \\ W &= \left(d_2\left(\lambda_2 U_1 + U_2\right) + d_0\right)\left(U_4\left(\frac{d_3}{d_0}U_3 + 1\right) - \frac{b_3}{d_0}U_3 - b_2 d_0\right)\,. \end{aligned} \tag{2.86}$$

The superpotential and its derivatives vanish at:

$$(U_1,U_2,U_3,U_4) = \left(-\frac{1}{\lambda_2}\left(U_2 + \frac{d_0}{d_2}\right), U_2, -\frac{d_0\left(d_2 U_4 - b_2\right)}{d_2\left(d_3 U_4 - b_3\right)}, U_4\right)\,. \tag{2.87}$$

Note that the residual moduli space is 2-dimensional and parametrised by U_2 and U_4. Let us present an explicit example. For $\lambda_2 = 1$, $b_2 = 0$, $b_3 = -4p$, $d_0 = 4p$, $d_2 = -2$ and $d_3 = 4p$ with $p \in \mathbb{Z}$, $N_{\text{flux}} = 8$ and the fluxes in (2.85) become:

$$\begin{aligned} \{a_0,a_1,a_2,a_3\} &= \{0,2,2,0\}\,, \quad \{b_0,b_1,b_2,b_3\} = \{0,0,0,-4p\}\,, \\ \{c_0,c_1,c_2,c_3\} &= \{0,-2,-2,0\}\,, \quad \{d_0,d_1,d_2,d_3\} = \{4p,-2,-2,4p\}\,. \end{aligned} \tag{2.88}$$

Clearly all fluxes are even. With these choices we get a cubic superpotential:

$$W = -2\left(U_3 U_4 + U_3 + U_4\right)\left(U_1 + U_2 - 2p\right)\,, \tag{2.89}$$

and the solution to $W = \partial_a W = 0\ \forall a = 1,\ldots,4$ is given by:

$$(U_1,U_2,U_3,U_4) = \left(2p - U_2, U_2, \frac{1}{U_4+1} - 1, U_4\right)\,. \tag{2.90}$$

All the solutions in this class, parametrised by an integer p, are shown to be dual to 1 physically distinct solution with representative $p = 0$ in App. A.1.

Dualities among solutions

The duality relations among the different classes of solutions presented above are analysed in detail in App. A.1 (for the case where $\lambda_\alpha \in \mathbb{Z}$). Here we just summarize the main results. The solutions of all 3 subfamilies in family $\mathcal{A}$ are dual to each other, and subfamily $\mathcal{A}_1$ features inequivalent solutions. Similarly, all the solutions in $\mathcal{B}_1$ are dual to solutions in $\mathcal{B}_2$, and $\mathcal{B}_1$ has physically different solutions. On the other hand, even if $\mathcal{C}_2$ and $\mathcal{C}_3$ are dual to each other, $\mathcal{C}_1$ is dual only to a subset of $\mathcal{C}_2$.[8] App. A.1 discusses the classification of inequivalent solutions within $\mathcal{C}_2$, together with an explicit example of a solution which is in $\mathcal{C}_2$ but not in $\mathcal{C}_1$.

An interesting fact is the presence of inter-family dualities despite distinct linear functional relations for the derivatives of the superpotential across the families $\mathcal{A}, \mathcal{B}, \mathcal{C}$. In App. A.1 we have found that $\mathcal{C}_2$ is dual to $\mathcal{B}_1$ and $\mathcal{A}_3$ is dual to a subset of $\mathcal{B}_1$. Hence, $\mathcal{B}_1$ is the subject of focus, for which physically distinct solutions have been classified in great detail in App. A.1.

Notice finally that in family $\mathcal{A}$ each subfamily gives a quadratic superpotential. Setting $p = q = 0$ in (2.45), (2.51) or (2.57) yields the superpotential discussed in [31] which can also be reproduced for suitable choices of fluxes in the cases $\mathcal{B}_1$, $\mathcal{C}_1$ and $\mathcal{C}_3$. On the other hand, in the cases $\mathcal{B}_2$ and $\mathcal{C}_2$ the superpotential is always cubic. In light of aforesaid dualities, a cubic superpotential can be mapped to a quadratic one in certain cases. However, we can find cases where cubic W can be made quadratic but not a homogeneous function of degree 2, e.g. setting $\lambda_3 = 1, b_2 = 2, c_3 = 4, d_0 = 4, d_1 = 0, d_2 = 4$ in (2.60).

Let us close this section commenting on some general features of the superpotential that we observe in these cases. W is always a product of 2 factors, each of which depends on 2 variables among $U_1, \ldots, U_4$. They also do not depend on the same U_a, and one of them is linear while the other is at most quadratic. Hence W can be written as:

$$W(U_1, \ldots, U_4) = f(U_{p(1)}, U_{p(2)})\, g(U_{p(3)}, U_{p(4)})\,, \tag{2.91}$$

where $(p(1), \ldots, p(4))$ is a permutation of $(1, \ldots, 4)$, f is linear, g is at most quadratic and the quadratic term in g (if any) is only the cross-term $U_{p(3)}U_{p(4)}$. Clearly:

$$\partial_{U_{p(1)}} W \propto g\,, \quad \partial_{U_{p(2)}} W \propto g\,, \quad \partial_{U_{p(3)}} W = f \partial_{U_{p(3)}} g\,, \quad \partial_{U_{p(4)}} W = f \partial_{U_{p(4)}} g\,. \tag{2.92}$$

Moreover, for $a = 3, 4$, $\partial_{U_{p(a)}} g$ is of the form $\mu_a U_{p(b \neq a)} + \nu_a$ for some real coefficients μ_a and ν_a at least one of which is non-zero. Hence $\partial_{U_{p(a)}} W = 0$ with $a = 3, 4$ sets $f = 0$ for complex solutions $\hat{U}_a$. Due to this, solving $f = g = 0$ automatically sets $W = \partial_{U_{p(a)}} W = 0$, $\forall a = 1, \ldots, 4$, reducing

[8] Precisely, $\mathcal{C}_3$ contains 2 copies of $\mathcal{C}_2$.

the number of linearly independent equation to 2. This shows clearly the existence of 2 flat direction since the number of moduli is 4. Notice also that in general neither f nor g (hereby W) has any scaling property. However, each class of fluxes presented above includes examples where at least one of the f and g, or both, can be made homogeneous in their arguments by suitably setting some fluxes to zero. For example, for a given class, $f(U_{p(1)}, U_{p(2)}) = \mu_1 U_{p(1)} + \mu_2 U_{p(2)} + \nu$ can be made homogeneous in $U_{p(1)}$, $U_{p(2)}$ by setting $\nu = 0$ whenever allowed.

2.2.4 Supersymmetries of the solutions

The solutions in the previous sections correspond to situations where the F-terms of the axio-dilaton and the complex structure moduli vanish and $W = 0$. Additionally, to be supersymmetric solutions of the 10-dimensional equations of motion, G_3 needs to be primitive. In this section we present a suitable Kähler form for all cases so that G_3 is primitive. The analysis is along the lines of [142].

The fluxes considered are diagonal, and so their expansion in the basis elements defined in (1.95) is of the form:

$$F_3 = a_0\alpha_0 + a_1\alpha_{11} + a_2\alpha_{22} + a_3\alpha_{33} + b_1\beta^{11} + b_2\beta^{22} + b_3\beta^{33} + b_0\beta^0 \,,$$
$$H_3 = c_0\alpha_0 + c_1\alpha_{11} + c_2\alpha_{22} + c_3\alpha_{33} + d_1\beta^{11} + d_2\beta^{22} + d_3\beta^{33} + d_0\beta^0 \,. \quad (2.93)$$

The period matrix is also diagonal for all the solutions obtained. Thus $\mathrm{d}z^j = \mathrm{d}x^j + \tau_j \mathrm{d}y^j$, $d\bar{z}^j = \mathrm{d}x^j + \bar{\tau}_j \mathrm{d}y^j$, $j = 1, 2, 3$. Now, taking the Kähler form to be:

$$J = \sum_{j=1}^{3} r_j^2 \, \mathrm{d}z^j \wedge d\bar{z}^j = -2 \sum_{j=1}^{3} \mathrm{Im}\,(\tau_j)\, r_j^2 \, \mathrm{d}x^j \wedge \mathrm{d}y^j \,, \quad (2.94)$$

it is easy to see that $J \wedge G_3 = 0$, i.e. G_3 is primitive.

Solutions with extended supersymmetry in 4 dimensions have been useful laboratories for developing our understanding of string theory. Some of our solutions with 2 flat directions preserve $N = 2$ supersymmetry in 4 dimensions. Being warped flux Minkowski compactifications with extended supersymmetry where the string coupling can be tuned to arbitrarily small values, they should be of interest for various theoretical studies.

The number of supersymmetries that a solution preserves can be determined by examining the decomposition of the G_3 flux under $SU(2)_L \times SU(2)_R \times U(1) \subset SO(6)$ (where $SO(6)$ is the group of rotations of the internal torus) [142]. In the charge convention of [142], a general 3-form decomposes as:

$$[6 \times 6 \times 6]_A \to (2,2)_0 + (2,2)_0 + (3,0)_2 + (3,0)_{-2} + (0,3)_2 + (0,3)_{-2} \,. \quad (2.95)$$

The requirement of extended supersymmetry is that G_3 must take values so that only the $(0,3)_2$ component is present. This implies that when G_3 is written as:

$$G_3 = \omega \wedge \mathrm{d}z^\alpha \,, \tag{2.96}$$

where z^α is the 'complex direction' with $U(1)$ charge 2, then ω has to be self dual in the remaining 4 (real) directions, with the orientation choice for Hodge duality which is consistent with (1.94). We present 2 explicit solutions which preserve $N = 2$ supersymmetry. In all our computations we will consider a metric of the form:

$$g_{i\bar{\jmath}} = r_i^2 \, \delta_{i\bar{\jmath}} \,, \tag{2.97}$$

that will ensure primitivity of the solutions.

Example 1: This solution lies in subfamily $\mathcal{A}_1$ of Sec. 2.2.3. Choosing $\lambda_1 = \lambda_2 = \lambda_3 = 1$, $b_3 = d_0 = 0$ and $d_3 = 2$ in the expressions for the flux quanta in (2.41) we obtain:

$$\begin{aligned} \{a_0, a_1, a_2, a_3\} &= \{0, 2, 0, -2\} \,, \qquad & \{b_0, b_1, b_2, b_3\} &= \{0, 0, 0, 0\} \,, \\ \{c_0, c_1, c_2, c_3\} &= \{0, 0, 0, 0\} \,, \qquad & \{d_0, d_1, d_2, d_3\} &= \{0, -2, 0, 2\} \,, \end{aligned} \tag{2.98}$$

together with $N_{\text{flux}} = 8$, and so the tadpole bound is satisfied. The superpotential is given by

$$W = 2(U_1 - U_3)(U_2 - U_4) \,. \tag{2.99}$$

The residual moduli space can be parametrised as:

$$(U_1, U_2, U_3, U_4) = (U_3, U_4, U_3, U_4) \,, \tag{2.100}$$

where we take U_3, U_4 to be in the fundamental domain of the upper half plane modulo modular transformations. Thus, all U_a have positive imaginary parts. The 3-form fluxes are:

$$\begin{aligned} F_3 &= -2\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}y^3 + 2\mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}y^1 \,, \\ H_3 &= 2\mathrm{d}x^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}y^3 - 2\mathrm{d}x^3 \wedge \mathrm{d}y^1 \wedge \mathrm{d}y^2 \,, \end{aligned} \tag{2.101}$$

leading to the complexified 3-form:

$$G_3 = -\frac{2}{U_3 - \overline{U}_3} (\mathrm{d}z^1 \wedge \mathrm{d}\overline{z}^3 + \mathrm{d}z^3 \wedge \mathrm{d}\overline{z}^1) \wedge \mathrm{d}z^2 \equiv \omega \wedge \mathrm{d}z^2 \,. \tag{2.102}$$

Identifying z^2 as the $U(1)$ coordinate, we see that G_3 has hypercharge $+2$. Furthermore, computing the $SO(4) \supset SU(2)_L \times SU(2)_R$ dual we get:

$$\star_4 \omega = \omega \,. \tag{2.103}$$

Thus the solution preserves $N = 2$ supersymmetry. Notice that this corresponds to the case studied in [31].

Example 2: This solution lies in subfamily $\mathcal{B}_1$ of Sec. 2.2.3. Choosing $\lambda_3 = 1$, $b_2 = 2$, $d_1 = 0$ and $c_3 = d_0 = d_2 = 4$ in the expressions for the flux quanta in (2.60) we obtain:

$$\{a_0, a_1, a_2, a_3\} = \{-4, 4, 0, 2\}\,, \qquad \{b_0, b_1, b_2, b_3\} = \{2, 0, 2, 4\}\,,$$
$$\{c_0, c_1, c_2, c_3\} = \{0, 0, 0, 4\}\,, \qquad \{d_0, d_1, d_2, d_3\} = \{4, 0, 4, 0\}\,, \tag{2.104}$$

together with $N_{\text{flux}} = 8$. The superpotential is given by

$$W = -2(U_1 U_2 + U_2 + 1)(2U_3 + U_4 + 1)\,. \tag{2.105}$$

This is an example where using dualities the superpotential cannot be brought to a degree-2 homogeneous polynomial. The residual moduli space can be parametrised as:

$$(U_1, U_2, U_3, U_4) = \left(-\frac{U_2 + 1}{U_2}, U_2, U_4 - \frac{1}{2}, U_4\right), \tag{2.106}$$

where we take U_2 and U_4 to be in the fundamental domain of the upper half plane modulo modular transformations. Thus, all U_a have positive imaginary parts. The 3-form fluxes are:

$$\begin{aligned} F_3 =& -4\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + 2\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}y^3 + 4\mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}y^1 \\ &+ 2\mathrm{d}x^2 \wedge \mathrm{d}y^1 \wedge \mathrm{d}y^3 - 4\mathrm{d}x^3 \wedge \mathrm{d}y^1 \wedge \mathrm{d}y^2 + 2\mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}y^3\,, \\ H_3 =& 4\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}y^3 + 4\mathrm{d}x^2 \wedge \mathrm{d}y^1 \wedge \mathrm{d}y^3 + 4\mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}y^3\,, \end{aligned} \tag{2.107}$$

leading to the complexified 3-form:

$$G_3 = -\frac{4}{U_2 - \overline{U}_2}\,(U_2\,\mathrm{d}z^1 \wedge \mathrm{d}\overline{z}^2 + \overline{U}_2\,\mathrm{d}z^2 \wedge \mathrm{d}\overline{z}^1) \wedge \mathrm{d}z^3 \equiv \omega \wedge \mathrm{d}z^3\,. \tag{2.108}$$

Identifying z^3 as the $U(1)$ coordinate, we see that G_3 has hypercharge $+2$. Furthermore, computing the $SO(4) \supset SU(2)_L \times SU(2)_R$ dual we get:

$$\star_4 \omega = \omega\,. \tag{2.109}$$

Thus the solution preserves $N = 2$ supersymmetry.

2.3 Solutions in Calabi-Yau Orientifolds

In this section we turn to CYs in the large complex structure limit. A detailed study will be carried out using the CY obtained by considering a degree-18 hypersurface in $\mathbb{CP}_{[1,1,1,6,9]}$ (first studied in the context of mirror symmetry in [186]). Then, we also briefly discuss another CY with more moduli.

2.3.1 Type IIB Calabi-Yau flux compactifications at large complex structure

In this section we first recapitulate some basic material on type IIB flux compactifications in the large complex structure limit[9] and the $\mathbb{CP}_{[1,1,1,6,9]}$ example.[10] Given that our discussion shall be quite brief, we refer the reader to [10, 12, 202, 203] for further details.

Type IIB flux compactifications have an internal manifold that is conformally an orientifolded CY X. To describe these in the language of special geometry, one works with a symplectic basis for $H_3(X, \mathbb{Z})$, $\{A_a, B^a\}$ for $a = 0, ..., h^{1,2}_-(X)$ with $A_a \cap A_b = 0$, $A_a \cap B^b = \delta_a{}^b$, and $B^a \cap B^b = 0$, and projective coordinates on the complex structure moduli U^a (in what follows, we will take $U^0 = 1$). The central object is the prepotential $\mathcal{F}$, which is degree-2 and homogeneous in the projective coordinates. The period vector is given by:

$$\Pi = \begin{pmatrix} \int_{B^a} \Omega \\ \int_{A_a} \Omega \end{pmatrix} = \begin{pmatrix} \mathcal{F}_a \\ U^a \end{pmatrix} , \tag{2.110}$$

where $\mathcal{F}_0 = 2\mathcal{F} - U^a \mathcal{F}_a$ with $\mathcal{F}_a \equiv \partial_{U^a} \mathcal{F}$. Similarly, (integer valued) flux vectors F and H are obtained by performing integrals of the 3-form field strengths over the A_a and B_a cycles. The flux superpotential, which is classically exact, is given by:

$$W = (F - \phi H)^t \cdot \Sigma \cdot \Pi \, , \tag{2.111}$$

where:

$$\Sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \tag{2.112}$$

is the symplectic matrix. The tree-level Kähler potential (for the complex structure moduli and the axio-dilaton) is:

$$\mathcal{K} = - \ln \left(-\Pi^\dagger \cdot \Sigma \cdot \Pi \right) - \ln \left(-(\phi - \bar{\phi}) \right) \, . \tag{2.113}$$

In the large complex structure limit, the prepotential is a sum of perturbative terms which are at most degree-3 and instanton corrections, i.e. $\mathcal{F}(U) = \mathcal{F}_{\text{pert}}(U) + \mathcal{F}_{\text{inst}}(U)$ with:

$$\mathcal{F}_{\text{pert}}(U) = -\frac{1}{3!} \mathcal{K}_{abc} U^a U^b U^c + \frac{1}{2} \mathrm{a}_{ab} U^a U^b + b_a U^a + \xi \, , \tag{2.114}$$

[9]For detailed studies of flux vacua in the large complex structure limit see e.g. [134, 174, 196-201].

[10]We follow the notation and conventions of [12] but with 2 exceptions: (i) in the definition of the GVW superpotential, the paper has an overall factor of $\sqrt{2/\pi}$ which we set equal to unity to be consistent with our earlier discussion. (ii) the chapter uses τ to denote the axio-dilaton, while we will continue to use ϕ.

where $\mathcal{K}_{abc}$ are the triple intersection numbers of the mirror CY, $\mathbf{a}_{ab}$ and b_a are rational, and $\xi = -\frac{\zeta(3)\chi}{2(2\pi i)^3}$, with χ the CY Euler number. The instanton corrections are:

$$\mathcal{F}_{\text{inst}}(U) = \frac{1}{(2\pi)^3} \sum_{\vec{q}} A_{\vec{q}}\, e^{2\pi \vec{q}\cdot\vec{U}} , \tag{2.115}$$

where the sum runs over effective curves in the mirror CY. The form of the perturbative part of the prepotential implies that it leads to a superpotential that is at most degree-3 polynomial in the complex structure moduli and the fluxes. Thus the search for supersymmetric minima with flat directions can be carried out using the method we have put forward in Sec. 2.1.

2.3.2 The $\mathbb{CP}_{[1,1,1,6,9]}[18]$ example

In this section we implement our method to find supersymmetric minima with flat directions focusing on the example of the degree-18 hypersurface in $\mathbb{CP}_{[1,1,1,6,9]}$. Let us record some basic facts about this CY which has 272 complex structure moduli and a $\mathcal{G} = \mathbb{Z}_6 \times \mathbb{Z}_{18}$ symmetry. By considering fluxes which are $\mathcal{G}$-invariant, one stabilizes on the $\mathcal{G}$-symmetric locus (see [19]). Thus the stabilization problem can be effectively reduced to a 2-moduli one. For this, the relevant geometric data are:

$$\mathcal{K}_{111} = 9 , \qquad \mathcal{K}_{112} = 3 , \qquad \mathcal{K}_{122} = 1 , \qquad \mathbf{a} = \frac{1}{2}\begin{pmatrix} 9 & 3 \\ 3 & 0 \end{pmatrix} , \qquad \vec{b} = \frac{1}{4}\begin{pmatrix} 17 \\ 6 \end{pmatrix} , \tag{2.116}$$

and the instanton corrections are $(2\pi)^3\mathcal{F}_{\text{inst}} = \mathcal{F}_1 + \mathcal{F}_2 + \cdots$ with:

$$\mathcal{F}_1 = -540\, q_1 - 3\, q_2 , \qquad \mathcal{F}_2 = -\frac{1215}{2}\, q_1^2 + 1080\, q_1 q_2 + \frac{45}{8}\, q_2^2 , \tag{2.117}$$

where $q_a = \exp(2\pi U^a)$ with $a = 1, 2$. We will consider the orientifold described in [155] with the D7 tadpole cancelled by 4 D7-branes on top of each O7-plane. This setup yields a D3-charge $Q_{D3} = 138$.

Neglecting exponentially small corrections in the prepotential, the F-flatness conditions are a set of 3 polynomial equations in 3 variables. We examine both cases, in which the superpotential vanishes or assumes a non-zero value at the minimum. As described earlier, our ansatz will involve looking for solutions where there is a linear relation between the derivatives of the superpotential.

Following the algorithm described in Sec. 2.1, we start by writing the flux vectors as:

$$F = (f_1\ f_2\ f_3\ f_4\ f_5\ f_6)^t , \qquad H = (h_1\ h_2\ h_3\ h_4\ h_5\ h_6)^t , \qquad f_i, h_i \in \mathbb{Z} . \tag{2.118}$$

For simplicity in this chapter we will take $f_4 = h_4 = 0$. As a result, the contribution to the superpotential of the term involving the CY Euler number in the prepotential (2.114) vanishes and the superpotential is polynomial with rational coefficients. This simplifies the search for solutions. Now, defining $\mathcal{N}_{\text{flux}} \equiv \frac{1}{2} N_{\text{flux}} = -\frac{1}{2} H^t \cdot \Sigma \cdot F$ and denoting (U^1, U^2, ϕ) by (U_1, U_2, U_3), we have:

$$\begin{aligned}
\mathcal{N}_{\text{flux}} &= \frac{1}{2}(f_2 h_5 + f_3 h_6 - f_5 h_2 - f_6 h_3)\,, \\
W &= f_1 + U_1(f_2 - h_2 U_3) + U_2(f_3 - h_3 U_3) \\
&\quad + \frac{1}{4}\left(2(3U_1 + U_2)^2 - 18U_1 - 6U_2 - 17\right)(f_5 - h_5 U_3) \\
&\quad + \frac{1}{2}\left(U_1(3U_1 + 2U_2 - 3) - 3\right)(f_6 - h_6 U_3) - h_1 U_3\,.
\end{aligned} \tag{2.119}$$

Solutions with $W = 0$

Given (2.119), consider the 4 polynomial equations in 3 variables: $W(U_a) = \partial_a W = 0\ \forall a = 1,2,3$. The degree of each of these equations is 3 or less, depending upon the choices of fluxes. For this system of equations to admit a solution, one of them should be dependent on the others. Here, we examine cases when this dependence is linear:[11]

(i) When $\partial_1 W = \lambda_2\, \partial_2 W + \lambda_3\, \partial_3 W$ with at least one of λ_2 and λ_3 which is non-zero and subject to $\mathcal{N}_{\text{flux}} \neq 0$, we find only 1 family of fluxes (details are given below) for which $W = \partial_a W = 0\ \forall a = 1,2,3$ admit solutions in the large complex structure limit. Here, we do not need to impose any further conditions ensuring the existence of a flat direction since it turns out that we always have a flat direction (parametrised by the axio-dilaton) with the above family of fluxes;

(ii) When $\partial_2 W = \lambda_3\, \partial_3 W$, $\lambda_3 \neq 0$, the conditions on the fluxes have no solution in keeping with $\mathcal{N}_{\text{flux}} \neq 0$.

We now provide the aforesaid family of fluxes which are dependent

[11] As the derivatives of the polynomial W are of lower degree, W can never be equal to a linear combination of its derivatives.

on the 5 parameters λ_2, λ_3, f_3, h_1 and h_3:

$$\{f_1, f_2, f_3, f_4, f_5, f_6\} = \tag{2.120}$$
$$\Bigg\{\frac{4f_3h_1 - \frac{\lambda_3((2\lambda_2-3)(2h_1^2+6h_1h_3+h_3^2)-6\lambda_2^2h_3^2)}{(\lambda_2-3)\lambda_2}}{4h_3}, \lambda_2 f_3 - \lambda_3 h_1 - \frac{3\lambda_3 h_3}{2},$$
$$f_3, 0, \frac{(2\lambda_2-3)\lambda_3 h_3}{(\lambda_2-3)\lambda_2}, \frac{(\lambda_2-3)\lambda_3 h_3}{\lambda_2}\Bigg\},$$
$$\{h_1, h_2, h_3, h_4, h_5, h_6\} = \{h_1, \lambda_2 h_3, h_3, 0, 0, 0\}\,, \qquad \lambda_2 \neq 0, 3\,, \qquad \lambda_3, h_3 \neq 0\,.$$

In this case we have:

$$\mathcal{N}_{\text{flux}} = -\frac{3\left[(\lambda_2-3)\lambda_2+3\right]\lambda_3 h_3^2}{2(\lambda_2-3)\lambda_2}\,,$$
$$W = \frac{1}{2(\lambda_2-3)\lambda_2 h_3}\left[h_1 + h_3(\lambda_2 U_1 + U_2)\right] \tag{2.121}$$
$$\times\Big[2(\lambda_2-3)\lambda_2(f_3 - h_3 U_3) + (3-2\lambda_2)\lambda_3 h_1$$
$$+ \lambda_3 h_3(3\lambda_2(U_1-2) + (2\lambda_2-3)U_2 + 9)\Big]\,.$$

Now, solving $W = \partial_a W = 0$, $\forall a = 1,2,3$, we see that U_1 and U_2 depend linearly on U_3 with slopes $-1/\lambda_3$ and λ_2/λ_3 respectively. Thus, by requiring $\lambda_2, \lambda_3 < 0$, we may obtain Im U_a, $\forall a = 1,2,3$ to be of the same sign. This keeps $\mathcal{N}_{\text{flux}}$ positive and also ensures that U_1 and U_2 are in the large complex structure limit when Im U_3 is taken large to be in the weak string coupling regime.

Let us notice the arguments of $f_5(\lambda_2, \lambda_3, h_3)$, $f_6(\lambda_2, \lambda_3, h_3)$, $h_2(\lambda_2, h_3)$ and $\mathcal{N}_{\text{flux}}(\lambda_2, \lambda_3, h_3)$. There are only 488 triples $(\lambda_2, \lambda_3, h_3)$, $\lambda_2, \lambda_3 \in \mathbb{Q}^-$, $h_3 \in \mathbb{Z}$, securing $f_5, f_6, h_2 \in \mathbb{Z}$ and $\mathcal{N}_{\text{flux}} \in \mathbb{Z}/2$ with $0 < \mathcal{N}_{\text{flux}} \leq 138$. For 420 of them there are no $f_3, h_1 \in \mathbb{Z}$ that keep all other fluxes in (2.120) integers. For each of the remaining 68 triples $(\lambda_2, \lambda_3, h_3)$, we get a subfamily of integer fluxes (2.120) parametrised by f_3 and h_1. All the members in any of the aforementioned subfamilies have the same $\mathcal{N}_{\text{flux}}(\lambda_2, \lambda_3, h_3)$ which happens to be an integer. In Tab. 2.1 and 2.2 we list a representative from each of these 68 subfamilies. Then, we also discuss one of these subfamilies in detail. Let us stress that among the above 68 values of $\mathcal{N}_{\text{flux}}(\lambda_2, \lambda_3, h_3)$ only 13 are distinct.

In all 68 cases in Tab. 2.1 and 2.2, W is a non-homogeneous function of degree 2. We need to check if these cases are dual to cases where W is homogeneous (as in [12, 23]). To do this, we can employ integer shifts of the complex structure moduli U_1 and U_2 and $SL(2,\mathbb{Z})$ transformations on the axio-dilaton U_3.[12] Note that $h_5, h_6 \neq 0$ in (2.119) yield a cubic W. In all cases in Tab. 2.1 and 2.2, $h_5 = h_6 = 0$ and $f_5 \neq 0$. From (A.1) we see that an $SL(2,\mathbb{Z})$ duality transformation with non-zero c and f_5 leads to a non-zero h_5, yielding a non-zero coefficient for $U_2^2 U_3$ in W. Thus, for the above check we must keep $c = 0$, and only integer shifts of U_a $\forall\, a = 1, 2, 3$ are useful. We find that in 53 of these 68 cases, appropriate integer shifts of U_a can transform W into a homogeneous function of degree 2. Interestingly, after including instanton corrections to the superpotential, it can be checked that only 2 out of these 53 solutions feature a weak string coupling and an instanton expansion which is definitely under control, corresponding to the 2 old vacua already found in [23, 26] (1 of them has been originally discovered in [12]).

On the other hand, W remains non-homogeneous in the remaining 15 cases, with the following case numbers in Tab. 2.1 and 2.2: $3, 4, 21, 22, 23, 24, 25, 26, 33, 34, 38, 41, 55, 65, 66$. These 15 solutions represent therefore novel perturbatively flat vacua which are qualitatively different from the ones studied in [12, 23]. In order to check if these can be solutions with small g_s, one should perform a careful study of dilaton stabilization via instantons which we leave however for future research.

Now, we consider one of the above 68 triples, $(\lambda_2, \lambda_3, h_3) = (-3, -72, 1)$. For this, $\mathcal{N}_{\text{flux}} = 126$ and (2.120) becomes:

$$\begin{aligned} \{f_1, f_2, f_3, f_4, f_5, f_6\} &= \{-63 + h_1(f_3 - 18(3 + h_1)), 108 - 3f_3 + 72h_1, f_3, \\ &\qquad\qquad\qquad\qquad 0, 36, -144\}\,, \\ \{h_1, h_2, h_3, h_4, h_5, h_6\} &= \{h_1, -3, 1, 0, 0, 0\}\,, \end{aligned} \tag{2.122}$$

and (2.121) gives:

$$W = -\left(U_2 - 3U_1 + h_1\right)\left(U_3 - 18(-3 + U_1 + U_2) - f_3 + 18h_1\right). \tag{2.123}$$

Clearly, every $f_3, h_1 \in \mathbb{Z}$ retain all the fluxes integers and the solution to $W = \partial_a W = 0$, $\forall a = 1, 2, 3$ is given by:

$$(U_1, U_2, U_3) = \left(\frac{1}{72}\left(U_3 - f_3 + 36h_1 + 54\right), \frac{1}{24}\left(U_3 - f_3 + 12h_1 + 54\right), U_3\right). \tag{2.124}$$

Now, choosing $h_1 = 0$ and $f_3 = 54$, W becomes a homogeneous function of degree 2. For this subfamily, although we obtain a non-homogeneous W with other choices of h_1 and , f_3, W can always be made homogeneous by appropriate integer shifts of U_1, U_2 and U_3.

[12] See App. A for the transformation rules.

	$(\lambda_2, \lambda_3, h_3)$	F	H	$\mathcal{N}_{\text{flux}}$	W
1	(-3,-72,1)	{-9,-9,-9,0,36,-144}	{-2,-3,1,0,0,0}	126	$-(3U_1 - U_2 + 2)(18U_1 + 18U_2 - U_3 - 27)$
2	(-3,-72,-1)	{-3,-9,15,0,-36,144}	{2,3,-1,0,0,0}	126	$(3U_1 - U_2 + 2)(18U_1 + 18U_2 - U_3 - 33)$
3	(-3,-8,3)	{-11,-10,2,0,12,-48}	{-5,-9,3,0,0,0}	126	$-(9U_1 - 3U_2 + 5)(2(-1 + U_1 + U_2) - U_3)$
4	(-3,-8,-3)	{-14,-13,11,0,-12,48}	{7,9,-3,0,0,0}	126	$(9U_1 - 3U_2 + 7)(2U_1 + 2U_2 - U_3 - 5)$
5	(-1,-48,1)	{-33,72,0,0,60,-192}	{0,-1,1,0,0,0}	126	$-(U_1 - U_2)(18U_1 + 30U_2 - U_3 - 90)$
6	(-1,-48,-1)	{-13,-10,-14,0,-60,192}	{1,1,-1,0,0,0}	126	$(U_1 - U_2 + 1)(18U_1 + 30U_2 - U_3 - 46)$
7	$(-1, -\frac{16}{3}, 3)$	{-3,-14,6,0,20,-64}	{-6,-3,3,0,0,0}	126	$-(U_1 - U_2 + 2)(6U_1 + 10U_2 - 3U_3 - 4)$
8	$(-1, -\frac{16}{3}, -3)$	{-10,-9,1,0,-20,64}	{3,3,-3,0,0,0}	126	$(U_1 - U_2 + 1)(6U_1 + 10U_2 - 3(7 + U_3))$
9	$(-3, -\frac{8}{9}, 9)$	{-9,-14,6,0,4,-16}	{-9,-27,9,0,0,0}	126	$-(3U_1 - U_2 + 1)(2(1 + U_1 + U_2) - 9U_3)$
10	$(-3, -\frac{8}{9}, -9)$	{1,-10,2,0,-4,16}	{9,27,-9,0,0,0}	126	$(3U_1 - U_2 + 1)(2(-3 + U_1 + U_2) - 9U_3)$
11	$(-\frac{3}{4}, -\frac{5}{2}, 4)$	{-14,-13,14,0,16,-50}	{-7,-3,4,0,0,0}	124	$-(3U_1 - 4U_2 + 7)(U_1 + 2U_2 - U_3 + 1)$
12	$(-\frac{3}{4}, -\frac{5}{2}, -4)$	{-11,-12,6,0,-16,50}	{3,3,-4,0,0,0}	124	$(3U_1 - 4U_2 + 3)(U_1 + 2U_2 - U_3 - 6)$
13	(-12,-80,1)	{-99,120,0,0,12,-100}	{0,-12,1,0,0,0}	122	$-(12U_1 - U_2)(-18 + 8U_1 + 6U_2 - U_3)$
14	(-12,-80,-1)	{99,-120,0,0,-12,100}	{0,12,-1,0,0,0}	122	$(12U_1 - U_2)(-18 + 8U_1 + 6U_2 - U_3)$
15	(-6,-72,1)	{-77,108,0,0,20,-108}	{0,-6,1,0,0,0}	114	$-(6U_1 - U_2)(-30 + 12U_1 + 10U_2 - U_3)$
16	(-6,-72,-1)	{77,-108,0,0,-20,108}	{0,6,-1,0,0,0}	114	$(6U_1 - U_2)(-30 + 12U_1 + 10U_2 - U_3)$
17	$(-\frac{1}{2}, -7, 2)$	{-3,-13,12,0,32,-98}	{-4,-1,2,0,0,0}	114	$-(4 + U_1 - 2U_2)(-2 + 3U_1 + 8U_2 - U_3)$
18	$(-\frac{1}{2}, -7, -2)$	{-9,-14,0,0,-32,98}	{1,1,-2,0,0,0}	114	$(1 + U_1 - 2U_2)(-20 + 3U_1 + 8U_2 - U_3)$
19	(-3,-64,1)	{-10,-11,-7,0,32,-128}	{-2,-3,1,0,0,0}	112	$-(2 + 3U_1 - U_2)(-23 + 16U_1 + 16U_2 - U_3)$
20	(-3,-64,-1)	{-4,-10,14,0,-32,128}	{2,3,-1,0,0,0}	112	$(2 + 3U_1 - U_2)(-30 + 16U_1 + 16U_2 - U_3)$
21	(-3,-16,2)	{-10,-13,-1,0,16,-64}	{-4,-6,2,0,0,0}	112	$-(2 + 3U_1 - U_2)(-9 + 8U_1 + 8U_2 - 2U_3)$
22	(-3,-16,-2)	{-6,-11,9,0,-16,64}	{4,6,-2,0,0,0}	112	$(2 + 3U_1 - U_2)(-17 + 8U_1 + 8U_2 - 2U_3)$
23	(-3,-4,4)	{-10,-14,2,0,8,-32}	{-8,-12,4,0,0,0}	112	$-2(2 + 3U_1 - U_2)(-1 + 2U_1 + 2U_2 - 2U_3)$
24	(-3,-4,-4)	{-6,-10,6,0,-8,32}	{8,12,-4,0,0,0}	112	$2(2 + 3U_1 - U_2)(-5 + 2U_1 + 2U_2 - 2U_3)$
25	(-3,-1,8)	{-9,-14,6,0,4,-16}	{-8,-24,8,0,0,0}	112	$-2(1 + 3U_1 - U_2)(1 + U_1 + U_2 - 4U_3)$
26	(-3,-1,-8)	{1,-10,2,0,-4,16}	{8,24,-8,0,0,0}	112	$2(1 + 3U_1 - U_2)(-3 + U_1 + U_2 - 4U_3)$
27	(-3,-56,1)	{-9,-10,-6,0,28,-112}	{-2,-3,1,0,0,0}	98	$-(2 + 3U_1 - U_2)(-20 + 14U_1 + 14U_2 - U_3)$
28	(-3,-56,-1)	{-5,-11,13,0,-28,112}	{2,3,-1,0,0,0}	98	$(2 + 3U_1 - U_2)(-27 + 14U_1 + 14U_2 - U_3)$
29	$(-3, -\frac{8}{7}, 7)$	{-9,-13,3,0,4,-16}	{-14,-21,7,0,0,0}	98	$-(2 + 3U_1 - U_2)(1 + 2U_1 + 2U_2 - 7U_3)$
30	$(-3, -\frac{8}{7}, -7)$	{-9,-14,6,0,-4,16}	{14,21,-7,0,0,0}	98	$(2 + 3U_1 - U_2)(2(-4 + U_1 + U_2) - 7U_3)$
31	(-3,-48,1)	{-10,-12,-4,0,24,-96}	{-2,-3,1,0,0,0}	84	$-(2 + 3U_1 - U_2)(-16 + 12U_1 + 12U_2 - U_3)$
32	(-3,-48,-1)	{-4,-9,11,0,-24,96}	{2,3,-1,0,0,0}	84	$(3U_1 - U_2 + 2)(-23 + 12U_1 + 12U_2 - U_3)$
33	(-3,-12,2)	{-12,-9,3,0,12,-48}	{-3,-6,2,0,0,0}	84	$-(6U_1 - 2U_2 + 3)(3(-1 + U_1 + U_2) - U_3)$
34	(-3,-12,-2)	{-14,-9,11,0,-12,48}	{5,6,-2,0,0,0}	84	$(6U_1 - 2U_2 + 5)(-7 + 3U_1 + 3U_2 - U_3)$
35	(-1,-32,1)	{-22,48,0,0,40,-128}	{0,-1,1,0,0,0}	84	$-(U_1 - U_2)(-60 + 12U_1 + 20U_2 - U_3)$
36	(-1,-32,-1)	{-12,-10,-6,0,-40,128}	{1,1,-1,0,0,0}	84	$(1 + U_1 - U_2)(-34 + 12U_1 + 20U_2 - U_3)$
37	(-1,-8,2)	{-3,-14,6,0,20,-64}	{-4,-2,2,0,0,0}	84	$-2(2 + U_1 - U_2)(-2 + 3U_1 + 5U_2 - U_3)$
38	(-1,-8,-2)	{-10,-9,1,0,-20,64}	{2,2,-2,0,0,0}	84	$(1 + U_1 - U_2)(-21 + 6U_1 + 10U_2 - 2U_3)$
39	$(-3, -\frac{16}{3}, 3)$	{-10,-14,2,0,8,-32}	{-6,-9,3,0,0,0}	84	$-(2 + 3U_1 - U_2)(-2 + 4U_1 + 4U_2 - 3U_3)$
40	$(-3, -\frac{16}{3}, -3)$	{-6,-10,6,0,-8,32}	{6,9,-3,0,0,0}	84	$(2 + 3U_1 - U_2)(-10 + 4U_1 + 4U_2 - 3U_3)$
41	$(-3, -\frac{4}{3}, 6)$	{-9,-13,3,0,4,-16}	{-12,-18,6,0,0,0}	84	$-(2 + 3U_1 - U_2)(1 + 2U_1 + 2U_2 - 6U_3)$
42	$(-3, -\frac{4}{3}, -6)$	{-9,-14,6,0,-4,16}	{12,18,-6,0,0,0}	84	$2(2 + 3U_1 - U_2)(-4 + U_1 + U_2 - 3U_3)$
43	(-2,-40,1)	{3,-14,-3,0,28,-100}	{-2,-2,1,0,0,0}	78	$-(2 + 2U_1 - U_2)(-17 + 12U_1 + 14U_2 - U_3)$
44	(-2,-40,-1)	{8,-10,-5,0,-28,100}	{1,2,-1,0,0,0}	78	$(1 + 2U_1 - U_2)(-23 + 12U_1 + 14U_2 - U_3)$
45	$(-\frac{3}{2}, -9, 2)$	{-9,-9,12,0,16,-54}	{-2,-3,2,0,0,0}	78	$-(2 + 3U_1 - 2U_2)(-2 + 3U_1 + 4U_2 - U_3)$
46	$(-\frac{3}{2}, -9, -2)$	{-2,3,-2,0,-16,54}	{3,3,-2,0,0,0}	78	$(3 + 3U_1 - 2U_2)(-5 + 3U_1 + 4U_2 - U_3)$
47	(-3,-40,1)	{-9,-11,-3,0,20,-80}	{-2,-3,1,0,0,0}	70	$-(2 + 3U_1 - U_2)(-13 + 10U_1 + 10U_2 - U_3)$
48	(-3,-40,-1)	{-5,-10,10,0,-20,80}	{2,3,-1,0,0,0}	70	$(2 + 3U_1 - U_2)(10(-2 + U_1 + U_2) - U_3)$
49	$(-3, -\frac{8}{5}, 5)$	{-9,-13,3,0,4,-16}	{-10,-15,5,0,0,0}	70	$-(3U_1 - U_2 + 3)(2U_1 + 2U_2 - 5U_3 + 1)$
50	$(-3, -\frac{8}{5}, -5)$	{-14,-9,7,0,-4,16}	{15,15,-5,0,0,0}	70	$(3U_1 - U_2 + 3)(2U_1 + 2U_2 - 5U_3 - 7)$

Table 2.1. Representatives of families of integer fluxes for solutions with $W = 0$ and 1 flat direction, Part 1.

	$(\lambda_2,\lambda_3,h_3)$	F	H	$\mathcal{N}_{\text{flux}}$	W
51	(-3,-32,1)	{-10,-13,-1,0,16,-64}	{-2,-3,1,0,0,0}	56	$-(2+3U_1-U_2)(-9+8U_1+8U_2-U_3)$
52	(-3,-32,-1)	{-6,-11,9,0,-16,64}	{2,3,-1,0,0,0}	56	$(2+3U_1-U_2)(-17+8U_1+8U_2-U_3)$
53	(-3,-8,2)	{-10,-14,2,0,8,-32}	{-4,-6,2,0,0,0}	56	$-2(2+3U_1-U_2)(-1+2U_1+2U_2-U_3)$
54	(-3,-8,-2)	{-6,-10,6,0,-8,32}	{4,6,-2,0,0,0}	56	$2(2+3U_1-U_2)(-5+2U_1+2U_2-U_3)$
55	(-3,-2,4)	{-9,-13,3,0,4,-16}	{-8,-12,4,0,0,0}	56	$-(2+3U_1-U_2)(1+2U_1+2U_2-4U_3)$
56	(-3,-2,-4)	{-9,-14,6,0,-4,16}	{8,12,-4,0,0,0}	56	$2(2+3U_1-U_2)(U_1+U_2-2(2+U_3))$
57	(-3,-24,1)	{-9,-12,0,0,12,-48}	{-2,-3,1,0,0,0}	42	$-(2+3U_1-U_2)(6(-1+U_1+U_2)-U_3)$
58	(-3,-24,-1)	{3,3,3,0,-12,48}	{2,3,-1,0,0,0}	42	$(2+3U_1-U_2)(-9+6U_1+6U_2-U_3)$
59	(-1,-16,1)	{-3,-14,6,0,20,-64}	{-2,-1,1,0,0,0}	42	$-(2+U_1-U_2)(-4+6U_1+10U_2-U_3)$
60	(-1,-16,-1)	{-10,-9,1,0,-20,64}	{1,1,-1,0,0,0}	42	$(1+U_1-U_2)(-21+6U_1+10U_2-U_3)$
61	$(-3,-\frac{8}{3},3)$	{-9,-13,3,0,4,-16}	{-6,-9,3,0,0,0}	42	$-(2+3U_1-U_2)(1+2U_1+2U_2-3U_3)$
62	$(-3,-\frac{8}{3},-3)$	{-14,-9,7,0,-4,16}	{9,9,-3,0,0,0}	42	$(3+3U_1-U_2)(-7+2U_1+2U_2-3U_3)$
63	(-3,-16,1)	{-2,-2,-2,0,8,-32}	{-2,-3,1,0,0,0}	28	$-(2+3U_1-U_2)(-6+4U_1+4U_2-U_3)$
64	(-3,-16,-1)	{0,-1,3,0,-8,32}	{2,3,-1,0,0,0}	28	$(2+3U_1-U_2)(-7+4U_1+4U_2-U_3)$
65	(-3,-4,2)	{-9,-13,3,0,4,-16}	{-4,-6,2,0,0,0}	28	$-(2+3U_1-U_2)(1+2U_1+2U_2-2U_3)$
66	(-3,-4,-2)	{-14,-9,7,0,-4,16}	{6,6,-2,0,0,0}	28	$(3U_1-U_2+3)(2U_1+2U_2-2U_3-7)$
67	(-3,-8,1)	{-1,-1,-1,0,4,-16}	{-2,-3,1,0,0,0}	14	$-(3U_1-U_2+2)(2U_1+2U_2-U_3-3)$
68	(-3,-8,-1)	{-1,-2,2,0,-4,16}	{2,3,-1,0,0,0}	14	$(3U_1-U_2+2)(2(-2+U_1+U_2)-U_3)$

Table 2.2. Representatives of families of integer fluxes for solutions with $W=0$ and 1 flat direction, Part 2.

Solutions with $W \neq 0$

As already pointed out in Sec. 2.1, supersymmetric solutions require $\partial_a W = -W\partial_a K\ \forall a = 1,2,3$. Thus, if $W \neq 0$, solving the global supersymmetry flatness conditions $\partial_a W = 0$ does not lead in general to F-flat solutions in supergravity. However, if the solutions to $\partial_a W = 0$ feature a flat direction parametrised by U_3, it is easy to realize that along the flat direction $\partial_1 K \sim \partial_2 K \sim \partial_3 K \sim 1/\text{Im}\,(U_3) \to 0$ for $\text{Im}\,(U_3) = g_s^{-1} \to \infty$. This limit corresponds to weak string couplings and, as can be seen in the explicit solution (2.124), to large complex structure where the non-perturbative contributions to the prepotential can be ignored. Therefore solving the global supersymmetry flatness conditions can be a useful starting point to construct solutions in a perturbative expansion. When U_3 is flat, this approximation can be made exact by taking by hand $\text{Im}\,(U_3)$ arbitrarily large, while when U_3 is lifted by instanton corrections, one has to make sure that at the minimum $W\partial_a K$ is infinitesimally small.

Below, we discuss some solutions to the global supersymmetry flatness conditions in the $\mathbb{CP}_{[1,1,1,6,9]}[18]$ example. Given (2.119), consider the 3 polynomial equations in 3 variables: $\partial_a W = 0\ \forall a = 1,2,3$. The degree of each of these equations is 3 or less, depending upon the choices of fluxes. For this system of equations to admit a solution with at least 1 flat direction, one of them should be dependent on the others. Here we examine cases when this dependence is linear:

(i) When $\partial_1 W = \lambda_2\,\partial_2 W + \lambda_3\,\partial_3 W$ with at least one of λ_2 and λ_3 which is non-zero and subject to $\mathcal{N}_{\text{flux}} \neq 0$, we find only 1 family of fluxes (details are given below) for which $\partial_a W = 0 \ \forall a = 1,2,3$ admit complex solutions $\hat{U}_a$, i.e. Im $\hat{U}_a \neq 0 \ \forall a$. With the above family of fluxes we have only 1 flat direction which is the dilaton;

(ii) When $\partial_2 W = \lambda_3 \, \partial_3 W$ with $\lambda_3 \neq 0$, the conditions on the fluxes have no solution in keeping with $\mathcal{N}_{\text{flux}} \neq 0$.

Let us provide the aforesaid family of fluxes which are dependent on 6 parameters λ_2, λ_3, f_1, f_3, h_1 and h_3:

$$\{f_1, f_2, f_3, f_4, f_5, f_6\} = \left\{ f_1, \lambda_2 f_3 - \lambda_3 h_1 - \frac{3\lambda_3 h_3}{2}, f_3, 0, \frac{(2\lambda_2 - 3)\lambda_3 h_3}{(\lambda_2 - 3)\lambda_2}, \frac{(\lambda_2 - 3)\lambda_3 h_3}{\lambda_2} \right\},$$
$$\{h_1, h_2, h_3, h_4, h_5, h_6\} = \{h_1, \lambda_2 h_3, h_3, 0, 0, 0\}, \qquad \lambda_2 \neq 0, 3, \qquad \lambda_3, h_3 \neq 0. \tag{2.125}$$

In this case we have:

$$\begin{aligned} \mathcal{N}_{\text{flux}} &= -\frac{3\left((\lambda_2 - 3)\lambda_2 + 3\right)\lambda_3 h_3^2}{2(\lambda_2 - 3)\lambda_2}, \\ W &= (f_1 + (f_3 - h_3 U_3)(\lambda_2 U_1 + U_2) - h_1(\lambda_3 U_1 + U_3)) \\ &\quad + \frac{\lambda_3 h_3}{4(\lambda_2 - 3)\lambda_2}\Big[2\lambda_2^2(U_1(3U_1 + 2U_2 - 6) - 3) \\ &\quad + 2\lambda_2(9U_1 + 2(U_2 - 3)U_2 + 1) - 6(U_2 - 3)U_2 - 3\Big]. \end{aligned} \tag{2.126}$$

Now, solving $\partial_a W = 0 \ \forall a = 1,2,3$, we see that U_1 and U_2 depend linearly on U_3 with slopes $-1/\lambda_3$ and λ_2/λ_3 respectively. Thus, only by requiring $\lambda_2, \lambda_3 < 0$, we may obtain Im U_a, $\forall a = 1,2,3$ to be of the same sign. This also keeps $\mathcal{N}_{\text{flux}}$ positive. At the solution we have:

$$W = f_1 - \frac{f_3 h_1}{h_3} + \frac{\lambda_3\left((4\lambda_2 - 6)h_1^2 + 6(2\lambda_2 - 3)h_1 h_3 + \left(-6\lambda_2^2 + 2\lambda_2 - 3\right)h_3^2\right)}{4(\lambda_2 - 3)\lambda_2 h_3}, \tag{2.127}$$

which we require not to vanish. In this case, by integer translations of U_1, U_2 and U_3, W cannot be made a degree-2 homogeneous function since a necessary condition for doing so is the same as the vanishing condition of W at the minimum. Note the arguments of $f_5(\lambda_2, \lambda_3, h_3)$, $f_6(\lambda_2, \lambda_3, h_3)$, $h_2(\lambda_2, h_3)$ and $\mathcal{N}_{\text{flux}}(\lambda_2, \lambda_3, h_3)$. There are only 488 triples $(\lambda_2, \lambda_3, h_3)$, $\lambda_2, \lambda_3 \in \mathbb{Q}^-$, $h_3 \in \mathbb{Z}$, securing $f_5, f_6, h_2 \in \mathbb{Z}$ and $\mathcal{N}_{\text{flux}} \in \mathbb{Z}/2$ with $0 < \mathcal{N}_{\text{flux}} \leq 138$. For each of the triples $(\lambda_2, \lambda_3, h_3)$, there exist $f_1, f_3, h_1 \in \mathbb{Z}$ that keep all other fluxes in (2.125) integers, as well as $W \neq 0$. In fact, for each of the triples, we get a subfamily of integer fluxes

(2.125) parametrised by f_1, f_3 and h_1. All the members in any of the aforementioned subfamilies have the same $\mathcal{N}_{\text{flux}}(\lambda_2, \lambda_3, h_3)$. Let us point out that among these 488 values of $\mathcal{N}_{\text{flux}}(\lambda_2, \lambda_3, h_3)$ only 64 are distinct. Below, we discuss one of these subfamilies in detail.

We consider one of the 488 triples given by $(\lambda_2, \lambda_3, h_3) = (-4, -56, -1)$. For this, $\mathcal{N}_{\text{flux}} = 93$ and (2.125) becomes:

$$\{f_1, f_2, f_3, f_4, f_5, f_6\} = \{f_1, -4(f_3 - 14h_1 + 21), f_3, 0, -22, 98\}\,,$$
$$\{h_1, h_2, h_3, h_4, h_5, h_6\} = \{h_1, 4, -1, 0, 0, 0\}\,, \tag{2.128}$$

and (2.126) gives:

$$W = f_1 - 4U_1(f_3 - 14h_1 - 8U_2 + U_3 + 33) \tag{2.129}$$
$$+ U_2(f_3 - 11U_2 + U_3 + 33) - h_1 U_3 + 48U_1^2 - \frac{107}{2}\,.$$

Clearly, every $f_1, f_3, h_1 \in \mathbb{Z}$ retain all fluxes integers and the solution to $\partial_a W = 0$, $\forall a = 1, 2, 3$ is given by:

$$(U_1, U_2, U_3) = \left(\frac{1}{56}\left(U_3 - 22h_1 + 33 + f_3\right), \frac{1}{14}\left(U_3 - 8h_1 + 33 + f_3\right), U_3\right)\,. \tag{2.130}$$

At the solution we have:

$$W = f_1 + h_1\left(f_3 - 11h_1 + 33\right) - \frac{107}{2}\,, \tag{2.131}$$

which is non-zero since $f_1, f_3, h_1 \in \mathbb{Z}$.

2.3.3 Flat directions in a Calabi-Yau with 4 moduli

In this section we search for flat directions using the CY discussed in [134] which features effectively 3 complex structure moduli at the $\mathcal{G}$-symmetric locus. We begin by quoting the large complex structure expansion of the prepotential (denoting (U^1, U^2, U^4, ϕ) by (U_1, U_2, U_3, U_4)):

$$\mathcal{F}(U_a) = -3U_1U_2U_4 - 3U_1U_3U_4 - 3U_2U_3U_4 - 3U_2U_4^2 - 3U_3U_4^2 - \frac{3}{2}U_1^2U_4$$
$$-\frac{9}{2}U_1U_4^2 - \frac{5}{2}U_4^3 + 3U_1U_4 + \frac{3}{2}U_2U_4 + \frac{3}{2}U_3U_4 + \frac{15}{4}U_4^2$$
$$+\frac{3}{2}U_1 + U_2 + U_3 + \frac{11}{4}U_4 + \xi\,, \tag{2.132}$$

where ξ (that involves the CY Euler number) is imaginary with irrational imaginary part. The period vector Π is given by (2.110) and the superpotential can be written explicitly using (2.111), together with the fluxes

$F = (f_1\ f_2\ f_3\ f_4\ f_5\ f_6\ f_7\ f_8\ f_9\ f_{10})^t$ and $H = (h_1\ h_2\ h_3\ h_4\ h_5\ h_6\ h_7\ h_8\ h_9\ h_{10})^t$ that are integer-valued. For simplicity we set $f_6 = h_6 = 0$ which eliminates the ξ dependence in W. Furthermore, due to the $\mathcal{G}$-symmetry, we identify:

$$f_4 = f_3\,, \qquad h_4 = h_3\,, \qquad f_9 = f_8\,, \qquad h_9 = h_8\,, \qquad U_2 = U_3\,. \tag{2.133}$$

Moreover the orientifold and brane setup discussed in [134] give a tadpole bound $\mathcal{N}_{\text{flux}} = -\frac{1}{2}H^t \cdot \Sigma \cdot F \le \frac{1}{2}\left(20 + 3(1+2n_b)^2\right)$ with $n_b \in \mathbb{Z}$. For definiteness, we shall choose $n_b = -2$ which yields $\mathcal{N}_{\text{flux}} \le 47/2$. With the above, we have:

$$\begin{aligned}
\mathcal{N}_{\text{flux}} =& \frac{1}{2}(f_5 h_{10} - f_7 h_2 - 2 f_8 h_3 - f_{10} h_5 + f_2 h_7 + 2 f_3 h_8)\,,\\
W =& (h_{10}U_4 - f_{10})\left[\frac{11}{4} + 3(U_1+U_2) + \frac{15}{2}U_3 - \frac{3}{2}\Big((U_1+2U_2+3U_3)^2\right.\\
&\left.- (U_2+2U_3)^2 - U_2^2\Big)\right] + U_1(f_2 - h_2U_4) + 2U_2(f_3 - h_3U_4)\\
&+ U_3(f_5 - h_5U_4) + \frac{3}{2}(f_7 - h_7U_4)(U_3(2U_1+4U_2+3U_3-2)-1) + f_1\\
&+ (f_8 - h_8U_4)(3U_3(2U_1+2U_2+2U_3-1)-2) - h_1U_4\,.
\end{aligned} \tag{2.134}$$

In this case we will not perform a systematic search for supersymmetric solutions with approximate flat directions. As a preliminary analysis, we note however that, given (2.134) and considering $\partial_1 W = \lambda_2\,\partial_2 W$, $\lambda_2 \neq 0$, $\mathcal{N}_{\text{flux}} \neq 0$, there exists a class of fluxes for which $\partial_a W = 0$, $\forall a = 1,\ldots,4$ admit complex solutions $\hat{U}_a$, i.e. Im $\hat{U}_a \neq 0\ \forall a$ with 2 flat directions. The aforesaid class of fluxes and corresponding $\mathcal{N}_{\text{flux}}$ are given by:

$$\begin{aligned}
&\{f_1, f_2, f_3, f_5, f_7, f_8, f_{10}, h_1, h_2, h_3, h_5, h_7, h_8, h_{10}\} =\\
&\left\{f_1, -2f_3, f_3, -\frac{f_8(h_1+h_3)}{h_3}, -\frac{4}{3}\,f_8, f_8, 0, h_1, -2h_3, h_3, 0, 0, 0, 0\right\}\,,\\
&\mathcal{N}_{\text{flux}} = -\frac{7}{3}\,f_8 h_3\,,
\end{aligned} \tag{2.135}$$

where $f_8, h_3 \neq 0$ and $\lambda_2 = -1$. Let us present an explicit example. For $f_8 = -3$ and $h_3 = 3$, $\mathcal{N}_{\text{flux}} = 21$. In this case the fluxes become:

$$\begin{aligned}
&\{f_1, f_2, f_3, f_5, f_7, f_8, f_{10}, h_1, h_2, h_3, h_5, h_7, h_8, h_{10}\} =\\
&\{f_1, -2f_3, f_3, 3+h_1, 4, -3, 0, h_1, -6, 3, 0, 0, 0, 0\}\,,
\end{aligned} \tag{2.136}$$

and the superpotential reads:

$$W = f_1 + 2f_3\,(U_2 - U_1) + (U_3 - U_4)\,(h_1 - 6U_1 + 6U_2)\,. \tag{2.137}$$

Clearly, $f_1, f_3, h_1 \in \mathbb{Z}$ retain all fluxes integer and the solution to $\partial_a W = 0$, $\forall a = 1, \ldots, 4$ is given by:

$$(U_1, U_2, U_3, U_4) = \left(U_2 + \frac{h_1}{6}, U_2, U_4 - \frac{f_3}{3}, U_4\right). \tag{2.138}$$

Notice that when the flat directions U_2 and U_4 are in the large complex structure limit, the same is necessarily true for U_1 and U_3. Moreover, the superpotential at the supersymmetric minimum is:

$$W = f_1 - \frac{f_3 h_1}{3}. \tag{2.139}$$

As in the $\mathbb{CP}_{[1,1,1,6,9]}[18]$ case, by integer translations of U_1, U_2, U_3 and U_4, a necessary condition for making W a degree-2 homogeneous function is the same as the vanishing condition of W at the minimum. Hence, for solutions with $W \neq 0$ at the minimum, the superpotential is a non-homogeneous polynomial of degree 2, while for solutions with vanishing W at the minimum, the superpotential in some cases can be brought to a homogeneous function of degree 2. An example where it can be done is the case with $f_1 = 6$, $f_3 = 3$ and $h_1 = 6$. However, in the case with $f_1 = -14$, $f_3 = -6$ and $h_1 = 7$, W cannot be brought to a degree-2 homogeneous function by integer shifts of U_a, although it vanishes at the minimum. Detailed explorations in various CYs will be carried out in the future.

Chapter III

Higher Derivative Corrections to Inflationary Potentials

In this chapter we present general classes of divisor topologies which are relevant for making such corrections naturally vanish for the inflaton direction. In particular, we find that blow-up inflation is protected against such higher derivative corrections if the inflaton corresponds to the volume of a dP_3 divisor, i.e. a del Pezzo surface of degree six. Fibre inflation is instead shielded if the inflaton is the volume of a $\mathbb{T}^4$-divisor, while poly-instanton inflation is naturally safe only for the inflaton being the volume of a so-called 'Wilson' divisor (W), i.e. a rigid divisor with a Wilson line and $h^{1,1}(W) = 2$. We present an explicit CY orientifold setting for each of these three classes of models. Moreover, we find that there are additional divisor topologies for which such F^4 corrections vanish.

For generic topologies with non-vanishing Π, we perform a numerical estimate of the effect of these F^4 corrections on inflation, paying particular attention to the study of reheating from moduli decay to determine the exact number of efoldings of inflation which is relevant to match observations. We find that higher derivative α'^3 effects do not substantially change the conclusions of fibre, blow-up and poly-instanton inflationary scenarios, therefore making those scenarios more robust under these corrections.

3.1 Divisor topologies in LVS

In this section we present a brief review of the role of divisor topologies in the context of the LVS scheme of moduli stabilisation. It has been well established that some divisor topologies play a central role in LVS model building. These are, for example, del Pezzo (dP) and K3 surfaces. Such studies and suitable CY scans have been presented at several different occasions with different sets of interests [28, 65, 145, 204–210], and we recollect some of the ingredients from [28, 102] which are relevant for the present work.

3.1.1 Generic LVS scalar potential

In the standard approach of moduli stabilisation in 4D type IIB effective supergravity models, one follows a so-called two-step strategy. In the first step, the axio-dilaton S and the complex structure moduli U^α are stabilized by the superpotential W_{flux} induced by background 3-form fluxes (F_3, H_3). This flux-dependent superpotential can fix all complex structure moduli and the axio-dilaton supersymmetrically at leading order by enforcing:

$$D_{U^\alpha} W_{\text{flux}} = D_S W_{\text{flux}} = 0\,. \tag{3.1}$$

After fixing S and the U-moduli, the flux superpotential can effectively be considered as constant: $W_0 = \langle W_{\text{flux}} \rangle$. At this leading order, the Kähler moduli T_i remain flat due to the no-scale cancellation. Using non-perturbative effects is one of the possibilities to fix these moduli. In this context, if we assume n non-perturbative contributions to W which can be generated by using rigid divisors, such as shrinkable dP 4-cycles, or by rigidifying non-rigid divisors using magnetic fluxes [155, 211, 212], the superpotential takes the following form:

$$W = W_0 + \sum_{i=1}^{n} A_i\, e^{-a_i T_i}\,, \tag{3.2}$$

where:

$$T_i = \tau_i + i\theta_i \qquad \text{with} \qquad \tau_i = \frac{1}{2}\int_{D_i} J \wedge J \quad \text{and} \quad \theta_i = \int_{D_i} C_4\,. \tag{3.3}$$

For the current work we consider CY orientifolds with trivial odd sector in the $(1,1)$-cohomology and subsequently orientifold-odd moduli are absent in our analysis (interested readers may refer to [170, 208, 213]). Note that in (3.2) there is no sum in the exponents $(a_i\, T_i)$, and summations are to be understood only when upper indices are contracted with lower indices; otherwise we will write an explicit sum as in (3.2). We will suppose that, out of $h^{1,1}_+ = h^{1,1}$ Kähler moduli, only the first n appear in W, i.e. $i = 1, ..., n \leq h^{1,1}_+$.

The Kähler potential including α'^3 corrections takes the form [214]:

$$K = -\ln\left[-i\int \Omega\left(U^\alpha\right)\wedge\bar{\Omega}\left(\bar{U^\alpha}\right)\right] - \ln\left(S+\bar{S}\right) \\ -2\ln\left[\mathcal{V}\left(T_i+\bar{T}_i\right)+\frac{\xi}{2}\left(\frac{S+\bar{S}}{2}\right)^{3/2}\right], \tag{3.4}$$

where Ω denotes the nowhere vanishing holomorphic 3-form which depends on the complex-structure moduli, while $\mathcal{V}$ denotes the CY volume which receives α'^3 corrections through $\xi = -\frac{\chi(X)\,\zeta(3)}{2\,(2\pi)^3}$ where $\chi(X)$ is the CY Euler characteristic and $\zeta(3)\simeq 1.202$.

Assuming that S and the U-moduli are stabilized as in (3.1), considering a superpotential given by (3.2) and an α'^3-corrected Kähler potential given by (3.4), one arrives at the following master formula for the scalar potential [102]:

$$V = V_{\alpha'^3} + V_{\mathsf{np1}} + V_{\mathsf{np2}}\,, \tag{3.5}$$

where (defining $\hat{\xi}\equiv \xi g_s^{-3/2}$ with $g_s = \langle \mathrm{Re}(S)\rangle^{-1}$):

$$\begin{aligned} V_{\alpha'^3} &= e^K\,\frac{3\,\hat{\xi}(\mathcal{V}^2+7\,\mathcal{V}\,\hat{\xi}+\hat{\xi}^2)}{(\mathcal{V}-\hat{\xi})(2\mathcal{V}+\hat{\xi})^2}\,|W_0|^2\,, \\ V_{\mathsf{np1}} &= e^K\sum_{i=1}^{n} 2\,|W_0|\,|A_i|\,e^{-a_i\tau_i}\,\cos(a_i\,\theta_i+\phi_0-\phi_i) \\ &\quad\times\left[\frac{(4\mathcal{V}^2+\mathcal{V}\,\hat{\xi}+4\,\hat{\xi}^2)}{(\mathcal{V}-\hat{\xi})(2\mathcal{V}+\hat{\xi})}\,(a_i\,\tau_i)+\frac{3\,\hat{\xi}(\mathcal{V}^2+7\,\mathcal{V}\,\hat{\xi}+\hat{\xi}^2)}{(\mathcal{V}-\hat{\xi})(2\mathcal{V}+\hat{\xi})^2}\right], \\ V_{\mathsf{np2}} &= e^K\sum_{i=1}^{n}\sum_{j=1}^{n}|A_i|\,|A_j|\,e^{-(a_i\tau_i+a_j\tau_j)}\,\cos(a_i\,\theta_i-a_j\,\theta_j-\phi_i+\phi_i) \\ &\quad\times\left[-4\left(\mathcal{V}+\frac{\hat{\xi}}{2}\right)(k_{ijk}\,t^k)\,a_i\,a_j+\frac{4\mathcal{V}-\hat{\xi}}{(\mathcal{V}-\hat{\xi})}\,(a_i\,\tau_i)\,(a_j\,\tau_j)\right. \\ &\quad\left.+\frac{(4\mathcal{V}^2+\mathcal{V}\,\hat{\xi}+4\,\hat{\xi}^2)}{(\mathcal{V}-\hat{\xi})(2\mathcal{V}+\hat{\xi})}\,(a_i\,\tau_i+a_j\,\tau_j)+\frac{3\,\hat{\xi}(\mathcal{V}^2+7\,\mathcal{V}\,\hat{\xi}+\hat{\xi}^2)}{(\mathcal{V}-\hat{\xi})(2\mathcal{V}+\hat{\xi})^2}\right], \end{aligned} \tag{3.6}$$

where we have introduced phases into the parameters as $W_0 = |W_0|\,e^{\mathrm{i}\,\phi_0}$ and $A_i = |A_i|\,e^{\mathrm{i}\,\phi_i}$. The good thing about the master formula (3.6) is the fact that it determines the complete form of V simply by specifying topological quantities such as the intersection numbers k_{ijk}, the CY Euler number and the number n of non-perturbative contributions to W.

Note that $V_{\alpha'^3}$ vanishes for $\hat{\xi}=0$ and reproduces the standard no-scale structure in the absence of a T-dependent non-perturbative W. On the other hand, for very large volume $\mathcal{V} \gg \hat{\xi}$, this term takes the standard form which plays a crucial rôle in LVS models [27]:

$$V_{\alpha'^3} \simeq \left(\frac{g_s\, e^{K_{cs}}}{2\,\mathcal{V}^2}\right) \frac{3\,\hat{\xi}\,|W_0|^2}{4\,\mathcal{V}}\,. \tag{3.7}$$

Let us also stress that $V_{\alpha'^3}$ depends only on the overall volume $\mathcal{V}$, while V_{np1} depends on $\mathcal{V}$ and the 4-cycle moduli τ_i (with the additional dependence on the axions θ_i). Hence these two contributions to V could be minimized by taking derivatives with respect to $\mathcal{V}$ and $(h^{1,1}-1)$ 4-cycle moduli. However V_{np2} depends on the quantity $k_{ijk}\,t^k$ which in general cannot be inverted to be expressed as an explicit function of the τ_i's. It has been observed that using the master formula (3.6) one can efficiently perform moduli stabilisation in terms of the 2-cycle moduli t^i as shown in [102, 146].

For example, considering $h^{1,1} = 2$, $n = 1$ and $\hat{\xi} > 0$ in the master formula (3.6) along with using the large volume limit, one can immediately read-off the following three terms:

$$V \simeq \frac{g_s\, e^{K_{cs}}}{2}\left[\frac{3\,\hat{\xi}\,|W_0|^2}{4\mathcal{V}^3} + \frac{4a_1\tau_1|W_0||A_1|}{\mathcal{V}^2}\, e^{-a_1\tau_1}\cos\left(a_1\theta_1+\phi_0-\phi_1\right) - \frac{4a_1^2|A_1|^2k_{111}t_1}{\mathcal{V}}\, e^{-2a_1\tau_1}\right]. \tag{3.8}$$

If the CY X has a Swiss-cheese form, one can find a basis of divisors such that the only non-zero intersection numbers are k_{111} and k_{222}. This leads to the relation $t^1 = -\sqrt{2\tau_1/k_{111}}$, where the minus sign is dictated from the Kähler cone conditions as the divisor D_1 in this Swiss-cheese CY is an exceptional 4-cycle. Using this in ((3.8)) one gets [27]:[1]

$$V \simeq \frac{g_s\, e^{K_{cs}}}{2}\left(\frac{\beta_{\alpha'}}{\mathcal{V}^3} + \beta_{\text{np1}}\,\frac{\tau_1}{\mathcal{V}^2}\, e^{-a_1\tau_1}\cos\left(a_1\theta_1+\phi_0-\phi_1\right) + \beta_{\text{np2}}\,\frac{\sqrt{\tau_1}}{\mathcal{V}}\, e^{-2a_1\tau_1}\right), \tag{3.9}$$

with:

$$\beta_{\alpha'} = \frac{3\hat{\xi}|W_0|^2}{4}\,, \qquad \beta_{\text{np1}} = 4a_1|W_0||A_1|\,, \qquad \beta_{\text{np2}} = 4a_1^2|A_1|^2\sqrt{2k_{111}}\,. \tag{3.10}$$

3.1.2 Scanning results for LVS divisor topologies

Let us start by briefly reviewing the generic methodology for analysing the divisor topologies which is widely adopted for scanning useful CY ge-

[1] Ref. [102] has shown that LVS moduli fixing can be realized also for generic cases where the CY threefold does not have a Swiss-cheese structure.

ometries suitable for phenomenology, e.g. see [28, 209]. Subsequently we will continue following the same in our current analysis. The main idea is to consider the CY threefolds arising from the four-dimensional reflexive polytopes listed in the Kreuzer-Skarke (KS) database [64], and classify the divisors based on their relevance for phenomenological model building aiming at explicit orientifold constructions. For that purpose, we rather have a very nice collection of the various topological data of CY threefolds available in the database of [65] which can be directly used for further analysis. In this regard, Tab. 3.1 presents the number of (favorable) polytopes along with the corresponding (favorable) triangulations and (favorable) geometries for a given $h^{1,1}(X)$ in the range $1 \leq h^{1,1}(X) \leq 5$.

$h^{1,1}$	Polytopes	Favorable Polytopes	Triangs.	Favorable Triangs.	Geometries	Favorable Geoms.
1	5	5	5	5	5	5
2	36	36	48	48	39	39
3	244	243	569	568	306	305
4	1197	1185	5398	5380	2014	2000
5	4990	4897	57132	56796	13635	13494

Table 3.1. Number of (favorable) triangulations and (favorable) distinct CY geometries arising from the (favorable) polytopes listed in the Kreuzer-Skarke database.

For a given CY geometry, the main focus is limited to:

- looking at the topology of the so-called 'coordinate divisors' D_i which are defined through setting the toric coordinates to zero, i.e. $x_i = 0$. This means that there is a possibility of missing a huge number of divisors, e.g. those which could arise via considering some linear combinations of the coordinate divisors, and some of such may have interesting properties. However, it is hard to make an exhaustive analysis including all the effective divisors of a given CY threefold.

- focusing on scans using the so-called 'favorable' triangulations (Triang*) and 'favorable' geometries (Geom*) for a given polytope. This could be justified in the sense that for non-favorable CY threefolds, the number of toric divisors in the basis is less than $h^{1,1}(X)$, and subsequently there is always at least one coordinate divisor which is non-smooth, and one usually excludes such spaces from the scan. However, the number of such CY geometries is almost negligible in the sense that there are just 1, 14 and 141 for $h^{1,1}(X)$ being 3, 4 and 5 respectively.

The role of divisor topologies in the LVS context can be appreciated by noting that the Swiss-cheese structure of the CY volume can be correlated with the presence of del Pezzo (dP$_n$) divisors D_s. These dP$_n$ divisors are defined for $0 \leq n \leq 8$ having degree $d = 9 - n$ and $h^{1,1} = 1 + n$, such that dP$_0$ is a $\mathbb{P}^2$ and the remaining 8 del Pezzo's are obtained by blowing up eight generic points inside $\mathbb{P}^2$. It turns out that they satisfy the following two conditions [145]:

$$\int_X D_s^3 = k_{sss} > 0\,, \qquad \int_X D_s^2\, D_i \leq 0 \qquad \forall\, i \neq s\,. \tag{3.11}$$

Here the self-triple-intersection number k_{sss} corresponds to the degree of the del Pezzo 4-cycle dP$_n$ where $k_{sss} = 9 - n > 0$, which is always positive as $n \leq 8$ for del Pezzo surfaces. In addition, one imposes the so-called 'diagonality' condition on such a del Pezzo divisor D_s via the following relation satisfied by the triple intersection numbers [145, 205]:

$$k_{sss}\, k_{sij} = k_{ssi}\,,\, k_{ssj} \qquad \quad \forall\;\; i, j. \tag{3.12}$$

It turns out that whenever this diagonality condition is satisfied, there exists a basis of coordinates divisors such that the volume of each of the 4-cycles D_s becomes a complete-square quantity as illustrated from the following relations:

$$\tau_s = \frac{1}{2}\, k_{sij} t^i\, t^j = \frac{1}{2\, k_{sss}}\, k_{ssi}\, k_{ssj} t^i\, t^j = \frac{1}{2\, k_{sss}}\, \left(k_{ssi}\, t^i\right)^2\,. \tag{3.13}$$

Subsequently what happens is that one can always shrink such a 'diagonal' del Pezzo ddP$_n$ to a point-like singularity by squeezing it along a single direction. A systematic analysis on counting the CY geometries which could support (standard) LVS models, in the sense of having at least one diagonal del Pezzo divisor, has been performed in [28] and the results are summarized in Tab. 3.2. Moreover, it is worth mentioning that the scanning result presented in Tab. 3.2 is quite peculiar in the sense that for all the CY threefolds with $h^{1,1} \leq 5$, one does not have any example having a 'diagonal' dP$_n$ divisor for $1 \leq n \leq 5$, which has been subsequently conjectured to be true for all the CY geometries arising from the KS database.

Let us mention that the classification of CY geometries relevant for LVS as presented in Tab. 3.2 corresponds to having a 'standard' LVS in the sense of having at least one 'diagonal' del Pezzo divisor in a Swiss-cheese

$h^{1,1}$	Poly*	Geom* (n_{CY})	ddP_0	$d\mathbb{F}_0$	ddP_n $1 \leq n \leq 5$	ddP_6	ddP_7	ddP_8	n_{LVS} ($\mathrm{ddP}_n \geq 1$)
1	5	5	0	0	0	0	0	0	0
2	36	39	9	2	0	2	4	5	22
3	243	305	59	16	0	17	40	39	132
4	1185	2000	372	144	0	109	277	157	750
5	4897	13494	2410	944	0	624	827	407	4104

Table 3.2. Number of CY geometries with a 'diagonal' del Pezzo divisor suitable for LVS. Here we have extended the notation to denote a $\mathbb{P}^2$ surface as ddP_0 and a diagonal $\mathbb{P}^1 \times \mathbb{P}^1$ surface as $d\mathbb{F}_0$.

like model. However, it has been found in some cases that one can still have alternative moduli stabilisation schemes realizing an exponentially large CY volume, e.g. using the underlying symmetries of the CY threefold in the presence of a non-diagonal del Pezzo [102], and in the framework of the so-called perturbative LVS [101, 156, 215, 216].

3.2 F^4 corrections

In addition to the α'^3 correction (3.7) derived in [214], generically there can be many other perturbative corrections to the 4D effective scalar potential induced from various sources (see [17, 217] for a classification of potential contributions at different orders in α' exploiting higher dimensional rescaling symmetries and F-theory techniques). One such effect is given by F^4 corrections which cannot be captured by the two-derivative ansatz for the Kähler and superpotentials. In this section we shall discuss the topological taming of such corrections in the context of LVS inflationary model building.

3.2.1 F^4 corrections to the scalar potential

The higher derivative F^4 contributions to the scalar potential for a generic CY orientifold compactification take the following simple form [46]:

$$V_{F^4} = -\left(\frac{e^{K_{cs}}\, g_s}{8\pi}\right)^2 \frac{\lambda\, |W_0|^4}{g_s^{3/2}\mathcal{V}^4} \sum_{i=1}^{h^{1,1}} \Pi_i\, t^i \;\equiv \frac{\gamma}{\mathcal{V}^4} \sum_{i=1}^{h^{1,1}} \Pi_i\, t^i, \tag{3.14}$$

where the topological quantities Π_i are given by:

$$\Pi_i = \int_X c_2(X) \wedge \hat{D}_i\,, \tag{3.15}$$

and λ is an unknown combinatorial factor which in the single modulus case is rather small in absolute value [218]:

$$\lambda = -\frac{11}{24}\,\frac{\zeta(3)}{(2\pi)^4} = -3.5 \cdot 10^{-4}\,. \tag{3.16}$$

Its value is not known for $h^{1,1} > 1$ but we expect it to remain small, in analogy with the $h^{1,1} = 1$ case. In fact, one can argue that the factor $\zeta(3)/(2\pi)^4$ in λ is expected to be always present for generic models with several Kähler moduli as well. This is because the coupling tensor $\mathcal{T}_{\bar{\imath}\bar{\jmath}kl}$ appearing in this correction through the following higher derivative piece [46]:

$$V_{F^4} = -e^{2K}\,\mathcal{T}^{\bar{\imath}\bar{\jmath}kl}\,\overline{D}_{\bar{i}}\overline{W}\,\overline{D}_{\bar{j}}\overline{W}\,D_kW\,D_lW\,, \tag{3.17}$$

can be schematically written as:

$$\mathcal{T}_{\bar{\imath}\bar{\jmath}kl} = \frac{c}{\mathcal{V}^{8/3}}\,\frac{\zeta(3)\,\mathcal{Z}}{g_s^{3/2}}\,, \tag{3.18}$$

where c can be considered as some combinatorial factor, which for example, in the single modulus case turns out to be $11/384$ [218], and:

$$\mathcal{Z} = (2\pi)^2 \int_X c_2(X) \wedge J\,, \tag{3.19}$$

where we stress that we are working with the convention $\ell_s = (2\pi)\sqrt{\alpha'} = 1$. Subsequently, we have

$$\mathcal{T}_{\bar{\imath}\bar{\jmath}kl} = c\,\frac{\zeta(3)}{(2\pi)^4\,\mathcal{V}^{8/3}\,g_s^{3/2}} \int_X c_2(X) \wedge J\,. \tag{3.20}$$

Note that the $\mathcal{V}^{-8/3}$ factor in the above expression cancels off with a $\mathcal{V}^{8/3}$ contribution coming from 4 inverse Kähler metric factors needed to raise the 4 indices of the coupling tensor $\mathcal{T}_{\bar{\imath}\bar{\jmath}kl}$ to go to (3.17).

Here, let us mention that the higher derivative F^4 correction under consideration appears at α'^3 order, like the BBHL-correction [214], and both are induced at string tree-level, resulting in a factor of $g_s^{-3/2}$. For explicitness, let us also note that the leading order BBHL correction [214] appearing at the two-derivative level takes the following form:[2]

$$V_{\alpha'^3} = \left(\frac{e^{K_{cs}}\,g_s}{8\pi}\right)\frac{3\,\xi\,|W_0|^2}{4\,g_s^{3/2}\,\mathcal{V}^3}, \qquad \xi = -\frac{\zeta(3)\chi(X)}{2\,(2\pi)^3}\,. \tag{3.21}$$

[2] In this regard, it may be worth noticing that the original result [214] has been obtained with the convention $(2\pi\alpha') = 1$ which removes the $(2\pi)^{-3}$ factor from the denominator of the $\hat{\xi}$ parameter.

Now, comparing these two α' corrections one finds that:

$$\frac{V_{F^4}}{V_{\alpha'^3}} = \tilde{c}\left(\frac{g_s}{8\pi}\right) e^{K_{cs}}\,|W_0|^2 \left(\frac{\Pi_i\, t^i}{\chi(X)\mathcal{V}}\right), \tag{3.22}$$

where $\tilde{c}$ is some combinatorial factor, which for the case of a single Kähler modulus is

$$\tilde{c} = \frac{11}{9(2\pi)} \simeq 0.2\,. \tag{3.23}$$

One can observe that each factors in (3.22) can be of a magnitude less than one in typical models. For example, demanding large complex-structure limit in order to ignore instanton effects can typically result in having $e^{K_{cs}} \sim 0.01$ [155], the string coupling g_s needs to be small and the CY volume large to trust the low-energy EFT, and the ratios between Π_i's and $\chi(X)$ are typically of $\mathcal{O}(1)$ [209]. Having these aspects in mind, it is very natural to anticipate that higher-derivative F^4 effects are subdominant as compared to the two-derivative corrections. Note that (3.14) can also be rewritten as:

$$V_{F^4} = -V_{\alpha'^3}\,\frac{\sqrt{g_s}}{3\pi}\left(\frac{\lambda}{\xi}\right)\sum_{i=1}^{h^{1,1}}\Pi_i\left(\frac{m_{3/2}}{M_{\text{KK}}^{(i)}}\right)^2 \tag{3.24}$$

where the gravitino mass is:

$$m_{3/2}^2 = \left(\frac{g_s}{8\pi}\right)\frac{|W_0|^2}{\mathcal{V}^2}\,M_p^2\,, \tag{3.25}$$

and $M_{\text{KK}}^{(i)}$ is the Kaluza-Klein scale associated to the i-th divisor:

$$\left(M_{\text{KK}}^{(i)}\right)^2 = \frac{M_s^2}{t_i} = \frac{\sqrt{g_s}}{4\pi}\frac{M_p^2}{t_i\mathcal{V}}\,. \tag{3.26}$$

In the above equation we have used the relation between the string scale and the Planck mass in the convention where $\mathcal{V}_s = \mathcal{V}\,g_s^{3/2}$ (with $\mathcal{V}_s$ the volume in string frame and $\mathcal{V}$ the volume in Einstein frame):

$$M_s^2 = \frac{1}{(2\pi)^2\alpha'} = \sqrt{g_s}\,\frac{M_p^2}{4\pi\mathcal{V}}\,. \tag{3.27}$$

Note that (3.24) makes clear that V_{F^4} is an $\mathcal{O}(F^4)$ correction since $V_{\alpha'^3}$ is an $\mathcal{O}(F^2)$ effect and [11]:

$$\left(\frac{m_{3/2}}{M_{\text{KK}}}\right)^2 \sim g\,\frac{|F|^2}{M_{\text{KK}}^2} \ll 1\,, \tag{3.28}$$

where $g \sim M_{\text{KK}}/M_p \sim \mathcal{V}^{-2/3} \ll 1$ is the coupling of heavy KK modes to light states.

3.2.2 Classifying divisors with vanishing F^4 terms

Two important quantities characterizing the topology of a divisor D are the Euler characteristic $\chi(D)$ and the holomorphic Euler characteristic (also known as arithmetic genus) $\chi_h(D)$ which are given by the following useful relations [52, 219, 220]:

$$\chi(D) \equiv \sum_{i=0}^{4} (-1)^i \, b_i(D) = \int_X \hat{D} \wedge \left(\hat{D} \wedge \hat{D} + c_2(X)\right) , \tag{3.29}$$

$$\chi_h(D) \equiv \sum_{i=0}^{2} (-1)^i \, h^{i,0}(D) = \frac{1}{12} \int_X \hat{D} \wedge \left(2\, \hat{D} \wedge \hat{D} + c_2(X)\right) , \tag{3.30}$$

where $b_i(D)$ and $h^{i,0}(D)$ are respectively the Betti and Hodge numbers of the divisor. Using these two relations we find that $\Pi(D)$ is related with the Euler characteristics and the holomorphic Euler characteristic as follows:

$$\Pi(D) = \chi(D) - \int_X \hat{D}\wedge\hat{D}\wedge\hat{D}, \qquad \Pi(D) = 12\, \chi_h(D) - 2 \int_X \hat{D}\wedge\hat{D}\wedge\hat{D}\,, \tag{3.31}$$

which also give another useful relation:

$$\Pi(D) = 2\, \chi(D) - 12\, \chi_h(D) \,. \tag{3.32}$$

Therefore, the topological quantity $\Pi(D)$ vanishes for a generic smooth divisor D if the following simple relation holds,

$$\Pi(D) = 0 \quad \Longleftrightarrow \quad \chi(D) = 6\, \chi_h(D) \,. \tag{3.33}$$

Now, using the relations $\chi(D) = 2\, h^{0,0} - 4\, h^{1,0} + 2\, h^{2,0} + h^{1,1}$ and $\chi_h(D) = h^{0,0} - h^{1,0} + h^{2,0}$, we find another equivalent relation for vanishing $\Pi(D)$:

$$h^{1,1}(D) = 4\, h^{0,0}(D) - 2\, h^{1,0}(D) + 4\, h^{2,0}(D) \,. \tag{3.34}$$

Any divisor satisfying the vanishing Π relation (3.34) will be denoted as D_Π. After knowing the topology of a generic divisor D, it is easy to check if $h^{1,1}$ satisfies this condition or equivalently $\chi = 6\, \chi_h$. To demonstrate it, let us quickly consider the following two examples:

$$\mathbb{T}^4 \equiv \begin{matrix} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{matrix} \qquad \text{and} \qquad \text{K3} \equiv \begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix} \;.$$

Now it is obvious that $\mathbb{T}^4$ has $\Pi(\mathbb{T}^4) = 0$ as it satisfies $\chi = 0 = 6\,\chi_h$. However, K3 has $\Pi(\text{K3}) = 24$ and $6\chi_h = 12 = \chi/2$. Alternatively, it can be also checked that the Hodge number condition in (3.34) is satisfied for $\mathbb{T}^4$ but not for K3.

Therefore, we can generically formulate that a divisor D of a Calabi-Yau threefold having the following Hodge Diamond results in a vanishing $\Pi(D)$:

$$D_\Pi \equiv \begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{1,0} & \\ h^{2,0} & & \left(4h^{0,0} - 2h^{1,0} + 4h^{2,0}\right) & & h^{2,0} \\ & h^{1,0} & & h^{1,0} & \\ & & h^{0,0} & & \end{array}\,, \tag{3.35}$$

and if we consider that the D_Π divisor is smooth and connected, then we have $h^{0,0}(D_\Pi) = 1$. Subsequently we can identify three different classes of vanishing Π divisors:

1. **dP$_3$ divisors:** For connected rigid 4-cycles with no Wilson lines we have $h^{1,0}(D) = h^{2,0}(D) = 0$, and hence a vanishing $\Pi(D)$ results in the following Hodge diamond:

$$D_\Pi \equiv \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 4 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \equiv \text{dP}_\Pi\,. \tag{3.36}$$

   This topology corresponds to the dP surface of degree six, i.e. a dP$_3$. Moreover, this class of D_Π which singles out a dP$_3$ surface, also includes the possibility of the 'rigid but not del Pezzo' 4-cycle denoted as NdP$_n$ for $n \geq 9$ [145]. These surfaces are blow-up of line-like singularities and have similar Hodge diamonds as those of the usual dP surfaces dP$_n$ defined for $0 \leq n \leq 8$.

2. **Wilson divisors:** For connected rigid 4-cycles with Wilson lines we have $h^{2,0}(D) = 0$ but $h^{1,0}(D) > 0$, resulting in the following Hodge diamond for D_Π:

$$D_\Pi \equiv \begin{array}{ccccc} & & 1 & & \\ & h^{1,0} & & h^{1,0} & \\ 0 & & \left(4 - 2h^{1,0}\right) & & 0 \\ & h^{1,0} & & h^{1,0} & \\ & & 1 & & \end{array} \equiv W_\Pi\,. \tag{3.37}$$

Given that all Hodge numbers are non-negative integers, the only possibility compatible with $h^{1,1} \geq 1$ (to be able to a have a proper definition of the divisor volume) is $h^{1,0} = 1$ which, in turn, corresponds to $h^{1,1} = 2$. This is a so-called 'Wilson' divisor with vanishing $\Pi(W)$ which we denote as W_Π. This W_Π divisor corresponds to a subclass of 'Wilson' divisors, characterized by the Hodge numbers $h^{0,0} = h^{1,0} = 1$ and arbitrary $h^{1,1}$, that have been introduced in [221] to support poly-instanton corrections.

3. **Non-rigid divisors:** Now let us consider the third special class which can have deformation divisors, i.e. $h^{2,0}(D) > 0$. When the divisor does not admit any Wilson line, i.e. $h^{1,0}(D) = 0$, the Hodge diamond for D_Π simplifies to:

$$D_\Pi \equiv \begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ h^{2,0} & & (4 + 4\,h^{2,0}) & & h^{2,0} \\ & 0 & & 0 & \\ & & 1 & & \end{matrix} \,. \tag{3.38}$$

To our knowledge, so far there are no known examples in the literature which have such a topology. The simplest of its kind will have $h^{2,0}(D) = 1$ and $h^{1,1}(D) = 8$. In this regard, it is worth mentioning that the topology of the so-called 'Wilson' divisors which are $\mathbb{P}^1$ fibrations over $\mathbb{T}^2$s, has been argued to be useful in [222] and some years later it was found to be the case while studying the generation of poly-instanton effects [221]. So it would be interesting to know if such non-rigid divisor topologies of vanishing Π exist in explicit CY constructions, and further if they could be useful for some phenomenological applications.

The last possibility is to consider the most general situation with deformations and Wilson lines, i.e. $h^{2,0}(D) > 0$ and $h^{1,0}(D) > 0$. As already mentioned, the simplest case is $\mathbb{T}^4$ with $h^{2,0}(\mathbb{T}^4) = 1$, $h^{1,0}(\mathbb{T}^4) = 2$ and $h^{1,0}(\mathbb{T}^4) = 4$ which however never shows up in our search through the KS list, as well as more general divisors with both deformations and Wilson lines.

Before coming to the scan of such divisor topologies of vanishing Π, let us mention a theorem of [223, 224] which states that if the CY intersection polynomial is linear in the homology class $\hat{D}_f$ corresponding to a divisor D_f, then the CY threefold has the structure of a K3 or a $\mathbb{T}^4$ fibration over a $\mathbb{P}^1$ base. Noting the following relation for the self-triple-intersection number of a generic smooth divisor D:

$$\int_X \hat{D} \wedge \hat{D} \wedge \hat{D} = 12\, \chi_h(D) - \chi(D)\,, \tag{3.39}$$

and subsequently demanding the absence of such cubics for D_f in the CY intersection polynomial, results in $\chi(D) = 12\chi_h(D)$ or the following equivalent relation:

$$h^{1,1}(D_f) = 10\, h^{0,0}(D_f) - 8\, h^{1,0}(D_f) + 10\, h^{2,0}(D_f)\,. \tag{3.40}$$

This relation is clearly satisfied for K3 and $\mathbb{T}^4$ divisors, and can be satisfied for some other possible topologies as well. For example, another non-rigid divisor for which the self-cubic-intersection is zero is given by the following Hodge diamond:

$$\text{SD} \equiv \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 2 & & 30 & & 2 \\ & 0 & & 0 & \\ & & 1 & & \end{array}, \qquad \chi(SD) = 36, \qquad \chi_h(SD) = 3\,.$$

This is also a very well known surface frequently appearing in CY threefolds, e.g. it appears in the famous Swiss-cheese CY threefold defined as a degree-18 hypersurface in WCP$^4[1,1,1,6,9]$ where the divisors corresponding to the first three coordinates with charge 1 are such surfaces.

Moreover, interestingly one can see that for the 'Wilson' type divisor the relation in (3.40) is indeed satisfied for $h^{1,1}(D) = 2$ which is exactly something needed for the generation of poly-instanton effects on top of having vanishing $\Pi(D)$ as we have discussed before. In this regard, let us also add that the simultaneous vanishing of $\Pi(D)$ and $D^3_{|X}$ results in the vanishing of $\chi(D)$ and $\chi_h(D)$ and vice-versa, and so, besides a particular type of 'Wilson' divisor, there can be more such divisor topologies satisfying the following if and only if condition:

$$\Pi(D) = 0 = \int_X \hat{D} \wedge \hat{D} \wedge \hat{D} \qquad \Longleftrightarrow \qquad \chi(D) = 0 = \chi_h(D)\,. \tag{3.41}$$

Thus, if a divisor D is connected and has $\Pi(D) = 0 = D^3_{|X}$, then its Hodge diamond is:

$$D_\Pi \equiv \begin{array}{ccccc} & & 1 & & \\ & 1+n & & 1+n & \\ n & & 2+2n & & n \\ & 1+n & & 1+n & \\ & & 1 & & \end{array} \equiv D_\Pi^{\text{cubic}}\,, \tag{3.42}$$

where n is the number of possible deformations for the divisor D. For $n = 0$ this corresponds to a W_Π divisor, and for $n = 1$ this corresponds to a $\mathbb{T}^4$. Although we are not aware of any such examples with $n \geq 2$, it would be interesting to know what topology they would correspond to.

3.2.3 Scan for divisors with vanishing F^4 terms

In this section we discuss the scanning results for divisors with $\Pi = 0$ using the favorable CY geometries arising from the four-dimensional reflexive polytopes of the KS database [64] and its pheno-friendly collection in [65]. As pointed out earlier, we will consider only the 'coordinate divisors' and the 'favorable' CY geometries listed in Tab. 3.1. For finding divisors with vanishing Π, we consider the following two different strategies in our scan:

1. One route is to directly compute Π by using the second Chern class of the CY threefold and the intersection tensor available in the database [65].

2. A second route is to compute the divisor topology using cohomCalg [225, 226] and subsequently to check the Hodge number condition (3.34), or the equivalent relation $\chi(D) = 6\,\chi_h(D)$, for vanishing Π.

Tab. 3.3 presents the scanning results for the number of CY geometries with vanishing Π divisors, and their suitability for realizing LVS models. On the other hand, Tab. 3.4 and 3.5 show the same results split for the cases where the divisors with $\Pi = 0$ are respectively dP_Π (i.e. dP_3) and Wilson divisors W_Π. These distinct CY geometries and their scanning results correspond to the favorable geometries arising from the favorable polytopes.

$h^{1,1}$	Poly*	Geom* (n_{CY})	single D_Π	two D_Π	three D_Π	four D_Π	n_{LVS} & 1 D_Π	n_{LVS} & 2 D_Π	n_{LVS} & 3 D_Π
1	5	5	0	0	0	0	0	0	0
2	36	39	0	0	0	0	0	0	0
3	243	305	23	0	0	0	4	0	0
4	1185	2000	322	24	0	0	78	1	0
5	4897	13494	3306	495	27	1	732	104	1

Table 3.3. CY geometries with vanishing Π divisors and a ddP_n to support LVS.

To appreciate the scanning results presented in Tab. 3.3, 3.4 and 3.5 corresponding to all CY threefolds with $1 \leq h^{1,1}(X) \leq 5$ in the KS database, let us make the following generic observations:

$h^{1,1}$	Poly*	Geom* (n_{CY})	At least one dP_Π	Single dP_Π	Two dP_Π	Three dP_Π	n_{LVS} & dP_Π	n_{LVS} & 1 dP_Π	n_{LVS} & 2 dP_Π
1	5	5	0	0	0	0	0	0	0
2	36	39	0	0	0	0	0	0	0
3	243	305	4	4	0	0	0	0	0
4	1185	2000	143	134	9	0	16	16	0
5	4897	13494	2236	2035	197	4	336	290	46

Table 3.4. CY geometries with vanishing Π divisors of the type $\mathrm{dP}_\Pi \equiv \mathrm{dP}_3$, and a ddP_n for LVS.

$h^{1,1}$	Poly*	Geom* (n_{CY})	At least one W_Π	Single W_Π	Two W_Π	Three W_Π	n_{LVS} & 1 W_Π	n_{LVS} & 2 W_Π	n_{LVS} & 3 W_Π
1	5	5	0	0	0	0	0	0	0
2	36	39	0	0	0	0	0	0	0
3	243	305	19	19	0	0	4	0	0
4	1185	2000	210	202	8	0	62	1	0
5	4897	13494	1764	1599	154	11	442	79	1

Table 3.5. CY geometries with vanishing Π divisors of the type W_Π, and a ddP_n to support LVS.

- We do not find any CY threefold in the KS database which has a $\mathbb{T}^4$ divisor or any divisor with vanishing $\Pi(D)$ and $h^{2,0}(D) \neq 0$. The only possible vanishing Π divisors we encountered in our scan are either a dP_3 divisor or a Wilson divisor with $h^{1,1}(W) = 2$. However, going beyond the coordinate divisors in an extended scan as compared to ours may have more possibilities.

- For $h^{1,1}(X) = 1$ and 2, there are no CY threefolds with a vanishing Π divisor.

- Although there are some dP_3 divisors for CY threefolds with $h^{1,1}(X) = 3, 4$ and 5, none of them are diagonal in the sense of being shrinkable to a point by squeezing along a single direction [205] - something in line with the conjecture of [28].

- There are no CY threefolds with $h^{1,1}(X) = 3$ which have (at least) one diagonal dP_n and a (non-diagonal) dP_3 with $\Pi(\mathrm{dP}_3) = 0$. Hence, in order to have a dP_3 divisor in LVS, we need CY threefolds with $h^{1,1}(X) \geq 4$. For $h^{1,1}(X) = 4$ there are 16 CY threefolds in the 'favorable' geometry which are suitable for LVS and feature a dP_3.

- For $h^{1,1}(X) \leq 4$, there is only one CY geometry which can lead to LVS and has two vanishing Π divisors which are of Wilson-type. Similarly, there is only one CY geometry with a ddP for LVS and 3 vanishing Π divisors.

3.3 Corrections to inflationary potentials

3.3.1 Blow-up inflation

The minimal LVS scheme of moduli stabilisation fixes the CY volume $\mathcal{V}$ along with a small modulus τ_s controlling the volume of an exceptional del Pezzo divisor. Therefore any LVS model with 3 or more Kähler moduli, $h^{1,1} \geq 3$, can generically have flat directions at leading order. These flat directions are promising inflaton candidates with a potential generated at subleading order. Blow-up inflation [49] corresponds to the case where the inflationary potential is generated by non-perturbative superpotential contributions. In this inflationary scenario the inflaton is a (diagonal) del Pezzo divisor wrapped by an ED3-instanton or supporting gaugino condensation. In addition, the CY has to feature at least one additional ddP$_n$ divisor to realize LVS.

On these lines, we present the scanning results in Tab. 3.6 corresponding to the number of CY geometries n_{CY} with their suitability for realizing LVS along with resulting in the standard blow-up inflationary potential, in the sense of having at least two ddP divisors, one needed for supporting LVS and the other one for driving inflation.

$h^{1,1}$	Poly*	Geom* (n_{CY})	$n_{\text{ddP}} = 1$	$n_{\text{ddP}} = 2$	$n_{\text{ddP}} = 3$	$n_{\text{ddP}} = 4$	n_{LVS}	Blow-up infl.
1	5	5	0	0	0	0	0	0
2	36	39	22	0	0	0	22	0
3	243	305	93	39	0	0	132	39
4	1185	2000	465	261	24	0	750	285
5	4897	13494	3128	857	106	13	4104	976

Table 3.6. Number of LVS CY geometries suitable for blow-up inflation.

3.3.2 Inflationary potential

The simplest blow-up inflation model is based on a two-hole Swiss-cheese CY threefold. Such a CY threefold has two diagonal del Pezzo divisors, say D_1 and D_2, which after considering an appropriate basis of divisors result in the following intersection polynomial:

$$I_3 = I_3'(D_{i'}) + k_{111}\, D_1^3 + k_{222}\, D_2^3\,, \quad \text{for } i' \neq \{1,2\}\,, \tag{3.43}$$

where $I_3'(D_{i'})$ is such that D_1 and D_2 do not appear in this cubic polynomial. Further, k_{111} and k_{222} are the self-intersection numbers which are fixed by the degrees of the two del Pezzo divisors, say dP_{n_1} and dP_{n_2}, as $k_{111} = 9 - n_1 > 0$ and $k_{222} = 9 - n_2 > 0$. This generically provides the following expression for the volume form:

$$\mathcal{V} = \frac{1}{6}\, k_{i'j'k'}\, t^{i'}\, t^{j'}\, t^{k'} + \frac{k_{111}}{6}\,(t^1)^3 + \frac{k_{222}}{6}\,(t^2)^3\,, \tag{3.44}$$

where the 2-cycle volume moduli $t^{i'}$ are such that $i' \neq \{1, 2\}$. Subsequently, the volume can be rewritten in terms of the 4-cycle volume moduli as:

$$\mathcal{V} = f_{3/2}(\tau_{i'}) - \beta_1\, \tau_1^{3/2} - \beta_2\, \tau_2^{3/2}\,, \tag{3.45}$$

where $\beta_1 = \frac{1}{3}\sqrt{\frac{2}{k_{111}}}$ and $\beta_2 = \frac{1}{3}\sqrt{\frac{2}{k_{222}}}$. Furthermore, under our choice of the intersection polynomial, $\tau_{i'}$ does not depend on the del Pezzo volumes τ_1 and τ_2. Now we can simplify things to the minimal three-field case with $h^{1,1}_+ = 3$ by taking $f_{3/2}(t^{i'}) = \frac{1}{6}\, k_{bbb}\,(t^b)^3$ and using the following relations between the 2-cycle moduli t^i and the 4-cycle moduli τ_i:

$$t^b = \sqrt{\frac{2\,\tau_b}{k_{bbb}}}\,, \qquad t^1 = -\sqrt{\frac{2\,\tau_1}{k_{111}}}\,, \qquad t^2 = -\sqrt{\frac{2\,\tau_2}{k_{222}}}\,. \tag{3.46}$$

The scalar potential of the minimal blow-up inflationary model [30, 49] can be reproduced by the master formula (3.6) via simply setting $h^{1,1}_+ = 3$, $n = 2$ and $\hat{\xi} > 0$, which leads to the following leading order terms in the large volume limit:

$$\begin{aligned} V = \frac{e^{K_{\rm cs}}}{2s}\Bigg[&\frac{3\hat{\xi}|W_0|^2}{4\mathcal{V}^3} + \sum_{i=1}^{2}\frac{4|W_0||A_i|a_i}{\mathcal{V}^2}\,\tau_i\, e^{-a_i\tau_i}\,\cos(a_i\theta_i + \phi_0 - \phi_i) \\ &- \sum_{i=1}^{2}\sum_{j=1}^{2}\frac{4|A_i||A_j|a_i a_j}{\mathcal{V}}\, e^{-(a_i\tau_i + a_j\tau_j)}\,\cos(a_j\theta_j - a_i\theta_i - \phi_j + \phi_i)\left(\sum_{k=1}^{3} k_{ijk}t_k\right)\Bigg]. \end{aligned} \tag{3.47}$$

Given that we are interested in a strong Swiss-cheese case where the only non-vanishing intersection numbers are k_{111}, k_{222} and k_{333}, we have:

$$\sum_{k=1}^{3} k_{iik}t_k = k_{iii}t_i = -\sqrt{2\,k_{iii}\,\tau_i} \quad \text{for } i = 1, 2 \qquad \text{and} \qquad \sum_{k=1}^{3} k_{ijk}t_k = 0 \quad \text{for } i \neq j\,.$$

Hence (3.47) reduces to the potential of known 3-moduli Swiss-cheese LVS models [30, 49]:

$$V = \frac{e^{K_{\rm cs}}}{2s}\left[\frac{\beta_{\alpha'}}{\mathcal{V}^3} + \sum_{i=1}^{2}\left(\beta_{\mathrm{np}1,i}\,\frac{\tau_i}{\mathcal{V}^2}\, e^{-a_i\tau_i}\,\cos(a_i\theta_i + \phi_0 - \phi_i) + \beta_{\mathrm{np}2,i}\,\frac{\sqrt{\tau_i}}{\mathcal{V}}\, e^{-2a_i\tau_i}\right)\right],$$

with:

$$\beta_{\alpha'} = \frac{3\hat{\xi}|W_0|^2}{4}\,, \qquad \beta_{\mathsf{np1},i} = 4a_1|W_0||A_1|\,, \qquad \beta_{\mathsf{np2},i} = 4a_1^2|A_1|^2\sqrt{2k_{111}}\,. \tag{3.48}$$

It has been found that such a scalar potential can drive inflation effectively by a single field after two moduli are stabilized at their respective minimum [49]. In fact, a three-field inflationary analysis has been also presented in [30, 50] ensuring that one can indeed have trajectories which effectively correspond to a single field dynamics.

3.3.3 F^4 corrections

In this three-field blow-up inflation model, higher derivative F^4 corrections to the scalar potential look like:[3]

$$\begin{aligned} V_{F^4} &= \frac{\gamma}{\mathcal{V}^4}\left(\Pi_b\, t^b + \Pi_1\, t^1 + \Pi_2\, t^2\right) \\ &= \frac{\gamma}{\mathcal{V}^4}\left(\Pi_b\, t^b - \Pi_1\sqrt{\frac{2\,\tau_1}{k_{111}}} - \Pi_2\sqrt{\frac{2\,\tau_2}{k_{222}}}\right) \\ &= \frac{\gamma}{\mathcal{V}^4}\left[\Pi_b\left(\frac{6}{k_{bbb}}\right)^{1/3}\left(\mathcal{V} + \beta_1\,\tau_1^{3/2} + \beta_2\,\tau_2^{3/2}\right)^{1/3}\right. \\ &\qquad \left. - \Pi_1\sqrt{\frac{2\,\tau_1}{k_{111}}} - \Pi_2\sqrt{\frac{2\,\tau_2}{k_{222}}}\right], \end{aligned} \tag{3.49}$$

where we have used the relations in (3.46). Assuming that inflation is driven by τ_2, only τ_2-dependent corrections can spoil the flatness of the inflationary potential. The leading correction is proportional to Π_2 and scales as $\mathcal{V}^{-4}$, while a subdominant contribution proportional to Π_b would scale as $\mathcal{V}^{-14/3}$. It is interesting to note that this subleading correction would be present even if $\Pi_2 = 0$, as in the case where the corresponding dP_n is a diagonal dP_3. As compared to the LVS potential, this inflaton-dependent F^4 correction is suppressed by a factor of order $\mathcal{V}^{-5/3} \ll 1$. Moreover, the ideal situation to completely nullify higher derivative F^4 corrections for blow-up inflation is to demand that:

$$\Pi_b = \Pi_2 = 0\,. \tag{3.50}$$

In this setting, making Π_b zero by construction appears to be hard and very unlikely since we have seen that vanishing Π divisors other than

[3] Additional perturbative corrections can arise from string loops but we assume that these contributions can be made negligible by either taking a small value of the string coupling or by appropriately small flux-dependent coefficients.

dP_3 could possibly be either a $\mathbb{T}^4$ or a Wilson divisor. However, for both divisors we have $\int_X D^3 = 0$ as they satisfy the condition (3.40) that implies vanishing cubic self-intersections, and so they do not seem suitable to reproduce the strong Swiss-cheese volume form that has been implicitly assumed in rewriting the scalar potential pieces in (3.49). Moreover, we have not observed any other kind of vanishing Π divisors in our scan involving the whole set of CY threefolds with $h^{1,1} \leq 5$ in the KS database.[4] Let us finally point out that a case with $\Pi_b = 0$ cannot be entirely ruled out as we have seen in a couple of non-generic situations that a non-fibred K3 surface can also appear as a 'big' divisor in a couple of strong Swiss-cheese CY threefolds, and so if there is a similar situation in which a non-fibred $\mathbb{T}^4$ appears with a ddP divisor it could possibly make Π_b identically zero.

3.3.4 Constraints on inflation

We are now going to study the effect of F^4 corrections in blow-up inflation, focusing on the case where their coefficients are in general non-zero, as suggested by our scan. In this analysis we shall follow the work of [227]. First of all, we will derive the value of the volume to subsequently analyse the effect of the F^4 corrections to the inflationary dynamics.

We start from the potential described in (3.47), stabilize the axions and set $e^{K_{\rm cs}}/(2s) = 1$, obtaining:

$$V_{\rm LVS} = \sum_{i=1}^{2}\left(\frac{8(a_iA_i)^2\sqrt{\tau_i}}{3\mathcal{V}\beta_i}e^{-2a_i\tau_i} - \frac{4a_iA_iW_0\tau_i}{\mathcal{V}^2}e^{-a_i\tau_i}\right) + \frac{3\hat{\xi}W_0^2}{4\mathcal{V}^3}\,, \tag{3.51}$$

where the volume has been expressed as:

$$\mathcal{V} = \tau_b^{3/2} - \beta_1\tau_1^{3/2} - \beta_2\tau_2^{3/2}\,. \tag{3.52}$$

The minimum condition of the LVS potential reads:

$$e^{-a_i\tau_i} = \frac{\Lambda_i}{\mathcal{V}}\sqrt{\tau_i}\,, \tag{3.53}$$

where the constants Λ_i are defined as:

$$\Lambda_i \equiv \frac{3|W_0|}{4}\frac{\beta_i}{a_i|A_i|}\,. \tag{3.54}$$

[4]However recall that our scan is limited to coordinate divisors only, and so may miss some possibilities.

Moreover, since we want to find an approximate Minkowski vacuum, we add an uplifting potential of the generic form:

$$V_{\mathsf{up}} = \frac{D}{\mathcal{V}^{4/3}}\,, \tag{3.55}$$

where the value of D will be computed in the next paragraph. Lastly, the F^4 corrections become:

$$V_{F^4} = \frac{\gamma}{\mathcal{V}^4}\left[\Pi_b\left(\mathcal{V} - \sum_{i=1}^{2}\beta_i\tau_i^{3/2}\right)^{1/3} - 3\sum_{i=1}^{2}\Pi_i\beta_i\sqrt{\tau_i}\right]. \tag{3.56}$$

Volume after inflation

We start by fixing in the LVS potential (3.51) the small moduli at their minimum given by (3.53):

$$V_{\mathsf{LVS}} = -\frac{3|W_0|^2}{2\mathcal{V}^3}\left(\sum_{i=1}^{2}\beta_i\tau_i^{3/2} - \frac{\hat{\xi}}{2}\right). \tag{3.57}$$

Defining $\psi \equiv \ln\mathcal{V}$, the minimum condition for τ_i can be approximated as:

$$\tau_i = \frac{1}{a_i}\left(\psi - \ln\Lambda_i - \ln\sqrt{\tau_i}\right) \simeq \frac{1}{a_i}\left(\psi - \ln\Lambda_i\right)\,, \tag{3.58}$$

leading to:

$$V_{\mathsf{LVS}}^{(\mathsf{PI})} = -\frac{3|W_0|^2}{4}e^{-3\psi}\left(\sum_{i=1}^{2}P_i\left(\psi - \ln\Lambda_i\right)^{3/2} - \hat{\xi}\right)\,, \tag{3.59}$$

where $P_i \equiv 2\beta_i a_i^{-3/2}$ and the superscript (PI) indicates that we consider the 'post inflation' situation where all the moduli reach their minimum. Analogously, the uplifting term reads:

$$V_{\mathsf{up}}^{(\mathsf{PI})} = De^{-\frac{4}{3}\psi}\,, \tag{3.60}$$

while the F^4 correction becomes:

$$\begin{aligned} V_{F^4}^{(\mathsf{PI})} = \gamma e^{-4\psi}\Bigg[&\Pi_b\left(e^{\psi} - \sum_{i=1}^{2}P_i\left(\psi - \ln\Lambda_i\right)^{3/2}\right)^{1/3} \\ &- 3\sum_{i=1}^{2}\Pi_i P_i a_i\left(\psi - \ln\Lambda_i\right)^{1/2}\Bigg]. \end{aligned} \tag{3.61}$$

The full post-inflationary potential for the field ψ is therefore:

$$V_{\mathsf{PI}}(\psi) = V_{\mathsf{LVS}}^{(\mathsf{PI})} + V_{\mathsf{up}}^{(\mathsf{PI})} + V_{F^4}^{(\mathsf{PI})}\,. \tag{3.62}$$

We are now able to calculate the factor D in order to have a Minkowski minimum, by imposing:

$$V'_{\text{PI}}(\tilde{\psi}) = V_{\text{PI}}(\tilde{\psi}) = 0\,, \tag{3.63}$$

which gives:

$$D = \frac{27|W_0|^2}{20} e^{-\frac{5}{3}\tilde{\psi}} \sum_i P_i \left(\tilde{\psi} - \ln \Lambda_i\right)^{1/2} + \delta D_{F^4}\,, \tag{3.64}$$

$$\begin{aligned}
\delta D_{F^4} =& \gamma e^{-\frac{8}{3}\tilde{\psi}} \Bigg[\Pi_b \left(e^{\tilde{\psi}} - \sum_i P_i \left(\tilde{\psi} - \ln \Lambda_i\right)^{3/2} \right)^{1/3} \\
& - 3 \sum_i \Pi_i P_i a_i \left(\tilde{\psi} - \ln \Lambda_i\right)^{1/2} - \frac{\Pi_b}{3} \frac{e^{\tilde{\psi}} - \frac{3}{2}\sum_i P_i \left(\tilde{\psi} - \ln \Lambda_i\right)^{1/2}}{\left(e^{\tilde{\psi}} - \sum_i P_i \left(\tilde{\psi} - \ln \Lambda_i\right)^{3/2}\right)^{2/3}} \\
& + \frac{3}{2} \sum_i \Pi_i P_i a_i \left(\tilde{\psi} - \ln \Lambda_i\right)^{-1/2} \Bigg]\,,
\end{aligned} \tag{3.65}$$

where $\tilde{\psi}$ solves the following equation:

$$\sum_i P_i \left(\tilde{\psi} - \ln \Lambda_i\right)^{1/2} \left(\tilde{\psi} - \ln \Lambda_i - \frac{9}{10}\right) - \hat{\xi} + \delta\tilde{\psi}_{F^4} = 0\,, \tag{3.66}$$

$$\begin{aligned}
\delta\tilde{\psi}_{F^4} =& -\frac{4\gamma}{5W_0^2} e^{-\tilde{\psi}} \Bigg[\frac{8\Pi_b}{3} \left(e^{\tilde{\psi}} - \sum_i P_i \left(\tilde{\psi} - \ln \Lambda_i\right)^{3/2} \right)^{1/3} \\
& - 8 \sum_i \Pi_i P_i a_i \left(\tilde{\psi} - \ln \Lambda_i\right)^{1/2} - \frac{\Pi_b}{3} \frac{e^{\tilde{\psi}} - \frac{3}{2}\sum_i P_i \left(\tilde{\psi} - \ln \Lambda_i\right)^{1/2}}{\left(e^{\tilde{\psi}} - \sum_i P_i \left(\tilde{\psi} - \ln \Lambda_i\right)^{3/2}\right)^{2/3}} \\
& + \frac{3}{2} \sum_i \Pi_i P_i a_i \left(\tilde{\psi} - \ln \Lambda_i\right)^{-1/2} \Bigg]\,,
\end{aligned} \tag{3.67}$$

from which we obtain the post inflation volume $\mathcal{V}_{\text{PI}} \equiv e^{\tilde{\psi}}$.

Volume during inflation

We now move on to determine the value of the volume modulus during inflation. In order to do so, we focus on the region in field space where the inflaton τ_2 is away from its minimum. In this region, the inflaton-dependent contribution to the volume potential becomes negligible due to the large exponential suppression from (3.51). Hence, the inflationary potential for the volume mode is given only by:

$$V_{\mathsf{inf}}(\psi) = -\frac{3|W_0|^2}{4} e^{-3\psi} \left(s P_1 \left(\psi - \ln \Lambda_1\right)^{3/2} - \hat{\xi} \right) s + D e^{-\frac{4}{3}\psi} \,, \qquad (3.68)$$

where we ignore F^4 corrections since the volume during inflation is bigger than the post-inflationary one. At this point we can again minimize the ψ field to a value $\hat{\psi}$, imposing the vanishing of the first derivative:

$$P_1 \left(\hat{\psi} - \ln \Lambda_1 \right)^{1/2} \left(s 2 \left(\hat{\psi} - \ln \Lambda_1 \right) - 1 \right) s - 2\hat{\xi} + \frac{16}{9|W_0|^2} D e^{\hat{\psi}} = 0 \,, \qquad (3.69)$$

and the volume during inflation is given as $\mathcal{V}_{\mathsf{inf}} \equiv e^{\psi}$.

Inflationary dynamics

During inflation all the moduli, except τ_2, sit at their minimum, including the volume mode which is located at $\mathcal{V} \equiv \mathcal{V}_{\mathsf{inf}}$. From now on, we will drop the subscript and always refer to the volume as the one during inflation, unless otherwise explicitly stated. The inflaton potential with higher derivative effects reads:

$$\begin{aligned} V(\tau_2) = V_0 &- \frac{4 a_2 |A_2||W_0|}{\mathcal{V}^2} \tau_2 e^{-a_2 \tau_2} + \frac{8\, a_2^2\, |A_2|^2 \sqrt{\tau_2}}{3 \mathcal{V} \beta_2} e^{-2 a_2 \tau_2} \\ &+ \frac{\gamma}{\mathcal{V}^4} \left[\Pi_b \left(\mathcal{V} - P_1 \left(\ln \frac{\mathcal{V}}{\Lambda_1} \right)^{3/2} - \beta_2 \tau_2^{3/2} \right)^{1/3} - 3 \sum_i \Pi_i \beta_i \sqrt{\tau_i} \right] , \end{aligned} \qquad (3.70)$$

where the explicit definition of γ in terms of λ is

$$\gamma = -\frac{\lambda |W_0|^4}{16 \pi^2 g_s^{3/2}} \,. \qquad (3.71)$$

Canonically normalizing the inflaton field as:

$$\tau_2 = \left(\langle \tau_2 \rangle^{3/4} + \sqrt{\frac{3\mathcal{V}}{4\beta_2}}\, \phi \right)^{4/3} , \qquad (3.72)$$

we find the inflaton effective potential:

$$\begin{aligned} V(\phi) = & V_0 - \frac{4a_2|A_2||W_0|}{\mathcal{V}^2}\left(\sqrt{\frac{3\mathcal{V}}{4\beta_2}}\phi + \langle\tau_2\rangle^{3/4}\right)^{4/3} e^{-a_2\left(\sqrt{\frac{3\mathcal{V}}{4\beta_2}}\phi+\langle\tau_2\rangle^{3/4}\right)^{4/3}} \\ & + \frac{8\,a_2^2\,|A_2|^2}{3\mathcal{V}\beta_2}\left(\sqrt{\frac{3\mathcal{V}}{4\beta_2}}\phi + \langle\tau_2\rangle^{3/4}\right)^{2/3} e^{-2a_2\left(\sqrt{\frac{3\mathcal{V}}{4\beta_2}}\phi+\langle\tau_2\rangle^{3/4}\right)^{4/3}} \\ & + \frac{\gamma}{\mathcal{V}^4}\Bigg[\Pi_b\left(\mathcal{V} - P_1\left(\ln\frac{\mathcal{V}}{\Lambda_1}\right)^{3/2} - \beta_2\left(\sqrt{\frac{3\mathcal{V}}{4\beta_2}}\phi + \langle\tau_2\rangle^{3/4}\right)^2\right)^{1/3} \\ & - 3\Pi_2\beta_2\left(\sqrt{\frac{3\mathcal{V}}{4\beta_2}}\phi + \langle\tau_2\rangle^{3/4}\right)^{2/3} - 3\Pi_1 P_1 a_1\left(\ln\frac{\mathcal{V}}{\Lambda_1}\right)^{1/2}\Bigg]. \end{aligned} \tag{3.73}$$

To simplify the notation, we introduce:

$$A \equiv \frac{4a_2|A_2|\,|W_0|}{\mathcal{V}^2}\,, \qquad B \equiv \frac{8\,a_2^2|A_2|^2}{3\mathcal{V}\beta_2}\,, \qquad C \equiv \mathcal{V} - P_1\left(\ln\frac{\mathcal{V}}{\Lambda_1}\right)^{3/2}\,, \tag{3.74}$$

$$\gamma_2 \equiv \frac{3\gamma\Pi_2\beta_2}{\mathcal{V}^4}\,, \qquad \gamma_b \equiv \frac{\gamma\Pi_b}{\mathcal{V}^4}\,, \qquad \alpha \equiv \sqrt{\frac{3\mathcal{V}}{4\beta_2}}\,, \tag{3.75}$$

$$\varphi \equiv \sqrt{\frac{3\mathcal{V}}{4\lambda_2}}\phi + \langle\tau_2\rangle^{3/4} = \alpha\phi + \langle\tau_2\rangle^{3/4}\,, \tag{3.76}$$

and we absorb the constant F^4 correction proportional to Π_1 inside V_0 as:

$$V_0 \to V_0 - \frac{3\gamma\Pi_1 P_1 a_1}{\mathcal{V}^4}\left(\ln\frac{\mathcal{V}}{\Lambda_1}\right)^{1/2}\,. \tag{3.77}$$

The potential therefore simplifies to:

$$V(\varphi) = V_0 - A\,\varphi^{4/3}\,e^{-a_2\,\varphi^{4/3}} + B\varphi^{3/2}e^{-2a_2\varphi^{4/3}} + \gamma_b(C - \beta_2\,\varphi^2)^{1/3} - \gamma_2\varphi^{2/3}\,. \tag{3.78}$$

Given that φ is different from the canonically normalized inflaton ϕ, we define the following notation for differentiation:

$$f'(\varphi) \equiv \frac{\mathrm{d}f(\varphi)}{\mathrm{d}\phi} = \sqrt{\frac{3\mathcal{V}}{4\lambda_2}}\frac{\mathrm{d}f(\varphi)}{\mathrm{d}\varphi} \equiv \alpha\dot{f}(\varphi)\,, \tag{3.79}$$

with the slow-roll parameters calculated as follows:

$$\epsilon = \frac{1}{2}\left(\frac{V'}{V}\right)^2 = \frac{1}{2}\alpha^2\left(\frac{\dot{V}}{V}\right)^2 \qquad \text{and} \qquad \eta = \frac{V''}{V} = \alpha^2\frac{\ddot{V}}{V}\,. \tag{3.80}$$

The next step is to find the value of ϕ at the end of inflation, which we denote as ϕ_{end}, where $\epsilon(\phi_{\text{end}}) = 1$. Moreover, the number of efoldings from horizon exit to the end of inflation can be computed as:

$$N_e(\phi_{\text{exit}}) = \int_{\phi_{\text{end}}}^{\phi_{\text{exit}}} \frac{\mathrm{d}\phi}{\sqrt{2\epsilon}} = \int_{\varphi_{\text{end}}}^{\varphi_{\text{exit}}} \frac{\mathrm{d}\varphi}{\alpha\sqrt{2\epsilon}} \,. \tag{3.81}$$

This value has to match the number of efoldings of inflation N_e computed from the study of the post-inflationary evolution which we will perform in the next section, i.e. ϕ_{exit} is fixed by requiring $N_e(\phi_{\text{exit}}) = N_e$. The observed amplitude of the density perturbations has to be matched at ϕ_{exit}, which typically fixes $\mathcal{V} \sim 10^{5-6}$. The predictions for the main cosmological observable are then be inferred as follows:

$$n_s = 1 + 2\eta(\phi_{\text{exit}}) - 6\epsilon(\phi_{\text{exit}}) \qquad \text{and} \qquad r = 16\epsilon(\phi_{\text{exit}}) \,. \tag{3.82}$$

Reheating

In order to make predictions that can be confronted with actual data, we need to derive the number of efoldings of inflation which, in turn, are determined by the dynamics of the reheating epoch. Assuming that the Standard Model is realized on a stack of D7-branes, a crucial term in the low-energy Lagrangian to understand reheating is the loop-enhanced coupling of the volume mode to the Standard Model Higgs h which reads [228]:

$$\mathcal{L} \subset c_{\text{loop}} \frac{m_{3/2}^2}{M_p} \phi_b h^2 \,, \tag{3.83}$$

where c_{loop} is a 1-loop factor and ϕ_b the canonically normalized volume modulus. Two different scenarios for reheating can arise depending on the presence or absence of a stack of D7-branes wrapped around the inflaton del Pezzo divisor:

- **No D7s wrapped around the inflaton:** The inflaton τ_2 is not wrapped by any D7 stack and the Standard Model is realized on D7-branes wrapped around the blow-up mode τ_1. This case has been studied in [228]. The volume mode, despite being the lightest modulus, decays before the inflaton due to the loop-enhanced coupling (3.83). Reheating is therefore caused by the decay of the inflaton which occurs with a width:

$$\Gamma_{\tau_2} \simeq \frac{1}{\mathcal{V}} \frac{m_{\tau_2}^3}{M_p^2} \simeq \frac{M_p}{\mathcal{V}^4} \,, \tag{3.84}$$

leading to a matter dominated epoch after inflation which lasts for the following number of efoldings:

$$N_{\tau_2} = \frac{2}{3}\ln\left(\frac{H_{\text{inf}}}{\Gamma_{\tau_2}}\right) = \frac{5}{3}\ln\mathcal{V}\,. \tag{3.85}$$

Thus, the total number of efoldings for inflation is given by:

$$N_e = 57 + \frac{1}{4}\ln r - \frac{1}{4}N_{\tau_2} \simeq 50 - \frac{1}{4}N_{\tau_2} = 50 - \frac{5}{12}\ln\mathcal{V}\,, \tag{3.86}$$

where we have focused on typical values of the tensor-to-scalar ratio for blow-up inflation around $r \sim 10^{-10}$. Thus, due to the long epoch of inflaton domination before reheating, the total number of required efoldings can be considerably reduced, resulting in a potential tension with the observed value of the spectral index, as we will point out in the next section. Note that the inflaton decay into bulk axions can lead to an overproduction of dark radiation which is however avoided by the large inflaton decay width into Standard Model gauge bosons, resulting in $\Delta N_{\text{eff}} \simeq 0.13$ [228].

- **D7s wrapped around the inflaton:** The inflaton is wrapped by a D7 stack which can be either the Standard Model or a hidden sector. These different cases have been analysed in [229–231]. The localization of gauge degrees of freedom on the inflaton divisor increases the inflaton decay width, so that the last modulus to decay is the volume mode. However the naive estimate of the number of efoldings of the matter epoch dominated by the oscillation of $\mathcal{V}$ is reduced due to the enhanced Higgs coupling (3.83). The early universe history is then given by a first matter dominated epoch driven by the inflaton which features now an enhanced decay rate:

$$\Gamma_{\tau_2} \simeq \mathcal{V}\,\frac{m_{\tau_2}^3}{M_p^2} \simeq \frac{M_p}{\mathcal{V}^2}\,. \tag{3.87}$$

Hence the number of efoldings of inflaton domination is given by:

$$N_{\tau_2} = \frac{2}{3}\ln\left(\frac{H_{\text{inf}}}{\Gamma_{\tau_2}}\right) = \frac{1}{3}\ln\mathcal{V}\,. \tag{3.88}$$

The volume mode starts oscillating during the inflaton dominated epoch. Redshifting both as matter, the ratio of the energy densities of the inflaton and the volume mode remains constant from the start of the volume oscillations to the inflaton decay:

$$\theta^2 \equiv \left.\frac{\rho_{\tau_b}}{\rho_{\tau_2}}\right|_{\text{osc}} = \left.\frac{\rho_{\tau_b}}{\rho_{\tau_2}}\right|_{\text{dec},\tau_2} \ll 1\,, \tag{3.89}$$

since the energy density after inflation is dominated by the inflaton. Assuming that the inflaton dumps all its energy into radiation when it decays, we can estimate:

$$\left.\frac{\rho_{\tau_b}}{\rho_\gamma}\right|_{\text{dec}} = \theta^2 \,. \tag{3.90}$$

The radiation dominated era after the inflaton decay ends when ρ_γ becomes comparable to ρ_{τ_b}, which occurs when:

$$\left.\rho_\gamma\right|_{\text{dec},\tau_2} \left(\frac{a_{\text{dec},\tau_2}}{a_{\text{eq}}}\right)^4 = \left.\rho_{\tau_b}\right|_{\text{dec},\tau_2} \left(\frac{a_{\text{dec},\tau_2}}{a_{\text{eq}}}\right)^3 \quad \Rightarrow \quad a_{\text{dec},\tau_2} = a_{\text{eq}}\,\theta^2 \,, \tag{3.91}$$

giving the dilution at equality:

$$\rho_{\text{eq}} = \left.\rho_\gamma\right|_{\text{dec}} \theta^8 \,. \tag{3.92}$$

Moreover, the Hubble scale at the inflaton decay is given by:

$$H_{\text{dec},\tau_2} = H_{\text{inf}}\, e^{-\frac{3}{2}N_{\tau_2}} \,, \tag{3.93}$$

allowing us to calculate the Hubble scale at radiation-volume equality:

$$H_{\text{eq}} = H_{\text{dec},\tau_2}\,\theta^4 = H_{\text{inf}}\, e^{-\frac{3}{2}N_{\tau_2}}\,\theta^4 \,. \tag{3.94}$$

Using the fact that the decay rate of the volume mode is:

$$\Gamma_{\tau_b} \simeq c_{\text{loop}}^2 \left(\frac{m_{3/2}}{m_{\tau_b}}\right)^4 \frac{m_{\tau_b}^3}{M_p^2} \simeq c_{\text{loop}}^2 \frac{M_p}{\mathcal{V}^{5/2}} \,, \tag{3.95}$$

we can now estimate the number of efoldings of the matter epoch dominated by volume mode as:

$$N_{\tau_b} = \ln\left(\frac{a_{\text{dec},\tau_b}}{a_{\text{eq}}}\right) \simeq \frac{2}{3}\ln\left(\frac{H_{\text{eq}}}{\Gamma_{\tau_b}}\right) \simeq \frac{2}{3}\ln\mathcal{V} - N_{\tau_2} \,, \tag{3.96}$$

where we considered $\theta^4 c_{\text{loop}}^{-2} \sim \mathcal{O}(1)$. Therefore, the total number of efoldings of inflation becomes:

$$N_e = 57 + \frac{1}{4}\ln r - \frac{1}{4}N_{\tau_2} - \frac{1}{4}N_{\tau_b} \simeq 50 - \frac{1}{4}N_{\tau_2} - \frac{1}{4}N_{\tau_b} \simeq 50 - \frac{1}{6}\ln\mathcal{V} \,. \tag{3.97}$$

Note that this estimate gives a longer period of inflation with respect to the scenario where the inflaton is not wrapped by any D7 stack, even if there are two epochs of modulus domination. The reason is that both epochs, when summed together, last less that the single epoch of inflaton domination of the case with no D7-branes wrapped around the inflaton. As we shall see, this results in a better agreement with the observed value of the scalar spectral index. Lastly, we stress that the loop-enhanced volume mode coupling to the Higgs sector suppresses the production of axionic dark radiation. As stressed above, this coupling is however effective only when the Standard Model lives on D7-branes since it becomes negligible in sequestered scenarios where the visible sectors is localized on D3-branes at dP singularities. In this case the volume would decay into Higgs degrees of freedom via a Giudice-Masiero coupling [58, 232, 233] and a smaller decay width $\Gamma_{\tau_b} \sim M_p/\mathcal{V}^{9/2}$ that would make the number of efoldings of inflation much shorter.

3.3.5 Numerical examples

No D7s wrapped around the inflaton

To quantitatively study the effect of higher derivative corrections, let us consider an explicit example characterized by the following choice of parameters:

W_0	g_s	ξ	a_1	a_2	A_1	A_2	β_1	β_2
0.1	0.13	0.1357	2π	2π	0.2	3.4×10^{-7}	0.4725	0.01

For simplicity, we fix $\Pi_1 = \Pi_b = 0$ and the model is studied by varying Π_2 and λ. Let us stress that this assumption does not affect the main result since the leading F^4 correction is the one proportional to Π_2. Fig. 3.1 shows the plot of the uncorrected inflationary potential (gray line) which is compared with the corrected potential obtained by setting $\Pi_2 = -1$ and choosing $\lambda \sim \mathcal{O}(10^{-4} - 10^{-3})$.

Knowing the explicit expression of the potential, we determine the spectral index (shown in Fig. 3.2 as function of ϕ) and, by integration, the number of efoldings. In this scenario the inflaton is the longest-living particle and the number of efoldings to consider for inflation is $N_e = 45.34$. Given the relations (3.86) and (3.81), we find the value of the field at horizon exit ϕ_{exit}, and then the value of the spectral index $n_s(\phi_{\text{exit}})$ which is reported in Tab. 3.7 for each value of λ.

In order for $n_s(\phi_{\text{exit}})$ to be compatible with Planck measurements [234]:

$$n_s = 0.9649 \pm 0.0042 \qquad (68\% \text{ CL}) \,, \tag{3.98}$$

we need to require $|\lambda| \lesssim 1.1 \times 10^{-3}$ for compatibility within 2σ. This bound might be satisfied by actual multi-field models since, as can be

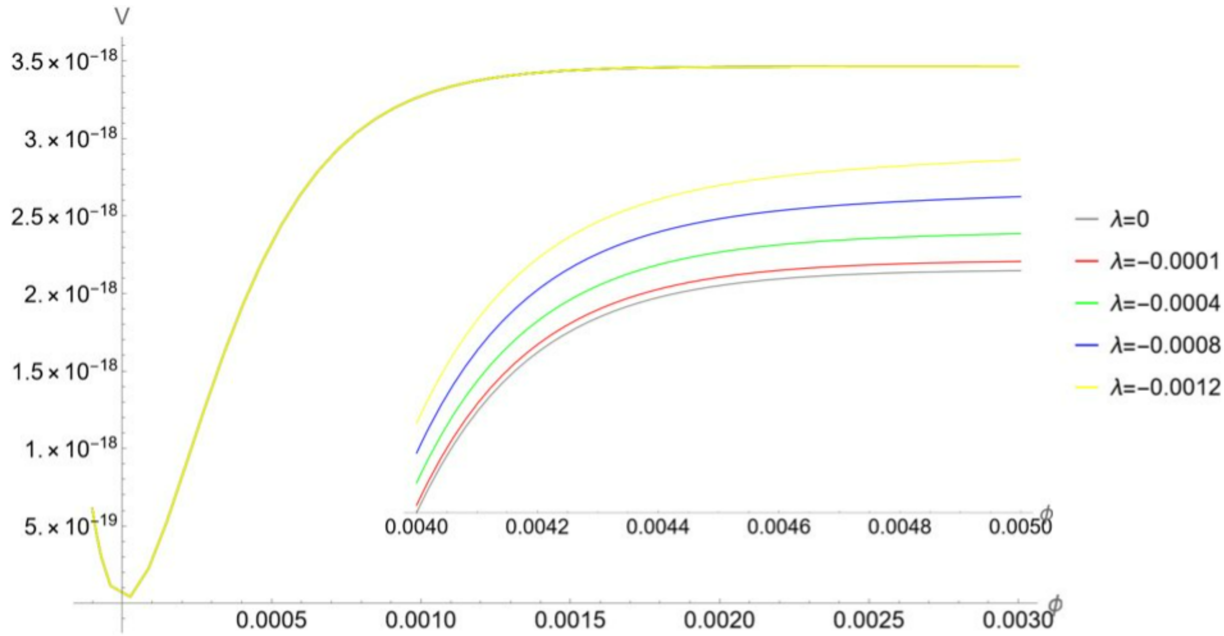

Figure 3.1. Potential of blow-up inflation with $\Pi_2 = -1$ and different values of λ. The difference between the corrections is visible in the zoomed region with $\phi \in [0.004; 0.005]$

seen from (3.16), the single-field case features $|\lambda| = 3.5 \cdot 10^{-4}$ and, as already explained, we expect a similar suppression to persist also in the case with several moduli.

$\lvert\lambda\rvert$	ϕ_{exit}	n_s	A_s
0	4.494899×10^{-3}	0.956386	2.11146×10^{-9}
1.0×10^{-4}	4.95668×10^{-3}	0.958164	1.94664×10^{-9}
4.0×10^{-4}	4.98039×10^{-3}	0.963219	1.53505×10^{-9}
8.0×10^{-4}	5.01349×10^{-3}	0.969316	1.13485×10^{-9}
1.2×10^{-3}	5.04829×10^{-3}	0.974691	8.53024×10^{-10}

Table 3.7. Values of the inflaton at horizon exit ϕ_{exit}, the spectral index n_s and the amplitude of the scalar perturbations A_s for different choices of λ.

By comparing in Tab. 3.7 the $\lambda = 0$ case with the cases with non-zero λ, it is clear that F^4 corrections are a welcome effect, if $|\lambda|$ is not too large, since they can increase the spectral index improving the matching with CMB data. This is indeed the case when Π_2 is negative, as we have chosen. On the other hand, when Π_2 is positive, higher derivative α'^3 corrections would induce negative corrections to n_s that would make the comparison with actual data worse. Such analysis therefore suggests that geometries with negative Π_2 would be preferred in the context of blow-up

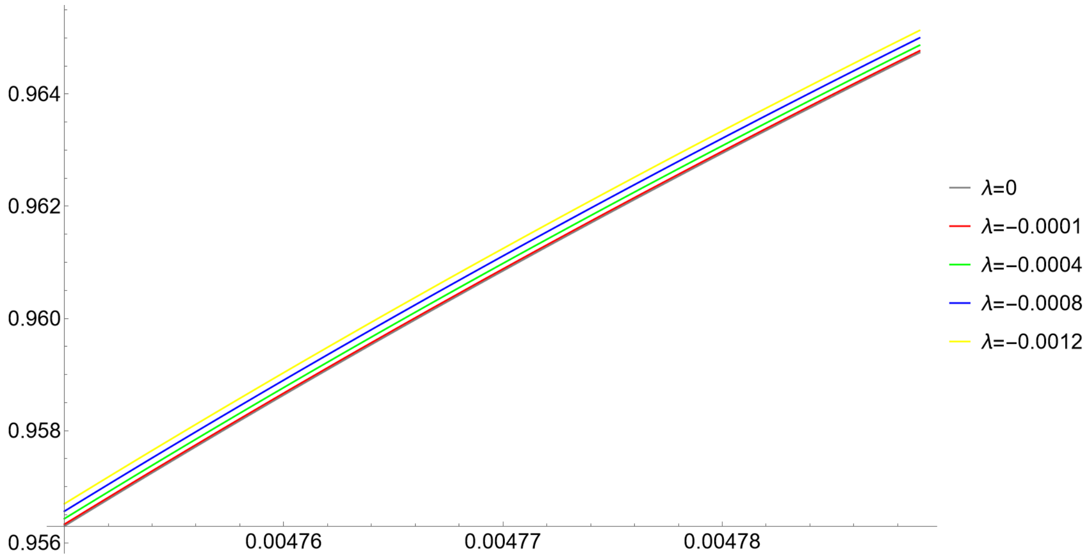

Figure 3.2. Spectral index for different values of λ. The field at horizon exit is given in Tab. 3.7.

inflation.

D7s wrapped around the inflaton

Let us now consider the scenario where the inflaton is wrapped by a stack of N D7-branes supporting a gauge theory that undergoes gaugino condensation. As illustrative examples, we choose the following parameters:

W_0	g_s	ξ	a_1	a_2	A_1	A_2	β_1	β_2
0.1	0.13	0.1357	2π	$2\pi/N$	0.19	3.4×10^{-7}	$\simeq$ 0.5	0.01

Considering $N = 2, 3, 5$, the total number of efoldings is now given by $N_e = 47.90$ for $N = 2$, $N_e = 47.93$ for $N = 3$, and $N_e = 48.02$ for $N = 5$. Repeating the same procedure as before for $\Pi_2 = -1$, we find the results shown in Tab. 3.8.

Due to a larger number of efoldings with respect to the case where the inflaton is not wrapped by any D7-stack, now the prediction for the spectral index falls within 2σ of the observed value also for $\lambda = 0$. Non-zero values of λ can improve the agreement with observations if $|\lambda| < |\lambda|_{\text{max}}$ where:

	$N=2$	$N=3$	$N=5$
$\lvert\lambda\rvert_{\text{max}}$	3.48×10^{-3}	7.15×10^{-3}	1.82×10^{-2}

N	$\lvert\lambda\rvert$	ϕ_{exit}	n_s	A_s
$N=2$	0	8.55743×10^{-3}	0.957692	2.20009×10^{-9}
	1.0×10^{-3}	8.59968×10^{-3}	0.962656	1.73612×10^{-9}
	2.0×10^{-3}	8.64364×10^{-3}	0.967233	1.38248×10^{-9}
	3.0×10^{-3}	8.68932×10^{-3}	0.971425	1.11088×10^{-9}
	4.0×10^{-3}	8.73669×10^{-3}	0.975239	9.00672×10^{-10}
$N=3$	0	1.22049×10^{-2}	0.957679	2.28554×10^{-9}
	2.0×10^{-3}	1.22649×10^{-2}	0.96252	1.81483×10^{-9}
	4.0×10^{-3}	1.23273×10^{-2}	0.966995	1.45345×10^{-9}
	6.0×10^{-2}	1.23921×10^{-2}	0.971106	1.174×10^{-9}
	8.0×10^{-3}	1.24593×10^{-2}	0.97486	9.56344×10^{-10}
$N=5$	0	1.78617×10^{-2}	0.957678	2.14345×10^{-9}
	5.0×10^{-3}	1.7953×10^{-2}	0.962446	1.70842×10^{-9}
	1.0×10^{-2}	1.80479×10^{-2}	0.966862	1.37293×10^{-9}
	1.5×10^{-2}	1.81464×10^{-3}	0.970927	1.11241×10^{-9}
	2.0×10^{-2}	1.82484×10^{-2}	0.974647	9.08706×10^{-10}

Table 3.8. Values of the inflaton at horizon exit ϕ_{exit}, the spectral index n_s and the amplitude of the scalar perturbations A_s for different choices of λ and $N = 2, 3, 5$.

In this case, given the larger number of efoldings, geometries with positive Π_2 can also be viable even if the corrections to the spectral index would be negative. Imposing again accordance with (3.98) at 2σ level for $\Pi_2 = 1$, we would obtain for example $\lvert\lambda\rvert_{\text{max}} = 2.29 \times 10^{-4}$ for $N = 2$.

3.3.6 Fibre inflation

Similarly to blow-up inflation, the minimal version of fibre inflation [51–58] involves also three Kähler moduli: two of them are stabilized via the standard LVS procedure and the remaining one can serve as an inflaton candidate in the presence of perturbative corrections to the Kähler potential. However, fiber inflation requires a different geometry from the one of blow-up inflation since one needs CY threefolds which are K3 fibrations over a $\mathbb{P}^1$ base. The simplest model requires the addition of a blow-up mode such that the volume can be expressed as:

$$\mathcal{V} = \frac{1}{6}\left(k_{111}(t^1)^3 + 3k_{233}t^2(t^3)^2\right) = \alpha\left(\sqrt{\tau_2}\tau_3 - \tau_1^{3/2}\right). \tag{3.99}$$

The requirement of having a K3 fibred CY threefold with at least a ddP$_n$ divisor for LVS moduli stabilisation is quite restrictive. The corresponding scanning results for the number of CY geometries suitable for realizing fibre inflation are presented in Tab. 3.9.

$h^{1,1}$	Poly*	Geom* (n_{CY})	n_{LVS}	K3 fibred CY	n_{LVS} with K3 fib. (fibre inflation)	n_{LVS} with K3 fib. & D_{Π}
1	5	5	0	0	0	0
2	36	39	22	10	0	0
3	243	305	132	136	43	0
4	1185	2000	750	865	171	28
5	4897	13494	4104	5970	951	179

Table 3.9. Number of LVS CY geometries suitable for fibre inflation.

It is worth mentioning that the scanning results presented in Tab. 3.9 are consistent with the previous scans performed in [52, 145]. To be more specific, the number of distinct K3 fibred CY geometries supporting LVS was found in [52] to be 43 for $h^{1,1} = 3$, and ref. [145] claimed that the number of polytopes giving K3 fibred CY threefolds with $h^{1,1} = 4$ and at least one diagonal del Pezzo ddP$_n$ divisor is 158.

3.3.7 Inflationary potential

The leading order scalar potential of fibre inflation turns out to be:

$$V(\mathcal{V}, \tau_1) = a_1^2 |A_1|^2 \frac{\sqrt{\tau_1}}{\mathcal{V}} e^{-2a_1\tau_1} - a_1 |A_1| |W_0| \frac{\tau_1}{\mathcal{V}} e^{-a_1\tau_1} + \frac{\xi |W_0|^2}{g_s^{3/2} \mathcal{V}^3} , \tag{3.100}$$

with a flat direction in the (τ_2, τ_3) plane which plays the role of the inflaton (the proper canonically normalized inflationary direction orthogonal to the volume mode is given by the ratio between τ_2 and τ_3). The inflaton potential is generated by subdominant string loop corrections:

$$\delta V_{\mathcal{O}(\mathcal{V}^{-10/3})}(\tau_2) = \left(g_s^2 \frac{A}{\tau_1^2} - \frac{B}{\mathcal{V}\sqrt{\tau_2}} + g_s^2 \frac{C\tau_2}{\mathcal{V}^2} \right) \frac{|W_0|^2}{\mathcal{V}^2} , \tag{3.101}$$

where A, B, C are flux-dependent coefficients that are expected to be of $\mathcal{O}(1)$. The minimum of this potential is approximately located at:

$$\langle \tau_2 | \tau_2 \rangle \simeq g_s^{4/3} \left(\frac{4A}{B} \right)^{2/3} \langle \mathcal{V} | \mathcal{V} \rangle^{2/3} . \tag{3.102}$$

Writing the canonically normalised inflaton field ϕ as:

$$\tau_2 = \langle \tau_2 | \tau_2 \rangle \, e^{\frac{2\hat{\phi}}{\sqrt{3}}} \simeq g_s^{4/3} \left(\frac{4A}{B} \right)^{2/3} \langle \mathcal{V} | \mathcal{V} \rangle^{2/3} \, e^{\frac{2\hat{\phi}}{\sqrt{3}}} , \tag{3.103}$$

where $\hat{\phi}$ is the shift with respect to the minimum, i.e. $\phi = \langle \phi | \phi \rangle + \hat{\phi}$, the potential (3.101) becomes:

$$V_{\text{inf}}(\hat{\phi}) = V_0 \left[3 - 4e^{-\hat{\phi}/\sqrt{3}} + e^{-4\hat{\phi}/\sqrt{3}} + R \left(e^{2\hat{\phi}/\sqrt{3}} - 1 \right) \right] , \tag{3.104}$$

where (introducing a proper normalization factor $g_s/(8\pi)$ from dimensional reduction):

$$V_0 \equiv \frac{g_s^{1/3}\,|W_0|^2 A}{8\pi\,\langle\mathcal{V}|\mathcal{V}\rangle^{10/3}}\left(\frac{B}{4A}\right)^{4/3} \qquad \text{and} \qquad R \equiv 16 g_s^4 \frac{AC}{B^2} \ll 1\,. \tag{3.105}$$

Note that we added in (3.104) an uplifting term to obtain a Minkowski vacuum. The slow-roll parameters derived from the inflationary potential look like:

$$\epsilon(\hat{\phi}) = \frac{2}{3}\frac{\left(2e^{-\hat{\phi}/\sqrt{3}} - 2e^{-4\hat{\phi}/\sqrt{3}} + Re^{2\hat{\phi}/\sqrt{3}}\right)^2}{\left(3 - R + e^{-4\hat{\phi}/\sqrt{3}} - 4e^{-\hat{\phi}/\sqrt{3}} + Re^{2\hat{\phi}/\sqrt{3}}\right)^2}\,, \tag{3.106}$$

$$\eta(\hat{\phi}) = \frac{4}{3}\frac{4e^{-4\hat{\phi}/\sqrt{3}} - e^{-\hat{\phi}/\sqrt{3}} + Re^{2\hat{\phi}/\sqrt{3}}}{\left(3 - R + e^{-4\hat{\phi}/\sqrt{3}} - 4e^{-\hat{\phi}/\sqrt{3}} + Re^{2\hat{\phi}/\sqrt{3}}\right)}\,, \tag{3.107}$$

and the number of efoldings is:

$$N_e(\hat{\phi}_{\text{exit}}) = \int_{\hat{\phi}_{\text{end}}}^{\hat{\phi}_{\text{exit}}} \frac{1}{\sqrt{2\epsilon(\hat{\phi})}} \simeq \int_{\hat{\phi}_{\text{end}}}^{\hat{\phi}_{\text{exit}}} \frac{\left(3 - 4e^{-\hat{\phi}/\sqrt{3}} + Re^{2\hat{\phi}/\sqrt{3}}\right)}{\left(2e^{-\hat{\phi}/\sqrt{3}} + Re^{2\hat{\phi}/\sqrt{3}}\right)}\,, \tag{3.108}$$

where $\hat{\phi}_{\text{end}}$ and $\hat{\phi}_{\text{exit}}$ are respectively the values of the inflaton at the end of inflation and at horizon exit.

3.3.8 F^4 corrections

Explicit CY examples of fibre inflation with chiral matter have been presented in [54] that has already stressed the importance to control F^4 corrections to the inflationary potential since they could spoil its flatness. This is in particular true for K3 fibred CY geometries since $\Pi(\text{K3}) = 24$, and so the coefficient of F^4 effects is non-zero. On the other hand, the theorem of [223, 224] allows in principle also for CY threefolds that are $\mathbb{T}^4$ fibrations over a $\mathbb{P}^1$ base. This case would be more promising to tame F^4 corrections since their coefficient would vanish due to $\Pi(\mathbb{T}^4) = 0$. However, in our scan for CY threefolds in the KS database we did not find any example with a $\mathbb{T}^4$ divisor. Thus, in what follows we shall perform a numerical analysis of fibre inflation with non-zero F^4 terms to study in detail the effect of these corrections on the inflationary dynamics.

Case 1: a single K3 fibre

The minimal fibre inflation case is a three field model based on a CY threefold that features a K3-fibration structure with a diagonal del Pezzo divisor.

Considering an appropriate basis of divisors, the intersection polynomial can be brought to the following form:

$$I_3 = k_{111}\, D_1^3 + k_{233}\, D_2\, D_3^2\,. \tag{3.109}$$

As the D_2 divisor appears linearly, from the theorem of [223, 224], this CY threefold is guaranteed to be a K3 or $\mathbb{T}^4$ fibration over a $\mathbb{P}^1$ base. Furthermore, the triple-intersection number k_{111} is related to the degree of the del Pezzo divisor $D_1 = dP_n$ as $k_{111} = 9 - n$, while k_{233} counts the intersections of the K3 surface D_2 with D_3. This leads to the following volume form:

$$\mathcal{V} = \frac{k_{111}}{6}\,(t^1)^3 + \frac{k_{233}}{2}\, t^2\,(t^3)^2 = \beta_2\,\sqrt{\tau_2}\,\tau_3 - \beta_1\,\tau_1^{3/2}\,, \tag{3.110}$$

where $\beta_1 = \frac{1}{3}\sqrt{\frac{2}{k_{111}}}$ and $\beta_2 = \frac{1}{\sqrt{2\,k_{233}}}$, and the 2-cycle moduli t^i are related to the 4-cycle moduli τ_i as follows:

$$t^1 = -\sqrt{\frac{2\,\tau_1}{k_{111}}}\,, \qquad t^2 = \frac{\tau_3}{\sqrt{2\,k_{233}\,\tau_2}}\,, \qquad t^3 = \sqrt{\frac{2\,\tau_2}{k_{233}}}\,. \tag{3.111}$$

The higher derivative α'^3 corrections can be written as:

$$\begin{aligned} V_{F^4} &= \frac{\gamma}{\mathcal{V}^4}\left(\Pi_1\, t^1 + \Pi_2\, t^2 + \Pi_3\, t^3\right) \\ &= \frac{\gamma}{\mathcal{V}^4}\left[\Pi_3\sqrt{\frac{2\,\tau_2}{k_{233}}} + \Pi_2\left(\frac{\mathcal{V}}{\tau_2} + \frac{1}{3}\sqrt{\frac{2}{k_{111}}}\,\frac{\tau_1^{3/2}}{\tau_2}\right) - \Pi_1\sqrt{\frac{2\,\tau_1}{k_{111}}}\right]. \end{aligned} \tag{3.112}$$

In the inflationary regime, $\mathcal{V}$ is kept constant at its minimum while τ_2 is at large values away from its minimum, as can be seen from (3.103) for $\hat{\phi} > 0$. Thus, the leading order term in (3.112) is the one proportional to Π_3. Therefore, a leading order protection of the fibre inflation model can be guaranteed by demanding a geometry with $\Pi_3 = 0$. However, the subleading contribution proportional to Π_2 would still induce an inflaton-dependent correction that might be dangerous. The ideal situation to completely remove higher derivative F^4 corrections to fibre inflaton is therefore characterized by:

$$\Pi_2 = \Pi_3 = 0\,, \tag{3.113}$$

where, as pointed out above, Π_2 would vanish for $\mathbb{T}^4$ fibred CY threefolds. Interestingly, such CY examples with $\mathbb{T}^4$ divisors have been found in the CICY database, without however any ddP for LVS [208]. It is also true that all K3 fibred CY threefolds do not satisfy $\Pi_2 = 0$.

Case 2: multiple K3 fibres

More generically, fibre inflation could be realized also in CY threefolds which admit multiple K3 or $\mathbb{T}^4$ fibrations together with at least a diagonal del Pezzo divisor. The corresponding intersection polynomial would look like (see [54] for explicit CY examples):

$$I_3 = k_{111}\, D_1^3 + k_{234}\, D_2\, D_3\, D_4\,. \tag{3.114}$$

As the divisors D_2, D_3 and D_4 all appear linearly, from the theorem of [223, 224], this CY threefold is guaranteed to have three K3 or $\mathbb{T}^4$ fibrations over a $\mathbb{P}^1$ base. As before, D_1 is a diagonal dP$_n$ divisor with $k_{111} = 9 - n > 0$. The volume form becomes:

$$\mathcal{V} = \frac{k_{111}}{6}\,(t^1)^3 + k_{234}\, t^2\, t^3\, t^4 = \beta_2\, \sqrt{\tau_2\, \tau_3\, \tau_4} - \beta_1\, \tau_1^{3/2}\,, \tag{3.115}$$

where $\beta_1 = \frac{1}{3}\sqrt{\frac{2}{k_{111}}}$ and $\beta_2 = \frac{1}{\sqrt{k_{234}}}$, and the 2-cycle moduli t^i are related to the 4-cycle moduli τ_i as:

$$t^1 = -\sqrt{\frac{2\,\tau_1}{k_{111}}}\,, \qquad t^2 = \frac{\sqrt{\tau_3\,\tau_4}}{\sqrt{k_{234}\,\tau_2}}, \qquad t^3 = \frac{\sqrt{\tau_2\,\tau_4}}{\sqrt{k_{234}\,\tau_3}}, \qquad t^4 = \frac{\sqrt{\tau_2\,\tau_3}}{\sqrt{k_{234}\,\tau_4}}\,. \tag{3.116}$$

This case features two flat directions which can be parametrised by τ_2 and τ_2. Moreover, the higher derivative F^4 corrections take the form:

$$\begin{aligned} V_{F^4} = & \quad \frac{\gamma}{\mathcal{V}^4}\,\left(\Pi_1\, t^1 + \Pi_2\, t^2 + \Pi_3\, t^3 + \Pi_4\, t^4\right) \\ = & \quad \frac{\gamma}{\mathcal{V}^4}\,(\mathcal{V} + \beta_1\, \tau_1^{3/2})\left(\frac{\Pi_2}{\tau_2} + \frac{\Pi_3}{\tau_3} + \frac{\Pi_4\,\tau_2\,\tau_3}{\beta_2\,(\mathcal{V} + \beta_1\,\tau_1^{3/2})^2}\right) - \Pi_1\,\frac{\gamma}{\mathcal{V}^4}\sqrt{\frac{2\,\tau_1}{k_{111}}}\,. \end{aligned} \tag{3.117}$$

In the explicit model of [54], non-zero gauge fluxes generate chiral matter and a moduli-dependent Fayet-Iliopoulos term which lifts one flat direction, stabilizing $\tau_3 \propto \tau_2$. After performing this substitution in (3.117), this potential scales as the one in the single field case given by (3.112). Interestingly, ref. [54] noticed that, in the absence of winding string loop corrections, F^4 effects can also help to generate a post-inflationary minimum. Note finally that if all the divisors corresponding to the CY multi-fibre structure are $\mathbb{T}^4$, the F^4 terms would be absent. However, incorporating a diagonal del Pezzo within a $\mathbb{T}^4$-fibred CY is yet to be constructed (e.g. see [208]).

3.3.9 Constraints on inflation

Let us focus on the simplest realization of fibre inflation, and add the dominant F^4 corrections (3.112) to the leading inflationary potential (3.104). The total inflaton-dependent potential takes therefore the form:

$$V_{\text{inf}}(\hat{\phi}) = V_0 \left[e^{-4\hat{\phi}/\sqrt{3}} - 4e^{-\hat{\phi}/\sqrt{3}} + 3 + R(e^{2\hat{\phi}/\sqrt{3}} - 1) - R_2\, e^{-2\hat{\phi}/\sqrt{3}} - R_3\, e^{\hat{\phi}/\sqrt{3}} \right], \tag{3.118}$$

where R is given by (3.105) while R_2 and R_3 are defined as:

$$R_2 \equiv \frac{|W_0|^2}{(4\pi)^2 A g_s^{3/2}} \frac{\lambda \Pi_2}{\mathcal{V}} \ll 1 \qquad \text{and} \qquad R_3 \equiv \frac{4\,|W_0|^2 \sqrt{g_s}}{B} \frac{\lambda \Pi_3}{\mathcal{V}} \ll 1\,. \tag{3.119}$$

Note that the most dangerous term that could potentially spoil the flatness of the inflationary plateau is the one proportional to R_3 since it multiplies a positive exponential. The term proportional to R_2 is instead harmless since it multiplies a negative exponential.

As we have seen for blow-up inflation, the study of reheating after the end of inflation is crucial to determine the number of efoldings of inflation which are needed to make robust predictions for the main cosmological observables. Reheating for fibre inflation with the Standard Model on D7-branes has been studied in [55], while ref. [58] analysed the case where the visible sector is realized on D3-branes. In both cases, a radiation dominated universe is realized from the perturbative decay of the inflaton after the end of inflation. In what follows we shall focus on the D7-brane case and include the loop-induced coupling between the inflaton and the Standard Model Higgs, similarly to volume-Higgs coupling found in [228]. The relevant term in the low-energy Lagrangian is the Higgs mass term which can be expanded as:

$$m_h^2 h^2 = m_{3/2}^2 \left[c_0 + c_{\text{loop}} \ln \left(\frac{M_{KK}}{m_{3/2}} \right) \right] h^2\,, \tag{3.120}$$

where $\ln\left(M_{KK}/m_{3/2}\right) \propto \ln \mathcal{V}$. Using the fact that [235]:

$$\mathcal{V} = \langle \mathcal{V} | \mathcal{V} \rangle \left(1 + \kappa \hat{\phi}\right), \tag{3.121}$$

with $\kappa \sim \langle \mathcal{V} \rangle^{-1/3}$, the Higgs mass term (3.120) generates a coupling between $\hat{\phi}$ and h that leads to the following decay rate:

$$\Gamma_{\phi \to hh} \simeq \frac{c_{\text{loop}}^2}{\mathcal{V}^{2/3}} \frac{m_{3/2}^4}{M_p^2 m_{\text{inf}}} \simeq \frac{c_{\text{loop}}^2}{\mathcal{V}^{2/3}} \left(\frac{m_{3/2}}{m_{\text{inf}}} \right)^4 \Gamma_{\phi \to \gamma\gamma}\,. \tag{3.122}$$

It is then easy to realise that the inflaton decay width into Higgses is larger than the one into gauge bosons for $\mathcal{V} \gg 1$ since:

$$\frac{\Gamma_{\phi\to hh}}{\Gamma_{\phi\to\gamma\gamma}} \simeq (c_{\text{loop}}\mathcal{V})^2 \gg 1\,. \tag{3.123}$$

The number of efoldings of inflation is then determined as:

$$N_e \simeq 56 - \frac{1}{3}\ln\left(\frac{m_{\text{inf}}}{T_{\text{rh}}}\right), \tag{3.124}$$

where the reheating temperature T_{rh} scales as:

$$T_{\text{rh}} \simeq \sqrt{\Gamma_{\phi\to hh}\, M_p}\,. \tag{3.125}$$

Substituting this expression in (3.124), and using (3.122), we finally find:

$$N_e \simeq 53 + \frac{1}{6}\ln\left[1 + \frac{c_{\text{loop}}^2}{\mathcal{V}^{2/3}}\left(\frac{m_{3/2}}{m_{\text{inf}}}\right)^4\right]. \tag{3.126}$$

This is the number of efoldings of inflation used in the next section for the analysis of the inflationary dynamics in some illustrative numerical examples.

3.3.10 Numerical examples

Let us now perform a quantitative study of the effect of higher derivative α'^3 corrections to fibre inflation for reasonable choices of the underlying parameters. In order to match observations, we follow the best-fit analysis of [57] and set $R = 4.8 \times 10^{-6}$, which can be obtained by choosing:

$$A = 1\,, \qquad B = 8\,, \qquad C = 0.19\,. \tag{3.127}$$

Moreover, given that D_2 is a K3 divisor, we fix $\Pi_2 = 24$, while we leave Π_3 and λ as free parameters that we constrain from phenomenological data.

Fig. 3.3 shows the potential of fibre inflation with F^4 corrections corresponding to $\Pi_3 = 1$ and different negative values of λ.

As for blow-up inflation, we find numerically the range of values of λ which are compatible with observations. In Tab. 3.10 we show the values for the spectral index evaluated at horizon exit, with $N_e = 53.81$ fixed from (3.126), for $\Pi_3 = 1$ and different values of λ. In order to reproduce the best-fit value of the scalar spectral index [57, 234]:

$$n_s = 0.9696^{+0.0010}_{-0.0026} \tag{3.128}$$

the numerical coefficient λ has to respect the bound $|\lambda| \lesssim 6.1 \times 10^{-4}$, which seems again compatible with the single-field result (3.16) that gives $|\lambda| = 3.5 \cdot 10^{-4}$.

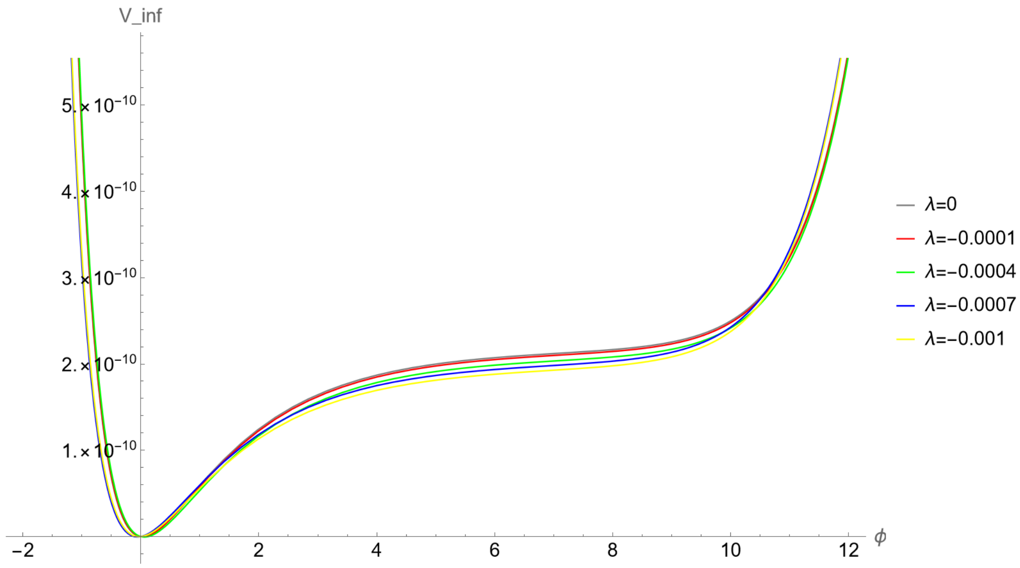

Figure 3.3. Potential of fibre inflation with F^4 corrections with $\Pi_3 = 1$ and different values of λ.

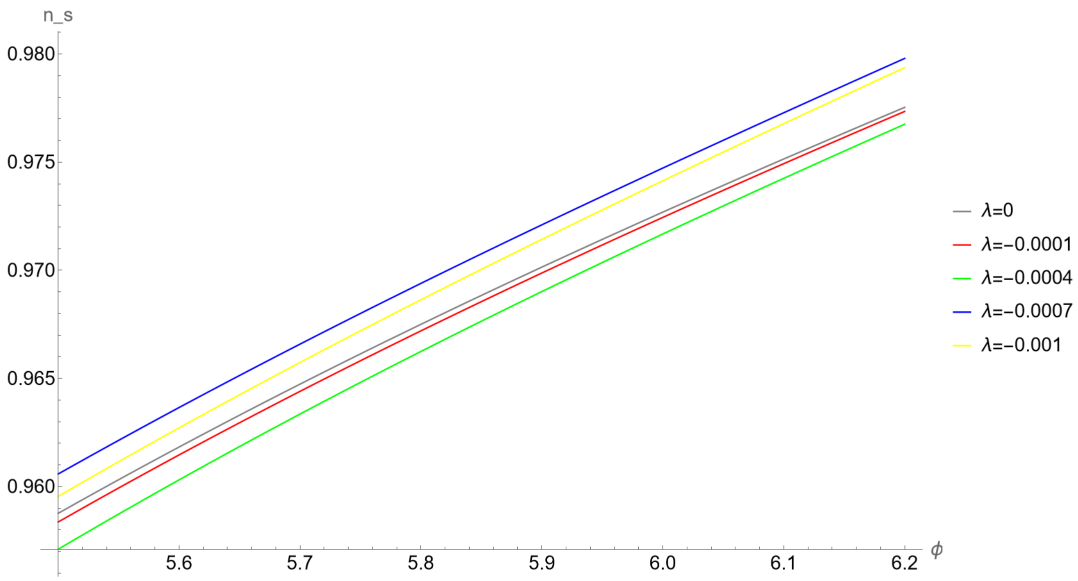

Figure 3.4. Spectral index for different values of λ in fibre inflation. The value of the inflaton at horizon exit is given in Tab. 3.10.

3.3.11 Poly-instanton inflation

Let us finally analyse higher derivative α'^3 corrections to poly-instanton inflation, focusing on its simplest realization based on a three-field LVS model [59, 61]. This model involves exponentially suppressed correc-

$\|\lambda\|$	ϕ_{exit}	n_s	A_s
0	5.91328	0.97049	2.13082×10^{-9}
0.1×10^{-3}	5.93005	0.970657	2.09702×10^{-9}
0.4×10^{-3}	5.98203	0.971207	1.99576×10^{-9}
0.7×10^{-3}	5.88793	0.97178	1.90293×10^{-9}
1.0×10^{-3}	5.93552	0.972399	1.81416×10^{-9}

Table 3.10. Values of the inflaton at horizon exit ϕ_{exit}, the spectral index n_s and the amplitude of the scalar perturbations A_s for different choices of λ in fibre inflation.

tions appearing on top of the usual non-perturbative superpotential effects arising from E3-instantons or gaugino condensation wrapping suitable rigid cycles of the CY threefold. In this three-field model, two Kähler moduli correspond to the volumes of the 'big' and 'small' 4-cycles (namely D_b and D_s) of a typical Swiss-cheese CY threefold, while the third modulus controls the volume of a Wilson divisor D_w which is a $\mathbb{P}^1$ fibration over $\mathbb{T}^2$ [221]. Moreover, such a divisor has the following Hodge numbers for a specific choice of involution: $h^{2,0}(D_w) = 0$ and $h^{0,0}(D_w) = h^{1,0}(D_w) = h^{1,0}_+(D_w) = 1$. For this model one can consider the following intersection polynomial:

$$I_3 = k_{sss}\, D_s^3 + k_{ssw}\, D_s^2\, D_w + k_{sww}\, D_s\, D_w^2 + k_{bbb}\, D_b^3\,, \tag{3.129}$$

where, as argued earlier, the self triple-intersection number of the Wilson divisor is zero, i.e. $k_{www} = 0$. This is because Wilson divisors are of the kind given in (3.42) for $n = 0$. We also have selected a basis of divisors where the large four-cycle D_b does not mix with the other two divisors to keep a strong Swiss-cheese structure. This leads to the following form of the CY volume:

$$\mathcal{V} = \frac{k_{bbb}}{6}\,(t^b)^3 + \frac{k_{sss}}{6}\,(t^s)^3 + \frac{k_{ssw}}{2}\,(t^s)^2\,t^w + \frac{k_{sww}}{2}\,t^s\,(t^w)^2\,, \tag{3.130}$$

which subsequently gives to the following 4-cycle volumes:

$$\tau_b = \frac{1}{2}\,k_{bbb}\,(t^b)^2, \quad \tau_s = \frac{1}{2}\,k_{sss}\left((t^s)^2 + 2\,\frac{k_{ssw}}{k_{sss}}\,(t^s)^2 + \frac{k_{sww}}{k_{sss}}\,(t^w)^2\right),$$
$$\tau_w = \frac{1}{2}\,\left(k_{ssw}\,(t^s)^2 + 2\,k_{sww}\,t^s\,t^w\right)\,. \tag{3.131}$$

Now it is clear that in order for the 'small' divisor to be diagonal, the above intersection numbers have to satisfy the following relation:

$$k_{sss} = \pm\,k_{ssw} = k_{sww}\,, \tag{3.132}$$

which is indeed the case when the divisor basis is appropriately chosen in the way we have described above. This leads to the following expression of the CY volume:

$$\mathcal{V} = \beta_b \, \tau_b^{3/2} - \beta_s \, \tau_s^{3/2} - \beta_s \, (\tau_s \mp \tau_w)^{3/2} \,, \tag{3.133}$$

where $\beta_s = \frac{1}{3}\sqrt{\frac{2}{k_{sss}}}$ and $\beta_b = \frac{1}{3}\sqrt{\frac{2}{k_{bbb}}}$, and the 4-cycle volumes τ_s, τ_w and τ_b are given by:

$$\tau_b = \frac{1}{2}\, k_{bbb}\, (t^b)^2, \quad \tau_s = \frac{1}{2}\, k_{sss}\, (t^s \pm t^w)^2, \quad \tau_w = \pm\frac{1}{2}\, k_{sss}\, (t^s \pm 2\, t^w)\,. \tag{3.134}$$

The $\pm$ sign is decided by the Kähler cone conditions, like for example in the case of D_s being a del Pezzo divisor where the corresponding two-cycle in the Kähler form J satisfies $t^s < 0$ in an appropriate diagonal basis. Looking at explicit CY examples [221], the sign is fixed through the Kähler cone conditions such that $k_{sss} = -\, k_{ssw} = k_{sww}$, leading to the following peculiar structure of the volume form [221]:

$$\mathcal{V} = \beta_b \, \tau_b^{3/2} - \beta_s \, \tau_s^{3/2} - \beta_s \, (\tau_s + \tau_w)^{3/2} \,, \tag{3.135}$$
$$\tau_b = \frac{1}{2}\, k_{bbb}\, (t^b)^2, \quad \tau_s = \frac{1}{2}\, k_{sss}\, (t^s - t^w)^2, \quad \tau_w = -\,\frac{1}{2}\, k_{sss}\, (t^s - 2\, t^w)\,.$$

3.3.12 Divisor topologies for poly-instanton inflation

In principle, one should be able to fit the requirements for poly-instanton inflation on top of having LVS in a setup with three Kähler moduli. Indeed we find that there are four CY threefold geometries with $h^{1,1}(X) = 3$ in the KS database which have exactly one Wilson divisors and a $\mathbb{P}^2$ divisor. However, as mentioned in [221], in order to avoid all vector-like zero modes to have poly-instanton effects, one should ensure that the rigid divisors wrapped by the ED3-instantons, should have some orientifold-odd $(1,1)$-cycles which are trivial in the CY threefold. Given that $\mathbb{P}^2$ has a single $(1,1)$-cycle, it would certainly not have such additional two-cycles which could be orientifold-odd and then trivial in the CY threefold. Hence one has to look for CY examples with $h^{1,1}(X) = 4$ for a viable model of poly-instanton inflation as presented in [61, 221]. In this regard, we present the classification of all CY geometries relevant for LVS poly-instanton inflation in Tab. 3.11.

Let us stress that in all our scans we have only focused on the minimal requirements to realise explicit global constructions of LVS inflationary models. However, every model has to be engineered in a specific way on top of fulfilling the first order topological requirements, as we do.

For example, merely having a K3-fibred CY threefold with a diagonal del Pezzo for LVS does not guarantee a viable fibre inflation model until one ensures that string loop corrections can appropriately generate the right form of the scalar potential after choosing some concrete brane setups.

$h^{1,1}$	Poly*	Geom* (n_{CY})	Single W	Two W	Three W	n_{LVS}	n_{LVS} & W (poly-inst.)	n_{LVS} & W_Π (topol. tamed)
1	5	5	0	0	0	0	0	0
2	36	39	0	0	0	22	0	0
3	243	305	19	0	0	132	4	4
4	1185	2000	221	8	0	750	75	63
5	4897	13494	1874	217	43	4104	660	522

Table 3.11. Number of LVS CY geometries suitable for poly-instanton inflation. Here W denotes a generic Wilson divisors, while W_Π a Wilson divisor with $\Pi = 0$.

As a side remark, let us recall that for having poly-instanton corrections to the superpotential one needs to find a Wilson divisor W with $h^{2,0}(W) = 0$ and $h^{0,0}(W) = h^{1,0}(W) = h^{1,0}_+(W) = 1$ for some specific choice of involution, without any restriction on $h^{1,1}(W)$ [221]. On these lines, a different type of 'Wilson' divisor suitable for poly-instanton corrections has been presented in [62], which has $h^{1,1}(W) = 4$ instead of 2, and so it has a non-vanishing Π. As we will discuss in a moment, this means that any poly-instanton inflation model developed with such an example would not have leading order protection against higher-derivative F^4 corrections for the inflaton direction τ_w. Tab. 3.11 and 3.12 show the existence of several Wilson divisors which fail to have vanishing Π since they have $h^{1,1}(W) \neq 2$.

$h^{1,1}$	Poly*	Geom*	at least one W	single W	two W	three W	at least one W_Π	single W_Π	two W_Π	three W_Π
1	5	5	0	0	0	0	0	0	0	0
2	36	39	0	0	0	0	0	0	0	0
3	243	305	19	19	0	0	19	19	0	0
4	1185	2000	229	221	8	0	210	202	8	0
5	4897	13494	2134	1874	217	43	1764	1599	154	11

Table 3.12. CY geometries with Wilson divisors W and vanishing Π Wilson divisors W_Π without demanding a diagonal del Pezzo divisor.

3.3.13 Comments on F^4 corrections

The higher-derivative F^4 corrections to the potential of poly-instanton inflation can be written as:

$$\begin{aligned} V_{F^4} &= \frac{\gamma}{\mathcal{V}^4}\left(\Pi_b\, t^b + \Pi_s\, t^s + \Pi_w\, t^w\right) \\ &= \frac{\gamma}{\mathcal{V}^4}\left[\Pi_b \left(\frac{6}{k_{bbb}}\right)^{1/3}\left(\mathcal{V} + \beta_s\, \tau_s^{3/2} + \beta_s\, (\tau_s+\tau_w)^{3/2}\right)^{1/3}\right. \\ &\quad \left. - \Pi_s \sqrt{\frac{2\,\tau_s}{k_{sss}}} - \Pi_w \sqrt{\frac{2\,(\tau_s+\tau_w)}{k_{sss}}}\right], \end{aligned} \tag{3.136}$$

where we have used:

$$t^b = \sqrt{\frac{2\,\tau_b}{k_{bbb}}}, \quad t^s = -\sqrt{\frac{2}{k_{sss}}}\left(\sqrt{\tau_s} + \sqrt{\tau_s+\tau_w}\right), \quad t^w = -\sqrt{\frac{2}{k_{sss}}}\sqrt{\tau_s+\tau_w}\,. \tag{3.137}$$

Now we know that for our Wilson divisor case, $\Pi_w = 0$, and so the last term in (3.136) automatically vanishes. This gives at least a leading order protection for the potential of the inflaton modulus τ_w after stabilizing the $\mathcal{V}$ and τ_s moduli through LVS. However the τ_w-dependent term proportional to Π_b would still induce a subleading inflaton-dependent correction that scales as $\mathcal{V}^{-14/3}$. As compared to the LVS potential, this F^4 correction is suppressed by a $\mathcal{V}^{-5/3}$ factor which for $\mathcal{V} \gg 1$ should be small enough to preserve the predictions of poly-instanton inflation studied in [61, 63, 236]. Interestingly, we have found that F^4 corrections to poly-instanton inflation can be topologically tamed, unlike the case of blow-up inflation. In fact, the topological taming of higher derivative corrections to blow-up inflation would require the inflaton to be the volume of a diagonal dP_3 divisor which, according to the conjecture formulated in [28], is however very unlikely to exist in CY threefolds from the KS database.

Chapter IV

Early Dark Energy in String Theory

In this chapter, we seek to identify and address the challenges to building a phenomenologically viable EDE model within the context of string theory. The first steps have been already provided in [71], in the context of KKLT compactifications, with the EDE field identified as a C_2 axion. Its potential is derived from non-perturbative corrections to the superpotential W generated by gaugino condensation on D5-branes. Besides the need to tune the prefactors of these non-perturbative effects to reproduce the correct EDE scale, it remains unclear if gaugino condensation on D5-branes can actually yield a non-zero contribution to the superpotential for cycles in the geometric regime[1] [170, 182].

Here we go beyond what achieved in [71] and perform a deeper analysis of EDE model building in type IIB string theory which is one of the most promising corners of string theory for moduli stabilization. We propose string embeddings of EDE in the moduli stabilization frameworks of KKLT [18] and the Large Volume Scenario (LVS) [27, 99]. Moreover, we identify different choices of axion as the EDE candidate. In particular, we try to realize the EDE potential $V = V_0 \left[1 - \cos(\varphi/f)\right]^3$ with the phenomenologically relevant parameters $V_0 \sim$ eV^4 and $f \simeq 0.2\, M_P$, while satisfying the following conditions:

1. **Controlled de Sitter moduli stabilization:** All string moduli should be stabilized in a dS vacuum where the effective field theory is under control. In particular the compactification volume should be large enough to trust the α' expansion, the string coupling should be small enough to remain in the perturbative regime, and the instanton expansion should be well behaved. One of the main obstacles against achieving moduli stabilization with full control is the fact that the decay constant f of the EDE field has to be relatively close to the Planck scale. This can intuitively be seen as follows. Explicit string

[1] See however [237] for cosmological applications.

computations [166, 167, 169], as well as the weak gravity conjecture applied to axions [238-241], give $fS \simeq \lambda M_P$ where S is the instanton action and λ an $\mathcal{O}(1)$ constant. Hence, $f \simeq 0.2\,M_P$ implies $S \sim \mathcal{O}(10)$. Given that in string compactifications S is set by the volume of the internal cycle wrapped by the instanton which generates the EDE potential, $f \simeq 0.2\,M_P$ implies volumes of $\mathcal{O}(10)$ in string units which might not be large enough to control the effective field theory.

2. **Decoupling of non-EDE modes:** All moduli different from the EDE field should be stabilized at an energy scale much larger than V_0, so that they become much heavier than the EDE field whose mass is of order $m \sim 10^{-27}$ eV. This requirement is needed for two reasons: (i) saxions with masses below about 1 meV would mediate unobserved fifth-forces; (ii) the dynamics of ultra-light axions with masses around $m \sim 10^{-27}$ eV could play a significant role around matter-radiation equality, potentially modifying the cosmological evolution of the EDE model[2]. The EDE scale V_0 should also be decoupled from the scale of supersymmetry breaking and the gravitino mass.

3. **Absence of fine-tuning:** The main phenomenological features of the model, namely the desired EDE energy scale V_0, the typical $[1 - \cos(\varphi/f)]^3$ shape of the potential and the decoupling of the non-EDE modes, should be realized without the need to fine-tune the underlying microscopic parameters. If instead some parameters need to take unnatural values, the UV completion should provide enough tuning freedom.

4. **Explicit Calabi-Yau realization:** A full-fledged string model of EDE should feature a globally consistent compactification with an explicit Calabi-Yau orientifold involution and brane setup which allow for tadpole cancellation and the realization of all perturbative and non-perturbative ingredients needed to fix the moduli and generate the EDE potential.

These challenges are not independent, but instead exhibit a rich interplay. For example, large volume can help achieve a convergent α' and instanton expansion and the desired EDE energy scale, but at the cost of lowering both the decay constant and all mass scales. On the other hand, at moderate volume, the EDE energy scale can be adjusted simply by lowering

[2]On the other hand, an ultra-light axion component of dark matter may in fact help the model to be in agreement with Large Scale Structure data [93, 94].

the prefactors of non-perturbative terms (this was the approach of [71]), at the cost of introducing an exponential fine-tuning in the model.

We elucidate and address the first 3 challenges within the context of various string theory realizations of EDE. We consider moduli stabilization both in the context of KKLT and LVS (a concise summary of the string theory background is provided in Sec. 1.3). These scenarios have been studied in depth in the literature and are two of the most promising frameworks for controlled moduli stabilization.

In order to ensure the decoupling of non-EDE modes, we consider both C_4 and C_2 axions as potential EDE candidates. In fact, both of these fields enjoy a continuous shift symmetry, which is exact at the perturbative level. Hence, any perturbative correction would generate the required hierarchy by fixing the corresponding saxions while leaving the C_4 and C_2 axions flat. This is what happens typically for C_2 axions and the bulk C_4 axion in LVS. The situation is somewhat different for C_4 axions in KKLT where moduli stabilization relies only on non-perturbative effects. In this case the C_4 axion cannot therefore play the role of the EDE field.

Regarding the task to reproduce the EDE scale, C_2 axions seem more promising than C_4 axions. An intuitive explanation concerns the fact that matching $V_0 \sim$ eV^4 without fine-tuning requires a violation of the weak gravity conjecture applied to axions. In fact, writing again $fS \simeq \lambda M_P$, the EDE scale can be written as

$$V_0 = A\, e^{-S}\, M_P^4 \simeq A\, e^{-\lambda M_P/f}\, M_P^4 \simeq A\, e^{-5\lambda}\, M_P^4 \qquad \text{for} \qquad f \simeq 0.2\, M_P\,. \tag{4.1}$$

Demanding $V_0 \sim$ eV$^4 \sim 10^{-108}\, M_P^4$ corresponds to $A\, e^{-5\lambda} \sim 10^{-108}$, which clearly requires $A \ll 1$ for $\lambda \sim \mathcal{O}(1)$.[3] On the other hand, cases with $\lambda \gg 1$ (i.e. violation of the weak gravity conjecture) could reproduce the correct EDE scale without the need to tune the prefactor A. Ref. [183] found $\lambda \sim \mathcal{V}^{1/3} \gg 1$ (where $\mathcal{V} \gg 1$ is the compactification volume in string units) for C_2 axions with superpotential from fluxed ED3-instantons/gaugino condensation on D7s, while $\lambda \sim \mathcal{O}(1)$ for C_2 axions with ED1-instanton corrections to the Kähler potential, and C_4 axions with superpotential generated by ED3-instantons/gaugino condensation on D7-branes. Therefore, we identify C_2 axions, with a potential generated by fluxed ED3-instantons or gaugino condensation on D7-branes with non-zero world-volume fluxes [183], as in principle the most promising candidates to match the EDE scale with minimal fine-tuning of the model.

[3] Here our logic is different from the one of [95], which set $A \simeq 10^{-8}$ to get an overall $M_{\rm GUT}^4$ scale and fixed $\lambda \sim \mathcal{O}(1)$ to infer the value of f needed to match the EDE scale. This logic yields $f \simeq 0.008\, M_P$, which is below the best fit value $f \simeq 0.2\, M_P$.

However, as we shall see, matching the required EDE scale without any tuning of the prefactors of non-perturbative effects leads to a compactification volume of order $\mathcal{V} \sim \mathcal{O}(10^4 - 10^5)$. In turn, obtaining $f \simeq 0.2\,M_P$ requires $\mathcal{O}(100)$ D7-branes supporting the gaugino condensates which generate the EDE potential. This number is relatively large but still achievable in F-theory compactifications [155]. Moreover, $\sim \mathcal{O}(10^4 - 10^5)$ can be obtained easily in LVS models, while in KKLT it requires scenarios with $\mathcal{O}(1000)$ D7-branes supporting the gaugino condensate that yields the leading KKLT potential, otherwise the mass of the volume modulus would be below the cosmological moduli problem bound, $m_{\mathcal{V}} \lesssim 50$ TeV. Such a large number of D7-branes might be hard to achieve in a way compatible with D7 tadpole cancellation and a controlled backreaction. Thus, our analysis indicates that the most promising candidates to realize EDE from type IIB string theory are C_2 axions in LVS with a potential generated by gaugino condensation on D7-branes with non-vanishing gauge fluxes. Fluxed ED3-instantons would instead not be compatible with $f \simeq 0.2\,M_P$ for $\mathcal{V} \sim \mathcal{O}(10^4 - 10^5)$.

The main challenge left is to construct an explicit Calabi-Yau embedding of these models where all the needed non-perturbative effects are explicitly shown to arise with the exact coefficients needed to reproduce the $[1 - \cos(\varphi/f)]^3$ shape of the EDE potential.

Let us also point out that the KKLT and LVS implementations of EDE are distinguished in part by the mass of the gravitino. The most natural LVS models predict a gravitino mass far above the energy scale of any particle physics experiment. In particular, typical Swiss-cheese Calabi-Yau models lead to $m_{3/2} \sim \mathcal{O}(10^{13})$ GeV, while K3-fibred compactifications feature $m_{3/2} \sim \mathcal{O}(10^{10})$ GeV. On the other hand, KKLT models with the lowest possible number of D7-branes correlate with a TeV-scale gravitino mass. This suggests that experimental searches for the gravitino may be complementary to cosmological searches for EDE as it emerges from KKLT. An additional complementary direction is to interface the LVS EDE models with LVS inflation models [47–49, 51, 52, 54, 59], wherein the volume $\mathcal{V}$ is fixed by matching to the amplitude of the CMB power spectrum, even if in some models the volume can evolve from inflation to today [242, 243].

These analyses suggest that EDE can be a viable cosmological model from the perspective of string theory. The more difficult model building task is to realize multiple non-perturbative contributions to W with precise coefficients that reproduce the EDE potential, even if other features of the model (like moduli stabilization, the EDE scale and decay constant, and the decoupling of non-EDE modes) can be achieved in LVS without fine-tuning. Hence we do not consider the theory challenges so different in difficulty in comparison to those faced by other cosmological models, such as dark energy [33, 34, 148, 244-246], or fuzzy dark matter [183, 247], and thus one expects an eventual plethora of model realizations, of which we have only scratched the surface. By expanding the playground of model-building frameworks for EDE, this work will enable future efforts to target specific aspects of phenomenology that may be of observational interest, such as the coupling to photons [169] and the associated particle and gravitational wave production [248].

Notation: In this chapter $M_P = 2.435 \times 10^{18}$ GeV is the reduced Planck mass.

4.1 The Hubble Tensions

The Hubble tension is sharpest between Planck 2018 CMB data and SH0ES cosmic distance ladder measurement. Here we focus on these two experiments, but re-emphasize that the tension exists between varied data sets - see [68] for a recent review. What follows is intended to be a non-technical review of the essential physics of the Hubble tension and the EDE model (see [73] for a detailed review), with a particular focus on the constraints from data that guide the model-building process. This is complementary but distinct from the review given in [71].

Key to understanding the EDE approach to the Hubble tension is that the CMB data, namely the distribution of CMB anisotropies on the sky, is an intrinsically two-dimensional picture of the universe. Thus, while one may directly measure the angular scale of features in the CMB, to translate this into length scales one must assume a cosmological model. The Hubble length, H_0^{-1}, is one such length scale that one may try to infer.

Indeed the most precise cosmological measurement to date is the Planck 2018 measurement of the angular extent of the comoving sound horizon at last scattering, $100\ \theta_s = 1.0411 \pm 0.0003$ [234]. This is defined by a ratio of length scales, as

$$\theta_s = \frac{r_s(z_*)}{D_A(z_*)}\,, \tag{4.2}$$

where r_s measures distances between points in the surface of last scattering[4], while D_A corresponds to the distance from an observer to the

[4] Last scattering surface refers to the time of last scattering of photons and electrons

CMB last scattering surface. More precisely, $r_s(z_*)$ is the comoving sound horizon at last scattering, defined as

$$r_s(z_*) = \int_{z_*}^{z_{\rm re}} \frac{{\rm d}z}{H(z)} c_s(z), \tag{4.3}$$

with z_* the redshift of last scattering, $z_{\rm re}$ is the redshift of reheating after cosmic inflation, and c_s the sound speed of the photon-baryon plasma, whereas $D_A(z_*)$ is the angular diameter distance to the surface of last scattering,

$$D_A(z_*) = \int_0^{z_*} {\rm d}z \, \frac{1}{H(z)} \, , \tag{4.4}$$

which is sensitive to $H(z = 0)$, i.e. the Hubble constant H_0. These expressions suggest a path forward for resolving the Hubble tension: The 0.03% measurement of the angle θ_s can accommodate the $\approx 10\%$ increase in H_0 if there is a commensurate increase in $H(z \sim z_*)$. This approach, which acts to reduce the sound horizon at last scattering, has been extensively studied (see [68] for a review). A popular model realization is Early Dark Energy [72].

4.2 The Early Dark Energy solution

The reduction of the sound horizon can be easily achieved by an ultralight scalar, satisfying the Klein-Gordon equation,

$$\ddot{\varphi} + 3H\dot{\varphi} + V' = 0 \, . \tag{4.5}$$

At early times, when the Hubble drag term dominates the dynamics, the scalar is nearly frozen in place and contributes a dark energy-like component to the universe. This phase eventually terminates, as the contents of the universe redshift and the Hubble parameter decreases, releasing the field from Hubble drag and triggering the decay of the EDE. This occurs around the time at which $H^2 \sim V''$. The decay of the EDE is necessary to avoid any unintended impact on post-CMB physics. On the other hand, in order to have any sizeable effect on the sound horizon, the decay of the EDE must happen within the decade of redshift preceding last scattering [70]. This fixes the mass of the ultralight scalar to $m \sim 10^{-27}$ eV.

The sound horizon is not the only scale probed the CMB, and likewise the dark energy -like phase of the EDE is not the only aspect of the dynamics that is constrained by data. The dissipation of CMB anisotropies

before the recombination of electrons and protons into hydrogen. For a review of CMB physics and terminology, see [249].

on small angular scales (high multipole moment ℓ), known as 'Silk Damping', provides another characteristic scale - the damping scale r_d. The damping scale constrains the decay of the EDE via its impact on the relative size r_s/r_d. CMB data selects as the best EDE-like model the one that maximizes the decrease in r_s and minimizes the change in r_s/r_d [72].

Putting these puzzle pieces together, one may build a concrete model. A well studied example is given by [72]

$$\begin{aligned} V(\varphi) &= V_0 \left[1 - \cos\left(\frac{\varphi}{f}\right)\right]^3 \\ &= V_0 \left[\frac{5}{2} - \frac{15}{4}\cos\left(\frac{\varphi}{f}\right) + \frac{3}{2}\cos\left(\frac{2\varphi}{f}\right) - \frac{1}{4}\cos\left(\frac{3\varphi}{f}\right)\right] , \end{aligned} \tag{4.6}$$

with $V_0 \equiv m^2 f^2$. The EDE potential (4.6) may be thought of as a generalization of the usual axion potential. The unconventional exponent is selected by data, which can be understood as largely due to the ability of the model to reduce the sound horizon while minimizing the impact on the damping scale, as described above. The exponent determines the shape of the potential near the minimum as locally $V \propto \varphi^6$, such that the energy density redshifts as $a^{-9/2}$ in the decaying phase. This can be contrasted with the conventional axion potential, $V \sim 1 - \cos(\varphi/f)$, which has a quadratic minimum, leading to a dark matter-like evolution in the decaying phase. An ultra-light axion component of dark matter is tightly constrained by data [250, 251] and can not resolve the Hubble tension. This model can also be contrasted with a monomial EDE potential $V = V_0 \left(\varphi/M_P\right)^{2n}$ [78], which, due to the convexity of the potential and the dynamics of perturbations, is strongly disfavoured by CMB data relative to a cosine-type potential [74]. We note that similar generalizations of an axion potential have been studied as an inflation model in [252–254].

The parameter values relevant to the Hubble tension in the EDE model, eq. (4.6), follow from simple considerations. Electrons and protons recombine when the temperature of the primordial plasma drops below $T \sim$ eV, selecting $V_0 \sim$ eV^4 as the benchmark energy scale if the EDE is to play a cosmologically relevant role around that time. The mass of the EDE scalar field should be comparable to the Hubble parameter at that time, $H \sim T^2/M_P$, to trigger the decay of the EDE, which fixes $m \sim 10^{-27}$ eV. From (4.6), these determine the decay constant as $f \sim M_P$. These order of magnitude estimates are born out in the fit to data, which selects out a near- but sub-Planckian decay constant, $f \simeq 0.2\, M_P$, as the preferred value [89].

The EDE model is a promising candidate to replace ΛCDM as the concordance model of cosmology [255]. However, the model faces serious challenges from both theory and data, as discussed in the introduction, that bring this privileged status into question [76]. On the data side, chief among these is the tension of EDE with large scale structure (LSS) data [75, 76, 256–259] (see also [260–263]). As discussed in detail in [75, 76, 256–258] and reviewed in [71], the addition of an EDE-like component necessitates a commensurate increase in the amount of dark matter, to compensate the impact of the EDE on the redshifting of CMB photons, as encoded in the height of the first peak of the CMB temperature anisotropy angular power spectrum. This increased dark matter is in tension with observations of weak gravitational lensing and galaxy clustering [75, 76], such as data from the Dark Energy Survey [264], and from BOSS [265]. The tension with LSS can be ameliorated by adding in additional degrees of freedom, such as in [89, 92–94]. We also note the preference for a non-zero EDE component from the Atacama Cosmology Telescope, see Ref. [266, 267]. These results add to the motivation to study the EDE in a UV complete framework, such as string theory.

4.3 EDE in KKLT

First, let us note that, in the present chapter, we will focus on type IIB compactifications, where the tree-level Kähler potential reads

$$K = K_{\text{Kähler}} + K_{\text{cs}} + K_{\text{dilaton}} \,, \tag{4.7}$$

and one has that the overall factor, both in the gravitino mass and in the F-term potential, factorizes as $e^K = e^{K_{\text{Kähler}}} e^{K_{\text{cs}}} e^{K_{\text{dilaton}}}$. Given that we will focus exclusively on the Kähler moduli sector, we will henceforth omit the $e^{K_{\text{cs}}} e^{K_{\text{dilaton}}}$ factors, though the reader should be aware that they are implicit throughout. In our estimates of the energy scales relevant for phenomenology, we will simply set $e^{K_{\text{cs}}} e^{K_{\text{dilaton}}} = 1$.

As our first string theory realization of EDE, we consider the KKLT scenario. In this case the EDE field has to be a C_2 axion since C_4 axions would be too heavy given that in KKLT the moduli are stabilized by non-perturbative effects, and so C_4 axions are as heavy as the Kähler moduli.

We therefore focus on C_2 axions and build our model following the recipe given in [71] to ensure the correct shape of the EDE potential. The

Kähler potential and superpotential are given by

$$K = -3\ln\left[T+\bar{T}-\gamma(G+\bar{G})^2\right]+3\,\frac{\bar{X}X}{T+\bar{T}}\,, \tag{4.8}$$

$$\begin{aligned} W =& W_0 + MX + A\,e^{-\mathfrak{a}T} \\ &+ A_1\,e^{-\tilde{\mathfrak{a}}(T+k\mathfrak{f}_1\,G)} + A_2\,e^{-\tilde{\mathfrak{a}}(T+k\mathfrak{f}_2\,G)} + A_3\,e^{-\tilde{\mathfrak{a}}(T+k\mathfrak{f}_3\,G)}\,, \end{aligned} \tag{4.9}$$

with

$$\mathfrak{a} = \frac{2\pi}{N} < \tilde{\mathfrak{a}} = \frac{2\pi}{M} \qquad \Leftrightarrow \qquad M < N\,, \tag{4.10}$$

to ensure that the EDE scale, $\sim$ eV, is naturally suppressed with respect to the standard KKLT potential. This exponential suppression removes the EDE fine-tuning to obtain the correct EDE scale previously argued in [71] since the parameters A and A_i ($i=1,2,3$) can take natural $\mathcal{O}(1)$ values in Planck units. Moreover, in order to generate the desired periodicity of the EDE potential, we have to impose

$$\mathfrak{f}_1 = \mathfrak{f}\,, \qquad \mathfrak{f}_2 = 2\mathfrak{f}\,, \qquad \mathfrak{f}_3 = 3\mathfrak{f}\,. \tag{4.11}$$

According to our discussion of non-perturbative effects in type IIB compactifications presented in Sec. 1.5.3, this situation can be reproduced at the microscopic level in two different possible ways:

1. **ED3-instantons on D_+:** the superpotential (4.9) can arise from fluxed ED3-instantons wrapped around D_+ if this divisor does not intersect with a D7-stack supporting the Standard Model. For $\mathfrak{a} = 2\pi n$ and $\tilde{\mathfrak{a}} = 2\pi m$, the condition (4.10) can be met if $m > n$. On the other hand, the condition (4.11) can be satisfied if different terms in the instanton expansion compete among each other. Clearly, the underlying assumption is that all the other terms in the expansion are either absent or suppressed.

2. **Gaugino condensation on D7-branes:** as we have seen, in the presence of gaugino condensation on D7-branes, ED3-instanton contributions are subdominant and can be safely neglected. In this case, the superpotential (4.9) can be reproduced by gaugino condensation on 4 stacks of D7-branes wrapping the same cycle. This can arise, for example, if there are 4 distinct representatives of the same homology class or if different branes of the same stack are differently magnetized. 1 D7-stack is not fluxed and consists of N D7-branes with $\mathfrak{a} = 2\pi/N$. On the other hand, the other 3 stacks have the same number of D7-branes $M < N$, so that $\tilde{\mathfrak{a}} = 2\pi/M$ and the condition (4.10) is met. Moreover, these 3 D7-stacks should carry different world-volume fluxes to satisfy the periodicity condition (4.11).

Note that, as explained in Sec. 1.5.3, each of the 3 fluxed non-perturbative effects in ((4.9) should come along with extra dilaton-dependent exponential suppressions of the form:

$$A_i = \tilde{A}_i\, e^{-\tilde{\mathfrak{a}}\,\mathfrak{f}_i^2 \bar{S}/2} \ll 1 \qquad \forall\, i = 1,2,3\,. \tag{4.12}$$

If present, these exponential suppression factors would require some fine tuning to obtain the correct $[1-\cos(\varphi/f)]^3$ shape of the EDE potential. In fact, the condition (4.11), when inserted in (4.12), implies that the 3 prefactors A_i are not all of the same order if all $\tilde{A}_i$ are $\mathcal{O}(1)$ coefficients, implying the need to tune the $\tilde{A}_i$ appropriately.

We shall however exploit model building to avoid this tuning by noticing that (1.181) and (1.182) allow for a cancellation of the $\bar{S}$-dependent part of the gauge kinetic function if even fluxes are turned on. This is always possible for gaugino condensation on D7-branes and ED3-instantons with rank $m > 1$ [179], which is actually forced to be the case due to the $m > n \geq 1$ condition. When the ED3/D7-stack is wrapping D_+, if $\tilde{k} = -p^2 k$ with $p \in \mathbb{N}$ (where in particular $p \neq 0$), (1.181) reduces to

$$f_{\rm D7} = T + k\mathfrak{f}_-\, G + \frac{k}{2}\left(\mathfrak{f}_-^2 - p^2\mathfrak{f}_+^2\right)\bar{S} = T + k\mathfrak{f}_-\, G\,, \tag{4.13}$$

if the fluxes are chosen such that $\mathfrak{f}_- = \pm p\,\mathfrak{f}_+$. Similar considerations apply to the case when the ED3/D7-stack is wrapping D_1.

We shall therefore focus on the effective field theory defined by (4.8) and (4.9). The resulting scalar potential can be separated into terms that scale with $e^{-\mathfrak{a}\tau}$ and a series of corrections suppressed by powers of $e^{-\tilde{\mathfrak{a}}\tau}$. We expand the scalar potential in small $e^{-\tilde{\mathfrak{a}}\tau}$, to arrive at

$$V = V_{\rm KKLT} + V_{\rm EDE}\,, \tag{4.14}$$

where $V_{\rm KKLT}$ is the standard KKLT potential defined in (1.136) and[5]

$$V_{\rm EDE} = \tilde{V}_0\left[A_1\cos(\tilde{\mathfrak{a}}|k|\mathfrak{f}\,c) + A_2\cos(2\tilde{\mathfrak{a}}|k|\mathfrak{f}\,c) + A_3\cos(3\tilde{\mathfrak{a}}|k|\mathfrak{f}\,c)\right]. \tag{4.15}$$

The potential (4.15) can reproduce the $[1-\cos(\varphi/f)]^3$ EDE potential if $A_1 = 15\,\tilde{A}/4$, $A_2 = -3\,\tilde{A}/2$ and $A_3 = \tilde{A}/4$. The EDE scale is then given by

$$V_0 \equiv -\tilde{A}\,\tilde{V}_0 = \frac{A\,\tilde{A}\,(2\tilde{\mathfrak{a}} - 3\mathfrak{a})\; e^{-(\mathfrak{a}+\tilde{\mathfrak{a}})\tau}}{6\tau^2}\,. \tag{4.16}$$

Let us point out that a $\cos(2\tilde{\mathfrak{a}}|k|\mathfrak{f}\,c)$ term would also arise from the mixed term between the 2 non-perturbative contributions in (4.9) proportional to

[5] Here and it what follows we will not include in $V_{\rm EDE}$ the constant term in (4.6). This contribution can be obtained by an appropriate tuning of the uplifting contribution.

A_1 and A_3, suggesting that the potential (4.15) could also be generated for $A_2 = 0$. However, this is not the case since it can be proven that, under the condition that the EDE field is hierarchically lighter than the Kähler modulus τ, this mixed term has always to be negligible. Hence $A_2 \neq 0$ is indeed needed to generate the $\cos(2\mathfrak{a}|k|\mathfrak{f}\,c)$ term in the EDE potential (4.15) and keep the correct hierarchy of scales.

Notice that we are computing the EDE potential at the Minkowski minimum of KKLT, namely enforcing conditions (1.137), and that, at leading order, we have the stabilization $b = g_s\text{Re}(G) = 0$ (which implies $c = \text{Im}(G)$) and $\theta = \text{Im}(T) = 0$. As discussed in Sec. 1.5.4, when b is stabilized at zero, the C_2 axion is not eaten up by an anomalous $U(1)$.

Furthermore, recalling the canonical normalization for the C_2 axion given in (1.171), we obtain a term in the potential of the form

$$\cos\left[\tilde{\mathfrak{a}}|k|\mathfrak{f}\,c\right] = \cos\left[\tilde{\mathfrak{a}}|k|\mathfrak{f}\sqrt{\frac{\tau}{6\gamma}}\,\varphi\right] \equiv \cos\left[\frac{\varphi}{f}\right]\,, \tag{4.17}$$

giving the following decay constant

$$f \equiv \sqrt{\frac{6\,\gamma}{\tau}}\,\frac{1}{\tilde{\mathfrak{a}}|k|\mathfrak{f}} = \sqrt{\frac{3g_s}{2|k|\,\tau}}\,\frac{1}{\tilde{\mathfrak{a}}\mathfrak{f}}\,. \tag{4.18}$$

Upon switching to the canonically normalized EDE field φ, we may write

$$V_{\text{EDE}} = V_0\left[-\frac{15}{4}\cos\left[\frac{\varphi}{f}\right] + \frac{3}{2}\cos\left[\frac{2\varphi}{f}\right] - \frac{1}{4}\cos\left[\frac{3\varphi}{f}\right]\right]\,, \tag{4.19}$$

where the overall scale V_0 can be expressed in terms of the decay constant f and the gravitino mass $m_{3/2}$ as (reinstating powers of M_P)

$$V_0 = \frac{N\,\tilde{A}}{\sqrt{2}\,\tau^{3/2}}\left(\frac{2}{M} - \frac{3}{N}\right)\left(\frac{m_{3/2}}{M_P}\right)\,e^{-\frac{3}{4\pi}\frac{g_s M}{|k|\mathfrak{f}^2}\left(\frac{M_P}{f}\right)^2}\,M_P^4\,. \tag{4.20}$$

Note that, contrary to general expectations from the weak gravity conjecture, V_0 is exponentially suppressed in terms of $g_s M\,(M_P/f)^2$, instead of just (M_P/f). For $M \gg 1$ and $f < M_P$, this helps to suppress the EDE scale and to reduce the required fine tuning on $\tilde{A}$, even if the presence of the small factor $g_s \ll 1$ does not allow to remove the tuning completely. Setting $f = 0.2\,M_P$ and $\mathfrak{f} = |k| = 1$, the EDE scale V_0 scales as

$$V_0 \simeq \tilde{A}\left(\frac{m_{3/2}}{M_P}\right)\,e^{-\frac{75}{4\pi}\,g_s M}\,M_P^4\,. \tag{4.21}$$

This relation depends on the gravitino mass. In KKLT models this is related to the mass of the Kähler modulus $m_\tau \simeq m_{3/2}\ln\left(M_P/m_{3/2}\right)$ which has to be above $\mathcal{O}(50)$ TeV in order to avoid any cosmological moduli problem. This implies $m_{3/2} \gtrsim \mathcal{O}(1)$ TeV. Imposing therefore a gravitino mass at the TeV-scale to maximize the suppression in (4.21) (larger values of $m_{3/2}$ would also require a larger value of N), $V_0 \simeq 10^{-108} M_P^4$ gives the value of M for a given string coupling. In turn, (4.18) yields the value of τ at the minimum that, when substituted in (1.108), sets the value of N for natural $\mathcal{O}(1)$ values of A.

g_s	$\tilde{A}$	M	N	τ	$V_0\, 10^{108}\, M_P^{-4}$	$m_{3/2}$ (TeV)	m_τ (TeV)
0.1	1	340	3200	10980.7	1.4	4.6	155.5
0.3	1	114	1000	3703.4	2.0	4.6	155.2
0.3	10^{-11}	100	750	2849.7	1.6	3.8	129.6
0.3	10^{-27}	80	470	1823.8	0.9	4.6	154.8
0.3	10^{-61}	36	85	369.3	2.2	3.0	104.0

Table 4.1. Benchmark parameters that realize the EDE potential (4.6) with $f = 0.2\, M_P$, $|A| = 1$ and $m_\tau = m_{3/2} \ln\left(M_P/m_{3/2}\right)$. Recall that $\mathfrak{a} = 2\pi/N$, $\tilde{\mathfrak{a}} = 2\pi/M$, with $M < N$.

We present in Tab. 4.1 some selected numerical examples for the present model for various choices of g_s, $\tilde{A}$, M and N, all of which give rise to the correct EDE scale, decay constant and a gravitino mass at the TeV-scale. If the string coupling is kept in the regime where perturbation theory does not break down, i.e. $g_s \lesssim 0.3$, natural $\mathcal{O}(1)$ values of $\tilde{A}$ correlate with $N \sim \mathcal{O}(1000) \gg M \sim \mathcal{O}(100) \gg 1$ and larger values of τ. Such large values of the ranks of the condensing gauge groups are very likely to be incompatible with D7 tadpole cancellation and to induce an uncontrolled backreaction on the internal geometry (see [155] for a study of the maximal rank of condensing gauge groups as a function of $h^{1,1}$ for F-theory compactifications). On the other hand, tuned values of the overall prefactor of the EDE potential of order $\tilde{A} \sim \mathcal{O}(10^{-50})$, can allow for viable models with acceptably smaller numbers of D7-branes, $N \sim \mathcal{O}(100) \gg M \sim \mathcal{O}(10) \gg 1$ and smaller values of τ. Note that, at fixed g_s, larger values of $m_{3/2}$, as can be seen from (4.21), would require larger values of M, and so from (4.18) larger values of τ, which imply even larger values of N, as can be seen from (1.108). Hence, cases with $m_{3/2}$ considerably above the TeV-scale are highly disfavoured. Let also point out that in general ED3-instantons would not give the required EDE decay constant since viable models with $\tau \gg 1$ require large values of M.

Summarizing, our analysis shows that EDE can be realized in KKLT with a C_2 axion whose potential is generated by gaugino condensates on D7-branes with world-volume fluxes. When the string coupling is small enough to trust the string loop expansion and $m_{3/2} \gtrsim \mathcal{O}(1)$ TeV, matching the EDE scale seems to require a substantial tuning of the prefactors of these non-perturbative effects. If instead $\tilde{A}$ takes natural $\mathcal{O}(1)$ numbers, then the number of D7-branes becomes too large to be compatible with a controlled effective field theory. The best scenarios correlate with a TeV-scale gravitino mass.

4.4 EDE in the Large Volume Scenario

We will now turn our attention to another class of models, built using the LVS. We will analyse the possibility of building the EDE potential from C_2 axions, as in the previous KKLT construction, and the new option of using C_4 axions. This last option is possible in LVS but not in KKLT. In fact, in LVS models the big cycle τ_b is much heavier than the corresponding axion θ_b since τ_b is stabilized by perturbative α' effects which do not lift θ_b.

4.4.1 EDE from C_4 axions

Consider a 'Swiss cheese' manifold with a large 4-cycle with size τ_b and a small 4-cycle with size τ_s. The low energy supergravity action is defined by the following Kähler potential and superpotential:

$$K = -2\ln\left[\tau_b^{3/2} - \tau_s^{3/2} + \frac{\hat{\xi}}{2}\right] + \frac{\bar{X}X}{\mathcal{V}^{2/3}}\,, \tag{4.22}$$

$$W = W_0 + MX + A_s\, e^{-\mathfrak{a}_s T_s} + A_1\, e^{-\mathfrak{a}_1 T_b} + A_2\, e^{-\mathfrak{a}_2 T_b} + A_3\, e^{-\mathfrak{a}_3 T_b}\,. \tag{4.23}$$

As explained in Sec. 1.5.3, the non-perturbative corrections to W could arise from either gaugino condensation on D7-branes or ED3-instantons, where in this case we are focusing on situations without orientifold-odd moduli and vanishing world-volume fluxes (more precisely, in the case of ED3-instantons, flux-dependent contributions would be exponentially suppressed in the dilaton with respect to the leading fluxless term). Moreover, in order to engineer the desired EDE periodic potential, we proceed similarly to (4.11) and require

$$\mathfrak{a}_1 = \mathfrak{a}_b \qquad \mathfrak{a}_2 = 2\mathfrak{a}_b\,, \qquad \mathfrak{a}_3 = 3\mathfrak{a}_b\,. \tag{4.24}$$

The scalar potential turns out to be

$$V = V_{\mathsf{LVS}} + V_{\mathsf{EDE}}\,, \tag{4.25}$$

with V_{LVS} as in (1.140) and the EDE part given by

$$V_{\mathsf{EDE}} = V_0\left[\tilde{A}_1\,\cos[\mathfrak{a}_b\theta_b] + \tilde{A}_2\, e^{-\mathfrak{a}_b\tau_b}\cos[2]\,\mathfrak{a}_b\theta_b\right] + \tilde{A}_3\, e^{-2\mathfrak{a}_b\tau_b}\cos[3\mathfrak{a}_b\theta_b]\,, \tag{4.26}$$

where the EDE scale reads (for $\tau_b \simeq \mathcal{V}^{2/3}$)

$$V_0 = \frac{4\mathfrak{a}_b\,|W_0|}{\mathcal{V}^{4/3}}\,A_b\,e^{-\mathfrak{a}_b\tau_b}\,, \tag{4.27}$$

and we have redefined $A_i \equiv A_b\tilde{A}_i$ $(i = 1, 2, 3)$ to factor out in V_0 and overall coefficient A_b. This example displays a crucial difference with respect to the KKLT C_2 example of Sec. 4.3: the moduli dependence, namely on τ_b, cannot be included completely into the overall normalization V_0. Instead, the 3 periodic terms appear in $V_{\rm EDE}$ with different powers of $e^{-\mathfrak{a}_b\tau_b}$, which must be compensated with an exponential hierarchy in $\tilde{A}_i$ $(i = 1, 2, 3)$ if one is to recover (4.6) for the EDE potential.[6] This situation is also different from the model of [71], where each harmonic has the same modulus-dependent suppression, which can in fact be reabsorbed into V_0.

This model requires also an exponential tuning of the overall prefactor A_b in order to match the EDE scale since C_4 axions do not give rise to any violation of the weak gravity conjecture (see Introduction). To see this more in detail, let us compute the EDE decay constant. The kinetic terms for the EDE field look like

$$_{\rm kin} = K_{T_b\bar{T}_b}\partial_\mu T_b\partial^\mu\bar{T}_b \supset K_{T_b\bar{T}_b}\partial_\mu\theta_b\partial^\mu\theta_b = \frac{3}{4\tau_b^2}\left[\partial\theta_b\right]^2\,, \tag{4.28}$$

and thus we canonically normalize the field as

$$\varphi = \sqrt{\frac{3}{2}}\frac{\theta_b}{\tau_b}\,. \tag{4.29}$$

We then obtain in the potential the term

$$\cos\left[\mathfrak{a}_b\theta_b\right] = \cos\left[\sqrt{\frac{2}{3}}\mathfrak{a}_b\tau_b\,\varphi\right] \equiv \cos\left[\frac{\varphi}{f}\right]\,, \tag{4.30}$$

from which we can easily see that the decay constant of the EDE field φ is

$$f = \sqrt{\frac{3}{2}}\frac{1}{\mathfrak{a}_b\tau_b} \simeq 0.2\,\frac{N_b}{\tau_b} \qquad \text{for} \qquad \mathfrak{a}_b = \frac{2\pi}{N_b}\,. \tag{4.31}$$

Setting $f \simeq 0.2\,M_P$, (4.31) clearly implies $\tau_b \simeq N_b$. Given that the α' expansion is controlled by $\mathcal{V}^{-1/3} = \tau_b^{-1/2} \ll 1$, the big modulus should be at least $\tau_b \gtrsim \mathcal{O}(100)$ which requires a large number of D7-branes $N_b \gtrsim \mathcal{O}(100)$.

[6] The different scaling of each term with τ_b would also backreact on the vacuum expectation value of τ_b, even if this effect is tiny since $V_{\rm EDE}$ is hierarchically smaller than $V_{\rm LVS}$.

Moreover, the EDE scale (4.27) can be rewritten down as (reinstating appropriate powers of M_P)

$$V_0 = \frac{64\pi^3}{3}\left(\frac{|W_0|\,A_b}{N_b^3}\right)\left(\frac{f}{M_P}\right)^2 M_P^4\, e^{-\sqrt{\frac{3}{2}}\frac{M_P}{f}}\,, \tag{4.32}$$

which for $f \simeq 0.2\,M_P$ reduces to

$$V_0 \simeq 0.06\left(\frac{|W_0|\,A_b}{N_b^3}\right) M_P^4\,. \tag{4.33}$$

From this expression it is clear that $V_0 \sim 10^{-108}\,M_P^4$ can be achieved only by fine-tuning A_b to exponentially small values since the flux superpotential $|W_0|$ cannot be taken too small, otherwise the volume modulus would become lighter than $\mathcal{O}(50)$ TeV. In fact, the volume modulus mass scales as

$$m_{\mathcal{V}} \simeq \frac{|W_0|\,M_P}{\tau_b^{9/4}} \gtrsim 50\,\text{TeV} \qquad \Leftrightarrow \qquad |W_0| \gtrsim 2\times 10^{-14}\,\tau_b^{9/4}\,. \tag{4.34}$$

For $\tau_b \gtrsim 100$, this gives also a lower bound on the gravitino mass of order:

$$m_{3/2} = \frac{|W_0|}{\tau_b^{3/2}}\,M_P \gtrsim 2\times 10^{-14}\,\tau_b^{3/4}\,M_P \gtrsim 1.5\times 10^6\,\text{GeV}\,. \tag{4.35}$$

In Tab. 4.2 we show two benchmark examples for $N_b = 100$ and $N_b = 1000$, which give $f \simeq 0.2\,M_P$ and the right EDE scale for the smallest possible value of $|W_0|$. At fixed N_b and τ_b, larger values of $|W_0|$ would give larger moduli masses and would require smaller A_b and larger N_s, as can be seen from (1.120).

N_b	N_s	τ_b	$\lvert W_0\rvert$	A_b	A_s	$V_0\,10^{108}\,M_P^{-4}$
100	3	97.5	6.0×10^{-10}	5×10^{-92}	0.29	1.8
1000	4	974.6	1.1×10^{-7}	2×10^{-91}	0.28	1.3

Table 4.2. Benchmark parameters for LVS EDE with C_4 axions. Each parameter set gives $f = 0.2\,M_P$ and $m_{\mathcal{V}} \simeq 50$ TeV, and features $\mathfrak{a}_s = 2\pi/N_s$, $\tau_s = 10$ for $g_s = 0.1$ and $\xi = 2$.

The need to perform a double fine-tuning on the 3 prefactors of the T_b-dependent non-perturbative effects, to get both the right EDE scale and periodicity, suggests that the C_4 axion associated with the volume modulus in LVS is not an optimal candidate for building an EDE model in string theory.

This conclusion continues to hold when we add more complication to the geometry, e.g. by considering a Calabi-Yau manifold with a fibred structure, as we will do in the following. Consider a manifold with volume given by

$$\mathcal{V} = \sqrt{\tau_1}\tau_2 - \tau_s^{3/2}\,, \tag{4.36}$$

and focus on the case $\tau_s \simeq \tau_1 \ll \tau_2$ such that the volume is predominantly set by τ_2. The scalar potential derived from

$$K = -2\ln\left[\mathcal{V} + \frac{\hat{\xi}}{2}\right] + \frac{\bar{X}X}{\mathcal{V}^{2/3}} \tag{4.37}$$

$$W = W_0 + MX + A_s\, e^{-\mathfrak{a}_s T_s} + A_1\, e^{-\mathfrak{a}_1 T_1} + A_2\, e^{-2\mathfrak{a}_1 T_1} + A_3\, e^{-3\mathfrak{a}_1 T_1}\,, \tag{4.38}$$

would take again the form $V = V_{\text{LVS}} + V_{\text{EDE}}$ with

$$V_{\text{EDE}} = V_0\left[A_1\,\cos\left[\mathfrak{a}_1\theta_1\right] + A_2\, e^{-\mathfrak{a}_1\tau_1}\cos\left[2\right]\mathfrak{a}_1\theta_1\right] + A_3\, e^{-2\mathfrak{a}_1\tau_1}\cos\left[3\mathfrak{a}_1\theta_1\right]\,, \tag{4.39}$$

where the EDE scale scales as (for $A_i = \mathcal{A}_1\tilde{A}_i$)

$$V_0 = \frac{4\mathfrak{a}_1\tau_1|W_0|}{\mathcal{V}^2}\,\mathcal{A}_1\, e^{-\mathfrak{a}_1\tau_1}\,. \tag{4.40}$$

Just like in the previous case, this scenario exhibits an explicit dependence of V_{EDE} on 4-cycle moduli, in this case τ_1, requiring an exponential hierarchy between $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{A}_3$.

However, the fibred model provides one advantage, in the form of increased flexibility in setting the EDE decay constant. Following the same procedure as before, we compute the kinetic terms at leading order in $1/\mathcal{V}$

$$_{\text{kin}} = K_{T_1\bar{T}_1}\partial_\mu T_1\partial^\mu \bar{T}_1 \supset K_{T_1\bar{T}_1}\partial_\mu\theta_1\partial^\mu\theta_1 = \frac{1}{4\tau_1^2}\left[\partial\theta_1\right]^2\,. \tag{4.41}$$

Thus, canonically normalizing as

$$\varphi = \frac{1}{\sqrt{2}\tau_1}\theta_1\,, \tag{4.42}$$

we have that the potential will contain terms like

$$\cos\left[\mathfrak{a}_1\theta_1\right] = \cos\left[\sqrt{2}\mathfrak{a}_1\tau_1\varphi\right] \equiv \cos\left[\frac{\varphi}{f}\right]\,, \tag{4.43}$$

finding

$$f = \frac{1}{\sqrt{2}\mathfrak{a}_1\tau_1} \simeq 0.1\,\frac{N_1}{\tau_1} \qquad \text{for} \qquad \mathfrak{a}_1 = \frac{2\pi}{N_1}\,, \tag{4.44}$$

which depends only on τ_1 and not τ_2. Given that in anisotropic compactifications with $\tau_2 \gg \tau_1$, the overall internal volume is controlled mainly by τ_2, an α' expansion under control can be compatible with $\tau_1 \sim \mathcal{O}(10)$

which removes, in turn, the need to go to a large number of D7-branes N_1 to reproduce $f \simeq 0.2\,M_P$. Clearly, in this case, more natural values $N_1 \simeq \mathcal{O}(10)$ can be allowed. However, the system still needs a very large tuning of the prefactor $\mathcal{A}_1$ in (4.40) since the EDE scale can be rewritten as (showing explicit powers of M_P)

$$V_0 = \frac{4}{\sqrt{2}|W_0|^{1/3}}\,\mathcal{A}_1 \left(\frac{m_{\mathcal{V}}}{M_P}\right)^{4/3} \left(\frac{M_P}{f}\right) M_P^4\, e^{-\frac{1}{\sqrt{2}}\frac{M_P}{f}}\,, \tag{4.45}$$

which, setting $f \simeq 0.2\,M_P$ and taking in (4.34) $m_{\mathcal{V}} \simeq 50$ TeV, reduces to

$$V_0 = 2.4 \times 10^{-19}\,\frac{\mathcal{A}_1}{|W_0|^{1/3}}\,M_P^4\,. \tag{4.46}$$

For $|W_0| \sim \mathcal{O}(1)$, clearly $V_0 \sim 10^{-108}\,M_P^4$ requires to tune $\mathcal{A}_1$ down to $\mathcal{A}_1 \sim \mathcal{O}(10^{-90})$.

The lesson to learn from these attempts of building EDE potentials by means of non-perturbative effects in W is twofold: (i) models, where the real part of the chiral superfield used for EDE is stabilized at zero, require less tuning of the underlying parameters to reproduce the correct periodicity of the EDE potential; (ii) matching the EDE scale without fine-tuning any prefactor of the non-perturbative effects, which generate the EDE potential, requires a violation of the weak gravity conjecture. This singles out C_2 axions since they can violate the weak gravity conjecture and belong to the chiral superfield $G = \bar{S}b + \mathrm{i}\,c$ where the B_2 axion is fixed at $b = 0$. With this in mind let us explore C_2 models in the framework of LVS.

4.4.2 EDE from C_2 axions

We now return to the C_2 axion case, studied previously in the context of KKLT in Sec. 4.3. We consider two possibilities for generating the EDE potential, namely gaugino condensation on D7-branes (or fluxed ED3-instantons) and gaugino condensation on D5-branes (or fluxed ED1-instantons).

Gaugino condensation on D7-branes

We focus on a situation with $h^{1,1}_+ = 2$ and $h^{1,1}_- = 1$ where the orientifold even moduli describe a typical Swiss-cheese Calabi-Yau, while the orientifold odd modulus mixes just with the big modulus. The volume form therefore looks like

$$\mathcal{V} = \tau_b^{3/2} - \tau_s^{3/2} = \frac{1}{2\sqrt{2}}\left[\left(T_b + \bar{T}_b - \gamma(G + \bar{G})^2\right)^{3/2} - \left(T_s + \bar{T}_s\right)^{3/2}\right]\,. \tag{4.47}$$

The low-energy effective action is determined by the following Kähler potential and superpotential

$$K = -2\ln\left[\mathcal{V} + \frac{\hat{\xi}}{2}\right] + \frac{\bar{X}X}{\mathcal{V}^{2/3}}\,, \tag{4.48}$$

$$W = W_{\mathsf{LVS}} + A_1\, e^{-\tilde{\mathfrak{a}}(T_b + k\mathfrak{f}_1 G)} + A_2\, e^{-\tilde{\mathfrak{a}}(T_b + k\mathfrak{f}_2 G)} + A_3\, e^{-\tilde{\mathfrak{a}}(T_b + k\mathfrak{f}_3 G)}\,, \tag{4.49}$$

with

$$W_{\mathsf{LVS}} = W_0 + MX + A_s\, e^{-\mathfrak{a}_s T_s} + A_b\, e^{-\mathfrak{a}_b T_b}\,. \tag{4.50}$$

Similarly to (4.11), we also impose

$$\mathfrak{f}_1 = \mathfrak{f}\,, \qquad \mathfrak{f}_2 = 2\mathfrak{f}\,, \qquad \mathfrak{f}_3 = 3\mathfrak{f}\,, \tag{4.51}$$

in order to match the periodicity of the EDE potential. W_{LVS} is the standard LVS superpotential with the inclusion of the T_b-dependent non-perturbative effect which stabilizes the C_4 axion θ_b. The last 3 terms in (4.49) are instead responsible for the generation of the EDE potential. As explained in Sec. 4.3, these can arise from branes wrapping the big divisor which can be either fluxed ED3-instantons or D7-branes with non-zero world-volume fluxes which support gaugino condensation. Similarly to Sec. 4.3, we will see that ED3-instantons cannot reproduce the correct EDE decay constant for $\mathcal{V} \gg 1$, as well as the correct EDE scale without fine-tuning the prefactors A_i $(i = 1, 2, 3)$ to exponentially small values. Moreover, when both effects are present, ED3-instantons are always subdominant with respect to gaugino condensation. In the following, we shall therefore focus mainly on gaugino condensation on D7-branes, keeping in mind however that the EDE potential could also be realized by ED3-instantons (at the price of introducing fine-tuning and working at small internal volume) if gaugino condensation effects are not generated.

As explained in Sec. 4.3, the last 3 non-perturbative effects in (4.49) receive also $\bar{S}$-dependent contributions in the exponents, which might destroy the required periodicity of the EDE potential if the corresponding prefactors take natural $\mathcal{O}(1)$ numbers. However, we have seen that these dilaton-dependent contributions can be cancelled by an appropriate choice of even and odd fluxes. In the LVS case, there is another intriguing possibility if initially the number of odd moduli is $h^{1,1}_- = 2$. Note that this is not possible in KKLT since in cases where the orientifold involution exchanges two non-identical divisors [204, 206], $0 \leq h^{1,1}_- \leq r$ for $h^{1,1} = 2r$ or $h^{1,1} = 2r+1$ with $r \in \mathbb{N}$, implying that $h^{1,1}_+ = 1$ is incompatible with $h^{1,1}_- = 2$. On the other hand, LVS models with $h^{1,1}_+ = 2$ can feature $h^{1,1}_- = 2$.

In this case, if the initial divisors are D_i ($i = 1, 2, 3, 4$) and the involution exchanges $D_1 \leftrightarrow D_2$ and $D_3 \leftrightarrow D_4$, the even and odd divisors are

$$D_+^{(b)} = D_1 \cup D_2\,, \qquad D_-^{(\tilde{b})} = D_1 \cup (-D_2)\,, \tag{4.52}$$
$$D_+^{(s)} = D_3 \cup D_4\,, \qquad D_-^{(\tilde{s})} = D_3 \cup (-D_4)\,, \tag{4.53}$$

where we have assumed to have a big and a small orientifold-even modulus. For 2 orientifold-odd moduli the expression (1.182) for the gauge kinetic function of a D7-stack wrapping D_1 generalizes to [177] (focusing for simplicity just on the case with odd world-volume fluxes)

$$\begin{aligned} f_{\text{D7}} &= T_b + \left(k_{b\tilde{b}\tilde{b}}\mathfrak{f}_{\tilde{b}} + k_{b\tilde{b}\tilde{s}}\mathfrak{f}_{\tilde{s}}\right) G_{\tilde{b}} + \left(k_{b\tilde{b}\tilde{s}}\mathfrak{f}_{\tilde{b}} + k_{b\tilde{s}\tilde{s}}\mathfrak{f}_{\tilde{s}}\right) G_{\tilde{s}} \\ &+ \frac{1}{2}\left[\left(k_{b\tilde{b}\tilde{b}}\mathfrak{f}_{\tilde{b}} + k_{b\tilde{b}\tilde{s}}\mathfrak{f}_{\tilde{s}}\right)\mathfrak{f}_{\tilde{b}} + \left(k_{b\tilde{b}\tilde{s}}\mathfrak{f}_{\tilde{b}} + k_{b\tilde{s}\tilde{s}}\mathfrak{f}_{\tilde{s}}\right)\mathfrak{f}_{\tilde{s}}\right]\bar{S}\,, \end{aligned} \tag{4.54}$$

where the indices with a tilde denote odd-moduli. If $k_{b\tilde{s}\tilde{s}} = 0$ and the flux quantum $\mathfrak{f}_{\tilde{b}}$ is set to zero, this expression simplifies to

$$f_{\text{D7}} = T_b + k_{b\tilde{b}\tilde{s}}\mathfrak{f}_{\tilde{s}}\, G_{\tilde{b}}\,, \tag{4.55}$$

implying that the superpotential would not depend on $G_{\tilde{s}}$. The B_2-axion $b_{\tilde{s}}$ would however appear in the FI-term since (1.192) would generalize to

$$\xi_{\text{FI}} \sim \frac{t_b}{\mathcal{V}}\left[k_{b\tilde{b}\tilde{b}} b_{\tilde{b}} + k_{b\tilde{b}\tilde{s}}\left(b_{\tilde{s}} - \mathfrak{f}_{\tilde{s}}\right)\right]. \tag{4.56}$$

If $k_{b\tilde{b}\tilde{b}} = 0$, the FI-term would simply depend on $b_{\tilde{s}}$ and, as explained in Sec. 1.5.4, D-term stabilization would fix $b_{\tilde{s}} = \mathfrak{f}_{\tilde{s}}$ if the charged matter fields do not acquire tachyonic masses from supersymmetry breaking. This implies that the $G_{\tilde{s}}$ axion is eaten up and disappears from the effective field theory. The B_2 axion $b_{\tilde{b}}$ could instead be fixed at zero by subleading effects.

The total scalar potential can be written as

$$V = V_{\text{LVS}}(\mathcal{V}, \tau_s, \theta_s) + V_b(\theta_b, b) + V_{\text{EDE}}(c)\,, \tag{4.57}$$

where in each term we have written down explicitly just the dependence on the moduli which get frozen by each type of contribution. V_{LVS} is the uplifted LVS potential (1.140) which stabilizes $\mathcal{V}$, τ_s and the axion θ_s, V_b is the contribution included in (1.202) which fixes θ_b and $b = 0$, while the EDE potential reads

$$V_{\text{EDE}} = V_0\left\{-\frac{15}{4}\cos\left[\tilde{\mathfrak{a}}\left(k\mathfrak{f}c + \theta_b\right)\right] + \frac{3}{2}\cos\left[\tilde{\mathfrak{a}}\left(2k\mathfrak{f}c + \theta_b\right)\right] - \frac{1}{4}\cos\left[\tilde{\mathfrak{a}}\left(3k\mathfrak{f}c + \theta_b\right)\right]\right\}, \tag{4.58}$$

with

$$V_0 = \frac{4|W_0|\tilde{\mathfrak{a}}}{\mathcal{V}^{4/3}}\,\tilde{A}\,e^{-\tilde{\mathfrak{a}}\tau_b}. \tag{4.59}$$

where we have set $A_1 = -15\tilde{A}/4$, $A_2 = 3\tilde{A}/2$ and $A_3 = -\tilde{A}/4$. In order to obtain the correct EDE potential we require $\mathfrak{a}_b < \tilde{\mathfrak{a}}$, so that $V_{\text{EDE}} \ll V_b$ and the stabilization of θ_b is completely determined by V_b for $b = 0$. The overall sign of (1.114) plays an important role in this EDE realization: if $A_b > 0$ the bulk C_4 axion is stabilized at $\theta_b = \pi/\mathfrak{a}_b$, whereas if $A_b < 0$ the minimum is at $\theta_b = 0$. Given the θ_b dependence of (4.58) it is evident that only $A_b < 0$ can lead to the desired EDE potential[7]

$$V_{\text{EDE}} = V_0\left[-\frac{15}{4}\cos(\tilde{\mathfrak{a}}|k|\mathfrak{f}\,c) + \frac{3}{2}\cos(2\tilde{\mathfrak{a}}|k|\mathfrak{f}\,c) - \frac{1}{4}\cos(3\tilde{\mathfrak{a}}|k|\mathfrak{f}\,c)\right]. \tag{4.60}$$

Let us now determine the EDE decay constant. Using the canonically normalized field defined in (1.171), the cosine terms in the EDE potential behave as

$$\cos\left[\tilde{\mathfrak{a}}|k|\mathfrak{f}\,c\right] = \cos\left[\tilde{\mathfrak{a}}|k|\mathfrak{f}\sqrt{\frac{\tau_b}{6\gamma}}\,\varphi\right] \equiv \cos\left[\frac{\varphi}{f}\right], \tag{4.61}$$

finding

$$f = \frac{1}{\tilde{\mathfrak{a}}|k|\mathfrak{f}}\sqrt{\frac{6\gamma}{\tau_b}} = \sqrt{\frac{3g_s}{8\pi^2|k|\mathfrak{f}^2}}\,\frac{M}{\sqrt{\tau_b}} \qquad \text{for} \qquad \tilde{\mathfrak{a}} = \frac{2\pi}{M}. \tag{4.62}$$

The overall EDE scale (4.59) therefore becomes (reinstating powers of M_P)

$$V_0 = \frac{16(2\pi)^5|k|^2\mathfrak{f}^4}{9g_s^2M^5}|W_0|\,\tilde{A}\left(\frac{f}{M_P}\right)^4 e^{-\frac{3}{4\pi}\frac{g_sM}{|k|\mathfrak{f}^2}\left(\frac{M_P}{f}\right)^2} M_P^4\,. \tag{4.63}$$

Note that the exponential suppression is the same as in (4.20) since we are again using C_2 axions whose potential is generated by gaugino condensation on D7-branes with non-zero fluxes. Contrary to the KKLT EDE case discussed in Sec. 4.3, however this case can realize EDE without the need to go to an excessively large number of D7-branes. The main reason is that in LVS, as can be seen from the minimization relations (3.53), large τ_b does not require a very large number of D7-branes to avoid ultralight moduli that would induce cosmological problems. Let us see this crucial point more in detail. For $f = 0.2\,M_P$ and $\mathfrak{f} = |k| = 1$, the EDE scale V_0 reduces to

$$V_0 \simeq \frac{27.85}{g_s^2M^5}\,|W_0|\,\tilde{A}\,e^{-\frac{75}{4\pi}g_sM}\,M_P^4\,, \tag{4.64}$$

where the lowest possible value of $|W_0|$ that maximizes the suppression is given by (4.34) in terms of τ_b which is fixed by (4.31) for a given M and g_s.

[7] More in general, if $\theta_b = \pi/\mathfrak{a}_b$, the correct EDE potential could still be obtained if $\tilde{\mathfrak{a}} = p\mathfrak{a}_b$ with $p \in \mathbb{N}$.

g_s	$\tilde{A}$	N_s	M	$\lvert W_0\rvert$	$\mathcal{V}=\tau_b^{3/2}$	τ_s	$V_0\, 10^{108}\, M_P^{-4}$	A_s	ξ	$m_{\mathcal{V}}$ (TeV)
0.3	1	1	128	1	3.2×10^5	2.37	2.6	1.70	1.2	3.2×10^7
0.3	1	1/2	121	2.8×10^{-6}	2.7×10^5	2.24	2.7	1.51	1.1	50
0.1	1	2	362	3.3×10^{-5}	1.4×10^6	8.25	2.2	2.96	1.5	50

Table 4.3. Benchmark parameters that realize the EDE potential (4.60) with $f=0.2\,M_P$. We have used the LVS minimization relations (1.120) with $\mathfrak{a}_s=2\pi/N_s$ and $\tilde{\mathfrak{a}}=2\pi/M$. The case with $N_s=1/2$ corresponds to a rank-2 ED3-instanton [179].

We present in Tab. 4.3 three numerical examples with a different value of g_s which reproduce the correct EDE scale without the need to tune the prefactor $\tilde{A}$. Contrary to the KKLT case discussed in Sec. 4.3, there is no need to have $\mathcal{O}(1000)$ D7-branes. If $\tilde{A}$ is kept of order unity, the number of D7-branes M has to be $M\sim\mathcal{O}(100)$ which is however realizable in F-theory compactifications [155]. Smaller values of M would require an exponentially small $\tilde{A}$ and would also reduce the value of τ_b due to the need to reproduce $f\simeq 0.2\,M_P$ from (4.62). The first case in Tab. 4.3 is the most generic since the flux superpotential takes the natural value $|W_0|=1$ which correlates with $m_{\mathcal{V}}\sim\mathcal{O}(10^{10})$ GeV and $m_{3/2}\sim\mathcal{O}(10^{13})$ GeV. On the other hand, the last two cases in Tab. 4.3 are characterized by lower moduli masses, $m_{\mathcal{V}}\simeq 50$ TeV and $m_{3/2}\simeq 10^6$ GeV, due to the tuning of $|W_0|$ to small values. Let us stress that in all cases the CY volume is large enough to trust the effective field theory.

To complete our analysis, let us consider also K3-fibred LVS compactifications with volume $\mathcal{V}=\sqrt{\tau_1}\tau_2-\tau_s^{3/2}$ since, as we have already seen in Sec. 4.4.1, they give more freedom in matching the EDE energy scale and decay constant if $\mathcal{V}$ is anisotropic with $\tau_2\gg\tau_1\gg 1$. If the G-modulus mixes only with T_1, the superpotential would still be given by (4.49) but with the substitution $T_b\to T_1$. The stabilization of θ_1 would proceed as the stabilization of θ_b above, while θ_2 would in practice remain as a massless spectator field. The EDE decay constant would still be given by (4.62) but again with the substitution $\tau_b\to\tau_1$. The EDE potential would take the same form as in (4.58) but the EDE scale would become (for $f=0.2\,M_P$ and $\mathfrak{f}=|k|=1$)

$$V_0\simeq 23.9\,\frac{|W_0|}{\mathcal{V}^2}\,g_s\,M\,\tilde{A}\,e^{-\frac{75}{4\pi}\,g_s M}\,M_P^4\,. \tag{4.65}$$

The difference with the previous case is that in fibred CY models, the lightest modulus is the direction u orthogonal to the volume mode which is stabilized beyond leading LVS order. Imposing that its mass is above the bound from the cosmological moduli problem, we find [55]

$$m_u \simeq \frac{|W_0|}{\mathcal{V}^{3/2}\tau_1^{1/4}} M_P \gtrsim 50\,\text{TeV} \qquad \Leftrightarrow \qquad \mathcal{V} \lesssim 1.3 \times 10^9\, |W_0|^{2/3}\, \tau_1^{-1/6}\,. \quad (4.66)$$

When τ_1 is fixed around $\tau_1 \sim \mathcal{O}(10^4)$ by the requirement to obtain $f \simeq 0.2\, M_P$, this condition can clearly be compatible with $|W_0| \sim \mathcal{O}(1)$ since it would just require $\mathcal{V} \lesssim \mathcal{O}(10^8)$. Setting $\mathcal{V} \sim \mathcal{O}(10^8)$ would indeed correspond to an anisotropic extra-dimensional volume with $\tau_2 \sim \mathcal{O}(10^6) \gg \tau_1 \sim \mathcal{O}(10^4) \gg 1$. We present in Tab. 4.4 two numerical examples with a different value of g_s which reproduce the correct EDE scale without the need to tune the prefactors $\tilde{A}$ and $|W_0|$. Both examples feature $m_u \simeq 50$ TeV, $m_\mathcal{V} \sim \mathcal{O}(5 \times 10^5)$ GeV and $m_{3/2} \sim \mathcal{O}(10^{10})$ GeV.

g_s	$\tilde{A}$	N_s	M	$\lvert W_0\rvert$	τ_1	$\mathcal{V}$	τ_s	$V_0\, 10^{108}\, M_P^{-4}$	A_s	ξ
0.1	1	3	362	1	1.24×10^4	2.74×10^8	9.32	1.7	1.2	1.8
0.3	2	1	121	1	4.17×10^3	3.29×10^8	3.33	1.3	0.83	2

Table 4.4. Benchmark parameters that realize the EDE potential (4.60) with $f = 0.2\, M_P$ and $m_u \simeq 50$ TeV for K3 fibred CY models. We have used the LVS minimization relations (1.120) with $\mathfrak{a}_s = 2\pi/N_s$ and $\tilde{\mathfrak{a}} = 2\pi/M$.

Summarizing, our analysis shows that EDE can be realized in LVS with a C_2 axion whose potential is generated by gaugino condensates on D7-branes with non-vanishing world-volume fluxes. When the string coupling is small enough to trust the string loop expansion, matching the EDE scale requires $\mathcal{O}(100)$ D7-branes, if $\tilde{A}$ takes natural $\mathcal{O}(1)$ numbers. Swiss-cheese models with natural $\mathcal{O}(1)$ values of $|W_0|$ are characterized by $\mathcal{V} \sim \mathcal{O}(10^5)$ and $m_{3/2} \sim \mathcal{O}(10^{13})$ GeV, which can be lowered down at most to $m_{3/2} \sim \mathcal{O}(10^6)$ GeV by tuning $|W_0|$ (otherwise the volume modulus would cause cosmological problems). On the other hand, K3-fibred CY examples can realize EDE for larger values of the internal volume, $\mathcal{V} \sim \mathcal{O}(10^8)$, improving the control over the effective field theory. In turn, the resulting gravitino mass for $|W_0| \sim \mathcal{O}(1)$ is lower, $m_{3/2} \sim \mathcal{O}(10^{10})$ GeV.

Gaugino condensation on D5-branes

As explained in Sec. 1.5.3, C_2 axions can develop a potential also due to non-perturbative corrections to the Kähler potential arising from ED1-instantons or gaugino condensation on D5-branes. However, similarly to situation of the C_4 axions, this case implies a severe tuning on the prefactors of the non-perturbative effects to match the correct EDE scale. This fact is related to consistency of the model with the weak gravity conjecture as explained in [183]. We shall therefore be brief in the description of this case.

Focusing on the case where these non-perturbative effects correct the big modulus τ_b, the Kähler potential and the superpotential would still be given by the standard LVS expressions (1.138) and (1.139) but now with the replacement:

$$\tau_b \to \tau_b - \gamma(G+\bar{G})^2 + e^{-\tilde{\mathfrak{a}}t_b/\sqrt{g_s}}\left[A_1\,\text{Re}\left[e^{-\tilde{\mathfrak{a}}k\mathfrak{f}G}\right] + A_2\,\text{Re}\left[e^{-2\tilde{\mathfrak{a}}k\mathfrak{f}G}\right] + A_3\,\text{Re}\,e^{-3\tilde{\mathfrak{a}}k\mathfrak{f}G}\right] \tag{4.67}$$

where $\tilde{\mathfrak{a}} = 2\pi/M$ and $G = \bar{S}b + c$, as defined in Sec. 1.5. The scalar potential admits 3 contributions of the form

$$V = V_{\text{LVS}}(\mathcal{V}, \tau_s) + V_{\text{EDE}}(c) + V_b(\theta_b)\,, \tag{4.68}$$

where V_{LVS} and V_b are given by (1.140) and (1.114), while the EDE potential is (for $b = 0$)

$$V_{\text{EDE}} = V_0\left[\tilde{A}_1\cos(\tilde{\mathfrak{a}}|k|\mathfrak{f}\,c) + \tilde{A}_2\cos(2\tilde{\mathfrak{a}}|k|\mathfrak{f}\,c) + \tilde{A}_3\cos(3\tilde{\mathfrak{a}}|k|\mathfrak{f}\,c)\right]\,, \tag{4.69}$$

where we have set $A_i = \tilde{A}\tilde{A}_i$ $(i = 1, 2, 3)$ and

$$V_0 = \frac{3\tilde{A}\,|W_0|^2\tilde{\mathfrak{a}}^2}{2g_s^2\mathcal{V}^2}\,e^{-\tilde{\mathfrak{a}}t_b/\sqrt{g_s}} \equiv \Lambda\,\tilde{A}\,e^{-\tilde{\mathfrak{a}}t_b/\sqrt{g_s}}\,. \tag{4.70}$$

Note that the potential V_b for the axion θ_b is decoupled from the EDE dynamics, and so θ_b can be safely set to zero, as in a standard LVS model. Using the canonically normalized field defined in (1.171), we obtain in the potential the term

$$\cos\left(\tilde{\mathfrak{a}}|k|\mathfrak{f}\,c\right) = \cos\left(\tilde{\mathfrak{a}}|k|\mathfrak{f}\sqrt{\frac{\tau_b}{6\gamma}}\,\varphi\right) \equiv \cos\left(\frac{\varphi}{f}\right)\,, \tag{4.71}$$

from which we can obtain the decay constant of the EDE field φ

$$f = \frac{1}{\tilde{\mathfrak{a}}\mathfrak{f}}\sqrt{\frac{3g_s}{2|k|\tau_b}} = \frac{1}{\mathfrak{f}}\sqrt{\frac{3g_s}{8\pi^2|k|}}\frac{M}{\sqrt{\tau_b}}\,. \tag{4.72}$$

Obtaining $f = 0.2$ in Planck units for $g_s \sim \mathcal{O}(0.1)$ and $\tau_b \gtrsim \mathcal{O}(100)$, clearly requires $M \gtrsim \mathcal{O}(30)$, suggesting that gaugino condensation on D5-branes is better than ED1-instantons which would anyway be volume-suppressed if both effects are present. Moreover, for $\mathfrak{f} = |k| = \tilde{k} = 1$ and $f = 0.2\,M_P$, the EDE scale becomes

$$V_0 = \Lambda\,\tilde{A}\,e^{-\frac{1}{\mathfrak{f}}\sqrt{\frac{3}{\tilde{k}|k|}}\left(\frac{M_P}{f}\right)}M_P^4 \sim 10^{-4}\,\Lambda\,\tilde{A}\,M_P^4\,, \tag{4.73}$$

which shows that $V_0 \sim 10^{-108}\, M_P^4$ can be obtained only by tuning $\tilde{A}$ to exponentially small values, in complete analogy with the C_4 axion case (see (4.32)).

We therefore conclude that realizing EDE with C_2 axions and ED1/D5 non-perturbative effects requires always an exponential tuning of the prefactors. Given that we have shown instead that models with C_2 axions and ED3/D7 non-perturbative effects can realize EDE in a more natural way, it is important to check that ED3/D7 contributions to the scalar potential dominate over ED1/D5 effects. This is guaranteed if the non-perturbative corrections to K are characterized by $\tilde{\mathfrak{a}} = 2\pi/M$, with $M \leq 2$, since in this case (4.70) would give $V_0 \ll$ eV^4 for the values of t_b and g_s found in Sec. 4.4.2 which reproduce the correct EDE scale for ED3/D7 effects.

Chapter V

Joint Statistics of Cosmological Constant and the scale of SUSY breaking

The goal of this chapter is to study the joint distribution of the gravitino mass (which sets the scale of supersymmetry breaking in the visible sector in these models) and the cosmological constant in the two most well developed scenarios for Kähler moduli stabilization in type IIB: KKLT [18] and LVS [27] models. In addition to the scenario for moduli stabilization, the quantities of interest also depend on the uplift sector of the models. This chapter will focus on models where the uplift sector is an anti-brane at the tip of a warped throat[1] (as in the construction of [18]) and a preliminary analysis for small complex structure F-term uplift [40].

The key ingredients for our analysis will be moduli stabilization and a systematic incorporation of the effect of the anti-brane via the nilpotent goldstino formalism, while for the second method we will employ the standard supergravity treatment. Let us describe them in detail.

- **Incorporation of moduli stabilization (both complex structure and Kähler):** The values of the cosmological constant and the gravitino mass (and other observables) in a vacuum are determined by the expectation value of the moduli fields. Thus, in order to study the distribution of observables, it is important to incorporate moduli stabilization and compute the distributions sampling only over points corresponding to the minima of the moduli potential. Early statistical analysis of observables incorporated the stabilization of complex structure moduli; the importance of Kähler moduli stabilization was instead emphasized recently in [129]. Here, the effect of Kähler

[1] There has been much discussion in the literature on the (meta)stability of anti-D3 branes in warped throats, see e.g. [268–284].

moduli stabilization on the distribution of the gravitino mass was studied. It was found that this has a significant effect on the statistics. Following [129] we will sample only over points corresponding to minima of the moduli potential.

- **Incorporation of the uplift sector**: In both KKLT and LVS, a crucial contribution to the cosmological constant comes from the so-called uplift sector. Before the incorporation of this sector into the effective action, the vacua obtained are necessarily AdS. Furthermore, the cosmological constant and the gravitino mass are correlated in these vacua. Before considering the uplift sector, KKLT vacua are supersymmetric. Thus (setting $M_P = 1$)

$$\hat{V}_{\text{KKLT}} = -3\hat{m}_{3/2}^2 \,, \tag{5.1}$$

where the hat indicates a quantity computed in the effective field theory before adding the uplift sector. Similarly for LVS vacua (before uplifting)

$$\hat{V}_{\text{LVS}} \simeq -\hat{m}_{3/2}^3 \,, \tag{5.2}$$

The relations (5.1) and (5.2) are broken solely due to introduction of the uplift sector. Thus it is crucial to incorporate it while computing the joint distribution of the cosmological constant and the gravitino mass[2].

Of course, there are various effects that can lead to an uplift. We shall focus on an anti-D3 brane at the bottom of a warped throat in the first section, and small complex structure F-term in the second. The reasons for this are the following:

  - **The nilpotent superfield formalism:** The nilpotent superfield formalism allows for explicit computation of the effect of an anti-D3 brane in a warped throat. The effects of the anti-brane are captured by the introduction of a nilpotent chiral superfield X such that $X^2 = 0$ (see for instance [160, 285–289] and references therein). X has a single propagating component, the Volkov-Akulov goldstino [290], and supersymmetry is broken by its F-term. The effective couplings of such a field X were studied in [291–300] and the KKLT uplifting term was reproduced within the supergravity framework in [161–163]. Finally,

[2] Reference [129] focused on the gravitino mass distribution before the inclusion of the uplift sector.

in [163] explicit string constructions were presented in which an anti-D3-brane at the bottom of a warped throat has only the goldstino as its light degree of freedom, justifying the use of the nilpotent field X to describe the anti-brane. Soft masses in KKLT and LVS were computed using this framework in [164].

- **Distribution of throat hierarchies:** For uplift with anti-D3 branes in warped throats, the magnitude of the uplift term is set by the hierarchy associated with the warped throat. Thus an understanding of the distribution of throat hierarchies is needed to understand the distribution of physical observables in this setting. The distribution of throat hierarchies has been studied in detail in [117]. We will make heavy use of these results.

Our main finding is that the distribution of the gravitino mass at zero cosmological constant is tilted towards lower values. This is not the same as the result of [124] (see also [123]) which carried out generic estimates on supersymmetry breaking F- and D-terms and concluded that there is a preference for high scale breaking. Note that this result is different also from the one of [129] which found a logarithmic preference for high scales of supersymmetry breaking after stabilizing the Kähler moduli but before uplifting. The difference arises due to the presence of the relations (5.1) and (5.2) and the form of the distribution for throat hierarchies. Our results should not be taken as giving the generic picture for string vacua. In fact, in some class of models such as [158] high scale breaking is essentially in-built. At the same time, we find it very interesting that the best understood models have distributions favouring lower masses of the gravitino.

It was argued in [117] that if one is dealing with a CY with a large number of flux quanta, the joint distribution of 2 (or a small number of) quantities which are not related through a functional relation is proportional to the product of the 2 individual distributions. For example, if one considers 2 conifold moduli $|z_1|$ and $|z_2|$, then these are determined by independent fluxes (hence are functionally independent), and so the joint distribution is

$$\mathcal{N}(|z_1|,|z_2|)\,\mathrm{d}|z_1|\,\mathrm{d}|z_2| \propto \mathcal{N}(|z_1|)\mathcal{N}(|z_2|)\,\mathrm{d}|z_1|\,\mathrm{d}|z_2| \ . \tag{5.3}$$

Similar considerations also apply when one is considering the joint distribution of a conifold modulus and a quantity that is a function of a large number of fluxes. The product structure in the joint distribution essentially follows from the fact that when there is a large number of fluxes, fixing one quantity should not affect the distribution of another quantity significantly unless there is a functional relation between them. See [117] for a more detailed discussion.

5.1 Anti D3-brane uplift

In this section we will evaluate the joint distribution of the gravitino mass and the cosmological constant for KKLT and LVS models with anti-brane uplifting. As described earlier, we will make use of the distributions of the axio-dilaton, W_0 and y (the hierarchy) to evaluate these. We start by writing the expression for the number of vacua in an infinitesimal volume in these coordinates. Making use of the results discussed in details in App. C, we have

$$\mathrm{d}\mathcal{N} = \frac{\eta |W_0|}{y(\ln y)^2 s^2}\,\mathrm{d}|W_0|\,\mathrm{d}S\,\mathrm{d}y\,, \tag{5.4}$$

where we have made use of the fact that the hierarchy is related to the size of the shrinking conifold modulus by $y = |z|^{4/3}$ ($\mathrm{d}S$ is the measure for integration over the axio-dilaton: $\mathrm{d}S = \mathrm{d}s\,\mathrm{d}C_0$). We have also taken the number of fluxes to be large, so justifying the factorized form of the density. In both KKLT and LVS the cosmological constant and the gravitino mass can be expressed in terms of $|W_0|$, y and s. We will use these expressions to carry out a change of variables in (5.4). This will provide us with the required densities.

5.1.1 KKLT

With the complex structure moduli and the dilaton stabilized by fluxes, we will take the low energy fields to be the Kähler moduli, chiral matter and the nilpotent superfield. For simplicity, we will consider one Kähler modulus T and matter fields living on D3-branes which we will collectively denote as ϕ. Then

$$K = -2\ln\mathcal{V} + \tilde{K}_i\;\phi\bar{\phi} + \tilde{Z}_i\;X\bar{X} + \tilde{H}_i\;\phi\bar{\phi}\;X\bar{X} + \ldots\,, \tag{5.5}$$

where $\tilde{K}_i$ and $\tilde{Z}_i$ are the Kähler metrics for the matter field on D3-branes and the nilpotent goldstino. $\tilde{H}_i$ is the quartic interaction between the matter field and the nilpotent goldstino. As per the discussion in Sec. 1.4.1, these are

$$\tilde{K}_i = \frac{\alpha}{\tau}\,, \qquad \tilde{Z}_i = \frac{\beta}{\tau}\,, \qquad \tilde{H}_i = \frac{\gamma}{\tau^2}\,, \tag{5.6}$$

where the scaling of $\tilde{K}_i$ with τ is due to the modular weight of the matter field on D3-branes [301, 302]. The superpotential is

$$W \;=\; W_0 + \rho X + A\, e^{-aT}\,, \tag{5.7}$$

where we have included a non-perturbative contribution in T which is needed to stabilize the Kähler modulus. The KKLT construction requires $|W_0| \ll 1$ (see e.g. [4, 12, 14, 20, 23] and references therein for recent works on explicit construction of vacua with low values of $|W_0|$). We will study the statistical distributions for fixed values of a and A.

Recall that the supergravity scalar potential is determined in terms of the Kähler potential and the superpotential as

$$V = e^K \left(K^{I\bar{J}} D_I W D_{\bar{J}}\bar{W} - 3|W|^2\right) \qquad \text{with} \qquad D_I W = \partial_I W + K_I W\,, \tag{5.8}$$

where the indices I and J run over the chiral superfield T, ϕ, X. $K^{I\bar{J}}$ is the inverse of the matrix $K_{I\bar{J}} \equiv \partial_I \partial_{\bar{J}} K$ and $K_I \equiv \partial_I K$. It is often convenient to write (5.8) as

$$V = F^I F_I - 3m_{3/2}^2\,, \tag{5.9}$$

where $F_I \equiv e^{K/2} D_I W$ and $F^I \equiv e^{K/2} K^{I\bar{J}} D_{\bar{J}} \bar{W}$ are the F-terms. The gravitino mass is instead given by $m_{3/2} \equiv e^{K/2}|W|$.

By making use of (5.5) and (5.7) in (5.8), one finds

$$V = (V_{\text{KKLT}} + V_{\text{up}}) + \frac{2}{3}\left[(V_{\text{KKLT}} + V_{\text{up}}) + \frac{1}{2} V_{\text{up}}\left(1 - \frac{3\gamma}{\alpha\beta}\right)\right]\; |\hat{\phi}|^2\,, \tag{5.10}$$

where the scalar component of the nilpotent field is set to zero (assuming no condensation of fermions). V_{KKLT} is the KKLT potential in the absence of the uplifting term:

$$V_{\text{KKLT}} \;=\; \frac{2\,e^{-2a\tau} a A^2}{s\,\mathcal{V}^{4/3}} + \frac{2\,e^{-2a\tau} a^2 A^2}{3s\,\mathcal{V}^{2/3}} \;-\; \frac{2\,e^{-a\tau} a A\,W_0}{s\,\mathcal{V}^{4/3}}\,. \tag{5.11}$$

To reduce clutter in the equations we have followed the standard practice of writing both W_0 and A real and positive. For general values of W_0 and A the expression for the potential can be obtained from the above by taking $W_0 \to |W_0|$ and $A \to |A|$. We have also neglected the overall contribution from the Kähler potential for the complex structure moduli which is an order one multiplicative factor. The imaginary part of T (the axion ψ) has its minimum at $\psi = \pi/a$. This is responsible for the negative sign in the third term in (5.11). The uplift term V_{up} arises from $F^X F_X$ and is given by

$$V_{\text{up}} = \frac{\rho^2}{2\beta s\tau^2} \equiv \frac{y}{2s\tau^2} \qquad \text{with} \qquad y = \frac{\rho^2}{\beta}\,. \tag{5.12}$$

The third contribution in (5.10) is proportional to $|\hat{\phi}|^2$ with $\hat{\phi}$ denoting the canonically normalized matter scalar field ϕ. This contribution corresponds to its soft mass. For non-tachyonic scalars, the vacuum expectation value of $\hat{\phi}$ would vanish, and so we will proceed by setting $\hat{\phi} = 0$. Soft masses will be discussed in Chap. 6.[3]

Minimizing the scalar potential, one finds that the following holds at the minimum

$$W_0 = e^{-a\tau} A \left(1 + \frac{2}{3}a\tau + \frac{y\, e^{2a\tau}}{2a^2 A^2 \tau}\right) , \tag{5.13}$$

where we have dropped subleading terms in the $(a\tau)^{-1}$ expansion multiplying the term proportional to y. In what follows, we shall drop such subleading terms, but shall include their effect in the numerical results that we will present later. Using (5.13) in the expression for the scalar potential (5.10), its value at the minimum is found to be

$$\Lambda = -\frac{2e^{-2a\tau} a^2 A^2}{3s\ \tau} + \frac{y}{2s\tau^2} \equiv V^{(0)}_{\mathsf{KKLT}} + V_{\mathsf{up}}. \tag{5.14}$$

Note that the hierarchy y can be tuned to make the cosmological constant zero or extremely small and positive. For a Minkowski vacuum one needs

$$y = \frac{4}{3}\tau\, e^{-2a\tau} a^2 A^2\, . \tag{5.15}$$

The gravitino mass becomes

$$m_{3/2} = e^{K/2}|W| = \frac{1}{\sqrt{2s}}\frac{1}{\tau^{3/2}}\left(W_0 - A\, e^{-a\tau}\right). \tag{5.16}$$

Next, we turn to evaluating the joint distribution for the gravitino and cosmological constant. For this, we will perform the change of variables $(W_0, y, S) \to (m_{3/2}, \Lambda, S)$ in (5.4). We need to compute the Jacobian of this transformation. The expressions for the cosmological constant and the gravitino mass ((5.14) and (5.16)) have explicit dependence on (W_0, y, s) and also implicit dependence via τ. To compute the partial derivatives of the cosmological constant and the gravitino mass with respect to W_0 and y we need to compute the partial derivatives of τ with respect to these variables. Making use of (5.13), we find

$$\frac{\partial\tau}{\partial y} = -\frac{e^{a\tau}}{2Aa^2\tau f(\tau)}\, , \tag{5.17}$$

[3] If the log hypothesis (1.133) holds, at leading order in the α' expansion, the soft masses for D3-brane matter vanish in KKLT. α' corrections and anomaly mediation contributions are relevant. α' corrections always make a positive contribution to the square masses.

with

$$f(\tau) = Ae^{-a\tau}\left(-\frac{2}{3}a^2\tau\left(1+\frac{1}{2a\tau}\right)+\frac{y\,e^{2a\tau}}{2aA^2\tau}\left(1-\frac{1}{a\tau}\right)\right)\,. \tag{5.18}$$

Now let us turn to the entries of the Jacobian. These are

$$\frac{\partial m_{3/2}}{\partial W_0} = \frac{1}{\sqrt{2s}}\frac{1}{\tau^{3/2}} + \left(\frac{1}{\sqrt{2s}}\frac{1}{\tau^{3/2}}aAe^{-a\tau} - \frac{3}{2\tau}m_{3/2}\right)\frac{\partial\tau}{\partial W_0}\,, \tag{5.19}$$

$$\frac{\partial m_{3/2}}{\partial y} = \left(\frac{1}{\sqrt{2s}}\frac{1}{\tau^{3/2}}aAe^{-a\tau} - \frac{3}{2\tau}m_{3/2}\right)\frac{\partial\tau}{\partial y}\,. \tag{5.20}$$

We also have

$$\frac{\partial\Lambda}{\partial W_0} = \frac{\partial\Lambda}{\partial\tau}\frac{\partial\tau}{\partial W_0}\,, \qquad \frac{\partial\Lambda}{\partial y} = \frac{\partial\Lambda}{\partial\tau}\frac{\partial\tau}{\partial y} + \frac{1}{2s\tau^2}\,, \tag{5.21}$$

with

$$\frac{\partial\Lambda}{\partial\tau} = \frac{4e^{-2a\tau}a^3A^2}{3s\tau}\left(1+\frac{1}{2a\tau}\right) - \frac{y}{s\tau^3}\,, \tag{5.22}$$

Analytical estimate

The expressions in (5.19), (5.20) and (5.21) are rather cumbersome. While we will use them for our numerical analysis in Sec. 5.1.1, we continue our analytic investigation in a regime that leads to considerable simplifications. For this, we define the quantity x as

$$x \equiv \frac{y\,e^{2a\tau}}{2a^2A^2\tau}\,, \tag{5.23}$$

which allows to write (5.14) as

$$\Lambda = -\frac{2e^{-2a\tau}a^2A^2}{3s\,\tau}\left(1-\frac{3}{2}x\right) = -3m_{3/2}^2\left(1-\frac{3}{2}x\right)\left(1+\frac{3}{2a\tau}x\right)^{-2}\,. \tag{5.24}$$

We shall now focus on the regime in which y is varied such that $0 < x \lesssim \mathcal{O}(1) \ll (a\tau)$. Comparing with (5.24) we see that this allows for uplift to zero and positive values of the cosmological constant. Hence, the regime covers the region of most interest since $x \gg 1$ corresponds to an unstable situation where the uplifting contribution would yield a runaway. Also note that in this regime, the contribution of the term involving y in right hand side of (5.13) becomes subdominant. We can therefore take the approximation

$$W_0 = e^{-a\tau}A\left(1+\frac{2}{3}a\tau + x\right) \simeq \frac{2}{3}A(a\tau)e^{-a\tau}\,, \tag{5.25}$$

and so, to leading order in the $(a\tau)^{-1}$ expansion, we have

$$\tau \simeq -\frac{1}{a}\ln\left(\frac{3}{2A}W_0\right), \tag{5.26}$$

and the gravitino mass looks like

$$m_{3/2} \simeq \frac{W_0}{\sqrt{2s}\tau^{3/2}} = \frac{2}{3}\frac{(a\tau)A}{\sqrt{2s}\tau^{3/2}}e^{-a\tau}\,. \tag{5.27}$$

This allows us to write

$$\tau \simeq -\frac{1}{a}\ln\left(3\sqrt{\frac{s}{2}}\frac{m_{3/2}}{a^{3/2}A}\right)\,. \tag{5.28}$$

In the regime we are considering, we can write (5.14) as

$$y \simeq 2s\tau^2\left(\Lambda + 3m_{3/2}^2\right)\,. \tag{5.29}$$

Combining this with (5.28) we can express y in terms of Λ, $m_{3/2}$ and s. Similarly, making use of (5.28) in (5.27) gives W_0 in terms of the same quantities.

The Jacobian entries undergo significant simplifications in this regime where the function $f(\tau)$ defined in (5.18) becomes $f(\tau) \simeq -\frac{2}{3}a^2Ae^{-a\tau}\tau$. We then have

$$\frac{\partial\tau}{\partial W_0} \simeq -\frac{3}{2Aa^2\tau}e^{a\tau}\,, \qquad \frac{\partial\tau}{\partial y} \simeq \frac{3}{4A^2a^4\tau^2}e^{2a\tau}\,. \tag{5.30}$$

Let us turn to the partial derivatives of $m_{3/2}$. First note that by making use of (5.13) and (5.16) one can write

$$m_{3/2} = \frac{Ae^{-a\tau}}{\tau^{3/2}(2s)^{1/2}}\left(\frac{2}{3}a\tau + \frac{y\,e^{2a\tau}}{2a^2A^2\tau}\right) \equiv \frac{Ae^{-a\tau}}{\tau^{3/2}(2s)^{1/2}}\left(\frac{2}{3}a\tau + x\right)\,. \tag{5.31}$$

Recall now that

$$\frac{\partial m_{3/2}}{\partial W_0} = \frac{1}{\sqrt{2s}}\frac{1}{\tau^{3/2}} + \left(\frac{1}{\sqrt{2s}}\frac{1}{\tau^{3/2}}aAe^{-a\tau} - \frac{3}{2\tau}m_{3/2}\right)\frac{\partial\tau}{\partial W_0}\,. \tag{5.32}$$

Making use of (5.31), we see that

$$\left(\frac{1}{\sqrt{2s}}\frac{1}{\tau^{3/2}}aAe^{-a\tau} - \frac{3}{2\tau}m_{3/2}\right) = -\frac{3}{2}\frac{Ae^{-a\tau}}{\tau^{5/2}(2s)^{1/2}}\,x\,. \tag{5.33}$$

Using (5.30) and (5.33) in (5.32), and the fact that in the above equation $0 < x \lesssim \mathcal{O}(1)$, we see that the term proportional to $\frac{\partial\tau}{\partial W_0}$ is always subleading in (5.32). Therefore we have

$$\frac{\partial m_{3/2}}{\partial W_0} \simeq \frac{1}{\sqrt{2s}}\frac{1}{\tau^{3/2}}\,. \tag{5.34}$$

Similarly, using the above formulae, one finds

$$\frac{\partial m_{3/2}}{\partial y} \simeq -\frac{9\sqrt{a}}{8A}\frac{1}{\sqrt{2s}}\frac{1}{(a\tau)^{9/2}}\, e^{a\tau} x\,. \tag{5.35}$$

Now, let us come to the derivatives of Λ. Note that the ratio of the two terms in (5.14) is $-\frac{3}{4}x$. Given this (and the fact that $0 < x \lesssim \mathcal{O}(1)$), comparing the various terms in (5.22) gives

$$\frac{\partial \Lambda}{\partial \tau} \simeq \frac{4e^{-2a\tau}a^3A^2}{3s\tau}\,. \tag{5.36}$$

Combining this with (5.21) and (5.30) one finds

$$\frac{\partial \Lambda}{\partial W_0} \simeq -\frac{2Aa^3}{s}\frac{e^{-a\tau}}{(a\tau)^2} \qquad \text{and} \qquad \frac{\partial \Lambda}{\partial y} \simeq \frac{1}{2s\tau^2}\,. \tag{5.37}$$

Combining (5.34), (5.35) and (5.37), the Jacobian is

$$J \simeq \frac{1}{(2s)^{3/2}}\frac{1}{\tau^{7/2}}\,. \tag{5.38}$$

Finally, we have

$$\mathrm{d}\mathcal{N} = \frac{\eta\, W_0}{y(\ln y)^2 s^2}\,\mathrm{d}W_0\,\mathrm{d}y\,\mathrm{d}s = \frac{\eta\, W_0}{y(\ln y)^2 s^2}|J|^{-1}\,\mathrm{d}m_{3/2}\,\mathrm{d}\Lambda\,\mathrm{d}S\,. \tag{5.39}$$

Equations (5.27), (5.28) and (5.29) can be used to express this density in terms of the desired quantities. We find

$$\mathrm{d}\mathcal{N} = \frac{2\tau^3 m_{3/2}}{\left(3m_{3/2}^2+\Lambda\right)\left[\ln\left(2s\tau^2(3m_{3/2}^2+\Lambda)\right)\right]^2 s}\,\mathrm{d}m_{3/2}\,\mathrm{d}\Lambda\,\mathrm{d}S\,. \tag{5.40}$$

To get the expression for the joint distribution of $m_{3/2}$ and Λ, we need to perform the integration over the axio-dilaton moduli space. However, the important features can be extracted by analysing the above density at fixed values of the axio-dilaton (s, C_0). Note that in the $|\Lambda| \ll m_{3/2}^2$ limit (which is physically most interesting) the density scales as

$$\rho_{\text{KKLT}}(\Lambda \simeq 0) \simeq \frac{\ln m_{3/2}}{m_{3/2}}\,, \tag{5.41}$$

implying that it is tilted favourably towards lower values of $m_{3/2}$.

This result can be understood as follows. Let us think of uplifting various AdS vacua obtained before the introduction of the uplift term. These vacua are supersymmetric and $\Lambda = -3m_{3/2}^2$. The introduction of the uplift term has a very small effect on the value of $m_{3/2}$. Thus, when we consider vacua with cosmological constant close to zero, a vacuum with a low value of $m_{3/2}$ also has a low value of the hierarchy y (as the associated supersymmetric AdS vacuum before the introduction of the uplift has a Λ of small magnitude). The distribution of throats is such that it is tilted in favour of lower values of y, and this makes the distribution of $m_{3/2}$ favourable towards lower values of $m_{3/2}$.

Numerical results

Let us now present the numerical results which we have obtained for the joint distribution of the cosmological constant and the gravitino mass without making any approximation. Fig. 5.1 shows the density of states as a function of $m_{3/2}$ for different fixed values of Λ corresponding to AdS, Minkowski and dS vacua. In the physically interesting region with $|\Lambda| \ll m_{3/2}^2$, the 3 curves approach each other, reproducing the analytical estimate (5.41) where ρ_{KKLT} becomes independent of Λ and is inversely proportional to $m_{3/2}$ (up to a logarithmic dependence). Note that the green curve with positive Λ features a raising behaviour of ρ_{KKLT} for $m_{3/2}^2 \lesssim \Lambda$. However this regime can be ignored for the following two reasons: (i) it is valid only for a small window of values of $m_{3/2}$ close to $m_{3/2}^2 \lesssim \Lambda$ since, as can be seen from (5.24), $m_{3/2}^2 \ll \Lambda$ would require $x \gg 1$ that yields a runaway; (ii) the region characterized by $m_{3/2}^2 \lesssim \Lambda$ is phenomenologically irrelevant since it would correspond to an essentially massless gravitino with $m_{3/2}$ below the Hubble parameter.

For completeness, in Fig. 5.2 we have also plotted the density of vacua as a function of $m_{3/2}$ and Λ. Note that the two blank regions corresponds respectively to the runaway limit (for positive Λ) and to the violation of the supergravity lower limit $\Lambda \geq -3m_{3/2}^2$ (for negative Λ).

5.1.2 LVS

We now turn to the LVS models. Using the same notation as in the previous subsection, the Kähler potential can be written as

$$K = -2\ln\left(\mathcal{V} + \xi\, s^{3/2}\right) + \tilde{K}_i\, \phi\bar{\phi} + \tilde{Z}_i\, X\bar{X} + \tilde{H}_i\, \phi\bar{\phi}\, X\bar{X} + \dots . \tag{5.42}$$

We will focus on the simplest LVS example, with two Kähler moduli T_s and T_b with real parts τ_b and τ_s, with the CY volume having a Swiss cheese structure: $\mathcal{V} = \tau_b^{3/2} - \tau_s^{3/2}$. The coefficients $\tilde{K}_i$, $\tilde{Z}_i$ and $\tilde{H}_i$ are matter metric and quartic interaction coefficients. The superpotential is

$$W \;=\; W_0 + \rho X + A\, e^{-a_s T_s} \tag{5.43}$$

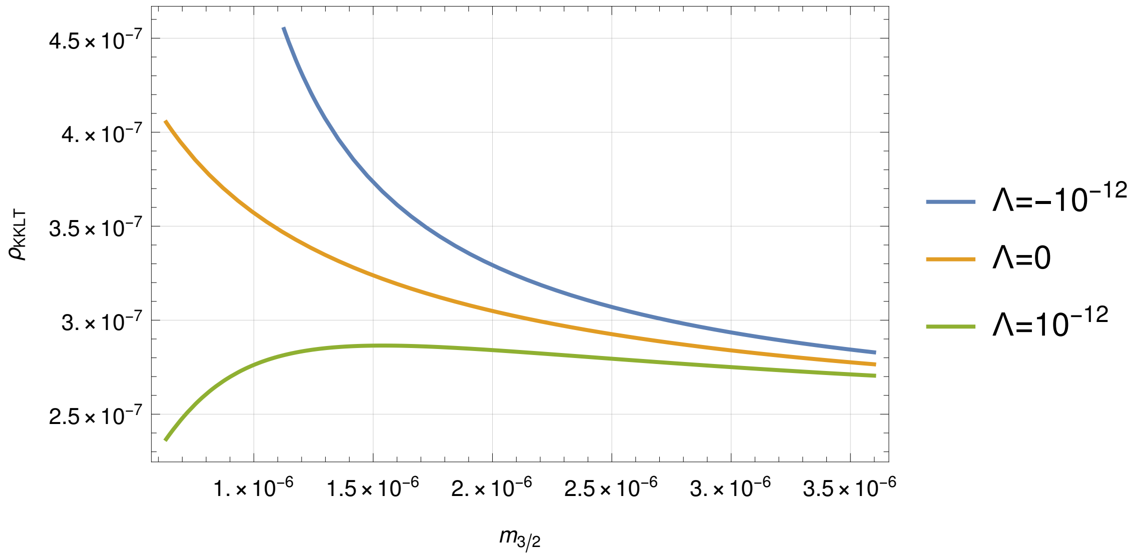

Figure 5.1. Density of KKLT flux vacua at fixed values of the cosmological constant Λ as a function of the gravitino mass $m_{3/2}$ in Planck units.

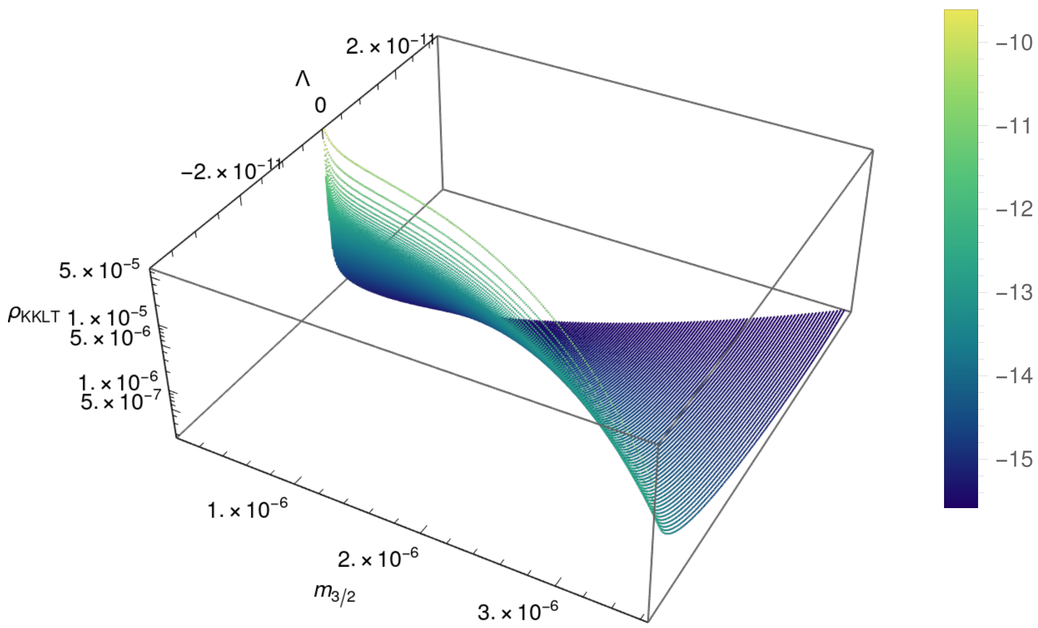

Figure 5.2. Density of KKLT flux vacua as a function of the gravitino mass $m_{3/2}$ and the cosmological constant Λ in Planck units.

where $W_0 \sim \mathcal{O}(1-10)$ in LVS. The supergravity scalar potential (after setting the matter field expectation values to zero) takes the form:

$$V = V_{\text{LVS}} + V_{\text{up}}\,, \tag{5.44}$$

where V_{LVS} is the LVS potential in the absence of the uplift sector

$$V_{\text{LVS}} = \frac{4}{3}\frac{e^{-2a_s\tau_s}\sqrt{\tau_s}\;a_s^2A^2}{s\;\mathcal{V}} - \frac{2e^{-a_s\tau_s}\tau_s\;a_sA\;W_0}{s\;\mathcal{V}^2} + \frac{3\sqrt{s}\;\xi\;W_0^2}{8\;\mathcal{V}^3}\,. \tag{5.45}$$

In LVS the minimum is non-supersymmetric before the introduction of the uplifting term. Minimizing the LVS potential V_{LVS}, one obtains the conditions that the Kähler moduli have to satisfy in the minimum[4]:

$$e^{-a_s\tau_s} = \frac{3\;\tau_s^{1/2}\;W_0}{4a_sA_s\;\mathcal{V}}\,, \tag{5.46}$$

and

$$\tau_s^{3/2} = \frac{s^{3/2}\xi}{2}\,. \tag{5.47}$$

The value of the potential at this minimum is

$$V_{\text{LVS}}^{(0)} = -\frac{3\sqrt{s}\;\xi\;W_0^2}{16a_s\tau_s\;\mathcal{V}^3}\,. \tag{5.48}$$

Next, we consider the potential (5.44) that includes the X-contribution responsible for uplifting the AdS minimum. The minimum condition (5.46) is not modified, while (5.47) is changed to

$$\tau_s^{3/2} = \frac{s^{3/2}\xi}{2}\left(1+\frac{16}{27}\frac{\mathcal{V}^{5/3}y}{W_0^2s^{3/2}\xi}\right)\,. \tag{5.49}$$

For later use, we define

$$x_{\text{LVS}} \equiv \frac{16}{27}\frac{y\,(a_s\tau_s)\mathcal{V}^{5/3}}{W_0^2s^{3/2}\xi}\,. \tag{5.50}$$

The value of the potential at the minimum now becomes

$$\Lambda = V_{\text{LVS}}^{(0)} + \frac{5}{9}V_{\text{up}} = -\frac{3\sqrt{s}\xi W_0^2}{16a_s\tau_s\mathcal{V}^3} + \frac{5}{9}\frac{y}{2s\mathcal{V}^{4/3}} = -\frac{3\sqrt{s}\xi W_0^2}{16a_s\tau_s\mathcal{V}^3}\left(1-\frac{5}{2}x_{\text{LVS}}\right)\,, \tag{5.51}$$

which gives a Minkowski minimum for

$$y = \frac{27}{40}\frac{W_0^2s^{3/2}\xi}{\mathcal{V}^{5/3}a_s\tau_s} \qquad \Leftrightarrow \qquad x_{\text{LVS}} = \frac{2}{5}\,. \tag{5.52}$$

[4]We will work to leading order in the $(a_s\tau_s)^{-1}$ expansion.

Thus we realize that (5.51) can allow for uplift to zero and positive values of the cosmological constant provided y is varied so that x_{LVS} lies in the $0 < x_{\text{LVS}} \lesssim \mathcal{O}(1) \ll (a_s\tau_s)$ regime, with $x_{\text{LVS}} \gg 1$ corresponding to the runaway limit. Hence from now on we shall focus just on the interesting region where the potential is stable and the gravitino mass is given by

$$m_{3/2} = \frac{W_0}{\sqrt{2s}\,\mathcal{V}}\,. \tag{5.53}$$

Let us now perform a variable change in (5.4) to obtain the joint distribution of the gravitino mass and the cosmological constant: $(W_0, y, s, C_0) \to (W_0, \Lambda, m_{3/2}, C_0)$. Λ and $m_{3/2}$, given in (5.51) and (5.53), have explicit dependence on s and y and also implicitly depend on the dilaton and the hierarchy via τ_s and $\mathcal{V}$. Note that, at leading order in the $(a_s\tau_s)^{-1}$ expansion, we have $\frac{\partial\mathcal{V}}{\partial\tau_s} = a_s\mathcal{V}$ and

$$\frac{\partial\tau_s}{\partial s} = \frac{\left(\frac{\xi}{2}\right)^{2/3}}{1-\frac{10}{9}x_{\text{LVS}}} \equiv \frac{\left(\frac{\xi}{2}\right)^{2/3}}{\Psi}\,, \qquad \frac{\partial\tau_s}{\partial y} = \frac{32}{81}\frac{\mathcal{V}^{5/3}\tau_s}{W_0^2 s^{3/2}\xi}\frac{1}{\Psi} = \frac{2}{3}\frac{x_{\text{LVS}}}{y\,a_s\,\Psi}\,. \tag{5.54}$$

Now we turn to the entries of the Jacobian (again, to leading order in the $(a_s\tau_s)^{-1}$ expansion). They read

$$\frac{\partial m_{3/2}}{\partial s} = -a_s m_{3/2}\frac{\partial\tau_s}{\partial s}\,, \qquad \frac{\partial m_{3/2}}{\partial y} = -a_s m_{3/2}\frac{\partial\tau_s}{\partial y}\,, \tag{5.55}$$

$$\frac{\partial\Lambda}{\partial s} = -3V^{(0)}_{\text{LVS}}a_s\Psi\frac{\partial\tau_s}{\partial s} = -3V^{(0)}_{\text{LVS}}a_s\left(\frac{\xi}{2}\right)^{2/3}\,, \qquad \frac{\partial\Lambda}{\partial y} = \frac{1}{2s\mathcal{V}^{4/3}}\,, \tag{5.56}$$

where we have used (5.54). These give a Jacobian of the form

$$J = -\frac{5}{18}\frac{m_{3/2}a_s}{\Psi}\left(\frac{\xi}{2}\right)^{2/3}\frac{1}{s\mathcal{V}^{4/3}} = -\frac{5}{2^{1/3}\,9}\left(\frac{\xi}{2}\right)^{2/3}\frac{a_s m_{3/2}^{7/3}}{\Psi s^{1/3}W_0^{4/3}}\,. \tag{5.57}$$

From (5.51) we can write

$$y = \frac{18}{5}s\mathcal{V}^{4/3}\left(\Lambda + \frac{3\sqrt{s}\xi W_0^2}{16a_s\tau_s\mathcal{V}^3}\right) = \frac{2^{1/3}\,9}{5}\frac{s^{1/3}W_0^{4/3}}{m_{3/2}^{4/3}}\left(\Lambda + \frac{3\sqrt{2}\xi m_{3/2}^3 s^2}{8a_s\tau_s W_0}\right)\,. \tag{5.58}$$

In summary, we find

$$\mathrm{d}\mathcal{N} = \frac{\eta\,W_0}{y(\ln y)^2 s^2}|J|^{-1}\,\mathrm{d}W_0\,\mathrm{d}m_{3/2}\,\mathrm{d}\Lambda\,\mathrm{d}C_0\,, \tag{5.59}$$

with J and y given in (5.57) and (5.58) and $\tau_s = s\left(\frac{\xi}{2}\right)^{2/3} \simeq -\frac{1}{a_s}\ln m_{3/2}$. To get the final form of the distribution, one should integrate over C_0 and W_0. Even if this is not tractable, the interesting features of the joint distribution can be obtained by considering fixed values of C_0 and W_0 (which is an $\mathcal{O}(1-10)$ quantity in LVS). In the physically interesting regime where $|\Lambda| \ll m_{3/2}^3$, we can also make the following approximations

$$|J|^{-1} \simeq \frac{(\ln m_{3/2})^{1/3}}{m_{3/2}^{7/3}}\,, \qquad y^{-1} \simeq \frac{1}{(\ln m_{3/2})^{4/3} m_{3/2}^{5/3}}\,, \tag{5.60}$$

which give

$$\rho_{\mathrm{LVS}}(\Lambda \simeq 0) \simeq \frac{1}{(\ln m_{3/2})^3 m_{3/2}^4}\,. \tag{5.61}$$

Interestingly, we find that the distribution of flux vacua with cosmological constant close to zero is highly tilted towards lower values of $m_{3/2}$. We have confirmed this analytical estimate with an exact numerical evaluation of the density of LVS flux vacua as a function of $m_{3/2}$ and Λ. Two plots showing these results are presented in Fig. 5.3 and Fig. 5.4. As stressed already in the KKLT case, the blank region in Fig. 5.4 for $\Lambda > 0$ would lead to a runaway, while the blank region for $\Lambda < 0$ would correspond to a violation of the supergravity lower limit $\Lambda \geq -3m_{3/2}^2$.

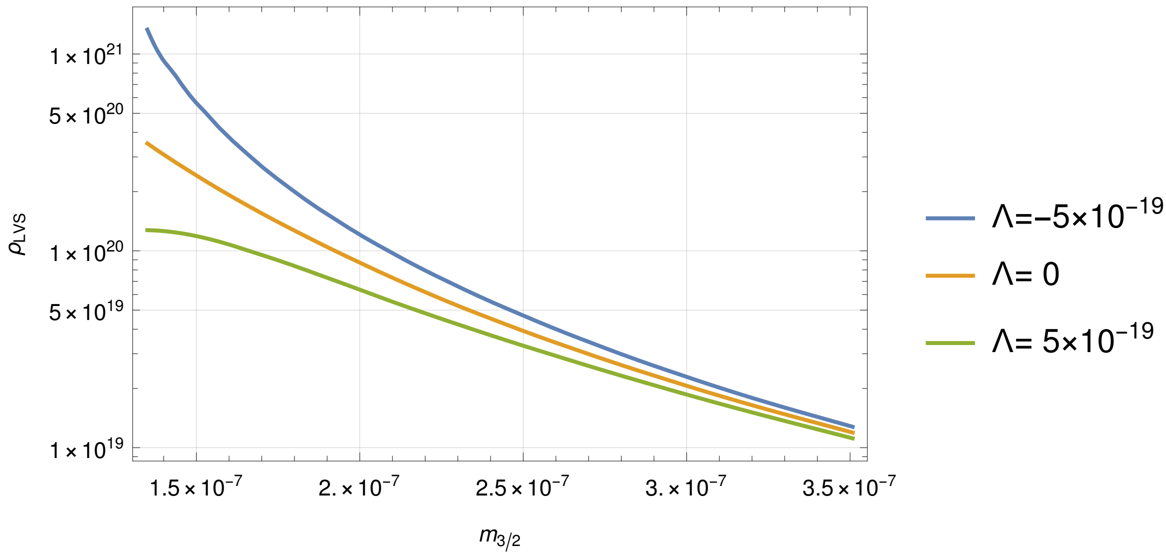

Figure 5.3. Density of LVS flux vacua at fixed values of the cosmological constant Λ as a function of the gravitino mass $m_{3/2}$ in Planck units.

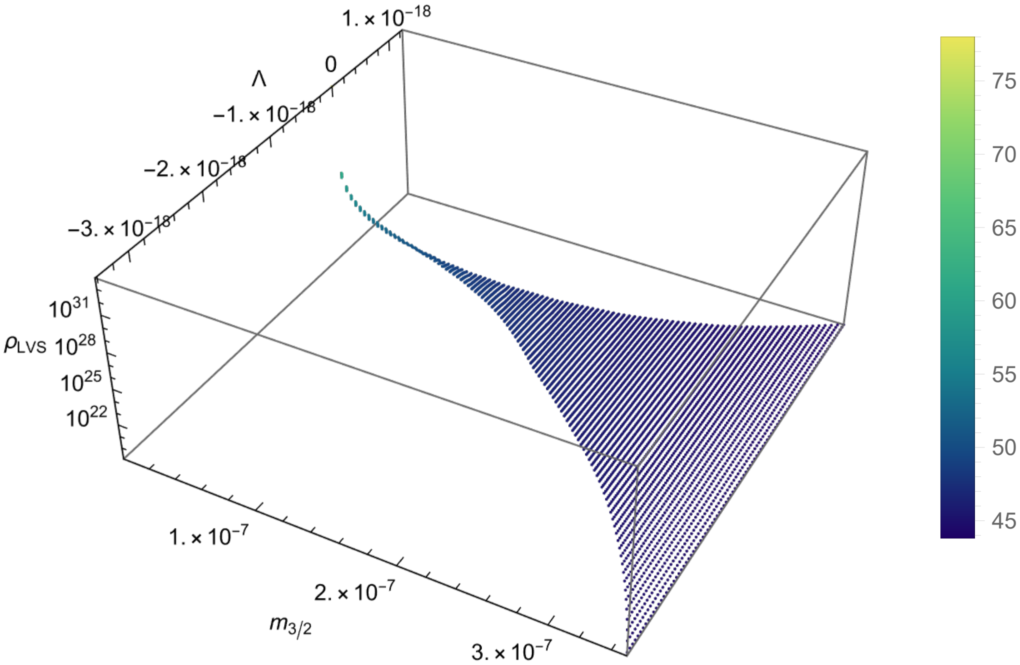

Figure 5.4. Density of LVS flux vacua as a function of the gravitino mass $m_{3/2}$ and the cosmological constant Λ in Planck units.

5.2 Complex structure uplifting

Let us now provide the preliminary results of our statistical studies using small complex structure F-term uplift.

As described earlier, we will make use of the distributions of the axio-dilaton, W_0 and $|F|^2$ to evaluate these. We start by writing the expression for the number of vacua in an infinitesimal volume in these coordinates. Making use of the results discussed in details in App. C, we have

$$\mathrm{d}\mathcal{N} = \frac{\eta |W_0|^2}{s^2}\,\mathrm{d}|W_0|\,\mathrm{d}S\,\mathrm{d}|F|^2\ . \tag{5.62}$$

($\mathrm{d}S$ is the measure for integration over the axio-dilaton: $\mathrm{d}S = \mathrm{d}s\,\mathrm{d}C_0$). We have also taken the number of fluxes to be large, so justifying the factorized form of the density. In both KKLT and LVS the cosmological constant and the gravitino mass can be expressed in terms of $|W_0|$, $|F|^2$ and s. We will use these expressions to carry out a change of variables in (5.62). This will provide us with the required densities.

5.2.1 KKLT

Recalling the uplift potential discussed in Sec. 1.4

$$V_{\text{up}} = \frac{|F|^2}{\tau^3}\,, \tag{5.63}$$

we obtain the total potential

$$V = \frac{4a^2A^2e^{-2a\tau}}{3\tau} + \frac{aA^2e^{-2a}\tau}{\tau^2} - \frac{aA|W_0|e^{-a\tau}}{\tau^2} + \frac{|F|^2}{\tau^3}\,. \tag{5.64}$$

Minimizing it, one finds that the following holds at the minimum

$$W_0 = Ae^{-a\tau}\left(1 + \frac{2}{3}a\tau + \frac{3|F|^2e^{2a\tau}\tau^2}{2a^2A^2}\right)\,, \tag{5.65}$$

where we have dropped subleading terms in the $(a\tau)^{-1}$ expansion multiplying the term proportional to $|F|^2$. In what follows, we shall drop such subleading terms. Using (5.65) in the expression for the scalar potential (5.64), its value at the minimum is found to be

$$\Lambda = -\frac{2e^{-2a\tau}a^2A^2}{3\ \tau} + \frac{|F|^2}{\tau^3} \equiv V^{(0)}_{\text{KKLT}} + V_{\text{up}}. \tag{5.66}$$

Note that the value of $|F|^2$ can be tuned to make the cosmological constant zero or extremely small and positive. For a Minkowski vacuum one needs

$$|F|^2 = \frac{4}{3}\,e^{-2a\tau}a^2A^2\tau^2\,. \tag{5.67}$$

The gravitino mass becomes

$$m_{3/2} = e^{K/2}|W| = \frac{W_0 - A\,e^{-a\tau}}{\tau^{3/2}}\,. \tag{5.68}$$

Next, we again turn to evaluating the joint distribution for the gravitino and cosmological constant. For this, we will perform the change of variables $(W_0, |F|^2, S) \to (m_{3/2}, \Lambda, S)$ in (5.62). We need to compute the Jacobian of this transformation. The procedure is completely analogous to the anti D3-brane uplift. The expressions for the cosmological constant and the gravitino mass ((5.66) and (5.68)) have explicit dependence on $(W_0, |F|^2, s)$ and also implicit dependence via τ. To compute the partial derivatives of the cosmological constant and the gravitino mass with respect to W_0 and $|F|^2$ we need to compute the partial derivatives of τ with respect to these variables. Making use of (5.65), we find

$$\frac{\partial\tau}{\partial|W_0|} = \frac{1}{f(\tau)}\,, \qquad \frac{\partial\tau}{\partial|F|^2} = -\frac{3e^{a\tau}}{4Aa^2\tau^2f(\tau)}\,, \tag{5.69}$$

with

$$f(\tau) = Ae^{-a\tau}\left(-\frac{2}{3}a^2\tau\left(1+\frac{1}{2a\tau}\right)+\frac{3|F|^2e^{2a\tau}}{4aA\tau^2}\left(1-\frac{2}{a\tau}\right)\right). \tag{5.70}$$

Now let us turn to the entries of the Jacobian. These are

$$\frac{\partial m_{3/2}}{\partial W_0} = \frac{1}{\tau^{3/2}} + \left(\frac{1}{\tau^{3/2}}aAe^{-a\tau} - \frac{3}{2\tau}m_{3/2}\right)\frac{\partial\tau}{\partial W_0}, \tag{5.71}$$

$$\frac{\partial m_{3/2}}{\partial|F|^2} = \left(\frac{1}{\tau^{3/2}}aAe^{-a\tau} - \frac{3}{2\tau}m_{3/2}\right)\frac{\partial\tau}{\partial|F|^2}. \tag{5.72}$$

We also have

$$\frac{\partial\Lambda}{\partial W_0} = \frac{\partial\Lambda}{\partial\tau}\frac{\partial\tau}{\partial W_0}, \qquad \frac{\partial\Lambda}{\partial|F|^2} = \frac{\partial\Lambda}{\partial\tau}\frac{\partial\tau}{\partial|F|^2} + \frac{1}{\tau^3}, \tag{5.73}$$

with

$$\frac{\partial\Lambda}{\partial\tau} = \frac{8e^{-2a\tau}a^3A^2}{3\tau}\left(1+\frac{1}{2a\tau}\right) - \frac{3|F|^2}{\tau^4}, \tag{5.74}$$

Analytical estimate

The expressions in (5.71), (5.72) and (5.73) are rather cumbersome. Let us now continue our analytic investigation in a regime that leads to considerable simplifications. For this, we define the quantity x as

$$x \equiv \frac{3|F|^2e^{2a\tau}}{4A^2a^2\tau^2}, \tag{5.75}$$

which allows to write (5.66) as

$$\Lambda = -\frac{4e^{-2a\tau}a^2A^2}{3\,\tau}(1-x) = -3m_{3/2}^2(1-x)\left(1+\frac{3}{2a\tau}x\right)^{-2}. \tag{5.76}$$

We shall now focus on the regime in which $|F|^2$ is varied such that $0 < x \lesssim \mathcal{O}(1) \ll (a\tau)$. Comparing with (5.76) we see that this allows for uplift to zero and positive values of the cosmological constant. Hence, the regime covers the region of most interest since $x \gg 1$ corresponds to an unstable situation where the uplifting contribution would yield a runaway. Also note that in this regime, the contribution of the term involving $|F|^2$ in right hand side of (5.65) becomes subdominant. We can therefore take the approximation

$$W_0 = e^{-a\tau}A\left(1+\frac{2}{3}a\tau+x\right) \simeq \frac{2}{3}A(a\tau)e^{-a\tau}, \tag{5.77}$$

and so, to leading order in the $(a\tau)^{-1}$ expansion, we have

$$\tau \simeq -\frac{1}{a}\ln\left(\frac{3}{2A}W_0\right), \tag{5.78}$$

and the gravitino mass looks like

$$m_{3/2} \simeq \frac{W_0}{\tau^{3/2}} = \frac{2}{3}\frac{(a\tau)A}{\tau^{3/2}}e^{-a\tau}. \tag{5.79}$$

This allows us to write

$$\tau \simeq -\frac{1}{a}\ln\left(\frac{3m_{3/2}}{2a^{3/2}A}\right). \tag{5.80}$$

In the regime we are considering, we can write (5.66) as

$$|F|^2 \simeq \tau^3\left(\Lambda + 3m_{3/2}^2\right). \tag{5.81}$$

Combining this with (5.80) we can express $|F|^2$ in terms of Λ and $m_{3/2}$. Similarly, making use of (5.80) in (5.79) gives W_0 in terms of the same quantities.

The Jacobian entries undergo significant simplifications in this regime where the function $f(\tau)$ defined in (5.70) becomes $f(\tau) \simeq -\frac{2}{3}a^2Ae^{-a\tau}\tau$. We then have

$$\frac{\partial\tau}{\partial W_0} \simeq -\frac{3}{2Aa^2\tau}e^{a\tau}, \qquad \frac{\partial\tau}{\partial|F|^2} \simeq \frac{9}{8A^2a^4\tau^3}e^{2a\tau}. \tag{5.82}$$

Let us turn to the partial derivatives of $m_{3/2}$. First note that by making use of (5.13) and (5.68) one can write

$$m_{3/2} = \frac{Ae^{-a\tau}}{3\tau^{3/2}}(2a\tau + 3x). \tag{5.83}$$

Recall now that

$$\frac{\partial m_{3/2}}{\partial W_0} = \frac{1}{\tau^{3/2}} + \left(\frac{aAe^{-a\tau}}{\tau^{3/2}} - \frac{3}{2\tau}m_{3/2}\right)\frac{\partial\tau}{\partial W_0}. \tag{5.84}$$

Making use of (5.83), we see that

$$\left(\frac{aAe^{-a\tau}}{\tau^{3/2}} - \frac{3}{2\tau}m_{3/2}\right) = -\frac{3Ae^{-a\tau}}{2\tau^{5/2}}x. \tag{5.85}$$

Using (5.82) and (5.85) in (5.84), and the fact that in the above equation $0 < x \lesssim \mathcal{O}(1)$, we see that the term proportional to $\frac{\partial\tau}{\partial W_0}$ is always subleading in (5.84). Therefore we have

$$\frac{\partial m_{3/2}}{\partial W_0} \simeq \frac{1}{\tau^{3/2}}. \tag{5.86}$$

Similarly, using the above formulae, one finds

$$\frac{\partial m_{3/2}}{\partial |F|^2} \simeq -\frac{27 x e^{a\tau}}{16 a^4 A \tau^{11/2}} \,. \tag{5.87}$$

Now, let us come to the derivatives of Λ. Note that the ratio of the two terms in (5.66) is $-x$. Given this (and the fact that $0 < x \lesssim \mathcal{O}(1)$), comparing the various terms in (5.74) gives

$$\frac{\partial \Lambda}{\partial \tau} \simeq \frac{8 e^{-2a\tau} a^3 A^2}{3\tau} \,. \tag{5.88}$$

Combining this with (5.21) and (5.30) one finds

$$\frac{\partial \Lambda}{\partial W_0} \simeq -\frac{4 a A e^{-a\tau}}{\tau^2} \qquad \text{and} \qquad \frac{\partial \Lambda}{\partial |F|^2} \simeq \frac{1}{\tau^3} \,. \tag{5.89}$$

Combining (5.86), (5.87) and (5.89), the Jacobian is

$$|J| \simeq \frac{1}{\tau^{9/2}} \,. \tag{5.90}$$

Finally, we have

$$\mathrm{d}\mathcal{N} = \frac{\eta\, W_0}{s^2}\, \mathrm{d}W_0\, \mathrm{d}|F|^2\, \mathrm{d}s = \frac{\eta\, W_0}{s^2} |J|^{-1}\, \mathrm{d}m_{3/2}\, \mathrm{d}\Lambda\, \mathrm{d}S \,. \tag{5.91}$$

Equations (5.79), (5.80) and (5.81) can be used to express this density in terms of the desired quantities. We find

$$\mathrm{d}\mathcal{N} \simeq \eta \frac{2 \left| \ln \left(\frac{3 m_{3/2}}{2 a^{3/2} A} \right) \right|^{11/2}}{a^6} m_{3/2}\, \mathrm{d}m_{3/2}\, \mathrm{d}\Lambda\, \mathrm{d}S \,. \tag{5.92}$$

This implies that the distribution is tilted favourably towards higher values of $m_{3/2}$. Moreover it is at leading order independent on the value of the cosmological constant.

5.2.2 LVS

Let us consider again the simplest LVS example, with two Kähler moduli T_s and T_b as in previous subsections. We again work at leading order in $a_s\tau_s \gg 1$, with uncorrected minimum conditions given by (5.46) and (5.47), which gives the AdS minimum in (5.48). Next, we consider the potential (5.44) that includes the $|F|^2$-contribution responsible for uplifting the AdS minimum

$$V_{\text{up}} = \frac{|F|^2}{\mathcal{V}^2} \,, \tag{5.93}$$

leading to

$$V = \frac{4a_s^2 A_s^2 e^{-2a_s\tau_s}\sqrt{\tau_s}}{3s\mathcal{V}} - \frac{2a_s A_s \tau_s |W_0|\, e^{-a_s\tau_s}}{s\mathcal{V}^2} + \frac{3|W_0|^2\xi\sqrt{s}}{8\mathcal{V}^3} + \frac{|F|^2}{\mathcal{V}^2}. \quad (5.94)$$

The minimum condition (5.46) is not modified, while (5.47) is changed to

$$\tau_s^{3/2} = \frac{\xi}{2}s^{3/2}\left(1+\frac{x}{s}\right)\left(1-\frac{1}{16a_s\tau_s}-\frac{9}{16}\frac{1}{1-a_s\tau_s}\right). \quad (5.95)$$

where we have defined

$$x \equiv \frac{16|F|^2\sqrt{s}\mathcal{V}}{9W_0^2\xi}. \quad (5.96)$$

Let us specify here the relation between the gravitino mass and the AdS minimum as follows

$$V_{\mathrm{LVS}}^{(0)} \equiv \Lambda_{\mathrm{AdS}} = -\frac{3s}{2\sqrt{2}a_s W_0}\left(\frac{\xi}{2}\right)^{1/3} m_{3/2}^3, \quad (5.97)$$

where

$$m_{3/2} = \frac{W_0}{\sqrt{2s}\mathcal{V}}. \quad (5.98)$$

The value of the potential at the minimum now becomes

$$\Lambda = \Lambda_{\mathrm{AdS}} + \frac{|F|^2}{3\mathcal{V}^2} = \Lambda_{\mathrm{AdS}}\left(1 - a_s x\left(\frac{\xi}{2}\right)^{2/3}\right), \quad (5.99)$$

which gives a Minkowski minimum for

$$|F|^2 = \frac{9W_0^2}{8a_s\mathcal{V}\sqrt{s}}\left(\frac{\xi}{2}\right)^{1/3} \quad \Leftrightarrow \quad x = \frac{1}{a_s}\left(\frac{2}{\xi}\right)^{2/3}. \quad (5.100)$$

Thus we realize that (5.99) can allow for uplift to zero and positive values of the cosmological constant provided $|F|^2$ is varied so that x lies in the $0 < x \lesssim \mathcal{O}(\frac{1}{a_s}\left(\frac{2}{\xi}\right)^{2/3}) \ll (a_s\tau_s)$ regime, with $x \gg 1$ corresponding to the runaway limit. Hence from now on we shall focus just on the interesting region where the potential is stable. Let us now perform a variable change in (5.62) to obtain the joint distribution of the gravitino mass and the cosmological constant: $(W_0, |F|^2, s, C_0) \to (W_0, \Lambda, m_{3/2}, C_0)$. Λ and $m_{3/2}$, given in (5.99) and (5.98), have explicit dependence on s and $|F|^2$ and also implicitly depend on the dilaton and the F-term via τ_s and $\mathcal{V}$. Note that, at leading order in the $(a_s\tau_s)^{-1}$ expansion, we have $\frac{\partial\mathcal{V}}{\partial\tau_s} = a_s\mathcal{V}$ and we get

$$\frac{\partial\tau_s}{\partial s} = \frac{2\tau_s}{3s}\frac{1+\frac{x}{2}}{1-\frac{2}{3}x\frac{a_s\tau_s}{s}}, \qquad \frac{\partial\tau_s}{\partial|F|^2} = \frac{2x\tau_s}{3|F|^2 s}\frac{1}{1-\frac{2}{3}x\frac{a_s\tau_s}{s}}. \quad (5.101)$$

Now we turn to the entries of the Jacobian (again, to leading order in the $(a_s\tau_s)^{-1}$ expansion). They read

$$\frac{\partial m_{3/2}}{\partial s} = -a_s m_{3/2}\frac{\partial \tau_s}{\partial s}\,, \qquad \frac{\partial m_{3/2}}{\partial |F|^2} = -a_s m_{3/2}\frac{\partial \tau_s}{\partial |F|^2}\,, \tag{5.102}$$

$$\frac{\partial \Lambda}{\partial s} = \frac{2\Lambda}{m_{3/2}}\frac{\partial m_{3/2}}{\partial s} + \Lambda_{\mathsf{AdS}}\left(\frac{1}{m_{3/2}}\frac{\partial m_{3/2}}{\partial s} + \frac{1}{s}\right)\,, \tag{5.103}$$

$$\frac{\partial \Lambda}{\partial |F|^2} = \frac{\Lambda}{|F|^2}\left(1 + \frac{2|F|^2}{m_{3/2}}\frac{\partial m_{3/2}}{\partial |F|^2}\right) - \frac{\Lambda}{|F|^2}\left(1 - \frac{|F|^2}{m_{3/2}}\frac{\partial m_{3/2}}{\partial |F|^2}\right)\,, \tag{5.104}$$

where we have used (5.101). These give a Jacobian of the form, at the leading order

$$|J| = \frac{a_s m_{3/2}}{3|F|^2}\left(\frac{\xi}{2}\right)^{2/3}\frac{x+2}{1-\frac{2}{3}a_s x\left(\frac{\xi}{2}\right)^{2/3}}\left(\Lambda_{\mathsf{AdS}} - \Lambda\right)\,. \tag{5.105}$$

Knowing that

$$\tau_s \simeq -\frac{1}{a_s}\ln\frac{3m_{3/2}}{2\sqrt{2}a_s^2A_s^2\left(\frac{\xi}{2}\right)^{1/3}} \qquad \mathcal{V} \simeq \frac{W_0}{m_{3/2}\sqrt{\frac{2}{a_s}\ln\frac{2\sqrt{2}a_s^2A_s^2\left(\frac{\xi}{2}\right)^{1/3}}{3m_{3/2}}}}\left(\frac{\xi}{2}\right)^{1/3}\,, \tag{5.106}$$

and from (5.99) we can write

$$|F|^2 = 3\mathcal{V}^2(\Lambda - \Lambda_{\mathsf{AdS}}) \sim \frac{\Lambda + 3m_{3/2}^3\ln m_{3/2}}{m_{3/2}^2\ln m_{3/2}} \xrightarrow{\Lambda\to 0} 3m_{3/2}\,. \tag{5.107}$$

In summary, we find

$$\mathrm{d}\mathcal{N} = \frac{\eta W_0}{s^2}|J|^{-1}\,\mathrm{d}W_0\,\mathrm{d}m_{3/2}\,\mathrm{d}\Lambda\,\mathrm{d}C_0\,, \tag{5.108}$$

with J given in (5.105) and $\tau_s = s\left(\frac{\xi}{2}\right)^{2/3} \simeq -\frac{1}{a_s}\ln m_{3/2}$. To get the final form of the distribution, one should integrate over C_0 and W_0. Even if this is not tractable, the interesting features of the joint distribution can be obtained by considering fixed values of C_0 and W_0 (which is an $\mathcal{O}(1-10)$ quantity in LVS). In the physically interesting regime where $|\Lambda| \ll m_{3/2}^3$, we can also make the following approximation

$$|J| \sim \frac{m_{3/2}}{|F|^2}|\Lambda_{\mathsf{AdS}}| \sim m_{3/2}^3\ln m_{3/2}\,, \tag{5.109}$$

which gives

$$\rho_{\mathsf{LVS}}(\Lambda \simeq 0) \simeq \frac{1}{m_{3/2}^3(\ln m_{3/2})^3}\,. \tag{5.110}$$

Interestingly, we find, as in the throat case, that the distribution of flux vacua with a cosmological constant close to zero is highly tilted towards lower values of $m_{3/2}$.

Chapter VI

Conclusions

In this work, we have thoroughly analysed various aspects of moduli stabilization in flux compactification, uncovering several new results and insights on the topic.

In Chap. 2 we have presented a novel method to obtain type IIB flux vacua with flat directions at tree level. The key idea is to make choices for flux quanta so that there are relations between the flux superpotential and its derivatives. These relations ensure that the equations of motion are satisfied. We implemented this method in toroidal and Calabi-Yau compactifications in the large complex structure limit. Explicit solutions were obtained and classified on the basis of duality equivalences. In the toroidal setting we presented solutions with both $N = 1$ and $N = 2$ supersymmetry. For the $\mathbb{CP}_{[1,1,1,6,9]}[18]$ CY, on top of solutions which were already known in the literature, we found 15 novel perturbatively flat vacua with approximate flat directions where the superpotential is not a homogeneous function of degree 2. We also presented solutions with $W \neq 0$ which might lead to an explicit realization of winding dS uplift. We also performed a preliminary analysis of flux vacua for the CY considered in [134] finding supersymmetric solutions with 2 approximate flat directions.

The flat directions studied in this chapter have interesting phenomenological implications. Before mentioning some of them, let us stress that these flat directions are approximate since they are expected to be lifted by subleading effects at either perturbative or non-perturbative level. In the $T^6/\mathbb{Z}_2$ case, a non-zero W should be generated by non-perturbative effects which depend on the Kähler moduli. A non-zero scalar potential for the leading order flat directions is then generated by the U-dependence of the prefactor of non-perturbative effects and the coefficients of α' and string loop corrections to the Kähler potential. Moreover one should carefully check potential modifications of the primitivity condition by quantum corrections.

For the CY cases, the flat direction of perturbative flat vacua with $W = 0$ should be lifted by instantons along the lines of [12]. On the other hand, for solutions with $W \neq 0$, the imaginary part of the approximate flat direction would be lifted already at perturbative level by including U-dependent effects that arise from the supergravity contribution to the Kähler covariant derivative WK_U.

Let us now briefly discuss several potential applications to phenomenology of approximate flat directions:

1. **Kähler moduli stabilization:** There are various mechanisms for stabilizing the Kähler moduli in type IIB (see for example [18, 27, 40, 48, 97–102]). In particular, an exponentially low W_0 is crucial for KKLT constructions [18]. Flux vacua with $W = 0$ and flat directions have been shown to be a promising starting point to realise these scenarios [12–16, 23, 24].

   Even if not strictly required, very low values of W_0 might be needed also in some dS LVS constructions where the visible sector lives on D3-branes at singularities [28]. In these models consistency conditions, like D7-tadpole and Freed-Witten anomaly cancellation, in general induce a T-brane background which yields a positive contribution to the scalar potential in the presence of background 3-form fluxes [29]. In [28], this contribution has been shown to be able to give a dS minimum for exponentially small values of W_0. On the other hand, [30] presented a different global model with D3-branes at singularities where dS moduli stabilization with T-brane uplifting can be achieved also for $W_0 \sim \mathcal{O}(1)$.

2. **Winding uplift:** Solutions with $W \neq 0$ could be a promising starting point for explicit realizations of dS uplifting via exponentially small F-terms of the complex structure moduli [38]. The idea is to have at leading order supersymmetric solutions in the large complex structure limit with $W \neq 0$ and 1 axionic flat direction. In turn, this axion is lifted by instantons which induce exponentially suppressed but non-zero F-terms for the complex structure moduli, so leading to a tunable (via flux choices) and positive uplifting contribution to the scalar potential.

   To be more explicit, the $W \neq 0$ solutions discussed in Sec. 2.3.2 for the $\mathbb{CP}_{[1,1,1,6,9]}[18]$ case, feature $\partial_1 W = \lambda_2 \partial_2 W + \lambda_3 \partial_3 W$. Thus solving the full supergravity F-term equations $\partial_a W + W \partial_a K = 0 \; \forall a = 1, 2, 3$, is equivalent to solving:

$$\partial_1 W = 0 - W\partial_1 K\,, \tag{6.1}$$
$$\partial_2 W = 0 - W\partial_2 K\,, \tag{6.2}$$
$$\partial_1 K = \lambda_2 \partial_2 K + \lambda_3 \partial_3 K\,. \tag{6.3}$$

Eq. (6.1) and (6.2) are 2 complex equations, and so would fix the 2 complex moduli U_1 and U_2 in terms of U_3. Moreover their solutions would very well be approximated by the solutions to $\partial_1 W = \partial_2 W = 0$ already found in Sec. 2.3.2, if the minimum is such that the supergravity corrections are infinitesimally small in the large complex structure limit. Finally (6.3) is a real equation since K is just a function of the imaginary parts of the complex structure moduli and the axio-dilaton in the large complex structure limit (denoting Im $(U_a) \equiv u_a$ $\forall a = 1, 2, 3$):

$$K = -\ln\left[4u_1\left(3u_1^2 + 3u_1u_2 + u_2^2\right) - 4\text{Im}\,(\xi)\right] - \ln(2u_3)\,. \tag{6.4}$$

Thus (6.3) should fix only Im (U_3), leaving Re (U_3) as the only axionic flat direction that is expected to be lifted by instanton corrections to the prepotential. In the large complex structure limit these contributions would be exponentially suppressed by $e^{-\text{Im}\,(U_1)} \ll 1$ and $e^{-\text{Im}\,(U_2)} \ll 1$.

3. **Cosmology:** Approximate flat directions have natural applications to cosmology where inflaton fields are required to be lighter than the Hubble scale during inflation to be in the slow-roll regime. In fact, flat directions in the type IIB flux superpotential have already been used in [31] to enlarge the inflaton field range, and more recently in [32] to build models of sequestered inflation. Approximate flat directions could be promising candidates also to drive the present day accelerated expansion of our universe since quintessence fields need to be very light to reproduce the observed cosmological constant scale. Moreover, leading order flat directions can help to avoid any destabilization problem coming from contributions to the dark energy potential due to the large inflationary energy scale [33, 34].

4. **Supersymmetry breaking:** Leading order flat directions can also play a relevant role in any model of supersymmetry breaking if $W = 0$ at classical level. In fact, in this case the F-terms of the Kähler moduli are vanishing at leading order and the effective field theory after integrating out the heavy complex structure moduli has

to include the Kähler moduli and all the complex structure moduli, including the axio-dilaton, which are massless at leading order [35]. The dynamics which stabilizes the Kähler moduli and the leading order flat directions is expected to break supersymmetry and to develop non-zero F-terms for all these fields which will play an important role in generating soft supersymmetry breaking terms. The F-term of the dilaton would be particularly important in D3-brane models with sequestered supersymmetry breaking where it is the main source for generating non-zero gaugino masses [36, 37].

5. **Statistics in the landscape:** The statistical approach to string phenomenology has received a lot of attention during the last two decades (see e.g. [103–107, 123–131, 303]). Trying to achieve a complete classification of flux vacua with exponentially small W_0 is crucial to understand the statistical significance of these vacua. The analysis of [129, 130] implies that if W_0 is uniformly distributed at very small values, then the scale of supersymmetry breaking has a power-law distribution, while if W_0 is exponentially small in the dilaton, as in the models of [12], then the gravitino mass has a logarithmic distribution. Preliminary steps in understanding the statistical significance of perturbatively flat vacua were taken in [23] which found that they represent a small fraction of the full set of vacua at low W_0 as estimated in [105]. Our work goes in the direction to explore novel classes of vacua at low W_0 to enrich their knowledge.

6. **CRG Conjecture:** The solutions found should be interesting in the context of studies on the consistency conditions of 4 graviton scattering in the classical limit (see [304, 305]). The solutions obtained are warped Minkowski compactifications in which the string coupling can be tuned to arbitrarily small values. The solutions are in the supergravity approximation. Developing a precise understanding of the fate of the solutions beyond the supergravity approximations, i.e. checking if there can be a solution where the flat direction survives to all orders in α',[1] (with the solution remaining Minkowski) and the study of 4 graviton scattering in these backgrounds is relevant in the context of the classical Regge growth conjecture.

In Chap. 3 we presented a general discussion of the quantitative effect of higher derivative F^4 corrections to the scalar potential of type IIB flux

[1] Warping dependent corrections would also have to be incorporated, see e.g. [306–309].

compactifications. In particular, we discussed the topological taming of these corrections which a priori might appear to have an important impact on well-established LVS models of inflation such as blow-up inflation, fibre inflation and poly-instanton inflation.

These F^4 corrections are not captured by the two-derivative approach where the scalar potential is computed from the Kähler potential and the superpotential, since they directly arise from the dimensional reduction of 10D higher derivative terms. In addition, such a contribution to the effective 4D scalar potential turns out to be directly proportional to topological quantities, Π_i, which are defined in terms of the second Chern class of the CY threefold and the (1,1)-form dual to a given divisor D_i. The fact that these higher derivative F^4 terms have topological coefficients has allowed us to perform a detailed classification of all possible divisor topologies with $\Pi = 0$ that would lead to a topological taming of these corrections. In particular, we have found that the divisors with vanishing Π satisfy $\chi(D) = 6\chi_h(D)$ which is also equivalent to the following relation among their Hodge numbers: $h^{1,1}(D) = 4\,h^{0,0}(D) - 2\,h^{1,0}(D) + 4\,h^{2,0}(D)$. In order to illustrate our classification, we presented some concrete topologies with $\Pi = 0$ which are already familiar in the literature. These are, for example, the 4-torus $\mathbb{T}^4$, the del Pezzo surface of degree-6 dP$_3$, and the so-called 'Wilson' divisor with $h^{1,1}(W) = 2$.

In search of seeking for divisors of vanishing Π, we investigated all (coordinate) divisor topologies of the CY geometries arising from the 4D reflexive polytopes of the Kreuzer-Skarke database. This corresponds to scanning the Hodge numbers of around 140000 divisors corresponding to roughly 16000 distinct CY geometries with $1 \leq h^{1,1}(X) \leq 5$. In our detailed analysis, we have found only two types of divisors of vanishing Π: the dP$_3$ surface and the 'Wilson' divisor with $h^{1,1}(W) = 2$.

In addition to presenting the scanning results for classifying the divisors of vanishing Π, we have also presented a classification of CY geometries suitable to realise LVS moduli stabilization and three different inflationary models, namely blow-up inflation, fibre inflation and poly-instanton inflation. Subsequently, we studied numerically the effect of F^4 corrections on these inflation models in the generic case where the inflaton is not a divisor with vanishing Π. In this regards, we performed a detailed analysis of the post-inflationary evolution to determine the exact number of efoldings of inflation to make contact with actual CMB data. When the coefficients of the F^4 corrections are non-zero, we found that they generically do not spoil the predictions for the main cosmological observables. A crucial help comes from the $(2\pi)^{-4}$ suppression factor present in (3.16) which gives the coefficient of higher derivative corrections for the $h^{1,1}(X) = 1$ case. However, we argued that this suppression factor should be universally present in all F^4 corrections of the kind presented in this work, even for cases with $h^{1,1}(X) > 1$.

Let us finally mention that our detailed numerical analysis shows that all the three LVS inflationary models, namely blow-up inflation, fibre inflation and poly-instanton inflation, turn out to be robust and stable against higher derivative α'^3 corrections, even for the cases when such effects are not completely absent thanks to appropriate divisor topologies in the underlying CY orientifold construction. In some cases, like in blow-up inflation, we have even found that such corrections can help to improve the agreement with CMB data of the prediction of the scalar spectral index.

It is however important to stress that these are not the only corrections which can spoil the flatness of LVS inflationary potentials. To make these models more robust, one should study in detail the effect of additional corrections, like for example string loop corrections to the potential of blow-up and poly-instanton inflation. In this chapter, we have assumed that these corrections can be made negligible by considering values of the string coupling which are small enough, or tiny flux-dependent coefficients. However, this assumption definitely needs a deeper analysis since in LVS the overall volume is exponentially dependent on the string coupling, and $\mathcal{V}$ during inflation is fixed by the requirement of matching the observed value of the amplitude of the primordial density perturbations. Therefore taking very small values of g_s to tame string loops might lead to a volume which is too large to match A_s. We leave this interesting analysis for future work.

In Chap. 4 we have performed a detailed analysis of the theoretical and phenomenological requirements to realize a viable EDE model [72] from string theory. We have focused on KKLT and LVS models in type IIB flux compactifications which are the best developed scenarios for moduli stabilization. Following the idea proposed in [71], we have tried to reproduce the EDE potential by exploiting 3 non-perturbative corrections to the effective action, considering both C_4 and C_2 axions. The outcome of our investigation is a set of working models, amongst which the most promising candidates to realize EDE in type IIB string theory are C_2 axions with a potential generated by gaugino condensation on D7-branes with non-zero world-volume fluxes. In this case the EDE scale and decay constant can be matched without tuning any of the underlying parameters and with the effective field theory approach under control. Let us explain in simple terms how we got to this conclusion by discussing the challenges outlined in Chap. 4:

1. **Controlled de Sitter moduli stabilization:** As already pointed out, KKLT and LVS are well-studied frameworks for moduli stabilization. However, the main requirement, in both cases, to trust the low-energy supergravity approximation is that the internal volume $\mathcal{V}$ is stabilized at large values to keep control over α' corrections. More precisely, the dimensionful CY volume can be expressed as Vol $= \mathcal{V}\,\ell_s^6$ (with $\ell_s = 2\pi\sqrt{\alpha'}$), implying that the parameter controlling the α' expansion is $\epsilon_{\alpha'} = \alpha'\,\text{Vol}^{-1/3} \simeq \mathcal{V}^{-1/3}$. In the simplest compactification with just a single Kähler modulus $\tau \simeq \mathcal{V}^{2/3}$, we need therefore to ensure that $\tau \gtrsim \mathcal{O}(100)$ so that $\epsilon_{\alpha'} \lesssim 0.1$. Writing the EDE decay constant f in terms of the instanton action $S = 2\pi\tau/M$, with M the number of branes, as $f\,S \simeq \lambda M_P$, as we did in Chap. , we easily see that matching $f \simeq 0.2\,M_P$ implies

$$\tau \simeq \frac{\lambda\,M}{2\pi}\left(\frac{M_P}{f}\right) \simeq \lambda\,M\,. \tag{6.5}$$

   As found in [183], C_4 axions with potential generated by ED3/D7 effects and C_2 axions with potential generated by fluxed ED1/D5 effects feature $\lambda \sim \mathcal{O}(1)$, in agreement with expectations from the weak gravity conjecture applied to axions [238–241]. In this case, $\tau \gtrsim \mathcal{O}(100)$ can be achieved only by considering $M \gtrsim \mathcal{O}(100)$. On the other hand, C_2 axions with potential generated by fluxed ED3/D7 effects can lead to a violation of the weak gravity conjecture since they are characterized by $\lambda \simeq \sqrt{g_s\tau}$ [183]. In this case, (6.5) reduces to $\tau \simeq g_s\,M^2$ which could give $\tau \gtrsim \mathcal{O}(100)$ for $M \gtrsim \mathcal{O}(30)$ if the string coupling is fixed (by an appropriate choice of background 3-form fluxes) at $g_s \lesssim \mathcal{O}(0.1)$ so that string perturbation theory does not break down. Hence, in all cases we are forced to consider situations with a relatively large number of branes.

2. **Decoupling of non-EDE modes:** This requirement is crucial to ensure that the EDE dynamics is not affected by any other field. The cleanest situation is therefore the one where all the non-EDE modes are stabilized at an energy scale which is higher than the EDE one. This observation implies that C_0 and B_2 axions are not well-suited to realize EDE since their shift symmetry is broken at perturbative level. Best candidates are instead C_2 and C_4 axions whose shift symmetry is broken only at non-perturbative level. More precisely, C_2 axion are in principle good EDE candidates in both KKLT and LVS models,

while C_4 axions can play the role of the EDE field only in LVS models since in KKLT they would be as heavy as the corresponding saxions, thus inducing a cosmological moduli problem.

3. **Absence of fine-tuning:** Two levels of fine-tuning can be necessary to reproduce the EDE potential: a tuning to get the right periodicity, and an additional tuning to match the EDE scale. We found that C_4 axions in LVS require both tunings, and so appear to be the worst EDE candidates. C_2 axions with potential generated by ED1/D5 corrections to the Kähler potential can instead reproduce the required periodicity naturally but need tuning to obtain the correct EDE scale, and so do not seem to be optimal EDE fields. The best EDE candidates are instead C_2 axions with potential generated by fluxed ED3/D7 corrections to the superpotential since they can, in principle, avoid both tunings.

   These results can be intuitively understood as follows. As explained in Sec. 4.4.1, the EDE periodicity can be naturally realized only when the saxionic partner of the EDE axion is stabilized at zero. The saxion associated to C_4 controls the volume of a 4-cycle which cannot be set to zero since it would cause a deviation from the supergravity approximation. This implies that EDE models based on C_4 axion require tuning. On the other hand, the saxion associated to C_2 is the B_2 axion which is naturally fixed at $b = 0$, implying that C_2 axions can realize the EDE periodicity in a more natural way. Regarding instead the matching of the EDE scale V_0 without any tuning of the UV parameters, as already explained in the introduction, this requires a violation of the weak gravity conjecture. In fact, (4.1) with $f \simeq 0.2\, M_P$, becomes

$$V_0 \simeq A\, e^{-\lambda M_P/f}\, M_P^4 \simeq A\, e^{-5\lambda}\, M_P^4 \simeq 10^{-108}\, M_P^4 \quad \text{for} \quad \lambda \simeq 50 \quad \text{if} \quad A \simeq 1\,. \tag{6.6}$$

   As we have already seen, C_4 axions with ED3/D7 effects and C_2 axions with fluxed ED1/D5 effects have $\lambda \sim \mathcal{O}(1)$, and so can match $V_0 \sim 10^{-108}\, M_P^4$ only by tuning A to exponentially small values. On the contrary, C_2 axions with fluxed ED3/D7 effects feature

$$\lambda \simeq \sqrt{g_s \tau} \simeq \left(\frac{g_s\, M_P}{f}\right) \frac{3M}{4\pi} \simeq 0.2\, M \quad \text{for} \quad g_s \simeq 0.2 \quad \text{and} \quad f \simeq 0.2\, M_P\,. \tag{6.7}$$

   Hence, $\lambda \simeq 50$ can be achieved for $A \simeq 1$ and $M \sim \mathcal{O}(100)$, implying that V_0 can be realized without tuning only for gaugino condensation on D7-branes since ED3-instantons, if the corresponding action is written as $S = 2\pi\tau/M$, can allow only for $M = 1/p$ with $p \in \mathbb{N}$. In turn, such a relatively large number of D7-branes ensures that the effective field theory is fully under control since (6.5) combined with (6.7) implies $\tau \simeq 0.2\, M^2 \sim \mathcal{O}(5 \times 10^3)$. Such a large value of τ can be easily realized in LVS models, while it would imply a very low gravitino mass in KKLT scenarios where $m_{3/2} \sim e^{-2\pi\tau/N} M_P$ where N is the number of D7-branes supporting the gaugino condensate that lifts the volume modulus. Requiring $m_{3/2} \gtrsim \mathcal{O}(1)$ TeV to ensure $m_{\mathcal{V}} \gtrsim 50$ TeV, implies $N \gtrsim \mathcal{O}(1000)$ which is very difficult to achieve in controlled CY orientifold compactifications with D7 tadpole cancellation. Thus, the only way-out in KKLT models to avoid such a huge number of D7-branes seems to involve again an exponential tuning of the prefactor A.

This problem is absent in LVS models. As explained in Sec. 4.4.2, C_2 axions with potential generated by 3 gaugino condensates on D7-branes with non-zero gauge fluxes can realize EDE without any tuning of the microscopic parameters. Two scenarios arise, depending on the topology of the underlying CY threefold. If the compactification space has a Swiss-cheese structure, τ is identified with the overall volume $\mathcal{V} \sim \tau^{3/2} \sim \mathcal{O}(10^5)$, leading to $m_{3/2} \sim \mathcal{O}(10^{13})$ GeV and $m_{\mathcal{V}} \sim \mathcal{O}(10^{10})$ GeV. If instead the internal space is a K3-fibred CY, the overall volume is controlled by 2 divisors $\mathcal{V} \simeq \sqrt{\tau_1}\tau_2$ and τ can be identified with the fibre modulus τ_1. Thus, matching $f \simeq 0.2\, M_P$ fixes only $\tau_1 \sim \mathcal{O}(5 \times 10^3)$, but not $\mathcal{V}$, which can therefore be larger than in the Swiss-cheese case if moduli stabilization yields an anisotropic CY with $\tau_2 \gg \tau_1 \gg 1$. In fact, we have obtained $\mathcal{V} \sim \mathcal{O}(10^8)$, which improves the control over the effective field theory and leads to lower moduli masses: $m_{3/2} \sim \mathcal{O}(10^{10})$ GeV, $m_{\mathcal{V}} \sim \mathcal{O}(10^6)$ GeV and $m_u \sim 50$ TeV (where u is the direction in the (τ_1-τ_2)-plane orthogonal to the volume mode $\mathcal{V}$).

4. **Explicit Calabi-Yau realization:** Our analysis outlines the features that a globally consistent compactification should have to realize a viable EDE model. The next step, to build a full-fledged string model, would be to provide a rigorous description of the underlying Calabi-Yau threefold, orientifold involution and brane setup in a way compatible with tadpole cancellation. The setup should also explicitly realize 3 gaugino condensates on fluxed D7-branes wrapping different homologous representatives of the same divisor, or on a

single stack of D7-branes where however some subsets of branes are differently magnetized. Such a detailed construction is beyond the scope of this study and we leave it for future work.

We conclude that our analysis establishes EDE as a viable model of string cosmology. While the model-building presented here is to some degree contrived, being designed to yield the $[1-\cos(\varphi/f)]^3$ EDE potential, it makes use of well-known and well-studied ingredients and does not rely on any exponential tuning of UV parameters for C_2 axions in LVS with gaugino condensation on fluxed D7-branes. In this sense, realizing our EDE models does not seem considerably harder than constructing other scenarios in string cosmology such as quintessence and inflation.

This is an important step towards understanding and developing the predictions of the model: with a concrete model realization in hand, one may then investigate the interactions of the EDE field with other fields which generate complementary indirect signals of the EDE dynamics, e.g. gravitational waves from the coupling to gauge fields [248], and may identify other ingredients in the model that play a cosmological role, such as an ultra-light axion component of dark matter. An additional interesting future direction of investigation is to unify these EDE constructions with other epochs of cosmic acceleration, namely with cosmic inflation or late-time dark energy. In the context of LVS, EDE seems particularly well suited to incorporating fibre inflation [47, 48, 51–54], and the use of a C_2 axion to generate an observable level of chiral primordial gravitational waves [310]. There is also a natural compatibility of EDE presented here with fuzzy dark matter as presented in [183], wherein one C_2 axion in the compactification could play the role of fuzzy dark matter and another the EDE. It would be interesting to revisit in this context some of the observables of fuzzy dark matter, such as dark matter substructure [311]. Moreover, an orthogonal approach would be to consider other possible EDE candidates, such as ultralight scalars that are composite states of fermions, following related work on ultralight dark matter, e.g. [312–314]. We leave these interesting avenues to future research.

In Chap. 5 we have studied the joint distribution of the gravitino mass and cosmological constant in KKLT and LVS models with the uplift sector arising from an anti-brane at the tip of a warped throat. We have found that, at values of the cosmological constant close to zero, the gravitino mass distribution is tilted favourably towards lower scales. This result is different from that based on generic expectations for the size of F- and D-terms [123, 124], including also moduli stabilization [129]. The form of the distribution of the throat hierarchies and the nature of the AdS vacua before the introduction of the uplift sector[2] leads to this difference. This work gives strong motivation for similar studies in related setups so that a better understanding of the joint statistics of the scale of supersymmetry breaking and the cosmological constant in string vacua can be obtained.

Our results have several interesting implications which we now briefly discuss:

- **Distribution of soft terms:** Supersymmetry breaking in a hidden sector leads to the generation of soft terms in the visible sector. The strength of supersymmetry breaking in the visible sector is characterized by the size of the scalar masses m_0^2, gaugino masses $M_{1/2}$ and trilinear couplings A_{ijk}. In both KKLT and LVS, these depend on how the Standard Model is realise on either D3- or D7-branes. Soft terms for both realizations were analysed in [164] (see also [35, 37, 163, 315–318]). In the case of D7-branes, all soft terms tend to be of order the gravitino mass. On the other hand, in the case of D3-branes, the models can exhibit sequestering at tree level with non-zero soft masses generated by α' and quantum corrections. One can find interesting patterns in the structure of soft masses, allowing for the realization of MSSM-like spectra or (mini) split supersymmetry. All the soft parameters are of the form $m_{3/2}^{1+p}$ with $p \geq 0$ or $m_{3/2}/(\ln(m_{3/2}))^q$ with $q > 0$. This implies that the tilt in the distribution favouring lower values of $m_{3/2}$ also corresponds to the same for the scale of supersymmetry breaking in the visible sector.

- **Comparison with data:** Statistical distributions of vacua have been used to confront classes of string vacua with data (see e.g. [319–326]) by examining the implications of the distributions of UV parameters for low energy observables. These studies have focused on power-law and logarithmic distributions with a preference for high scale supersymmetry. It will be interesting to carry out similar analysis with our results given that the tilt in favour of low scale supersymmetry is likely to help to make contact with observations or even to generate some tension with data. Of course, this line of study assumes that the distributions computed by studying the distribution of vacua should be translated to distributions for predic-

[2] This is one of the factors that contributes to the difference in the result for KKLT and LVS models.

tions for experiments[3].

- **Multiple throats:** The simplest generalization of our setup is to consider multiple throats (this was studied in the context of supersymmetry breaking in [123]). We examine it to understand the effect of multiple supersymmetry breaking sectors.

  Let us consider n throats, each with a single anti-brane. Taking n to be small, so that the distribution in the throat sector factorizes,[4] we have

$$\mathrm{d}\mathcal{N}_{\text{th}} \propto \prod_{i=1}^{n} \frac{1}{y_i (\ln y_i)^2} \prod_{i=1}^{n} \mathrm{d}y_i \,. \tag{6.8}$$

  The uplift term is given by

$$V_{\text{up}} = \frac{1}{2s\tau^2} \sum_{i=1}^{n} y_i \,. \tag{6.9}$$

  It is useful to make a series of variable changes: $y_i = w_i^2$, then to spherical coordinates in w_i (with $r^2 = \sum_{i=1}^{n} (w_i)^2$) and angular variables ($\theta_j, j = 1, ..., (n-1)$) and finally $\tilde{r} = r^2$. With this, the distribution (6.10) takes the form

$$\mathrm{d}\mathcal{N}_{\text{th}} \propto \frac{1}{2\tilde{r} g(\theta_i) \prod_{i=1}^{n} (\ln w_i)^2} \mathrm{d}\Omega_{n-1} \, \mathrm{d}\tilde{r} \,, \tag{6.10}$$

  where $g(\theta_i)$ is a function of the angular variable. In these coordinates, the uplift term is given by

$$V_{\text{up}} = \frac{\tilde{r}}{2s\tau^2} \,, \tag{6.11}$$

  it is independent of the angular coordinates and has exactly the same form as the single variable case. To obtain the joint distribution of the gravitino mass and the cosmological constant, consider the full distribution function

$$\mathrm{d}\mathcal{N} = \frac{\eta' \, W_0}{s^2} \, \mathrm{d}S \, \mathrm{d}W_0 \, \mathrm{d}\mathcal{N}_{\text{th}} \,, \tag{6.12}$$

  and make the variable change $(W_0, \tilde{r}, s, C_0, \theta_i) \to (W_0, \Lambda, m_{3/2}, C_0, \theta_i)$ for LVS and $(W_0, \tilde{r}, S, \theta_i) \to (m_{3/2}, \Lambda, S, \theta_i)$ for KKLT. Since the uplift

[3] Emergence of selection principle(s) in the space of vacua (which can arise from early universe cosmology) can invalidate this assumption.

[4] Factorization will break down if there are many throats. At present, we do not have the tools to compute the distributions in such cases.

term has the same functional form as in the single variable case, also the functional form of the Jacobian is the same as in the single variable case. As a result, the $\tilde{r}$ dependence of the density is similar to the y dependence in the single variable case. Thus in the $\Lambda \to 0$ limit, the functional dependence on $m_{3/2}$ is the same as in the single throat case (up to logarithmic terms). We conclude that the distribution function of throats is such that adding multiple sectors (but small in number) preserves the tilt in the distribution function in favour of lower values of $m_{3/2}$.

- **Sensitivity to uplift:** The tilt in the distribution for $m_{3/2}$ towards lower values is tied to the distribution for throat hierarchies. In particular, the y^{-1} factor in the distribution for hierarchies plays a crucial role. Thus our result should not be taken as providing the general picture for the distribution of $m_{3/2}$ and the cosmological constant. In order to gain a general understanding, the distribution has to be studied for various uplift mechanisms (see e.g. [29, 40, 98, 100, 101, 152]). To gain a general understanding of the joint distribution of the supersymmetry breaking and the cosmological constant scales in the whole flux landscape, one should then be able to determine the relative abundance of vacua characterized by different uplifting mechanisms. Our work represents just the first step forward in this direction. We have thus provided preliminary results involving a second uplift scheme, the small F-term uplift, which confirms this insight. We leave this important direction for future work.

- **Quantum corrections:** The distributions that we have derived make use of the Wilsonian effective field theory obtained at the high compactification scale after integrating out stringy and Kaluza-Klein states. Physical quantities will be however affected by low energy quantum loops. These corrections can be large for the cosmological constant, i.e. $\mathcal{O}(m_{3/2}^4)$, but the conditions used to obtain the small Λ limit of the distributions, i.e. $\Lambda \ll m_{3/2}^2$ in KKLT and $\Lambda \ll m_{3/2}^3$ in LVS, are stable against such corrections. Hence the form of the distribution functions obtained, is expected to be stable against the incorporation of quantum effects.

Appendix A

Duality transformations in type IIB

In this Appendix we briefly summarize the dualities relevant for our discussion and record our conventions for Chap 2.

$SL(2,\mathbb{Z})$ symmetry: The type IIB theory enjoys an $SL(2,\mathbb{Z})$ symmetry. Under this, the 3-form flux and the axio-dilaton transform as:

$$\begin{pmatrix} H_3 \\ F_3 \end{pmatrix} \to \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} H_3 \\ F_3 \end{pmatrix}, \qquad \phi \to \phi' = \frac{a\phi + b}{c\phi + d}, \tag{A.1}$$

where:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}). \tag{A.2}$$

Note that $\int F_3 \wedge H_3$ is invariant under this transformation, implying that the D3-charge of a flux configuration is invariant. However, the superpotential transforms as:

$$W[U^a, \phi] \qquad \to \qquad W'[U^a, \phi'] = \frac{W[U^a, \phi(\phi')]}{c\phi(\phi') + d}, \tag{A.3}$$

where U^a are the complex structure moduli and $\phi(\phi')$ is obtained inverting (A.1).

$SL(6,\mathbb{Z})$ symmetry for T^6: T^6 is obtained as the quotient of $\mathbb{R}^6$ by a 6D lattice, and $SL(6,\mathbb{Z})$ matrices relate the different choices of basis of the same lattice. An $SL(6,\mathbb{Z})$ action transforms the fluxes as well as the period matrix. 2 flux configurations are equivalent (or dual) if the fluxes and the solution for the complex structure moduli are related by an $SL(6,\mathbb{Z})$ transformation. In our solutions the T^6 is factorized into $T^2 \times T^2 \times T^2$. The relevant $SL(6,\mathbb{Z})$ transformations are the ones that permute the 3 2-tori and the $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$ subgroup that acts on each of the 3 tori. The action of each of these $SL(2,\mathbb{Z})$ on their respective tori is as follows. The coordinates on the 2-torus transform as:[1]

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \tag{A.4}$$

where we can think of the primed coordinates as the new coordinates and the unprimed ones as the old ones. For the complex structure of the 2-torus we have:[2]

$$U' = \frac{dU + b}{cU + a}. \tag{A.5}$$

An $SL(2,\mathbb{Z})$ transformation can be generated by the successive action of $\mathcal{T}$- and $\mathcal{S}$-transformations given by:

$$\mathcal{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \mathcal{T} : U \to U + 1, \tag{A.6}$$

$$\mathcal{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathcal{S} : U \to -\frac{1}{U}. \tag{A.7}$$

In what follows we often use a product of n $\mathcal{T}$-transformations given by:

$$\mathcal{T}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \qquad \mathcal{T}^n : U \to U + n. \tag{A.8}$$

Note that configurations $\{a_i, b_i, c_i, d_i\}$ and $\{-a_i, -b_i, -c_i, -d_i\}$ are dual by an action of $\mathcal{S}^2 \times \mathcal{S}^2 \times \mathcal{S}^2$ on $T^2 \times T^2 \times T^2$, which helps to classify inequivalent solutions. This action preserves N_{flux} and the solution to $W = \partial_a W = 0$ $\forall a = 1, \ldots, 4$, since N_{flux} and W are respectively quadratic and linear in $\{a_i, b_i, c_i, d_i\}$.

$Sp(2h_-^{2,1} + 2, \mathbb{Z})$ symmetry for Calabi-Yaus: The perturbative Kähler potential (2.113) for CY compactifications is independent of the axions $\text{Re}(U^a)$, $a = 1, \ldots, h_-^{2,1}$. Due to this, the discrete gauge symmetries of the theory are the integer shifts of the complex structure moduli:

$$U^a \to U^a + n^a, \qquad n^a \in \mathbb{Z}, \qquad a = 1, \ldots, h_-^{2,1}, \tag{A.9}$$

causing the period and flux vectors to undergo a monodromy transformation:

$$\{\Pi, H, F\} \to M_{\{n^a\}} \{\Pi, H, F\}, \qquad M_{\{n^a\}} \in Sp(2h_-^{2,1} + 2, \mathbb{Z}). \tag{A.10}$$

[1] The transformation of the fluxes follows from this via the usual transformation rule of 3-forms. One can check that under the action of $SL(6,\mathbb{Z})$ the transformed flux quanta are even integers as long as the original ones are.

[2] To be consistent with our notations, here we denote the τ parameter of a 2-torus by U.

Furthermore, the monodromy matrix is required to be unipotent:

$$\left(M_{\{n^a\}} - I\right)^p \neq 0, \qquad \left(M_{\{n^a\}} - I\right)^{p+1} = 0, \qquad 1 \leq p \leq 3\,. \tag{A.11}$$

We can compute the monodromy matrix $M_{\{n^a\}}$ as follows. Notice that

$$\Pi^i(U^a + n^a) = \sum\nolimits_j \left(M_{\{n^a\}}\right)^i_j \Pi^j(U^a),\ \ i = 1, \ldots, 2h^{2,1}_- + 2\,, \tag{A.12}$$

are a set of functional relations. Using the definition of the period vector (2.110), the above relations can be evaluated at multiple values $\hat{U}^a$ to generate independent linear equations in the elements of the monodromy matrix. Inverting the latter we obtain the matrix elements uniquely. For example, in the $\mathbb{CP}_{[1,1,1,6,9]}[18]$ case (discussed in Sec. 2.3.2) we get:

$$M_{\{n_1,n_2\}} = \begin{pmatrix} 1 & -n_1 & -n_2 & 3n_2 + \frac{1}{2}n_1\left(3n_1^2 + 3n_2 n_1 + n_2^2 + 17\right) & \frac{1}{2}\left(3n_1+n_2\right)\left(3n_1+n_2+3\right) & \frac{3}{2}n_1\left(n_1+1\right) + n_1 n_2 \\ 0 & 1 & 0 & -\frac{1}{2}\left(3n_1+n_2-3\right)\left(3n_1+n_2\right) & -3\left(3n_1+n_2\right) & -3n_1 - n_2 \\ 0 & 0 & 1 & -\frac{1}{2}n_1\left(3n_1+2n_2-3\right) & -3n_1 - n_2 & -n_1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & n_1 & 1 & 0 \\ 0 & 0 & 0 & n_2 & 0 & 1 \end{pmatrix}. \tag{A.13}$$

It is easy to see that the above matrix belongs to $Sp(6,\mathbb{Z})$, i.e. with Σ as given in (2.112) we obtain: $M^T_{\{n_1,n_2\}} \cdot \Sigma \cdot M_{\{n_1,n_2\}} = \Sigma$. Also, it is unipotent as per requirement. Moreover note that the shift (A.9) keeps $\mathcal{N}_{\mathsf{flux}} = -\frac{1}{2}\, H^t \cdot \Sigma \cdot F$ invariant.

A.1 Duality in toroidal solutions

In this appendix we discuss the duality relations among the solutions with flat directions of the toroidal compactification case.

A.1.1 Solutions with 1 flat direction

Let us now discuss in detail the duality among the solutions (2.39) with 1 flat direction. They are parametrised by an integer p, and $N_{\mathsf{flux}} = 24$ irrespective of p. Below we show that the $p = 0$ case is dual to any $p \neq 0$ case via an $SL(6,\mathbb{Z})$ transformation.

Let us use unprimed and primed coordinates for $p = 0$ and $p \neq 0$ respectively. We act with an $SL(6,\mathbb{Z})$ matrix M on the coordinates of $T^2 \times T^2 \times T^2$ in accordance with (A.4), where M is given by:

$$M = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_2 \end{pmatrix}, \qquad M_1 = \begin{pmatrix} 1 & 2p \\ 0 & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}. \tag{A.14}$$

This transforms the period matrix as:

$$M \; : \; \mathrm{diag}\{U_1, U_2, U_3\} \rightarrow \mathrm{diag}\{U_1 + 2p, U_2 + p, U_3 + p\}\,. \tag{A.15}$$

Under this, the solution (2.39) with $p = 0$ is clearly mapped to a solution with $p \neq 0$. Now we need to show that the fluxes (2.36) map between the $p = 0$ and $p \neq 0$ cases. Indeed, using (1.95) and (A.4), we have:[3]

$$\begin{aligned} F_3 = 4\alpha_{11} - 2\alpha_{22} - 2\alpha_{33} \quad &\rightarrow \quad 4\alpha'_{11} - 2\alpha'_{22} - 2\alpha'_{33} + 4p\beta'^{11} - 4p^2\beta'^0 = F'_3\,, \\ H_3 = -4\beta^{11} + 2\beta^{22} + 2\beta^{33} \quad &\rightarrow \quad -4\beta'^{11} + 2\beta'^{22} + 2\beta'^{33} + 4p\beta'^0 = H'_3\,. \end{aligned} \tag{A.16}$$

A.1.2 Solutions with 2 flat directions

Dualities of family $\mathcal{A}$

First we show that $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ are dual via permutations of the 3 2-tori. Then the question to classify the inequivalent solutions in family $\mathcal{A}$ essentially boils down to that of subfamily $\mathcal{A}_1$, which we address subsequently.
Duality between $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$: The fluxes in subfamilies $\mathcal{A}_1$ and $\mathcal{A}_2$, given respectively by (2.41) and (2.47), depend on the 6 parameters

$$\lambda_1, \lambda_2, \lambda_3, b_3, d_0, d_3\,,$$

while those of $\mathcal{A}_3$, given in (2.53), depend on the 6 parameters

$$\lambda_1, \lambda_2, \lambda_3, b_2, d_0, d_2\,.$$

Under the permutation between the first and the second tori of $T^2 \times T^2 \times T^2$, the fluxes of $\mathcal{A}_1$ map to those of $\mathcal{A}_2$ when we identify $\{\lambda_1, \lambda_2, \lambda_3, b_3, d_0, d_3\}$ of $\mathcal{A}_1$ with $\{\lambda_2, \lambda_1, \lambda_3, b_3, d_0, d_3\}$ of $\mathcal{A}_2$. Moreover the respective transformation of the period matrix, $\mathrm{diag}\{U_1, U_2, U_3\} \rightarrow \mathrm{diag}\{U_2, U_1, U_3\}$, along with the above identification, relate their solutions. Similarly, under the permutation between the second and the third tori of $T^2 \times T^2 \times T^2$, the fluxes of $\mathcal{A}_1$ map to those of $\mathcal{A}_3$ when we identify $\{\lambda_1, \lambda_2, \lambda_3, b_3, d_0, d_3\}$ of $\mathcal{A}_1$ with $\{\lambda_1, \lambda_3, \lambda_2, b_2, d_0, d_2\}$ of $\mathcal{A}_3$. The respective transformation of the period matrix, $\mathrm{diag}\{U_1, U_2, U_3\} \rightarrow \mathrm{diag}\{U_1, U_3, U_2\}$, along with the above identification, relate their solutions as well.
Inequivalent solutions in $\mathcal{A}_1$: The requirement that a_1, b_2 and b_3 in (2.41) be even integers results in the parametrization shown below:

$$\begin{aligned} &b_3 = 2p\,, \qquad d_0 = 2q\lambda_2\,, \qquad d_3 = 2r\lambda_2\,, \qquad r \neq 0, \quad p, q, r \in \mathbb{Z}\,, \\ &N_{\text{flux}}\left(r, \lambda_2, \frac{\lambda_3}{\lambda_1}\right) = \frac{8r^2\lambda_2\lambda_3}{\lambda_1}\,. \end{aligned} \tag{A.17}$$

[3] α' and β' denote the basis of 3-forms (1.95) with respect to the primed coordinates (x'^i, y'^i).

The dependence of the fluxes (2.41) on λ_1 and λ_3 are only through the ratio λ_3/λ_1. For the present analysis we confine to integer values of λ_2 and λ_3/λ_1. It can be shown that whenever it is possible to find a triple $(r, \lambda_2, \lambda_3/\lambda_1)$ with $\frac{8r^2\lambda_2\lambda_3}{\lambda_1}, \frac{\lambda_3 r}{\lambda_1}, \frac{\lambda_2\lambda_3 r}{\lambda_1} \in \mathbb{Z}$[4] and $0 < N_{\text{flux}} \leq 32$, there exist infinitely many pairs (p, q) so that all the fluxes (2.41) are even integers. For example $q = r$ and any $p \in \mathbb{Z}$ always work. Therefore we first need to find all possible integer triples $(r, \lambda_2, \lambda_3/\lambda_1)$. This will provide all allowed values of N_{flux}. Then, among the different flux configurations corresponding to each of those triples (i.e. given an N_{flux}) we need to find the distinct equivalence classes (using duality).

Denoting the integer λ_3/λ_1 by s ($\neq 0$), we have $N_{\text{flux}} = 8r^2 s\lambda_2$. Clearly N_{flux} takes values in $\{8, 16, 24, 32\}$. The possible values of r are $\pm1, \pm2$. The requirement that all the fluxes (2.41) be even integers results in:

$$\begin{aligned} &\text{when } r = \pm 1\,, \qquad p, q \in \mathbb{Z}\,; \\ &\text{when } r = \pm 2\,, \qquad \{p \in 2\mathbb{Z}, q \in \mathbb{Z}\} \quad \text{or} \quad \{p \in \mathbb{Z}, q \in 2\mathbb{Z}\}\,. \end{aligned} \tag{A.18}$$

Replacing (r, p, q) by $(-r, -p, -q)$ maps the fluxes to minus themselves. Hence, in order to obtain the inequivalent solutions, it would be sufficient to consider $r > 0$. Now there are only 4 classes whose respective parametrizations, N_{flux} and the solutions are as follows.

Class 1:

$$\begin{aligned} &\frac{\lambda_3}{\lambda_1} = s\,, \qquad b_3 = 2p\,, \qquad d_0 = 2q\lambda_2\,, \qquad d_3 = 2\lambda_2\,, \\ &s = 1, \ldots, 4\,, \qquad \lambda_2 = 1, \ldots, \left[\frac{4}{s}\right], \quad p, q \in \mathbb{Z}\,, \\ &N_{\text{flux}} = 8s\lambda_2\,, \qquad (U_1, U_2, U_3, U_4) = \left(\frac{q}{s} + \frac{U_3}{s}, \lambda_2 U_4 - p, U_3, U_4\right), \end{aligned} \tag{A.19}$$

where $[n]$ denotes the greatest integer $\leq n$ and N_{flux} takes values in $\{8, 16, 24, 32\}$.

Class 2:

$$\begin{aligned} &\frac{\lambda_3}{\lambda_1} = 1\,, \qquad \lambda_2 = 1\,, \qquad b_3 = 2p\,, \qquad d_0 = 2q\,, \qquad d_3 = 4\,, \\ &\{p \in 2\mathbb{Z}, q \in \mathbb{Z}\} \qquad \text{or} \qquad \{p \in \mathbb{Z}, q \in 2\mathbb{Z}\}\,, \\ &N_{\text{flux}} = 32\,, \qquad (U_1, U_2, U_3, U_4) = \left(\frac{q}{2} + U_3, U_4 - \frac{p}{2}, U_3, U_4\right). \end{aligned} \tag{A.20}$$

[4] These respectively ensure that N_{flux} takes integer values and a_3, d_1 are even integers.

Class 3:

$$\frac{\lambda_3}{\lambda_1} = s\,, \qquad b_3 = 2p\,, \qquad d_0 = 2q\lambda_2\,, \qquad d_3 = 2\lambda_2, \quad s, \lambda_2 < 0\,,$$
$$|s| = 1, \ldots, 4\,, \qquad |\lambda_2| = 1, \ldots, \left[\frac{4}{|s|}\right], \quad p, q \in \mathbb{Z}\,,$$
$$N_{\text{flux}} = 8s\lambda_2\,, \qquad (U_1, U_2, U_3, U_4) = \left(\frac{q}{s} + \frac{U_3}{s}, \lambda_2 U_4 - p, U_3, U_4\right), \tag{A.21}$$

where N_{flux} takes values in $\{8, 16, 24, 32\}$.
Class 4:

$$\frac{\lambda_3}{\lambda_1} = -1\,, \qquad \lambda_2 = -1\,, \qquad b_3 = 2p\,, \qquad d_0 = -2q\,, \qquad d_3 = -4\,,$$
$$\{p \in 2\mathbb{Z}, q \in \mathbb{Z}\} \qquad \text{or} \qquad \{p \in \mathbb{Z}, q \in 2\mathbb{Z}\}\,,$$
$$N_{\text{flux}} = 32\,, \qquad (U_1, U_2, U_3, U_4) = \left(-\frac{q}{2} - U_3, -\frac{p}{2} - U_4, U_3, U_4\right). \tag{A.22}$$

A duality may exist between 2 flux configurations with the same N_{flux}. After incorporating such dualities, we find that each of the 4 classes has only a finite number of physically distinct flux configurations. To check aforesaid dualities, the solution space for the moduli in all the 4 classes suggests that only $SL(2, Z)$-actions on the first and the second tori of $T^2 \times T^2 \times T^2$ may help. Thus the $SL(6, \mathbb{Z})$ matrix in our considerations will be:

$$M = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & I \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}, \quad k, l \in \mathbb{Z}\,. \tag{A.23}$$

For all 4 classes the action of M transforms the fluxes keeping N_{flux} unaltered. The following details depend on the class in consideration.
For the case of Class 1, the new solution with the transformed fluxes is:

$$(U_1, U_2, U_3, U_4) = \left(k + \frac{q}{s} + \frac{U_3}{s},, \lambda_2 U_4 - p + l, U_3, U_4\right). \tag{A.24}$$

When $q = m$ modulo s (i.e. $m - q$ is a multiple of s) with the choices:

$$k = \frac{m - q}{s},\ l = p\,, \tag{A.25}$$

the transformed fluxes as well as the new solution respectively coincide with the fluxes and solution of the case with $p = 0$ and $q = m$ for each $\lambda_2 = 1, \ldots, [4/s]$. In the later case N_{flux} and the solution are given by:

$$N_{\text{flux}} = 8s\lambda_2\,, \quad (U_1, U_2, U_3, U_4) = \left(\frac{m}{s} + \frac{U_3}{s}, \lambda_2 U_4, U_3, U_4\right), m = 0, \ldots, s-1\,. \tag{A.26}$$

For the case of Class 2, the new solution with the transformed fluxes is:

$$(U_1, U_2, U_3, U_4) = \left(k + \frac{q}{2} + U_3, U_4 - \frac{p}{2} + l, U_3, U_4\right). \tag{A.27}$$

When $p = m, q = n$ modulo 2 (i.e. $m - p$ and $n - q$ are multiples of 2) with the choices:

$$k = \frac{n-q}{2}, \qquad l = \frac{p-m}{2}, \tag{A.28}$$

the transformed fluxes as well as the new solution respectively coincide with the fluxes and solution of the case with $p = m$ and $q = n$. In the later case N_{flux} and the solution are given by:

$$N_{\text{flux}} = 32, \qquad (U_1, U_2, U_3, U_4) = \left(\frac{n}{2} + U_3, U_4 - \frac{m}{2}, U_3, U_4\right),$$
$$\{m = 0,\ n = 0, 1\} \qquad \text{or} \qquad \{m = 0, 1,\ n = 0\}. \tag{A.29}$$

For Classes 3 and 4, the analysis is similar to that for classes 1 and 2 respectively.

Dualities of family $\mathcal{B}$

First we show that $\mathcal{B}_1$ is dual to $\mathcal{B}_2$ via an $SL(6, \mathbb{Z})$ transformation. Then the question to classify the inequivalent solutions in family $\mathcal{B}$ essentially boils down to that of subfamily $\mathcal{B}_1$, which we address subsequently.
Duality between $\mathcal{B}_1$ and $\mathcal{B}_2$: To prove the duality between $\mathcal{B}_1$ and $\mathcal{B}_2$, we act with an $\mathcal{S}$-transformation only on the first 2-torus of $T^2 \times T^2 \times T^2$, transforming the period matrix as:

$$M : \text{diag}\{U_1, U_2, U_3\} \quad \to \quad \text{diag}\{-\frac{1}{U_1}, U_2, U_3\},$$
$$M = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}, \qquad M_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{A.30}$$

This transforms the fluxes (2.4) as:

$$\{a_0, a_1, a_2, a_3\} \to \{-a_1, a_0, b_3, b_2\}, \qquad \{b_0, b_1, b_2, b_3\} \to \{-b_1, b_0, -a_3, -a_2\},$$
$$\{c_0, c_1, c_2, c_3\} \to \{-c_1, c_0, d_3, d_2\}, \qquad \{d_0, d_1, d_2, d_3\} \to \{-d_1, d_0, -c_3, -c_2\}. \tag{A.31}$$

It is straightforward to check that, under the above action, the fluxes of $\mathcal{B}_1$, given by (2.60), map to those of $\mathcal{B}_2$, given by (2.66), when we identify $\{b_2, d_2, d_1, d_0, c_3\}$ of $\mathcal{B}_1$ with $\{a_3, c_3, -d_0, d_1, -d_2\}$ of $\mathcal{B}_2$. Such identification relates the crucial condition $d_2 \neq 0$ of (2.60) to the condition $c_3 \neq 0$ of

(2.66), and leaves $N_{\text{flux}} = \frac{2}{\lambda_3}(c_3 d_0 - d_1 d_2)$ invariant. With this identification now (A.30) maps the solution (2.62) to (2.68), establishing the duality.
Inequivalent solutions in $\mathcal{B}_1$: The fluxes (2.60) depend on $\lambda_3, b_2, c_3, d_0, d_1, d_2$. For the present analysis we confine to integer values of λ_3. There are only 4 classes consistent with even integer fluxes and $0 < N_{\text{flux}} \leq 32$. Their respective parametrizations, N_{flux} and the solutions are as follows.
Class 1:

$$
\begin{gathered}
b_2 = 2ks\,, \qquad c_3 = 2p\lambda_3\,, \qquad d_0 = 2q\lambda_3\,, \qquad d_1 = 2r\lambda_3\,, \qquad d_2 = 2s\lambda_3\,, \\
pq - rs = 1, \ldots, 4\,, \qquad \lambda_3 = 1, \ldots, \left[\frac{4}{pq-rs}\right]\,, \qquad k,p,q,r,s \in \mathbb{Z}\,, \\
N_{\text{flux}} = 8(pq-rs)\lambda_3\,, \qquad (U_1,U_2,U_3,U_4) = \left(-\frac{sU_2+q}{pU_2+r}, U_2, -k+\lambda_3 U_4, U_4\right)\,,
\end{gathered} \tag{A.32}
$$

where N_{flux} takes values in $\{8, 16, 24, 32\}$.
Class 2:

$$
\begin{gathered}
\lambda_3 = 1\,, \qquad b_2 = 2ks\,, \qquad c_3 = 4p\,, \qquad d_0 = 4q\,, \qquad d_1 = 4r\,, \qquad d_2 = 4s\,, \\
pq - rs = 1\,, \qquad k \in 2\mathbb{Z}+1\,, \qquad p,q,r,s \in \mathbb{Z}\,, \\
N_{\text{flux}} = 32\,, \qquad (U_1,U_2,U_3,U_4) = \left(-\frac{sU_2+q}{pU_2+r}, U_2, -\frac{k}{2}+U_4, U_4\right)\,.
\end{gathered} \tag{A.33}
$$

Class 3:

$$
\begin{gathered}
b_2 = 2ks\,, \quad c_3 = 2p\lambda_3\,, \quad d_0 = 2q\lambda_3\,, \quad d_1 = 2r\lambda_3\,, \quad d_2 = 2s\lambda_3\,, \quad pq-rs, \lambda_3 < 0\,, \\
|pq - rs| = 1, \ldots, 4\,, \qquad |\lambda_3| = 1, \ldots, \left[\frac{4}{|pq-rs|}\right]\,, \qquad k,p,q,r,s \in \mathbb{Z}\,, \\
N_{\text{flux}} = 8(pq-rs)\lambda_3\,, \qquad (U_1,U_2,U_3,U_4) = \left(-\frac{sU_2+q}{pU_2+r}, U_2, -k+\lambda_3 U_4, U_4\right)\,.
\end{gathered} \tag{A.34}
$$

Class 4:

$$
\begin{gathered}
\lambda_3 = -1\,, \qquad b_2 = 2ks\,, \qquad c_3 = -4p\,, \qquad d_0 = -4q\,, \qquad d_1 = -4r\,, \qquad d_2 = -4s\,, \\
pq - rs = -1\,, \qquad k \in 2\mathbb{Z}+1\,, \qquad p,q,r,s \in \mathbb{Z}\,, \\
N_{\text{flux}} = 32\,, \qquad (U_1,U_2,U_3,U_4) = \left(-\frac{sU_2+q}{pU_2+r}, U_2, -\frac{k}{2}-U_4, U_4\right)\,.
\end{gathered} \tag{A.35}
$$

A duality may exist between 2 flux configurations with the same N_{flux}. After incorporating such dualities, we find that each of the 4 classes has only a finite number of physically distinct flux configurations. To check aforesaid dualities, the solution space for the moduli in all the 4 classes suggests that only $SL(2,Z)$-actions on the first and the third tori of $T^2 \times T^2 \times T^2$ may help. Thus the $SL(6,\mathbb{Z})$ matrix in our considerations will be:

$$M = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & M_3 \end{pmatrix}, \qquad M_1 = \begin{pmatrix} g & h \\ i & j \end{pmatrix}, \qquad M_3 = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix},$$
$$gj - hi = 1\,, \quad g,h,i,j,l \in \mathbb{Z}\,. \quad \text{(A.36)}$$

For all 4 classes the action of M transforms the fluxes keeping N_{flux} unaltered. The following details depend on the class in consideration.
For the case of Class 1, the new solution with the transformed fluxes is:

$$(U_1,U_2,U_3,U_4) = \left(\frac{(hp-js)U_2 + (hr-jq)}{(gp-is)U_2 + (gr-iq)}, U_2, -k+l+\lambda_3 U_4, U_4 \right). \quad \text{(A.37)}$$

When $pq - rs = 1$, with the choices:

$$g = -s\,, \qquad h = q\,, \qquad i = -p\,, \qquad j = r\,, \qquad l = k\,, \qquad \text{(A.38)}$$

the transformed fluxes as well as the new solution respectively coincide with the fluxes and solution of the case with $p = 0$, $q = 0$, $r = 1$, $s = -1$ and $k = 0\ \forall\, \lambda_3 = 1,2,3,4$. In the later case N_{flux} and the solution are given by:

$$N_{\text{flux}} = 8\lambda_3\,, \quad (U_1,U_2,U_3,U_4) = (U_2,U_2,\lambda_3 U_4,U_4)\,. \qquad \text{(A.39)}$$

When $pq - rs = 2$, depending on each of p,q,r,s even (e) or odd (o), the transformed fluxes and solution coincide with those of some specific configuration. In keeping with $pq - rs = 2$, p,q,r,s can only be:

eeeo, eeoe, eoee, eoeo, eooe, oeee, oeeo, oeoe, oooo. (A.40)

For $p,q,r,s =$ eoeo, oeoe, oooo, with the choices:

$$g = q\,, \qquad h = \frac{s-q}{2}\,, \qquad i = r\,, \qquad j = \frac{p-r}{2}\,, \qquad l = k\,, \qquad \text{(A.41)}$$

the transformed fluxes as well as the new solution respectively coincide with the fluxes and solution of the case with $p = 2$, $q = 1$, $r = 0$, $s = 1$ and $k = 0$ for each $\lambda_3 = 1,2$. In the later case N_{flux} and the solution are given by:

$$N_{\text{flux}} = 16\lambda_3\,, \quad (U_1,U_2,U_3,U_4) = \left(\frac{-U_2 - 1}{2U_2}, U_2, \lambda_3 U_4, U_4 \right). \qquad \text{(A.42)}$$

For $p,q,r,s =$ eeoe, eoee, eooe, with the choices:

$$g = q\,, \qquad h = \frac{s}{2}\,, \qquad i = r\,, \qquad j = \frac{p}{2}\,, \qquad l = k\,, \qquad \text{(A.43)}$$

the transformed fluxes as well as the new solution respectively coincide with the fluxes and solution of the case with $p = 2$, $q = 1$, $r = 0$, $s = 1$ and $k = 0$ for each $\lambda_3 = 1, 2$. In the later case N_{flux} and the solution are given by:

$$N_{\text{flux}} = 16\lambda_3\,, \quad (U_1, U_2, U_3, U_4) = \left(-\frac{1}{2U_2}, U_2, \lambda_3 U_4, U_4\right). \tag{A.44}$$

For $p, q, r, s =$ eeeo, oeee, oeeo, with the choices:

$$g = -s\,, \qquad h = \frac{q}{2}\,, \qquad i = -p\,, \qquad j = \frac{r}{2}\,, \qquad l = k\,, \tag{A.45}$$

the transformed fluxes as well as the new solution respectively coincide with the fluxes and solution of the case with $p = 0$, $q = 0$, $r = 2$, $s = -1$ and $k = 0$ for each $\lambda_3 = 1, 2$. In the later case N_{flux} and the solution are given by:

$$N_{\text{flux}} = 16\lambda_3\,, \quad (U_1, U_2, U_3, U_4) = \left(\frac{U_2}{2}, U_2, \lambda_3 U_4, U_4\right). \tag{A.46}$$

When $pq - rs = 3$, we need to analyse cases where each of $p, q, r, s = 0, 1, 2$ modulo 3.[5] Out of 3^4 possibilities, only 32 cases are consistent with $pq - rs = 3$ where p, q, r, s can be:

$$\begin{aligned}
&0001,\ 0002,\ 0010,\ 0020,\ 0100,\ 0101,\ 0102,\ 0110\,,\\
&0120,\ 0200,\ 0201,\ 0202,\ 0210,\ 0220,\ 1000,\ 1001\,,\\
&1002,\ 1010,\ 1020,\ 1111,\ 1122,\ 1212,\ 1221,\ 2000\,,\\
&2001,\ 2002,\ 2010,\ 2020,\ 2112,\ 2121,\ 2211,\ 2222\,.
\end{aligned} \tag{A.47}$$

For $p, q, r, s = 0001,\ 0002,\ 1000,\ 1001,\ 1002,\ 2000,\ 2001,\ 2002$, with the choices:

$$g = -s\,, \qquad h = \frac{q}{3}\,, \qquad i = -p\,, \qquad j = \frac{r}{3}\,, \qquad l = k\,, \tag{A.48}$$

the transformed fluxes as well as the new solution respectively coincide with the fluxes and solution of the case with $p = 0$, $q = 0$, $r = 3$, $s = -1$, $k = 0$ and $\lambda_3 = 1$. In the later case N_{flux} and the solution are given by:

$$N_{\text{flux}} = 24\,, \quad (U_1, U_2, U_3, U_4) = \left(\frac{U_2}{3}, U_2, U_4, U_4\right). \tag{A.49}$$

For $p, q, r, s = 0010,\ 0020,\ 0100,\ 0110,\ 0120,\ 0200,\ 0210,\ 0220$, with the choices:

$$g = -\frac{s}{3}\,, \qquad h = q\,, \qquad i = -\frac{p}{3}\,, \qquad j = r\,, \qquad l = k\,, \tag{A.50}$$

[5] 2 integers n_1 and n_2 are equal modulo 3 if there exists an integer n_3 such that $n_1 = 3n_3 + n_2$. For example, note that $-2 = 1$ and $-1 = 2$ modulo 3.

the transformed fluxes as well as the new solution respectively coincide with the fluxes and solution of the case with $p = 0$, $q = 0$, $r = 1$, $s = -3$, $k = 0$ and $\lambda_3 = 1$. In the later case N_{flux} and the solution are given by:

$$N_{\text{flux}} = 24\,, \quad (U_1, U_2, U_3, U_4) = (3U_2, U_2, U_4, U_4)\,. \tag{A.51}$$

For $p, q, r, s = 0101,\ 0202,\ 1010,\ 1111,\ 1212,\ 2020,\ 2121,\ 2222$, with the choices:

$$g = q\,, \qquad h = \frac{s-q}{3}\,, \qquad i = r\,, \qquad j = \frac{p-r}{3}\,, \qquad l = k\,, \tag{A.52}$$

the transformed fluxes as well as the new solution respectively coincide with the fluxes and solution of the case with $p = 3$, $q = 1$, $r = 0$, $s = 1$, $k = 0$ and $\lambda_3 = 1$. In the later case N_{flux} and the solution are given by:

$$N_{\text{flux}} = 24\,, \quad (U_1, U_2, U_3, U_4) = \left(\frac{-U_2 - 1}{3U_2}, U_2, U_4, U_4\right). \tag{A.53}$$

For $p, q, r, s = 0102,\ 0201,\ 1020,\ 1122,\ 1221,\ 2010,\ 2112,\ 2211$, with the choices:

$$g = q\,, \qquad h = \frac{1}{3}(s - 2q)\,, \qquad i = r\,, \qquad j = \frac{1}{3}(p - 2r)\,, \qquad l = k\,, \tag{A.54}$$

the transformed fluxes as well as the new solution respectively coincide with the fluxes and solution of the case with $p = 3$, $q = 1$, $r = 0$, $s = 2$, $k = 0$ and $\lambda_3 = 1$. In the later case N_{flux} and the solution are given by:

$$N_{\text{flux}} = 24\,, \quad (U_1, U_2, U_3, U_4) = \left(\frac{-2U_2 - 1}{3U_2}, U_2, U_4, U_4\right). \tag{A.55}$$

When $pq - rs = 4$, a similar analysis can be done.
<u>For the case of Class 2</u>, the new solution with the transformed fluxes is:

$$(U_1, U_2, U_3, U_4) = \left(\frac{(hp - js)U_2 + (hr - jq)}{(gp - is)U_2 + (gr - iq)}, U_2, -\frac{k}{2} + l + U_4, U_4\right). \tag{A.56}$$

Now with the choices:

$$g = -s\,, \qquad h = q\,, \qquad i = -p\,, \qquad j = r\,, \qquad l = \frac{k-1}{2}\,, \tag{A.57}$$

the transformed fluxes as well as the new solution respectively coincide with the fluxes and solution of the case $p = 0,\ q = 0,\ r = 1,\ s = -1,\ k = 1$. In the later case N_{flux} and the solution are given by:

$$N_{\text{flux}} = 32\,, \quad (U_1, U_2, U_3, U_4) = \left(U_2, U_2, U_4 - \frac{1}{2}, U_4\right). \tag{A.58}$$

<u>For Classes 3 and 4</u>, the analysis is similar to that for classes 1 and 2 respectively.

Dualities of family $\mathcal{C}$

As per (2.73), (2.79) and (2.85), the fluxes of $\mathcal{C}_1$ and $\mathcal{C}_2$ have 5 independent parameters, whereas it is 6 in case of $\mathcal{C}_3$. Despite this, we are able to prove that $SL(6,\mathbb{Z})$ transformations relate $\mathcal{C}_3$ to $\mathcal{C}_2$, while $\mathcal{C}_1$ to a subset of $\mathcal{C}_2$. Below we provide the details. Then the question to classify the inequivalent solutions in family $\mathcal{C}$ essentially boils down to that of $\mathcal{C}_2$, which we address subsequently.

Duality between $\mathcal{C}_2$ and $\mathcal{C}_3$: The fluxes of $\mathcal{C}_3$, given by (2.85), depend on the 6 parameters $\lambda_2, b_2, b_3, d_0, d_2$ and d_3 with $\lambda_2, d_0, d_2 \neq 0$ and $b_2 d_3 \neq b_3 d_2$. We divide $\mathcal{C}_3$ in 2 complementary subsets with $d_3 = 0$ and $d_3 \neq 0$ respectively. Each of these is shown to be dual to $\mathcal{C}_2$.

To prove the duality between the subset of $\mathcal{C}_3$ with $d_3 = 0$ and $\mathcal{C}_2$, we act with an $\mathcal{S}$-transformation only on the third 2-torus of $T^2 \times T^2 \times T^2$ transforming the period matrix as:

$$M \,:\, \mathrm{diag}\{U_1, U_2, U_3\} \quad \rightarrow \quad \mathrm{diag}\{U_1, U_2, -\frac{1}{U_3}\},$$
$$M = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & M_3 \end{pmatrix}, \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad M_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.59}$$

This transforms the fluxes (2.4) as:

$$\begin{aligned} \{a_0, a_1, a_2, a_3\} &\rightarrow \{-a_3, b_2, b_1, a_0\}, \quad \{b_0, b_1, b_2, b_3\} \rightarrow \{-b_3, -a_2, -a_1, b_0\}, \\ \{c_0, c_1, c_2, c_3\} &\rightarrow \{-c_3, d_2, d_1, c_0\}, \quad \{d_0, d_1, d_2, d_3\} \rightarrow \{-d_3, -c_2, -c_1, d_0\}. \end{aligned} \tag{A.60}$$

It is straightforward to check that, under the above action, the fluxes of $\mathcal{C}_3$, given by (2.85) with $d_3 = 0$, map to those of $\mathcal{C}_2$, given by (2.79), when we identify $\{-\frac{b_3 d_2}{d_0}, \frac{b_2 d_0}{d_2}, d_2\lambda_2, d_0\}$ of $\mathcal{C}_3$ with $\{b_2, b_3, c_2, d_3\}$ of $\mathcal{C}_2$. Such identification relates the crucial conditions $b_3, d_0, d_2 \neq 0$ of (2.85) (when $d_3 = 0$) to the conditions $b_2, c_2, d_3 \neq 0$ of (2.79). With this identification now (A.59) maps the solution (2.87) with $d_3 = 0$ to (2.81), establishing the duality.

To prove the duality between the subset of $\mathcal{C}_3$ with $d_3 \neq 0$ and $\mathcal{C}_2$, we act with an $SL(2,\mathbb{Z})$-transformation only on the third 2-torus of $T^2 \times T^2 \times T^2$

transforming the period matrix as:

$$M \,:\, \mathrm{diag}\{U_1, U_2, U_3\} \quad \rightarrow \quad \mathrm{diag}\{U_1, U_2, \frac{jU_3+h}{iU_3+g}\}\,,$$

$$M = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & M_3 \end{pmatrix}\,, \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,, \qquad M_3 = \begin{pmatrix} g & h \\ i & j \end{pmatrix}\,,$$

$$gj - hi = 1\,, \quad g,h,i,j \in \mathbb{Z}\,. \tag{A.61}$$

This transforms the fluxes (2.4) as:

$$\begin{aligned}
\{a_0, a_1, a_2, a_3\} \quad &\rightarrow \quad \{a_0 g + a_3 i,\; a_1 g - b_2 i,\; a_2 g - b_1 i,\; a_0 h + a_3 j\}\,,\\
\{b_0, b_1, b_2, b_3\} \quad &\rightarrow \quad \{-b_3 h + b_0 j, -a_2 h + b_1 j, -a_1 h + b_2 j,\; b_3 g - b_0 i\}\,,\\
\{c_0, c_1, c_2, c_3\} \quad &\rightarrow \quad \{c_0 g + c_3 i,\; c_1 g - d_2 i,\; c_2 g - d_1 i,\; c_0 h + c_3 j\}\,,\\
\{d_0, d_1, d_2, d_3\} \quad &\rightarrow \quad \{-d_3 h + d_0 j, -c_2 h + d_1 j, -c_1 h + d_2 j,\; d_3 g - d_0 i\}\,.
\end{aligned} \tag{A.62}$$

It is straightforward to check that, under the above action, the fluxes of $\mathcal{C}_3$, given by (2.85) with $d_3 \neq 0$, map to those of $\mathcal{C}_2$, given by (2.79), when we implement the following steps:

1. Given non-zero even integer fluxes d_0 and d_3 in $\mathcal{C}_3$ find 4 integers g, h, i, j satisfying:

$$h = \frac{d_0 j}{d_3}\,, \qquad i = \frac{d_3(gj-1)}{d_0 j}\,, \quad j \neq 0\,. \tag{A.63}$$

   This can be done if the following holds. Given 2 integers $(p, q) \neq (0,0)$ (i.e., taking $d_0 = 2p$ and $d_3 = 2q$) one can always find other 2 integers (g, j), $j \neq 0$ such that $(\frac{jp}{q}, \frac{q(gj-1)}{jp})$ are integers. We have verified this numerically for $p, q = -1000, \ldots, 1000$.

2. Identify $\left\{j(b_2 - \frac{b_3 d_2}{d_3}), \frac{b_2 d_3(1-gj)}{d_2 j} + b_3 g, \frac{d_2 d_3 \lambda_2}{d_0 j}, \frac{d_3}{j}\right\}$ of $\mathcal{C}_3$ with $\{b_2, b_3, c_2, d_3\}$ of $\mathcal{C}_2$. As (2.85) are even integer fluxes and g, h, i, j are chosen to be integers, clearly $j(b_2 - \frac{b_3 d_2}{d_3}) = -a_1 h + b_2 j$, $\frac{b_2 d_3(1-gj)}{d_2 j} + b_3 g = b_3 g - b_0 i$, $\frac{d_2 d_3 \lambda_2}{d_0 j} = c_2 g - d_1 i$, $\frac{d_3}{j} = d_3 g - d_0 i$ are even integers. Alternatively, in the transformed fluxes of $\mathcal{C}_3$ one can substitute b_2, b_3, d_2, d_3 in terms of b_2, b_3, c_2, d_3 of $\mathcal{C}_2$ and d_0 of $\mathcal{C}_3$ (obtained by inverting the above identification map) to get the fluxes of $\mathcal{C}_2$, i.e. the explicit dependence on d_0 of $\mathcal{C}_3$ goes away. The above identification also relates the crucial conditions $d_0, d_2 \neq 0$, $b_2 d_3 \neq b_3 d_2$ of (2.85) (with $d_3 \neq 0$) to the conditions $b_2, c_2, d_3 \neq 0$ of (2.79).

Now (A.61) maps the solution (2.87) with $d_3 \neq 0$ to (2.81), establishing the duality.

Duality between $\mathcal{C}_1$ and a subset of $\mathcal{C}_2$: Consider a $\mathcal{T}^l$- and an $SL(2,\mathbb{Z})$-action respectively on the first and the third 2-tori of $T^2 \times T^2 \times T^2$, i.e. the $SL(6,\mathbb{Z})$ matrix is:

$$M = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & M_3 \end{pmatrix}, \qquad M_1 = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}, \qquad M_3 = \begin{pmatrix} g & h \\ i & j \end{pmatrix},$$
$$gj - hi = 1\,, \quad g,h,i,j,l \in \mathbb{Z}\,. \tag{A.64}$$

This action, together with an appropriate choice for g,h,i,j, transforms the fluxes of $\mathcal{C}_1$, given by (2.73), to those of $\mathcal{C}_2$, given by (2.79), with $\frac{d_3}{c_2} = -l$ (i.e. integer) only. The appropriate choices depend on the flux quanta (2.73) as follows:

$$\begin{aligned} &g = 1\,, \quad h = 0\,, \quad j = 1\,, \quad \text{when} \quad d_2 = 0\,, \\ &h = 1\,, \quad i = -1\,, \quad j = 0\,, \quad \text{when} \quad d_2 \neq 0\,,\, c_2 = 0\,, \\ &j = \frac{c_2 h}{d_2 \lambda_2}\,, \quad g = \frac{d_2 \lambda_2 (1 + hi)}{c_2 h}\,, \quad h \neq 0\,, \quad \text{when} \quad d_2\,,\, c_2 \neq 0\,. \end{aligned} \tag{A.65}$$

For non-zero even integer fluxes $c_1 = \frac{c_2}{\lambda} = 2p$ and $d_2 = 2q$ in $\mathcal{C}_1$, the last choice can always be made (which we checked numerically when $p, q = -1000, \ldots, 1000$). The period matrix transforms in a way that in all the above cases the solution for the moduli in $\mathcal{C}_1$ maps to that of the corresponding subset of $\mathcal{C}_2$.

Clearly, there are flux configurations in $\mathcal{C}_2$ for which $\frac{d_3}{c_2}$ is non-integer. For example $b_2 = 4, b_3 = 2, c_2 = 4, d_3 = 2$ with $N_{\text{flux}} = 32$ is not dual to any flux configuration in $\mathcal{C}_1$.

Inequivalent solutions in $\mathcal{C}_2$: The fluxes (2.79) depend on $\lambda_2, b_2, b_3, c_2, d_3$. The requirement that b_2, b_3, c_1, d_3 be even integers results in the parametrization shown below:

$$b_2 = 2p\,, \quad b_3 = 2q\,, \quad c_2 = 2r\lambda_2\,, \quad d_3 = 2s\,, \quad p,r,s \neq 0\,, \quad p,q,r,s \in \mathbb{Z}\,,$$
$$N_{\text{flux}} = 8pr\lambda_2\,. \tag{A.66}$$

For the present analysis we confine to integer values of λ_2. This allows N_{flux} to take values in $\{8, 16, 24, 32\}$ and one can show that, whenever we find a triple (p, r, λ_2) corresponding to a given N_{flux} value, there exist infinitely many pairs (q, s) so that all the fluxes (2.79) are even integers.

For example $s = r$ and any $q \in \mathbb{Z}$ always work. Therefore, given an N_{flux}, we first need to find all possible integer triples (p, r, λ_2). Then, among the different flux configurations corresponding to each of those triples, we need to find the distinct equivalence classes (using duality). The number of possible triples is 4 when $N_{\text{flux}} = 8$, 12 for both cases with $N_{\text{flux}} = 16$ and $N_{\text{flux}} = 24$, and 24 when $N_{\text{flux}} = 32$. To demonstrate the aforesaid dualities, we consider below only the $N_{\text{flux}} = 8$ case with $p = -1$, $r = 1$ and $\lambda_2 = -1$.

With $(p, r, \lambda_2) = (-1, 1, -1)$ more generally one can take $q = ks,\ k \in \mathbb{Z}$ that leads to even integer fluxes (2.79). In this case the solution (2.81) reads:

$$(U_1, U_2, U_3, U_4) = \left(s + U_2, U_2, \frac{1}{k - U_4}, U_4\right). \tag{A.67}$$

Now the above fluxes and solution with $(s, k) \neq (0, 1)$ can be mapped to those with $(s, k) = (0, 1)$ by acting with $\mathcal{T}^{1-s}$ and $\mathcal{S}\mathcal{T}^k\mathcal{S}$ respectively on the first and the third 2-tori of $T^2 \times T^2 \times T^2$.

Dualities between families

The linear relation that the derivatives of the superpotential satisfy differs across the families $\mathcal{A}, \mathcal{B}, \mathcal{C}$, see (2.40), (2.59) and (2.72). Despite this, below we find certain dualities among them. In summary, we show that $\mathcal{B}_1$ contains $\mathcal{A}_3$. Also, we know from the previous subsection that $\mathcal{C}_3$ contains 2 copies of $\mathcal{C}_2$, one of which is shown here to be dual to $\mathcal{B}_1$.

Duality between $\mathcal{A}_3$ and a subset of $\mathcal{B}_1$: The fluxes of $\mathcal{B}_1$, given by (2.60), with $c_3 = 0$ map to those of $\mathcal{A}_3$, given by (2.53), when we identify $\{\lambda_3, b_2, d_0, d_1, d_2\}$ of $\mathcal{B}_1$ with $\{\lambda_3, b_2, d_0, -\frac{d_2\lambda_2}{\lambda_1}, d_2\}$ of $\mathcal{A}_3$.[6] Such identification relates the crucial conditions $\lambda_3, d_1, d_2 \neq 0$ of (2.60) (when $c_3 = 0$) to the conditions $\frac{\lambda_2}{\lambda_1}, \lambda_3, d_2 \neq 0$ of (2.53). Furthermore, with this identification, the solution (2.62) with $c_3 = 0$ is same as the solution (2.55), establishing the duality.

Duality between $\mathcal{B}_1$ and a subset of $\mathcal{C}_3$: The fluxes of $\mathcal{C}_3$, given by (2.85), depend on the parameters $\lambda_2, b_2, b_3, d_0, d_2$ and d_3 with $\lambda_2, d_0, d_2 \neq 0$ and $b_2 d_3 \neq b_3 d_2$. We take the subset of $\mathcal{C}_3$ for which $d_3 = 0$ and show that it is dual to $\mathcal{B}_1$. To prove this, we act with an $SL(2, \mathbb{Z})$-transformation only on

[6] Note that the fluxes of $\mathcal{A}_3$ depend on λ_1, λ_2 via the ratio $\frac{\lambda_2}{\lambda_1}$.

the first 2-torus of $T^2 \times T^2 \times T^2$ transforming the period matrix as:

$$M \ : \ \mathrm{diag}\{U_1, U_2, U_3\} \quad \to \quad \mathrm{diag}\{\frac{jU_1 + h}{iU_1 + g}, U_2, U_3\}\,,$$

$$M = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}, \qquad M_1 = \begin{pmatrix} g & h \\ i & j \end{pmatrix}, \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$gj - hi = 1\,, \quad g, h, i, j \in \mathbb{Z}\,. \qquad \text{(A.68)}$$

This action, together with an appropriate choice for g, h, i, j, transforms the fluxes of $\mathcal{C}_3$, given by (2.85), with $d_3 = 0$ to those of $\mathcal{B}_1$, given by (2.60). The appropriate choices depend on the flux quanta (2.60) as follows:[7]

$$\begin{aligned} &j = \frac{d_2\lambda_2 h}{d_0}\,, \qquad g = \frac{d_0(1+hi)}{d_2\lambda_2 h}\,, \qquad h \neq 0\,, \qquad \text{when} \quad d_0' = 0\,, \\ &g = j = 1\,, \qquad h = i = 0\,, \qquad \text{when} \quad d_0' \neq 0\,,\, c_3' = 0\,, \\ &g = i = j = 1\,, \qquad h = 0\,, \qquad \text{when} \quad d_2'\,,\, c_3' \neq 0\,. \end{aligned} \qquad \text{(A.69)}$$

For non-zero even integer fluxes $d_0 = 2p$ and $d_1 = d_2\lambda_2 = 2q$ in $\mathcal{C}_3$, the last choice can always be made (which we checked numerically when $p, q = -1000, \ldots, 1000$). The period matrix transforms in a way that in all the above cases the solution for the moduli in $\mathcal{C}_3$ (when $d_3 = 0$) maps to that of the corresponding subset of $\mathcal{B}_1$.

Thus, we conclude that $\mathcal{B}_1$ is the master family which contains all distinct solutions.

[7] To distinguish, here we use prime for the flux quanta (2.60).

Appendix B

Warped throats in Type IIB compactification

In this subsection we review some aspects of warped throats in flux compactifications that will be useful for this work. Type IIB flux compactifications have 3-form fluxes (NSNS and RR) threading the 3-cycles of an orientifolded Calabi-Yau (CY). The back-reaction of fluxes has the effect of generating warping (as in [327]). The 10D metric takes the form [7, 10]

$$\mathrm{d}s_{10}^2 = e^{2A(y)}\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + e^{-2A(y)}g_{mn}\,\mathrm{d}y^m\,\mathrm{d}y^n\,, \tag{B.1}$$

where $e^{2A(y)} \equiv h^{-1/2}(y)$ is the warp factor and g_{mn} the CY metric. The warp factor satisfies a Poisson-like equation which is sourced by 3-form fluxes and localized objects carrying D3-charge. For non-vanishing fluxes, the warp factor varies over the compact directions. The warp factor acts like a redshift factor for the objects localized in the compact directions. In regions where e^{-2A} is large, it can be used to generate hierarchies in physical scales. Regions of large warping arise when fluxes thread the 3-cycle associated with a conifold modulus and its dual cycle. The geometry in this region is close to that of the Klebanov-Strassler (KS) throat [328]. Our primary interest will be in the regime in which the warp factor is almost constant over the whole compact space, except for a single throat where the geometry is highly warped.

As pointed out in [306], a constant shift of e^{4A} maps solutions of the Poisson equation to solutions, and this freedom is to be identified with (a power of) the volume modulus. Furthermore, a pure scaling of the CY metric $\mathrm{d}s^2_{CY}$ to a unit-volume fiducial metric $\mathrm{d}s^2_{CY_0}$, given by $\mathrm{d}s^2_{CY} \to \lambda\mathrm{d}s^2_{CY}$, implies a rescaling of the warp factor $e^{2A} \to \lambda e^{2A}$, and hence has no effect on the physical geometry. Given this, a useful parametrization of the 10D geometry is

$$\mathrm{d}s_{10}^2 = \mathcal{V}^{1/3}\left(e^{-4A_0} + \mathcal{V}^{2/3}\right)^{-1/2}\mathrm{d}s_4^2 + \left(e^{-4A_0} + \mathcal{V}^{2/3}\right)^{1/2}\mathrm{d}s^2_{CY_0}\,, \tag{B.2}$$

which is equivalent to:

$$\mathrm{d}s_{10}^2 = \left(1+\frac{e^{-4A_0}}{\mathcal{V}^{2/3}}\right)^{-1/2}\mathrm{d}s_4^2 + \left(1+\frac{e^{-4A_0}}{\mathcal{V}^{2/3}}\right)^{1/2}\mathrm{d}s_{CY}^2\ . \tag{B.3}$$

The factor $\left(1+\frac{e^{-4A_0}}{\mathcal{V}^{2/3}}\right)^{-1/4} \equiv \Omega$ is the redshift factor. In a highly warped region, $e^{-4A_0} \gg \mathcal{V}^{2/3}$ and $\Omega \sim e^{A_0}\mathcal{V}^{1/6} \ll 1$. It is important to keep the following in mind:

1. In highly warped regions generated where the local geometry of the underlying CY is close to that of a conifold, the spacetime metric is close to the KS geometry:

$$\mathrm{d}s_{10}^2 = e^{2A(r)}\,\mathrm{d}s_4^2 + e^{-2A(r)}\left(\mathrm{d}r^2 + r^2\,\mathrm{d}s_{T^{1,1}}^2\right)\ , \tag{B.4}$$

   The presence of the fluxes resolves the conifold singularity, and one has a minimal area 3-sphere at the bottom of the throat. The warp factor takes its minimal value on this 3-sphere [10]

$$e^{4A_{\mathsf{min}}} \sim e^{-\frac{8\pi K}{3g_s M}} \equiv y\ , \tag{B.5}$$

   where g_s is the string coupling and K and M are the integral fluxes that thread the 3-sphere and its dual cycle.

2. The hierarchy in (B.5) is related to the value at which the (shrinking) conifold modulus $|z|$ is stabilized [10]. They are related as follows:

$$y = |z|^{4/3}.$$

   Note that this relation implies that the statistical distribution of the stabilized value of $|z|$ determines the distribution of the hierarchy y.

3. The warped volume $\mathcal{V}_W$ which relates the 10D and 4D Planck masses is given by

$$\mathcal{V}_W = \int \mathrm{d}^6y\,\sqrt{g_{CY}}\,e^{-4A} = \mathcal{V}\int \mathrm{d}^6y\,\sqrt{g_{CY_0}}\left(1+\frac{e^{-4A_0}}{\mathcal{V}^{2/3}}\right) \sim \mathcal{V}\,, \tag{B.6}$$

   where the last approximation is valid if the volume of the throat region is small compared to the (large) CY volume. Note that the warped volume remains finite even when regions of large warping are present. This is related to the finiteness of the Kähler potential in the complex structure moduli space.

4. The 10D action of an anti-D3 brane can be used to compute its contribution to the 4D scalar potential. This crucially depends on the anti-D3 position $\vec{r}_{\mathrm{D3}}$ in the internal dimensions, i.e. whether it is in a warped or unwarped region:

$$V_{\overline{\mathrm{D3}}} = 2T_3 \int \mathrm{d}^4x \sqrt{-g_4} \sim \frac{2M_s^4\,\mathcal{V}^{2/3}}{e^{-4A(r_{\mathrm{D3}})} + \mathcal{V}^{2/3}} \tag{B.7}$$

$$\sim \left\{ \begin{array}{ll} \frac{e^{4A(r_{\mathrm{D3}})}}{\mathcal{V}^{4/3}} & \text{for} \quad e^{-4A(r_{\mathrm{D3}})} \gg \mathcal{V}^{2/3} \\ \frac{1}{\mathcal{V}^2} & \text{for} \quad \mathcal{V}^{2/3} \gg e^{-4A(r_{\mathrm{D3}})} \end{array} \right\} \tag{B.8}$$

where T_3 is the tension of the brane.

5. Note that in the absence of warping one has a string scale contribution to the potential which typically will lead to a run away. Thus warped throats are necessary to obtain stable vacua in the presence of anti-D3 branes. On the other hand, a large volume is necessary to keep the α' expansion valid. In the presence of both large warping and large volume it is important to understand the interplay between these two large quantities and the regime of validity of the effective field theory. This has been analysed in detail in [100, 306, 329]. One finds the requirement

$$e^{-A_0} \ll \mathcal{V}^{2/3} \ll e^{-4A_0}\,. \tag{B.9}$$

We emphasize that in the very large radius limit, i.e. $\mathcal{V}^{2/3} \gg e^{-4A_0(y)}$ at all points in the compact directions, the metric becomes the standard unwarped CY metric $\mathrm{d}s_{10}^2 = \mathrm{d}s_4^2 + \mathcal{V}^{1/3}\,\mathrm{d}s_{CY_0}^2 = \mathrm{d}s_4^2 + \mathrm{d}s_{CY}^2$. In this limit there are no throats, and hence this region of moduli space is not appropriate if one wants to uplift KKLT and LVS AdS vacua.

Appendix C

Statistical Distributions

As emphasized in the introduction and in Chap. 5, the properties of a string vacuum are determined by the values at which moduli are stabilized. Our focus will be on KKLT and LVS models. Here fluxes stabilize the complex structure moduli. The Kähler moduli remain flat after the introduction of fluxes. The stabilization of the Kähler moduli can be studied in a low energy effective theory where the complex structure moduli are integrated out. Although the number of fluxes can be large, their effect on the low energy effective field theory of the Kähler moduli is encoded in terms of a small number of parameters. The observables that interest us are: the expectation value of the Gukov-Vafa-Witten superpotential W_0, the dilaton g_s, the hierarchy associated with the bottom of the warped throat y and the small complex structure F-terms. The statistical distributions of these quantities will serve as input for determining the distributions of the observables. The distributions of W_0, g_s, y and $|F|^2$ have been well studied. Below, we describe them.

C.1 Gukov-Vafa superpotential

The expectation value of the Gukov-Vafa-Witten superpotential W_0 is uniformly distributed as a complex variable [105] and physical quantities will be functions of $|W_0|$. Given the flat distribution of W_0, the distribution for $|W_0|$ is proportional to $|W_0|$. The distribution of W_0 and its physical implications were analysed in detail in [11].

C.2 Dilaton

The distribution of the dilaton is known to be uniform [104, 105]. In terms of the axio-dilaton $S = s - iC_0$ with $s = 1/g_s$

$$\mathrm{d}\mathcal{N} = \frac{\mathrm{d}S}{s^2}\,, \tag{C.1}$$

where $\mathrm{d}S = \mathrm{d}s\,\mathrm{d}C_0$ is the integration measure over the axio-dilaton. This has been confirmed by various numerical and analytical studies (see e.g. [129, 330]).

C.3 Hierarchy of throats

We turn now, to the hierarchy y. As discussed in App. B, the hierarchy is determined by the vacuum expectation value of the shrinking conifold modulus $|z|$. The distribution for $|z|$ (as determined by stabilization from fluxes) was studied in [105]. The fraction of vacua in which the conifold modulus takes value below $|z|$ was found to be

$$f(|z|) = -\frac{C}{\ln|z|}\,, \tag{C.2}$$

where C is a positive constant. The corresponding density is

$$\mathcal{N}(|z|)\,\mathrm{d}|z| = \frac{C\,\mathrm{d}|z|}{|z|(\ln|z|)^2}\,. \tag{C.3}$$

It is important to keep in mind that the singularity in the density for small $|z|$ is benign. Arbitrarily small $|z|$ corresponds to arbitrary large fluxes, which would be in conflict with the D3-tadpole cancellation condition. The statistical description is expected to break down before the singularity. Related is the fact that the fraction of states as given in (C.2) vanishes in the limit of $|z| \to 0$. Although, note that there is a significant enhancement of states in comparison with the expectation from the canonical metric of $\mathbb{C}$.

This distribution was used to study the distribution of throat hierarchies and the expectations for the number of throats in [117]. It was found that throats are ubiquitous.

C.4 Complex structure F-terms

Lastly, we address the distribution for small F-terms. This was described in details in [107], where it was found that for a generic distribution of superpotentials, described in previous section, one expects that $|F|^2$ is uniformly distributed.

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