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Black hole scattering from Worldline Quantum Field Theory and the Classical Double Copy of Spinning particles

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Abstract

This PhD thesis focuses on studying the classical scattering of massive/massless particles toward black holes, and investigating double copy relations between classical observables in gauge theories and gravity. This is done in the Post-Minkowskian (PM) approximation i.e. a perturbative expansion of observables controlled by the gravitational coupling constant $\kappa = \sqrt{32\pi G_N}$, with G_N being the Newtonian coupling constant. The investigation is performed by using the path integral formulation of the exact QFT amplitude describing such processes, then extracting the classical limit. This leads to the Worldline Quantum Field Theory (WQFT) formulation, displaying a worldline path integral describing the scattering objects and a QFT path integral in the Born approximation, describing the intermediate bosons exchanged in the scattering event by the massive/massless particles.

We first start by studying how to propagate a bi-adjoint scalar on the worldline. Such theory is at the basis of the quantum and classical double copy, since it encodes the locality structure of amplitudes, and will be used to study the classical double copy in the subsequent analysis. We describe two equivalent formulations of the worldline theory which use a matrix-valued action and the introduction of auxiliary variables on the worldline, taking care of the color indices of the scalar particle. Next we build up a worldline path integral for such theory on the circle. This allows us to study the one loop effective action for the bi-adjoint and multi-adjoint scalar. As an application we recover the beta function and the self-energy of the theory from a worldline path integral view point.

Next we introduce the WQFT, by explicitly deriving a relation between the classical limit of a scattering amplitudes for scalar particles in gauge/gravity background and a WQFT path integral. We derive such relation for the case of off-shell tree-level currents, for a two-fold reason. Firstly it makes easier to derive a relation between the Kosower-Maybee-O'Connell (KMOC) limit of amplitudes and the WQFT, then, we want to use the path integral representation of the current to study the classical Compton amplitude and higher point amplitudes. We also present a nice application of our formulation to the case of Hard Thermal Loops (HTL), by explicitly evaluating hard thermal currents in gauge theory and gravity.

Next we move to the investigation of the classical double copy (CDC), which is a powerful tool to generate integrands for classical observables related to the binary inspiralling problem in General Relativity. In order to use a Bern-Carrasco-Johansson (BCJ) like prescription, straight at the classical level, one has to identify a double copy (DC) kernel, encoding the locality structure of the classical amplitude. Such kernel is evaluated by using a theory where scalar particles interacts through bi-adjoint scalars. We show here how to push forward the classical double copy so to account for spinning particles, in the framework of the WQFT. Here the quantization procedure on the worldline allows us to fully reconstruct the quantum theory on the gravitational side. To test our double copy prescription, we evaluate the 2PM eikonal phase for three worldlines and the leading gravitational bremsstrahlung for two worldlines, by using our double copy prescription, finding agreement with a direct calculation from the gravitational theory.

Finally we investigate the case where light scatters off a black hole, in the optical regime. In this case we show how, starting from a quantum field theoretical formulation of the problem allows us to cleanly derive a WQFT path integral, suitable to extract the classical and optical limit of the observables related to the scattering event, particularly

the impulse and the deflection angle of the light ray. Ultimately, this allows us to show that the light ray turns out to be a massless scalar in the optical limit, as stated by the Equivalence Principle. This efficiently allows us to evaluate classical observables.

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Contents

Chapter 1

Introduction

The direct observation of gravitational wave signal on earth [1, 2, 3, 4, 5] required for higher precision theoretical predictions, so to account for data comparison and particularly for the calculation of waveform templates. Part of the LIGO-VIRGO collaboration follows binary systems of compact objects during their evolution, measuring the radiation emitted during the encounter. The complete evolution is made up by the inspiral, merger and ring-down phase. In the former, the two objects starts orbiting around each other at high angular frequency, the merger refers to the actual collision, while the ring-down phase is where the final black hole recovers from the titanic event from which it was formed. This experimental success increased the developments of theoretical methods so to tackle this new events. Particularly, the so called Post-Newtonian (PN) approach is well suited to describe the inspiral phase. Assuming r is the black hole separation and M the total mass of the system, one has by the viral theorem that $G_N M/c^2 r \sim v^2/c^2 << 1$, where v is the relative velocity of the two bodies, resulting that the PN expansion is an expansion in the Newtonian coupling constant G_N and in the relative velocity of the two bodies, w.r.t the speed of light c in vacuum. A different approximation, also recovering the PN result, is the Post-Minkowskian (PM) approximation, which is an expansion of classical quantities w.r.t. the gravitational coupling constant $\kappa = \sqrt{32\pi G_N}$. It is applied for the unbound scattering problem in the limit of large impact parameter and small deviation angles. In contrast to the PN, in the PM expansion, the velocity of the particle is not taken to be small w.r.t. the speed of light. Both the above approximations are performed by assuming one can describe the compact objects encounter, in the inspiral phase, by using an effective field theory treating the objects as point particle interacting through gravitons. This was proposed by Goldberger and Rothstein in [6]. In this approach a worldline action is used to describe the compact objects, while gravity is described by the Einstein-Hilbert action, treated perturbatively in the gravitational coupling constant. This boils down to write down the equations of motion for the point particle and the Einstein field equations, solving them perturbatively so to get classical observables. It was then extended to include spin and finite size effects [7, 8].

An alternative route to attack the two body problem consists in using an interacting quantum EFT, as initiated by Donoghue [9]. In such a framework one introduces matter fields interacting through gravity, treated perturbatively at low energies, identifying the quanta of such matter fields with the compact objects. Then, scattering amplitudes from such theory can be used to generate classical results, once taking the large mass limit.

Later on this process has been systematized and various formalisms are nowaday used to tackle the problem (see [10, 11, 12] for reviews of amplitude methods to PM scattering). The advantage in using amplitudes lies in that one can use on-shell techniques (BCFW, unitarity cuts for instance)[13] and mostly integration methods for Feynman integrals (IBP relations, differential equations for instance) [14], so to generate classical quantities.

Among these framework, it was proposed recently by Mogull, Plefka and Steinhoff, the worldline quantum field theory (WQFT) [15] further upgraded so to include spin and radiation reaction effects in [16, 17]. The WQFT allows for the extraction of the classical limit of amplitudes, by using the path integral approach to QFT. Particularly, as investigated in a collaboration of the author [18] and then also studied in [19], by starting from scattering amplitudes dressed with coherent wavefunctions of Poincarè [20], one can rephrase the classical limit of a gauge-invariant quantum observable as an insertion of a phase space function of some worldline variables inside a path integral, thus showing a direct relation between the classical limit of scattering amplitudes and the WQFT. The WQFT path integral is the product of a QFT functional integral and a worldline path integral. The effective theory so obtained describes the interaction between two worldlines through gauge bosons, in the Born approximation i.e. neglecting the propagation of gauge and mixed gauge-matter loops among the interacting worldlines. The main advantage results in that the WQFT bypasses the $\hbar \to 0$ limit, and, through the use of a Feynman diagrammatic expansion of the WQFT action, fastly delivers the classical result.

The QFT-like component of the WQFT functional integral is widely recognized, involving a functional integral over a gauge field, treated perturbatively in the coupling constant. This gauge field describes the intermediate particles exchanged between two or more massive external particles. The latter are implemented by the use of a worldline theory, whose quantization propagates the massive external states. The worldline-like component of the WQFT functional integral does correspond to a particular representation of a two point Green function known as the dressed propagator, describing the sub-set of 1PI Feynman diagrams related to the field theory realized once quantizing the worldline theory. As already stressed above, the dressed propagator is introduced in the WQFT context when considering the classical limit of the four point amplitude with massive external states, at the level of functional integrals, so to neglect intermediate loops of matter field and mixed gauge bosons matter field. Let us briefly describe how to reconstruct a field theory by starting by the simple case of a massive point particle of charge Q, and mass m, interacting with an abelian gauge field A_μ. The worldline phase space action for such a theory can be easily written

$$S[x,p;(e,A)] = -\int_0^1 d\tau(\dot{x}_\mu p^\mu - e(\tau)\underbrace{\frac{1}{2}\left(p - QA - m\right)^2}_{H})) \tag{1.1}$$

once introducing the einbein $e(\tau)$ so to gauge translational invariance whose generator is represented by the point particle Hamiltonian H. The phase space action allows to read out the Poisson brackets between canonical coordinates $\{x_\mu, p^\nu\} = \delta_\mu{}^\nu$, while the equation on motion for the einbein $\frac{\delta S}{\delta e(\tau)} = H = 0$ imposes a constraint on the particle phase space. Such a worldline particle is trivially of first class i.e. the constraint algebra can be easily check to be $\{H, H\} = 0$. In order to reconstruct the quantum theory implemented by the worldline particle, we proceed by quantizing the model promoting the Poisson brackets to commutator, namely $\{\bullet, \bullet\} \to -i[\bullet, \bullet]$. The equation of motion for the einbein then is interpreted as a constraint selecting physical states $|\Phi\rangle$ of the quantum theory

$$\frac{\delta S}{\delta e(\tau)} = (p - QA)^2 - m^2 = 0 \to \left(D^2 + m^2\right)|\Phi\rangle = 0 \tag{1.2}$$

where $D_\mu = \partial_\mu - iQA_\mu$ is the covariant derivative. This reveals that, in such a case, the physical states propagated by the worldline theory do represent massive charged scalar fields, once projecting the state into the configuration space. Thus we see how from the quantization of a simple worldline particle, a field theory emerges. Particularly, the construction of the path integral on the line, for the worldline particle, allows to evaluate a sub-set of Feynman diagrams related to the charged scalar field minimally coupled to Maxwell. For the path integral quantization, we send the reader to Chapter 3, where also the case of gravity is taken into account.

Not only the amplitude or worldline based approach are useful for the calculation of classical observables, but they also allow to investigate gauge-gravity duality. Such duality was first found to hold between tree amplitudes of closed and open strings by Kawai, Lewellen, and Tye (KLT)[21]. Particularly, such relation was found to hold for closed and Type II strings. Later on, Bern, Carrasco and Johansson (BCJ) [22, 23] investigated the above relation at the level of scattering amplitudes between Yang-Mills and SUGRA, finding that duality can be realized by rearranging the numerators of the Yang-Mills amplitude so to obey the Jacobi identity, as the related color factors The latter feature is the so called color-kinematics duality, and, leads to the double copy relation (see [24] for recent reviews on the double copy). More precisely, the full color-dressed n-point tree amplitude in Yang-Mills theory can be organized in terms of trivalent diagrams as

$$\mathcal{A}_n^{\text{tree}} = \sum_{i \in \text{trivalent}} \frac{c_i n_i}{D_i}, \tag{1.3}$$

where c_i denotes the color factors, while n_i, D_i are the corresponding kinematic numerator and propagator, respectively. In the case where the color factors satisfy the Jacoby identity $c_i + c_j + c_k = 0$, one can always arrange the kinematical numerators n_i to obey the same algebraic equations $n_i + n_j + n_k = 0$. This is termed "color-kinematics duality" (CKD). Once such relations are satisfied, the color factors can be replaced with the corresponding kinematic numerators,

$$\mathcal{M}_n^{\text{tree}} = \sum_{i \in \text{trivalent}} \frac{n_i n_i}{D_i} \tag{1.4}$$

yielding an amplitude of a gravitational theory. This is the so-called double copy relation. More specifically, the resulting theory is the $\mathcal{N} = 0$ supergravity (SUGRA or NS-NS gravity), which describes Einstein-Hilbert gravity coupled to the dilaton field ϕ and the Kalb-Ramond two-form B. In the Einstein frame, the action[1] in D dimensions is given as [25]:

$$S_{N=0} = \frac{2}{\kappa^2} \int \mathrm{d}^D x \sqrt{-g(x)} \left(-R + \frac{4}{D-2}\partial_\mu \phi \partial^\mu \phi + \frac{1}{12} e^{-8\phi/(D-2)} H_{\lambda\mu\nu} H^{\lambda\mu\nu}\right), \tag{1.5}$$

[1]We use the mostly minus convention for the metric.

where $H_{\lambda\mu\nu} = \partial_\lambda B_{\mu\nu} + \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu}$ is the field strength related to $B_{\mu\nu}$. A relation of this kind was expected since, in the low energy limit, the closed string propagates the $\mathcal{N} = 0$ supergravity (SUGRA) which is Einstein-Hilbert action coupled to the dilaton scalar and the Kalb-Ramond field, while the Type IIB contains the Yang-Mills field, thus, the KLT relations must hold from these quantum field theories. Color-kinematics was proven at tree level [26, 27] and, although it is not striktly demonstradet at loop level, there are evidences of color-kinematics duality between loop amplitudes integrands [23]. Double copy relations have also been investigated for a great variety of theories including scalar [28, 29, 24, 30, 31] and spinning external matter particles [32, 33, 34, 35]. This line of research, not only shades light on gauge-gravity duality, but makes the double copy an efficient tool to generates amplitudes needed for applications to black hole scattering, in the classical limit. Particularly, as obtained by following the upper arrow in Fig. 1.1, one evaluates the amplitude with external massive particles of spin s in Yang-Mills, then uses the double copy to get the SUGRA partner (after projecting out Kalb-Ramond and dilaton [36]), yielding in the classical limit to the result needed to extract classical observables.

An interesting question is how we can apply the BCJ perturbative double copy relation straight at the classical level, leading to the so called classical double copy (CDC). It should be noted that differing from amplitudes where the locality structure is encoded by the poles in the Feynman propagators, classical integrands do not enjoy such a feature, making it difficult to recast the amplitude in a BCJ form. It is well-known that some exact solutions of YM theory can be related to general relativity. This was first discovered by Monteiro, O'Connell, and White, and is known as the Kerr-Schild double copy [37], which was then developed to a great extent [38, 39, 40, 41, 42, 43, 44, 45, 46, 36, 47, 48]. However what one is also interested in, is described by following the bottom arrow in Fig. 1.1 i.e. starting from the classical Yang-Mills amplitude with external spinning particles, reaching directly the classical SUGRA amplitude, thus bypassing the $\hbar \to 0$ limit. In addition, a worth question would be whether, starting from the classical amplitude, we could get informations about the quantum theory generating such result in the $\hbar \to 0$ limit, corresponding to follow backward the rightmost (leftmost) arrow in the diagram in Fig.1.1.

CDC was first studied in [49], and later on extended to bound states [50], as well as to incorporate spin effects [51, 52]. Specifically, in these approaches, the basic idea is to iteratively solve the equations of motion (Wong's equations in the case of a charged particle in YM field [53]) to obtain classical observables such as the radiation. Then, by adopting appropriate replacement rules of the color factors, they recover the corresponding quantities in dilaton-gravity from YM theory. However such replacement rules are not suited to hold to higher orders, and, moreover, do not make manifest any color-kinematics relation at the classical level. A way to tackle such a question was provided by Shen [54], who showed that the double copy at the classical level could be realized in an analogous way to the BCJ one from amplitudes by identifying the so-called "double copy kernel", encoding locality of the classical integrands. This kernel can be evaluated by considering the scattering of massive scalar particles interacting with bi-adjoint scalars, whose worldline theory and quantization is fully described in [55]. This was realized by solving perturbatively the Wong equations [53], obtaining the classical integrands in both gauge and gravity theories, then checking that the CDC of the gauge observables would match the ones evaluated directly in gravity. Although the above approaches gives prescriptions on how to implement the CDC, they

$$
\begin{tikzcd}
\mathcal{A}^{s}_{\text{YM}} \arrow[r, "\text{DC}"] \arrow[d, "\hbar\to 0"] & \mathcal{M}^{s'}_{\text{SUGRA}} \arrow[d, "\hbar\to 0"] \\
\bar{\mathcal{A}}^{s}_{\text{YM}} \arrow[r, "\text{CDC}"] & \bar{\mathcal{M}}^{s'}_{\text{SUGRA}}
\end{tikzcd}
$$

Figure 1.1: Diagrammatic path showing the calculation of classical amplitudes through double copy. Here $\mathcal{A}^{s}_{\text{YM}}, \bar{\mathcal{A}}^{s}_{\text{YM}}$ are respectively the quantum Yang-Mills amplitudes with external massive spin s particle and its classical limit, while $\mathcal{M}^{s'}_{\text{SUGRA}}, \bar{\mathcal{M}}^{s'}_{\text{SUGRA}}$ represents respectively the quantum SUGRA amplitude with external massive s' particles and its classical limit.

only work for scalar particles and does not tell us anything about the quantum theory generating the classical integrands used for the double copy. Recently it was shown by Plefka and Shi in [56] that this prescription can be implemented nicely in the WQFT. In particular, the authors double copied the eikonal and leading radiation for massive scalar particles interacting through gluons into the same matter content interacting through the dilaton-gravity sector of the supergravity. This gave room to use the WQFT as a framework to investigate the CDC for spinning particles, as done in a collaboration of the author in [57], showing that the one can move freely in the diagram from Fig.1.1, for the case $s = 1/2, s' = 1$. Particularly, the quantization of the worldline sector of the WQFT gives crucial informations about the quantum theory describing the double copied integrands, allowing to write down the lagrangian propagating the matter particles entering as external lines in the double copied amplitude. As a last remark, we stress that the text in such a thesis closely follows the published work of the author in various collaborations [55, 58, 18, 57].

Chapter 2

Worldline description of a Bi-Adjoint scalar

The duality between color and kinematics [59, 23] reveals a deep structure relating amplitudes of gauge and gravity theories, see [24] for a review. An interesting model, used for studying color/kinematic relations, is that of a bi-adjoint scalar [60, 61]. Its basic structure is that of a ϕ^3 theory, a standard QFT toy model which is super-renormalizable in 4 dimensions and renormalizable in 6 dimensions (where it becomes asymptotically free), but with the scalar field extended to be symmetric under the action of a compact group of the form $G \times \tilde{G}$. In particular, the scalar is taken to transform in the adjoint representation for each factor, so that its Feynman rules carry corresponding symmetry factors (named color factors). The latter may be substituted by kinematical factors having the same symmetries, thus producing Feynman rules and amplitudes of non-abelian gauge fields and gravity.

In a purely classical setting, the bi-adjoint scalar has also served as the basis to find maps between classical solutions in gauge and gravity at both non-perturbative [37, 47, 36, 62] and perturbative [49, 63, 50, 54] levels. The perturbative solutions of the bi-adjoint scalar theory coupled with point sources (parametrized by their worldline coordinates) are obtained by iteratively solving the equations of motion. The kinematic structure of these solutions is crucial for a systematic amplitude-like implementation of the double copy [54]. At classical perturbative level, the form of the action leading to equations of motion is unimportant as long as it reproduces the proper equations of motion, but its precise form is central to undertake quantization.

Seeds of color/kinematic relations, and corresponding double copy structure that reproduces gravity amplitudes from the gauge ones, can be traced back to the origin of string theory, when it was noticed that the Veneziano amplitude (an open-string scattering amplitude) could be related in a simple way to the Virasoro-Shapiro amplitude (a closed-string amplitude). Open strings give rise to gauge fields while closed strings give rise to gravitons, so one notices a signal of the gauge/gravity relations [21]. The string results were obtained and interpreted in a first quantized picture. Similarly, in this paper we wish to investigate the color/kinematic relations using a first quantized approach to particle theory [64].

Motivated by the above reasons we construct a worldline particle which in first quantization allows us to propagate a bi-adjoint a scalar. To accomplish this we introduce color variables on the worldline, using the set-up described in [65, 66, 67], where color variables are coupled to a worldline $U(1)$ gauge field which carries an additional Chern-Simons term. The gauge field implements a projection to the desired representation of the symmetry group assigned to the particle. The choice of the representation is obtained by tuning the Chern-Simons coupling to a suitable discrete value. A form of this projection was described also in [68], and later recognized to arise from the coupling to a $U(1)$ gauge field. The projection mechanism was seen at work in the worldline path integral treatments of differential forms [69, 70, 71], and extended shortly afterwards to color degrees of freedom [65]. Further extensions have been discussed in [72, 73]. On the other hand, worldline color variables had been introduced originally much earlier [74, 75].

After having identified the correct worldline action for the bi-adjoint particle, we use it to build up a path integral representation of the one-loop effective action of the bi-adjoint scalar field. Then, we evaluate the bi-adjoint and multi-adjoint self-energy diagram and extract the one-loop beta function of the theory in 6 dimensions, which was recently shown to vanish [76]. We also study the one-loop self-energy correction to the bi-adjoint propagator. Eventually, we indicate how our model can be extended to that of a particle carrying an arbitrary representation of direct products of global symmetry groups, and compute the one-loop beta function of a multi-adjoint scalar particle in six dimensions. After this, we highlight a double copy relation holding at the level of the vertex operators on the worldline, for the bi-adjoint scalar, Yang-Mills and Einstein-Hilbert theory.

2.1 Effective action in the matrix valued approach

Let us start by using a pure QFT approach to derive the worldline action for a scalar particle coupled to the bi-adjoint scalar $\Phi^{a\alpha}$. We consider directly the special case of a particle interpreted as the quantum of the bi-adjoint scalar field itself. For that purpose, we start from the QFT of a bi-adjoint scalar field $\Phi^{a\alpha}$, charged under a $G\times\tilde{G}$ global symmetry group and transforming in the adjoint for each factor[1]. The Euclidean Lagrangian of the model is [47, 62]

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi^{a\alpha}\partial^\mu\Phi^{a\alpha} + \frac{m^2}{2}\Phi^{a\alpha}\Phi^{a\alpha} + \frac{y}{3!}f^{abc}\tilde{f}^{\alpha\beta\gamma}\Phi^{a\alpha}\Phi^{b\beta}\Phi^{c\gamma}\,, \tag{2.1}$$

where m is the mass and y the coupling constant. This model is the massive version of a massless one, whose amplitudes in the Cachazo-He-Yuan (CHY) representation were considered in [60, 61]. The massive generalization of these amplitudes was considered e.g., in [77, 78, 79]. In a quantum/background split ($\Phi^{a\alpha}\to\phi^{a\alpha}+\Phi^{a\alpha}$) one finds, after a partial integration, a quadratic action for the quantum field $\phi^{a\alpha}$

$$\mathcal{L}_q^{(2)} = \frac{1}{2}\phi^{a\alpha}\Big(\delta^{ab}\delta^{\alpha\beta}(-\partial^2+m^2) + yf^{abc}\tilde{f}^{\alpha\beta\gamma}\Phi^{c\gamma}\Big)\phi^{b\beta}\,, \tag{2.2}$$

[1] The Lie algebra for each factor has the form $[T^a, T^b] = if^{ab}{}_cT^c$ and the adjoint representation is given by the structure constants $(T^a_{\rm A})^b{}_c = -if^{ab}{}_c = -if^{abc}$. The Killing metric is normalized to δ^{ab}, so that upper and lower adjoint indices are equivalent. We use greek indices to label generators of the second group $\tilde{G}$.

which is coupled to the background $\Phi^{a\alpha}$. It contains a quadratic differential operator of the form (setting $p^2 = -\partial^2$)

$$2H = \delta^{ab}\delta^{\alpha\beta}(p^2 + m^2) + y f^{abc}\tilde{f}^{\alpha\beta\gamma}\Phi^{c\gamma} \ , \tag{2.3}$$

which is interpreted as a particle Hamiltonian H— the factor 2 being conventional — which acts on wave functions $\phi^{a\alpha}(x)$ that carry indices of the bi-adjoint. The inverse of this Hamiltonian gives the bi-adjoint propagator in the $\Phi^{a\alpha}$ background, and the functional trace of the corresponding heat kernel is related to the one-loop effective action [80]. As it stands, this Hamiltonian has a matrix-valued potential, and it produces a matrix-valued action. In a path integral, a path ordering must be used to properly define its exponential.

To present the particle Hamiltonian in a more compact way, let us introduce a matrix notation to cast (2.2) in the form

$$\mathcal{L}_q^{(2)} = \frac{1}{2}\phi^T\left(-\partial^2 + m^2 + y\hat{\Phi}\right)\phi \ , \tag{2.4}$$

where $\phi = \phi^A = \phi^{a\alpha}$ has to be considered as a column vector with components labeled by the multi-index $A = (a, \alpha)$, while $\hat{\Phi}$ is a matrix-valued background potential

$$\hat{\Phi} = \hat{\Phi}^{A,B} = \hat{\Phi}^{a\alpha,b\beta} = f^{abc}\tilde{f}^{\alpha\beta\gamma}\Phi^{c\gamma} \ . \tag{2.5}$$

The matrix $\hat{\Phi}$ is symmetric in the indices A and B. The corresponding matrix-valued Hamiltonian, eq. (2.3), is then cast as

$$2H = p^2 + m^2 + y\hat{\Phi}(x) \ . \tag{2.6}$$

It acts on wave functions of the form $\phi(x) = \phi^A(x)$, which have components labeled by the multi-index A. The Hamiltonian leads classically to an action that is also matrix-valued. The latter, after some simple redefinitions, can be written in the form

$$S[x] = \int_0^1 \mathrm{d}\tau \Big[\frac{1}{4T}\dot{x}^2 + Tm^2 + Ty\hat{\Phi}(x)\Big], \tag{2.7}$$

where T is the Fock-Schwinger proper-time. The corresponding path integral needs a time ordering prescription, that makes sure that it solves the correct Schrödinger equation (just as for time dependent Hamiltonians). Then, following Schwinger [80], we find that the worldline path integral for the effective action takes the form

$$\Gamma[\Phi] = -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}T}{T} e^{-m^2 T}\,\mathrm{tr}\int_{\mathcal{P}} Dx\,\mathbf{T}\,e^{-S[x]} \ , \tag{2.8}$$

where $\mathbf{T}$ denotes the time ordering, 'tr' the additional trace on the finite color degrees of freedom, and with the constant mass term explicitly taken out of the worldline action S. The symbol $\mathcal{P}$ indicates that the coordinates x must have periodic boundary conditions in order to produce the trace in the corresponding Hilbert space.

As already discussed above, the matrix-valued action requires a path ordering prescription to define properly the corresponding path integral. The Schwinger proper-time representation of the one-loop effective action of the bi-adjoint field has already been anticipated in (2.8), where the trace of the heat kernel is computed by the path integral with periodic boundary conditions. The periodic paths $x^\mu(\tau)$ with $x^\mu(0) = x^\mu(1)$ can be split as

$$x^\mu(\tau) = x_0^\mu + q^\mu(\tau) \ , \tag{2.9}$$

where x_0^μ is any point on the loop traced by $x^\mu(\tau)$, while $q^\mu(\tau)$ are the remaining fluctuations that must satisfy vanishing Dirichlet boundary conditions (i.e. $q^\mu(0) = q^\mu(1) = 0$). Then, one integrates over x_0^μ and $q^\mu(\tau)$ separately, and the effective action takes the form

$$\Gamma[\Phi] = -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}T}{T} e^{-m^2 T} \int d^D x_0 \ \mathrm{tr} \int_{\mathcal{D}} Dq \, \mathbf{T} \, e^{-S[x]} \ , \tag{2.10}$$

where $S[x]$ is the action in (2.7), and $\mathcal{D}$ indicates the Dirichlet boundary conditions on $q^\mu(\tau)$ mentioned above. Other methods for extracting the constant x_0 are possible, and will be discussed in the next section.

The effective action (2.10) may now be computed perturbatively in terms of the Wick contractions of the q-propagators (see appendix A for further details)

$$\Gamma[\Phi] = \int d^D x_0 \left[-\frac{1}{2}\int_0^\infty \frac{\mathrm{d}T}{T} \frac{e^{-m^2 T}}{(4\pi T)^{\frac{D}{2}}} \ \mathrm{tr} \, \langle \mathbf{T} \, e^{-S_{\mathrm{int}}} \rangle \right] \tag{2.11}$$

with

$$S_{\mathrm{int}} = Ty \int_0^1 \mathrm{d}\tau \, \hat{\Phi}(x(\tau)) \ . \tag{2.12}$$

The perturbative computation gives an answer of the form

$$\mathrm{tr} \, \langle \mathbf{T} e^{-S_{\mathrm{int}}} \rangle = \sum_{n=0}^{\infty} \mathrm{tr} \, a_n(x_0) T^n \ , \tag{2.13}$$

where $a_n(x_0)$ are the so-called Seeley-DeWitt (or heat kernel) coefficients. Inserting this expansion in (2.11), and renaming $x_0 \to x$, gives

$$\begin{aligned}
\Gamma[\Phi] &= \int d^D x \left[-\frac{1}{2}\sum_{n=0}^{\infty} \frac{\mathrm{tr} \, a_n(x)}{(4\pi)^{\frac{D}{2}}} \int_0^\infty \frac{dT}{T} e^{-m^2 T} T^{n-\frac{D}{2}} \right] \\
&= \int d^D x \left[-\frac{1}{2}\sum_{n=0}^{\infty} \frac{\mathrm{tr} \, a_n(x)}{(4\pi)^{\frac{D}{2}}} \frac{1}{(m^2)^{n-\frac{D}{2}}} \int_0^\infty \frac{dT}{T} e^{-T} T^{n-\frac{D}{2}} \right] \\
&= \int d^D x \left[-\frac{1}{2}\sum_{n=0}^{\infty} \frac{\mathrm{tr} \, a_n(x)}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma[n-\frac{D}{2}]}{(m^2)^{n-\frac{D}{2}}} \right] \ .
\end{aligned} \tag{2.14}$$

We have integrated the proper time to obtain the usual gamma function. At $D = 6$, we see divergences for $n = 0, 1, 2, 3$ in the effective action, as captured by the poles of the gamma function. The one for $n = 3$ is mass independent and corresponds to the usual logarithmic divergence seen in dimensional regularization. The specific calculation of the coefficients in (2.13) can be done as in [81, 82, 83, 84], and gives the answer

$$a_0 = 1 \ , \quad a_1 = -y\hat{\Phi} \ , \quad a_2 = \frac{y^2}{2}\hat{\Phi}^2 \ , \quad a_3 = -\frac{y^2}{6}\hat{\Phi}^3 + \frac{y^2}{12}\hat{\Phi}\partial^2\hat{\Phi} \ , \tag{2.15}$$

which we have simplified by adding total derivatives, and neglecting matrix orderings that are inconsequential under the trace. The calculation of these coefficients is briefly outlined in appendix A.

2.1.1 Beta function for bi-adjoint scalar in $D = 6$

The $n = 3$ pole in the effective Lagrangian at $D = 6 - \epsilon$ is

$$\mathcal{L}_{\text{eff, div}} = -\frac{1}{2}\frac{\operatorname{tr} a_3(x)}{(4\pi)^3}\Gamma[\tfrac{\epsilon}{2}] = -\frac{\operatorname{tr} a_3(x)}{(4\pi)^3}\frac{1}{\epsilon} \ , \tag{2.16}$$

Then, using (2.15) the diverging part of the effective Lagrangian (2.16) becomes

$$\mathcal{L}_{\text{eff, div}} = \frac{1}{(4\pi)^3}\frac{1}{\epsilon}\left(\frac{y^3}{6}\operatorname{tr}\hat{\Phi}^3 - \frac{y^2}{12}\operatorname{tr}\hat{\Phi}\partial^2\hat{\Phi}\right) \ . \tag{2.17}$$

Let us now relate the terms in (2.17) to the one in (2.1), and recognize the counterterms needed to renormalize the theory at one-loop. We find

$$\begin{aligned} \operatorname{tr}\hat{\Phi}^2 &= T(A)\tilde{T}(A)(\Phi^{a\alpha})^2 \\ \operatorname{tr}\hat{\Phi}^3 &= \frac{1}{4}T(A)\tilde{T}(A)f^{abc}\tilde{f}^{\alpha\beta\gamma}\Phi^{a\alpha}\Phi^{b\beta}\Phi^{c\gamma} \ , \end{aligned} \tag{2.18}$$

where $T(A)$ indicates the index in the adjoint representation[2]. These divergences ask for wave function and coupling constant renormalizations, obtained by adding to (2.1) the counterterms

$$\mathcal{L}_{\text{ct}} = (Z_\Phi - 1)\tfrac{1}{2}\Phi^{a\alpha}(-\partial^2)\Phi^{a\alpha} + (Z_y - 1)\tfrac{y}{3!}f^{abc}\tilde{f}^{\alpha\beta\gamma}\Phi^{a\alpha}\Phi^{b\beta}\Phi^{c\gamma} \tag{2.19}$$

with

$$Z_\Phi - 1 = -\frac{1}{6}T(A)\tilde{T}(A)\frac{y^2}{(4\pi)^3}\frac{1}{\epsilon} \ , \qquad Z_y - 1 = -\frac{1}{4}T(A)\tilde{T}(A)\frac{y^2}{(4\pi)^3}\frac{1}{\epsilon} \ . \tag{2.20}$$

These counterterms produce a vanishing one-loop beta function, as recently discovered in [76]. In more details, parameterizing the counterterms with coefficients a_1 and c_1 as

$$\begin{aligned} Z_\Phi - 1 &= a_1\frac{y^2}{(4\pi)^3}\frac{1}{\epsilon} \ , \qquad & a_1 &= -\frac{1}{6}T(A)\tilde{T}(A) \ , \\ Z_y - 1 &= c_1\frac{y^2}{(4\pi)^3}\frac{1}{\epsilon} \ , \qquad & c_1 &= -\frac{1}{4}T(A)\tilde{T}(A) \ , \end{aligned} \tag{2.21}$$

the one-loop beta function is computed by

$$\beta(y) = \left(c_1 - \frac{3}{2}a_1\right)\frac{y^3}{(4\pi)^3} = 0 \ . \tag{2.22}$$

---

[2]The index $T(R)$ of a representation R is defined by $\operatorname{tr}(T_R^a T_R^b) = T(R)\,\delta^{ab}$. It coincides with its quadratic Casimir if R is the adjoint representation. In particular, one has $T(A) = C_2(A) = N$ for the adjoint representation of $SU(N)$.

In [76] the calculation of the beta function has been pushed to four loops (and actually to five loops in [85]), with the two-loop result indicating the asymptotic freedom of the theory. The latter would furnish a nice test on the structure of worldline methods at higher loops [86, 64].

There is also a mass renormalization that we have ignored so far. To complete the one-loop renormalization of the theory, let us identify the counterterm needed for renormalizing the mass. To achieve that, we have to treat the mass perturbatively in (2.11). Expanding the term $e^{-m^2 T}$ we find (up to total derivatives and up to a constant that renormalizes the vacuum energy) an extra contribution to $\mathrm{tr}\, a_3(x)$ of the form

$$\Delta \mathrm{tr}\, a_3(x) = -\frac{1}{2} m^2 y^2 \mathrm{tr}\, \hat{\Phi}^2 = -\frac{1}{2} m^2 y^2 T(A) \tilde{T}(A) (\Phi^{a\alpha})^2 \tag{2.23}$$

giving an extra divergence of the effective Lagrangian

$$\Delta \mathcal{L}_{\text{eff, div}} = -\frac{\Delta \mathrm{tr}\, a_3(x)}{(4\pi)^3} \frac{1}{\epsilon} = \frac{1}{(4\pi)^3} \frac{1}{\epsilon} \left(\frac{y^2 m^2}{2} T(A) \tilde{T}(A) (\Phi^{a\alpha})^2 \right) . \tag{2.24}$$

This is canceled by adding to (2.1) the additional counterterm

$$\Delta \mathcal{L}_{\text{ct}} = (Z_m - 1) \tfrac{1}{2} m^2 (\Phi^{a\alpha})^2 \tag{2.25}$$

with

$$Z_m - 1 = -T(A) \tilde{T}(A) \frac{y^2}{(4\pi)^3} \frac{1}{\epsilon} , \tag{2.26}$$

which leads to the mass anomalous dimension γ_m (defined as usual by $\gamma_m = \frac{1}{m} \frac{dm}{d \ln \mu}$)

$$\gamma_m(y) = -\frac{5}{12} T(A) \tilde{T}(A) \frac{y^2}{(4\pi)^3} . \tag{2.27}$$

2.1.2 Beta function for the multi-adjoint particle

As an extension, one may consider the n-adjoint scalar particle, quantum of the n-adjoint scalar field $\Phi^A = \Phi^{a_1 a_2 \cdots a_n}$ where the multi-index A contains an adjoint index a_i for each factor of the symmetry group $G_1 \times G_2 \cdots \times G_n$. A suitable interacting QFT Lagrangian is given by

$$\mathcal{L} = \frac{1}{2} \Phi^A (-\partial^2 + m^2) \Phi^A + \frac{y}{3!} F^{ABC} \Phi^A \Phi^B \Phi^C , \tag{2.28}$$

where

$$F^{ABC} = f^{a_1 b_1 c_1} f^{a_2 b_2 c_2} \cdots f^{a_n b_n c_n} . \tag{2.29}$$

The theory is nontrivial for n even, otherwise the potential vanishes as the F^{ABC} are then completely antisymmetric. In a quantum-background split $\Phi \to \phi + \Phi$, the quadratic part of the quantum field Lagrangian takes the same form of (2.4), but with a matrix-like background potential

$$\hat{\Phi}^{AB} = F^{ABC} \Phi^C . \tag{2.30}$$

The formula for the one-loop logarithmic divergences (2.17) still applies, now with

$$\begin{aligned} \mathrm{tr}\,\hat{\Phi}^2 &= \alpha \Phi^A \Phi^A \\ \mathrm{tr}\,\hat{\Phi}^3 &= \frac{\alpha}{2^n} F^{ABC} \Phi^A \Phi^B \Phi^C \ , \end{aligned} \tag{2.31}$$

where

$$\alpha = \prod_{i=1}^{n} T_i(A) \tag{2.32}$$

is the product of the indices of the adjoint representation for each factor G_i. Here we have taken n to be an even integer. Note that α is always a positive number. These divergences are renormalized by adding to the Lagrangian (2.28) the counterterms

$$\mathcal{L}_{\rm ct} = (Z_\Phi - 1)\tfrac{1}{2}\Phi^A(-\partial^2)\Phi^A + (Z_y - 1)\tfrac{y}{3!} F^{ABC}\Phi^A\Phi^B\Phi^C \tag{2.33}$$

with

$$Z_\Phi - 1 = -\frac{\alpha}{6}\frac{y^2}{(4\pi)^3}\frac{1}{\epsilon} \ , \qquad Z_y - 1 = -\frac{\alpha}{2^n}\frac{y^2}{(4\pi)^3}\frac{1}{\epsilon} \ , \tag{2.34}$$

producing the one-loop beta function

$$\beta(y) = \frac{\alpha}{4}\left(1 - \frac{1}{2^{n-2}}\right)\frac{y^3}{(4\pi)^3} \ . \tag{2.35}$$

Evidently, the beta function for $n > 2$ is positive, while it vanishes for $n = 2$, and it is negative for $n = 0$ (setting $\alpha = 1$ in such a case), consistently with the asymptotic freedom of the latter case. These results are in agreement with those of ref. [76].

2.1.3 Beta function for symmetric multi-adjoint scalar

As a final example, let us consider a different theory for the n-adjoint scalar by taking only groups of the type $SU(N)$ with $N \geq 3$, which admit the completely symmetric invariant tensor d^{abc}. Models for a single symmetric tensor were introduced long time ago to study critical phenomena in condensed matter physics, see, e.g., refs. [87, 88] and refs. [89, 90, 85] for more recent discussions. Thus, let us consider the replacement in (2.28)

$$F^{ABC} \quad \to \quad D^{ABC} = d^{a_1b_1c_1}d^{a_2b_2c_2}\cdots d^{a_nb_nc_n} \ , \tag{2.36}$$

which is consistent for all integers $n \in \mathbb{N}$, as D^{ABC} is now always totally symmetric. The corresponding one-loop effective action is again represented as in (2.8), but now with

$$\hat{\Phi}^{AB} = D^{ABC}\Phi^C \tag{2.37}$$

which produces

$$\begin{aligned} \mathrm{tr}\,\hat{\Phi}^2 &= x_1 \delta^{AB}\Phi^A\Phi^B \ , \\ \mathrm{tr}\,\hat{\Phi}^3 &= x_2 D^{ABC}\Phi^A\Phi^B\Phi^C \ , \end{aligned} \tag{2.38}$$

with coefficients x_1 and x_2 that depend on the chosen groups. They are computed by

$$x_1 = \prod_{i=1}^{n}\frac{N_i^2 - 4}{N_i} \ , \quad x_2 = \prod_{i=1}^{n}\frac{N_i^2 - 12}{2N_i} \ , \tag{2.39}$$

with N_i being the dimension of the fundamental representation of each copy of the $SU(N_i)$ group, see [91] whose normalization for the d^{abc} tensor we adopt. From the one-loop logarithmic divergences of the effective action (2.17) we extract the counterterms

$$\mathcal{L}_{\rm ct} = (Z_\Phi - 1)\tfrac{1}{2}\Phi^A(-\partial^2)\Phi^A + (Z_y - 1)\tfrac{y}{3!}D^{ABC}\Phi^A\Phi^B\Phi^C \tag{2.40}$$

with

$$Z_\Phi - 1 = -\frac{x_1}{6}\frac{y^2}{(4\pi)^3}\frac{1}{\epsilon}\,, \qquad Z_y - 1 = -x_2\frac{y^2}{(4\pi)^3}\frac{1}{\epsilon}\,, \tag{2.41}$$

producing the one-loop beta function

$$\beta(y) = \frac{y^3}{(4\pi)^3}\left(\frac{x_1}{4} - x_2\right) \tag{2.42}$$

that is

$$\beta(y) = \frac{y^3}{(4\pi)^3}\left\{\frac{1}{4}\left(\prod_{i=1}^{n}\frac{N_i^2-4}{N_i}\right)\left[1 - \frac{1}{2^{n-2}}\prod_{i=1}^{n}\frac{N_i^2-12}{N_i^2-4}\right]\right\}. \tag{2.43}$$

This formula includes the well-known asymptotic freedom case for $n = 0$ (with $x_1 = x_2 = 1$), while for $n = 1$ the result is consistent with [89] upon mapping different conventions, showing a positive beta function for $N \leq 4$, and a negative beta function (and thus asymptotic freedom) for $N \geq 5$. Further, the bi-adjoint particle ($n = 2$) has beta function

$$\beta(y) = \frac{y^3}{(4\pi)^3}\frac{2}{N_1N_2}\left(N_1^2 + N_2^2 - 16\right)\,, \tag{2.44}$$

which is non-vanishing and always positive, independently of the dimension of the $SU(N)$ groups ($N_i \geq 3$, as otherwise the d^{abc} symbols would vanish). Quite generally, the one-loop beta function is positive for $n \geq 2$ in these models.

2.2 Effective action in the color variables approach

As we have seen above, starting from the QFT lagrangian, identifying the dressed kinetic term with the background field method, and then, finding a path integral representation for the inverse kinetic term, generates a matrix valued path integral. Then, in order to preserve gauge invariance, the construction of the path integral requires a path ordering prescription of the exponentiated action appearing inside the path integral. However the path ordering prescription might be cumbersome if one need to identify Feynman rules for the worldline theory. As we will see the derivation of such rules is a crucial part of the WQFT, thus, with an eye on classical applications, we need to find a different approach to build up the worldline bi-adjoint theory.

A way to get rid of path ordering it to use color variables. The quantization of the latter gives rise to creation/annihilation operators that create color degrees of freedom in the Hilbert space of the particle. At the same time their worldline propagator reconstructs the path ordering. Eventually, a projection on a fixed occupation number of the color degrees of freedom selects the precise representation of the symmetry group (the color charge) that one is willing to assign to the particle. This last step is achieved by coupling the color variables to a $U(1)$ worldline gauge field with an additional discrete Chern-Simons coupling, that fixes the chosen color occupation number in the Hilbert space [65]. Thus, recalling that for a relativistic particle the Hamiltonian must be gauged[3], one is led to the following worldline action in phase space (subscript 'ps') for the bi-adjoint scalar in real time τ

$$S = \int \mathrm{d}\tau \Big[p_\mu \dot{x}^\mu + \mathrm{i}\bar{c}_a \dot{c}^a + \mathrm{i}\bar{d}_\alpha \dot{d}^\alpha - eH - aJ - \tilde{a}\tilde{J} \Big] \; , \tag{2.45}$$

where $H, J, \tilde{J}$ denote first class constraints

$$H = \frac{1}{2}(p^2 + m^2 - y\, Q^a \Phi^{a\alpha}(x) \tilde{Q}^\alpha) \; , \qquad Q^a = -\mathrm{i} f^{abc} \bar{c}^b c^c \; , \quad \tilde{Q}^\alpha = -\mathrm{i} \tilde{f}^{\alpha\beta\gamma} \bar{d}^\beta d^\gamma \; , \tag{2.46}$$

$$J = \bar{c}_a c^a - s \; , \tag{2.47}$$

$$\tilde{J} = \bar{d}_\alpha d^\alpha - \tilde{s} \; , \tag{2.48}$$

gauged by the fields $e, a, \tilde{a}$, i.e. the einbein and the two independent worldline $U(1)$ gauge fields, respectively, while the constants s and $\tilde{s}$ are suitable Chern-Simons couplings. The color variables can be taken to be either commuting or anticommuting. Here we choose the first option, which as a bonus may allow to study a classical limit of the color charge, as described by Wong's equations [53]. The kinetic term defines the phase space symplectic form, leading to the Poisson brackets. The nonvanishing Poisson brackets are given by $\{x^\mu, p_\nu\} = \delta^\mu_\nu$, $\{c^a, \bar{c}_b\} = -i\delta^a_b$, $\{d^\alpha, \bar{d}_\beta\} = -i\delta^\alpha_\beta$, and used to verify that the constraints are indeed first class

$$\{H, J\} = \{H, \tilde{J}\} = \{J, \tilde{J}\} = 0 \; . \tag{2.49}$$

Thus, the constraints can be gauged consistently. Notice also the following Poisson brackets

$$\{Q^a, Q^b\} = f^{abc} Q^c \; , \quad \{\tilde{Q}^\alpha, \tilde{Q}^\beta\} = \tilde{f}^{\alpha\beta\gamma} \tilde{Q}^\gamma \; , \quad \{Q^a, \tilde{Q}^\alpha\} = 0 \; , \quad \{J, Q^a\} = 0 \; , \quad \text{etc.} \tag{2.50}$$

Notice how the generator $Q^a, \tilde{Q}^\alpha$ obeys the gauge group color algebra, and, in addition, by studying its equation of motions, it is easy to see that it is the color charge of point particle. This will be important later on in the studying of the classical limit of observables related to colored particles. The precise value of the Chern-Simons couplings s and $\tilde{s}$ will be specified after transition to the quantum theory, which we discuss next using canonical quantization.

The quantum theory of the particle has an associated Hilbert space of "wave-functions" on which the fundamental quantum operators act. These operators are the particle position and momentum, $\hat{x}^\mu, \hat{p}_\mu$, and two pairs of color variables, $\hat{c}^a, \hat{c}^\dagger_a$ and $\hat{d}^\alpha, \hat{d}^\dagger_\alpha$. The latter act as sets of creation and annihilation operators. Considering only the first set of color variables, $\hat{c}^a$, $\hat{c}^\dagger_a$, they satisfy

$$[\hat{c}^a, \hat{c}^\dagger_b] = \delta^a_b, \quad [\hat{c}^a, \hat{c}^b] = 0 = [\hat{c}^\dagger_a, \hat{c}^\dagger_b] \tag{2.51}$$

with indices running up to $N_A = \dim G$, the dimension of the adjoint representation. They are naturally represented by $\hat{c}^\dagger_a \sim \bar{c}_a$ and $\hat{c}^a \sim \partial/\partial\bar{c}_a$ when acting on wave-functions of the form $\phi(x, \bar{c})$. The Taylor expansion of the latter reads

$$\phi(x, \bar{c}) = \phi(x) + \phi^a(x)\bar{c}_a + \frac{1}{2!}\phi^{ab}(x)\bar{c}_a\bar{c}_b + \ldots , \tag{2.52}$$

[3]The corresponding gauge field (the einbein) will eventually lead to the Fock-Schwinger proper time of the previous section.

and exposes sectors with different occupation numbers for the color variables, as measured by the number operator $\hat{N} = \hat{c}_a^\dagger \hat{c}^a \sim \bar{c}^a \partial/\partial \bar{c}^a$. If one is interested only on wave-functions transforming in the adjoint, and not on tensor products of the adjoint, one must impose a constraint that fixes the occupation number to 1, i.e.,

$$(\hat{N} - 1)\phi(x, \bar{c}) = \Big(\bar{c}^a \frac{\partial}{\partial \bar{c}^a} - 1\Big)\phi(x, \bar{c}) = 0 \ . \tag{2.53}$$

This is precisely the restriction that the quantum version of the first class constraint (2.47) should impose for meeting our purposes. The quantization of this constraint must resolve ordering ambiguities, which we fix as in the harmonic oscillator by requiring a symmetric ordering

$$J = \bar{c}_a c^a - s = \frac{1}{2}(\bar{c}_a c^a + c^a \bar{c}_a) - s \ , \tag{2.54}$$

leading to

$$\hat{J} = \frac{1}{2}(\hat{c}_a^\dagger \hat{c}^a + \hat{c}^a \hat{c}_a^\dagger) - s = \hat{c}_a^\dagger \hat{c}^a + \frac{N_A}{2} - s = \bar{c}^a \frac{\partial}{\partial \bar{c}^a} + \frac{N_A}{2} - s \ . \tag{2.55}$$

It is now simple to fix the Chern-Simons coupling s to achieve occupation number 1

$$\frac{N_A}{2} - s = -1 \qquad \to \qquad s = 1 + \frac{N_A}{2} \ . \tag{2.56}$$

A similar coupling with $\tilde{s} = 1 + \frac{\tilde{N}_A}{2}$ must be used for the other color sector as well, so to have physical wave-functions transforming in the bi-adjoint, as desired. Thus, the particle wave-function $\phi(x, \bar{c}, \bar{d})$ that satisfies the quantum constraints corresponding to J and $\tilde{J}$ has the form

$$\phi(x, \bar{c}, \bar{d}) = \phi^{a\alpha}(x) \bar{c}_a \bar{d}_\alpha \tag{2.57}$$

and describes a wave-function carrying indices of the bi-adjoint. The remaining quantum constraint is the one corresponding to the Hamiltonian H in (2.46), whose operator version takes the form

$$\hat{H} = \frac{1}{2}\Big(-\partial^2 + m^2 + y\Phi^{a\alpha}(x) f^{abc} \bar{c}^b \frac{\partial}{\partial \bar{c}^c} \tilde{f}^{\alpha\beta\gamma} \bar{d}^\beta \frac{\partial}{\partial \bar{d}^\gamma}\Big) \ . \tag{2.58}$$

The constraint equation $\hat{H}\phi(x, \bar{c}, \bar{d}) = 0$ reproduces precisely the equations of motion of the bi-adjoint field obtained from the usual Lagrangian (2.1). Note that simple Lie groups have traceless generators, so that in this case the ordering of the color operators in the quantum version of C^a and $\tilde{C}^\alpha$ is inconsequential. We have thus completed the derivation of the worldline model for the bi-adjoint field theory.

In order for the theory to be used to perform calculations we must gauge fix the worldline gauge fields. A covariant gauge fixing can be imposed by setting $(e(\tau), a(\tau), \tilde{a}(\tau)) = (2T, \theta, \phi)$, with T identified with the usual Fock-Schwinger proper time, and (θ, ϕ) two additional moduli, corresponding to two angles, related to the projection on occupation number 1 for each color sector. This gauge fixing is valid both for worldlines with the topology of a circle, as appropriate for the one-loop effective action, and for worldlines with the topology of an interval, as appropriate for the propagator. The difference is just on the measure on the moduli space. After going in configuration space, by inverting the momenta in the phase space action we finally get the gauge fixed Euclidean worldline action

$$S_E = Tm^2 - is\theta - i\tilde{s}\phi + \int_0^1 \mathrm{d}\tau \Big[\frac{1}{4T}\dot{x}^2 + \bar{c}_a(\partial_\tau + \mathrm{i}\theta)c^a + \bar{d}_\alpha(\partial_\tau + \mathrm{i}\phi)d^\alpha - TyQ^a\Phi^{a\alpha}(x)\tilde{Q}^\alpha\Big]. \quad (2.59)$$

where, using reparametrization invariance, we allowed the worldline parameter τ to move in the interval $[0,1]$.

For completeness, let us consider again the case of a particle symmetric under the product of an arbitrary number of simple Lie groups, $G_1 \times G_2 \times \cdots \times G_n$, with the wave-function transforming in an arbitrary representation R_i with generators $(T^{a_i}_{R_i})^{\alpha_i}{}_{\beta_i}$ for each factor G_i. To propagate such particle on the worldline, using the color variables formulation, we introduce a set of color variables $c_i^{\alpha_i}(\tau)$ and complex conjugate $\bar{c}^i_{\alpha_i}(\tau)$, with $i = 1, \dots, n$, and indices that match those of the chosen representation, to construct the color charges (no sum over the index i)

$$Q_i^{a_i}(\tau) = \bar{c}^i_{\alpha_i}(\tau)(T^{a_i}_{R_i})^{\alpha_i}{}_{\beta_i} c_i^{\beta_i}(\tau) \quad (2.60)$$

and write down the action for the phase space coordinates $(x, p, c_i, \bar{c}_i)$ and worldline gauge variables (e, a_i) as follows

$$S = \int_0^1 d\tau \Big[p_\mu \dot{x}^\mu - \frac{e}{2}(p^2 + m^2 + U(\Phi(x), Q_i)) + \sum_{i=1}^n (\mathrm{i}\bar{c}^i_{\alpha_i}(\partial_\tau + \mathrm{i}a_i)c_i^{\alpha_i} + s_i a_i)\Big]. \quad (2.61)$$

The potential $U(\Phi(x), Q_i)$ is taken to depend on suitable scalar background fields, collectively denoted by Φ, and color charges Q_i as given above. Of course, the potential is required to be invariant under the global symmetry group. Our previous bi-adjoint particle (2.45) is a special case of this more general model with $n = 2$, color variables $c_i, \bar{c}_i$ transforming in the adjoint representation of the corresponding group, and scalar potential $U(\Phi(x), Q_i)$ taken as in (2.46).

Let us now come back to the case of the bi-adjoint scalar, moving to the construction of the path integral on the loop, from the above action. The general structure of the one-loop effective action is

$$\Gamma[\Phi] = \int_{S^1} \frac{DxDeDaD\tilde{a}DcD\bar{c}DdD\bar{d}}{\mathrm{Vol(Gauge)}} e^{-S_E} \quad (2.62)$$

with S_E given in eq. (2.59), with $\tau \in [0,1]$ describing the circle. To bring it in a computable form, we must gauge-fix it. Let us review the main steps adapted to the circle S^1.

As anticipated, on the circle one may fix the worldline gauge fields to constants $(e(\tau), a(\tau), \tilde{a}(\tau)) = (2T, \theta, \phi)$, with the latter playing the role of moduli, i.e. gauge invariant configurations that must be integrated over a suitable moduli space [69]. The measure on the moduli space is given by Faddeev-Popov determinants, which are constant in our case (i.e. they do not depend on the moduli), except that there is a factor $\frac{1}{T}$ that takes into account the symmetry generated by constant translations on the circle: this factor avoids relative overcountings of paths with different proper time T. Then, fixing the overall normalization to match that of a real scalar, we have

$$\Gamma[\Phi] = -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}T}{T}\int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\phi}{2\pi}\int_{\mathcal{P}} DxDcD\bar{c}DdD\bar{d}\; e^{-S_E} \;, \tag{2.63}$$

where $\mathcal{P}$ denotes periodic boundary conditions for the remaining path integration variables. The integration over the moduli space is set to cover the moduli space only once. The remaining path integral is normalized to that of a free particle, arising when the interactions are set to vanish (more on this later).

Let us further manipulate the remaining path integral to unearth the zero modes that may be present on the circle. Let us first address the zero modes of the coordinates x^μ. They are present when the potential is treated perturbatively. These zero modes satisfy $\partial_\tau^2 x^\mu(\tau) = 0$ and are therefore constant configurations. Their separation is generically achieved by splitting

$$x^\mu(\tau) = x_0^\mu + q^\mu(\tau) \;, \tag{2.64}$$

where x_0^μ are the constant zero modes, and $q^\mu(\tau)$ are the remaining quantum fluctuations without the zero modes. There is a variety of ways of achieving this split, with the most common ones corresponding to setting Dirichlet boundary conditions on $q^\mu(\tau)$ (i.e. imposing $q(0) = q(1) = 0$) or using the "string-inspired" boundary conditions (that is requiring $\int_0^1 \mathrm{d}\tau\, q^\mu(\tau) = 0$). They give equivalent results, see [84] for a recent application and comparison of the two methods. The integration of the zero modes factorizes, as perturbatively there is no action for them, and there remains the path integral over the quantum fluctuations q^μ. At this stage, we may extract the zero modes x_0 to rewrite the path integral (2.63) as

$$\Gamma[\Phi] = -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}T}{T}\int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\phi}{2\pi}\int d^D x_0 \int_{\bar{\mathcal{P}}} Dq \int_{\mathcal{P}} DcD\bar{c}DdD\bar{d}\; e^{-S_E} \;, \tag{2.65}$$

where now $\bar{\mathcal{P}}$ denotes periodic boundary conditions with a constraint that eliminates the zero modes (it depends on the chosen method). We have considered arbitrary dimensions D, also in view of the application of dimensional regularization, though the case $D = 6$ is the most interesting one as in such dimensions the model is renormalizable.

Let us now address the color variables c and d. For non-vanishing constant gauge fields θ and ϕ, and periodic boundary conditions, the color variables do not have any zero mode. Indeed, θ and ϕ act as a kind of mass term, i.e. a Wick rotated frequency-squared term for these harmonic-like oscillators, see their action in (2.59). Zero modes appear only for $\theta = 0$ and $\phi = 0$, and we will consider their effect later on. It is also possible to remove the couplings to θ and ϕ by shifting those couplings to the boundary conditions by field redefinitions

$$c(\tau) \to e^{-\mathrm{i}\theta\tau}c(\tau),\;\; \bar{c}(\tau) \to e^{\mathrm{i}\theta\tau}\bar{c}(\tau)\;, \qquad d(\tau) \to e^{-\mathrm{i}\phi\tau}d(\tau),\;\; \bar{d}(\tau) \to e^{\mathrm{i}\phi\tau}\bar{d}(\tau)\;, \tag{2.66}$$

which implies that the new fields c and d thus obtained satisfy twisted boundary conditions (which we denote by $\mathcal{T}$)

$$c(1) = e^{i\theta}c(0), \quad \bar{c}(1) = e^{-i\theta}\bar{c}(0)\,, \qquad d(1) = e^{i\phi}d(0), \quad \bar{d}(1) = e^{-i\phi}\bar{d}(0)\;. \tag{2.67}$$

The effective action then takes the form

$$\Gamma[\Phi] = -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}T}{T} e^{-m^2T}\int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi}e^{\mathrm{i}s\theta}\int_0^{2\pi}\frac{\mathrm{d}\phi}{2\pi}e^{\mathrm{i}\tilde{s}\phi}\int d^Dx_0\int_{\bar{\mathcal{P}}}Dq\int_{\mathcal{T}}DcD\bar{c}DdD\bar{d}\; e^{-S_0}, \tag{2.68}$$

where the worldline action S_0 now reads

$$S_0 = \int_0^1 \mathrm{d}\tau\Big[\frac{1}{4T}\dot{q}^2 + \bar{c}_a\dot{c}^a + \bar{d}_\alpha\dot{d}^\alpha - TyQ^a\Phi^{a\alpha}(x)\tilde{Q}^\alpha\Big] \tag{2.69}$$

since we have extracted the mass term e^{-m^2T} and the Chern-Simons couplings $e^{\mathrm{i}s\theta+\mathrm{i}\tilde{s}\phi}$.

From this expression we see that in a perturbative expansion we may treat the potential as a perturbation, while recognizing from the kinetic term the free propagators of the particle coordinates q^μ and auxiliary color variables $c,\bar{c},d,\bar{d}$. Extracting the normalization due to the free path integral we find

$$\Gamma[\Phi] = \int d^Dx_0\left[-\frac{1}{2}\int_0^\infty\frac{\mathrm{d}T}{T}\frac{e^{-m^2T}}{(4\pi T)^{\frac{D}{2}}}\int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi}\frac{e^{\mathrm{i}s\theta}}{\left(2\mathrm{i}\sin\frac{\theta}{2}\right)^{N_A}}\int_0^{2\pi}\frac{\mathrm{d}\phi}{2\pi}\frac{e^{\mathrm{i}\tilde{s}\phi}}{\left(2\mathrm{i}\sin\frac{\phi}{2}\right)^{\tilde{N}_A}}\langle e^{-S_{\mathrm{int}}}\rangle\right] \tag{2.70}$$

with the term in square bracket representing the QFT one-loop effective Lagrangian $\mathcal{L}_{\mathrm{eff}}$ (that, of course, must be renormalized, as for example by treating the arbitrary dimension D as in dimensional regularization). The perturbation is here given by

$$S_{\mathrm{int}} = -Ty\int_0^1\mathrm{d}\tau\, Q^a\Phi^{a\alpha}(x)\tilde{Q}^\alpha\ . \tag{2.71}$$

The free propagators that go along with this representation, and needed to compute the average $\langle e^{-S_{\mathrm{int}}}\rangle$ by Wick contractions, take into account the boundary conditions. They are given by

$$\begin{aligned}\langle q^\mu(\tau)q^\nu(\sigma)\rangle &= -2T\delta^{\mu\nu}\Delta(\tau,\sigma)\ ,\\ \langle c^a(\tau)\bar{c}_b(\sigma)\rangle &= \delta^a_b\Delta_{\mathcal{T}}(\tau,\sigma;\theta)\ ,\\ \langle d^\alpha(\tau)\bar{d}_\beta(\sigma)\rangle &= \delta^\alpha_\beta\Delta_{\mathcal{T}}(\tau,\sigma;\phi)\ ,\end{aligned} \tag{2.72}$$

where[4]

$$\begin{aligned}\Delta(\tau,\sigma) &= \frac{1}{2}|\tau-\sigma| - \frac{1}{2}(\tau-\sigma)^2 - \frac{1}{12}\ ,\\ \Delta_{\mathcal{T}}(\tau,\sigma;\theta) &= \frac{1}{2\mathrm{i}\sin\frac{\theta}{2}}\left[e^{\mathrm{i}\frac{\theta}{2}}\Theta(\tau-\sigma)+e^{-\mathrm{i}\frac{\theta}{2}}\Theta(\sigma-\tau)\right]\ ,\end{aligned} \tag{2.73}$$

where $\Theta(x)$ is the standard Heaviside step-function with $\Theta(0)=\frac{1}{2}$.

Eq. (2.70) is our final form of the worldline representation of the one-loop effective action of the bi-adjoint scalar, with the additional color variables. It is readily calculable in perturbation theory.

[4]We use the string inspired method to factor out the zero modes from q. The constant term $-\frac{1}{12}$ could also be dropped from the propagator $\Delta(\tau,\sigma)$ [64]. As for the propagator $\Delta_{\mathcal{T}}(\tau,\sigma;\theta)$, one may check that it satisfies the Green's equation $\partial_\tau\Delta_{\mathcal{T}}(\tau,\sigma;\theta)=\delta(\tau-\sigma)$, and the twisted boundary conditions $\Delta_{\mathcal{T}}(1,\sigma;\theta)=e^{\mathrm{i}\theta}\Delta_{\mathcal{T}}(0,\sigma;\theta)$ and $\Delta_{\mathcal{T}}(\tau,1;\theta)=e^{-\mathrm{i}\theta}\Delta_{\mathcal{T}}(\tau,0;\theta)$.

2.2.1 Bi-adjoint scalar self-energy

It is easy to see that the color variables formulation can be equivalently used to reproduce the beta functions computed in the previous section, thus here we move to a different application of (2.70): the calculation of the self-energy of the bi-adjoint scalar, as described by Fig. 2.1.

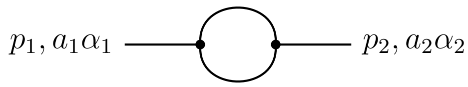


Figure 2.1: Self-energy contribution to the bi-adjoint propagator at one-loop.

To start with, let us first check the normalization of the effective action (2.70), by setting $S_{\rm int}$ to vanish, and verifying that it contains the correct number of degrees of freedom circulating in the loop. To do that, and to perform in a more convenient way the integration over the angular moduli, we change variables from θ and ϕ to the Wilson loop variables $z = e^{-{\rm i}\theta} = e^{-{\rm i}\int_0^1 {\rm d}\tau\, a(\tau)}$ and $w = e^{-{\rm i}\phi} = e^{-{\rm i}\int_0^1 {\rm d}\tau\, \tilde{a}(\tau)}$, so that the original integration is mapped to a contour integration, the unit circle γ of the complex plane for each variable, e.g. for θ

$$\int_0^{2\pi} \frac{d\theta}{2\pi} = \oint_\gamma \frac{dz}{2\pi {\rm i} z}\ , \quad 2{\rm i}\sin\tfrac{\theta}{2} = \frac{1-z}{\sqrt{z}}\ , \quad \int_0^{2\pi} \frac{{\rm d}\theta}{2\pi}\frac{e^{{\rm i}s\theta}}{\left(2{\rm i}\sin\frac{\theta}{2}\right)^{N_A}} = \oint_\gamma \frac{{\rm d}z}{2\pi{\rm i}}\frac{1}{z^2}\frac{1}{(1-z)^{N_A}}\ , \tag{2.74}$$

where we used the value of the Chern-Simons coupling $s = 1 + \frac{N_A}{2}$. One may notice that at $\theta = 0$ the auxiliary color variables $c, \bar{c}$ develop zero modes, which correspond to the pole at $z = 1$. This divergence is due to the determinant of the free path integral that vanishes at such a point because of the zero modes. Previously we postponed the discussion of these zero modes, but now we see their effect and the way we should treat them. They cause a singularity on the integration contour at $z = 1$, which we push out of the integration contour (indicated by γ_-, see Fig. 2.2). This prescription gives the expected answer. With it, only the poles at $z = 0$ are responsible for the projection and we get

$$\oint_{\gamma_-} \frac{{\rm d}z}{2\pi{\rm i}}\frac{1}{z^2}\frac{1}{(1-z)^{N_A}} = \frac{\rm d}{{\rm d}z}(1-z)^{-N_A}\Big|_{z=0} = N_A\ , \tag{2.75}$$

which is the expected number.

Note that inclusion of the pole $z = 1$ inside the contour would give a total vanishing result, which means that the previous answer can be obtained also by viewing the complex plane as the Riemann sphere, and considering only the poles outside the contour γ_-. This fact is related to a redefinition of the Wilson loop variable $z \to z' = \frac{1}{z} = e^{{\rm i}\theta}$, (or equivalently to a different gauge fixing for $a(\tau)$, namely $\theta \to -\theta$), that must give equivalent results. This change of variables maps the contour γ_- of the z-plane to the contour γ_+ of the z'-plane, which now includes the pole at $z' = 1$, see Fig. 2.3 (a change of sign has been taken care of by reversing the orientation of the contour).

Then eq. (2.70) reduces to

$$\Gamma[\Phi] = -\frac{1}{2}\int d^D x_0 \int_0^\infty \frac{{\rm d}T}{T}\frac{e^{-m^2 T}}{(4\pi T)^{\frac{D}{2}}}\left(N_A \tilde{N}_A + \dots\right)\ , \tag{2.76}$$

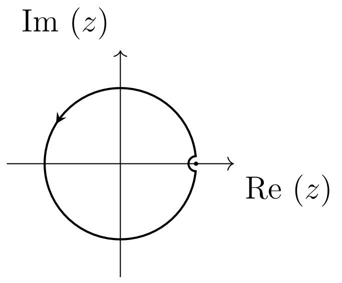


Figure 2.2: Integration contour γ_- in z-plane. Pole at $z = 1$ is excluded.

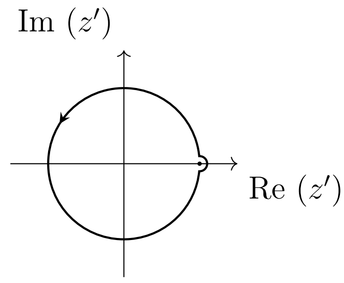


Figure 2.3: Integration contour γ_+ in z'-plane. Pole at $z' = 1$ is included.

which reproduces the expected degrees of freedom of the bi-adjoint scalar[5].

After this check, we are ready to study the self-energy of Fig. 2.1. For that we consider a background given by the sum of two-plane waves with quantum numbers a_1, α_1, p_1 and a_2, α_2, p_2

$$\Phi^{a\alpha}(x) = \delta^{aa_1}\delta^{\alpha\alpha_1}e^{\mathrm{i}p_1\cdot x(\tau)} + \delta^{aa_2}\delta^{\alpha\alpha_2}e^{\mathrm{i}p_2\cdot x(\tau)} \,. \tag{2.77}$$

Expanding S_{int} in (2.70) and keeping the contribution linear in each plane wave, we find the contribution to the self-energy is given by the vacuum expectation value

$$\langle e^{-S_{\text{int}}}\rangle \;\to\; y^2T^2\langle V(p_1, a_1, \alpha_1)V(p_2, a_2, \alpha_2)\rangle \,, \tag{2.78}$$

with the vertex operator on the worldline given by

$$V(p_1, a_1, \alpha_1) = y\int_0^1 d\tau\, Q^{a_1}(\tau)\tilde{Q}^{\alpha_1}(\tau)\, e^{ip_1\cdot q(\tau)}. \tag{2.79}$$

The integration over the zero modes produce a delta function for momentum conservation and one finds the self-energy correction

$$\Gamma[\Phi] \;\to\; (2\pi)^D\delta^D(p_1+p_2)\Pi^{a_1\alpha_1,a_2\alpha_2}(p) \,, \tag{2.80}$$

where $p \equiv p_1 = -p_2$ and

$$\begin{aligned}\Pi^{a_1\alpha_1,a_2\alpha_2}(p) = -\frac{y^2}{2}\int_0^\infty \mathrm{d}T\frac{e^{-m^2T}T}{(4\pi T)^{\frac{D}{2}}}\oint_{\gamma_-}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{z^{-2}}{(1-z)^{N_A}}\oint_{\gamma_-}\frac{\mathrm{d}w}{2\pi\mathrm{i}}\frac{w^{-2}}{(1-w)^{\tilde{N}_A}} \\ \times \langle V_q(p, a_1, \alpha_1)V_q(-p, a_2, \alpha_2)\rangle\end{aligned} \tag{2.81}$$

with the vertex operators V_q depending only on the q coordinates without the zero modes. Carrying out the Wick contractions, one finds

$$\begin{aligned}&\langle V_q(p, a_1, \alpha_1)V_q(-p, a_2, \alpha_2)\rangle \\ &= \int_0^1 d\tau\int_0^1 d\sigma\, \langle Q^{a_1}(\tau)Q^{a_2}(\sigma)\rangle\langle\tilde{Q}^{\alpha_1}(\tau)\tilde{Q}^{\alpha_2}(\sigma)\rangle\langle e^{ip\cdot q(\tau)}\, e^{-ip\cdot q(\sigma)}\rangle \\ &= f^{a_1bc}f^{a_2bc}\tilde{f}^{\alpha_1\beta\gamma}\tilde{f}^{\alpha_2\beta\gamma}\frac{zw}{(1-z)^2(1-w)^2}\int_0^1 d\tau\int_0^1 d\sigma\, e^{-2Tp^2\Delta_0(\tau,\sigma)} \\ &= \delta^{a_1a_2}\delta^{\alpha_1\alpha_2}T(A)\tilde{T}(A)\frac{zw}{(1-z)^2(1-w)^2}\int_0^1 d\tau\, e^{-Tp^2(\tau-\tau^2)} \,,\end{aligned} \tag{2.82}$$

[5]A factor of 1 instead of $N_A\tilde{N}_A$ corresponds to a free real scalar.

where $\Delta_0(\tau,\sigma) = \Delta(\tau,\sigma) - \Delta(\tau,\tau)$, and we used translational invariance of the string inspired propagator to get the last line. Next, using that the complex integration gives one, we find that the proper time integration yields

$$\Pi^{a_1\alpha_1,a_2\alpha_2}(p) = -\frac{y^2}{2(4\pi)^{\frac{D}{2}}}\delta^{a_1a_2}\delta^{\alpha_1\alpha_2}T(A)\tilde{T}(A)\,(P^2)^{\frac{D}{2}-2}\,\Gamma\left(2-\tfrac{D}{2}\right)\,, \tag{2.83}$$

where we defined

$$(P^2)^x \equiv \int_0^1 d\tau\,\left(m^2 + p^2(\tau-\tau^2)\right)^x\ . \tag{2.84}$$

This is the unrenormalized contribution to the self-energy. As a check, one may extract the UV divergences related to wave-function and mass renormalizations using dimensional regularization, which are easily seen to match the ones calculated in the previous Section.

We have used the Chern-Simons couplings to select occupation number 1 in each colored sector, thus making sure that there is precisely a bi-adjoint scalar particle circulating in the loop. However, one could modify our worldline theory by selecting different occupation numbers, say r and $\tilde{r}$, for the color variables. This corresponds to a differently charged particle circulating in the loop, but coupled to the same background bi-adjoint field $\Phi^{a\alpha}$, see Fig. 2.4.

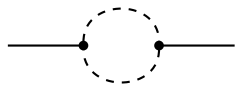
$p_1, a_1\alpha_1$  $p_2, a_2\alpha_2$

Figure 2.4: Self-energy contribution to the bi-adjoint propagator at one-loop due to a virtual scalar particle. The charge of the particle in the loop is specified by occupation numbers r and $\tilde{r}$.

This is achieved by setting the Chern-Simons couplings to

$$s = r + \frac{N_A}{2}\,, \qquad \tilde{s} = \tilde{r} + \frac{\tilde{N}_A}{2}\,, \tag{2.85}$$

as compared to our previous eq. (2.56). For example, setting $(r,\tilde{r}) = (0,0)$ gives an uncharged particle that should decouple from the loop, setting $(r,\tilde{r}) = (2,0)$ would give a particle that transforms in the symmetric products of two adjoints for the group G and scalar under $\tilde{G}$, which should also decouple, while setting $(r,\tilde{r}) = (2,1)$ would give a particle that transforms in the symmetric products of two adjoints for the group G and in the adjoint for $\tilde{G}$, which should give a nontrivial contribution to the self-energy, as depicted in Fig. 2.4. Thus, using the above Chern-Simons couplings, which modify eq. (2.74), we find the following contribution to the self-energy

$$\Pi^{a_1\alpha_1,a_2\alpha_2}_{r\tilde{r}}(p) = -\frac{y^2_{r\tilde{r}}}{2(4\pi)^{\frac{D}{2}}}\delta^{a_1a_2}\delta^{\alpha_1\alpha_2}T(A)\tilde{T}(A)\binom{N_A+r}{N_A+1}\binom{\tilde{N}_A+\tilde{r}}{\tilde{N}_A+1}(P^2)^{\frac{D}{2}-2}\,\Gamma\left(2-\tfrac{D}{2}\right)\,, \tag{2.86}$$

where we have denoted by $y_{r\tilde{r}}$ the coupling constant of the new particle to the bi-adjoint field, and where the definition of $(P^2)^x$ in eq. (2.84) should contain a different mass $m_{r\tilde{r}}$ instead of m. The binomial coefficients are defined to vanish for $r = 0$ and $\tilde{r} = 0$, as usual, as one may check that in those cases there are no poles in the Cauchy integrals over the moduli. The particle with charge $r, \tilde{r}$ corresponds to a scalar field $\varphi^{i,x}$ of mass $m_{r\tilde{r}}$ that would couple to the bi-adjoint $\Phi^{a\alpha}$ with a potential of the form

$$\mathcal{L}_{\text{int}} = -\frac{y_{r\tilde{r}}}{2}(T^a_r)^{ij}(\tilde{T}^\alpha_{\tilde{r}})^{xy}\Phi^{a\alpha}\varphi^{ix}\varphi^{jy} \ , \tag{2.87}$$

where by T^a_r we indicate the generators in the symmetric tensor product of r adjoint representations. Note that such a representation has an index $T(r) = T(A)\binom{N_A+r}{N_A+1}$, which indeed is what appears in (2.86). For instance, in the case where the color group is $SU(N)$ we have that $T(A) = N$ and $N_A = N^2 - 1$.

2.2.2 Multi-adjoint scalar self-energy and effective vertices

One can now repeat the exercise of Section 2.2.1, and calculate the one-loop self-energy correction to the propagator of the n-adjoint scalar due to particles that sits in the r_i representation (meaning the symmetric tensor product of r_i copies of the adjoint) of the group G_i. Denoting by $N_{A,i}$ the dimension of the group G_i, we can write the self-energy as

$$\Pi^{BC}_r(p) = -\frac{y_r^2}{2(4\pi)^{\frac{D}{2}}}\delta^{BC}(P^2)^{\frac{D}{2}-2}\Gamma\left(2-\tfrac{D}{2}\right)\prod_{i=1}^{n} T_i(A)\binom{N_{A,i}+r_i}{N_{A,i}+1} \ , \tag{2.88}$$

where by r we indicate the vector with components r_i. The definition of $(P^2)^x$ given in eq. (2.84) now depends on a new mass m_r. The case of the pure n-adjoint field propagating in the loop is obtained by setting $r_i = 1$ for all i. Again, the group theory factor $\prod_{i=1}^n T_i(A)\binom{N_{A,i}+r_i}{N_{A,i}+1}$ can be rewritten as the product of the indices of the r_i representations, namely $\prod_{i=1}^n T_i(r_i)$. Similarly one could proceed as before to compute the beta functions for the couplings y and y_r, showing in this case a nontrivial mixing.

The methods we have developed in this section can be generalized to higher point correlation functions at 1-loop, taking (2.70) as starting point. Let us now discuss briefly how to get a general formula for the 1PI higher-point correlation functions at one-loop in momentum space, by taking (2.70) as starting point. The generalization of the plane wave expansion of the scalar field (2.77) is given by

$$\Phi^{a\alpha}(x_0+q) = \sum_{\ell=1}^{N}\delta^{aa_\ell}\delta^{\alpha\alpha_\ell}e^{\mathrm{i}p_\ell\cdot x_0}e^{\mathrm{i}p_\ell\cdot q(\tau)} \ , \tag{2.89}$$

which can be inserted in (2.70), with the constraint that each plane wave should appear only once. The integration over the loop base point produces a momentum conservation delta function which enables us to rewrite the 1-loop effective action as

$$\Gamma[\Phi] = (2\pi)^D\delta^D\left(\sum_{\ell=1}^{N}p_\ell\right)\Gamma^{a_1\alpha_1\cdots a_N\alpha_N}_{r\tilde{r}}(p_1\cdots p_n) \ . \tag{2.90}$$

Stripping off the momentum conservation Dirac delta function we define the 1PI N-point function as

$$\Gamma^{a_1\alpha_1\dots a_N\alpha_N}_{r\tilde{r}}(p_1\dots p_n) = -\frac{1}{2}y^N_{r\tilde{r}}\oint_{\gamma_-}\frac{dz}{2\pi\mathrm{i}}\frac{1}{z^{r+1}(1-z)^{N_A}}\oint_{\gamma_-}\frac{dw}{2\pi\mathrm{i}}\frac{1}{w^{\tilde{r}+1}(1-w)^{\tilde{N}_A}}$$
$$\int_0^\infty\frac{dT}{T}\frac{e^{-m^2_{r\tilde{r}}T}}{(4\pi T)^{\frac{D}{2}}}T^N\left(\prod_{\ell=1}^N\int_0^1 d\tau_\ell\right)\left\langle e^{\mathrm{i}\sum_{k=1}^N p_k q(\tau_k)}\right\rangle\Big\langle\prod_{k=1}^N Q^{a_k}(\tau_k)\Big\rangle\Big\langle\prod_{\ell=1}^N\tilde{Q}^{\alpha_\ell}(\tau_\ell)\Big\rangle\,, \tag{2.91}$$

where we also switched to the Wilson loop variables defined in (2.74). The v.e.v of the kinematical part of the vertex operator can be evaluated using the identity

$$\left\langle e^{\mathrm{i}\sum_{k=1}^N p_k\cdot q(\tau_k)}\right\rangle = e^{T\sum_{k,\ell=1}^N p_k\cdot p_\ell\Delta(\tau_k,\tau_\ell)}. \tag{2.92}$$

The integral over proper time T can the be performed exactly

$$\begin{aligned}&\int_0^\infty dT\, e^{-T\left(m^2_{r\tilde{r}}-\sum_{k,\ell=1}^N p_k\cdot p_\ell\Delta(\tau_k,\tau_\ell)\right)}T^{N-\frac{D}{2}-1}\\ &\qquad=\Gamma\left(N-\tfrac{D}{2}\right)\left(m^2_{r\tilde{r}}-\sum_{k,\ell=1}^N p_k\cdot p_\ell\Delta(\tau_k,\tau_\ell)\right)^{\frac{D}{2}-N}.\end{aligned} \tag{2.93}$$

Finally, putting all of the pieces together we get our desired master formula

$$\Gamma^{a_1\alpha_1\cdots a_N\alpha_N}_{r\tilde{r}}(p_1,p_2,\dots,p_N) = -\frac{1}{2}\frac{y^N_{r\tilde{r}}}{(4\pi)^{\frac{D}{2}}}\oint_{\gamma_-}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{1}{z^{r+1}(1-z)^{N_A}}\oint_{\gamma_-}\frac{\mathrm{d}w}{2\pi\mathrm{i}}\frac{1}{w^{\tilde{r}+1}(1-w)^{\tilde{N}_A}}$$
$$\Gamma(N-\tfrac{D}{2})\left(\prod_{\ell=1}^N\int_0^1 d\tau_\ell\right)\left(m^2_{r\tilde{r}}-\sum_{k,\ell=1}^N p_k\cdot p_\ell\,\Delta(\tau_k,\tau_\ell)\right)^{\frac{D}{2}-N}\Big\langle\prod_{k=1}^N Q^{a_k}(\tau_k)\Big\rangle\Big\langle\prod_{\ell=1}^N\tilde{Q}^{\alpha_\ell}(\tau_\ell)\Big\rangle, \tag{2.94}$$

which holds in the case where the particle propagating in the loop has charge $r,\tilde{r}$ with respect to both Lie groups. The (amputated) external lines correspond instead to the plane waves of the bi-adjoint field $\Phi^{a\alpha}$.

2.2.3 Coupling to gauge fields and double copy of vertex operators

Introducing gauge fields to the theory is straightforward. Starting with the phase space action (2.61) we add gauge field interactions by the minimal substitution

$$p_\mu \to p_\mu - \sum_i g_i A^i_\mu \tag{2.95}$$

in the constraint, with the non-abelian gauge fields A^i_μ now given by (no sum over i)

$$A^i_\mu = A^{a_i,i}_\mu(x)Q^{a_i}_i = A^{a_i,i}_\mu(x)\,\bar{c}^i_{\alpha_i}(\tau)(T^{a_i}_{R_i})^{\alpha_i}{}_{\beta_i}c^{\beta_i}_i(\tau)\,, \tag{2.96}$$

which also uses the composite color charge (2.60). In order to gauge only a subset of the symmetry groups one may simply set the remaining gauge couplings to zero. For instance, considering again the bi-adjoint case given by the action (2.61) with $n=2$ and generators in the adjoint, the coupling to a single gauge field can be implemented by, say, setting $g_1 \equiv g$ and $g_2 = 0$. Eliminating the momenta through their equations of motion, using $(T^a_{\rm A})^b{}_c = -if^{abc}$, and Wick rotating, produces the action

$$S[x, c, \bar{c}, d, \bar{d}, e, a, \tilde{a}; \Phi, A_\mu] = \int_0^1 d\tau \Bigg[\frac{1}{2}e^{-1}\dot{x}^2 - gf^{abc}\dot{x}_\mu A^{\mu a}(x)\bar{c}^b c^c + \frac{e}{2}(m^2 - yQ^a\Phi^{a\alpha}(x)\tilde{Q}^\alpha) + \bar{c}_a(\partial_\tau + {\rm i}a)c^a + \bar{d}_\alpha(\partial_\tau + {\rm i}\tilde{a})d^\alpha - {\rm i}s(a + \tilde{a})\Bigg] , \tag{2.97}$$

where we used again the notation of previous sections. It is easy to see that the action (2.97) leads to two types of vertices, namely the spin one vertex

$$V^{(1)}[k, \varepsilon, A; x, c, \bar{c}] = \int_0^1 d\tau \ \varepsilon \cdot \dot{x}(\tau) \ \bar{c}_{a'}(\tau)(T^a)^{a'}{}_{b'} \, c^{b'}(\tau) \ e^{{\rm i}k\cdot x(\tau)} , \tag{2.98}$$

representing the insertion of an external gluon on the worldline of a massive scalar particle, and the vertex operator (2.79), inserting a bi-adjoint scalar on the worldline. Such vertex operators must be considered separately. Using dimensional reduction, similar types of interactions and corresponding amplitudes have been considered in the context of the CHY representation [92] and the double copy for supersymmetric theories [93, 94]. There, quartic scalar potentials are also included. Starting from our general approach these potentials may be easily added to the theory.

An interesting feature of the action (2.97) lies in a double copy relation holding at the level of vertex operators in different worldline theories. In the spirit of the double copy, we may define the kinematic factor $K(\tau) = \varepsilon \cdot \dot{x}(\tau)$ and the color factor $Q^a(\tau) = \bar{c}_{a'}(\tau)(T^a)^{a'}{}_{b'} \, c^{b'}(\tau)$, and consider in (2.98) the replacement

$$Q^a(\tau) \ \to \ \tilde{K}(\tau) , \tag{2.99}$$

where the tilde indicates dependence on another polarization vector $\tilde{\varepsilon}$. We obtain the following vertex operator

$$V^{(2)}[k, \varepsilon; x] = \int_0^1 d\tau \ \varepsilon \cdot \dot{x}(\tau) \ \tilde{\varepsilon} \cdot \dot{x}(\tau) \ e^{{\rm i}k\cdot x(\tau)} , \tag{2.100}$$

which should correspond to a vertex operator for the emission/absorption of a graviton [95]. This is indeed correct, after identifying the graviton polarization by $\varepsilon_\mu\tilde{\varepsilon}_\nu \to \varepsilon_{\mu\nu}$, and eventually taking into account prescriptions for regulating UV ambiguities on the worldline (this is typical for the coupling of a particle to gravity [96, 97]).

Now, inspired by the zeroth copy, let us consider instead the replacement

$$K(\tau) \ \to \ \tilde{Q}^\alpha(\tau) , \tag{2.101}$$

with a color factor associated to a different symmetry group $\tilde{G}$, carrying its own color variables $d^{\alpha'}(\tau)$ and $\bar{d}_{\alpha'}(\tau)$ taken in the fundamental and antifundamental representation, respectively. This replacement produces the vertex operator (2.79), clearly confirming a double copy structure at the level of vertex operators on the worldline.

In the case of scattering amplitudes it is well-known that the replacement of kinematic numerators by color ones leads to the bi-adjoint scalar theory [98, 99]. Here, we see that this replacement produces a vertex operator for the coupling of the particle to the plane wave of a bi-adjoint scalar, reminding the classical double copy of [37]. This vertex operator can be used to obtain an effective actions of the form given in the previous sections, for a particle coupled to a background bi-adjoint scalar field $\Psi^{(0)} \sim \Phi^{a\alpha}$.

Chapter 3

WQFT from off-shell currents in the classical limit

The WQFT formalism has been introduced recently by Mogull, Plefka and Steinhoff in [15]. It is designated to model classical scatterings of compact objects in general relativity, and it has been successfully extended to incorporate spin and finite-size effects [16]. More recently the authors extended the WQFT formalism so to include radiation-reaction effects due to the recoil of the compact objects, once emitting gravitons [17], further reaching a high level of precision in the PM calculations [100, 101, 102, 103]. The WQFT has also served as a framework to understand gauge-gravity duality in the classical limit, in the case where the external matter, exchanging gluons-gravitons, has also spinning degrees of freedom [56, 57].

WQFT is based on a relation between elastic (or inelastic) scattering amplitudes in the absence of matter loops and a worldline path integral representation of the dressed Feynman propagator.[1] The relation between the S-matrix, and dressed propagators requires a procedure to put the latter on-shell after having removed the external legs. The prescription for obtaining such propagator was pioneered by Fradkin long time ago [109], and applied in [110] to study high energy scattering in gravity. Once the worldline path integral is under control and the correspondence to the S-matrix made explicit, expectation values can be computed from a partition function expressed as a worldline path integral. One can then derive Feynman rules of the theory which allow the calculation of these expectation values directly. This is the WQFT approach to classical scattering observables. WQFT shares some similarities with the Effective Field Theory (EFT) approach to gravitational dynamics [6, 111, 112] with the important difference that in WQFT worldine degrees of freedom are also quantized.

In this chapter we start by discussing the basic aspects of the WQFT and particularly its relation to scattering amplitudes, considering the case of a scalar point particle coupled to gravity. To derive the WQFT from amplitudes we consider the case of a single matter line scattering in a gravitational background, and study the classical limit of the related off-shell current. The reason why we will consider currents here is twofold. They allow for an easy and clean derivation of the WQFT path integral from QFT, also highlighting

[1] Roughly speaking, in perturbation theory, a dressed propagator represents a resummation of tree-level Feynman diagrams of a particle propagating in a background (see Fig.3.1). Dressed propagators have been developed in a worldline representation for a variety of models, see e.g. [104, 105, 67, 106, 107, 108].

a direct relation between the KMOC formalism and the WQFT. Particularly, as we will see, they can also be applied in the context of Thermal field theory, for the calculations of Hard Thermal Loops (HTL), delivering higher point results in a very efficient way. Secondly, currents in the classical limit encode solutions to the classical equation of motion for the point particle in a gauge/gravity background thus they are relevant in classical context. In addition, the case of the two point current with spinning external particles is particularly important since related to ongoing discussion on the classical limit of the Compton amplitude, the latter is known to give details on the structure of interactions in massive higher spin theory [113, 34, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123].

3.1 Worldline quantum field theory

In order to introduce our notation and conventions, let us consider first the case of scalar massive particle of mass m interacting through gravitons. We will use the mostly minus signature for the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ and set the gravitational coupling to $\kappa^2 = 32\pi G_N$, where G_N is the Newton constant. The gravitational action is given by the usual Einstein-Hilbert action

$$S_{\mathrm{EH}} = -\frac{2}{\kappa^2}\int \mathrm{d}^4x\sqrt{-g}R\,, \tag{3.1}$$

whereas the action for the massive scalar field including a non-minimal coupling of the scalar field to the background curvature is given by

$$S_m = \int \mathrm{d}^4x\sqrt{-g}\left[g^{\mu\nu}\partial_\mu\varphi^*\partial_\nu\varphi + (\xi R - m^2)\varphi^*\varphi\right]. \tag{3.2}$$

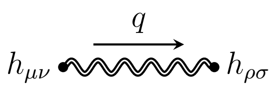
Here ξ is a free dimensionless coupling. Requiring Weyl invariance in the massless case fixes this coupling to $\xi = \frac{1}{6}$ ($\xi = \frac{d-2}{4(d-1)}$ in arbitrary dimensions), but here we shall keep it arbitrary. The Einstein-Hilbert action is treated perturbatively in the gravitational coupling constant κ. by expanding the exact metric as $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, with $h_{\mu\nu}$ being the graviton field. Then, we add to the Einstein-Hilbert action the gauge-fixing term

$$S_{\mathrm{gf}} = \int \mathrm{d}^4x\,(\partial^\nu h_{\mu\nu} - 1/2\partial_\mu h^\nu{}_\nu)^2\,, \tag{3.3}$$

which imposes a weighted version of the de Donder gauge $\partial^\nu h_{\mu\nu} = 1/2\partial_\mu h^\nu{}_\nu$. The full action[2] is then

$$S_g = S_{\mathrm{EH}} + S_{\mathrm{gf}} \tag{3.4}$$

while in de Donder gauge, the momentum space graviton propagator in $D = 4$ reads as



$$= \frac{i}{q^2}\mathcal{P}_{\mu\nu\rho\sigma}, \quad \mathcal{P}_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\rho\nu} - \eta_{\mu\nu}\eta_{\rho\sigma}\right)\,. \tag{3.5}$$

[2] We neglect the BRST ghost interaction in the full action, since interested in classical applications

being flexible about the $i\epsilon$ prescription: either write the denominator as $q^2+i\epsilon$, making it a time-symmetric Feynman propagator, or $(q^0 \pm i\epsilon)^2 - \vec{q}^2$, making it retarded/advanced.

A crucial object for the WQFT is the dressed propagator[3], which, in such a case is defined as

$$G(x,y;g) = \int \mathcal{D}\varphi \mathcal{D}\varphi^* e^{iS_m[\varphi,\varphi^*,h]} \varphi(x)\varphi^*(y) \tag{3.6}$$

which, in momentum space, delivers Feynman diagrams with two external off-shell scalars in a graviton background. In order to relate scattering amplitudes and path integrals, we first rewrite the scalar propagator in an external gravitational field in a proper time representation

$$iG(x,y;g) = \langle y| \frac{1}{\hat{H} - i\epsilon} |x\rangle = i \int_0^\infty \mathrm{d}T \langle y| e^{-iT(\hat{H}-i\epsilon)} |x\rangle, \tag{3.7}$$

where T is the Schwinger proper time. The Hamiltonian operator $\hat{H}$ corresponds to the Klein-Gordon operator fixed by the action (3.2) and is given by

$$\hat{H} = g^{\mu\nu}\nabla_\mu\nabla_\nu + m^2 - \xi R = \frac{1}{\sqrt{-g}}\partial_\mu \sqrt{-g} g^{\mu\nu}\partial_\nu + m^2 - \xi R\,. \tag{3.8}$$

It can be viewed as arising from a classical particle Hamiltonian obtained by setting $\partial_\mu \to -ip_\mu$ in the last expression, finding $H = -g^{\mu\nu}p_\mu p_\nu + m^2 - \xi R$. The particle action in hamiltonian form can be written as

$$S_\mathrm{p} = \int_0^1 \mathrm{d}\tau (p_\mu \dot{x}^\mu - eH), \tag{3.9}$$

where e is the einbein that gauges translations on the worldline, leading to a reparametrization invariant description of the worldline. The einbein reproduces the effect of the proper time T upon gauge fixing $e(\tau) = T$. Then, rescaling the proper time τ to range in the interval $[0,T]$, we obtain the particle action in configuration space

$$S_\mathrm{p} = \int_0^T \mathrm{d}\tau \left[-\frac{1}{4} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - m^2 + \xi R \right]\,. \tag{3.10}$$

In order to define a path integral free of spurious UV divergences and of regularization ambiguities, one must introduce auxiliary worldline ghost variables and a finite counterterm to the worldline action (3.10). Let us mention that these issues are only related with the one-dimensional worldline theory, and are not related with regularization of spacetime. The case of UV divergences on the worldline can be addressed by defining the path integral measure as follows

$$\mathcal{D}x := Dx \prod_{0<\tau<T} \sqrt{-g(x(\tau))} = Dx \int DaD\tilde{b}Dc\ \exp\left[-i \int_0^T \mathrm{d}\tau \frac{1}{4} g_{\mu\nu}(a^\mu a^\nu + \tilde{b}^\mu c^\nu) \right], \tag{3.11}$$

[3]Normalization constant in the definition of the dressed propagator have been absorbed in the path integral measure

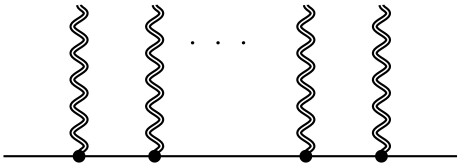


Figure 3.1: Dressed propagator with external massive particles off-shell

where the final form contains the standard translationally invariant measures, indicated by the symbol D as opposed to the symbol $\mathcal{D}$. In the second equality we introduced the Lee-Yang ghosts $(a, \tilde{b}, c)$, so to exponentiate the determinant factor, making the measure translational invariant, leading to the standard perturbation theory on the worldline. As regularization ambiguities play no role in the upcoming discussion, let us just mention that three options to fix such ambiguities are known, and correspond to the time slicing (TS) regularization, mode regularization (MR), and worldline dimensional regularization (DR) [96]. The appropriate counterterms in these schemes can be written as follows

$$S_{\mathrm{CT}} = \int_0^T \mathrm{d}\tau \left(-\frac{1}{4} R - V_{\mathrm{TS/MR/DR}} \right) , \tag{3.12}$$

where the additional terms $V_{\mathrm{TS/MR/DR}}$ are scheme dependent[4]. The path integral in configuration space associated with the propagator is finally given by (see Fig.3.1).

$$\begin{aligned} G(x,y;g) = \int_0^\infty \mathrm{d}T e^{-im^2 T} \int_{x(0)=x}^{x(T)=y} Dx \int Da D\tilde{b} Dc \qquad\qquad (3.13) \\ \exp\left\{ -i \int_0^T \mathrm{d}\tau \left[\frac{1}{4} g_{\mu\nu} \left(\dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + \tilde{b}^\mu c^\nu \right) + \left(\frac{1}{4} - \xi \right) R + V_{\mathrm{TS/MR/DR}} \right] \right\} . \end{aligned}$$

It can be solved in perturbation theory with standard gaussian integration [95, 64], delivering tree level Feynman diagrams with two off-shell massive particle, in a gravitational background.

3.1.1 From Green functions to the WQFT

Let us now see how to derive the WQFT formulation of a scattering event, by starting from its amplitude formulation. For the sake of simplicity in the derivation, we consider the case where one spinless particle is moving in a background of off-shell gravitons. The case where two pairs of different flavored scalar particles scatter by exchanging gravitons, can be obtained by following the same steps discussed below, with slight modifications, as discussed in [19].

Let $p = (p_1, p_2)$ and $k = (k_1, \cdots, k_n)$ denote respectively the momenta of the massive scalars, and the external gravitons, all of them assumed to be outgoing. Then, the natural quantum object associated to such a process is the $n+2$ off-shell current

$$\mathcal{A}_n^{I_1,\ldots,I_n}(p,k) := \hat{\delta}^4(p_1 + p_2 + \sum_{i=1}^n k_i) A_n^{I_1,\ldots,I_n}(p,k), \tag{3.14}$$

[4]They are given by $V_{\mathrm{TS}} = -1/4\, g^{\mu\nu}\Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\nu\beta}$, $V_{\mathrm{MR}} = 1/12\, g^{\mu\nu} g^{\alpha\beta} g_{\rho\sigma}\Gamma^\rho_{\mu\alpha}\Gamma^\sigma_{\nu\beta}$, and $V_{\mathrm{DR}} = 0$. Counterterms for supersymmetric versions of the nonlinear sigma model can be found in [97].

where, without loss of generality the massive scalars obey $p_i^2 = m^2$, while for the massless particles $k_i^2 \neq 0$. We have introduced the notation $\hat{\delta}^n(x) := (2\pi)^n \delta^n(x)$ and for later use we define $\hat{d}^n x := (2\pi)^n \mathrm{d}^n x$. The upper indices denote collectively Lorentz indices associated with the massless particles.

We are interested in the classical limit of Eq.(3.14) understood as the limit $\hbar \to 0$ or more precisely as a Laurent expansion in powers of some dimensionless parameter ξ. In the spirit of the KMOC one would restore $\hbar$ into the current and perform dimensional analysis. One should also distinguish the momenta of massive scalar particles and massless gauge bosons. The latter being described by wavenumbers through the rescaling $k \to \hbar k$, while the former obeys $p_i^\mu = m_i u_i^\mu$, where $u_i^2 = 1$. To achieve definite classical momenta the initial states must be dressed with appropriate coherent[5] wavefunctions, giving us the notion of sharply peaked position and momenta. The most general coherent relativistic wavefunctions associated with the restricted Poincare group have the form [125]

$$f_z(p) := \langle e_z | f \rangle = \int \mathrm{d}\Phi(p) e^{-\mathrm{i} z \cdot p} f(p) \,, \tag{3.15}$$

where $\mathrm{d}\Phi(p) := (2\pi)^{d-1} \mathrm{d}^d p \delta(p^2 - m^2) \Theta(p_0)$ and $\langle e_z | p \rangle := \mathcal{C}_z e^{-\mathrm{i}p \cdot z}$. Here $z = x - \mathrm{i}y$ is a complex vector, which in general is time-dependent. The classical phase space is obtained by setting $t = 0$ [126]. The normalization of states can be derived from

$$\langle e_z | e_w \rangle = \int \mathrm{d}\Phi(p) e^{-\mathrm{i}p \cdot (z - \bar{w})} = \mathcal{C}_z \mathcal{C}_w^* \left(\frac{m}{2\pi\eta} \right)^{d/2-1} K_{d/2-1}(\eta m), \tag{3.16}$$

where $\eta = \sqrt{-(z - \bar{w})^2}$ and $K_\nu(x)$ is the modified Bessel function. For $z = w$, $\eta = 2|y|$, one obtains

$$\mathcal{C}_z = \left[\left(\frac{4\pi|y|}{m} \right)^{d/2-1} \frac{1}{K_{d/2-1}(2m|y|)} \right]^{1/2} . \tag{3.17}$$

Wavefunctions employed by KMOC correspond to the case where one chooses the complex vector to be

$$z_i^\mu = -b_i^\mu + \mathrm{i} \frac{u_i^\mu}{m\xi}, \tag{3.18}$$

where ξ is a dimensionless parameter, which can be thought as the square of the ratio of the Compton wavelength to the intrinsic spread of the wavepacket. Here u_i is the classical four velocity of the particle of mass m_i. Therefore, it is natural to consider the current[6] weighted with coherent wavefunctions as

$$\mathcal{C}_n(p, k) := \int \mathrm{d}\Phi(p_1, p_2) \phi_{z_1}(p_1) \phi_{z_2}(p_2) \mathcal{A}_n(p_1, p_2, k_1, \ldots, k_n) \,, \tag{3.19}$$

[5]Here we think of coherent states in the sense of Perelemov [124]. We refer the interested reader to [20] for details of the Perelomov formalism.

[6]We will keep the indices explicit for calculations but otherwise suppress them to avoid cluttered expressions.

where $\mathrm{d}\Phi(p_1, p_2) := \mathrm{d}\Phi(p_1)\mathrm{d}\Phi(p_2)$. Now the classical limit of the current can be computed from a Laurent expansion of the formal expression

$$\mathcal{C}_n(p,k) = \int \mathrm{d}\Phi(p_1,p_2)\phi(p_1)\phi(p_2)e^{\mathrm{i}b\cdot\sum_{i=1}^n k_i}\mathcal{A}_n(p_1,p_2,\hbar k_1,\ldots,\hbar k_n)\,, \tag{3.20}$$

where we have used momentum conservation, $k \to \hbar k$ and $b \to b/\hbar$. The rescaling has an important effect into the structure of (3.20). Indeed writing explicitly the momentum conservation Dirac-delta we have

$$\begin{aligned}\mathcal{C}_n(p,k) &= \int \mathrm{d}\Phi(p_1,p_2)\phi(p_1)\phi(p_2)e^{\mathrm{i}b\cdot\sum_{i=1}^n k_i}\hat{\delta}^4(p_1+p_2+\hbar\sum_{i=1}^n k_i)A_n(p,\hbar k) \\ &= \int \mathrm{d}\Phi(p_1)\ \phi(p_1)\phi(-p_1-\hbar\sum_{i=1}^n k_i)\hat{\delta}(\hbar^2\sum_{i,j=1}^n k_i\cdot k_j+2\hbar p_1\cdot\sum_{i=1}^n k_i)e^{\mathrm{i}b\cdot\sum_{i=1}^n k_i}A_n(p,\hbar k),\end{aligned} \tag{3.21}$$

where we have used the momentum conservation Dirac-delta to perform the phase-space integral over p_2. Making the identifications $q \to \sum_{i=1}^n k_i$ the remaining integral has the form

$$\int \mathrm{d}\Phi(p)\phi(p)\phi(-p-\hbar q)\hat{\delta}(2\hbar p\cdot q+\hbar^2q^2)f(p,q), \tag{3.22}$$

which is sharply peaked around $p^\mu = mu^\mu$, where u^μ is the classical velocity of the particle with mass m. The analysis of the above integral by KMOC (see Appendix B of Ref.[127]) does not depend on the on-shell properties of q, which plays the role of the momentum mismatch in KMOC, so we can apply it here as well. The current then simplifies to

$$\mathcal{C}_n(p,k) = \frac{1}{2}\hat{\delta}\left(p\cdot\sum_{i=1}^n k_i\right)e^{\mathrm{i}b\cdot\sum_{i=1}^n k_i}\bar{A}_n(p,k)\,, \tag{3.23}$$

where $\bar{A}(p,k)$ denotes the non-vanishing term in the Laurent expansion, where by abuse of notation we have set $p_1 = p$. From a practical point of view we are done. We can already perform calculations following the KMOC algorithm and reach Eq.(3.23) for the theory under study.

A few comments are in order. Strictly speaking Eq.(3.23) should be understood as the average of the RHS over wavefunction of p, which in KMOC is denoted by a double bracket. The net effect of weighting over coherent wavefunctions is producing an overall factor depending on the external soft momenta, which is analogous to the momentum mismatch in classical observables. The attentive reader might ask about the presence of singular terms produced by the series expansion. The current is not an observable so one might expect such terms. However, the current is an off-shell tree-level object so we may safely ignore Feynman's iϵ prescription in calculations, thus leading to cancellation of those singular terms. In QED we have checked this up to 7-points. In the worldline formulation of the current the absence of those singular terms will become clear as we discuss now.

As it stands, (3.23) does not display any relation with the WQFT. To derive that we need to find a functional representation for $\bar{A}_n(p,k)$. Let us see how it arises, by starting from a field theory path integral definition of the off-shell current in momentum space

$$\mathcal{P}^{\rm m}_n(p,k) = \frac{1}{\mathcal{N}}\int \mathcal{D}h\; e^{iS_g[h]} \int \mathcal{D}\varphi\mathcal{D}\varphi^\dagger e^{iS_m[\varphi,\varphi^\dagger,h]} \qquad (3.24)$$
$$ik_1^2\cdots ik_n^2\;\varphi(p_1)\varphi^\dagger(p_2)h_{\mu_1\nu_1}(k_1)\cdots h_{\mu_n\nu_n}(k_n)\Big|_{k^2\neq 0}.$$

Before LSZ reduce the external scalar let us consider the classical limit of Eq.(3.24). In the classical approximation we can neglect loops mediated by scalars and replace the path integral over scalar fields by the graviton-dressed scalar propagator, after using the definition (3.6). Going in momentum space, we can write the dressed propagator as

$$G(x,y;g) \to \hat{\delta}\left(p_1+p_2+\sum_{i=1}^n k_i\right) G(p,k) \qquad (3.25)$$

whose explicit form is not required for our purposes. Now, we need to amputate the dressed propagator w.r.t. the external massive legs. At the level of the worldline integral we can achieve the LSZ reduction by Fradkin's prescription of exchanging the limit of integration in the worldline action to $(-\infty,+\infty)$ as a consequence of performing the Schwinger proper time integration after amputating the external scalar propagators. Thus, once plugging it in (3.24) we obtain

$$\mathcal{A}_n(p,k) = \frac{1}{\tilde{\mathcal{N}}}\hat{\delta}^4\left(p_1+p_2+\sum_{i=1}^n k_i\right) \qquad (3.26)$$
$$\int \mathcal{D}h\, e^{iS_g[h]}G(p,k)\; ik_1^2\cdots ik_n^2\; h_{\mu_1\nu_1}(k_1)\cdots h_{\mu_n\nu_n}(k_n)\Big|^{\rm trees}_{k^2\neq 0},$$

which is a path integral representation for the off-shell current (3.14) in the classical limit. Particularly, stripping off the momentum conservation delta function we get the classical limit of $\bar{A}_n(p,k)$ in (3.23).

Now, the last step needed to fully get a path integral representation of the current (3.23), in the classical limit, is encoded in the functional relation

$$\frac{\Sigma(b,p,h)}{\Sigma_0} = e^{ib\cdot\sum_{j=1}^n k_j}\hat{\delta}\left(p\cdot\sum_{j=1}^n k_j\right) G(p,k)\,, \qquad (3.27)$$

which was explicitly demonstrated for the graviton-dressed scalar propagator in [15]. Here Σ_0 is some overall factor that we can absorb into the normalization of the correlation function. Notice that both sides depend only on $p_1 = p$ on the support of the Dirac-delta in (3.26). The left hand side of (3.27) is given by the worldline path integral

$$\Sigma(b,p;h) = \int \mathcal{D}x \exp\left[-i\int_{-\infty}^{\infty} d\tau\,\frac{1}{2}g_{\mu\nu}(x(\tau))\left(\dot{x}^\mu(\tau)\dot{x}^\nu(\tau) + a^\mu(\tau)a^\nu(\tau) + \tilde{b}^\mu(\tau)c^\nu(\tau)\right)\right] \qquad (3.28)$$

where b and p arise from the background expansion $x^\mu(\tau) = b^\mu + p^\mu\tau + q^\mu(\tau)$. Notice how, the effect of the $b,c-$ghost field in the dressed propagator has been neglected since they do not contribute to classical calculations, as we will see in the next applications. Finally, gathering all of the informations we arrive at the WQFT representation of the off-shell current

$$\mathcal{C}_n(p,k) = \frac{1}{\mathcal{Z}} \int \mathcal{D}h\, e^{iS_g[h]} \int \mathcal{D}x\, e^{-i\int_{-\infty}^{\infty} d\tau \frac{1}{2} g_{\mu\nu}(\dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu)}\, ik_1^2 h_{\mu_1\nu_1}(k_1)\cdots ik_n^2 h_{\mu_n\nu_1}(k_n), \tag{3.29}$$

which concludes our derivation[7]. The factor $\mathcal{Z}$ ensures that the current is normalized to one when there are no gauge fields in the path integral and defines the WQFT partition function

$$\mathcal{Z} = \int \mathcal{D}h\, e^{iS_g[h]} \int \mathcal{D}x e^{-i\int_{-\infty}^{\infty} d\tau \frac{1}{2} g_{\mu\nu}(\dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu)}\,. \tag{3.30}$$

Collecting all of the integration constants into $\mathcal{Z}$ the classical off-shell current in WQFT can be succinctly written as

$$\mathcal{C}_n(p,k) = ik_1^2 \cdots ik_n^2\, \langle h_{\mu_1\nu_1}(k_1)\dots h_{\mu_n\nu_n}(k_n)\rangle_{\text{WQFT}} \tag{3.31}$$

which turns our current in a vacuum expectation value (vev) of fields inside the WQFT partition function.

Let us conclude this section by extending the above results to the case of the scattering of two different flavoured worldlines. The derivation of the WQFT partition function from the $2 \to 2$ amplitude follows exactly the same steps as above, with slight modifications due to the choice of the wave-packets. At the end of the day the partition function is written as the product of two single lines partition function, namely

$$\mathcal{Z}_{\text{WQFT}} = \int \mathcal{D}h\, e^{iS_g[h]} \mathcal{Z}_1\, \mathcal{Z}_2 \tag{3.32}$$

with the single line partition functions $\mathcal{Z}_i, i = 1,2$ defined in (3.30) and labeled by the subscript i running over the flavour of the worldlines. This easily generalizes to the case of n different flavour lines, exactly as it happens in thermodynamics, where the partition function for a canonical system made up of different species of particles, is simply the product of single species partition functions.

3.1.2 WQFT Feynman rules

Let us now move to the perturbative evaluation of the path integral (3.29). In de Donder gauge the propagator for the theory is given by (3.5), thus we move to graviton interactions vertices. As known the Feynman diagrammatic expansion of the Einstein-Hilbert action is involved, given that gauge invariance, represented in such a case by the difeomorphism on a Riemanian manifold, generates an infinite tower of graviton self-interactions. Lower points graviton vertices have been evaluated by various authors [9, 128, 129, 130] and higher point vertices can be efficiently derived by using *Mathematica*, with the recently developed package *FeynGrav* [131], an extension of the usual *FeynCalc* [132, 133, 134], the latter often used for Standard Model calculations.

[7]Another alternative to generate graviton insertions inside the path integral, is to consider the functional with a factor $\exp(Jh)$ and take derivatives over J, thought as a source field.

Let us now move to the derivation of Feynman rules from the worldline action in (3.30). After having introduced the graviton field as in the previous section, we background expand the configuration space worldline variables as

$$x^{\mu}(\tau) = b^{\mu} + p^{\mu}(\tau) + z^{\mu}(\tau) \tag{3.33}$$

with $p^{\mu} = mu^{\mu}$, where u^{μ} is the point particle four velocity. For the case of a single line we interpret b as the asymphtotic position of the particle, however for the elastic scattering b will be interpreted later on as the impact parameter i.e. the asymphtotic distance between the two scattering bodies The terms $z^{\mu}(\tau)$ are quantum fluctuations around the straight line, treated perturbatively inside the path integral, thus they are path integrated over the free theory. We refer to them as kinematic fluctuations since related to the configuration space variables. The background expansion relies on translational invariance of the path integral measure, which holds in such a case, given the introduction of the $\tilde{b}, c$ variables on the worldline. Once using the background expansion in the action from (3.30), the kinetic and interacting action turns out to be

$$S_{Kin}[z] = -\int_{-\infty}^{\infty} d\tau \, \frac{1}{2}\dot{z}^2 \quad S_{int}[z;h] = -\frac{\kappa}{2}\int_{-\infty}^{\infty} d\tau \, h_{\mu\nu}(x(\tau)) \left(p^{\mu}p^{\nu} + 2p^{(\mu}\dot{z}^{\nu)}(\tau) + \dot{z}^{\mu}(\tau)\dot{z}^{\nu}(\tau)\right) \tag{3.34}$$

with the symmetrization prescription $a^{(\mu}b^{\nu)} = \frac{1}{2}\left(a^{\mu}b^{\nu} + a^{\nu}b^{\mu}\right)$. In order to derive Feynman rules we move to momentum and energy space for the graviton and the fluctuations, by defining the related Fourier transforms as

$$h_{\mu\nu}(x) = \int_k e^{-i\ell\cdot x} h_{\mu\nu}(\ell), \quad z^{\mu}(\tau) = \int_{\omega} e^{-i\omega\tau} z^{\mu}(\omega) \tag{3.35}$$

where, for convenience, we defined the short-hand notation

$$\int_{\omega} := \int \hat{d}\omega \quad \int_k := \int \hat{d}^4 k \tag{3.36}$$

so to avoid proliferation of π factors. The first object we are interested in to set-up the perturbative expansion, is the worldline propagator for the quantum fluctuations. It is defined by the Green equation

$$\eta^{\mu\nu}\frac{d^2}{d\tau^2}\Delta_{\nu\alpha}(\tau - \sigma) = \delta^{\mu}{}_{\alpha}\delta(\tau - \sigma) \tag{3.37}$$

which, one solves in energy space, by choosing a $i\epsilon$ prescription on the energy poles of the solution. The choice of the of the $i\epsilon$ prescription also determines the physical interpretation of the background parameters b, p. This is understood from that one has to pick up boundary conditions on the fluctuations so to solve the Green equation. Such conditions are of the form $z^{\mu}(\tau_0) = \dot{z}^{\mu}(\tau_0) = 0$. Here on we fix $\tau_0 = -\infty$, fixing then retarded boundary conditions on the worldline variables, so to identify the straight line as the exact trajectory of the particle in the past-infinity. As a consequence this forces us to pick up retarded worldline propagators and to interpret b, p as the far past distance between the two black holes and the initial momentum of the particle. Recently it has been shown by the WQFT collaboration [17] that this choice corresponds to the so called “in-in” formalism (or Schwinger-Keldysh formulation) and is suitable to include back-reaction effects in classical integrands. Thus, from here on, we pick up the retarded worldline propagator for the worldline fluctuations. In energy space the worldline propagator reads as

$$z^\mu \bullet \overset{\xrightarrow{\omega}}{\rule{4em}{0.4pt}} \bullet z^\nu \quad = -i\frac{\eta_{\mu\nu}}{\omega^2} \tag{3.38}$$

where we dropped the $i\epsilon$ prescription, understood as the retarded one.

Let us now move to interactions. To derive worldline interaction vertices we need to expand the interacting action in (3.34) wrt the quantum fluctuations. To accomplish that we first expand the graviton field on the worldline

$$\begin{aligned} h_{\mu\nu}(x(\tau)) &= \int_k e^{ik\cdot(b+p\tau+z(\tau))}h_{\mu\nu}(-k) = \sum_{n=0}^{\infty}\frac{i^n}{n!}\int_k e^{ik\cdot(b+p\tau)}(k\cdot z(\tau))^n h_{\mu\nu}(-k) \\ &= \sum_{n=0}^{\infty}\frac{i^n}{n!}\int_{k,\omega_1,\cdots,\omega_n} e^{ik\cdot b}\, e^{i\tau\left(k\cdot p+\sum_{i=1}^n\omega_i\right)}\left(\prod_{i=1}^n k\cdot z(-\omega_i)\right)h_{\mu\nu}(-k). \end{aligned} \tag{3.39}$$

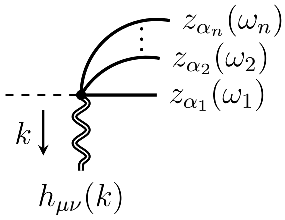
Once plugged into the interacting action (3.34), the expansion of the graviton field in powers of the fluctuations yielding to the interacting action

$$\begin{aligned} S_{int}[z;h] = -\kappa\sum_{n=0}^{\infty}\frac{i^n}{n!}\int_{k,\omega_1,\cdots,\omega_n} & e^{ik\cdot b}\hat{\delta}\left(k\cdot p+\sum_{i=1}^n\omega_i\right)h_{\mu\nu}(-k)\left(\prod_{i=1}^n z^{\rho_i}(-\omega_i)\right)\times \\ \frac{1}{2}\left(\left(\prod_{i=1}^n k_{\rho_i}\right)p^\mu p^\nu + \sum_{i=1}^n\omega_i\left(\prod_{j\neq i}^n k_{\rho_j}\right)p^{(\mu}\delta^{\nu)}_{\rho_i} \right. & \left. + \sum_{i<j}\omega_i\omega_j\left(\prod_{l\neq i,j}^n k_{\rho_l}\right)\delta^{(\mu}_{\rho_i}\delta^{\nu)}_{\rho_l}\right) \end{aligned} \tag{3.40}$$

which is ready to be used to derive Feynman rules with a fixed number of quantum fluctuations. Vertices can be easily obtained by differentiation from the above action, once stripped off the gravtion in the interacting action $S_{int}[z;h] = S^{int}_{\mu\nu}[z]h^{\mu\nu}$, namely

$$V_{\mu\nu\rho_1\ldots\rho_n}(k,\omega_1,\ldots,\omega_n) = \frac{\delta S^{int}_{\mu\nu}[z]}{\delta z^{\rho_1}(-\omega_1)\ldots\delta z^{\rho_n}(-\omega_n)} \tag{3.41}$$

which is equivalent to use the Wick theorem to evaluate the time ordered correlation function $\langle\Omega|\mathrm{T}\prod_{i=1}^n z_{\rho_i}(-\omega_i)|\Omega\rangle$ and then amputating the worldline propagators generated by Wick contractions. We implemented this procedure in *Mathematica*. In such a case the simplicity of the interaction vertices in the worldline action allows us to write down a quite compact expression for the $n-$fluctuations vertex



$$\begin{aligned} &= i^{n-1}\kappa\, e^{ik\cdot b}\hat{\delta}\left(k\cdot p+\sum_{i=1}^n\omega_i\right)\times \\ &\frac{1}{2}\left(\left(\prod_{i=1}^n k_{\rho_i}\right)p^\mu p^\nu + \sum_{i=1}^n\omega_i\left(\prod_{j\neq i}^n k_{\rho_j}\right)p^{(\mu}\delta^{\nu)}_{\rho_i} + \sum_{i<j}\omega_i\omega_j\left(\prod_{l\neq i,j}^n k_{\rho_l}\right)\delta^{(\mu}_{\rho_i}\delta^{\nu)}_{\rho_l}\right) \end{aligned} \tag{3.42}$$

completing the derivation of Feynman rules from the WQFT action.

The algorithm used above, to generate Feynman rules from the worldline action, is general and can be applied independently of the worldline variables used to build up the theory. As we will see in the next sections, the same treatment can be used in the case where the particle has color or spin degrees of freedom. To deal with that, one has to use different background expansions of such worldline variables, generating new propagators in the theory, with a different pole structure and, moreover, different interaction vertices which are a consequence of gauge invariance at the level of the worldline action, as we will see.

3.2 WQFTs for colored scalar particles

Let us move to the construction of the WQFT and the derivation of the Feynman rules for various theories, which then will be used to evaluate off-shell currents, relevant for classical calculations.

As we have seen from the analysis in Sec.3.1.1, the crucial step to move to the WQFT action is to shift the boundary of the off-shell worldline action to $(-\infty, \infty)$. However, in the case of the bi-adjoint we also have color dofs, generating, after the gauge fixing procedure on the worldline, a gauge moduli integration (see (2.74) for instance). Thus one need to deal with that, in order to fully take the classical limit of the color dofs. A direct calculation [28], then proved in [18] shows that one can get ride of the moduli integration, in the classical limit. Formally, the moduli integration appears once fixing the Chern-Simmons coupling so to project on a specific color representation for the scalar particle. Thus getting rid of that means that one is propagating all of the color representations, encoded in the spectrum of the worldline particle. This results in a higher value of the representation index of the color, implicitly taking the large color limit. Thus explaining why one can get rid of the moduli integration for classical applications. This is equivalente to the amplitude procedure developed in [135] using the KMOC, and then also applied in the context of the EFT in [136].

3.2.1 Bi-adjoint scalar

Based on the above reasoning, starting from the action (2.69), the worldline sector of the WQFT action for the bi-adjoint scalar can be written as

$$S_{\text{BW}}[\mathsf{X};\varphi] = -\int_{-\infty}^{\infty} d\sigma \left(\frac{1}{2}\dot{x}^2 + \bar{c}_a \dot{c}^a + \bar{d}_\alpha \dot{d}^\alpha - \frac{y}{2} Q^a \varphi^{a\alpha} \tilde{Q}^\alpha\right) \tag{3.43}$$

with $X = (x, c, d, \bar{c}, \bar{d})$ denoting a vector of worldline variables. Once in hand the action, we can easily write down the partition function for the theory as

$$\mathcal{Z} = \int \mathcal{D}\varphi \, e^{iS_{BS}[\varphi]} \int \mathcal{D}\mathsf{X} \, e^{iS_{\text{BW}}[\mathsf{X};\varphi]} \tag{3.44}$$

where, as usual, normalization constants are absorbed inside the path integral measure.

In order to derive Feynman rules we need to background expand the color variables, beside the configuration space ones, expanded using (3.33). For such a task, we expand the color variables as

$$\bar{c}^a(\tau) = \bar{u}^a + \bar{\lambda}^a(\tau), \qquad c^a(\tau) = u^a + \lambda^a(\tau), \tag{3.45}$$
$$\bar{d}^\alpha(\tau) = \bar{v}^\alpha + \bar{\gamma}^\alpha(\tau), \qquad d^\alpha(\tau) = v^\alpha + \gamma^\alpha(\tau). \tag{3.46}$$

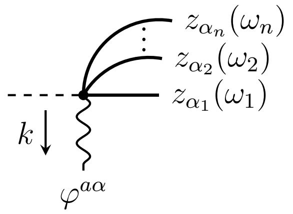
picking retarded boundary conditions for the fluctuations λ, ω. The background expansion allows us to identify the far-past[8] value of the color charge for the point particle, namely

$$C^a = \bar{u} \cdot T^a \cdot u, \quad \tilde{C}^a = \bar{v} \cdot \tilde{T}^a \cdot v \tag{3.47}$$

which will appear in classical integrands. Once going in energy space this generates the color propagator[9]

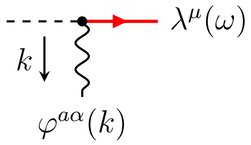
$$= \frac{i}{\omega}\delta_{\alpha\beta} \tag{3.48}$$

with the variables γ having exactly a copy of the same propagator above. Proceeding as in Sec.3.1.2, we get the $n-$kinematic fluctuation vertex



$$= \frac{y}{2} C^a \, \tilde{C}^\alpha (i)^{n+1} e^{\mathrm{i} b \cdot k} \hat{\delta}\left(k \cdot p + \sum_{j=1}^{n} \omega_j\right) \prod_{i=1}^{n} k^{\mu_i} \tag{3.49}$$

alongside with vertices propagating fluctuations of the color variables, namely



$$= \mathrm{i}\frac{y}{2}\tilde{C}^\alpha e^{\mathrm{i} b \cdot k} \, \hat{\delta}\left(k \cdot p + \omega\right) (\bar{u} \cdot T^a)_\mu. \tag{3.50}$$

The the rule for the vertex propagating $\gamma(\omega)$ is the same as the above one once using the generator of the tilded color group, while the rule for $\bar{\lambda}_\mu(\omega)/\bar{\gamma}_\mu(\omega)$ is obtained by reversing the arrow and replacing $(\bar{u} \cdot T^a)_\mu \to (T^a \cdot u)^\mu$. The same holds for the vertex propagating $\bar{\eta}_\mu(\omega)$, after exchanging $u \to v$. The above Feynman rules are enough for the calculations we are interested in.

3.2.2 Scalar QCD

Let us now move to the WQFT for a massive scalar particle coupled to a Yang-Mills field, the so called scalar chromodynamis (sQCD). For such a task we use the worldline

[8]We are picking up retarded boundary conditions also for the color variables

[9]To distinguish between λ, γ fluctuations, when drawing diagrams, we use red and blue lines respectively

formulation developed in (2.97), switching off the bi-adjoint scalar coupling[10]. This yeilds to the worldline action

$$S_{\text{sQCD}}[\mathsf{X};A] = -\int_{-\infty}^{\infty} \mathrm{d}\tau \left(\frac{1}{2}\dot{x}^2 + i\bar{c}\cdot\dot{c} + g\dot{x}\cdot A^a Q_a\right), \tag{3.51}$$

where now $\mathsf{X} = (x, c, \bar{c})$, while here we explicitly fix our gauge group $G = SU(N)$. The action can now be used to build up the partition function

$$\mathcal{Z} = \int \mathcal{D}A\, e^{iS^{gf}_{\text{YM}}[A]} \int \mathcal{D}\mathsf{X}\, e^{iS_{\text{sQCD}}[x;A]} \tag{3.52}$$

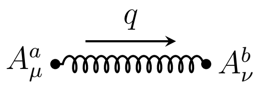
where $S^{gf}_{\text{YM}}[A]$ is the Yang-Mills field theory action[11]

$$S_{\text{YM}}[A] = -\frac{1}{4}\int d^D x\, F^a_{\mu\nu}(x) F^{a\mu\nu}(x) \tag{3.53}$$

with the Yang-Mills field strength defined as $F^a_{\mu\nu} = 2\partial_{[\mu}A^a_{\nu]} - igf^{abc}A^b_\mu A^c_\nu$, gauge fixed by including the term

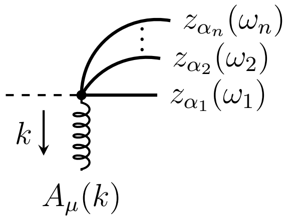
$$S_{gf} = -\frac{1}{2}\int d^D x (\partial_\mu A^a_\nu)^2 \tag{3.54}$$

corresponding to the Feynman gauge choice. This delivers the gluon propagator



$$= -\frac{i}{q^2}\eta_{\mu\nu}\delta^{ab}\,. \tag{3.55}$$

Feynman rules can be obtained analogously as for the bi-adjoint, using the background expansion (3.33), (3.45) for the configuration space and color variables respectively, delivering the same propagators as above. This allows us to write down the $n-$fluctuation vertex for the kinematical fluctuations



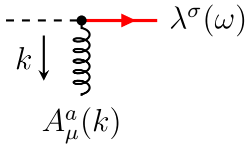
$$= g\, i^{n-1} C^a e^{ib\cdot k}\hat{\delta}\left(k\cdot p + \sum_{i=1}^n \omega_i\right)\left(p^\mu \prod_{i=1}^n k^{\alpha_i} + \sum_{i=1}^n \omega_i \eta^{\mu\alpha_i}\prod_{j\neq i}^n k^{\alpha_j}\right), \tag{3.56}$$

We will also need fluctuations of the background color variables, which we require in QCD,

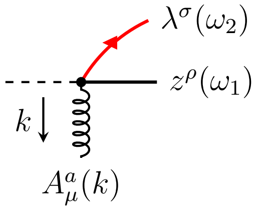
[10] From here we adapt to the conventions in [57]

[11] In our convention, we choose the commutator of the generators as $[T^a, T^b] = f^{abc}T^c$. While the adjoint representation is given by $(T^a_{Adj})^b{}_c = -f^{abc}$.

namely



$$= - \mathrm{i}ge^{\mathrm{i}b\cdot k}\, \hat{\delta}\left(k\cdot p + \omega\right)\left(p^\mu\right)\left(\bar{u}\cdot T^a\right)_\sigma, \tag{3.57}$$



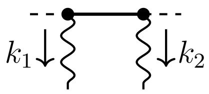
$$= ge^{\mathrm{i}b\cdot k}\hat{\delta}(k\cdot p + \omega_1 + \omega_2)\left(p^\mu k^\rho + \omega_1 \eta^{\mu\rho}\right)\left(\bar{u}\cdot T^a\right)_\sigma, \tag{3.58}$$

where the second mixes kinematic and color fluctuations. The vertices propagating fluctuations of $\bar{\lambda}_\sigma(\omega)$ can be obtained by reversing the arrows and replacing $(\bar{u}\cdot T^a)_\sigma \to (T^a\cdot u)^\sigma$ in the above ones.

From the above formulation it is straightforward to get a WQFT formulation for sQED. In such a case, the gauge group generator turns out to be $(T^a)_\alpha{}^\beta = (T^1)_\alpha{}^\beta = \delta_\alpha{}^\beta$. The color charge then turns out to be $Q = \bar{c}\cdot c$, which we set to unity given that in sQED it is conserved along the worldline. This further implies that the kinetic term for the color variables vanishes, thus, all of the Feynman rules are encoded in (3.56), once setting $C \to 1, g \to e$, with e being the electron charge.

3.2.3 Higher point off-shell current in sQED

Let us consider the $n = 2$ off-shell current for scalar electrodynamics. Two equivalent diagrams with symmetry factor $\frac{1}{2}$ are generated by the wordline path integration. Hence it is enough to consider the following diagram



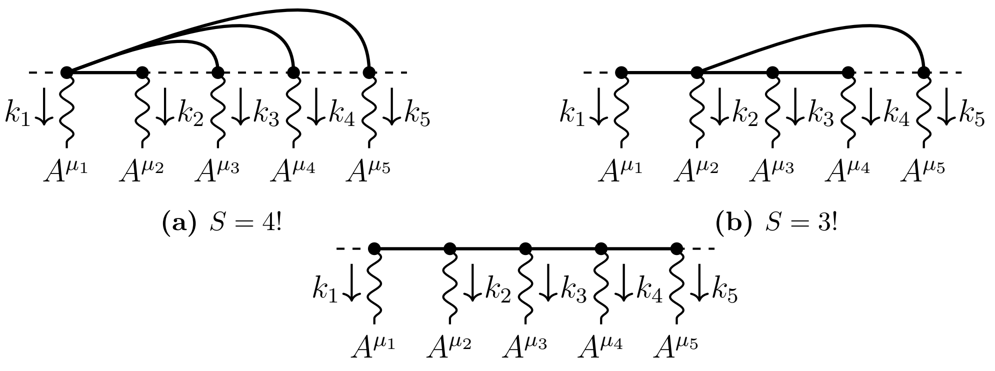
$$= -ie^2\left(\eta^{\mu_1\mu_2} - \frac{k_1\cdot k_2 p^{\mu_1}p^{\mu_2}}{\left(p\cdot k_1\right)^2} + \frac{k_2^{\mu_1}p^{\mu_2}}{p\cdot k_1} - \frac{k_1^{\mu_2}p^{\mu_1}}{p\cdot k_1}\right) \tag{3.59}$$

with symmetry factor equals unity. An easy calculation then gives

$$\begin{aligned}\mathcal{C}^{\mu\nu}_{\mathrm{sQED}}(p,k) &= e^2\, e^{\mathrm{i}b\cdot(k_1+k_2)}\hat{\delta}\left(p\cdot(k_1+k_2)\right)\bar{A}^{\mu\nu}_{\mathrm{sQED}}(k_1,k_2)\\ &= e^2\, e^{\mathrm{i}b\cdot(k_1+k_2)}\hat{\delta}\left(p\cdot(k_1+k_2)\right)\mathrm{i}\left(\eta^{\mu\nu} + \frac{k_2^\mu p^\nu}{p\cdot k_1} - \frac{k_1^\nu p^\mu}{p\cdot k_1} - \frac{k_1\cdot k_2 p^\mu p^\nu}{(p\cdot k_1)^2}\right),\end{aligned} \tag{3.60}$$

which satisfies the Ward identity $k_{i,\mu}\mathcal{C}^{\mu\nu}_{\mathrm{sQED}}(p,k) = 0$ and matches what one calculates using Eq.(3.23).

The case of the two and three point current can be trivially accounted without any particular technique for the generation of diagrams in the WQFT. Thus we move straight to the non trivial case of the five point current, corresponding to the classical limit of seven point tree level amplitude with two external on-shell scalars and five off-shell photons. The Feynman diagrammatic calculation requires 450 diagrams which then, must be Laurent expanded once having reintroduced all the $\hbar$ factors, while our WQFT formulation requires



(a) $S = 4!$ **(b)** $S = 3!$ **(c)** $S = 2 \times 3!$

Figure 3.2: Examples of worldline topologies required for the calculation of 7-point sQED current and their symmetry factors S.

only 12 diagrams, summed over the permutations of the external photons. Particularly, the most time consuming step, in the amplitude approach is the Laurent expansion, which, in the WQFT is completely bypassed by the use of the WQFT Feynman rules. To generate the Feynman diagrams for the amplitude calculation we used *Feyncalc*, while for the generation of the WQFT diagrams we used a Wick theorem implementation of the worldline contractions in *Mathematica*, generating all of the topologies needed for the calculation (see Fig.3.2). Schematically the result[12] can written as

$$\mathcal{C}^{\mu_1\cdots\mu_5}_{\text{sQED}} = e^5\, e^{ib\cdot(k_1+\cdots+k_5)}\hat{\delta}\left(p\cdot\sum_{i=1}^{5}k_i\right)\bar{A}^{\mu_1\cdots\mu_5}_{\text{sQED}} = e^5\, e^{ib\cdot(k_1+\cdots+k_5)}\hat{\delta}\left(p\cdot\sum_{i=1}^{5}k_i\right)\sum_{i=1}^{2451}a_i\mathsf{T}_i^{\mu_1\cdots\mu_5}, \tag{3.61}$$

where the sum runs over independent tensor structures. We have compared this result against the Feynman diagram calculation and found agreement. Such higher point calculation also reveals that there are no any singular terms of the form $\hbar^{-n}$ in the expansion of the current, confirming the classical nature of our formulation.

3.2.4 Classical Compton amplitude from currents

A further interesting application of our WQFT representation for the off-shell current is the calculation of the classical limit of Compton amplitude in gravity. The physical situation consists in studying the scattering of gravitational waves off a black hole, in the limit where the graviton wavelength $\lambda_W \gg R_S$, where R_S is the Schwarzschild radius of the black hole, being the characteristic scale of the black hole. In such a limit we are allowed to treat the black hole as a massive point particle, since it is delocalized in the gravitational wave, thus we can use the WQFT to model such a scattering event.

[12] The Mathematica file with the full result has been attached to the published version of [18] on the arXiv

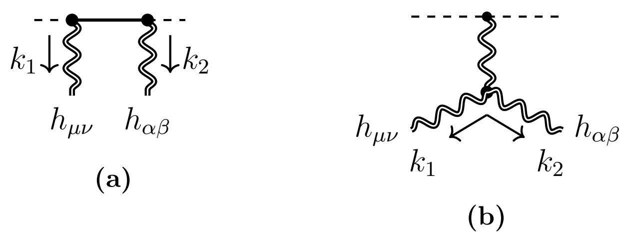


Figure 3.3: Worldline diagrams contributing to the on-shell Compton amplitude.

By using (3.23) for the two point current we get that

$$\mathcal{C}_2^{\mu\nu\alpha\beta}(p,k_1,k_2) = \frac{1}{2}\kappa^2 e^{\mathrm{i}b\cdot(k_1+k_2)}\hat{\delta}(p\cdot(k_1+k_2))\bar{M}_2^{\mu\nu\alpha\beta}(p,k_1,k_2) \tag{3.62}$$

where now, we take the external gravitons to be on-shell.

The relation (3.23) then tells us that $\bar{M}_{\mathrm{GR}}^{\mu\nu\alpha\beta}(p,k_1,k_2)$ is the classical on-shell Compton current, thus to get an amplitude we just need to contract with graviton polarization tensors

$$\mathcal{M}^{h_1h_2}(p,k_1,k_2) = \bar{M}_2^{\mu\nu\alpha\beta}(p,k_1,k_2)\epsilon_{\mu\nu}^{h_1}(k_1)\epsilon_{\alpha\beta}^{h_2}(k_2)\Big|_{k_i^2=0}, \tag{3.63}$$

where h_i is the helicty of the external graviton. A direct calculation of the current, from the diagrams in Fig.3.3 shows that it can be recast in a Kawai-Lewellen-Tye [21] (KLT) fashion

$$\mathcal{C}_2^{\mu_1\nu_1,\mu_2\nu_2} = -\kappa^2\mathrm{i}\, e^{\mathrm{i}b\cdot(k_1+k_2)}\hat{\delta}(p\cdot(k_1+k_2))\frac{k_1\cdot p\, k_2\cdot p}{k_1\cdot k_2}\bar{A}_{\mathrm{sQED}}^{\mu_1\mu_2}\bar{A}_{\mathrm{sQED}}^{\nu_1\nu_2}, \tag{3.64}$$

with the QED current given in (3.60), after replacing $e^2 \to \kappa^2/4$. This is remarkable since the same holds for the quantum version of the above current. It is the first example of double copy relation appearing straight at the classical level. We will came back on that in the next chapter.

In order to fuse the full current into the amplitude (3.63), we use physical polarization tensors $\varepsilon_{\pm\pm}^{\mu\nu}(k_i) = \varepsilon_{\pm}^{\mu}(k_i)\varepsilon_{\pm}^{\nu}(k_i)$ written as a product of null transverse photon polarizations. We set k_1 as incoming momentum and k_2 as outgoing and choose the rest frame of the worldline, i.e.,

$$p^\mu = mu^\mu = (m,0,0,0), \quad k_1^\mu = E(1,0,0,1), \quad k_2^\mu = E(1,\sin\theta,0,\cos\theta), \tag{3.65}$$

where E is the energy of the graviton.

Explicit polarization vectors $\varepsilon_{\pm}^{\mu}$ follow from the transversality and traceless conditions. Therefore, we can evaluate (3.63) for the independent set of helicity configurations $(++), (+-)$ yielding to

$$|\mathcal{M}_{++}|_0 = |\mathcal{M}_{--}|_0 = \frac{\kappa^2m^2}{4}\frac{\cos^4\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}, \quad |\mathcal{M}_{+-}|_0 = |\mathcal{M}_{-+}|_0 = \frac{\kappa^2m^2}{4}\sin^2\frac{\theta}{2}. \tag{3.66}$$

The above helicity amplitudes are enough to evaluate the unpolarized differential cross section for the classical scattering of gravitational waves off spinless black holes, at leading order in perturbation theory

$$\frac{d\sigma}{d\Omega} = \frac{G^2m^2}{\sin^4\frac{\theta}{2}}\left(\cos^8\frac{\theta}{2}+\sin^8\frac{\theta}{2}\right) \tag{3.67}$$

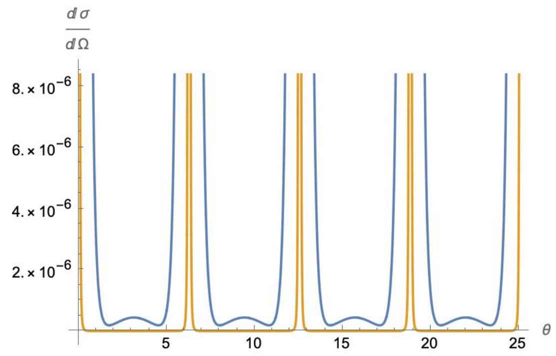


Figure 3.4: Unpolarized differential cross section for the scattering of gravitational waves off a black hole at $O(G^2)$. The blue line is the unpolarized differential cross section for a supermassive black hole $m = 10^7 M_\odot$ while the yellow line for an intermediate mass black hole $m = 10^5 M_\odot$. Here $M_\odot$ is the sun mass.

confirming that helicity is not preserved in gravitational waves scattering. Further our calculation is in agreement with the well known results in [137, 138, 139, 140] recently reproduced by [114, 115] using amplitudes.

3.3 HTL and off-shell currents

A nice application of the ideas presented in the past sections is the case of Hard Thermal Loops (HTLs). These are currents in the high temperature limit which can be resumed and incorporated into an effective theory known as HTL effective theory [141, 142, 143, 144, 145]. It is well known that the high temperature regime is equivalent to a classical regime. Using a KMOC-like approach this was explicitly demonstrated in Ref.[146], where HTLs were computed as the limit $\hbar \to 0$. Schematically HTL currents can be written as

$$\Pi_n(k) = \int \mathrm{d}\Phi(p) f(p_0) \bar{A}_n(p, k)\,, \tag{3.68}$$

where $f(p_0)$ is a distribution function at equilibrium and $\bar{A}_n(p, k)$ is the classical limit of the current in the regularized forward limit. The regularization is required since the same diagrams that contribute to the currents contribute to amplitudes so in general the forward limit is singular. Let $\mathcal{F}$ be the set of all Feynman diagrams contributing to the current (3.14). Diagrammatically the regularization consists on dropping the set of all diagrams producing zero momentum internal edges $\mathcal{X}$ in the forward limit[13]. It is defined by

$$A_n(p, k) := \sum_{G \in \mathcal{F} \setminus \mathcal{X}} d(G)\,, \tag{3.69}$$

[13]These problematic diagrams appear e.g., Eq.(3.31) so the regularization should also be implemented in the worldline formalism.

where $d(G)$ is a rational expression of the form $N(G)/D(G)$. In the forward limit $p_1 = -p_2$ so momentum conservation becomes

$$\sum_{i=1}^{n} k_i = 0. \tag{3.70}$$

The classical limit of Eq.(3.69) is obtained through Eq.(3.23). These currents have been considered in Refs.[146, 147]. The WQFT approach gives us a new way of obtaining these currents. In QED the equivalence between (3.23) and (3.31) implies that the $n-$point HTL can be read off from

$$\hat{\delta}\left(p \cdot \sum_{i=1}^{n} k_i\right) \frac{1}{2}\bar{A}_n(p,k) = ik_1^2 ik_2^2 \cdots ik_n^2 \langle A_{\mu_1}(k_1)A_{\mu_2}(k_2)\cdots A_{\mu_n}(k_n)\rangle_{\text{WQFT}}, \tag{3.71}$$

where the regularization is understood in both sides. Since we are interested in $\bar{A}_n(p,k)$ we will strip-off the Dirac-delta produced by WQFT. Inserting the RHS of this equation side into Eq.(3.68) gives and alternative worldline path integral representation of the HTL resumed current. A similar matching can be used to obtain HTLs in other theories.

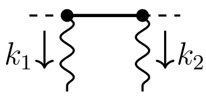
When classical color factors are recovered, thermal currents are obtained after phase-space integration over color. Phase space integration over classical color factor is defined by

$$\mathrm{d}C := \mathrm{d}^8C\, c_R \delta(C^a C^b \delta^{ab} - q_2)\delta(d^{abc}C^a C^b C^c - q_3), \tag{3.72}$$

where q_2 and q_3 are Casimir invariants. The factor c_R ensure that the color measure is normalized to unity and we have set the gauge group to be $SU(3)$. For bi-adjoint scalars we will take two copies of the phase-space integration measure.

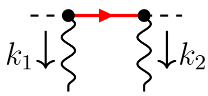
3.3.1 Bi-adjoint

Let us start by the simple calculation of the two point HTL for a bi-adjoint scalar. Given the presence of a three point scalar vertex, one should use the regularization (3.69) to get rid of that. Thus contributions to the current arises from two diagrams: one related to the kinematic fluctuations of the worldline, with a symmetry factor of 1/2 to account for Bose-Einstein symmetry, namely



$$= \mathrm{i}\frac{y^2}{4} C^{a_1}C^{a_2}\tilde{C}^{\alpha_1}\tilde{C}^{\alpha_2}\frac{k_1 \cdot k_2}{(k_1 \cdot p)^2}\,, \tag{3.73}$$

and the remaining ones related to the color fluctuations on the worldline



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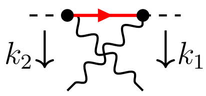
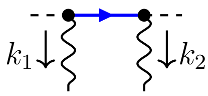


$$= \mathrm{i}\frac{y^2}{4}\frac{\tilde{C}^{\alpha_1}\tilde{C}^{\alpha_2}}{k_1 \cdot p}\bar{u}\cdot(T^{a_1}\cdot T^{a_2} - T^{a_2}\cdot T^{a_1})\cdot u\,, \tag{3.74}$$



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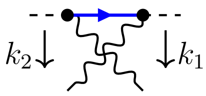


$$= \mathrm{i}\frac{y^2}{4}\frac{C^{a_1}C^{a_2}}{k_1 \cdot p}\bar{v}\cdot(\tilde{T}^{\alpha_1}\cdot\tilde{T}^{\alpha_2} - \tilde{T}^{\alpha_2}\cdot\tilde{T}^{\alpha_1})\cdot v\,. \tag{3.75}$$

It should be noticed how adding up the two topologies in each of Eq.(3.74) and (3.75) generates the structure constants of the Lie algebra

$$\bar{u} \cdot (T^{a_1} \cdot T^{a_2} - T^{a_2} T^{a_1}) \cdot u = f^{a_1 a_2 a_3} C^{a_3}. \tag{3.76}$$

Then, the current simplifies to

$$\bar{A}_2^{a_1\alpha_1, a_2\alpha_2} = \frac{y^2}{2} \left(C^{a_1} C^{a_2} \tilde{C}^{\alpha_1} \tilde{C}^{\alpha_2} \frac{k_1^2}{(k_1 \cdot p)^2} + \frac{\tilde{C}^{\alpha_1} \tilde{C}^{\alpha_2} f^{a_1 a_2 a_3} C^{a_3} + C^{a_1} c^{a_2} \tilde{f}^{\alpha_1 \alpha_2 \alpha_3} \tilde{C}^{\alpha_3}}{k_1 \cdot p} \right). \tag{3.77}$$

The phase-space integration over color can be done using the identities

$$\int \mathrm{d}C \; C^a = 0, \qquad \int \mathrm{d}C \; C^a C^b = \delta^{ab}, \tag{3.78}$$

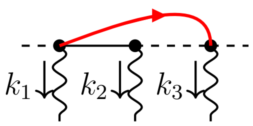
which follow from Eq.(3.72). Hence after some relabeling

$$\Pi^{a_1\alpha_1, a_2\alpha_2}(k) = \delta^{a_1 a_2} \delta^{\alpha_1 \alpha_2} \frac{q_2^2 y^2}{2} \int \mathrm{d}\Phi(p) \frac{k^2}{(k \cdot p)^2}, \tag{3.79}$$

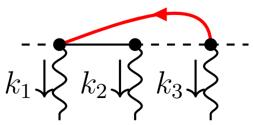
which is in agreement with kinetic theory of [147].

3.3.2 Gauge and gravity

Let us now move to sQCD, for the calculation of the three point HTL. After using the regularization (3.69), we need to evaluate two diagrams, plus permutations of the external gluons. For example, for the permutation $\sigma = (1, 2, 3)$ for the external gluons one has



$$= g^3 \; \bar{u} \cdot T^{a_1} \cdot T^{a_3} u \; C^{a_2} \frac{p^{\mu_3}}{p \cdot k_3} \bar{A}^{\mu_2 \mu_1}_{\mathrm{sQED}}(k_2, k_1), \tag{3.80}$$



$$= -g^3 \; \bar{u} \cdot T^{a_3} \cdot T^{a_1} u \; C^{a_2} \frac{p^{\mu_3}}{p \cdot k_3} \bar{A}^{\mu_2 \mu_1}_{\mathrm{sQED}}(k_2, k_1), \tag{3.81}$$

which generate the $SU(N)$ structure constants as in Eq.(3.76). Performing the phase space integration (3.78) and summing over all permutations the final answer can be written as

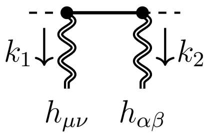
$$\bar{A}^{\mu_1 \mu_2 \mu_3}_{a_1 a_2 a_3} = -2g^3 \sum_{\sigma \in S_3} f^{a_{\sigma_1} a_{\sigma_2} a_{\sigma_3}} \frac{p^{\mu_{\sigma_3}}}{p \cdot k_{\sigma_3}} \bar{A}^{\mu_{\sigma_2} \mu_{\sigma_1}}_{\mathrm{sQED}}(k_{\sigma_2}, k_{\sigma_1}) \tag{3.82}$$

where S_3 is the set of all permutations of $\{1, 2, 3\}$. The above result satisfies the identity

$$k_{3\mu_3} \bar{A}^{\mu_1 \mu_2 \mu_3}_{a_1 a_2 a_3} = 2g^3 f^{a_1 a_2 a_3} \left(\bar{A}^{\mu_1 \mu_2}_{\mathrm{sQED}}(k_1, -k_1) - \bar{A}^{\mu_1 \mu_2}_{\mathrm{sQED}}(k_2, -k_2) \right) \tag{3.83}$$

and can be straightforwardly brought into the form given in Ref.[146]. The form of Eq.(3.82) shows the direct connection between QED and QCD in the high temperature regime.

Let us now move to the evaluation of the two point HTL in gravity, suing the results in the previous section. Here the situation goes similar to the bi-adjoint scalar and Yang-Mills case, where the regularization introduced in (3.69) given the three point graviton diagram is divergent in the forward limit. The contribution can be easily computed by the following diagram after taking the forward limit



$$
\begin{aligned}
=\frac{1}{2}\bar{A}_2^{\mu\nu\alpha\beta} &= \frac{1}{4} i p^\beta p^\nu \eta^{\alpha\mu} + \frac{1}{4} i p^\alpha p^\nu \eta^{\beta\mu} + \frac{1}{4} i p^\beta p^\mu \eta^{\alpha\nu} + \frac{1}{4} i p^\alpha p^\mu \eta^{\beta\nu} \\
&- \frac{i k_1^\alpha p^\beta p^\mu p^\nu}{4 p\cdot k_1} - \frac{i k_1^\beta p^\alpha p^\mu p^\nu}{4 p\cdot k_1} - \frac{i k_1^\mu p^\alpha p^\beta p^\nu}{4 p\cdot k_1} - \frac{i k_1^\nu p^\alpha p^\beta p^\mu}{4 p\cdot k_1} + \frac{i k_1^2 p^\alpha p^\beta p^\mu p^\nu}{4\,(p\cdot k_1)^{\,2}}\,.
\end{aligned}
\tag{3.84}
$$

which is in complete agreement with a simple amplitude calculation [146] and the literature on HTL [148].

Chapter 4

Classical double copy for spinning particles

Gauge theories and gravity features many differences at the quantum level, from the renormalizability to the Feynman diagrammatic expansion, the latter under control in a gauge theory, while more involved in gravity, given that diffeomorphism invariance generates an infinite tower of graviton self-interactions. However Bern, Carrasco and Johansson (BCJ) showed that, perturbatively one can relate scattering amplitudes in quantum gravity to that in gauge theories [22, 23], as the low energy field theory version of the Kawai, Lewellen, and Tye (KLT) relation for open and closed string amplitudes [149]. More specifically, the resulting theory is the $\mathcal{N} = 0$ supergravity [25] (SUGRA or NS-NS gravity), which describes Einstein-Hilbert gravity coupled to the dilaton field ϕ and the Kalb-Ramond two-form B. Such investigation has then been pushed forward for a great variety of theories including scalar [28, 29, 24, 30, 31] and spinning external matter particles [32, 33, 34, 35]. This line of research, not only shades light on gauge-gravity duality, but makes the double copy an efficient tool to generates amplitudes needed for applications to black hole scattering, in the classical limit.

All these lines of research study the double copy at the level of quantum amplitudes, however, the question we are asking in this chapter is whether double copy relations appears also at the classical level, and if so, how can we sistematize perturbatively double copy, so to use that as a tool to generate integrands related to black hole scattering. More importantly we also wonder if, from the knowledge of the perturbative classical double copy, it is possible to gain informations on the quantum theory, which, in the classical limit generates the related classical results. Thus, we aks ourselfs if there exists a classical double copy, relating amplitudes/observables related to black hole scattering in gauge theory, to the corresponding partner in gravity (or SUGRA). A first study of such double copy relation was performed by Plefka et. al [150, 151], using the EFT formalims, at the level of the effective action describing the full classical theory of a point particle in gauge/gravity background. However, such double copy turns out to breakdown at next-to-leading order due to the gauge dependence and off-shell nature of the effective action. Seeds of such classical double copy relations working at the level of on-shell solutions to the point particle equation of motions (EOMs) have appeared in the works of Goldberger et. al [49, 51, 52], however, they are based on a set of replacements rules between color and kinematics, working specifically at leading order in perturbation theory, thus it is not obvious that they would also work at higher orders.

The difficulty of implementing a classical double copy for scalar particles, mainly lies in that, differing from amplitudes, the locality structure of classical observables is not manifest. This can be explained in that moving from quantum to classical, Feynman propagators undergo a soft expansion (Laurent expansion in $\hbar \to 0$), generating also numerator structures, making ambiguous the identification of propagators. A way to tackle such a question was provided by Shen [54], who showed that the double copy at the classical level could be realized in an analogous way to the BCJ one from amplitudes by identifying the so-called "double copy kernel", arising from a theory where scalar particles interact through bi-adjoint scalar. This was shown by perturbatively solving the EOMs for a point particle in gauge and gravity backgrounds. Recently Plefka and Shi [56] showed that Shen's prescpription can be nicely implemented in the WQFT, double copying the eikonal and the radiation sourced by scalar particles.

Inspired from such results here we show how to extend them to the case of spinning particles, using the WQFT. Particularly we will show how to double copy classical integrands related to classical observables at linear order in spin, into the corresponding gravitational ones, at quadratic order in spin! From an amplitude view point this corresponds to the double copy of fermionic amplitudes interacting through gluons into scattering amplitudes of massive vector fields [152] interacting through the dilaton-gravity (DG) sector of the SUGRA. Not only we will be able to show that the classical double copy can be extended also to spinning particle, but, exploiting the quantization procedure on the worldline, we will also reconstruct the full quantum theory behind the gravitational integrands obtained with our double copy prescription.

We start by reviewing the classical double copy and then we move to the inclusion of spin on the worldline by coupling the $\mathcal{N} = 1$ susy model to a Yang-Mills background, writing down the WQFT action. We then compute the NLO eikonal and LO radiation sourced by a spinning particle, capturing only the spin-orbit terms. We show that the related integrands can be recasted so to make manifest CKD at the classical level. Then, we move to the $\mathcal{N} = 2$ model and work out the coupling with dilaton-gravity, which we anticipate is the double copied theory of the $\mathcal{N} = 1$ coupled to YM. We study how to double copy integrands from the latter and show how our DC prescription generates, from the $\mathcal{N} = 1$, integrands from the $\mathcal{N} = 2$. In addition, we quantize the latter, gaining informations about the quantum theory behind the double copy. In addition, we use our DC and the WQFT formulation of currents in the previous chapter, so to evaluate the gravitational Compton amplitude, capturing quadratic in spin contributions.

4.1 Classical double copy of sQCD

Let us start by reviewing some of the salient aspects of the classical double copy and how it has been implemented in the WQFT framework in [56]. Particularly we will focus on the double copy of the eikonal phase [153, 153] and the radiation sourced by scalar particle. To study the classical double copy we consider the scattering of two different flavoured scalar particles interacting through the Yang-Mills(YM) field. The WQFT has been developed in Sec.3.2.2. In such a case, it is known that the double copy of classical observables for sQCD leads to observables in a theory where the scalar particle is coupled to the dilaton-gravity sector of the SUGRA action

$$S_{N=0} = \frac{2}{\kappa^2} \int \mathrm{d}^D x \sqrt{-g(x)} \left(-R + \frac{4}{D-2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{12} e^{-8\phi/(D-2)} H_{\lambda\mu\nu} H^{\lambda\mu\nu} \right). \tag{4.1}$$

where ϕ is the dilaton scalar, while $H_{\lambda\mu\nu} = \partial_\lambda B_{\mu\nu} + \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu}$ is the field strength related to $B_{\mu\nu}$. The worldline action for the scalar coupled to dilaton gravity is

$$S^{\mathrm{pm}} = \int_{-\infty}^{\infty} \mathrm{d}\tau \left(-\frac{m}{2} e^{2\kappa\phi} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right), \tag{4.2}$$

where ϕ denotes the dilaton scalar. A crucial observation for the study of the classical double copy is that at higher orders perturbation theory, some of the color factors in YM theory will be vanishing due to the anti-symmetry of the structure constant, whereas the corresponding numerators are required for the double copy. To avoid this problem, we will use as many different worldlines as worldline- field interactions occur. For example, at leading order of the eikonal phase, two worldlines are sufficient, but at next-to-leading order, we will use three worldlines.

The WQFT partition function for such a scattering can be easily written as explained in the previous chapter. A useful relation that we will use later on lies in that the WQFT partition function is related to the so called eikonal phase χ by exponentiation, namely

$$\mathcal{Z}_{\mathrm{WQFT}} = e^{i\chi} = \int_q e^{iq\cdot b} \, \hat{\delta}(q \cdot p_1) \hat{\delta}(q \cdot p_2) \lim_{\hbar \to 0} \mathcal{M}_{\phi_1 \phi_1' \to \phi_1 \phi_1'}(q) \tag{4.3}$$

meaning that connected diagrams arising from the partition function do contributes to the eikonal. Physically, the latter represent the classical limit of the scattering amplitude $\mathcal{M}_{\phi_1 \phi_1' \to \phi_1 \phi_1'}(q)$ between two different flavoured scalars in the impact parameter space.

Let us now see how to double copy the eikonal and radiation in the YM theory. The double copy relation between the eikonal of Yang-Mills theory χ^{YM} and dilaton-gravity χ^{DG} at $\mathrm{N}^{(\mathrm{n}-1)}$LO can be expressed as

$$\chi_n^{\mathrm{YM}} = -(ig)^{2n} \int \mathrm{d}\mu_{1,2,\ldots,(n+1)}(0) \sum_{i,j} \mathcal{C}_i \, \mathcal{K}_{ij} \, \mathcal{N}_j, \tag{4.4a}$$

$$\chi_n^{\mathrm{DG}} = -\left(\frac{\kappa}{2}\right)^{2n} \int \mathrm{d}\mu_{1,2,\ldots,(n+1)}(0) \sum_{i,j} \mathcal{N}_i \, \mathcal{K}_{ij} \, \mathcal{N}_j, \tag{4.4b}$$

where $\mathcal{C}_i, \mathcal{N}_j$ are the arrays of color factors and kinematic numerators, respectively, which should be arranged to satisfy color-kinematic duality. $\mathcal{K}_{ij}$ is the BCJ double copy kernel that are derived from bi-adjoint scalar theory coupled to spinless worldlines, which, for brevity, will not be presented here. We have also defined the integral measure as

$$\mathrm{d}\mu_{1,2,\ldots,n}(k) = \prod_{i=1}^{n} \left(\frac{\mathrm{d}^4 q_i}{(2\pi)^4} e^{iq_i\cdot b_i} \hat{\delta}\left(q_i \cdot p_i\right) \right) \hat{\delta}^{(4)} \left(\sum_{i=1}^{n} q_i^\mu - k^\mu \right), \tag{4.5}$$

with $p_i^\mu = m_i v_i^\mu$ being the kinetic momentum, and q_i is the total outgoing momentum of gluons or gravitons attached to a worldline. Relation (4.4a) exemplifies color-kinematics duality at the classical level i.e. if a classical observable obeys color kinematics duality at the classical level, it means that it can be recasted in a form like (4.4a). Once the observable is written in a color-kinematic satisfying fashion, as stated by (4.4b), it can be double copied into its gravitational partner by replacing the color vector with the related vector of BCJ numerators.

Let us now move to calculations in gauge theory. At LO, the double copy structure is simple since there is only one color factor. We now present the calculation at NLO as an example to show how WQFT double copy works. The color factors can be arranged as

$$\mathcal{C}_i^{(123)} = \big\{(C_1\cdot C_2)(C_1\cdot C_3), \quad (C_1^{ab}C_2^aC_3^b), \quad (C_1^{ba}C_2^aC_3^b)\big\} \qquad \mathcal{C}_i^{(0)} = f^{abc}C_1^aC_2^bC_3^c, \tag{4.6}$$

where we have used $C_1^{ab} = \bar{u}_\alpha (T^a)^\alpha{}_\beta (T^b)^\beta{}_\gamma u^\gamma$, and f^{abc} is the structure constant. Note that there are also $\mathcal{C}_i^{(231)}, \mathcal{C}_i^{(312)}$ which can be obtained by simply rotating the indices $(1,2,3)$ in $\mathcal{C}_i^{(123)}$. Together, they compose a 10-dimensional array of color factors. Due to the commutation relation of the group generators, the color factors satisfy

$$C_1^{ab}C_2^aC_3^b - C_1^{ba}C_2^aC_3^b = f^{abc}C_1^cC_2^aC_3^b. \tag{4.7}$$

The BCJ kernel $\mathcal{K}_{ij}$ is block-diagonal, and the blocks corresponding to the color factors in (4.6) are

$$\mathcal{K}_{ij}^{(123)} = \frac{1}{q_2^2 q_3^2}\begin{pmatrix} \frac{q_2\cdot q_3}{\omega_1^2} & \frac{1}{\omega_1} & -\frac{1}{\omega_1} \\ \frac{1}{\omega_1} & 0 & 0 \\ -\frac{1}{\omega_1} & 0 & 0 \end{pmatrix}, \qquad \mathcal{K}_{ij}^{(0)} = \frac{2}{q_1^2 q_2^2 q_3^2}. \tag{4.8}$$

For further convenience, we define[1]

$$\omega_1 = p_1\cdot q_2, \quad \omega_2 = p_2\cdot q_3, \quad \omega_3 = p_3\cdot q_1. \tag{4.9}$$

Explicit calculations shows that both the eikonal phases in YM and dilaton-gravity can be expressed in terms of the same numerators

$$\mathcal{N}_j^{(123)} = \left\{n_0\,,\ \frac{n_1}{2}\,,\ \frac{-n_1}{2}\right\} \qquad \mathcal{N}_j^{(0)} = n_1, \tag{4.10}$$

$$n_0 = p_1\cdot p_2\, p_1\cdot p_3 \qquad n_1 = q_2\cdot p_3\, p_1\cdot p_2 - q_3\cdot p_2\, p_1\cdot p_3 - q_2\cdot p_1\, p_2\cdot p_3. \tag{4.11}$$

Obviously, they satisfy the color-kinematic duality since $\frac{n_1}{2} - \frac{-n_1}{2} = n_1$. Similarly, we can obtain the other two blocks of the kernel $\mathcal{K}_{ij}^{(231)}, \mathcal{K}_{ij}^{(312)}$ and the corresponding numerators $\mathcal{N}_j^{(231)}, \mathcal{N}_j^{(312)}$ by relabeling the labels $(1,2,3)$. Thus the eikonals can be decomposed as

$$\chi_2^{\mathrm{YM}} = -g^4\int \mathrm{d}\mu_{1,2,3}(0)\sum_{i,j}\Big(\mathcal{C}_i^{(0)}\mathcal{K}_{ij}^{(0)}\mathcal{N}_j^{(0)} + \big(\mathcal{C}_i^{(123)}\mathcal{K}_{ij}^{(123)}\mathcal{N}_j^{(123)} + \text{cyclic}\big)\Big) \tag{4.12}$$

$$\chi_2^{\mathrm{DG}} = -\frac{\kappa^4}{16}\int \mathrm{d}\mu_{1,2,3}(0)\sum_{i,j}\Big(\mathcal{N}_i^{(0)}\mathcal{K}_{ij}^{(0)}\mathcal{N}_j^{(0)} + \big(\mathcal{N}_i^{(123)}\mathcal{K}_{ij}^{(123)}\mathcal{N}_j^{(123)} + \text{cyclic}\big)\Big).$$

[1] Note that due to different convention, the off-diagonal components of the BCJ kernel are different by a minus sign from [56].

This prescription agrees with the double copy of scattering amplitudes in scalar QCD, given that the eikonal is directly related to the classical limit of the 6-scalar amplitude [28]. This completes the story of the double copy of eikonal phase at NLO in the WQFT formalism.

We note that the 10-dimensional BCJ kernel is actually reducible, as a consequence of the fact that the color factors form a over-completed basis. Specifically, from $\mathcal{C}_i^{(123)}$ and $\mathcal{K}_{ij}^{(123)}$ we see that the contributions to the YM eikonal from the color factors $C_1^{ab}C_2^aC_3^b$ and $C_1^{ba}C_2^aC_3^b$ are different only by a minus sign. We can thus use the Jacobi identity (4.7) to simplify their contributions to $\chi_2^{\rm YM}$

$$C_1^{ab}C_2^aC_3^b\frac{n_0}{q_2^2q_3^2\omega_1}+C_1^{ba}C_2^aC_3^b\frac{-n_0}{q_2^2q_3^2\omega_1}=f^{abc}C_1^cC_2^aC_3^b\frac{n_0}{q_2^2q_3^2\omega_1}. \tag{4.13}$$

This is guaranteed by the fact that C_1^{ab}, C_1^{ba} do not appear in the classical equations of motion, so they must be removed in the final solutions [53]. After the reduction, the YM eikonal can be rewritten as

$$\begin{aligned}\chi_2^{\rm YM}=-g^4\int{\rm d}\mu_{1,2,3}(0)\Big\{&f^{abc}C_1^cC_2^aC_3^b\bigg(\Big(\frac{n_0}{q_2^2q_3^2\omega_1}+\text{cyclic}\Big)+\frac{2n_1}{q_1^2q_2^2q_3^2}\bigg)\\&+\bigg((C_1\!\cdot\!C_2)(C_1\!\cdot\!C_3)\Big(\frac{q_2\!\cdot\!q_3\,n_0}{q_2^2q_3^2\omega_1^2}+\frac{n_1}{q_1^2q_2^2\omega_1}\Big)+\text{cyclic}\bigg).\Big\}\end{aligned} \tag{4.14}$$

We can reduce the full 10-dimensional BCJ double copy kernel to 4 dimensions

$$\mathcal{K}_{ij}=\begin{pmatrix}\frac{q_2\cdot q_3}{q_2^2q_3^2\omega_1^2} & 0 & 0 & \frac{1}{q_2^2q_3^2\omega_1}\\ 0 & \frac{q_1\cdot q_3}{q_1^2q_3^2\omega_2^2} & 0 & \frac{1}{q_1^2q_3^2\omega_2}\\ 0 & 0 & \frac{q_1\cdot q_2}{q_1^2q_2^2\omega_3^2} & \frac{1}{q_1^2q_2^2\omega_3}\\ \frac{1}{q_2^2q_3^2\omega_1} & \frac{1}{q_1^2q_3^2\omega_2} & \frac{1}{q_1^2q_2^2\omega_3} & \frac{2}{q_1^2q_2^2q_3^2}\end{pmatrix}, \tag{4.15}$$

The associated arrays of color factors and numerators are

$$\mathcal{C}_i=\big\{(C_1\!\cdot\!C_2)(C_1\!\cdot\!C_3),\quad (C_1\!\cdot\!C_2)(C_2\!\cdot\!C_3),\quad (C_1\!\cdot\!C_3)(C_2\!\cdot\!C_3),\quad f^{abc}C_1^aC_2^bC_3^c\big\} \tag{4.16}$$
$$\mathcal{N}_j=\big\{n_0,\quad n_0',\quad n_0'',\quad n_1\big\}, \tag{4.17}$$

where $n_0'=p_1\!\cdot\!p_2\,p_2\!\cdot\!p_3$ and $n_0''=p_1\!\cdot\!p_3\,p_2\!\cdot\!p_3$ are obtained by relabeling $(1,2,3)$ in n_0.

In this new basis of the color factors, we no longer explicitly have the Jacobi identities, whereas the color-kinematics duality is hidden in that such a decomposition is possible. The number of independent color factors also agrees with that of the 6-quark amplitude in QCD in the Melia basis [154]. It is straightforward to check that the double copy relation (4.4) still works.

The advantage of the reduced BCJ kernel is not only the lower dimension, but, more importantly, that it is invertible. Therefore, one can easily do the KLT-like double copy by inverting the BCJ kernel, as we will show later on. However, this is not true for the radiation kernel, which is degenerate. As we will see in the next sections, this has to do with gauge invariance.

4.2 QCD on the worldline

Let us start by including spin on the worldline. With an eye on double copy applications, we are interested in a worldline theory propagating a massive spin half fermion, coupled to a Yang-Mills background. The proper worldline model needed for such a purpose is known as the $\mathcal{N}=1$ susy particle. The free theory was originally formulated in [155], then coupled to gravity on the loop in [156] for computing gravitational anomalies, and recently used to compute Feynman diagrams in QED in [108] where a path integral on the line has been implemented to accomplish such a task.

The model can be formulated by introducing a set of real Grassmann variables $\psi_M = (\psi_\mu, \theta)$ alongside the usual bosonic variables $x^M = (x^\mu, x^5), P_{_M} = (P_\mu, P_5)$, with μ being a Lorentz index while the raising and lowering procedure is done by $\eta_{_{MN}} = \text{diag}\,(\eta_{\mu\nu}, -1)$. As we will see, the Grassmann variables will take care of the spinning degrees of freedom of the particle propagated in first quantization by the worldline model, while the auxiliary fifth component has been introduced so to be able to give a mass to such a particle by the Kaluza-Klein dimensional reduction. We consider the following phase space action

$$S_{\text{ph}} = -\int_0^1 \mathrm{d}\tau \left(\dot{x}^M P_M + \frac{i}{2}\psi_M \dot{\psi}^M + i\bar{c}_\alpha \dot{c}^\alpha - eH - i\chi Q \right), \tag{4.18}$$

where we gauge the reparametrization invariance through the gauge field e, known as the einbein, and its generator H representing the point particle Hamiltonian. Further, we also gauge the worldline supersymmetry through the Grassmann-valued gauge field χ, known as the gravitino, and the correspondent generator Q which is the SUSY charge, i.e., the conserved charge under supersymmetry transformation of the worldline variables.

One could even gauge a $U(1)$ worldline symmetry on the color sector, including a Chern-Simmons like we did prevously for the case of the bi-adjoint scalar, so to project on a specific field that has s indices of the fundamental representation of the color group. However, as already explained in the previous chapter, such a gauging is not necessary for classical applications, meaning that propagating all of the color representations in the WQFT consistently implements the classical limit for the color degrees of freedom.

The phase space action (4.18) allows us to read out Poisson brackets between canonical coordinates, namely

$$\{x^\mu, P_\nu\} = \delta^\mu{}_\nu, \quad \{\psi^\mu, \psi_\nu\} = -i\delta^\mu{}_\nu, \quad \{\theta, \theta\} = i, \quad \{c^\alpha, \bar{c}_\beta\} = -i\delta^\alpha{}_\beta \tag{4.19}$$

with all of the remaining Poisson brackets vanishing. The Kaluza-Klein reduction here is simply implemented by fixing $P_5 = m$ and gauge away x_5 since, even after coupling to background fields, it will appear as a total derivative.

In order to couple to a Yang-Mills background we define the SUSY charge as

$$Q = \psi^\mu \left(P_\mu - gA^a_\mu q_a\right) - m\,\theta, \tag{4.20}$$

such that, by using the SUSY algebra we can fix the point particle Hamiltonian

$$\{Q, Q\} = -2iH = -2i\left(\frac{1}{2}\left(\pi^2 - m^2\right) - \frac{g}{2} S^{\mu\nu} F^a_{\mu\nu} q_a\right), \tag{4.21}$$

where $\pi_\mu = P_\mu - gA^a_\mu q_a$ is the covariant momentum of the particle while, $F^a_{\mu\nu} = 2\partial_{[\mu}A^a_{\nu]} - igf^{abc}A^b_\mu A^c_\nu$ is the Yang-Mills field strength. We have also defined the spin tensor as

$$S^{\mu\nu} = -i\psi^\mu\psi^\nu, \tag{4.22}$$

which is the conserved current under the Lorentz symmetry on the fermionic sector of the worldline i.e. $\psi \to e^{\omega_{\mu\nu}J^{\mu\nu}}\psi$ with $\omega_{\mu\nu}$ and $J^{\mu\nu}$ being respectively the Lorentz group parameters and the generators in the fundamental representation. One can check that the Possion bracket $\{S^{\mu\nu}, S^{\rho\sigma}\}$ agrees with the Lorentz algebra.

Let us now turn to the quantization of the model, so to explicitly see which is the particle propagated by the worldline in first quantization. In order to recover unitarity at the quantum level, we need to check that the constraint algebra is first class, namely that

$$\{Q, Q\} = -2iH, \quad \{Q, H\} = 0, \tag{4.23}$$

which is straightforward in such a case and holds by construction. Then we move to the quantization by promoting the Poisson brackets to graded commutators as $\{\bullet, \bullet\} \to -i[\bullet, \bullet\}$. This allows us to get the quantum algebra

$$[\hat{x}^\mu, \hat{P}_\nu] = i\delta^\mu_\nu, \quad \{\hat{\psi}^\mu, \hat{\psi}^\nu\} = \eta^{\mu\nu}, \quad \{\hat{\theta}, \hat{\theta}\} = -1, \quad [\hat{c}^\alpha, \hat{c}^\dagger_\beta] = \delta^\alpha_\beta\,. \tag{4.24}$$

We represent $\hat{P}_\mu = -i\partial_\mu$ and $\hat{x}_\mu$ acting as a multiplication on states, further, once rescaling $\psi^\mu \to \frac{1}{\sqrt{2}}\psi^\mu, \theta \to \frac{i}{\sqrt{2}}\theta$ we can realize the Grassmann variables as $\hat{\psi}^\mu = \gamma^\mu$, $\hat{\theta} = i\gamma_5$, generating then the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ alongside with $\{\gamma^\mu, \gamma_5\} = 0$. Instead, the color variables can be naturally realized at the quantum level as $\hat{c}^\dagger = \bar{c}$, $\hat{c} = \partial/\partial\bar{c}$, thus being creation and annihilation operators for a set of oscillators. Then, one can use the coherent state basis related to such operators, to expand a generic wave function propagated by the worldline as

$$\Phi(x, \bar{c}) = \sum_{n=0}^{\infty} \frac{1}{n!}\Phi_{\alpha_1\alpha_2\cdots\alpha_n}(x)\bar{c}^{\alpha_1}\bar{c}^{\alpha_2}\cdots\bar{c}^{\alpha_n}\,. \tag{4.25}$$

Recalling the color variables are charged under the $SU(N)$ global symmetry, as said previously, implies that the worldline particle is propagating totally symmetric tensor products of the fundamental representation. To project out on a s-tensor product, we use the gauged $U(1)$ constraint[2] at the quantum level

$$\left(\bar{c}^\alpha \frac{\partial}{\partial\bar{c}^\alpha} - s\right)\Phi(x, \bar{c}) = 0, \tag{4.26}$$

uniquely selecting the component of the wave function with s indices in the fundamental representation of the color group

$$\Phi_s(x, \bar{c}) = \frac{1}{s!}\Phi_{\alpha_1\alpha_2\cdots\alpha_s}(x)\bar{c}^{\alpha_1}\bar{c}^{\alpha_2}\cdots\bar{c}^{\alpha_s}\,. \tag{4.27}$$

[2]We use normal ordering to solve the ordering ambiguities arising when writing quantum constraints from classical ones.

Then, we impose the equations of motion of the worldline gauge fields (e, χ) as operator constraints on the above wave function. In particular, the equation of motion for the gravitino delivers

$$\frac{\delta S_{\rm ph}}{\delta \chi} = 0 \quad \rightarrow \quad \hat{Q}\Phi_s(x) = 0 \quad \rightarrow \quad \big(\gamma_5 \not{D} + m\big)\, \Phi_s(x, \bar{c}) = 0 \tag{4.28}$$

with the gauge covariant derivative $D_\mu = \partial_\mu - igA^a_\mu q_a$ acting in the s representation. Particularly, choosing $s = 1$, allows us to project on a colored Dirac Fermion, as stressed by the above equation of motion. The latter can be recast as the standard Dirac equation from textbooks, by performing a change of basis in the spinor space, keeping invariant the Clifford algebra, namely defining $i\tilde{\gamma}_\mu = -\gamma_5\gamma_\mu$.

This analysis reveals that the worldline supersymmetric $\mathcal{N} = 1$ model, with the SUSY charge deformation (4.20), allows to propagate a colored Dirac spinor in the fundamental representation of $SU(N)$. Thus, the path integral quantization allows to resum Feynman diagrams with two external spin-half particles in a non-Abelian gauge background. Noticeably, recalling that the expansion of classical observables at linear order in the spin tensor is reproduced by scattering amplitudes of Dirac fermions [32, 157, 33, 158], the above analysis makes it clear why one must use the $\mathcal{N} = 1$ SUSY model to compute classical observables at linear order in spin.

After this digression we can come back to the action principle needed for our classical applications. Now we can plug the Hamiltonian and the supercharge (4.21) inside the phase space action (4.18). Then, eliminating the momentum, by using its equation of motion

$$\frac{\delta S_{\rm ph}}{\delta P_\mu} = 0 \quad \rightarrow \quad P_\mu = e^{-1}\left(\dot{x}_\mu + egA^a_\mu q_a - i\chi\psi_\mu\right) \tag{4.29}$$

and plugging it back in the phase space action (4.18), we can write down the worldline action for a color-charged point particle in configuration space as follows

$$\begin{aligned} S_{\rm pc} = \int_0^1 \mathrm{d}\tau \Big(&- \frac{1}{2}e^{-1}\left(\dot{x}^2 + e^2m^2\right) - \frac{i}{2}\psi\cdot\dot{\psi} + \frac{i}{2}\theta\dot{\theta} - i\bar{c}\cdot\dot{c} \\ &- g\dot{x}\cdot A^a q_a - \frac{eg}{2}S^{\mu\nu}F^a_{\mu\nu}q_a + ie^{-1}\chi\dot{x}\cdot\psi + im\chi\theta\Big). \end{aligned} \tag{4.30}$$

Then, following [16], imposing the constraints $\theta = 0, \chi = 0$ yields $\dot{x}\cdot\psi = 0$, which implies the spin supplementary condition (SSC), $\dot{x}_\mu S^{\mu\nu} = 0$. Furthermore, we gauge fix $e = 1/m$, then we change the integration boundaries to $(-\infty, \infty)$ as a consequence of the LSZ reduction procedure on the external legs [15, 55]. Further, we rescale $\tau \rightarrow m\tau$ such that the worldline action, ready to be used to perform classical calculations, reads as

$$S_{\rm pc} = -\int_{-\infty}^{\infty} \mathrm{d}\tau \left(\frac{1}{2}\dot{x}^2 + \frac{i}{2}\psi\cdot\dot{\psi} + i\bar{c}\cdot\dot{c} + g\dot{x}\cdot A^a q_a + \frac{g}{2}S^{\mu\nu}F^a_{\mu\nu}q_a\right). \tag{4.31}$$

We note that due to the rescaling of the integration variable, $\dot{x}^\mu$ has the dimension of momentum, so that in perturbative calculations of the WQFT, it is easier to make contact with integrands arising from the classical limit of scattering amplitudes.

Before ending this subsection, let us give some comments on the supersymmetry on the worldline and the double copy. The action (4.31) is in agreement with the one used by

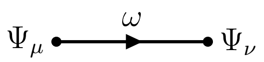
Goldberger, Li and Prabhu in [51] to study the double copy of classical spinning particles at linear order in spin. In such a case they showed that, in order to have a double copy radiation satisfying the linearized Ward identities for gravity, one must fix the coupling of the Pauli interaction $S_{\mu\nu}F^{\mu\nu}$ to be minus one half (in such conventions). However, in our consideration we get such a coupling, very surprisingly, completely by the SUSY algebra on the worldline, thus implying that SUSY, at linear level in spin, allows for a consistent double copy of spinning worldlines! This may also be tracked back to the fact that the constraint algebra is of first class, thus the model can be consistently quantized, propagating a Dirac spinor coupled to Yang-Mills, which leads to consistent double copy of amplitudes in four dimensions at the quantum level as shown in [33, 34].

4.2.1 Feynman rules

Let us now move to the derivation of the vertices arising from the above action. The configuration space and color variables are expanded as already discussed in the previous chapter, the new variables to deal with here are the fermionic ψ^μ. We background expand them as

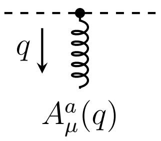
$$\psi^\mu(\tau) = \zeta^\mu + \Psi^\mu(\tau) \tag{4.32}$$

picking as usual retarded boundary conditions on the fermionic fluctuations $\Psi^\mu(\tau)$. Particularly these expansions allows us to identify the classical values of the spin tensor of the worldline particle, namely $\mathcal{S}_{\mu\nu} = -i\zeta_\mu\zeta_\nu$, which, squares to zero. This is expected since the $\mathcal{N}=1$ particle propagates a spin half fermion, whose amplitudes in the classical limit only encode spin-orbit terms. As for the color variables, even the spin variables propagator only contains a simple pole in the enrgy of the particle

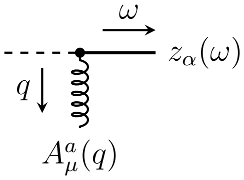


$$= i\frac{\eta_{\mu\nu}}{\omega}\,. \tag{4.33}$$

Here we list only the interactions needed for the calculations of the next section

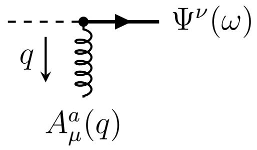


$$= -igC^a e^{iq\cdot b}\hat{\delta}(q\cdot p)\left(p^\mu + i(q\cdot\mathcal{S})^\mu\right)$$

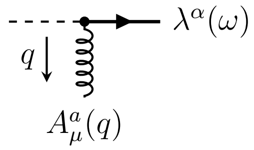


$$= gC^a e^{iq\cdot b}\hat{\delta}(q\cdot p+\omega)\left(p^\mu q^\alpha + \omega\eta^{\mu\alpha} + i(q\cdot\mathcal{S})^\mu q^\alpha\right) \tag{4.34}$$

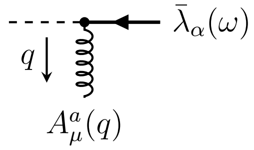
where $(q\cdot\mathcal{S})^\mu = q_\nu\mathcal{S}^{\nu\mu}$. One can notice that, up to on-shell terms, the first vertex above corresponds to the classical piece in the three point amplitude for two spin half fermions emitting a gluon, as expected by the previous quantization procedure. Differing from the scalar particle, here, the use of SUSY on the worldline introduces vertices propagating fluctuations of the Grassman variables



$$= -2igC^a e^{iq\cdot b}\hat{\delta}(q\cdot p+\omega)\zeta^\rho q_{[\rho}\eta_{\nu]\mu}$$

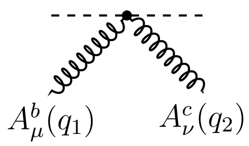


$$= -ige^{iq\cdot b}\hat{\delta}(q\cdot p+\omega)\left(p^\mu + i(q\cdot\mathcal{S})^\mu\right)\bar{u}_\beta (T^a)^\beta{}_\alpha \tag{4.35}$$



$$= -ige^{iq\cdot b}\hat{\delta}(q\cdot p-\omega)\left(p^\mu + i(q\cdot\mathcal{S})^\mu\right)(T^a)^\alpha{}_\beta u^\beta$$

along side with the usual color vertices modified by the spin structure. Lastly, we also get a vertex with the emission of two gluons from the worldline, namely



$$= -g^2 f^{bcd} C^d e^{i(q_1+q_2)\cdot b}\hat{\delta}((q_1+q_2)\cdot p)\mathcal{S}^{\mu\nu}, \tag{4.36}$$

as a direct consequence of SUSY and non-abelianity of the gluon. In the calculation of Feynman diagrams, we follow the arrow of (Ψ,λ) propagators to combine the vertices.

4.2.2 Yang-Mills eikonal phase

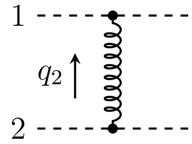
For a model with n worldlines described by the action (4.31), the partition function reads as

$$\mathcal{Z}_{\text{YM}} = \int \mathcal{D}A\, e^{iS^{gf}_{\text{YM}}[A]} \int \prod_{k=1}^{n} \mathcal{D}X_k\, e^{iS[X_k;A]}, \tag{4.37}$$

where we collected all of the worldline variables in $X=(x,\psi,c,\bar{c})$. Let us recall that the particion function is related to the eikonal phase through (4.3). With the background expansion of the worldline variables and the Feynman rules provided in the previous subsection, the eikonal can be calculated perturbatively.

Leading order eikonal: At leading order (LO), only one diagram contributes to the eikonal. Using the vertex in (4.34), we obtain

$$i\chi_1^{\text{YM}} =$$



$$= ig^2 C_1\cdot C_2 \int \frac{\mathrm{d}\mu_{1,2}(0)}{q_2^2}\big(\gamma - iq_2\cdot\mathcal{S}_1\cdot p_2 + iq_2\cdot\mathcal{S}_2\cdot p_1 - q_2\cdot\mathcal{S}_1\cdot\mathcal{S}_2\cdot q_2\big), \tag{4.38}$$

where we have defined $\gamma = p_1\cdot p_2$, and used the notation $a\cdot\mathcal{S}_i\cdot\mathcal{S}_j\cdot b = a_\mu(\mathcal{S}_i)^{\mu\nu}(\mathcal{S}_j)_{\nu\rho}b^\rho$ for arbitrary vectors a^μ, b^μ. We can recover the scalar WQFT result by simply setting

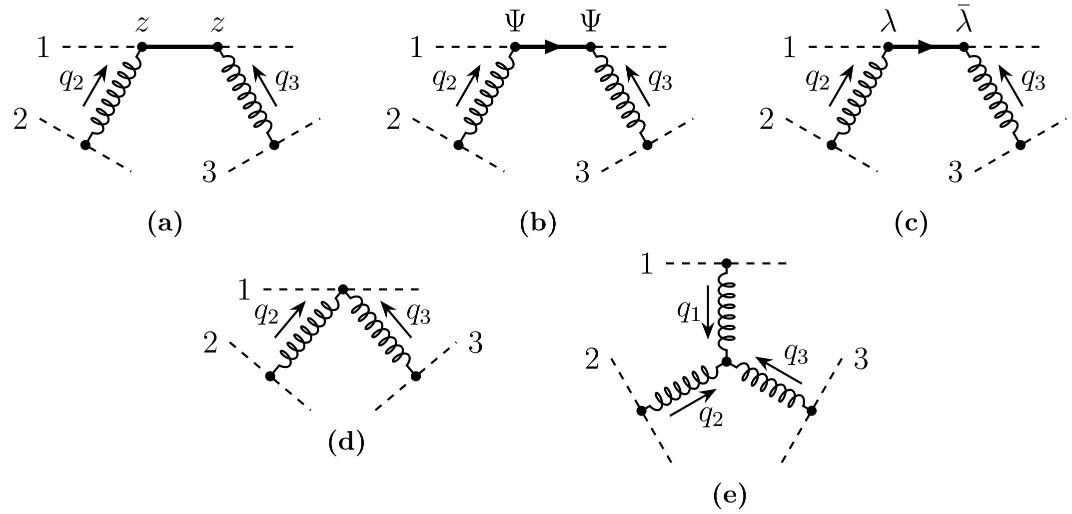


Figure 4.1: The diagrams for the NLO eikonal with three worldlines. For diagrams involving the propagators of Ψ (4.1b) and λ (4.1c), we also need to include the crossed diagrams that can be obtained by simply reversing the arrows on the worldline. We only display diagrams with worldline propagator and contact interaction of particle 1.

the spin to zero, and it agrees with the result in [56]. In the spirit of the double copy, we identify the color factor, the double copy kernel and the numerator in the form (4.4) as

$$\mathcal{C} = C_1 \cdot C_2, \qquad \mathcal{K} = \frac{1}{q_2^2}, \tag{4.39}$$

$$\mathcal{N} = \gamma - i\big(q_2 \cdot \mathcal{S}_1 \cdot p_2 - q_2 \cdot \mathcal{S}_2 \cdot p_1\big) - q_2 \cdot \mathcal{S}_1 \cdot \mathcal{S}_2 \cdot q_2. \tag{4.40}$$

Next-to-leading order eikonal: For next-to-leading order (NLO) eikonal of two worldlines, some of the color factors will be vanishing due to the anti-symmetry of structure constant, whereas the corresponding numerators are needed in the double copy. To circumvent this problem, we will consider three bodies at NLO. One can easily retrieve the binary system by identifying two of the three worldlines.

The diagrams with three worldlines are collected in Fig.4.1. The full result can be formally expressed as

$$\chi_2^{\mathrm{YM}} = -g^4 \int \mathrm{d}\mu_{1,2,3}(0)\Big(f^{abc}C_1^a C_2^b C_3^c\,\mathcal{A}^{(0)} + \big(C_1{\cdot}C_2\, C_1{\cdot}C_3\,\mathcal{A}^{(1)} + \text{cyclic}\big)\Big), \tag{4.41}$$

where $\mathcal{A}^{(i)}$ are gauge-invariant "partial eikonals" akin to color-ordered amplitudes at the quantum level. Explicitly, they can be written as

$$\mathcal{A}^{(0)} = \Big(\frac{2}{q_1^2 q_2^2 q_3^2} n^{(0)} + \frac{1}{q_2^2 q_3^2 \omega_1} n^{(123)} + \frac{1}{q_1^2 q_3^2 \omega_2} n^{(231)} + \frac{1}{q_1^2 q_2^2 \omega_3} n^{(312)}\Big) \tag{4.42}$$

$$\mathcal{A}^{(1)} = \Big(\frac{1}{q_2^2 q_3^2 \omega_1} n^{(0)} + \frac{q_2 \cdot q_3}{q_2^2 q_3^2 \omega_1^2} n^{(123)}\Big) \tag{4.43}$$

where ω_i is defined in the same way as the spinless case (4.9), and the numerators read

$$
\begin{aligned}
n^{(123)} =& \tfrac{1}{2} p_1 \cdot p_2\, p_1 \cdot p_3 + i\big(p_1 \cdot p_3\, (q_2 \cdot \mathcal{S}_2 \cdot p_1 - q_2 \cdot \mathcal{S}_1 \cdot p_2) + \tfrac{1}{2} q_3 \cdot p_1\, p_2 \cdot \mathcal{S}_1 \cdot p_3\big) \\
&+ q_2 \cdot \mathcal{S}_2 \cdot p_1 \left(q_3 \cdot \mathcal{S}_1 \cdot p_3 - \tfrac{1}{2} q_3 \cdot \mathcal{S}_3 \cdot p_1\right) - p_1 \cdot p_3\, q_2 \cdot \mathcal{S}_1 \cdot \mathcal{S}_2 \cdot q_2 - q_3 \cdot p_1\, q_2 \cdot \mathcal{S}_2 \cdot \mathcal{S}_1 \cdot p_3 \\
&+ i\big(\tfrac{1}{2} q_3 \cdot p_1\, q_2 \cdot \mathcal{S}_2 \cdot \mathcal{S}_1 \cdot \mathcal{S}_3 \cdot q_3 - q_2 \cdot \mathcal{S}_1 \cdot \mathcal{S}_2 \cdot q_2\, q_3 \cdot \mathcal{S}_3 \cdot p_1\big) + (2 \leftrightarrow 3) \qquad (4.44) \\
n^{(0)} =& p_1 \cdot p_2\, q_2 \cdot p_3 - i\,(p_2 \cdot p_3\, q_3 \cdot \mathcal{S}_1 \cdot q_1 + q_1 \cdot p_3\, q_1 \cdot \mathcal{S}_1 \cdot p_2 + q_3 \cdot p_2\, q_1 \cdot \mathcal{S}_1 \cdot p_3) \\
&+ q_2 \cdot p_3\, q_1 \cdot \mathcal{S}_1 \cdot \mathcal{S}_2 \cdot q_2 + q_2 \cdot \mathcal{S}_1 \cdot q_3\, q_2 \cdot \mathcal{S}_2 \cdot p_3 + q_2 \cdot \mathcal{S}_2 \cdot q_3\, q_1 \cdot \mathcal{S}_1 \cdot p_3 \\
&- i q_1 \cdot \mathcal{S}_3 \cdot q_2\, q_1 \cdot \mathcal{S}_1 \cdot \mathcal{S}_2 \cdot q_2 + \text{cyclic}, \qquad (4.45)
\end{aligned}
$$

and $n^{(231)}$ and $n^{(312)}$ are obtained by relabeling $(1, 2, 3)$ in (4.44). We stress that the $\mathcal{N} = 1$ model captures only linear terms in spin, so the numerators are truncated up to linear order in each of the spin tensor.

In (4.41) we have arranged the numerators to satisfy color-kinematic duality, in the sense that the eikonal can be decomposed into form (4.4a). At NLO, the BCJ double copy kernel and the corresponding array of color factors are the same as the spinless case given in (4.15) and (4.16), respectively, whereas the array of numerators read

$$
\mathcal{N}_j = \begin{pmatrix} n^{(123)} & n^{(231)} & n^{(312)} & n^{(0)} \end{pmatrix}. \qquad (4.46)
$$

As stated in subsection 4.1, such a decomposition is always possible since the double copy kernel (4.15) is invertible. Moreover, we note that the kinematic numerators are uniquely fixed by the "partial eikonals" $\mathcal{A}^{(i)}$,

$$
\mathcal{A}^{(i)} = \mathcal{K}_{ij} \mathcal{N}_j \qquad \Rightarrow \qquad \mathcal{N}_j = (\mathcal{K}^{-1})_{ji} \mathcal{A}^{(i)} \qquad (4.47)
$$

exactly as it happens to its quantum counterpart - the 6-point amplitude for 3 distinguishable quark-antiquark pairs in QCD [154].

4.2.3 Leading Yang Mills radiation

Let us now use the Feynman rules to evaluate the leading order ($O(g^3)$) Yang-Mills radiation emitted in the process of the scattering of two colored and spinning particles. In the WQFT we define the radiation as

$$
\mathcal{R}^a_\mu(\sigma) = \langle ik^2 A^a_\mu(k) \rangle_{\text{WQFT}}\big|_{k^2=0} = \mathcal{Z}^{-1}_{\text{YM}} \int \mathcal{D}A\, e^{iS_{\text{YM}}} \int \left(\prod_{j=1}^{2} \mathcal{D}X_j\, e^{iS[X_j;A]} \right) ik^2 A^a_\mu(k) \qquad (4.48)
$$

where the external gluon has to be considered on-shell while $\sigma = (k, b_{1,2}, p_{1,2}, \mathcal{S}_{1,2})$ packages all of the scattering data, including the external momentum of the radiated gluon. The diagrams contributing at the leading order to the radiation are shown in Fig.4.2. The diagram contributing the the leading radiation are displayed in Fig 4.2. After performing simple shifts in the numerators of the partial amplitudes, preserving gauge invariance of the partial amplitudes, we can rearrange the radiation so that the it satisfies color

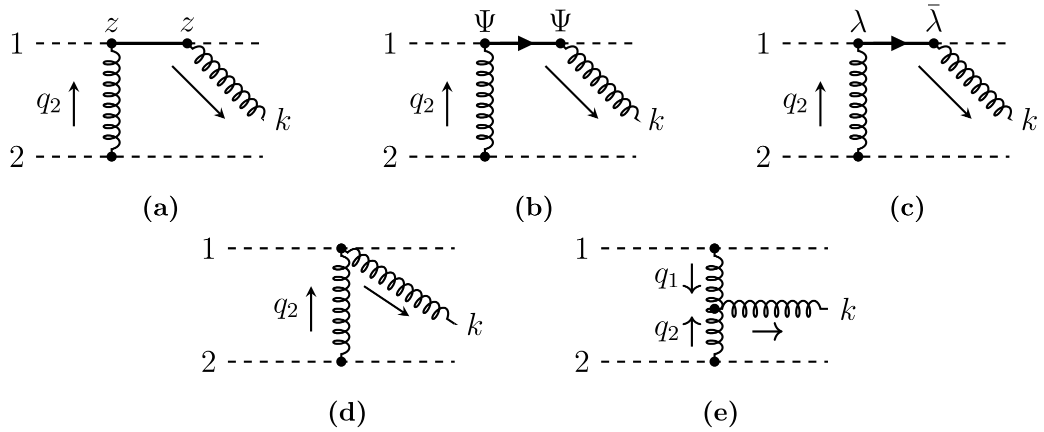


Figure 4.2: Topologies contributing to the leading Yang-Mills radiation for spinning worldlines. For (4.2c), we also need to include the crossed diagrams that can be obtained by simply reversing the arrows on the worldline. For (4.2a)-(4.2d), we also need to include diagrams with 1 and 2 exchanged.

kinematics duality

$$\mathcal{R}^{a\mu}(\sigma) = ig^3 \int \mathrm{d}\mu_{1,2}(k) \Bigg(\tilde{C}_1^a \left(-\frac{k\cdot q_2}{q_2^2\omega_1^2} n_0^\mu + \frac{1}{q_2^2\omega_1} n_1^\mu \right) + \tilde{C}_2^a \left(-\frac{k\cdot q_1}{q_1^2\omega_2^2} \bar{n}_0^\mu + \frac{1}{q_1^2\omega_2} n_1^\mu \right) \\ + \tilde{C}_3^a \left(\frac{1}{q_2^2\omega_1} n_0^\mu + \frac{1}{q_1^2\omega_2} \bar{n}_0^\mu + \frac{2}{q_1^2 q_2^2} n_1^\mu \right) \Bigg), \tag{4.49}$$

where in this case, we have $\omega_1 = p_1\cdot q_2$, and $\omega_2 = -p_2\cdot k$, which are consistent with (4.9) upon identifying $k \to -q_3$. For convenience, we also define the color factors as

$$\tilde{C}_1^a = C_1\cdot C_2\, C_1^a, \quad \tilde{C}_2^a = C_1\cdot C_2\, C_2^a, \quad \tilde{C}_3^a = f^{abc} C_1^b C_2^c, \tag{4.50}$$

and the numerators as

$$\begin{aligned} n_0^\mu =& \gamma\, p_1^\mu + i\gamma\,(k\cdot\mathcal{S}_1)^\mu - ip_1^\mu(q_2\cdot\mathcal{S}_1\cdot p_2 - q_2\cdot\mathcal{S}_2\cdot p_1) - ip_1\cdot q_2(p_2\cdot\mathcal{S}_1)^\mu \\ & - q_2\cdot\mathcal{S}_2\cdot p_1(k\cdot\mathcal{S}_1)^\mu - q_2\cdot\mathcal{S}_1\cdot\mathcal{S}_2\cdot q_2\, p_1^\mu + p_1\cdot q_2(q_2\cdot\mathcal{S}_2\cdot\mathcal{S}_1)^\mu \\ n_1^\mu =& \gamma q_2^\mu - p_1\cdot q_2\, p_2^\mu + p_2\cdot q_1\, p_1^\mu + q_1\cdot\mathcal{S}_1\cdot\mathcal{S}_2\cdot q_2\, q_2^\mu + \big[ip_1\cdot\mathcal{S}_2\cdot q_2\, q_1^\mu \\ & - iq_1\cdot\mathcal{S}_2\cdot q_2\, p_1^\mu + ip_1\cdot q_2(q_2\cdot\mathcal{S}_2)^\mu + q_1\cdot\mathcal{S}_2\cdot q_2(q_1\cdot\mathcal{S}_1)^\mu - (1\leftrightarrow 2) \big] \end{aligned} \tag{4.51}$$

with $\bar{n}_0 = n_0|_{1\leftrightarrow 2}$. We can see that the leading-order radiation can be obtained from the NLO eikonal (4.41) by cutting off worldline 3 and putting the external gluon on shell. Specifically, we send $q_3 \to -k, \mathcal{S}_3 \to 0$, strip off C_3^a and p_3^μ, multiply the result by $-ik^2/g$, and finally set the outgoing momentum k^μ on-shell. We stress that in (4.49), we have already arranged the numerators to satisfy color-kinematics duality, in the sense that the Yang-Mills radiation can be written in a similar form as (4.4a),

$$\mathcal{R}^{a\mu}(\sigma) = ig^3 \int \mathrm{d}\mu_{1,2}(k) \sum_{i,j} \tilde{C}_i^a \mathcal{K}_{ij} \mathcal{N}_j^\mu, \tag{4.52}$$

where the array of numerators is

$$\mathcal{N}^\mu = (n_0^\mu \quad \bar{n}_0^\mu \quad n_1^\mu) \tag{4.53}$$

and the double copy kernel for the LO radiation reads

$$\mathcal{K}_{ij} = \begin{pmatrix} -\frac{k\cdot q_2}{q_2^2\omega_1^2} & 0 & \frac{1}{q_2^2\omega_1} \\ 0 & -\frac{k\cdot q_1}{q_1^2\omega_2^2} & \frac{1}{q_1^2\omega_2} \\ \frac{1}{q_2^2\omega_1} & \frac{1}{q_1^2\omega_2} & \frac{2}{q_1^2 q_2^2} \end{pmatrix}, \tag{4.54}$$

which is obtained by considering the same diagrams as in Fig 4.2 but in a theory of scalar worldlines interacting through the bi-adjoint scalar field [56, 55]. Noticeably, gauge invariance implies the Ward idnetity $\mathcal{K}_{ij}(k_\mu \mathcal{N}_j^\mu) = 0$, such that the double copy kernel has zero determinant, i.e., it is not an invertible matrix in the usual algebraic sense.[3]

4.3 Susy in the sky with dilatons

Let us now move to the gravitational case. We anticipate that our classical double copy prescription on integrands generated by the $\mathcal{N} = 1$ susy particle will lead to integrands produced by using the so called $\mathcal{N} = 2$ susy particle coupled to the dilaton-gravity sector of the SUGRA. Thus, here we work out the coupling of the $\mathcal{N} = 2$ particle to DG. The free model was first proposed by [155], then, it was consistently coupled to gravity by Bastianelli, Benincasa, and Giombi in [69, 160] and used by Mogull, Jakobsen, Plefka and Steinhoff in [16] for application to the scattering of compact objects. The construction we will perform later on will allow us to reproduce such results while extending them with the inclusion of the dilaton.

We start by describing the free theory, building on the results from the $\mathcal{N} = 1$ model. Let us consider two independent free SUSY charges from two copies of the $\mathcal{N} = 1$ model. We label the SUSY charges as

$$Q_L = \psi_L^\mu P_\mu - m\theta_L, \quad Q_R = \psi_R^\mu P_\mu - m\theta_R \tag{4.55}$$

with their related real fermionic variables carrying the same label. What we see is that, upon defining complex Grassmann variables

$$\psi^\mu = \frac{1}{\sqrt{2}}(\psi_L^\mu - i\psi_R^\mu), \qquad \theta^\mu = \frac{1}{\sqrt{2}}(\theta_L^\mu - i\theta_R^\mu) \tag{4.56}$$

we are able to identify a new SUSY charge $Q = \frac{1}{\sqrt{2}}(Q_L - iQ_R) = \psi^\mu P_\mu - m\theta$ alongside with its complex conjugate. We can gauge the charges so to write the free $\mathcal{N} = 2$ phase space action as

$$S_{\rm ph} = -\int_0^1 \mathrm{d}\tau \left(\dot{x}^\mu P_\mu + i\bar{\psi}_\mu\dot{\psi}^\mu - i\bar{\theta}\dot{\theta} - \frac{e}{2}P^2 - i\bar{\chi}Q - i\chi\bar{Q} + a(J-s)\right), \tag{4.57}$$

[3] Instead the kernel can be inverted in the sense of pseudo-inverse matrices [159], so to allow for a KLT-like double copy.

where $\chi, \bar{\chi}$ are Grassmann-valued Lagrange multipliers gauging the supersymmetry. We also gauge a $U(1)$ symmetry of the complex Grassmann variable through the gauge field $a(\tau)$ and the current $J = \bar{\psi}_\mu \psi^\mu - \bar{\theta}\theta$, with s being the Chern-Simons integer parameter. This way we have a R-symmetry on the worldline, under which the Grassmann variables and the $U(1)$ gauge field transform as follows

$$\delta\psi^\mu = -i\alpha\psi^\mu, \quad \delta\psi^\nu = i\alpha\bar{\psi}^\nu, \quad \delta a = \dot{\alpha} \tag{4.58}$$

with α being a gauge parameter. The Poisson brackets between worldline variables are defined as

$$\{x^\mu, P_\nu\} = \delta^\mu{}_\nu, \quad \{\psi^\mu, \bar{\psi}_\nu\} = -i\delta^\mu{}_\nu, \quad \{\theta, \bar{\theta}\} = i. \tag{4.59}$$

Let us for a moment focus on the quantization of such a free particle. Firstly one has that the algebra is of first class, namely

$$\{Q, \bar{Q}\} = -2iH, \quad \{Q, J\} = iQ, \quad \{\bar{Q}, J\} = -i\bar{Q} \tag{4.60}$$

with all of the remaining brackets vanishing. This allows to use the equations of motion for the gauge fields as constraints at the quantum level. From the above Poisson brackets, we can implement $\psi \sim \hat{\psi}, \bar{\psi} \sim \partial/\partial\psi$, $\theta \sim \hat{\theta}, \bar{\theta} \sim -\partial/\partial\theta$ and use a coherent state basis to expand a generic wavefunction of the Hilbert space as

$$\begin{aligned}\Phi(x, \psi, \theta) = F(x) + F_\mu\psi^\mu + \frac{1}{2}F_{\mu\nu}(x)\psi^\mu\psi^\nu + \ldots \frac{1}{D!}F_{\mu_1\cdots\mu_D}(x)\psi^{\mu_1}\ldots\psi^{\mu_D} \\ + im\theta\left(W(x) + W_\mu(x)\psi^\mu + \ldots \frac{1}{D!}W_{\mu_1\cdots\mu_D}(x)\psi^{\mu_1}\ldots\psi^{\mu_D}\right) .\end{aligned} \tag{4.61}$$

Thus, we clearly see that the worldline model is propagating totally antisymmetric tensor fields that, once worked out the first class algebra at the quantum level will be identified as $p-$forms. However, in this way, the model is propagating all of the $p-$forms. To project on a specific sub-space we need to use the equation of motion of $a(\tau)$ as a quantum constraint on the wave function

$$\frac{\delta S_{\text{ph}}}{\delta a} = 0 \quad \rightarrow \quad \left(\hat{J} - s\right)\Phi(x, \psi, \theta) = \left(\psi^\mu\frac{\partial}{\partial\psi^\mu} + \theta\frac{\partial}{\partial\theta} - s\right)\Phi(x, \psi, \theta) = 0 \tag{4.62}$$

such that picking an arbitrary value of s, allows to project on a s-form and the related $(s-1)-$form gauge field, namely

$$\Phi_s(x, \phi, \theta) = \frac{1}{s!}F_{\mu_1\cdots\mu_s}(x)\psi^{\mu_1}\cdots\psi^{\mu_s} + \frac{im\theta}{(s-1)!}W_{\mu_1\cdots\mu_{s-1}}(x)\psi^{\mu_1}\cdots\psi^{\mu_{s-1}} . \tag{4.63}$$

Then, acting with the Hamiltonian $H = i/2\{Q, \bar{Q}\} = p^2/2$ on the selected wave function gives the mass-shell condition on each component of the wave function, while the SUSY charges $\hat{Q}, \hat{\bar{Q}}|\Phi_s\rangle = 0$ constraints give Bianchi identity on the field strength, transversality condition on the $(s$–$1)-$form, alongside with the Proca equation of motions for the $s-$form. Particularly, in $D = 4$ one can see that the model is propagating a massive vector boson and a scalar field (after dualization of the massive forms), exactly as observed in [34] from a field theory viewpoint, when studying the double copy of QCD. On the classical side, this is in agreement with that the classical double copy of QCD amplitudes leads to quadratic effects in the black hole spin [32, 157] which, on the worldline, can only be accounted by a model having at least two pairs of real Grassmann variables.

Let us now move to the coupling with the dilaton-gravity background. From a field theory viewpoint we expect such a worldline model to describe $p-$forms in a curved space coupled to the dilaton. Inspired by the results in [161], where the $\mathcal{N}=4$ worldline SUSY model has been used to propagate the supergravity spectrum after quantization, we deform the SUSY charges as

$$\begin{aligned} Q &= e^{-\kappa\phi} e^{\mu}_{a} \psi^{a} \left(P_{\mu} - i\Sigma_{\mu cd}\bar{\psi}^{c}\psi^{d}\right) - m\theta \\ \bar{Q} &= e^{-\kappa\phi} e^{\mu}_{a} \bar{\psi}^{a} \left(P_{\mu} - i\Sigma_{\mu cd}\bar{\psi}^{c}\psi^{d}\right) - m\bar{\theta}. \end{aligned} \tag{4.64}$$

Since now we are in a curve space, it is necessary to differentiate the local flat tangent space, denoted by the Latin indices $a, b, c, \dots,$ from the usual covariant curve space denoted by the Greek indices $\mu, \nu, \rho, \dots$. They are related via the vielbein e^{a}_{μ} defined as $\eta_{ab} e^{a}_{\mu} e^{b}_{\nu} = g_{\mu\nu}$. For example, we have $\psi^{\mu} = e^{\mu}_{a}\psi^{a}$. Due to the presence of the dilaton, we use $\Sigma_{\mu ab} = \omega_{\mu ab} - 2\kappa\partial_{[a}\phi\, e_{b]\mu}$ as a modification of the spin connection $\omega_{\mu}{}^{ab} = e^{a}_{\nu}(\partial_{\mu} e^{\nu b} + \Gamma^{\nu}{}_{\mu\rho} e^{\rho b})$. Let us now go through this deformation. The Poisson bracket $\{Q, Q\}$ identically vanishes when requiring our manifold to be torsionless and upon invoking Bianchi identity on the Riemann tensor. Indeed, the deformation is designed such that this Poisson bracket would deliver the torsion and the Riemann tensor in what is known as the Einstein frame. The same happens for the bracket $\{\bar{Q}, \bar{Q}\}$. More details on this can be seen in Appendix A. This means that our coupling preserves supersymmetry, thus allowing for a consistent quantization of the model.

Let us inspect the quantum theory implemented by the worldline particle using the SUSY charges as constraints on the wave function (4.61). Choosing the Chern-Simons parameter $s = 2$ so to propagate a massive vector boson and its related field strength

$$\Phi_2(x, \psi, \theta) = \frac{1}{2} F_{\mu\nu}(x)\psi^{\mu}\psi^{\nu} + im\theta W_{\mu}(x)\psi^{\mu} \tag{4.65}$$

and acting on such a state with the SUSY charges (4.64) one gets

$$\hat{Q}^{\dagger}|\Phi_2\rangle = 0 \to \begin{cases} \nabla^{\mu} F_{\mu\nu} = m^2 e^{\kappa\phi} W_{\nu} \\ \nabla_{\mu} W^{\mu} = 0 \end{cases} \qquad \hat{Q}|\Phi_2\rangle = 0 \to \begin{cases} \nabla_{[\mu} F_{\nu\rho]} = 0 \\ e^{\kappa\phi} F_{\mu\nu} = 2\nabla_{[\mu} W_{\nu]} \end{cases} \tag{4.66}$$

in agreement with the equations of motion from the $D = 4$ limit of the theory $\frac{1}{2} \otimes \frac{1}{2}$ in [33], where the dilaton is only sourced to the mass term for the vector boson. Working out the SUSY algebra from the SUSY charges in (4.64), whose details can be found in Appendix B, one is able to write down the following WQFT action for a massive point particle coupled to dilaton-gravity

$$S_{\text{pm}} = \int_{-\infty}^{\infty} \mathrm{d}\tau \left(-\frac{1}{2} e^{2\kappa\phi} g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} - i\bar{\psi}_a \dot{\psi}^a + \frac{1}{2}\Sigma_{\mu}{}^{ab}\dot{x}^{\mu} S_{ab} + \frac{1}{8} e^{-2\kappa\phi} \mathcal{R}_{abcd} S^{ab} S^{cd} \right) \tag{4.67}$$

where in this $\mathcal{N} = 2$ case the spin tensor is defined as $S^{ab} = -2i\bar{\psi}^{[a}\psi^{b]}$, and the deformed Riemann tensor in the flat space is

$$\mathcal{R}_{abcd} = e^{\mu}_{a} e^{\nu}_{b} \left(e^{\rho}_{c} e^{\sigma}_{d} R_{\mu\nu\rho\sigma} - 4\kappa \nabla_{[\mu}\nabla_{[c}\phi e_{d]\nu]} + 2\kappa^2 \left(2\partial_{[c}\phi\partial_{[\mu}\phi e_{\nu]d]} - \partial^2\phi e_{[c[\mu} e_{\nu]d]} \right) \right). \tag{4.68}$$

Let us conclude this section by giving some comment on the above results. First, one can notice that the action (4.67) is really a Weyl rescaling $g_{\mu\nu} \to e^{2\kappa\phi} g_{\mu\nu}$ of the action of a $\mathcal{N}=2$ particle coupled to pure gravity used in [16]. This is a direct consequence of that even the deformation of the SUSY charges (4.64) is a Weyl rescaling of the one used in [160, 16] to couple the model to gravity. This nice property implies that the dilaton can be completely disentangled from the worldline by reversing the Weyl transformation, yielding an action in the string frame. As we will see in the next subsection, the Feynman rules will be much simpler and less Feynman diagrams will be needed to compute the eikonal and the radiation.

4.3.1 WQFT expansion and Feynman rules

As explained before, in order to simplify as much as possible the perturbative expansion, we move back to the string frame by performing the Weyl rescaling

$$g_{\mu\nu} = e^{-2\kappa\phi} \tilde{g}_{\mu\nu}, \tag{4.69}$$

which allows to rewrite the worldline action (4.67) as

$$S_{\rm pm} = \int_{-\infty}^{\infty} \mathrm{d}\tau \biggl(-\frac{1}{2}\tilde{g}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - i\bar{\psi}_a \dot{\psi}^a + \frac{1}{2}\dot{x}^\mu \tilde{\omega}_\mu{}^{ab} S_{ab} + \frac{1}{8}\tilde{R}_{abcd} S^{ab} S^{cd} \biggr), \tag{4.70}$$

where objects in the string frame are tilded. The spin connection and the Riemann tensor are given in terms of the tilded quantities,

$$\tilde{\omega}_\mu{}^{ab} = \tilde{e}^a_\nu (\partial_\mu \tilde{e}^{\nu b} + \Gamma^\nu{}_{\mu\rho} \tilde{e}^{\rho b}) \tag{4.71}$$

$$\tilde{R}_{abcd} = 2\tilde{e}^\mu_c \tilde{e}^\nu_d \left(\partial_{[\mu} \tilde{\omega}_{\nu]ab} + \tilde{\omega}_{[\mu a}{}^f \tilde{\omega}_{\nu]fb} \right). \tag{4.72}$$

In the bulk, we expect to probe the dilaton-gravity sector of the $\mathcal{N}=0$ supergravity, known to arise from the double copy of pure Yang-Mills, and the action reads

$$S_{\rm dg} = -\frac{2}{\kappa^2} \int d^D x \sqrt{-\tilde{g}(x)} e^{-2\phi} \left(\tilde{R} + 4\tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right). \tag{4.73}$$

We find it convenient to expand the metric as

$$\tilde{g}_{\mu\nu} = e^{\kappa h_{\mu\nu}} = \eta_{\mu\nu} + \kappa h_{\mu\nu} + \frac{\kappa^2}{2} h_{\mu\rho} h^\rho{}_\nu + \dots, \tag{4.74}$$

where for perturbative quantities such as $h_{\mu\nu}$ the Greek indices are now raised and lowered by the flat metric $\eta_{\mu\nu}$. In this expansion, to decouple the kinetic terms of ϕ and $h_{\mu\nu}$, we perform a field redefinition $\phi \to \tilde{\phi} + \frac{1}{4} h_\mu{}^\mu$, and add to the bulk action the following gauge fixing term

$$S_{\rm gf} = \int \mathrm{d}^D x \left(\partial_\nu h^{\mu\nu} + 2\partial^\mu \tilde{\phi} \right) \left(\partial_\nu h_\mu{}^\nu + 2\partial_\mu \tilde{\phi} \right). \tag{4.75}$$

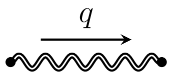
In the end, the gauge-fixed dilaton-gravity action simply reads

$$\begin{aligned} S_{\rm dg} = \int \mathrm{d}^D x \biggl(&\frac{1}{2}\partial_\rho h_{\mu\nu} \partial^\rho h^{\mu\nu} - 4\partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} \\ &- \kappa \Bigl(\frac{1}{2} h^{\mu\nu} \partial_\mu h^{\rho\sigma} \partial_\nu h_{\rho\sigma} - h^{\mu\nu} \partial_\nu h_{\rho\sigma} \partial^\sigma h_\mu{}^\rho \Bigr) \biggr) + \mathcal{O}(\kappa^2), \end{aligned} \tag{4.76}$$

where we have neglected interaction terms involving the dilaton $\tilde{\phi}$. Thus, we can write down the WQFT partition function for n worldlines as follows

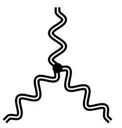
$$\mathcal{Z}_{\rm dg} = \int D[h_{\mu\nu}, \tilde{\phi}]\, e^{iS_{\rm dg}[h,\tilde{\phi}]} \int \prod_{k=1}^{n} \mathcal{D}X_k\, e^{iS_{\rm pm}[X_k;h]} \tag{4.77}$$

with the worldline variables $X_k = (x_k, \psi_k, \bar{\psi}_k)$ and the path integral measure defined as $\mathcal{D}X_k = Dx_k D\psi_k D\bar{\psi}_k$, where we implicitly included the Lee-Yang ghost term as introduced in (3.11). Let us move now to the Feynman rules needed for our calculations. In this way, the Feynman graviton propagator turns out to be



$$= \frac{i}{2q^2}\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}\right)\ . \tag{4.78}$$

From the second line in (4.76), we can extract the three-point graviton vertex, which turns out to be extremely simple and can be written as

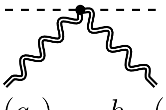


$$= \frac{i\kappa}{2} q_1^\alpha q_2^\beta \eta^{\rho\mu}\eta^{\sigma\nu} - i\kappa q_1^\beta q_2^\nu \eta^{\mu\sigma}\eta^{\rho\alpha} + \mathrm{S}_3(1,2,3),$$

where $\mathrm{S}_3(1,2,3)$ is the set of all of the permutations of the list $(1,2,3)$ labeling the Lorentz indices and momenta of the external gravitons. We have also implicitly symmetrize in $(\mu,\nu), (\rho,\sigma), (\alpha,\beta)$, separately. Moving on to the worldline action, we see that, in the string frame, the leading PM expansion of the action generates the same Feynman rules given in [16], as a consequence of (4.74). The sub-leading expansion reads as

$$\begin{aligned} S_{\rm pm} = \int d^D x \Big(&-\frac{\dot{x}^2}{2} + \kappa\Big(-\frac{1}{2}h_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \frac{1}{2}S^{\nu\rho}\dot{x}^\mu\partial_\rho h_{\mu\nu} - \frac{1}{4}S^{\mu\nu}S^{\rho\sigma}\partial_\sigma\partial_\nu h_{\mu\rho}\Big) \\ &+ \kappa^2\Big(-\frac{1}{4}h_\mu{}^\rho h_{\nu\rho}\dot{x}^\mu\dot{x}^\nu - \frac{1}{8}h^{\nu\rho}S_\nu{}^\sigma\dot{x}^\mu\partial_\mu h_{\rho\sigma} + \frac{1}{4}h^{\nu\rho}S_\nu{}^\sigma\dot{x}^\mu\partial_\rho h_{\mu\sigma} + \frac{1}{4}h_\mu{}^\nu S^{\rho\sigma}\dot{x}^\mu\partial_\sigma h_{\nu\rho} \\ &\quad - \frac{1}{8}S^{\mu\nu}S^{\rho\sigma}\partial_\rho h_\mu{}^\lambda\partial_\sigma h_{\nu\lambda} - \frac{1}{8}S^{\mu\nu}S^{\rho\sigma}\partial_\nu h_\mu{}^\lambda\partial_\sigma h_{\rho\lambda} - \frac{1}{4}h^{\mu\nu}S_\mu{}^\rho S^{\sigma\lambda}\partial_\lambda\partial_\nu h_{\rho\sigma} \\ &\quad + \frac{1}{4}S^{\mu\nu}S^{\rho\sigma}\partial_\sigma h_{\nu\lambda}\partial^\lambda h_{\mu\rho} - \frac{1}{16}S^{\mu\nu}S^{\rho\sigma}\partial_\lambda h_{\nu\sigma}\partial^\lambda h_{\mu\rho}\Big)\Big) + \mathcal{O}(\kappa^3). \end{aligned} \tag{4.79}$$

which allows to write down the vertex with the emission of two gravitons from the worldline



$$\begin{aligned} = -\frac{1}{16}i\kappa^2\Big(&4p^\mu p^\rho\eta^{\nu\sigma} - 4iq_2^\nu p^\rho S^{\mu\sigma} + 4ip^\mu(q_2\cdot S)^\sigma\eta^{\nu\rho} \\ &+ 2i(p\cdot q_2)S^{\mu\sigma}\eta^{\nu\rho} + 4q_2^\nu S^{\mu\rho}((q_1+q_2)\cdot S)^\sigma - (q_1\cdot q_2)S^{\mu\rho}S^{\nu\sigma} \\ &- 2S^{\mu\rho}(q_1\cdot S\cdot q_2)\eta^{\nu\sigma} - 2(q_1\cdot S)^\mu(q_2\cdot S)^\rho\eta^{\nu\sigma}\Big) + (1\leftrightarrow 2). \end{aligned} \tag{4.80}$$

where, again, we need to symmetrize the indices (μ, ν) and (ρ, σ), separately. Thus, what we see is that our field redefinition combined with the gauge fixing term (4.75) allows to disentangle the dilaton from the graviton kinetic term, to drastically simplify the three graviton vertex, and to decouple the dilaton from the worldline action, such that, for our purposes, we should only focus on the interaction concerning the graviton.

4.3.2 Double copy of the eikonal phase

Let us start by considering the simplest case of the leading order eikonal for the scattering of two spinning worldlines in a gauge background. Firstly, let us clarify how to identify the spin tensor after the double copy. Given that we expect to probe quadrupole, the underlying WQFT should be constructed with complex Grassmann variables, related to the real ones from the $\mathcal{N}=1$ model, by (4.56). This is already selecting the $\mathcal{N}=2$ model as a free theory. In particular, using (4.56) and (4.32) this also fixes the background expansion of the complex Grassmann variables of the $\mathcal{N}=2$ model as

$$\begin{aligned}\psi^\mu(\tau) &= \frac{1}{\sqrt{2}}\left(\psi_L^\mu(\tau) - i\psi_R^\mu(\tau)\right) \\ &= \frac{1}{\sqrt{2}}\left(\zeta_L^\mu - i\zeta_R^\mu\right) + \frac{1}{\sqrt{2}}\left(\Psi_L^\mu(\tau) - i\Psi_R^\mu(\tau)\right) = \zeta^\mu + \Psi^\mu(\tau),\end{aligned} \tag{4.81}$$

further implying that the Grassmann variables from the two copies should be treated differently, so that we have two real Grassmann variables at the gravity side. For convenience, we label them as L and R. We thus propose the following relation between the spin tensor in the gravity and the YM side,

$$\mathcal{S}^{ab} = \mathcal{S}_L^{ab} + \mathcal{S}_R^{ab}, \tag{4.82}$$

with $\mathcal{S}_{L,R}^{ab} = -i\zeta_{L,R}^a \zeta_{L,R}^b$, so that when written in terms of the complex Grassmann variables in (4.81), the spin tensor is $\mathcal{S}^{ab} = -2i\zeta^{[a}\bar{\zeta}^{b]}$. This is also consistent with the fact that the Lorentz generator for spin-s particles is basically the sum of s vector representations in the spinor-helicity formalism [157].

Let us now move to studying the double copy of the leading eikonal phase. At LO, it has been calculated in (4.38). We take two copies of the numerators, and label the spins by L and R, interpreted as labeling the left and a right copy of the $\mathcal{N}=1$ theory. For example, if both particles are labeled with L, the numerator reads

$$\mathcal{N}_{L_1 L_2} = \gamma - i\big(q_2 \cdot \mathcal{S}_{1,L} \cdot p_2 - q_2 \cdot \mathcal{S}_{2,L} \cdot p_1\big) - q_2 \cdot \mathcal{S}_{1,L} \cdot \mathcal{S}_{2,L} \cdot q_2 \,. \tag{4.83}$$

In such a case, following (4.4), the naive double copy procedure yields

$$\chi_1^{\mathrm{DC}} \sim \frac{\mathcal{N}_{L_1 L_2} \mathcal{N}_{R_1 R_2}}{q_2^2}. \tag{4.84}$$

The question now is how to select a worldline theory generating such numerator and whether it enjoys supersymmetry or R-symmetry, the former being crucial to fix all of the interactions on the wordline, while the latter, allowing to propagate a specific particle among the worldline spectrum. Let us now discuss the consequences of R-symmetry at the level of classical integrands from the WQFT. After gauge fixing the $U(1)$ gauge field $a(\tau)$ on the worldline, the theory enjoys the global version of the symmetry (4.58) on the background parameters $\zeta, \bar{\zeta}$, which can be used to check global $U(1)$ invariance of the double copy integrand. This amounts to check that the integrand can be expressed entirely in terms of global R-invariant objects i.e. in terms of the spin tensor $\mathcal{S}^{ab}$.

Then, this is signaling that (4.84), cannot be generated by a R-invariant theory! To keep $U(1)$ invariance, we perform the replacement above, then symmetrizing over the left and right indices, such that, the double copied eikonal turns out to be

$$\chi_1^{\mathrm{DC}} = -\frac{\kappa^2}{4}\int \mathrm{d}\mu_{1,2}(0)\frac{\mathcal{N}_L \otimes \mathcal{N}_R}{q_2^2} \tag{4.85}$$

where $\otimes$ defines symmetrization over L_1, R_1 and L_2, R_2 separately. This yields to the double copied numerator

$$\begin{aligned}\mathcal{N}_L \otimes \mathcal{N}_R =&\gamma^2 - i\gamma\, q_2\cdot\mathcal{S}_1\cdot p_2 + i\gamma\, q_2\cdot\mathcal{S}_2\cdot p_1\\ &-\frac{1}{2}\left(q_2\cdot\mathcal{S}_1\cdot p_2\right)^2 - \frac{1}{2}\left(q_2\cdot\mathcal{S}_2\cdot p_1\right)^2 - \frac{1}{2}i\, q_2\cdot\mathcal{S}_1\cdot\mathcal{S}_2\cdot q_2\, q_2\cdot\mathcal{S}_2\cdot p_1\\ &+\frac{1}{2}i\, q_2\cdot\mathcal{S}_1\cdot\mathcal{S}_2\cdot q_2\, q_2\cdot\mathcal{S}_1\cdot p_2 + \frac{1}{2}\, q_2\cdot\mathcal{S}_1\cdot p_2\, q_2\cdot\mathcal{S}_2\cdot p_1\\ &-\frac{1}{2}\gamma\, q_2\cdot\mathcal{S}_1\cdot\mathcal{S}_2\cdot q_2 + \frac{1}{4}\left(q_2\cdot\mathcal{S}_1\cdot\mathcal{S}_2\cdot q_2\right)^2\end{aligned} \tag{4.86}$$

which is manifestly background R-invariant, given that it can be recast in terms of the $\mathcal{N}=2$ spin tensor. We stress that a crucial role in obtaining the above numerator from the double copy of the $\mathcal{N}=1$ model, is played by the real Grassmann variables $\psi^\mu_{L,R}$. Indeed, some of the terms one gets vanish once using that $\psi^2_{L,R}=0$ and anti-symmetry of the string $\psi^\mu_{L,R}\psi^\nu_{L,R}$. For instance, a factor quadratic in $\mathcal{S}_1$ will be simplified as

$$\begin{aligned}&\left(q_2\cdot\mathcal{S}_{1,L}\right)^\mu\left(q_2\cdot\mathcal{S}_{1,R}\right)^\nu + \left(q_2\cdot\mathcal{S}_{1,R}\right)^\mu\left(q_2\cdot\mathcal{S}_{1,L}\right)^\nu\\ =&\left(q_2\cdot\mathcal{S}_{1,L}\right)^\mu\left(q_2\cdot\mathcal{S}_{1,L}\right)^\nu + \left(q_2\cdot\mathcal{S}_{1,L}\right)^\mu\left(q_2\cdot\mathcal{S}_{1,R}\right)^\nu\\ +&\left(q_2\cdot\mathcal{S}_{1,R}\right)^\mu\left(q_2\cdot\mathcal{S}_{1,L}\right)^\nu + \left(q_2\cdot\mathcal{S}_{1,R}\right)^\mu\left(q_2\cdot\mathcal{S}_{1,R}\right)^\nu\\ =&\left(q_2\cdot\mathcal{S}_1\right)^\mu\left(q_2\cdot\mathcal{S}_1\right)^\nu,\end{aligned} \tag{4.87}$$

where we have used the fact that $\left(q_2\cdot\mathcal{S}_{1,L}\right)^\mu\left(q_2\cdot\mathcal{S}_{1,L}\right)^\nu \sim q_{2\alpha}q_{2\beta}\zeta_L^\alpha\zeta_L^\beta\zeta_L^\mu\zeta_L^\nu = 0$ due to the anti-symmetry of the Grassmann variables. The same holds for the right copy of such term. By inspection, it can be seen that the double copied numerator (4.86) is in agreement with the one extracted from the leading eikonal evaluated using the Feynman rules in subsection 4.3.1, thus automatically checking SUSY invariance of the double copied eikonal phase. We note that the Grassmann nature of the spin tensor is critical in matching the double copy eikonal to the direct calculation from dilaton gravity.

Let us now proceed to the NLO. At this order, again, the double copy multiplication rule is the same as the spinless case, with the numerators given in (4.46). Written with

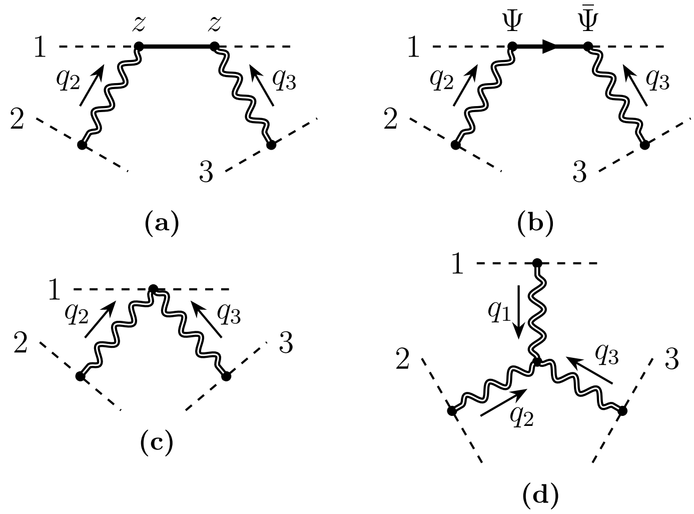


Figure 4.3: The diagrams needed for the DG eikonal at NLO with three worldlines. For diagrams involving the propagators of Ψ (4.3b), we also need to include the crossed diagrams that can be obtained by simply reversing the arrows on the worldline. We only display diagrams with worldline propagator and contact interaction of particle 1.

the $\otimes$ symmetrization prescription, the double copy eikonal (4.4b) can be expressed as

$$
\begin{aligned}
\chi_2^{\mathrm{DC}} &= -\frac{\kappa^4}{16}\int \mathrm{d}\mu_{1,2,3}(0)\sum_{i,j}\mathcal{K}_{ij}\mathcal{N}_{i,L}\otimes\mathcal{N}_{j,L} \\
&= -\frac{\kappa^4}{16}\int \mathrm{d}\mu_{1,2,3}(0)\Bigg[\Big(\frac{2}{q_2^2q_3^2\omega_1}n_L^{(123)}\otimes n_R^{(0)}+\frac{q_2\cdot q_3}{q_2^2q_3^2\omega_1^2}n_L^{(123)}\otimes n_R^{(123)}+\text{cyclic}\Big) \\
&\qquad\qquad + \frac{2}{q_1^2q_2^2q_3^2}n_L^{(0)}\otimes n_R^{(0)}\Bigg].
\end{aligned}
\tag{4.88}
$$

One can check that, with the symmetrization of L_i and R_i labels, this agrees with the direct computation of the Feynman diagrams (see Fig. 4.3) from DG using the Feynman rules in subsection 4.3.1. The final double copy numerators of the NLO eikonal (as well as the radiation) are too lengthy to fit into this article, instead they are provided in an attached ancillary file.

As stated in subsection 4.1, the double copy kernel for the NLO eikonal is invertible, so alternatively we can easily perform the double copy in a KLT-like fashion. Specifically, according to the relation between the "partial eikonals" and the kinematic numerators (4.47), the eikonal phase at the gravity side of the double copy theory can be re-expressed as

$$
\chi_2^{\mathrm{DC}} = -\frac{\kappa^4}{16}\int \mathrm{d}\mu_{1,2,3}(0)\sum_{i,j}(\mathcal{K}^{-1})_{ij}\mathcal{A}_L^{(i)}\otimes\mathcal{A}_R^{(j)}, \tag{4.89}
$$

where again, the $\otimes$ denotes the symmetrization of the L and R labels for each individual worldline. We have checked that this yields exactly the same result as the BCJ double copy.

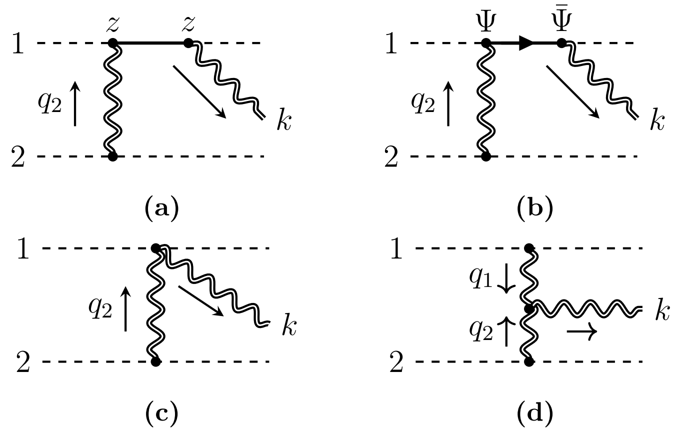


Figure 4.4: Topologies contributing to the leading-order gravitational radiation for spinning worldlines. For diagrams (4.4b), we also need to include the crossed diagrams that can be obtained by simply reversing the arrows on the worldline. For (4.4a)-(4.4c), we also need to include diagrams with 1 and 2 exchanged.

4.3.3 Double copy of the radiation

Let us now move to the double copy of the leading radiation. We have seen that the radiation can be recasted in a manifest color-kinematic fashion, from the analysis in Sec.(4.2.3). Then, the double copy procedure would require us to replace the color factors $\tilde{C}_i^a \to \mathcal{N}_{iR}^\mu$, the latter being the numerator computed from the right copy of the $\mathcal{N}=1$ model. However, as pointed out previously, this procedure does not preserve R-symmetry. Then, as for the eikonal phase, we perform the above replacement, then using the $\otimes$ symmetrization prescription on the labels of the two independent copies of the $\mathcal{N}=1$ particle. This yields the double copied radiation

$$\begin{aligned}
\mathcal{R}^{\mu\nu}(\sigma) &= i\frac{\kappa^3}{8}\int \mathrm{d}\mu_{1,2}(k)\sum_{i,j}\mathcal{K}_{ij}\,\mathcal{N}_{iL}^\mu\otimes\mathcal{N}_{jR}^\mu \\
&= i\frac{\kappa^3}{8}\int \mathrm{d}\mu_{1,2}(k)\Bigg(-\frac{k\cdot q_2}{q_2^2\omega_1^2}n_{0L}^\mu\otimes n_{0R}^\nu+\frac{2}{q_2^2\omega_1}n_{1L}^{(\mu}\otimes n_{0R}^{\nu)}-\frac{k\cdot q_1}{q_1^2\omega_2^2}\bar{n}_{0L}^\mu\otimes\bar{n}_{0R}^\nu \\
&\qquad\qquad +\frac{2}{q_1^2\omega_2}n_{1L}^{(\mu}\otimes\bar{n}_{0R}^{\nu)}+\frac{2}{q_1^2q_2^2}n_{1L}^\mu\otimes n_{1R}^\nu\Bigg)
\end{aligned} \tag{4.90}$$

with the shorthand notation $L=L_1,L_2$, $R=R_1,R_2$. As for the eikonal, so to rewrite the above result in terms of the spin tensor (5.3.1) one needs to use anti-symmetry of the real Grassmann variables belonging to the left and right copy. One can check that the above radiation correctly reproduces the result obtained by evaluating the diagrams in Fig.4.4 on the gravitational side.

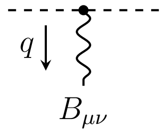
4.3.4 B-field in the double copy spectrum

In the above derivation of the double copy of radiation and eikonal, we mostly relied on the R-symmetry as a guideline. We found that, in this case, the $\otimes$ prescription preserves even

SUSY in the double copied integrands. Here we wonder if this is not the case when the R-symmetry is not preserved during the double copy. To investigate this, let us study the consequences of the double copy replacement (4.84) which does not preserve R-symmetry in the integrands. In such a case, given that we cannot use the R-symmetry as a guideline, we study the double copy features of the three point worldline vertex from the first line in (4.34). The replacement (4.84) can be implemented at this level by sending the color factor to a right copy of the vertex, carrying a polarization $\bar{\varepsilon}(q)$. Then, we can introduce the NS-NS spectrum by using the expansion

$$\varepsilon_\mu(q)\bar{\varepsilon}_\nu(q) = \varepsilon^h_{\mu\nu}(q) + \varepsilon^B_{\mu\nu}(q) + \varepsilon\cdot\bar{\varepsilon}\left(-\eta_{\mu\nu} + \frac{2q_{(\mu}r_{\nu)}}{r\cdot q}\right) \tag{4.91}$$

with r^μ being a arbitrary null vector. Separately contracting the double copied vertex with the above expansion and using (5.3.1) with the SSC, we reproduce the graviton and dilaton vertices which can be checked against (4.67), with, in addition, the contribution from the B field, whose three point vertex can be reconstructed as



$$= \frac{i\kappa}{2} e^{iq\cdot b}\hat{\delta}(q\cdot p)\left(ip^{[\mu} q\cdot(\mathcal{S}_L - \mathcal{S}_R)^{\nu]} + (q\cdot\mathcal{S}_L)^{[\mu}(q\cdot\mathcal{S}_R)^{\nu]}\right) \tag{4.92}$$

which manifestly breaks R-invariance, given the mixing of the left and right copies of the spin tensor. By studying the structure of such vertex we can reproduce this interaction by deforming the SUSY charges (4.64) as $Q\to Q - \frac{1}{4}\kappa e^{\kappa\phi} e^\mu_a \psi^a\, H_{\mu cd}T^{cd}$, with the twisted Lorentz generator $T^{ab} = -i\psi^a_L\psi^b_L + i\psi^a_R\psi^b_R = S^{ab}_L - S^{ab}_R$ as coming from each copies of the $\mathcal{N}=1$ model. As it can be checked by working out the SUSY algebra, this deformation breaks supersymmetry, then showing in our case that, if R-symmetry is not preserved neither SUSY is on the double copied theory. In addition, given the constraint algebra is not of first class $\{Q,Q\}\neq 0$ we cannot quantize the model as previously, and, particularly, this does not allow to write down a path integral to access classical calculations. However, using (4.67) as a base theory, one can write down an effective WQFT action, capturing such effects, namely

$$\begin{aligned} S = \int_{-\infty}^{\infty} \mathrm{d}\tau \Bigg(&-\frac{1}{2}e^{2\kappa\phi} g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - i\bar{\psi}_a\dot{\psi}^a + \frac{1}{2}\Sigma_\mu{}^{ab}\dot{x}^\mu S_{ab} + \frac{1}{8}e^{-2\kappa\phi}\mathcal{R}_{abcd}S^{ab}S^{cd} \\ &-\frac{1}{4}\kappa\dot{x}^\mu H_{\mu ab}T^{ab} - \frac{1}{8}\kappa\nabla_\mu H_{\nu ab}S^{\mu\nu}T^{ab}\Bigg) . \end{aligned} \tag{4.93}$$

This theory agrees up to terms linear in spin with the action proposed by Goldberger and Li in [162] to study the double copy of spinning worldlines by solving the classical equation of motions. We check that (4.93) reproduces the leading binary radiation from [162], confirming their string interpretation of the above action by extending these results up to quadratic order in spin.

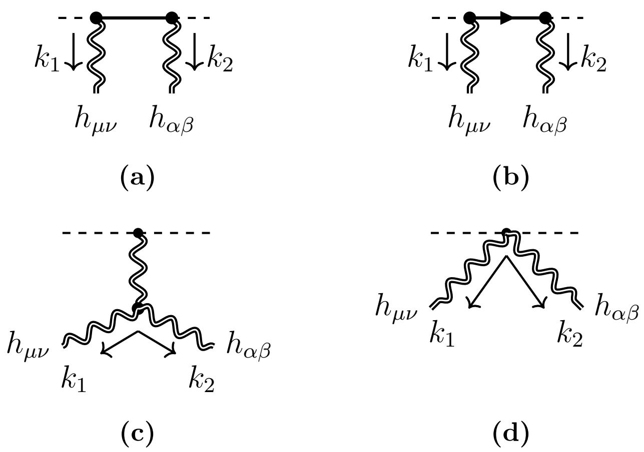


Figure 4.5: Worldline diagrams contributing to the on-shell Compton amplitude up to quadratic in spin. 4.5a,4.5b, with crossed topologies are associated with fluctuations of kinematic and spin variables, respectively.

4.4 Double copy of the spinning YM Compton

Let us now use our double copy prescription to evaluate the classical limit of the gravitational Compton amplitude up to quadratic in spin. As already explained in 3.2.4, such amplitude can be used as an input to write down lagrangians for massive higher spin particles interacting with gravity, which, in the classical limit, would allow to reproduce Kerr black hole observables.

For such a task we use a current-like formulation of the $\mathcal{N}=1$ model coupled to YM. It is easy then to recast the amplitude[4] in a color-kinematic fashion as

$$\mathcal{A}^{ab}_{\text{YM}} = \sum_{(i,j)=(1,2)} C^{ab}_i \mathcal{K}_{i,j} N_j, \quad \mathcal{K} = \begin{pmatrix} -\frac{is}{2\omega^2} & \frac{i}{\omega} \\ \frac{i}{\omega} & -\frac{2i}{s} \end{pmatrix} \tag{4.94}$$

with the color array $C^{ab} = \left(C^a C^b, f^{ab\ell} C^\ell\right)$ and the array of BCJ numerators $N = (n_1, n_2)$ whose components are given by.

$$\begin{aligned} n_1 &= ip\cdot\varepsilon_2 k_1\cdot\mathcal{S}\cdot\varepsilon_1 + p\cdot\varepsilon_1 p\cdot\varepsilon_2 + i\omega\varepsilon_1\cdot\mathcal{S}\cdot\varepsilon_2 \\ n_2 &= -\frac{isp\cdot\varepsilon_1 k_2\cdot\mathcal{S}\cdot\varepsilon_2}{2\omega} - i\varepsilon_2\cdot k_1 k_1\cdot\mathcal{S}\cdot\varepsilon_1 + i\varepsilon_1\cdot k_2 k_1\cdot\mathcal{S}\cdot\varepsilon_2 - ie_1\cdot e_2 k_1\cdot\mathcal{S}\cdot k_2 \\ &\quad - i\varepsilon_2\cdot k_1 k_2\cdot\mathcal{S}\cdot\varepsilon_1 + i\varepsilon_1\cdot k_2 k_2\cdot\mathcal{S}\cdot\varepsilon_2 - k_1\cdot\varepsilon_2 p\cdot\varepsilon_1 + k_2\cdot\varepsilon_1 p\cdot\varepsilon_2 + \omega\,\varepsilon_1\cdot\varepsilon_2 \end{aligned} \tag{4.95}$$

where we defined $\omega = p\cdot k_1, s = 2k_1\cdot k_2$. As it also happens for the two body radiation, the double copy kernel in (4.94) is degenerate, as a consequence of gauge invariance. Here we could apply our double copy prescription obtaining the correct amplitude, as it is easy to check. However, in order to make manifest KLT-like relations at the classical level let us massage (4.94).

[4]The topologies for the YM Compton are the same as in Fig.4.5, with in addition a diagram like 4.5b, propagating color fluctuations.

The kernel is not invertible, but it is hermitian, thus we can diagonalize that. Exploiting diagonalizability of the kernel we can recast the YM Compton as

$$\mathcal{A}^{ab}_{\text{YM}} = \sum_{(i,j)=(1,2)} C^{ab}_i \left(P^{-1}D\right)_{ij} (PN)_j = -\frac{2i}{s} f^{\ell ab}(-i\omega\mathcal{A}_{\text{QED}}) + \frac{i}{\omega} C^a C^b (-i\omega\mathcal{A}_{\text{QED}}) \quad (4.96)$$

where P and D are respectively the eigenvector and diagonal matrix related to the double copy kernel, namely

$$P = \begin{pmatrix} \frac{2\omega}{s} & 1 \\ -\frac{s}{2\omega} & 1 \end{pmatrix} \quad D = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{i\left(\frac{s^2}{\omega^2}+4\right)}{2s} \end{pmatrix} \quad (4.97)$$

while we identified the new double copy kernel reads as

$$\mathcal{K}' = P^{-1}D = \begin{pmatrix} 0 & \frac{i}{\omega} \\ 0 & -\frac{2i}{s} \end{pmatrix}. \quad (4.98)$$

Notice how in such a basis, (4.96) only depends on the classical QED Compton amplitude

$$\begin{aligned} \mathcal{A}_{\text{QED}} = & \frac{i\varepsilon_1 \cdot k_2 p \cdot e_2}{\omega} - \frac{i\varepsilon_2 \cdot k_1 p \cdot e_1}{\omega} - \frac{isp \cdot \varepsilon_1 p \cdot e_2}{2\omega^2} + i\varepsilon_1 \cdot \varepsilon_2 \\ & \frac{sp \cdot \varepsilon_2 k_1 \cdot \mathcal{S} \cdot e_1}{2\omega^2} + \frac{\varepsilon_2 \cdot k_1 k_1 \cdot \mathcal{S} \cdot \varepsilon_1}{\omega} - \frac{\varepsilon_1 \cdot k_2 k_1 \cdot \mathcal{S} \cdot \varepsilon_2}{\omega} + \frac{s\varepsilon_1 \cdot \mathcal{S} \cdot \varepsilon_2}{2\omega} \\ & \frac{sp \cdot \varepsilon_1 k_2 \cdot \mathcal{S} \cdot \varepsilon_2}{2\omega^2} + \frac{\varepsilon_1 \cdot \varepsilon_2 k_1 \cdot \mathcal{S} \cdot k_2}{\omega} + \frac{e_2 \cdot k_1 k_2 \cdot \mathcal{S} \cdot \varepsilon_1}{\omega} - \frac{\varepsilon_1 \cdot k_2 k_2 \cdot \mathcal{S} \cdot \varepsilon_2}{\omega}. \end{aligned} \quad (4.99)$$

Thus, we can further simplify (4.96) by recasting that as

$$\mathcal{A}^{ab}_{\text{YM}} = \sum_{(i,j)=(1,2)} C^{ab}_i \mathcal{K}'_{i,j} N'_j \quad (4.100)$$

where, we used the freedom in writing down (4.96) to choose the BCJ array as $N' = \left(0, -i\omega\mathcal{A}_{\text{QED}}\right)$, so to make manifest gauge invariance. Now, using our double copy prescription on the above result yields to

$$\mathcal{M}_{\text{DC}} = \frac{2i\omega^2}{s} \mathcal{A}^L_{\text{QED}} \otimes \mathcal{A}^R_{\text{QED}} \quad (4.101)$$

which, by a direct calculation of the diagrams in Fig.4.5, with the Feynman rules from the $\mathcal{N} = 2$ susy particle coupled to gravity[5], can be checked to be the gravitational Compton amplitude, up to quadrupole. We also notice that (4.101) is a generalization of the classical KLT-like relation (3.64), which is not known in the literature, as far as we are aware.

Using the polarization vectors (3.65) it is easy to evaluate the gravitational Compton, for independent helicity configurations. The spinless contributions are listed in (3.66).

[5]The Feynman rules can be found in [16], or alternatively derived from (4.67), once switching off the dilaton scalar

Recalling that the spin tensor is related to the Pauli-Lubanski pseudovector by $s^a = -\frac{1}{2}\varepsilon^{abcd}p_b\mathcal{S}_{cd}$, the spin-orbit terms can be recast as

$$|\mathcal{M}_{++}|_{O(\mathcal{S})} = |\mathcal{M}_{++}|_0 \left| s\cdot(k_1+k_2)\tan^2\frac{\theta}{2} + \mathrm{i}\frac{k_1\cdot\mathcal{S}\cdot k_2}{mE\cos^2\frac{\theta}{2}} \right|, \quad |\mathcal{M}_{+-}|_{O(\mathcal{S})} = |\mathcal{M}_{+-}|_0 \left|s\cdot(k_1-k_2)\right|, \tag{4.102}$$

while the quadratic in spin contributions can be recast in the suggestive way

$$|\mathcal{M}_{++}|_{O(\mathcal{S}^2)} = \frac{1}{2}\frac{|\mathcal{M}_{++}|^2_{O(\mathcal{S})}}{|\mathcal{M}_{++}|_0}, \quad |\mathcal{M}_{+-}|_{O(\mathcal{S}^2)} = \frac{1}{2}\frac{|\mathcal{M}_{+-}|^2_{O(\mathcal{S})}}{|\mathcal{M}_{+-}|_0}. \tag{4.103}$$

The remaining helicity configurations can be obtained by replacing $\mathcal{S}^{ab} \to -\mathcal{S}^{ab}$, $s^a \to -s^a$. The above result is in agreement with [115] up to quadratic in spin upon matching with our conventions. In addition, the classical amplitude (4.101) is in agreement with [163].

Chapter 5

Light bending from eikonal in the WQFT

The bending of light due to black holes (compact objects) is a well known test of General relativity. Despite that, the studying about light deflection and null geodesics, is nowaday ongoing, using both amplitudes and GR techniques (see [164, 165, 166] for instance). Recently it was found, by using scattering amplitudes with external massless particles, [9, 167, 168] that there are small quantum corrections differentiating the scattering of a massless particle from that of light off a heavy mass object [169, 128, 170], thus testing the equivalence principle [171]. Since these differences are absent in the classical limit, one can recover deflection angles by taking the massless limit of the scattering of massive particles, see e.g. [172]. In particular, as studied a long time ago by Amati, Ciafaloni and Veneziano [153], one can employ the eikonal phase to extract classical deflection angles in gravity through differentiation. More recently, this approach has received a renewed interest [173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 120, 185, 117, 136, 186, 187]. Another route to the problem was taken in Ref.[188], where the Kosower-Maybee-O'Connell (KMOC) formalism [127, 32, 135] for classical observables was generalized to describe the classical limit of massless particles. In one of the applications of Ref.[188], the geometric-optics regime was used to extract the deflection angle from scattering amplitudes, since it is in this limit in which the amplitude can be related to a beam of light through the precise definition of the classical observable.

In this chapter we construct the appropriate WQFT to deal with photons with a first goal of realizing the equivalence between the scattering of a photon and a massive scalar with that of a massless and a massive scalar in the geometric-optics regime. The second goal is a simplified approach to the calculation of the scattering angle. Unlike the case of a matter particle propagating in a gravitational background, the dressed photon propagator depends on a matrix-valued action and therefore the worldline path integral must include a path ordering. As already seen in the previous chapters, the path ordering can be avoided by rewriting the path integral in terms of auxiliary variables at the cost of introducing additional integrals.

Once the dressed propagator is obtained and put on-shell, we can follow the WQFT setup to derive its Feynman rules and compute the eikonal phase. The Feynman rules that we find depend in general on the auxiliary variables. Building on the insights of Ref.[188], we consider the geometric-optics regime, which leads to a great simplification of the calculations. As we shall see, this regime implies the vanishing of terms proportional to the spin-tensor in the worldline action and serves as a check of our setup. This shows how, in the optical regime, the worldline propagating the photon reduces to the one for a massless scalar field, exactly as predicted by the equivalence principle. Then, the calculation of the eikonal phase and derived quantities, such as the deflection angle and the impulse, dramatically simplifies.

5.1 Derivation of the gravitationally dressed photon propagator

In order to build up the WQFT for a photon in a gravitational background, we need to derive the dressed propagator for such scattering even. Our starting point is the Einstein-Maxwell action, consisting in Maxwell theory minimally coupled to gravity

$$S_\gamma = -\frac{1}{4}\int \mathrm{d}^4x\sqrt{-g}\, g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}\ . \tag{5.1}$$

Its gauge symmetry, $\delta A_\mu(x) = \partial_\mu\alpha(x)$, $\delta g_{\mu\nu}(x) = 0$, can be covariantly gauge-fixed using standard BRST methods. The procedure is as follows. One replaces the gauge parameter $\alpha(x)$ by the anticommuting ghost $c(x)$, and from the gauge algebra obtains the anticommuting BRST variation s (the so-called Slavnov variation). It is required to be nilpotent, and is then extended to the non-minimal fields needed for gauge fixing, the antighost $\bar{c}(x)$ and auxiliary $B(x)$, which are Graßmann odd and even, respectively. The BRST symmetry is

$$sA_\mu = \partial_\mu c\ ,\quad sc = 0\ ,\quad s\bar{c} = B\ ,\quad sB = 0, \tag{5.2}$$

and can be easily verified to be nilpotent ($s^2 = 0$). It is used to obtain the gauge-fixed total action by adding to the lagrangian contained in (5.1) the manifestly BRST invariant term $s\Psi_\xi$, where $\Psi_\xi = \sqrt{-g}\,\bar{c}\left(\nabla^\mu A_\mu + \frac{\xi}{2}B\right)$ is the gauge fermion chosen to produce a R_ξ gauge in curved space. The fields c, $\bar{c}$ and B are all taken to be scalars under change of coordinates, so to keep covariance manifest. One finds

$$s\Psi_\xi = \sqrt{-g}\left(B\nabla^\mu A_\mu + \frac{\xi}{2}B^2 - \bar{c}\nabla^\mu\partial_\mu c\right) \sim \sqrt{-g}\left(-\frac{1}{2\xi}(\nabla^\mu A_\mu)^2 - \bar{c}\nabla^\mu\partial_\mu c\right), \tag{5.3}$$

where in the last step the auxiliary field B has been eliminated by its own algebraic equations of motion. We choose the value $\xi = 1$, that implements the Feynman gauge. The total gauge-fixed BRST invariant action, $S_{\mathrm{tot}} = S_\gamma + \int \mathrm{d}^4x\, s\Psi_\xi$, contains a ghost action that we disregard (at tree-level ghosts do not contribute) and terms that identify the photon propagator reads

$$S_{\gamma,gf} = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\nabla^\mu A_\mu)^2\right] = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}A^\mu \hat{H}_\mu{}^\nu A_\nu\right], \tag{5.4}$$

where the second form is obtained by performing partial integrations. It produces the second order differential operator

$$\hat{H}_\mu{}^\nu = \delta_\mu{}^\nu\nabla^2 - R_\mu{}^\nu\ , \tag{5.5}$$

whose inverse gives the photon propagator in a curved background, i.e. the dressed propagator of our interest. The operator $\hat{H}_\mu{}^\nu$ may be interpreted as a first-quantized Hamiltonian. For that purpose, we find it convenient to use flat indices by introducing a vielbein $e^a_\mu(x)$, so the metric is given by $g_{\mu\nu}(x) = \eta_{ab} e^a_\mu(x) e^b_\nu(x)$. This allows to present the Hamiltonian as

$$\hat{H}_a{}^b = \delta_a{}^b \nabla^2 - R_a{}^b, \tag{5.6}$$

where the covariant derivative contains also the spin connection $\omega_{\mu a}{}^b(x)$ (previously defined in Sec.4.3), as it acts on vectors $A_a(x) = e^\mu_a(x) A_\mu(x)$ with flat indices, so the covariant derivative is $\nabla_\mu A_a = \partial_\mu A_a + \omega_{\mu a}{}^b A_b$. On a more general Lorentz tensor, the covariant derivative takes the form $\nabla_\mu = \partial_\mu - \frac{i}{2}\omega_\mu{}^{ab} S_{ab}$, with S_{ab} the Lorentz generators in the representation of the tensor. For the photon, the spin tensor in the spin-1 representation is $(S_{cd})_a{}^b = i\left(\eta_{ca}\delta_d{}^b - \eta_{da}\delta_c{}^b\right)$. Now, for the purpose of constructing a WQFT, we prefer to use auxiliary variables $(c, \bar{c})$[1] on the worldline, taking care of the spin indices of the Hamiltonian, entering in the phase space action as a first class constraint. The construction follows the same steps as the ones used in Sec. 2.2 to introduce color variables. At the end of the day, the gravitationally dressed photon propagator turns out to be

$$D(x_0, y_0, u, \bar{u}; g) = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{e^{z\bar{u}\cdot u}}{z^2} \int_0^\infty \mathrm{d}T \int_{x(0)=x_0}^{x(1)=y_0} \mathcal{D}x \int_{\lambda(0)=0}^{\bar{\lambda}(1)=0} D\lambda D\bar{\lambda}\, e^{iS}, \tag{5.7}$$

where the auxiliary variables have been decomposed as

$$\bar{c}^a(\tau) = z\bar{u}^a + \bar{\lambda}^a(\tau), \qquad c_a(\tau) = u_a + \lambda_a(\tau) \tag{5.8}$$

with λ_a and $\bar{\lambda}^a$ denoting the quantum fluctuations, and with the remaining classical parts u_a and $\bar{u}^a$ describing the initial and final polarization of the photon depending on the modulus z as indicated. It requires a worldline action that now reads

$$S = \int_0^1 \mathrm{d}\tau \Big(-\frac{1}{4T} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + i\bar{\lambda}^a \dot{\lambda}_a - \frac{1}{2}\dot{x}^\mu \omega_\mu^{cd} (S_{cd})_a{}^b \bar{c}^a c_b + T R_a{}^b \bar{c}^a c_b - \frac{3}{4} T R \Big). \tag{5.9}$$

where the last term arises once choosing DR to write down the path integral. The above expression for the dressed propagator is purely quantum and off-shell for all of the external lines, thus it can be used to evaluate Feynman diagrams propagating photons and gravitons as external states, as we will see in the next paragraphs.

Free photon propagator The free photon propagator is obtained by switching off the interactions in (5.9) and setting $g_{\mu\nu} = \eta_{\mu\nu}$. Then, the worldline action reduces to

$$S_{\text{free}} = \int_0^1 \mathrm{d}\tau \Big(-\frac{1}{4T} \eta_{\mu\nu} \left(\dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu \right) + i\bar{\lambda}^a \dot{\lambda}_a \Big). \tag{5.10}$$

Expanding the coordinates as $x = x_0 + (y_0 - x_0)\tau + q(\tau)$, the free path integral on quantum fluctuations produces the measure $(4\pi T)^{-2}$, while the classical trajectory has the simple action $S_{\text{cl}} = -(x_0 - y_0)^2/4T$, and we are left with

[1]We used the same symbol also for the fermionic Lee-Yang ghosts, since their difference is clear from the context

$$D(x_0, y_0, u, \bar{u}; g) = \oint \frac{\mathrm{d}z}{2\pi i} \frac{e^{zu\cdot\bar{u}}}{z^2} \int_0^\infty \frac{\mathrm{d}T}{(4\pi T)^2}\, e^{-i\frac{(x_0-y_0)^2}{4T}} = \bar{u}^\mu \Delta_{\mu\nu}(x_0 - y_0) u^\nu, \tag{5.11}$$

which leads to the causal photon propagator in Feynman gauge once stripping off the auxiliary variables, i.e.,

$$\Delta_{\mu\nu}(x-y) = \eta_{\mu\nu} \int_0^\infty \frac{\mathrm{d}T}{(4\pi T)^2}\, e^{-i\frac{(x-y)^2}{4T}} = \int \hat{\mathrm{d}}^4 p \frac{i\eta_{\mu\nu}}{p^2} e^{ip\cdot(x-y)}. \tag{5.12}$$

Photon-photon-graviton vertex

Let us now consider the more interesting case of a single graviton. In this case we have to take in to account quantum fluctuations, which we implement by performing a background expansion of the metric tensor and the configuration space variables as

$$g_{\mu\nu}(x(\tau)) = \eta_{\mu\nu} + \kappa\varepsilon_{\mu\nu}(\ell) e^{i\ell\cdot x(\tau)}, \qquad x^\mu(\tau) = x_0^\mu + (y_0 - x_0)^\mu \tau + q^\mu(\tau) \tag{5.13}$$

with the quantum fluctuations $q^\mu(\tau)$ acquiring DBC, i.e. $q^\mu(0) = q^\mu(1) = 0$. Notice that we have Fourier expanded the graviton field as a unique plane wave to insert just one graviton in a photon line. The contributions to the three point amplitude arise from the interactions which we organize as follows:

$$S_{\text{kin}} := \int_0^1 \mathrm{d}\tau \Big(-\frac{1}{4T} h_{\mu\nu}(x) \left(\dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu\right) \Big), \tag{5.14}$$

$$S_{\text{spin}} := -\frac{1}{2} \int_0^1 \mathrm{d}\tau \dot{x}^\mu \omega_\mu^{cd} (S_{cd})_a{}^b \bar{c}^a c_b, \tag{5.15}$$

$$S_{\text{Ric}} := T \int_0^1 \mathrm{d}\tau R_a{}^b \bar{c}^a c_b, \tag{5.16}$$

$$S_{\text{ct}} := -\frac{3}{4} T \int_0^1 \mathrm{d}\tau R. \tag{5.17}$$

Except for S_{kin} the remaining interactions have to be background-field expanded up to $\mathcal{O}(\kappa)$ in order to give contributions to the three point amplitude. Next, we rewrite the dressed propagator in momentum space introducing the external momenta of the photons p_1, p_2, so that

$$\begin{aligned} \widetilde{D}(p_1, p_2, u, \bar{u}) &= \int \mathrm{d}^d x_0 \mathrm{d}^d y_0\, e^{ip_1\cdot x_0} e^{-ip_2\cdot y_0} D(x_0, y_0, u, \bar{u}; g) \\ &= \hat{\delta}\left(p_1 - p_2 + \ell\right) \oint \frac{\mathrm{d}z}{2\pi i} \frac{e^{zu\cdot\bar{u}}}{z^2} \int_0^\infty \frac{\mathrm{d}T}{(4\pi T)^{\frac{d}{2}}} e^{-\frac{i\xi^2}{4T}} \int \mathrm{d}^d\xi\, e^{-i\xi\cdot p_2} \left\langle e^{i(S_{\text{kin}}+S_{\text{spin}}+S_{\text{Ric}}+S_{\text{ct}})} \right\rangle, \end{aligned} \tag{5.18}$$

where the expectation value evaluated on the free theory, i.e., w.r.t the action (5.10). In the second equality we have performed the change of variables $2x_+ = y_0 + x_0, \xi = y_0 - x_0$ to factor out the momentum conservation delta function. We can now perform the perturbative expansion of the path integral[2]. After evaluating all the contributions from each term we strip-off the auxiliary variables as in the free photon propagator example and amputate the external legs (see also Ref.[67] for a similar treatment in the case of Yang-Mills).

We list here the final results of the on-shell procedure for each contribution from the perturbative expansion. For the kinetic term (including the regulating ghost interaction) and the spin connection vertex we obtain

$$A^{\alpha\beta}_{\text{kin}} = -\frac{i\kappa}{4}\,(p_1+p_2)^\mu\,(p_1+p_2)^\nu\,\varepsilon_{\mu\nu}(\ell)\,\eta^{\alpha\beta}, \quad A^{\alpha\beta}_{\text{spin}} = \frac{i\kappa}{2}\left(p_1^\beta\eta^{\alpha(\mu}p_2^{\nu)} + p_2^\alpha\eta^{\beta(\mu}p_1^{\nu)}\right)\varepsilon_{\mu\nu}(\ell), \tag{5.19}$$

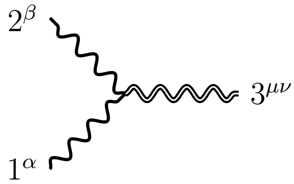
respectively. The Ricci tensor vertex contribution reads

$$\begin{aligned} A^{\alpha\beta}_{\text{Ric}} = \frac{i\kappa}{2}\Big[&-\eta^{\mu\nu}\left(p_1^\beta p_2^\alpha + s_{12}\eta^{\alpha\beta}\right) + 2\eta^{\alpha\beta}p_1^{(\mu}p_2^{\nu)} \\ &+p_1^\beta\eta^{\alpha(\mu}p_2^{\nu)} + p_2^\alpha\eta^{\beta(\mu}p_1^{\nu)} - s_{12}\eta^{\alpha(\mu}\eta^{\nu)\beta}\Big]\varepsilon_{\mu\nu}(\ell), \end{aligned} \tag{5.20}$$

where $s_{12} = 2p_1\cdot p_2$. Finally, the counter-term vertex generates

$$A^{\alpha\beta}_{\text{ct}} = \frac{3\mathrm{i}\kappa}{4}\left(s_{12}\eta^{\mu\nu}\eta^{\alpha\beta} - 2\eta^{\alpha\beta}p_1^{(\mu}p_2^{\nu)}\right)\varepsilon_{\mu\nu}(\ell). \tag{5.21}$$

In all of these contributions the free indices contract the external photon polarization vectors. In this way, adding the above terms one obtains the three point vertex



$$\begin{aligned} = \frac{i\kappa}{2}\Big(&\frac{1}{2}s_{12}\eta^{\alpha\beta}\eta^{\mu\nu} - s_{12}\eta^{\alpha(\mu}\eta^{\nu)\beta} - 2\eta^{\alpha\beta}p_1^{(\mu}p_2^{\nu)} \\ &+ 2p_1^\beta p_2^{(\mu}\eta^{\nu)\alpha} + 2p_2^\alpha p_1^{(\mu}\eta^{\nu)\beta} - \eta^{\mu\nu}p_1^\beta p_2^\alpha\Big), \end{aligned} \tag{5.22}$$

where the momentum conservation delta function has been stripped off. Higher point examples follow a similar procedure but with a more involved structure. However it is obvious already that the dressed propagator contains much more information than required in the classical limit. Moreover, it raises the question of what is the role, if any, of the auxiliary variables in the classical limit. Remember that they were introduced only to keep track of time-ordering which is requirement forced upon us by the quantization via path integrals.

5.2 Partition function and derivation of Feynman rules

Based on the results of the previous chapters, it is now easy to write down the WQFT partition function for the scattering of a photon off a spinless black hole

$$\mathcal{Z}[u,\bar{u}] = \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{e^{z\bar{u}\cdot u}}{z^2}\int \mathcal{D}g_{\mu\nu}\int \mathcal{D}x_1\int\mathcal{D}x_2\int D\lambda\int D\bar{\lambda}\,e^{iS}, \tag{5.23}$$

[2]In order to perform such a task one need to use the DBC propagator $\Delta(\tau,\sigma) = (\tau-1)\sigma\theta(\tau-\sigma) + (\sigma-1)\tau\theta(\sigma-\tau)$ alongside with the two point function of the auxiliary variables $\langle\lambda_a(\tau)\bar{\lambda}^b(\sigma)\rangle = \theta(\tau-\sigma)$, with $\theta(0) = 1/2$.

where the full action reads

$$S = S_g[g] + S_{\mathrm{p}_1}[x_1, \lambda, \bar{\lambda}, z; g] + S_{\mathrm{p}_2}[x_2; g], \tag{5.24}$$

and S_g was defined in Eq.(3.4). After rescaling the wordline parameter as $\tau \to \sigma/m$ we write the scalar massive point particle action as

$$S_{\mathrm{p}_2}[x_2; g] = -\int_{-\infty}^{+\infty} \mathrm{d}\sigma \, \frac{1}{2} g_{\mu\nu} \dot{x}_2^\mu \dot{x}_2^\nu. \tag{5.25}$$

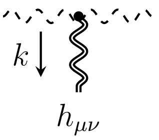
Now the point particle action can be read off from the dressed propagator of the photon in a gravitational background and gives

$$S_{\mathrm{p}_1}[x_1, \bar{\lambda}, \lambda; g] = -\int_{-\infty}^{+\infty} \mathrm{d}\sigma \left(\frac{1}{2} g_{\mu\nu} \dot{x}_1^\mu \dot{x}_1^\nu - i\bar{\lambda} \cdot \dot{\lambda} + \frac{1}{2} \dot{x}_1^\mu \omega_\mu^{cd} (S_{cd})_a{}^b \bar{c}^a c_b \right). \tag{5.26}$$

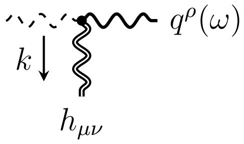
We have also rescaled the worldline parameter of the photon action

$$\tau = \frac{\sigma}{E} \tag{5.27}$$

introducing a parameter E with dimension of energy, whose physical meaning will become clear later on. For the photon worldline, we have neglected the $R_a{}^b$ vertex both with the counter-terms since they are sub-leading in the classical limit (see example below). Notice here that the integration regions are now from $\sigma \in (-\infty, +\infty)$ in agreement with the procedure to set dressed propagators on-shell. Following the steps in the previous chapters we can easily derive interaction vertices. The Feynman rules associated with the photon worldline interactions read

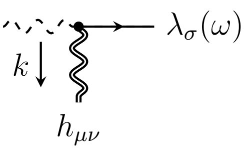


$$= \frac{i\kappa}{2} e^{ik\cdot b} \hat{\delta}(k \cdot p) \left(-p^\mu p^\nu + izk^\alpha p^{(\nu} \eta^{\mu)\beta} (S_{\alpha\beta})_\rho{}^\sigma \bar{u}^\rho u_\sigma \right), \tag{5.28}$$

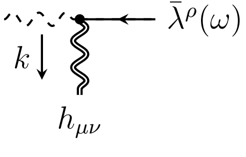


$$\begin{aligned} = \frac{\kappa}{2} e^{ik\cdot b} \hat{\delta}\left(k \cdot p + \omega\right) \Big[& \left(p^\mu p^\nu k_\rho + 2\omega p^{(\mu} \delta_\rho^{\nu)} \right) \\ & - izk^\alpha \left(\eta^{\beta(\mu} (p^{\nu)} k_\rho + \omega \delta_\rho^{\nu)}) \right) (S_{\alpha\beta})_\lambda{}^\delta \bar{u}^\lambda u_\delta \Big], \end{aligned} \tag{5.29}$$

where the solid photon line represents a fluctuation of the worldline photon line . At this order we also have vertices related to the fluctuations of auxiliary variables which we obtain by expanding the spin connection vertex in (5.26) as well, leading to

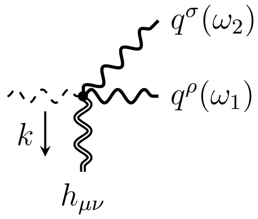


$$= -\frac{\kappa}{2} e^{\mathrm{i}k\cdot b} \hat{\delta}(\omega + k \cdot p) z \, k^\alpha p^{(\nu} \eta^{\mu)\beta} (S_{\alpha\beta})_\rho{}^\sigma \bar{u}^\rho, \tag{5.30}$$



$$= -\frac{\kappa}{2} e^{\mathrm{i}k\cdot b} \hat{\delta}(\omega + k \cdot p) k^\alpha p^{(\nu} \eta^{\mu)\beta} (S_{\alpha\beta})_\rho{}^\sigma u_\sigma, \tag{5.31}$$

where the arrow distinguishes between $\bar{\lambda}^a$ and λ^a which scale differently with z. Other rules can be easily derived, e.g., the rule for a 2-fluctuation reads



$$
\begin{aligned}
&= \frac{i\kappa}{2} e^{ik\cdot b}\hat{\delta}\left(k\cdot p+\omega_1+\omega_2\right)\bigg[\frac{1}{2}p^\mu p^\nu \ell_\rho k_\sigma + \omega_1 k_\sigma p^{(\mu}\delta^{\nu)}_\rho + \omega_2 k_\rho p^{(\mu}\delta^{\nu)}_\sigma \\
&\quad + \omega_1\omega_2 \delta^{(\mu}_\rho \delta^{\nu)}_\sigma - \frac{i}{2} z k^\alpha (S_{\alpha\beta})_\lambda{}^\delta \bar{u}^\lambda u_\delta \left(p^{(\mu}\eta^{\nu)\beta} k_\rho k_\sigma + \omega_1 \eta^{\beta(\nu}\delta^{\mu)}_\rho k_\sigma\right)\bigg].
\end{aligned}
\tag{5.32}
$$

The above interactions still contains informations about the feature of the wave scattering off the black hole. With an eye on the equivalence principle, this should not happen. As it stands, (5.23) is yet semi-classical, in the above sense. To fully take the classical limit we need to restrict the partition function to the optical regime. From the calculation of the photon-graviton vertex we see that we can interpret the coherent variables u and $\bar{u}$ as spin-1 polarization vectors

$$
u_\rho \bar{u}_\sigma \to \varepsilon^*_\rho(p)\varepsilon_\sigma(p'), \tag{5.33}
$$

this is consistent with the previous calculation of the photon-graviton vertex . Based on this, to implement the optical regime at the level of integrands generated from (5.23) we give the prescription

$$
\mathcal{Z}_{\text{geom-opt}} := -\mathcal{Z}(u,\bar{u})\Big|_{\bar{u}\to u}, \tag{5.34}
$$

where the minus sign is due $\bar{u}\cdot u = -1$, which is consistent with our metric signature. For the moment (5.34) is just a prescription to fully achieve classical results from the photon WQFT, however, we will give a diagrammatic proof showing that such prescription realizes the equivalence principle at the level of worldlines, yielding to the proper classical results.

Example

In order to illustrate our methods let us give a simple example. The three-point amplitude $\mathcal{M}(\gamma\gamma h)$ is formally vanishing in the optical regime so let us consider its off-shell version $\mathcal{M}^{\mu\nu}(\gamma\gamma h)$ instead. Consider the 3-point vertex of (5.22) and suppose we are interested in the limit where the graviton is soft. Let $p_2^\mu = p_1^\mu + \hbar\bar{q}^\mu$ where $\bar{q}$ is interpreted in the classical limit as a wave-number related to the graviton. The polarization vector thus satisfies $\varepsilon_\mu(p_2) = \varepsilon_\mu(p_1) + \mathcal{O}(\hbar)$. We also parameterize the incoming photon momentum as $p_1 = Ev^\mu = (E,0,0,E)$ where E is the energy of the photon. Then, up to terms proportional of $\hbar$, we obtain

$$
\mathcal{M}^{\mu\nu}(p_1,p_2) = i\kappa\, p_1^\mu p_1^\nu + \mathcal{O}(\hbar) = i\kappa E^2 v^\mu v^\nu + \mathcal{O}(\hbar), \tag{5.35}
$$

where we have used the on-shell condition on the momenta p_1 and p_2 and the transversality of the photon polarization vectors.

The equivalent object in WQFT is obtained from the LO Feynman rule (5.28). The partition function then simply reads

$$\mathcal{Z}(z,u,\bar{u}) = -\frac{i\kappa}{2}\int \hat{\mathrm{d}}^4 q\, e^{iq\cdot b}\hat{\delta}(q\cdot p)\left[p^\mu p^\nu + z p^\nu (q\cdot \bar{u}\, u^\mu - \bar{u}^\mu\, q\cdot u)\right]. \tag{5.36}$$

Clearly, in order to match this expression with the previous it must be that the extra terms must vanish. To see that this is the case let us consider the integration of the kernel over z. Exchanging the integration orders and using the identity

$$\frac{1}{2\pi i}\oint \mathrm{d}z \frac{e^{z\bar{u}\cdot u}}{z^{k+1}} = \begin{cases} \frac{(u\cdot\bar{u})^k}{k!}, & k\geq 0, \\ 0, & \text{otherwise}, \end{cases} \tag{5.37}$$

we obtain

$$\mathcal{Z}(z,u,\bar{u}) = -\frac{i\kappa}{2}u_\rho \bar{u}_\sigma \int \hat{\mathrm{d}}^4 q\, e^{iq\cdot b}\hat{\delta}(q\cdot p)p^\mu \left[p^\nu \eta^{\rho\sigma} - 2iq_\lambda (S^{\lambda\nu})^{\rho\sigma}\right]. \tag{5.38}$$

Notice that the dependence on $(\bar{u},u)$ on the first term appears due to this identity. Furthermore, using the anti-symmetry of $(S^{\lambda\nu})_\rho{}^\sigma$ yields to the following relation

$$\mathcal{Z}^{\mu\nu}_{\text{geom-opt}} = \int \hat{\mathrm{d}}^4 q\, e^{iq\cdot b}\,\hat{\delta}(q\cdot p)\left(-\frac{i\kappa}{2}p^\mu p^\mu\right), \tag{5.39}$$

where we stripped off the graviton polarization vector. Therefore, up to an irrelevant sign, we see that in the geometric optics regime the leading order Feynman rule matches the classical limit computed in Eq.(5.35) as expected.

5.2.1 Massless limit vs photon worldline in WQFT

On general grounds we expect that the contributions related with the spin-terms should not contribute in the geometric optics regime since assuming the equivalence principle one may replace the photon worldline by the worldline of a massless scalar. We had a taste of this in the calculation of the lowest order partition function (5.39). For the general case let us consider the diagram shown in Fig.5.1 which appears as a sub-topology of our worldline diagrams. Let $\mathcal{K}$ denote the mathematical expression of this diagram and let k_i be the momenta associated with each graviton line which we will consider as outgoing. The momentum of the photon worldline is labeled by p^μ. This diagram is still to be

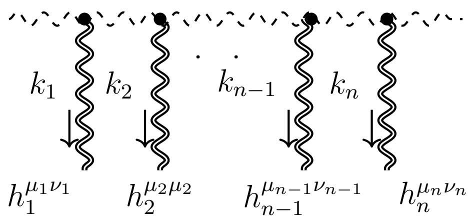


Figure 5.1: WQFT current dressed with external photon lines on-shell

integrated over z and hence the identity (5.37) implies that after integration all terms proportional to z^i vanish for $i > 1$. The same identity produces the factor $u \cdot \bar{u}$, which gives a non-vanishing contribution associated with the term independent of z. In other words only the terms independent of z and proportional to z contribute. Now let us rewrite the Feynman rule of a single vertex in the form

$$V_\ell^{\mu_\ell\nu_\ell}(p, k_\ell) := \frac{i\kappa}{2} e^{-ik_\ell \cdot p}(-p^{\mu_\ell}p^{\nu_\ell} + zB_\ell^{\mu_\ell\nu_\ell}), \tag{5.40}$$

where

$$B_\ell^{\mu_\ell\nu_\ell} := ik_\ell^\alpha p^{(\nu_\ell}\eta^{\mu_\ell)\beta}(S_{\alpha\beta})_\rho{}^\sigma \bar{u}^\rho u_\sigma.$$

From the Feynman rules and on the support of the Dirac delta functions this diagram can be organized as follows

$$\mathcal{K}^{\mu_1\cdots\mu_n\nu_1\ldots\nu_n} = \frac{i^n}{2^n}\kappa^n \left(\prod_{i=1}^n \hat{\delta}(k_i \cdot p)e^{ik_i \cdot b}\right) K^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}, \tag{5.41}$$

where the only non-vanishing contributions are proportional to z due to (5.37). Hence

$$K^{\mu_1\nu_1\cdots\mu_n\nu_n} = (-1)^n p^{\mu_1}p^{\nu_1}\cdots p^{\mu_n}p^{\nu_n} - z\sum_{i=1}^n p^{\mu_1}p^{\nu_1}\cdots\widehat{p^{\mu_i}p^{\nu_i}}\cdots p^{\mu_1}p^{\nu_1}B_i^{\mu_i\nu_i}, \tag{5.42}$$

where the hat means that the factor should be excluded. Since the coefficient $B_i^{\mu_i\nu_i}$ depends on a single factor of the spin tensor we can use the anti-symmetry of $(S^{\lambda\nu})_\rho{}^\sigma$ and that $\bar{u} \to u$ thus concluding that all terms proportional to z vanish as expected by the geometric optics regime. The same exercise can be done in the case where fluctuations on the photon worldlines or auxiliary variables are considered obtaining the same result. A realization of this will appear in the calculation of the 2PM eikonal in the next section.

Let us conclude by stressing that it is only the combination of the geometric optics regime and the identity (5.37) that makes these terms vanishing. This provides an alternative path to show the equivalence between the massless limit of scattering amplitudes involving two massive particles and the amplitudes of photons and a single massive particle in the classical limit, which in WQFT can be understood as disregarding the spin tensor. Deviations from the geometric-optics regime are known as the gravitational spin Hall effect[3] of light and have been studied from first principles in Ref[190] for the case of propagation of light and Ref.[191] for propagation of gravitational waves.

5.3 Eikonal phase and deflection angle

Let us now move to the evaluation of the light ray deflection angle once scattering off a spinless black hole. The light ray deflection angle can be extracted by differentiation from the eikonal phase, the latter evaluated as usual by considering the connected diagrams from the optical regime of (5.23) i.e. $\mathcal{Z}_{\text{geom-opt}} = e^{i\chi}$. Following [128] we define the deflection angle in the small angle approximation at each order in perturbation theory by

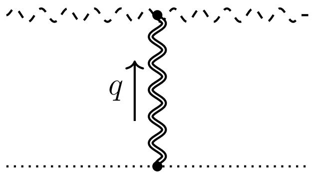
$$\theta_i = -\frac{1}{E}\frac{\partial\chi_i}{\partial|b|}. \tag{5.43}$$

[3]See. Ref-[189] for a review and References therein.

where E is the energy of the light ray while $|b| \equiv \sqrt{-b^2}$ is the modulus of the space-like impact parameter b^μ. We will use the result of Sec.5.2.1 which implies that the spin tensor plays no role in our computations. Therefore the integration over the modulus z is trivial (see example in Sec.(5.2)) and produces the contraction of the auxiliary variables $\bar{u} \cdot u = -1$ in the geometric optics regime. For calculations of the Feynman one loop integrals we will parametrize the momenta of the particles as $p_1 = Ev_1$ and $p_2 = mv_2$ where $v_1^2 = 0$ and $v_2^2 = 1$. It would also be useful to choose the rest frame of the massive particle such that $v_1 = (1, 0, 0, 1)$ and $v_2 = (1, 0, 0, 0)$.

At 1PM the calculation is pretty easy since there is only one diagram, namely

$i\chi_1 =$



$$= -i\kappa^2 \frac{(p_1 \cdot p_2)^2}{4} \int \hat{\mathrm{d}}^4 q \, \hat{\delta}(q \cdot p_1) \hat{\delta}(q \cdot p_2) \frac{e^{iq \cdot b}}{q^2}. \tag{5.44}$$

To regulate the divergent integral in this expression we find convenient to use a cut-off regulator. The regulated integral then reads

$$\mathcal{I}_{\succ\prec} = \int \hat{\mathrm{d}}^4 q \frac{\hat{\delta}(q \cdot p_1) \hat{\delta}(q \cdot p_2)}{q^2} e^{iq \cdot b} = \frac{1}{4\pi p_1 \cdot p_2} \log \left(\frac{|b|^2}{L^2} \right). \tag{5.45}$$

To obtain this result we have set up a differential equation for $\mathcal{I}_{\succ\prec}$ such that the derivative produces a finite expression following similar steps as in Ref.[135] with minor changes related to the parametrization of momenta. Remember that in $d = 4$ we set $p_1 = E(1, 0, 0, 1)$ and $p_2 = m(1, 0, 0, 0)$. Therefore we obtain

$$\chi_1 = -2G_N (p_1 \cdot p_2) \log \left(\frac{|b|^2}{L^2} \right), \tag{5.46}$$

where we have used $\kappa^2 = 32\pi G_N$. Finally, using the rest frame of the heavy massive particle we have

$$\theta_1 = -\frac{1}{E} \frac{\partial \chi_1}{\partial |b|} = \frac{4G_N m}{|b|}, \tag{5.47}$$

which matches the result from general relativity.

Let us now move to the 2PM calculation, whose diagrams are shown in Fig.5.2. We start with the diagrams involving the 3-graviton vertex[4]. Diagram 5.2a vanishes identically. In order to see this first notice that this diagram contains as a subtopology the diagram in Fig.5.1. Therefore, from Eq.(5.41) the integrand of this subtopology is proportional to

$$\hat{\delta}(p_1 \cdot q) \hat{\delta}(p_1 \cdot k) p_1^{\mu_1} p_1^{\mu_2} p_1^{\nu_1} p_1^{\nu_2}. \tag{5.48}$$

Upon using the 3-graviton vertex Feynman rule and $p_1^2 = 0$, we obtain identically zero due to the Dirac-delta constraints. This result is independent of the other subtopology

[4]We use the conventions of Ref.[130]. See also Ref.[150].

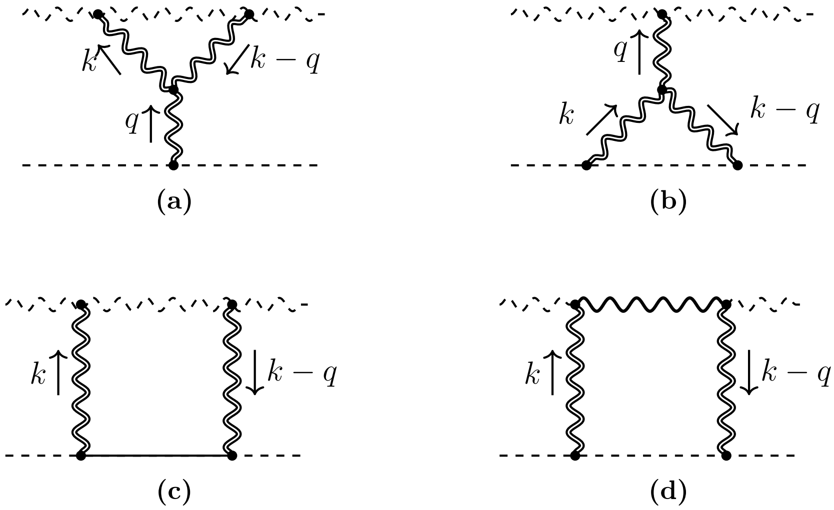
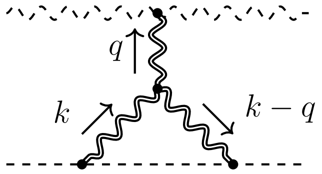


Figure 5.2: 2PM diagrams

containing the matter worldline. Then, we move to the flipped topology 5.2b



$$= -i\frac{\kappa^4 m^2}{32}\int \hat{\mathrm{d}}^4 q\,\hat{\delta}(q\cdot p_1)\hat{\delta}(q\cdot p_2)\frac{e^{iq\cdot b}}{q^2}\int \hat{\mathrm{d}}^4 k\,\hat{\delta}(k\cdot p_2)\frac{N_1(q,k,p_1)}{k^2(k-q)^2}, \tag{5.49}$$

where $N_1(q,k,p_1) = (p_1\cdot p_2)^2(k^2+(k-q)^2) + m^2(k\cdot p_1)^2$. The terms multiplying $(p_1\cdot p_2)^2$ lead to tadpole integrals which are vanishing. Using our parametrization of momenta let us focus on the integral

$$I_1 = \int \hat{\mathrm{d}}^4 k\hat{\delta}(k\cdot v_2)\frac{(k\cdot v_1)^2}{k^2(k-q)^2} = v_1^\mu v_1^\nu \int \hat{\mathrm{d}}^4 k\,\hat{\delta}(k\cdot v_2)\frac{k_\mu k_\nu}{k^2(k-q)^2}. \tag{5.50}$$

On the support of the Dirac-delta constraints $\hat{\delta}(q\cdot v_1)\hat{\delta}(q\cdot v_2)$ this integral can be reduced by performing a simple Passarino-Veltman reduction leading to

$$I_1 = \frac{v_1^\mu v_1^\nu}{8}(3q_\mu q_\nu + q^2(v_{2\mu}v_{2\nu} - \eta_{\mu\nu}))\int \hat{\mathrm{d}}^4 k\frac{\hat{\delta}(k\cdot v_2)}{k^2(k-q)^2} = \frac{q^2\sigma^2}{8}\int \hat{\mathrm{d}}^4 k\frac{\hat{\delta}(k\cdot v_2)}{k^2(k-q)^2}, \tag{5.51}$$

where we have defined $\sigma = v_1\cdot v_2$ to write the result in a Lorentz invariant form. Then we are left to calculate the integral

$$I_{\triangleright} := \int \hat{\mathrm{d}}^4 q\,\hat{\delta}(q\cdot v_1)\hat{\delta}(q\cdot v_2)e^{\mathrm{i}q\cdot b}\int \hat{\mathrm{d}}^4 k\frac{\hat{\delta}(k\cdot v_2)}{k^2(k-q)^2}, \tag{5.52}$$

which can be computed following Ref.[172] adapted to our case. The result is

$$I_{\triangleright} = \frac{1}{16\pi|b|}. \tag{5.53}$$

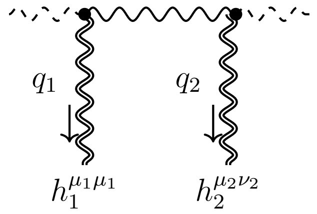
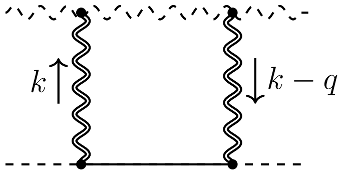


Figure 5.3: Subtopology of diagram with photon fluctuation

Let us now move on to diagrams with the topology of a box. 5.2c generates



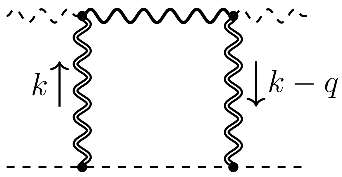
$$= i\frac{(p_1\cdot p_2)^4\kappa^4}{32}\int \hat{\mathrm{d}}^4q\,\hat{\delta}(q\cdot p_1)\hat{\delta}(q\cdot p_2)e^{ib\cdot q}\int \hat{\mathrm{d}}^4k\frac{\hat{\delta}(k\cdot p_1)k\cdot(k-q)}{(k\cdot p_2)^2k^2(k-q)^2}\,. \tag{5.54}$$

The integral reduction produces integrals with double poles on the same side of the complex plane. Following Ref.[192] these integrals vanish after closing the contour in the opposite direction[5].

Finally we have 5.2d. Let us first show that the spin tensor does not contribute in the geometric optics regime. For that consider a simpler version of the exercise in Section 5.2.1 but now including a single fluctuation as shown in Fig.5.3. Focusing on the terms proportional to z we find that the integrand is proportional to the tensor structures

$$\eta^{\mu_1\nu_2}p_1^{\mu_2}(S_\alpha^{\ \nu_1})^{\rho\sigma}u_\rho\bar{u}_\sigma q_i^\alpha,\qquad p_1^{\ \mu_1}p_1^{\ \nu_1}q_i^{\ \nu_2}(S_\alpha^{\ \mu_2})^{\rho\sigma}u_\rho\bar{u}_\sigma q_i^\alpha,\quad i=1,2, \tag{5.55}$$

which vanish in the geometric optics regime. The other contributions simplify to



$$= i\frac{\kappa^4(p_1\cdot p_2)^2}{32}\int \hat{\mathrm{d}}^4q\hat{\delta}(q\cdot p_1)\hat{\delta}(q\cdot p_2)e^{iq\cdot b}\int \hat{\mathrm{d}}^4k\frac{\hat{\delta}(k\cdot p_2)\,N_2(q,k,p_1)}{(k\cdot p_1)^2k^2(k-q)^2}, \tag{5.56}$$

where $N_2=(p_1\cdot p_2)^2(k^2-k\cdot q)+2m^2(k\cdot p_1)^2$. The integral reduction produces finite integrals with double poles on the same side of the complex plane which we can set to zero. Therefore the only surviving term is the one which cancels the double, which is proportional to (5.52).

Therefore after adding up the contributing diagrams (b) and (d) the result of the eikonal phase reads

$$\chi_2=\kappa^4\frac{15}{256}m(p_1\cdot p_2)\frac{1}{16\pi|b|}, \tag{5.57}$$

and using our parametrization of momenta the scattering angle reads

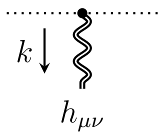
$$\theta_2=-\frac{1}{E}\frac{\partial\chi_2}{\partial|b|}=\frac{15\pi}{4}\frac{G_N^2m^2}{|b|^2}, \tag{5.58}$$

[5]For time symmetric propagators one simple applies this argument twice for each $i\epsilon$ prescription.

in agreement with the massless limit of the scattering angle of two massive objects in which one of the masses goes to zero.

5.3.1 Spinning massive particle

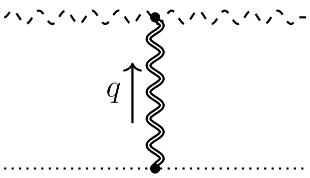
The case of a spinning massive particle can be treated along the same lines. The description in Ref.[16] is based on the inclusion of supersymmetry at the level of the the wordline action. The WQFT thus constructed is valid at quadratic order in spin. Here we conform ourselves with summarizing the LO Feynman rules for this case. Performing the same rescaling as in the previous Section we have the Feynman rule



$$= -i\frac{\kappa}{2}e^{\mathrm{i}k\cdot b}\hat{\delta}(k\cdot p)\left(p^\mu p^\nu + \mathrm{i}m(k\cdot\mathcal{S})^{(\mu}p^{\nu)} - \frac{1}{2}m^2(k\cdot\mathcal{S})^\mu(k\cdot\mathcal{S})^\nu\right), \quad (5.59)$$

where $(k\cdot\mathcal{S})^\mu := k_\nu\mathcal{S}^{\nu\mu}$. Therefore at LO the eikonal phase is computed from a single diagram obtaining

$$i\chi_1 =$$



$$= -i\kappa^2\frac{(p_1\cdot p_2)^2}{4}\int \hat{\mathrm{d}}^4q\,\hat{\delta}(q\cdot p_1)\hat{\delta}(q\cdot p_2)\frac{e^{iq\cdot b}(1+N_{\mathcal{S}})}{q^2}, \quad (5.60)$$

with the numerator $N_{\mathcal{S}}$ is given by

$$N_{\mathcal{S}} = -\frac{im}{p_1\cdot p_2}(p_1\cdot\mathcal{S}\cdot q) - \frac{m^2}{2(p_1\cdot p_2)^2}(p_1\cdot\mathcal{S}\cdot q)^2\ . \quad (5.61)$$

Specializing the spin tensor defined in Ref.[16] for the case at hand, let us parametrize it as

$$\mathcal{S}^{\mu\nu} = \frac{2s}{|b|}\left(b^{[\mu}(v_1-\sigma v_2)^{\nu]}\right)\ . \quad (5.62)$$

Hence, after reintroducing the dimensionless velocities we can rewrite the numerator as

$$N_{\mathcal{S}} = -is\frac{b\cdot q}{|b|} - s^2\frac{1}{2|b|^2}(b\cdot q)^2\ . \quad (5.63)$$

To complete the calculation we need to evaluate the following type of integrals

$$\mathcal{I}^{\mu_1\mu_2\cdots\mu_n} := \int \hat{\mathrm{d}}^4q\,\hat{\delta}(q\cdot v_1)\hat{\delta}(q\cdot v_2)e^{iq\cdot b}\frac{q^{\mu_1}\cdots q^{\mu_n}}{q^2}. \quad (5.64)$$

The simplest case $\mathcal{I}^{\mu_1}$ can be computed from (5.45). For the case $\mathcal{I}^{\mu_1\mu_2}$ we adapt the procedure of Ref.[32] to our case. Noting that the results must lie in the plane orthogonal to the four velocities the integral $\mathcal{I}^{\mu_1\mu_2}$ can be written as

$$\mathcal{I}^{\mu_1\mu_2} = c_1 b^{\mu_1}b^{\mu_2} + c_2\Pi^{\mu_1\mu_2}, \quad (5.65)$$

where $\Pi^{\mu_1\mu_2}$ is a projector which explicitly reads

$$\Pi^{\mu_1\mu_2} = \eta^{\mu_1\mu_2} + v_1^{\mu_1} v_1^{\mu_2} - v_2^{\mu_1} v_1^{\mu_2} - v_1^{\mu_1} v_2^{\mu_2} \tag{5.66}$$

Therefore employing the traceless property of $\mathcal{I}^{\mu_1\mu_2}$ and (5.45) we obtain

$$\mathcal{I}^{\mu_1\mu_2} = \frac{1}{\pi b^4 \sigma} \left(b^{\mu_1} b^{\mu_2} - \frac{1}{2} b^2 \Pi^{\mu_1\mu_2} \right). \tag{5.67}$$

After some algebra we compute

$$\chi_1 = \kappa^2 \frac{p_1 \cdot p_2}{8\pi} \left(-\frac{1}{2} \log \left(\frac{|b^2|}{L^2} \right) - \frac{s}{|b|} + \frac{s^2}{2|b|^2} \right), \tag{5.68}$$

which leads to the scattering angle

$$\theta_1 = 4 \left(\frac{1}{|b|} - \frac{s}{|b|^2} + \frac{s^2}{|b|^3} \right) G_N m, \tag{5.69}$$

in agreement with the massless limit of the Kerr-result of Ref.[16] (See also Refs. [193, 194]).

Chapter 6

Summary and Outlook

The use of double copy techniques can boost the calculations of perturbative observables related to black hole scattering, both from the QFT and worldline like approaches. In particular, the examination of the classical-level double copy proves to be an effective tool for this purpose, although it has not yet been thoroughly explored. When the double copy is studied at the classical level, two crucial questions must be addressed: how to miantain locality manifest in calculations and how to get informations on the underlying quantum theory, reproducing the classical integrands. The former can be rephrased by asking if it is possible to have a BCJ-like representation of the classical amplitude. The answer to that lies in using the bi-adjoint worldline, so to identify the propagator matrix implementing locality at the classical level. The worldline particle implementing such field theory has been built up in Chapter 2, where it has been showed how, starting from the bi-adjoint scalar field and using the background field method, one can build up a worldline action for a scalar particle interacting with the bi-adjoint scalar. For such a worldline particle and also other theories, including spinning particles, the construction of the WQFT action was performed in Chapter 3, where, in addition, we derived the main Feynman rules needed for the calculations in the subsequent chapters. Next, in Chapter 4, we addressed the investigation of the classical double copy for the case of a Dirac fermion, showing that the worldline formalism, not only allows to easily maintain locality manifest at the classical level, but, by exploiting the quantization procedure on the worldline, it also allows to write down the lagrangian describing the double copied theory. We found that, in $D = 4$, such theory is propagating a massive vector boson interacting with the DG sector of the $\mathcal{N} = 0$ supergravity, through the interactions read out from (4.66). This was accomplished by using the quantized version of the susy charges for the $\mathcal{N} = 2$ susy particle coupled to DG, as a first class constraint on the sub-Hilbert space of the worldline spectrum, propagating a massive vector boson. We showed also how, the use of real Grassmann variables combined with a symmetrization procedure on the copies of the classical integrands, allows to take care of the spinning d.o.f. when double copying classical integrands. We stress that such procedure is general, it could be used independently of the spin carried out by the worldline. Finally, in Chapter 5 we have also seen that, starting from first principles, one can propagate a photon by using worldline degrees of freedom and then, one can couple the photon to background gravity so to study the scattering of electromagnetic waves off black holes. In such a case, the WQFT Feynman rules are not enough to get classical results. This is expected since the worldline action arises from a gauge fixing procedure, introducing coupling between the spin tensor of the worldline particle and the background gravity. Such interactions seems to generate a tension with the Equivalence Principle i.e. they introduce spin contributions to the well known deflection angle of light evaluated by using Einstein field equations. To solve this tension, one need to implement the optical regime at the level of the WQFT integrands. This finally generates classical results for massless particles consistent with the literature.

There are many future directions one could investigate, starting from the results in this thesis. The main one concerns the use of the classical double copy as a tool to detect WQFT interaction vertices between massive higher spinning particles and background gravity. Such theories are known to generate amplitudes which, once used in the classical limit, do generate observables related to the scattering of Kerr black holes [195], making them a worth line of investigation. On the amplitude side, massive higher spin theories are known to violate tree level unitarity [196] at energy well below the Plank scale, however, as it has been shown in [195, 113] only a suitable truncation of the related lagrangian is needed to evaluate amplitudes related to Kerr black holes observables. On the worldline side, the main issue to deal with higher spin particles is super-symmetry, since we would expect a susy breaking even at low spin (it happens for instance in the case of the $\mathcal{N} = 2$ particle coupled to Maxwell). In this regard we think that exploring further the relation among Grassmann variables and massive spinor helicity variables [197] could help to achieve some results on the higher spin side. Indeed, as seen in Chapter 4, to generate spin one integrands, we defined a symmetrized product acting on the left and right copies of spin half integrands. In such a case, the symmetrization symbol hits the real Grassmann variables, generating new fermionic variables on the worldline having two flavours (L,R), which we reorganized into complex fermionic variables on the worldline, making easier the calculations and the identification of the double copied theory with the $\mathcal{N} = 2$ susy particle. This is similar to what happens on the amplitude side when using massive spinor helicity variables to compute amplitudes [34]. This investigation might allow to study the coupling of massive higher spin particles on the worldline, since the double copy could be used as a tool to generate the WQFT vertices propagating massive higher spin worldlines and gravitons. Next, it may be worthwhile to explore methods needed to remove the dilaton scalar from the double copied integrands, to all order in perturbation theory. On the amplitude side ghost fields were introduced to remove the dilaton [198], however, this only seems to work at leading order in the gravitational coupling constant. An efficient and more elegant way consists in using ad-hoc projectors to BCJ double copying gauge theory amplitudes [199]. This allows to remove the dilaton from the intermediate states exchanged during the double copy procedure, order by order in perturbation theory. The latter seems to be on the same foot of the symmetrization procedure used in Chapter 4 to preserve the $R-$symmetry of the double copy worldline integrands. Indeed, using worldline degrees of freedom, one could build up projectors, order by order in perturbation theory, in the form of contour integrals, allowing to remove the dilaton contribution from calculations.

Lastly, we think that, WQFT techniques, could be used to investigate the scattering of massive particles on strong backgrounds, which has recently been studied in [200] and reveals relevant to evaluate classical observables in the self-force expansion, meant as an expansion of classical observables around a non-flat gravitational background. In this regard, we think that the background field method on the worldline and QFT action could be used to introduce a non flat background in the calculations, generating Feynman rules suitable to evaluate self-force observables, even in the case where the worldline particle has spinning degrees of freedom.

Appendix A

Seeley-DeWitt coefficients

Let us briefly review the perturbative calculation of the Seeley-DeWitt coefficients in (2.15), starting from the representation of the effective action in (2.11), which contains the time ordering. At the perturbative order needed to obtain eq. (2.15), the effect of the time ordering is inconsequential, but it is important at higher orders.

Using the split (2.9), we Taylor expand the interaction term $S_{\rm int}$ in (2.12) around x_0

$$S_{\rm int} = Ty \int_0^1 \mathrm{d}\tau \, \left(\hat{\Phi}(x_0) + q^\mu(\tau)\partial_\mu \hat{\Phi}(x_0) + \cdots \right) \, , \tag{A.1}$$

insert it into the exponential in (2.13), and expand the exponential. Then, collecting all terms of the same power in T, and taking into account that the q-propagator with the vanishing Dirichlet boundary conditions goes like T

$$\langle q^\mu(\tau) q^\nu(\sigma) \rangle = -2T\delta^{\mu\nu} \left(\frac{1}{2}|\tau - \sigma| - \frac{1}{2}(\tau + \sigma) + \tau\sigma \right) \, , \tag{A.2}$$

we easily find from the constant $\hat{\Phi}(x_0)$ term

$$a_0 = 1 \, , \quad a_1 = -y\hat{\Phi}(x_0) \, , \quad a_2 = \frac{y^2}{2}\hat{\Phi}^2(x_0) \, , \quad a_3 = -\frac{y^2}{6}\hat{\Phi}^3(x_0) + \cdots \tag{A.3}$$

where the dots denote three additional terms arising from using the q-propagator. They are as follows. A first term is

$$\begin{aligned} &\left\langle \mathbf{T} \, \frac{1}{2} \left(Ty \int_0^1 \mathrm{d}\tau \, q^\mu(\tau)\partial_\mu \hat{\Phi}(x_0) \right) \left(Ty \int_0^1 \mathrm{d}\sigma \, q^\nu(\sigma)\partial_\nu \hat{\Phi}(x_0) \right) \right\rangle \\ &= \frac{1}{2} T^2 y^2 \partial_\mu \hat{\Phi}(x_0) \partial_\nu \hat{\Phi}(x_0) \int_0^1 \mathrm{d}\tau \int_0^1 \mathrm{d}\sigma \, \langle q^\mu(\tau) q^\nu(\sigma) \rangle \\ &= \frac{1}{12} T^3 y^2 (\partial_\mu \hat{\Phi}(x_0))^2 \end{aligned} \tag{A.4}$$

where the time ordering $\mathbf{T}$ is again inconsequential as the matrices at different times commute. A second term reads

$$\begin{aligned}
&\left\langle \mathbf{T}\, \frac{1}{2}\left(Ty\hat{\Phi}(x_0)\right)\left(Ty\int_0^1 \mathrm{d}\tau\, \frac{1}{2} q^\mu(\tau)q^\nu(\tau)\partial_\mu\partial_\nu\hat{\Phi}(x_0)\right)\right\rangle \\
&= \frac{1}{4}T^2y^2\hat{\Phi}(x_0)\int_0^1 \mathrm{d}\tau\, \langle q^\mu(\tau)q^\nu(\tau)\rangle\partial_\mu\partial_\nu\hat{\Phi}(x_0) \\
&= \frac{1}{12}T^3y^2\hat{\Phi}(x_0)\partial^2\hat{\Phi}(x_0)\ .
\end{aligned} \tag{A.5}$$

A third term is similar, but with the matrices $\hat{\Phi}(x_0)$ and $\partial^2\hat{\Phi}(x_0)$ interchanged. These matrices commute under the trace, so that this last term just doubles the previous one. This completes the derivation of a_3, which is equivalent to the one in (2.15) by adding a total derivative, that anyway drops out from the effective action.

Appendix B

SUSY algebra and DG background

Here we work out the coupling of the $\mathcal{N}=2$ SUSY model to the dilaton and gravity, obtained by deforming the free SUSY charges as

$$\begin{aligned} Q &= e^{-\kappa\phi} e^{\mu}_{a} \psi^{a} \left(P_{\mu} - i\Sigma_{\mu cd} \bar{\psi}^{c} \psi^{d} \right) - m\theta = e^{-\kappa\phi} \psi^{a} \pi_{a} - m\theta \\ \bar{Q} &= e^{-\kappa\phi} e^{\mu}_{a} \bar{\psi}^{a} \left(P_{\mu} - i\Sigma_{\mu cd} \bar{\psi}^{c} \psi^{d} \right) - m\bar{\theta} = e^{-\kappa\phi} \bar{\psi}^{a} \pi_{a} - m\bar{\theta} \end{aligned} \tag{B.1}$$

where ϕ is the dilaton field and $\Sigma_{\mu cd} = \omega_{\mu cd} - 2\kappa \partial_{[c}\phi e_{d]\mu}$. In the above lines we defined the generalized momentum

$$\pi_{a} = e^{\mu}_{a} \pi_{\mu} = e^{\mu}_{a} \left(P_{\mu} - i\Sigma_{\mu cd} \bar{\psi}^{c} \psi^{d} \right), \tag{B.2}$$

where the spin connection $\Sigma_{\mu cd}$ introduced in the deformation, exactly corresponds to the the spin connection written in the Einstein frame, the latter reached through the Weyl rescaling

$$\tilde{e}^{\mu}_{a} = e^{-\kappa\phi} e^{\mu}_{a} \qquad \tilde{g}_{\mu\nu} = e^{2\kappa\phi} g_{\mu\nu} \tag{B.3}$$

with tilded objects defined in the string frame. Next, so to consistently quantize the model and write down a path integral, we need to ensure that the constraint algebra is of first class, particularly that $\{Q,Q\} = \{\bar{Q},\bar{Q}\} = 0$. To this aim we first list here the free theory Poisson brackets

$$\{x^{\mu}, P_{\nu}\} = \delta^{\mu}{}_{\nu} \quad \{\psi^{a}, \bar{\psi}_{b}\} = -i\delta^{a}{}_{b} \quad \{\theta, \bar{\theta}\} = i, \tag{B.4}$$

then, we can evaluate the bracket $\{Q,Q\}$, yielding

$$\begin{aligned} \{Q,Q\} =& 2e^{-\kappa\phi} \psi^{a} \left(\{\pi_{a}, e^{-\kappa\phi}\} \psi^{b} + e^{-\kappa\phi} \{\pi_{a}, \psi^{b}\} \right) e^{\nu}_{b} \pi_{\nu} + e^{-2\kappa\phi} \psi^{a} \psi^{b} \{\pi_{a}, \pi_{b}\} \\ =& 2e^{-2\kappa\phi} \left(\psi^{a} \{\pi_{a}, e^{-\kappa\phi}\} \psi^{b} e^{\nu}_{b} + e^{-\kappa\phi} \psi^{a} \{\pi_{a}, \psi^{b}\} e^{\nu}_{b} - \psi^{a} \psi^{b} e^{\mu}_{[a} \partial_{\mu} e^{\nu}_{b]} \right) \pi_{\nu} \\ &+ e^{-2\kappa\phi} e^{\mu}_{a} e^{\nu}_{b} \psi^{a} \psi^{b} \{\pi_{\mu}, \pi_{\nu}\} \\ =& 2e^{-2\kappa\phi} \left(\kappa \partial_{a}\phi \psi^{a} \psi^{b} e^{\nu}_{b} - \psi^{a} \psi^{d} e^{\nu}_{b} \omega_{[ad]}{}^{b} - \kappa \partial_{a}\phi \psi^{a} \psi^{b} e^{\nu}_{b} - \psi^{a} \psi^{b} e^{\mu}_{[a} \partial_{\mu} e^{\nu}_{b]} \right) \pi_{\nu} \\ &+ e^{-2\kappa\phi} e^{\mu}_{a} e^{\nu}_{b} \psi^{a} \psi^{b} \{\pi_{\mu}, \pi_{\nu}\} \\ =& -2e^{-2\kappa\phi} \left(\omega_{[ab]}{}^{d} e^{\nu}_{d} + e^{\mu}_{[a} \partial_{\mu} e^{\nu}_{b]} \right) \psi^{a} \psi^{b} \pi_{\nu} + e^{-2\kappa\phi} e^{\mu}_{a} e^{\nu}_{b} \psi^{a} \psi^{b} \{\pi_{\mu}, \pi_{\nu}\}\ . \end{aligned} \tag{B.5}$$

What we see is that, a direct consequence of our SUSY charge deformation is that, the modification of the spin connection term in (B.1) allows us to generate the torsion tensor in the Einstein frame

$$T^{\nu}_{[ab]} = e^{-2\kappa\phi}\left(\omega_{[ab]}{}^{d} e^{\nu}_{d} + e^{\mu}_{[a}\partial_\mu e^{\nu}_{b]}\right) \tag{B.6}$$

which we set to zero assuming our connection is symmetric. This way, once evaluating the last term in (B.5), we can recast the bracket as

$$\{Q, Q\} = i\psi^\mu\psi^\nu\bar{\psi}^c\psi^d e^{-2\kappa\phi}\mathcal{R}_{\mu\nu cd} \tag{B.7}$$

$$\text{with}\quad \mathcal{R}_{\mu\nu cd} = \left(R_{\mu\nu cd} - 4\kappa\nabla_{[\mu}\nabla_{[c}\phi e_{d]\nu]} + 2\kappa^2\left(2\partial_{[c}\phi\partial_{[\mu}\phi e_{\nu]d]} - \partial^2\phi e_{[c[\mu}e_{\nu]d]}\right)\right) \tag{B.8}$$

where, from the last line, we can read out the Riemann tensor in the Einstein frame. Thus, invoking Bianchi identity and assuming no torsion imply that the above bracket must vanish. A similar calculation holds for $\{\bar{Q}, \bar{Q}\}$. Now one can evaluate the point particle Hamiltonian by using that $\{Q, \bar{Q}\} = -2iH$. The calculation delivers the following Hamiltonian

$$H = \frac{1}{2}e^{-2\kappa\phi}\left(g^{\mu\nu}\pi_\mu\pi_\nu - m^2 e^{2\kappa\phi} - \mathcal{R}_{abcd}\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d\right) \tag{B.9}$$

where the deformed Riemann tensor $\mathcal{R}_{abcd}$ is given in (4.68). Once switching off the dilaton field, H reduces to the Hamiltonian in the pure gravity case [16]. Now we can write down the worldline action in configuration space. We consider the phase space action gauging the reparametrization invariance, supersymmetry, and R-symmetry

$$S_{\rm ph} = -\int_0^1 d\tau\left(\dot{x}^\mu P_\mu + i\bar{\psi}_a\dot{\psi}^a - i\bar{\theta}\dot{\theta} - eH - i\bar{\chi}Q - i\chi\bar{Q} + a(J-s)\right). \tag{B.10}$$

Then, we eliminate the momentum using the equation of motion

$$\frac{\delta S_{\rm ph}}{\delta P_\mu} = 0 \to P_\mu = e^{-1}\,e^{2\kappa\phi}\left(g_{\mu\nu}\dot{x}^\nu - i\chi e^{-\kappa\phi}\bar{\psi}_\mu - i\bar{\chi}e^{-\kappa\phi}\psi_\mu - \frac{e}{2}e^{-2\kappa\phi}\Sigma_{\mu ab}S^{ab}\right) \tag{B.11}$$

once defining the spin tensor as $S^{ab} = -2i\bar{\psi}^a\psi^b$. Plugging it back in the Hamiltonian and the SUSY charges one is able to write down a configuration space action, ready to be gauge fixed

$$\begin{aligned} S = \int_0^1 d\tau\Big(&-\frac{1}{2}e^{-1}e^{2\kappa\phi}\left(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - e^2m^2\right) - i\bar{\psi}_a\dot{\psi}^a + i\bar{\theta}\dot{\theta} + \frac{1}{2}\dot{x}^\mu\Sigma_{\mu ab}S^{ab} \\ &+ \frac{e}{8}e^{-2\kappa\phi}\mathcal{R}_{abcd}S^{ab}S^{cd} + ie^{-1}e^{\kappa\phi}g_{\mu\nu}\dot{x}^\mu(\bar{\chi}\psi^\nu + \chi\bar{\psi}^\nu) \\ &+ im(\chi\bar{\theta} + \bar{\chi}\theta) - \bar{\chi}\chi e^{-1}\bar{\psi}_a\psi^a - a(J-s)\Big) \end{aligned} \tag{B.12}$$

which should be gauge fixed according to the topology one would like to evaluate the path integral. For our classical application, we set $\theta = \bar{\theta} = 0$, while using the equations of motion for the gravitinos $(\chi, \bar{\chi})$ and setting them to zero implement the spin supplementary condition. In addition, the constraint arising from the gauge field $a(\tau)$ allows to recover the normalization of the spin tensor $S^{\mu\nu}S_{\mu\nu} = 2s^2$ through the condition $\bar{\psi}\cdot\psi = s$.

Analogous to the YM case, we fix the einbein by choosing $e = 1/m$, then change the integration boundaries to $(-\infty, \infty)$ by the LSZ reduction procedure. Upon rescaling the integration variable $\tau \to m\tau$, we obtain the following $\mathcal{N} = 2$ worldline action coupled to dilaton-gravity

$$S = \int_{-\infty}^{\infty} d\tau\left(-\frac{1}{2}e^{2\kappa\phi}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - i\bar{\psi}_a\dot{\psi}^a - i\dot{x}^\mu\Sigma_\mu{}^{ab}\bar{\psi}_a\psi_b + \frac{1}{8}e^{-2\kappa\phi}\mathcal{R}_{abcd}S^{ab}S^{cd}\right). \tag{B.13}$$

Appendix C

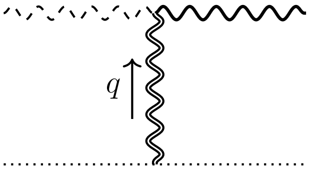
2PM photon impulse

In the WQFT one can define the observables related to the point particle mechanics, by making use of Noether theorem which in such case is nothing else but Ehrenfest theorem. Let us first define the impulse of the photon as

$$\Delta p^\mu = \int_{-\infty}^{+\infty} \mathrm{d}\sigma \left\langle \frac{\mathrm{d}^2 q^\mu}{\mathrm{d}\sigma^2} \right\rangle = \int_{-\infty}^{+\infty} \hat{\mathrm{d}}\omega \left(-\omega^2 \hat{\delta}(\omega) \langle \tilde{q}^\mu(\omega) \rangle \right) . \tag{C.1}$$

The leading order calculation is straightforward and can be obtained by the evaluation of a solely diagram which unlike the eikonal phase has two kinematical fluctuations, namely

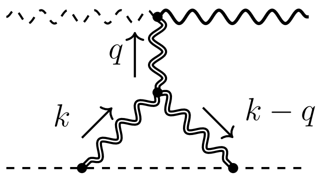
$$\Delta^{(0)} p^\mu =$$



$$= -\kappa^2 \frac{(p_1 \cdot p_2)^2}{4} \frac{\partial I_{\succ\!\prec}}{\partial b_\mu} = 4 G_N \, (p_1 \cdot p_2) \, \frac{b^\mu}{|b|^2} . \tag{C.2}$$

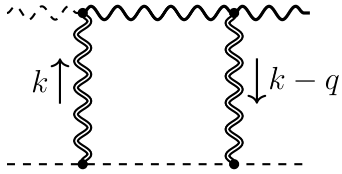
where the tree level integral $I_{\succ\!\prec}$ has been evaluated in (5.45).

Let us move now to the next to leading (2PM) photon impulse. The topologies which are vanishing for the eikonal phase do also vanish here in the same way. Particularly, the diagram with the three-graviton vertex with the two gravitons starting from the photon line is exactly zero once using momentum conservation and the delta constraints. Thus the only non-vanishing contributions arise from



$$= -\frac{i m^2 \kappa^4}{32} \int \hat{\mathrm{d}}^d q \, \hat{\delta}(q \cdot p_1) \hat{\delta}(q \cdot p_2) \frac{e^{iq\cdot b}}{q^2} \int \hat{\mathrm{d}}^4 k \, \hat{\delta}(k \cdot p_2) \frac{N_1(q,k,p) q^\mu}{k^2 (k-q)^2} \tag{C.3}$$

with the numerator $N_1 = (p_1 \cdot p_2)^2 \, (k^2 + (k-q)^2) + m^2 (k \cdot p_1)^2$, while the remaining diagram contributing is



$$= \frac{i (p_1 \cdot p_2)^2 \kappa^4}{16} \int \hat{\mathrm{d}}^4 q \, \hat{\delta}(q \cdot p_1) \hat{\delta}(q \cdot p_2) e^{iq\cdot b} \int \hat{\mathrm{d}}^4 k \, \hat{\delta}(k \cdot p_2) \frac{N_2(q,k,p) \, (q-k)^\mu}{k^2 (k-q)^2 (k \cdot p_1)^2} , \tag{C.4}$$

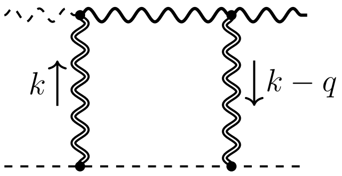
where the numerator $N_2 = (p_1 \cdot p_2)^2 k \cdot (k-q) + 2m^2 (k \cdot p_1)^2$, exactly as for the eikonal phase. Let us briefly review the integration procedure. We first focus on Eq.(C.3). For this one we just need to evaluate the integral

$$\mathcal{I}_1 = \int \hat{\mathrm{d}}^4 q\, \hat{\delta}(q \cdot v_1) \hat{\delta}(q \cdot v_2) \frac{e^{iq \cdot b}}{q^2} \int \hat{\mathrm{d}}^4 k\, \hat{\delta}(k \cdot v_2) \frac{(k \cdot v_1)^2}{k^2 (k-q)^2} = \frac{1}{128\pi |b|} \tag{C.5}$$

since we can rewrite the whole expression in the RHS of Eq.(C.3) as

$$-\frac{iEm^2\kappa^4}{32} \frac{1}{i} \frac{\partial}{\partial b^\mu} \mathcal{I}_1 = -\frac{1}{4} \pi G_N^2 m (p_1 \cdot p_2) \frac{b^\mu}{|b|^3} \,. \tag{C.6}$$

Let us move now to the next diagram, which can be recast as follows



$$= \frac{im^2 k^4 E^2}{16} \int \hat{\mathrm{d}}^4 q\, \hat{\delta}(q \cdot v_1) \hat{\delta}(q \cdot v_2) e^{iq \cdot b} \left(2\mathcal{I}_2^\mu + \mathcal{I}_3^\mu \right), \tag{C.7}$$

where we have defined the vector integrals

$$\mathcal{I}_2^\mu = \int \hat{\mathrm{d}}^4 k \frac{\hat{\delta}(k \cdot v_2)(q-k)^\mu}{k^2 (q-k)^2}, \quad \mathcal{I}_3^\mu = \int \hat{\mathrm{d}}^4 k \frac{\hat{\delta}(k \cdot v_2) k \cdot (k-q)(q-k)^\mu}{k^2 (q-k)^2 (k \cdot v_1)^2} \,. \tag{C.8}$$

Let us start performing the tensor reduction of the above integrals. The first one can be decomposed as $\mathcal{I}_2^\mu = A q^\mu + B v_2^\mu$ such that, using the delta constraint $q \cdot v_1 = q \cdot v_2 = 0$ one finds that $B = 0$ while contracting with q^μ one finds that

$$A = \frac{1}{2} \int \hat{\mathrm{d}}^4 k \frac{\hat{\delta}(k \cdot v_2)}{k^2 (k-q)^2} \,. \tag{C.9}$$

In this way one is able to evaluate the first contribution in (C.7) as

$$\frac{im^2 \kappa^4 E}{8} \int \hat{\mathrm{d}}^4 q\, \hat{\delta}(q \cdot v_1) \hat{\delta}(q \cdot v_2)\, e^{iq \cdot b} \mathcal{I}_2^\mu = 4\pi G_N^2 m (p_1 \cdot p_2) \frac{b^\mu}{|b|^3} \,. \tag{C.10}$$

Before proceeding further notice that adding up the two contributions just evaluated i.e. (C.6) and (C.10) one obtains

$$\frac{15\pi}{4} G_N^2 m (p_1 \cdot p_2) \frac{b^\mu}{|b|^3}, \tag{C.11}$$

which corresponds to the impulse obtained from the NLO eikonal phase, without any iteration of the scattering data. However, the evaluation here is not finished yet since we still need to evaluate the first piece in (C.7). For this task we first decompose the integral

$$\mathcal{I}_3^\mu = A q^\mu + B v_2^\mu + C v_1^\mu \tag{C.12}$$

next, using that $v_2^\mu M_\mu = 0$ one gets that $B = -\sigma C$ which allows to rewrite $\mathcal{I}_3^\mu = A q^\mu + C \left(v_1^\mu - \sigma v_2^\mu \right)$. This way, after contracting with q^μ one finds that the integral obtained are tadpoles, thus $A = 0$, while contracting with v_1^μ and using integral reduction allows to fix the remaining coefficients

$$C = \frac{1}{2}q^2 \int \hat{\mathrm{d}}^4 k \frac{\hat{\delta}(k \cdot v_2)}{(k^2 + i\epsilon)((k-q)^2 + i\epsilon)(k \cdot v_1 + i\epsilon)} = -\frac{i}{2}q^2 \int \hat{\mathrm{d}}^4 k \frac{\hat{\delta}(k \cdot v_1)\hat{\delta}(k \cdot v_2)}{k^2 (k-q)^2} \tag{C.13}$$

where, in order to get the last equality, we performed the change of variables $k - q \to -k'$ using then the Dirac delta representation

$$\hat{\delta}(x) = i\left(\frac{1}{x + i\epsilon} - \frac{1}{x - i\epsilon}\right) , \tag{C.14}$$

which enables us to rewrite the full contribution related ti $\mathcal{I}_3$ from (C.7), as the square of $I_{\succ\!\prec}$ which has been evaluated previously (5.45).

Finally, putting all pieces together and performing the extra momentum integration, one gets the next to leading order photon impulse

$$\Delta p^\mu = 4G_N(p_1 \cdot p_2)\frac{b^\mu}{|b|^2} + G_N^2 m \frac{(p_1 \cdot p_2)}{|b|}\left(\frac{15\pi}{4}\frac{b^\mu}{|b|^2} - \frac{8}{|b|}\left(v_1^\mu - \sigma v_2^\mu\right)\right) \tag{C.15}$$

where we reintroduced $\sigma = v_1 \cdot v_2$ delivering also the result in a Lorentz invariant form.

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