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QUANTUM PROPERTIES OF COHERENT BLACK HOLES

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Abstract

The corpuscular model describes black holes as leaky bound states of gravitons. To account for the role of matter, an improved description of nonuniform geometries can be obtained by employing coherent states of gravitons, which recovers the Newtonian potential (with necessary departures) from a coherent state for a scalar field of gravitons in flat spacetime.

Given that the majority of black holes in nature are very likely to spin, we study the quantum hair associated with coherent states describing slowly rotating black holes and show how it can be naturally related with the Bekenstein-Hawking entropy and with 1-loop quantum corrections of the metric for the (effectively) non-rotating case. We also estimate corrections induced by such quantum hair to the temperature of the Hawking radiation through the tunnelling method.

We then provide a concise review of the key features of the horizon quantum mechanics formalism. This formalism is then applied to electrically neutral and spherically symmetric black hole geometries emerging from coherent quantum states of gravity to compute the probability that the matter source is inside the horizon. We find that quantum corrections to the classical horizon radius become significant if the matter core has a size comparable to the Compton length of the constituents and the system is indeed a black hole with probability very close to one unless the core radius is close to the (classical) gravitational radius.

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Chapter 1

Introduction

1.1 Motivations and outline

The contemplation and discussion of space and time have run through nearly the entire history of human thought and civilization. As early as the pre-Qin period of ancient China (around 300BC), Qu Yuan raised a profound question in “天问” (‘Tian Wen’ in Chinese, referring to Heavenly Questions) “Before the formation of heaven and earth, how can their origins be examined?” During the Warring States period in ancient China, Shi Jiao, in his work “Shi Zi,” concluded with the statement “The four directions and above and below are called “宇” (‘Yu’ in Chinese, referring to the infinite space), the past and present are called “宙” (‘Zhou’ in Chinese, referring to the infinite time)” suggesting that our universe (“宇宙” in Chinese) constitutes a holistic entity encompassing all time and space. During the same period, ancient Greek philosophers such as Parmenides, Heraclitus, and Zeno began philosophical inquiries into time and space. Parmenides posited that time and space are immutable, whereas Heraclitus argued for perpetual change in all things. In the 17th century, Isaac Newton [1] proposed the concept of absolute space and time, suggesting that the nature of space and time is independent of any object or motion, thereby postulating the existence of absolute space and absolute time. By the early 20th century, Albert Einstein’s theory [2, 3] had profoundly altered our understanding of spacetime. He proposed that time and space are not absolute but rather relative, intertwining to form a continuum of spacetime. Simultaneously, research in Quantum Mechanics (QM) has revealed the peculiar properties of time and space in the microscopic world, challenging many fundamental notions of classical physics.

General Relativity, formulated by Einstein in 1915, has been widely accepted as a cornerstone of gravitational theory and it successfully predicted various phenomena, including gravitational time dilation, gravitational lensing, and the gravitational redshift of light. One of the most remarkable predictions is the existence of black holes [4–7]. A black hole is a region of spacetime, where gravity is so strong that nothing, not even light, can escape from it. Black holes are very simple objects, at the classical level, since they are completely characterized by their mass, their charge, and their angular momentum as stated by the No Hair Theorem [8–14]. Some remarkable validations has arrived about a century after the formulation of Einstein's theory. The release of the first image of the supermassive black hole M87* by the Event Horizon Telescope (EHT) Collaboration in 2019 provided direct visual evidence of black holes in our universe [15–20], demonstrating that black holes are no longer merely theoretical models. The breakthrough in gravitational wave astronomy from the LIGO and Virgo collaboration opened up another new observational window [21–23], allowing us to directly learn more about black holes. Gravitational waves are predicted by General Relativity and described as "ripples" of the gravitational field caused by some of the most violent and energetic processes in the Universes. Those multiple direct channels provide us access to regions of black hole characterized by previously unexplored energy scales and spacetime curvature, enabling us to probe the strong gravity regime for the first time. The information obtained may lead to the discovery of new and unexpected phenomena.

As one of the most important predictions of general relativity, black holes are regions of spacetime exhibiting an extremely strong gravitationoal field such that no light or other radiation can escape from their interior. The boundary of such a region is called the event horizon of the black hole. One can easily find that the location of the horizon represents a coordinate singularity [24,25], which can be removed by an appropriate choice of coordinates. However, the center of the black hole displays a singular behaviour which cannot be removed by a coordinate transformation, which is referred to as curvature singularity. The emergence of curvature singularities is the benchmark for the breakdown of the classical general relativity, and contradicts one of the basic principle of quantum mechanics, that is, a concentration of a finite amount of energy in a point-like region clearly violates the Heisenberg uncertainty principle. One may expect the quantum theory will fix this inconsistent classical picture of the gravitational interaction. Many quantum models of black holes have been proposed in the past years (for a very partial list, see Refs. [26–42]). Some approaches, like the corpuscular picture [43–46], assume that the geometry should only emerge at suitable (macroscopic) scales from the underlying (microscopic) quantum field theory of gravitons [47–49]. It is demonstrated that attempting to provide a quantum mechanical description of the background itself could offer novel insights into some of the most mysterious aspects of gravitational phenomena. Based on this idea, the task of this thesis is to explore an improved description of nonuniform geometries, which can be achieved by employing coherent states of gravitons [50, 51], leading to necessary departures from the classical Schwarzschild metric [52] (and thermodynamics [53, 54]).

This thesis dissertation is organized as follows. In chapter 2, we will introduce the corpuscular model of a black hole as a bound state of gravitons, and the emerging nature of the Einstein geometrical view of gravity is presented. In chapter 3 coherent states for a quantum field are briefly introduced. In chapter 4 presents the coherent state approach to the corpuscular model. A coherent state for the scalar graviton field is built with the aim of reproducing the Newtonian configuration coming from the bound state. In chapter 5, we first review the classical solutions of the Klein-Gordon equation and show how coherent states of a massless scalar field on a reference flat spacetime associated to the vacuum can be used to reproduce a black hole geometry with small angular momentum in Section 5.1; Section 5.2 is devoted to studying the quantum hair of such coherent state black holes, whose existence implies that information about the interior state is present outside the horizon; the relation with the Bekenstein-Hawking entropy is derived in Section 5.3, where corrections to the Hawking temperature are also estimated using the semiclassical tunnelling methods. In chapter 6, we will first introduce the formalism of the Horizon Wave Function in Section 6.1; we then reconstruct the state $|\psi_{\rm S}\rangle$ from the effective energy density associated with the quantum corrected geometry (4.45) in Section 6.2; using that result, we will obtain the horizon wavefunction in Section 6.3. Finally, in chapter 7, we conclude with remarks and hints for future research.

1.2 Notation and conventions

In this thesis, greek indices run over the values $\mu = 0, 1, 2, 3,$ and latin indices over $i = 1, 2, 3$.

Repeated upper and lower Greek letters are summerd over,

$$A^\mu B_\mu \equiv \sum_{\mu=0}^{3} A^\mu B_\mu \ . \tag{1.1}$$

We will adopt the "mostly plus convention" foe all Lorentzian manifolds, meaning that the signature of the Minkowski metric is

$$\eta_{\mu\nu} = \mathrm{diag}(-, +, +. +) \ . \tag{1.2}$$

Where otherwise indicated, in this work the speed of light will be set equal to one, $c = 1$, while the Planck constant $\hbar$ and the Newton constant G will be left explicit.

The scalar product between two solutions of the Klein-Gordon equation, f_1 and f_2, is written as

$$(f_1|f_2) = i \int \mathrm{d}^3 x \left(f_1^* \, \partial_t f_2 - f_2 \, \partial_t f_1^* \right) \ . \tag{1.3}$$

Throughout the discussion of this thesis, we shall use the following notation.

R_{H}: event horizon,

R_{s}: size of the source,

R_∞: length associated to the time of collapse,

m_{p}: Planck mass,

ℓ_{p}: Planck length,

UV, IR: ultraviolet and infrared.

Chapter 2

Corpuscular picture

As one of the most fascinating entities in the cosmos predicted by General Relativity, black holes even point beyond the classical theory, due to the presence of singularities, which might just signal the breakdown of classical physics in strong field regime, and the intriguing phenomenon of black hole thermodynamics [11, 55–60], which appears when the predictions of Quantum Mechanics and General Relativity are combined. Moreover, the evolution of black holes is not completely well understood at present, because it is conceptually and technically involved. Since establishing a full quantum theory of gravity remains nowadays a challenge, it is natural to ask if the available mathematical tools and physical frameworks, can effectively describe the classical and semiclassical effects involving black holes. Those effects have already been described by General Relativity and Quantum Field Theory, but a more refined quantum description may provide additional information and insights into the underlying physics.

To better understand classical or quantum gravitational effects, it is crucial to determine the scale at which these effects become strong and therefore measurable. The classical General Relativity contains no intrinsic length-scale. The only existing parameter, the Universal gravitational constant $G_{\rm N}$, determines the intrinsic strength of gravity and has a dimension of length divided by mass,

$$[G_{\rm N}] = \frac{[length]}{[mass]} \ . \tag{2.1}$$

In order to establish a characteristic length of classical gravity, we need multiply $G_{\rm N}$ by a quantity with the dimension of mass (or energy), such as the energy (mass) of the source, M. Combining these two quantities yields a length,

$$R_{\rm H} = 2\, G_{\rm N}\, M \ , \tag{2.2}$$

known as Schwarzschild radius, which defines the distance at which the classical gravitational effects of a gravitational source become dominant. As an intrinsically classical length, the Schwarzschild radius $R_{\rm H}$ is the most important characteristics of the gravitational properties of the source in classical gravity.

It is important to emphasize that the above length scale is purely classical, but nature is fundamentally quantum. In order to explore quantum effects, we need consider the Planck constant $\hbar$, which has dimension

$$[\hbar] = [length]\,[mass] \ . \tag{2.3}$$

From this constant and the Newtonian constant $G_{\rm N}$, we can derive another length scale as

$$\ell_{\rm p}^2 = \hbar\, G_{\rm N} \ , \tag{2.4}$$

at which the quantum gravitational fluctuations of the spacetime metric become significant and can no longer be ignored. Unlike the schwarzschild radius $R_{\rm H}$, the length scale $\ell_{\rm p}$, called Planck length, is an intrinsically quantum concept. At this distance, any perturbative approach in Quantum Field Theory will break down and fail to produce accurate physical results.

Analogously, from the Newtonian constant $G_{\rm N}$ and the Planck constant $\hbar$, it is possible to find a quantity with the dimension of mass, known as the Planck mass,

$$m_{\rm p}^2 = \frac{\hbar}{G_{\rm N}} \ . \tag{2.5}$$

Other quantum length scales are the well known Compton (or de Broglie) wave-lengths of the source of mass M,

$$L_{\rm C} = \frac{\hbar}{M} \ . \tag{2.6}$$

The physical significance of $L_{\rm C}$ is that it sets the scale at which energy of quantum fluctuations becomes comparable to the energy of the source.

Notice that both the Planck length $\ell_{\rm p}$ and the Compton wave length $L_{\rm C}$, as well as the Planck mass $m_{\rm p}$, vanish in the limit $\hbar \to 0$ when the mass M and the Newtonian constant $G_{\rm N}$ are kept fixed, which leads that the sizes of black holes can be arbitrarily small in classical General Relativity ($\hbar = 0$). In reality, however, $\hbar$ and $G_{\rm N}$ are fixed constants, with the mass of the source M being the only variable parameter. Consequently, the criterion for the classicality of the gravitational filed produced by a source can be expressed as

$$\ell_{\rm p} \ll R_{\rm H} \ . \tag{2.7}$$

In principle, an effective theory of gravity can safely be constructed within this region.

Consider now a gravitating source with rest mass M, for which $R_{\rm H} \gg \ell_{\rm P}$. The entire space can be divided into three different regions for analyzing the interaction: the first one covers the range $r \in [R_{\rm H}, \infty)$, where gravity can be considered weak with respect to the other forces, as it can be seen by comparison of the coupling constants inherent in classical laws. The second one stays in the regime $r \in [\ell_{\rm p},\ R_{\rm H}]$, where gravitational effects become strong. With a certain amount of approximation at distances where Quantum Mechanics becomes relevant, a complete physical description about this region requires a semiclassical framework where matter fields obey quantum field theory while spacetime dynamics remain governed by general relativity under a certain level of approximation: the classical Newtonian theory requires corrections to take into account the nonlinear nature of gravity. The last one is sub-Planck length region, where a complete quantum theory of gravity is required.

As previously mentioned, since formulating quantum general relativity presents significant difficulties and remains a challenge, we will exclude the third region. Fortunately, the other two regions could be modelled within a quantum field framework, and it is crucial to understand how to recover classical physics from a quantum description of the configuration. In particular, the corpuscular picture [43–46], proposed by Dvali and Gomez, have offered a new perspective to effectively describe the interior of black holes as extended objects, and drawn more and more attention. The idea is very simple: the black hole is a leaky bound state in form of a cold Bose-condensate of N weakly interacting soft gravitons of characteristic Compton-de Broglie wavelength $\lambda \sim R_{\rm H}$. With completely neglecting the role of the precise quantum composition of the source and other interactions, this theory recovers classical and semiclassical effects arising from black hole mechanics based on the physics of condensates.

In the weak gravity regime, the General Relativity is viewed as a Quantum Field Theory that propagates a unique weakly coupled quantum particle, the graviton, with zero mass and spin-2. At low energies, a dimensionless quantum self-coupling of gravitons can be defined as the ratio of the two length scales

$$\alpha_{gr} = \frac{\ell_{\rm p}^2}{\lambda^2} \ , \tag{2.8}$$

where λ is the typical wavelength of the gravitational interacting particles. It is evident that at large distances (or, in other words, at low energies) the coupling constant becomes quite small, resulting in very weak interaction among the gravitational quanta composed the source. Intuitively, this condition of weak interaction should be satisfied if Eq. (2.7) holds true, equivalently, for macroscopic black holes. For wavelength $R_{\rm H} \ll \ell_{\rm p}$, the above coupling (2.8) becomes strong, and the theory violates perturbative (in α_{gr}) unitarity in graviton-graviton scattering. The traditional approach for addressing the issue involving assuming that Einstein gravity requires a Wilsonian UV-completion. However, the corpuscular model provided an alternative simple quantum description of non-Wilsonian UV-self-completion of Einstein gravity. In order to support this view, they consider the Newtonian potential at a distance r generated by a system with mass M as

$$V_{\rm N} \sim -\frac{G_{\rm N}\, M}{r} \ . \tag{2.9}$$

From the quantum field theory point of view, the above linearized metric perturbation about the flat space (2.9) represents a superposition of N gravitons, where N denotes the occupation number, indicating the number of field quanta and measure the level of classicality. In particular, the condition of classicality is typically represented as

$$N \gg 1 \ . \tag{2.10}$$

The effective mass m of each graviton is related to their characteristic quantum mechanical size via the Compton-de Broglie wavelength,

$$\lambda \simeq \frac{\hbar}{m} = \ell_{\rm p} \, \frac{m_{\rm p}}{m} \ . \tag{2.11}$$

These gravitons can superpose to form a Bose-condensate of radius λ. The corpuscular theory proposed herein characterises black holes as quantum condensates of gravitons. According to the physics of condensates, this framework allows for the recovery of classical and semiclassical effects coming from black holes mechanics, while entirely neglecting the role of matter and other interactions.

Let us now consider a source of radius $r \gg R_{\rm H}$. In this regime, General Relativity is well approximated by the Newtonian theory, and thus the gravitational component of the energy is

$$E_g \sim \frac{M \, R_{\rm H}}{r} \ . \tag{2.12}$$

The total energy, on the other hand, can be approximated as the sum of the energies of the individual gravitons with the wavelength λ and the occupation number N,

$$E_g \sim \sum_{\lambda} N_{\lambda} \frac{\hbar}{\lambda} \ . \tag{2.13}$$

Since the dominant contribution arises from the most occupied wavelength of the gravitational source, $\lambda \sim r$, and the contributions from the shorter wavelength are exponentially suppressed and can be ignored, the gravitons contributing to the energy are of very long wavelengths and thus interact weakly. The above expression can be further approximated as

$$E_g \sim N \, \frac{\hbar}{r} \ . \tag{2.14}$$

Comparing the two expressions given for the energy, we easily obtain the occupation number of gravitons,

$$N = \frac{M \, R_{\rm H}}{\hbar} \ , \tag{2.15}$$

which is a safe estimation until quantum gravitational effects will not become dominant.

However, it is important to note that in the regime $r \gg R_{\rm H}$, the gravitational self-sourcing (the interactions among the gravitons and between any individual graviton and the entire collective gravitational energy) can be ignored. Furthermore, any energetic contributions coming from matter are also neglected at the beginning. The only available energy in the configuration is the gravitational one, which has a negative sign since it is a binding energy. Therefore the condensate, where only the Newtonian potential is present, cannot be self-sustained and must necessary collapse. Then it is important to notice that by continuing ignoring matter contributions, when the size of the condensate reaches $r \sim R_{\mathrm{H}}$, although the interactions among the individual gravitons remain to be negligible, the gravitational energy becomes comparable to the energy of the source. At this point, the collapse of the condensate can be stopped and the condensate becomes self-sustained. Moreover, the occupation number of gravitons can still be estimated as given by (2.15), since the interactions among the individual gravitons are still negligible for the size $r \gg \ell_{\mathrm{p}}$. The equation (2.15) can be rewritten in a more useful way as

$$N = \frac{M^2}{m_{\mathrm{p}}^2} = \frac{\lambda^2}{\ell_{\mathrm{p}}^2} \ , \tag{2.16}$$

a result which reproduces Bekenstein's conjecture for the quantisation of horizon area [51]. Notice that in the classical limit $\hbar \to 0$, the number of gravitons diverges, so this quantity is "super-classical". Only when $\hbar \to 0$ does the system becomes truly classical, causing all possible quantum effects to vanish.

As stated before, the occupation number N is considered as the criterion for classicality. For example, an electron will never be regarded as a classical gravitational source, since for an electron, $N_e = (m_e^2/M_{\mathrm{P}}^2) \sim 10^{-44} \ll 1$ (and thus no gravitons at all), despite the fact that the electron can exchange gravitons in the scattering processes and does create a Newtonian gravitational potential.

By keeping $\lambda \sim R_{\mathrm{H}}$, we can rewrite the relations between the black hole parameters and the occupation number N from Eq.(2.16) as

$$M \sim \sqrt{N}\, M_{\mathrm{p}} \ , \quad \lambda \sim \sqrt{N}\, \ell_{\mathrm{p}} \ . \tag{2.17}$$

Furthermore, with equation (2.8), we immediately arrive at

$$\alpha_{gr} \sim \frac{1}{N} \ , \tag{2.18}$$

which means the interaction among the gravitons is extremely weak if the black hole is composed out of large N gravitons. To gain a better understanding of the self-sustained Bose-condensate of gravitons, we attempt to form a Bose-condensate of gravitons of characteristic wavelength λ by gradually increasing the occupation number N. Initially, external sources are required to maintain the condensate because gravitons interactions are negligible when N is small. However, with increasing N, the effects of these interactions become significant. Individual gravitons feel stronger and stronger collective binding potential and for the critical occupation number

$$N = N_c = \frac{1}{\alpha_{gr}} \ , \tag{2.19}$$

the potential is given by

$$U_{\rm G} \sim -\,\alpha_{gr}\, N\, \frac{\hbar}{r} \ , \tag{2.20}$$

and its kinetic energy will be

$$K \sim \frac{\hbar}{\lambda} \ . \tag{2.21}$$

Here we assume that the potential (2.9) becomes negligible for $r \geq \lambda$. For $r \sim \lambda$ and the critical occupation number (2.19), the self-sustainability condition, resulting from the equilibrium between the collective binding potential and kinetic energies of individual gravions, is reached, *i.e.*

$$K + U_{\rm G} \sim 0 \ . \tag{2.22}$$

Therefore, the graviton condensate becomes self-sustained for the critical value of N given by Eq.(2.19) [61]. A significant fact is that among all possible sources of some characteristic physical size λ, the occupation number N is maximized by a black hole. In other words, this energy balance yields the "maximal packing"

$$N\,\alpha_{gr} = 1 \ . \tag{2.23}$$

This implies that further increasing N without increasing λ is impossible. Any attempt to increase N will result in an increase in the size of the black hole, and consequently, in the particle wavelengths as $\sqrt{N}$.

An important fact about this black hole quantum N-portrait is that for a black hole with very high center of mass energy M, (or equivalently, the occupation number $N \gg 1$ according to Eq. (2.16)), the characteristic radius of its event horizon can be estimated, from the wavelength expression (2.17) and the expression of N as a function of the mass M, as

$$\lambda \sim R_{\rm H} \ . \tag{2.24}$$

This means that the geometric aspect of General Relativity emerges from the corpuscular picture by setting the event horizon is nothing but the dominant wavelength of gravitons composing the black holes. Furthermore, the occupation number N of gravitons does not change because with increasing the mass of the source M, the event horizon grows and this means that the wavelengths of the constituent gravitons increase, which lead them to become softer. This process happens since the coupling constant depends on the energy of the system and it also plays a key role in making gravity simple at the semiclassical level, which means that the physics can be described by few parameters. In particular, for the corpuscular model, the quantum physics of black holes (such as no-hair, thermality, and entropy) can be understood in terms of a single quantum quantity, N.

Applying the semiclassical approach of Quantum Field Theory in curved space time, Stephen Hawking found that a black hole would emit radiation [55]. This idea is that a virtual particle-antiparticle pair arises naturally near the event horizon if the Quantum Mechanics is taken into account and is pulled apart due to the gravitational field of black hole. Then one of them falls into the black hole while the other one escapes to future null infinity. However, such evaporation process is understood as scattering process since black holes are viewed as leaky bound states of weakly interacting gravitons on flat spacetime and the escape energy is just slightly above the energy of the condensate quanta. Thus the evaporation spontaneously occurs when one of gravitons gains energy above the condensation energy during scattering. The final state of the scattering process is another black hole with $N-1$ gravitons plus one graviton escaping to infinity. At first order, reciprocal $2 \to 2$ scattering inside the condensate will result in a depletion rate

$$\Gamma = \frac{1}{N^2} N^2 \frac{\hbar}{\sqrt{N}\,\ell_{\rm p}} \ , \tag{2.25}$$

where the first factor comes from the interaction vertices $N^{-2} \sim \alpha_{gr}^2$, the second factor is combinatoric (there are about N gravitons scattering with other $N-1 \sim N$ gravitons), and the last factor comes from the typical energy of he process $\Delta E \sim m$. The number of black hole constituents will then decrease according to

$$\dot{N} \simeq -\Gamma \simeq -\frac{1}{\sqrt{N}\,\ell_{\rm p}} \ . \tag{2.26}$$

The emission rate of the black hole mass is given by

$$\dot{M} \simeq -\frac{\hbar}{N\,\ell_{\rm p}^2} \ . \tag{2.27}$$

From this flux one can read off the "effect" Hawking temperature

$$T_{\rm H} \simeq \frac{\hbar}{\sqrt{N}\,\ell_{\rm p}} \simeq \frac{m_{\rm p}}{\sqrt{N}} \ . \tag{2.28}$$

To summarise, in the corpuscular portrait of gravitons, a black hole is modelled as a Bose condensate of marginally bound, self-interacting gravitons, trapped in a gravitational well described by the simple potential (2.20). This kind of black hole is characterized by only one parameter, the occupation number N of the gravitons. This system is at its quantum critical point, so it continuously depletes and leaks gravitons because of the graviton-graviton scattering, as one would expect for an interacting homogeneous Bose condensate. This leakage behave therefore could be interpreted as the well known Hawking radiation as a result of the quantum depletion. The black hole quantum N portrait offers us a simple way to understand black hole and leads to very "reasonable" properties, but its original form lacks some features that might make it even more appealing. For example, in the above picture, the geometry at suitable (macroscopic) scales emerges as an effective phenomenon from an underlying (microscopic) quantum field of gravitons with respect to a flat spacetime vacuum. From the phenomenological point of view, the question then arises naturally as whether the causal structure of spacetime indeed contains a trapping surface. Another issue is that, in the original corpuscular picture [43–45], only gravitons are considered and baryonic matter that initially collapsed and formed the black hole is argued to become essentially irrelevant. However, it is well known that, unless the black hole is formed from a primordial quantum fluctuation of the vacuum in the very early Universe, the only known mechanism that a black hole could form is the gravitational collapse of a star or other astrophysical source [62]. The question then arises whether neglecting the contribution of any matter source in the final state is reliable approximation. This picture can be improved by employing the Quantum Mechanics theory and reconciled with the usual geometric description of spacetime from the underlying quantum description of the gravitational field. This crucial aspect is the starting point of the *coherent state* description [50–54, 63–65]. This picture is based on identifying the quantum state of gravitational potential as a coherent state of (virtual) soft gravitons, which establishes a connection between the microscopic dynamics of gravity, understood in terms of interacting quanta, and the macroscopic description of a curved background.

Chapter 3

Coherent states in QFT

The topic of coherent state dates back to the seminal paper of Schrödinger in 1926 [66], in which he studied the dynamics of the harmonic oscillator to demonstrate how classical mechanics can be recovered from Quantum Mechanics. Since then, developments in the field of coherent states and their applications have been breathtaking, see e.g. Refs. [67, 68]. In fact, the term "coherent state" does not appear in the original paper, but what would convey his intuition better there probably is "semiclassical". Actually, the essential breakthrough in the concept of coherent state was achieved through the pioneering works of Glauber and Sudarshan in the early 1960s [69–72], where the term "coherent state" was first coined. In these seminal papers, they provided the first modern and specific applications in the context of quantum optics and launched this fruitful and important field of study. The coherent states for harmonic oscillator by historical definition refer to certain quantum states of the oscillator, in which the motion of particles is governed by the laws of Quantum Mechanics, and mean values of the dynamical variables of position and momentum that closely mimics their counterparts in classical mechanics. However, this property can be satisfied by any time-evolved state [73], as proved by the Ehrenfest Theorem. Therefore, coherent states are more than that: all coherent states conform to minimize the uncertainty of given observables, where "minimize" implies that the Uncertainty Principle involving these observables is saturated with the inequality. Furthermore, their coherence is maintained despite time dependence, although the underlying parameters could change with it.

According to Glauber [69–71], the field coherent states can be constructed from three equivalent perspectives. The first is the ladder-operator method where coherent states emerge as the eigenstates of the annihilation operator. The second is displacement-operator method, in which coherent states are defined by the application of a "displacement" operator on the vacuum state. The third is employment of minimum uncertainty condition where coherent states are considered as quantum states with a minimum Heisenberg uncertainty relationship concerning the product of the uncertainties of the canonical position and momentum assuming the value of $\frac{\hbar}{2}$. The three arguments appear as a first attempt to construct the coherent states and have been the basis of their later developments. However, as we mentioned above, a state is really called as coherent not just for the minimized uncertainties that it gives, but also because such quantities are constant in time. This face has been proven for the harmonic oscillator, owing to the special quadratic form of its Hamiltonian. It does not apply to arbitrary system: a Gaussian wave-packet built for a free particle will irrevocably spread with time and looses its initially compact shape, and with it, its uncertainty will spread as well. From this perspective, there is no "quasi-classical state" for the free particle. Whereas a coherent wavefunction of the harmonic oscillator will stay confined within a finite range and even come back periodically to its initial spreading since there is no irresistible increasing of spreading. The coherent states are shown to be the only states minimising the Heisenberg uncertainty inequality, and these states whose spreading in position is constant and minimal. Thus, the definition of coherent state is not only kinematical, but also dynamical, and the harmonic oscillator was the first system for which these states have been built.

It is also noteworthy that the subject of coherent states has a wide range of current interests. The important emerging fields in this direction are evident in recent developments in quantum gravity, which make use of notions rooted in geometry. We will first review the harmonic oscillator coherent states and touch upon their various properties.

3.1 Quantum harmonic oscillator

Let us consider a one-dimensional harmonic oscillator for a particle of mass m and pulsation ω, whose time evolution is dictated by the Hamiltonian [67, 74]

$$H = \frac{\hat{p}^2}{2\,m} + \frac{1}{2}\,m\,\omega^2\,\hat{x}^2 \ , \tag{3.1}$$

where the "hat" denotes the position ($\hat{x}$) and momentum ($\hat{p}$) operators. The operators $\hat{x}$ and $\hat{p}$ are postulated to satisfy the Heisenberg's uncertainty relation

$$[\hat{x},\,\hat{p}] = i\,\hbar \quad \Rightarrow \quad \Delta x\,\Delta p \geq \frac{\hbar}{2} \ . \tag{3.2}$$

Since $\hat{x}$ and $\hat{p}$ cannot be simultaneously diagonalized, we shall choose the Schrödinger representation, in which $\hat{x}$ is diagonal and $\hat{p}$ is given by a first-order differential operator

$$\hat{x} = x\,\mathbb{I} \quad \hat{p} = -i\,\hbar\,\frac{\mathrm{d}}{\mathrm{d}x} \ , \tag{3.3}$$

where $\mathbb{I}$ is the identity operator. With the Dirac's elegant operator method, we obtain the position eigenkets as [75–77]

$$\hat{x}|x\rangle = x|x\rangle, \quad \langle x|x'\rangle = \delta(x - x') \ , \quad \int \mathrm{d}x|x\rangle\langle x| = \mathbb{I} \ , \tag{3.4}$$

and a similar set of relations for the momentum eigenkets

$$\hat{p}|p\rangle = p|p\rangle, \quad \langle p|p'\rangle = \delta(p-p') \ , \quad \int \mathrm{d}p|p\rangle\langle p| = \mathbb{I} \ . \tag{3.5}$$

The forms of overlaps read as

$$\langle x|p\rangle = \frac{1}{\sqrt{2\,\pi\,\hbar}} e^{i\,p\,x} \ , \quad \langle p|x\rangle = \frac{1}{\sqrt{2\,\pi\,\hbar}} e^{-i\,p\,x} \ . \tag{3.6}$$

In order to find the eigenvectors and eigenvalues of the time independent Schrödinger equation,

$$-\frac{\hbar^2}{2\,m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \frac{m\,\omega^2\,x^2}{2}\psi = E\,\psi \ , \tag{3.7}$$

it is convenient to define two non-Hermitian operators,

$$a = \frac{1}{\sqrt{2\,m\,\hbar\,\omega}}\left(m\,\omega\,\hat{x} + i\,\hat{p}\right) \ , \quad a^\dagger = \frac{1}{\sqrt{2\,m\,\hbar\,\omega}}\left(m\,\omega\,\hat{x} - i\,\hat{p}\right) \ , \tag{3.8}$$

which are known as the annihilation and creation operators acting on a Fock space spanned by the eigenstates of the Hamiltonian, respectively. Then we have

$$\hat{x} = \sqrt{\frac{\hbar}{2\,m\,\omega}}\left(a^\dagger + a\right) \ , \quad \hat{p} = i\sqrt{\frac{m\,\omega\,\hbar}{2}}\left(a^\dagger - a\right) \ . \tag{3.9}$$

It is straightforward to obtain an important relation between the Hamiltonian operator, annihilation operator, and creation operator

$$H = \frac{1}{2}\hbar\,\omega\left(a^\dagger\,a + a\,a^\dagger\right) = \hbar\,\omega\left(a^\dagger\,a + \frac{1}{2}\right) = \hbar\,\omega\left(N + \frac{1}{2}\right) \ , \tag{3.10}$$

where we defined the number operator $N = a^\dagger\,a$. Using the canonical commutation relations, we obtain

$$\left[a,\, a^\dagger\right] = 1 \ , \quad [H,\, a] = -\hbar\,\omega\,a \ , \quad \left[H,\, a^\dagger\right] = \hbar\,\omega\,a^\dagger \ . \tag{3.11}$$

Since the Hamiltonian H is a liner function of the number operator N, N can be diagonalized simultaneously with H. It is convenient to find the eigenvectors and eigenvalues of the number operator N. Since the number operator N is a Hermitian operator, we can denote an energy eigenvector of it by its eigenvalue n as

$$N|n\rangle = n|n\rangle \ . \tag{3.12}$$

Multiplying above formula by $\langle n|$ on the left, we have

$$\langle n|\,a^\dagger\,a\,|n\rangle = \Big|a|n\rangle\Big|^2 = n \geq 0 \ , \tag{3.13}$$

which implies that n can never be negative.

Now applying the annihilation operator a to both sides of Eq. (3.12),

$$a\, a^\dagger\, a|n\rangle = \left(a^\dagger\, a + 1\right) a|n\rangle = n\, a|n\rangle \ , \tag{3.14}$$

we have

$$N\,(a|n\rangle) = (n-1)\,(a|n\rangle) \ , \tag{3.15}$$

which shows that $a|n\rangle$ is also an eigenvector of the number operator N with eigenvalue decreased by one. Similarly, it is easy to know that $a^\dagger|n\rangle$ is an eigenvector of the number operator N with eigenvalue increased by one, *i.e.*

$$N\left(a^\dagger|n\rangle\right) = (n+1)\left(a^\dagger|n\rangle\right) \ . \tag{3.16}$$

Because the decrease (increase) of n by one amounts to annihilation (creation) of one quantum unit of energy $\hbar\omega$, it identifies the operator a and $a^\dagger$ as "lowing" and "rasing" operators between adjacent eigenstates,

$$a|n\rangle = \sqrt{n}|n-1\rangle \ , \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \ , \tag{3.17}$$

where numerical constants $\sqrt{n}$ and $\sqrt{n+1}$ is determined from the requirement that $|n{-}1\rangle$, $|n{-}1\rangle$, and $|n{+}1\rangle$ are normalized. The creation operator is the adjoint of the annihilation operator and vice versa. Note that

$$\langle n|\, a^\dagger\, a\,|n\rangle = \langle n-1|n|n-1\rangle = n \ , \tag{3.18}$$

and

$$\langle n|\, a\, a^\dagger\,|n\rangle = \langle n+1|n+1|n+1\rangle = n+1 \ . \tag{3.19}$$

By applying the annihilation operator a on $|n\rangle$ repeatedly, equation (3.17) shows that a sequence of eigenvectors are generated as $|n-1\rangle$, $|n-2\rangle$, $|n-3\rangle$, ... Since $n \geq 0$ and since $a|0\rangle = 0$, this sequence has to terminate at $n = 0$. The state $|0\rangle$ is defined as the vacuum which is annihilated by the operator a,

$$a|0\rangle = 0 \ . \tag{3.20}$$

Because the smallest possible value of n is zero, the ground state of the quantum oscillator has

$$E_0 = \frac{1}{2}\,\hbar\,\omega \ . \tag{3.21}$$

Similarly, the repeated applications of $a^\dagger$ on the ground state $|0\rangle$ generates the excited states

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(a^\dagger\right)^n|0\rangle \ . \tag{3.22}$$

In this way, we construct the energy spectrum of a harmonic oscillator as

$$E_n = \left(n + \frac{1}{2}\right) \hbar\omega \ , \quad (n = 0,\, 1,\, 2,\, ...) \ . \tag{3.23}$$

It is obversely that the energy levels of the energy spectrum are equally spaced, and the lowest energy eigenvalue of the oscillator is not zero but is instead equal to $E_0 = \hbar\,\omega/2$.

3.2 Coherent states for the harmonic oscillator

From Eq.(3.17) and the orthonormality requirement of $|n\rangle$, we derive the matrix elements

$$\langle n'|\, a\, |n\rangle = \sqrt{n}\, \delta_{n',n-1} \ , \quad \langle n'|\, a^\dagger\, |n\rangle = \sqrt{n+1}\, \delta_{n',n+1} \ . \tag{3.24}$$

Using these together with Eq.(3.9), we obtain the matrix elements of the position operator x

$$\langle n'|\, x\, |n\rangle = \sqrt{\frac{\hbar}{2\, m\, \omega}} \left(\sqrt{n}\, \delta_{n',n-1} + \sqrt{n+1}\, \delta_{n',n+1}\right) \ , \tag{3.25}$$

and the momentum operator p

$$\langle n'|\, \hat{p}\, |n\rangle = i\sqrt{\frac{m\, \omega\, \hbar}{2}} \left(-\sqrt{n}\, \delta_{n',n-1} + \sqrt{n+1}\, \delta_{n',n+1}\right) \ . \tag{3.26}$$

It is clear that neither x nor p is diagonal in the N-representation. From above equations, it follows that on the ground state,

$$\langle x\rangle = \langle p\rangle = 0 \ , \tag{3.27}$$

which also applies for the excited states. Furthermore, this conclusion is valid with time evolution being considered, *i.e.*

$$\langle n|\, x(t)\, |n\rangle = \langle n|\, p(t)\, |n\rangle = 0 \ , \tag{3.28}$$

which means that the expectation values of position and momentum vanish.

We have observed that the behavior of oscillating expectation values for position x and momentum p does not resemble that of a classical oscillator, no matter how large n may be. The question then arises naturally that how can we construct a superposition of energy eigenstates that most closely imitates the behavior of a classical oscillator? In wave function language, a wave packet that bounces back and forth without spreading in shape is needed.

For a one-dimensional classical harmonic oscillator, its motion is described by the function

$$x_c(t) = x_0 \cos\left(\omega\, t + \varphi\right) \ , \tag{3.29}$$

and the potential energy stored in it at position x_0 is

$$E_c = \frac{1}{2} m\,\omega^2\, x_0^2 \ , \tag{3.30}$$

where x_0 is the constant amplitude, ω is the angular frequency, and φ is initial phase. In order to facilitate comparison with Quantum Mechanics, we can rewritten above equations as

$$\begin{aligned} x_c(t) &= \frac{1}{2}\left(x_0\, e^{-i\varphi}\, e^{-i\omega t} + x_0\, e^{i\varphi}\, e^{i\omega t}\right) \\ &= \lambda\left(z\, e^{-i\omega t} + z^*\, e^{i\omega t}\right) \ , \end{aligned} \tag{3.31}$$

and

$$E_c = 2\, m\,\omega^2\, \lambda^2\, |z|^2 \ , \tag{3.32}$$

where $\lambda\, z = \frac{1}{2} x_0\, e^{-i\varphi}$, with z being a complex number and λ being an appropriate real constant.

In the Schrödinger picture, the normalized coherent state of the harmonic oscillator is assumed as $|z(t)\rangle$, and we can abbreviate $|z(0)\rangle$ as $|z\rangle$. For the time-developed coherent state $|z\rangle$, we have

$$|z(t)\rangle = e^{-\frac{i}{\hbar} t\, H} |z\rangle \ . \tag{3.33}$$

The condition that mean values for the dynamical variables, position and momentum, closely mimic their counterparts in classical mechanics then can be expressed as

$$\langle z(t)|\, \hat{x}\, |z(t)\rangle = x_c(t) \ , \tag{3.34}$$

and

$$\langle z(t)|\, H\, |z(t)\rangle = E_c \ . \tag{3.35}$$

Notice that here the Hamiltonian of quantum harmonic oscillator is set to be $H = \hbar\,\omega\, a^\dagger\, a$, since the classical energy starts at zero. We can construct the coherent states based on the above two conditions. From (3.9) and (3.33), the left side of equation (3.34) can be rewritten as

$$\langle z(t)|\, \hat{x}\, |z(t)\rangle = \sqrt{\frac{\hbar}{2\, m\,\omega}} \left(\langle z|\, a^\dagger\, |z\rangle\, e^{i\omega t} + \langle z|\, a\, |z\rangle\, e^{-i\omega t}\right) \ . \tag{3.36}$$

By Comparing with the classical equation of motion (3.31), we find that assuming $\lambda = \sqrt{\frac{\hbar}{2\, m\,\omega}}$ transforms the first condition (3.34) into

$$\langle z|\, a\, |z\rangle = z \ . \tag{3.37}$$

Here z is "quantum number" of coherent states, and it is related with the classical amplitude by

$$|z| = \sqrt{\frac{m\,\omega}{2\,\hbar}}\,x_0 \ . \tag{3.38}$$

However, the condition (3.34) cannot uniquely determine $|z\rangle$. This can be seen from the following discussion. By performing a unitary transformation $D(z)$, we can transform a into $\tilde{a} = a + z$, *i.e.*

$$\begin{aligned} a \to \tilde{a} &= D^\dagger(z)\,a\,D(z) = a + z \ , \\ a^\dagger \to \tilde{a}^\dagger &= D^\dagger(z)\,a^\dagger\,D(z) = a^\dagger + z^* \ . \end{aligned} \tag{3.39}$$

The transformation of $|z\rangle$ is

$$|z\rangle \to |\tilde{z}\rangle = D^\dagger(z)|z\rangle \ . \tag{3.40}$$

As a consequence, the equation (3.37)

$$\langle\tilde{z}|\,\tilde{a}\,|\tilde{z}\rangle = \langle z|\,a\,|z\rangle = z \tag{3.41}$$

becomes

$$\langle\tilde{z}|\,a\,|\tilde{z}\rangle = \langle\tilde{z}|\ (\tilde{a} - z)\ |\tilde{z}\rangle = 0 \ . \tag{3.42}$$

The left side of above equation shows that equation (3.42) is satisfied as long as $|\tilde{z}\rangle$ is an eigenstate of the harmonic oscillator. Therefore, the coherent states that satisfy condition (3.34) are

$$|z\rangle = D(z)|\tilde{z}\rangle = D(z)|n\rangle \ , \tag{3.43}$$

which implies that the first condition (3.34) is insufficient to uniquely determine the coherent state $|z\rangle$ given a classical amplitude x_0.

Now we shall consider the second condition (3.35),

$$\begin{aligned} \langle H\rangle &= \langle z(t)|\,H\,|z(t)\rangle = \langle z|\,H\,|z\rangle = \langle\tilde{z}|\,\tilde{H}\,|\tilde{z}\rangle \\ &= \langle\tilde{z}|\,D^\dagger(z)\,H\,D(z)|\tilde{z}\rangle = \hbar\,\omega\langle\tilde{z}|\ \left(a^\dagger + z^*\right)(a + z)\,|\tilde{z}\rangle \\ &= \langle\tilde{z}|\,H\,|\tilde{z}\rangle + \hbar\,\omega\,|z|^2 \ , \end{aligned} \tag{3.44}$$

where we have used equation (3.41). From equations (3.30) and (3.9), for the second condition (3.35) to be satisfied, we must require $\langle\tilde{z}|\,H\,|\tilde{z}\rangle = 0$, *i.e.*

$$0 = \langle\tilde{z}|\,H\,|\tilde{z}\rangle = \hbar\,\omega\langle\tilde{z}|\,a^\dagger\,z|\tilde{z}\rangle \ . \tag{3.45}$$

In order to ensure the above equation holds, we must have $a|\tilde{z}\rangle = 0$, *i.e.* $|\tilde{z}\rangle = 0$, which shows that $|\tilde{z}\rangle$ must be the ground state of harmonic oscillator. Thus the second condition (3.35) eliminates the arbitrariness in the equation (3.43), and the state $|n\rangle$ on the right side of equation (3.43) according to the second condition (3.35) must be $|0\rangle$.

Therefore, the final form of the coherent state $|z\rangle$ is

$$|z\rangle = D(z)|0\rangle \ , \tag{3.46}$$

and

$$|z(t)\rangle = e^{-i\omega t a^\dagger a}|z\rangle \ . \tag{3.47}$$

The apparent form of unitary operator $D(z)$ is easy to obtain from its definition (3.39) as

$$D(z) = e^{z a^\dagger - z^* a} \ , \tag{3.48}$$

and, finally,

$$|z\rangle = e^{z a^\dagger - z^* a}|0\rangle = e^{-\frac{1}{2}|z|^2} e^{z a^\dagger} |0\rangle \ . \tag{3.49}$$

Furthermore, $|z\rangle$ can be rewritten to another form as

$$|z\rangle = e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle \ . \tag{3.50}$$

We have obtained a series coherent states of harmonic oscillator. From Eq.(3.31), *i.e.* $z = \frac{1}{\lambda}x_0 \, e^{-i\varphi}$, we observe that the value of z is related to both the amplitude and the initial phase of the classical harmonic oscillator, meaning that z can be any complex number. $\{|z\rangle\}$ are a series (infinity) normalized vectors in the Hilbert space of the harmonic oscillator, but they are not orthogonal to each other. The eigenvectors $|n\rangle$ $(n = 0, 1, 2, \ldots)$ of Hamiltonian of the harmonic oscillator form an orthogonal basis $\{|n\rangle\}$. Note the differences between the coherent states $\{|z\rangle\}$ and the orthogonal basis; only the ground state $\{|0\rangle\}$ is shared between them. The relation between coherent states and orthogonal basis is shown in Eq.(3.50).

From equations (3.39) and (3.43), we obtain

$$a|z\rangle = a \, D(z) \, |0\rangle = D(z) \, \tilde{a}|0\rangle = D(z) \, (a + z)|0\rangle = z \, D(z)|0\rangle \ , \tag{3.51}$$

that is

$$a|z\rangle = z|z\rangle \ . \tag{3.52}$$

Above equation demonstrates a significant property about coherent states: the state $|z\rangle$ is an eigenstate of the annihilation operator a. It can be shown that this equation is equivalent to the definition of coherent states used at the beginning of this section, therefore, Eq.(3.52) can be served as an alternative definition of coherent states.

3.3 Properties of coherent states

In this section, we first study the properties of a single coherent state of the harmonic oscillator, then discuss some basic properties of the set $\{|z\rangle\}$ of all coherent states.

For the coherent state $|z\rangle$, what we are most concerned with is the probability distribution of various physical quantities of the particle in this state, especially the probability of its position. For this reason, we use the coordinate representation, with its basis vector denoted by $|x\rangle$. For convenience, we introduce a new variable $\xi = \sqrt{\frac{m\,\omega}{\hbar}}x$, and rewrite $|x\rangle$ as $|\xi\rangle$; that is, under the condition $\xi = \sqrt{\frac{m\,\omega}{\hbar}}x$, we have $|x\rangle = |\xi\rangle$.

We now calculate $\langle\xi|z\rangle$ from Eq.(3.49). First, from Eq.(3.9), we get

$$a^\dagger = \sqrt{2}\sqrt{\frac{m\,\omega}{\hbar}}\hat{x} - a = \sqrt{2}\,\hat{x}' - a\ . \tag{3.53}$$

We also know that

$$\hat{x}'|x\rangle = \sqrt{\frac{m\,\omega}{\hbar}}x|x\rangle\ , \tag{3.54}$$

that is

$$\hat{x}'|\xi\rangle = \xi|\xi\rangle\ , \tag{3.55}$$

and using Baker-Campbell-Hausdorff formula

$$e^{A+B} = e^A\,e^B\,e^{-C/2}, \qquad \text{where} \quad C = [A, B]\ , \tag{3.56}$$

we yield

$$e^{\sqrt{2}\,z\,\hat{x}'-z\,a} = e^{\sqrt{2}\,z\,\hat{x}'}e^{-z\,a}e^{-\frac{1}{2}z^2}\ . \tag{3.57}$$

Using above equation and Eq.(3.49), we obtain

$$\begin{aligned}\langle\xi|z\rangle &= e^{-\frac{1}{2}\left(|z|^2+z^2\right)}\langle\xi|e^{\sqrt{2}\,z\,\hat{x}'}e^{z\,a}|0\rangle = e^{-\frac{1}{2}\left(|z|^2+z^2\right)}e^{\sqrt{2}\,z\,\xi}\langle\xi|0\rangle \\ &= \left(\frac{m\,\omega}{\pi\,\hbar}\right)^{\frac{1}{4}}e^{-\frac{1}{2}\left(|z|^2-z^2\right)}e^{-\frac{1}{2}\left(\xi-\sqrt{2}\,z\right)^2}\ ,\end{aligned} \tag{3.58}$$

where we have used

$$\langle\xi|0\rangle = \Psi_0(\xi) = \left(\frac{m\,\omega}{\pi\,\hbar}\right)^{\frac{1}{4}}e^{-\frac{1}{2}\xi^2}\ . \tag{3.59}$$

Eq.(3.58) are exactly the state functions of coherent states in the coordinate representation.

By comparing the coherent state (3.58) with the ground state of the harmonic oscillator (3.59), we observe that apart from a phase factor $\exp\left[-\frac{1}{2}\left(|z|^2 - z^2\right)\right]$ with a modulus of 1 in Eq. (3.58), the wavefunctions of both states exhibit Gaussian profiles and have identical amplitudes. The only difference between them is that the center of the ground state of the harmonic oscillator is at $x = 0$, while the coherent state $|z\rangle$ is centered at $\sqrt{2}\,z$. (In fact, note that z now is still a complex number, so such statement is not entirely accurate. Refer to the following discussion on the probability of position for further clarification.)

The position probability of a particle in coherent state $|z\rangle$ is obtained by taking the modulus squared of Eq. (3.58):

$$|\langle\xi|z\rangle|^2 = \sqrt{\frac{m\,\omega}{\pi\,\hbar}}e^{-\left(\xi-\sqrt{2}\,z\,\cos\varphi\right)^2} \; . \tag{3.60}$$

By comparing this with the position probability of the harmonic oscillator's ground state (which is also a coherent state, specifically the coherent state with $z = 0$.), it is found that the distributions of both probabilities are identical. The only difference is that one center is at $x = 0$, while the other is centered at $\xi = \sqrt{2}\,z\,\cos\varphi$, or equivalently $x = x_0\,\cos\varphi$, where x_0 reprensents the amplitude of a classical harmonic oscillator with the same energy (see Eq.(3.31)). Furthermore, the shape and amplitude of the position probability distribution, or probability cloud, remain unchanged for a coherent state, regardless of the value of z, but the overall position of the entire profile shifts.

Now let us briefly consider the uncertainties associated with coherent states. From Eq.(3.52),

$$a|z\rangle = z|z\rangle \tag{3.61}$$

and

$$\langle z|a^\dagger\,a|z\rangle = |z|^2 \tag{3.62}$$

we observe that

$$\langle z|\left(a+a^\dagger\right)^2|z\rangle = \langle z|\left(a+a^\dagger\right)\left(a+a^\dagger\right)|z\rangle = (z+z^*)^2 + 1 \; , \tag{3.63}$$

and similarly

$$\langle z|\left(a-a^\dagger\right)^2|z\rangle = \langle z|\left(a-a^\dagger\right)\left(a-a^\dagger\right)|z\rangle = (z-z^*)^2 - 1 \; . \tag{3.64}$$

We then get the following representation of the uncertainty $\langle\hat{x}\rangle_z$

$$\langle\hat{x}\rangle_z = \sqrt{\frac{\hbar}{2\,m\,\omega}}\langle a+a^\dagger\rangle_z = \sqrt{\frac{\hbar}{2\,m\,\omega}}\,(z+z^*) \;\; , \tag{3.65}$$

where the right side is a real number. Furthermore, we have

$$\langle\hat{x}^2\rangle_z = \frac{\hbar}{2\,m\,\omega}\langle\left(a+a^\dagger\right)^2\rangle_z = \frac{\hbar}{2\,m\,\omega}\left[(z+z^*)^2+1\right] \; . \tag{3.66}$$

We therefore obtain

$$\Delta\hat{x} = \langle\hat{x}^2\rangle_z - (\langle\hat{x}\rangle_z)^2 = \frac{\hbar}{2\,m\,\omega} \; , \tag{3.67}$$

where we used the notation $\Delta\hat{O} = \langle\hat{O}^2\rangle - \langle\hat{O}\rangle^2$, with $\langle\hat{O}\rangle$ corresponding to the expectation in the state vector $|z\rangle$.

Similarly for $\Delta\hat{p}$, we get

$$\Delta\hat{p} = \langle\hat{p}^2\rangle_z - \left(\langle\hat{p}\rangle_z\right)^2 = \frac{\hbar\, m\, \omega}{2}\ . \tag{3.68}$$

Taking the product of Eq. (3.67) and Eq. (3.68) yields

$$\Delta\hat{x}\,\Delta\hat{p} = \frac{\hbar^2}{4}\ , \tag{3.69}$$

which shows that the coherent state $|z\rangle$ satisfies the minimal uncertainty condition. A coherent state is therefore a state in which the uncertainties in both position and momentum simutaneously take their minimum values, as permitted by the uncertainty principle. Such state is a state that position and momentum are most localized. The most localized of position is shown by the probability cloud, which is most compact and smallest, and the position is most localized, meaning that the possible values of the momentum of the particle are closest to the mean value. We have now proven that all coherent states possess this property.

We now turn to the time evolution and classical behaviour of coherent states. In the Schrödinger picture, the state vector evolves with time but the Hamiltonian does not rely on time. From Eqs. (3.47) and (3.50), for the time-developed coherent state $|z\rangle$ we know

$$\begin{aligned} |z(t)\rangle &= e^{-\frac{i}{\hbar}t\,H}|z\rangle = e^{-\frac{1}{2}|z|^2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}e^{-i\,n\,\omega\,t}|n\rangle \\ &= e^{-\frac{1}{2}\left|z\,e^{-i\,\omega\,t}\right|^2}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\left(z\,e^{-i\,\omega\,t}\right)^n|n\rangle = |z\,e^{-i\,\omega\,t}\rangle\ , \end{aligned} \tag{3.70}$$

implying that the time evolution of a coherent state $|z\rangle$ can be obtained by simply replacing z with $z\,e^{-i\,\omega\,t}$ in its expression.

Therefore, the time evolution of the state function for a coherent state (the probability amplitude of position) can be immediately obtained as

$$\langle\xi|z(t)\rangle = \left(\frac{m\,\omega}{\pi\,\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\left(|z|^2 - z^2 e^{-2\,i\,\omega\,t}\right)} e^{-\frac{1}{2}\left(\xi-\sqrt{2}\,z\,e^{-i\,\omega\,t}\right)^2}\ , \tag{3.71}$$

where the modulus of the first exponential function is 1. Furthemore, the time evolution of the position probability can be obtained from Eq. (3.60) as

$$|\langle\xi|z(t)\rangle|^2 = \sqrt{\frac{m\,\omega}{\pi\,\hbar}}e^{-\left[\xi-\sqrt{2}\,|z|\,\cos(\omega\,t+\varphi)\right]^2}\ . \tag{3.72}$$

From above two equations, we can find that the minimal uncertainty characteristic of the coherent state remains unchanged under time evolution. The size and shape of the wave packet does not change; only its central position undergoes simple harmonic motion:

$$\sqrt{2}\, z\, \cos\left(\omega\, t + \varphi\right) = \sqrt{\frac{m\,\omega}{\hbar}} x_0\, \cos\left(\omega\, t + \varphi\right)\ , \tag{3.73}$$

which shows that coherent states, which are truly quantum states, depict a classical behavior.

We now study some basic properties of the set $\{|z\rangle\}$ of all coherent states of a harmonic oscillator. Coherent states for the harmonic oscillator are eigenstates of the annihilation operator acting on a Fock space. However, the annihilation operator is non-Hermitian, so its eigenstates $|z\rangle$ exhibit properties that differ significantly from those of a Hermitian operator. First of all, the eigenvalue z can be any complex number. Coherent states therefore belong to a continuum of eigenstates. Despite this, all the eigenstates are normalized, *i.e.*

$$\langle z|z\rangle = 1\ , \tag{3.74}$$

which can be directly proven by taking the modulus squared of Eq. (3.50).

Coherent states have three additional properties.

i) Any two coherent states are non-orthogonal.

This can also be proven from Eq. (3.50),

$$\begin{aligned}\langle z|z'\rangle\ &=\ e^{-\frac{1}{2}|z|^2}\, e^{-\frac{1}{2}|z'|^2} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{z^{*\,n}\, z'^{\,m}}{\sqrt{n!\, m!}} \langle n|m\rangle \\ &=\ e^{-\frac{1}{2}|z|^2-\frac{1}{2}|z'|^2+z^*\, z'}\ ,\end{aligned} \tag{3.75}$$

which is non-zero for any values of z and z'.

ii) All (infinity many) coherent states are linearly dependent, whereas a finite number of coherent states are not linearly dependent.

To prove the first part of this property, it suffices to provide an example where the sum (integral) of all coherent states, each multiplied by a specific constant, equals zero. By assuming $z = r\, e^{i\,\theta}$, and using $\mathrm{d}^2 z = r\,\mathrm{d}r\,\mathrm{d}\theta$ for the integral, with m being any non-vanishing integer, such an integral can be found as

$$\int z^m\, |z\rangle\, \mathrm{d}^2\, z = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}}\, |n\rangle \int_0^{\infty} e^{-\frac{1}{2}\, r^2}\, r^{n+m+1}\, \mathrm{d}r \int_0^{2\pi} e^{i(m+n)\theta} \mathrm{d}\theta\ . \tag{3.76}$$

The integral of the right side of above equation is bounded with respect to r and the integral vanishes with respect to θ, for any value of n. Therefore, we obtain

$$\int z^m\, |z\rangle\, \mathrm{d}^2\, z = 0\ , \tag{3.77}$$

which shows that all the coherent states are linearly dependent.

However, a finite number of coherent states are linearly independent. For instance, consider any arbitrary n coherent states $|z_k\rangle$, multiply each by a constant c_k, and then sum them all together such that the result is zero

$$\sum_{k=1}^{n} c_k |z_k\rangle = 0 \ . \tag{3.78}$$

We now need to determine the constant c_k using the above condition. Left-multiplying the equation (3.78) by $\langle z|$ yields

$$\sum_{k=1}^{n} c_k \langle z|z_k\rangle = 0 \ . \tag{3.79}$$

From Eq.(3.50), we have

$$\langle z|z_k\rangle = e^{-\frac{1}{2}|z|^2} e^{-\frac{1}{2}|z_k|^2} e^{z^* z_k} \ . \tag{3.80}$$

If we regard $\langle z|z_k\rangle$ as different functions of z, they then are linearly independent, which means that all of c_k in Eq.(3.79) must be zero. Since c_k must be zero, it follows from Eq.(3.78) that a finite number of coherent states $\{|z_k\rangle\}$ are linearly independent.

The linear independence in Eq.(3.80) can be understood as follows. In Eq.(3.80), for different values of k, they can be considered as various functions of z, with the first exponential functions being the same for all. The second exponential functions can be viewed as constants. The key point lies in the third exponential functions, which can be expressed as the product of two real exponential functions, and we already know that the functions $e^{a_1 x}$, $e^{a_2 x}$, $e^{a_3 x}$, $\cdots$, are linearly independent.

iii) The set of states $|z\rangle$, for z varying, is a complete set of vectors, which means that, as a consequence of their non-orthogonality, any coherent state can be expanded in terms of this complete set of states. The mathematical representation of completeness is that the completeness relation

$$\frac{1}{\pi}\int |z\rangle\,\langle z|\mathrm{d}^2 z = 1 \tag{3.81}$$

holds. To prove the above expression, it suffices to demonstrate that Parseval's identity

$$\langle\Psi|\Phi\rangle = \frac{1}{\pi}\int \langle\Psi|z\rangle\,\langle z|\Phi\rangle \mathrm{d}^2 z \tag{3.82}$$

holds for arbitrary $|\Psi\rangle$ and $|\Phi\rangle$. The right side of above equation is

$$\begin{aligned} &\frac{1}{\pi}\sum_n\sum_m \langle\Psi|n\rangle \int_0^\infty \frac{1}{\sqrt{n!\,m!}} r^{n+m}\, e^{-r^2}\, \left(\frac{1}{2}\right) \mathrm{d}\left(r^2\right) \int_0^{2\pi} e^{i\,(m-n)\,\theta} \mathrm{d}\theta \langle m|\Phi\rangle \\ &= \sum_n \langle\Psi|n\rangle\langle n|\Phi\rangle = \langle\Psi|\Phi\rangle \ , \end{aligned} \tag{3.83}$$

which proves Eq.(3.82), thereby validating Eq.(3.81). In the above calculation, the result of the integral with respect to θ is $2\,\pi\delta_{m\,n}$, and substituting $r^2 = t$, $2\,r\,\mathrm{d}r = \mathrm{d}t$ to rewrite the integral

$$2\int_0^\infty r\,\mathrm{d}r e^{-r^2}\,r^{2\,n} = \int_0^\infty e^{-t}\,t^n\,\mathrm{d}t = \Gamma(n+1) = n!\ , \tag{3.84}$$

we recover the exact definition of the Gamma function.

Via the completeness relation, any coherent states can be expanded in terms of all the other coherent states,

$$\begin{aligned} |z\rangle &= \frac{1}{\pi}\int |z'\rangle\langle z'|z\rangle \mathrm{d}^2 z' \\ &= \frac{1}{\pi}\int |z'\rangle e^{-\frac{1}{2}|z|^2-\frac{1}{2}|z'|^2+z^*\,z'}\mathrm{d}^2 z'\ , \end{aligned} \tag{3.85}$$

which is consistent with the conclusion that the coherent states are not linearly independent.

Although the coherent states are not orthogonal, they are completed which means that it is possible to expand any vector in the Hilbert space of the harmonic oscillator in terms of a complete set of states. In fact, this property that the coherent states are not orthogonal but still form a complete set is referred to as “overcompleteness” [68].

3.4 Coherent states for a field

Let us now apply the previous considerations of a simple one dimensional harmonic oscillator to a free scalar field [52]. Considering a single mode operator, we can write the field as

$$\hat{\Phi}_k(x) = \hat{a}_k\,e^{-i\,k\,x} + \hat{a}_k^\dagger\,e^{i\,k\,x}\ , \tag{3.86}$$

and its conjugate momentum as

$$\Pi_k(x) = -\frac{i}{2}\left(\hat{a}_k\,e^{-i\,k\,x} - \hat{a}_k^\dagger\,e^{i\,k\,x}\right)\ . \tag{3.87}$$

For simplification, the field is considered as a general superposition of modes,

$$\hat{\Phi}(x) = \int \frac{\mathrm{d}^3 k}{(2\,\pi)^{\frac{3}{2}}}\sqrt{\frac{\hbar}{2\,\omega_k}}\left[\hat{a}_k\,e^{-i\,k\,x} + \hat{a}_k^\dagger\,e^{i\,k\,x}\right]\ . \tag{3.88}$$

Now we can calculate the Hamiltonian in terms of the Fourier modes:

$$H = \frac{1}{2}\int \mathrm{d}^3 k\,\hbar\,\omega_k\,\left(a_k\,a_k^\dagger + a_k^\dagger\,a_k\right)\ . \tag{3.89}$$

We then construct the eigenstates of the Hamiltonian to find the spectrum of states. We define the "vacuum" state as follows

$$a_k|0\rangle = 0 \ . \tag{3.90}$$

By convention, we call a_k an "annihilation " operator. We define a one-particle state via the "creation" operator $a_k^\dagger$

$$a_k^\dagger|0\rangle = |k\rangle \ . \tag{3.91}$$

Particle number operator is defined as

$$N_k = a_k^\dagger\, a_k \ , \quad N_k|n_k\rangle = n_k|n_k\rangle \ . \tag{3.92}$$

The operator H commutes with N_k.

By quantizing the field, we turn Φ and Π into operators obeying

$$[\Phi(x),\, \Pi(y)] = i\,\delta(x-y) \ , \tag{3.93}$$

and all other commutators are zero

$$[\Phi(x),\, \Phi(y)] = [\Pi(x),\, \Pi(y)] = 0 \ . \tag{3.94}$$

The only non vanishing canonical commutation relation is given by

$$\left[\hat{a}_k,\, \hat{a}_k^\dagger\right] = 1 \ , \tag{3.95}$$

which leads to the same algebra of the harmonic oscillator and the same definition of the coherent state

$$\hat{a}_k|\alpha_k\rangle = \alpha_k\,|\alpha_k\rangle \ . \tag{3.96}$$

By assuming $\alpha_k = |\alpha_k|e^{i\,\theta_k}$, we get

$$\langle\alpha_k|\hat{\Phi}_k(x)|\alpha_k\rangle = 2\,|\alpha_k|\cos\left(\vec{k}\cdot\vec{x} - \omega_k\,t + \theta_k\right) \ , \tag{3.97}$$

which shows that the expectation value of the quantum field over a coherent state reproduces the classical wave configuration.

The field (3.88) can split into a positive and a negative frequency part

$$\hat{\Phi}(x) = \hat{\Phi}^{(+)}(x) + \hat{\Phi}^{(-)}(x) \ , \tag{3.98}$$

with

$$\hat{\Phi}^{(+)}(x) = \int \frac{\mathrm{d}^3k}{(2\,\pi)^{\frac{3}{2}}}\sqrt{\frac{\hbar}{2\,\omega_k}}\hat{a}_k\,e^{-i\,k\,x} \ . \tag{3.99}$$

Note that a coherent state can be defined as the eigenstate of the positive frequency part of the field

$$\hat{\Phi}^{(+)}|\Phi\rangle = \Phi|\Phi\rangle \ , \tag{3.100}$$

which again shows that the dynamics of coherent state closely resemble the behavior of a classical system

$$\langle\hat{\Phi}(x)|\Phi\rangle = \Phi(x) \ . \tag{3.101}$$

By generalizing Eq. (3.49), the expression for this state can be found, with respect to the occupation number basis,

$$|\Phi\rangle = e^{-\frac{1}{2}\int \mathrm{d}^3k\,|\alpha_k|^2} e^{\int \mathrm{d}^3k\,\alpha_k\,\hat{a}_k^\dagger}|0\rangle \ . \tag{3.102}$$

3.5 Path integrals and coherent state

The concept of coherent states have been applied to various fields of Quantum Mechanics. In this section, we will study its application in path integrals [78].

As discussed in the pioneering work of Klauder [74], coherent states are highly useful in constructing a path-integral representation of quantum dynamics. Before delving into this, we will first revisit the ordinary path integral of Quantum Mechanics developed by Feynman, which can be obtained from the evolution operator by expressing it as

$$U\left(t_f,\, t_0\right) = e^{-\frac{i}{\hbar}\left(t_f - t_0\right)H} = \lim_{N\to\infty}\left[\exp\left\{-\frac{i}{\hbar}\,H\,\frac{t_f - t_0}{N}\right\}\right]^N \tag{3.103}$$

and then inserting a resolution of identity in terms of the position states

$$\int \mathrm{d}x\,|x\rangle\langle x| = I \tag{3.104}$$

between the terms of above product. From this, we get the familiar path integral of Quantum Mechanics,

$$\begin{aligned}\langle x'\left(t_f\right)|x\left(t_0\right)\rangle &= \langle x'|U\left(t_f,\, t_0\right)|x\rangle \\ &= \int [\mathrm{d}x(t)]\exp\left\{\frac{i}{\hbar}\int_{t_0}^{t_f}\mathrm{d}t\,\mathcal{L}\left(x(t),\, \dot{x}(t)\right)\right\} \ ,\end{aligned} \tag{3.105}$$

where $\mathcal{L}$ is a classical Lagrangian with a general form

$$\mathcal{L}\left(x,\, \dot{x}\right) = \frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - V(x) \ , \tag{3.106}$$

and

$$[\mathrm{d}x(t)] = \Pi_{t_0 \le t \le t_f} \mathrm{d}x(t) \tag{3.107}$$

is a functional measure of the path integration.

However, instead of inserting a complete set of states (3.104), we will now insert an overcomplete set of coherent states

$$\int |z\rangle\langle z| \, \frac{\mathrm{d}z\mathrm{d}z^*}{2\,\pi} = I \tag{3.108}$$

between the terms of product (3.103). We can then derive a phase space formulation of path integrals, initially proposed by Klauder,

$$\begin{aligned} \langle z'\,(t_f)\,|z\,(t_0)\rangle &= \langle z'|U\,(t_f,\,t_0)\,|z\rangle \\ &= \int [\mathrm{d}x\,(t)] \left[\frac{\mathrm{d}p\,(t)}{2\,\pi}\right] \exp\left\{\frac{i}{\hbar}\int_{t_0}^{t_f} \mathrm{d}t\,\mathcal{L}\,(x(t),\,p(t))\right\} \ , \end{aligned} \tag{3.109}$$

with

$$\begin{aligned} \mathcal{L}\,(x,\,p) &= \langle z|i\,\frac{\mathrm{d}}{\mathrm{d}t}|z\rangle - \langle z|H|z\rangle \\ &= \frac{1}{2}\left(p\,\frac{\mathrm{d}x}{\mathrm{d}t} - x\,\frac{\mathrm{d}p}{\mathrm{d}t}\right) - \mathcal{H}\,(x,\,p) \ , \end{aligned} \tag{3.110}$$

where $z = (x + i\,p)\,/\sqrt{2}$ and $z^* = (x - i\,p)\,/\sqrt{2}$, with the initial and final positions $x\,(t_0)$ and $x\,(t_f)$ fixed. This derivation of Feynman's path integral is particularly useful for obtaining a functional integral of quantum field theory.

As an example, let us consider the neutral scalar field $\Phi(x)$ (3.86) to illustrate how to derive a functional integral of quantum field theory. The corresponding Lagrangian density of such field is given by

$$\mathcal{L} = \frac{1}{2}\left[(\partial\phi)^2 - m^2\,\phi^2\right] - V(\phi) \ , \tag{3.111}$$

where $V(\phi)$ is a self-interacting potential.

The canonical momentum density conjugate to $\phi(\mathbf{x}, t)$ is determined by $\pi(\mathbf{x}, t) = \partial\mathcal{L}/\partial\dot{\phi}(\mathbf{x}, t)$. Then the canonical quantization leads to

$$[\Phi(\mathbf{x}, t), \Pi(\mathbf{x}, t)] = i\delta^3\,(\mathbf{x} - \mathbf{x}') \ . \tag{3.112}$$

Using Eqs. (3.88), we have

$$\Pi = -i\int \frac{\mathrm{d}^3k}{(2\,\pi)^{\frac{3}{2}}}\sqrt{\frac{\omega_k}{2}}\left[\hat{a}_k\,e^{-i\,k\,x} - \hat{a}_k^\dagger\,e^{i\,k\,x}\right] \ , \tag{3.113}$$

and the quantum Hamiltonian is given by

$$H(t) = \int \mathrm{d}^3\mathbf{x} : \left\{ \frac{1}{2} \left[\Pi^2 + (\nabla\Phi)^2 + m^2\,\Phi^2 \right] + V(\Phi) \right\} : , \tag{3.114}$$

where $::$ is the normal ordering with respect to the creation and annihilation operators $a_k^\dagger$ and a_k. The scalar field coherent state now can be defined at a given instant time t over the whole space $\{\mathbf{x}\}$ as

$$\begin{aligned} |\phi\pi\rangle &= \exp\left\{ i \int \mathrm{d}^3x \, [\pi(x)\,\Phi(x) - \phi(x)\,\Pi(x)] \right\} |0\rangle \\ &= \exp\left\{ -\frac{1}{2} \int \mathrm{d}^3k |z_k|^2 \right\} \exp\left\{ \int \mathrm{d}^3k \, \left(z_k\, a_k^\dagger \right) \right\} |0\rangle \ , \end{aligned} \tag{3.115}$$

from which a function integral of field theory can be obtained explicitly.

Note that the Hamiltonian formalism of the field theory is analogous to that in quantum mechanics. We can directly derive the time Green's function denfined as the matrix element of the evolution operator in coherent state basis,

$$G\,(t_f,\, t_0) = \langle \phi'\pi' | U\,(t_f,\, t_0)\, |\phi\,\pi\rangle = \langle \phi'\pi' | T\, \exp\left\{ -i \int_{t_0}^{t_f} \mathrm{d}t\, H(t) \right\} |\phi\,\pi\rangle \ , \tag{3.116}$$

where T is the time-ordering operator.

One may break up the time interval from t_0 to t_f into N equal pieces of duration $\epsilon = (t_f - t_0)\,/N$. In the limit $\epsilon \to 0$, the evolution operator can be expressed as a successive multiplication of the evolution operator over the interval ϵ,

$$\begin{aligned} U\,(t_f,\, t_0) &= \exp\{-i\,\epsilon\, H(t_n)\}\, \exp\{-i\,\epsilon\, H(t_{n-1})\} \\ &\qquad \cdots \exp\{-i\,\epsilon\, H(t_2)\}\, \exp\{-i\,\epsilon\, H(t_1)\} \ . \end{aligned} \tag{3.117}$$

Following the same procedure used in the derivation of Feynman's path integral in quantum mechanics, one should insert a complete set of intermediate states between each of these factors, in the form

$$I = \int [\mathrm{d}\phi\,(\mathbf{x})]\, [\frac{\mathrm{d}\pi\,(\mathbf{x})}{2\,\pi}] |\phi\,\pi\rangle\langle\phi\,\pi| \ , \tag{3.118}$$

where $[\mathrm{d}\phi\,(\mathbf{x})] \equiv \prod_{-\infty<\mathbf{x}<\infty} \mathrm{d}\phi\,(\mathbf{x})$, etc. are defined over the whole space. Then

$$G\,(t_f,\, t_0) = \lim_{N\to\infty} \int \left(\prod_{i=1}^{N-1} [\mathrm{d}\phi_i\,(\mathbf{x})]\, [\frac{\mathrm{d}\pi_i\,(\mathbf{x})}{2\,\pi}] \right) \prod_{i=1}^{N} \langle \phi_i\,\pi_i | \exp\{-i\,\epsilon\, H\,(t_i)\} |\phi_{i-1}\,\pi_{i-1}\rangle \tag{3.119}$$

The first order in ϵ can be obtained as

$$\langle\phi_i\,\pi_i|\exp\left(-i\,\epsilon\,H\left(t_i\right)\right)|\phi_{i-1}\,\pi_{i-1}\rangle \;\approx\; \langle\phi_i\,\pi_i|\phi_{i-1}\,\pi_{i-1}\rangle\exp\left(-i\,\epsilon\frac{\langle\phi_i\,\pi_i|H\left(t_i\right)|\phi_{i-1}\,\pi_{i-1}\rangle}{\langle\phi_i\,\pi_i|\phi_{i-1}\,\pi_{i-1}\rangle}\right)\;. \tag{3.120}$$

Note that the coherent state $|\phi\,\pi\rangle$ is normalized. In the limit of $\epsilon\to\infty$, we have

$$\begin{aligned}\langle\phi_i\,\pi_i|\phi_{i-1}\,\pi_{i-1}\rangle \;&=\; 1-\langle\phi_i\,\pi_i|\left(|\phi_i\,\pi_i\rangle-|\phi_{i-1}\,\pi_{i-1}\rangle\right)\\ &\simeq\; \exp\left\{i\,\epsilon\langle\phi_i\,\pi_i|i\frac{\Delta|\phi_i\,\pi_i\rangle}{\epsilon}\right\}\;,\end{aligned} \tag{3.121}$$

where $\Delta|\phi_i\,\pi_i\rangle\equiv|\phi_i\,\pi_i\rangle-|\phi_{i-1}\,\pi_{i-1}\rangle$. The Green's function then becomes

$$\begin{aligned}G\left(t_f,\,t_0\right) \;&=\; \lim_{N\to\infty}\int\left(\prod_{i=1}^{N-1}\left[\mathrm{d}\phi_i\left(\mathbf{x}\right)\right]\left[\frac{\mathrm{d}\pi_i\left(\mathbf{x}\right)}{2\,\pi}\right]\right)\\ &\qquad\times\,\exp i\sum_{i=1}^{N}\epsilon\left\{\langle\phi_i\,\pi_i|i\frac{\Delta|\phi_i\,\pi_i\rangle}{\epsilon}-\langle\phi_i\,\pi_i|H\left(t_i\right)|\phi_i\,\pi_i\rangle\right\}\\ &=\; \int\left[\mathrm{d}\phi\left(\mathbf{x}\right)\right]\left[\frac{\mathrm{d}\pi\left(\mathbf{x}\right)}{2\,\pi}\right]\exp\left\{i\int_{t_0}^{t_f}\mathrm{d}t\left[\langle\phi\,\pi|i\frac{\mathrm{d}}{\mathrm{d}t}|\phi\,\pi\rangle-\langle\phi\,\pi|H|\phi\,\pi\rangle\right]\right\}\\ &=\; \int\left[\mathrm{d}\phi\left(\mathbf{x}\right)\right]\left[\frac{\mathrm{d}\pi\left(\mathbf{x}\right)}{2\,\pi}\right]\exp\left\{i\int_{t_0}^{t_f}\mathrm{d}t\int\mathrm{d}^3x\left[\frac{1}{2}\left(\pi\,\dot{\phi}-\phi\,\dot{\pi}\right)-\mathcal{H}\left(x\right)\right]\right\},\end{aligned} \tag{3.122}$$

with

$$\mathcal{H}=\frac{1}{2}\left[\pi^2+\left(\nabla\phi\right)^2+m^2\,\phi^2\right]+V(\phi)\;. \tag{3.123}$$

This shows that the coherent state gives a natural derivation of path integrals in field theory.

Let $G^{(n)}\left(x_1,\cdots x_n\right)$ be the connected n-point function,

$$G^{(n)}\left(x_1,\cdots x_n\right)=\langle 0|T\left(\phi\left(x_1\right)\cdots\phi\left(x_n\right)\right)|0\rangle\;, \tag{3.124}$$

which can be derived from the generating functional $W(J)$, defined as the vacuum-to-vacuum amplitude in the presence of an external current $J(x)$,

$$W(J)=\langle 0|U\left(-\infty,\infty\right)|0\rangle_J\;. \tag{3.125}$$

By adding a term in the exponent and then assuming the time variable of integration in the exponent runs from $-T$ to T, with $T\to\infty$, we can express the generating functional in term of the time Green's function $G(t_f, t_0)$ as

$$\begin{aligned} W(J) &= \int [\mathrm{d}\phi(\mathbf{x})]\,[\frac{\mathrm{d}\pi(\mathbf{x})}{2\,\pi}]\exp\left\{\int \mathrm{d}^4x\left[\frac{1}{2}\left(\pi\,\dot{\phi}-\phi\,\dot{\pi}\right)-\mathcal{H}(x)+J(x)\,\phi(x)\right]\right\} \\ &= \int [\mathrm{d}\phi(\mathbf{x})]\exp\left\{\int \mathrm{d}^4x\,[\mathcal{L}(x)+J(x)\,\phi(x)]\right\} \\ &= \exp\{i\,Z(J)\}\ , \end{aligned} \tag{3.126}$$

where $Z(J)$ is a functional partition function in quantum field theory. The n-point Green's function, which includes only the connected graphs, is given by

$$\begin{aligned} G_c^{(n)}(x_1,\cdots x_n) &= \frac{(-i)^n}{W(J)}\frac{\delta^n W(J)}{\delta J(x_1)\cdots\delta J(x_n)}\bigg|_{J=0} \\ &= (-i)^{n-1}\frac{\delta^n Z(J)}{\delta J(x_1)\cdots\delta J(x_n)}\bigg|_{J=0}\ . \end{aligned} \tag{3.127}$$

The classical equations of motions can be naturally obtained by applying the stationary phase approximation to $W(J)$ [79, 80]. Additionally, the functional integrals $W(J)$ and $Z(J)$ become covariant after integrating out the $\pi(x)$ field. Finally, all physical quantities in field theory can, in principle, be derived from $W(J)$ or $Z(J)$ in a covariant form. Notice that the results derived above is applicable only to bosonic fields.

Chapter 4

Quantum coherent states for the gravitational field

The corpuscular black hole picture has been discussed in detail in Chapter 2. This intuitive model [43–45], which is based on the assumption that black holes are described as bound states of N identical gravitons, presents various qualities and puts a new light on gravitation. Such bound state reproduces the classical geometric aspect of gravity as an emerging feature, where gravitons play the role of spacetime quanta. This black hole quantum portrait is very simple and considers classical and semiclassical aspects of black hole physics from a new perspective. However, it does not delve into details and subtleties of some important issues. For example, the connection with the usual geometrical picture of General Relativity is not immediate, it entirely neglects the role of the matter from which the black hole formed, and the horizon "emergies" from a classical mechanical condition rather than from relativistic considerations.

Before presenting another semiclassical approach to the underlying quantum theory that quantizes a graviton field [52, 81–83], we first aim to review the quantization of a spin 2 graviton field emerging from the weak-field limit of the Einstein-Hilbert action [84–86],

$$S_{\text{EH}} = -\frac{1}{16\,\pi\,G_{\text{N}}} \int \mathrm{d}^4x\sqrt{-g}R \ . \tag{4.1}$$

In the weak field limit, where the source of the field is small if compared to other scales, the quantum fluctuations in the gravitational field can be expanded about a smooth background metric, which here is flat space time,

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu} \ , \tag{4.2}$$

with $\epsilon \ll 1$. The action (4.1) then can be rewritten as

$$\begin{aligned} S_{\rm EH} &= -\frac{1}{16\,\pi\,G_{\rm N}}\int {\rm d}^4x\sqrt{-g}R \\ &\simeq -\frac{1}{32\,\pi\,G_{\rm N}}\int {\rm d}^4x\left(\frac{1}{2}\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho} - \partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho} + \partial_\mu h\partial_\rho h^{\nu\rho} - \frac{1}{2}\partial_\mu h\partial^\mu h\right) \end{aligned} \quad (4.3)$$

where $h \equiv \eta^{\mu\nu}h_{\mu\nu}$. This implies that the free massless spin 2 theory (linearized approximation) can be presented by the massless Fierz-Pauli Lagrangian [48, 49],

$$\mathcal{L}_{\rm FP} = -\frac{1}{2}\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho} + \partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho} - \partial_\mu h\partial_\rho h^{\nu\rho} + \frac{1}{2}\partial_\mu h\partial^\mu h\ , \quad (4.4)$$

and performing the variation of the action (4.2) with respect to $h_{\mu\nu}$, one obtains the equations of motion

$$\begin{aligned} G_{\mu\nu} &\simeq \frac{1}{2}\left(-\Box h_{\mu\nu} + \eta_{\mu\nu}\Box h + \partial_\mu\partial^\rho h_{\rho\nu} + \partial_\nu\partial^\rho h_{\rho\mu} - \eta_{\mu\nu}\partial^\lambda\partial^\rho h_{\lambda\rho} - \partial_\mu\partial_\nu h\right) \\ &= 0\ , \end{aligned} \quad (4.5)$$

where we defined the d'Alembertian as $\Box = \partial_\mu\partial^\mu$. One important aspect of the above field equations is their invariance under a local gauge transformation of the type

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu\ , \quad (4.6)$$

involving an arbitrary gauge parameter $\varepsilon_\mu(x)$. In the quantum theory, it implies the existence of Wald identities. Furthermore, to simplify the field equations, we can consider the harmonic (or de Donder) gauge $g^{\mu\nu}\Gamma^\alpha_{\mu\nu} = 0$ or equivalently $\partial_\mu h^\mu_\nu - (1/2)\partial_\nu h = 0$. Now the equations of motion read

$$\Box h_{\mu\nu} = 0\ . \quad (4.7)$$

Since the pure gravitational field Lagrangian (4.4) only propagates transverse traceless modes, these correspond quantum mechanically to a particle of zero mass and spin two, with two helicity states $h = \pm 2$. We then find that a general plane wave expansion for $h_{\mu\nu}$

$$h_{\mu\nu} = \sum_{\lambda=+,-}\int {\rm d}\mu(\mathbf{p})\left[a(\mathbf{p},\lambda)\epsilon_{\mu\nu}(\mathbf{p},\lambda)e^{ip\cdot x/\hbar} + a(\mathbf{p},\lambda)^*\epsilon_{\mu\nu}(\mathbf{p},\lambda)e^{-ip\cdot x/\hbar}\right]\ , \quad (4.8)$$

will satisfy the wave equatoin and the harmonic gauge condition. Here the polarization tensor $\epsilon_{\mu\nu}(\mathbf{p},\lambda)$ is normalized as $\epsilon_{\mu\nu}(\mathbf{p},\lambda)\epsilon^{\mu\nu}(\mathbf{p},\lambda') = \delta_{\lambda\lambda'}$. In order to proceed with the quantization, we promote $a(\mathbf{p},\lambda)$ and $a(\mathbf{p},\lambda)^*$ to distribution-valued operators $a(\mathbf{p},\lambda)$ and $a(\mathbf{p},\lambda)^\dagger$ with canonical commutation relations

$$\left[a(\mathbf{p},\lambda),\, a(\mathbf{q},\lambda)^\dagger\right] = \delta^{(3)}(\mathbf{p}-\mathbf{q})\delta_{\lambda\lambda'} \ . \tag{4.9}$$

After inverting the kinetic term of the action (4.4), the graviton Feynman propagator is found to be of the form

$$i\, D^{(F)}_{\alpha\beta\mu\nu}(p) = \frac{i\, P_{\alpha\beta\mu\nu}}{p^2 - i\,\varepsilon} \ , \tag{4.10}$$

where $P_{\alpha\beta\mu\nu}$ is the projection operator, defined as

$$P_{\alpha\beta\mu\nu} \equiv \frac{1}{2}\left(\eta_{\alpha\mu}\eta_{\beta\nu} + \eta_{\alpha\nu}\eta_{\beta\mu} - \eta_{\alpha\beta}\eta_{\mu\nu}\right) \ . \tag{4.11}$$

The observables of greatest interest for experimental purposes involve the gravitational interactions of various types of matter. Therefore, it is useful to generalize the previous arguments to include the coupling between matter and gravity. Rather than describing the general case, we will only consider an interaction between a massive scalar field Φ and the graviton. The graviton interacts with matter fields through their stree-energy tensor, thus recalling that

$$T^{(\Phi)}_{\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}\eta_{\mu\nu}\left(\partial_\lambda\Phi\partial_\lambda\Phi + m_\Phi^2\Phi^2\right) \ , \tag{4.12}$$

one can introduce the coupling with matter in the action as

$$S = \int \mathrm{d}^4x \left(\frac{1}{32\,\pi\, G}\mathcal{L}_{\mathrm{FP}} + \frac{1}{2}h^{\mu\nu}T^{(\Phi)}_{\mu\nu}\right) \ . \tag{4.13}$$

From this action, one can then read off the Feynman rules for the graviton [87–90].

With the Feynman rules, we can compute the scattering of two scalar particles by a single graviton exchange. Considering the non-relativistic limit, $p^\mu \approx (m,\, \mathbf{0})$, the amplitude of this process is given by

$$\mathcal{M} = -16\,\pi\, G\frac{m^2}{q^2} \ , \tag{4.14}$$

Performing the Fourier transform, we obtain the non-relativistic potential

$$V(r) = -G\frac{m^2}{r} \ , \tag{4.15}$$

which is exactly the Newton's potential [91]. A significant problem arises with this approach. Pure gravity is finite at one loop, because the lowest order equation of motion is $R_{\mu\nu} = 0$ for pure gravity, causing the $\mathcal{O}(R^2)$ terms in the Lagrangian to vanish for all solutions to the Einstein equation. However, as shown by Goroff and Sagnatti in [92], even for pure gravity in four dimensions, there is a divergence which remains even after the renormalization of the Einstein-Hilbert action. This can be concluded by saying that Einstein's gravity is a non-renormalizable theory, which remains a significant obstacle in unifying Quantum Field Theory and General Relativity.

The previous quantization of the graviton field emerges from the weak field limit, where the metric is expanded around Minkowski space, with $h_{\mu\nu}$ representing the dynamical part of the metric. The vacuum is denoted as $|0\rangle$ and it satisfies the condition that no matter nor metric are excited. In this regime, the harmonic combined with residual gauge freedom reduces the polarization vector to two transverse traceless degrees of freedom, corresponding to a massless spin two degrees of freedom, and at tree level, the gravitational field is linearly coupled to the energy momentum tensor of the matter fields. This weak field approximation and pertubation method are indeed useful tools in addressing the difficulties arising from the non linearity of the self interactions and the gauge fixing problem of the full General Relativity theory.

The weak field perturbation theory, however, breaks down in regions of strong gravity, such as near black holes or in the early universe, particularly close to the event horizon or singularity. In these regimes, where the curvature of spacetime becomes large, the perturbation can no longer be treated as a small correction to flat spacetime and nonlinear terms in the metric perturbation $h_{\mu\nu}$ become significant and cannot be ignored. A possible solution one may consider is to treat the graviton field as a perturbation of a non flat background solution of the field equations $g^{(0)}_{\mu\nu}$,

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \epsilon h_{\mu\nu} \ , \tag{4.16}$$

where $g^{(0)}_{\mu\nu}$ can be any arbitrary known solution, such as the Schwarzschild metric. This consideration is indeed applicable in certain contexts, such as in gravitational wave physics. However, it appears conceptuarlly problematic from at least two perspectives. Firstly, the selection of $g^{(0)}_{\mu\nu}$ based on the arbitrary background results in ambiguity in the definition of excitations. Secondly, in the corpuscular picture, both the geometrical Einstein equations and their solutions are understood to emerge from the underlying pure quantum theory in an appropriate limit. Therefore, a classical solution cannot be assumed a priori as the basis to quantize the theory.

Solving such problem is challenging. Therefore, instead of seeking the complete non interacting quantum theory, we will focus on looking for a quantum state built out from the Fock space of gravitons for the linearized theory that can reproduce classical field configuraitons. In addition to the conceptural issues, there are also technical challenges arising from treating metric tensors. It is well known that the computational difficulties in quantum field theory increase with spin of fields, because higher-spin fields involve more components, more complicated propagators, intricate gauge symmetries, and impose more challenging consistence conditions. To simplify the calculations, a possible solution is to replace the metric tensor $h_{\mu\nu}$ with a scalar field Φ. At first glance, this choice of a scalar may seem strange since graviton is typically described as a massless spin 2 particle in a quantum field theory framework. But it is also true that, in the non relativistic limit, the gravitational interaction between two masses can be understood as being mediated by the exchange of a graviton, and the scalar Newtonian potential is recovered from the non propagating temporal component of the metric tensor [47]. Thus, after recognizing how the Newtonian potential is embedded in the metric tensor of the Schwarzschild solution at the classical level, a scalar mean-field approach [52] will be employed to extract as much information as possible by closely examining the characteristics of a metric function that arise from a quantization procedure. Thus, based on the requirement that the Newtonian potential (and consequently the Schwarzschild metric function) must be recovered, up to quantum corrections, by a scalar quantum field representing gravity via the expectation value over the quantum state, this state can be defined as

$$V_q = \langle g|\hat{\Phi}|g\rangle \ . \tag{4.17}$$

The geometrical description then is restored with a corrected Schwarzschild metric of the form

$$\mathrm{d}s^2 = -\left(1+2\,V_q\right)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1+2\,V_q} + r^2\,\mathrm{d}\Omega^2 \ , \tag{4.18}$$

It is important to stress that, from this perspective, the Newtonian configuration arises entirely from a quantized theory in Minkowski spacetime. The state $|g\rangle$, if it exists, should accurately reproduce the gravitational potential V_q, with the expectation of finding new features on the nature of gravity through Quantum Mechanics. Through a non perturbative mean field approach, these corpuscular corrections to the Newtonian potential will enter the metric function.

4.1 Quantum coherent states for classical static configurations

As we have discussed previously, the expectation value of the quantum field over a coherent state reproduces a semiclassical behaviour, where "semiclassical" means minimum and constant uncertainty of the field configuration. In our work, the coherent state is in particular built for a scalar field whose expectation value effectively describes the geometry emerging from the (longitudinal or temporal) polarisation of the graviton in the linearised theory. We shall find that the very existence of a quantum coherent state again requires departures from the classical geometry (at least) near the (would-be) classical central singularity, which will induce the presence of "quantum hair" [93, 94].

A metric of the form in Eq. (4.18) can be conveniently described as the mean field of the coherent state of a (canonically normalised) free massless scalar field $\sqrt{G_{\rm N}}\,\Phi = (f-1)/2 = V$ (see Refs. [51, 52, 81] for all the details). We first quantise this canonically normalised field $\sqrt{G_{\rm N}}\,\Phi$ as a massless field satisfying the Klein-Gordon equation in flat spacetime

$$\left[-\frac{\partial^2}{\partial t^2}+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right]\Phi(t,r)\equiv\left(-\frac{\partial^2}{\partial t^2}+\Delta\right)\Phi(t,r)=0\ . \tag{4.19}$$

Imposing the expected spherical symmetry of the system, we obtain the (positive frequency) eigenfunctions

$$u_k = e^{-i\,k\,t}\,j_0(k\,r)\ , \tag{4.20}$$

where $j_0 = \sin(k\,r)/k\,r$ with $k > 0$ are spherical Bessel functions, which allow us to write the field operator as

$$\hat{\Phi} = \int_0^\infty \frac{k^2\,{\rm d}k}{2\,\pi^2}\sqrt{\frac{\hbar}{2\,k}}\left[u_k\,\hat{a}(k)+u_k^*\,\hat{a}^\dagger(k)\right] \tag{4.21}$$

and its conjugate momentum as

$$\hat{\Pi} = i\int_0^\infty \frac{k^2\,{\rm d}k}{2\,\pi^2}\sqrt{\frac{\hbar\,k}{2}}\left[u_k\,\hat{a}(k)-u_k^*\,\hat{a}^\dagger(k)\right] \tag{4.22}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the usual annihilation and creation operators.

These operators satisfy the equal-time commutation relations,

$$\left[\hat{\Phi}(t,r),\hat{\Pi}(t,r')\right] = \frac{i\,\hbar}{4\,\pi}\,\frac{\delta(r-r')}{r^2}\ , \tag{4.23}$$

provided the creation and annihilation operators obey the commutation rules

$$\left[\hat{a}_k,\hat{a}_p^\dagger\right] = \frac{2\,\pi^2}{k^2}\,\delta(k-p)\ . \tag{4.24}$$

The vacuum state is first defined by $\hat{a}_k\,|0\rangle = 0$ for all allowed values of $k > 0$, and a basis for the Fock space is constructed by the usual action of creation operators.

Classical configurations of the scalar field that can be realised in the quantum theory must correspond to suitable states in this Fock space, and a natural choice is given by coherent states $|g\rangle$ such that

$$\hat{a}_k\,|g\rangle = g_k\,e^{i\,\gamma_k(t)}\,|g\rangle\ . \tag{4.25}$$

In particular, we are interested in those $|g\rangle$ for which the expectation value of the quantum field $\hat{\Phi}$ reproduces the classical potential, namely

$$\sqrt{\frac{\ell_{\rm p}}{m_{\rm p}}}\,\langle g|\,\hat{\Phi}(t,r)\,|g\rangle = V(r) \ . \tag{4.26}$$

From the expansion (4.21), we obtain

$$\langle g|\,\hat{\Phi}(t,r)\,|g\rangle = \int_0^\infty \frac{k^2\,{\rm d}k}{2\,\pi^2}\,\sqrt{\frac{2\,\ell_{\rm p}\,m_{\rm p}}{k}}\,g_k\,\cos[\gamma_k(t)-k\,t]\,j_0(k\,r) \ . \tag{4.27}$$

By expanding the function V in momentum space, we have

$$V = \int_0^\infty \frac{k^2\,{\rm d}k}{2\,\pi^2}\,\tilde{V}(k)\,j_0(k\,r) \ , \tag{4.28}$$

we immediately obtain

$$\gamma_k = k\,t \tag{4.29}$$

and

$$g_k = \sqrt{\frac{k}{2}}\,\frac{\tilde{V}(k)}{\ell_{\rm p}} \ . \tag{4.30}$$

The coherent state finally reads

$$|g\rangle = e^{-N_{\rm G}/2}\,\exp\left\{\int_0^\infty \frac{k^2\,{\rm d}k}{2\,\pi^2}\,g_k\,\hat{a}_k^\dagger\right\}|0\rangle \ , \tag{4.31}$$

where

$$N_{\rm G} = \int_0^\infty \frac{k^2\,{\rm d}k}{2\,\pi^2}\,g_k^2 \tag{4.32}$$

is identified with the graviton number because it is the result of $N = a^\dagger a$ on the coherent state, and thus the value of $N_{\rm G}$ measures the "distance" in the Fock Space of $|g\rangle$ from the vacuum $|0\rangle$ corresponding to $N_{\rm G} = 0$. Such a quantity also ensures the proper normalization for the coherent state itself. We also can define another quantity as

$$\langle\, k\,\rangle = \int_0^\infty \frac{k^2\,{\rm d}k}{2\,\pi^2}\,k\,g_k^2 \ , \tag{4.33}$$

from which one obtains the "average" wavelength $\lambda_{\rm G} = N_{\rm G}/\langle\, k\,\rangle$.

4.2 Quantum Schwarzschild black holes

We will now apply results from the previous section to the Schwarzschild metric (4.18). This geometry contains only the function $V_{\rm N} = \sqrt{G_{\rm N}}\,\Phi$, and all of the relevant expressions introduced can be explicitly computed from the coefficients g_k representing the occupation numbers of the modes u_k.

4.2.1 Pointlike source

Let us first consider the most simple solution for the spherically symmetric case, which comes from a point-like source described by a density function

$$\rho(r) = \frac{M}{4\,\pi\,r^2}\delta(r)\ , \tag{4.34}$$

where M is the mass of the source. By Fourier transforming the classical Poisson equation

$$\Delta V_N = 4\,\pi\,G_{\rm N}\,\rho\ , \tag{4.35}$$

we have

$$\tilde{V_{\rm N}} = -4\,\pi\,G_{\rm N}\,\frac{M}{k^2} \tag{4.36}$$

and the coefficients

$$g_k = -\frac{4\,\pi\,M}{\sqrt{2\,k^3}\,m_{\rm p}}\ . \tag{4.37}$$

This gives an explicit expression for the graviton number

$$N_{\rm G} = \frac{4\,M^2}{m_{\rm p}^2}\int_0^\infty \frac{{\rm d}k}{k}\ , \tag{4.38}$$

and

$$\langle\, k\,\rangle = \frac{4\,M^2}{m_{\rm p}^2}\int_0^\infty {\rm d}k\ . \tag{4.39}$$

The number of quanta $N_{\rm G}$ contains a logarithmic divergence both in the infrared (IR) and the ultraviolet (UV), whereas $\langle\, k\,\rangle$ only diverges (linearly) in the UV.

The meaning of such divergences was already explored in details in previous works [52, 81, 95, 96]. In particular, the UV divergence arises from demanding a Schwarzschild geometry for all values of $r > 0$ and can be formally regularised by introducing a cut-off

$k_{\rm UV} \sim 1/R_{\rm s}$, where $R_{\rm s}$ can be interpreted as the finite radius of a regular matter source. [1] Such a cut-off is just a mathematically simple way of accounting for the fact that the very existence of a proper quantum state $|g\rangle$ requires the coefficients g_k to depart from their purely classical expression (4.37) for $k \to \infty$. Likewise, we introduce a IR cut-off $k_{\rm IR} = 1/R_\infty$ to account for the necessarily finite lifetime $\tau \sim R_\infty$ of the system, and finally write

$$N_{\rm G} = \frac{4\,M^2}{m_{\rm p}^2} \int_{k_{\rm IR}}^{k_{\rm UV}} \frac{{\rm d}k}{k} = 4\,\frac{M^2}{m_{\rm p}^2}\,\ln\left(\frac{R_\infty}{R_{\rm s}}\right) \ , \tag{4.40}$$

and

$$\langle\, k\,\rangle = \frac{4\,M^2}{m_{\rm p}^2} \int_{k_{\rm IR}}^{k_{\rm UV}} {\rm d}k = 4\,\frac{M^2}{m_{\rm p}^2}\left(\frac{1}{R_{\rm s}} - \frac{1}{R_\infty}\right) \ . \tag{4.41}$$

We started from the condition in Eq. (4.26), which demands that the coherent state $|g\rangle$ reproduces the classical potential everywhere. We then found that acceptable occupation numbers g_k do not exist which satisfy this requirement for $k \to 0$ and $k \to \infty$. The above cut-offs imply that the quantum coherent state for a black hole does not need to include all possible (high and low) frequency modes. This conclusion is further supported by the fact that, for a black hole, the mean scalar field only needs to reproduce the classical metric function $V_{\rm N}$ with sufficient accuracy outside the horizon $R_{\rm H}$ to satisfy experimental constraints. Since the behaviour of the matter content insider the horizon remains unknown and experimental bounds can only be established in the outer conmunication region beyond the horizon, the focus is on this external region. This means that the coherent state $|g_{\rm BH}\rangle$ representing a black hole must give

$$\sqrt{\frac{\ell_{\rm p}}{m_{\rm p}}}\,\langle g_{\rm BH}|\,\hat{\Phi}(t,r)\,|g_{\rm BH}\rangle \simeq V_{\rm N}(r) \qquad \text{for } r \gtrsim R_{\rm H} \ , \tag{4.42}$$

where we recall that $V_{\rm N}(R_{\rm H}) = -1/2$ and the approximate equality is subject to experimental precision. In practice, this weaker condition means that $|g_{\rm BH}\rangle$ does not need to contain the modes of infinitely short wavelength that are necessary to resolve the classical singularity at $r = 0$.

It is important to remark that the expression (4.40) differs from the corpuscular scaling (2.16) due to the presence of the logarithmic term involving the profiles of the source, which leads to the violation of the no-hair theorem at the level of the occupation number. The resolution of the divergences explicitly depends on the choice of cut-offs, which was arbitrary. However, these cut-offs provide an indication of what may occur when Quantum Mechanics is used to describe a function within a gravitational framework. Note that

[1] For a (quantum) black hole, we must have $R_{\rm s} \lesssim R_{\rm H} = 2\,G_{\rm N}\,M$.

different shapes for the deviation from the classical potential $V_{\rm N}$ would be obtained if one employed a different UV cut-off in the integral (4.44). In particular, one could consider a smooth window function rather than the hard cut-off $k < R_{\rm s}^{-1}$. Assuming such a smooth function is physically related to the distribution of matter in the black hole interior, one could in principle detect different matter profiles from analysing (test particle motion in) the black hole exterior, whose geometry is going to be described in details next.

In fact, Eq. (4.42) can be satisfied by building the coherent state $|g_{\rm BH}\rangle$ according to Eq. (4.31) with modes of wavelength k^{-1} larger than some fraction of the size of the gravitational radius $R_{\rm H}$ of the source, which we can further identify with the UV cut-off $R_{\rm s}$. By momentarily considering also the IR scale $k_{\rm IR}$, we thus have that only the modes k satisfying

$$R_\infty^{-1} \sim k_{\rm IR} \lesssim k \lesssim k_{\rm UV} \sim R_{\rm s}^{-1} \tag{4.43}$$

are significantly populated in the quantum state $|g_{\rm BH}\rangle$. This yields an effective quantum potential

$$\begin{aligned} V_{\rm QN} &\simeq \int_{k_{\rm IR}}^{k_{\rm UV}} \frac{k^2\,{\rm d}k}{2\,\pi^2}\,\tilde V_{\rm N}(k)\,j_0(k\,r) \\ &\simeq -\frac{2\,\ell_{\rm p}\,M}{\pi\,m_{\rm p}\,r}\int_0^{r/R_{\rm s}} {\rm d}z\,\frac{\sin z}{z}\ , \end{aligned} \tag{4.44}$$

where we defined $z = k\,r$ and let $k_{\rm IR} = 1/R_\infty \to 0$ as mentioned above. We thus find

$$\begin{aligned} V_{\rm QN} &\simeq -\frac{2\,G_{\rm N}\,M}{\pi\,r}\,{\rm Si}\left(\frac{r}{R_{\rm s}}\right) \\ &\simeq V_{\rm N}\left\{1-\left[1-\frac{2}{\pi}\,{\rm Si}\left(\frac{r}{R_{\rm s}}\right)\right]\right\}\ , \end{aligned} \tag{4.45}$$

where Si denotes the sine integral function (see Fig. 4.1 for an example).

Therefore, inserting the cut-offs brings quantum corrections to the classical metric function through $V_{\rm QN}$, and the reconstructed Schwarzschild solution is

$${\rm d}s^2 = -\left(1+2\,V_{\rm QN}\right){\rm d}t^2 + \frac{{\rm d}r^2}{1+2\,V_{\rm QN}} + r^2\,{\rm d}\Omega^2\ , \tag{4.46}$$

where the dependence of $V_{\rm QN}$ on $R_{\rm s}$ therefore results in a quantum violation of the no-hair theorem [93].

Besides the presence of an event horizon even in this mean field metric tensor, it is interesting to look at the classical spacetime singularity of the Schwarzschild solution. The quantum corrected function $V_{\rm QN}$ is regular near the origin

$$V_{\rm QN} \simeq -\frac{2\,G_{\rm N}\,M}{\pi\,R_{\rm s}}\left[1-\frac{\pi\,r^2}{18\,R_{\rm s}^2}\right]\ , \tag{4.47}$$

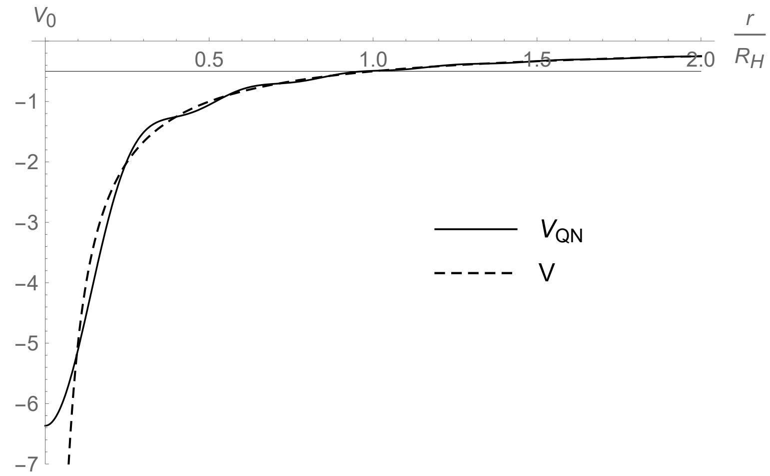

Figure 4.1: Quantum metric function $V_{\rm QN}$ in Eq. (4.45) (solid line) compared to $V_{\rm N}$ (dashed line) for $R_{\rm s} = R_{\rm H}/20$. The horizontal thin line marks the location of the horizon for $V = -1/2$.

so that it is bounded and its derivative vanishes for $r = 0$ (see Fig. 4.1). This suggests that gravitational tidal forces will not show any singular behaviour, as we shall show in more details next.

In the classical Schwarzschild spacetime (4.18), the Kretschmann scalar $R_{\alpha\beta\mu\nu}\,R^{\alpha\beta\mu\nu} \sim R^2 \sim r^{-6}$ for $r \to 0$, whereas for the above quantum corrected metric we have

$$R_{\alpha\beta\mu\nu}\,R^{\alpha\beta\mu\nu} \simeq R^2 \simeq \frac{64\,G_{\rm N}^2\,M^2}{\pi\,R_{\rm s}^2\,r^4} \ . \tag{4.48}$$

This ensures that tidal forces remain finite all the way to the centre, as can be seen more explicitly from the relative acceleration of radial geodesics approaching $r = 0$, to wit

$$\frac{\ddot{\delta r}}{\delta r} = -R^1{}_{010} \simeq \frac{8\,G_{\rm N}^2\,M^2}{9\,\pi^2\,R_{\rm s}^4}\left(1 - \frac{\pi\,R_{\rm s}}{4\,G_{\rm N}\,M}\right) \ , \tag{4.49}$$

where δr is the separation between two nearby radial geodesics and a dot denotes again the derivative with respect to the proper time. We recall that, in the Schwarzschild spacetime, $\ddot{\delta r}/\delta r \sim r^{-4}$, which causes the so-called "spaghettification" of matter approaching the central singularity. One can say that $r = 0$ is now an integrable singularity [97], where some geometric invariants still diverge but no harmful effects occur to matter.

The corpuscular scaling (2.16) for the number $N_{\rm G}$ with the square of the energy M of the system already appears at this stage, whereas the second crucial result

$$\lambda_{\rm G} = \frac{N_{\rm G}}{\langle\, k\,\rangle} \sim \ell_{\rm p}\,\frac{M}{m_{\rm p}} \ , \tag{4.50}$$

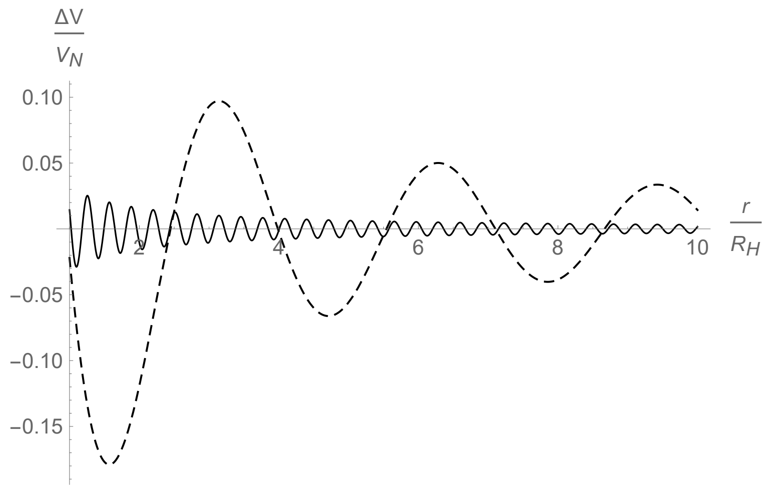

Figure 4.2: Oscillations of the quantum potential V_{QN} in Eq. (4.45) around the Schwarzschild expression V_{N} for $R_{\mathrm{s}} = G_{\mathrm{N}}\, M = R_{\mathrm{H}}/2$ (dashed line) and $R_{\mathrm{s}} = R_{\mathrm{H}}/20$ (solid line) in the region outside the horizon $R_{\mathrm{H}} = 2\, G_{\mathrm{N}}\, M$.

is obtained from Eqs. (4.40) and (4.41) only provided the cut-offs satisfy

$$\ln\left(\frac{R_\infty}{R_{\mathrm{s}}}\right) \simeq \frac{R_{\mathrm{H}}}{R_{\mathrm{s}}} \ . \tag{4.51}$$

Assuming $R_{\mathrm{s}} \lesssim R_{\mathrm{H}} \ll R_\infty$, the above yields

$$R_{\mathrm{s}} \simeq \frac{R_{\mathrm{H}}}{\ln\left(R_\infty/R_{\mathrm{H}}\right)} \ , \tag{4.52}$$

so that the size of the inner source and the radius of the outer region containing a gravitational field are actually connected at the quantum level.

4.2.2 Gaussian source

Let us consider the case of a Gaussian source by replacing the density function (4.34) with

$$\rho(r) = \frac{M}{(2\,\pi\,\delta^2)^{\frac{3}{2}}} e^{-\frac{r^2}{2\,\delta^2}} \ , \tag{4.53}$$

where δ is the width of the source, and

$$M = 4\,\pi \int_0^\infty \mathrm{d}r\, r^2\, \rho(r) \tag{4.54}$$

is the total mass of the source. Let us remark that the above density is essentially zero for $r \gtrsim R \equiv 3\,\delta$, which allows us to make contact with previous case. We immediately find the Fourier transform for the matter distribution $\rho(r)$

$$\tilde{\rho}(k) = M\, e^{-\frac{\delta^2 k^2}{4}} \tag{4.55}$$

from which the coefficients building the state $|g\rangle$ are obtained by the general formula

$$g_k = -\frac{4\,\pi\,\tilde{\rho}(k)}{\sqrt{2\,k^3}\,m_{\rm p}} = -\frac{4\,\pi\,M\,e^{-\frac{k^2\,\delta^2}{4}}}{\sqrt{2\,k^3}\,m_{\rm p}} \tag{4.56}$$

The coherent state $|V_M\rangle$ so defined corresponds to a quantum-corrected metric function [81]

$$V_{{\rm q}M} = \sqrt{G_{\rm N}}\,\langle V_M|\,\hat{\Phi}\,|V_M\rangle = -\frac{G_{\rm N}\,M}{r}\,{\rm erf}\left(\frac{r}{\delta}\right)\ , \tag{4.57}$$

where erf denotes the error function and we let $R_\infty^{-1} \to 0$. For a comparison with the analogous potential generated by a point-like source with the same mass M, see Fig. 4.3. For $r \gtrsim R \equiv 3\,\delta = 3\,R_{\rm H}/2$, the two potentials are clearly indistinguishable.

The total occupation number is obtained as

$$\begin{aligned} N_M &= 4\,\frac{M^2}{m_{\rm p}^2}\int_{R_\infty^{-1}}^{\infty}\frac{{\rm d}k}{k}\,e^{-\frac{k^2\,\delta^2}{2}} \\ &= 2\,\frac{M^2}{m_{\rm p}^2}\,\Gamma\left(0,\frac{\delta^2}{2\,R_\infty^2}\right) \\ &\simeq 4\,\frac{M^2}{m_{\rm p}^2}\,\ln\left(\frac{R_\infty}{\delta}\right)\ , \end{aligned} \tag{4.58}$$

where $\Gamma = \Gamma(a,x)$ is the incomplete gamma function and we assumed $\delta \ll R_\infty$.

It is important to note that the number N_M presents an IR divergence if the source contains modes of vanishing momenta (which would only be physically consistent with an eternal source). Therefore, the state $|V_M\rangle$ and the number are not mathematically well-defined in general. In complete analogy with the point-like source, we have also introduced an IR cut-off $k_{\rm IR} \sim R_\infty^{-1}$ that can cure the IR divergence. Furthermore, this divergence can be eliminated if the scalar field is massive.

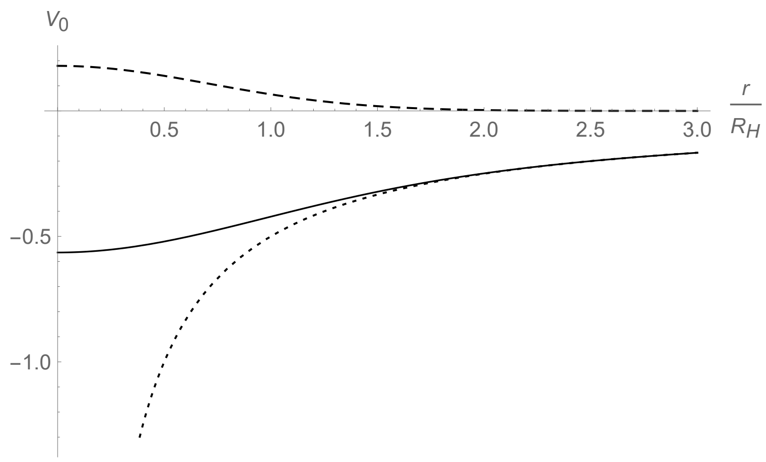

Figure 4.3: Newtonian potential (solid line) for Gaussian matter density with $\delta = 2\,G_{\rm N}\,M$ (dashed line) vs Newtonian potential (dotted line) for point-like source of mass M.

Chapter 5

Quantum hair and entropy for slowly rotating quantum black holes

So far we have discussed how the static and spherically symmetric Schwarzschild geometry can be reproduced by employing coherent states of a massless scalar field on a reference flat spacetime, which then leads to necessary departures from the classical Schwarzschild metric [52]. In particular, the central singularity of the Schwarzschild black hole is replaced by an integrable singularity [97]. The coherent state is built for a scalar field whose expectation value effectively describes the geometry emerging from the (longitudinal or temporal) polarisation of the graviton in the linearised theory.

It is well known that stellar black holes are formed from the collapse of astrophysically significant bodies under suitable conditions. Since the generator of black hole is considered as a typically rotating body, the resulting black hole is also expected to rotate. Additionally, infalling accreting material, black hole mergers, and the feedback from relativistic jets provide futher mechanisms to either create or maintain a spinning black hole. Therefore, the majority of black holes in nature are very likely to spin, which motivates investigating quantum descriptions of black holes with non-vanishing specific angular momentum $a = J/M$ [98]. A complete description of axisymmetric Kerr black holes [99] remains beyond our scope, but this (conceptually and phenomenologically) important issue can be addressed for slow rotation by considering coherent states of gravitons similarly to the spherically symmetric case. In particular, we will focus on the quantum description of the approximate Kerr metric for $|a| \ll G_\mathrm{N}\, M$, which can be written as [100]

$$\mathrm{d}s^2 \simeq -\,(1 + 2\,V)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1+2\,V} - \frac{4\,G_\mathrm{N}\,M\,a}{r}\,\sin^2\theta\,\mathrm{d}t\,\mathrm{d}\phi + r^2\,\mathrm{d}\Omega^2 \ , \tag{5.1}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$. In the above, the metric function

$$V = V_M + W_a \ , \tag{5.2}$$

where

$$V_M = -\frac{G_{\rm N}\, M}{r} \tag{5.3}$$

corresponds to the Schwarzschild metric [101] for $a = 0$, and

$$W_a = \frac{a^2}{2\, r^2}\ . \tag{5.4}$$

In the above stationary geometry, the possible event horizon is a sphere located at $r = r_{\rm H}$ defined as the largest (real) solution of $1+2\,V = 0$. We shall find that the very existence of a quantum coherent state again requires departures from the classical geometry (at least) near the (would-be) classical central singularity. This induces the presence of "quantum hair",which we will further connect with the Bekenstein-Hawking entropy [57], 1-loop quantum corrections to the metric (see [102] and references therein for earlier works), and the Hawking evaporation [55].

5.1 Coherent quantum states for slowly rotating geometry

Like in Ref. [52], the quantum vacuum is here assumed to correspond to a spacetime devoid of matter and gravitational excitations. Any classical metric should then emerge from a suitable (highly excited) quantum state. A standard approach for recovering classical behaviours employs coherent states, which is generically motivated by their property of minimising the quantum uncertainty, and is further supported by studies of electrodynamics [51, 103], linearised gravity [104, 105], and the de Sitter spacetime [83, 106].

In particular, we can again obtain the potential function in Eq. (5.1) as the expectation value of a free massless scalar field $\sqrt{G_{\rm N}}\,\Phi = V$ satisfying the Klein-Gordon equation

$$\Box\Phi = 0\ . \tag{5.5}$$

It is convenient to employ spherical coordinates in which a complete (normalised) set of positive frequency solutions is given by

$$u_{\omega\ell m} = \frac{e^{-i\,\omega\, t}}{\sqrt{2\,\omega}}\, j_\ell(\omega\, r)\, Y_{\ell m}(\theta,\varphi)\ , \tag{5.6}$$

where j_ℓ are spherical Bessel functions of the first kind, and

$$Y_\ell^m = (-1)^m \sqrt{\frac{(2\,\ell+1)(\ell-m)!}{4\,\pi\,(\ell+m)!}}\, P_\ell^m(\cos\theta)\, e^{i\,m\,\varphi}\ , \tag{5.7}$$

are spherical harmonics of degree ℓ and order m, P_ℓ^m being associated Legendre polynomials. We recall that these solutions are orthonormal, [1]

$$(u_{\omega\ell m}|u_{\omega'\ell' m'}) = \frac{\pi}{2\,\omega^2}\,\delta(\omega-\omega')\,\delta_{\ell\ell'}\,\delta_{mm'}\ , \quad (u_{\omega\ell m}|u^*_{\omega'\ell' m'}) = 0\ , \tag{5.8}$$

in the Klein-Gordon scalar product

$$(f_1|f_2) = i\int \mathrm{d}^3x\,(f_1^*\,\partial_t f_2 - f_2\,\partial_t f_1^*)\ . \tag{5.9}$$

The quantum theory is built by mapping the field Φ into an operator expanded in terms of the normal modes (5.6),

$$\hat{\Phi} = \sum_\ell \sum_{m=-\ell}^{\ell} \frac{2}{\pi}\int_0^\infty \omega^2\,\mathrm{d}\omega\,\sqrt{\hbar}\left[u_{\omega\ell m}\,\hat{a}_{\ell m}(\omega) + u^*_{\omega\ell m}\,\hat{a}^\dagger_{\ell m}(\omega)\right]\ . \tag{5.10}$$

Likewise, its conjugate momentum reads

$$\hat{\Pi} = i\sum_\ell \sum_{m=-\ell}^{\ell} \frac{2}{\pi}\int_0^\infty \omega^3\,\mathrm{d}\omega\,\sqrt{\hbar}\left[u_{\omega\ell m}\,\hat{a}_{\ell m}(\omega) - u^*_{\omega\ell m}\,\hat{a}^\dagger_{\ell m}(\omega)\right]\ . \tag{5.11}$$

These operators satisfy the equal-time commutation relations,

$$\left[\hat{\Phi}(t,r,\theta,\varphi),\hat{\Pi}(t,r',\theta',\varphi')\right] = i\,\hbar\,\frac{\delta(r-r')}{r^2}\,\frac{\delta(\theta-\theta')}{\sin\theta}\,\delta(\varphi-\varphi')\ , \tag{5.12}$$

provided the creation and annihilation operators obey the commutation rules

$$\left[\hat{a}_{\ell m}(\omega),\hat{a}^\dagger_{\ell' m'}(\omega')\right] = \frac{\pi}{2\,\omega^2}\,\delta(\omega-\omega')\,\delta_{\ell\ell'}\,\delta_{mm'}\ . \tag{5.13}$$

The vacuum state is first defined by $\hat{a}_{\ell m}(\omega)\,|0\rangle = 0$ for all allowed values of ω, ℓ and m, and a basis for the Fock space is constructed by the usual action of creation operators.

5.1.1 Semiclassical metric function

We seek a quantum state of Φ which effectively reproduces (as closely as possible) the expected slow-rotation limit of the Kerr geometry (5.1), that is

$$\sqrt{G_\mathrm{N}}\,\langle V|\,\hat{\Phi}(t,r,\theta,\varphi)\,|V\rangle \simeq V(r)\ . \tag{5.14}$$

[1] See Appendix A for more details about the notation.

We can build $|V\rangle$ as a superposition of coherent states satisfying

$$\hat{a}_{\ell m}(\omega)\,|g_{\ell m}(\omega)\rangle = g_{\ell m}(\omega)\,e^{i\,\gamma_{\ell m}(\omega)}\,|g_{\ell m}(\omega)\rangle \ , \tag{5.15}$$

where $g_{\ell m} = g^*_{\ell m}$ and $\gamma_{\ell m} = \gamma^*_{\ell m}$, so that

$$\sqrt{G_{\rm N}}\,\langle V|\,\hat{\Phi}\,|V\rangle \ = \ \ell_{\rm p}\sum_{\ell}\sum_{m=-\ell}^{\ell}\frac{2}{\pi}\int_0^\infty \omega^2\,{\rm d}\omega\, j_\ell(\omega\, r)\,\frac{(-1)^m}{\sqrt{2\,\omega}}\sqrt{\frac{(2\,\ell+1)(\ell-m)!}{4\,\pi\,(\ell+m)!}}$$
$$\times\, 2\,\cos(\omega\, t - \gamma_{\ell m} + m\,\varphi)\,P^m_\ell(\cos\theta)\,g_{\ell m}(\omega) \tag{5.16}$$

Since the Kerr metric is stationary and axially symmetric, we impose that the phases $\gamma_{\ell m} \simeq \omega\, t + m\,\varphi$. Indeed, one could argue that recovering exact spacetime symmetries with such a limiting procedure reflects the fact that no perfect isometries exist in nature [52].

The coefficients $g_{\ell m}$ can be determined by expanding the metric field V on the spatial part of the normal modes (5.6),

$$V(r,\theta) = \sum_{\ell}\sum_{m=-\ell}^{\ell}\frac{2}{\pi}\int_0^\infty \omega^2\,{\rm d}\omega\, j_\ell(\omega r)\,(-1)^m\sqrt{\frac{(2\,\ell+1)\,(\ell-m)!}{4\,\pi\,(\ell+m)!}}\,P^m_\ell(\cos\theta)\,\tilde{V}_{\ell m}(\omega)\ . \tag{5.17}$$

By comparing the expansions (5.16) and (5.17), we obtain

$$g_{\ell m} = \sqrt{\frac{\omega}{2}}\,\frac{\tilde{V}_{\ell m}(\omega)}{\ell_{\rm p}}\ . \tag{5.18}$$

The coherent state finally reads

$$|V\rangle = \prod_{\ell}\prod_{m=-\ell}^{\ell} e^{-N_{\ell m}/2}\exp\left\{\frac{2}{\pi}\int_0^\infty \omega^2\,{\rm d}\omega\, g_{\ell m}(\omega)\,\hat{a}^\dagger_{\ell m}(\omega)\right\}|0\rangle \ , \tag{5.19}$$

where

$$N_{\ell m} = \frac{2}{\pi}\int_0^\infty \omega^2\,{\rm d}\omega\,|g_{\ell m}(\omega)|^2 \ , \tag{5.20}$$

is the occupation number for the state $|g_{\ell m}(\omega)\rangle$. We note in particular that $N_V = \sum_{\ell m} N_{\ell m}$ measures the "distance" of $|V\rangle$ from the vacuum $|0\rangle$ in the Fock space and should be finite [52].

5.1.2 Schwarzschild geometry

For zero angular momentum, hence $a = W_a = 0$, the metric function (5.3) is obtained from

$$\tilde{V}_{00} = -\frac{2\sqrt{\pi}}{\omega^2}\, G_{\rm N}\, M \ , \tag{5.21}$$

so that the only contributions to the coherent state $|V_M\rangle$ are given by the eigenvalues [52]

$$g_{00} = -\sqrt{\frac{2\,\pi}{\omega^3}}\,\frac{M}{m_{\rm p}} \ , \tag{5.22}$$

yielding the total occupation number

$$N_M = N_{00} = 4\,\frac{M^2}{m_{\rm p}^2}\int_0^\infty \frac{{\rm d}\omega}{\omega} \ . \tag{5.23}$$

The number N_M diverges logarithmically both in the infrared (IR) and in the ultraviolet (UV). In particular, the UV divergence arises from demanding a Schwarzschild geometry for all values of $r > 0$ and can be formally regularised by introducing a cut-off $\omega_{\rm UV} \sim 1/R_{\rm s}$, where $R_{\rm s}$ can be interpreted as the finite radius of a regular matter source [81]. [2] Such a cut-off is just a mathematically simple way of accounting for the fact that the very existence of a proper quantum state $|V_M\rangle$ requires the coefficients $g_{00} = g_{00}(\omega)$ to depart from their purely classical expression (5.22) for $\omega \to \infty$. Likewise, we introduce a IR cut-off $\omega_{\rm IR} = 1/R_\infty$ to account for the necessarily finite lifetime $\tau \sim R_\infty$ of the system, and finally write

$$N_M = 4\,\frac{M^2}{m_{\rm p}^2}\ln\!\left(\frac{R_\infty}{R_{\rm s}}\right) \ . \tag{5.24}$$

Note that the boundaries $R_{\rm s}$ and R_∞ act as endpoints of the spacetime manifold which induce geodesic incompleteness, since their presence prevent geodesic from extending to arbitrarily small and large values of the affine parameter.

The coherent state $|V_M\rangle$ so defined corresponds to a quantum-corrected metric function

$$\begin{aligned} V_{{\rm q}M} &\simeq \sqrt{G_{\rm N}}\,\langle V_M|\,\hat{\Phi}\,|V_M\rangle = \frac{1}{\pi^{3/2}}\int_{\omega_{\rm IR}}^{\omega_{\rm UV}} \omega^2\,{\rm d}\omega\, j_0(\omega r)\,\tilde{V}_{00}(\omega) \\ &\simeq -\frac{2\,G_{\rm N}\,M}{\pi\,r}\int_{R_\infty^{-1}}^{R_{\rm s}^{-1}} {\rm d}\omega\,\frac{\sin(\omega\,r)}{\omega} \\ &\simeq -\frac{G_{\rm N}\,M}{r}\left\{1-\left[1-\frac{2}{\pi}\,{\rm Si}\!\left(\frac{r}{R_{\rm s}}\right)\right]\right\} \ , \end{aligned} \tag{5.25}$$

[2] For a (quantum) black hole, we must have $R_{\rm s} \lesssim R_{\rm H} = 2\,G_{\rm N}\,M$ [52, 95].

where we let $\omega_{\rm IR} = 1/R_\infty \to 0$ and Si denotes the sine integral function. [3] This result was already analysed in Ref. [52], to which we refer for further details.

5.1.3 Slowly rotating black hole

The classical metric (5.1) is characterised by an angular momentum of modulus $\hbar \ll J = |a|\, M \ll G_{\rm N}\, M^2$ oriented along the axis of symmetry, so that $J^z = J$ for $a > 0$, and by the metric function W_a in Eq. (5.4). We can now show that a quantum state that reproduces such a metric can be obtained by linearly combining the coherent state $|V_M\rangle$ of the Schwarzschild geometry with a suitable coherent state $|W_a\rangle$.

The normal modes (5.6) are eigenfunctions of the angular momentum operators $\hat{L}^2$ and $\hat{L}_z$ (in Minkowski spacetime) with eigenvalues $\hbar^2\, \ell\, (\ell + 1)$ and $\hbar\, m$, respectively. The expectation values of the angular momentum operators on the coherent state $|g_{\ell m}(\omega)\rangle$ are therefore given by (see Appendix B)

$$J_{\ell m} = \langle g_{\ell m}(\omega)|\, \sqrt{\hat{L}^2}\, |g_{\ell m}(\omega)\rangle = \hbar\, \sqrt{\ell\, (\ell + 1)}\, |g_{\ell m}(\omega)|^2 \ , \tag{5.26}$$

and

$$J^z_{\ell m} = \langle g_{\ell m}(\omega)|\, \hat{L}_z\, |g_{\ell m}(\omega)\rangle = \hbar\, m\, |g_{\ell m}(\omega)|^2 \ . \tag{5.27}$$

The total angular momentum for a superposition $|W\rangle$ of states $|g_{\ell m}(\omega)\rangle$ can be obtained as

$$J \equiv \langle W|\, \sqrt{\hat{L}^2}\, |W\rangle = \sum_{\ell>0} \sum_{m=-\ell}^{\ell} \frac{2}{\pi} \int_0^\infty \omega^2\, {\rm d}\omega\, J_{\ell m}(\omega) = \sum_{\ell>0} \hbar\, \sqrt{\ell\, (\ell + 1)} \sum_{m=-\ell}^{\ell} N_{\ell m} \ . \tag{5.28}$$

Likewise,

$$J^z \equiv \langle W|\, \hat{L}_z\, |W\rangle = \sum_{\ell>0} \sum_{m=-\ell}^{\ell} \frac{2}{\pi} \int_0^\infty \omega^2\, {\rm d}\omega\, J^z_{\ell m}(\omega) = \sum_{\ell>0} \sum_{m=-\ell}^{\ell} \hbar\, m\, N_{\ell m} \ . \tag{5.29}$$

Let us next consider coherent states defined by the eigenvalues

$$g_{\ell m} = C_{\ell m}\, \frac{\sqrt{2\, \pi}\, \ell_{\rm p}^\alpha\, M}{\omega^{3/2-\alpha}\, m_{\rm p}} \ , \tag{5.30}$$

[3] For $R_{\rm s} \to 0$, the term in square brackets vanishes at any $r > 0$ and the Schwarzschild metric is formally recovered.

where $C_{\ell m}$ are numerical coefficients that do not depend on ω and $\ell \geq 1$. The corresponding occupation numbers (5.20) are given by

$$N_{\ell m} \simeq \begin{cases} C_{\ell m}^2 \, N_M & \text{for } \alpha = 0 \\ 4\, C_{\ell m}^2 \, \dfrac{M^2}{m_{\rm p}^2} \left[\left(\dfrac{\ell_{\rm p}}{R_{\rm s}} \right)^{2\alpha} - \left(\dfrac{\ell_{\rm p}}{R_\infty} \right)^{2\alpha} \right] & \text{for } \alpha \neq 0 \ , \end{cases} \tag{5.31}$$

where $N_M \sim M^2/m_{\rm p}^2$ is given in Eq. (5.24). Note that the IR limit $R_\infty \to \infty$ is regular only for $\alpha > 0$, for which $N_{\ell m} \ll N_M$ if $R_{\rm s} \gg \ell_{\rm p}$. In this case, we can further approximate

$$N_{\ell m} \simeq 4\, C_{\ell m}^2 \, \frac{M^2}{m_{\rm p}^2} \left(\frac{\ell_{\rm p}}{R_{\rm s}} \right)^{2\alpha} \sim C_{\ell m}^2 \ , \tag{5.32}$$

where we considered $R_{\rm s} \sim R_{\rm H}$ for a black hole. [4] Moreover, the modification (5.17) to the metric function is given by

$$\begin{aligned} W_{\ell m} &\simeq \ell_{\rm p} \, \frac{2}{\pi} \int_{\omega_{\rm IR}}^{\omega_{\rm UV}} \omega^2 \, {\rm d}\omega \, j_\ell(\omega\, r) \, \frac{(-1)^m}{\sqrt{2\,\omega}} \sqrt{\frac{(2\,\ell+1)\,(\ell-m)!}{\pi\,(\ell+m)!}} \, P_\ell^m(\cos\theta) \, g_{\ell m}(\omega) \\ &\simeq \frac{G_{\rm N}\, M}{r} \left(\frac{\ell_{\rm p}}{r} \right)^{\alpha} \left[C_{\ell m} \, (-1)^m \sqrt{\frac{(2\,\ell+1)\,(\ell-m)!}{(\ell+m)!}} \, P_\ell^m(\cos\theta) \, \frac{2}{\pi} \int_0^{r/R_{\rm s}} z^\alpha \, {\rm d}z \, j_\ell(z) \right] \end{aligned} \tag{5.33}$$

where the integral in square brackets can be expressed in terms of regularised hypergeometric functions [see Eq. (A.11)]. We then see that the leading terms in the correction (5.33) are of the classical form $W_a \sim r^{-2}$ in Eq. (5.4) if $\alpha = 1$.

Finally, the contribution to the angular momentum satisfies the classicality conditions

$$\hbar \ll J_{\ell m} \simeq \hbar \sqrt{\ell\,(\ell+1)} \, N_{\ell m} \simeq \hbar\, m\, N_{\ell m} \simeq J_{\ell m}^z \ , \tag{5.34}$$

provided $m \simeq \ell$ and $\ell\, N_{\ell m} \sim \ell\, C_{\ell m}^2 \gg 1$. We can build a coherent state $|W_a\rangle$ that reproduces the geometry (5.1) by including different coherent states (5.30) with $\alpha = 1$ and angular momentum numbers satisfying these conditions. The rotation coefficient will then be given by

$$\frac{m_{\rm p}}{M} \ll \frac{a}{G_{\rm N}\, M} \sim \frac{m_{\rm p}^2}{M^2} \sum_{\ell=1}^{\ell_{\rm c}} \sqrt{\ell\,(\ell+1)} \, N_{\ell\ell} \sim \frac{1}{N_M} \sum_{\ell=1}^{\ell_{\rm c}} \sqrt{\ell\,(\ell+1)} \, N_{\ell\ell} \lesssim \delta_J \ll 1 \ , \tag{5.35}$$

where we introduced a parameter $\delta_J > 0$ to define the slow rotation regime in terms of a maximum value of ℓ, denoted by $\ell_{\rm c}$.

[4] All numerical factors can be included in $C_{\ell m}$.

5.2 Quantum hair

Black hole solutions in general relativity are determined only by the total mass, angular momentum, and electric charge (if present). These uniqueness theorems [107] strongly limit the information about the internal state of a black hole that can be obtained by outside observers. However, the situation changes when we consider the quantum description of black holes given by coherent states already for the spherically symmetric case of Section 5.1.2. In fact, the coherent states from which the geometry emerges as a mean field effect cannot accommodate for perfect Schwarzschild spacetimes [52], but they instead depend on the internal structure of the matter sources (classically) hidden inside the horizon.

The classical case of slow rotation was considered in Section 5.1.3, where we assumed that the quantum states of the geometry only include specific coherent states (5.30) with $\alpha = 1$ satisfying the relations in Eq. (5.34) for the angular momentum. However, the possibility that other states contribute can only be limited from the condition of recovering the classical metric (5.1) within the experimental bounds. Their presence, on the other hand, will constitute a further example of quantum hair [93, 94, 108–111], with departures from the classical geometry.

Instead of attempting a general analysis, we shall only consider states that violate one of the classicality conditions defined in Section 5.1.3 at a time. In particular, we will study a) quantum contributions with $J^z \simeq J$ but $\alpha > 1$ inducing departures from V_M smaller than W_a at large r in Section 5.2.1 and b) modes with $\alpha = 1$ and $a > 0$ given by Eq. (5.35) but such that $|J^z| \ll J$ in Section 5.2.2.

5.2.1 Quantum metric corrections

An explicit example of a coherent state which satisfies the classical conditions $J^z \simeq J$ for the angular momentum but leads to a geometry with terms that fall off at $r \gg R_{\rm H} = 2\, G_{\rm N}\, M$ faster than W_a in Eq. (5.4) is built from

$$g_{\bar{\ell}\bar{\ell}} = C_{\bar{\ell}}\, \frac{\sqrt{2\,\pi}\,\ell_{\rm p}^{\alpha}\, M}{\omega^{3/2-\alpha}\, m_{\rm p}} \ , \tag{5.36}$$

where $\alpha > 1$ and $\bar{\ell}$ is a fixed integer value. The hairy geometry can now be obtained from

$$W_{\bar{\ell}\bar{\ell}} \simeq \frac{\ell_{\rm p}}{\pi^2} \int\limits_{\omega_{\rm IR}}^{\omega_{\rm UV}} \omega^{3/2}\, {\rm d}\omega\, j_{\bar{\ell}}(\omega\, r)\, \frac{2\,\bar{\ell}+1}{2^{\bar{\ell}-1/2}\,\bar{\ell}!}\, (\sin\theta)^{\bar{\ell}}\, g_{\bar{\ell}\bar{\ell}}(\omega) \ , \tag{5.37}$$

where we used Eq. (A.8). We thus find

$$\begin{aligned}
W_{\bar\ell\bar\ell} &\simeq C_{\bar\ell}\,\frac{\ell_{\rm p}^\alpha\,G_{\rm N}\,M}{\pi^{3/2}}\,\frac{2\,\bar\ell+1}{2^{\bar\ell-1}\,\bar\ell!}\,(\sin\theta)^{\bar\ell}\int\limits_{\omega_{\rm IR}}^{\omega_{\rm UV}}\omega^\alpha\,{\rm d}\omega\,j_{\bar\ell}(\omega\,r)\\
&\simeq C_{\bar\ell}\,\frac{G_{\rm N}\,M}{\pi^{3/2}\,r}\left(\frac{\ell_{\rm p}}{r}\right)^\alpha\frac{2\,\bar\ell+1}{2^{\bar\ell-1}\,\bar\ell!}\,(\sin\theta)^{\bar\ell}\int_0^{r/R_{\rm s}}z^\alpha\,{\rm d}z\,j_{\bar\ell}(z)\\
&\sim \frac{G_{\rm N}\,M}{r}\left(\frac{\ell_{\rm p}}{r}\right)^\alpha\ ,
\end{aligned}\tag{5.38}$$

where the integral is given in Eq. (A.11).

We can in particular estimate the correction on the (unperturbed) Schwarzschild horizon at $r=R_{\rm H}$,

$$W_{\bar\ell\bar\ell}(R_{\rm H})\sim\left(\frac{m_{\rm p}}{M}\right)^\alpha(\sin\theta)^{\bar\ell}\ .\tag{5.39}$$

Such corrections with different $\bar\ell$ represent purely axial perturbations on the horizon, with vanishingly small amplitude for macroscopically large black holes of mass $M\gg m_{\rm p}$ provided $\alpha\gtrsim 1$.

5.2.2 Quantum angular momentum

States that lead to metric functions of the classical form W_a in Eq. (5.4) with a given by Eq. (5.35) but satisfy

$$J^z\sim\sum_{\ell=1}^{\ell_{\rm c}}\sum_{m=-\ell}^{\ell}m\,N_{\ell m}\simeq 0\tag{5.40}$$

can be simply obtained by assuming $|g_{\ell m}|=|g_{\ell -m}|$ so that $N_{\ell m}=N_{\ell -m}$. As an example, we consider

$$g_{\bar\ell\bar\ell}=g_{\bar\ell-\bar\ell}=C_{\bar\ell}\sqrt{\frac{2\,\pi}{\omega}\,\frac{M}{m_{\rm p}}}\ ,\tag{5.41}$$

where $\bar\ell$ is again a fixed integer value and $\bar\ell\,C_{\bar\ell}^2$ is of the correct size to yield a rotation parameter $a>0$ satisfying the bounds in Eq. (5.35). The metric correction is now given by

$$W_{\bar\ell\bar\ell}\simeq\frac{\ell_{\rm p}}{\pi^2}\int\limits_{\omega_{\rm IR}}^{\omega_{\rm UV}}\omega^2\,{\rm d}\omega\,j_{\bar\ell}(\omega\,r)\,\frac{(-1)^{\bar\ell}+1}{\sqrt{2\,\omega}}\,\frac{2\,\bar\ell+1}{2^{\bar\ell-1}\,\bar\ell!}\,(\sin\theta)^{\bar\ell}\,g_{\bar\ell\bar\ell}(\omega)\ ,\tag{5.42}$$

where we used the known relation (A.9). For $\bar{\ell}$ odd the above expression vanishes, whereas for $\bar{\ell}$ even we find twice the value in Eq. (5.38) with $\alpha = 1$, that is

$$W_{\bar{\ell}\bar{\ell}} \sim \frac{\ell_{\rm p}\, G_{\rm N}\, M}{r^2} \ . \tag{5.43}$$

From the above few examples, it should be clear that one can engineer many different axially symmetric configurations, all of which differ from the (slowly-rotating) Kerr geometry only by terms of order $(\ell_{\rm p}/r)^\alpha$ for $\alpha \geq 1$. Of course, this ambiguity would be removed by computing the coherent state generated by a given matter source, which is however supposedly hidden behind the horizon. Moreover, we remark that such terms would result in a (slight) shift in the position $r_{\rm H}$ of the event horizon with respect to the classical value $R_{\rm H}$.

5.3 Entropy and evaporation

In the previous Sections, for simplicity, we have modelled the dependence of the geometry from the internal structure of the black hole by introducing cut-offs $\omega_{\rm IR} \sim 1/R_\infty$ and $\omega_{\rm UV} \sim 1/R_{\rm s}$ in momentum space and allowing for contributions of angular momentum that have no classical counterpart. Were we able to test the gravitational field with sufficient accuracy, for instance from the motion of test particles and light in the outer region to the horizon, we could remove these uncertainties and gather information about the interior of the black hole.

5.3.1 Bekenstein-Hawking entropy

A common way to measure our ignorance about the actual state of a system is provided by the thermodynamic entropy, which is obtained by counting the possible microstates corresponding to a given macroscopic configuration. For a Schwarzschild black hole, the Bekenstein-Hawking entropy [57]

$$S_{\rm BH} = \frac{\pi\, R_{\rm H}^2}{\ell_{\rm p}^2} = \frac{4\,\pi\, M^2}{m_{\rm p}^2} \tag{5.44}$$

can be obtained [53] by supplementing a pure coherent state of the Schwarzschild geometry (5.22) with the Planckian distribution of Hawking quanta at the temperature [55]

$$T_{\rm H} = \frac{m_{\rm p}^2}{8\,\pi\, M} \ . \tag{5.45}$$

Given a black hole of mass M, instead of one pure coherent state, we could consider all possible states giving rise to (practically) indistinguishable semiclassical geometries with the mass M. We can employ the total occupation number (5.24) [5] of the correspond-

[5]We just mention that the same quantisation law is obtained for the ground state of a dust ball [112].

ing coherent state to estimate the total number of microstates available to build such configurations as

$$\mathcal{N}_M \sim \sum_{n=0}^{N_M} \binom{N_M}{n} = \sum_{n=0}^{N_M} \frac{N_M!}{(N_M - n)!\, n!} = 2^{N_M} \ . \tag{5.46}$$

The thermodynamic entropy is thus

$$S_M \propto \ln(\mathcal{N}_M) \sim \left(\frac{M}{m_{\rm p}}\right)^2 \ , \tag{5.47}$$

which is clearly proportional to the Bekenstein-Hawking entropy (5.44). One can therefore envisage that the coherent states giving rise to Schwarzschild black hole geometries contain the precursors (or proxies) of the Hawking particles, like in the original corpuscular picture [43–46].

5.3.2 Entropy and angular momentum

We can also estimate the number of quantum states with angular momentum corresponding to geometric configurations that cannot be observationally distinguished from a non-rotating Schwarzschild black hole. For this purpose, we can consider again a maximum angular momentum parameter $\delta_J \ll 1$ such that configurations with

$$\frac{J}{M\, R_{\rm H}} \simeq \frac{a}{G_{\rm N}\, M} \lesssim \delta_J \ , \tag{5.48}$$

cannot be distinguished from the coherent state reproducing the quantum-corrected Schwarzschild geometry (5.25). Furthermore, we shall include in this count only those contributions of the form in Eq. (5.30),

$$g_{\ell m} \sim C_{\ell m} \frac{\omega^\alpha\, \ell_{\rm p}^\alpha\, M}{\omega^{3/2}\, m_{\rm p}} \ , \tag{5.49}$$

that violate both of the classicality conditions considered in Sections 5.2.1 and 5.2.2, that is $\alpha \gtrsim 1$ and $0 \le |m| \ll \ell$.

In particular, the contribution of modes with $m \simeq 0$ to the angular momentum (5.28) is approximately given by

$$\frac{a_\ell}{G_{\rm N}\, M} \simeq \ell\, \frac{m_{\rm p}^2}{M^2}\, N_{\ell 0} \sim \ell \left(\frac{\ell_{\rm p}}{R_{\rm s}}\right)^{2\,\alpha} \ , \tag{5.50}$$

in which we assumed $N_{\ell 0} \sim C_{\ell 0}^2 \sim 1$ and $\ell \gg 1$. Imposing the constraint (5.48) on the total angular momentum,

$$\sum_{\ell=1}^{\ell_{\rm c}} \frac{a_\ell}{G_{\rm N}\, M} \sim \ell_{\rm c}^2 \left(\frac{\ell_{\rm p}}{R_{\rm s}}\right)^{2\,\alpha} \lesssim \delta_J \ , \tag{5.51}$$

yields

$$\ell_{\rm c} \lesssim \left(\frac{R_{\rm s}}{\ell_{\rm p}}\right)^{\alpha} \sqrt{\delta_J} \sim \left(\frac{M}{m_{\rm p}}\right)^{\alpha} \sqrt{\delta_J} \ , \tag{5.52}$$

where we again set $R_{\rm s} \sim R_{\rm H}$ for a black hole. Upon allowing for the inclusion of modes $|g_{\ell 0}\rangle$ with $1 \leq \ell \leq \ell_{\rm c}$, we can estimate the degeneracy of the quantum black hole given by the total number of possible combinations in angular momentum, that is

$$\mathcal{N}_{\rm c} = \sum_{\ell=0}^{\ell_{\rm c}} \binom{\ell_{\rm c}}{\ell} = \sum_{\ell=0}^{\ell_{\rm c}} \frac{\ell_{\rm c}!}{(\ell_{\rm c}-\ell)!\,\ell!} = 2^{\ell_{\rm c}} \ . \tag{5.53}$$

The corresponding thermodynamic entropy,

$$S \propto \ln(\mathcal{N}_{\rm c}) \sim \left(\frac{M}{m_{\rm p}}\right)^{\alpha} \sqrt{\delta_J} \ , \tag{5.54}$$

is also proportional to the Bekenstein-Hawking entropy (5.44) for $\alpha = 2$.

It is then interesting to notice that the metric corrections for $\alpha = 2$ are of the same order in $\ell_{\rm p}$ and $1/r$ as those obtained from 1-loop corrections to the Schwarzschild metric [102], that is

$$\begin{aligned} W_{\rm qa} &\simeq \sum_{\ell=1}^{\ell_{\rm c}} W_{\ell 0} \simeq \frac{G_{\rm N}\, M}{r} \left(\frac{\ell_{\rm p}}{r}\right)^{2} \sum_{\ell=1}^{\ell_{\rm c}} \left[C_{\ell 0}\, \sqrt{2\,\ell+1}\, P_\ell(\cos\theta)\, \frac{2}{\pi} \int_0^{r/R_{\rm s}} z^2\, {\rm d}z\, j_\ell(z)\right] \\ &\sim \frac{G_{\rm N}\, M}{r} \left(\frac{\ell_{\rm p}}{r}\right)^{2} \ , \end{aligned} \tag{5.55}$$

where $P_\ell = P_\ell^0$ are Legendre polynomials.

Finally, we can check that the condition (5.52) guarantees that the horizon does not shift significantly from the unperturbed Schwarzschild radius. In fact, for $\alpha = 2$, we can write

$$V \simeq V_M + \epsilon\, \frac{\ell_{\rm p}^2\, G_{\rm N}\, M}{r^3} \ , \tag{5.56}$$

where $\epsilon \sim \sqrt{\delta_J}$ now contains all the parameters (and angular dependence) shown in the first line of Eq. (5.55). The largest solution to $V = -1/2$ is then given by

$$r_{\rm H} \simeq 2\, G_{\rm N}\, M - \epsilon\, \ell_{\rm p} \ , \tag{5.57}$$

which represents a negligible correction to $R_{\rm H} = 2\, G_{\rm N}\, M$. Given the fast fall-off of the metric correction in Eq. (5.55), one could interpret these perturbations as being “confined” about the horizon $R_{\rm H}$, like in the membrane approach [113] and in derivation of the entropy (5.44) based on conformal symmetry [114]. In both approaches the essential physics is argued to reside very close to the horizon. Both approaches rely on a slight positional adjustment of the event horizon. This adjustment not only regularizes physical quantities (rendering entropy, temperature, etc., finite) but also provides a clear and operable boundary for describing the interactions between the black hole and external fields. The membrane paradigm and the conformal field theory approach, respectively from the perspectives of macroscopic physical description and microscopic statistical description, reveal the important role of black hole horizon regularization (or positional correction).

5.3.3 Hawking radiation

The Hawking evaporation has been studied with several methods since its discovery [55]. In particular, semiclassical approaches describe this effect as particles that tunnel out from within the event horizon on classically forbidden paths [115–124]. We will employ the WKB approach to compute corrections to the Hawking temperature for slowly rotating black holes described by the quantum-corrected Schwarzschild metric (5.25) with the metric modification in Eq. (5.55) that we showed can contribute to the Bekenstein-Hawking entropy.

We start by noting that, replacing the WKB ansatz

$$\Phi \simeq \exp\left[\frac{i}{\hbar}\, S(t, r, \theta, \phi)\right] \tag{5.58}$$

in the Klein-Gordon Eq. (5.5) at leading order in $\hbar$, yields the Hamilton-Jacobi equation

$$g^{\mu\nu}\,\partial_\mu S\,\partial_\nu S \simeq 0\ . \tag{5.59}$$

Solutions can be written in the form

$$S = -E\,t + \mathcal{W}(r) + \mathcal{J}(\theta, \phi) + K\ , \tag{5.60}$$

where E represents the energy of the emitted boson and K is a complex constant that will be fixed later. The ratio E/M regulates the magnitude of the backreaction of the emission on the black hole, which can alter the thermal nature of the Hawking radiation [115]. We only consider large black holes with mass $M \gg m_{\rm p}$, hence this effect can be neglected for $E \ll M$.

Given the inverse of the metric (5.1), Eq. (5.59) can be written as

$$(1+2\,V)\left[\left(\frac{\partial \mathcal{W}}{\partial r}\right)^2 + \frac{r^2}{w\,\sin\theta}\left(\frac{\partial \mathcal{J}}{\partial \phi}\right)^2\right] + \frac{1}{r^2}\left(\frac{\partial \mathcal{J}}{\partial \theta}\right)^2 + \frac{4\,a\,G_{\rm N}\,M\,r}{w} E\,\frac{\partial \mathcal{J}}{\partial \phi} \simeq \frac{r^4}{w} E^2\ , \tag{5.61}$$

where we used the form (5.60) for S and defined

$$w \equiv r^4\,(1+2\,V) + 4\,a^2\,G_{\rm N}^2\,M^2\,\sin^2\theta\ . \tag{5.62}$$

For Hawking particles in a quantum corrected Schwarzschild geometry, we can just consider purely radial trajectories [121–124], along which $\mathcal{J}$ is constant, and further approximate $w \simeq r^4 (1 + 2V)$. In this case, Eq. (5.61) is solved by

$$\mathcal{W}_\pm \simeq \pm E \int^r \frac{\mathrm{d}\bar{r}}{1 + 2V} \ , \tag{5.63}$$

with + (−) for outgoing (ingoing) particles.

Imaginary terms in the action S correspond to the Boltzmann factor for emission and absorption across the event horizon. Such terms can only arise due to the pole at $r = r_\mathrm{H} \simeq R_\mathrm{H}$, where $1+2V = 0$, and from the imaginary part of K in Eq. (5.60), resulting in the probabilities

$$P_\pm \propto \exp\left[-\frac{2}{\hbar}\left(\Im\mathcal{W}_\pm + \Im K\right)\right] \ , \tag{5.64}$$

where $\Im$ denotes the imaginary part. Assuming that ingoing particles necessarily cross the event horizon, that is $P_- \simeq 1$, one must set $\Im K = -\Im\mathcal{W}_-$. Since $\mathcal{W}_+ = -\mathcal{W}_-$, the probability of a particle tunnelling out then reads

$$P_+ \simeq \exp\left(-\frac{4}{\hbar}\Im\mathcal{W}_+\right) \ . \tag{5.65}$$

The integral (5.63) around the pole at $r \simeq R_\mathrm{H}$ with the Feynman prescription for the propagator [121–124] yields

$$\Im\mathcal{W}_+ \simeq \lim_{r \to R_\mathrm{H}} \frac{\pi E}{2\hbar V'(r)} \ , \tag{5.66}$$

where $V' = \partial_r V$. Finally,

$$P_+ \simeq \exp\left[-\frac{2\pi E}{V'(R_\mathrm{H})}\right] \tag{5.67}$$

which implies that the temperature must be given by

$$T_M \simeq \frac{\hbar}{2\pi} V'(R_\mathrm{H}) = \frac{\hbar}{2\pi}\left[V'_{\mathrm{q}M}(R_\mathrm{H}) + W'_{\mathrm{q}a}(R_\mathrm{H})\right] \ . \tag{5.68}$$

This expression with the metric function (5.25) and the contribution (5.55) with $\alpha = 2$ for the case of Section 5.3.1 gives

$$T_M \simeq T_\mathrm{H}\,\frac{2}{\pi}\left[\mathrm{Si}\left(\frac{R_\mathrm{H}}{R_s}\right) - \sin\left(\frac{R_\mathrm{H}}{R_s}\right) - \frac{3\,m_\mathrm{p}^2}{4\pi M^2}\sum_{\ell=1}^{\ell_c} C_{\ell 0}\sqrt{2\ell+1}\,P_\ell(\cos\theta)\int_0^{R_\mathrm{H}/R_s} z^2\,\mathrm{d}z\, j_\ell(z)\right] \tag{5.69}$$

where $T_{\rm H}$ is the standard Hawking temperature (5.45), which is therefore recovered asymptotically for $M \gg m_{\rm p}$ and $R_{\rm s} \ll R_{\rm H}$.

Using the metric function in Eq. (5.56), one analogously finds

$$T_M \simeq T_{\rm H} \left(1 - \frac{3\,\epsilon\, m_{\rm p}^2}{4\,M^2}\right) \ . \tag{5.70}$$

On equating the two corrections of order $m_{\rm p}^2/M^2$, we obtain

$$\epsilon \simeq \frac{1}{\sqrt{\pi^3}} \sum_{\ell=1}^{\ell_c} C_{\ell 0} \, \frac{\sqrt{2\,\ell+1}\,\Gamma(\ell/2+3/2)\,P_\ell(\cos\theta)}{2^\ell\,\Gamma\left(\ell+3/2\right)\,\Gamma(\ell/2+5/2)}\, {}_1F_2\left(\frac{\ell+3}{2}, \ell+\frac{3}{2}, \frac{\ell+5}{2}, -\frac{1}{4}\right) \ , \tag{5.71}$$

where we used Eq. (A.11) with $\alpha = 2$.

5.4 Conclusions

In this Chapter, the semiclassical metric function reproducing a Kerr geometry in the slow-rotation regime was shown to arise from suitable highly-excited coherent states, thus generalising previous results obtained for spherically symmetric geometries [52, 106, 125]. Quantum hair naturally emerges in this context, since the existence of the quantum coherent state does not allow for any possible IR and UV divergences in general.

An additional source of quantum hair was then identified in angular momentum modes that do not satisfy the conditions for giving rise to a classical rotating geometry described in Section 5.1.3. Such modes were further associated with the Bekenstein-Hawking entropy of Schwarzschild black holes and are therefore expected to play the role of precursors of the Hawking radiation, at least for very massive black holes. The Hawking evaporation was then studied with the Hamilton-Jacobi method, from which modes representing quantum hair in the geometry were related to metric corrections of the form that one expects from 1-loop quantum corrections [102].

Chapter 6

Horizon quantum mechanics for coherent quantum black holes

The general relativistic description of gravitational collapse, leading to the formation of black holes, was first investigated in the seminal papers by Oppenheimer and his co-workers [126, 127]. However, a fully understanding of the physics underlying these processes remains one of the most intriguing challenges in contemporary theoretical physics. What is unanimously accepted is that gravity becomes significant whenever a large enough amount of matter is "compacted" into a sufficiently small volume, and the effects of gravitational interaction on the causal structure of spacetime cannot be neglected. K. Thorne formulated this idea in the *hoop conjecture* [128]: A black holes forms when the impace parameter b of two colliding objects (for simplicity, of negligible spatial extension and total angular momentum) is shorter than the Schwarzschild radius of the system, that is for

$$b \lesssim 2\,\ell_{\rm p}\frac{E}{m_{\rm p}} \equiv R_{\rm H} \ , \tag{6.1}$$

where E is total energy in the centre-mass frame.

This hoop conjecture has been theoretically checked and verified in various situations, but it was initially formulated for black holes of astrophysical size [129–131], whose energy is orders of magnitude above the scale of quantum gravity, and can, therefore, be reasonably described by classical General Relativity. One of the most important questions then arises is whether the concepts underlying this conclusion can also be trusted for masses approaching the Planck scale. As previously discussed, at very short scales, *i.e*, lengths on the order of the Planck length or less, the spacetime description in terms of General Relativity breaks down, and a quantum theory of gravity must be invoked. A quantum gravitational theory would allow us to predict the behaviour of gravity at all scales, but, as of now, this goal has not yet been fully realized. In fact, the horizon wave function [132] raises as an effective approach to give us hints on what would be expected from black hole physics near the Planck scale. The main idea is to extend the fundamental principles of gravity and quantum mechanics beyond our current experimental limits. In doing so, one encounters the conceptual challenge of consistently describing classical and quantum objects, such as horizons and particles, within the same framework. This is achieved by assigning wave functions to the quantum black hole horizon. The horizon wave function is defined to be associated with any localised quantum mechanical particle, which makes it easy to formulate in a quantitative way a condition for distinguishing black holes from regular particles. This auxiliary wave function yields the probability of finding a horizon of a certain radius centred around the source, and one can therefore determine the probability that a quantum mechanical particle is a black hole depending on its mass.

6.1 The horizon wave functions formalism

The horizon quantum mechanics was introduced in Refs. [35,132] (see also Ref. [133] for a review) to compute the probability of the presence of horizons associated with static and spherically symmetric matter sources in a given quantum state $|\psi_{\rm S}\rangle$. Now we introduce the horizon wave function in the standard way. From classical General Relativity, we know that the horizons of black holes are described by trapping surfaces, whose locations are determined by [134]

$$0 = g^{ij}\nabla_i r \nabla_j r = 1 - \frac{2\, m(r)}{r} \ , \tag{6.2}$$

where $\nabla_i r$ is the covector perpendicular to the surfaces of constant area $\mathcal{A} = 4\,\pi\, r^2$. The function $m(r)$ is the active gravitational (or Misner-Sharp) [135,136] mass, representing the total energy enclosed within a sphere of radius r, and is given by

$$m(r) = 4\,\pi \int_0^r \rho(x)\, x^2\, {\rm d}x \ . \tag{6.3}$$

An horizon then exists if there are values of $r = r_{\rm H}$ such that $2\, G_{\rm N}\, m(r_{\rm H}) = r_{\rm H}$. A quantum mechanical description is obtained by replacing the classical energy density with the energy decomposition of the source wavefunction,

$$|\psi_{\rm S}\rangle = \sum_E C(E)\, |\psi_E\rangle \ , \tag{6.4}$$

where the sum represents the spectral decomposition in Hamiltonian eigenmodes,

$$\hat{H}\, |\psi_E\rangle = E\, |\psi_E\rangle \ , \tag{6.5}$$

and H will depend on the model we wish to consider. Upon expressing E in terms of the gravitational Schwarzschild radius, [1] $E = r_{\rm H}/2\,G_{\rm N}$, we obtain the horizon wavefunction

$$\psi_{\rm H}(r_{\rm H}) \equiv \langle\, r_{\rm H} \mid \psi_{\rm H}\,\rangle = \mathcal{N}_{\rm H} \sum_{E=r_{\rm H}/2\,G_{\rm N}} C(E) \ , \tag{6.6}$$

whose normalisation $\mathcal{N}_{\rm H}$ is fixed in the Schrödinger scalar product

$$\langle\, \psi_{\rm H} \mid \phi_{\rm H}\,\rangle = 4\,\pi \int_0^\infty \psi_{\rm H}^*(r_{\rm H})\,\phi_{\rm H}(r_{\rm H})\,r_{\rm H}^2\,{\rm d}r_{\rm H} \ . \tag{6.7}$$

The normalised wavefunction yields the probability density for the values of the gravitational radius $r_{\rm H}$ associated with the source in the quantum state $|\psi_{\rm S}\rangle$, namely

$$\mathcal{P}_{\rm H}(r_{\rm H}) = 4\,\pi\,r_{\rm H}^2\,|\psi_{\rm H}(r_{\rm H})|^2 \ . \tag{6.8}$$

Moreover, the probability density that the source lies inside its own gravitational radius will be given by

$$\mathcal{P}_<(r_{\rm H}) = P_{\rm S}(r_{\rm H})\,\mathcal{P}_{\rm H}(r_{\rm H}) \ , \tag{6.9}$$

where

$$P_{\rm S}(r_{\rm H}) = 4\,\pi \int_0^{r_{\rm H}} |\psi_{\rm S}(r)|^2\,r^2\,{\rm d}r \tag{6.10}$$

is the probability that the source is found inside a sphere of radius $r = r_{\rm H}$. Finally, the probability that the object described by the state $|\psi_{\rm S}\rangle$ is a black hole will be obtained by integrating Eq. (6.9) over all possible values of the gravitational radius, namely

$$P_{\rm BH} = \int_0^\infty \mathcal{P}_<(r_{\rm H})\,{\rm d}r_{\rm H} \ . \tag{6.11}$$

6.2 Coherent quantum states for Schwarzschild geometry

As presented in the previous Chapters, coherent quantum states can be employed to describe the static and spherically symmetric Schwarzschild black holes as emergent (semi)classical geometries [52]. [2] It is important to remark that this approach implies

[1] For the local version of the formalism, see Ref. [137].

[2] For studies of their thermodynamics and configurational entropy, see Refs. [53,64,138] and, for more phenomenological consequences, see Ref. [139].

the removal of the central singularities by the presence of a quantum matter core that could therefore lead to phenomenological signatures of the kinds analysed in Ref. [140]. It appears natural to apply the horizon quantum mechanics to black hole geometries described by coherent states and to verify under which conditions there exists a horizon with probability close to one.

The presence of horizons in the above approach can only be established from semiclassical arguments, that is by considering the quantum corrected metric

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\,\mathrm{d}\Omega^2 \ , \tag{6.12}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$ and

$$f = 1 + 2\,V_{\mathrm{q}}(r) \ . \tag{6.13}$$

In the above, the function $V_{\mathrm{q}} = \langle V|\,\hat{V}(r)\,|V\rangle$ is the expectation value of the relevant metric field on the coherent quantum state $|V\rangle$. The locations of horizons are then given by solutions $r = r_{\mathrm{H}}$ of the classical equation $f(r) = 0$.

We now first reconstruct the state $|\psi_{\mathrm{S}}\rangle$ from the effective energy density associated with the quantum corrected geometry (6.13); using that result, we will obtain the horizon wavefunction in next Section. A metric of the form in Eq. (6.12) can be conveniently described as the mean field of the coherent state of a (canonically normalised) free massless scalar field $\sqrt{G_{\mathrm{N}}}\,\Phi = (f-1)/2 = V$ (see Ref. [52] for all the details). From the Klein-Gordon equation

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right]\Phi(t,r) = 0 \ , \tag{6.14}$$

we obtain the (positive frequency) eigenfunctions

$$u_k = e^{-i\,k\,t}\,j_0(k\,r) \ , \tag{6.15}$$

where $j_0 = \sin(k\,r)/k\,r$ with $k > 0$ are spherical Bessel functions, which allow us to write the field operator as

$$\hat{\Phi} = \int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\sqrt{\frac{\hbar}{2\,k}}\left[u_k\,\hat{a}(k) + u_k^*\,\hat{a}^\dagger(k)\right] \tag{6.16}$$

and its conjugate momentum as

$$\hat{\Pi} = i\int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\sqrt{\frac{\hbar\,k}{2}}\left[u_k\,\hat{a}(k) - u_k^*\,\hat{a}^\dagger(k)\right] \tag{6.17}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the usual annihilation and creation operators.

In particular, we are interested in a coherent state

$$|V_M\rangle = e^{-N_M/2} \exp\left\{\int_0^\infty \frac{k^2\, \mathrm{d}k}{2\,\pi^2}\, g_k\, \hat{a}^\dagger(k)\right\} |0\rangle \ , \tag{6.18}$$

which effectively reproduces (as closely as possible) the Schwarzschild geometry, that is

$$\sqrt{G_\mathrm{N}}\, \langle V_M|\, \hat{\Phi}(t,r)\, |V_M\rangle \simeq V_M(r) = -\frac{2\, G_\mathrm{N}\, M}{r} \ . \tag{6.19}$$

From

$$\langle V_M|\, \hat{\Phi}\, |V_M\rangle \ = \int\limits_0^\infty \frac{k^2\, \mathrm{d}k}{2\,\pi^2} \sqrt{\frac{2\,\ell_\mathrm{p}\, m_\mathrm{p}}{k}}\, g_k\, \cos(k\,t - \gamma_k)\, j_0(k\,r) \ , \tag{6.20}$$

we impose $\gamma_k = k\,t$ for staticity and the coefficients g_k can be determined by expanding the metric function $V_M = V_M(r)$ on the spatial part of the normal modes (6.15), to obtain

$$g_k = -\frac{4\,\pi\, M}{\sqrt{2\, k^3}\, m_\mathrm{p}} \ . \tag{6.21}$$

However, the corresponding normalisation factor

$$N_M = 4\, \frac{M^2}{m_\mathrm{p}^2} \int\limits_0^\infty \frac{\mathrm{d}k}{k} \tag{6.22}$$

diverges both in the infrared and in the ultraviolet. The infrared divergence can be eliminated by embedding the geometry in a universe of finite Hubble radius $r = R_\infty$, whereas the ultraviolet divergence could be removed by assuming the existence of a matter core of finite size $r = R_\mathrm{s}$.

For the present work, it is convenient to regularise the ultraviolet divergence by replacing the coefficients in Eq. (6.21) with

$$g_k = -\frac{4\,\pi\, M\, e^{-\frac{k^2\, R_\mathrm{s}^2}{4}}}{\sqrt{2\, k^3}\, m_\mathrm{p}} \ . \tag{6.23}$$

This adjustment ensures that the effective energy density remains positive, as will be demonstrated below, and yields the total occupation number

$$\begin{aligned}N_M &= 4\,\frac{M^2}{m_{\rm p}^2}\int\limits_{R_\infty^{-1}}^{\infty}\frac{{\rm d}k}{k}\,e^{-\frac{k^2 R_{\rm s}^2}{2}}\\ &= 2\,\frac{M^2}{m_{\rm p}^2}\,\Gamma\left(0,\frac{R_{\rm s}^2}{2\,R_\infty^2}\right)\\ &\simeq 4\,\frac{M^2}{m_{\rm p}^2}\ln\left(\frac{R_\infty}{R_{\rm s}}\right)\ ,\end{aligned} \tag{6.24}$$

where $\Gamma = \Gamma(a, x)$ is the incomplete gamma function and we assumed $R_{\rm s} \ll R_\infty$. The coherent state $|V_M\rangle$ so defined corresponds to a quantum-corrected metric function

$$V_{{\rm q}M} = \sqrt{G_{\rm N}}\,\langle V_M|\,\hat{\Phi}\,|V_M\rangle = -\frac{G_{\rm N}\,M}{r}\,{\rm erf}\left(\frac{r}{R_{\rm s}}\right)\ , \tag{6.25}$$

where erf denotes the error function and we let $R_\infty^{-1} \to 0$.

6.2.1 Effective energy density

From the definition of the mass function in Eq. (6.3) and

$$1 + 2\,V_{{\rm q}M} = 1 - \frac{2\,G_{\rm N}\,m}{r}\ , \tag{6.26}$$

we easily obtain

$$\rho(r) = -\frac{V_{{\rm q}M}}{4\,\pi\,G_{\rm N}\,r^2}\left(1 + r\,\frac{V_{{\rm q}M}'}{V_{{\rm q}M}}\right)\ . \tag{6.27}$$

We next note that the quantum corrected potential (6.25) is of the form

$$V_{{\rm q}M} = V_M(r)\,v(r)\ , \tag{6.28}$$

where the function v has the asymptotic behaviours

$$v(r \to 0) \to 0 \qquad \text{and} \qquad v(r \gg R_{\rm s}) \to 1\ . \tag{6.29}$$

The effective energy density therefore reads

$$\rho = \frac{M\,v'}{4\,\pi\,r^2}\ , \tag{6.30}$$

so that Eq. (6.29) implies

$$m(r \to \infty) = M \int_0^\infty v'(x)\,\mathrm{d}x = M \ , \tag{6.31}$$

as expected.

In particular, we have $v = \mathrm{erf}(r/R_\mathrm{s})$ and

$$\rho = \frac{M\, e^{-\frac{r^2}{R_\mathrm{s}^2}}}{2\,\pi^{\frac{3}{2}}\, R_\mathrm{s}\, r^2} \ , \tag{6.32}$$

which is the same result one would obtain from the Einstein field equations $G^\mu{}_\nu = 8\,\pi\, G_\mathrm{N}\, T^\mu{}_\nu$, where $G^\mu{}_\nu$ is the Einstein tensor for the quantum corrected metric from Eq. (6.25).

6.3 Horizon quantum mechanics

We are interested in a matter source with energy density (6.32) made of a very large number N of particles. For simplicity, we assume that all particles are identical and have a mass $\mu = M/N$.

The (normalised) wavefunction of each particle in position space can be estimated as

$$\psi_\mathrm{S}(r_i) \propto \rho^{1/2} \propto \frac{e^{-\frac{r_i^2}{2\,R_\mathrm{s}^2}}}{\sqrt{2}\,\pi^{\frac{3}{4}}\, R_\mathrm{s}^{\frac{1}{2}}\, r_i} \ , \tag{6.33}$$

where $i = 1, \ldots, N$. In momentum space, we then have

$$\psi_\mathrm{S}(k_i) = \frac{2\,\pi^{\frac{3}{4}}\, R_\mathrm{s}^{\frac{1}{2}}}{k_i}\, \mathrm{erfi}\left(\frac{k_i\, R_\mathrm{s}}{\sqrt{2}}\right)\, e^{-\frac{k_i^2\, R_\mathrm{s}^2}{2}} \ , \tag{6.34}$$

where erfi is the imaginary error function. Notice that the wavefunction (6.34) peaks around $k = R_\infty^{-1}$, and the imaginary error function can be approximated for $k_i\, R_\mathrm{s} \ll 1$ as

$$\mathrm{erfi}\left(\frac{k_i\, R_\mathrm{s}}{\sqrt{2}}\right) \simeq \sqrt{\frac{2}{\pi}}\, k_i\, R_\mathrm{s} \ . \tag{6.35}$$

Each particle can therefore be assumed in a state described by [3]

$$|\psi_\mathrm{s}^{(i)}\rangle \simeq \mathcal{N}_k \int\limits_{R_\infty^{-1}}^{\infty} \mathrm{d}k_i\, e^{-\frac{k_i^2\, R_\mathrm{s}^2}{2}}\ |k_i\rangle \ , \tag{6.36}$$

[3] Given the approximation (6.35), the expression (6.36) "underestimates" the exact wavefunction at large $k \sim R_\mathrm{s}^{-1}$. The related error can be reduced by decreasing the value of R_s with respect to the (unknown) actual size of the core.

where $\mathcal{N}_k$ is a suitable normalisation factor.

The dynamics of each particle is determined by a Hamiltonian H_i with spectrum

$$\hat{H}_i \,|E_i\rangle = E_i \,|E_i\rangle \ , \tag{6.37}$$

where

$$E_i^2 = \mu^2 + \hbar^2 \, k_i^2 \ . \tag{6.38}$$

Thus, we can rewrite the state (6.36) of each particle as

$$|\psi_{\rm s}^{(i)}\rangle \ \simeq \ \mathcal{N}_E \int_{\mu}^{\infty} {\rm d}E_i \, e^{-\frac{\left(E_i^2-\mu^2\right) R_s^2}{2\, m_{\rm p}^2\, \ell_{\rm p}^2}} \ |E_i\rangle \ , \tag{6.39}$$

where $\mathcal{N}_E$ is also a normalisation factor.

The total wavefunction of the source will be given by the symmetrised product of N such states,

$$|\psi_N\rangle \simeq \frac{1}{N!} \sum_{\{\sigma_i\}}^{N} \left[\bigotimes_{i=1}^{N} |\psi_{\rm s}^{(i)}\rangle \right] \ , \tag{6.40}$$

where the sum is over all the permutations $\{\sigma_i\}$ of the N states.

6.3.1 Source spectral decomposition

The above $|\psi_N\rangle$ can be decomposed into eigenstates $|E\rangle$ of the total Hamiltonian [4]

$$H = \sum_{i=1}^{N} H_i = \sum_{i=1}^{N} \left(\mu^2 + \hbar^2\, k_i^2\right)^{1/2} \ . \tag{6.41}$$

The details of the (approximate analytical) calculation are shown in Appendix C, where we find that $C(E) \equiv \langle E\,|\psi_N\rangle \simeq 0$, for $E < M$, and

$$C(E) \simeq \mathcal{N}_c \left(\frac{E-M}{m_{\rm p}}\right)^{M/\mu} e^{-\frac{R_{\rm s}^2\, \mu\, (E-M)}{\ell_{\rm p}^2\, m_{\rm p}^2}} \ , \tag{6.42}$$

for $E > M$, with the normalisation constant $\mathcal{N}_c = \mathcal{N}_+$ given in Eq. (C.19). This result means that we can describe the quantum state $|\psi_N\rangle$ of our N-particle system by means of the effective one-particle state

$$|\Psi_{\rm S}\rangle \simeq \mathcal{N}_{\rm S} \int_{M}^{\infty} {\rm d}E \left(\frac{E-M}{m_{\rm p}}\right)^{M/\mu} e^{-\frac{R_{\rm s}^2\, \mu\, (E-M)}{\ell_{\rm p}^2\, m_{\rm p}^2}} \ |E\rangle \ , \tag{6.43}$$

[4] Notice that we are assuming that the total energy is just the sum of individual particle energies to parallel the expression (6.3) of the classical mass function.

with $E^2 = M^2 + \hbar^2 k^2$ and $\mathcal{N}_\mathrm{S}$ is a normalisation constant. For example, the expectation value of the total energy can be approximated with its upper bound computed in Eq. (C.27) and reads

$$\langle \hat{H} \rangle \simeq M \left(1 + \frac{m_\mathrm{p}^2 \, \ell_\mathrm{p}^2}{\mu^2 \, R_\mathrm{s}^2} \right) = M \left(1 + \frac{\lambda_\mu^2}{R_\mathrm{s}^2} \right) \ , \tag{6.44}$$

where λ_μ is the Compton length of the constituent particles of mass μ. We notice that the relative correction becomes negligibly small for $R_\mathrm{s} \gg \lambda_\mu$ and diverges for $R_\mathrm{s} \to 0$. This is another indication that no well-defined coherent state exists for a pure Schwarzschild geometry [52].

6.3.2 Horizon wavefunction

We can now obtain the horizon wavefunction from the effective single-particle wavefunction (6.43) by setting $r_\mathrm{H} = 2\, G_\mathrm{N}\, E$ and defining $|r_\mathrm{H}\rangle \propto |2\, \ell_\mathrm{p}\, E/m_\mathrm{p}\rangle$. This yields $\Psi_\mathrm{H}(r_\mathrm{H}) \simeq 0$, for $r_\mathrm{H} < R_\mathrm{H} = 2\, G_\mathrm{N}\, M$, and

$$\Psi_\mathrm{H}(r_\mathrm{H}) \simeq \mathcal{N}_\mathrm{H} \left(\frac{r_\mathrm{H} - R_\mathrm{H}}{\ell_\mathrm{p}} \right)^{\frac{m_\mathrm{p}\, R_\mathrm{H}}{2\, \mu\, \ell_\mathrm{p}}} e^{-\frac{\mu (r_\mathrm{H} - R_\mathrm{H}) R_\mathrm{s}^2}{2\, m_\mathrm{p}\, \ell_\mathrm{p}^3}} \ , \tag{6.45}$$

for $r_\mathrm{H} \geq R_\mathrm{H}$, where the normalisation constant $\mathcal{N}_\mathrm{H}$ is given in Eq. (D.2).

The expectation value of the gravitational radius is computed in Eq. (D.3) and can be written as

$$\langle \hat{r}_\mathrm{H} \rangle \simeq R_\mathrm{H} \left(1 + \frac{\lambda_\mu^2}{R_\mathrm{s}^2} \right) \ , \tag{6.46}$$

which is in perfect agreement with the expression of the energy given in Eq. (6.44). It is again noteworthy that $\langle \hat{r}_\mathrm{H} \rangle > R_\mathrm{H}$, although the correction with respect to the classical expression is negligible for an astrophysical black hole unless the core is of a size comparable to the Compton length λ_μ. It is also important to recall that $\langle \hat{r}_\mathrm{H} \rangle$ is the horizon radius only if the core is sufficiently smaller, as we will determine next.

By means of the effective single-particle wavefunction (6.43) and the horizon wavefunction (6.45), we can numerically compute the probability P_BH defined in Eq. (6.11) that the system lies inside its own gravitational radius and is a black hole, as reviewed in Section 6.1. More details of the calculation are given in Appendix D, where we show that the final expression of P_BH can only be estimated numerically. Some cases are displayed in Figs. 6.1-6.3, with values of R_H, R_s and μ chosen for clarity, albeit they fall far from any astrophysical regimes. From those graphs, it appears that the probability increases for decreasing size R_s of the core and for increasing (decreasing) mass M (μ) (equivalent

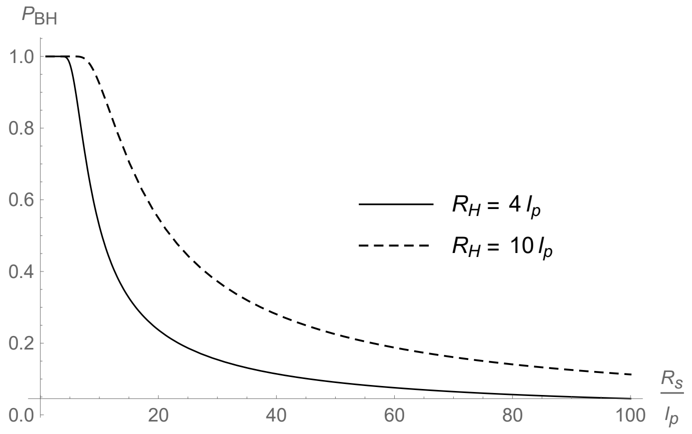

Figure 6.1: Probability that the coherent state is a black hole as a function of $R_{\rm s}$ for different values of $R_{\rm H}$ (and same value of $\mu = 0.2\, m_{\rm p}$).

to increasing $R_{\rm H} = 2\, G_{\rm N}\, M$ or the number $N = M/\mu$ of matter particles). For example, a core of size $R_{\rm s} = 10\, \ell_{\rm p}$ can be a black hole of radius $R_{\rm H} = 10\, \ell_{\rm p}$ with probability $P_{\rm BH} \gtrsim 0.9$ but this probability drops to $P_{\rm BH} \lesssim 0.5$ if $R_{\rm H} = 4\, \ell_{\rm p}$. This result is in qualitative agreement with the expectation (6.46) for very massive black holes with cores larger than λ_μ, but smaller than the classical gravitational radius $R_{\rm H}$.

6.4 Conclusions

We have here employed the formalism of the horizon quantum mechanics [132] in order to verify that coherent state black hole geometries of the Schwarzschild type sourced by a core of large mass M and with a size $R_{\rm s}$ larger than Planckian are very likely to display an outer horizon and be black holes in the usual sense. For that purpose, we needed to find an explicit description of the (electrically neutral and spherically symmetric) matter core in terms of a many-particle state that was then expressed as a superposition of total energy eigenstates. Our analysis supports the conclusion that the system is indeed a black hole of mass $M \gg m_{\rm p}$ if its core made of particles of mass μ has a size $R_{\rm s} \gtrsim \lambda_\mu \gg \ell_{\rm p}$ but (sufficiently) smaller than the classical gravitational radius $R_{\rm H} = 2\, G_{\rm N}\, M$.

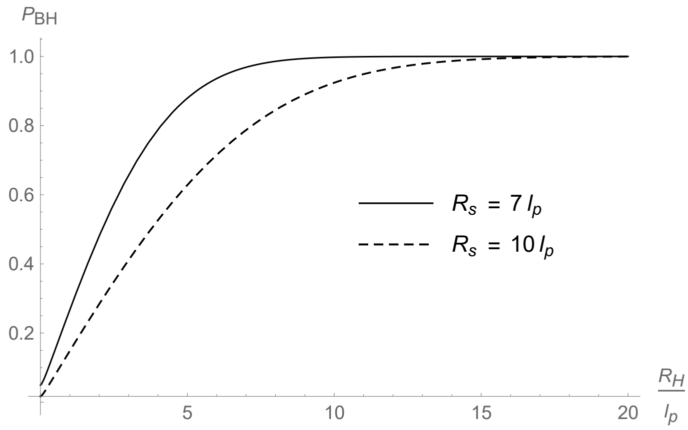

Figure 6.2: Probability that the coherent state is a black hole as a function of $R_{\rm H}$ for different values of $R_{\rm s}$ (and same value of $\mu = 0.2\, m_{\rm p}$).

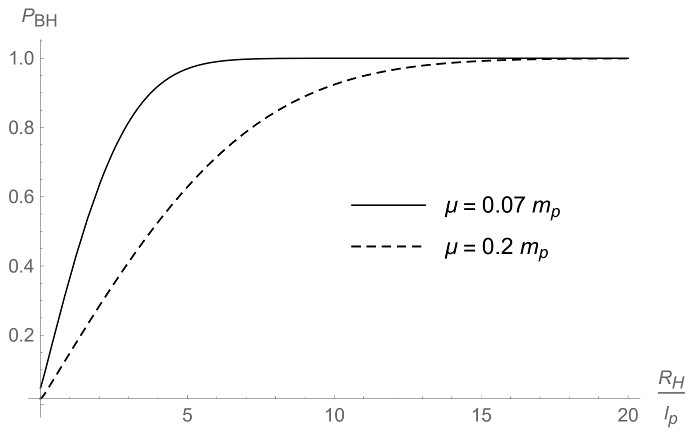

Figure 6.3: Probability that the coherent state is a black hole as a function of $R_{\rm H}$ for different values of μ (and same value of $R_{\rm s} = 10\, \ell_{\rm p}$).

Chapter 7

Conclusions and outlook

Among the various gravitational systems described by solutions to the Einstein field equations, black holes are the most intriguing. However, as one of the most outstanding predictions of General Relativity, these spacetimes contain singularities, which might indicate the breakdown of classical physics in the strong field regime, where physical quantities become infinite. Therefore, it is unavoidable to pursue a theory that resolves this singularity problem. It is widely believed that a theory of quantum gravity for the gravitational interaction will provide a mechanism to fix this inconsistency and enhance our understanding of the fundamental mechanisms governing the universe. Although many candidate theories of quantum gravity have been proposed, developing a fully ultraviolet-complete theory remains a significant challenge. Indeed, one could argue that if we could formulate an effective quantum theory of gravity that unifies all experimental evidence into a single consistent framework, then such a theory would carry a flavour of quantum gravity.

The so called corpuscular black hole model assumes that the geometrical description emerges at suitable scales from the underlying quantum field theory of gravitons. This picture regards black holes as bound states of gravitons. In this context, the number of gravitons N is the only parameter of the theory, which can be considered as a measure of the "classicality" of the system. The original proposal, however, totally neglects the role of matter, whose effects are argued to be unimportant, and views a black hole as a quantum state made of only gravitons with one typical wavelength, which cannot reproduce the gravitational field in the accessible outer spacetime, even in the simple Newtonian approximation. By following this perspective, an improved description of nonuniform geometries can be obtained by employing coherent states of gravitons, which recovers the Newtonian potential (with necessary departures) from a coherent state for a scalar field of gravitons in flat spacetime. The Newton potential could then used to reconstruct the full Schwarzschild metric function through a mean field approach.

In this thesis, we presented the coherent quantum black hole models and the horizon quantum mechanics formalism. Let us now summarise the main points addressed in this work.

First, in Chapter 2 we have reviewed the corpuscular pitcure which views a black hole as a self-sustained quantum state by means of an effective scalar theory for the toy gravitons that gives back the Einstein geometrical theory as emerging. In particular, it is demonstrated that a black hole is modellde as a Bose-Einstein condensate of marginally bound, self-interacting gravitons, whose size is given by the characteristic Compton-de Brogle wavelength $\lambda_{\mathrm{G}} \sim R_{\mathrm{H}}$ and whose depth is proportional to the very large number N of soft quanta in this condensate. The classical description of a Schwarzschild black hole is recovered when the number of constituents N is large. Furthernore, this picture offers a natural way to reproduce Hawking radiation as a quantum depletion of the graviton Bose-condensate, which reproduces a thermal spectrum of temperature $T = 1/\sqrt{N}$. Bekenstein's conjecture for the horizon area quantisation naturally follows for the occupation number of gravitons is proportional to the square of the ADM mass of the black hole in units of the Planck mass m_{p}.

In Chapter 3, following the historical development of coherent states, we introduced their unique properties that are fundamental to field theory. These coherent states statisfy the Heisenberg uncertainty relation with equality and the set of coherent states is an over-complete set of states. Since coherent states remain coherent under time evolution, mean values of the dynamical variables of position and momentum satisfy the equations of motion for the classical harmonic oscillator that closely mimics their counterparts in classical mechanics. Given the unique properties of coherent behaviour, coherent states may bring us some new lights to the context of the effective quantum theory of gravity.

Since the corpuscular pitcure is rather qualitative, and neglects the role of matter in the condensate formation, in Chapter 4, we employed the coherent state of a massless scalar field on a reference flat spacetime to describe the static and spherically sysmmetric Schwarzschild geometry. The expected behaviour in the weak field region outside the horizon is recovered, with quantum corrections to the Newtonian potential, and consequently, to the Schwarzschild metric function. In particular, the expectation value of the gravitational potential displays oscillations around the classical V_{N}. Besacse of the necessary departures from the classical Schwarzschild metric, the central singularity of the Schwarzschild black hole is relaced by an integrable singularity without Cauchy horizons. These deviations from the classical black hole geometry can be viewed as quantum hair and lead to a quantum corrected horizon radius.

Given that the majority of black holes in nature are very likely to spin, we investigated quantum descriptions of black holes with non-vanishing specific angular momentum in Chapter 5. In particular, we study the quantum hair associated with coherent states describing slowly rotating black holes and show how it can be naturally related with the Bekenstein-Hawking entropy and with 1-loop quantum corrections of the metric for the (effectively) non-rotating case. We also estimate corrections induced by such quantum hair to the temperature of the Hawking radiation through the tunnelling method.

In Chapter 6, we have provided a concise review of the key features of the horizon quantum mechanics formalism. This formalism is then applied to electrically neutral and spherically symmetric black hole geometries emerging from coherent quantum states of gravity to compute the probability that the matter source is inside the horizon. We find that quantum corrections to the classical horizon radius become significant if the matter core has a size comparable to the Compton length of the constituents and the system is indeed a black hole with probability very close to one unless the core radius is close to the (classical) gravitational radius.

Finally, we want to give some hints for future developments. There are different directions along which the results in Chapter 5 could be improved and developed. First of all, results regarding the Hawking evaporation can be straightforwardly generalised to massive bosons and fermions [124]. One could furthermore study other black hole solutions that can emerge from coherent quantum states and eventually attempt at a quantum description of black holes with arbitrary angular momentum [98]. It would also be interesting to generalise the analysis in Chapter 6 to include electric charge and rotation. Whereas the former case should be rather straightforward, including rotation is going to be much more problematic since it will require extending the horizon quantum mechanics beyond the perturbative regime considered in Ref. [141].

Appendix A

Normalisations and conventions

We summarise here the convention we use in the main text. Projections on the spatial part of the normal modes (5.6) are defined as

$$\tilde{f}_{\ell m}(\omega) = \int_{-1}^{+1} \mathrm{d}\cos\theta \int_{0}^{2\pi} \mathrm{d}\varphi \int_{0}^{\infty} r^2\, \mathrm{d}r\, j_\ell(\omega\, r)\, [Y_\ell^m(\theta,\varphi)]^*\, f(r,\theta,\varphi) \ . \tag{A.1}$$

The orthonormality relations (5.8) then follow from the orthonormality of spherical Bessel functions,

$$\int_{0}^{\infty} r^2\, \mathrm{d}r\, j_\ell(\omega\, r)\, j_{\ell'}(\omega'\, r) = \frac{\pi}{2\,\omega^2}\, \delta(\omega-\omega')\, \delta_{\ell\ell'} \ , \tag{A.2}$$

as well as the orthonormality of spherical harmonics,

$$\int_{-1}^{+1} \mathrm{d}\cos\theta \int_{0}^{2\pi} \mathrm{d}\varphi\, Y_\ell^m(\theta,\varphi) \left[Y_{\ell'}^{m'}(\theta,\varphi)\right]^* = \delta_{\ell\ell'}\, \delta_{mm'} \ . \tag{A.3}$$

The commutation relations (5.12) and (5.13) follow from the completeness relations

$$\frac{2}{\pi} \int_{0}^{\infty} \omega^2\, \mathrm{d}\omega\, j_\ell(\omega\, r)\, j_\ell(\omega\, r') = \frac{\delta(r-r')}{r^2} \tag{A.4}$$

and

$$\sum_{\ell} \sum_{m=-\ell}^{\ell} Y_\ell^m(\theta,\varphi)\, [Y_\ell^m(\theta',\varphi')]^* = \frac{\delta(\theta-\theta')}{\sin\theta}\, \delta(\varphi-\varphi') \ . \tag{A.5}$$

Other useful properties of spherical harmonics are given by

$$[Y_\ell^m]^* = (-1)^m \, Y_\ell^{-m} \tag{A.6}$$

and

$$P_\ell^{-m} = (-1)^m \, \frac{(\ell - m)!}{(\ell + m)!} \, P_\ell^m \ . \tag{A.7}$$

From

$$P_\ell^\ell = \frac{(-1)^\ell}{2^\ell \, \ell!} \sqrt{\frac{(2\,\ell + 1)!}{4\,\pi}} \, (\sin\theta)^\ell \ , \tag{A.8}$$

we then obtain

$$P_\ell^{-\ell} = \frac{1}{2^\ell \, \ell! \, (2\,\ell)!} \sqrt{\frac{(2\,\ell + 1)!}{4\,\pi}} \, (\sin\theta)^\ell \ . \tag{A.9}$$

In all of the above expressions, the Kronecker delta is defined by $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. The Dirac delta is defined by

$$\int \mathrm{d}z \, \delta(z - z_0) \, f(z) = f(z_0) \ , \tag{A.10}$$

where integration is assumed on the natural domain of the variable z.

Relevant integrals of the spherical Bessel functions are given by

$$\begin{aligned} \int_0^x z^\alpha \, \mathrm{d}z \, j_\ell(z) \ &= \ \frac{\sqrt{\pi}}{2^{\ell+2}} \, \frac{\Gamma((1 + \alpha + \ell)/2)}{\Gamma(3/2 + \ell) \, \Gamma((3 + \alpha + \ell)/2)} \\ &\quad \times {}_1F_2((1 + \alpha + \ell)/2, \ell + 3/2, (3 + \alpha + \ell)/2, -x^2/4) \ , \end{aligned} \tag{A.11}$$

where ${}_1F_2$ is the generalised hypergeometric function. In particular, for $\alpha = 1$, we have

$$\int_0^x z \, \mathrm{d}z \, j_\ell(z) = \frac{\sqrt{\pi}}{2^{\ell+2}} \, \frac{\Gamma(1 + \ell/2)}{\Gamma(3/2 + \ell) \, \Gamma(2 + \ell/2)} \, {}_1F_2(1 + \ell/2, \ell + 3/2, 2 + \ell/2, -x^2/4) \ . \tag{A.12}$$

Appendix B

Angular momentum

The normal modes (5.6) are eigenfunctions of the angular momentum, that is

$$\hat{L}^2\, u_{\omega\ell m} = \hbar^2\, \ell\,(\ell+1)\, u_{\omega\ell m} \qquad \text{and} \qquad \hat{L}_z\, u_{\omega\ell m} = \hbar\, m\, u_{\omega\ell m}\ . \tag{B.1}$$

It then follows that

$$\hat{L}^2\, |1_{\ell m}(\omega)\rangle = \hbar^2\, \ell\,(\ell+1)\, |1_{\ell m}(\omega)\rangle \qquad \text{and} \qquad \hat{L}_z\, |1_{\ell m}(\omega)\rangle = \hbar\, m\, |1_{\ell m}(\omega)\rangle\ , \tag{B.2}$$

where $|1_{\ell m}(\omega)\rangle = \hat{a}^\dagger_{\ell m}(\omega)\,|0\rangle$. We can also write the first relation as defining the operator

$$\sqrt{\hat{L}^2}\, |1_{\ell m}(\omega)\rangle = \hbar\, \sqrt{\ell\,(\ell+1)}\, |1_{\ell m}(\omega)\rangle\ . \tag{B.3}$$

Likewise, we have

$$\sqrt{\hat{L}^2}\, |n_{\ell m}(\omega)\rangle = \hbar\, \sqrt{\ell\,(\ell+1)}\, n_{\ell m}\, |n_{\ell m}(\omega)\rangle \quad \text{and} \quad \hat{L}_z\, |n_{\ell m}(\omega)\rangle = \hbar\, m\, n_{\ell m}\, |n_{\ell m}(\omega)\rangle\ , \tag{B.4}$$

where $|n_{\ell m}(\omega)\rangle = (n!)^{-1/2}\left[\hat{a}^\dagger_{\ell m}(\omega)\right]^n |0\rangle$ (with $n = n_{\ell m}$ for brevity).

Let us consider a coherent state of fixed ω (which we omit for simplicity), ℓ and m,

$$\begin{aligned} |g_{\ell m}\rangle &= e^{-N_{\ell m}/2}\, \exp\left\{g_{\ell m}\, \hat{a}^\dagger_{\ell m}\right\} |0\rangle \\ &= e^{-N_{\ell m}/2} \sum_n \frac{\left(g_{\ell m}\, \hat{a}^\dagger_{\ell m}\right)^n}{n!}\, |0\rangle \\ &= e^{-N_{\ell m}/2} \sum_n \frac{g^n_{\ell m}}{\sqrt{n!}}\, |n_{\ell m}\rangle\ . \end{aligned} \tag{B.5}$$

From $\langle\, n_{\ell m} \mid n'_{\ell m}\,\rangle = \delta_{nn'}$, the normalisation

$$\langle\, g_{\ell m} \mid g_{\ell m}\,\rangle = e^{-N_{\ell m}} \sum_n \frac{g^{2\,n}_{\ell m}}{n!} = 1 \tag{B.6}$$

implies $N_{\ell m} = g_{\ell m}^2$. From Eq. (B.4), we then find

$$\begin{aligned}
\langle g_{\ell m}| \sqrt{\hat{L}^2}\, |g_{\ell m}\rangle &= e^{-g_{\ell m}^2} \sum_{n,s} \frac{g_{\ell m}^s}{\sqrt{s!}} \frac{g_{\ell m}^n}{\sqrt{n!}} \langle s_{\ell m}| \sqrt{\hat{L}^2}\, |n_{\ell m}\rangle \\
&= e^{-g_{\ell m}^2} \sum_{n_{\ell m}} \frac{g_{\ell m}^{2\, n_{\ell m}}}{n_{\ell m}!} \hbar \sqrt{\ell\,(\ell+1)}\, n_{\ell m} \\
&= e^{-g_{\ell m}^2}\, \hbar \sqrt{\ell\,(\ell+1)} \sum_{n_{\ell m}} \frac{g_{\ell m}^{2\, n_{\ell m}}}{(n_{\ell m}-1)!} \\
&= \hbar \sqrt{\ell\,(\ell+1)}\, g_{\ell m}^2\, e^{-g_{\ell m}^2} \sum_{n} \frac{g_{\ell m}^{2\, n}}{n!} \\
&= \hbar \sqrt{\ell\,(\ell+1)}\, N_{\ell m} \ ,
\end{aligned} \tag{B.7}$$

which is Eq. (5.26) with $N_{\ell m} = g_{\ell m}^2(\omega)$. Likewise,

$$\langle g_{\ell m}| \hat{L}_z\, |g_{\ell m}\rangle = \hbar\, m\, N_{\ell m} \tag{B.8}$$

which is Eq. (5.27).

Appendix C

Spectral decomposition and total energy

Here we show how the total wavefunction (6.40), that is

$$|\psi_N\rangle \simeq \frac{1}{N!}\sum_{\{\sigma_i\}}^{N}\left[\bigotimes_{i=1}^{N}\mathcal{N}_E\int_{\mu}^{\infty}\mathrm{d}E_i\, e^{-\frac{\left(E_i^2-\mu^2\right)R_s^2}{2\,m_\mathrm{p}^2\,\ell_\mathrm{p}^2}}\;|E_i\rangle\right]\;, \tag{C.1}$$

can be decomposed in terms of the total energy eigenstates $|E\rangle$ by computing the spectral coefficients $C(E)\equiv\langle\, E\mid\psi_N\,\rangle$. From Eq. (C.1), we first find

$$\begin{aligned}C(E) &= \frac{1}{N!}\,\langle E|\sum_{\{\sigma_i\}}^{N}\left[\bigotimes_{i=1}^{N}\mathcal{N}_E\int_{\mu}^{\infty}\mathrm{d}E_i\, e^{-\frac{\left(E_i^2-\mu^2\right)R_s^2}{2\,m_\mathrm{p}^2\,\ell_\mathrm{p}^2}}\;|E_i\rangle\right]\\ &= \frac{\mathcal{N}_E^N}{N!}\int_{\mu}^{\infty}\mathrm{d}E_1\cdots\int_{\mu}^{\infty}\mathrm{d}E_N\left[\prod_{i=1}^{N}e^{-\frac{\left(E_i^2-\mu^2\right)R_s^2}{2\,m_\mathrm{p}^2\,\ell_\mathrm{p}^2}}\right]\delta\!\left(E-\sum_{i=1}^{N}E_i\right)\;.\end{aligned} \tag{C.2}$$

Since $\sum_{i=1}^{N}E_i\geq N\,\mu=M$, it follows that $C(E<M)=0$. For $E\geq M$, we can use $E_N=\sum_{i=1}^{N-1}E_i$ and write

$$C(E)\propto\int_{\mu}^{\infty}\mathrm{d}E_1\cdots\int_{\mu}^{\infty}\mathrm{d}E_{N-1}\exp\left\{-\sum_{i=1}^{N-1}\frac{\left(E_i^2-\mu^2\right)R_s^2}{2\,m_\mathrm{p}^2\,\ell_\mathrm{p}^2}-\frac{\left[\left(E-\sum_{i=1}^{N-1}E_i\right)^2-\mu^2\right]R_s^2}{2\,m_\mathrm{p}^2\,\ell_\mathrm{p}^2}\right\}\;. \tag{C.3}$$

It is now convenient to define the function

$$
\begin{aligned}
F(E,E_i) &\equiv \sum_{i=1}^{N-1}\left(E_i^2-\mu^2\right)+\left(E-\sum_{i=1}^{N-1}E_i\right)^2-\mu^2 \\
&= \sum_{i=1}^{N-1}(\mathcal{E}_i+\mu)^2-N\,\mu^2+\left[E-\sum_{i=1}^{N-1}\mathcal{E}_i-(N-1)\,\mu\right]^2 \ , \qquad \text{(C.4)}
\end{aligned}
$$

where $\mathcal{E}_i = E_i - \mu$. By recalling that $M = N\,\mu$, we then obtain

$$
\begin{aligned}
F(E,E_i) &= \sum_{i=1}^{N-1}(\mathcal{E}_i+\mu)^2-\mu\,M+\left[(E-M)-\sum_{i=1}^{N-1}\mathcal{E}_i+\mu\right]^2 \\
&= (E-M)^2+\left(\sum_{i=1}^{N-1}\mathcal{E}_i\right)^2+\mu^2-2\,(E-M)\sum_{i=1}^{N-1}\mathcal{E}_i+2\,\mu\,(E-M)-2\,\mu\sum_{i=1}^{N-1}\mathcal{E}_i \\
&\quad +\sum_{i=1}^{N-1}\mathcal{E}_i^2+2\,\mu\sum_{i=1}^{N-1}\mathcal{E}_i+(N-1)\,\mu^2-\mu\,M \\
&= (E-M)^2+2\,\mu\,(E-M)-2\,(E-M)\sum_{i=1}^{N-1}\mathcal{E}_i+\left(\sum_{i=1}^{N-1}\mathcal{E}_i\right)^2+\sum_{i=1}^{N-1}\mathcal{E}_i^2 \\
&= [E-(M-\mu)]^2-\mu^2-2\,(E-M)\sum_{i=1}^{N-1}\mathcal{E}_i+\left(\sum_{i=1}^{N-1}\mathcal{E}_i\right)^2+\sum_{i=1}^{N-1}\mathcal{E}_i^2 \ . \qquad \text{(C.5)}
\end{aligned}
$$

Plugging this result into Eq. (C.3) yields

$$
\begin{aligned}
C(E) &\propto e^{-\frac{R_s^2}{2\,\ell_p^2\,m_p^2}\left\{[E-(M-\mu)]^2-\mu^2\right\}}\int_0^\infty \mathrm{d}\mathcal{E}_1\cdots\int_0^\infty \mathrm{d}\mathcal{E}_{N-1} \\
&\qquad \times\exp\left\{\frac{R_s^2}{2\,\ell_p^2\,m_p^2}\left[2\,(E-M)\sum_{i=1}^{N-1}\mathcal{E}_i-\left(\sum_{i=1}^{N-1}\mathcal{E}_i\right)^2-\sum_{i=1}^{N-1}\mathcal{E}_i^2\right]\right\} \\
&\equiv e^{-\frac{R_s^2}{2\,\ell_p^2\,m_p^2}\left\{[E-(M-\mu)]^2-\mu^2\right\}}\,I(E,M) \ . \qquad \text{(C.6)}
\end{aligned}
$$

We next note that, since $\mathcal{E}_i \geq 0$ for $i = 1, \ldots, N-1$, we have

$$
0\le\sum_{i=1}^{N-1}\mathcal{E}_i^2\le\left(\sum_{i=1}^{N-1}\mathcal{E}_i\right)^2 \ . \qquad \text{(C.7)}
$$

A lower bound $I_- \leq I$ is obtained from the upper bound $\sum_i \mathcal{E}_i^2 = (\sum_i \mathcal{E}_i)^2$ in Eq. (C.7) and is given by

$$
\begin{aligned}
I_- &= \int_0^\infty \mathrm{d}\mathcal{E}_1 \cdots \int_0^\infty \mathrm{d}\mathcal{E}_{N-1} \exp\left[\frac{R_s^2\,(E-M)}{\ell_\mathrm{p}^2\,m_\mathrm{p}^2} \sum_{i=1}^{N-1} \mathcal{E}_i\right] \exp\left[-\frac{R_s^2}{\ell_\mathrm{p}^2\,m_\mathrm{p}^2} \left(\sum_{i=1}^{N-1} \mathcal{E}_i\right)^2\right] \\
&\propto \int_0^\infty \mathcal{E}^{N-2} \mathrm{d}\mathcal{E} \exp\left[-\frac{R_s^2\,\mathcal{E}^2}{\ell_\mathrm{p}^2\,m_\mathrm{p}^2} + \frac{R_s^2\,(E-M)\,\mathcal{E}}{\ell_\mathrm{p}^2\,m_\mathrm{p}^2}\right] \\
&= \frac{1}{2}\left(\frac{R_\mathrm{s}}{m_\mathrm{p}\,\ell_\mathrm{p}}\right)^{1-N} \Gamma\!\left(\frac{N-1}{2}\right) {}_1F_1\left[\frac{N-1}{2},\frac{1}{2},\frac{(E-M)^2\,R_\mathrm{s}^2}{4\,m_\mathrm{p}^2\,\ell_\mathrm{p}^2}\right] \\
&\quad +\frac{1}{2}\left(\frac{R_\mathrm{s}}{m_\mathrm{p}\,\ell_\mathrm{p}}\right)^{1-N} \frac{(E-M)\,R_\mathrm{s}}{m_\mathrm{p}\,\ell_\mathrm{p}} \Gamma\!\left(\frac{N}{2}\right) {}_1F_1\left[\frac{N}{2},\frac{3}{2},\frac{(E-M)^2\,R_\mathrm{s}^2}{4\,m_\mathrm{p}^2\,\ell_\mathrm{p}^2}\right] ,
\end{aligned}
\tag{C.8}
$$

where Γ is the Euler gamma function and ${}_1F_1$ the Kummer confluent hypergeometric function. An upper bound $I \leq I_+$ is likewise obtained from the lower bound $\sum_i \mathcal{E}_i^2 = 0$ in Eq. (C.7) and is given by

$$
\begin{aligned}
I_+ &= \int_0^\infty \mathrm{d}\mathcal{E}_1 \cdots \int_0^\infty \mathrm{d}\mathcal{E}_{N-1} \exp\left[\frac{R_s^2\,(E-M)}{\ell_\mathrm{p}^2\,m_\mathrm{p}^2} \sum_{i=1}^{N-1} \mathcal{E}_i\right] \exp\left[-\frac{R_s^2}{2\,\ell_\mathrm{p}^2\,m_\mathrm{p}^2} \left(\sum_{i=1}^{N-1} \mathcal{E}_i\right)^2\right] \\
&\propto \int_0^\infty \mathcal{E}^{N-2} \mathrm{d}\mathcal{E} \exp\left[-\frac{R_s^2\,\mathcal{E}^2}{2\,\ell_\mathrm{p}^2\,m_\mathrm{p}^2} + \frac{R_s^2\,(E-M)\,\mathcal{E}}{\ell_\mathrm{p}^2\,m_\mathrm{p}^2}\right] \\
&= \frac{1}{2^{\frac{N}{2}-\frac{3}{2}}}\left(\frac{R_\mathrm{s}}{m_\mathrm{p}\,\ell_\mathrm{p}}\right)^{1-N} \Gamma\!\left(\frac{N-1}{2}\right) {}_1F_1\left[\frac{N-1}{2},\frac{1}{2},\frac{(E-M)^2\,R_\mathrm{s}^2}{2\,m_\mathrm{p}^2\,\ell_\mathrm{p}^2}\right] \\
&\quad +\frac{1}{2^{\frac{N}{2}-1}}\left(\frac{R_\mathrm{s}}{m_\mathrm{p}\,\ell_\mathrm{p}}\right)^{1-N} \frac{(E-M)\,R_\mathrm{s}}{m_\mathrm{p}\,\ell_\mathrm{p}} \Gamma\!\left(\frac{N}{2}\right) {}_1F_1\left[\frac{N}{2},\frac{3}{2},\frac{(E-M)^2\,R_\mathrm{s}^2}{2\,m_\mathrm{p}^2\,\ell_\mathrm{p}^2}\right] .
\end{aligned}
\tag{C.9}
$$

For $R_\mathrm{s} \gg \ell_\mathrm{p}$ and $(E-M) \gtrsim m_\mathrm{p}$, we can employ the asymptotic behaviour of the Kummer confluent hypergeometric function,

$$
{}_1F_1\,(a,b,x) \sim x^{a-b}\,e^x ,
\tag{C.10}
$$

for $x \sim (E-M)^2\,R_\mathrm{s}^2/\ell_\mathrm{p}^2\,m_\mathrm{p}^2 \gg 1$, which leads to

$$
\begin{aligned}
I_- &\simeq \frac{1}{2^{N-2}}\,\frac{m_\mathrm{p}\,\ell_\mathrm{p}}{R_\mathrm{s}}\,(E-M)^{N-2}\,e^{\frac{(E-M)^2 R_\mathrm{s}^2}{4\,m_\mathrm{p}^2\,\ell_\mathrm{p}^2}} \left[2\,\Gamma\!\left(\frac{N-1}{2}\right) + \Gamma\!\left(\frac{N}{2}\right)\right] \\
&\simeq \Gamma\!\left(\frac{N-1}{2}\right) \frac{m_\mathrm{p}\,\ell_\mathrm{p}}{2^{N-1}\,R_\mathrm{s}}\,(E-M)^{N-2}\,e^{\frac{(E-M)^2 R_\mathrm{s}^2}{4\,m_\mathrm{p}^2\,\ell_\mathrm{p}^2}} ,
\end{aligned}
\tag{C.11}
$$

where we also used $2\,\Gamma\left((N-1)/2\right) > \Gamma\left(N/2\right)$ for $N \gg 1$ in the last step. Likewise,

$$
\begin{aligned}
I_+ &\simeq \frac{m_\mathrm{p}\,\ell_\mathrm{p}}{\sqrt{2}\,R_\mathrm{s}}\,(E-M)^{N-2}\,e^{\frac{(E-M)^2 R_\mathrm{s}^2}{2\,m_\mathrm{p}^2\,\ell_\mathrm{p}^2}}\left[\Gamma\left(\frac{N-1}{2}\right)+2\,\Gamma\left(\frac{N}{2}\right)\right] \\
&\simeq \sqrt{2}\,\Gamma\left(\frac{N}{2}\right)\frac{m_\mathrm{p}\,\ell_\mathrm{p}}{R_\mathrm{s}}\,(E-M)^{N-2}\,e^{\frac{(E-M)^2 R_\mathrm{s}^2}{2\,m_\mathrm{p}^2\,\ell_\mathrm{p}^2}}\ ,
\end{aligned}
\tag{C.12}
$$

where we used $2\,\Gamma(N/2) > \Gamma((N-1)/2)$ for $N \gg 1$.

Therefore, we have

$$
\Gamma\!\left(\frac{N-1}{2}\right)\frac{m_\mathrm{p}\,\ell_\mathrm{p}}{2^{N-1}\,R_\mathrm{s}}\,(E-M)^{N-2}\,e^{\frac{(E-M)^2 R_\mathrm{s}^2}{4\,m_\mathrm{p}^2\,\ell_\mathrm{p}^2}} \lesssim I \lesssim \sqrt{2}\,\Gamma\!\left(\frac{N}{2}\right)\frac{m_\mathrm{p}\,\ell_\mathrm{p}}{R_\mathrm{s}}\,(E-M)^{N-2}\,e^{\frac{(E-M)^2 R_\mathrm{s}^2}{2\,m_\mathrm{p}^2\,\ell_\mathrm{p}^2}}
\tag{C.13}
$$

We might note that the above approximation fails for $0 < (E-M) \lesssim m_\mathrm{p}$ at fixed value of $R_\mathrm{s} \gtrsim \ell_\mathrm{p}$, for which we instead find the Taylor expansion

$$
I_+ \sim I_- \propto 1 + \mathcal{O}\left(\frac{E-M}{m_\mathrm{p}}\right)\ .
\tag{C.14}
$$

However, for an astrophysical system of mass $M \gg m_\mathrm{p}$, this regime can be discarded overall.

By recalling that $N = M/\mu \gg 1$, we finally obtain the bounding functions

$$
C_-(E) = \mathcal{N}_-\left(\frac{E-M}{m_\mathrm{p}}\right)^{M/\mu} e^{-\frac{R_\mathrm{s}^2\,\mu\,(E-M)}{\ell_\mathrm{p}^2\,m_\mathrm{p}^2}}\ e^{-\frac{R_\mathrm{s}^2 (E-M)^2}{4\,m_\mathrm{p}^2\,\ell_\mathrm{p}^2}}
\tag{C.15}
$$

and

$$
C_+(E) \;=\; \mathcal{N}_+\left(\frac{E-M}{m_\mathrm{p}}\right)^{M/\mu} e^{-\frac{R_\mathrm{s}^2\,\mu\,(E-M)}{\ell_\mathrm{p}^2\,m_\mathrm{p}^2}}\ .
\tag{C.16}
$$

The normalizations $\mathcal{N}_\pm$ can be obtained from the condition

$$
1 = \int_M^\infty C_\pm^2(E)\,\mathrm{d}E\ ,
\tag{C.17}
$$

yielding

$$
\begin{aligned}
\mathcal{N}_-^{-2} &= \left(\frac{\ell_\mathrm{p}}{\sqrt{2}\,R_\mathrm{s}}\right)^{\frac{2\,M}{\mu}}\Gamma\!\left(1+\frac{2\,M}{\mu}\right)U\!\left(1+\frac{M}{\mu},\frac{3}{2},\frac{2\,\mu^2\,R_\mathrm{s}^2}{m_\mathrm{p}^2\,\ell_\mathrm{p}^2}\right) \\
&\simeq \left(\frac{\ell_\mathrm{p}}{\sqrt{2}\,R_\mathrm{s}}\right)^{\frac{2\,M}{\mu}}\Gamma\!\left(\frac{2\,M}{\mu}\right)U\!\left(\frac{M}{\mu},\frac{3}{2},\frac{2\,\mu^2\,R_\mathrm{s}^2}{m_\mathrm{p}^2\,\ell_\mathrm{p}^2}\right)\ ,
\end{aligned}
\tag{C.18}
$$

where $U = U(a, b, x)$ is the Tricomi confluent hypergeometric function, and

$$\begin{aligned} \mathcal{N}_{+}^{-2} &= m_{\rm p} \left(\frac{m_{\rm p}\, \ell_{\rm p}^2}{2\,\mu\, R_{\rm s}^2} \right)^{1+\frac{2\,M}{\mu}} \Gamma\!\left(1 + \frac{2\,M}{\mu}\right) \\ &\simeq m_{\rm p} \left(\frac{m_{\rm p}\, \ell_{\rm p}^2}{2\,\mu\, R_{\rm s}^2} \right)^{\frac{2\,M}{\mu}} \Gamma\!\left(\frac{2\,M}{\mu}\right) \ . \end{aligned} \tag{C.19}$$

In Section 6.3.1, we use the upper bounding function (C.16) in order to estimate the maximum corrections to the total energy. In fact, the bounds from the spectral coefficients can be used to bound the expectation value of the total energy as

$$M + H_{-} \lesssim \langle\, \hat{H}\, \rangle \lesssim M + H_{+} \ , \tag{C.20}$$

where

$$\begin{aligned} H_{+} &= \int_{M}^{\infty} C_{+}^{2}(E)\, E\, {\rm d}E - M \\ &= \int_{0}^{\infty} C_{+}^{2}(\mathcal{E})\, \mathcal{E}\, {\rm d}\mathcal{E} \\ &= \frac{\ell_{\rm p}^2\, m_{\rm p}^2\, (2\,M + \mu)}{2\,\mu^2\, R_{\rm s}^2} \\ &\simeq M\, \frac{m_{\rm p}^2\, \ell_{\rm p}^2}{\mu^2\, R_{\rm s}^2} \end{aligned} \tag{C.21}$$

and

$$H_{-} = \int_{0}^{\infty} C_{-}^{2}(\mathcal{E})\, \mathcal{E}\, {\rm d}\mathcal{E} = \frac{\ell_{\rm p}^2\, m_{\rm p}^2\, (2\,M + \mu)\, U\!\left(1 + \frac{M}{\mu}, \frac{1}{2}, \frac{2\,\mu^2\, R_{\rm s}^2}{m_{\rm p}^2\, \ell_{\rm p}^2}\right)}{2\,\mu^2\, R_{\rm s}^2\, U\!\left(1 + \frac{M}{\mu}, \frac{3}{2}, \frac{2\,\mu^2\, R_{\rm s}^2}{m_{\rm p}^2\, \ell_{\rm p}^2}\right)} \ . \tag{C.22}$$

Employing the definition of the Tricomi confluent hypergeometric function,

$$U(a,b,x) = \frac{\Gamma\,(1-b)}{\Gamma\,(a+1-b)}\, {}_1F_1\ (a,b,x) + \frac{\Gamma\,(b-1)}{\Gamma\,(a)}\, x^{1-b}\, {}_1F_1\ (a+1-b, 2-b, x) \ , \tag{C.23}$$

and the asymptotic behaviour of the Kummer confluent hypergeometric function (C.10), we find

$$U\!\left(1 + \frac{M}{\mu}, \frac{1}{2}, \frac{2\,\mu^2\, R_{\rm s}^2}{m_{\rm p}^2\, \ell_{\rm p}^2}\right) \simeq \left[\frac{1}{\Gamma\!\left(\frac{M}{\mu} + \frac{3}{2}\right)} - \frac{2}{\Gamma\!\left(\frac{M}{\mu} + 1\right)} \right] \Gamma\!\left(\frac{1}{2}\right) \left(\frac{\sqrt{2}\,\mu\, R_{\rm s}}{m_{\rm p}\, \ell_{\rm p}} \right)^{1+2\,\frac{M}{\mu}} e^{\frac{2\,\mu^2\, R_{\rm s}^2}{m_{\rm p}^2\, \ell_{\rm p}^2}} \tag{C.24}$$

and

$$U\left(1+\frac{M}{\mu},\frac{3}{2},\frac{2\,\mu^2\,R_{\rm s}^2}{m_{\rm p}^2\,\ell_{\rm p}^2}\right) \simeq \left[\frac{1}{\Gamma\left(\frac{M}{\mu}+1\right)}-\frac{2}{\Gamma\left(\frac{M}{\mu}+\frac{1}{2}\right)}\right]\Gamma\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}\,\mu\,R_{\rm s}}{m_{\rm p}\,\ell_{\rm p}}\right)^{2\,\frac{M}{\mu}-1} e^{\frac{2\,\mu^2\,R_{\rm s}^2}{m_{\rm p}^2\,\ell_{\rm p}^2}} \tag{C.25}$$

from which

$$\begin{aligned} H_- &\simeq (2\,M+\mu)\,\frac{\Gamma\left(\frac{M}{\mu}+\frac{1}{2}\right)\left[\Gamma\left(\frac{M}{\mu}+1\right)-2\,\Gamma\left(\frac{M}{\mu}+\frac{3}{2}\right)\right]}{\Gamma\left(\frac{M}{\mu}+\frac{3}{2}\right)\left[\Gamma\left(\frac{M}{\mu}+\frac{1}{2}\right)-2\,\Gamma\left(\frac{M}{\mu}+1\right)\right]} \\ &\simeq 2\,\mu\left(\frac{M}{\mu}\right)^{\frac{1}{2}} . \end{aligned} \tag{C.26}$$

Putting the above bounds together, we obtain

$$M\left[1+2\,\left(\frac{\mu}{M}\right)^{\frac{1}{2}}\right] \lesssim \langle\,\hat{H}\,\rangle \lesssim M\left(1+\frac{\lambda_\mu^2}{R_{\rm s}^2}\right) , \tag{C.27}$$

which shows that $\langle\,\hat{H}\,\rangle$ cannot be smaller than the classical ADM mass and the (maximum) relative correction is proportional to the Compton length $\lambda_\mu = \ell_{\rm p}\,m_{\rm p}/\mu$.

Appendix D

Horizon wavefunction and black hole probability

In Section 6.3.2, we continue to employ the upper bound on the spectral decomposition to obtain the horizon wavefunction in Eq. (6.45) and estimate the maximum possible correction to the gravitational radius and minimum probability P_{BH}. Its normalisation is given by

$$1 = 4\,\pi \int_{R_{\mathrm{H}}}^{\infty} |\Psi_{\mathrm{H}}(r_{\mathrm{H}})|^2\, r_{\mathrm{H}}^2\, \mathrm{d}r_{\mathrm{H}} = 4\,\pi\, \mathcal{N}_{\mathrm{H}}^2 \int_0^{\infty} \left(\frac{\tilde{r}_{\mathrm{H}}}{\ell_{\mathrm{p}}}\right)^{\frac{m_{\mathrm{p}}\,R_{\mathrm{H}}}{\mu\,\ell_{\mathrm{p}}}} e^{-\frac{\mu\,\tilde{r}_{\mathrm{H}}\,R_{\mathrm{s}}^2}{m_{\mathrm{p}}\,\ell_{\mathrm{p}}^3}}\, (\tilde{r}_{\mathrm{H}} + R_{\mathrm{H}})^2\, \mathrm{d}\tilde{r}_{\mathrm{H}}\ , \qquad \text{(D.1)}$$

where $\tilde{r}_{\mathrm{H}} = r_{\mathrm{H}} - R_{\mathrm{H}}$. The above expression yields

$$\begin{aligned}
\mathcal{N}_{\mathrm{H}}^{-2} &= \frac{8\,\pi\,\ell_{\mathrm{p}}^9\,m_{\mathrm{p}}^3}{\mu^3\,R_{\mathrm{s}}^6} \left(\frac{m_{\mathrm{p}}\,\ell_{\mathrm{p}}^2}{\mu\,R_{\mathrm{s}}^2}\right)^{\frac{m_{\mathrm{p}}\,R_{\mathrm{H}}}{\mu\,\ell_{\mathrm{p}}}} \left[\left(1+\frac{m_{\mathrm{p}}\,R_{\mathrm{H}}}{\mu\,\ell_{\mathrm{p}}}\right)\left(1+\frac{m_{\mathrm{p}}\,R_{\mathrm{H}}}{2\,\mu\,\ell_{\mathrm{p}}}+\frac{\mu\,R_{\mathrm{H}}\,R_{\mathrm{s}}^2}{m_{\mathrm{p}}\,\ell_{\mathrm{p}}^3}\right)+\frac{\mu^2\,R_{\mathrm{H}}^2\,R_{\mathrm{s}}^4}{2\,\ell_{\mathrm{p}}^6\,m_{\mathrm{p}}^2}\right] \\
&\quad \times \Gamma\left(1+\frac{m_{\mathrm{p}}\,R_{\mathrm{H}}}{\mu\,\ell_{\mathrm{p}}}\right) \\
&\simeq \frac{4\,\pi\,\ell_{\mathrm{p}}^3\,m_{\mathrm{p}}\,R_{\mathrm{H}}^2}{\mu\,R_{\mathrm{s}}^2} \left(\frac{m_{\mathrm{p}}\,\ell_{\mathrm{p}}^2}{\mu\,R_{\mathrm{s}}^2}\right)^{\frac{m_{\mathrm{p}}\,R_{\mathrm{H}}}{\mu\,\ell_{\mathrm{p}}}} \Gamma\left(\frac{m_{\mathrm{p}}\,R_{\mathrm{H}}}{\mu\,\ell_{\mathrm{p}}}\right)\ , \qquad \text{(D.2)}
\end{aligned}$$

where we again used $R_{\mathrm{s}} \sim R_{\mathrm{H}} \gg \ell_{\mathrm{p}}$ in the last approximation.

The expectation value of the gravitational radius is given by

$$
\begin{aligned}
\langle \Psi_{\rm H} | \, \hat{r}_{\rm H} \, | \Psi_{\rm H} \rangle &= 4\,\pi \int\limits_0^\infty |\Psi_{\rm H}(r_{\rm H})|^2 \, r_{\rm H}^3 \, {\rm d}r_{\rm H} \\
&= R_{\rm H} + R_{\rm H} \, \frac{m_{\rm p}^2 \, \ell_{\rm p}^2}{\mu^2 \, R_{\rm s}^2} \left(1 + \frac{3\,\ell_{\rm p}\,\mu}{m_{\rm p}\,R_{\rm H}} \right) \\
&\quad - R_{\rm H} \, \frac{\left(1 + \frac{m_{\rm p}\,R_{\rm H}}{\mu\,\ell_{\rm p}}\right) + \frac{\mu\,R_{\rm H}\,R_{\rm s}^2}{\ell_{\rm p}^3\,m_{\rm p}}}{\left(1 + \frac{m_{\rm p}\,R_{\rm H}}{\mu\,\ell_{\rm p}}\right)\left(1 + \frac{m_{\rm p}\,R_{\rm H}}{2\,\ell_{\rm p}\,\mu}\right) + \frac{\mu\,R_{\rm H}\,R_{\rm s}^2}{\ell_{\rm p}^3\,m_{\rm p}}\left(1 + \frac{m_{\rm p}\,R_{\rm H}}{\ell_{\rm p}\,\mu}\right) + \frac{\mu^2\,R_{\rm H}^2\,R_{\rm s}^4}{2\,\ell_{\rm p}^6\,m_{\rm p}^2}} \\
&\simeq R_{\rm H} \left[1 + \frac{\lambda_\mu^2}{R_{\rm s}^2} \left(1 - \frac{2\,\ell_{\rm p}^2}{R_{\rm H}\,\lambda_\mu} \right) \right] \\
&\simeq R_{\rm H} \left(1 + \frac{\lambda_\mu^2}{R_{\rm s}^2} \right) \ ,
\end{aligned}
\tag{D.3}
$$

with $R_{\rm H} = 2\,G_{\rm N}\,M \gg \lambda_\mu$ the classical Schwarzschild radius and $\lambda_\mu \gg \ell_{\rm p}$ the Compton length of the matter costituents.

The probability density (6.8) for the horizon to be located on the sphere of radius $r = r_{\rm H}$ vanishes for $0 \le r_{\rm H} < R_{\rm H}$, else is given by

$$
\mathcal{P}_{\rm H}(r_{\rm H}) \simeq 4\,\pi\,\mathcal{N}_{\rm H}^2\,r_{\rm H}^2 \left(\frac{r_{\rm H} - R_{\rm H}}{\ell_{\rm p}} \right)^{\frac{m_{\rm p}\,R_{\rm H}}{\mu\,\ell_{\rm p}}} e^{-\frac{\mu(r_{\rm H} - R_{\rm H})R_{\rm s}^2}{m_{\rm p}\,\ell_{\rm p}^3}} \ . \tag{D.4}
$$

The probability density (6.9) can be explicitly computed from the wavefunction (6.33) with $r_i = r$ and the horizon probability density (D.4),

$$
\begin{aligned}
\mathcal{P}_<(r < r_{\rm H}) &= \left(4\,\pi \int\limits_{R_{\rm H}}^{r_{\rm H}} |\psi_{\rm S}(r)|^2 \, r^2 \, {\rm d}r \right) \mathcal{P}_{\rm H}(r_{\rm H}) \\
&= {\rm erf}\left(\frac{r_{\rm H}}{R_{\rm s}} \right) \mathcal{P}_{\rm H}(r_{\rm H}) \ ,
\end{aligned}
\tag{D.5}
$$

where erf denotes the error function. The black hole probability (6.11) now reads

$$
P_{\rm BH} = \int\limits_{R_{\rm H}}^{\infty} \mathcal{P}_<(r < r_{\rm H}) \, {\rm d}r_{\rm H} \ , \tag{D.6}
$$

which however can only be computed numerically for specific values of $R_{\rm H}$, $R_{\rm s}$ and μ (see Figs. 6.1-6.3).

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